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# VORTICES IN THE GINZBURG-LANDAU SUPERCONDUCTIVITY MODEL

A Dissertation in
Mathematics
by
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# Abstract

In this dissertation we analyze the behavior of vortices in superconductors. The vortices might appear in a superconductor when it is immersed in a magnetic field and lead to the energy dissipation which makes them important to study.

In Chapter 1 we discuss the main concepts related to superconductivity and introduce the Ginzburg-Landau model that describes this phenomenon. We explain the nature of superconductivity and the Ginzburg-Landau model in both physical and mathematical aspects. The model itself describes the *ground state* of the superconductor that is the lowest energy state. Mathematically the ground state can be found via a minimization problem, where an energy functional is being minimized among all possible physical states to achieve the lowest possible value. The main focus of this work are the vortices that appear in superconductor as certain singularities when the applied magnetic field is strong enough. The appearance of vortices leads to the energy dissipation, therefore we introduce a way to suppress it via the *pinning* effect. Pinning happens when there is a columnar defect (CD) in a superconductor that traps nearby vortices to itself. In this work we consider a superconductor of a cylindrical shape so that it can be modeled by a 2D cross section. The columnar defects are then the cylindrical tunnels of damaged material, material with different conductivity, or an empty holes, that extend through the sample. Superconductivity is weakened in CDs and the energy cost of locating a vortex in this region is reduced.

In Chapter 2 (with L. Berlyand, V. Rybalko, and V. Vinokur [1]) we show that a particular configuration of the geometry of a superconducting sample as well as the strength of the magnetic field leads to the pinning effect which presents an unexpected distribution of pinned vortices. The CDs are arranged in a periodic lattice with a small period and mathematically modeled as holes with even smaller radii. We call the vortices in the CDs *hole vortices* as opposed to *bulk vortices* that appear outside of them. The degree of the hole vortex is a mathematical concept that essentially counts the number of vortices trapped in a CD. We assume that all vortices are pinned and none of them appear outside of CDs. This is

modeled using a constraint $|u| = 1$, where $u$ is the order parameter, the function that describes two conducting phases in a sample. Essentially this means that there is a perfect superconducting phase outside of CDs. Under this assumption, we find that a superconductor admits a hierarchical nested domain structure where these domains have a different average number of vortices pinned at each CD inside them. The average number of vortices follows the integer sequence starting at 0 in the outer subdomain with the only exception for the most inner subdomain where this number might be fractional. This ground state is completely different from what was observed before both experimentally and analytically.

Chapter 3 (with D. Golovaty, V. Rybalko, and L. Berlyand [2]) justifies the assumption used in Chapter 2. We consider a similar setting to the one used in Chapter 2 with CDs arranged randomly instead of periodically. The number of CDs is fixed and does not increase when their size becomes smaller. We show that this setting leads to the absence of vortices outside of CDs. Moreover, the absence of bulk vortices and the potential term in the Ginzburg-Landau energy suggest that the absolute value of the order parameter $u$ should be close to 1 at the majority of the domain. This is shown by considering a constraint $|u| = 1$ on the Ginzburg-Landau energy minimization problem. We prove that the energies of both unconstrained and constrained problems are close to each other and their minimizers have the same degrees of the hole vortices at the corresponding CDs. This result allows us to consider a simpler $S^1$-valued problem when we need to find the distribution of the degrees of the vortices that is exactly what was done in order to find the nested structure of subdomains in Chapter 2.

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# Chapter 1 Introduction

## 1.1 Phenomenon of Superconductivity

### 1.1.1 General Explanation

Superconductivity is a complete loss of resistance in conductors that appears at low temperatures. This phenomenon was first observed by a Dutch physicist Heike Kamerlingh Onnes on April 8, 1911 in Leiden [3]. He studied the behavior of mercury at low temperatures and found out that at 4.2K the resistance completely disappeared. This phenomenon is observed below a certain critical temperature and this value depends on the material of superconductor Figure 1.1.

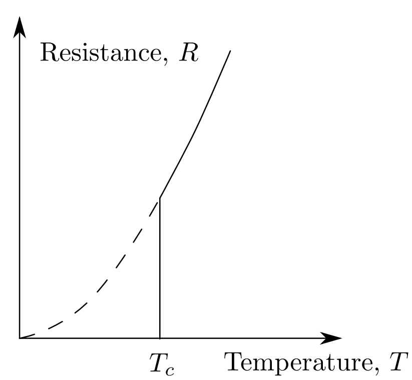



Figure 1.1: Complete loss of resistance in low temperatures.

Whereas superconductivity previously was only observed at very low temperatures, some materials experience the same behavior when cooled by liquid nitrogen. Their critical temperature is above $77K$ and thus they are called *high-temperature* *superconductors* (HTS) [4]. At the beginning these were only some cuprates (materials consisting of copper and oxygen such as BSCCO and YBCO). They experienced the superconducting state at temperatures as high as $137K$. The most recent results show that hydrogen sulfide $H_2S$ shows superconducting properties under very high pressure and the temperature of only $203K$ [5].

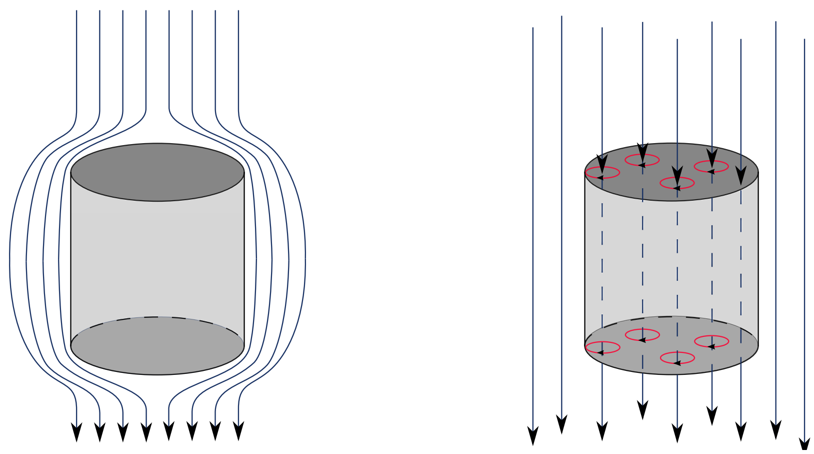
The mathematical study of superconductivity at a macroscopic scale is governed by the **Ginzburg-Landau theory** that is a phenomenological model derived in 1950. Later it was shown that this theory agrees with a microscopic BardeenâĂŞ-CooperâĂŞSchrieffer (BCS) theory of superconductivity named after three physicists that discovered it, John Bardeen, Leon Cooper, and John Robert Schrieffer.

### 1.1.2 Superconductors in the Magnetic Field

We model superconductivity using a complex-valued field $u(x)$ called the *order parameter* where $x$ varies throughout the bulk of a superconductor. The absolute value of it squared $|u|^2$ represents the local density of *Cooper pairs* of electrons which provide superconductivity. Thus the regions with $u = 0$ represent the normal, non-superconducting state, whereas the regions with $|u| = 1$ are superconducting.

The superconductors, that are put in a magnetic field, exhibit two qualitatively different types of behavior that are called **type-I** and **type-II** superconductors. Type-I superconductors have a perfect superconducting state with $|u| = 1$ when the applied field is weak. The magnetic field is expelled from the bulk resulting in what is known as the **Meissner effect** Figure 1.2a which was discovered by W. Meissner and R. Ochsenfeld in 1933. However, the superconducting state is lost completely and $u = 0$ when the field reaches some critical value $T = T_c$ called the **critical magnetic field**. Such qualitative change in behavior is called a first-order phase transition because the order parameter experiences a discontinuity when temperature passes through the critical value $T = T_c$.

Type-II superconductors exhibit second-order phase transitions and possess an intermediate **mixed** state and there are two critical magnetic fields Figure 1.3. Below the **first critical field** $H_{c1}$ one observes the same Meissner effect as in Type-I superconductors. In a region between $H_{c1}$ and the **second critical magnetic field** $H_{c2}$ the magnetic field starts penetrating the cross-section of the bulk at isolated points Figure 1.2b. The nonzero magnetic flux through these points generates the



(a) Meissner effect

(b) Mixed state with magnetic flux lines penetrating the bulk of a superconductor

Figure 1.2: Different states observed in superconductors

**vortices** of current surrounding them. The stronger magnetic field is, the larger number of vortices appear. When the magnetic field is greater than $H_{c2}$, the mixed state turns into a non-superconducting state with superconductivity left only near the boundary of the sample. With the further increase of the magnetic field the boundary superconductivity disappears too leaving the sample at the normal state.

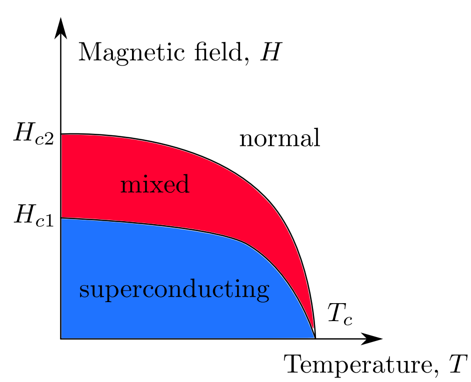



Figure 1.3: The phase diagram with 3 distinct states in Type-II superconductors.

In this work we consider the Type-II superconductors and our main objects of study are the vortices.

## 1.2 Mathematical Model

### 1.2.1 Ginburg-Landau Model

The phenomenological model for superconductivity was suggested by V. Ginzburg and L. Landau in 1950 [6]. Later it was shown this model agrees with a microscopic theory developed by John Bardeen, Leon Cooper, and John Robert Schrieffer (BCS theory) [7].

The ground state of a superconductor is determined by the Ginzburg-Landau (GL) functional:

$$\mathcal{F}[\mathbf{A},\Psi] = \int\limits_{\widetilde{\Omega}} \left[ \frac{\hbar^2}{4m} \left| \left( \nabla - \frac{2ie}{\hbar c}\mathbf{A} \right) \Psi \right|^2 + \alpha|\Psi|^2 + \frac{\beta}{2}|\Psi|^4 \right] dx + \frac{1}{8\pi} \int\limits_{\mathbb{R}^3} (\operatorname{curl}\mathbf{A} - \mathbf{H}_a)^2 dx. \tag{1.1}$$

In this work we consider a cylindrical sample $\widetilde{\Omega} = \Omega \times \mathbb{R}$ and the external magnetic field $\mathbf{H}_a = (0, 0, h_a)$ both aligned with the $z$-axis. This allows us later to consider a two-dimensional cross section of a superconductor. $\Psi(x)$ is the superconducting order parameter, $m$ and $-e$ are electron mass and charge, respectively, $\mathbf{A}(x)$ is the vector potential related to the magnetic induction by $\mathbf{B} = \operatorname{curl}\mathbf{A}$, and the integration is taken over the sample volume $\widetilde{\Omega}$. The characteristic parameters of a particular sample are coherence length and London penetration depth which are expressed through the coefficients of the GL functional:

$$\xi^2 = \frac{\hbar^2}{4m|\alpha|} \quad \text{and} \quad \lambda^2 = \frac{mc^2\beta}{8\pi e^2|\alpha|}, \tag{1.2}$$

respectively. The concept of penetration depth comes from the London equation for the magnetic field in superconductors:

$$\Delta\mathbf{B} = \frac{1}{\lambda^2}\mathbf{B}. \tag{1.3}$$

The solution of (1.3) decreases exponentially from the boundary. For example, for $\Omega = \{x \in \mathbb{R}^3 \mid x_3 > 0\}$ the solution is $B(x) = B(x_1, x_2, 0)e^{-x_3/\lambda}$. Thus penetration depth characterizes the exponential decay of the magnetic field inside superconductors. The coherence length exists because the density of Cooper pairs of superconducting electrons $|u|^2$ cannot change too fast. Therefore there should be a transition layer of finite thickness between superconducting and normal states characterized by the coherence length. Together these two parameters can be combined in a single Ginzburg-Landau parameter $\kappa = \lambda/\xi$. The two types of superconductors can be differentiated by $\kappa$ with Type-II superconductors having $\kappa > 1/\sqrt{2}$.

For the purpose of mathematical analysis of the GL functional and its minimizers, we use the dimensionless order parameter $u = \frac{\Psi}{\Psi_0}$ with $\Psi_0 = -\sqrt{\alpha/\beta}$, and measure length in the units of $\lambda$ and the magnetic fields in the units of $2H_{c1}/\ln\kappa = \Phi_0/(2\pi\lambda\xi)$. After the rescaling and dropping $z$-coordinate we obtain the following two-dimensional Ginzburg-Landau functional:

$$GL_\delta[u, A] = \frac{1}{2}\int_\Omega |\left(\nabla - iA\right)u|^2\,dx + \frac{\kappa^2}{4}\int_\Omega (1 - |u|^2)^2\,dx + \frac{1}{2}\int_{\mathbb{R}^2} (\mathrm{curl}\,A - h_{ext})^2\,dx \tag{1.4}$$

where $u(x) \in \mathbb{C}$ and $A(x) \in \mathbb{R}^2$.

A lot of basic results corresponding to the mathematical analysis of the GL functional are described in the major work of F. Bethuel, H. Brezis, and F. Helein [8] where they studied the so-called *simplified Ginzburg-Landau functional* with no magnetic field.

### 1.2.2 Vortices

Mathematically the vortices are characterized by the order parameter being zero at an isolated point with a nonzero winding number around it. The winding number is called the *degree of the vortex* and can be defined on any curve $\gamma$ surrounding vortex without any other zeros of $u$ inside it (see Figure 1.4):

$$\deg(u, \gamma) = \frac{1}{2\pi}\int_\gamma v \wedge \frac{\partial v}{\partial \tau}\,ds \tag{1.5}$$

where $a \wedge b = a_1 b_2 - a_2 b_1$ and $v = u/|u|$.

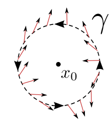



Figure 1.4: The vortex at $x_0$ is a zero of the order-parameter $u$ with a nonzero winding number.

### 1.2.3 Pinning

In the presence of an electro-magnetic field the vortices can move inside the sample which leads to the dissipation of energy [9]. This can be prevented by creating the pinning sites inside the sample which capture the surrounding vortices. This can be done by the drilling the micro-holes or damaging material with an ion bombardment technique [10]. Understanding the role of imperfections in a superconductor can then be used to design more efficient superconducting materials.

The columnar defects, that work as the pinning sites, can be modeled by several different ways: replacing the potential term with $(a(x) - |u|^2)^2$ where $a(x)$ varies throughout the sample [11–14]; considering a composite of two superconducting materials with different properties [15, 16]; and treating CDs as holes with no current inside. In this work we consider the latter. Thus we work with an arbitrary domain $\Omega \subset \mathbb{R}^2$ with holes $\{\omega_j\}_{j\in\mathfrak{J}}$ that are arranged either periodically in Chapter 2 or randomly in Chapter 3.

We say that the hole $\omega_j$ has a vortex of degree $d_j$ inside if there exists a curve $\gamma$ surrounding $\omega_j$ with no zeros of the order parameter $u$ on it and such that the $d_j = \deg(u, \gamma) \neq 0$. If $|u| = 1$ on $\partial\omega_j$, then we take $\gamma = \partial\omega_j$ and this definition is trivial. In the general complex-valued case this definition requires a special choice of the curve $\gamma$ which is explained in details in Chapter 3.

## 1.3 Results

In Chapter 2 we consider a two-dimensional cross section of a superconducting sample with a lattice of small holes in the regime when both the period of the lattice as well as hole size are shrinking. We show that under a certain relation between the geometry of a superconductor, the strength of the applied magnetic field, and the properties of the material of the sample, the hole vortices form a nonuniform nested structure of subdomains. Each subdomain is characterized by the average number of vortices trapped at each hole inside it. These numbers increase from 0 at the most outer subdomain towards most inner subdomain following the integer sequence with possible exception at the last subdomain. This structure is shown on Figure 1.5.

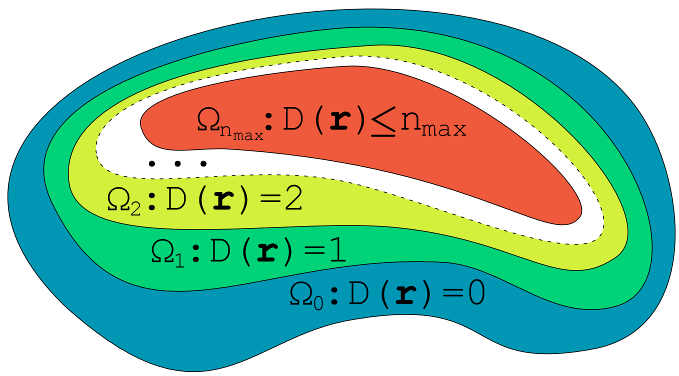



Figure 1.5: **Phase separation in distribution of vortices captured by the columnar defects**. Instead of homogeneous distribution all over the sample the vortices form a nested sequence of the domains characterized by the filling factor of the defects which grows from the borders towards the center of the sample.

This result is obtained using a conjecture that the GL functional (1.4) can be replaced by a simpler functional with the constraint $|u| = 1$. This conjecture is justified in Chapter 3 for domains with finitely many shrinking holes arranged without an order. The main result of Chapter 3 is in Theorem 1.

# Chapter 2
# Vortex phase separation in mesoscopic superconductors

## 2.1 Introduction

In this chapter we demonstrate that in mesoscopic type-II superconductors with the lateral size commensurate with London penetration depth the ground state of vortices pinned by homogeneously distributed columnar defects can form a hierarchical nested domain structure. Each domain is characterized by an average number of vortices trapped at a single pinning site within a given domain. Our study marks a radical departure from the current understanding of the ground state in disordered macroscopic systems and provides an insight into the interplay between disorder, vortex-vortex interaction, and confinement within finite system size. The observed vortex phase segregation implies the existence of the soliton solution for the vortex density in the finite superconductors and establishes a new class of nonlinear systems that exhibit the soliton phenomenon.

Vortex matter in the presence of structural defects forms a wide variety of phases with specific properties depending on the relation between the vortex-vortex and vortex-defect interactions [17, 18]. The findings of [19, 20], which revealed a significant enhancement of vortex pinning in high-temperature superconductors by ion irradiation, broke ground for a new direction in vortex physics. Heavy ions leave the tracks of the damaged amorphous material where superconductivity is suppressed. Thus the vortices penetrating the sample occupy columnar defects where the vortex energy is appreciably less than in the undamaged material. A theory of the resulting vortex Bose glass phase was developed in [21, 22], where the physics of flux lines in superconductors pinned by columnar defects was mapped onto boson localization in two dimensions. The distribution of vortices in the Bose glass state that forms in the infinite (i.e. thermodynamically large) samples, containing columnar defects, is a uniform one. A question about what happens to the Bose glass in the finite samples is most natural in view of explosively developing studies of small superconductors, i.e. superconductors with the lateral sizes $R_s$ comparable to the London screening length $\lambda$ or even with the coherence length $\xi$. Indeed even the samples without columnar defects reveal that the properties of the homogeneous vortex state change dramatically as $R_s \lesssim \lambda$. The boundaries start to affect the distribution of vortices and makes it nonuniform. Experimental study of mesoscopic superconducting discs with the total vorticity $L < 40$ revealed formation of the concentric shells of vortices [23] in accord with the results of numerical simulations [24]. The analysis of shell filling with increasing $L$ allowed the authors of [23] to identify magic numbers corresponding to the appearance of consecutive new shells. At the same time, vortex distribution over the sample remains "quasi-homogeneous" with the vortex density gradually changing with the distance from the sample center. For example, the experimental and numerical studies of the samples containing a macroscopic number of vortices showed that, almost everywhere, vortices arrange themselves into a nearly perfect Abrikosov lattice, containing the few disclinations necessary to match the cylindrical symmetry of the sample. Only within a few, 2-3, shells adjacent to the surface, vortex distribution differs noticeably from that in the bulk.

At the same time, theoretical consideration of the critical state in a superconducting slab containing a lattice of strong pins [25] predicted that instead of the expected in the critical state constant gradient in the vortex density a terraced piecewise vortex structure structure can form. This terraced vortex distribution, unexpected from the viewpoint of an orthodox concept of the critical state, is, formally, nothing but a standard soliton solution for the one-dimensional commensurate structures, which appeared first as a 1D model for dislocations [26, 27]. The physical reason for emerging such a structure is the competition between the effect of the critical current flowing uniformly through the slab and thus implying the constant gradient of the vortex density across the sample and the action of the lattice of strong pinning sites that tend to trap vortices enforcing them into a regular array with the commensurate period. As a result, a metastable structure forms, comprising vortex domains of a piecewise constant vortex density. The originally uniform current is compressed into the current filaments concentrated along the boundaries between the domains i.e. in the narrow regions of the maximal gradient of the vortex density. The terraced metastable critical state, which was indeed found experimentally [28], establishes a fruitful connection between the breaking down of a vortex configuration into the domains, characterized by the different vortex density, and a general concept of formation of the regular patterns in non-linear media, which often allows for a description in a general framework of the soliton physics.

Here we consider an *equilibrium* vortex system and report that the *ground state* of vortices pinned by homogeneously distributed large columnar defects can form a hierarchical nested domain structure, where each domain is characterized by its own filling factor, the average number of vortices trapped at a single pinning site. In view of the above connection between vortex phase segregation and soliton description of the commensurate structures, our finding also breaks ground to novel approach to soliton physics. Contrasting past models where soliton structures were established by the explicit writing-down of an analytical solution to a particular nonlinear 1D equation, our work rigorously proves the principal and fundamental existence of a soliton terraced solution for *equilibrium* vortex density. Significantly, by considering a cylindrical sample with an arbitrary base, our approach goes beyond the 1D physics.. We develop our approach in the context of vortex pinning by large columnar defects in a small superconductor with $R_{\rm s} \simeq \lambda$ in the low field range $H < H_{\rm c1}$, where $H_{\rm c1}$ is the lower critical field. Since the energy of trapped vortices is less than those in the bulk, vortices penetrate the superconductor even in this field range. The thermodynamics of the Bose glass at $H < H_{\rm c1}$ was investigated in [29], where the equation of states and the Bose-glass transition line were found, but the effects of the finite size were not addressed. Here we show that the interplay of vortex interaction, and pinning in a mesoscopic superconductor can result in a hierarchical domain structure of the *ground* vortex state. Importantly, after the coarse graining procedure described below, our model becomes quasicontinuous and the discreteness (or periodicity) of pinning arrays does not come explicitly into play.

## 2.2 Results

We consider a large but finite superconducting sample in the form of a generalized cylinder, with a base of arbitrary shape, see Fig. 1, with the characteristic linear size of the base $R_s \simeq \lambda$, where $\lambda \gg \xi$ is the London penetration length, containing a square array (with spacing $a$ such that $R \ll a \ll R_s$) of cylindrical (columnar) vortex traps with radii $R$ much exceeding the vortex core size $\xi$. This is an exemplary model system for superconductors that contain arrays of either columnar defects or artificially engineered arrays of holes. Such systems are extensively used in studies of the so-called vortex matching effect, one of the central avenues of contemporary vortex physics (see, for example, [30] and references therein). Since the energy of a vortex trapped by a cylindrical defect is less then its energy in the bulk of an undamaged material, vortices start to occupy the sample containing CDs at fields $H$ below than the thermodynamic lower critical field $H_{c1}$, at which the thermodynamically stable vortices start to exist in a superconductor without CDs [29]. It is this range of fields $H < H_{c1}$ we investigate in our work.

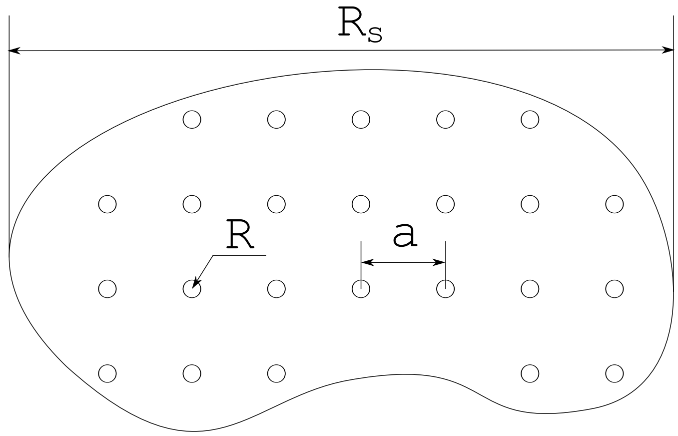



Figure 2.1: **Square array of columnar defects in a cylindrical superconducting sample**. The superconducting sample has a form of a generalized cylinder, with the base of an arbitrary shape with the characteristic linear size $R_s \simeq \lambda$. The period $a$ of the square array of holes of the radii $R \gg \xi$ satisfies the condition $R \ll a \ll R_s$.

To quantify the spatial distribution of trapped vortices we introduce the coarse-grained filling factor density $D(\mathbf{r}) = N_v a^2/A_{\mathbf{r}}$, where $\mathbf{r}$ is the coordinate perpendicular to the cylinder axis $z$, $A_{\mathbf{r}}$ is some area surrounding the point $\mathbf{r}$

and containing many CDs, and $N_v$ is the number of vortex quanta trapped by CDs within this area. So if, for example, each CD in this domain contains exactly one single-quantum vortex, $D(\mathbf{r}) = 1$; if CDs outnumber the trapped vortex quanta, then $D(\mathbf{r}) < 1$. We show that the vortex system confined within such a sample can break up into a sequence of the distinct nested domains (listing from the sample border inward): $\Omega_0\,, \Omega_1\,, \Omega_2\,...\Omega_{n_{\rm max}}$, with the filling factors $D(\mathbf{r} \in \Omega_0) = 0 < D_1 \equiv D(\mathbf{r} \in \Omega_1) < ... < D_{n_{\rm max}} \equiv D(\mathbf{r} \in \Omega_{n_{\rm max}})$, see Fig. 2, respectively. Our analysis shows that $D$ can be any positive integer or a fraction, depending on the relation between the radius of CDs, the superconducting coherence length $\xi$, and the strength of the magnetic field. Accordingly, there exists a sequence of characteristic fields $H_1 < H_2 ... < H_n < H_{\rm c1}$ such that at $H_a < H_1 < H_{\rm c1}$ the superconductor retains its vortex-free Meissner state, at $H_1 < H_a < H_2 < H_{\rm c1}$, appears the finite compact domain $\Omega_1$ with the filling factor $D_1 \leq 1$, and so on. The possible maximal number of the distinctly filled domains, $n_{\rm max}$ and the corresponding maximal filling factor, $D_{n_{\rm max}}$, are determined by the maximal number of vortices which CD can trap as given by the expression [31] $n_{\rm max} = [R/2\xi]$ ($n_{\rm max}$ is derived from the condition that the circular current around the CD due to trapped flux cannot exceed the superconducting pair-breaking current). Using a square array of CDs does not result in a loss of generality and ensures that the observed vortex phase separation is an inherent property of vortex systems that stems from the subtle balance between the confinement and vortex-vortex and vortex-defect interactions rather then a trivial consequence of fluctuations in the defect density.

We will be describing superconductivity of our system and the resulting vortex state within the Ginzburg-Landau (GL) formalism. The ground state is determined by the standard GL functional:

$$\mathcal{F} = \int\limits_{\Omega} \left[ \frac{\hbar^2}{4m} \left| \left( \nabla - \frac{2ie}{\hbar c} \mathbf{A} \right) \Psi \right|^2 + \alpha |\Psi|^2 + \frac{\beta}{2} |\Psi|^4 + \frac{1}{8\pi} (\mathrm{curl}\,\mathbf{A} - \mathbf{H}_a)^2 \right] d^2\mathbf{r} dz \,, \tag{2.1}$$

where the cylindrical sample and the applied magnetic field $\mathbf{H}_a$ are aligned with the $z$-axis, $\Psi(\mathbf{r})$ is the superconducting order parameter, $\mathbf{r}$ is the coordinate vector in the plain perpendicular to the $z$-axis, $m$ and $-e$ are electron mass and charge, respectively, $\mathbf{A}(\mathbf{r})$ is the vector potential related to the magnetic induction by $\mathbf{B} = \mathrm{curl}\,\mathbf{A}$, and the integration is taken over the sample volume $\Omega$. The coherence length and London penetration length are expressed through the coefficients of the

GL functional as $\xi^2 = \hbar^2/(4m|\alpha|)$ and $\lambda^2 = (mc^2\beta)/(8\pi e^2|\alpha|)$, respectively. The properties of a superconducting material are characterized by the Ginzburg-Landau parameter $\kappa = \lambda/\xi$. We consider an extreme type II superconductor such that $\kappa \gg 1$. As usual in the GL analysis, it is convenient to introduce the dimensionless order parameter $u = \Psi/\Psi_0$, where $\Psi_0 = -\sqrt{\alpha/\beta}$, and measure lengths in the units of $\lambda$ and the magnetic fields in the units of $2H_{\rm c1}/\ln\kappa = \Phi_0/(2\pi\lambda\xi)$. The dimensionless columnar defect radius is $\rho = R/\lambda \ll \varepsilon = a/\lambda \ll \rho_s = R_s/\lambda \simeq 1$.

To determine the conditions for the emergence of the vortex domain structure, we require that the CD spacing is not extremely small such that $1/\varepsilon^2 \ll \ln\kappa$. Further we let the CD radius be very small and parametrize it as $\rho = \exp(-\gamma/\varepsilon^2)$, where $\gamma$ is a constant of the order of unity. This means that the characteristic lengths separation is exponentially stronger as compared to the condition $\rho \ll \varepsilon$. And, finally, we parametrize the dimensionless magnetic field as $h = \sigma/\varepsilon^2$, where $\sigma \simeq 1$. The key point of our approach is the observation that under the chosen relations between the characteristic parameters of our system, for the purpose of the determination of the vorticity the amplitude of the order parameter can be taken $|u| = 1$ everywhere in the bulk of the sample except in CDs. This implies that the (dimensionless) GL free energy (depending solely on the distribution of the magnetic field $h = \mathrm{curl}\,\mathbf{A}$) can be reduced to the following form [see Supplementary Information (SI) section]:

$$\mathcal{F}_{\rm G}[u, A] = \frac{1}{2}\int\limits_{\Omega} \left[|(\nabla - i\mathbf{A})u|^2 + (\mathrm{curl}\,\mathbf{A} - \mathbf{h}_a)^2\right] d^2\mathbf{r}dz\,, \qquad (2.2)$$

which we call the *harmonic map functional*. In other words, the distribution of vortices, derived by the minimization of $\mathcal{F}_{\rm G}$ coincides with that obtained by the minimization of $\mathcal{F}$. Varying Eq. (2.2) with respect to $h$ and taking into account the boundary conditions at the boundaries of CDs, one finds the equation for the magnetic field:

$$-\Delta h + h = 2\pi\mu(\mathbf{r})\,, \qquad (2.3)$$

where $\mu(\mathbf{r}) = \sum_j d_j\delta(\mathbf{r} - \mathbf{r}_j)$, $\mathbf{r}_j$ is the coordinate of the $j$-th CD, and $d_j$ is its corresponding vorticity which can also be zero if there are no flux trapped at the particular CD. The remaining task is finding the configuration of the field, i.e. the unknown numbers $d_j$ which minimize the free energy $\mathcal{F}_{\rm G}$. To implement this we coarse grain Eq. (2.3) over the distances exceeding the CD spacing. As a result the r.h.s. of the equation for the coarse grained field $\bar{h}$ will admit the form $2\pi D(\mathbf{r})$, where the average vorticity introduced above can now be rigorously defined as $D(\mathbf{r}) = \lim_{\varepsilon\to 0}\mu(\mathbf{r})$. The boundary conditions for the field now become simply $\bar{h} = \sigma$ at the sample's boundary. The phase separation picture emerges from exploring the dependence of $D(\mathbf{r})$ on the parameter $\sigma$, characterizing the magnitude of the magnetic field. Our main finding can be formulated as follows (see Methods for more detail). Let the radii of the holes (in dimensionless units) be $\rho = \exp(-\gamma/\varepsilon^2)$ and the applied magnetic field be $h_a = \sigma/\varepsilon^2$. There exists a strictly increasing sequence of the critical values $\sigma_{\rm crj}$, $j = 1, 2...$ such that if $\sigma_{\rm crj} < \sigma < \sigma_{\rm cr(j+1)}$, then the average vorticity assumes the constant values in domains $\Omega_{\rm k}$, where $\Omega_{\rm k} \equiv \Omega_{\rm k}(\sigma)$, $k = 0, 1, 2, ..., j$ are strictly nested sets characterized by distinct vorticities. Namely, the vorticity $D(\mathbf{r}) = 0$ in $\Omega_0$ and $D(\mathbf{r}) = k$ in $\Omega_{\rm k}$, $k \leq j - 1$. We further show that when $\mathbf{r} \in \Omega_{\rm j}$ one of the two possibilities can realize: (i) if $\sigma < 2\pi j + (j - 1/2)\gamma$, then $j - 1 < D(\mathbf{r}) < j$, otherwise (ii) $D(\mathbf{r}) = j$.

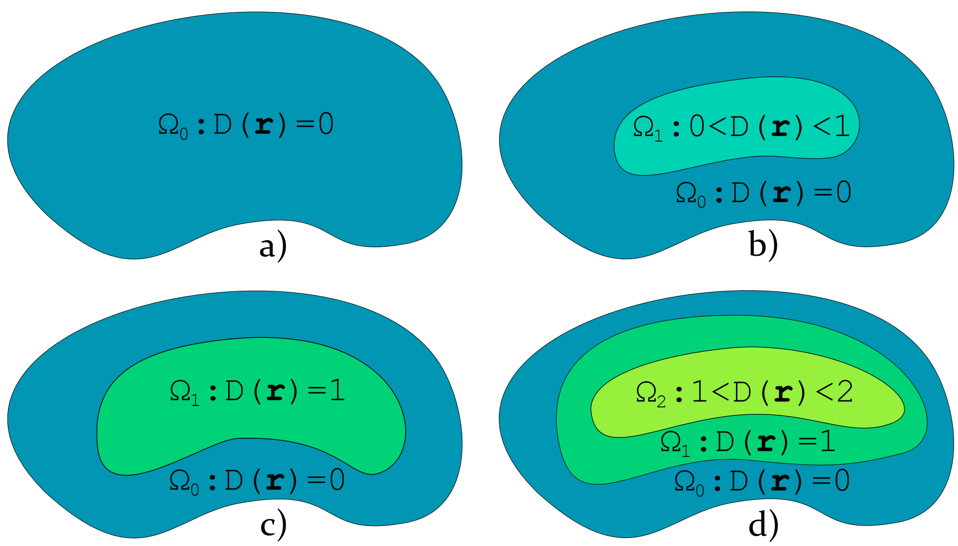



Figure 2.2: **Sequential vortex domain formation upon increasing magnetic field**. At smallest fields vortices are not trapped at all, upon increasing field the central domain where $D < 1$, i.e. where part of the defects captured one vortex forms. With further increasing field, all the traps in the central domain get filled by one vortex each; then, finally, the sequence of domains with $D = 0$, $D = 1$, and $1 < D < 2$, forms.

To illustrate this we consider the evolution of the distribution of the trapped vortices upon increasing magnetic field. Let $\sigma_{\rm k} = 2\pi k + \gamma/2$, $k = 0, 1, 2....$ One can show now, that if $0 < \sigma < \sigma_{\rm cr1} \equiv \gamma/(2 \max |f_1|)$, where $f_1$ is the solution to the equation $\Delta f_1 - f_1 = 1$, $\mathbf{r} \in \Omega$, and $f_1 = 0$ if $\mathbf{r} \in \partial\Omega$, where $\partial\Omega$ means the boundary of the sample, $D(\mathbf{r}) = 0$ and there are no trapped vortices at all (see, Fig. 3a). If $\gamma$ is such that $\sigma_{\rm cr1} < \sigma_1$, then for $\sigma_{\rm cr1} < \sigma < \sigma_1$ the superconductor breaks up into two phases, see Fig. 3b: the vortex-free phase in $\Omega_0$ and the phase of trapped vortices with $0 < D(\mathbf{r}) < 1$ in the domain $\Omega_1$. If now $\gamma$ is such that $\max\{\sigma_{\rm cr1}, \sigma_1\} < \sigma < \sigma_{\rm cr2}$ (the procedure for deriving $\sigma_{\rm crk}$ is described in the Methods section), then in the filled phase $D(\mathbf{r}) = 1$ (i.e. each columnar defect traps exactly one vortex), see Fig. 3c. Continuing this process we find that if $\gamma$ is such that $\sigma_{\rm cr2} < \sigma_2$ then for $\sigma_{\rm cr2} < \sigma < \sigma_2$ the superconductor comprises three nesting hierarchical phases: $\Omega_0$, with no vortices in it, $D(\mathbf{r}) = 0$, $\Omega_1$, where $D(\mathbf{r}) = 1$, and $\Omega_2$, where $1 < D(\mathbf{r}) < 2$. The latter means that some of the CDs in the "interior" trapped double-quanta vortices, but some CDs have only a single vortex captured, so the average filling factor in $\Omega_2$ is less then 2, see Fig. 3d. And if $\gamma$ is such that $\max\{\sigma_{\rm cr2}, \sigma_2\} < \sigma < \sigma_{\rm cr3}$, then the three nesting phases have the vorticity $D(\mathbf{r}) = 0, 1$, and 2 respectively. That is all the CDs of the internal phase are filled with double-quanta vortices. This process can be continued till the maximal possible multi-quanta vortices appear. The maximal multiplicity is determined by the condition $n_{\max} = [R/2\xi]$.

Finally, to complete our consideration we have to check the stability of the established domain structure. To this end we first estimate the typical sizes $R_{\rm D}$ of our domains. Considering for brevity the circular domain $\Omega = B(0, 1)$ and find the dimensionless radius $\tilde{R}_{\rm D}$ of a subdomain $\Omega_1$ in the case when there are just two phases $D(\mathbf{r}) = 0$ and $0 < D(\mathbf{r}) \leq 1$ (in the radially symmetrical case $\Omega_1$ is also a ball), one finds, coming back to dimensional units (see SI)

$$R_{\rm D} \approx 0.567\lambda. \tag{2.4}$$

Now we have to check that density of the current circulating at the boundary between the domains, $J_{\rm s}$ does not exceed the pairbreaking current density $J_0 \simeq \Phi_0/(\lambda^2\xi)$. Taking a single central domain of linear size $L$, with traps capturing exactly one vortex and making use the expression $v_{\rm s} = n\hbar/2mr$ for the Cooper pair velocity around the vortex for $r \gg \xi$ and the relation $J_{\rm s} = n_s e v_{\rm s}$, one easily arrives at the estimate $J_s \sim J_0 R_D \xi/a^2$. This implies that for the domain to be stable the inequality

$$R_D \xi/a^2 < 1 \tag{2.5}$$

must hold. Now let us write down the chain of inequalities which constitute the base of our consideration:

$$\xi \ll R \equiv \lambda \exp\{-\lambda^2\gamma/a^2\} \ll \frac{a^2}{\lambda}, \tag{2.6}$$

i.e. $a^2/\xi \gg \lambda$. Therefore

$$L < \lambda \ll \frac{a^2}{\xi} \tag{2.7}$$

so the condition (2.5) of the stability of the domains automatically follows from the assumed hierarchy of the lengths involved. Therefore, the domains are stable. This analysis can be straightforwardly generalized to an arbitrary number of domains.

## 2.3 Discussion

We have demonstrated that the *equilibrium* ground state of a cylindrical superconductor with a base of arbitrary shape, containing uniformly distributed columnar pins, can develop a hierarchical structure of nesting domains, where each distinct vortex domain is characterized by a sequence of different filling factors. This result takes us beyond the frontiers of conventional soliton physics, where soliton structures resulted from explicit solutions of a particular 1D nonlinear equation. For example, the terraced vortex distribution found in a nonequilibrium (metastable) distribution of vortices in a critical state carries a direct analogy with the soliton structure derived for a 1D system of atoms adsorbed on a periodic substrate [32]. Also, the obtained domain structure differs fundamentally from the nonuniform vortex distribution found in [33], which is generated by the nonuniform arrangement of pinning sites.

The essential feature of our model that ensures a sequence of nested domains is the large radii, $R \gg \xi$, of columnar defects which enable them to capture a large number of flux quanta. The formation of the multiply quantized vortices in the forest of large CDs was already discussed in [34], although the possibility of the formation of distinct domains with different multi-vorticity was not explored. The phase separation discussed above arises as a result of the subtle balance between the different logarithmic contributions to energy of the vortex system: repulsive interaction between the vortices favoring homogenization of their spatial distribution, Meissner currents pushing vortices towards the center of the sample, and interactions of vortices with their images that appear both outside the sample and within the columnar cavities. This suggests that although our results were proved in a mathematically rigorous way, only for specific parameters, one should expect that the main conclusion about vortex phase separation remains valid well beyond the restrictions of the particular model. Note, in this connection, an interesting experimental work [35] where the formation of the vortex clusters and multiquanta vortices was observed. Note that as we have already mentioned, the effect of vortex phase separation can realize in the vortex matching systems with regular arrays of large, $R \gg \xi$, holes and with CD spacing still much exceeding $R$. In this respect, direct scanning tunneling microscopy and spectroscopy (STM/STS) experiments, that have revealed strong confinement effects on the vortex arrangements in extreme type II superconductors and enabled to discriminate between the multi-vortex and multi-quanta vortex formations [36], seem to be a very adequate approach for the search of the vortex phase separation. Furthermore, in the case where vortices are pinned by weak point defects in the collective pinning regime, one can view a pinned vortex line as confined within the slightly curved tube-like potential well of radius $\xi$, which arises self-consistently from the interplay between the pinning and elastic energies [17]. One thus can anticipate that pinned vortices may cluster together to form an array of compact domains of pinned vortices separated by distances well exceeding the size of a domain. The remarkable possibility of searching for segregation of localized phases can be realized in the $^7$Li atomic gas [37, 38] where the interaction strength can be tuned by a Feshbach resonance, thus achieving the required balance between the competing repulsive, pinning, and confining forces.

## 2.4 Methods

The central point of our consideration is minimizing the free energy functional (2.2) with respect to vorticity numbers $d_j$, given the constraint (2.3). The first step in this process is the coarse graining procedure introducing the coarse-grained vorticity $D(\mathbf{r})$ and magnetic field $\bar{h}$. Then the problem reduces to minimization of the coarse-grained energy functional

$$\bar{F}(D(\mathbf{r})) = \frac{1}{2}\int_\Omega \left(|\nabla \bar{h}|^2 + (\bar{h}-\sigma)^2\right) d^2\mathbf{r} + \pi\gamma \int_\Omega \Phi(D(x)) d^2\mathbf{r}\,, \tag{2.8}$$

where

$$\Phi(D) = (2p+1)|D| - p^2 - p, \text{ when } p \le |D| < p+1,\ p = 0,1,2,\dots\,, \tag{2.9}$$

is a piecewise parabolic function, see Fig. 3, under the constraint $-\Delta\bar{h} + \bar{h} = 2\pi D(\mathbf{r})\,, \mathbf{r} \in \Omega$, and $\bar{h} = \sigma$ on $\partial\Omega$. Making use of the Legendre transform of the function $\pi\gamma\Phi(D/2\pi)$ such that $\Phi^*(f) = 2\pi\sup_D[Df - (\gamma/2)\Phi(D)]$, one arrives at the effective energy functional

$$\bar{F}(D) = \frac{1}{2}\int_\Omega \left(|\nabla f|^2 + f^2\right) d^2\mathbf{r} + \int_\Omega [\Phi^*(f) + \sigma f] d^2\mathbf{r}\,, \tag{2.10}$$

which is to be minimized with respect to the auxiliary field $f$ dual to $\bar{h} - \sigma$ so that for minimizing the configuration $f_m = \bar{h}_m - \sigma$. Function $\Phi^*$ is a piecewise linear function of $f$ enveloped by a parabola, see Fig. 4.

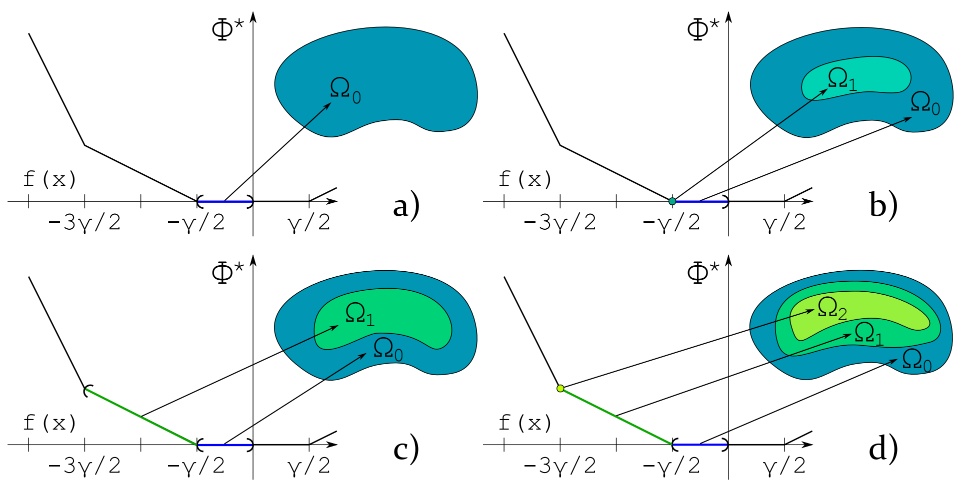



Figure 2.3: **Sequential phase segregation as a function of the auxiliary field $f$.** The stages of the domain formations are the same as in the Fig. 3

Now one can follow the evolution of the solution upon increasing $\sigma$. It follows from (2.10) that $f = 0$ for $\sigma = 0$ and therefore $f \in (-\gamma/2, 0)$ where $\Phi^* = 0$ for sufficiently small $\sigma$. The variation with respect to $f$ in (6) leads to $\Delta f - f = \sigma$ with $f = 0$ for $\mathbf{r} \in \partial\Omega$. Since $2\pi D(\mathbf{r}) = -\Delta f + f + \sigma$, we have $D(\mathbf{r}) = 0$. Function $\Phi^*$ remains zero until $\min f$ reaches value $-\gamma/2$, which defines $\sigma_{\mathrm{cr1}} = \gamma/(2\max|f_1|)$. Upon further increase in $\sigma$ beyond $\sigma_{\mathrm{cr1}}$, $\Phi^*$ acquires the first oblique linear piece, and (2.10) leads to the energy

$$\bar{F} = \int_\Omega \left[\frac{1}{2}\left(|\nabla f|^2 + f^2\right) + \sigma f + 2\pi(|f| - \gamma/2)_+\right] d^2\mathbf{r}. \tag{2.11}$$

Disregarding for the moment the gradient term in (2.11) (which only penalizes variations of $f$) we see a competition of the positive term $f^2 + 4\pi(|f| - \gamma/2)_+$ and the negative one $2\sigma f$ (note that always $f \leq 0$). Now if $\sigma_{\mathrm{cr1}} < 2\pi + \gamma/2$, then the competition eliminates $f < -\gamma/2$ for $\sigma_{\mathrm{cr1}} < \sigma < 2\pi + \gamma/2$. Then we get $f = -\gamma/2$ for $r \in \Omega_1$, where we obtain $0 < D(\mathbf{r}) < 1$. If $\sigma_{\mathrm{cr1}} \geq 2\pi + \gamma/2$, then the negative term wins for $f < -\gamma/2$ that corresponds to $\mathbf{r} \in \Omega_1$, where we have $\Delta f - f = \sigma - 2\pi$, and we get $D(\mathbf{r}) \equiv 1$. The continuation of this procedure further generates a sequence of critical values of $\sigma$ and defines the corresponding sequence of nested domains with the increasing (towards to the inner part of the sample) vorticity as described in the main text. The evolution of the CD filling and formation of the spatially inhomogeneous vortex state with the increase of the reduced field $\sigma$ is illustrated in Fig. 4.

# Chapter 3

# Approximating Ginzburg-Landau minimizers by $\mathbb{S}^1$-valued maps: equality of degrees

## 3.1 Introduction

We consider a two-dimensional Ginzburg-Landau problem on an arbitrary domain with a finite number of the vanishingly small holes. A special choice of scaling relation between the material and geometric parameters (Ginzburg-Landau parameter vs holes radii) is motivated by a striking phenomenon of vortex phase separation in superconducting composites found in Chapter 2. We show that for each hole the degrees of the minimizers of the Ginzburg-Landau problems in the classes of $S^1$-valued and $C$-valued maps, respectively, are the same. The presence of two parameters widely separated on logarithmic scale constitutes the principal difficulty of the analysis based on energy decomposition techniques.

The appearance and behavior of vortices for the minimizers of the Ginzburg-Landau functional

$$GL^{\varepsilon}[u, A] = \frac{1}{2}\int_{\Omega} |\,(\nabla - iA)\,u|^2\,dx + \frac{1}{4\varepsilon^2}\int_{\Omega}(1-|u|^2)^2\,dx + \frac{1}{2}\int_{\Omega}(\operatorname{curl} A - h_{ext})^2\,dx \tag{3.1}$$

have been studied, in particular, in [39, 40] where the existence of two critical magnetic fields, $H_{c1}$ and $H_{c2}$, was established rigorously for a small $\varepsilon > 0$. When the external magnetic field is weak ($h_{ext} < H_{c1}$) it is completely expelled from the bulk semiconductor (Meissner effect) and there are no vortices. When the field strength is between $H_{c1}$ and $H_{c2}$, the magnetic field begins penetrating the superconductor through isolated vortices while the superconductivity is destroyed everywhere once the field exceeds $H_{c2}$.

The pinning phenomenon that we consider in this paper is observed in non-simply-connected domains with holes that may or may not contain another material. If a hole "pins" a vortex the order parameter field has a nonzero winding number on the boundary of the hole. We refer to this situation a *hole vortex*. Note that degrees of the hole vortices increase along with the strength of the external magnetic field. This situation is in contrast with the regular bulk vortices that have degree $\pm 1$ and increase in number as the field becomes stronger.

An alternative way to model the impurities is to consider a potential term $(a(x) - |u|^2)^2$ where $a(x)$ varies throughout the sample. It was proven in [11] that the impurities corresponding to the weakest superconductivity (where $a(x)$ is minimal) pin the vortices first. This model was studied further in [12] and [13] to demonstrate the existence of nontrivial pinning patterns and in [14] to investigate the breakdown of pinning in an increasing external magnetic field, among other issues. A composite consisting of two superconducting samples with different critical temperatures was considered in [15, 16] where nucleation of vortices near the interface was shown to occur.

In our model we consider a superconductor with holes, similar to the setup in [41]. In that work, the authors considered the asymptotic limits of minimizers of $GL^\varepsilon$ as $\varepsilon \to 0$ and determined that holes act as pinning sites gaining nonzero degree for moderate but bounded magnetic fields. For magnetic fields below the threshold of order $|\ln \varepsilon|$ the degree of the order parameter on the holes continues to grow without bound, however beyond the critical field strength, the pinning breaks down and vortices appear in the interior of the superconductor. Since the contribution to the energy from the hole vortices has an inverse dependence on the diameter of the holes, the hole size can be used as an additional small parameter to enforce a finite degree of the hole vortex in the limit of small $\varepsilon$. The domain with finitely many shrinking (pinning) subdomains with weakened superconductivity was considered in [42] in the case of the simplified Ginzburg-Landau functional. The model with a potential term $(a(x) - |u|^2)^2$ with piecewise constant $a(x)$ was used to enforce pinning and it was observed that the vortices are localized within pinning domains and converge to their centers.

The problem considered in this chapter was inspired by the result of the Chapter 2 that dealt with a periodic lattice of vanishingly small holes. The analysis there relies on a conjecture that for small $\varepsilon$, the degrees of the hole vortices are the same for both $\mathbb{C}$- and $\mathbb{S}^1$-valued maps. The principal aim of the current chapter is to establish the validity of this conjecture in the case of finitely many vanishingly small holes.

Our approach builds on that of [41] combined with appropriately chosen lower bounds on the energy and the ball construction method [8], [43], [44]- [46].

Section 3.2 contains the formulation of the problem as well as the main result described in Theorem 1. In section 3.3 we prove the existence and uniqueness of the minimizing set of degrees $\{D_\delta^j\}_j$ of the $\mathbb{S}^1$-valued problem under certain condition. Section 3.4 uses the approach similar to the one in [41] to decompose the energy of $\mathbb{C}$-valued minimizer in terms of the $S^1$-valued minimizer. The difference from [41] occurs because in this paper the holes of finite radius are considered but we study the shrinking holes instead. Because of the presence of another small parameter we use a different ball construction method that incorporates both the Ginzburg-Landau parameter $\varepsilon$ and another small parameter $\delta$. In Section 3.5 we show that it is not possible to have regular bulk vortices with nonzero degrees outside holes in our setting. This section also includes sharp energy estimates that allow us to prove the main theorem. Finally, in Section 3.6 the equality of degrees in the general case of several holes follows from the estimates in the previous section.

## 3.2 Main Results

Let $B\left(x_0, R\right) \subset \mathbb{R}^2$ denote a disk of radius $R$ centered at $x_0$. Let $\Omega$ be an arbitrary smooth bounded simply connected domain and $\omega_\delta^j = B(a^j, \delta) \subset \Omega$, $j = 1 \ldots N$ are the holes in it where $a^j$ are their centers and the radius $\delta$ is a small parameter. Introduce the perforated domain

$$\Omega_\delta = \Omega \setminus \bigcup_{j=1}^{N} \omega_\delta^j \tag{3.2}$$

and consider the Ginzburg-Landau functional

$$GL_\delta^\varepsilon[u,A] = \frac{1}{2}\int\limits_{\Omega_\delta} |\,(\nabla - iA)\,u|^2\,dx + \frac{1}{4\varepsilon^2}\int\limits_{\Omega_\delta}(1-|u|^2)^2\,dx + \frac{1}{2}\int\limits_{\Omega}(\operatorname{curl} A - h_{ext})^2\,dx \tag{3.3}$$

in it. The domain $\Omega_\delta$ represents a cross-section of a superconducting sample. Here $u:\Omega_\delta \to \mathbb{C}$ is an order parameter, $A:\Omega\to\mathbb{R}^2$ is a vector potential of the induced magnetic field, and $h_{ext}$ is the strength of the external magnetic field. By $\varepsilon$ we denote the inverse of the Ginzburg-Landau parameter that determines the radius of a typical vortex core. In what follows, we will assume that the cores radii are much smaller than the radius of the holes $\omega_\delta^j$.

The functional $GL_\delta^\varepsilon[u,A]$ is gauge-invariant, i.e., for any $\varphi \in H^2(\Omega,\mathbb{R})$ and any admissible pair $(u,A)$ the equality $GL_\delta^\varepsilon[u,A] = GL_\delta^\varepsilon\left[u\,e^{i\varphi}, A+\nabla\varphi\right]$ always holds. This degeneracy can be eliminated by imposing the *Coulomb gauge*, that is requiring that

$$A \in H(\Omega,\mathbb{R}^2) := \left\{a \in H^1(\Omega,\mathbb{R}^2) \mid \operatorname{div} a = 0 \text{ in } \Omega,\ a\cdot\nu = 0 \text{ on } \partial\Omega\right\}, \tag{3.4}$$

where $\nu$ is an outward unit normal vector to $\partial\Omega$. We will fix the Coulomb gauge throughout the rest of this work.

We consider the minimizers of two variational problems

$$(u_\delta^\varepsilon, A_\delta^\varepsilon) := \arg\min\left\{GL_\delta^\varepsilon[u,A] \mid u \in H^1(\Omega_\delta;\mathbb{C}), A \in H(\Omega;\mathbb{R}^2)\right\}, \tag{3.5}$$

and

$$(u_\delta, A_\delta) := \arg\min\left\{GL_\delta^\varepsilon[u,A] \mid u \in H^1(\Omega_\delta;S^1), A \in H(\Omega;\mathbb{R}^2)\right\}. \tag{3.6}$$

Note that, trivially,

$$(u_\delta, A_\delta) := \arg\min\left\{GL_\delta[u,A] \mid u \in H^1(\Omega_\delta;S^1), A \in H(\Omega;\mathbb{R}^2)\right\}, \tag{3.7}$$

where

$$GL_\delta[u,A] = \frac{1}{2}\int\limits_{\Omega_\delta} |\nabla u - iAu|^2\,dx + \frac{1}{2}\int\limits_{\Omega}(\operatorname{curl} A - h_{ext})^2\,dx. \tag{3.8}$$

If there exists $R = \delta + o(\delta)$ such that the winding number $d = \deg\left(u/|u|, \gamma_R^j\right)$

of the order parameter $u$ over a circle $\gamma_R^j = \partial B(a^j, R)$ does not equal to zero, we say that $u$ has a hole vortex of the degree $d$ inside $\omega_\delta^j$. The existence of circles $\gamma_R^j$ is stated in the Theorem 1 and they are specified using the results of Theorem 3. Hole vortices may exist inside $\omega_\delta^j$ for the minimizers of both (3.5) and (3.7) and our principal goal is to prove that for each hole the degrees of the vortices in both problems coincide as long as the external magnetic field has the same strength and $\delta$ is sufficiently small. This result implies that the non-linear potential term can be effectively replaced by the constraint $|u| = 1$ when one is interested in studying the distribution of degrees of the hole vortices for the minimizer of the problem (3.5).

The main result of this work is the following theorem.

**Theorem 1.** *Assume that the parameters $\varepsilon$ and $\delta$ satisfy*

$$|\log \varepsilon| \gg |\log \delta|. \tag{3.9}$$

*Suppose*

$$\sigma \in \mathbb{R}_+ \setminus \Sigma \tag{3.10}$$

*where $\Sigma$ is a discrete set described below. Let*

$$h_{ext} = \sigma |\log \delta| \tag{3.11}$$

*and $(u_\delta^\varepsilon, A_\delta^\varepsilon)$ and $(u_\delta, A_\delta)$ be defined by (3.5) and (3.7), respectively.*

*Then for sufficiently small $\delta$ there exists $R_\delta \in [\delta, \delta + \delta^2]$ such that for any $R \geq R_\delta$ such that $\gamma_R^j = \partial B(a^j, R)$ do not intersect $\partial\Omega$ and each other, the degrees of the hole vortices $D_{\delta,\varepsilon}^j = \deg\left(\frac{u_\delta^\varepsilon}{|u_\delta^\varepsilon|}, \gamma_R^j\right)$ and $D_\delta^j = \deg\left(u_\delta, \gamma_R^j\right)$ coincide for all $j = 1 \ldots N$ when $D_{\delta,\varepsilon}^j$ are defined, e.g. when $u_\delta^\varepsilon \neq 0$ on $\gamma_R^j$.*

**Remark 1.** The set $\Sigma$ is responsible for the values of $\sigma$ that correspond to the cases when the degree of one of the hole vortices increments by one, i.e. from $d$ to $d+1$. At this threshold field strengths the first order approximation of the energy is the same for both degrees $d$ and $d+1$ and the degrees of the hole vortices of minimizers $u_\delta^\varepsilon$ and $u_\delta$ cannot be determined uniquely. The set $\Sigma$ is described as follows:

$$\Sigma = \bigcup_{j=1}^{N} \Sigma_j \quad \text{where} \quad \Sigma_j = \left\{\sigma > 0 \mid \sigma\left(1 - \xi_0(a^j)\right) \in \mathbb{Z} + \frac{1}{2}\right\} \tag{3.12}$$

describes the switching points for each hole $j = 1 \dots N$ and $\xi_0$ solves the boundary value problem

$$\begin{cases} -\Delta\xi_0 + \xi_0 = 0 & \text{in } \Omega, \\ \xi_0 = 1 & \text{on } \partial\Omega. \end{cases} \tag{3.13}$$

**Remark 2.** Notice that $u_\delta(x) \in \mathbb{S}^1$ therefore there are no vortices outside holes and

$$D_\delta^j = \deg\left(u_\delta, \Gamma_R^j\right) = \deg\left(u_\delta, \partial\omega_\delta^j\right). \tag{3.14}$$

**Remark 3.** As we will show in Section 3.5, although the external magnetic field satisfying the bound (3.11) is strong enough to generate hole vortices, it is too weak for vortices to appear inside the bulk superconductor $\Omega_\delta$, away from the boundary $\partial\Omega$.

We prove Theorem 1 in two steps. First, we consider minimizers $(u_{\delta D}, A_{\delta D})$ of the variational problem (3.8) in the class of $\mathbb{S}^1$-valued maps with prescribed degrees $\deg(u, \partial\omega_\delta^j) = D^j$, $j = 1 \dots N$:

$$(u_{\delta D}, A_{\delta D}) := \arg\min\left\{GL_\delta[u, A] \mid u \in H^1(\Omega_\delta; S^1), A \in H(\Omega; \mathbb{R}^2), \deg(u, \partial\omega_\delta^j) = D^j\right\}. \tag{3.15}$$

Then the degrees $D_\delta^j$ of the map $u_\delta$ minimize the energy

$$l_\delta(D) := GL_\delta\left[u_{\delta D}, A_{\delta D}\right] \tag{3.16}$$

where $D = (D^1, \dots, D^N)$. It turns out that the function $l_\delta(D)$ is a quadratic polynomial in $D^1, \dots, D^N$. Its minimum is attained at one of the integer points adjacent to the vertex of paraboloid $l_\delta(T)$ with $T \in \mathbb{R}^N$. We enforce the condition (3.10) to ensure that such minimizing integer point is unique.

We then express a minimizer $(u_\delta^\varepsilon, A_\delta^\varepsilon)$ of $GL_\delta^\varepsilon[u, A]$ as a sum of $(u_\delta, A_\delta)$ and an appropriate correction term and consider a corresponding energy decomposition in the spirit of the approach in [41] for finite-size holes. The analysis relies principally on the techniques developed in [41] and the ball construction method [46]. Compared to [41], new challenges arise due to the presence of the second small parameter and require additional estimates and sharper energy bounds.

## 3.3 $S^1$-valued Problem

The main goal of this section is to establish the relation between the energy of the minimizer $(u_{\delta D}, A_{\delta D})$ and the degrees $D$ of the hole vortices of $u_{\delta D}$. We approximate the minimizer $(u_{\delta D}, A_{\delta D})$, calculate its energy $l_\delta(D) = GL_\delta\left[u_{\delta D}, A_{\delta D}\right]$, and find the minimizing degrees $D_\delta = (D^1_\delta, \dots, D^N_\delta)$. The main result is formulated in the following theorem.

**Theorem 2.** *Let* $(u_{\delta D}, A_{\delta D})$ *be a minimizer of* (3.15) *with prescribed degrees* $D \in \mathbb{Z}^N$. *Then the Ginzburg-Landau energy* $GL_\delta\left[u_{\delta D}, A_{\delta D}\right]$ *as a function of D takes the following form:*

$$l_\delta(D) = \pi \sum_{j=1}^{N} \left[ \left(D^j\right)^2 - 2\sigma\left(1 - \xi_0(a^j)\right) D^j \right] |\log\delta| + C|\log\delta|^2 + |D|^2 O(1) \quad (3.17)$$

*where* $\xi_0$ *solves the boundary value problem* (3.13), $C = O(1)$, *and* $|D| = \max\limits_j |D^j|$.

**Proof.** The main idea of the proof is to decompose $(u_{\delta D}, A_{\delta D})$ into the influence of external magnetic field and the influence of the holes. First, prescribe the degrees of the order parameter

$$\deg(u, \partial\omega^j_\delta) = D^j, \quad j = 1 \dots N \quad (3.18)$$

and write down the Euler-Lagrange equation for (3.8) in terms of the induced magnetic field $h = \operatorname{curl} A$ with the corresponding boundary conditions:

$$\begin{cases} -\Delta h + h = 0, & \text{in } \Omega_\delta, \\ h = h_{ext}, & \text{on } \partial\Omega, \\ h = H_j, & \text{in } \omega^j_\delta, \quad j = 1 \dots N, \\ -\int_{\partial\omega^j_\delta} \frac{\partial h}{\partial\nu} ds = 2\pi D^j - \int_{\omega^j_\delta} h dx, & j = 1 \dots N. \end{cases} \quad (3.19)$$

The constants $H_j$ are a priori unknown and are defined through the solution $h_{\delta D} = h_\delta(D)$ of (3.19) where $D = (D^1, \dots, D^N)$ is the vector of prescribed degrees. The value of the energy (3.8) on the minimizer $(u_{\delta D}, A_{\delta D})$ can be expressed in terms of $h_{\delta D}$:

$$GL_\delta\left[u_{\delta D}, A_{\delta D}\right] = GL_\delta[h_{\delta D}] = \frac{1}{2}\int_{\Omega_\delta} |\nabla h_{\delta D}|^2 dx + \frac{1}{2}\int_\Omega (h_{\delta D} - h_{ext})^2 dx \qquad (3.20)$$

Decompose the solution of (3.19) $h_{\delta D}$ into

$$h_{\delta D} = h_1 + h_2 + h_3 \qquad (3.21)$$

where $h_1$ captures the influence of the external field $h_{ext}$, $h_2$ approximates vortices around holes, and $h_3$ is the remainder. More precisely,

$$h_1 = h_{ext}\xi_0 \qquad (3.22)$$

where $\xi_0$ solves the boundary value problem (3.13) in the domain $\Omega$ with no holes:

$$\begin{cases} -\Delta\xi_0 + \xi_0 = 0 & \text{in } \Omega, \\ \xi_0 = 1 & \text{on } \partial\Omega. \end{cases} \qquad (3.23)$$

The approximation of vortices $h_2$ is defined by

$$h_2(x) = \sum_{j=1}^{N} D^j \theta_j(x)\phi_j(x) \qquad (3.24)$$

where $D^j$ is the prescribed degree of the vortex, $\theta_j(x) = \theta(x - a^j)$ is the modified Bessel function of the second kind truncated in the hole:

$$\theta(x) = \begin{cases} K_0(\delta), & |x| \le \delta, \\ K_0(|x|), & |x| > \delta, \end{cases} \qquad (3.25)$$

and $\phi_j(x) = \phi(x - a^j) \in C^\infty(\mathbb{R}^2)$ is a smoothening function such that

$$\phi(x) = \begin{cases} 1, & |x| \le R/4, \\ 0, & |x| \ge R/2 \end{cases} \qquad (3.26)$$

with $R$ defined as a largest radius such that $B(a^j, R)$, $j = 1 \dots N$ do not intersect neither each other nor boundary $\partial\Omega$. The function $K_0(|x|)$ is chosen since it is a fundamental solution of the equation $-\Delta u + u = 2\pi\delta(x)$ in $\mathbb{R}^2$. The function $h_2$ satisfies the following conditions:

$$\begin{cases} -\Delta h_2 + h_2 = \sum_{j=1}^{N} D^j \left[-\Delta + I\right] (\theta_j \phi_j), & \text{in } \Omega_\delta, \\ h_2 = 0, & \text{on } \partial\Omega, \\ h_2 = D^j K_0(\delta), & \text{on } \partial\omega_\delta^j, \quad j = 1 \dots N, \\ -\int_{\partial\omega_\delta^j} \frac{\partial h_2}{\partial \nu} ds = 2\pi D^j - D^j K_0(\delta)|\omega_\delta^j| + D^j O(\delta^2), & j = 1 \dots N. \end{cases} \tag{3.27}$$

Notice that $f_j(x) := \left[-\Delta + I\right](\theta_j \phi_j)$ is nonzero in $\Omega_\delta$ only when $\phi_j$ is non-constant that is the annulus $T_j = B(a^j, R/2) \setminus \overline{B(a^j, R/4)}$. Since $T_j$ are far from the holes, the function $f_j$ is smooth and finite. Thus $h_2$ gives us the needed degrees $D^j$, each $\theta_j \phi_j$ is constant on $j$th hole and decays to zero on $\partial B(a^j, R/2)$. These approximations have disjoint supports for $j = 1 \dots N$ and capture the main influence of the hole vortices.

We show that the contribution of the error $h_3 = h - h_1 - h_2$ to the energy is small, so the energy of the vortex-vortex interactions is negligible for our analysis and disjoint vortex approximations are reasonable. Write down the boundary value problem for $h_3$ using the original problem (3.19), the problem (3.13) for $h_1 = h_{ext}\xi_0$, and the formula (3.24) for $h_2$:

$$\begin{cases} -\Delta h_3 + h_3 = -\sum_{j=1}^{N} D^j f_j(x), & \text{in } \Omega_\delta, \\ h_3 = 0, & \text{on } \partial\Omega, \\ h_3 = \widetilde{H}_j - h_{ext}(\xi_0(x) - \xi_0(a^j)), & \text{on } \partial\omega_\delta^j, \quad j = 1 \dots N, \\ -\int_{\partial\omega_\delta^j} \frac{\partial h_3}{\partial \nu} ds = -\widetilde{H}_j |\omega_\delta^j| + D^j O(\delta^2) + O(\delta^3 \log \delta), & j = 1 \dots N. \end{cases} \tag{3.28}$$

where $\widetilde{H}_j = H_j - h_{ext}\xi_0(a^j) - D^j K_0(\delta)$ are the unknown constants. The proof of estimates for $h_3$ is delicate and is put into a separate lemma.

**Lemma 1.** *The solution of* (3.28) $h_3$ *satisfies the following estimates:*

$$\|h_3\|_{L^\infty(\Omega)} \le C_1 \delta |\log \delta|^2 + C_2 |D|, \tag{3.29}$$

$$\|\nabla h_3\|_{L^\infty(\Omega)} \le C_1 |\log \delta|^2 + C_2 |D| |\log \delta|, \tag{3.30}$$

$$\left| \frac{\partial h_3}{\partial \nu} \right| \le C_1 |\log \delta| + C_2 |D| \;\; \textit{on } \partial\omega_\delta^j \textit{ for all } j = 1 \dots N. \tag{3.31}$$

*Proof.* In order to get the estimates for $h_3$ we split (3.28) into subproblems. Start with $\eta = \sum_{j=1}^{N} D^j \eta_j$ that corresponds to the non-homogeneity of the equation and where each $\eta_j$ solves

$$\begin{cases} -\Delta\eta_j + \eta_j = -\left[-\Delta + I\right](\theta_j\phi_j)\,\mathbb{1}_{T_j}, & \text{in } \Omega, \\ \eta_j = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.32}$$

Each $\eta_j$ is smooth and does not depend on $\delta$. Next, introduce $\eta_0$ that corresponds to non-constant values on $\partial\omega_\delta^j$ and solves the problem

$$\begin{cases} -\Delta\eta_0 + \eta_0 = 0, & \text{in } \Omega_\delta, \\ \eta_0 = 0, & \text{on } \partial\Omega, \\ \eta_0 = -h_{ext}(\xi_0(x) - \xi_0(a^j)) - (\eta(x) - \eta(a^j)), & \text{on } \partial\omega_\delta^j, \quad j = 1\dots N. \end{cases} \tag{3.33}$$

By Maximum Principle

$$\|\eta_0\|_{L^\infty} \le C\delta(|\log\delta| + \max_j |D^j|). \tag{3.34}$$

The Lemma 6 gives the estimate for the gradient:

$$\|\nabla\eta_0\|_{L^\infty} \le C(|\log\delta| + \max_j |D^j|). \tag{3.35}$$

The remainder $\zeta = h_3 - \sum_{j=0}^{N} \eta_j$ solves the following system:

$$\begin{cases} -\Delta\zeta + \zeta = 0, & \text{in } \Omega_\delta, \\ \zeta = 0, & \text{on } \partial\Omega, \\ \zeta = c_j, & \text{on } \partial\omega_\delta^j, \quad j = 1\dots N, \\ -\int_{\partial\omega_\delta^j} \frac{\partial\zeta}{\partial\nu} ds = -|\omega_\delta^j| c_j + A_\delta^j, & j = 1\dots N, \end{cases} \tag{3.36}$$

where $c_j = \widetilde{H}_j - \eta(a^j)$ are unknown constants and $A_\delta^j = |D|O(\delta) + O(\delta\log\delta)$ is an error. The first three equations in 3.36 set up the boundary value problem for $\zeta$ with unknown boundary values $c_j$. The fourth condition is the system of $N$ equations for $N$ unknowns $c_j$. Since the first three conditions are linear, we start with the estimates for the basis functions $\zeta_i$ that solve the problems

$$\begin{cases} -\Delta\zeta_i + \zeta_i = 0, & \text{in } \Omega_\delta, \\ \zeta_i = 0, & \text{on } \partial\Omega, \\ \zeta_i = \delta_{ij}, & \text{on } \partial\omega_\delta^j, \quad j = 1\dots N. \end{cases} \tag{3.37}$$

Then using representation $\zeta = \sum_i c_i\zeta_i$ we solve the linear system for $c_i$.

Use the method of sub- and supersolutions to get estimates for $\zeta_i$. By Maximum Principle $0 \le \zeta_i \le 1$. In the case of radially symmetric domain with one hole in the center the solutions of (3.37) are the modified Bessel functions. We show that they provide a good approximation for the actual solutions $\zeta_i$. First, fix $i \in 1\dots N$ and construct a supersolution for $\zeta_i$. Take $R_{\max} > 0$ such that $\Omega \in B(a^i, R_{\max})$ and set

$$\zeta_i^{\text{sup}} = \frac{K_0\left(\frac{|x-a^i|}{R_{\max}}\right)}{K_0\left(\frac{\delta}{R_{\max}}\right)}. \tag{3.38}$$

The function $\zeta_i^{\text{sup}}$ is strictly positive in $\Omega_\delta$, equals 1 on $\partial\omega_\delta^i$, and has $[-\Delta + I]\,\zeta_i^{\text{sup}} = 0$. Therefore it satisfies

$$\begin{cases} -\Delta\zeta_i^{\text{sup}} + \zeta_i^{\text{sup}} = 0 & \text{in } \Omega_\delta, \\ \zeta_i^{\text{sup}} > 0 & \text{on } \partial\Omega, \\ \zeta_i^{\text{sup}} = 1 & \text{in } \omega_\delta^i, \\ \zeta_i^{\text{sup}} > 0 & \text{in } \omega_\delta^j,\ j \ne i, \end{cases} \tag{3.39}$$

and is indeed a supersolution. This yields

$$0 \le \zeta_i \le \zeta_i^{\text{sup}}. \tag{3.40}$$

Now construct a subsolution. Take $R_{\min} > 0$ such that $B(a^i, 2R_{\min}) \in \Omega_\delta$ for any $i = 1\dots N$ and set

$$\zeta_i^{\text{sub}} = \frac{K_0\left(\frac{|x-a^i|}{R_{\min}}\right)}{K_0\left(\frac{\delta}{R_{\min}}\right)} \tag{3.41}$$

The Bessel function is a fundamental solution of $[-\Delta + I]\,u = \delta(x)$, it is decreasing, therefore $\zeta_i^{\mathrm{sub}}$ is negative outside $B(a^i, R_{\min})$. Thus it satisfies

$$\begin{cases} -\Delta\zeta_i^{\mathrm{sub}} + \zeta_i^{\mathrm{sub}} = 0 & \text{in } \Omega_\delta, \\ \zeta_i^{\mathrm{sub}} < 0 & \text{on } \partial\Omega, \\ \zeta_i^{\mathrm{sub}} = 1 & \text{in } \omega_\delta^i, \\ \zeta_i^{\mathrm{sub}} < 0 & \text{in } \omega_\delta^j,\ j \neq i, \end{cases} \tag{3.42}$$

and is indeed a subsolution. This together with (3.40) implies

$$\max(0, \zeta_i^{\mathrm{sub}}) \leq \zeta_i \leq \zeta_i^{\mathrm{sup}}, \tag{3.43}$$

that is a very sharp description of the behavior of $\zeta_i$ near $i$th hole. Notice that for $x \in \partial\omega_\delta^i$

$$\frac{L_1}{\delta \log \delta} \leq \frac{\partial \zeta_i^{\mathrm{sub}}}{\partial \nu}(x) \leq \frac{\partial \zeta_i^{\mathrm{sup}}}{\partial \nu}(x) \leq \frac{L_2}{\delta \log \delta} \tag{3.44}$$

with $L_1, L_2 > 0$ therefore

$$\frac{\partial \zeta_i}{\partial \nu}(x) \sim \frac{1}{\delta \log \delta} \text{ on } \partial\omega_\delta^i. \tag{3.45}$$

To estimate the normal derivative of $\zeta_i$ on $\partial\omega_\delta^j$ for $j \neq i$ we need a better supersolution that captures the Dirichlet boundary conditions. Outside of $B(a^i, R_{\min})$ we have

$$|\zeta_i(x)| \leq \frac{K_0\left(\frac{R_{\min}}{R_{\max}}\right)}{K_0\left(\frac{\delta}{R_{\max}}\right)} \leq C_R |\log \delta|^{-1}. \tag{3.46}$$

Construct $\zeta_{ij}^{\mathrm{sup}}$ that solves the following conditions:

$$\begin{cases} -\Delta\zeta_{ij}^{\mathrm{sup}} + \zeta_{ij}^{\mathrm{sup}} = 0 & \text{in } B(a^j, R_{\min}) \setminus \overline{B(a^j, \delta)}, \\ \zeta_{ij}^{\mathrm{sup}} = C_R |\log \delta|^{-1} & \text{on } \partial B(a^j, R_{\min}), \\ \zeta_{ij}^{\mathrm{sup}} = 0 & \text{on } \partial\omega_\delta^j. \end{cases} \tag{3.47}$$

This problem is radially symmetric in $B(a^j, R_{\min}) \setminus \overline{B(a^j, \delta)}$. The function

$$\zeta_{ij}^{\mathrm{sup}} = C_1 I_0(r) + C_2 K_0(r), \quad r = |x - a^j| \tag{3.48}$$

with

$$C_1 \sim -|\log\delta|^{-1} \quad \text{and} \quad C_2 \sim |\log\delta|^{-2}. \tag{3.49}$$

satisfies (3.47) because the modified Bessel functions $I_0$ and $K_0$ behave as 1 and $|\log r|$ respectively near the origin. Therefore

$$0 \leq \frac{\partial \zeta_i}{\partial \nu} \leq \frac{\partial \zeta_{ij}^{\text{sup}}}{\partial \nu} = \frac{C_{ij}}{\delta |\log\delta|^2} \text{ on } \partial\omega_\delta^j. \tag{3.50}$$

As a result

$$\int_{\partial\omega_\delta^j} \left|\frac{\partial \zeta_i}{\partial \nu}\right| ds \leq \frac{C}{|\log\delta|^2}. \tag{3.51}$$

for all $i \neq j$. Combine the precise behavior of $\zeta_i$ on $\partial\omega_\delta^i$ in (3.45) with (3.51) and estimate the constants $c_i$ using the fourth equation in (3.36):

$$\begin{aligned}\pi\delta^2|c_i| + \left|A_i^\delta\right| \geq \left|\int_{\partial\omega_\delta^i} \frac{\partial \zeta}{\partial \nu} ds\right| &\geq \left|c_i \int_{\partial\omega_\delta^i} \frac{\partial \zeta_i}{\partial \nu} ds\right| - \sum_{j\neq i} \left|c_j \int_{\partial\omega_\delta^i} \frac{\partial \zeta_j}{\partial \nu} ds\right| \\ &\geq |c_i| \frac{C_1}{|\log\delta|} - \sum_{j\neq i}^{N} |c_j| \frac{C_2}{|\log\delta|^2}\end{aligned} \tag{3.52}$$

or

$$|c_i| \left(\frac{C_1}{|\log\delta|} - \pi\delta^2\right) - \sum_{j\neq i}^{N} |c_j| \frac{C_2}{|\log\delta|^2} \leq \left|A_i^\delta\right| \tag{3.53}$$

with some positive $C_1, C_2 > 0$ for all $i = 1 \ldots N$. The coefficient matrix is a small perturbation of the identity matrix up to the factor $C_1|\log\delta|^{-1}$. This allows us to show that

$$|c_i| \leq |D| O(\delta \log\delta) + O(\delta \log^2 \delta) \tag{3.54}$$

for all $i = 1 \ldots N$. Let

$$c_i = \max_j |c_j|. \tag{3.55}$$

Then

$$|c_i| \leq \left|A_i^\delta\right| \left(\frac{C_1}{|\log\delta|} - \pi\delta^2 - (N-1)\frac{C_2}{|\log\delta|^2}\right)^{-1} \leq |D| O(\delta \log\delta) + O(\delta \log^2 \delta). \tag{3.56}$$

which leads to the estimate on $\zeta$:

$$\|\zeta\|_{L^\infty(\Omega_\delta)} \le \sum_j |c_j| \le C_1|D|\delta|\log\delta| + C_2\delta|\log\delta|^2. \tag{3.57}$$

In order to get the final result for

$$h_3 = \eta_0 + \sum_{j=1}^N D^j\eta_j + \sum_{j=1}^N c_j\zeta_j \tag{3.58}$$

we summarize all the estimates. □

*Proof of Theorem 2 continued.* Now we are able to write down the asymptotics for the energy $l_\delta(D) = GL_\delta[h_{\delta D}]$:

$$\begin{aligned} l_\delta(D) &= GL_\delta[h_1 + h_2 + h_3] \\ &= \frac{1}{2}\int_{\Omega_\delta} |\nabla h_1|^2 dx + \frac{1}{2}\int_{\Omega_\delta} |\nabla h_2|^2 dx + \frac{1}{2}\int_{\Omega_\delta} |\nabla h_3|^2 dx \\ &+ \frac{1}{2}\int_\Omega (h_1 - h_{ext})^2 dx + \frac{1}{2}\int_\Omega h_2^2 dx + \frac{1}{2}\int_{\Omega_\delta} h_3^2 dx \\ &+ \int_{\Omega_\delta} \left[\nabla(h_1 - h_{ext})\cdot\nabla\widehat{h} + (h_1 - h_{ext})\widehat{h}\right] dx + \int_{\Omega_\delta} \left[\nabla h_2\cdot\nabla h_3 + h_2h_3\right] dx \\ &+ |D|^2 O(\delta^2|\log\delta|^3) + O(\delta^2|\log\delta|^3) \end{aligned} \tag{3.59}$$

where $\widehat{h} = h_2 + h_3$ and the error comes from the integrals over holes $\omega_\delta^j$. Estimate each term in (3.59). The terms that involve $h_1$ only do not depend on degrees and do not play a role in the minimization of $l_\delta(D)$:

$$\begin{aligned} \frac{1}{2}\int_{\Omega_\delta} |\nabla h_1|^2 dx + \frac{1}{2}\int_\Omega (h_1 - h_{ext})^2 dx &= h_{ext}^2\frac{1}{2}\int_{\Omega_\delta} |\nabla\xi_0|^2 dx + h_{ext}^2\frac{1}{2}\int_\Omega (1-\xi_0)^2 dx \\ &= O(|\log\delta|^2). \end{aligned} \tag{3.60}$$

The gradient of $h_2$ gives the main quadratic term:

$$\begin{aligned} \frac{1}{2}\int_{\Omega_\delta} |\nabla h_2|^2 dx &= \frac{1}{2}\sum_{j=1}^N \left(D^j\right)^2 \int_{T_j} |\nabla(\theta_j(x)\phi_j(x))|^2 dx \\ &= \pi\sum_{j=1}^N \left(D^j\right)^2 \left[\int_\delta^{R/4} |K_0(r)'|^2 r dr + \int_{R/4}^R \left|\frac{d}{dr}(K_0(r)\phi(r))\right|^2 r dr\right] \end{aligned}$$

$$= \pi \sum_{j=1}^{N} \left(D^j\right)^2 \left[\int_\delta^{R/4} \left|-\frac{1}{r} + O(r\log r)\right|^2 rdr + O(1)\right] \tag{3.61}$$

$$= \pi \sum_{j=1}^{N} \left(D^j\right)^2 |\log\delta| + |D|^2 O(1). \tag{3.62}$$

The $L^2$-norm of $h_2$ has much smaller order:

$$\frac{1}{2}\int_\Omega h_2^2 dx = \pi \sum_{j=1}^{N} \left(D^j\right)^2 \int_0^{R/2} |\theta_j \phi|^2 rdr = |D|^2 O(1). \tag{3.63}$$

The most interesting part is the integral involving $\widehat{h}$ because it gives us the linear terms. Notice that since $h_{\delta D}$ and $h_1$ solve the homogeneous equation $[-\Delta + I]\, h = 0$, so does their difference $\widehat{h} = h_{\delta D} - h_1$:

$$\begin{aligned}
\left\langle h_1 - h_{ext}, \widehat{h}\right\rangle_{H^1(\Omega_\delta)} &= \int_{\Omega_\delta} (h_1 - h_{ext})\left(-\Delta\widehat{h} + \widehat{h}\right) dx - \int_{\partial\Omega_\delta} (h_1 - h_{ext}) \frac{\partial \widehat{h}}{\partial \nu} ds \\
&= \sum_{j=1}^{N} \int_{\partial\omega_\delta^j} (h_1 - h_{ext}) \frac{\partial (h_2 + h_3)}{\partial \nu} ds \\
&= \sum_{j=1}^{N} \int_{\partial\omega_\delta^j} (h_1 - h_{ext}) \left[D^j \left(\frac{1}{\delta} + O(\delta\log\delta)\right) + O(\log\delta) + |D| O(1)\right] ds \\
&= \sum_{j=1}^{N} D^j (h_1(a^j) - h_{ext}) 2\pi\delta \cdot \frac{1}{\delta} + O(\delta|\log\delta|^2) + |D| O(\delta\log\delta) \\
&= -2\pi\sigma |\log\delta| \sum_{j=1}^{N} D^j (1 - \xi_0(a^j)) + O(\delta|\log\delta|^2) + |D| O(\delta\log\delta)
\end{aligned} \tag{3.64}$$

Here for simplicity I use the notation $\langle u, v\rangle_{H^1} = \int [\nabla u \cdot \nabla v + uv]\, dx$. The other terms in (3.59) are small and are estimated using the similar trick with integration by parts:

$$\begin{aligned}
\|h_3\|^2_{H^1(\Omega_\delta)} &= \int_{\Omega_\delta} h_3 \left(-\Delta h_3 + h_3\right) dx - \int_{\partial\Omega_\delta} h_3 \frac{\partial h_3}{\partial \nu} ds = \sum_{j=1}^{N} \int_{\partial\omega_\delta^j} h_3 \frac{\partial h_3}{\partial \nu} ds \\
&= C\delta \left(C_1 \delta |\log\delta|^2 + C_2 |D|\right) \left(C_1 |\log\delta| + C_2 |D|\right) \\
&= O(\delta^2 |\log\delta|^3) + |D|^2 O(\delta|\log\delta|)
\end{aligned} \tag{3.65}$$

$$\langle h_2, h_3\rangle_{H^1(\Omega_\delta)} = \int_{\Omega_\delta} h_2\left(-\Delta h_3 + h_3\right)dx - \int_{\partial\Omega_\delta} h_2 \frac{\partial h_3}{\partial \nu} ds = \sum_{j=1}^{N} \int_{\partial\omega_\delta^j} h_2 \frac{\partial h_3}{\partial \nu} ds$$
$$= \sum_{j=1}^{N} 2\pi\delta D^j K_0(\delta)\left(C_1|\log\delta| + C_2|D|\right)$$
$$= |D|^2 O(\delta|\log\delta|^2) \tag{3.66}$$

Summarizing all the estimates we obtain the asymptotic expansion (3.17). □

**Corollary 1.** The leading part of the energy $l_\delta(Z)$ is a sum of $N$ one-dimensional parabolas with the real-valued vertexes at

$$Z_j = \sigma(1 - \xi_0(a^j)). \tag{3.67}$$

Since the degrees should be integer, the minimizing degrees $D^j$ are the closest integers to $Z_j$:

$$D^j = [\![\sigma(1 - \xi_0(a^j))]\!] \tag{3.68}$$

where $[\![\cdot]\!]$ denotes the nearest integer.

## 3.4 Energy Decomposition

Since $(u_{\delta D}, A_{\delta D})$ is an admissible pair for the problem (3.5), we can use the representation of $S^1$-valued energy (3.17) with $D = 0$ to obtain an upper bound

$$GL_\delta^\varepsilon\left[u_\delta^\varepsilon, A_\delta^\varepsilon\right] \le GL_\delta^\varepsilon\left[u_{\delta D}, A_{\delta D}\right] \le C|\log\delta|^2 \tag{3.69}$$

on the energy of the minimizer of (3.5). In order to obtain a matching lower energy bound, we need to localize the regions of the domain where the magnitude of the order parameter is small. To this end, we use the following theorem.

**Theorem 3** (Ball Construction Method [46])**.** *For any $\alpha \in (0,1)$ there exists $\varepsilon_0(\alpha) > 0$ such that, for any $\varepsilon < \varepsilon_0$, if $(u, A)$ is a configuration such that $GL_\delta^\varepsilon[u, A] < \varepsilon^{\alpha-1}$, where $\varepsilon$ is an inverse of the Ginzburg-Landau parameter, the following holds.*

*For any $1 > \rho > C\varepsilon^{\alpha/2}$, where $C$ is a universal constant, there exists a finite collection of disjoint closed balls $\mathfrak{B} = \{B_i = B(b^i, r_i)\}_{i\in\mathfrak{I}}$ such that*

1. *$r(\mathfrak{B}) = \rho$ where $r(\mathfrak{B}) = \sum_{i\in\mathfrak{I}} r(B_i)$*

2. *Letting* $V = \Omega_\delta \cap \cup_{i\in\mathfrak{I}} B_i$,

$$\left\{ x \in \Omega_\delta \;\; | \;\; ||u(x)| - 1| \geq \varepsilon^{\alpha/4} \right\} \subset V. \tag{3.70}$$

3. *Writing* $d_i = \deg(u, \partial B_i)$, *if* $B_i \subset \Omega_\delta$ *and* $d_i = 0$ *otherwise,*

$$\frac{1}{2}\int_V \left[ |\nabla_A u|^2 + \rho^2 |\mathrm{curl}\, A|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2 \right] dx \geq \pi d \left( \log \frac{\rho}{d\varepsilon} - C \right), \tag{3.71}$$

*where* $d = \sum_{i\in\mathfrak{I}} |d_i|$ *is assumed to be nonzero and* $C$ *is a universal constant.*

4. *There exists a universal constant* $C$ *such that*

$$d \leq C \frac{GL^\varepsilon_\delta[u, A]}{\alpha |\log \varepsilon|}. \tag{3.72}$$

We consider now a domain with $N$ holes $\omega^j_\delta = B(a^j, \delta)$ so that $\Omega_\delta = \Omega \setminus \cup_{j=1}^N \overline{\omega^j_\delta}$. Set $\alpha = 1/2$ and $\rho = \delta^2/2$ in the ball construction method. Assume that $\varepsilon$ is small enough so that $|u(x)| > 1 - \theta$ on $\Omega_\delta \cap (\cup_{i\in\mathfrak{I}} B_i)$. The parameter $\theta$ will be chosen later in the section 3.6.

**Lemma 2.** *Let* $(u^\varepsilon_\delta, A^\varepsilon_\delta)$ *be a minimizer of the full problem* (3.5)*. Then the following energy decomposition holds:*

$$GL^\varepsilon_\delta[u^\varepsilon_\delta, A^\varepsilon_\delta] = GL_\delta[u_{\delta D}, A_{\delta D}] + F_\delta[v, B] - \int_{\Omega_\delta} \nabla^\perp h_{\delta D} \cdot \mathrm{Im}\, \overline{v} \nabla v \, dx + o(1) \tag{3.73}$$

*where* $u^\varepsilon_\delta = v \cdot u_{\delta D}$, $A^\varepsilon_\delta = A_{\delta D} + B$, $h_{\delta D} = \mathrm{curl}\, A_{\delta D}$ *and*

$$F_\delta[v, B] = \frac{1}{2}\int_{\Omega_\delta} \left( |(\nabla - iB)v|^2 + \frac{1}{2\varepsilon^2}(1 - |v|^2)^2 \right) dx + \frac{1}{2}\int_\Omega (\mathrm{curl}\, B)^2 \, dx. \tag{3.74}$$

*Here* $(u_{\delta D}, A_{\delta D})$ *is the minimizer of the* $S^1$*-valued problem with prescribed degrees* $D$ (3.15).

*Proof.* Use the following Euler-Lagrange equation that $(u_{\delta D}, A_{\delta D})$ satisfies outside the holes:

$$\nabla^\perp h_{\delta D} = -\mathrm{Im}\, (\overline{u}_{\delta D} \nabla u_{\delta D} - i A_{\delta D}) \tag{3.75}$$

We also use the representation (3.20) of Ginzburg-Landau functional in terms of

$h_{\delta D}$. We start the proof with representing $GL^{\varepsilon}_{\delta}[u^{\varepsilon}_{\delta}, A^{\varepsilon}_{\delta}]$ as a sum of three terms:

$$GL_{\delta}[u^{\varepsilon}_{\delta}, A^{\varepsilon}_{\delta}] = I_1 + I_2 + I_3 \tag{3.76}$$

where

$$I_1 = \frac{1}{2}\int_{\Omega} |\nabla u^{\varepsilon}_{\delta} - iA^{\varepsilon}_{\delta} u^{\varepsilon}_{\delta}|^2\,dx, \quad I_2 = \frac{1}{4\varepsilon^2}\int_{\Omega_\delta} (1 - |u^{\varepsilon}_{\delta}|^2)^2\,dx, \quad I_3 = \frac{1}{2}\int_{\Omega} (\operatorname{curl} A^{\varepsilon}_{\delta} - h_{ext})^2\,dx \tag{3.77}$$

Observe that $|u^{\varepsilon}_{\delta}| = |v|$ as $u^{\varepsilon}_{\delta} = v \cdot u_{\delta D}$ and $|u_{\delta D}| = 1$. Hence we can rewrite $I_2$ as

$$I_2 = \frac{1}{4\varepsilon^2}\int_{\Omega_\delta} (1 - |u_{\delta D}|^2)^2\,dx = \frac{1}{4\varepsilon^2}\int_{\Omega_\delta} (1 - |v|^2)^2\,dx \tag{3.78}$$

that is the second term of $F_{\delta}[v, B]$. Now rewrite $I_3$:

$$\begin{aligned} I_3 &= \frac{1}{2}\int_{\Omega} (\operatorname{curl} A^{\varepsilon}_{\delta} - h_{ext})^2\,dx \\ &= \frac{1}{2}\int_{\Omega} (h_{\delta D} - h_{ext})^2\,dx + \frac{1}{2}\int_{\Omega} (\operatorname{curl} B)^2\,dx + \int_{\Omega} \operatorname{curl} B \cdot (h_{\delta D} - h_{ext})\,dx \end{aligned} \tag{3.79}$$

First term is a part of $GL_{\delta}[u_{\delta D}, A_{\delta D}]$ and the second one is a part of $F_{\delta}[v, B]$. The last term will cancel while rewriting $I_1$ below.

The last thing to do is to rewrite $I_1$. Start from the integrand:

$$\begin{aligned} |\nabla u^{\varepsilon}_{\delta} - iA^{\varepsilon}_{\delta} u^{\varepsilon}_{\delta}|^2 &= |v\,(\nabla u_{\delta D} - iA_{\delta D} u_{\delta D}) + u_{\delta D}\,(\nabla v - iBv)|^2 \\ &= |v|^2 |\nabla u_{\delta D} - iA_{\delta D} u_{\delta D}|^2 + |u_{\delta D}|^2 |\nabla v - iBv|^2 \\ &+ 2\mathrm{Re}\,(\overline{u}_{\delta D}\,(\nabla u_{\delta D} - iA_{\delta D} u_{\delta D}) \cdot v\,(\nabla \overline{v} + iB\overline{v})) \\ &= |\nabla v - iBv|^2 + |v|^2 |\nabla h_{\delta D}|^2 \\ &+ 2|v|^2 \nabla^{\perp} h_{\delta D} \cdot B - 2\nabla^{\perp} h_{\delta D} \cdot \mathrm{Im}\,(\overline{v} \nabla v) \end{aligned} \tag{3.80}$$

The first term is a part of $F_{\delta}[v, B]$. The last term is included in the right hand side of the decomposition. The sum of two other terms has the form $|v|^2 \cdot R(x)$, where

$$R(x) = |\nabla h_{\delta D}|^2 + 2\nabla^{\perp} h_{\delta D} \cdot B$$

Now add and subtract $\frac{1}{2}\int_{\Omega_\delta} R(x)\,dx$ to the energy $GL_{\delta}[u^{\varepsilon}_{\delta}, A^{\varepsilon}_{\delta}]$. The first term

$\frac{1}{2}\int_{\Omega_\delta}|\nabla h_{\delta D}|^2\,dx$ is a part of $GL_\delta[u_{\delta D}, A_{\delta D}]$. Using integration by parts we prove that the second term $\int_{\Omega_\delta}\nabla^\perp h_{\delta D}\cdot B\,dx$ cancels with the last term in the representation (3.79) of $I_3$ as we mentioned above:

$$\begin{aligned}\int_{\Omega_\delta}\nabla^\perp h_{\delta D}\cdot B\,dx &= \int_{\Omega_\delta}\nabla^\perp(h_{\delta D}-h_{ext})\cdot B\,dx\\ &= \int_{\partial\Omega_\delta}(h_\delta - h_{ext})B\cdot\tau\,dS - \int_{\Omega_\delta}(h_{\delta D}-h_{ext})\nabla^\perp\cdot B\,dx\\ &= -\sum_{j=1}^{N}(h_{\delta D}-h_{ext})|_{\partial B(a^j,\delta)}\int_{\partial B(a^j,\delta)}B\cdot\tau\,dS - \int_{\Omega_\delta}(h_{\delta D}-h_{ext})\,\mathrm{curl}\,B\,dx\\ &= -\sum_{j=1}^{N}(h_{\delta D}-h_{ext})|_{\partial B(a^j,\delta)}\int_{B(a^j,\delta)}\mathrm{curl}\,B\,dS - \int_{\Omega_\delta}(h_{\delta D}-h_{ext})\,\mathrm{curl}\,B\,dx\\ &= -\int_{\Omega}(h_{\delta D}-h_{ext})\,\mathrm{curl}\,B\,dx. \end{aligned} \tag{3.81}$$

Here we used the facts that $h_{\delta D}=h_{ext}$ on the boundary $\partial\Omega$ and $h_{\delta D}=\text{const}$ in $B(a^j,\delta)$ that follow from the equation that $h_{\delta D}$ satisfies.

Thus summarizing the results we obtain:

$$\begin{aligned}GL_\delta^\varepsilon[u_\delta^\varepsilon, A_\delta^\varepsilon] &= GL_\delta[u_{\delta D}, A_{\delta D}] + F_\delta[v,B]\\ &\quad -\int_{\Omega_\delta}\nabla^\perp h_{\delta D}\cdot\mathrm{Im}\,\overline{v}\nabla v\,dx + \int_{\Omega_\delta}(1-|v|^2)R(x)\,dx + o(1)\end{aligned} \tag{3.82}$$

The last thing is to show that

$$I = \int_{\Omega_\delta}(1-|v|^2)R(x)\,dx$$

goes to zero as $\delta\to 0$. Holder inequality implies

$$|I| \le \|1-|v|^2\|_{L^2(\Omega_\delta)}\cdot\left(2\|\nabla h_{\delta D}\|^2_{L^4(\Omega_\delta)} + \|B\|^2_{L^4(\Omega_\delta)}\right). \tag{3.83}$$

First multiplier is less then $M\varepsilon|\log\delta|$ when $\delta\to 0$ because of the a priori estimate on the energy. Using the relation between $\varepsilon$ and $\delta$

$$|\log\varepsilon| \gg |\log\delta| \tag{3.84}$$

we show that $\varepsilon$ is sufficiently small to compensate the growth of the other terms.

The function $h_{\delta D}$ is described in Theorem 2 and because of Lemma 6 satisfies the estimate

$$\|\nabla h_{\delta D}\|^2_{L^4(\Omega_\delta)} \le \frac{C|\log\delta|^2}{\delta^2}. \tag{3.85}$$

Estimate $\|B\|_{L_4(\Omega_\delta)}$. Due to the gauge invariance we have $\operatorname{div} A^\varepsilon_\delta = 0$. Therefore by Poincaré's lemma $A^\varepsilon_\delta$ has a potential, i.e. there exists $\Pi^\varepsilon_\delta$ such that $\nabla^\perp \Pi^\varepsilon_\delta = A^\varepsilon_\delta$. Substituting this into $h^\varepsilon_\delta = \operatorname{curl} A^\varepsilon_\delta$ we obtain the equality $\Delta\Pi^\varepsilon_\delta = h^\varepsilon_\delta$. The function $\Pi^\varepsilon_\delta$ is a potential so we are able to make it zero on the boundary $\partial\Omega$. From the theory of elliptic operators and the a priori energy estimate we obtain

$$\|\Pi^\varepsilon_\delta\|^2_{H^2(\Omega)} \le \|h^\varepsilon_\delta\|^2_{L_2(\Omega)} \le C|\log\delta|^2. \tag{3.86}$$

Since the embedding $H^1(\Omega) \subset L^4(\Omega)$ is continuous we have

$$\|A^\varepsilon_\delta\|_{L_4(\Omega_\delta)} \le C\|\Pi^\varepsilon_\delta\|_{H^2(\Omega)} \le C|\log\delta|.$$

The same estimate holds for $A_{\delta D}$. Using the decomposition $A^\varepsilon_\delta = B + A_{\delta D}$ we obtain this estimate for $B$:

$$\|B\|_{L_4(\Omega_\delta)} \le C|\log\delta|$$

Summarize all the estimates to obtain

$$|I| \le C\varepsilon|\log\delta|\left(\frac{|\log\delta|^2}{\delta^2} + |\log\delta|^2\right).$$

The condition $|\log\varepsilon| \gg |\log\delta|$ implies that $\varepsilon$ is much smaller than any power of $\delta$, therefore $I$ goes to zero as $\delta \to 0$ that completes the proof. □

## 3.5 Absence of Bulk Vortices

In this section we analyze further the energy decomposition (3.73). The energy of the unconstrained solution is minimal so

$$GL^\varepsilon_\delta[u^\varepsilon_\delta, A^\varepsilon_\delta] \le GL_\delta[u_{\delta D}, A_{\delta D}]. \tag{3.87}$$

Combine (3.87) with (3.73) to get

$$F_\delta[v,B] \le \int_{\Omega_\delta} \nabla^\perp h_{\delta D}\cdot \operatorname{Im}\overline{v}\nabla v\,dx + o(1). \tag{3.88}$$

First, get an upper bound for energy $F_\delta$ and the integral term in (3.88). We start with a simple fact that will also be used later. For any $R \in L^2(D)$, $v \in H^1(D)$, and $B \in H^1(D)$

$$\begin{aligned}\left|\int_D R(x)\cdot \operatorname{Im}\overline{v}\nabla v\,dx\right| &\le \left|\int_D R(x)\cdot \left(\operatorname{Im}\overline{v}(\nabla - iB)v + B|v|^2\right)\,dx\right| \\ &\le \|R\|_{L^2(D)}\cdot \left(\|(\nabla - iB)v\|_{L^2(D)} + \|B\|_{L^2(D)}\right)\end{aligned} \tag{3.89}$$

**Lemma 3.** *The following estimates hold:*

$$F_\delta[v,B] \le |\log\delta|^2, \tag{3.90}$$

$$\left|\int_{\Omega_\delta} \nabla^\perp h_{\delta D}\cdot \operatorname{Im}\overline{v}\nabla v\,dx\right| \le |\log\delta|^2. \tag{3.91}$$

*Proof.* Use (3.89) and Poincaré inequality to estimate the integral term in (3.88):

$$\begin{aligned}\left|\int_{\Omega_\delta} \nabla^\perp h_{\delta D}\cdot \operatorname{Im}\overline{v}\nabla v\,dx\right| &\le \|\nabla h_{\delta D}\|_{L^2(\Omega_\delta)}\cdot\left(\|(\nabla - iB)v\|_{L^2(\Omega_\delta)} + C_\Omega\|\operatorname{curl}B\|_{L^2(\Omega)}\right) \\ &\le \frac{1}{2\alpha}\|\nabla h_{\delta D}\|^2_{L^2(\Omega_\delta)} + \frac{\alpha}{2}\left(\|(\nabla - iB)v\|^2_{L^2(\Omega_\delta)} + C_\Omega^2\|\operatorname{curl}B\|^2_{L^2(\Omega)}\right) \\ &\le O(|\log\delta|^2) + \frac{1}{2}F_\delta[v,B]\end{aligned} \tag{3.92}$$

where $\alpha = \min(1, C_\Omega^{-2})$. Combine this inequality with (3.88) to get

$$F_\delta[v,B] \le O(|\log\delta|^2). \tag{3.93}$$

The estimates (3.92) and (3.93) imply (3.91). □

The bound (3.93) allows to apply the ball construction method for $F_\delta$. Theorem 3 gives the following lower bound on the energy inside "bad" disks:

$$F_\delta[v,B;B_i] \ge \pi|d_i|\left(\log\frac{\delta^2}{|d_i|\varepsilon} - C\right) \text{ for every } i \in \mathfrak{I}. \tag{3.94}$$

Here $F_\delta[v,B;B_i]$ is the energy $F_\delta[v,B]$ where first two integrals are taken over the domain $B_i = B(b^i, r_i)$. To continue working with (3.88) we prove the following lemma.

**Lemma 4.** *The following representation holds:*

$$\int_{\Omega_\delta} \nabla^\perp h_{\delta D} \cdot \operatorname{Im} \overline{v} \nabla v \, dx = 2\pi \sum_{i \in \mathfrak{I}_1} (h_{ext} - h_{\delta D}(b^i)) d_i + 2\pi \sum_{j=1}^{N} D_v^j (h_{ext} - H_R^j) + O(1) \tag{3.95}$$

*where* $D_v^j = \deg(v, \Gamma_R^j) = D_{\delta,\varepsilon}^j - D^j$, $\Gamma_R^j = \partial B(a^j, R)$ *are the circles enclosing* $\omega_\delta^j$ *with* $R = \delta + O(\delta^2)$, $H_R^j = D^j K_0(R) + h_{ext} \xi_0(a^j)$, *and* $\mathfrak{I}_1$ *describes only balls that are inside* $\Omega_\delta \setminus \cup_{j=1}^N \overline{\omega_\delta^j}$ *and do not intersect the boundary* $\partial\Omega_\delta$.

*Proof.* We divide the domain $\Omega_\delta$ into three disjoint parts:

$$\Omega_\delta = S \cup V \cup G \tag{3.96}$$

where $S = \cup_{j=1}^N S_j$ is the set of annuli around holes between $\partial\omega_\delta^j$ and $\Gamma_R^j$, $V = [(\cup_{i\in\mathfrak{I}} B_i) \setminus S] \bigcap \Omega_\delta$ are the "bad" disks, and $G$ is everything left.

Consider the subdomains $S$, $V$, and $G$ separately. The balls $B_i$ as well as stripes $S_j$ are very small and do not contribute a lot:

$$\begin{aligned} \int_{V\cup S} \nabla^\perp h_{\delta D} \cdot \operatorname{Im} \overline{v} \nabla v \, dx &\le \operatorname{meas}(V \cup S)^{1/4} \cdot \|\nabla^\perp h_{\delta D}\|_{L^4(V\cup S)} \\ &\quad \cdot \Big( \|(\nabla - iB)v\|_{L^2(V\cup S)} + \|B\|_{L^2(V\cup S)} \Big) \\ &\le C\delta^{3/4} \cdot |\log\delta| \cdot |\log\delta| = o(1) \end{aligned} \tag{3.97}$$

Introduce the function $w = v/|v|$. Then

$$\begin{aligned} \int_G \nabla^\perp h_{\delta D} \cdot \operatorname{Im} \overline{v} \nabla v \, dx &= \int_G \nabla^\perp h_{\delta D} \cdot \operatorname{Im} \overline{w} \nabla w \, dx + \int_G \nabla^\perp h_{\delta D} \cdot (\operatorname{Im} \overline{v} \nabla v - \operatorname{Im} \overline{w} \nabla w) \, dx \\ &= I_1 + I_2. \end{aligned} \tag{3.98}$$

To estimate the second integral, use the following:

$$\begin{aligned} \operatorname{Im} \overline{v} \nabla v - \operatorname{Im} \overline{w} \nabla w &= \operatorname{Im} \left( \overline{w}|v|(w\nabla|v| + |v|\nabla w) - \overline{w}\nabla w \right) \\ &= \operatorname{Im} \Big( |v|\nabla|v| + (|v|^2 - 1)\overline{w}\nabla w \Big) = (|v|^2 - 1) \operatorname{Im} \overline{w} \nabla w \end{aligned} \tag{3.99}$$

and

$$|\nabla v|^2 = |v|^2|\nabla w|^2 + |\nabla|v|| \geq (1-\theta)^2|\nabla w|^2 \geq \frac{1}{4}|\nabla w|^2. \tag{3.100}$$

since by Theorem 3 we have $|v| \geq 1-\theta$ outside $B_i$. The function $v$ admits the same estimate as $u_\delta^\varepsilon$. Add and subtract $iBv$ to get

$$\begin{aligned}\frac{1}{2}\|\nabla v\|^2_{L^2(G)} = \frac{1}{2}\int_G |\nabla v|^2\,dx &\leq \int_G \left(|(\nabla - iB)v|^2 + |v|^2|B|^2\right)\,dx \\ &\leq \int_{\Omega_\delta} |(\nabla - iB)v|^2\,dx + C_\Omega \int_\Omega |\operatorname{curl} B|^2\,dx \leq C|\log\delta|^2. \end{aligned} \tag{3.101}$$

This leads to the following estimate:

$$\begin{aligned}|I_2| &\leq \int_G \nabla^\perp h_{\delta D}\cdot(|v|^2-1)\operatorname{Im}\overline{w}\nabla w\,dx \\ &\leq \|\nabla^\perp h_{\delta D}\|_{L^\infty(G)}\cdot\int_G (|v|^2-1)\cdot|\nabla w|\,dx \\ &\leq C\delta^{-1}\cdot\int_G(|v|^2-1)\cdot 2|\nabla v|\,dx \\ &\leq C\delta^{-1}\cdot\||v|^2-1\|_{L^2(G)}\cdot\|\nabla v\|_{L^2(G)} \\ &\leq C\delta^{-1}\cdot\varepsilon|\log\delta|\cdot|\log\delta| = o(1) \end{aligned} \tag{3.102}$$

since (3.9).

Now rewrite the integral $I_1$. Use Divergence theorem:

$$\begin{aligned} I_1 = &\int_G \nabla^\perp(h_{\delta D}-h_{ext})\cdot\operatorname{Im}\overline{w}\nabla w\,dx = -\int_G (h_{\delta D}-h_{ext})\nabla^\perp\cdot\operatorname{Im}\overline{w}\nabla w\,dx \\ &+\int_{\partial\Omega}(h_{\delta D}-h_{ext})\operatorname{Im}\overline{w}\nabla w\cdot\tau\,ds - \int_{\partial V}(h_{\delta D}-h_{ext})\operatorname{Im}\overline{w}\nabla w\cdot\tau\,ds \\ &-\int_{\cup_j\gamma_j}(h_{\delta D}-h_{ext})\operatorname{Im}\overline{w}\nabla w\cdot\tau\,ds \\ &= -\sum_{i\in\mathfrak{I}} I_{1i} - \sum_{j=1}^N \int_{\Gamma_R^j}(h_{\delta D}-h_{ext})\operatorname{Im}\overline{w}\nabla w\cdot\tau\,ds \end{aligned} \tag{3.103}$$

where $I_{1i} = \int_{\partial V_i}(h_{\delta D}-h_{ext})\operatorname{Im}\overline{w}\nabla w\cdot\tau\,ds$ and $V_i = B_i\cap\Omega_\delta$. The term $\nabla^\perp\cdot\operatorname{Im}\overline{w}\nabla w = \operatorname{curl}\nabla\Phi = 0$, where $\Phi$ is a phase of $w$, disappears.

Since the curves $\Gamma_R^j$ are small, we can approximate $h_{\delta D}$ by a constant:

$$\int_{\Gamma_R^j}(h_{\delta D}-h_{ext})\operatorname{Im}\overline{w}\nabla w\cdot\tau\,ds = 2\pi D_v^j(H_R^j - h_{ext}) + \int_{\Gamma_R^j}(h_{\delta D}-H_R^j)\operatorname{Im}\overline{w}\nabla w\cdot\tau\,ds$$

for a constant $H_R^j$. Take $H_R^j = h_{ext}\xi_0(a^j) + D^j K_0(R)$. Using the decomposition of $h_{\delta D}$ (3.21) we get

$$|h_{\delta D}(x) - H_R^j| \le h_{ext}|\xi_0(x) - \xi_0(a^j)| + |h_3(x)| \le C_1\delta|\log\delta|^2 + C_2|D| \tag{3.104}$$

for $x \in \Gamma_R^j$. This yields

$$\left|\int_{\Gamma_R^j}(h_{\delta D} - H_R^j)\operatorname{Im}\overline{w}\nabla w \cdot \tau\, ds\right| \le \Big(C_1\delta|\log\delta|^2 + C_2|D|\Big)\cdot D_v^j = O(1). \tag{3.105}$$

As a result we get

$$I_1 = -\sum_{i\in\mathfrak{I}} I_{1i} - \sum_{j=1}^{N} 2\pi D_v^j(H_R^j - h_{ext}) + O(1). \tag{3.106}$$

Consider two cases. First, consider the set $\mathfrak{I}_1 \subset \mathfrak{I}$ such that $B_i \subset \Omega_\delta \setminus S$ for $i \in \mathfrak{I}_1$. We estimate the integrals $I_{1i}$ in a similar way as we did for the hole vortices. Approximate $h_{\delta D}(x)$ by a constant value in the center of $B_i$:

$$I_{1i} = \int_{\partial V_i}(h_{\delta D} - h_{\delta D}(b^i))\operatorname{Im}\overline{w}\nabla w\cdot\tau\, ds + \int_{\partial V_i}(h_{\delta D}(b^i) - h_{ext})\operatorname{Im}\overline{w}\nabla w\cdot\tau\, ds = J_{1i} + J_{2i} \tag{3.107}$$

Second integral directly gives the degree of the possible bulk vortex $d_i$:

$$J_{2i} = 2\pi d_i(h_{\delta D}(b^i) - h_{ext}) \tag{3.108}$$

To estimate $J_{1i}$ we introduce subdomains $U_i = V_i \cap \{x \mid |v(x)| \le 1/2\}$ so that their boundaries are the level sets of $v$. We add and subtract the integral over $\partial U_i$:

$$\sum_{i\in\mathfrak{I}} J_{1i} = J_1 + J_2 \tag{3.109}$$

where

$$J_1 = \int_{\cup_{i\in\mathfrak{I}_1}\partial U_i}(h_{\delta D} - h_{\delta D}(b^i))\operatorname{Im}\overline{w}\nabla w\cdot\tau ds, \tag{3.110}$$

$$\begin{aligned} J_2 &= \int_{\cup_{i\in\mathfrak{I}_1}\partial V_i}(h_{\delta D} - h_{\delta D}(b^i))\operatorname{Im}\overline{w}\nabla w\cdot\tau ds - \int_{\cup_{i\in\mathfrak{I}_1}\partial U_i}(h_{\delta D} - h_{\delta D}(b^i))\operatorname{Im}\overline{w}\nabla w\cdot\tau ds \\ &= \int_{\cup_{i\in\mathfrak{I}_1}(V_i\setminus U_i)}\nabla^\perp\cdot\Big[(h_{\delta D} - h_{\delta D}(b^i))\operatorname{Im}\overline{w}\nabla w\Big]\, dx \end{aligned}$$

$$= \int_{\cup_{i\in\mathfrak{I}_1}(V_i\setminus U_i)} \nabla^\perp h_{\delta D}\cdot \mathrm{Im}\,\overline{w}\nabla w dx. \tag{3.111}$$

since $\nabla^\perp \cdot \mathrm{Im}\,\overline{w}\nabla w = 0$. The term $J_2$ is small:

$$|J_2| \le \mathrm{meas}\,(\mathfrak{B})^{1/2}\cdot\|\nabla^\perp h_{\delta D}\|_{L^\infty(\mathfrak{B})}\cdot 2\|\nabla v\|_{L^2(\mathfrak{B})} \le O(\delta^2)\cdot O\left(\frac{1}{\delta}\right)\cdot O(|\log\delta|) = o(1). \tag{3.112}$$

Estimate $J_1$. Notice, that $|v| = 1/2$ on $\partial U_i$, so $\nabla w\cdot\tau = 2\nabla v\cdot\tau$ there:

$$\begin{aligned}
J_1 &= \int_{\cup_{i\in\mathfrak{I}_1}\partial U_i}(h_{\delta D}-h_{\delta D}(b^i))\mathrm{Im}\,\overline{w}\nabla w\cdot\tau\, ds = 4\int_{\cup_{i\in\mathfrak{I}_1}\partial U_i}(h_{\delta D}-h_{\delta D}(b^i))\mathrm{Im}\,\overline{v}\nabla v\cdot\tau\, ds\\
&= 4\int_{\cup_{i\in\mathfrak{I}_1}U_i}\nabla^\perp h_{\delta D}\cdot\mathrm{Im}\,\overline{v}\nabla v\, dx + 4\int_{\cup_{i\in\mathfrak{I}_1}U_i}(h_{\delta D}-h_{\delta D}(b^i))\mathrm{Im}\,(\nabla^\perp\overline{v}\cdot\nabla v)\, dx\\
&= L_1 + L_2.
\end{aligned}$$

The first integral $L_1$ admits the same estimate as in (3.112). To estimate $L_2$ notice, that

$$|\mathrm{Im}\,(\nabla^\perp\overline{v}\cdot\nabla v)| \le |\nabla^\perp\overline{v}|\cdot|\nabla v| = |\nabla v|^2. \tag{3.113}$$

Then

$$\begin{aligned}
|L_2| &\le 4\sum_{i\in\mathfrak{I}_1}\|h_{\delta D}-h_{\delta D}(b^i)\|_{L^\infty(U_i)}\cdot\|\nabla v\|^2_{L^2(\Omega)}\\
&\le 4\sum_{i\in\mathfrak{I}_1}\|\nabla h_{\delta D}\|_{L^\infty(U_i)}\cdot r_i\cdot|\log\delta|^2 \le O\left(\frac{1}{\delta}\right)\cdot\delta^2\cdot|\log\delta|^2 = o(1).
\end{aligned} \tag{3.114}$$

Thus all integrals $L_1$, $L_2$, and therefore $J_1$, $J_2$, and $J_{1i}$ are small. The only thing left is the set $\mathfrak{I}_2$ consisting of balls that intersect the boundary $\partial\Omega$. The estimates do not change a lot from the estimates in the balls from $\mathfrak{I}_1$ if we recall the boundary condition $h_{\delta D} = h_{ext}$ on $\partial\Omega$:

$$\begin{aligned}
\sum_{i\in\mathfrak{I}_2} I_{1i} &= \int_{\cup_{i\in\mathfrak{I}_2}\partial V_i}(h_{\delta D}-h_{ext})\mathrm{Im}\,\overline{w}\nabla w\cdot\tau\, ds\\
&= 4\int_{\cup_{i\in\mathfrak{I}_2}\partial U_i}(h_{\delta D}-h_{ext})\mathrm{Im}\,\overline{v}\nabla v\cdot\tau\, ds + \int_{\cup_{i\in\mathfrak{I}_2}(V_i\setminus U_i)}\nabla^\perp h_{\delta D}\cdot\mathrm{Im}\,\overline{w}\nabla w\, dx\\
&= 4\int_{\cup_{i\in\mathfrak{I}_2}U_i}\nabla^\perp(h_{\delta D}-h_{ext})\cdot\mathrm{Im}\,\overline{v}\nabla v\, dx\\
&+ 4\int_{\cup_{i\in\mathfrak{I}_2}U_i}(h_{\delta D}-h_{ext})\mathrm{Im}\,(\nabla^\perp\overline{v}\cdot\nabla v)\, dx + o(1)
\end{aligned}$$

$$= o(1). \tag{3.115}$$

The external magnetic field here plays the same role as $h_{\delta D}(b^i)$ in (3.114):

$$|h_{\delta D}(x) - h_{ext}| \leq \|\nabla h_{\delta D}\|_{L^\infty(\Omega)} \cdot 2r_i \leq O(\delta) \tag{3.116}$$

in $B_i$ for $B_i \cap \partial\Omega \neq \emptyset$ because $h_{\delta D} = h_{ext}$ on $\partial\Omega$.

Summarizing the estimates we end up with the formula

$$\sum_{i \in \mathfrak{I}} I_{1i} = \sum_{i \in \mathfrak{I}_1} 2\pi d_i (h_{\delta D}(b^i) - h_{ext}) + o(1) \tag{3.117}$$

and finish the proof. □

Combining (3.88), (3.94), and (3.95) we get

$$F_\delta[v, B; G] + \pi d \left( \log \frac{\delta^2}{d\varepsilon} - C \right) \leq 2\pi \sum_{i \in \mathfrak{I}_1} (h_{ext} - h_{\delta D}(b^i)) d_i + 2\pi \sum_{j=1}^{N} D_v^j (h_{ext} - H_R^j) + O(1) \tag{3.118}$$

where $d = \sum_{i \in \mathfrak{I}} |d_i|$ as before. This inequality holds under the assumption that $d$ is nonzero. If it does equal to zero, the term $\pi d \left( \log \frac{\delta^2}{d\varepsilon} - C \right)$ should be dropped.

In the following lemma we obtain the lower bound for $F_\delta$ that allows us to show the absence of bulk vortices $d_i = 0$ and get a quadratic inequality for $D_v^j$.

**Lemma 5.** *There is $\delta_0$ such that for any $\delta \leq \delta_0$ there are no bulk vortices inside the domain $\Omega \setminus \overline{S}$. Moreover, there exist $\alpha > 1$ and $\delta \ll R' \ll 1$ such that the following inequality holds:*

$$\sum_{j=1}^{N} \left[ \pi(1-\theta)^2 (|\log \delta| - |\log R'| + O(\delta)) \left( D_v^j \right)^2 - 2\pi D_v^j (h_{ext} - H_R^j) \right] \leq O(1) \tag{3.119}$$

*Proof.* Fix $\alpha > 1$ and consider two cases:

1. $\sum_{j=1}^{N} |D_v^j| \leq \alpha \sum_{i \in \mathfrak{I}} |d_i|$. The leading term in (3.118) is $\pi d |\log \varepsilon|$ in the left hand side and it cannot be bounded by the right hand side if $d \neq 0$ since the leading term there is of order $d \cdot O(|\log \delta|)$. Therefore $d = 0$, there are no bulk vortices and all $D_v = 0$ implying the desired result.

2. $\sum_{j=1}^{N} |D_v^j| > \alpha \sum_{i\in\mathfrak{I}} |d_i|$. We need an additional lower bound on the energy $F_\delta[v, B; G]$.

To estimate $F_\delta[v, B; G]$ we integrate over circles $\gamma_r^j = \partial B(a^j, r)$ with $r > R$ around the holes $\omega_\delta^j$. If $|u| \neq 0$ on $\gamma_r^j$ for some $r > R$, we can define the degree on it:

$$D_r^j = \deg(u, \gamma_r^j) = \deg(v, \gamma_r^j) \tag{3.120}$$

Denote

$$\mathfrak{R} = \{r \in (R, R_{max}) \ : \ |u| > 1 - \theta \text{ on } \gamma_r^j \text{ for all } j = 1 \dots N\}, \tag{3.121}$$

where $\theta$ comes from the Ball Construction Method and $R_{max}$ plays the same role as in Lemma 1: it is the maximal radius $r$ such that $B(a^j, r)$ are disjoint and do not intersect $\partial\Omega$. The total degree on $\partial\Omega$ is the sum of the degrees of all vortices. Since by definition of $D_v^j$ we have $D_r^j = D_v^j$, it implies

$$\sum_{j=1}^{N} |D_r^j| \geq \sum_{j=1}^{N} |D_v^j| - \sum_{i\in\mathfrak{I}} |d_i| \geq \frac{\alpha - 1}{\alpha} \sum_{j=1}^{N} |D_v^j|. \tag{3.122}$$

Using the definition of the degree and the Divergence Theorem for $r \in \mathfrak{R}$ we get

$$2\pi D_r^j - \int_{B_r^j} \operatorname{curl} B \, dx = \int_{\gamma_r^j} \nabla\Phi \cdot \tau - B \cdot \tau dS = \int_{\gamma_r^j} (\nabla\Phi - B) \cdot \tau dS \tag{3.123}$$

or

$$2\pi D_r^j = \int_{\gamma_r^j} (\nabla\Phi - B) \cdot \tau dS + \int_{B_r^j} \operatorname{curl} B \, dx = I_1(r) + I_2(r) \tag{3.124}$$

for any $j = 1 \dots N$. Here $B_r^j = B(a^j, r)$ and $v = |v|e^{i\Phi}$. Estimate the integrals:

$$\begin{aligned} I_1^2 &\leq \operatorname{meas}(\gamma_r^j) \int_{\gamma_r^j} |\nabla\Phi - B|^2 dS \leq 2\pi r \int_{\gamma_r^j} \frac{|(\nabla - iB)v|^2}{|v|^2} dS \\ &\leq \frac{2\pi r}{(1-\theta)^2} \int_{\gamma_r^j} |(\nabla - iB)v|^2 dS, \end{aligned} \tag{3.125}$$

$$I_2^2 \leq \operatorname{meas}(B_r^j) \int_{B_r^j} |\operatorname{curl} B|^2 dx \leq C_1 |\log\delta|^2 r^2, \tag{3.126}$$

since $|v| > 1 - \theta$ by the Ball Construction Method. Now square and estimate

(3.124) for $r \in \mathfrak{R}$:

$$4\pi^2 \left(D_r^j\right)^2 = (I_1(r) + I_2(r))^2$$
$$\leq \frac{2\pi r}{(1-\theta)^2} \int_{\gamma_r^j} |(\nabla - iB)v|^2 dS + 2C_1 |\log \delta|^2 r^2 \cdot I_1 + C_1 |\log \delta|^2 r^2. \qquad (3.127)$$

Divide both sides by $r$ and integrate (3.127) outside "bad" disks from $R$ to $R'$ with a $R' \ll R_{max}$ that will be prescribed later:

$$\begin{aligned}
4\pi^2 \int_{(R,R')\cap\mathfrak{R}} \frac{(D_r^j)^2}{r} dr \leq{}& \frac{2\pi}{(1-\theta)^2} \int_{(R,R')\cap\mathfrak{R}} \int_{\gamma_r^j} |(\nabla - iB)v|^2 dS dr \\
&+ 2C_1 |\log \delta|^2 \cdot \int_{(R,R')\cap\mathfrak{R}} I_1 r dr + C_1 |\log \delta|^2 \left.\frac{r^2}{2}\right|_R^{R'} \\
&\leq \frac{4\pi}{(1-\theta)^2} F_\delta[v, B; B_{R'}^j] + \frac{C_1}{2} |\log \delta|^2 R'^2 \\
&+ 2C_1 |\log \delta|^2 \cdot R' \cdot \sqrt{\pi R'^2} \cdot \left( \int_{(R,R')\cap\mathfrak{R}} \int_{\gamma_r} \frac{|(\nabla - iB)v|^2}{|v|^2} dS dr \right)^{1/2} \\
&\leq \frac{4\pi}{(1-\theta)^2} F_\delta[v, B; K^j] + \frac{C_1}{2} |\log \delta|^2 R'^2 \\
&+ \frac{C_3}{1-\theta} |\log \delta|^3 R'^2 \qquad (3.128)
\end{aligned}$$

since $|v| > 1 - \theta$ by the definition of $\mathfrak{R}$. Here $K^j$ is a union of concentric rings around $j$th hole:

$$K^j = \bigcup_{r \in (R,R')\cap\mathfrak{R}} \gamma_r^j = \bigcup_{r \in (R,R')\cap\mathfrak{R}} \partial B(a^j, r). \qquad (3.129)$$

Notice that all $K^j$ are disjoint since $R' \ll R_{max}$ and $K^j \subset G$ for all $j = 1 \ldots N$.

In order to obtain the lower bound for $F_\delta$ we divide both sides in (3.128) by $4\pi/(1-\theta)^2$:

$$\begin{aligned}
\pi(1-\theta)^2 \int_{(R,R')\cap\mathfrak{R}} \frac{(D_r^j)^2}{r} dr \leq{}& F_\delta[v, B; K^j] + \frac{C_1(1-\theta)^2}{8\pi} |\log \delta|^2 R'^2 \\
&+ \frac{C_3(1-\theta)}{4\pi} |\log \delta|^3 R'^2. \qquad (3.130)
\end{aligned}$$

We can choose

$$R' = C\zeta^{1/2}|\log\delta|^{-2} \gg R \tag{3.131}$$

and an appropriate constant $C$ such that for $\zeta = |\log\delta|^{-1} = o(1)$ the sum of last two terms in (3.130) is less than $\zeta$ for small $\delta$. Notice, that meas $((R, R') \setminus \mathfrak{R}) < \delta^2$ by the Ball Construction Method and $R \le \delta + \delta^2$. Therefore

$$\begin{aligned}\sum_{j=1}^{N} \int_{(R,R')\cap\mathfrak{R}} \frac{(D_r^j)^2}{r}\, dr &\ge \frac{(\alpha-1)^2}{\alpha^2}\frac{1}{N}\sum_{j=1}^{N} |D_v^j|^2 \log r\Big|_{\delta+2\delta^2}^{R'} \\ &\ge \frac{(\alpha-1)^2}{\alpha^2}\frac{1}{N}\sum_{j=1}^{N} |D_v^j|^2(|\log\delta| - |\log R'| + O(\delta)).\end{aligned} \tag{3.132}$$

Thus we can combine (3.130) and (3.132) to write the lower estimate for $F_\delta[v, B; G]$ in terms of the additional degrees $D_v^j$:

$$\begin{aligned}F_\delta[v, B; G] &\ge \sum_{j=1}^{N} F_\delta[v, B; K^j] \\ &\ge \pi(1-\theta)^2\frac{(\alpha-1)^2}{\alpha^2}\frac{1}{N}\sum_{j=1}^{N} |D_v^j|^2(|\log\delta| - |\log R'| + O(\delta)) - \zeta.\end{aligned} \tag{3.133}$$

Substitute $\zeta = |\log\delta|^{-1}$ and combine (3.133) with (3.118) to get

$$\begin{aligned}&\sum_{j=1}^{N}\left(\frac{1}{N}\pi(1-\theta)^2\frac{(\alpha-1)^2}{\alpha^2}(|\log\delta| - |\log R'| + O(\delta))\left(D_v^j\right)^2 - 2\pi(h_{ext} - H_R^j)D_v^j\right) \\ &\le -\pi\sum_{i\in\mathfrak{I}_1} |d_i|\left(|\log\varepsilon| - 2|\log\delta| + |\log d| - C\right) + 2\pi\sum_{i\in\mathfrak{I}_1}(h_{ext} - h_{\delta D}(b^i))d_i + O(1)\end{aligned} \tag{3.134}$$

Compare the order of the leading terms in (3.134):

$$\sum_{j=1}^{N}\left(A|\log\delta|\left(D_v^j\right)^2 - O(|\log\delta|)D_v^j\right) \le -d|\log\varepsilon| + O(1) \tag{3.135}$$

with $A > 0$. The left hand side of (3.135) is a sum of quadratic functions in $D_v^j$

with positive leading coefficients:

$$q_j(D_v^j) = A|\log\delta|\left(D_v^j\right)^2 - O(|\log\delta|)D_v^j. \tag{3.136}$$

The values of parabolas $q_j$ are bounded from below by the values at their vertexes

$$t^j = \frac{O(|\log\delta|)}{2A|\log\delta|} = O(1) \tag{3.137}$$

that are bounded. Therefore

$$-d|\log\varepsilon| + o(1) \geq \sum_{j=1}^{N} q_j(D_v^j) \geq \sum_{j=1}^{N} q_j(t_j) = O(|\log\delta|) \tag{3.138}$$

Since $|\log\varepsilon| \gg |\log\delta|$ the inequality (3.138) can hold only if $d = 0$, i.e. there are no bulk vortices. This in turn implies that $D_r^j = D_v^j$ and the inequality (3.122) is no longer needed. It simplifies the lower bound (3.132) and yields the desired inequality. □

## 3.6 Proof of Theorem 1: Equality of the Degrees

*Proof.* To finish the proof of Theorem 1 we need to show that all $D_v^j = 0$. Start with the quadratic inequality for $D_v^j$ obtained in Lemma 5:

$$\sum_{j=1}^{N}\left[\pi(1-\theta)^2(|\log\delta| - |\log R'| + O(\delta))\left(D_v^j\right)^2 - 2\pi D_v^j(h_{ext} - H_R^j)\right] \leq O(1). \tag{3.139}$$

where $H_R^j = h_{ext}\xi_0(a^j) + D^j K_0(R)$. This inequality has the same structure as the quadratic functional in $S^1$-valued case: there are no mixed terms $D_v^i D_v^j$. Therefore we can find zeros for each $j = 1 \dots N$ separately.

Fix $1 \leq j \leq N$. Clearly, $D_v^j = 0$ is one of two roots of

$$\pi(1-\theta)^2(|\log\delta| - |\log R'| + O(\delta))\left(D_v^j\right)^2 - 2\pi D_v^j(h_{ext} - H_R^j) = 0 \tag{3.140}$$

Since $K_0(R) = |\log\delta| + O(1)$ and

$$D^j = \left[\!\left[\sigma\left(1 - \xi_0(a^j)\right)\right]\!\right] \tag{3.141}$$

we can calculate the coefficient for the linear term in (3.139):

$$\begin{aligned}-2\pi(h_{ext} - H_R^j) &= -2\pi(\sigma|\log\delta| - \sigma|\log\delta|\xi_0(a^j) - \left[\!\left[\sigma\left(1-\xi_0(a^j)\right)\right]\!\right]|\log\delta|) + O(1)\\ &= -2\pi|\log\delta|\Big(\sigma\left(1-\xi_0(a^j)\right) - \left[\!\left[\sigma\left(1-\xi_0(a^j)\right)\right]\!\right]\Big) + O(1).\end{aligned} \tag{3.142}$$

Since $[\![\cdot]\!]$ is the nearest integer, we have

$$\left|\sigma\left(1-\xi_0(a^j)\right) - \left[\!\left[\sigma\left(1-\xi_0(a^j)\right)\right]\!\right]\right| \le \frac{1}{2} - \xi \tag{3.143}$$

assuming the uniqueness condition (3.10) and taking

$$\xi = \min_{j=1\dots N} \operatorname{dist}\left(\sigma\left(1-\xi_0(a^j)\right), \mathbb{Z} + \frac{1}{2}\right) > 0. \tag{3.144}$$

Find the second zero of (3.140) as a negative ratio of the linear coefficient to quadratic coefficient:

$$|t_j| = \left|\frac{-2\pi\Big(\sigma\left(1-\xi_0(a^j)\right) - [\![\sigma\left(1-\xi_0(a^j)\right)]\!]\Big) + o(1)}{\pi(1-\theta)^2 + o(1)}\right| < \frac{1-2\xi}{(1-\theta)^2 + o(1)} + o(1). \tag{3.145}$$

Having $\xi$ fixed and $\delta < \delta_0$ sufficiently small, we can always take $\theta > 0$ small enough to make sure $|t_j| < 1 - \xi$.

Since $D_v^j$ can take only integer values, then if at least one $D_v^j$ is nonzero, the left hand side of (3.139) becomes strictly positive with order $O(\log\delta)$. This contradiction finishes the proof of main theorem yielding the result

$$D_v^j = 0 \text{ or } D_{\delta,\varepsilon}^j = D^j \tag{3.146}$$

for all $j = 1 \dots N$. □

# Chapter 4
# Conclusions and Prospectives

This dissertation is devoted to the study of the behavior of vortices and their degrees in the superconductors that experience pinning effect. In this chapter we summarize our findings, discuss the challenges that we met and the prospectives of this research.

The major contributions of Chapter 2 are the following.

- We establish that a periodic lattice of columnar defects with a particular relation between its geometry, the magnetic field, and the material parameters, experiences a nonuniform pattern of vortices differentiated by their degrees.
- This pattern divides the cross section of the sample into a set of nested subdomains with different average vorticity which is radically different from the current understanding of the ground states of superconductors.

One of the difficulties is to analyze the original nonlinear functional in such a complex domain with the radii of holes as well as the distances between them tending to zero. This obstacle is overcome by studying the degrees of the simpler functional with $S^1$-valued minimizer.

The justification of this transition is suggested in Chapter 3. The finite number of shrinking holes distributed randomly is considered insted of fine periodic lattice. The major findings of it are:

- Under the same conditions as in Chapter 2 we show that no vortices appear outside of holes and therefore the assumption $|u| = 1$ outside of holes is reasonable.

- Moreover, the constrained functional with $|u| = 1$ and the original unconstrained functional have the same degrees of the vortices at the corresponding holes which justify the study of $S^1$-valued functional instead of the original GL functional when one is interested in the degrees of the hole vortices.

The main difficulty that we face in Chapter 3 is the vanishingly small radius of the holes. Because of their size we need much sharper and delicate estimates and analysis than what was done in any similar stuies before. Such analysis can provide insight for the other scientits working with Ginzburg-Landau functional or any other nonlinear elliptic problems.

The next reasonable study would be to prove similar transitions as in Chapter 3 but in more general cases, when the number of holes does not need to be constant when their sizes decrease. The logical continuation of the findings in Chapter 2 would be to study different shapes of the lattice (hexagonal instead of rectangular) as well as a random distribution of holes with their number going to infinity as usually considered in homogenization problems. The random distribution was studied numerically in [47] and the authors obtained similar nested domain structure as described in Chapter 2. The homogenization approach can be used to prove this result analytically.

# Appendix

# Appendix

## 1 Estimate of the typical size of the nested subdomains

Consider the circular domain $\Omega = B(0,1)$ and find $\tilde{R}_{\mathrm{D}}$ that is the dimensionless radius of subdomain $\Omega_1$ in the case when there are just two phases $D(x) = 0$ and $0 < D(x) \leq 1$ (in radially symmetrical case $\Omega_1$ is also a ball). Take $\sigma = \gamma/2 + 2\pi$ (the case when the minimizer $\bar{f}_\sigma$ is still equal to $-\gamma/2$ inside $\Omega_1$ but $D(x)$ already turned to 1) and $\gamma = 1$.

We have the following equation in $\Omega_0$:

$$\begin{cases} -\Delta f + f + \sigma = 0 \text{ in } \Omega_0 \\ f = 0 \text{ on } \partial\Omega \end{cases} \tag{.1}$$

This equation is nothing but the modified Bessel equation in the polar coordinates and it has solution

$$\bar{f}_\sigma(r) = C_1 I_0(r) + C_2 K_0(r) - \sigma. \tag{.2}$$

Using the Dirichlet boundary conditions on $\partial\Omega$, continuity of $\bar{f}_\sigma$ and its derivative at $r = \tilde{R}_{\mathrm{D}}$ one arrives at:

$$C_1 I_0(R_{\mathrm{D}}) + C_2 K_0(\tilde{R}_{\mathrm{D}}) - \sigma = \frac{\gamma}{2} \tag{.3}$$

$$C_1 I_0'(\tilde{R}_{\mathrm{D}}) + C_2 K_0'(\tilde{R}_{\mathrm{D}}) = 0 \tag{.4}$$

$$C_1 I_0(1) + C_2 K_0(1) - \sigma = 0 \tag{.5}$$

Solving it, we obtain:

$$\frac{K_0(\tilde{R}_{\mathrm{D}})I_1(\tilde{R}_{\mathrm{D}})+K_1([\tilde{R}_{\mathrm{D}})I_0(\tilde{R}_{\mathrm{D}})}{K_0(1)I_1(\tilde{R}_{\mathrm{D}})+K_1(\tilde{R}_{\mathrm{D}})I_0(1)}=\frac{\gamma}{2\sigma}+1. \tag{.6}$$

Given our choice of $\gamma$ and $\sigma$ the RHS of (.6) is equal to $(4\pi)/(4\pi+1)$. Solving this equation numerically, we find $\tilde{R}_{\mathrm{D}}$:

$$\tilde{R}_{\mathrm{D}}=0.567\,. \tag{.7}$$

# 2 Gradient estimate for elliptic maps

**Lemma 6.** *Let $u$ solve the Poisson equation with Dirichlet boundary conditions in $\Omega_\delta=\Omega\setminus\cup_{j=1}^{N}\omega_\delta^j$ with $\omega_\delta^j=B(a^j,\delta)$:*

$$\begin{cases}-\Delta u=f & \text{in } \Omega_\delta,\\ u=g & \text{on } \partial\Omega,\\ u=g_j & \text{on } \partial\omega_\delta^j.\end{cases} \tag{.8}$$

*where $g$ and $g_j$ are smooth functions that are actually defined in the whole $\Omega_\delta$. Then*

$$\|\nabla u\|_{L^\infty(\Omega_\delta)}\le C\left(\frac{1}{\delta}\|u\|_{L^\infty(\Omega_\delta)}+\|f\|_{L^\infty(\Omega_\delta)}+\|\Delta g\|_{L^\infty(\Omega)}+\delta\sum_{j=1}^{N}\|\Delta g_j\|_{L^\infty(\Omega_\delta)}\right). \tag{.9}$$

**Remark 4.** We use lemmas A.1 and A.2 from [48] and extend their results to a domain with fine boundaries.

*Proof.* Consider three cases: the point $x_0\in\Omega_\delta$ is far from the boundaries of $\partial\Omega_\delta$, it is close to $\partial\Omega$, and it is close to $\partial\omega_\delta^j$ for some $j=1\dots N$. The first case when $x_0\in K\subset\subset\Omega_\delta$ is resolved in Lemma A.1 [48] and the second case when $x_0$ is close to $\partial\Omega$ can be answered by Lemma A.2 using $\widetilde{u}=u-g$. The results of both lemmas can be merged together in the following estimate:

$$|\nabla u(x_0)|\le C\left(\|u\|_{L^\infty}+\|f\|_{L^\infty}+\|\Delta g\|_{L^\infty}\right)\quad \text{a.e.} \tag{.10}$$

when $\operatorname{dist}(x_0,\partial\omega_\delta^j)>m>0$ with some fixed $m$ independent of $\delta$.

The third case is specific for our setting. Let $x_0$ be close to one of the holes: dist $(x_0, \partial\omega_\delta^j) \leq m$ for some $j = 1 \dots N$. Without loss of generality assume $a^j = 0$. We introduce the new spatial variable $y = \frac{x}{\delta}$ to rescale the domain so that the $\omega_\delta^j$ becomes $B(0,1)$ and $x_0$ becomes $y_0$. The Poisson equation in new coordinates becomes

$$-\Delta_y u = \delta^2 f. \tag{.11}$$

If dist $(y_0, \partial B(0,1)) > m$, we apply Lemma A.1 from [48] again. It gives us the estimate for $|\nabla_y u(y_0)|$:

$$|\nabla_y u(y_0)| \leq C\left(\|u\|_{L^\infty} + \delta^2\|f\|_{L^\infty}\right) \tag{.12}$$

that in turn implies the estimate for $|\nabla_x u(x_0)|$:

$$|\nabla_x u(x_0)| = \frac{1}{\delta}|\nabla_y u(y_0)| \leq \frac{C}{\delta}\|u\|_{L^\infty(\Omega_\delta)} + C\delta\|f\|_{L^\infty(\Omega_\delta)}. \tag{.13}$$

Finally, we apply Lemma A.2 to $\widetilde{u}_j = u - g_j$ that satisfies the problem

$$\begin{cases} -\Delta_y \widetilde{u}_j = \delta^2 f + \Delta_y g_j & \text{in } B(0,1+m) \setminus \overline{B(0,1)}, \\ \widetilde{u}_j = h_j & \text{on } \partial B(0,2+m), \\ \widetilde{u}_j = 0 & \text{on } \partial B(0,1). \end{cases} \tag{.14}$$

where $h_j(y) = u(y) - g_j(y)$. Since the proof of Lemma A.2 uses only local estimates and $y_0$ is far from the $\partial B(0,2+m)$, the function $h_j$ does not play a role for the estimate of $|\nabla_y u(y_0)|$. It yields the estimate

$$|\nabla_y u(y_0)| \leq C\left(\|u\|_{L^\infty} + \delta^2\|f\|_{L^\infty} + \|\Delta_y g_j\|_{L^\infty}\right). \tag{.15}$$

Going back to $x$ we obtain

$$|\nabla_x u(x_0)| \leq \frac{C}{\delta}\|u\|_{L^\infty} + C\delta\left(\|f\|_{L^\infty} + \|\Delta_x g_j\|_{L^\infty}\right). \tag{.16}$$

Merging all the estimates we finish the proof. □

# Bibliography


[1] Iaroshenko, O., Rybalko, V., Vinokur, V.M., and Berlyand, L., “Vortex phase separation in mesoscopic superconductors,” *Scientific Reports*, volume 3, 2013

[2] Berlyand, L., Golovaty, D., Iaroshenko, O., and Rybalko, V., “On approximation of Ginzburg–Landau minimizers by S1-valued maps in domains with vanishingly small holes,” *Journal of Differential Equations*, 2017

[3] de Bruyn Ouboter, Rudolf, “Heike Kamerlingh Onnes’s discovery of superconductivity,” *Scientific American*, volume 276, no. 3, pp. 98–103, 1997

[4] Ford, P. J. and Saunders, G. A., *The rise of the superconductors*, CRC press, 2004

[5] Drozdov, AP, Eremets, MI, Troyan, IA, Ksenofontov, Vadim, and Shylin, SI, “Conventional superconductivity at 203 kelvin at high pressures in the sulfur hydride system,” *Nature*, volume 525, no. 7567, p. 73, 2015

[6] Ginzburg, V. L. and Landau, L. D., “Collected Papers of L. D. Landau,” *ed D Ter Haar*, 1965

[7] Bardeen, J., Cooper, L. N., and Schrieffer, J. R., “Microscopic Theory of Superconductivity,” *Phys Rev*, volume 106, pp. 162–164, Apr 1957

[8] Bethuel, F., Brezis, H., and Hélein, F., *Ginzburg–Landau Vortices*, Springer, 1994

[9] Chen, B., Halperin, W. P., Guptasarma, P., Hinks, D. G., Mitrovich, V. F., Reyes, A. P., and Kuhns, P. L., “Two-dimensional vortices in superconductors,” *Nature Physics*, volume 3, pp. 239–242, 2007

[10] Nakajima, Y, Tsuchiya, Y, Taen, T, Tamegai, T, Okayasu, S, and Sasase, M, “Enhancement of critical current density in Co-doped BaFe 2 As 2 with columnar defects introduced by heavy-ion irradiation,” *Physical Review B*, volume 80, no. 1, p. 012510, 2009

[11] Chapman, S. J. and Richardson, G., “Vortex pinning by inhomogeneities in type-II superconductors,” *Physica D Nonlinear Phenomena*, volume 108, no. 4, pp. 397–407, 1997

[12] Aftalion, A., Sandier, E., and Serfaty, S., “Pinning phenomena in the Ginzburg–Landau model of superconductivity,” *Journal de mathematiques pures et appliquees*, volume 80, no. 3, pp. 339–372, 2001

[13] Andre, N., Bauman, P., and Phillips, D., “Vortex pinning with bounded fields for the Ginzburg–Landau equation,” *Annales de lInstitut Henri Poincare C Non Linear Analysis*, volume 20, no. 4, pp. 705–729, 2003

[14] Alama, S. and Bronsard, L., “Pinning effects and their breakdown for a Ginzburg–Landau model with normal inclusions,” *Journal of Mathematical Physics*, volume 46, no. 9, p. 095102, 2005

[15] Aydi, H. and Kachmar, A., “Magnetic vortices for a Ginzburg–Landau type energy with discontinuous constraint. II,” *Commun Pure Appl Anal*, volume 8, pp. 977–998, 2009

[16] Kachmar, A., “Magnetic vortices for a Ginzburg–Landau type energy with discontinuous constraint,” *ESAIM Control Optim Calc Var*, volume 16, pp. 545–580, 2010

[17] Blatter, Gianni, Feigel’man, Mikhail V, Geshkenbein, Vadim B, Larkin, Anatoly I, and Vinokur, Valerii M, “Vortices in high-temperature superconductors,” *Reviews of Modern Physics*, volume 66, no. 4, p. 1125, 1994

[18] Nattermann, Thomas and Scheidl, Stefan, “Vortex-glass phases in type-II superconductors,” *Advances in Physics*, volume 49, no. 5, pp. 607–704, 2000

[19] Konczykowski, M, Rullier-Albenque, F, Yacoby, ER, Shaulov, A, Yeshurun, Y, and Lejay, P, “Effect of 5.3-GeV Pb-ion irradiation on irreversible magnetization in Y-Ba-Cu-O crystals,” *Physical Review B*, volume 44, no. 13, p. 7167, 1991

[20] Civale, L, Marwick, AD, Worthington, TK, Kirk, MA, Thompson, JR, Krusin-Elbaum, L, Sun, Y, Clem, JR, and Holtzberg, F, “Vortex confinement by columnar defects in YBa 2 Cu 3 O 7 crystals: Enhanced pinning at high fields and temperatures,” *Physical Review Letters*, volume 67, no. 5, p. 648, 1991

[21] Nelson, David R and Vinokur, VM, “Boson localization and pinning by correlated disorder in high-temperature superconductors,” *Physical review letters*, volume 68, no. 15, p. 2398, 1992

[22] Nelson, David R and Vinokur, VM, “Boson localization and correlated pinning of superconducting vortex arrays,” *Physical Review B*, volume 48, no. 17, p. 13060, 1993

[23] Grigorieva, IV, Escoffier, W, Richardson, J, Vinnikov, LY, Dubonos, S, and Oboznov, V, "Direct observation of vortex shells and magic numbers in mesoscopic superconducting disks," *Physical review letters*, volume 96, no. 7, p. 077005, 2006

[24] Bedanov, Vladimir M and Peeters, FrancÌğois M, "Ordering and phase transitions of charged particles in a classical finite two-dimensional system," *Physical Review B*, volume 49, no. 4, p. 2667, 1994

[25] Cooley, LD and Grishin, AM, "Pinch effect in commensurate vortex-pin lattices," *Physical review letters*, volume 74, no. 14, p. 2788, 1995

[26] Kontorova, T and Frenkel, J, "On the theory of plastic deformation and twinning. II." *Zh Eksp Teor Fiz*, volume 8, pp. 1340–1348, 1938

[27] Frank, FC and van der Merwe, Jan H, "One-dimensional dislocations. I. Static theory," *Proc. R. Soc. Lond. A*, volume 198, The Royal Society, 1949

[28] Silhanek, AV, Gutierrez, J, Kramer, RBG, Ataklti, GW, Van de Vondel, Joris, Moshchalkov, VV, and Sanchez, A, "Microscopic picture of the critical state in a superconductor with a periodic array of antidots," *Physical Review B*, volume 83, no. 2, p. 024509, 2011

[29] Larkin, AI and Vinokur, VM, "Bose and vortex glasses in high temperature superconductors," *Physical review letters*, volume 75, no. 25, p. 4666, 1995

[30] Thakur, Ajay D, Ooi, Shuuichi, Chockalingam, Subbaiah P, Jesudasan, John, Raychaudhuri, Pratap, and Hirata, Kazuto, "Vortex matching effect in engineered thin films of NbN," *Applied Physics Letters*, volume 94, no. 26, p. 262501, 2009

[31] Mkrtchyan, GS and Shmidt, VV, "Interaction Between a Cavity and a Vortex in a Superconductor of the Second Kind," *Soviet Journal of Experimental and Theoretical Physics*, volume 34, p. 195, 1972

[32] Burkov, SE and Talapov, AL, "One-dimensional incommensurate structures at finite temperatures," *Journal de Physique Lettres*, volume 41, no. 16, pp. 387–390, 1980

[33] Misko, VR and Nori, Franco, "Magnetic flux pinning in superconductors with hyperbolic-tessellation arrays of pinning sites," *Physical Review B*, volume 85, no. 18, p. 184506, 2012

[34] Buzdin, Alexandre I, "Multiple-quanta vortices at columnar defects," *Physical Review B*, volume 47, no. 17, p. 11416, 1993

[35] Grigorieva, IV, Escoffier, W, Misko, VR, Baelus, BJ, Peeters, FM, Vinnikov, L Ya, and Dubonos, SV, “Pinning-induced formation of vortex clusters and giant vortices in mesoscopic superconducting disks,” *Physical review letters*, volume 99, no. 14, p. 147003, 2007

[36] Cren, Ten, Serrier-Garcia, L, Debontridder, François, and Roditchev, D, “Vortex fusion and giant vortex states in confined superconducting condensates,” *Physical review letters*, volume 107, no. 9, p. 097202, 2011

[37] Pollack, Scott E, Dries, D, Junker, Mark, Chen, YP, Corcovilos, TA, and Hulet, RG, “Extreme Tunability of Interactions in a Li 7 Bose-Einstein Condensate,” *Physical Review Letters*, volume 102, no. 9, p. 090402, 2009

[38] Haller, Elmar, Gustavsson, Mattias, Mark, Manfred J, Danzl, Johann G, Hart, Russell, Pupillo, Guido, and Nägerl, Hanns-Christoph, “Realization of an excited, strongly correlated quantum gas phase,” *Science*, volume 325, no. 5945, pp. 1224–1227, 2009

[39] Sandier, E. and Serfaty, S., “Global minimizers for the Ginzburg–Landau functional below the first critical magnetic field,” *Annales de lInstitut Henri Poincare C Non Linear Analysis*, volume 17, no. 1, pp. 119–145, 2000

[40] Sandier, E. and Serfaty, S., “Ginzburg–Landau minimizers near the first critical field have bounded vorticity,” *Calculus of Variations and Partial Differential Equations*, volume 17, no. 1, pp. 17–28, 2003

[41] Alama, S. and Bronsard, L., “Vortices and pinning effects for the Ginzburg–Landau model in multiply connected domains,” *Communications on pure and applied mathematics*, volume 59, no. 1, pp. 36–70, 2006

[42] Dos Santos, M. and Misiats, O., “Ginzburg–Landau model with small pinning domains,” *Netw Heterog Media*, volume 6, no. 4, 2011

[43] Jerrard, R., “Lower Bounds for Generalized Ginzburg–Landau Functionals,” *SIAM Journal on Mathematical Analysis*, volume 30, no. 4, pp. 721–746, 1999

[44] Sandier, E., “Lower bounds for the energy of unit vector fields and applications,” *Journal of Functional Analysis*, volume 152, no. 2, pp. 379–403, 1998

[45] Sandier, E. and Serfaty, S., “On the energy of type-II superconductors in the mixed phase,” *Reviews in Mathematical Physics*, volume 12, no. 09, pp. 1219–1257, 2000

[46] Sandier, E. and Serfaty, S., *Vortices in the Magnetic Ginzburg–Landau Model*, Birkhäuser Boston, 2007

[47] Ryan, SD, Mityushev, V, Vinokur, VM, and Berlyand, L, "Rayleigh approximation to ground state of the Bose and Coulomb glasses," *Scientific reports*, volume 5, p. 7821, 2015

[48] Bethuel, Fabrice, Brezis, Haïm, and Hélein, Frédéric, "Asymptotics for the minimization of a Ginzburg-Landau functional," *Calculus of Variations and Partial Differential Equations*, volume 1, no. 2, pp. 123–148, 1993

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## Publications

1. *Image compression: sparse coding vs. bottleneck autoencoders*, with Y. Watkins, M. Sayeh, and G. Kenyon, NICE Workshop (2018).

2. *Random on-board pixel sampling (ROPS) X-ray Camera*, with Z. Wang, S. Li, T. Liu, N. Parab, W.W. Chen, P. Chu, G. Kenyon, R. Lipton, K.-X. Sun, submitted to Journal of Instrumentation, arXiv preprint arXiv:1709.08659 (2017).

3. *On approximation of Ginzburg–Landau minimizers by S1-valued maps in domains with vanishingly small holes*, with L. Berlyand, D. Golovaty, and V. Rybalko, Journal of Differential Equations (2017).

4. *Vortex phase separation in mesoscopic superconductors*, with V. Rybalko, V.M. Vinokur, and L. Berlyand, Scientific Reports, 3 (2013).