DENSITY PROPERTIES OF CERTAIN GROUP ACTIONS AND
JOINING RIGIDITY OF SOME SMOOTH SINGULAR FLOWS

A Dissertation in
Mathematics
by
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Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Doctor of Philosophy

August 2018
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This dissertation investigates two problems in dynamical systems.

One is about density properties for certain group actions. More precisely, let $Y$ be a compact metric space, $G$ be a group acting by transformations on $Y$. For any infinite subset $A \subset Y$, we study the density of $gA$ for $g \in G$ and quantitative density of the set $\bigcup_{g \in G} gA$ by the Hausdorff semimetric $d^H$. As an example, it is proven that for any integer $n \geq 2$, $\epsilon > 0$, any infinite subset $A \subset \mathbb{T}^n$, there is a $g \in SL(n, \mathbb{Z})$ such that $gA$ is $\epsilon$-dense.

The other one is a joint work with Adam Kanigowski, on joining rigidity of von Neumann flows with one singularity. They are given by a smooth vector field $X$ on $\mathbb{T}^2 \setminus \{a\}$, where $X$ is not defined at $a \in \mathbb{T}^2$. The phase space can be decomposed into a (topological disc) $D_X$ and an ergodic component $E_X = \mathbb{T}^2 \setminus D_X$. Let $\omega_X$ be the 1-form associated to $X$. We show that if $|\int_{E_{X_1}} d\omega_X| \neq |\int_{E_{X_2}} d\omega_X|$, then the corresponding flows $(v^X_t)$ and $(v^{X_2}_t)$ are disjoint. Moreover, for every $X$ there is a uniquely associated frequency $\alpha \in \mathbb{T}$ and it follows that for a full measure set of $\alpha \in \mathbb{T}$ the class of smooth time changes of $(v^X_t\alpha)$ is joining rigid, i.e. every two smooth time changes are either cohomologous or disjoint. This gives a natural class of flows for which the answer to Problem 3 in [29] is positive.
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Acknowledgments

This is finally the end of my study from kindergarten to graduate school. Along this unforgettable trip, there are so many people that I would like to thank.

I am grateful to each one of the teachers that taught me Mathematics, Chinese, English etc. These include but are not limited to Linmei Zhu, Xinhua Li, Xinzhong Cheng, Fuhai Zhu, Jun Li, Chungen Liu, Ke Liang, Guimei An, Jia Guo and Huagui Duan.

Over the past six years I have received support and encouragement from a great number of professors. I must first thank my adviser, Professor Anatole Katok, for his continuous guidance, endless encouragement and generous help during my graduate study and research. This dissertation could not have been finished without his advices and support. His guidance and friendship have made my graduate study a thoughtful and rewarding journey. It is unfortunate that he could not be present during my defense.

I would also like to thank my dissertation committee members: Svetlana Katok, Federico Roderiguez Hertz and William Brandt for generously offering their time, insightful comments and support. I learned a lot from Professor Roderiguez Hertz and benefited a lot from his precious suggestions and endless patience. I own my thanks to Professor Svetlana Katok for her invaluable help which makes it possible for Xiaofei to study in the Ph.D. program at Penn State. I am very grateful to Professor Brandt for his valuable time and care.

I would like to offer my thanks to Professor Svetlana Katok, and later Professor Anna Mazzucato and Professor Nathanial Brown for directing such a wonderful Ph.D. program, as well as for their help during these six years. I also thank the many staffs in Department of Mathematics for their kindly assistance.

I am also grateful to Professor Anatole Katok, Professor Federico Rodriguez Hertz, Professor Omri Sarig, Professor Jean-Paul Thouvenot and Dr. Misha Guysinsky for writing reference letters for me.

I should also thank Sharon Childs, Professor Yakov Pesin, Professor Nigel Hig-
son, Professor Zhiren Wang and Professor Yuxi Zheng, who helped me a lot in various aspects. In particular, I am grateful to Professor Federico Rodriguez Hertz for temporarily being my adviser, and to Professor Yakov Pesin and Professor Zhiren Wang for temporarily being in my dissertation committee.

I must also thank my fellow students, including but not limited to Shilpak Banerjee, Dong Chen, Alena Erchenko, Qiao Liu, Kurt Vinhage, Daren Wei and Weisheng Wu. I am grateful to Gabriel Ponce and Régis Varão for having opportunity to discuss problems together.

I should thank Adam Kanigowski for his patience and tolerance along the way we work together.

I am most grateful and indebted to my parents (Gongliang Dong and Bojiao Geng) and the rest of my family for their unconditional love. Lastly, I must thank Xiaofei for her unwavering love, patience and support.

There are certainly many others who deserve mentioning to whom I offer a simple message: Thank You!
Dedication

To my beloved grandfather, Chengshan Geng (11/23/1943–08/21/2017)
Density of infinite subsets I

Let $Y$ be a compact metric space, $G$ be a group acting by transformations on $Y$. For any infinite subset $A \subset Y$, we study the density of $gA$ for $g \in G$ and quantitative density of the set $\bigcup_{g \in G_n} gA$ by the Hausdorff semimetric $d^H$. It is proven that for any integer $n \geq 2$, $\epsilon > 0$, any infinite subset $A \subset \mathbb{T}^n$, there is a $g \in SL(n, \mathbb{Z})$ such that $gA$ is $\epsilon$-dense. We also show that, for any infinite subset $A \subset [0, 1]$, for generic rotation and generic 3-IET,

$$\lim \inf_n n \cdot d^H \left( \bigcup_{k=0}^{n-1} T^k A, [0, 1] \right) = 0.$$

1.1 Introduction and Results

Let $Y$ be a compact metric space, $G$ be a locally compact second countable (lcsc) (semi-)group. Let $\alpha$ be a $G$ action on $Y$ by transformations. If $\alpha$ admits an ergodic probability measure with full support, then by Poincaré recurrence Theorem, it is easy to see that the orbit of almost every point in $Y$ is dense. For some special group actions, it is already an interesting question to consider the effectivization of the density of an orbit. For example in [6] and [33], they investigate how fast the orbit of a generic point can become dense in the torus for certain higher rank abelian actions. In order to obtain the effective result, one has to make some restriction on the single orbit (or finite many orbits). Without making any generic assumptions, it is natural to consider at first a subset of infinite many points, and
study the density of the iterations under the group action. Along this way, we are going to describe and study two types of density problems. Throughout this note, $d^H$ is the Hausdorff semimetric, and $d_L$ is the standard metric.

1.1.1 Dense iterations of infinite subset

Let $A$ be an infinite subset of $Y$, we can consider the set containing all subsets of the form $gA := \{\alpha(g)x | x \in A\}$ for a $g \in G$. For the fixed $A$, we would like to know: for any $\epsilon > 0$, whether there exists a $g \in G$ such that $gA$ is $\epsilon$-dense in $Y$, or equivalently $d^H(gA,Y) < \epsilon$. We will call this dense iteration problem simply D.I. problem. Before we give some definitions, let’s first state a nontrivial result in this direction.

Let $S^1 = \mathbb{R}/\mathbb{Z}$ be the standard circle, and $T_\alpha : S^1 \to S^1$ be the translation map: $x \mapsto x + \alpha \pmod{1}$. A theorem of Glasner [12] asserts that if $X$ is an infinite subset of $S^1$, then for any $\epsilon > 0$, there exists an integer $n$ such that the dilation $nX := \{nx \pmod{1} : x \in X\}$ is $\epsilon$-dense. This gives an affirmative answer to the D.I. problem in the case of the natural action by multiplication of $\mathbb{N}$ on the circle $S^1$.

In view of this result, we make the following definitions.

**Definition 1.1.1.** Given a $G$ action on a metric space $Y$, if an infinite subset $A$ satisfies that for any $\epsilon > 0$, there exists a $g \in G$ such that $gA$ is $\epsilon$-dense in $Y$, then $A$ is called Glasner set with respect to $(Y,d,G)$.

Let’s remark that in [4] and [25], the authors have already used the term Glasner set, but that is defined on the acting group. Our definition here is obviously different from theirs.

**Definition 1.1.2.** Given a $G$ action on a metric space $Y$, if any infinite subset $A$ is a Glasner set, then we say the dynamical system $(Y,d,G)$ has Glasner property.

Using our definition, the natural action $(S^1,d_L,\mathbb{N})$ has the Glasner property. In [21], Kelly and Lê generalize the result to higher dimensional torus. In particular, they proved that for any positive integer $N$, the system $(\mathbb{T}^N,d_L,M(N \times N,\mathbb{Z}))$ has Glasner property.

Our first result is a partial strengthening of the above mentioned one.
**Theorem 1.1.3.** For any integer \( n \geq 2 \), the system \((\mathbb{T}^n, d_L, SL(n, \mathbb{Z}))\) has Glasner property.

In [21], the Glasner property mainly comes from uniform dilation. In contrast, the Glasner property in Thoerem 2.1.3 should be considered from the dilation in certain directions. Theorem 1.1.3 can also be regarded as a strengthen of the result in [13], [5] and [3], which asserts that any infinite \( SL(n, \mathbb{Z}) \)-invariant subset of \( \mathbb{T}^n \) is dense. Although our method is not different from [21], we use highly nontrivial classification results of stationary measures for the lattice action from [3].

We say that, the Lyapunov exponents of an ergodic invariant measure for a \( \mathbb{Z}^k \) action are in general position if they are all simple and nonzero, and if the Lyapunov hyperplanes are distinct hyperplanes in general position. We say, a subset \( A \) is subordinate to a \( \mathbb{Z}^k \) action, if \( A \) is contained in a leaf of a single Lyapunov foliation.

**Theorem 1.1.4.** Let \( k \geq 2 \), and \( \alpha \) be an action of \( \mathbb{Z}^k \) on \( \mathbb{T}^n \) by ergodic automorphisms. Suppose that the Lyapunov exponents of the Lebesgue measure are in general position. Let \( A \) be an infinite subset which is subordinate to \( \alpha \), then \( A \) is a Glasner set with respect to \((\mathbb{T}^n, d_L, \mathbb{Z}^k))\).

In particular, Theorem 1.1.4 applies to Cartan actions on torus. Simply speaking, Cartan action is a \( \mathbb{R} \) split algebraic abelian action on torus with maximal rank.

In [?], we will make further progress on Glasner property. We will prove, in particular, that parabolic subgroup actions on certain homogeneous spaces have Glasner property.

### 1.1.2 Quantitative density for orbits of infinite subset

Let’s first consider a \( \mathbb{Z} \) action, that is a single transformation. Given the system \((Y, d, T)\), for any infinite subset \( A \subset Y \), we can also study the quantitative density of the set \( \bigcup_{k=0}^{n-1} T^k A \) by the Hausdorff semimetric \( d^H \). It will be natural to expect that the density decays much faster than \( \frac{1}{n} \), namely

\[
\inf_n n \cdot d^H \left( \bigcup_{k=0}^{n-1} T^k A, Y \right) = 0.
\]
Consider arbitrary countable discrete group $G$, and the system $(Y, d, G)$, we can define an increasing family $\{F_n\}$ of subsets of $G$, and consider the density of the set $\bigcup_{g \in F_n} gA$. Similarly, one may expect
\[
\inf_n |F_n| \cdot d^H \left( \bigcup_{g \in F_n} gA, Y \right) = 0.
\]
We will refer this quantitative density problem simply as **Q.D. problem**.

**Definition 1.1.5.** Given a system $(Y, d, G)$ and an increasing family $\{F_n\}$ of subsets of $G$, let $A$ be an infinite subset of $Y$. We say, $A$ is a **Q.D. set** with respect to $\{F_n\}$, if
\[
\inf_n |F_n| \cdot d^H \left( \bigcup_{g \in F_n} gA, Y \right) = 0.
\]

**Definition 1.1.6.** Given a system $(Y, d, G)$ and an increasing family $\{F_n\}$ of subsets of $G$. If any infinite subset of $Y$ is a Q.D. set with respect to $\{F_n\}$, then we say, $(Y, d, G, \{F_n\})$ has **Q.D. property**.

Our first result in this direction is about circle rotation.

**Theorem 1.1.7.** If $A$ is an infinite subset of $S^1$, then there is a full measure subset $X \subset S^1$, such that for the circle rotation $T_\alpha$ with $\alpha \in X$, $A$ is a Q.D. set with respect to $\{F_n := [0, n-1]\}$. Namely, we have
\[
\liminf_n n \cdot d^H \left( \bigcup_{k=0}^{n-1} T_\alpha^k A, S^1 \right) = 0.
\]

Let $0 < \alpha < \beta < 1$. A nondegenerate 3 interval exchange transformations (3-IET) $P_{\alpha, \beta}$ is defined by
\[
P_{\alpha, \beta}(x) = \begin{cases} 
  x + 1 - \alpha, & 0 \leq x < \alpha, \\
  x + 1 - \alpha - \beta, & \alpha \leq x < \beta, \\
  x - \beta, & \beta \leq x \leq 1.
\end{cases}
\]

It is well known [19] that $P_{\alpha, \beta}$ is the induced map on $[0, 1]$ of the translation map
$T_{1-\alpha}$ on $[0, 1 + \beta - \alpha]$. The rotation number of the translation map is $\frac{1-\alpha}{1+\beta-\alpha}$. For 3-IETs, by Theorem 2.1.3 we have

**Corollary 1.1.8.** If $A$ is an infinite subset of $I = [0, 1]$, then for generic 3 IET $T$, $A$ is a Q.D. set with respect to $\{F_n := [0, n-1]\}$. Namely, we have

$$\lim inf_n n \cdot d_H\left(\bigcup_{k=0}^{n-1} T^k A, I\right) = 0.$$

If the circle rotation is fixed at first, then $(S^1, d_L, \mathbb{Z})$ does not have Q.D. property.

**Theorem 1.1.9.** For any $\alpha$, there exists an infinite subset $A$ of $S^1$, such that

$$\inf_n n \cdot d_H\left(\bigcup_{k=0}^{n-1} T^k_{\alpha} A, S^1\right) > 0.$$

It seems from the above results that, a $\mathbb{Z}$ action could rarely have Q.D. property. However, if the acting group is large, then one may expect a lot of actions with this property. We will study this in a forthcoming paper.

### 1.1.3 Remarks about Glasner property and Q.D. property

It is natural to compare the two properties defined here with certain notion in topological dynamics. We consider a single transformation $T$ on a metric space $Y$ below.

As is well known, topological transitivity asserts that for any open subset $U$, any $\epsilon > 0$, there is an $n$ such that $T^n U$ is $\epsilon$-dense, namely $\inf_n d_H(T^n U, Y) = 0$. Except in the case of discrete topology, an open subset contains infinite many points. Thus by definition, Glasner property is stronger than topological transitivity. Another notion is topological mixing. It is obvious that topological mixing does not imply Glasner property. Indeed, consider any hyperbolic automorphism on torus, it is topological mixing, but never has Glasner property. However, it seems not quite clear whether Glasner property implies topological mixing.

It is still a question whether there exists a dynamical systems ($\mathbb{Z}$ action) with Glasner property or Q.D. property. We can prove the nonexistence under local
connectedness condition, see Theorem 1.3.1 below. In addition, we cannot make any assertion on the relation between these two properties. Simply speaking, Glasner property means that any infinite subset will be “spreaded” to the whole space, and Q.D. property means that the orbit of an infinite subset will be “spreaded” to the whole space at certain rate. While it is quite possible that these two properties fail for \( \mathbb{Z} \) action, we emphasize that our study will mostly be focused on “large” group actions.

1.2 Some abstract results

We start with some abstract results, some of which will be used later.

**Proposition 1.2.1.** Let \((X, T, \mathcal{B})\) be a topological dynamical system, \(Y\) be a closed subset of \(X\), \(T_Y\) be the transformation on \(Y\) induced from the map \(T\). Let \(d\) be a metric on \(X\), \(d_Y\) be the induced metric on \(Y\). Denote \(d^H_Y\) and \(d^H_X\) as the Hausdorff semimetric by \(d_Y\) and \(d\) respectively. If there exists a constant \(M > 0\) such that \(d^H_Y(A, B) \leq M \cdot d^H_X(A, B)\) for any \(A, B \subset Y\), then for any \(C \subset Y\),

\[
\inf_n n \cdot d^H_Y \left( \bigcup_{k=0}^{n-1} T^k_Y C, Y \right) \leq M \inf_n n \cdot d^H_X \left( \bigcup_{k=0}^{n-1} T^k_X C, X \right).
\]

**Proof.** The proof is straightforward because \(\bigcup_{k=0}^{n-1} T^k_X C \cap Y \subset \bigcup_{k=0}^{n-1} T^k_Y C\) for any subset \(C\).

Below we show some necessary conditions for a system generated by one transformation which has Glasner property, although we do not know any example so far.

**Proposition 1.2.2.** If \((Y, d, T)\) is a topological system with Glasner property, then the following are true.

(1) \(T\) can not be an isometry.

(2) The orbit of any point is either discrete or dense.

(3) Every nonatomic ergodic measure of \(T\) must have full support.
(4) If $T$ is uniformly continuous, then $\forall k \neq 0$, $(Y, d, T^k)$ also has Glasner property. In particular, this applies if $Y$ is compact.

Proof. The proof of (1), (2) and (3) are straightforward. For (4), fix the $k$. As $T$ is uniformly continuous, then for any $\delta > 0$, there is a $\epsilon > 0$, such that if $d(x, y) \leq \epsilon$, then $\max_{1 \leq i \leq k} d(T^i x, T^i y) \leq \delta$. Let $A$ be an infinite subset, then $\forall \epsilon > 0$, there exists $n \in \mathbb{Z}$ such that $T^n A$ is $\epsilon$-dense. Hence there is an $i$, $1 \leq i \leq k$ such that $k|(n + i)$ and $T^{n+i}A$ is $\delta$-dense. Since $\delta$ is chosen arbitrarily, this completes the proof.

We also give a sufficient condition. Of course, this is far away to be necessary.

**Proposition 1.2.3.** Given a topological system $(Y, d, T)$, consider the infinite (direct) product system $(Y \otimes \mathbb{N}, D, T \otimes \mathbb{N})$, here the metric $D$ is defined by

$$D(\bar{x} := (x_1, x_2, \cdots), \bar{y} := (y_1, y_2, \cdots)) = \sup_i d(x_i, y_i).$$

(1) If for a point $\bar{x} \in Y \otimes \mathbb{N}$ such that $x_i \neq x_j \forall i, j \in \mathbb{N}, i \neq j$, the orbit of $\bar{x}$ under $T \otimes \mathbb{N}$ is dense, then $A := \{x_i| i \in \mathbb{N}\}$ is a Glasner set w.r.t. $(Y, d, T)$.

(2) Assume that for any point $\bar{x} \in Y \otimes \mathbb{N}$ such that $x_i \neq x_j \forall i, j \in \mathbb{N}, i \neq j$, the orbit of $\bar{x}$ under $T \otimes \mathbb{N}$ is dense, then $(Y, d, T)$ has Glasner property.

Proof. Let $K(Y)$ be the space of subsets of $Y$. Define a map $\pi : Y \otimes \mathbb{N} \to K(Y)$ as:

$$\pi(\bar{x}) = \{x_i| i \in \mathbb{N}\}.$$

(1) If the orbit of $\bar{x}$ is dense, then for any $\epsilon > 0$, there exist $k$ such that $\pi((T \otimes \mathbb{N})^k \bar{x})$ is $\epsilon$-dense.

(2) This is essentially the same as (1). We omit the proof.

### 1.3 $\mathbb{Z}$ action with Glasner property

We will show that under local connectedness, no $\mathbb{Z}$ action admits Glasner property. We do not know whether it is also the case without the additional assumption.
Theorem 1.3.1. Let $(Y, d)$ be a compact metric space which is locally connected. Then there does not exist a homeomorphism $T$ such that $(Y, d, T)$ has Glasner property.

During the proof, we will use several results. We state them first. In [17], Kato introduced a generalization of expansivity, which is called continuum-wise expansive (cw-expansive). Recall that $T$ is cw-expansive, if there is $\eta > 0$ such that if $C \subset Y$ is connected and $\text{diam}(T^n(C)) < \eta$ for all $n \in \mathbb{Z}$ then $C$ is a singleton. Here, $\eta$ is called cw-expansive constant. Combine this with the definition of Glasner property, the following proposition follows.

Proposition 1.3.2. For a $\mathbb{Z}$ action, Glasner property implies cw-expansivity.

For the homeomorphism $T$, define the $\epsilon$-local stable set of a point $x$ in $Y$ as the set

$$W^s_\epsilon(x) = \{ y \in Y : d(T^n x, T^n y) \leq \epsilon, \\forall n \geq 0\}.$$ 
Define similarly the $\epsilon$-local unstable set $W^u_\epsilon(x)$. Denoting by $CW^\sigma_\epsilon(x)$ the connected component of $x$ in the set $W^\sigma_\epsilon(x)$ for $\sigma = s, u$. The following useful result guarantees the existence of non trivial local stable/unstable sets.

Theorem 1.3.3 ([30]). If $T$ is a cw-expansive homeomorphism on a locally connected compact metric space $Y$, then for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\inf_{x \in Y'} \text{diam}(CW^s_\epsilon(x)) \geq \delta, \quad \inf_{x \in Y'} \text{diam}(CW^u_\epsilon(x)) \geq \delta.$$ 

Here $Y'$ is the set of accumulation points of $Y$.

Now we are ready to give the proof.

Proof of Theorem 1.3.1. By contradiction, assume there is a homeomorphism $T$ such that $(Y, d, T)$ has Glasner property. By Proposition 1.3.2, $T$ is cw-expansive. In fact, it is cw-expansive with cw-expansive constant $\eta$ as long as $\eta < \sup d(x, y)$.

Fix $\epsilon > 0$ small enough, by Theorem 1.3.3, there is a $\delta > 0$, such that for any $x \in Y'$, $\text{diam}(CW^s_\epsilon(x)) \geq \delta$. Let’s regard $4\delta$ as the cw-expansive constant.

It is known [17, Corollary 2.4] that there is $N > 0$ such that if $\text{diam}(CW^s_\epsilon(x)) = \delta$, then

$$\text{diam}(T^{-N}(CW^s_\epsilon(x))) > 4\delta.$$
Let \( g = T^{-N} \), by Proposition 1.2.2, \( (Y, d, g) \) also has Glasner property.

Now choose an \( x \in Y' \), let \( C_0 = CW_s^*(x) \) and \( U = \{ y \in Y : d(x, y) \leq \delta/4 \} \). Since \( \text{diam}(C_0) \geq \delta \), then \( \text{diam}(g(C_0)) > 3\delta \). Hence we can choose two disjoint connected component \( C_1^1 \) and \( C_2^1 \) from \( g(C_0) - U \) such that \( \text{diam}(C_1^{1/2}) = \delta \).

Replace \( C_0 \) by \( C_1^1 \) and also by \( C_2^1 \), we can get four connected component \( C_2^i \) from \( g(C_0) - U \) for \( 1 \leq i \leq 4 \) such that \( \text{diam}(C_2^i) = \delta \). Continuing in this way, we can get at the \( k \)th step, \( 2^k \) connected component \( C_k^i \) for \( 1 \leq i \leq 2^k \) such that \( \text{diam}(C_k^i) = \delta \). Let

\[
\hat{C} = \bigcap_{k=0}^{\infty} \bigcup_{1 \leq i \leq 2^k} g^{-k}C_k^i.
\]

It is obvious from the construction, that \( \hat{C} \) is Cantor set, and hence contains infinite many points.

However, since \( \hat{C} \subset C_0 \subset CW_s^*(x) \), \( g^n(\hat{C}) \subset g^n(C_0) \subset g^n(CW_s^*(x)) \subset B(g^n x, \epsilon) \) for \( n \leq 0 \), and by the construction, \( g^n(\hat{C}) \cap U = \emptyset \) for \( n > 0 \). Therefore

\[
\inf_{n \in \mathbb{Z}} d^H(g^n(C), Y) \geq \delta/4 > 0.
\]

This is a contradiction to the Glasner property for \( g \). \( \square \)

### 1.4 Proofs

#### 1.4.1 Proof of Theorem 1.1.3

Theorem 1.1.3 follows easily from the following quantitative result, whose proof is carried over through this subsection. The idea of the proof comes originally from [1], which is generalized for much more broader cases in [21].

**Theorem 1.4.1.** For any set \( A \) of \( k \) distinct points in \( \mathbb{T} \), if there is no \( \gamma \in \Gamma \) such that \( \gamma A \) is \( \epsilon \)-dense, then

\[
k \leq \frac{C_n}{\epsilon C_n},
\]

provided \( \epsilon > 0 \) small enough.
Let \( \{x_1, x_2, \ldots, x_k\} \) be a set of \( k \) distinct points in \( \mathbb{T}^n \). Define 
\[
h_m := \#\{(i, j) | 1 \leq i, j \leq k \text{ such that } m(x_i - x_j) \in \mathbb{Z}^n\},
\]
let \( H_m := h_1 + h_2 + \cdots + h_m \). The estimates of \( h_m \) and \( H_m \) already appear in the work [1] and [21]. Using the same idea as the proof of Proposition 1 in [21], we can get

**Proposition 1.4.2.** For any positive integers \( k \) and \( m \), \( H_m \leq km^{n+1} \).

**Lemma 1.4.3.** Let \( r > 1 \). If \( s_2, s_3, \ldots \) is a sequence of nonnegative integers such that \( S_b = s_2 + s_3 + \cdots + s_b \leq H_b \), and \( S_b \leq k^2 \). Then 
\[
\sum_{b=2}^{\infty} s_b b^{-r} \leq C_{n,r} k^{2-r/(n+1)}.
\]

**Proof.** The proof is carried over similarly as that of [21, Corollary 1]. Let \( S_1 = 0 \), note that 
\[
\sum_{b=2}^{\infty} s_b b^{-r} = \sum_{b=2}^{\infty} (S_b - S_{b-1}) b^{-r} = \sum_{b=2}^{\infty} S_b (b^{-r} - (b+1)^{-r})
\]
\[
\leq \sum_{b=2}^{[k^{1/(n+1)}]} S_b (b^{-r} - (b+1)^{-r}) + k^2 \cdot k^{-r/(n+1)}
\]
\[
\leq \sum_{b=2}^{[k^{1/(n+1)}]} H_b b^{-r-1} + k^{2-r/(n+1)}
\]
\[
\leq \sum_{b=2}^{[k^{1/(n+1)}]} kb^{n+1} b^{-r-1} + k^{2-r/(n+1)} \leq Ck \cdot k^{\frac{n-r+1}{n+1}} + k^{2-r/(n+1)}
\]
\[
= Ck^{2-r/(n+1)}.
\]

We will apply a special case of a general theorem about stationary measure for discrete group actions proven by Benoist and Quint. We reformulate it for our case. Let \( \nu \) be a probability measure supported on a finite set of generators of \( \Gamma \).
Theorem 1.4.4 ([3]). For any \( \phi \in C_c(\mathbb{T}^n) \) and \( x \in \mathbb{T}^n \), then
\[
\frac{1}{N} \sum_{\gamma \in \Gamma} \sum_{k=0}^{N-1} \nu^k(\gamma) \phi(\gamma^{-1}x) \to \int_{\mathbb{T}^n} \phi d\mu_x
\]
as \( N \to \infty \). Here \( \nu^k \) means convolution of \( \nu \) \( k \) times.

Remark 1.4.5. By results from [13], [5] and [3], either \( \Gamma x \) is discrete or it is dense. When \( \Gamma x \) is discrete, then \( x \in \mathbb{Q}^n \), and \( \mu_x \) is an atomic measure depending on \( x \) which is invariant under \( \Gamma \) action. When \( \Gamma x \) is dense, then \( \mu_x \) is the Lebesgue measure.

Let \( \mathbf{m} = (m_1, \cdots, m_n) \in \mathbb{Z}^n \) be a nonzero integer vector, \( q \) be a positive integer. Let
\[
c_q(\mathbf{m}) := \sum_k e\left( \frac{\langle \mathbf{m}, \mathbf{k} \rangle}{q} \right),
\]
here \( \langle \cdot, \cdot \rangle \) is the usual inner product and \( e(z) = e^{2\pi iz} \). The sum is taken over all \( \mathbf{k} = (k_1, \cdots, k_n) \) such that \( 1 \leq k_i \leq q \) and \( \gcd(k_1, \cdots, k_n, q) = 1 \). Let \( \phi_q \) be the number of such \( \mathbf{k} \). It is easy to see that \( \phi_q \) grows like \( (q/\log q)^n \). Therefore there is a constant \( C_0 > 1 \) such that \( \phi_q \geq C_0 q^{n-1} \) for any \( q \geq 1 \). We would like to have an upper bound for \( c_q(\mathbf{m}) \). Notice that when \( n = 1 \), \( c_q(\mathbf{m}) \) is the famous Ramanujan sum.

Lemma 1.4.6. For \( \mathbf{m} \neq 0 \), \( |c_q(\mathbf{m})| \leq (\gcd(m_1, \cdots, m_n))^n \).

Proof. By [14], \( c_q(\mathbf{m}) = \prod_{p^r || q} c_{p^r}(\mathbf{m}) \) and
\[
c_{p^r}(\mathbf{m}) = \begin{cases} p^{(r-1)n}(p^n - 1) & \text{if } p^r \mid \gcd(m_1, \cdots, m_n), \\ -p^{(r-1)n} & \text{if } p^r-1 \mid \gcd(m_1, \cdots, m_n), \\ -1 & \text{if } p^{r-1} \nmid \gcd(m_1, \cdots, m_n). \end{cases}
\]
From here, the lemma follows easily. \( \square \)

We will also need the following fact about “bump” functions.

Lemma 1.4.7. For \( 0 < \epsilon < 1 \), there exists a nonnegative function \( g_\epsilon : \mathbb{T}^n \to \mathbb{R} \), such that
(1) \( \int_{\mathbb{T}^n} g_\epsilon(x) \, dx = 1 \),

(2) \( g_\epsilon(x) = 0 \) if \( d_L(x, e) \geq \epsilon \), here \( d_L \) is the standard metric on \( \mathbb{T}^n \), and \( e \in \mathbb{T}^n \) is the identity,

(3) there exists a positive constant \( C_1 \) such that, for any \( m \in \mathbb{Z}^n \), the Fourier coefficient \( \hat{g}_\epsilon(m) = \int_{\mathbb{T}^n} g_\epsilon(x) e_m(x) \, dx \) satisfies

\[
|\hat{g}_\epsilon(m)| \leq C_1 e^{-\sqrt{\epsilon |m|}}.
\]

Here \( |m| = \sum |m_i| \) is a norm on \( \mathbb{Z}^n \).

Proof. This follows from the circle version, which appears as Lemma 6.2 in [1].
Indeed, let \( \tilde{g}_\epsilon \) be the function obtain from Lemma 6.2 [1], for \( x = (x_1, \ldots, x_n) \), let \( g_\epsilon(x) = \prod_{i=1}^n \tilde{g}_\epsilon(x_i) \), then one can easily check (1), (2) and (3). Let’s remark that

\[
\int_{\mathbb{T}^n} g_\epsilon^2(x) \, dx = \prod_{i=1}^n \int_{\mathbb{T}} \tilde{g}_\epsilon^2(x_i) \, dx_i \leq C \frac{1}{\epsilon^n}.
\]

\[\square\]

Proof of Theorem 1.4.1. We use the idea of the proof of Proposition 6.1 in [1].

For \( \epsilon > 0 \) small, let \( g_\epsilon \) be a function from Lemma 1.4.7. Let \( \{x_1, x_2, \ldots, x_k\} \) be a set of \( k \) distinct points in \( \mathbb{T}^n \), and suppose there is no \( \gamma \in \Gamma \) such that \( \gamma A \) is \( \epsilon \)-dense. Then for each \( \gamma \), there is a point \( x_\gamma \in \mathbb{T}^n \) such that \( g_\epsilon(\gamma x_i - x_\gamma) = 0 \) for any \( i \).

Hence we have, for any \( N > 0 \),

\[
0 = \frac{1}{N} \sum_{\gamma \in \Gamma} \sum_{j=0}^{N-1} \nu^{\gamma j}(\gamma) \sum_{i=1}^k g_\epsilon(\gamma x_i - x_\gamma) =
\]

\[
\frac{1}{N} \sum_{\gamma \in \Gamma} \sum_{j=0}^{N-1} \nu^{\gamma j}(\gamma) \sum_{i=1}^k \hat{g}_\epsilon(m) e_m(\gamma^{-1} x_i - x_\gamma). \quad (1.1)
\]

Since \( \hat{g}_\epsilon(0) = \int_{\mathbb{T}^n} g_\epsilon(x) \, dx = 1 \), then from (1.1),
Now by (3) of Lemma 1.4.7,

\[
\sum_{m \in \mathbb{Z}^n, |m| > M} |\hat{g}_\ell(m)| \leq C_1 \sum_{\ell=M+1}^{\infty} C_{\ell+1} \leq 2 C_1 \sum_{\ell=M+1}^{\infty} \ell^{-1} e^{-\sqrt{\ell}} \\
\leq \frac{2C_1}{\epsilon^n-1} \int_M^{\infty} x^{n-1} e^{-\sqrt{x}} dx = \frac{4C_1}{\epsilon^n-1} \int_\sqrt{M\epsilon}^{\infty} u^{2n-1} e^{-u} du. 
\]  

(1.3)

As the integral \(\int_0^{\infty} u^{2n-1} e^{-u} du = \Gamma(2n) < \infty\), there exists a great enough number \(a\) (depends on \(\epsilon\)) such that for \(M := \frac{1}{\epsilon^a}\),

\[
\int_\sqrt{M\epsilon}^{\infty} u^{2n-1} e^{-u} du \leq \frac{\epsilon^n-1}{8C_1},
\]

and hence

\[
\sum_{m \in \mathbb{Z}^n, |m| > M} |\hat{g}_\ell(m)| \leq \frac{1}{2}.
\]

Hence from (1.2), we have for \(M \geq M_0\),

\[
\frac{k}{2} \leq \left| \frac{1}{N} \sum_{\gamma \in \Gamma} \sum_{j=0}^{N-1} \nu^j(\gamma) \sum_{i=1}^{k} \sum_{m \in \mathbb{Z}^n \setminus 0, |m| \leq M} \hat{g}_\ell(m) e_m (\gamma^{-1} x_i - x_\gamma) \right|.
\]

(1.4)

Now square both sides of the above inequality, then apply Cauchy-Schwartz inequality twice,

\[
\frac{k^2}{4} \leq \left| \left( \frac{1}{N} \sum_{\gamma \in \Gamma} \sum_{j=0}^{N-1} \nu^j(\gamma) \sum_{i=1}^{k} \sum_{m \in \mathbb{Z}^n \setminus 0, |m| \leq M} \hat{g}_\ell(m) e_m (\gamma^{-1} x_i - x_\gamma) \right)^2 \right|
\]

\[
= \sum_{\gamma \in \Gamma} \left( \frac{1}{N} \sum_{j=0}^{N-1} \nu^j(\gamma) \right)^2 \left( \frac{1}{N} \sum_{j=0}^{N-1} \nu^j(\gamma) \sum_{i=1}^{k} \sum_{m \in \mathbb{Z}^n \setminus 0, |m| \leq M} \hat{g}_\ell(m) e_m (\gamma^{-1} x_i - x_\gamma) \right)^2
\]
\[
\left( \sum_{\gamma \in \Gamma} \frac{1}{N} \sum_{j=0}^{N-1} \nu^{*j}(\gamma) \right) \cdot \left( \sum_{\gamma \in \Gamma} \frac{1}{N} \sum_{j=0}^{N-1} \left| \hat{\gamma}(m) \right|^2 \right)
\leq \sum_{\gamma \in \Gamma} \frac{1}{N} \sum_{j=0}^{N-1} \nu^{*j}(\gamma) \left( \sum_{m \in \mathbb{Z}^n \setminus \{0\}, |m| \leq M} \left| \sum_{i=1}^{k} e_m(\gamma^{-1}x_i - x_\gamma) \right|^2 \right)
\leq C_2 \epsilon^n. \quad (1.5)
\]

By Bessel’s inequality and the construction in Lemma 1.4.7,
\[
\left( \sum_{m \in \mathbb{Z}^n \setminus \{0\}, |m| \leq M} \left| \hat{\gamma}(m) \right|^2 \right) \leq \frac{C_2}{\epsilon^n}. \quad (1.6)
\]

Combining (1.5) and (1.6), we obtain
\[
k^2 \leq \frac{4C_2}{\epsilon^n} \left( \sum_{m \in \mathbb{Z}^n \setminus \{0\}, |m| \leq M} \sum_{1 \leq i, \ell \leq k} \lim_{N \to \infty} \sum_{\gamma \in \Gamma} \frac{1}{N} \sum_{j=0}^{N-1} \nu^{*j}(\gamma) e_m(\gamma^{-1}(x_i - x_\gamma)) \right). \quad (1.7)
\]

By Theorem 1.4.4 and the remark after it, as \(N \to \infty\),
\[
\sum_{\gamma \in \Gamma} \frac{1}{N} \sum_{j=0}^{N-1} \nu^{*j}(\gamma) e_m(\gamma^{-1}(x_i - x_\ell))
\]
contributes to the right hand sum in (1.7) only when \((x_i - x_j) \in \mathbb{Q}^n/\mathbb{Z}^n\). When \((x_i - x_j) \in \mathbb{Q}^n/\mathbb{Z}^n \setminus \{0\}\), let \(q_{i,\ell}\) be the least positive integer that \(q_{i,\ell}(x_i - x_\ell) \in \mathbb{Z}^n\).
Hence by Theorem 1.4.4,

\[
\sum_{\gamma \in \Gamma} \frac{1}{N} \sum_{j=0}^{N-1} \nu^{*j}(\gamma) e_{m}(\gamma^{-1}(x_i - x_\ell)) \rightarrow \frac{1}{\phi_{\ell}} \sum_{k} e\left(\frac{\langle m, k \rangle}{q_i, \ell}\right) = \frac{1}{\phi_{\ell}} c_{q_i, \ell}(m).
\]

By Lemma 1.4.6, \( c_{q_i, \ell}(m) \leq M^n \) because \(|m| \leq M\). Combine this with \( \phi_q \geq C_0 q^{n-1} \), we have

\[
\left| \sum_{\gamma \in \Gamma} \frac{1}{N} \sum_{j=0}^{N-1} \nu^{*j}(\gamma) e_{m}(\gamma^{-1}(x_i - x_\ell)) \right| \leq \frac{M^n}{C_0 q^{n-1}}
\]
as \( N \to \infty \).

Recall that \( M = \frac{1}{e^a} \), by Proposition 2.3.2 and Lemma 3.1.2, let \( N \to \infty \)

\[
\sum_{m \in \mathbb{Z}^n \setminus \{0\}, |m| \leq M} \sum_{1 \leq i, \ell \leq k} \left| \sum_{\gamma \in \Gamma} \frac{1}{N} \sum_{j=0}^{N-1} \nu^{*j}(\gamma) e_{m}(\gamma^{-1}(x_i - x_\ell)) \right| \leq \left( 2^m M^n k + C_n M^n \sum_{q=1}^{\infty} h_q q^{-n+1} \right)
\]

\[
\leq \left( 2^m M^n k + C_n M^n k^{2-(n-1)/(n+1)} \right)
\]

\[
\leq \left( C_n M^n k^{1+1/(n+1)} \right)
\]

\[
= \frac{C_n M^n k^{1+1/(n+1)}}{e^{na}}.
\]  
(1.8)

Combine (1.7) and (1.8), we obtain

\[
k \leq \frac{C(n)}{e^{2(a+1)n}}.
\]

This finishes the proof. \( \Box \)

A direct corollary of Theorem 1.1.3 is the following concerning to finite index subgroups of \( SL(n, \mathbb{Z}) \).

**Corollary 1.4.8.** Let \( n \geq 2 \), and \( \Gamma \) be a finite index subgroup of \( SL(n, \mathbb{Z}) \). Then the system \((\mathbb{T}^n, d_L, \Gamma)\) has Glasner property.
Remark 1.4.9. We will extend the above results in [?] via a new method, proving that the system \((\mathbb{T}^n, d_L, \Gamma)\) has Glasner property if \(\Gamma < SL(n, \mathbb{Z})\) is Zariski dense in \(SL(n, \mathbb{R})\). Unfortunately, we could not obtain this here.

1.4.2 Proof of Theorem 1.1.4

We will use some of the results and notations from the previous subsection, and modify the previous argument to prove Theorem 1.1.4. We start with the following useful property of algebraic abelian actions.

Proposition 1.4.10. Let \(k \geq 2\), and \(\alpha\) be an action of \(\mathbb{Z}^k\) on \(\mathbb{T}^n\) by ergodic automorphisms. Suppose that the Lyapunov exponents of the Lebesgue measure are in general position. Then for any Lyapunov exponent \(\chi\), \(\{\chi(n) | n \in \mathbb{Z}^k\}\) is dense in \(\mathbb{R}\).

Proof. Since \(\chi : \mathbb{Z}^k \to \mathbb{R}\) is a linear functional, it suffices to prove that for any \(\epsilon > 0\), there exists an \(n \in \mathbb{Z}^k\) such that \(|\chi(n)| \leq \epsilon\). Now since the Lyapunov exponents are in general position, each Lyapunov hyperplane (a \(k-1\) dimensional hyperplane in \(\mathbb{R}^k\)) has no intersection with \(\mathbb{Z}^k\). Therefore, for any \(\epsilon > 0\), there always exists an element \(n \in \mathbb{Z}^k\), which is very close to the hyperplane corresponding to \(\chi\), satisfying that \(|\chi(n)| \leq \epsilon\). This finishes the proof.

We will prove the following quantitative version, which indicates Theorem 1.1.4.

Theorem 1.4.11. Let \(k \geq 2\), and \(\alpha\) be an action of \(\mathbb{Z}^k\) on \(\mathbb{T}^n\) by ergodic automorphisms. Suppose that the Lyapunov exponents of the Lebesgue measure are in general position. Let \(A\) be a finite subset which is subordinate to \(\alpha\). If for an \(\epsilon > 0\) small enough, there exists no \(n \in \mathbb{Z}^k\) such that \(\alpha(n)A\) is \(\epsilon\)-dense, then \(#A \leq \frac{C}{\epsilon^{\ell + \gamma}}\).

Proof. Let \(A = \{x_1, \ldots, x_{\ell}\}\) with \(\ell = \#A\). Assume that \(A\) lies in the leaf of the foliation corresponding to the Lyapunov exponent \(\chi\). Let \(p_{j,r} \in \mathbb{R}\) be such that \(x_j - x_r = p_{j,r}v\mod \mathbb{Z}^n\), here \(v\) is a fixed vector on \(\mathbb{T}^n\) generating the leaf where \(A\) lies in.

First, fix an integer \(N > 1\). By Proposition 1.4.10, we can choose \(n_i \in \mathbb{Z}^k\) for \(i \in \mathbb{N}\), so that \(e^{\chi(n_i)}\) is uniformly distributed in \([1, N]\). Since there exists no \(n \in \mathbb{Z}^k\) such that \(\alpha(n)A\) is \(\epsilon\)-dense, then for each \(i\), there is a \(y_i \in \mathbb{T}^n\) such that \(g_\epsilon(\alpha(n_i)x_j - y_i) = 0\) for any \(j\).
Therefore for any $L > 1$,
\[
0 = \frac{1}{L} \sum_{i=1}^{L} \sum_{j=1}^{\ell} g_r(\alpha(n_i)x_j - y_i) = \frac{1}{L} \sum_{i=1}^{L} \sum_{j=1}^{\ell} \sum_{m \in \mathbb{Z}^n} \hat{g}_r(m)e_m(\alpha(n_i)x_j - y_i).
\]

From here, analyzing in the same way as to obtain Equation (1.7) in the proof of Theorem 1.4.1, we can get
\[
\ell^2 \leq \frac{C_2}{\epsilon^n} \left( \sum_{m \in \mathbb{Z}^n \setminus \{0\}, |m| \leq M} \sum_{j=1}^{\ell} \sum_{r=1}^{\ell} \sum_{m} \frac{1}{N-1} \int_1^N e_m(tp_{j,r}\mathbf{v})dt \right).
\]

Observe that $\alpha(n_i)(x_j - x_r) = e^{\chi(n_i)}(x_j - x_r) \mod \mathbb{Z}^n = e^{\chi(n)}p_{j,r}\mathbf{v} \mod \mathbb{Z}^n$, hence $e_m(\alpha(n_i)(x_j - x_r)) = e_m(e^{\chi(n)}p_{j,r}\mathbf{v})$. By the choice of $n$, let $L \to \infty$, thus we have
\[
\ell^2 \leq \frac{C_2}{\epsilon^n} \left( 2^n M^n \ell + \sum_{m \in \mathbb{Z}^n \setminus \{0\}, |m| \leq M} \sum_{1 \leq j \neq r \leq \ell} \frac{1}{N-1} \int_1^N e_m(tp_{j,r}\mathbf{v})dt \right). \quad (1.9)
\]

Notice that for $j \neq r$, $p_{j,r} \neq 0$, the flow on $\mathbb{T}^n$ generated by the vector $p_{j,r}\mathbf{v}$ is uniquely ergodic, thus
\[
\lim_{N \to \infty} \frac{1}{N-1} \int_1^N e_m(tp_{j,r}\mathbf{v})dt = \int_{\mathbb{T}^n} e_m(x)dx = 0.
\]

Therefore let $N \to \infty$ in (1.9), we have
\[
\ell^2 \leq \frac{C_3}{\epsilon^n} M^n \ell = \frac{C_3 \ell}{\epsilon^{(a+1)n}}.
\]

Hence $\ell \leq \frac{C_3}{\epsilon^{(a+1)n}}$. \qed
1.4.3 Proofs of the other results

We are going to use the subsequent two results, as they play an important role in the proof. The first one is a strengthen of the aforementioned Glasner’s result.

**Theorem 1.4.12** (Theorem 1.3, [4]). *For any infinite subset $X \subset S^1$, there exists a sequence $\{n_i\}$ with density 1 in $\mathbb{N}$, such that $E_nX \to S^1$ in the Hausdorff semimetric. Here $E_nx = nx \mod 1$.*

Namely, there exists a sequence $\{n_i\}$ with density 1 in $\mathbb{N}$, such that

$$\lim_{n_i} n_i \cdot d^H \left( \bigcup_{k=0}^{n_i-1} T^{k}_{n_i} X, S^1 \right) = 0.$$  \hspace{1cm} (1.10)

This is because if $E_nX$ is $\epsilon$-dense, then its preimage $E_n^{-1}(E_nX)$ is $\frac{\epsilon}{n}$-dense, and the fact

$$E_n^{-1}(E_nX) = \bigcup_k T^k_{n} X.$$

**Theorem 1.4.13** (Theorem III, [8]). *Let $Q$ be an increasing sequence of integers with a positive lower density. Let $a_1, a_2, \cdots$ be a sequence of positive numbers such that $\sum_q a_q = \infty$, and for some real number $c$, $\frac{a_q}{q^c}$ is a decreasing function of $q$. Then for almost all $x$, there exist arbitrarily many relative prime $p$ and $q$, such that

$$\left| x - \frac{p}{q} \right| \leq \frac{a_q}{q}, q \in Q.$$*

We are now ready to give the proof.

**Proof of Theorem 2.1.3.** Let $X$ be an arbitrary infinite subset of $S^1$, then by Theorem 1.4.12, there is an increasing sequence $\{n_i\}$ of natural numbers with density 1 in $\mathbb{N}$, such that $\mathbb{N}$, such that

$$\lim_{n_i} n_i \cdot d^H \left( \bigcup_{k=0}^{n_i-1} T^k_{n_i} X, S^1 \right) = 0.$$ \hspace{1cm} (1.10)

Now apply Theorem 1.4.13 with $Q = \{n_i\}$ and $a_q = \frac{1}{q \log q}$, we have that there is a subset $U \subset S^1$ of full measure, such that for any $\alpha \in U$, there exist infinitely
many \( n_i \in Q \), such that

\[
\left| \alpha - \frac{p}{n_i} \right| \leq \frac{1}{n_i^2 \log n_i} \quad \text{for some } p, (p, n_i) = 1. \tag{1.11}
\]

Now for any \( \alpha \in U \), we claim that

\[
\liminf_n n \cdot d_H \left( \bigcup_{k=0}^{n-1} T_{\alpha}^k X, S^1 \right) = 0.
\]

Let \( k_i \) be the subsequence of \( Q \) such that (1.11) holds for \( \alpha \). Then we have

\[
k_i d_H \left( \bigcup_{j=0}^{k_i-1} T_{\alpha}^j X, S^1 \right) \leq k_i d_H \left( \bigcup_{j=0}^{k_i-1} T_{\alpha}^j X, \bigcup_{j=0}^{k_i-1} T_{\frac{p}{k_i}}^j X \right) + k_i d_H \left( \bigcup_{j=0}^{k_i-1} T_{\frac{p}{k_i}}^j X, S^1 \right)
\]

\[
\leq k_i^2 \left| \alpha - \frac{p}{k_i} \right| + k_i d_H \left( \bigcup_{j=0}^{k_i-1} T_{\frac{p}{k_i}}^j X, S^1 \right)
\]

\[
\leq \frac{1}{\log k_i} + k_i d_H \left( \bigcup_{j=0}^{k_i-1} T_{\frac{p}{k_i}}^j X, S^1 \right).
\]

Combine the above inequality with (1.10), the claim follows. \( \square \)

We can use the same idea to prove the following

**Theorem 1.4.14.** If \( A_i \) is an infinite subset of \( S^1 \) for \( i = 1, 2 \), then for almost every \( \alpha \),

\[
\liminf_n n \cdot d_H \left( \bigcup_{k=0}^{n-1} T_{\alpha}^k A_1, \bigcup_{k=0}^{n-1} T_{\alpha}^k A_2 \right) = 0.
\]

**Proof.** Note that, by Theorem 1.4.12, there is an increasing sequence \( \{n_i\} \) of natural numbers with density 1 in \( \mathbb{N} \), such that \( \mathbb{N} \), such that (1.10) holds for both \( A_1 \) and \( A_2 \). Using the triangle inequality, the rest of the argument will be essentially the same as the proof of Theorem 2.1.3. We omit the details here. \( \square \)

**Proof of Corollary 2.1.4.** The proof follows from Proposition 1.2.1 and the fact in [19]. We give the details here.

Let \( X \) be an arbitrary infinite subset of \( I = [0, 1] \). We can also think of \( X \) as a subset of \([0, 1 + \beta - \alpha]\). Thus by Theorem 2.1.3, for fixed \( \beta - \alpha := \ell \), and generic
\( \alpha \in (0, 1-\ell) \) (Namely, \( \frac{1-\alpha}{1+\beta-\alpha} \) satisfies a generic condition), then

\[
\liminf_n n \cdot d^H \left( \bigcup_{k=0}^{n-1} T^k_{1-\alpha} X, [0, 1+\beta-\alpha] \right) = 0.
\]

Now note that

\[
\left( \bigcup_{k=0}^{n-1} T^k_{1-\alpha} X \right) \cap I \subset \bigcup_{k=0}^{n-1} P^k_{\alpha,\beta} X,
\]

hence by Proposition 1.2.1,

\[
\liminf_n n \cdot d^H \left( \bigcup_{k=0}^{n-1} P^k_{\alpha,\beta} X, I \right) = 0.
\]

By applying Fubini’s Theorem on the pair \((\alpha, \ell)\), the claim follows. \(\square\)

**Proof of Theorem 1.1.9.** If \( \alpha \in \mathbb{Q} \), then it suffices to let \( X = [0, \epsilon] \) for \( \epsilon > 0 \) small enough.

Assume that \( \alpha \notin \mathbb{Q} \). Let the integers \( a_k \) be the coefficients of the continued fraction of \( \alpha \), and \( \frac{p_k}{q_k} \) be the partial convergence. Let \( ||q\alpha|| = \text{dist}(q\alpha, \mathbb{Z}) \), \( |x| \) be the fraction part of \( x \). Then

1. \( \frac{1}{q_{k+1}+q_k} < ||q_k\alpha|| < \frac{1}{a_k+1q_k} \),
2. \( d^H(\bigcup_{i=0}^{q_k+1-1} \{|i\alpha|\}, S^1) \geq ||q_k\alpha||. \)

From (1) and (2), it is easy to see that

\[
\liminf_n n \cdot d^H \left( \bigcup_{k=0}^{n} \{T^k(0)\}, S^1 \right) > 0. \tag{1.12}
\]

Choose an increasing subsequence \( \{n_k\} \) of \( \mathbb{N} \), such that \( \log \log(q_{n_{k+1}}) \geq q_{n_k} \), and \( 0 < |q_{n_k+1}\alpha| < |q_{n_k}\alpha| \). Let \( X = \{|q_{n_k}\alpha| : k \in \mathbb{N}\} \cup \{0\} \). We claim that

\[
\liminf_n n \cdot d^H \left( \bigcup_{k=0}^{n-1} T^k_\alpha X, S^1 \right) > 0.
\]
Let $n$ be any integer, and $k$ such that $q_{n_k} - q_{n_k-1} \leq n \leq q_{n_k+1} - q_{n_k}$. Then

$$\bigcup_{i=0}^{n-1} T^i_{\alpha} X \subset \bigcup_{i=0}^{q_{n_k} + n} \{T^i_{\alpha}(0)\} \cup \bigcup_{i=0}^{n-1} \{T^i_{\alpha} + q_{n_{k+1}}(0)\} \cup \bigcup_{j=k+2}^{\infty} \bigcup_{i=0}^{n-1} \{T^{i+q_{n_j}}_{\alpha}(0)\}.$$ 

Now by the choice of $n_k$, when $j \geq k+2$, $\bigcup_{i=0}^{n-1} \{T^i_{\alpha} + q_{n_j}(0)\}$ is at least $|q_{n_j} \alpha|$ close to $\bigcup_{i=0}^{n-1} \{T^i_{\alpha}(0)\}$ because $|q_{n_j} \alpha| \ll |q_{n_{k+1}} \alpha|$. Hence there is a positive constant $c$, such that

$$d^H \left( \bigcup_{i=0}^{n-1} T^i_{\alpha} X, S^1 \right) \geq cd^H \left( \bigcup_{i=0}^{q_{n_k} + n} \{T^i_{\alpha}(0)\} \cup \bigcup_{i=0}^{n-1} \{T^i_{\alpha} + q_{n_{k+1}}(0)\}, S^1 \right)$$

$$\geq \frac{1}{2} cd^H \left( \bigcup_{i=0}^{q_{n_k} + n} \{T^i_{\alpha}(0)\}, S^1 \right)$$

$$\geq \frac{1}{2} cd^H \left( \bigcup_{i=0}^{3n} \{T^i_{\alpha}(0)\}, S^1 \right).$$

Thus

$$n \cdot d^H \left( \bigcup_{i=0}^{n-1} T^i_{\alpha} X, S^1 \right) \geq \frac{1}{2} cn \cdot d^H \left( \bigcup_{i=0}^{3n} \{T^i_{\alpha}(0)\}, S^1 \right).$$

Hence by (1.12),

$$\liminf_n n \cdot d^H \left( \bigcup_{i=0}^{n-1} T^i_{\alpha} X, S^1 \right) \geq \frac{c}{6} \liminf_n 3n \cdot d^H \left( \bigcup_{i=0}^{3n} \{T^i_{\alpha}(0)\}, S^1 \right) > 0.$$

1.5 Concluding remarks

We are not aware of a direct proof of Theorem 2.1.4 without using the property of circle rotation. It will be good to find one, as it possibly can be generalized to prove that for all IETs. In fact, another motivation of this work is the following two conjectures.
Conjecture 1.5.1. If $X$ is an infinite subset of $I = [0, 1]$, then for a.e. IET $T$,  

$$\inf_n d^H(T^n X, I) = 0.$$  

Conjecture 1.5.2. If $X$ is an infinite subset of $I$, then for a.e. IET $T$,  

$$\inf_n n \cdot d^H \left( \bigcup_{k=0}^{n-1} T^k X, I \right) = 0.$$  

It is also natural to ask  

Question 1.5.3. If $X$ is an infinite subset of $I$, is it true that for a.e. IET $T$,  

$$\sup_n n \cdot d^H \left( \bigcup_{k=0}^{n-1} T^k X, I \right) = 0?$$  

As we illustrated before, it will be interesting to know whether one can find a $\mathbb{Z}$ action with Glasner property or Q.D. property. It will be surprising to really have such an example with either property. We hope to make further progress towards these questions in the subsequent work.
Chapter 2

On density of infinite subsets II: dynamics on homogeneous spaces

Let $G$ be a noncompact semisimple Lie group, $\Gamma$ be an irreducible cocompact lattice in $G$, and $P < G$ be a minimal parabolic subgroup. We consider the dynamics of $P$ acting on $G/\Gamma$ by left translation. For any infinite subset $A \subset G/\Gamma$, we show that, for any $\epsilon > 0$, there is a $g \in P$ such that $gA$ is $\epsilon$-dense. We also prove a similar result for certain discrete group actions on $T^n$.

2.1 Introduction and Results

In this note, we make further progress on density of infinite subset initiated in [?]. We will in particular focus on the D.I. problem.

To be more precise, let $Y$ be a compact metric space, and $G$ be a locally compact second countable topological (semi-)group which acts on $Y$ by homeomorphisms. Let $A$ be an infinite subset of $Y$, we can consider the set containing all subsets of the form $gA := \{\alpha(g)x \mid x \in A\}$ for a $g \in G$. For the fixed $A$, we would like to know: for any $\epsilon > 0$, whether there exists a $g \in G$ such that $gA$ is $\epsilon$-dense in $Y$, or equivalently $d_H^Y(gA, Y) < \epsilon$. We call this dense iteration problem simply D.I. problem.

Here is a nontrivial result in this direction. Let $S^1 = \mathbb{R}/\mathbb{Z}$ be the standard circle, and $T_\alpha : S^1 \to S^1$ be the translation map: $x \mapsto x + \alpha \ (mod \ 1)$. A theorem of Glasner [12] asserts that if $X$ is an infinite subset of $S^1$, then for any $\epsilon > 0$,
there exists an integer $n$ such that the dilation $nX := \{nx \ (mod\ 1) : x \in X\}$ is $\epsilon$-dense. This gives an affirmative answer to the D.I. problem in the case of the natural action by multiplication of $\mathbb{N}$ on the circle $S^1$.

In view of this result, we made the following definitions in [?].

**Definition 2.1.1.** Given a $G$ action on a metric space $Y$, if an infinite subset $A$ satisfies that for any $\epsilon > 0$, there exists a $g \in G$ such that $gA$ is $\epsilon$-dense in $Y$, then $A$ is called **Glasner set** with respect to $(Y, d, G)$.

**Definition 2.1.2.** Given a $G$ action on a metric space $Y$, if any infinite subset $A$ is a Glasner set, then we say the dynamical system $(Y, d, G)$ has **Glasner property**.

Using our definition, the system $(S^1, d_L, \mathbb{N})$ has the Glasner property. We also proved in [?] that for any positive integer $N \geq 2$, the system $(\mathbb{T}^N, d_L, SL(N, \mathbb{Z}))$ has Glasner property.

In this note, we consider “large” group acting on homogeneous spaces. Recall that, a subgroup $F$ of a real algebraic group $G$ is called **epimorphic** in $G$ if any $F$-fixed vector is also $G$-fixed for any finite dimensional algebraic linear representation of $G$. As an example, the parabolic group of a semisimple real Lie group without compact factor is epimorphic. Our first result is

**Theorem 2.1.3.** Let $G$ be a connected semisimple real Lie group with trivial center and no compact factor, $\Gamma$ be an irreducible cocompact lattice in $G$, and $P < G$ be an epimorphic subgroup. Consider $P$ acting on $G/\Gamma$ by left translation. Then $(G/\Gamma, d, P)$ has Glasner property.

Here, a lattice $\Gamma$ in a connected semisimple Lie group $G$ with finite center is **irreducible** if the projection of $\Gamma$ to $G/H$ is dense for every nontrivial connected normal subgroup $H \leq G$.

Our second result is a generalization of [?, Theorem 1.1].

**Theorem 2.1.4.** Let $n \geq 2$, and $\Gamma$ be a subgroup of $GL(n, \mathbb{Z})$. Assume the Zariski closure of $\Gamma$ is semisimple, Zariski connected and with no compact factor, and acts irreducibly on $\mathbb{Q}^n$. Then the system $(\mathbb{T}^n, d_L, \Gamma)$ has Glasner property.

**Remark 2.1.5.** A particular case of Theorem 2.1.4 is when $\Gamma < SL(n, \mathbb{Z})$ is Zariski dense in $SL(n, \mathbb{R})$. 
Our main ingredient is the classification of orbit closure of certain group action. We heavily use the orbit closure results in [3],[31].

2.2 Facts from homogeneous dynamics

2.2.1 Orbit closure

The action of epimorphic subgroups on homogeneous spaces is well understood either in the case of invariant measure classification [24] or in the case of orbit closure [31]. Here we will use the result on orbit closure.

**Theorem 2.2.1** (Corollary 1.3,[31]). Let $F < G < L$ be an inclusion of connected real algebraic groups such that $F$ is epimorphic in $G$. Then any closed $F$-invariant subset in $L/\Lambda$ is $G$-invariant, where $\Lambda$ is a lattice in $L$.

Hence we have the following

**Corollary 2.2.2.** Let $G, \Gamma, P$ be given as in Theorem 2.1.3. For any integer $k$, consider the $P$ (or $G$) action on $(G/\Gamma)^k$ defined by $g(x_1, \ldots, x_k) = (gx_1, \ldots, gx_k)$ for $g \in P$ (or $G$) and $(x_1, \ldots, x_k) \in (G/\Gamma)^k$. Then for any $\bar{x} \in (G/\Gamma)^k$, the closure of $P$ orbit of $\bar{x}$ coincides with the closure of $G$ orbit of $\bar{x}$.

**Proof.** Apply Theorem 2.2.1 with $L = G^k$, $F = P$ and $\Lambda = \Gamma^k$, the result follows. \hfill \Box

2.2.2 Commensurability group of $\Gamma$

Let $\gamma \in G$, $\gamma$ is an element of the commensurator of $\Gamma$ in $G$ if $\Gamma \cap \gamma \Gamma \gamma^{-1}$ has finite index in both $\Gamma$ and $\gamma \Gamma \gamma^{-1}$. We write $Comm(\Gamma)$ for the commensurator of $\Gamma$ in $G$, namely, $Comm(\Gamma) = \{ \gamma \in G : [\Gamma : \Gamma \cap \gamma \Gamma \gamma^{-1}] < \infty, [\gamma \Gamma \gamma^{-1} : \Gamma \cap \gamma \Gamma \gamma^{-1}] < \infty \}$. It is known that $Comm(\Gamma)$ is a subgroup of $G$. Moreover, $Comm(\Gamma)$ satisfies a dichotomy (see [34]): either $Comm(\Gamma)$ contains $\Gamma$ as a subgroup of finite index, or $Comm(\Gamma)$ is dense in $G$. In fact, it is a celebrated theorem of Margulis that this is precisely the dichotomy of arithmeticity v.s. non-arithmeticity.

**Theorem 2.2.3** (Margulis, [34],[23]). Let $G$ be a connected semisimple real Lie group with trivial center and no compact factor, $\Gamma < G$ be an irreducible cocompact
lattice. Then either $\Gamma$ is arithmetic and $\text{Comm}(\Gamma)$ is dense in $G$ (w.r.t. Hausdorff topology), or $\Gamma$ is not arithmetic and $\Gamma$ is a finite index subgroup of $\text{Comm}(\Gamma)$.

The commensurators of $\Gamma$ play an important role in analyzing the dynamics on $G/\Gamma$. In fact, as we will describe later, they will give nontrivial self joinings of the $G$ action on $G/\Gamma$.

### 2.2.3 Benoist-Quint Theorems

We are going to use several results from [3]. In order to be self contained, we collect in the following those which will be used in the proofs.

**Theorem 2.2.4** (Benoist-Quint,[3]). Let $G$ be a connected semisimple real Lie group with trivial center and no compact factor, $\Gamma < G$ be an irreducible cocompact lattice. Let $\Lambda < G$ be a Zariski dense subgroup. Consider $\Lambda$ acting on $G/\Gamma$ by left translations, then

1. every $\Lambda$ orbit closure is either discrete (and hence finite) or $G/\Gamma$. In particular, this is true for the action of any finite index subgroup of $\Gamma$,
2. any sequence of distinct finite $\Lambda$ orbits has $G/\Gamma$ as the only limit in the Hausdorff topology.

**Theorem 2.2.5** (Benoist-Quint,[3]). Let $n \geq 2$, and $\Gamma$ be a subgroup of $\text{GL}(n, \mathbb{Z})$. Assume the Zariski closure of $\Gamma$ is semisimple, Zariski connected and with no compact factor. Consider $\Gamma$ acting on $\mathbb{T}^n$ naturally by automorphisms, then every $\Gamma$-orbit closure is a finite homogeneous union of affine submanifolds.

**Remark 2.2.6.** These affine submanifolds are defined over $\mathbb{Q}$, by which we mean they are given by some affine equations with coefficients in $\mathbb{Q}$.

**Theorem 2.2.7** (Benoist-Quint,[3]). Let $n \geq 2$, and $\Gamma$ be a subgroup of $\text{GL}(n, \mathbb{Z})$. Assume the Zariski closure of $\Gamma$ is semisimple, Zariski connected and with no compact factor, and acts irreducibly on $\mathbb{Q}^n$. Consider $\Gamma$ acting on $\mathbb{T}^n$ naturally by automorphisms, then

1. every $\Gamma$ orbit closure is either discrete (and hence finite) or $\mathbb{T}^n$. In particular, this is also true for the action of any finite index subgroup of $\Gamma$, 

(2) any sequence of distinct finite $\Gamma$ orbits has $\mathbb{T}^n$ as the only limit in the Hausdorff topology.

Remark 2.2.8. The above theorem applies when $\Gamma < SL(n, \mathbb{Z})$ is Zariski dense in $SL(n, \mathbb{R})$.

2.3 Orbit closure of $G$ action on products of $(G/\Gamma, Haar)$

Let $L$ be an arbitrary group. Consider two measure preserving systems $(L, X_1, \mu)$ and $(L, X_2, \nu)$, a joining is a measure on $X_1 \times X_2$ which is invariant under the $L$ action, and coincides with $\mu$ (respectively $\nu$) when projects to $X_1$ (respectively $X_2$). A self joining of $(L, X, \mu)$ is a joining for $(L, X, \mu)$ and $(L, X, \mu)$. In this subsection, we describe all ergodic self joinings of $G$ action on $(G/\Gamma, Haar)$.

As $G$ is generated by unipotent elements, applying Ratner rigidity Theorems, any ergodic self joining either coincides with the product Haar measure, or it reduces to a Haar measure supported on a closed $G$ invariant homogeneous submanifold. The latter is related to the elements in $Comm(\Gamma)$, and is essentially a finite extension of Haar measure on $G/\Gamma$. There are many ways to describe such self joinings. We present a description via $G$ equivariant maps.

For any $\gamma \in Comm(\Gamma)$, let $\hat{\Gamma} = \Gamma \cap \gamma \Gamma \gamma^{-1}$, we have a series of $G$ equivariant maps:

$$G/\hat{\Gamma} \hookrightarrow G/\hat{\Gamma} \times G/\hat{\Gamma} \rightarrow G/\Gamma \times G/(\gamma \Gamma \gamma^{-1}) \rightarrow G/\Gamma \times G/\Gamma$$

defined by

$$(x\hat{\Gamma}) \mapsto (x\hat{\Gamma}, x\hat{\Gamma}) \mapsto (x\Gamma, x\gamma \Gamma \gamma^{-1}) \mapsto (x\Gamma, x\gamma \Gamma).$$

Then the Haar measure on $G/\hat{\Gamma}$ will be mapped to a $G$ invariant measure on $G/\Gamma \times G/\Gamma$. We will call this self joining supported on a graph.

**Lemma 2.3.1.** For any $\gamma \in Comm(\Gamma)$, the $\Gamma$ orbit of point $\gamma \Gamma$ in $G/\Gamma$ contains finite many points. On the other hand, if $\Gamma$ orbit of a point $x \in G/\Gamma$ contains finite many points, then $x = \gamma \Gamma$ for some $\gamma \in Comm(\Gamma)$.

**Proof.** For $\gamma \in Comm(\Gamma)$, let $\hat{\Gamma} = \Gamma \cap \gamma \Gamma \gamma^{-1}$, then $\hat{\Gamma}$ is the stabilizer of $\gamma \Gamma$. 

Combine this with the fact that $[\Gamma : \hat{\Gamma}] < \infty$, we obtain the first claim. The second claim follows similarly by considering the stabilizer.

**Proposition 2.3.2.** Combining with the product Haar measure, these exhaust all ergodic self joinings on $G/\Gamma \times G/\Gamma$.

*Proof.* Let $\mu$ be an ergodic self joining on $G/\Gamma \times G/\Gamma$, and assume that $\mu \neq \text{Haar} \times \text{Haar}$. By Theorem 2.2.4, $\mu$ is a Haar measure supported on a $G$-invariant homogeneous space. Let $W$ be the support of $\mu$. Then $W \cap (\{\Gamma\} \times G/\Gamma)$ is finite. Indeed, notice that the $G$ action on $\{\Gamma\} \times G/\Gamma$ reduces to a $\Gamma$ action on $G/\Gamma$, then if $W \cap (\{\Gamma\} \times G/\Gamma)$ is not finite, by Theorem 2.2.4, the $\Gamma$ orbit must be dense, this contradicts to the finiteness of $\mu$ and $\mu \neq \text{Haar} \times \text{Haar}$.

Now by Lemma 2.3.1, there is a $\gamma \in \text{Comm}(\Gamma)$ such that $W \cap (\{\Gamma\} \times G/\Gamma) = \Gamma \circ (\Gamma, \gamma \Gamma)$. In particular, $(\Gamma, \gamma \Gamma) \in W$. From here, it is easy to see that the measure $\mu$ is supported on a graph just as what we described before. 

By Proposition 2.3.2, we have

**Corollary 2.3.3.** The orbit closure of any point will be given by the support of some ergodic self joining.

Let $(x\Gamma), (y\Gamma)$ be two points on $G/\Gamma$. Define a relation $\sim$: $(x\Gamma) \sim (y\Gamma)$ if there exists a $\gamma \in \text{Comm}(\Gamma)$ such that $x = y\gamma$. It is straightforward to see that $\sim$ is an equivalence relation.

**Theorem 2.3.4.** Let $(a_1, \ldots, a_\ell) \in (G/\Gamma)^\ell$, $\ell \geq 1$. If there is no pair $i, j$ with $i \neq j$ such that $a_i \sim a_j$, then the $G$-orbit closure of $(a_1, \ldots, a_\ell) \in (G/\Gamma)^\ell$ is $(G/\Gamma)^\ell$.

*Proof.* By induction on $\ell$. When $\ell = 1$, it is true because $G$ action is minimal. When $\ell = 2$, this is a corollary of Proposition 2.3.2.

Now assume it is true for $\ell = 1, 2, \ldots, k$, we want to prove the case that $\ell = k + 1$. Since the theorem is true for $\ell = k$, apply Ratner’s results on measure rigidity and orbit closure, the $G$-orbit closure of $(a_1, \ldots, a_{k+1})$ is algebraic. Let $H$ be an algebraic group such that

$$W := G.(a_1, \ldots, a_{k+1}) = H.(a_1, \ldots, a_{k+1}).$$
Then it follows that $G^k \subset H \subset G^{k+1}$ and $\text{vol}(H/(H \cap \Gamma^{k+1})) < \infty$. If $H = G^{k+1}$, then we are done.

Now if $H \subsetneq G^{k+1}$, then $H = G^k$. Let $\pi_k : (G/\Gamma)^{k+1} \to (G/\Gamma)^k$ be the projection map to the first $k$ coordinates. By assumption, $\pi_k(W) = (G/\Gamma)^k$. Then by algebraicity of $W$,

$$\#(W \cap \pi_k^{-1}(\bar{x})) < \infty$$

for any $\bar{x} \in (G/\Gamma)^k$. This enables us to take finite extension of $(G/\Gamma)^{k+1}$ to obtain $(G/\Gamma')^{k+1}$, such that the orbit closure of $(a_1, \ldots, a_{k+1})$ intersects the fibre built by the corresponding projection map $\pi_k'$ with exactly one point. Let $W'$ be the orbit closure. It is given by $(x, \omega(x)) \in (G/\Gamma')^k \times G/\Gamma'$ for some $G$ equivariant map $\omega : (G/\Gamma')^k \to G/\Gamma'$. In fact, $\omega$ comes from a group homomorphism from $G^k$ to $G$ such that $\omega(\Gamma^k) = \Gamma'$. From here, one have that $\omega$ maps one coordinate of $(G/\Gamma')^k$ to its image. Let it be the $i$th coordinate. Then combine $i$th and $(k + 1)$th coordinate of $(G/\Gamma')^{k+1}$, the corresponding $G$ orbit is supported on a graph in $(G/\Gamma)^2$. Therefore by Proposition 2.3.2, we have $a_i \sim a_{k+1}$, a contradiction to our assumption. This finishes the proof.

\[\square\]

### 2.4 Orbit closure of certain group actions on products of $\mathbb{T}^n$

The space of self joinings of discrete group actions on $\mathbb{T}^n$ is a little bit complicated than that of the $G$ action described in previous subsection. One reason is that there are infinitely many finite orbits on $\mathbb{T}^n$.

**Lemma 2.4.1.** Let $n \geq 2$, and $\Gamma$ be a subgroup of $GL(n, \mathbb{Z})$. Assume the Zariski closure of $\Gamma$ is semisimple, Zariski connected and with no compact factor, and acts irreducibly on $\mathbb{Q}^n$. Let $C(\Gamma) = \{ \lambda \in M(n \times n, \mathbb{Z}) : \det \lambda \neq 0, \lambda \circ \gamma = \gamma \circ \lambda, \forall \gamma \in \Gamma \}$ be the space of centralizers of $\Gamma$. Then $C(\Gamma) = \{ kI_n : k \neq 0, k \in \mathbb{Z} \}$.

**Proof.** Assume $\eta \in C(\Gamma)$. Let $H$ be the Zariski closure of $\Gamma$. Then by assumptions, $H$ is a semisimple group in $GL(n, \mathbb{R})$ and $\eta \circ h = h \circ \eta$ for any $h \in H$. Since they are matrix Lie groups, then after conjugation simultaneously, $\eta$ is a diagonal block
matrix of the diagonal form as $H$. For each simple block matrix, the corresponding $\eta$ must be a constant multiple of Identity. By the irreducibility on $\mathbb{Q}^n$, the multiplying constants for different blocks should be equal. Therefore $\eta$ is a constant multiple of $I_n$. Since $\eta \in M(n \times n, \mathbb{Z})$, $\eta = kI_n$ for some nonzero $k \in \mathbb{Z}$. 

We first consider orbit closures on product spaces. For any $r \geq 1$, we say $x_1, \cdots, x_r$ are rationally dependent, if there exists $a_1, \cdots, a_r \in \mathbb{Z}$ such that

$$\sum_{i=1}^{r} a_i x_i \in \mathbb{Q}^n / \mathbb{Z}^n;$$

otherwise, $x_1, \cdots, x_r$ are rationally independent.

**Theorem 2.4.2.** Let $n, \Gamma, C(\Gamma)$ be as in Lemma 2.4.1. Consider $\Gamma$ acting on $\mathbb{T}^n$ naturally by automorphisms. Let $x, y$ be any two points in $\mathbb{T}^n$, then exactly one of the following holds:

1. $x \in \mathbb{Q}^n / \mathbb{Z}^n$, and $y \in \mathbb{Q}^n / \mathbb{Z}^n$. The $\Gamma$ orbit closure of $(x, y)$ is discrete and hence finite;

2. only one of $x, y$ is in $\mathbb{Q}^n / \mathbb{Z}^n$. The $\Gamma$ orbit closure of $(x, y)$ is a direct product of a finite orbit with $\mathbb{T}^n$;

3. $x, y$ are rationally dependent. The $\Gamma$ orbit closure of $(x, y)$ is a finite union of rational translations of $(\phi_{\lambda, \theta})(\mathbb{T}^n)$ for some $\lambda, \theta \in C(\Gamma)$, where $\phi_{\lambda, \theta} : \mathbb{T}^n \to \mathbb{T}^n \times \mathbb{T}^n$ is defined by $\phi_{\lambda, \theta}(x) = (\lambda x, \theta x)$;

4. $x, y$ are rationally independent. The $\Gamma$ orbit closure of $(x, y)$ is $\mathbb{T}^n \times \mathbb{T}^n$.

**Proof.** By Theorem 2.2.5, it is known that the $\Gamma$ orbit closure of $(x, y)$ is a finite union of affine manifold. By replacing $\Gamma$ by its finite index subgroup $\Gamma'$, we have that the $\Gamma'$ orbit closure of $(x, y)$ is an affine manifold. The cases (1) and (2) are straightforward.

Now we turn to (3) first. When $x, y$ are rationally dependent, then there is a $z \in \mathbb{T}^n$ such that $x = az$ and $y = bz + q_1$ where $a, b \in \mathbb{Z}$ with $(a, b) = 1$ and $q_1 \in \mathbb{Q}^n / \mathbb{Z}^n$. Then the $\Gamma$ orbit closure of $(x, y)$ reduces to $\Gamma'$ orbit closure of $(az, bz)$. As $z$ is not a rational point, it is easy to see that the latter is $(\phi_{\lambda, \theta})(\mathbb{T}^n)$ with $\lambda = aI_n$ and $\theta = bI_n.$
When \( x, y \) are rationally independent, the orbit closure is the product space, since there is no \( \Gamma \)-invariant affine submanifold containing \( (x, y) \). This yields (4).

\[ \square \]

**Corollary 2.4.3.** Let \( n, C(\Gamma) \) be as in Lemma 2.4.1. Consider \( \Gamma \) acting on \( \mathbb{T}^n \) naturally by automorphisms with the Lebesgue measure \( m \), then there are 2 types of ergodic self joinings of this action:

1. \( m \times m \), the product of Lebesgue measures on \( \mathbb{T}^n \times \mathbb{T}^n \);
2. average of finitely many translations of \( (\phi_{\lambda, \theta})_*(m) \), where \( \lambda, \theta \in C(GL(n, \mathbb{Z})) \), \( \phi_{\lambda, \theta} : \mathbb{T}^n \rightarrow \mathbb{T}^n \times \mathbb{T}^n \) is defined by \( \phi_{\lambda, \theta}(x) = (\lambda x, \theta x) \).

**Theorem 2.4.4.** Let \( n, \Gamma, C(\Gamma) \) be as in Lemma 2.4.1. Consider \( \Gamma \) acting on \( \mathbb{T}^n \) naturally by automorphisms. For any \( k \), if \( x_1, x_2, \ldots, x_k \) are rationally independent, then the orbit closure of \( (x_1, x_2, \ldots, x_k) \) is \( (\mathbb{T}^n)^k \).

**Proof.** It follows from Theorem 2.2.7 and the fact that there is no invariant affine submanifold containing the point \( (x_1, x_2, \ldots, x_k) \), when \( x_1, x_2, \ldots, x_k \) are rationally independent. \( \square \)

### 2.5 Proof of Theorem 2.1.3

Let \( A \) be an arbitrary infinite subset.

Let \( (x\Gamma), (y\Gamma) \) be two points on \( G/\Gamma \). Consider the equivalence relation \( \sim : (x\Gamma) \sim (y\Gamma) \) if there exists a \( \gamma \in Comm(\Gamma) \) such that \( (x\Gamma) = (y\gamma\Gamma) \). Notice that by Theorem 2.3.4, only if \( (x\Gamma) \sim (y\Gamma) \), the orbit closure of \( (x\Gamma, y\Gamma) \) under \( G \) will be a graph as described before. Now, we can partite \( A \) into subsets \( \{A_1, A_2, \ldots, A_i, \ldots\} \), such that each \( A_i \) contains points in one equivalence class.

If \( Card(I) = \infty \), then we can get an infinite subset \( \hat{A} \subset A \) by simply choosing one point from each subset, say choose \( a_i \in A_i \). For any \( \ell > 0 \), the orbit closure of \( (a_1, \ldots, a_\ell) \in (G/\Gamma)^\ell \) is \( (G/\Gamma)^\ell \). Now for any \( \epsilon > 0 \), let \( \ell \) be great enough, then there exists \( g \in G \) such that the subset \( g\{a_1, \ldots, a_\ell\} = \{ga_1, \ldots, ga_\ell\} \) is \( \epsilon \)-dense. Therefore the set \( g(A) \) is also \( \epsilon \)-dense. We are done in this case. Let’s remark that if \( \Gamma \) is not arithmetic, then \( Card(I) = \infty \).
If \( \text{Card}(I) < \infty \), since \( A \) is an infinite subset, there exists \( i \in I \) such that \( A_i \) also contains infinite many points. Thus without loss of generality, afterwards assume \( A \) contains points in one equivalence class. As \( G \) acts transitively on \( G/\Gamma \), assume that \( A = \{(\Gamma), (\gamma_1 \Gamma), \ldots.\} \) where \( \gamma_i \in \text{Comm}(\Gamma) \), and the point \((\Gamma) \in G/\Gamma \) is the only accumulating point of \( A \).

**Lemma 2.5.1.** For any \( \ell > 0 \), any open subset \( U \subset (G/\Gamma)^\ell \), there exist a \( g \in G \) and \((b_1 \Gamma, \ldots, b_\ell \Gamma) \in (G/\Gamma)^\ell \) with \((b_i \Gamma) \in A \), such that \( g(b_1 \Gamma, \ldots, b_{\ell-1} \Gamma) \in U \).

**Proof.** It suffices to prove the case when \( U = U_1 \times \cdots \times U_\ell \), where \( U_i \subset G/\Gamma \) is an open subset. We prove this by induction on \( \ell \).

When \( \ell = 1 \), since \( G \) action on \( G/\Gamma \) is minimal, any point in \( A \) works.

Assume that when \( \ell = k-1 \geq 1 \), the lemma is true. Now we prove it for \( \ell = k \).

Let \( U = U_1 \times \cdots \times U_k \) be an arbitrary open set. Apply the case \( \ell = k-1 \) for the first \( k-1 \) product \( U_1 \times \cdots \times U_{k-1} \), we thus obtain \((b_1 \Gamma, \ldots, b_{k-1} \Gamma) \in (G/\Gamma)^{k-1} \) with \((b_i \Gamma) \in A \), and \( g_0(b_1 \Gamma, \ldots, b_{k-1} \Gamma) \in U_1 \times \cdots \times U_{k-1} \) for some \( g_0 \in G \). Notice that the \( G \) orbit closure of \((b_1 \Gamma, \ldots, b_{k-1} \Gamma) \) is essentially a homogeneous \( G \)-space, then the stabilizer of the point \( g_0(b_1 \Gamma, \ldots, b_{k-1} \Gamma) \) is a discrete group \( g_0 \Gamma_{k-1} g_0^{-1} \), where \( \Gamma_{k-1} \) is a finite index subgroup of \( \Gamma \). Here \( g_0 \Gamma_{k-1} g_0^{-1} \) is still a cocompact lattice.

Let \( A_k := A \setminus \{b_i \Gamma : 1 \leq i \leq k-1\} \), then \( \text{Card}(A_k) = \infty \). By Theorem 2.2.4, it follows that there is a \((b_k \Gamma) \in A_k \) such that

\[
(g_0 \Gamma_{k-1} g_0^{-1}(g_0 b_k \Gamma)) \cap U_k \neq \emptyset.
\]

That is there is an element \( g_1 \in g_0 \Gamma_{k-1} g_0^{-1} \) such that \( g_1 g_0 b_k \Gamma \in U_k \). Hence we have

\[
g_1 g_0(b_1 \Gamma, \ldots, b_{k-1} \Gamma, b_k \Gamma) \in U_1 \times \cdots \times U_{k-1} \times U_k = U,
\]

which completes the induction. \( \square \)

We continue the proof of Theorem 2.1.3. Let \( \pi_\ell \) be the map from \((G/\Gamma)^\ell \) to \( \mathcal{K}(G/\Gamma) \), the space of subsets of \( G/\Gamma \), defined by \( \pi_\ell(x_1, \ldots, x_\ell) = \{x_1, \ldots, x_\ell\} \). Observe that for any \( \epsilon > 0 \), as \( \ell \) large enough, there is an open subset \( U \subset (G/\Gamma)^\ell \) such that \( \pi_\ell(\bar{x}) \) is \( \epsilon \)-dense for any \( \bar{x} \in U \). Therefore applying Lemma 2.5.1, Theorem 2.1.3 follows.
2.6 Proof of Theorem 2.1.4

The proof is similar to that of Theorem 2.1.3. However, since the orbit closure is quite involved, the argument is much more complicated.

Let \( A \) be an arbitrary infinite subset of \( \mathbb{T}^n \). Without loss of generality, assume \( A \) is countable, and denote \( A = \{a_1, \cdots, a_i, \cdots\} \). For any \( \ell \geq 1 \), let \( d_\ell \) be the dimension of the linear \( \mathbb{Q} \)-spanning space of \( \{a_i, \cdots, a_\ell\} \). If for all \( 1 \leq i \leq \ell \), \( a_i \in \mathbb{Q}^n/\mathbb{Z}^n \), then \( d_\ell = 0 \). Note that \( d_\ell \) is increasing if \( \ell \) increases. Therefore the limit \( \lim_{\ell \to \infty} d_\ell \) exists (possibly \( \infty \)). Let \( r = r(A) = \lim_{\ell \to \infty} d_\ell \), we have \( r \in \mathbb{N} \cup \{0, \infty\} \).

We split the proof in the following three cases.

**Case 1:** \( r = \infty \). Then for any \( \ell \geq 1 \), one can pick a subset \( \{b_1, \cdots, b_\ell\} \) from \( A \), such that the points \( b_1, \cdots, b_\ell \) are rationally independent. By Theorem 2.4.4, the \( \Gamma \) orbit closure of \( (b_1, \cdots, b_\ell) \) is \( (\mathbb{T}^n)^\ell \). Therefore, for any \( \epsilon > 0 \), one can choose \( \ell \) large enough and the points \( b_1, \cdots, b_\ell \) from \( A \) such that, there is a \( \gamma \in \Gamma \) with the property that the set \( \gamma \{b_1, \cdots, b_\ell\} \) is \( \epsilon \)-dense. We are done.

**Case 2:** \( r = 0 \). In this case \( A \subset \mathbb{Q}^n/\mathbb{Z}^n \). We will need the following useful result.

**Lemma 2.6.1.** For any \( \ell > 0 \), any open subset \( U \subset (\mathbb{T}^n)^\ell \), there exist a \( g \in \Gamma \) and \( (b_1, \cdots, b_\ell) \in (\mathbb{T}^n)^\ell \) with \( (b_i) \in A \), such that \( g(b_1, \cdots, b_\ell) \in U \).

**Proof.** The proof is similar to that of Lemma 2.5.1. It suffices to prove the case when \( U = U_1 \times \cdots \times U_\ell \), where \( U_i \subset \mathbb{T}^n \) is an open subset. We prove this by induction on \( \ell \).

When \( \ell = 1 \), since the orbit of any point in \( A \) is finite and \( \text{Card}(A) = \infty \), by Theorem 2.2.7, there is a \( \Gamma \) orbit that intersects the fixed \( U_1 \). Therefore, one can pick this point and find an element of \( \Gamma \), satisfying the lemma.

Assume that when \( \ell = k - 1 \geq 1 \), the lemma is true. Now we prove it for \( \ell = k \). Let \( U = U_1 \times \cdots \times U_k \) be an arbitrary open set. Apply the case \( \ell = k - 1 \) for the first \( k - 1 \) product \( U_1 \times \cdots \times U_{k-1} \), we thus obtain \( (b_1, \ldots, b_{k-1}) \in (\mathbb{T}^n)^{k-1} \) with \( b_i \in A \), and \( g_0(b_1, \ldots, b_{k-1}) \in U_1 \times \cdots \times U_{k-1} \) for some \( g_0 \in \Gamma \). Notice that the \( \Gamma \) orbit closure of \( (b_1, \ldots, b_{k-1}) \) is finite, then the stabilizer of the point \( g_0(b_1, \ldots, b_{k-1}) \) is a discrete group \( g_0 \Gamma_{k-1} g_0^{-1} \), where \( \Gamma_{k-1} \) is a finite index subgroup of \( \Gamma \). Here \( g_0 \Gamma_{k-1} g_0^{-1} \) is still a subgroup of \( \Gamma \).
Let \( \mathcal{A}_k := A \setminus \{b_i : 1 \leq i \leq k-1\} \), then Card(\( \mathcal{A}_k \)) = \( \infty \). Since \( U_k \) is an open set, by Theorem 2.2.7, it follows that there is a \( b_k \in \mathcal{A}_k \) such that
\[
(g_0\Gamma_{k-1}g_0^{-1}(g_0b_k)) \cap U_k \neq \emptyset.
\]
That is there is an element \( g_1 \in g_0\Gamma_{k-1}g_0^{-1} \) such that \( g_1g_0b_k \in U_k \). Hence we have
\[
g_1g_0 \in \Gamma, \text{ and } g_1g_0(b_1, \ldots, b_{k-1}, b_k) \in U_1 \times \cdots \times U_{k-1} \times U_k = U,
\]
which completes the induction. \( \square \)

Let \( \pi_\ell \) be the map from \((\mathbb{T}^n)^\ell \) to \( \mathcal{K}(\mathbb{T}^n) \), the space of subsets of \( \mathbb{T}^n \), defined by \( \pi_\ell(z_1, \ldots, z_\ell) = \{z_1, \ldots, z_\ell\} \). For any \( \epsilon > 0 \), let \( \ell \) be large enough, then there exists an open subset \( U \subset (\mathbb{T}^n)^\ell \), such that the subset \( \pi_\ell(z_1, \ldots, z_\ell) \) is \( \epsilon \)-dense for any \( (z_1, \ldots, z_\ell) \in U \). By applying Lemma 2.6.1 with the \( \ell \) and \( U \), we are done.

**Case 3**: \( 1 \leq r < \infty \). One can pick a subset \( \{z_1, \cdots, z_r\} \) of \( r \) elements from \( A \) such that \( z_1, \cdots, z_r \) are rationally independent and any other point in \( A \) is a \( \mathbb{Q} \) combination of \( z_1, \cdots, z_r \) and \( \mathbb{Q}^n/\mathbb{Z}^n \). Without loss of generality, assume that \( \{z_1, \cdots, z_r\} = \{a_1, \cdots, a_r\} \), and let \( \mathcal{A}_r = A \setminus \{a_1, \cdots, a_r\} \). Denote \( \mathbf{a} = (a_1, \cdots, a_r) \), then we can rewrite \( a_i \) as \( q_i^0 + \langle \mathbf{q}_i, \mathbf{a} \rangle := q_i^0 + \sum_{j=1}^r q_i^ja_j \), where \( q_i^0 \in \mathbb{Q}^n/\mathbb{Z}^n \) and \( \mathbf{q}_i = (q_i^1, \cdots, q_i^r) \in \mathbb{Q}^r \).

If \( \mathcal{A}_r \cap \mathbb{Q}^n/\mathbb{Z}^n \) is infinite, then we can play the game as in **Case 2** and obtain the proof. On the other hand, if \( \mathcal{A}_r \cap \mathbb{Q}^n/\mathbb{Z}^n \) is finite, we may remove the finitely many rational points which will not affect our result. Therefore, we assume afterwards that \( \mathcal{A}_r \cap \mathbb{Q}^n/\mathbb{Z}^n = \emptyset \). We assume also that \( \{q_i\}_{i \in \mathbb{N}} \) does not intersect any \( \mathbb{Q} \)-hyperplane \( q_0 + \mathbb{Q}^{r-1}(q_0 \in \mathbb{Q}^n) \) with infinitely many points. Otherwise, we may get a case of \( r-1 \), from where we can start over again.

**Lemma 2.6.2.** For any positive integer \( \ell \), and \( (b_1, \ldots, b_\ell) \in (\mathbb{T}^n)^\ell \) with \( b_j \in \mathcal{A}_r \) for \( 1 \leq j \leq \ell \), then the \( \Gamma \) orbit closure of \((a_1, \ldots, a_r, b_1, \ldots, b_\ell) \) in \((\mathbb{T}^n)^{r+\ell}\) is a finite union of affine manifolds, and each one of the affine manifolds is the image of an affine map from \((\mathbb{T}^n)^r\) to \((\mathbb{T}^n)^{r+\ell}\). In particular, the dimension of the affine manifold is \( nr \).

**Proof.** This follows from Theorem 2.2.5, Theorem 2.4.4 and the assumption on \( a_1, \ldots, a_r \) and \( \mathcal{A}_r \). \( \square \)
We now describe the affine map appeared above. Consider the point $a_k = q_k^0 + \langle q_k, a \rangle$, let $q_k^j = \frac{s^j_k}{t^j_k}$, with $s^j_k, t^j_k \in \mathbb{Z}$ and $(s^j_k, t^j_k) = 1$, $t^j_k \geq 1$. If $q_k^j = 0$, then set $s^j_k = 0$, $t^j_k = 1$. Then the affine map $\phi_h : (\mathbb{T}^n)^r \to (\mathbb{T}^n)^r \times \mathbb{T}^n$ is defined by

$$
\phi_h(x_1, \ldots, x_r) = (t^1_k x_1, \ldots, t^r_k x_r, h + \sum_{j=1}^{r} s^j_k x_j),
$$

where $h \in \{\Gamma.q_k^0\}$. The corresponding orbit closure of $(a_1, \ldots, a_r, a_k)$ is given by

$$
\bigcup_{h \in \{\Gamma.q_k^0\}} \phi_h((\mathbb{T}^n)^r).
$$

Next, if there is another point $a_l = q_l^0 + \langle q_l, a \rangle$ with $q_l^j = \frac{s^j_l}{t^j_l}$. Then the affine map is defined by

$$
\phi_h(x_1, \ldots, x_r) = (t^1_l x_1, \ldots, t^r_l x_r, h + \sum_{j=1}^{r} \bar{s}^j_k x_j + \sum_{j=1}^{r} \bar{s}^j_l x_j),
$$

where $h \in \{\Gamma.(q_k^0, q_l^0)\} \subset (\mathbb{T}^n)^2$, $t_j = \frac{s^j_k t^j_l}{(t^j_k, t^j_l)}$, $\bar{s}^j_k = \frac{s^j_k t^j_l}{(t^j_k, t^j_l)}$ and $\bar{s}^j_l = \frac{s^j_l t^j_k}{(t^j_k, t^j_l)}$. The corresponding orbit closure of $(a_1, \ldots, a_r, a_k, a_l)$ is given by

$$
\bigcup_{h \in \{\Gamma.(q_k^0, q_l^0)\}} \phi_h((\mathbb{T}^n)^r).
$$

One can define similarly for the case when $\ell \geq 3$, which is even more complicated. We choose not to do the cumbersome work here but hope the construction is clear enough.

**Lemma 2.6.3.** If $B \subset \mathcal{A}_r$ is an infinite subset, then for any open subset $V \subset (\mathbb{T}^n)^r$ and any open subset $U \subset \mathbb{T}^n$, there exist two points $\hat{b}, \bar{b} \in B$ such that

- the orbit closure of $(a_1, \ldots, a_r, \hat{b}, \bar{b})$ has non empty intersection with $(\mathbb{T}^n)^r \times U \times \mathbb{T}^n$;

- the preimage of the intersection under the affine map has non empty intersection with $V$.  

Proof. Since by assumption that $A_r \cap \mathbb{Q}^n / \mathbb{Z}^n = \emptyset$, $B$ contains only irrational points. Pick any one of them, say $a_k = q^0_k + (q_k, a) \notin \mathbb{Q}^n / \mathbb{Z}^n$. Then by Lemma 2.6.2, the orbit closure of $(a_1, \ldots, a_r, a_k)$ is a graph defined by some affine map $\phi : (\mathbb{T}^n)^r \to (\mathbb{T}^n)^r \times \mathbb{T}^n$, and must have nontrivial intersection with $(\mathbb{T}^n)^r \times U$ since the $\Gamma$ orbit closure of $a_k$ is $\mathbb{T}^n$. This intersection is open in the orbit closure because $U$ is open.

Now have the construction of affine maps in mind, the second assertion is equivalent to: for some $a_k$, there is a $a_l$ such that

$$\exists (x_1, \ldots, x_r) \in V, \text{ such that } \bigcup_{h \in \{\Gamma.(q_k^0,q^0_l)\}} \{h + (\sum_{j=1}^{r} s^j_k x_j, \sum_{j=1}^{r} s^j_l x_j)\} \subset U \times \mathbb{T}^n.$$ 

As $V$ is open, this is true when $\max_j \{|s^j_k|\}$ is large enough. By assumption (3), since $B$ is an infinite subset, we can choose an $a_l$ so that some $t^j_l$ is large enough (so $|s^j_k| = \frac{|s^j_l|}{(t^j_k,t^j_l)}$ is large enough). The proof is complete by making $\hat{b} = a_k$ and $\bar{b} = a_l$. \hfill \Box

**Lemma 2.6.4.** For any positive integer $\ell$, and any open subset $U \subset (\mathbb{T}^n)^\ell$, there exist $(b_1, \ldots, b_\ell) \in (\mathbb{T}^n)^\ell$ and $(c_1, \ldots, c_\ell) \in (\mathbb{T}^n)^\ell$ with $b_j, c_j \in A_r$ for $1 \leq j \leq \ell$, such that

- the $\Gamma$ orbit closure of $(a_1, \ldots, a_r, b_1, \ldots, b_\ell, c_1, \ldots, c_\ell)$ in $(\mathbb{T}^n)^{r+2\ell}$ has non empty intersection with $(\mathbb{T}^n)^r \times U \times (\mathbb{T}^n)^\ell$;

- the intersection is open when restricted in the orbit closure (affine submanifold).

Proof. Firstly, note that by the assumption on $a_1, \ldots, a_r$, the $\Gamma$ orbit closure of $(a_1, \cdots, a_r)$ is $(\mathbb{T}^n)^r$. Next, as in the previous two lemmas, it suffices to prove the case when $U = U_1 \times \cdots \times U_\ell$, where $U_i \subset \mathbb{T}^n$ is an open subset. We prove this by induction on $\ell$.

When $\ell = 1$, this is the content of Lemma 2.6.3.

Assume that when $\ell = k - 1 \geq 1$, the lemma is true. Now we prove it for $\ell = k$. Let $U = U_1 \times \cdots \times U_k$ be an arbitrary open set. Apply the case $\ell = k - 1$ for the first $k - 1$ product $U_1 \times \cdots \times U_{k-1}$, and let $W$ be the intersection resulted. By Lemma 2.6.2, $W$ is the intersection of the image of an affine map
with $(\mathbb{T}^n)^r \times U_1 \times \cdots \times U_{k-1} \times (\mathbb{T}^n)^{k-1}$. Let $V \subset (\mathbb{T}^n)^r$ be the preimage. Since $W$ is open in the orbit closure, it follows that $V$ is an open set of $(\mathbb{T}^n)^r$. Now apply Lemma 2.6.3 for $V$, $U_k$ and $B = \mathcal{A}_r \setminus \{b_1, \ldots, b_{k-1}, c_1, \ldots, c_{k-1}\}$, we have two points $\hat{b}$ and $\bar{b}$ satisfying that

- the orbit closure of $(a_1, \ldots, a_r, \hat{b}, \bar{b})$ has non empty intersection with $(\mathbb{T}^n)^r \times U_k \times \mathbb{T}^n$;
- the preimage of the intersection under the affine map has non empty intersection with $V$.

Let $b_k = \hat{b}$ and $c_k = \bar{b}$, then $(b_1, \ldots, b_k)$ and $(c_1, \ldots, c_k)$ satisfies the lemma. Hence the induction is complete and the proof is done.

Continue the proof of Case 3. Let $\pi_\ell$ be the map from $(\mathbb{T}^n)^\ell$ to $\mathcal{K}(\mathbb{T}^n)$, the space of subsets of $\mathbb{T}^n$, defined by $\pi_\ell(z_1, \ldots, z_\ell) = \{z_1, \ldots, z_\ell\}$. For any $\epsilon > 0$, let $\ell$ be large enough, then there exists an open subset $U \subset (\mathbb{T}^n)^\ell$, such that the subset $\pi_\ell(z_1, \ldots, z_\ell)$ is $\epsilon$-dense for any $(z_1, \ldots, z_\ell) \in U$. Apply Lemma 2.6.4 with the $\ell$ and $U$, there exists a $\gamma \in \Gamma$ such that $\gamma A$ is $\epsilon$-dense. The proof is complete.
Chapter 3

Rigidity of a class of smooth singular flows on $\mathbb{T}^2$

In this chapter, we study joining rigidity in the class of von Neumann flows with one singularity. They are given by a smooth vector field $\mathcal{X}$ on $\mathbb{T}^2 \setminus \{a\}$, where $\mathcal{X}$ is not defined at $a \in \mathbb{T}^2$. The phase space can be decomposed into a (topological disc) $D_\mathcal{X}$ and an ergodic component $E_\mathcal{X} = \mathbb{T}^2 \setminus D_\mathcal{X}$. Let $\omega_\mathcal{X}$ be the 1-form associated to $\mathcal{X}$. We show that if $|\int_{E_{\mathcal{X}_1}} d\omega_{\mathcal{X}_1}| \neq |\int_{E_{\mathcal{X}_2}} d\omega_{\mathcal{X}_2}|$, then the corresponding flows $(v_{t_1}^{\mathcal{X}_1})$ and $(v_{t_2}^{\mathcal{X}_2})$ are disjoint. Moreover, for every $\mathcal{X}$ there is a uniquely associated frequency $\alpha \in \mathbb{T}$ and it follows that for a full measure set of $\alpha \in \mathbb{T}$ the class of smooth time changes of $(v_{t}^{\mathcal{X}_\alpha})$ is joining rigid, i.e. every two smooth time changes are either cohomologous or disjoint. This gives a natural class of flows for which the answer to Problem 3 in [29] is positive.

3.1 Introduction

This paper deals with disjointness properties of von Neumann flows. Von Neumann flows were introduced in [32] as the first systems with continuous spectrum (weakly mixing systems). They are given by a smooth vector field $\mathcal{X}$ on $\mathbb{T}^2 \setminus \{a_1, \ldots, a_k\}$, where the vector field is not defined (singular) at $a_i$, $i = 1, \ldots, k$. We will be interested in the situation where $\mathcal{X}$ has just one singularity $a \in \mathbb{T}^2$. More precisely (see e.g. [10]), let $p : \mathbb{T}^2 \to \mathbb{R}$ be a $C^\infty$, positive function on $\mathbb{T}^2 \setminus \{a\}$ and $p(a) = 0$. The vector field $\mathcal{X}$ is given by $\mathcal{X} := \frac{\mathcal{X}_H}{p(a)}$, where $\mathcal{X}_H$ is a Hamiltonian vector field.
(generating a smooth flow \((h_t)\)), \(X_H = \left( \frac{\partial H}{\partial y}, -\frac{\partial H}{\partial x} \right)\), where \(H : \mathbb{R}^2 \to \mathbb{R}\) is 1-periodic and \(a \in \mathbb{T}^2\) is (the only) critical point for \(H\) (on \(\mathbb{T}^2\)). Then the von Neumann flow \((v_t^X)\) is given by the solution of

\[
\frac{d\bar{x}}{dt} = X(\bar{x}).
\]

Notice that the orbits of \((v_t^X)\) and \((h_t)\) are the same (modulo the fixed point of \((h_t)\)). Therefore, by [2], it follows that phase space decomposes into one region \(D_X\) (homeomorphic to the disc) filled with periodic orbits and an ergodic component \(E_X = \mathbb{T}^2 \setminus D_X\). Let \(\omega_X(Y) = \frac{\langle X,Y \rangle}{\langle X,X \rangle}\) (notice that \(\omega_X\) is \(C^\infty(\mathbb{T}^2 \setminus \{a\})\)). Our main theorem is the following:

**Theorem 3.1.1.** Let \((v_t^{X_1})\) and \((v_t^{X_2})\) be such that \(|\int_{E_{X_1}} d\omega_{X_1}| \neq |\int_{E_{X_2}} d\omega_{X_2}|\). Then \((v_t^{X_1})\) and \((v_t^{X_2})\) are disjoint.

An important consequence of Theorem 3.1.1 is related to *joining rigidity* of *time changes*. Recall that if \((\phi_t)\) is a flow on \(M\) generated by a vector field \(Z_\phi\) and \(\tau : M \to \mathbb{R}_{>0}\), then the time changed flow \((\phi_t^\tau)\) is generated by the vector field \(\tau(\cdot)Z_\phi\). In [29], M. Ratner established strong rigidity phenomena for \(C^1\) time changes of horocycle flows. Namely, Ratner showed that if \(\tau_1\) and \(\tau_2\) are time changes of \((h_t^1)\) and \((h_t^2)\) acting respectively on \(SL(2,\mathbb{R})\setminus \Gamma\) and \(SL(2,\mathbb{R})\setminus \Gamma'\), then either the time changed flows \((h_t^1)\) and \((h_t^2)\) are disjoint or \(\tau_1\) and \(\tau_2\) are *jointly cohomologous* (see Definition 2 in [29]). Moreover, M. Ratner posed a problem (see Problem 3 in [29]) asking whether there are other classes of measure preserving flows for which the class of smooth functions is *joining rigid*, i.e. any joining between any smooth time changes is of algebraic nature (Definition 2 in [29]).

Recall that the only natural class beyond horocycle flows for which the class of smooth functions is joining rigid is the class of linear flows on \(\mathbb{T}^2\) with diophantine frequencies. Indeed, it follows by [22] that every two smooth time changes are cohomologous. In this case however, the first part of the alternative (disjointness) can never be observed. Theorem 3.1.1 gives an answer to Ratner’s problem in a strong sense, moreover one can observe non-trivial joining rigidity phenomena (both cases, i.e. disjointness and cohomology are realizable). Namely we have the following corollary:
Corollary 3.1.2. There exists a full measure set $D \subset \mathbb{T}$ such that for every $\alpha \in D$ the flow\(^1\) $(v_1^{X_\alpha})$ is (strongly) joining rigid; i.e. for any $\psi, \phi \in C^\infty(\mathbb{T}^2)$ with $\int_{\mathbb{T}^2} \psi = \int_{\mathbb{T}^2} \phi$, either $\psi$ and $\phi$ are cohomologous\(^2\), or the time changed flows $(v_1^{X_\alpha, \psi})$ and $(v_1^{X_\alpha, \phi})$ are disjoint.

We will give a proof of Corollary 3.1.2 in Section 3.5. Moreover, Theorem 3.1.1 has another consequence related to Sarnak’s conjecture on Möbius disjointness. We say that a continuous flow $(T_1) : (Z, d) \to (Z, d)$ is Möbius disjoint, if for every $f \in C(Z)$, every $x \in X$ and every $t_0 \in \mathbb{R}$, we have
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n \leq N} \mu(n) f(T_{nt_0}x) = 0, \tag{3.1}
\]
where $\mu$ denotes the Möbius function\(^3\).

Corollary 3.1.3. For every $\alpha \in \mathbb{T}$ and every $p, q > 0$ with $p \neq q$, the flows $(v_{pt}^{X_\alpha})$ and $(v_{qt}^{X_\alpha})$ are disjoint. Therefore, by [7], every uniquely ergodic topological model of $(v_1^{X_\alpha})$ is Möbius disjoint and moreover the convergence (3.1) is uniform in $x$.

We will give a proof of Corollary 3.1.3 in Section 3.5.

It turns out that von Neumann flows (with one singularity) can be represented (on the ergodic component) as special flows over irrational rotations and roof functions of bounded variation which are absolutely continuous except one point at which there is a jump discontinuity, which comes from the singularity of the vector field $X$. More precisely, let $R_\alpha x = x + \alpha \mod 1$ and let $f : \mathbb{T} \to \mathbb{R}_+$ be given by
\[
f(x) = A_f \{x\} + f_{ac}(x),
\]
where $f_{ac} \in C^1(\mathbb{T})$. Then the von Neumann flow $(v_1^{X_\alpha})$ is isomorphic to the special flow $(T_1^{R_{\alpha, f}})$, where $A_f := \int_{E_{X_\alpha}} d\omega_{X_\alpha}$. Using the language of special flows, Theorem 3.1.1 is a straightforward consequence of the following theorem:

Theorem 3.1.4. If $|A_f| \neq |A_g|$, then the flows $\mathcal{T} = (R_{t}^{\alpha, f})$ and $\mathcal{R} = (R_{t}^{\beta, g})$ are disjoint.

---

\(^1\)On the ergodic component $E_{X_\alpha}$.

\(^2\)This is equivalent to $\int_{\partial D_x} \psi(v_1^{X_\alpha})dt = \int_{\partial D_x} \phi(v_1^{X_\alpha})dt$.

\(^3\)\(\mu(n) = (-1)^k\), if $n$ is a product of $k$ distinct primes and $\mu(n) = 0$ otherwise.
Notice that there are no assumptions on the irrationals $\alpha, \beta \in \mathbb{T}$ in Theorem 3.1.4. The statistical orbit growth of von Neumann flows is linear and hence they exhibit features both from elliptic and parabolic paradigm. On the one hand they are never mixing, [18], and have singular maximal spectral type. On the other hand as shown in [9] and [11], if $\alpha$ is of bounded type, they are mildly mixing. Moreover, from [15] it follows that they are never of finite rank, in particular they don’t have fast approximation property, [20]. Our methods rely on the parabolic features of von Neumann flows. On of the main ingredients in the proof is a variant of parabolic disjointness criterion introduced in [16].

**Plan of the paper.** In Section 3.2 we recall basic definitions. In Section 3.3 we recall a variant of disjointness criterion from [16]. Section 3.4 is devoted for the proof of Theorem 3.1.4, which we divide in two subsections (Subsections 3.4.1 and 3.4.2) depending on the diophantine type of $\alpha$ and $\beta$. Finally, in Section 3.5 we give proofs of Corollaries 3.1.2 and 3.1.3.

### 3.2 Definitions and notations

#### 3.2.1 Time changes of flows

Let $(T_t)$ be a flow on $(Z, \mathcal{D}, \kappa)$ and let $v \in L^1(Z, \mathcal{D}, \kappa)$ be a positive function. Then the time change of $(T_t)$ along $v$ is given by

$$T^v_t(x) = T_{u(t,x)}(x),$$

where $u : Z \times \mathbb{R} \to \mathbb{R}$ is the unique solution to

$$\int_0^{u(t,x)} \tau(T_s x) ds = t.$$

Note that the function $u = u(t, x)$ satisfies the cocycle identity: $u(t_1 + t_2, x) = u(t_1, x) + u(t_2, T^v_{t_1} x)$. The new flow $(T^v_t)$ has the same orbits as the original flow. We say that $\psi, \phi \in L^1(Z, \mathcal{D}, \kappa)$ are cohomologous if there exists $\xi \in L^1(Z, \mathcal{D}, \kappa)$
such that for every $t \in \mathbb{R}$
\[
\int_0^t \psi(T_s x) - \phi(T_s x) ds = \xi(x) - \xi(T_t x).
\]

It follows that if $\psi, \phi$ are cohomologous, then the flows $(T^\psi_t)$ and $(T^\phi_t)$ are isomorphic.

### 3.2.2 Disjointness, special flows

Let $(T_t) : (X, \mathcal{B}, \lambda) \to (X, \mathcal{B}, \lambda)$ and $(S_t) : (Y, \mathcal{C}, \nu) \to (Y, \mathcal{C}, \nu)$ be two ergodic flows. A *joining* between $(T_t)$ and $(S_t)$ is any $(T_t \times S_t)$ invariant probability measure on $X \times Y$ such that
\[
\rho(A \times Y) = \lambda(A) \quad \text{and} \quad \rho(X \times B) = \nu(B).
\]

We denote the set of joinings by $J((T_t), (S_t))$. We say that $(T_t)$ and $(S_t)$ are disjoint (denoted by $(T_t) \perp (S_t)$), if $J((T_t), (S_t)) = \{\lambda \otimes \nu\}$.

We will be interested in disjointness in the class of special flows over irrational rotations. For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, let $[0.a_1, a_2, ...]$ denote the continued fraction expansion of $\alpha$ and let $(q_n)$ denote the sequence of denominators, i.e.
\[
q_{n+1} = a_n q_n + q_{n-1}, \quad \text{with } q_0 = q_1 = 1.
\]

We say that $\alpha$ is of *bounded type* if $\sup_{n \in \mathbb{N}} a_n < M$ for some constant $M > 0$ (equivalently, if $q_{n+1} \leq C_\alpha q_n$ for every $n \in \mathbb{N}$), otherwise we say of *unbounded type*.

The following set will be important in the proof of Corollary 3.1.2. Let
\[
DC := \bigcup_{\tau > 0} DC(\tau) \subset \mathbb{T}, \tag{3.2}
\]
where
\[
DC(\tau) := \left\{ \alpha \in \mathbb{T} : \left| \alpha - \frac{p}{q} \right| > \frac{C_\alpha}{q^\tau}, \text{ for every } p, q \in \mathbb{N} \right\}.
\]

Let $R_\alpha(x) = x + \alpha \mod 1$ and $f \in L^1(\mathbb{T}, \mathcal{B}, \lambda)$. We define the $\mathbb{Z}$-cocycle given
by
\[ f^{(n)}(x) = \sum_{i=0}^{n-1} f(R_i^nx) \] and
\[ f^{(-n)}(x) = -f^{(n)}(R_i^{-n}x), \]
for \( n \geq 1 \) and we set \( f^{(0)}(x) = 0 \). Let \( T^f := \{(x, s) : 0 \leq s < f(x)\} \) and let \( \mathcal{B}^f \) and \( \lambda^f \) denote respectively the \( \sigma \)-algebra \( \mathcal{B} \otimes \mathbb{R} \) and the measure \( \lambda \otimes \text{Leb} \) restricted to \( T^f \). We define the special flow \((T^f_t) : (T^f, \mathcal{B}^f, \lambda^f) \to (T^f, \mathcal{B}^f, \lambda^f)\), by
\[ T^f_t(x, s) := (x + N(x, s, t)\alpha, s + t - f^{(N(x,s,t))}(x)). \]

We will consider the product metric on the space \( T^f \), i.e.
\[ d^f((x, s), (y, r)) := \|x - y\| + |s - r|. \]
For a set \( A \subset T \), we denote \( A^f := \{(x, s) \in T^f : x \in A, 0 \leq s < f(x)\} \).

The following general remark follows from the definition of special flow and will be useful in the proof of Theorem 3.1.4.

**Remark 3.2.1.** Let \((G^h_t)\) be a special flow over an irrational rotation \( G : T \to T \) and under \( h \in \text{BV}(T) \), \( h \in C^1(T \setminus \{0\}) \). For every \( \epsilon > 0 \) there exists \( \tilde{\epsilon}_\epsilon > 0 \) such that for every \((z, w), (z', w') \in T^h\), \( d^h((z, w), (z', w')) < \tilde{\epsilon}_\epsilon\), we have
\[ d^h\left(G^h_t(z, w), G^h_t+h(N(z,w,t))(z)\right)(z', w') < \epsilon^2, \]
for every \( t \in \mathbb{R} \) for which
\[ G^h_t(z, w) \notin \partial(T^h, \epsilon), \]
where \( \partial(T^h, \epsilon) := \{(z, w) \in T^h : \|z\| < \epsilon^2 \text{ and } \epsilon^2 < w < f(x) - \epsilon^2\} \).

Moreover, since \( h \in \text{BV}(T) \cap C^1(T \setminus \{0\}) \), for every \( \epsilon > 0 \) there exists \( \bar{\kappa}_\epsilon > 0 \) and \( t_\epsilon > 0 \) such that for every \( |T| \geq t_\epsilon \) and every \((z, w) \in T^h\), we have
\[ |U_{z,w}| > (1 - \epsilon^2)|T|, \quad (3.3) \]
where
\[ U_{z,w} := \{t \in [0, T] : G^h_t(z, w) \notin \partial(T^h, \bar{\kappa}_\epsilon)\}. \]
Notice also that \( U_{z,w} \) consists of at most \((\inf_T h)^{-1}|T|\) disjoint intervals.
3.2.3 Diophantine lemmas

Let $\beta \in \mathbb{T}$ and let $(q'_n)$ denote the sequence of denominators of $\beta$. We have the following lemma:

Lemma 3.2.2 ([15], Lemma 3.3). Fix $y, y' \in \mathbb{T}$, and let $n \in \mathbb{N}$ be any integer such that

$$\|y - y'\| < \frac{1}{6q'_n}.$$  

Then at least one of the following holds:

(i) $0 \notin \bigcup_{k=0}^{\frac{n+1}{6}} R^k_{\beta}[y, y']$;

(ii) $0 \notin \bigcup_{k=0}^{\frac{n+1}{6}} R^{-k}_{\beta}[y, y']$;

(iii) $0 \in \bigcup_{k=0}^{q'_n-1} R^k_{\beta}[y, y']$.

We will also need the following lemma, which is a simple consequence of Denjoy-Koksma inequality:

Lemma 3.2.3. Let $\phi : \mathbb{T} \to \mathbb{R}$ be a function of bounded variation. For every $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$, such that for every $n \in \mathbb{Z}$ with $|n| \geq n_\epsilon$ and every $x \in \mathbb{T}$

$$\left| \phi^{(n)}(x) - n \int_{\mathbb{T}} \phi d\lambda \right| < \epsilon |n|.$$

Proof. By cocycle identity, it is enough to consider the case $n > 0$. Notice that by Denjoy-Koksma inequality, there exists a constant $c_\phi > 0$ such that for every $s \in \mathbb{N}$ and every $z \in \mathbb{T}$, we have

$$\left| \phi^{(n_s)}(z) - q_s \int_{\mathbb{T}} \phi d\lambda \right| < c_\phi.$$

The proof then follows by Ostrovski expansion since for $n \in \mathbb{N}$ we can write

$$n = \sum_{i=0}^{k} b_i q_i,$$

where $b_i \leq \frac{n+1}{q_i}$ and, by cocycle identity:

$$\phi^{(n)}(x) = \sum_{i=0}^{k} \sum_{r=1}^{b_i} \phi^{(q_k-1)}(x + z_{i,r} \alpha),$$

where $z_{i,r} = \sum_{h=0}^{i} b_h q_h + r q_{k-1}$. This finishes the proof. \qed

3.3 Disjointness criterion

We will use a variant of disjointness criterion introduced in [16].

Proposition 3.3.1. Let $P \subset \mathbb{R}^2 \setminus \{(x, x) : x \in \mathbb{R}\}$ be a compact subset and fix $c \in (0, 1)$. Assume there exist $(A_k) \subset \text{Aut}(X_k, \mathcal{B}|_{X_k}, \lambda|_{X_k})$ for $k \geq 1$, such that $\lambda(X_k) \rightarrow \lambda(X)$, $A_k \rightarrow \text{Id}$ uniformly. Assume moreover that for every $\epsilon > 0$ and $N \in \mathbb{N}$ there exist $(E_k = E_k(\epsilon)) \subset \mathcal{B}$, $\lambda(E_k) \geq c\lambda(X)$, $0 < \kappa = \kappa(\epsilon) < \epsilon$, $\delta = \delta(\epsilon, N) > 0$, a set $Z = Z(\epsilon, N) \subset Y$, $\nu(Z) \geq (1 - \epsilon)\nu(Y)$ such that for all $y, y' \in Z$ satisfying $d_2(y, y') < \delta$, every $k$ such that $d_1(A_k, \text{Id}) < \delta$ and every $x \in E_k$, $x' := A_k x$ there are $M \geq N$, $L \geq 1$, $\frac{L}{M} \geq \kappa$ and $(p, q) \in P$ for which at least one of the following holds:

$$d_1(T_t x, T_{t+p} x') \neq 0 \text{ for } t \in U \subset [M, M + L]$$

(3.4)

or

$$d_1(T_t x, T_{t+p} x') \neq 0 \text{ for } -t \in U \subset [M, M + L],$$

(3.5)

where $U$ is a union of at most $[c^{-1}L] + 1$ intervals and $|U| \geq (1 - \epsilon)L$. Then $(T_t)$ and $(S_t)$ are disjoint.

Proof. The proof is a consequence of Theorem 3 in [16]. Namely, for $x, x', y, y'$ as in the statement of Proposition 3.3.1 we define a function $a = a_{x, x', y, y'} : [M, M + L] \rightarrow \mathbb{R}$ given by $a(t) = t + q$. Then by Theorem 3 in [16], we just need to verify that $(a, U, c)$ is $\epsilon$-good (see Definition 3.1. in [16], (P1)). This however follows straightforward from the definition of $a$ and $U$. The proof is thus finished. \(\square\)

3.4 Proof of Theorem 3.1.4

In this section we will use Proposition 3.3.1 to prove Theorem 3.1.4. For simplicity, we will use the following notation: $(T'_t)$ is a special flow over $T(x) = x + \alpha$ (with sequence of denominators $(q_n)$) and under $f(x) = A_f\{x\} + f_{ac}(x)$, where $f_{ac} \in C^1(\mathbb{T})$ and $(S'_t)$ is a special flow over $S(x) = x + \beta$ (with sequence of denominators $(q'_n)$) and under $g(x) = A_g\{x\} + g_{ac}(x)$, where $g_{ac} \in C^1(\mathbb{T})$. Moreover, we assume that $|A_f| \neq |A_g|$. Note that $(T'_t)$ is isomorphic to $(R'_t)_{t \in \mathbb{R}}$ built over $Rx = x - \alpha$.
with roof function \( h(x) = A_f \{1 - x\} + f_{ac}(1 - x) = (-A_f) \{x\} + (A_f + f_{ac}(1 - x)) \). This allows to assume that \( A_f, A_g > 0 \). We will divide the proof in two cases depending on the diophantine types of \( \alpha \) and \( \beta \).

### 3.4.1 Proof in case at least one of \( \alpha \) and \( \beta \) is of unbounded type

We will without loss of generality assume that \( \alpha \) is of unbounded type.

**Proof of Theorem 3.1.4.** We will verify that the assumptions of Proposition 3.3.1 are satisfied. Since Proposition 3.3.1 has many quantifiers, we will divide the proof into paragraphs, in which we indicate what quantity we are defining. Let \( \xi := \int T f \, d\lambda \int T g \, d\lambda \) and \( \Delta := 1000 \max(\xi, A_f A_g, A_g A_f^{-1}, A_f^{-1} A_g^{-1}) \).

**Definition of \( c \) and \( P \):** Let \( c := \min \{A_f, A_g, |A_f - A_g|, \xi, \xi^{-1} \} \) and \( \Delta := 100 \max(\xi, A_f A_g, A_g A_f^{-1}, A_f^{-1} A_g^{-1}) \).

**Definition of \( A_k \) and \( X_k \):** Recall that since \( \alpha \) is of unbounded type, there exists an increasing sequence \( \{n_k\}_{k \in \mathbb{N}} \) such \( \frac{q_{n_k+1}}{q_{n_k}} \to \infty \). Define

\[
X_k := \left\{ (x, s) \in T^f : \left( x - \frac{c}{q_{n_k}}, s \right) \in T^f \right\},
\]

and \( A_k(x, s) = (x - \frac{c}{q_{n_k}}, s) \). It follows from the definition that \( \lambda^f(X_k) \to \lambda(T^f) \), \( A_k \in \text{Aut}(X_k, B|X_k, \lambda^f|X_k) \) and \( A_k \to \text{Id} \) uniformly.

**Definition of \( E_k(\epsilon) \):** Fix \( \epsilon < \min \{\frac{q_{n_k} + 1}{q_{n_k}}, \frac{A_f}{72}, \frac{A_g}{72}, c\} \). Let

\[
\tilde{E}_k := \bigcup_{i = -n_k}^{c^2 q_{n_k}} T^i \left[ \frac{2}{q_{n_k+1}}, \frac{c^2}{q_{n_k}} \right].
\]

Notice that \( \lambda(\tilde{E}_k) \geq c^4 \). Let \( E_k = E_k(\epsilon) = \tilde{E}_k \cap X_k \).

Notice that \( \lambda^f(\tilde{E}_k) \geq (\inf_T f) \lambda(\tilde{E}_k) \), and hence \( \lambda^f(E_k) > c^5 \). The following
property of the set \( E_k \) will be crucial in the proof: Let \((x, s) \in E_k \) and denote 
\((x', s) = A_k(x, s) \). Then there exists \( i_x \in [n_k, c^2q_{n_k}] \) such that

\[
0 \in [x + i_x \alpha, x' + i_x \alpha] \quad \text{and} \quad 0 \in [x - (q_{n_k} - i_x)\alpha, x' - (q_{n_k} - i_x)\alpha]. \tag{3.6}
\]

Indeed, notice that since \((x, s) \in E_k \subset E^f_k \) it follows that there exists \( i_x \in [n_k, c^2q_{n_k}] \)
\( x + i_x \alpha \in \left[ \frac{2}{q_{n_k + 1}}, \frac{c^2}{q_{n_k}} \right] \). By the definition of \( A_k \), we have \( x' + i_x \alpha = x + i_x \alpha - \frac{c}{q_{n_k}} \) and so
\( 0 \in [x' + i_x \alpha, x + i_x \alpha] \) (since, by assumptions \( \frac{c}{q_{n_k}} > \frac{2}{q_{n_k + 1}} \)).

Moreover, since \( \|q_{n_k}\alpha\| \leq \frac{1}{q_{n_k + 1}} \), we have

\[
x - (q_{n_k} - i_x)\alpha = x - i_x \alpha - q_{n_k} \alpha \in \left[ \frac{1}{q_{n_k + 1}}, \frac{c^2}{q_{n_k}} \right]
\]

and

\[
x' - (q_{n_k} - i_x)\alpha = x + i_x \alpha - \frac{c}{q_{n_k}} - q_{n_k} \alpha < 0.
\]

This finishes the proof of (3.6).

**Definition of \( \kappa, \delta \) and \( Z \):** Let \( \kappa := \kappa(\epsilon) = c\epsilon^3 \).

Notice that for every \( k \in \mathbb{Z} \) there exists \( \theta_k \in [x, y] \) such that \( \psi^{(k)}(x) - \psi^{(k)}(y) = \psi^{(k)}(\theta_k)(x - y) \), for \( \psi \in \{f_{ac}, g_{ac}\} \). By Lemma 3.2.3 for \( \phi = f'_{ac} \) and \( \phi = g'_{ac} \), there exists \( \delta_\epsilon > 0 \) such that if \( z, z' \in T \) satisfy \( \|z - z'\| < \delta_\epsilon \), then

\[
|f_{ac}^{(k)}(z) - f_{ac}^{(k)}(z')| < \frac{\epsilon^3}{20} \max\{1, k\|z - z'\|\}, \tag{3.7}
\]

\[
|g_{ac}^{(k)}(z) - g_{ac}^{(k)}(z')| < \frac{\epsilon^3}{20} \max\{1, k\|z - z'\|\}.
\]

Let \( \delta = \delta(\epsilon, N) := \min\{\delta_\epsilon, \frac{\epsilon}{10A_y}, \frac{12\epsilon}{20A_yq_{n_0}^2}\} \), where \( n_0 \) is such that

\[
q_{n_0}' > \max \left\{ \frac{12N(\inf g)^{-1}}{\frac{N}{2}}, \frac{N^2(\inf g)^{-2}}{2} \right\}.
\]

We will now define the set \( Z = Z(\epsilon, N) \). First let

\[
Z_1 := \{(y, r) \in T^g : \delta < s < g(y) - \delta\}.
\]

Notice that by the definition of \( \delta \), we have \( \lambda^g(Z_1) > (1 - \epsilon/2)\lambda^g(T^g) \). By the definition of \( Z_1 \) it follows that for every \((y, r), (y', r') \in Z_1 \) with \( d^g((y, r), (y', r')) < \)
\[ d_g((y, r), (y', r')) = \|y - y'\| + |r - r'|. \]

Let \( B_n := \bigcup_{i=\left\lfloor \sqrt{q_n} \right\rfloor}^{\left\lceil \sqrt{q_n} \right\rceil} R_i \left[ -\frac{1}{6q_n}, \frac{1}{6q_n} \right], \) then \( \lambda(B_n) < \frac{1}{\sqrt{q_n}}. \) Since \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{q_n}} < \infty \) (\( q_n' \) grows exponentially), there is an \( m \in \mathbb{N} \) such that \( \lambda(\bigcup_{n=m}^{\infty} B_n) < \frac{\epsilon}{4g_{\max}}. \)

Define
\[
Z_2 := \left\{ (x, s) \in T^g : x \notin \bigcup_{n=m}^{\infty} B_n \right\},
\]
then \( \lambda^g(Z_2) > (1 - \epsilon/2) \lambda^g(T^g). \) Finally, let \( Z = Z_1 \cap Z_2. \) Notice that we have \( \lambda^g(Z) > (1 - \epsilon) \lambda^g(T^g). \)

**Main estimates:** Take \((x, s) \in E_k \) and \((x', s) = A_k(x, s), \) where \( k \in \mathbb{N} \) is such that \( d((A_k, Id) < \epsilon \) and let \((y, r), (y', r') \in Z \) with \( d_g((y, r), (y', r')) \leq \delta. \) Let \( n \in \mathbb{N} \) be the unique integer such that
\[
\frac{1}{6q_{n+1}} \leq \|y - y'\| < \frac{1}{6q_n}. \tag{3.8}
\]

We will show that in the forward case, there exists an interval \([M', M' + L'] \) with \( \frac{L'}{M'} \geq \epsilon^3 \) and such that for every \( n \in [M', M' + L'] \) and some \((p, q) \in P, \) we have
\[
|f^{(n)}(x) - f^{(n)}(x') - p| < \epsilon^2 \tag{3.9}
\]
and
\[
|g^{(\xi^{-1}n)}(y) - g^{(\xi^{-1}n)}(y') - q| < \epsilon^2. \tag{3.10}
\]
The backward case is analogous.

**Claim:** (3.9) and (3.10) (together with the backward version) imply the statement of Proposition 3.3.1.

**Proof of the Claim.** Let
\[
M := \max(f^{(M')}(x), g^{(\xi^{-1}M')}(y))
\]
and
\[
L := \min \left( f^{(M' + L')}(x) - f^{(M')}(x), g^{(\xi^{-1}(M' + L'))}(y) - g^{(\xi^{-1}M')}(y) \right).
\]
Notice that by Remark 3.2.1 for \((T^f_t)\) and \((S^g_t)\) it follows that for every \( t \in [M, M +\)
\[ T^f_t(x, s) \notin \partial(\mathbb{R}^f, \epsilon^2) \text{ and } S^g_t(y, r) \notin \partial(\mathbb{R}^g, \epsilon^2), \]

we have by (3.9) and (3.10) that

\[ d^f(T^f_t(x, s), T^f_{t+p}(x', s')) < \epsilon \text{ and } d^g(S^g_t(y, r), S^g_{t+q}(y', r')) < \epsilon. \]

Moreover, by (3.3) for \((T^f_t)\) and \((S^g_t)\) it follows that if \(U := U_{x,s} \cap U_{y,r}\), then \(|U| \geq (1 - \epsilon)L\) and \(U\) consists of at most \(3\) max(\(\inf_T f\)^{-1}, \(\inf_T g\)^{-1}) intervals.

Hence it only remains to show that \(\frac{L}{M} \geq \kappa\). Let \(\xi_f = \int_T f d\lambda\). By Lemma 3.2.3 for \(f\) and \(g\), it follows that

\[ M \in [(1 - \kappa^2)\xi_f M', (1 + \kappa^2)\xi_f M'] \]

and

\[ M + L \in [(1 - \kappa^2)\xi_f (M' + L'), (1 + \kappa^2)\xi_f (M' + L')]. \]

Therefore (recall that \(L' \geq \epsilon^3 M'\)),

\[ \frac{M + L}{M} \geq \frac{1 - \kappa^2}{1 + \kappa^2} (1 + \epsilon^3) \geq 1 + \kappa, \]

and so indeed \(\frac{L}{M} \geq \kappa\). This gives (3.4) in the statement of Proposition 3.3.1. \(\square\)

Hence we only need to show that (3.9) and (3.10) hold in the forward case. We will split the proof in three cases according to Lemma 3.2.2.

**Case 1.** Assume (i) in Lemma 3.2.2 holds for \(y, y'\).

By (i) and (3.7) for \(g_{ac}\) and \(y, y'\), we get that for any \(i \in \left[1, \frac{q_{n+1}}{6}\right]\),

\[ |g^{(i)}(y) - g^{(i)}(y') - iA_g(y - y')| \leq |g^{(i)}_{ac}(y) - g^{(i)}_{ac}(y')| \leq \frac{\epsilon^3}{20} \max\{1, i\|y - y'\|\}. \]  

(3.11)

Now, let \(\ell = \ell_{x, x'} \in \mathbb{N}\) be the smallest integer such that \(0 \in T^f_\alpha[x, x']\). Since \((x, s) \in E_k\) and by (3.6), \(\ell = i_x \in [n_k, c^2 q_{n_k}]\). By the definition of \(\ell\) it follows that for any \(0 \leq j < \ell\), \(0 \notin [T^f_\alpha x, T^f_\alpha x']\) and therefore and by (3.7) for \(f\) and \(x, x'\), we
get

\[ |f^{(j)}(x) - f^{(j)}(x') - jA_f(x - x')| \leq \frac{\epsilon^3}{20} \max\{1, j\|x - x'\|\} < \frac{\epsilon^3}{10}, \quad (3.12) \]

the last inequality, since (by the definition of \( A_k \)) \( \|x - x'\| = \frac{c}{q_{nk}} \) and \( j \leq \ell \leq c^2q_{nk} \).

Since \( 0 \in [T^k_\alpha x, T^k_\alpha x'] \) and \( \|x - x'\| = \frac{c}{q_{nk}} \) it follows that for any \( j \in [\ell + 1, \ell + q_{nk}) \),
\( 0 \notin [T^j_\alpha x, T^j_\alpha x'] \). Therefore for any \( j \in [\ell + 1, \ell + q_{nk}) \), using (3.7) for \( f \), and \( x, x' \), we have

\[ |f^{(j)}(x) - f^{(j)}(x') + A_f - jA_f(x - x')| \leq \frac{\epsilon^3}{20} \max\{1, j\|x - x'\|\} < \frac{\epsilon^3}{10}; \quad (3.13) \]

where the last inequality since \( j \leq \ell + q_{nk} \leq c^2q_{nk} + q_{nk} \) and \( \|x - x'\| = \frac{c}{q_{nk}} \). We consider the following subcases:

**Subcase 1.** \( \ell A_g\|y - y'\| > 4c \). Let then \( \ell' = 2c \) be defined by \( \ell' := \left\lceil \frac{2c}{A_g\|y - y'\|} \right\rceil \).

By definition \( \ell' \leq \frac{\ell}{2} \). Then by (3.12) for any \( j \in [\ell', (1 + \epsilon^3)\ell'] \), we have

\[ |f^{(j)}(x) - f^{(j)}(x') - \ell' A_f(x - x')| \leq \epsilon^3\ell' A_f\|x - x'\| + \epsilon^3/10 \]
\[ \leq \epsilon^3\ell A_f \frac{c}{q_{nk}} + \epsilon^3/10 \leq \epsilon^2; \quad (3.14) \]

since \( \ell \leq c^2q_{nk} \). Notice that

\[ |p| := \ell' A_f\|x - x'\| \leq \ell A_f\|x - x'\| \leq \epsilon^3 A_f \leq c\xi^{-1} \quad (3.15) \]

(by taking a smaller \( c > 0 \) if necessary). Moreover, by (3.11) and the definition of \( \ell' \), for \( j \in [\ell', (1 + \epsilon^3)\ell'] \), we have

\[ \left| g^{([\xi^{-1}\ell_j])}(y) - g^{([\xi^{-1}\ell_j])}(y') - [\xi^{-1}\ell'] A_g(y - y') \right| \leq \epsilon^3 \]
\[ ([\xi^{-1}\ell'] - [\xi^{-1}(1 + \epsilon^3)\ell']) A_g\|y - y'\| + \frac{\epsilon^3}{20} \max(1, [\xi^{-1}(1 + \epsilon^3)\ell']\|y - y'\|) \leq \epsilon^2, \quad (3.16) \]

the last inequality by the definition of \( \ell' \). We also have

\[ |q| := [\xi^{-1}\ell'] A_g\|y - y'\| \in [2c\xi^{-1} - A_g\delta_c, 2c\xi^{-1}]. \quad (3.17) \]
Notice that by (3.14), (3.15) and (3.16) (3.17) it follows that (3.9) and (3.10) hold for \( M' = \ell', L' = e^3 M' \). Moreover, by (3.15) and (3.17) it follows that \((p,q) \in P\). This finishes the proof of **Subcase 1**.

**Subcase 2.** \( \ell_A \|y - y'\| \leq 4c \). Then by (3.13) for any \( j \in [\ell + 1, (1 + e^3)(\ell + 1)] \), we have (recall that \( \ell \leq c^2 q_{n_k} \))

\[
|f^j(x) - f^{(j)}(x') + A_f - \ell A_f (x - x')| \leq \epsilon^3 \ell A_f \|x - x'\| + \epsilon^3 / 10 \\
\leq \epsilon^3 \ell A_f \frac{c}{q_{n_k}} + \epsilon^3 / 10 \leq \epsilon^2
\]  

(3.18)

Notice that if \( p := -A_f + \ell A_f (x - x') \), then

\[
-A_f \leq p \leq -A_f + \ell A_f \|x - x'\| \leq -A_f + \epsilon^2.
\]

Since \( \ell_A \|y - y'\| < 4c \), we get (see (3.8)) \( \xi^{-1}(1 + e^3)(\ell + 1) \leq q_{n_k+1}^2 \) and therefore, for every \( j \in [\ell + 1, (1 + e^3)(\ell + 1)] \), we have

\[
|g^{([\xi^{-1} j])}(y) - g^{([\xi^{-1} j])}(y') - [\xi^{-1} \ell] A_g (y - y')| \\
\leq ([\xi^{-1} \ell] - [\xi^{-1}(1 + e^3) \ell]) A_g \|y - y'\| + \frac{e^3}{20} \max(1, [\xi^{-1}(1 + e^3) \ell] \|y - y'\|) \leq \epsilon^2,
\]

(3.20)

since \( \ell_A \|y - y'\| < 4c \). Let \( q := [\xi^{-1} \ell] A_g (y - y') \), then

\[
|q| \leq \xi^{-1} \ell A_g \|y - y'\| \leq 4c \xi^{-1} \leq \frac{A_f}{2},
\]

(3.21)

(by taking smaller \( c > 0 \) if necessary). We define \( M' := \ell + 1 \) and \( L' = (1 + e^3) M' \), then by (3.19) and (3.21), we have that \((p,q) \in P\) and by (3.18) and (3.20) we get that (3.9) and (3.10) hold on \([M', M' + L']\). This finishes the proof of **Subcase 2.** and hence also the proof of **Case 1.**

**Case 2.** Assume (ii) holds. The proof is analogous to the proof in **Case 1.** by considering backward iterations.

**Case 3.** Assume (iii) holds. Let \( k_0 \) be the least real number such that \([\xi^{-1} k_0] \in [0, q_{n_0} - 1] \) and \( 0 \in R^{[\xi^{-1} k_0]}[y, y'] \). Then by the definition of \( Z \), \( \xi^{-1} k_0 \geq \sqrt{q_{n_0}} \), and \( 0 \notin R^k[y, y'] \) for any \( k \in [0, [\xi^{-1} k_0] + q_{n_0}], k \neq \xi^{-1} k_0 \). Hence if \( \xi^{-1} k < [\xi^{-1} k_0] \), by
Lemma 3.2.3, for $\phi = g_{ac}'$

\[ \left| g^{(\xi^{-1}k)}(y) - g^{(\xi^{-1}k)}(y') - [\xi^{-1}k]A_g\|y - y'\| \right| \leq |g_{ac}^{(\xi^{-1}k)}(y) - g_{ac}^{(\xi^{-1}k)}(y')| \leq \frac{\epsilon^3}{20}. \]

(3.22)

Moreover for $k_0 + \xi < k < k_0 + q_n'$,

\[ \left| g^{(\xi^{-1}k)}(y) - g^{(\xi^{-1}k)}(y') + A_g - [\xi^{-1}k]A_g(y - y') \right| \leq |g_{ac}^{(\xi^{-1}k)}(y) - g_{ac}^{(\xi^{-1}k)}(y')| \]

\[ \leq \frac{\epsilon^3}{20}. \]

(3.23)

where $\pm$ depends only on $y, y'$.

Let $\ell \in \mathbb{N}$ be the smallest integer such that $0 \in R^\ell[x, x']$. By the definition of $E_k, n_k \leq \ell \leq c^2q_{n_k}$. It is clear that for any $j < \ell$,

\[ \left| f^{(j)}(x) - f^{(j)}(x') - jA_f(x - x') \right| \leq \frac{\epsilon^3}{20} \max\{1, j\|x - x'\| \} < \frac{\epsilon^3}{10}, \]

(3.24)

and for any $j \in [\ell, \ell + q_{n_k})$,

\[ \left| f^{(j)}(x) - f^{(j)}(x') - A_f - jA_f(x - x') \right| \leq \frac{\epsilon^3}{20} \max\{1, j\|x - x'\| \} < \frac{\epsilon^3}{10}. \]

(3.25)

**Subcase 1.** $k_0 \leq (1 - \epsilon)\ell$. notice that for every $T \in [0, (1 + \epsilon^3)k_0]$ (note that $(1 + \epsilon^3)k_0 < \ell$) and every $j \in [T, (1 + \epsilon^3)T]$, by (3.24), we have

\[ \left| f^{(j)}(x) - f^{(j)}(x') - TA_f(x - x') \right| \leq \epsilon^3A_f\|x - x'\| + \epsilon^3/10 \leq \epsilon^2, \]

(3.26)

and if $p(T) := TA_f(x - x')$, then

\[ |p(T)| \leq \ell A_f\|x - x'\| \leq \epsilon^3A_f. \]

(3.27)

Moreover, by (3.22) for every $j \in [(1 - \epsilon^3)k_0, k_0)$, by (3.8) and $k_0 \leq q_n'$, we have

\[ \left| g^{(\xi^{-1}j)}(y) - g^{(\xi^{-1}j)}(y') - [\xi^{-1}k_0]A_g\|y - y'\| \right| \leq \xi^{-1}\epsilon^3k_0A_g\|y - y'\| + \epsilon^3/20 \leq \epsilon^2. \]

(3.28)

Similarly, by (3.23), for every $j \in [k_0 + 1, (1 + \epsilon^3)(k_0 + 1)]$, by (3.8) and $k_0 \leq q_n'$,
we have
\[
\left| g^{(\xi^{-1}i)}(y) - g^{(\xi^{-1}i)}(y') \pm (A_g - [\xi^{-1}k_0]A_g(y - y')) \right| \leq \epsilon^2. \tag{3.29}
\]

Notice that if \( q_1 := [\xi^{-1}k_0]A_g\|y - y'\| \) and \( q_2 := \pm(A_g - [\xi^{-1}k_0]A_g(y - y')) \), then
\[
\max(|q_1|, |q_2|) \geq \frac{A_g}{2}. \tag{3.30}
\]

By (3.26), (3.28) and (3.29) it follows that (3.9) and (3.10) hold on
\[
[M'_1, M'_1 + L'_1] := [(1 - \epsilon^3)k_0, (1 - \epsilon^6)k_0]
\]
with \( p_1 := p((1 - \epsilon^3)k_0) \) and \( q_1 \), and also on
\[
[M'_2, M'_2 + L'_2] := [k_0 + 1, (1 + \epsilon^3)(k_0 + 1)]
\]
with \( p_2 := p(k_0 + \xi) \) and \( q_2 \). Moreover, by (3.27) and (3.30), at least one of \((p_1, q_1)\), \((p_2, q_2)\) belongs to \( P \).

**Subcase 2.** \( k_0 \geq (1+\epsilon)\ell \). In this case let \( M'_1 := \ell + 1 \) and \( L'_1 = \epsilon^3(\ell + 1) \). Notice that \( M'_1 + L'_1 < k_0 \) and hence for every \( j \in [M'_1, M'_1 + L'_1] \) (reasoning analogously to Case 1), by (3.25), we have
\[
| f^{(j)}(x) - f^{(j)}(x') - p_1 | < \epsilon^2,
\]
where \( p_1 = -A_f + M'_1A_f(x - x') \) and similary by (3.22)
\[
| g^{(\xi^{-1}i)}(y) - g^{(\xi^{-1}i)}(y') - q_1 | < \epsilon^2,
\]
where \( q_1 := [\xi^{-1}M'_1]A_g(y - y') \). Let \( M'_2 := \lfloor \ell/2 \rfloor \) and \( L'_2 = \epsilon^3M'_2 \). Then for every \( j \in [M'_2, M'_2 + L'_2] \) (reasoning analogously to Case 1), by (3.24), we have
\[
| f^{(j)}(x) - f^{(j)}(x') - p_2 | < \epsilon^2,
\]
where $|p_2| = |-M_2' A_f(x - x')| \leq c^3 A_f$ and similarly by (3.22)

$$\left| g^{(k^{-1}j)}(y) - g^{(k^{-1}j)}(y') - q_2 \right| \leq \varepsilon^2,$$

where $q_2 := [\xi^{-1} M_2] A_g(y - y')$. We will show that (3.9) and (3.10) hold either on $[M_1', M_1' + L_1']$ or on $[M_2', M_2' + L_2']$.

By the above, we only need to show that at least one of $(p_1, q_1)$ and $(p_2, q_2)$ belongs to $P$. Notice that $q_2 \geq \frac{q_2}{3}$, $p_1 \geq 10p_2$ and $|p_1| \geq \frac{A_f}{2}$ (since $c > 0$ is small). Therefore if $|p_2 - q_2| < c^2$, then $|p_1 - q_1| > c^2$. This finishes the proof in Subcase 2.

Subcase 3. $(1 - \varepsilon)\ell \leq k_0 \leq (1 + \varepsilon)\ell$. The proof is similar to the proof in Subcase 2.

Let $M_1 := (1 - \varepsilon) \min(k_0, \ell)$ and $L_1 := \varepsilon^3 M_1$, $M_2 := (1 + \varepsilon) \max(k_0, \ell)$ and $L_2 := \varepsilon^3 M_2$. Notice that by (3.24) for every $j \in [M_1', M_1' + L_1']$, we have

$$\left| f^{(j)}(x) - f^{(j)}(x') - p_1 \right| < \varepsilon^2,$$  \hspace{2cm} (3.31)

where $p_1 = M_1 A_f(x - x')$ and similarly by (3.22)

$$\left| g^{(k^{-1}j)}(y) - g^{(k^{-1}j)}(y') - q_1 \right| \leq \varepsilon^2,$$  \hspace{2cm} (3.32)

where $q_1 := [\xi^{-1} M_1] A_g(y - y')$. Similarly, by (3.25) for every $j \in [M_2, M_2 + L_2]$, we have

$$\left| f^{(j)}(x) - f^{(j)}(x') - p_1 \right| < \varepsilon^2,$$  \hspace{2cm} (3.33)

where $p_2 = -A_f + M_2 A_f(x - x')$ and similarly by (3.22)

$$\left| g^{(k^{-1}j)}(y) - g^{(k^{-1}j)}(y') - q_1 \right| \leq \varepsilon^2,$$  \hspace{2cm} (3.34)

where $q_2 = \pm (A_g - [\xi^{-1} M_2] A_g(y - y'))$.

Notice that $|p_1| < c^3 A_f$, $|q_1| < \xi^{-1} A_g$ and $|p_2| < A_f + c^3 A_f$ and $|q_2| < A_g + \xi^{-1} A_g$. If $(p_1, q_1) \in P$, then (3.9) and (3.10) hold on $[M_1, M_1 + L_1]$ (see (3.31) and (3.32)). If $(p_1, q_1) \not\in P$, then

$$|p_1 - q_1| < c^2,$$
and so |q_1| < 2c^2 \max(1, A_f). We also have |q_2 - A_g| \leq 2|q_1| \leq 4c^2 \max(1, A_f). Therefore

\[|p_2 - q_2| \geq |A_f - A_g| - c^2 A_f - 4c^2 \max(1, A_f) \geq \frac{1}{2} |A_f - A_g| \geq c.\]

Therefore (p_2, q_2) \in P and (3.9) and (3.10) hold on [M_2, M_2 + L_2] (see (3.33) and (3.34)).

This finishes the proof of Subcase 3. and hence also the proof of Theorem 3.1.4.

\[\square\]

### 3.4.2 The proof when both \(\alpha\) and \(\beta\) are of bounded type

The argument here is similar to that in previous subsection. Recall that \(A_f > 0, A_g > 0\).

**Proof of Theorem 3.1.4.** We will verify that the assumptions of Proposition 3.3.1 are satisfied. Let \(\xi := \frac{f \cdot g \cdot \alpha}{\int T_f} \) and \(\Delta := 1000 \max(\xi, \frac{A_f}{A_g}, A_f^{-1}, A_g^{-1})\).

**Definition of \(c\) and \(P\):** Let \(a_0 := 10 \max\left\{\sup\left\{\frac{q_{n+1}}{q_n}\right\}, \sup\left\{\frac{q'_{n+1}}{q'_n}\right\}\right\}\), and

\[c := \frac{\min\{A_f, A_g, a_0^{-1}, |A_f - A_g|, \xi, \xi^{-1}\}}{100 \Delta^2} > 0.\]

Let \(P := \{(p, q) : \max(|p|, |q|) \leq 10 \max\{A_f, A_g\}, |p - q| \geq c^2\}\).

**Definition of \(A_k\) and \(X_k\):** Since \(\alpha\) is of bounded type, we have \(\frac{g_{n+1}}{q_n} < \frac{1}{2c}\) for all \(n\). Define

\[X_k := \left\{(x, s) \in T^f : (x - \frac{c}{q_k}, s) \in T^f\right\},\]

and \(A_k(x, s) = (x - \frac{c}{q_k}, s)\). It follows from the definition that \(\lambda^f(X_k) \to \lambda^f(T^f)\), \(A_k \in Aut(X_k, B_{(X_k, \lambda^f_{X_k})})\) and \(A_k \to Id\) uniformly.

**Definition of \(E_k(\epsilon)\):** Fix \(\epsilon < \min\left\{\frac{g_{\min}}{4}, \frac{A_g}{T_2}, \frac{g_{\max}}{72}, \frac{A_g}{72}, c\right\}\). Let

\[E_k := \bigcup_{i=-k}^{c^2 q_k} T^i_{\alpha} \left[\frac{c^2}{q_k}, \frac{2c^2}{q_k}\right].\]
Define \( \lambda \). Notice that \( \lambda(\tilde{E}_k) \geq \frac{c^4}{T} \). Let

\[
E_k := E_k(\epsilon) = \tilde{E}_k^f \cap X_k.
\]

Notice that \( \lambda^f(\tilde{E}_k^f) \geq (\inf_T f)\lambda(\tilde{E}_k) \), and hence \( \lambda^f(E_k) > c^5 \). By the definition of \( E_k \subset \tilde{E}_k^f \) it follows that if \((x, s) \in E_k \) and \((x', s) = A_k(x, s) \), then there exists a unique \( i_x \in [k, c^2q_k] \) such that

\[
0 \in [x + i_x \alpha, x' + i_x \alpha]. \quad (3.35)
\]

**Definition of \( \kappa, \delta \) and \( Z \):** Let \( \kappa := \kappa(\epsilon) = c\epsilon^3 \).

Notice that for every \( k \in \mathbb{Z} \) there exists \( \theta_k \in [x, y] \) such that \( \psi^{(k)}(x) - \psi^{(k)}(y) = \psi^{(k)}(\theta_k)(x - y) \), for \( \psi \in \{f_{ac}, g_{ac}\} \). By Lemma 3.2.3 for \( \phi = f_{ac} \) and \( \phi = g_{ac} \), there exists \( \delta_\epsilon > 0 \) such that if \( z, z' \in \mathbb{T} \) satisfy \( \|z - z'\| < \delta_\epsilon \), then

\[
|f_{ac}^{(k)}(z) - f_{ac}^{(k)}(z')| < \frac{\epsilon}{20} \max\{1, k\|z - z'\|\},
\]

\[
|g_{ac}^{(k)}(z) - g_{ac}^{(k)}(z')| < \frac{\epsilon}{20} \max\{1, k\|z - z'\|\}. \quad (3.36)
\]

Let \( \delta = \delta(\epsilon, N) := \min\{\delta_\epsilon, \frac{\epsilon}{10\lambda^2}, \frac{\epsilon}{20AC}, \frac{12\epsilon}{\lambda^2q_{n_0}}\} \), where \( n_0 \) is such that

\[
q_{n_0} > \max \left\{ 12N(\inf_{\mathbb{T}} g)\lambda^{-1}, N^2(\inf_{\mathbb{T}} g)\lambda^{-2} \right\}.
\]

We will now define the set \( Z = Z(\epsilon, N) \). First let

\[
Z_1 := \{(y, r) \in \mathbb{T}^y : \delta < s < g(y) - \delta\}.
\]

Notice that by the definition of \( \delta \), we have \( \lambda^y(Z_1) > (1 - \epsilon/2)\lambda^y(\mathbb{T}^y) \). By the definition of \( Z_1 \) it follows that for every \((y, r), (y', r') \in Z_1 \) with \( d^y((y, r), (y', r')) < \delta \), we have \( d^y((y, r), (y', r')) = \|y - y'\| + |r - r'| \).

Let \( B_n := \bigcup_{i=\left[-\frac{1}{q_n}, \frac{1}{q_n}\right]} R^i \), then \( \lambda(B_n) < \frac{1}{\sqrt{q_n}} \). Since \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{q_n}} < \infty \) (\( (q_n) \) grows exponentially), there is an \( m \geq N \) such that \( \lambda(\bigcup_{n=m}^{\infty} B_n) < \frac{\epsilon}{4g_{\max}} \).

Define

\[
Z_2 := \left\{(x, s) \in \mathbb{T}^y : x \notin \bigcup_{n=m}^{\infty} B_n \right\},
\]
then \( \lambda^g(Z_2) > (1 - \epsilon/2) \lambda^g(\mathbb{T}^d) \). Finally, let \( Z = Z_1 \cap Z_2 \). Notice that we have \( \lambda^g(Z) > (1 - \epsilon) \lambda^g(\mathbb{T}^d) \).

**Main estimates:** Take \((x, s) \in E_k \) and \((x', s) = A_k(x, s)\), where \( k \in \mathbb{N} \) is such that \( d^f(A_k, Id) < \epsilon \) and let \((y, r), (y', r') \in Z \) with \( d^g((y, r), (y', r')) \leq \delta \). Let \( n \in \mathbb{N} \) be the unique integer such that

\[
\frac{1}{2q_{n+1}} \leq \|y - y'\| < \frac{1}{2q_n}. \tag{3.37}
\]

Let \( \ell_0 \) be the smallest positive integer that \( \{\ell_0 \alpha\} \in [y, y'] \). It is clear by the definition of \( Z \) that \( \sqrt{q_n} \leq \ell_0 \leq q_{n+1} \), and \( \{j \alpha\} \notin [y, y'] \) for any \( j \in [1, \ell_0 + q_n] \) with \( j \neq \ell_0 \). Hence if \( j \in [1, \ell_0 - 1] \), by (3.36)

\[
|g^{(j)}(y) - g^{(j)}(y') - jA_g y - y'| \leq |g^{(j)}_{ac}(y) - g^{(j)}_{ac}(y')| \leq \frac{\epsilon^3}{20}; \tag{3.38}
\]

and if \( \ell_0 < j < \ell_0 + q_n \),

\[
|g^{(j)}(y) - g^{(j)}(y') \pm A_g| - jA_g y - y'| \leq |g^{(j)}_{ac}(y) - g^{(j)}_{ac}(y')| \leq \frac{\epsilon^3}{20}, \tag{3.39}
\]

where \( \pm \) depends only on \( y, y' \).

For \((x, s), (x', s)\), let \( m \in \mathbb{N} \) be the smallest integer such that \( 0 \in R^m[x, x'] \). By the construction of \( E_i \), \( n_k \leq m \leq c^2 q_{b_k} \). Similarly, by (3.36) for any \( j \in [1, m - 1] \),

\[
|f^{(j)}(x) - f^{(j)}(x') - jA_f(x - x')| \leq \frac{\epsilon}{20} \max\{1, j\|x - x'\|\} < \frac{\epsilon^3}{20}, \tag{3.40}
\]

and for any \( j \in [m + 1, m + q_{b_k}] \),

\[
|f^{(j)}(x) - f^{(j)}(x') - A_f - jA_f(x - x')| \leq \frac{\epsilon}{20} \max\{1, j\|x - x'\|\} < \frac{\epsilon^3}{10}. \tag{3.41}
\]

Reasoning as in the previous subsection (see (3.9), (3.10) and the proof of the **Claim**) it suffices to show that there exists an interval \([M', M' + L']\) with \( \frac{L'}{M'} \geq \epsilon^3 \), such that for every \( n \in [M', M' + L'] \) and some \((p, q) \in P\), we have

\[
|f^{(n)}(x) - f^{(n)}(x') - p| < \epsilon^2. \tag{3.42}
\]
and

\[ |g^{([\xi^{-1}])}(y) - g^{([\xi^{-1}])}(y')| < \epsilon^2. \]  \quad (3.43)

We consider three cases, which are analogous to Subcase 1–Subcase 3 in Case 3, in Subsection 3.4.1.

**A.** \( \ell_0 \leq (1 - \epsilon)m\xi^{-1} \). Notice that for every \( j \in [(1 - \epsilon^3)\xi\ell_0, \xi\ell_0) \) and \( j \in (\xi\ell_0, (1 + \epsilon^3)\xi\ell_0] \), (note that \( (1 + \epsilon^3)\xi\ell_0 < m \)) by (3.40), we have

\[ |f^{(j)}(x) - f^{(j)}(x') - \xi\ell_0Af(x - x')| \leq \epsilon^3\xi\ell_0A_f\|x - x'\| + \epsilon^3/10 \leq \epsilon^2, \]  \quad (3.44)

and if \( p := \xi\ell_0Af(x - x') \), then

\[ |p| \leq \xi\ell_0A_f\|x - x'\| \leq \epsilon^3A_f. \]  \quad (3.45)

Moreover, by (3.38) and (3.37), for every \( j \in [(1 - \epsilon^3)\xi\ell_0, \xi\ell_0) \), and the fact \( \ell_0 \leq a_0q'_1 \), we have

\[ \left| g^{([\xi^{-1}])}(y) - \left[ \xi\ell_0\right]A_g \|y - y'\| - \left[ \xi\ell_0\right]A_g \|y - y'\| \right| \leq \epsilon^3\xi\ell_0A_g\|y - y'\| + \epsilon^3/20 \leq \epsilon^2. \]  \quad (3.46)

Similarly, by (3.39) and (3.37), for every \( j \in [\xi\ell_0 + 1, (1 + \epsilon^3)\xi\ell_0] \), and \( \ell_0 \leq a_0q'_1 \), we have

\[ \left| g^{([\xi^{-1}])}(y) - \left[ \xi\ell_0\right]A_g \|y - y'\| \right| \leq \epsilon^2. \]  \quad (3.47)

Notice that if \( q_1 := \left[ \xi\ell_0\right]A_g \|y - y'\| \) and \( q_2 := \pm A_g - \left[ \xi\ell_0\right]A_g \|y - y'\| \), then

\[ \max(|q_1|, |q_2|) \geq A_g/2. \]  \quad (3.48)

By (3.44), (3.46) and (3.47) it follows that (3.42) and (3.43) hold with \( p, q_1 \) on

\[ [M'_1, M'_1 + L'_1] = [(1 - \epsilon^3)\xi\ell_0, \xi\ell_0 - 1] \]

and also with \( p, q_2 \) on

\[ [M'_2, M'_2 + L'_2] = [(\xi\ell_0) + 1, (1 + \epsilon^3)\xi\ell_0]. \]

By (3.45) and (3.48), at least one of \((p, q_1), (p, q_2)\) belongs to \( P \).
B. $\ell_0 \geq (1 + \epsilon)m\xi^{-1}$.

In this case let $M_1' := m + 1$ and $L_1' = \epsilon^3 m - 1$. Notice that $M_1' + L_1' < \xi \ell_0$ and hence for every $j \in [M_1', M_1' + L_1']$ (reasoning analogously to A.), by (3.41), we have

$$|f^{(j)}(x) - f^{(j)}(x') - p_1| < \epsilon^2,$$

where $p_1 := A_f - M_1'A_f(x - x')$ and similarly by (3.38)

$$|g^{(\ell^{-1}j)}(y) - g^{(\ell^{-1}j)}(y') - q_1| \leq \epsilon^2,$$

where $q_1 := [\xi^{-1}M_1']A_y(y - y')$. Let $M_2' := \lceil m/2 \rceil$ and $L_2' = \epsilon^3 M_2'$. Then for every $j \in [M_2', M_2' + L_2']$ (reasoning analogously to A.), by (3.40), we have

$$|f^{(j)}(x) - f^{(j)}(x') - p_2| < \epsilon^2,$$

where $p_2 = -M_2'A_f(x - x') \leq \epsilon^3 A_f$ and similarly by (3.38)

$$|g^{(\ell^{-1}j)}(y) - g^{(\ell^{-1}j)}(y') - q_2| \leq \epsilon^2,$$

where $q_2 := [\xi^{-1}M_2']A_y(y - y')$. We will show that (3.42) and (3.43) hold either on $[M_1', M_1' + L_1']$ or on $[M_2', M_2' + L_2']$.

By the above, we only need to show that at least one of $(p_1, q_1)$ and $(p_2, q_2)$ belongs to $P$. Notice that $|q_2| \geq \frac{|q_1|}{3}$, $|p_1| \geq 10|p_2|$ and $|p_1| \geq \frac{A_f}{2}$ (since $c > 0$ is small). Therefore if $|p_2 - q_2| < \epsilon^2$, then $|p_1 - q_1| \geq \epsilon^2$. This finishes the proof in B..

C. $(1 - \epsilon)m\xi^{-1} \leq \ell_0 \leq (1 + \epsilon)m\xi^{-1}$. The proof is similar to the proof in B..

Let $M_1' := (1 - \epsilon)\min(\xi \ell_0, m)$ and $L_1' := \epsilon^3 M_1'$, $M_2' := (1 + \epsilon)\max(\xi \ell_0, m)$ and $L_2' := \epsilon^3 M_2'$. Notice that for every $j \in [M_1', M_1' + L_1']$, by (3.40) we have

$$|f^{(j)}(x) - f^{(j)}(x') - p_1| < \epsilon^2,$$  \hspace{1cm} (3.49)

where $p_1 = M_1'A_f(x - x')$ and similarly by (3.38)

$$|g^{(\ell^{-1}j)}(y) - g^{(\ell^{-1}j)}(y') - q_1| \leq \epsilon^2,$$  \hspace{1cm} (3.50)
where \( q_1 := [\xi^{-1} M_1'] A_g (y - y') \). Similarly, for every \( j \in [M_2', M_2' + L_2'] \), by (3.41) we have
\[
|f^{(j)}(x) - f^{(j)}(x') - p_1| < \epsilon^2, \tag{3.51}
\]
where \( p_2 = -A_f + M_2' A_f (x - x') \) and similarly by (3.39)
\[
\left| |g^{(k^{-1} j)}(y) - g^{(k^{-1} j)}(y')| - q_1 \right| \leq \epsilon^2, \tag{3.52}
\]
where \( q_2 = \pm A_g - [\xi^{-1} M_2'] A_g (y - y') \).

Notice that \( |p_1| < c^3 A_f, |q_1| < \xi^{-1} A_g \) and \( |p_2| < A_f + c^3 A_f \) and \( |q_2| < A_g + \xi^{-1} A_g \). If \((p_1, q_1) \in P\), then (3.42) and (3.43) hold on \([M_1', M_1' + L_1']\) (see (3.49) and (3.50)). If \((p_1, q_1) \notin P\), then
\[
|p_1 - q_1| < c^3,
\]
and so \( |q_1| < 2c^3 \max(1, A_f) \). We also have \( ||q_2| - A_g| \leq 2|q_1| \leq 4c^3 \max(1, A_f) \).

Therefore
\[
||p_2| - |q_2|| \geq |A_f - A_g| - c^3 A_f - 4c^3 \max(1, A_f) > \frac{1}{2} |A_f - A_g| \geq c^2,
\]
if \( c \) is small enough. Therefore \((p_2, q_2) \in P\) and (3.42) and (3.43) hold on \([M_2, M_2 + L_2]\) (see (3.51) and (3.52)).

This finishes the proof of \( C \), and hence also the proof of Theorem 3.1.4. \( \square \)

### 3.5 Proof of Corollaries 3.1.2 and 3.1.3

**Proof of Corollary 3.1.2.** Let \( D := DC \) (see (3.2)) and let \( \alpha \in D \) and \( \phi, \psi \in C^\infty(\mathbb{T}^2) \). Recall that for \( \tau \in \{\psi, \phi\} \), the flow \((v_t^{\alpha, \tau})\) is isomorphic to \((T_t^{\alpha, f_\tau})\), where \( A_{f_\tau} = \int_{\partial D} \tau(v_t^{\alpha}) dt > 0 \). Moreover, \( \psi, \phi \) are cohomologous for \((v_t^{\alpha})\) if and only if \( f_\psi, f_\phi \) are cohomologous for \( R_\alpha \)\(^4\). By Theorem 3.1.4 and [22], it follows that the latter holds if and only if \( A_{f_\psi} = A_{f_\phi} \). Indeed, the fact that it is necessary follows from Theorem 3.1.4, and that it is sufficient from [22], as in this case (since the

\(^4\)This follows from the fact that \((f_\tau, R_\alpha)\) is the first return map and the first return time of the flow \((v_t^{\alpha, \tau}), \tau \in \{\psi, \phi\}\).
jumps cancel out)

\[ f_\psi - f_\phi \in C^\infty_0(\mathbb{T}), \]

here we also use the fact that \( \int_{\mathbb{T}^2} \phi d\lambda_{\mathbb{T}^2} = \int_{\mathbb{T}^2} \psi d\lambda_{\mathbb{T}^2} \) to know that \( \int_{\mathbb{T}} (f_\psi - f_\phi) d\lambda_{\mathbb{T}} = 0. \)

\[ \square \]

Proof of Corollary 3.1.3. From the special representation it follows that \( (v_t^\alpha) \) is isomorphic to \( (T_t^{\alpha,f}) \), where \( A_f \neq 0 \). So for \( r > 0 \), \( (v_t^\alpha) \) is isomorphic to \( (T_t^{\alpha,r^{-1}f}) \).

Using this for \( r \in \{p, q\} \), it is enough to show that \( (T_t^{\alpha,p^{-1}f}) \) and \( (T_t^{\alpha,q^{-1}f}) \) are disjoint. This however follows from Theorem 3.1.4 since \( |A_{p^{-1}f}| = p^{-1}|A_f| \neq q^{-1}|A_f| = |A_{q^{-1}f}| \). This finishes the proof. \( \square \)
Bibliography


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