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MULTI-LEADER-FOLLOWER GAMES OF FREIGHT SERVICE PRICING

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Abstract

Recently, Use of Stackelberg game and game theory concepts to solve dynamic competitive pricing, Nonlinear pricing, supply chain management and transportation network has been increased. This dissertation also focuses on Stackelberg game and EPEC formulation to deal with a linear and nonlinear pricing problem in freight service companies.

We are interested in computing freight service prices when an oligopoly of freight service providers compete with one another to carry cargo for large and complex, dynamic network markets. This dissertation presents two projects as linear and nonlinear pricing on multi-leader-follower games of freights Service companies based on the Stackelberg-Cournot-Nash behavioral assumption. We also consider oligopoly of producers of a single abstract homogeneous commodity that is brought to market by the aforementioned freight service providers. For such an environment, we study dynamic freight service pricing from the point of view of multi-leader-follower games.

The focus of the first project is on the linear pricing decision model for an oligopoly of carrier as leaders and an oligopoly of shippers as followers who compete in product's price, production quantity output and shipments pattern. This problem is a bi-level game problem with carriers at the upper level and shippers at the lower level. To formulate the problem we review the dynamic Stackelberg games and how the levels of such games can be described as a differential variational inequalities (DVI). We also show how the DVI of the lower level can be rewritten as the mathematical complementarity formulation. Hence we are able to convert the bi-level optimization problem into a single-level problem. An application of this differential Stackelberg game will be shown in revenue management of freight services after proper time-discretization,

The second topic aim to apply a more realistic, applicable and complicated model by adding non-linear pricing decision to the problem. For each carrier, the combined pricing-routing problem is a mathematical program with equilibrium constraints (MPEC). On top of this, we aim to find a Nash equilibrium among the leaders, thereby coupling multiple MPECs into a single equilibrium problem with equilibrium constraints (EPEC). We show the computability of this EPEC model by proposing novel yet practical algorithms called double adjoint approach based on computational intelligence and high performance computing.

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Dedication

This thesis is dedicated to my parents who gave up on having their daughter beside them in order to support me and giving me the opportunity to receive my education from one of the best institutions.

Chapter 1 Introduction

1.1 Introduction and Motivation

1.1.1 Differential Shipper-Carrier Problem

In this thesis we are interested in computing urban freight service prices when an oligopoly of freight service providers compete with one another to carry cargo for large and complex, dynamic network markets. We also consider multiple producers of a single abstract homogeneous commodity that is brought to market by the aforementioned freight service providers. Such produces, acting as shippers, also constitute an oligopoly in the market. For such an environment, we study dynamic freight service pricing from the point of view of multi-leader-follower(Stackelberg Game) games.

Concept of Stackelberg game has been first introduced by Heinrich Freiherr von Stackelberg within the context of static economic competition(1952) [1]. Such game is a two level optimization problem while the leader on the upper level minimize his/her objective function and followers on the lower level react to the optimal strategies set by the upper-level leader in order to maximize their own utilities. In such problems, the leader make optimal decisions by predicting the reactions of the followers and the resulting equilibrium states at the lower level.

Simaan [2] presented a very popular method to solve static Stacklbereg game. He added an artificial constraint to the upper level problem to force the duality gap between primal and dual problem of follower equal to zero. He then used the penalty function to add the duality gap constraint to the upper level objective function. This approach is widely used by many other researches such as Aiyoshi and Shmizu [3]; and Shimizu and Aiyoshi [4]. Also Hoesel [5] presents a enumeration procedure which is based on a branch-and-bound method to solve static Stackelberg games.

Stackelberg games has been widely used in different application related to this dissertation such as supply chain, inventory management and network design. Application of Stackelber game in the static shipper-carrier problem has been studied extensively in the literature. One of the first models considering multiple agents has been developed by Friesz et al. [6] with a sequential modeling framework. Xiao and Yang [7] studied a three-player noncooperative game among shippers, carriers, and infrastructure companies. On the other hand, there are relatively few studies that emphasize computability in studying of the shipper-carrier relationship in a dynamic setting. One such model leads the dynamic shipper-carrier problem as a differential Stackelberg game when there is only one carrier acting as the leader (see [8] and [9]).

Also Recently, the multiple-leader-followers games has become an important part of game theory ([10], [11]) . This type of model arise from those oligopoly markets with several leading firms such as the electricity energy market (See [12], [13]). To the best of our knowledge, This thesis is the first in modeling the freight service competition as a dynamic multi-leader- follower game. A related literature employs the notion of a spatial computable general equilibrium (SCGE) as the primary framework for modeling. Brocker et al. [14] bases an SCGE model on an optimal savings model and uses transport cost added to sales price to represent delivered price. Tavasszy et al. [15] uses the notion of a generalized cost to model transportation within an SCGE. In both cases transportation is handled in a rather coarse way. The freight model we propose herein, by contrast, presents greater detail regarding freight service provision.

In addition, the general solution methodology which has been used by many researchers is writing the necessary conditions for the lower level and impose the maximum principle necessary condition of the followers' problem to the leaders' objective function. Therefore, the adjoint variables of followers' problem would be considered as state variables for the leader. This approach convert the bi-level differential problem to a single level two-point boundary problem.

Abou-Kandil [16] presented an approach to find a closed-form solution for discrete-time linear-quadratic Stackelberg Games. Moreover, Wie [17] used Hooke Jeeves algorithm to find the Stackelberg equilibrium. Kleimenov et al. [18] developed a method for finding Stackelberg equilibrium for linear system dynamic and in 2009, a heuristic method developed by Wenyong et al. [19] using Genetic Algorithm to solve Stackelberg game in tariff network problem.

This research presents a bi-level Stackelberg game formulation for a combined pricing-routing game among oligopoly of the urban freight services on the upper level and oligopoly of several retailers, referred to as shippers, who maximize their own utilities on the lower level. There have been researchers who widely applied theory of oligopolistic market in their recherches such as Greenhut et al. [20] and Greenhut and Lane [21]. Also monotonistic market was introduced by Chamberlin [22] and extended by Salop [23], Singh and Vives [24] In monopolistic market companies provide similar products that can be substitute for each other.

In addition, our multi-leader-follower game is formulated in the form of an Equilibrium problems with Equilibrium Constraints (EPEC), where several MPECs are solved simultaneously. Solving MPECs and EPECs in an efficient way are still popular research questions.

There are multiple ways proposed by researches for solving mathematical problems with equilibrium constraints. Some researchers have shown that MPEC can be reformulated to an equivalent problem which can be solved by any of the nonlinear optimization algorithms. The solution to these NLP will be the solution of the original MPEC ([25], [26], [27]). In this approach local convergence is guaranteed as there are some types of MPEC constraints qualification such as linear independence (MPEC-LICQ) which guarantee the existence of the Lagrangian multipliers at the local optimal ([25], [27]). Facchinei et al. and Fukushima et al. applied a so called perturbed Fischer-Burmeister function to design a sequence of smooth and regular NLPs which approximate the MPEC ([28], [29]).

Also Scholtes investigated a series of NLPs which can be solved to attain the optimal point of MPECs [30] His approach finds a sequence of stationary points of a parametric NLP which regularizes MPEC in the form of complementarity conditions. There are also exact penalization approaches which move the complementarity constraints to the objective function and solve the resulted problem instead on the original MPEC ([30], [31]).

On the other hand, due to non-convexity of EPEC, its solution is considered as a difficult problem. One of the solution strategies for EPECs is the (Jaccobi/Gauss-Seidel) diagonalization method ([32], [10]), where the underlying MPECs are solved in turns until an equilibrium point could be obtained. Also to solve an EPEC, Hu suggested to reformulate each leader's MPEC to NLP and find the first order KKT condition. Then the system of all first order conditions can be solved to find the equilibrium solution [33].

Other solution strategies for EPECs include sequential nonlinear complementarity algorithm presented in [25] where Su suggests to relax the complementarity constraints in each MPEC and perturb the coefficient of objective function and constraints simultaneously. The problem then becomes solving a sequence of perturbed NLPs while any sequence tends to zero.

Other EPEC algorithms include mixed complementarity formulation using big-M method by Ehrenmann [34] and parametric smoothing approach by Bouza Allende with the focus on existence results and more efficient rate of the convergence [35]. More comprehensive studies about solution methodologies for MPECs and EPECs can be found in ([36], [34]).

In addition, in pricing focused researches, price can be discussed as two questions: what the optimal price is (demand-based price) and what price each group of customers should pay (price discrimination). The most effective applications of revenue management are generally found in industries in which both duration and price can be managed. Also, a few industries, do not know the exact duration of the customer use, even though some may try to control that duration.

There is a wide literature on dynamic pricing policies and their connection to revenue management. Rothstein and Littlewood were the pioneers of the revenue management with the application in airline and hotel overbooking. In addition, the revenue management field has been given a lot of attention after the work of Belobaba (1987a, b; 1989) and the American Airlines success (Smith et al. 1992). Also, the main focus of RM has been on capacity management and overbooking.

After 1990, the interest in RM has been increased and researchers have started to apply RM in more complex problems such as airlines, hotels, or retail stores with a higher degree of complexity. In the last decade, the pricing policies started to get more attention and became a popular topic in revenue management(e.g., Gallego and van Ryzin 1994; Bitran and Mondschein 1997; Feng and Gallego 1995, 2000).

Moreover, dynamic pricing practices are particularly useful for those industries having high start-up costs and useful to recognize "selling the right product to the right customer at the right time." In addition to airlines industry who were the first users of revenue management techniques and dynamic pricing, this field is being used in other industries such as retailers (e.g., Bitran and Mondschein 1997), hotels (e.g., Bitran and Mondschein 1995,), bandwidth and Internet providers (e.g., Nair and Bapna 2001), passenger railways (e.g., Ciancimino et al. 1999) and electric power supply (e.g., Schweppe et al. 1987).

In general, pricing in revenue management has been discussed from two perspectives: price discrimination and demand-based pricing. The multi-leader-follower game of freight service in chapter 6 focuses on demand-based pricing and chapter 7 will focus on nonlinear pricing setting of oligopolistic networks which considers price discrimination to answer the question of "who should pay which price". In summary, shipper-carrier pricing models can be considered as an application of revenue management since in their framework, the carriers/shippers are interested in finding an optimal pricing strategy that maximizes the revenue collected over the selling horizon.

In this dissertation we designed a revenue management freight model for the shippers (lower level) who are selling their product to a price sensitive group of customers. They optimized production level and price to maximize their revenue. Also, the focus of the upper level problem is on the strategic role of the price in optimizing carriers' revenue.

This research aims to develop and make contributions to the existing literature in dynamic bi-level optimization problems by developing a model for computing urban freight service prices within an oligopoly of freight service providers and oligopoly of the producers.

The motivation for this work is our belief that pricing policies are, more than ever before, a critical component of the operations of manufacturing and service companies. Price of the products and services are now a very important variable that companies can manipulate to increase or decrease the demand in the market in the short run. In addition to the financial point of view, price can be considered as a tool to regulate inventory and production pressures. Dynamic pricing, also called real-time pricing, helps to set the cost for a product or service that is highly flexible. our motivation to include dynamic pricing is to allow carriers/shippers to sell their services/goods while adjusting prices in response to market demands. Also carriers and shippers in our problem are forming oligopolies and therefore they need to know the optimal price at which they should offer a product, while maintaining a good profit margin and keeping up with the competitors. Game theory has been widely applied to this types of problems to consider the behavior of freight service companies/producers in a network of their competitor.

In addition, consumers intuitively intend to buy a product/service which maximizes their surplus. Non-linear pricing can help to increase the consumer surplus. It actually considers how much customers are willing to pay for the product and they charge them based on this information. Consequently, our motivation for the nonlinear pricing research on carrier/shipper problem is to address the question of "selling the right product to the right customer at the right time" correctly.

In this dissertation we are interested in computing urban freight service prices when an oligopoly of freight service providers compete with one another to carry cargo for large and complex, dynamic network markets. Thus this thesis presents two projects as 1/linear and 2/nonlinear pricing on multi-leader-follower games of freight service based on the Stackelberg-

Cournot-Nash behavioral assumption. We also consider multiple producers of a single abstract homogeneous commodity that is brought to market by the aforementioned freight service providers. Such producers, acting as shippers, also constitute an oligopoly in the market for their output of the single homogeneous good. For such an environment, we study dynamic freight service pricing from the point of view of multi-leader-follower games.The resulting organizational behavior can be called an equilibrium problem with equilibrium constraints (EPEC). In that the decision environment we study is dynamic, we refer to the model we present below as a differential-EPEC, which we abbreviate as D-EPEC.

The contributions of this dissertation could be specified as followings:

- there are few studies that consider computability in studying of the shipper-carrier relationship in a dynamic setting. One such model leads the dynamic shipper-carrier problem as a differential Stackelberg game when there is only one carrier acting as the leader. However the contribution of this thesis is to consider an oligopoly of carriers. This assumption leads to a competition in pricing among freight companies which makes it possible to quantify the benefits gained through regulated pricing.

- the other contribution of this paper is to assume nonlinear pricing in its fully general setting as in [37]. This is a more realistic model in contrast to the limited pricing schemes which only consider two-part or three-part pricing, etc. This problem is complicated since adding this feature requires a careful modeling of the demand profile which introduces one more dimension to the decision layer. As we mentioned, most of the researches in non-linear transportation pricing field have considered special cases such as two-part or three-part tariff. However the structure of nonlinear pricing in this paper is of a so-called quantity discount pricing which makes us able to capture a more general form of price discrimination.

- Solving EPECs are generally complicated and time consuming due to the non-convexity associated with MPECs. Also popular diagonalization algorithms might not be able to find the optimal solution [25]. To overcome this difficulty we will introduce dual adjoint algorithms to guarantee finding optimal solution and reducing the computational time.

Rest of this dissertation is organized as follows: Chapter 1/discuses the existing literature of shipper-carrier problems and the foundations of the optimal control theory and nonlinear pricing. Chapter 2/ represents the concept, foundation and basic formulation of Mathematical Program with Equilibrium Constraints as well as the Equilibrium Problem with Equilibrium

Constraints. Chapter 3/discuses all the fundamental definitions and theorems of differential Nash games, differential Stackelberg game and differential variational inequalities used in this thesis. Chapter 4/ expresses some examples of static and dynamic equilibrium problems and chapter 5/ introduces the mathematical programming algorithms and their convergence theorems.Chapter 6/ proposes a model on linear multi-leader-follower games of freight service pricing. Yet we have to extend the model to the non-linear structure of the urban freight supply chains and find the best computational algorithm to solve it Chapter 7/extends the model to non-linear pricing. The numerical example will be solved in Chapter 7/to show the applicability of the model and its solution methodology. Then we will close the dissertation by conclusion remarks in chapter 8/.

1.2 Foundations of the Optimal Control Theory

1.2.1 Some Basic Notions

We briefly review some mathematical notions which are going to be used widely in this thesis:

Definition 1.2.1. *(Space of square-integrable functions)*
Space of square-integrable functions, which is denoted by $\mathcal{L}^2[t_0, t_f]$, is a Banach space consists of all real-valued functions on the segment of the real line $[t_0; t_f] \in \mathbb{R}^1_+$. It obeys the following form:

$$||\mathcal{Z}||_2 = \Big(\int_{t_0}^{t_f} |\mathcal{Z}(t)|^2 dt \Big)^{\frac{1}{2}} \leq \infty$$

It is also possible to define the space of square-integrable functions, $\mathcal{L}^2[t_0, t_f]$ in a direct way by the use of the Lebesgue integral and Lebesgue measurable functions $\mathcal{Z}$ on $[t_0; t_f] \in \mathbb{R}^1_+$ such that the Lebesgue integral of $|\mathcal{Z}|^2$ over $[t_0; t_f]$ exists and is finite.

Definition 1.2.2. *(Cauchy problem)*
For a defined partial differential equation on R^n, the Cauchy problem is the one of finding

the solution y of the following differential equation of order k

$$y(x) = f_0(x) \quad \forall x \in S$$
$$\frac{\partial y(x)}{\partial t^i} = f_i(x) \quad \forall i = 1, 2, .., k-1 \text{ and } \forall x \in S$$

S is a smooth manifold $S \subset \mathbb{R}^n$ of dimension $t-1$ which is called the Cauchy surface, where t is a normal vector to S. Also f_i are given functions defined on the surface S and are known as the Cauchy data of the problem

Definition 1.2.3. *(Hilbert Space)*
In mathematics, given an absolutely continuous v, a vector space $\mathcal{H}^1[t_0, t_f]$ is defined as follows:

$$\mathcal{H}^1[t_0,\ t_f] = \{v|v \in (\mathcal{L}^2[t_0,\ t_f])^m; \frac{\partial v}{\partial x_i} \in \mathcal{L}^2[t_0.t_f]\ \forall i = 1, 2, \cdots m\}$$

Definition 1.2.4. *(G-differentiable)*
The Gateaux differential generalizes the idea of a direc- tional derivative. Suppose V and U are locally convex topological vector spaces and x is a vector in V. Assuming functional $f : V \to U$ and a direction $h \neq 0$, the Gateaux differential $d_h f$ is:

$$d_h f = \lim_{\theta \to 0} \frac{f(x + \theta h) - f(x)}{\theta}$$

In addition functional $f : V \to U$ is called Gateaux diffe rentiable or G-differentiable at $x \in V$ in the direction h while $d_h f$ exist.

Definition 1.2.5. *(Riesz representation theorem)*
Let V be a Hilbert space and let $J \in V^*$ be a continuous linear form on V. Then there exists

a unique element $m_J \in V$ *such that* $\forall x \in V$

$$J(x) = \langle m_J,\ x\rangle,\ \forall x \in V$$
$$||J||_{V*} = ||m_J||_V$$

Conversely, we can associate with each $y \in V$ *the continuous linear form* J_y *defined by*

$$J_y(x) = \langle y,\ x\rangle \forall x \in V$$

In addition there is an alternative form of the Riesz representation theorem developed Lueberger [88]. This alternative can be also helpful in some applications and the complete theorem can be find in [38].

Definition 1.2.6. *(Gradient in Hilbert space)*
Suppose V *is a Hilbert space with associated scalar product* $\langle ., .\rangle$*. Given* x *as a vector in* V *and* f *a functional which is G-diffe rentiable on* V*, there exist an element* $\nabla f(x) \in V$ *such that*

$$\partial f(x, h) = \langle \nabla f(x), h\rangle,\ \forall h \in V$$

$\nabla f(x)$ *is called the gradient of* f *at* x

1.2.2 Optimal Control Problem

Optimal control theory has been first developed by Pontryagin, Boltyanskii, Gamkrelidze Mishchenko in 1958 in Russian and 1962 in English with the publication of The mathematical theory of optimal processes [39]. In this section, we introduce continuous-time optimal control problem and we consider the following canonical form of the continuous-time optimal control problem:

$$
\begin{aligned}
&min\ J = \Gamma(x(t_f), u(t_f)) + \int_{t_0}^{t_f} f_0(x(t),\ u(t),\ t)dt && (1.1)\\
&s.t\\
&State\ Dynamics : \frac{dx}{dt} = f(x(t),\ u(t),\ t)\forall t \in [t_0,\ t_f] && (1.2)\\
&Initial\ condition : x(t_0) = x_0 && (1.3)\\
&Feasible\ set : u \in U && (1.4)
\end{aligned}
$$

Where x is state variables and u control variables. t_0, t_f, x_0 are referred to the initial time, terminal time and initial value for stat variable at time t_0 and are all fixed;

We also assume:

$$
\begin{aligned}
&u\ \in \{u : [t_0,\ t_f] \to U,\ U \subseteq \mathbb{R}^m\} \subseteq (\mathcal{L}^2[t_0,\ t_f])^m\\
&x\ \in (\mathcal{H}^1[t_0,\ t_f])^n\\
&x_0\ \in \mathbb{R}^n\\
&f_0\ :\ (\mathcal{H}^1[t_0,\ t_f])^n \times (\mathcal{L}^2[t_0,\ t_f])^m \times \mathbb{R}_+^1 \to \mathcal{L}^2[t_0,\ t_f]\\
&f\ : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n\\
&\Gamma : \mathbb{R}^n \times \mathbb{R}_+^m \to \mathbb{R}^1
\end{aligned}
$$

Note that $(\mathcal{H}^1[t_0,\ t_f])^n$ is the n-fold product of the Sobolev space $\mathcal{H}^1[t_0,\ t_f]$. Additionally if the introduced optimal control problem meets the regularity condition, therefore the operator $x(u,\ t)$ exists and would be implicitly found by state dynamics $\frac{dx}{dt} = f(x,\ u,\ t)$

Theorem 1.2.1. *(Regularity)*
The optimal control problem $OCP(f_0,\ f,\ \Gamma,\ U,\ x_0,\ t_0,\ t_f)$ *is regular, if the following conditions hold:*

- $f_0(x,u,t) : (\mathcal{H}^1[t_0,\ t_f])^n \times (L^2[t_0,\ t_f])^m \times \mathbb{R}_+^1 \to L^2[t_0,t_f]$ *is convex and continuously* *differentiable with respect to x and u;*

- *$f(x,\ u,\ t)$ is continuously differentiable with respect to $x; u$;*
- *$f(x,\ u,\ t)$ and $\frac{\partial f(x,u,t)}{\partial x}$ are bounded*
- *$\Gamma(x,\ t)$ is is continuously differentiable with respect to x;*
- *$x_0 \in \mathbb{R}^n, t_0, t_f$ are known and fixed*
- *U is non-empty, convex and compact;*

Theorem 1.2.2. *Suppose that optimal control problem 1.1-1.4 is regular in sense of definition 1.2. 1. If $u(t) \in U$ is the control whose solution of state variable is $x(u;t)$ defined on $[t_0,\ t_f]$. Then, for every $m(t)$ which is bounded and measurable, the map $\sigma \rightarrow x(u + \sigma m; t)$ is differentiable for $\forall t \in [t_0,\ t_f]$. Therefore, the operator $x(u,\ t)$ is G-differentiable.*

Now we are ready to prove the necessary conditions for optimal control problem in 1.1-1.4.

Theorem 1.2.3. *The objective functional f_0 in 1.1-1.4 is continuously differentiable in the sense of Gateaux, and*

$$\nabla L(u) = \frac{\partial}{\partial u} H(x,\ u,\ \lambda,\ t)$$

Where H is the Hamiltonian and follows:

$$H = f_0(x,\ u,\ t) + \lambda^T f(x,\ u,\ t)$$

Also λ is the adjoint variable which is the solution to the following problem:

$$-\frac{d\lambda}{dt} = (\frac{\partial f}{\partial x})^T \lambda + (\frac{\partial f_0}{\partial x})^T$$
$$\lambda(t_f) = \frac{\partial \Gamma(x(t_f), t_f)}{\partial x(t_f)}$$

Proof. We first take a derivative of f_0 in direction h as

$$\partial L = \frac{\partial \Gamma(x(t_f), t_f)}{\partial t_f} y(t_f) + \int_{t_0}^{t_f} \frac{\partial f_0}{\partial x} y + \frac{\partial f_0}{\partial u} dt \tag{1.5}$$

where $y = \delta x$ is a variation in x and it depends on direction h. Also

$$x(t) = x_0 + \int_{t_0}^{t_f} f(x,\ u,\ t) ds$$

Hence:

$$y = \delta x = \int_{t_0}^{t} [\frac{\partial f}{\partial x} y + \frac{\partial f}{\partial u} h] ds$$

Then let us introduce the λ as the adjoint variable obeying the following definition

$$-\frac{d\lambda}{dt} = (\frac{\partial f_0}{\partial x})^T + (\frac{\partial f}{\partial x})^T \lambda$$
$$\lambda(t_f) = \frac{\partial \Gamma(x(t_f), t_f)}{\partial x(t_f)}$$

Then by substitution, the derivative in (1.5) will become:

$$\delta L(u,\ h) = \lambda(t_f) y(t_f) + \int_{t_0}^{t_f} \{[-\frac{d\lambda}{dt} - \lambda^T \frac{\partial f}{\partial x}] y + \frac{\partial f_0}{\partial u} h\} dt$$

And

$$
\begin{aligned}
\int_{t_0}^{t_f} -\frac{d\lambda}{dt} y dt\tau &= -\lambda(t_f)y(t_f) + \int_{t_0}^{t_f} \lambda^T \frac{dy}{dt} dt \\
&= \Big[-\lambda(t_f)y(t_f) + \int_{t_0}^{t_f} \lambda^T \frac{\partial f}{\partial x}[y + \frac{\partial f_0}{\partial u} h]\Big] dt
\end{aligned}
$$

Which concludes that

$$
\begin{aligned}
\delta L(u,\ h) &= \lambda(t_f)y(t_f) + \int_{t_0}^{t_f} \lambda^T [\frac{df}{dx} y + \frac{df}{du} h] - \lambda^T \frac{\partial f}{\partial x} y + \frac{\partial f_0}{\partial u} h dt \\
&= \int_{t_0}^{t_f} \{\lambda^T \frac{\partial f_0}{\partial u} + \frac{\partial f}{\partial u}\} h dt \\
&= < \lambda^T \frac{\partial f}{\partial u} + \frac{\partial f_0}{\partial u},\ h >
\end{aligned}
$$

Hence, the the gradient of $L(u)$ *becomes*

$$
\begin{aligned}
\nabla L(u) &= \lambda^T \frac{\partial f}{\partial u} + \frac{\partial f_0}{\partial u} \\
&= \frac{\partial H}{\partial u}
\end{aligned}
$$

We then introduce the following necessary conditions for optimality of optimal control problem $OCP(f_0,\ f,\ \Gamma,\ U,\ x_0,\ t_0,\ t_f)$.

- state dynamics:

$$
\frac{dx}{dt} = f(x(t, u(t),\ t)
$$

- initial time conditions:

$$
\begin{aligned}
f_0[x(t_0),\ u(t_0),\ t)] = 0 \\
x(t_0) = x_0
\end{aligned}
$$

- adjoint equations:

$$\frac{\partial \lambda}{\partial u} = -H_x = -f_{0x} - \lambda^T f_x$$

- transversality conditions:

$$\lambda(t_f) = \frac{\partial \Gamma_x[x(t_f), t_f]}{\partial x(t_f)}$$

- minimum principle:

$$H_u(x,\ u,\ \lambda,\ t) = 0 \tag{1.6}$$

□

To solve the the minimum principle in (1.6) we define the following theorem:

Theorem 1.2.4. *A necessary condition for $u^* \in U$ to solve the $OCP(f_0,\ f,\ \Gamma^1,\ U,\ x_0,\ t_0,\ t_f)$ in (1.1-1.4) is:*

$$\langle \frac{\partial H(x^*, u^*, \lambda^*, t)}{\partial u},\ u - u^* \rangle \geq 0,\ \forall u \in U$$

Proof. Note that U is convex. therefore:

$$u^* + h(u - u^*) \in U \forall h \in [0,\ 1\ ] \tag{1.7}$$

The condition for u^* to be the minimum of L on U is :

$$[\frac{d}{dh}L(u^* + h(u - u^*))]_{h=0} = \delta L(u^*,\ u - u^*) \geq 0 \tag{1.8}$$

L is well-defined and G-differentiable at u, Therefore

$$\delta L(u^*,\ u - u^*) = \langle \lambda^T \frac{\partial f(x^*, u^*, t)}{\partial u} + \frac{\partial f_0(x^*, u^*, t)}{\partial u},\ u - u^* \rangle \tag{1.9}$$

$$= \langle \frac{\partial H(x^*, u^*, t)}{\partial u},\ u - u^* \rangle \geq 0,\ \forall u \in U \tag{1.10}$$

In next chapter we will see that 1.8 is a variational inequality, which can be used to solve the minimum principle of the optimal control problem $OCP(f_0,\ f,\ \Gamma,\ U,\ x_0,\ t_0,\ t_f)$. □

1.3 Foundation of Nonlinear Pricing

1.3.1 Introduction of Price Discrimination and Profile Demand

Grouping customers to apply nonlinear pricing as the second-degree price discrimination has been widely popular among researchers and economists. The earliest discussion has been placed by Dupuit [40] during his study on railroad pricing. According to economists such as Musaa and Rosen [41], Maskin and Riley [42] and Goldman et al. [43], quality or quantity distortion are sometimes optimal to gain more money from types of customers who have higher demands compared to other group of customers. They considered the general model of price discrimination in nonlinear pricing for an oligopoly structure. In their setting firms are spatially differentiated; They characterize the nature of the optimal solution which depends on type of private information of the customers [44].

In addition, there are good examples of researches working on monopoly price discrimination model in a competitive environment such as Katz [45], Spulber [46], Champsaur et al. [47] and specially Wilson [48]

Nowadays, developed freight services need to be treated in a more complicated and efficient way. Most of the transportation companies will compete in price over demand while optimize their routing plan on the network. There are many studies in which the companies compete given a linear tariff rate. However, many of the price schemes in economical systems are not linear. In general, the term Nonlinear Pricing is used to refer to the pricing scheme when the total price of the contract/product is not proportional to the quantity of the commodity/service [37]

Classic examples of nonlinear pricing are public utilities, e.g. electricity supply. A good daily example of this price scheme is city's taxi rides, while the cost of the taxi does not grow linearly and mostly depends on other criteria such as distance and the existing traffic congestion on the roads. In the freight service industry, such pricing is more often observed and more complicated, not only counting the weight of the item to be shipped, but also volume, express service, etc.

Examples of recent studies include: Hurley and Peterson's paper on carrier competitions which allows carriers to choose two-part tariffs depending on volume [49]. Xiao and Yang have done a research on a three agents-three layer problem which emphasizes on a linear pricing. The carriers and infrastructure companies are assumed to behave cooperatively while [50]. In addition Lawphongpanich and Yin established a model on tolling in which the toll price follows a piecewise linear functions to determine tolls user equilibrium distribution [51].

This research applies a general version of nonlinear pricing to the case of freight service. Compared to previous approaches which rely on specific, and very often essentially piecewise affine, functional forms of the pricing schemes, our approach enjoys the capability of fully capturing the freight companies capabilities to employ price discrimination. Inevitably, such general assumption requires the careful modeling of the demand profile hence it introduces one more dimension to the decision layer.

In general price discrimination is a helpful way for the company to deliver the right product to the right person at the right price. This concept is mostly helpful when sellers can not distinguish groups of customers. In first order price discrimination, companies charge the price that each consumer is willing to pay while the second order discrimination happens when sellers can not distinguish between customers because of their unobservable facts such as income. In this type sellers takes the strategy of quantity discounts for bulk purchases. The third order degree price discrimination gives different prices to different consumer groups. Two-part, three-part, block, fixed-fee and non-linear pricing can be refereed as different types of price discrimination.

The following chart shows the difference between these types of pricing [37]

The most important concept in nonlinear pricing is demand profile. Aggregate demand

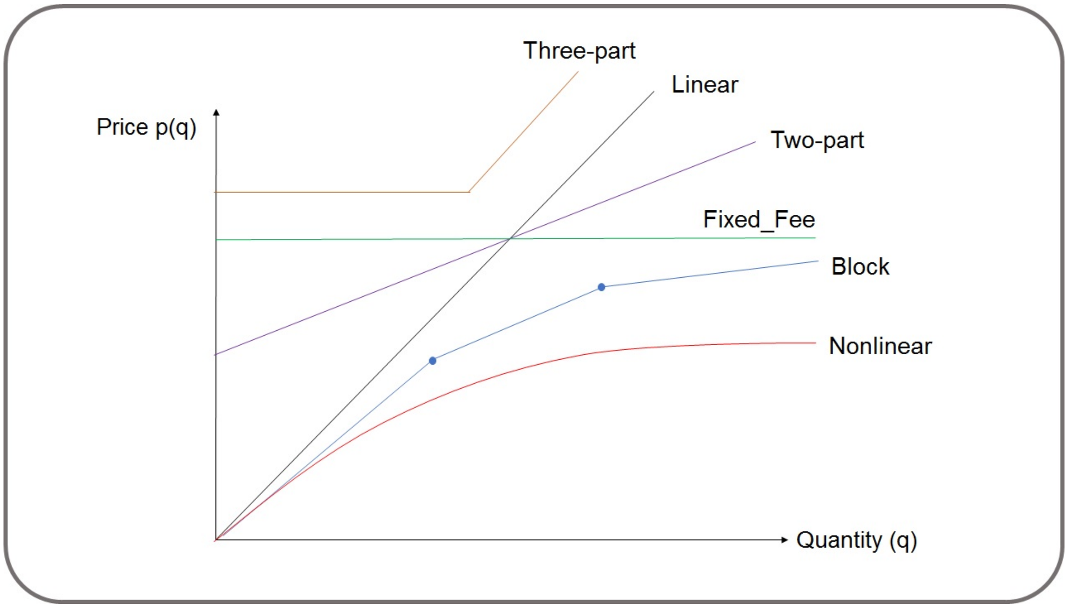

Figure 1.1. Illustration of the different types of pricing

which has been widely used in the literature for individual and market segments can not be very informative to the companies. According to Wilson, aggregate demand is in form of the total number of items sold at each price and it sometimes does not provide enough information to companies. For example the question about if the demand is bought to the company by one customer or many can not be answered by the aggregate demand concept. The demand profile which has been introduced by Wilson and denoted by $N(p(q),\ q)$ is a bivariate function providing data to determine the demand and show the number of customers (shippers) wiling to purchase q^{th} units at the marginal price γ [37]. The relationship between aggregated demand and demand profile can be expressed as follows:

$$D = \int_0^\infty N(p(q),\ q)dq \tag{1.11}$$

Where $p(q)$ is the marginal price schedule, D is the aggregated demand and $N(p(q),\ q)$ is the demand profile.

1.3.2 Interpretation for demand profile

There are two interpretations for the demand profile

The first interpretation considers the demand profile as the distribution of purchase size in response to each price/number of fraction purchasing at least q units:

$$N(p,\ q) = \#t|d(p,\ q) \geq q$$

The second interpretation considers the demand profile as the number of customers willing to pay the price p for each q^{th} unit:

$$N(p,\ q) = \#t|v(p,\ t) \geq p$$

Where $v(p,\ t)$ the marginal benefit customers that are willingness to pay for the q^{th} unit increment in the purchase size.

1.3.3 Example of Price Schedule of a Monopolist

The following example has been introduced by Wilson [37] to show the price schedule of a monopolist and calculate the optimal tariff. In the following table rows show the candidates

for the marginal price of the product p and columns show the quantity offering by the company as q. In addition the numbers in the cells provide information about the demand profile, $N(p,\ q)$ of quantity q at each price p

Example of Price Schedule of a Monopolist						
p	q	1	2	3	4	5 units
$ 2/units		90	75	55	30	5
$ 3		80	65	45	20	0
$ 4		65	50	30	5	0
$ 5		45	30	10	0	0

Table 1.1. Example of Price Schedule of a Monopolist

To define the demand in a discrete price set we need to use the following formulations

$$D = \sum_{q=1}^{\infty} N(p,\ q)\sigma = \sum_{q=1}^{\infty} n(p,\ q)q$$

While σ is the quantity increment, $N(p,\ q)$ is the demand profile and $n(p,\ q)$ is the exact demand at price p for quantity q. Therefore the demand for the first row, price $\$2/unit$ would be calculated in the one of the following ways:

$$D = \sum_{q=1}^{\infty} n(p,\ q)q = (5 \times 5) + (25 \times 4) + (25 \times 3) + (20 \times 2) + (15 \times 1) = 255$$

$$D = \sum_{q=1}^{\infty} N(p,\ q)\sigma = (5 \times 1) + (30 \times 1) + (55 \times 1) + (75 \times 1) + (90 \times 1) = 255$$

In addition, the profit of the company will be defined as follows

$$R(p(q),\ q) = N(p(q),\ q).[p(q) - c]\sigma \tag{1.12}$$

Where c is the unit cost including manufacturing cost, shipping cost, etc. For simplicity, the cost in this example is considered as \$ 1. Defining the profit function for each price candidate at each quantity provides us with the information about what marginal price, $p(q)$ is optimal to offer for quantity purchase q. Also let $P(q)$ define the exact cost for quantity purchase q as $P(q) = \Sigma_{i=1}^{q} p(i)$. Therefore the optimal price value at each quantity will be defined in the following table.

The optimal total profit of the company is \$480. The concept of demand profile plays an important role to differential multi-leader-follower games applied to nonlinear freight service pricing research in our dissertation.

Optimal solution to the Schedule of a Monopolist						
p	q	1	2	3	4	5 units
\$ 2/units		90	75	55	30	5
\$ 3		80	65	45	20	0
\$ 4		65	50	30	5	0
\$ 5		45	30	10	0	0
\$ $p(q)$		\$ 4	\$ 4	\$ 3	\$ 3	\$ 2/unit
\$ $P(q)$		\$ 4	\$ 8	\$ 11	\$ 14	\$ 16
\$ $R(p(q), q)$		\$ 195	\$ 90	\$ 40	\$ 140	\$ 5

Table 1.2. Optimal solution to the Schedule of a Monopolist

Chapter 2
Introduction and Foundation of MPECs and EPECS

2.1 Mathematical Program with Equilibrium Constraints

In this section, we define the foundation of the mathematical programs with equilibrium constraints(MPECs). Note that we borrowed many of our notation from [25].

Definition 2.1.1. *(mathematical programs with equilibrium constraints)*
An MPEC is a formulation of a nonlinear program with complementarity constraints:

$$\min f(x) \tag{2.1}$$

$$subject\ to: \tag{2.2}$$

$$g(x) \leq 0,\ \ h(x) = 0 \tag{2.3}$$

$$0 \leq G(x) \perp H(x) \geq 0 \tag{2.4}$$

where $f : R^n \to R, g : R^n \to R^p, h : R^n \to R^q, G : R^n \to R^m$*, and* $H : R^n \to R^m$ *are twice continuously differentiable functions.*

In addition, given a feasible vector x^f of the MPEC (2.1), we define the following index sets of active and inactive constraints

$$\begin{aligned} I_G(x^f) &:= \{i | G_i(x^f) = 0\}, \\ I_G^c(x^f) &:= \{i | G_i(x^f) > 0\} \end{aligned}$$

$$
\begin{aligned}
&I_H(x^f) \ := \{i|H_i(x^f) = 0\} \\
&I_H^c(x^f) \ := \{i|H_i(x^f) > 0\} \\
&I_{GH}(x^f) \ := \{i|G_i(x^f) = H_i(x^f) = 0\} \\
&I_g(x^f) := \{i|g_i(x^f) = 0\}
\end{aligned}
$$

Also, there is a nonlinear program, associated with any given feasible vector x^f of MPEC (2.1). This NLP is called the tightened NLP (TNLP (x^f)) ([52], [53]) .

$$
\begin{aligned}
&minf(x) \\
&Subject\ to: \\
&g(x) \leq 0, \\
&h(x) = 0, \\
&G_i(x) = 0,\ i \in \mathcal{I}_G(x^f) \\
&G_i(x) \geq 0,\ i \in \mathcal{I}_G^c(x^f) \\
&H_i(x) = 0,\ i \in \mathcal{I}_H(x^f) \\
&H_i(x) \geq 0,\ i \in \mathcal{I}_H^c(x^f)
\end{aligned}
\tag{2.5}
$$

In addition, there is a relaxed $NLP(RNLP(x^f))$ associated with MPEC (2.1)([52], [53]) .

$$
\begin{aligned}
&\min\ f(x) \\
&Subject\ to: \\
&g(x) \leq 0, \\
&h(x) = 0, \\
&G_i(x) = 0,\ i \in \mathcal{I}_H^c(x^f) \\
&G_i(x) \geq 0,\ i \in \mathcal{I}_H(x^f) \\
&H_i(x) = 0,\ i \in \mathcal{I}_G^c(x^f) \\
&H_i(x) \geq 0,\ i \in \mathcal{I}_G(x^f)
\end{aligned}
\tag{2.6}
$$

It is well known that the assumption that LICQ or MFCQ should be satisfied at local minimizers is required for KKT necessary optimality conditions. However, since MPECs cannot satisfy such constraint qualifications, such as linear independence constraint qualification (LICQ) or MangasarianFromovitz constraint qualification (MFCQ), at any feasible point, one need to develop appropriate variants of CQs for MPECs. ([54], [53])

Theorem 2.1.1. *Given a feasible point x^f, the MPEC (2.1) is said to satisfy the MPEC-LICQ (MPEC-MFCQ) if the corresponding RNLP (x^f)(2.6) satisfies the LICQ (MFCQ) at x^f.*

Following, we define various stationarity concepts for MPECs introduced in [25].

Definition 2.1.2. *(Bouligand-or B-stationary point of an MPEC)*
Given a feasible point x^f for the MPEC (2.1), we say that x^f is a Bouligand- or B-stationary point if $u = 0$ solves the following linear program with equilibrium constraints (LPEC),· where the vector $u \in \mathcal{R}^n$ is the decision variable

$$\begin{aligned}
&\min \nabla f(x^f)^T u \\
&\textit{Subject to} \\
&g(x^f) + \nabla g(x^f)^T u \leq 0 \\
&h(x^f) + \nabla h(x^f)^T u = 0 \\
&0 \leq G(x^f) + \nabla G(x^f)^T u \| H(x^f) + \nabla H(x^f)^T d \geq 0
\end{aligned} \tag{2.7}$$

B-*stationary points are good candidates for local minimizers of the MPEC problems. However,* B-*stationarity is difficult to check, since it may need checking the optimality of* $2^{|I_{GH}(x^f)|}$ *linear program ([55], [53]).*

Theorem 2.1.2. *(Strong Stationary Point)*
If feasible point x^f is a strong stationary point for the MPEC (2.1), then it is a B-stationary point. Conversely, if x^f is a B-stationary point for the MPEC, and if it satisfies the MPEC-LICQ constraint qualification at x^f, then it is a strongly stationary point.

Theorem 2.1.3. *([56], [57], [25])*
If a local optimal point x^* of the MPEC (2.1) satisfies the MPEC-LICQ, then x^* is a

strongly stationary point. i.e. There exist unique MPEC Lagrangian multipliers $\lambda = (\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,\ \lambda_5)$.

In addition, Some researchers have shown that MPECs can be reformulated as the following nonlinear problem ([26], [58], [25]).

$$\begin{aligned}
&\textit{minimize } f^k(x^k,\ y; x^{-k}) \\
&st:\ \ g^k(x^k,\ y; x^{-k}) \leq 0, \\
&h^k(x^k,\ y; x^{-k}) = 0, \\
&G(x^k,\ y; x^{-k}) \geq 0, \\
&H(x^k,\ y; x^{-k}) \geq 0, \\
&G(x^k,\ y; x^{-k}) \circ H(x^k,\ y; x^{-k}) \leq 0
\end{aligned} \tag{2.8}$$

To provide the convergence of this approach, one need to show that strong stationarity is equivalent to the KKT conditions of the equivalent NLP (2.8). This is presented in the following theorem.

Theorem 2.1.4. *([26], [58], [25])*
A vector x^* *is the strong stationary of the MEPC (2.1) if and only if it is equivalent to the KKT point of* $NLP(2.8)$. *Therefore, there exist unique multipliers as* $\lambda = (\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,\ \lambda_5)$ *that satisfies the follows KKT equations:*

$$\begin{aligned}
&\nabla f(x^*) + \lambda_1^T \nabla g(x^*) + \lambda_2^T \nabla h(x^*) - [\lambda_3 - H(x^*) \circ \lambda_5]^T \nabla G(x^*) - [\lambda_4 - G(x^*) \circ \lambda_5]^T \nabla H(x^*) = 0 \\
&G(x^*) \geq 0,\ H(x^*) \geq 0,\ G(X^*) \circ H(x^*) \leq 0,\ h(x^*) = 0,\ g(x^*) \leq 0 \\
&\lambda_1^T g(x^*) \geq 0,\ \lambda_3^T G(x^*) \geq 0,\ \lambda_4^T G(x^*) \geq 0,\ \lambda_5^T [G(x^*) \circ H(x^*)] \geq 0 \\
&\lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = 0,\ \lambda_4 = 0,\ \lambda_5 = 0
\end{aligned} \tag{2.9}$$

For the proof please refer to [5].

2.2 Equilibrium Problem with Equilibrium Constraints

In this section, we review EPEC concept and its definitions. As we will see later, diagonalization methods has been the most popular method in solving EPECs. However, The absence of convergence results for diagonalization methods is one of their main drawbacks. In next chapters, we will propose a creative method denoted as double adjoint lagorithm which can be a practical, efficient algorithm for EPEC formulation of optimal control problems.

An EPEC is a problem of finding an equilibrium point that couple multiple MPECs simultaneously into a single equilibrium problem with equilibrium constraints (EPEC). EPECs are considered as multiple MPECs with shared decision variables and shared equilibrium constraints.

2.2.1 Formulation

Let us assume an EPEC consists of M MPECs. Also assume for each $m = 1, .., M$, the m^{th} MPEC is defined based on the following formulation.

$$\begin{aligned} &Min\ f^m(x^m, y; \bar{x}^{-m}) \\ &\quad Subject\ to \\ &\quad g^m(x^m, y; \bar{x}^{-m}) \leq 0, \\ &\quad h^m(x^m, y; \bar{x}^{-m}) = 0 \\ &\quad 0 \leq G(x^m, y; \bar{x}^{-m}) \perp H(x^m, y; \bar{x}^{-m}) \geq 0 \end{aligned} \tag{2.10}$$

Where $f^m : \mathcal{R}^n \to \mathcal{R}$, $g^m : \mathcal{R}^n \to \mathcal{R}^{p_m}$, $h^m : \mathcal{R}^n \to \mathcal{R}^{q_m}$, $G : \mathcal{R}^n \to \mathcal{R}^m$ and $H : \mathcal{R}^n \to \mathcal{R}^m$ are twice continuously differentiable functions in both x and y. Also $x^m \in \mathcal{R}^{n_m}$ is the independent decision variable for each MPEC m and $y \in \mathcal{R}^{n_k}$ is the shared decision variable. Also the notion $\bar{x}^{-m}$ expresses that $x^{-m} = (x^j) \backslash x^m,\ j = 1, .., M$ is not a variable and is fixed. We denote (2.10) by $MPEC(\bar{x}^{-m})$.
Also with the assumption that the solution set of the m^{th} $MPEC$ is not empty, we denote such a set by $SOL(MPEC(\bar{x}^{-m}))$. Then, the shared equilibrium constraints between $MPECs$ is represented as the following complementarity system

$$0 \leq G(x,y) \perp H(x,y) \geq 0$$

Now, we are ready to present the EPEC formulation, associated with m introduced $MPECS$. Such EPEC tried to find a Nash equilibrium $(x^*, y^*) \in R^n$ such that

$$(x^{m*}, y^*) \in SOL(MPEC(\bar{x}^{-m*})) \ \ \forall \ m = 1, ..., M \tag{2.11}$$

To review necessary optimality conditions of EPECs in the context of multi-objective optimization with constraints governed by parametric variational systems in infinite-dimensional space, please see [59].

Since we only review finite-dimensional optimization problems, we continue with defining stationary conditions for EPECs by applying those for MPECs [33]

Definition 2.2.1. *A vector (x^*, y^*) is called B-stationary, strongly stationary, Mstationary, C-stationary, weakly stationary point of the $EPEC$ (2.11) if for each $m = 1, .., M$, (x^{m*}, y^*) is a B-stationary, strongly stationary, Mstationary, C-stationary, weakly stationary point of the $MPEC$ (x^{-m*})*

Theorem 2.2.1. *Let us assume (x^*, y^*) is a local equilibrium point of $EPEC$ (2.11). If the MPEC-LICQ holds at $(x^{m*},\ y^*)$ for $MPEC(x^{-m*})$ (2.10), at each $m = 1, ..., M$, Then (x^*, y^*) is an EPEC strongly stationary point.*
In addition, there exist vectors $\lambda^* = (\lambda^{1*}, ..., \lambda^{M*})$ with $\lambda^{m*} = (\lambda^{g,m*}, \lambda^{h,m*}, \lambda^{G,m*}, \lambda^{H,m*}, \lambda^{GH,m*})$ such that $(x^*,\ y^*,\ \lambda^*)$ solves the system

$$\begin{aligned}
&\nabla_{x^m} f^m(x^m,\ y;\ x^{-m}) \\
&+\nabla_{x^m} g^m(x^m,\ y;\ x^{-m})\lambda^{g,m} + \nabla_{x^m} h^m(x^m,\ y;\ x^{-m})\lambda^{h,m} \\
&-\nabla_{x^m} G(x^m,\ y;\ x^{-m})\lambda^{G,m} - \nabla_{x^m} H(x^m,\ y;\ x^{-m})\lambda^{H,m} \\
&+\nabla_{x^m} G(x^m,\ y;\ x^{-m})^T[H(x^m,\ y;\ x^{-m}) \circ \lambda^{GH,m}] \\
&+\nabla_{x^m} H(x^m,\ y;\ x^{-m})^T[G(x^m,\ y;\ x^{-m}) \circ \lambda^{GH,m}] = 0
\end{aligned}$$

$$\begin{aligned}
&\nabla_y f^m(x^m,\ y;\ x^{-m}) \\
&+\nabla_y g^m(x^m,\ y;\ x^{-m})^T \lambda^{g,m} + \nabla_y h^m(x^m,\ y;\ x^{-m})^T \lambda^{h,m} \\
&-\nabla_y G(x^m,\ y;\ x^{-m})^T \lambda^{G,m} - \nabla_y H(x^m,\ y;\ x^{-m})^T \lambda^{H,m} \\
&+\nabla_y G(x^m,\ y;\ x^{-m})^T [H(x^m,\ y;\ x^{-m}) \circ \lambda^{GH,m}] \\
&+\nabla_y H(x^m,\ y;\ x^{-m})^T [G(x^m,\ y;\ x^{-m}) \circ \lambda^{GH,m}] = 0 \\
&h^m(x^m,\ y;\ x^{-m}) = 0 \\
&0 \geq g^m(x^m,\ y;\ x^{-m}) \perp \lambda^{g,m} \geq 0 \\
&0 \leq G(x^m,\ y;\ x^{-m}) \perp \lambda^{G,m} \geq 0 \\
&0 \leq H(x^m,\ y;\ x^{-m}) \perp \lambda^{H,m} \geq 0 \\
&0 \leq -G(x^m,\ y;\ x^{-m}) \ \circ\ H(x^m,\ y;\ x^{-m}) \perp \lambda^{GH,m} \geq 0 \\
&\forall\ m = 1, .., M
\end{aligned} \tag{2.12}$$

In addition, we can conversely specifies that if (x^*, y^*, λ^*) *is the solution to (2.12), then* (x^*, y^*) *is also a B-stationary point of the EPEC (2.11).*

Proof. As we mentioned (x^*, y^*) is a local equilibrium point of the EPEC (2.11). Therefore, (x^{m*}, y^*) is a local minimizer for $MPEC(\bar{x}^{-m*})$ for each $m = 1, .., M$.

By the theorems in the last section, we can show existence of a vector $\lambda^m = (\lambda^{g,m*}, \lambda^{h,m*},$ $\lambda^{G,m*}\lambda^{H,m*}, \lambda^{GH,m*})$ such that $(x^{m*}, y^*, \lambda^{m*})$ satisfies the conditions in (2.12), for $\forall\ m = 1, ..., M$.

Now let us assume $\lambda^* = (\lambda^{1*}, ..., \lambda^{M*})$. Therefore the vector (x^*, y^*, λ^*) is a solution for problem (2.12).

Conversely, we can simply show that vector (x^*, y^*) is a strongly stationary point for each $m = 1, 2, .., M$, and consequently B-stationary point for the $MPEC(\bar{x}^{-m*})$. Therefore, we can conclude that vector (x^*, y^*) is a B-stationary point for $EPEC$ (2.11)

□

2.2.2 Diagonalization Algorithm

$EPECs$ are formulated in the form of an Equilibrium Problem with Equilibrium Constraints (EPEC), where several MPECs are solved simultaneously. Due to the non-convexity of the EPEC, its solution is considered a difficult problem. Due to this difficulty, efficient algorithms specifically designed for solving EPECs have not been very well developed in the literature.

However, because of simplicity and ease of use, the most popular solution strategies is the (Jaccobi/ Gauss-Seidel) diagonalization methods. Such algorithm mostly rely on NLP solvers, or more appropriately, MPEC algorithms where the underlying MPECs are solved in turns until an equilibrium point could be obtained.

There exist two types of diagonalization method: nonlinear Jacobi and nonlinear Gauss-Seide and these two algorithms will be disused in this section. Such algorithms has been applied in many researches such as ([60], [61], [62], [33], [63])

Other solution strategies include sequential nonlinear complementarity algorithm as in [25], and a mixed complementarity formulation using big-M method by Ehrenmann [34]. This new method is based on simultaneously relaxing the complementarity constraints in each MPEC, and solves EPECs by solving a sequence of nonlinear complementarity problems.

Harker was the first to propose diagonalization methods to find a solution to a variational inequality formulation of the Nash equilibrium problem in an oligopolistic market. we describe two diagonalization methods: nonlinear Jacobi and nonlinear Gauss-Seidel. Below is the detailed nonlinear Jacobi algorithm description:

Step 0: (Initialization)

Determine an initial feasible solution of each agent ($\gamma^{(0)} = (\gamma^{1,(0)}, \cdots, \gamma^{|\mathcal{K}|,(0)})$) and set the maximum number of outer iterations to be L, the current iteration to be $l = 0$ and an accuracy tolerance $\epsilon \in \mathcal{R}_{++}$.

Step 1: (Solve diagonalized problem, loop over each MPEC)

Suppose the current iteration point of the problem is $\gamma^{(l)}$. For each $k = 1, 2, \cdots, |\mathcal{K}|$ solve the MPEC faced by each individual agent while fixing $\bar{\gamma}^{-k,(l)} = (\gamma^{1,(l)}, \cdots, \gamma^{k-1,(l)}, \gamma^{k+1,(l)}, \cdots, \gamma^{|\mathcal{K}|,(l)})$. Denote the γ part of the optimal solution by $\gamma^{k,(l+1)}$ That is,

$$\gamma^{k,(l+1)} \in MPEC_k(\bar{\gamma}^{-k,(l)})$$

Step 2: (Stopping test and updating)

Let ($\gamma^{(l+1)} = (\gamma^{1,(l+1)}, \cdots, \gamma^{|\mathcal{K}|,(l+1)})$). If $l < L$, increase the counter by 1 and repeat Step

1. Otherwise, stop and for $\epsilon \in \mathcal{R}_{++}$, a present tolerance, check if $||\gamma^{k,(l+1)} - \gamma^{k,(l)}|| < \epsilon$ for $k = 1, \cdots, \mathcal{K}$ then report the solution $\gamma^{(J)}$; otherwise report no equilibrium point.

The difference between two mentioned diagonalization algorithms is that the nonlinear Jacobi method does not use the most recently available information when computing $\gamma^{k,l+1}$. As an example we can mention that vector $\gamma^{1,j+1}$ is known, while calculating $\gamma^{2,j+1}$. However it is not used in the calculation and instead, $\gamma^{1,j}$ is used in the calculation of $\gamma^{2,j+1}$.

Gauss − Seidel method is another diagonalization method which use such recent information which exist in the problem. The details of the nonlinear *Gauss − Seidel* method is the same as nonlinear Jacobi method, with a difference in Step 1. Step 1 of the *Gauss − Seidel* method can be expressed as follows

$$\bar{\gamma}^{-k,(l)} = (\gamma^{1,(l+1)}, \cdots, \gamma^{k-1,(l+1)}, \gamma^{k+1,(l)} \cdots, \gamma^{|\mathcal{K}|,(l)})$$

Solving the step 1 of the diagonalization method can be equivalent to solve the following nonlinear problem For each $k = 1, ..., K$,

$$\begin{aligned} & Min\ f^k(\gamma^k; \bar{\gamma}^{-k,(l)}) \\ & Subject\ to \\ & \qquad g^k(\gamma^k; \bar{\gamma}^{-k,(l)}) \leq 0 \\ & \qquad h^k(\gamma^k; \bar{\gamma}^{-k,(l)}) = 0 \\ & \qquad G(\gamma^k; \bar{\gamma}^{-k,(l)}) \geq 0 \\ & \qquad H(\gamma^k; \bar{\gamma}^{-k,(l)}) \geq 0 \\ & \qquad G(\gamma^k; \bar{\gamma}^{-k,(l)}) \circ H(\gamma^k; \bar{\gamma}^{-k,(l)}) \leq 0 \end{aligned} \tag{2.13}$$

This mentioned equivalent NLP to the k^{th} $MPEC$ is denoted by $NLP^k(\bar{\gamma}^{-k,(l)})$

Theorem 2.2.2. *Let us assume $(x^{(l)}, y^{(l)})$ is a sequence of solutions generated by a diagonalization (nonlinear Jacobi or nonlinear Gauss-Seidel) method, in which each MPEC is reformulated and solved as an equivalent NLP (2.13). Also assume the sequence $(x^{(l)}, y^{(l)})$ converges to (x^*, y^*) as $l \to \infty$. If (x^{k*}, y^*) satisfies the MPEC-LICQ for each $\forall k = 1, ..., K$, then (x^*, y^*) is B-stationary for the EPEC (2.11).*

Chapter 3
Foundations of the Differential Nash Games, Differential Stackelberg Games

A non-cooperative game is one in which users make unilateral decisions to maximize their individual benefits. The theory of noncooperative games provides a framework to observe and analyze systems of noncooperative agents. Game theory provides a framework for modeling the interaction between groups of players whose utility functions and set of feasible strategies are related.

In this chapter we will present two important types of noncooperative games as Nash [64] and Stackelberg games [65]. In both games users make unilateral decisions to maximize their individual benefits wherein no agent may enhance its payoff without diminishing the payoff experienced by one or more other agents. This idea has been especially useful to model a wide variety of decision environments including vehicular traffic networks, revenue management, nonlinear pricing, supply chains, and facility location. Such models can be found in researches by Dafermos and Nagurney [66] ; Henderson and Quandt [67]; and Friesz et al. [68]. In this chapter we focus on the introduction of this games as well as reviewing notion of a deterministic non-cooperative Nash equilibrium to a dynamic, continuous-time setting.

Nash game presented first by John Nash [64], refers to the game in where N players are competing while no agent may enhance its payoff by changing his/her own strategy from the Nash equilibrium without diminishing the payoff experienced by one or more other agents, given that the other players use their equilibrium strategies. Note that in this chapter, we will focus on the open-loop games in which the initial information is perfectly prepared and the complete solution can be calculated from t_0 to t_f without a feedback. Also, we borrow most of our notations from Friesz [38].

3.1 Nash Game and Differential Nash Game

Definition 3.1.1. *(Nash equilibrium)*
Let us assume N agents which choose their feasible strategy vector u^i from the strategy set ω_i which is independent of the other agent$_S$) strategy sets. and let us define the disutility functional for each agent $i \in [1,\ N]$ as $\Phi_i(u) : \omega \to \mathcal{R}^1$ Note that $\Phi_i(u)$ depends on all agents' strategies where:

$$\omega = \prod_{i=1}^{N} \omega_i$$
$$u = (u^i : i = 1,\ N)$$

While every agent $i \in [1,\ N]$ is trying to optimize the following optimization problem

$$Min\ \Phi_i(u^i,\ u^{-i}) s.t.\ u^i \in \omega_i$$

where

$$u^{-i} = (u^j : j \neq i)$$

is the non-own tuple for each agent $i \in N$. Therefore, a Nash equilibrium is a tuple of strategies u, one for each agent $i \in N$, such that each u^i solves the mathematical program 3.1. In other words no agent may enhance its payoff without diminishing the payoff experienced by one or more other agents. Such Nash equilibrium problem is denoted as NE $(\Phi,\ \omega)$.

The extension of the definition of a Nash equilibrium has been presented for the case in which the strategy set of any agent $i \in N$ depends on non-own strategies $x^j, i \neq j$. This extension is called a generalized Nash equilibrium.

To define the differential Nash game and differential variational inequality (DVI) in this section, we need to introduce the Hilbert space. Hilbert space is a complete vector space with well-defined norm and well-defined inner product that induces a norm. Also, The specific function spaces we employ are those that allow optimal control problem to be analyzed as infinite dimensional mathematical programming, and are chosen setting of an infinite dimensional variational inequality.

Let us begin by the following notations:

$$
\begin{aligned}
&u^i \in (\mathcal{L}^2[t_0,\ t_f])^{m_i}\\
&x^i \in (\mathcal{H}^1[t_0,\ t_f])^{n_i}\\
&x_0^i \in \mathcal{R}^{n_i}\\
&m = m_1 + \ \ + m_N\\
&n = n_1 + n_2 + \ \ + n_N\\
&\Phi_i:\ \ (\mathcal{H}^1[t_0,\ t_f])^n \times (\mathcal{L}^1[t_0,\ t_f])^m \times \mathbb{R}^1_+ \to \mathcal{L}^2[t_0,\ t_f]\\
&f^i:(\mathcal{H}^1[t_0,\ t_f])^{n_i} \times (\mathcal{L}^1[t_0,\ t_f])^{m_i} \times \mathbb{R}^1_+ \to (\mathcal{L}^2[t_0,\ t_f]^{n_i})\\
&\mathcal{Z}:\mathbb{R}^{n_i} \times \mathbb{R}^1_+ \to \mathbb{R}^1\\
&\mathcal{K}:\mathbb{R}^{n_i} \times \mathbb{R}^1_+ \to \mathbb{R}^{r_i}
\end{aligned}
$$

Where $(\mathcal{H}^1[t_0,\ t_f])^n$ is the n-fold product of the Sobolev space $(\mathcal{H}^1[t_0,\ t_f])$ and $(\mathcal{L}^2[t_0,\ t_f])^m$ is the m-fold product of the space non-negative of square-integrable functions $(\mathcal{H}^1[t_0,\ t_f])$ with inner product.

$$
x = (x^i,\ x^{-i}); u = u^i,\ u^{-i} \tag{3.1}
$$

where x^{-i} is the non-own state vector and u^{-i} is the non-own control vector.

Also, one should note that, a differential game follows this structure:

- There are state variables which determine the state of the dynamic system at any time t. As we see later the state variable for our pricing problem is the inventories for shippers.
- Controls decision variables are selected by the game players: In our problem control variables are introduced as production output rates, allocation of output and shipping pattern
- Each game player has an objective function and they try to selfishly optimize it by taking the optimal choice of decisions
- There are set of differential equations which represents the change of the state variables over time. This equations involve both state and control variables.

Definition 3.1.2. *(Differential Nash Equilibrium Problem)*
Let us assume N agents which choose their feasible strategy vector u^i from the strategy set

ω_i which is independent of the other agents' strategy sets. Also, let us define the disutility functional for each agent $i \in [1,\ N]$ as $\Phi_i(u) : \omega \to \mathcal{R}^1$ Note that $\Phi_i(u)$ depends on all agents' strategies where:

$$\omega = \prod_{i=1}^{N} \omega_i$$
$$u = (u^i : i = 1,\ N)$$

While every agent $i \in [1,\ N]$ is trying to optimize the following optimization problem

$$Min\ \Phi_i(u^i,\ u^{-i}) = Z_i[x^i(t_f),\ t_f] + \int_{t_0}^{t_f} \Phi_i(x^i,\ u^i,\ x^{-i},\ u^{-i},\ t)dt \tag{3.2}$$
$$s.t.$$
$$\frac{\partial x^i}{\partial t} = f^i(x^i,\ u^i,\ t) \tag{3.3}$$
$$x^i(t_0) = x_0^i \tag{3.4}$$
$$u^i \in \omega_i \tag{3.5}$$
$$K^i[x^i(t_f,\ t_f] = 0 \tag{3.6}$$

Where u_i are control and x_i are state variables. In addition, 3.4 is the state dynamic, (3.5) is the initial condition, and (3.7) is the terminal condition for the state variable.

For each non-own control tuple

$$u^{-i} = (u^i : j \neq i)$$

We define

$$x^{-i} = (x^i : j \neq i)$$

as the non-own state tuple. A differential Nash equilibrium is a tuple of strategies $u = (u_i$: $i = 1,$ 2, .., $N)$ such that each u_i is the solution for the optimal control problem 3.3-3.7. Such differential Nash equilibrium problem is denoted as $DNE(\Phi,\ f,\ Z,\ K,\ \omega,\ x_0,\ t_0,\ t_f)$.

3.2 Variational Inequality And Differential Variational Inequality

Concept of Differential Variational Inequality(DVI) has been first introduced and comprehensively discussed by Pang and Stewart [69]. They studied mathematical approaches solution of a sequence of finite-dimensional variational inequalities with inequalities and discontinuities. The DVI has been an important modeling approach for many areas in engineering and economics which presents dynamics, variational inequalities and equilibrium conditions ([70], [71], [72], [69], [73], [74]).

In this section, we review the notion of variational inequality and differential variational inequality and then prove that differential variational inequality is equivalent to differential Nash equilibrium.

3.2.1 Variational Inequality

Definition 3.2.1. *(Finite dimensional variational inequality problem)*

$VI(U,\ F)$ is to find a vector $u \in V$ such that the following condition hold:

$$\begin{aligned} & u \in U \\ & \langle F(u)^T(x-u) \geq 0\rangle \ \forall x \in U \end{aligned} \tag{3.7}$$

Where $U \subseteq \mathcal{R}^n$ is a non-empty subset of $\mathcal{R}^n$ and $F : U \subseteq \mathcal{R}^n \rightarrow \mathcal{R}^n$ is a mapping from U into U

Note that under the regularity conditions VIs and nonlinear complementarity problem (NCP) are equivalent. Also, under more strict situation, the solution to VIP is the solution to a NCP.

Definition 3.2.2. *(Infinite dimensional variational inequality problem)*

$VI(U,\ F)$ is to find a vector $x^* \in U$ such that the following condition hold:

$$\begin{aligned} & u^* \in V \\ & \langle F(u^*)^T(u-u^*) \geq 0\rangle \ \forall u \in U \subseteq V \end{aligned} \tag{3.8}$$

Where $U \subseteq V$ is a non-empty subset of Hilbert space V and $F : U \subseteq V \to V$ is a mapping from V into itself.

Studies by Harker [75] and Harker and Pang [76] described the generalized linear algorithms for variational inequalities. Also, Ferris and Pang [77] have done a complete review on the finite dimensional VI applications and algorithms in their book.

In addition more reviews on the algorithms and applications for infinite dimensional VI can be found in Dafermos [78], Baiocchi and Capelo citebaiocchil984variational, Harker [79], Hammond [80], Friesz et al. [81], Nagurney [82], Nagurney [83], Nagurney et al. [84] and Goldsman and Harker [85].

3.3 Alternative Formulations for Nash Games

3.3.1 Differential Variational Inequality

The same specific function spaces we defined in previous definitions are again employed in this section since we are extending the notion of an optimal control problem to the more general setting of an infinite-dimensional variational inequality.

According to Friesz et al. [68]. we define the following differential variational inequality denoted as $DVI(F,\ f,\ K,\ U,\ x\text{O},\ t_0,\ t_f)$:

$$\begin{aligned} & Find\ u^* \in U \\ & s.t \\ & \langle F(x(u^*,\ t),\ u^*,\ t)\ ,\ u^* - u\rangle \geq 0,\ \forall u \in U \end{aligned} \tag{3.9}$$

where

$$
\begin{aligned}
& u \in U \subset (\mathcal{L}^2[t_0,\ t_f])^m \\
& x(u,\ t) = arg\{\frac{dx}{dt} = f(x,\ u,\ t)\ ,\ x(\mathrm{O}) = x_0,\ K[y,\ x(t_f),\ t_f] = 0\} \in (\mathcal{H}^1[t_0,\ t_f])^n \\
& x_0 \in R^n \qquad (3.10) \\
& F : (\mathcal{H}^1[t_0,\ t_f])^n) \times (L^2[t_0,\ t_f])^m \times \mathbb{R}^1_+ \to (\mathcal{L}^2[t_0,\ t_f])^m \\
& f :\ (\mathcal{H}^1[t_0,\ t_f])^n) \times (L^2[t_0,\ t_f])^m \times \mathbb{R}^1_+ \to (\mathcal{L}^2[t_0,\ t_f])^n \\
& K : (\mathcal{H}^1[t_0,\ t_f])^n) \times \mathbb{R}^1_+ \to \mathbb{R}^1
\end{aligned}
$$

$(L^2[t_0,\ t_f])^m$ is the m-fold product of the space of square-integrable functions $(L^2[t_0,\ t_f])$. In addition the inner product is defined by

$$\langle x,\ y\rangle = \int_{t_0}^{t_f} x^T y dt \qquad (3.11)$$

According to the inner product definition in 3.14, the differential variational inequality 3.11 can be stated as

$$\langle F(x(u^*,\ t),\ u^*,\ t), u^*, t), u - u^*\rangle = \int_{t_0}^{t_f} [F(x(u^*,\ t),\ u^*,\ t)]^T (u^* - u) dt \geq 0 \forall u \in U \qquad (3.12)$$

Definition 3.3.1. *(Regularity of the Differential Variational Inequality)*

The regularity assumption is needed to analyze the optimal control problem, and its necessary conditions from the point of view of infinite dimensional mathematical programming. $DVI(F,\ f,\ K,\ U,\ x\mathrm{O},\ t_0,\ t_f)$ *is considered regular if*

- $F(x; u; t)$ *is convex and continuously differentiable with respect to* $x, u,$ $f(x; u; t)$ *is continuously differentiable with respect to* x *and* u
- $f(x; u; t)$ *and* $\frac{\partial f(x; u; t)}{\partial x}$ *are bounded*
- $K(x; t)$ *is continuously differentiable with respect to* x

- *U is non-empty, convex and compact; and*
- *$x_0^i \in R^n$ is known and fixed.*

Theorem 3.3.1. *(Differential variational inequality equivalent to differential Nash equilibrium)*
For fixed $t_0, x(t_0)$ and t_f, there exist a differential variational inequality equivalent to the differential Nash equilibrium $DNE(\Phi,\ f,\ Z,\ K,\ \omega,\ x_0,\ t_0,\ t_f)$ when $f_i(x^i,\ u^i,\ t)$ and $\Phi(x^i,\ u^i,\ x^{-i},\ u^{-i},\ t)$ are convex and continuously differentiable with respect to $(x^i,\ u^i)$ for all fixed non-own tuples $(x^{-i},\ u^{-i})$, $\forall i \in [1,\ N]$.

Proof. (differential variational inequality theorem)
To prove the differential variational inequality theorem we first need to introduce the following notations for agent $i \in [1,\ N]$

$$y^i = \begin{bmatrix} x^i \\ \lambda^i \end{bmatrix}$$

and

$$y^{-i} = \begin{bmatrix} x^{-i} \\ \lambda^{-i} \end{bmatrix}$$

And λ^i is the adjoint variable for player $i \in [1,\ N]$ determined by

$$\begin{aligned}
&-\frac{\partial \lambda^i}{\partial t} = \nabla_{x^i}(\lambda^i)^T f^i(x,\ u,\ t) \\
&\lambda^i(t_f) = \frac{\partial \Psi^i[x(t_f), t_f]}{\partial x(t_f)} \\
&\Psi^i[x(t_f),\ t_f] = Z^i[x(t_f),\ t_f] + (\gamma^i)^T K^i[x(t_f),\ t_f]
\end{aligned}$$

The Hamiltonian for agent $i \in [1,\ N]$ associated with optimal control problem (3.3-3.7) is:

$$H_i(x^i,\ u^i,\ \lambda^i; x^{-i},\ u^{-i}) = \Phi^i(x^i,\ u^i,\ x^{-i},\ u^{-i},\ t) + (\lambda^i)^T f^i(x^i,\ u^i,\ t)$$

While the necessary and sufficient condition represents the minimum principle as

$$[\nabla_{u^i} H_i(x^i,\ u^i,\ \lambda^i; x^{-i},\ u^{-i})]^T (v^i - u^i) \geq 0,\ \forall v^i \in \omega_i \tag{3.13}$$

In addition we define the following tuples:

$$g^i = \begin{bmatrix} f^i \\ -\nabla_{x^i} \lambda^{i^T} f^i(x, u, t) \end{bmatrix}$$

and

$$\eta^i[x(t_f),\ t_f] = \begin{bmatrix} \Psi^i[x(t_f), t_f] \\ \lambda^i(t_f) - \frac{\partial \Psi^i[x(t_f), t, f]}{\partial x(t_f)} \end{bmatrix} = 0$$

for each agent $i \in [1,\ N]$, so that

$$\begin{aligned} y &= (y^i : i = 1,\ N) \\ g &= (g^i : i = 1,\ N) \\ \Psi &= (\Psi^i : i = 1,\ N) \end{aligned}$$

In addition we define:

$$y(t_0) = y_0 = \begin{pmatrix} x(t_0) \\ \lambda(t_0) free \end{pmatrix}$$

and

$$\begin{aligned} &G^i(y^i,\ u^i,\ t; y^{-i},\ u^{-i}) = \nabla_{u^i} H_i(x^i,\ u^i,\ \lambda^i,\ t; x^{-i},\ u^{-i}) \\ &G = (G^i : i = 1,\ \ldots,\ N) \end{aligned}$$

From the minimum principle in (3.8) as well as the above notations we have

$$\begin{aligned}&Find\ u^* \in \omega \\ &such\ that: \\ &\int_{t_0}^{t_f} [G(y(u^*,\ t),\ u^*,\ t)]^T (v-u)^* dt \geq 0\ \ \forall v \in \omega\end{aligned} \tag{3.14}$$

Where

$$y(u,\ t) = arg\{\frac{dy}{dt} = g(y,\ u,\ t)\ ,\ y(t_0) = y_0,\ \Psi[y(t_f),\ t_f] = 0\} \tag{3.15}$$

If given differential variational inequality (3.15) and (3.16), by selecting $v^j = u^{*j}$ for all $i \neq j$, the minimum principle is recovered for each individual $i \in [1,\ N]$. □

Theorem 3.3.2. *(Solution of the DVI)*
Any solution of the DVI (3.15-3.16) is a differential game equilibrium when the regularity conditions 3.3.1 hold.

$DVI(\Phi,\ f,\ K,\ U,\ x_0,\ t_0,\ t_f)$ is an abstract form of an finite set of constraints. There have been several numerical methods to solve DVI including Gap function, Nonlinear Complementarity, and Fixed-Point method (see Friesz [38]).

For the first project in this research we will reformulate the DVI into a nonlinear complementarity problem in finite dimensions. Reformulation from DVI to Nonlinear complementarity is going to be introduced in the next section.

3.3.2 Nonlinear Complementarity Formulation

Definition 3.3.2. *(Nonlinear complementarity problem)*
Given a (nonlinear) function $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$, the nonlinear complementarity problem $NCP(G)$ is to find a vector z such that

$$\langle [G(z)]^T . z \rangle = 0 \tag{3.16}$$

$$G(z) \geq 0 \tag{3.17}$$

$$z \geq 0 \tag{3.18}$$

Also, from the Geometric perceptive, a vector z is a solution of $NCP(G)$ if and only if.·

- *z is nonnegative*
- *$G(z)$ is nonnegative*
- *$G(z)$ is orthogonal to z*

Alternatively, a vector z is a solution of $NCP(z)$ if and only if.·

- *Every element of z is nonnegative,*
- *Every element of $G(z)$ is nonnegative*
- *For each positive element of z, denoted by z_i, then $G_i(z)$ is zero (and vice versa)*

3.3.2.1 Nonlinear Complementarity Reformulation of the OCP

Let us assume a linear control set U

$$U = u \geq 0 : \ Au \leq b$$

Where $b \in \mathcal{R}^l$ is a constant vector and $A = (a_{i,j})$ is a constant $l \times m$ matrix.

By following KKT conditions for each player, we will end up with the nonlinear complementarity formulation in the finite dimensions

$$\nabla_u H(x,\ u,\ \lambda, t) + \sum_{i=1}^{m} \rho_j \nabla_u(-u_j) + \sum_{j=1}^{l} \zeta_j \nabla_u (AU - b)_j = 0 \tag{3.19}$$

$$\rho_j u_j = 0 \tag{3.20}$$

$$\zeta_j (Au - b)_j = 0 \tag{3.21}$$

$$\zeta_j \geq 0 \tag{3.22}$$

for all $j \in [1,\ m]$. Therefore for each $j \in [1,\ m]$ we have

$$F_i(x,\ u,\ t) + \sum_{j=1}^{n} \lambda_j \frac{\partial}{\partial u_i} f_j(x,\ u,\ t) + \sum_{j=1}^{l} \zeta_i \frac{\partial}{\partial u_i} (Au - b)_j = \rho_i \geq 0 \tag{3.23}$$

Based on the fact that $\frac{\partial}{\partial u_i}(Au-b)_j = a_{ij}$ and complementary slackness in 3.22 we have

$$[F_i(x,\ u,\ t) + \sum_{j=1}^{n} \lambda_j \frac{\partial}{\partial u_i} f_i(x,\ u,\ t) + \sum_{j=1}^{l} a_{ij}\zeta_j]u_j = \rho_j u_j = 0 \tag{3.24}$$

Thus, for each $i = 1, 2, m$, the KKT condition turns to the following nonlinear complementarity formulation

$$\langle G(z,\ t)\ ,\ z\rangle\rangle = 0 \tag{3.25}$$

$$G(z,\ t) \geq 0 \tag{3.26}$$

$$z \geq 0 \tag{3.27}$$

Where

$$z \in (L^2[t_0,\ t_f])^{2m+l})$$
$$G\ :\ (L^2[t_0,\ t_f])^{2m+l} \times \mathcal{R}^1_+ \rightarrow (L^2[t_0,\ t_f])^{2m+l})$$

and

$$G = \begin{pmatrix} F(x,u,t) + [\nabla_u f(x,u,t)]^T.\lambda + [\nabla_u(Au-b)]^T.\zeta \\ Au-b \\ u \end{pmatrix} \zeta = \begin{pmatrix} u \\ \zeta \\ \rho \end{pmatrix}$$

One should note that $x(.)$ and $\lambda(.)$ are operators obeying (3.16). Also $F(x;u;t)$ and $f(x;u;t)$ are linear in u and formulation (3.26), (3.27) and (3.28) produces a linear complementarity problem.

3.3.3 Fixed point

To be able to compute the solutions to differential variational inequalities, there is often an equivalent functional fixed-point problem corresponding to a given differential Nash game. Fixed-point problem formulation is a simple, efficient algorithm which provides solution for $DVI(\Phi,\ f,\ K,\ U,\ x_0,\ t_0,\ t_f)$.

Theorem 3.3.3. *($Fixed-point\ formulation\ of\ DVI(\Phi,\ f,\ K,\ U,\ x_0,\ t_0,\ t_f)$)*
When regularity in the sense of definition 3.3.1 holds and $f(x;u)$ is convex in $(x;u)$, then

$DVI(\Phi,\ f,\ K,\ U,\ x_0,\ t_0,\ t_f)$ is equivalent to the following fixed-point problem:

$$u = P_U[u - \alpha\Phi(x(u,t),u,t)]$$

P_U is the minimum norm projection onto $U \subseteq (L^2[t_0,\ \tau])^m$ and $\alpha \in R^1_{++}$ is arbitrary positive constant.

Proof. *The mentioned fixed-point problem requires that*

$$u = argmin\{\frac{1}{2}||u - \alpha\Phi(x(u,\ t),\ u,\ t) - v||^2 : v \in U\} \tag{3.28}$$

$\alpha \in \mathbb{R}^1_{++}$ is a positive constant. Therefore, control problem

We are trying to solve the following optimal

$$\min \gamma^T K[x(t_f),\ t_f] + \int_{t_0}^{t_f} \frac{1}{2}[u - \alpha\Phi(x,\ u,\ t) - v]^2 dt$$

subject to:

$$\frac{dx}{dt} = f(x,\ v,\ t)\ ;$$

$$x(t_0) = x_0$$

$$v \in U$$

For the purpose of projection, u is treated as fixed.

Let us write the necessary conditions of the mentioned optimal control problem, which are also sufficient by due to of the convexity

$$[\nabla_v H_1(x^*,\ v^*,\ \eta^*,\ t)]^T(v - v^*) \geq 0 \forall v \in U \tag{3.29}$$

Where

$$H_1(x,\ v,\ \eta,\ t) = \frac{1}{2}[u - \alpha\Phi(x,\ u,\ t) - v]^2 + \eta^T f(x,\ v,\ t) \tag{3.30}$$

and for given x and v

$$\eta = arg\{(-1)\frac{d\eta}{dt} = \nabla_x H_1(x,\ v,\ \eta,\ t)\ ,\ \eta(t_f) = \gamma^T \frac{\partial K[x(t_f), t_f]}{\partial x(t_f)}\}$$

One should note that we have

$$\nabla_u H_1(x,\ v,\ \eta,\ t) = -u + \alpha\Phi(x,\ u,\ t) + v + \nabla_u[\eta^T f(x,\ u,\ t)] \tag{3.31}$$

Because u equals v by 3.30, then

$$\nabla_u H_1(x,\ v,\ \eta,\ t) = \alpha\Phi(x,\ u,\ t) + \nabla_u[\eta^T f(x,\ u,\ t)] \tag{3.32}$$

Let us consider the following notation

$$\lambda = \frac{\eta}{\alpha}$$

Therefore,

$$[\Phi(x^*,\ u^*,\ t^*) + \nabla_u(\lambda^*)^T f(x^*,\ u^*,\ t^*)]^T(u - u^*) \geq 0,\ \forall v \in U \tag{3.33}$$

This is identical to the necessary conditions of the finite-dimensional variational inequality. In addition, since the other optimality conditions are also identical, this completes the proof □

3.3.3.1 Unembellished fixed-point algorithm

Naturally there is a fixed point algorithm associated with Theorem 3.3.3; The iterative scheme of the fixed-point algorithm is expressed as follows

$$u^{k+1} = P_U[u^k - \alpha\Phi(x(u^k;t);u^k;t)]$$

The positive constraint α should be chosen empirically and can be changed as the algorithm processes to help the convergence. The detailed steps of the algorithm are

Step 0. **Initialization:** Identify an initial feasible solution as $u^0 \in U$, Set $k = 0$

Step 1. Solve the optimal control subproblem: Solve the following optimal control subproblem

$$
\begin{aligned}
minJ^k(v) = \gamma^T K[x(t_f),\ t_f] \\
+ \int_{t_0}^{t_f} \frac{1}{2}[u^k - \alpha\Phi(x^k,\ u^k,\ t) - v]^2 dt
\end{aligned} \tag{3.34}
$$

Subject to

$$\frac{dx}{dt} = f(x,\ v,\ t) \tag{3.35}$$

$$x(t_0) = x_0 \tag{3.36}$$

$$v \in U \tag{3.37}$$

And call the solution to this step as u^{k+1}

Step 2. Termination: If $||u^{k+1} - u^k|| \leq \epsilon$, where $\epsilon \in \mathcal{R}^1_{++}$ is the error tolerance, then stop. The solution is $u^* \approx u^{k+1}$ Otherwise go to Step 1 and set $k = k + 1$.

The mentioned algorithm is called unembellished fixed-point algorithm and following conditions guarantees its convergence

Theorem 3.3.4. *((Convergence of the unembellished fixed-point algorithm)) When $DVI(\Phi,\ f,\ K,\ U,\ x_0,\ t_0,\ t_f)$ is regular in the sense of definition 3.3.1 and $\Phi(x(u),\ u,\ t)$ is strongly monotonically increasing and satisfies the following Lipschitz condition*

$$||\Phi(x(u),\ t)\ ,\ u,\ t) - \Phi(x(v,\ t),\ v,\ t)|| \leq \kappa_0 ||u - v||$$

for some $\kappa_0 \in \mathbb{R}^1_{++}$ and all $u, v \in U$ and for appropriate $\alpha \in (0,\ \alpha)$, then the introduced unembellished fixed-point algorithm converges.

One should note that the strictly monotonic condition mentioned in 3.3.4 is unlikely to be verifiable for the problems of realistic size. Hence, The fixed point algorithm becomes a heuristic approach

3.3.3.2 Descent Algorithm in Hilbert Space for the Projection Sub-Problem

The introduced fixed-point algorithm may be applied in continuous time if we employ a continuous-time representation of the solution of each subproblem. This may be done using a continuous-time gradient projection method. Therefore, descent in Hilbert space algorithm for projection subproblems may be defined as

Step 0**. Initialization:**
Pick an initial solution as $v^{k;0}(t) \in U$ and set $j = 0$.

Step 1. Finding state variables:
Solve the following state dynamics

$$\frac{dx}{dt} = f(x,\ v^{k,j},\ t) \tag{3.38}$$

$$x(t_0) = x_0 \tag{3.39}$$

A discrete-time method should be used to solve (3.39) and (3.40), then curve fitting is used to obtain the continuous-time state vector. Let us call the current solution $x^{k,j}(t)$.

Step 2. Finding adjoint variables:
Given $v = v^{k,j}$ and $x = x^{k,j}$, solve the adjoint dynamics

$$-\frac{d\lambda}{dt} = \nabla_x H_1^k \tag{3.40}$$

$$\lambda(t_f) = \gamma^T \frac{\partial K[x^{k,j}(t_f), t_f]}{\partial x(t_f)} \tag{3.41}$$

Where

$$H_1^k = \frac{1}{2}[u^k - \alpha\Phi(x^k,\ u^k,\ t) - v]^2 + \lambda^T f(x,\ v^{k,j},\ t)$$

A discrete-time method should be used to solve (3.41) and (3.42), then curve fitting is used to obtain the continuous-time adjoint vector. Let us call the current solution $\lambda^{k,j}(t)$.
Step 3. Finding the gradient: Determine

$$\nabla_v J^{k,j}(t) = \nabla_v H_1^k$$

Step 4. Step determination: For a fixed and suitably small fixed step size

$$\theta_k \in \mathbb{R}^1_{++}$$

determine

$$v^{k,j+1}(t) = P_U[v^{k,j}(t) - \theta_k \nabla_v J^{k,j}] \tag{3.42}$$

A discrete-time method should be used to solve the presented projection subproblem, then curve fitting is used to obtain the continuous-time control vector (3.43).

Step 5. Termination:

If $||v^{k,j+1} - v^{k,j}|| \leq \epsilon_1$, where $\epsilon_1 \in \mathcal{R}^1_{++}$ is the error tolerance, then stop. The solution will be declared as $v^{k*} \approx v^{j,k+1}$ Otherwise go to Step 1 and set $j = j + 1$

The unembellished fixed-point algorithm presented in this section is not completely reliable. There exist some drawbacks with such an algorithm as it sometimes converges slowly even when the regularity conditions which assures convergence are satisfied. The algorithm may also fail to converge in the case that regularity conditions are not satisfied.

There are different algorithms which can be applied to solve a DVI. One of them is to reformulate the DVI of interest as a nonlinear complementarity problem in function space. Proper time discretization helps to approximate the nonlinear complementarity problem. Penalty function then may be employed to attach the nonlinear constraints to the objective function and produce a nonlinear program with linear constraints.

In addition, there are some other algorithms such as gap function to create an equivalent optimal control problem that may be solved for infinite-dimensional mathematical programs. One may use the a gap function to convert a variational inequality problem to an equivalent optimization problem. Objective function of such problem is always nonnegative and optimal objective function value is zero if and only if the optimal solution solves the original variational inequality problem.

3.3.4 Gap Functions

To consider using gap function for analyzing the differential variational inequality, $DVI(F, f, K, U, x_0, t_0, t_f)$, such DVI should belongs to the class of infinite-dimensional variational inequalities considered by Konnov et al. [86]. This class of DVI is the ones which satisfy the regularity conditions, wherein U is a nonempty closed and convex subset and F is a continuously differentiable mapping of u.

Definition 3.3.3. *(Gap function definition)*

A function $G : U \rightarrow \mathbb{R}_+$ is called a gap function for $DVI(F, f, K, U, x_0, t_0, t_f)$ when the following hold:

- $G(u) \geq 0$ *for all* $u \in U$
- $G(u) = 0$ *if and only if* u *is the solution of* $DVI(F, f, K, U, x_0, t_0, t_f)$.

We will consider gap functions in the following form

$$G_\alpha(u) = max\ \Psi_\alpha(u, v) \quad \forall v \in U \tag{3.43}$$

Where

$$\Psi_\alpha(u, v) = \langle F[x(u, t), u, t], u - v\rangle - \alpha\Gamma(u, v) \tag{3.44}$$

$$x(u, t) = arg\left\{\frac{dy}{dt} = f(y, u, t),\ y(t_0) = y_0,\ K[y(t_f), t_f] = 0\right\}$$

$$\in (H^1[t_0, t_f])^n \tag{3.45}$$

$$U \subseteq (L^2[t_0, t_f])^m \tag{3.46}$$

$$\alpha \in \mathbb{R}^1_{++} \tag{3.47}$$

Where, function Γ is a function satisfying the following conditions:

- Γ is continuously differentiable on $(L^2[t_0, t_f])^{2m}$;
- Γ is nonnegative on $(L^2[t_0, t_f])^{2m}$
- $\Gamma(u, v) = 0$ if and only if $u = v$; and
- $\Gamma(u, v)$ is strongly convex in $v \in U$ with $s > 0$ for any $u \in (L^2[t_0, t_f])^m$; that follows

$$\Gamma(u, v) + \langle\nabla_v\Gamma(u, v), u - v\rangle + \frac{1}{2}s||u - v||^2 \leq \Gamma(u, u) = 0 \quad \forall u \in U \tag{3.48}$$

Also for finite-dimensional spaces, Yamashita et al. [87] suggested a Γ function which satisfies the following conditions

- $\Gamma_1(u, v) = \beta_1(u - v)$, where β_1 is nonnegative, continuously differentiable, strongly convex, and $\beta_1(0) = 0$;

- $\Gamma_1(u,v) = \beta_2(v) - \beta_2(u) - \langle \nabla_{\beta 2}(u), u - v\rangle$ where β_2 is twice continuously differentiable, and strongly convex; and

- $\Gamma_3(u,v) = \langle u - v, M(u)(u - v)\rangle$ where $M(u)$ is a continuously differentiable, symmetric, and uniformly positive-definite matrix.

Theorem 3.3.5. *(Gap function for $DVI(F,f,K,U,x_0,t_0,t_f)$)*
The function $G_\alpha(u)$ in (3.43) is a gap function for $DVI(F,f,K,U,x_0,t_0,t_f)$. In particular, we say that u is the solution of $DVI(F,f,K,U,x_0,t_0,t_f)$, if and only if $u = v_\alpha(u)$.

For the proof please see [38]

3.3.4.1 D-gap Equivalent Optimal Control Problem

One of the concerns for defining the gap function is no guarantee that $G_\alpha(u)$ is in general differentiable. Therefore, the so-called D-gap function has been introduced as an extension to the gap function.

D-gap function has the following form:

$$\eta_{\alpha\beta}(u) = G_\alpha(u) - G_\beta(u) \quad for\ 0 \leq \alpha \leq \beta \tag{3.49}$$

Where in general, $G_\alpha(u)$ is not differentiable and $\eta_{\alpha\beta}(u)$ is Gateaux differential. We show that $\eta_{\alpha\beta}$ is a gap function by only showing that non-negativity property holds.

Let us invoke the assumption introduced for the gap function, therefore

$$\begin{aligned}\eta_{\alpha\beta}(u) =& G_\alpha(u) - G_\beta(u)\\ =&\Psi_\alpha(u, v_\alpha) - \Psi_\beta(u, v_\beta)\\ \geq&\Psi_\alpha(u, v_\beta) - \Psi_\beta(u, v_\beta)\\ =&\langle F(x,u,t), u - v_\beta)\rangle - \alpha\Gamma(u, v_\beta) - \langle F(x,u,t), u - v_\beta\rangle + \beta\Gamma(u, v_\beta)\\ =&(\beta - \alpha)\Gamma(u, v_\beta)\end{aligned} \tag{3.50}$$

This proves that $\eta_{\alpha\beta}$ is non-negative, and that $G_\alpha(u) \geq G_\beta(u),\ \forall u \in U$. So (3.49) does in fact define a gap function.

3.4 User Equilibrium

This section is concerned with a specific type of traffic assignment (DTA) known as dynamic user equilibrium (DUE). In this assignment, travel cost such as delay, early and late arrival penalties are the same for the routes and travel agents select the routes and the departure time between given origin-destination pairs.

In this section we review the formulation for a dynamic user equilibrium (DUE) with elastic demand as a differential variational inequality (DVI). The DUE model is articulated and formulated as a variational inequality in Friesz et al. [88] and then as a differential variational inequality in Friesz and Mookherjee [89]. It is also solved by a fixed-point algorithm in a Hilbert space by Friesz et al. [90].

3.4.1 Dynamic User Equilibrium

The most fundamental concept of noncooperative game theory is the notion of a Nash equilibrium, wherein no agent may enhance its payoff without diminishing the payoff experienced by one or more other agents. This idea has been especially useful in modeling vehicular traffic on urban road networks where it is referred to as Wardrop's First Principle of traffic assignment. A network flow pattern computed according Wardrop's First Principle is often referred to as a user equilibrium (UE). In particular, for a UE flow pattern, no user may lower his/her unit cost of travel on a path connecting any given origin-destination pair without causing the unit cost on some other path (connecting the same OD pair) to increase. For a comprehensive reference we refer the readers to Friesz [38].

increasingly sophisticated traffic monitoring, control, and route guidance technology has produced a need for some sort of dynamic notion of noncooperative competition among vehicles competing for limited road capacity. That notion, known as dynamic user equilibrium (DUE), extended UE to consider not only initial path choice but also departure time choice, and route updating.

A DUE flow pattern is modeled as a differential Nash game in either discrete or continuous time. The differential Nash games used to study DUE are extensions of optimal control theory to a problem class known as a differential variational inequality (DVI) with state-dependent time shifts. For a comprehensive review we refer our readers to Friesz et al. [8].

Let us first consider a fixed time interval $[t_0,\ t_f] \subset \mathbb{R}$ and define the most important components of a dynamic user equilibrium model as the path delay operator. Path delay operator shows the travel delay for path $p \in \mathcal{P}$ per unit of flow departing from the origin of that path; where $\mathcal{P}$ is the set of paths selected by travelers. Path delay operator will be denoted as follows:

$$D_p(t,\ h)\ \forall p \in \mathcal{P} \tag{3.51}$$

One should note that the path delay operators usually do not have any closed form, instead they can only be evaluated numerically through the dynamic network loading (DNL) procedure (For definition and review on dynamic network loading (DNL) please see [91]). Where t is the departure time and h is a vector of departure flows. Now we are ready to construct unit path delay operators as

$$\Psi_p(t,\ h) = D_p(t,\ h) + F[t + D_p(t,\ h) - T_A]\ \forall p \in \mathcal{P} \tag{3.52}$$

Where T_A is the desired travel time. Additionally, we denoted the vector of path flows by $\{h = h_p : p \in \mathcal{P}\}$, which are square integrable:

$$h \in (\mathcal{L}_+^2[t_0,\ t_f])^{|7^{\supset}|}$$

where $(\mathcal{L}_+^2[t_0,\ t_f])^{|7^{\supset}|}$ denotes the positive cone of the $|P|$ *−fold* product of the space $L_+^2[t_0,\ t_f]$ consisting of square-integrable functions on $[t_0,\ t_f]$.

Also, Let us introduce the fixed trip matrix $(Q_{ij}\ :\ (i,\ j) \in \mathcal{W})$. Each $Q_{ij} \in \mathcal{R}^+$ in the matrix is the fixed travel demand expressing the volume, between origin-destination pair $(i,\ j) \in \mathcal{W}$; where $\mathcal{W}$ is the set of all origin-destination pairs. In addition, we denote the subset of paths that link origin-destination pair $(i,\ j) \in \mathcal{W}$ as $\mathcal{P}_{ij}, \forall (i,\ j) \in \mathcal{W}$. Therefore, the flow conservation constraints will be expresses as

$$\sum_{p \in 7^{\supset}_{ij}} \int_{t_0}^{t_f} h_p(t)dt = Q_{ij}\ \forall (i,\ j) \in \mathcal{W} \tag{3.53}$$

Using all the notations introduced, the feasible region for the path flow can be expressed as follows

$$\Omega_0 = \{h \geq 0;\ \sum_{p \in 7^{\supset}_{ij}} \int_{t_0}^{t_f} h_p(t)dt = Q_{ij} \forall (i,\ j) \in \mathcal{W}\} \subseteq (\mathcal{L}_+^2[t_0,\ t_f])^{|7^{\supset}|} \tag{3.54}$$

The essential infimum of effective travel delays is

$$v_{ij} = essinf[\Psi_p(t,\ h)] :\ p \in \mathcal{P}_{ij}]\forall(i,\ j) \in \mathcal{W} \tag{3.55}$$

Now we are ready to define the dynamic user equilibrium which was first formulated by Friesz et al. [88].

Definition 3.4.1. *(Dynamic user equilibrium)*
A vector of departure rates (path flows) $h^* \in \Omega_0$ is a dynamic user equilibrium if

$$h_p^*(t) > 0,\ p \in \mathcal{P}_{ij} \to \Psi_p[t,\ h^*(t)] = v_{ij}$$

This equilibrium is denoted by DUE $(\Psi,\ \Omega_0,\ [t_0,\ t_f])$

In addition, Friesz et al. [88] has used measure theoretic arguments to prove that a dynamic user equilibrium is equivalent to the following variational inequality under suitable regularity conditions

$$\begin{aligned}
&find\ h^* \in \Omega_0 \\
&such\ that \\
&\sum_{p \in 7^{\supset}} \int_{t_0}^{t_f} \Psi_p(t,\ h^*)(h_p - h_p^*)dt \geq 0 \\
&\forall h \in \Omega_0
\end{aligned} \tag{3.56}$$

It has been mentioned in [90] that (3.57) is equivalent to a differential variational inequality. This is most clear by noting that the flow conservation constraints may be re-stated as a two-point boundary value problem

$$\begin{aligned}
&\frac{dy_{ij}}{dt} = \sum_{pin7^{\supset}ij} h_p(t) \\
&y_{ij}(t_0) = 0\ \forall(i,\ j) \in \mathcal{W} \\
&y_{ij}(t_f) = Q_{ij}
\end{aligned}$$

Where y_{ij} is the the cumulative traffic that has departed between origin-destination pair $(i,\ j) \in \mathcal{W}$. Eventually, (3.57) can be expressed as the following differential variational inequality

$$\begin{aligned} &\textit{find } h^* \in \Omega \\ &\textit{such that} \\ &\sum_{p\in 7^{\supset}} \int_{t_0}^{t_f} \Psi_p(t,\ h^*)(h_p - h_p^*)dt \geq 0 \\ &\forall h \in \Omega \end{aligned} \tag{3.57}$$

And

$$\Omega = \{h \geq 0 : \frac{dy_{ij}}{dt} = \sum_{p\in 7^{\supset}ij} h_p(t)\ ,\ y_{ij}(t_0) = 0,\ y_{ij}(t_f) = Q_{ij}\ \forall (i,\ j) \in \mathcal{W}\} \tag{3.58}$$

The formulation 3.58 is denoted as $DVI(\Psi,\ \Omega,\ [t_0,\ t_f])$

Because of the optimal control framework inherent in the DVI problems, analysis and computation of dynamic user equilibrium is tremendously simplified by stating it as a differential variational inequality (DVI). Finally, we are ready to show how the solution of the DVI 3.50 may be obtained by solving a fixed point problem:

Theorem 3.4.1. *(Fixed point re-statement of* $DVI(\Psi, \Omega, [t_0, t_f$
Let us assume $\Psi_p(,\ .h) : [t_0,\ t_f] \to \mathcal{R}_+$ *as a measurable function for* $\forall p \in \mathcal{P},\ \forall h \in \Omega.$ *Then the fixed point problem is*

$$h = P_\Omega[h - \alpha\Psi(t,\ h)] \tag{3.59}$$

is equivalent to $DVI(\Psi,\ \Omega,\ [t_0,\ t_f])$ *where* P_Ω *is the minimum norm projection onto* Ω *and* $\alpha \in \mathbb{R}_+$.

Proof of this theorem can be found in Friesz et al. [32].

Theorem 3.1.5 guides to solve the dynamic user equilibrium problem by an iterative scheme of the following form

$$h^{k+1} = P_\Omega[h^k - \alpha\Psi(t,\ h^k)]$$

Where $h^{k+1}, h^k \in \Omega$ are two consecutive iterates. Convergence of such scheme requires monotonicity, or a weaker notion of monotonicity, of the effective delay operator. These theorems are discussed in Nagurney [83] and Friesz et al. [90].

3.4.2 Price of Anarchy

the "Price of anarchy" is the name given by Roughgarden [92] to a measure of inefficiency associated with selfish behavior of Nash players in a network resource allocation games .
The price of anarchy (PoA) has been initially introduced by Koutsoupias and Papadimitriou [93] and extended by Papadimitriou [94] as a concept which helps to observe how poor a result can get in a competitive situation. It clearly proves how the result is worse in the situation, where the players act rationally and in their own interests, compared to the optimal situation in which the players agree or be forced to behave in a collaborative way.
Price of anarchy in differential user equilibrium is expressed by the ratio of total congestion arising from user equilibrium (worst-case objective function value of a Nash equilibrium of a game), and minimum total congestion arising from a system optimal traffic assignment.

Roughgarden and Tardos's famous paper "How bad is selfish routing?" includes great results which shows the price of anarchy in routing is upper bounded by 33%, when the linear latency assumption holds [92].

Later on, Papadimitriou has disused the disagreement of Roughgarden and Tardos's model with Internet reality. in this paper, they argued that nowadays, paths are not chosen by Internet flows, in fact, players direct the traffic based on local information. The paper, in fact, argues about what happens to address these issues [94].

On the other hand, significantly good bounds are introduced on price of anarchy in a wide range of applications. These discussions can be found in Nisan et al. survey in [95]. However, the necessary condition for these bounds is that the game's players should successfully reach a Nash equilibrium. This drawback motives Roughgarden [96] to introduce a general connection between the price of anarchy and its more general relatives. He has identified a sufficient condition for an upper bound of the price of anarchy of pure Nash equilibria.

To define the price of anarchy in user equilibrium, we first introduce the following notation for traffic assignment. In particular, we will employ the following by now familiar set notation

- $\mathcal{N}$ the set of nodes of the network
- $\mathcal{A}$ the set of arcs of the network

- $\mathcal{P}$ the set of paths of the network
- P_{ij} the set of paths connecting OD pair $(i, j) \in \mathcal{W}$
- $h \in \mathbb{R}_{+}^{\mathcal{P}}$ the vector of path flows
- $f \in \mathbb{R}_{+}^{\mathcal{A}}$ the vector of arc flows
- $\mathcal{Q} \in \mathbb{R}_{++}^{\mathcal{W}}$ the vector of fixed travel demands
- $c(f) \in \mathbb{R}_{++}^{\mathcal{A}}$ the vector of arc cost functions.
- $c(h) \in \mathbb{R}_{++}^{\mathcal{A}}$ the vector of path cost functions

In addition, we denote the network associates the graph $\mathcal{G}(\mathcal{N}, \mathcal{A})$ with the fixed vector of demands $\mathcal{Q}$ and the arc cost function vector $c(f)$, as $[\mathcal{G}(\mathcal{N}, \mathcal{A}), \mathcal{Q}, c(f)]$ or simply $[\mathcal{G}, \mathcal{Q}, c(f)]$. Now we are ready to define the price of anarchy for user equilibrium problems:

Definition 3.4.2. *(Price of Anarchy)*
The price of anarchy(PoA) for network $[\mathcal{G}, \mathcal{Q}, c(f)]$ is defined as follows:

$$p[\mathcal{G}, \mathcal{Q}, c(f)] = \frac{\sum_{a \in \mathcal{A}} c_a(f_a^{ue}) f_a^{ue}}{\sum_{a \in \mathcal{A}} c_a(f_a^{so}) f_a^{ue}}$$

$f^{ue} \in \omega_1$ is the user equilibrium flow vector and $f^{ue} \in \omega_1$ is the system optimal flow vector; where ω_1 define the set of feasible solutions.

Since the congestion arising from user equilibrium is always worse than the one resulting from system optimal, clearly the price of anarchy should never be less than unity.

3.4.2.1 Bounding the Price of Anarchy

There have been a lot of study on bounding the price of anarchy. The main question in literature is whether the price of anarchy may be bounded. Answering to this question may then express that the inefficiency associated with a Nash equilibrium is bounded.

Theorem 3.4.2. *(Price of anarchy for $[\mathcal{G}, \mathcal{Q}, c(f)]$ with separable linear arc costs)*
Consider $f^{ue} \in \omega_1$ and $f^{so} \in \omega_1$, as respectively, the user equilibrium flow vector and the

system optimal flow vector for the network $[\mathcal{G}, \mathcal{Q}, c(f)]$. Then the following bound exists on the price of anarchy for a user equilibrium:

$$p[\mathcal{G}, \mathcal{Q}, c(f)] = \frac{\sum_{a \in \mathcal{A}} c_a(f_a^{ue}) f_a^{ue}}{\sum_{a \in \mathcal{A}} c_a(f_a^{so}) f_a^{ue}} \leq \frac{3}{4}$$

For the proof please see [97]

3.5 Stackelberg Equilibrium

Stackelberg game was first presented by Stackelberg as a system with two types of players [65]. The leader on the upper level choses his/her own strategy first to optimize his/her own objective functions and the followers make decisions as the second mover in response to the leader. A Stackelberg equilibrium is obtained when the leader anticipates his/her strategy taking into account the best response of the followers.

Let $x \in \omega_F$ be the vector of the leader's strategy where ω_F is the leader's feasible strategy set. Also let $y = \phi(x)$ be the followers' strategy where ω_L is the feasible strategy set for followers. Therefore the bi-level Stackelberg can be introduced as follows:

$$\min \phi_L(x, y) \tag{3.60}$$

$$Subject\ to: \tag{3.61}$$

$$y = argmax\ \phi_F(x,\ y) \tag{3.62}$$

$$y \in \omega_F \tag{3.63}$$

$$x \in \omega_L \tag{3.64}$$

We denote the introduced Stackelberg equilibrium by $SE\ (\phi_L,\ \phi_F,\ \omega_L,\ \omega_F)$

3.5.1 Single Level Problem Formulation of Stackelberg Game

As we introduced earlier Stackelberg-Cournot-Nash games can be converted to a single level problem as nonlinear complementarity formulation. In such attempt the uniqueness of solution for the lower level problem must be guaranteed. Otherwise, the single level problem leads to the solution of lower level problem which is the most beneficial one for the upper level problem which is unlikely to happen in non-cooperative games.

3.5.1.1 Numerical Example

In this section we employ the method of converting the bi-level stachelberg problem to a single level problem and its solution. This example has been originally proposed and solved by Dockner and Jorgensen [98].

Consider the leaders' optimization problem as follows

$$J(v,\ x) = \int_0^1 [v(t) - (\frac{1}{2}(v^2(t) + x^2(t))]dt \tag{3.65}$$

And followers try to solve the following problem to optimize their own objective function

$$\begin{aligned}
&(u,\ x) = \int_0^1 [u(t) - \frac{u^2(t)}{2}] - \frac{x^2(t)}{2} dt \\
&Subject\ to: \\
&\frac{dx}{dt} = v(t) + u(t) \\
&x(\mathrm{O}) = 5
\end{aligned} \tag{3.66}$$

Let us write the Hamiltonian for the lower level as

$$H(u,\ x,\ \lambda,\ t) = u(t) - \frac{u^2(t)}{2} - \frac{x^2(t)}{2} + \lambda(t)[v(t) + u(t)] \tag{3.67}$$

Therefore, the necessary and sufficient conditions for the followers can be summarized as:

- Minimum principle

$$\nabla_u H = 0 \tag{3.68}$$

$$1 - u(t) + \lambda(t) = 0 \tag{3.69}$$

- Adjoint equations:

$$\frac{d\lambda}{dt} = -\nabla_x H = x(t) \tag{3.70}$$

- transversality conditions

$$\lambda(1) = 0 \tag{3.71}$$

- Dynamic

$$\frac{dx}{dt} = v(t) + u(t) \tag{3.72}$$

- Terminal condition

$$x(0) = 5 \tag{3.73}$$

Therefore, the bi-level problem has been converted into the following single level problem

$$Max\ J(v,\ x) = \int_0^1 [v(t) - (\frac{1}{2}(v^2(t) + x^2(t))]dt \tag{3.74}$$

$$Subject\ to:$$

$$1 - u(t) + \lambda(t) = 0 \tag{3.75}$$

$$\frac{d\lambda}{dt} x(t) \tag{3.76}$$

$$\lambda(1) = 0 \tag{3.77}$$

$$\frac{dx}{dt} = v(t) + u(t) \tag{3.78}$$

$$x(0) = 5 \tag{3.79}$$

Considering $(x(t),\ \lambda(t))$ as the pair of state variables, the single level problem can be solved by time-discrete approximation:.

$$MaxJ = \sum [v_t - (\frac{1}{2})(v_t^2 + x_t^2)] \tag{3.80}$$

$$Subject\ to$$

$$1 - u_t + \lambda_t = 0 \tag{3.81}$$

$$x_{t+1} = x_t + v_t + u_t \tag{3.82}$$

$$\lambda_{t+1} = \lambda_t + x_t \tag{3.83}$$

$$x_0 = 1 \tag{3.84}$$

$$\lambda_1 = 0 \tag{3.85}$$

3.5.2 Stackelberg Differential Games and Equilibrium Programs with Equilibrium Constraints

Stackelberg differential game models have been used to study sequential decision making in noncooperative games in various fields. Such games have mostly been used to study a bi-level dynamic interaction between the leader on the upper level and players act as followers on the lower level of the problem. In fact, the Stackelberg strategy is the optimal strategy for the leader on the upper level while the follower reacts by playing optimally.

The differential games was first introduced by Isaacs(1965). He applied such games in warfare and pursuit-evasion problems. More recently, Stackelberg differential games have been applied to the hierarchical or sequential decision making situations, and for which a reasonable solution concept is that of Stackelberg equilibrium.

There have been a wide literature on Stackelberg differential games in various fields. Such game has been discuses in many papers which studied conflicts and coordination issues associated with inventory and production policies, outsourcing, capacity , dynamic competitive advertising strategies, and dynamic competitive pricing and revenue management.

In general, there are two different information structures defining a Stackelberg equilibrium:

- Open-loop information structure:
  This information structure provides perfect and complete initial information. Solution from the start time t_0 to the end time t_f can be computed, independent of any feedback. In such structure, the players should be aware of the initial condition of the state at time zero in order to take their decisions at time t.

- Closed-loop information structure:
  Closed-loop games or feedback information structure involves the explicit consideration of feedback. In such structure we assume that the players use their knowledge of the current state at current time t in order to take their decisions at time t.

Recent applications of Stackelberg differential game models can be extensively found in supply chain management, marketing, operations management, economics, pricing and revenue management.

X He, et al. have a complete study on application of Stackelberg differential game in supply chain management and marketing channels as a decentralized channels consist of a manufacturer and independent retailers. They considered a sequential decision procedure with demand and supply dynamics and coordination issues. [99]. Also, there are number of papers and books, which provide surveyed applications in supply chain management and marketing, (e.g., Jorgensen [100], Feichtinger et al. [101] and Dockner et al. [102]).

Nerlove and Arrow has been published a paper in the marketing area, where the demand dynamics are usually advertising capital models [103]. They proposed a model for a situation in which, present advertising expenditures affect the future demand for the product. They have also introduced the necessary conditions for a maximum of the present value of future net revenues under some assumptions. In addition, there are papers such as sethi's [104] which considered the demand dynamics usually as sales-advertising response models. In this paper, sales-advertising response is defined as a direct correlation between the rate of change in sales and advertising. Also, the stochastic version of the problem is considered in this paper in which the dynamics is an Ito equation.

Eliashberg and Steinberg [105] and Pekelman have papers on Stackelberg differential game in which the manufacturer acts as a leader and decides about the wholesale price and/or production rate, while the retailers act as followers deciding about the retail price or shelf-space allocation. Eliashberg and Steinberg's paper includes a two-part processing strategy as well as two-part pricing strategy for the distributor. The manufacturer's policies are also following a two-part production policy. Moreover, Pekelman applied a general time-varying demand function with parameters as the market potential, price sensitivity, and the distributor's price.

A different paper is presented by Desai [106] in which a quadratic holding cost function is assumed for a manufacturer who produces the goods and sells them through a retailer within a network. In his model, manufacturer is allowed to update the wholesale price over time, while, the retailer is not allowed to carry inventory. Manufacturer's decision is on the production and pricing decisions while the retailer decides on the processing rate and pricing strategies. They also extended the paper in (1996) by assuming that retailer would process the goods after receiving them from the manufacturer and before selling them in the market. [107]

In more recent studies, He and Sethi [108] studied pricing and slotting decisions to find out the impact of promotional devices such as shelf space allocation on retail demand. The assumption is that the retail demand is a concave function which is increasing on the shelf-space of merchandise displayed on the shelf.

Also a supply chain with manufacturer on the upper level (leader) and a retailer on the lower level (follower) has been considered by kogan et al. [109]. They introduced a time-dependent endogenous demand which depends on the price strategies taken by the retailer. Finite processing capacity is assumed for the retailer, which requires consideration of the effect of inventory. They also extended the work to SDG between a supplier (the leader) with the wholesale price as his/her decision and a retailer (the follower) with a retail product price as his/her decision. The authors incorporate learning process when the supplier's production cost declines as his/her produces more units.

In another category researchers discus the necessary conditions for optimality of the Stackelberg differential game. Shell [110] have questioned the necessity of a transversality condition for optimality in infinite horizon. In addition, Weitzman has challenged such conditions in discrete time [111] and Benveniste and Scheinkman in continuous time [112]. Similarly, Danyang Xie, et al. have pointed out a problem of the necessity for optimality of one boundary condition in the existing literature. The boundary condition is necessary for the time inconsistency results which exist in the literature, however, They have found that such condition is not necessary in some cases which undermines the validity of the existing conclusions [113].

Moreover, Stackelberg differential game has been used to model the interaction of the government as the leader and private agents as the followers. The government on the upper level often set monetary policies while private players, responding optimally to government's policy in their decision, investment, labor supply and so on. In this case the leader takes the private agents' best response and decide about his/her own optimal strategy. Examples of this topic can be found in researches by Kydland and Prescott [114], Turnovsky and Brock [115], Lucas and Stokey [116] and Persson and Svensson [117]. Also, Simaan has published a paper on necessary and sufficient conditions for its existence of the Stackelberg solution in static and dynamic nonzero-sum two-player games [118]. In this paper, Several situation for the players are investigated and necessary and sufficient conditions for the existence are determined. In other words, these cases include the ones in where one of the two layers is not aware of the other's performance criterion or games with different speeds in computing the strategies. In fact the main question of their research is the following: what will be the best policy to choose, if a player has to announce his/her policy first?

The following examples from [118], presents the basic idea and related properties of the Stackelberg strategies:

3.5.2.1 Example

Let us consider the simple matrix game shown in figure 3.1. Assume that the leader P2 seeks to select a tax rate from the following set of possible strategies: $\{a1\%, a2\%, a3\%\}$ for taxing a certain firm P1. And consider P1 firm as a follower who tries to decide on manufacturing decision from the strategy set $\{v1, v2, v3\}$ of the products that it can manufacture.Also, the objectives function representing net income for the firm and the leader. This objective function of the leader and the follower is quantitatively computed for every pair of tax rate and product variety by the entries in Fig 3.1.
Now, consider two players as P1 and P2 in Fig 3.1. Consider P2 as the leader while both P1 and P2 try to maximize their benefits, $J1$ and $J2$. Let us denote the strategy set of P1 as the set of pairs $S = \{(a1, v1), (a2, v2), (a3, v3)\}$. The Stackelberg strategy for the leader P2 is the element of D that maximizes $J2$, which are clearly pair $(a1, v1)$.
This shows that, selecting a tax rate of $a1\%$, by the upper leader would force the firm(follower) to manufacture the product $v1$. Thus, in this case, the leader P2 achieves more for $J2$ compared to the case in which $a2$ or $a3$ have been chosen instead. In addition, the Stackelberg strategy with $P1$ as a leader, $(a3, v2)$ and the Nash strategy $(a2, v3)$, are also easily computed. It is obvious that, in this example, both leaders in the Stackelberg solution achieve better results than Nash solution case and that the followers achieve a result.

		P2		
		a1	a2	a3
P1	v1	8,10	5,10	8,11
	v2	7,5	8,6	11,7
	v3	5,6	9,9	12,6

Figure 3.1. Basic idea and related properties of the Stackelberg strategies

3.5.3 Stability and Lyapunov's Theorem

Stability theory is a very old subject and gives this opportunity to conclude about the behavior of a system with no need to compute the direct solution trajectories. stability in the modern sense was first studied by Lagrange [119]. He used Lagrangian mechanics to analyze the mechanical systems and concluded that in the absence of external forces, the equilibrium of the mechanical system is stable. Today following definition is being used for stability

Definition 3.5.1. *Let* 0 *be an equilibrium point. The equilibrium point* 0 *is called stable if for every* $\epsilon > 0$ *and each* $t_0 \in R_+$ *there exist* $\delta(x) > 0$ *such that:*

$$x(\mathrm{O}) \in \beta(0,\ \delta)\ ,\ t_0 \geq 0 \rightarrow x(t) \in \beta(0,\ \epsilon)\ ,\ \forall t \geq t_0$$

The equilibrium is unstable if it is not stable. In addition other types of stability will be defined as follows:

Definition 3.5.2. *To extend definition 3.5.1 we define the equilibrium solution* 0 *asymptotic stable if* $\delta(x)$ *can be chosen to also satisfy* $x \rightarrow 0$ *as* $t \rightarrow 0$*.*

Definition 3.5.3. *The equilibrium solution* 0 *is exponentially stable for each* $t \in R_+$ *if there exist:*

$$\|x\| \leq me^{-\alpha(t-t_0)}\|x_0\|\ \forall x_0 \in \beta,\ t \geq t_0 \geq 0$$

The constant α *is called the rate of convergence*

Even though stabilizing nonlinear system is challenging, Lyapunov developed a theory on stability which has been the most popular and successful method so far [120]. Lyapunov functions introduced first by Aleksandr Lyapunov to prove stability of equilibrium nonlinear. Since there is not a standard approach for Lyapunov functions, it is still difficult to obtain it for general nonlinear systems. However, the idea is still being applied extensively in the literature of control for nonlinear systems [121].

Lyapunov theorem has been largely used by researches. Examples of this studies include: Letov [122], Kalman et L. [123], LaSalle et al. [124], Parks [125] and Kalman [126] who use this theorem in control and systems literature. In addition, Friesz et al. [127] presented a qualitative analysis of stability in static Wardropian user equilibria on congested networks with fully general demand and cost structures. James A. Primbs et al. [121] reviewed the use of the Lyapunov method in control systems which is known as a control Lyapunov function (CLF).They presented the use of CI⌟F as well as another approach, known as receding horizon control (RHC) to nonlinear control. Their comparison suggest that the strength of both methods are complementary to suggest ideas for control design.

Hassan K. Khalil [128] presented a complete review on Lyapunov's theory and how to use it in studying the stability of equilibrium points for time-invariant and time-varying systems modeled by ordinary differential equations. On the other hand, a complete review and presentation of Lyapunov stability can be found in chapter 5 of Sastry's book [129]. The book contains the theories needed to study the stability of a differential game(DVI). However, those subjects are not discussed directly in the book.

Other application of Lyapunov stability method can be found in traffic assignment problems [130].

Definition 3.5.4. *(Lyapunov's theorem)*

For the dynamic system

$$x(t) = f(x(t)) \tag{3.86}$$

let $x = 0$ *be the equilibrium point and* $D \subseteq R_n$ *be the domain of* f *containing* $x = 0$*. let* $V : D \to R$ *be a contentious differential function such that:*

$$V(\mathrm{O}) = 0, \text{ and } V(x) > 0, \in D - \{\mathrm{O}\}$$
$$\dot{V}(x) \leq 0 \in D$$

Then the equilibrium point is stable.

Further if $\dot{V}(x) < 0$ in $D - \{\mathrm{O}\}$ then the equilibrium point is asymptotically stable. in other words, the equilibrium point is stable if there is a continuous differential positive definite function $V(x)$ such that $\dot{V}(x)$ is negative $semidefinite_2$, and is asymptotically stable if $\dot{V}(x)$ is negative definite.

Proof. ([129])
Lyapunov's theorem can be proven by constructing a $\delta \geq 0$ *for a given* $\epsilon \geq 0$ *such that the trajectories starting in* $\beta(0,\ \delta)$ *does not leave* $\beta(0,\ \delta)$ *. Therefore the proof takes the following steps*:

- *We will construct set* ω_β *such that* $\omega_\beta \in \beta_\epsilon$ *and show positivity invariant by using properties of V*
- *Using continuity of* V *and* $V = 0$, *we deduce existence of* $\beta(0,\ \delta)$ *in* ω_β

Proof of the first step

For given $\epsilon \geq 0$, *if* β_ϵ *is contained in* D, *then the Lyapunov function of all points on boundary is strictly positive because based on definition for* V, V *is positive everywhere else than zero. If we consider* $\alpha : \mathrm{Min} V(x)$ *on* $||x|| = \epsilon$ *Then* $\alpha > 0$.

Let us choose $\beta \subset (0,\ \alpha)$ *and define* $\omega_\beta = \{X \in \beta_\epsilon | V(x) \leq \beta\}.\omega_\beta$ *is the interior of* β_ϵ *and any trajectory starts in* ω_β *for* $\forall t \geq 0$, *will stay within it.*

Proof of the second step

We need to find a $\delta > 0$ *such that* $\beta_\delta < \omega_\beta$. *We know* $V(x)$ *is continuous, therefore, for every* $\beta > 0$ *there exist a* $\delta > 0$ *such that*

$$\begin{aligned} x \in \beta_\delta \rightarrow |V(x) - V(0)| < \beta \\ \rightarrow x \in \beta_\delta \rightarrow |V(x)| < \beta \end{aligned}$$

Lyapunov theorem's conditions are only sufficient. Failure of finding a function to satisfy such conditions does not mean the equilibrium is not stable. It perhaps means that stability cannot be established by using these Lyapunov's function candidate.

Finding the Lyapunov's function is very complicated and challenging for the introduced multi carrier-shipper problem. However, we partially developed the following stability results of our problem □

Chapter 4
Examples of static and dynamic equilibrium problems

4.1 Static Models

4.1.1 Static Spatial Price Equilibrium Problem

It is generally accepted that accurate models of transportation flow would use elastic supply and demand function for each commodity at each market. Also transportation cost function depends on the commodity flow on each link of the network in order to represent the traffic congestion. Spatial price equilibrium problem refers to types of problem in which origin-to-destination transportation demand would be considered as a by-product of the spatial price equilibrium solution rather than articulated in the model [131].

Assume spatially separated markets producing and consuming a single homogeneous product. When demand is more than the market supply, the market tries to import commodities from other markets. In addition, if demand is below the supply for a market, the market will try to export commodity to other markets. Markets keep importing and exporting in different situations until they reach an equilibrium. Equilibrium occurs when the local market price is equal to the price of any import at the latter's market of origin plus the unit cost of the transportation between the two market.

Let us review the mathematical situation of equilibrium in the following compact form. Assume the following definitions:

- i, j, m, l = Nodes of the network
- P_{ij} = Available paths goes from node i to node j

- $p \in P_{ij}$ = A path goes from node i to node j
- h_p = The commodity flow on path p
- $\beta_p(h)$ = The average transportation unit cost on path p
- γ_l = The price of a single commodity at market l
- D_l = The l^{th} market's demand
- S_l = The l^{th} market's supply
- $\Pi_l(D_l)$ = The inverse demand function for l^{th} market
- $\theta_l(S_l)$ = The inverse supply function for the single commodity for l^{th} market
- $A(l)$ = Set of all arcs with tail as node l
- $B(l)$ = Set of all arcs with head as node l

Now we call any solution (γ, h), a spatial price equilibrium while it satisfies conditions (i)-(iv).

1. Non-negativity of price and flows:

$$h_p \geq 0 \quad \forall i, j, p \in P_{ij}$$
$$\gamma_l \geq 0 \quad \forall i$$

2. Local prices and final delivered price should be equal for non-trivial flows

$$h_p > 0, p \in P_{ij} \rightarrow \gamma_i + \beta_p = \gamma_j \quad \forall i, j, p \in P_{i,j}$$

3. Trivial flows for delivered prices which are above the local prices

$$\gamma_i + \beta_p > \gamma_j, p \in P_{ij} \leftarrow h_p = 0 \quad \forall i, j, p \in P_{i,j}$$

4. flow conservation

$$K_l(\gamma_l, h) = D_l - S_l + \sum_m \sum_{p \in P_{lm}} h_p - \sum_m \sum_{p \in P_{lm}} h_p = 0 \quad \forall l$$

One should note that the traditional demand could be calculated from the formula $\sum_{p\in P_{ij}} h_p = Q_{ij}$ where Q_{ij} is the transportation demand between node i and node j. For more details about these conditions please see [131]

4.1.1.1 The Equivalent Optimization Problem

The equivalent optimization problem of the Spatial price equilibrium problem can be considered as follows. In such problem firms seek to minimize the cost by optimizing flow, f of the commodity, demand,Dd and supply,S. The notation introduced in last section are used in the following optimization problem.

$$
\begin{aligned}
Min \quad & \sum_a \int_0^{f_a} \beta_a(x_a)dx_a - \sum_l \int_0^{D_l} \Pi_l(y_l)dy_l + \sum_l \int_0^{S_l} \theta_l(y_l)dy_l && (4.1)\\
& Subject\ to: && (4.2)\\
& K_l = D_l - S_l + \sum_{a\in A(l)} f_a - \sum_{b\in B(l)} f_b = 0 \quad \forall l && (4.3)\\
& f \geq 0 && (4.4)\\
& D \geq 0 && (4.5)\\
& S \geq 0 && (4.6)
\end{aligned}
$$

Where f_a is the flow of the single commodity on arc a, h is the full vector of path flows in network:

$$f_a = \sum_i \sum_j \sum_{p\in P_{ij}}$$

And

$$\sigma_{ap} = \begin{Bmatrix} 1 \;\; \text{if arc } a \text{ is on path p} \\ 0 \;\; \text{otherwise} \end{Bmatrix}$$

Also the objective function is in such a way that meet all the conditions mentioned for spatial price equilibrium in section 4.1.1. We will refer to the mathematical model 4.6 as an equivalent optimization problem of spatial price equilibrium or EOP for short.

4.1.2 Static Classic Plant/Warehouse Location Problem

Locating a new plant along with producing and shipping product to markets on a network usually causes reactions from other markets on the network. One of the consequences is that the new plants increase the capacity of the overall market and then unbalance the equilibrium of supplies, demands and flows. In this case, competitors would react to new capacity of a new market or to an completely new competitor on the network. Therefore, we can say that the existing equilibrium of the market would be affected by the decision that the firms make for their new location. Therefore an optimal location decision is necessary to be taken to maximize the profit while anticipating the reaction of the competitors in regard to the location decision.

In general, location model is about competitive facilities when it explicitly incorporates the fact that there are other existing facilities on the network and that the new facility(ies) will have to compete with them to get market share.

The time at which a firm establishes a facility at a node is the location decision. Hence, the location problem is a mathematical programming with integer variables. The formulation requires the fixed cost of a facility as a function of the "start time." Such a notion has been used in the economics literature a lot.

In this section we introduce the classic static plant/warehouse location problem and we introduce the dynamic notation of such problem in the next sections. By classical model, we mean one for which the locating firm is either a price taker or a monopolist, so that the analysis of competition is especially simple

4.1.2.1 Uncapacitated Plant Location

formulation of the plant location problem on a network without production capacity constraints has been first introduced by Balinski [132]. This formulation assumes no shipping congestion and a single homogeneous output. In addition, locating decision by the firm have no ability to impact price, because the firm is small relative to the market. The firm seeks the minimum cost of location, production and transportation while ensuring that the demand constraint is met.

location problem formulation deals with a subset of possible candidates for opening new facilities or increasing capacity of the existing facilities. In this section we introduce a simple plant location formulation which satisfies the demand of a single commodity for a set of customers with predefined demand for the commodity. We assume that the capacity of each facility is unlimited. This optimization model seeks to minimize the production/transportation cost while selecting the optimal solution for number of facilities, their locations and amount shipped by each facility to each customer.

We borrow most of our notations from Krarup [133]. Assume the following notations

- i: The number of facilities, $i \in 1, ..., I$
- j: Number of customers, $j \in 1, ..., J$
- p_i: Facility i unit cost, production cost and administrative cost
- F_i: Fixed cost to locate facility i
- D_j: Demand of customer j
- s_{ij}: Unit transportation cost from facility i to facility j

Also

- y_i: 1 if facility i is open and o otherwise
- X_{ij}: The fraction of produced units at facility i shipped to customer j

There is zero cost to send no unit from the facility, (i.e. the facility is closed) while there is a fixed cost F_i when sending positive shipment from the i^{th} facility plus costs $p_i + s_{ij}$ per unit produced at facility i^{th} and shipped to facility j^{th}. The formulation is as follows:

$$
\begin{aligned}
& min \ \sum_i \sum_j (p_i + s_{ij}) X_{ij} + \sum_i F_i y_i \\
& subject \ to: \\
& \sum_i X_{ij} = 1, \quad j \in 1, ..., J \\
& M_i y_i - \sum_j X_{ij} \geq 0, \quad i \in 1, ..., I \\
& X_{ij} \geq 0, \quad i \in 1, ..., I, j \in 1, ..., J \\
& y_i \in 0, 1, \quad i \in 1, ..., I
\end{aligned}
\tag{4.7}
$$

The objective function clearly seeks to minimize the cost of location, production and transportation. The second constraint ensures that the total fixed cost for a facility happens when a non-zero shipments are made. M is a positive constraint, which is less than the maximum outflow from the corresponding facility. If $p_i \geq 0$ and $s_{ij} \geq 0$ then there is no need for the facility to send more than the total demand and then Ms might be replaced by $\sum_j D_j$. In addition, in the case than $p_i + s_{ij} \geq 0$, then never larger amount of shipment than the demand would be shipped to customers so the inequalities can be replaced by equations.

4.1.2.2 Capacitated Plant Location

Capacitated Plant Location assumes certain capacities and physical limits for facilities. This constraint can dramatically affects the optimal location of the plants. Davis and Ray [134] introduced a version of the capacitated plant location problem on a network which has been widely used in the literature.

The formulation is identical to that of the uncapacitated problem described in last section, however it includes the constraint for maximal limit of capacity.

The following notation along with the ones used in uncapacitated problem will be used in such problems:

- L_i: The capacity limit for facility i
- N_{ij}: Number of produced units at facility i shipped to customer j

$$
\begin{aligned}
& min \quad \sum_i \sum_j (p_i + s_{ij}) N_{ij} + \sum_i F_i y_i \\
& subject\ to: \\
& \sum_i N_{ij} \geq D_j, \quad j \in 1, ..., J \\
& \sum_j N_{ij} \leq L_i, \quad j \in 1, ..., J \\
& M_i y_i - \sum_j N_{ij} \geq 0, \quad i \in 1, ..., I \\
& N_{ij} \geq 0, \quad i \in 1, ..., I, j \in 1, ..., J \\
& y_i \in 0, 1, \quad i \in 1, ..., I
\end{aligned}
\tag{4.8}
$$

The second constraints assures that all demand are satisfied, while the third one prevents to send more shipment than the plant capacity at each plant site.

4.1.2.3 Plant Location with Elastic Demands

The models introduced in previous sections clearly assume that demand is fixed and inelastic. There have been some normative plant location models in the literature that consider a more realistic case in which the demand is price-sensitive (i.e. Wagner and Falkson [135], Hansen and Thisse [136] and Erlenkotter [137]). We use Hansen and Thisse's plant location mode to describe such setting:

Assume the following notations:

- i: The number of facilities, $i \in 1, ..., I$
- j: Number of customers, $j \in 1, ..., J$
- N_{ij}: Number of produced units at facility i shipped to customer j
- D_j: Demand of customer j
- $\theta_j(D_j)$: Inverse demand function
- s_{ij}: Unit transportation cost from facility i to facility j
- p_i: Facility i unit cost, production cost and administrative cost
- F_i: Fixed cost to locate facility i

$$
\begin{aligned}
& Max \quad \sum_j \theta_j(D_j)D_j - \sum_i \sum_j (p_i + s_{ij})N_{ij} - \sum_i F_i y_i \\
& Subject\ to \\
& \sum_i N_{ij} \geq D_j, \quad j \in 1, ..., J \\
& M_i y_i - \sum_j N_{ij} \geq 0, \quad i \in 1, ..., I \\
& N_{ij} \geq 0, \quad i \in 1, ..., I, j \in 1, ..., J \\
& D_j \geq 0 \quad j \in 1, ..., J \\
& y_i \in (0, 1) \quad i \in 1, ..., I
\end{aligned} \tag{4.9}
$$

The first term of the objective function shows the gross revenue received by the firm and the other terms are the transportation and production costs which should be minimized. One should note that the formulations presented here are based on either a price taking or a monopolistic firm. Therefore, all are single level in nature, which means there is not any distinction among network agents or access to information.

4.2 Dynamic Models

4.2.1 Dynamic Spatial Price Equilibrium

The spatial price equilibrium models has been used in variety of applications in the fields of agriculture, regional science, and energy markets. Many of the recent researches in model formulation and algorithm development in general, have considered static spatial price equilibrium problem (see, e.g.,Florian and Los [138], Friesz et al. [139], Friesz, Harker and Tobin [140], Pang[141, 142], Dafermos and Nagurney [143, 144]). However, there are some exemptions such as works from Takayama and Uri and Takayama, Hashimoto and Uri who considered the dynamic spatial price equilibrium.

Dynamic models can be viewed as static models if the associations of carry-over costs between time periods which were assumed fixed in static models are made. However, it is noted that the direct replication of static models over time might not be enough to express such important issues such as inventorying at supply and at demand markets and backordering [145].

In this section we present a generalized formation of dynamic spatial price equilibrium in network based on Takayama and Judge [146] temporal spatial price equilibrium model.

The time period is a discrete time $t \in [t_0,\ t_f] \subset \Re^1_+, t_0 \in \Re^1_{++}$ denotes the initial time and $t_f \in \Re^1_{++}$ denotes the terminal time. We denote a supply market by i and demand market by j while $i \in 1, ..., I$ and $j \in I+1, ..., I+J$. We assume that the commodity can be stored at both supply and the demand markets.

To define the mathematical representation of dynamic spatial price equilibrium we use the notations and formulation from Aronson and Chen [147]:

- S_{it} = Quantity of the commodity produced at supply market i in the t^{th} period
- D_{jt} = Demand for demand market j in the t^{th} period
- S = T-tuples of vector $S_1, ..., S_T$ of all supplies

- D = T-tuples of vector $D_1, ..., D_T$ of all demands
- x_{itjt} = Quantity of the commodity shipped from supply market i to demand market j at time t
- x_{itit+1} = Quantity of commodity stored at supply market i from time period t to $t+1$
- x_{jtjt+1} = Quantity of commodity stored at demand market j from time period t to $t+1$
- x_{itit-1} = Quantity of backorder stored at market i from time period t to $t+1$
- PS_{it} = Supply price for supply market i at time period t
- PD_{jit} = Demand price for demand market i at time period t
- PS = T-tuples of vector $PS_1, ..., PS_T$ of all supply price
- PD = T-tuples of vector $PD_1, ..., PD_T$ of all demand price
- c_{itjt} = Transportation price from market i to market j
- c_{itit+1} = Inventory cost at supply market i from time period t ro $t+1$
- c_{jtjt+1} = Inventory cost at demand market j from time period t ro $t+1$
- c_{jtjt-1} = Backordering cost at demand market j from time period t ro $t-1$
- a_{itjt} = Link Originating at a node it and terminating at a node $jt.$ the total number of transportation links in the network is IJT
- x_r = Flow on a path r by x,
- C_r = Associated cost on path r
- n_p = Number of path in the network
- P^{it} = The set of paths originating in supply market node it, $P^{it} \in 1, ..., n_{p^{it}}$
- $P_{jt'}$ = The set of paths terminating in demand market node jtâĂŹ, $P_{jt'} \in 1, ..., n_{p_{jt}}$
- $P_{itit'}$ = The set of paths originating in supply market node it and terminating in demand market node $jt\prime$, $P_{itit'} \in n_{p_{itjt'}}$

Also commodity shipment x_{itjt} will be grouped into a vector x_1 in $\mathcal{R}^{IJT}$, commodity stored at the supply market x_{itit+1} will be grouped into a vector x_2 in $\mathcal{R}^{\mathcal{I}(\mathcal{T}-\infty)}$, the quantities stored at the demand market, x_{jtjt+1}, into a vector x_3 in $\mathcal{R}_{J(T-1)}$ and the quantities back-ordered x_{jtjt-1} into a vector x_4 in $\mathcal{R}^{J(T-1)}$. Then x_1, x_2, x_3, x_4 will be grouped into a vector x in $\mathcal{R}^{IJT+I(T-1)+2J(T-1)}$. Similarly, transportation cost c_itjt will be grouped into a vector c_1 in $\mathcal{R}^{IJT}$, the supply market inventory cost c_{itit+1} will be grouped into a vector c_2 in $\mathcal{R}^{\mathcal{I}(\mathcal{T}-\infty)}$, the demand market inventory cost, c_{jtjt+1}, into a vector c_3 in $\mathcal{R}_{J(T-1)}$ and the backordering cost c_{jtjt-1} into a vector c_4 in $\mathcal{R}^{J(T-1)}$. Then c_1, c_2, c_3, c_4 will be grouped into a single vector c in $\mathcal{R}^{IJT+I(T-1)+2J(T-1)}$.

In addition, A sequence of links which starts from supply market it and terminates in demand market $jt\prime$ builds a path. A path from a supply market node to a demand market node is denoted by r. Then, the dynamic spatial price equilibrium conditions here are as follows:

$$PS_{it} + C_r = \left\{ \begin{array}{c} PD_{jt'} \quad \text{if } x_r > 0 \\ \geq PD_{jt'} \quad \text{if } x_r = 0 \end{array} \right\}$$

If we replace t' with t and makes the problem single period time period, then we reach the static price equilibrium conditions in which C_r is only the transportation cost. The amount of produced and consumed commodity must meet the following conditions:

$$S_{it} = \sum_{R \in p^{IT}} x_r \tag{4.10}$$

$$D_{jt'} = \sum_{r \in P_{jt'}} x_r \tag{4.11}$$

$$x_{itjt'} = \sum_r x_r \sigma_{(itjt')r} \tag{4.12}$$

Where

$$\sigma_{itjt'} = \left\{ \begin{array}{c} 1 \quad \text{if arc } itjt' \text{ is on path r} \\ 0 \quad \text{otherwise} \end{array} \right\}$$

And

$$C_r = \sum_{itjt'} c_{(itjt')r}$$

Also supply, demand, transportation, inventory and backorder cost should follow the below conditions:

$$PS = \hat{PS}(s)$$
$$PD = \hat{PD}(d)$$
$$c = \hat{c}(x)$$

where $\hat{PS}, \hat{PD}$ and $\hat{c}$ are known smooth functions.

4.2.2 Dynamic Plant/Warehouse Location and Differential Stackelberg-Cournot-Nash Competitive Facility Location

Static location problem has been already introduced in section 4.1.1. However, considering the influence of a location decision over a multi-period horizon represents an important capability. The multi-period planning horizon employs the proper timing of location decisions, in addition to the determination of the best location(s). In addition, it gives the firms a better opportunity to meet forecast growth or in general the market's demand over time. We describe the location decisions of a leader entering a competitive oligopoly in which spatially separated Cournot-Nash firms are connected by a transportation network and produce a homogeneous product. The models employed are fully dynamic and expressed as Stackelberg leader follower games constrained by differential variational inequalities.

4.2.2.1 Introductory Remarks

We imagine a price-setting Stackelberg firm with foresight that enters a competitive oligopoly in which spatially separated Cournot-Nash firms are connected by a transportation network and produce a homogeneous product. The times at which the Stackelberg firm establishes facilities at specific nodes of a shared transportation network constitute the location decisions of interest.

4.2.2.2 Market Dynamics

We start by letting i and j denote nodes of the transportation network and assume that all production and consumption occurs at such nodes, although some nodes may be pure transshipment nodes. We will also use $\mathcal{T}$ the set of Cournot-Nash firms, $\mathcal{W}_f$ the set of origin-destination pairs connecting production sites to consumption sites where firm $\mathcal{N}_f$ has a presence, and $\mathcal{N}_f$ the set of network nodes where firm $f \in \mathcal{T}$ produces and/or sells its output. Naturally, the set of all nodes $\mathcal{N}$ obeys

$$\mathcal{N} = \bigcup_{f \in \mathcal{F}} \mathcal{N}_f$$

and the set of all origin-destination pairs $\mathcal{W}$ obeys

$$\mathcal{W} = \bigcup_{f \in \mathcal{F}} \mathcal{W}_f$$

Furthermore, we take $I_i^f \in \Re^{|\mathcal{N}|}$ to be inventory held by firm $f \in \mathcal{F}$ at node $i \in \mathcal{N}_f$. The concatenation of all inventories held by firm $f \in \mathcal{F}$ is denoted by the vector $I^f \in \Re^{|\mathcal{N}|}$. Naturally the concatenation of firm-specific inventories is $I \in \Re^{|\mathcal{N}| \times |\mathcal{F}|}$. We also use the notation $\mathcal{P}_{ij}$ to denote the network paths that connect $(i,j) \in \mathcal{W}$. We will denote the set of all network paths by $\mathcal{P}$ and observe that

$$\mathcal{P} = \bigcup_{(i,j) \in \mathcal{W}} \mathcal{P}_{ij}$$

In order to describe the flow of output between production and consumption sites, we use s_{ij}^f to denote shipment pattern (flow) of firm $f \in \mathcal{F}$ between $(i,j) \in \mathcal{W}$. The vectors s_i^f and s^f are concatenations of the s_{ij}^f. Another piece of critical notation is q_i^f, which denotes production, expressed as a flow, by firm $f \in \mathcal{F}$ at node $i \in \mathcal{N}_f$. Naturally the concatenations of interest are $q^f \in \Re^{|\mathcal{N}|}$ and $q \in \Re^{|\mathcal{N}| \times |\mathcal{F}|}$. Note that

$$s_{ij}^f = \sum_{p \in \mathcal{P}_{ij}} h_p^f \quad \forall f \in \mathcal{F}, (i,j) \in \mathcal{W}$$

In light of the chosen notation we have the following dynamics of the market for the single commodity of interest:

$$\frac{dI_i^f}{dt} = q_i^f + \sum_{j:(j,i)\in\mathcal{W}_f} s_{ji}^f - \sum_{j:(i,j)\in\mathcal{W}_f} s_{ij}^f - c_i^f \quad \forall f \in \mathcal{F}, i \in \mathcal{N}_f \tag{4.13}$$

4.2.2.3 Nash Best Response Problem

The best response problem for each Cournot-Nash follower is based on the instantaneous profit of firm $f \in \mathcal{F}$ at node $i \in \mathcal{N}$ expressed as

$$\begin{aligned}\Phi_i^f\left(c_i^f, q_i^f, s_i^f; c_i^{-f}\right) \;&=\; \pi_i\left(c_i^f + \sum_{g\in\mathcal{F}-\{f\}} c_i^g\right)c_i^f \\ &\quad -V_i^f\left(q_i^f\right) - \sum_{j:(i,j)\in\mathcal{W}_f} r_{ij}s_{ij}^f - \Psi_i^f\left(I_i^f\right)\end{aligned} \tag{4.14}$$

where $\pi_i(\cdot)$ is the inverse demand function for the market at node $i \in \mathcal{N}$ and $\Psi_i^f(.)$ is the inventory holding cost at node $i \in \mathcal{N}$. The notation c_i^g will be used for consumption, at node $i \in \mathcal{N}$, of the output of firm $g \in \mathcal{F}$, while the consumption vector for firm $g \in \mathcal{F}$ is the obvious concatenation c^g. Moreover, concatenation of the c^g gives the complete vector of firm-specific nodal consumption rates c, while c^{-f} denotes the vector of all consumption rates other than those associated with $f \in \mathcal{F}$. We further define c_i^{-f} to be a vector of non-own (from the point of view of firm f) consumption flows at node i. We will use a similar notation, called q^{-f}, for non-own production and shipment flows. The vector s^{-f}

If the nominal discount rate is a constant $\rho \in \Re^1_{++}$ then the present value of profits for firm $f \in \mathcal{F}$ across all its facilities may be expressed as

$$J_f\left(c^f, q^f, s^f; c^{-f}\right) = \sum_{i\in\mathcal{N}_f} V_i^f\left[T, I_i^f(T)\right] + \int_{t_0}^{T} dt \sum_{i\in\mathcal{N}_f} e^{-\rho t}\Phi_i^f\left(c^f, q^f, s^f; c^{-f}\right) \tag{4.15}$$

where t_0 and T are, respectively, the beginning and the end of the continuous-time planning horizon $[t_0, T] \subset \Re^1_+$. Also appearing in (4.15) is the salvage value $V_i^f(\cdot,\cdot)$ of the physical plant employed by firm $f \in \mathcal{F}$ at node $i \in \mathcal{N}_f$. Combining (4.15) with the dynamics (4.13) and adding upper and lower bounds gives the following optimal control problem faced by each firm $f \in \mathcal{F}$:

$$\max J_f\left(c^f, q^f, s^f; c^{-f}\right) \tag{4.16}$$

subject to

$$\frac{dI_i^f}{dt} = q_i^f + \sum_{j:(j,i)\in\mathcal{W}_f} s_{ji}^f - \sum_{j:(i,j)\in\mathcal{W}_f} s_{ij}^f - c_i^f \quad \forall i \in \mathcal{N}_f \tag{4.17}$$

$$Q_i^f \geq q_i^f \geq 0 \quad \forall i \in \mathcal{N}_f \tag{4.18}$$

$$S_{ij}^f \geq s_{ij}^f \geq 0 \quad \forall (i,j) \in \mathcal{W}_f \tag{4.19}$$

$$C_i^f \geq c_i^f \geq 0 \quad \forall i \in \mathcal{N}_f \tag{4.20}$$

Note that in the best response problem (4.16)-(4.20) Q_i^f, S_i^f, and C_i^f are respectively upper bounds on the output, shipping. and consumption rates for firm $f \in \mathcal{F}$ at node $i \in \mathcal{N}_f$. Firm $f \in \mathcal{F}$ employs the control variables c^f, q^f, and s^f and considers c^{-f} as fixed yet arbitrary for each instant of time $t \in [t_0, T]$. It will be useful to express the set of admissible solutions available to firm $f \in \mathcal{F}$ as

$$\Omega_f = \left\{ \begin{pmatrix} c^f \\ q^f \\ s^f \end{pmatrix} : (4.18), (4.19), (4.20) \text{ hold} \right\}$$

The best response problem (4.16)-(4.20) allows expression of a Nash game as: each firm $f \in \mathcal{F}$ maximizes the present value of its profits over a set of admissible solutions treating the decisions c^{-f}, q^{-f} and $s^{-f}$of other followers as fixed yet arbitrary. That is, for every $f \in \mathcal{F}$, we re-express (4.16)-(4.20) in the following concise form:

$$\max J_f \left(c^f, q^f, s^f; c^{-f} \right) \tag{4.21}$$

subject to

$$\frac{dI_i^f}{dt} = q_i^f + \sum_{j:(j,i)\in\mathcal{W}_f} s_{ji}^f - \sum_{j:(i,j)\in\mathcal{W}_f} s_{ij}^f - c_i^f \quad \forall i \in \mathcal{N}_f \tag{4.22}$$

$$\begin{pmatrix} c^f \\ q^f \\ s^f \end{pmatrix} \in \Omega_f \tag{4.23}$$

The simultaneous solution of each problem (4.21)-(4.23) constitutes the Nash game played by the oligopolistic who alone contest the market for their homogeneous product prior to entrance of the leader.

4.2.2.4 The Entering Firm

We now consider a single entering firm seeking to locate one or more production facilities at nodes of the transportation network that connects the oligopolistic followers. Its location criterion is maximization of its profits on the time interval $[t_0, T]$. Some additional notation is required. In particular, we define $\mathcal{N}_L$ to be the set of markets (nodes) contested by the leader. Additionally, we take $\mathcal{W}_L$ to the set of origin-destination pairs for which the leader employs contracted logistics services that do change over time. Naturally these sets obey

$$\begin{aligned} \mathcal{N}_L &\subseteq \mathcal{N} \\ \mathcal{W}_L &\subseteq \mathcal{W} \end{aligned}$$

Moreover, the entering firm knows that there are "fixed costs" associated with establishing a new production and or warehouse facility at node $i \in \mathcal{N}_L$. These costs actually depend on the time at which the facility is ready to process orders, and we shall use $F_i(t)$ to denote the cost for the leader to establish a new facility at node $i \in \mathcal{N}_L$ at time $t \in [t_0, T]$. Next we introduce additional new control variables that express the decision of whether to establish a new facility; they are

$$y_i^L(t_i) = \begin{cases} 1 & \text{if a facility at node } i \in \mathcal{N}_L \text{ placed into service at time } t_i \in [t_0, T] \\ 0 & \text{otherwise} \end{cases}$$

The instants of time t_i are those at which the leader's facility at node $i \in \mathcal{N}_f$ is put into service. Initially we will treat the y_i^L as continuous variables obeying the constraints

$$0 \leq y_i^L(t_i) \leq 1 \quad \forall i \in \mathcal{N}_f$$

However, we will use of the following alternative formulation of the binary constraints:

$$\left[y_i^L(t_i) - 1\right] y_i^L(t_i) = 0 \quad \forall i \in \mathcal{N}_f \tag{4.24}$$

We must also insist that no production occurs for a facility that has not yet been located. To that end, we impose the following constraints:

$$q_i^L(t) \leq M y_i^L(t_i) \qquad \forall i \in \mathcal{N}_L, t \geq t_i \tag{4.25}$$

If the location variables have the binary character we intend, then constraints (4.25) will assure the consistency of location-output-consumption decisions related to the leader.

We also take c_i^L to be the consumption at node $i \in \mathcal{N}$ of the leader's output, while q_i^L is the output flow of the leader for node $i \in \mathcal{N}_L$. Concatenation of the aforementioned controls yields the vectors c^L and q^L. Likewise, we take y^L and C^L to be the obvious concatenations of the y_i^L and C_i^L, respectively, for all $i \in \mathcal{N}_L$. The vector s_i^L arises from the appropriate concatenation of the scalar controls s_{ij}^L; naturally concatenation of the s_i^L gives the shipping pattern s^L. As an obvious extension of our prior notation, we let c_i^{-L} be the vector of non-leader consumption at node $i \in \mathcal{N}_L$. Similarly we employ the notation s_i^{-L} for the vector of non-leader shipping flows originating from node $i \in \mathcal{N}_L$ node i.

Exploiting the notation introduced immediately above, we express the leader's instantaneous profits at node $i \in \mathcal{N}_L$ as

$$\begin{aligned} \Phi_i^L\left(c^L, q_i^L, s_i^L; c_i^{-L}\right) &= \pi_i\left(c_i^L + \sum_{g \in \mathcal{F}} c_i^g\right) c_i^L \\ &\quad - V_i^L\left(q_i^L\right) - \sum_{j:(i,j)\in\mathcal{W}_L} r_{ij} s_{ij}^L - \Psi_i^L\left(I_i^L\right) \end{aligned} \tag{4.26}$$

where the leader's variable costs $V_i^L(\cdot)$ and holding costs $\Psi_i^L(\cdot)$ are trivial extensions of previously introduced notation. The leader maximizes its own profits

$$\begin{aligned} J_L\left(c^L, q^L, s^L, y^L; c^{-L}\right) &= \\ &+ \int_{t_0}^{T} dt \sum_{i\in\mathcal{N}_L} \left[e^{-\rho t}\Phi_i^L\left(c^L, q^L, s^L; c_i^{-L}\right)\right] - \int_{t_0}^{T} dt \sum_{i\in\mathcal{N}_L} F_i(t_i)\, y_i^L(t_i) \end{aligned} \tag{4.27}$$

The pertinent dynamics and constraints for the leader are

$$\frac{dI_i^L}{dt} = q_i^L + \sum_{j:(j,i)\in\mathcal{W}_L} s_{ji}^L - \sum_{j:(i,j)\in\mathcal{W}_L} s_{ij}^L - c_i^L \quad \forall i \in \mathcal{N}_L \tag{4.28}$$

$$q_i^L \leq M y_i^L \qquad \forall i \in \mathcal{N}_L \tag{4.29}$$

$$\left[y_i^L(t_i) - 1\right] y_i^L(t_i) = 0 \quad \forall i \in \mathcal{N}_f \tag{4.30}$$

$$Q_i^L \geq q_i^L \geq 0 \quad \forall i \in \mathcal{N}_L \tag{4.31}$$

$$S_{ij}^L \geq s_{ij}^L \geq 0 \quad \forall (i,j) \in \mathcal{W}_L \tag{4.32}$$

$$C_i^L \geq c_i^L \geq 0 \quad \forall i \in \mathcal{N}_L \tag{4.33}$$

Again we define an admissible set:

$$\Omega_f = \left\{ \begin{pmatrix} c^L \\ q^L \\ s^L \\ y^L \end{pmatrix} : (4.28) - (4.33) \text{ hold} \right\}$$

4.2.3 The Dynamic Nash Game for Oligopoly Pricing and Production Planning

In this section we review a version of the dynamic oligopolistic network competition problem due to [148]. We review oligopolistic firms in a network economy and in oligopolistic game-theoretic competition described by a Nash equilibrium. In this type of problem there are also multiple producers of a single abstract homogeneous commodity that is brought to market by the freight service providers. Such produces, acting as shippers, also constitute an oligopoly in the market. This problem includes a dynamics that represent the trajectories of inventories/backorder and correspond to flow conservation for each firm at each node of the network of interest.

The following notation will be used to express the mathematical formulation. We use the notation used in Friesz [148], to handle temporal considerations.

The time period is a continuous time $t \in [t_0,\ t_f] \subset \Re^1_+, t_0 \in \Re^1_{++}$ denotes the initial time and $t_f \in \Re^1_{++}$ denotes the terminal time. There are $f \in \mathcal{F}$ shippers which decide on their price to gain the maximum benefit out of the market. The network consists of $\mathcal{A}$ arcs, $\mathcal{N}$ nodes and $\mathcal{W}$ (OD) pairs. (OD) pairs will be denoted by $(i,\ j)$ too. Each carrier $k \in \mathcal{K}$ has the objective function of maximizing the revenue as the difference between benefit and cost while deciding about the production rates and shipment patterns. The necessary notations for this section's formulation will be as follows:

- ρ: The constant nominal rate of discount

- r_w: The freight tariff per unit flow
- $f \in \mathcal{F}$: Specific shipper
- Π_i^f: Price set on node i by firm f,
- V_i: variable cost of production for firm $f \in \mathcal{F}$
- ψ_i^f: Firm f's inventory cost at node i
- I_i^f: Inventory/backorder of firm f at node i.
- $\mathcal{D}_i(p_i^f + \gamma_{ji}^{f,k}; \mathrm{p}^{-f},\ \gamma)$: Demand on node i for given price p_i^f and transportation cost γ

Moreover, each firm controls production output rates:

- $q^f = (q_i^f : i \in \mathcal{N})$ as the production output rates for firm f
- $c^f = (c_i^f : i \in \mathcal{N})$ As the allocation of output to meet demand and
- $s^f = (s_{ij}^f : i, j \in \mathcal{W})$ as the shipping pattern for firm f

In addition inventories $I^f = (I_i^f : i \in N)$ are the state variables which is determined by the control variables q^f, c^f and s^f.

Each producer has the objective of maximizing net profit expressed as revenue less cost and taking the form of an operator acting on allocations of output to meet demands, production rates and shipment patterns. Each firm $f \in \mathcal{F}$, the net profit is:

$$\begin{aligned}\Phi_f(c^f, q^f, s^f, q^{-f}) = \int_{t_0}^{t_f} e^{-\rho t} \Bigg\{ &\sum_{i \in \mathcal{N}} \Pi_i(\sum_{f \in \mathcal{F}} c_i^f, t) \\ &- \sum_{i \in \mathcal{N}_{\{}} V_i^f(q^f, t) - \sum_{ij \in W_f} t_{ij}(t) s_{ij}^f - \sum_{i \in \mathcal{N}} \psi_i^f(I_i^f, t) \Bigg\} dt \end{aligned} \tag{4.34}$$

Where N is the set of nodes in the network. Note that $\Phi_f(p^f,\ q^f,\ s^f; p^{-f},\ q^{-f})$ is a functional that is completely determined by the controls p^f, q^f and s^f when p^{-f} and q^{-f} are non-own price and non-own production rates and taken as exogenous data by firm f. The first term of the functional $\Phi_f(p^f,\ q^f,\ s^f; p^{-f},\ q^{-f})$ in expression 4.34 is the firm's revenue; the second term is the firm's cost of production; the third term is the firm's shipping costs; and the last term is the firm's inventory or holding cost.

All consumption, production and shipping variables are non-negative and bounded from above; that is

$$\begin{aligned} P^f &\geq p^f \geq 0 \\ Q^f &\geq q^f \geq 0 \\ S^f &\geq s^f \geq 0 \end{aligned} \tag{4.35}$$

with $P^f \in \Re_{++}^{|\mathcal{F}|}, Q^f \in \Re_{++}^{|\mathcal{F}|}$ and $S^f \in \Re_{++}^{|\mathcal{W}_f|}$ above constraints are recognized as pure control constraints. In addition the terminal time inventory obeys the following constraint

$$I_i^f(t^f) \geq M_i^f \quad \forall f \in \mathcal{F}, i \in \mathcal{N}_f \tag{4.36}$$

where $M_i^f \in \mathcal{R}_{++}^1$ are exogenous.

Firm f solves an optimal control problem to determine its production q^f, allocation of production to meet demand c^f, and shipping pattern s^f, thereby also determining inventory I^f via dynamics we articulate momentarily–by maximizing its profit functional $\Phi_f(p^f,\ q^f,\ s^f; p^{-f},\ q^{-f})$ subject to inventory dynamics expressed as flow balance equations and pertinent production and inventory constraints. The inventory dynamics for firm $f \in \mathcal{F}$, expressing simple flow conservation, obey

$$\frac{dI_i^f}{dt} = q_i^f + \sum_{(j,i)\in\mathcal{W}} s_{ji}^f - \sum_{(i,j)\in\mathcal{W}} s_{ij}^f - c_i^f, \quad \forall i \in \mathcal{N}_f \tag{4.37}$$

$$I_i^f(t_0) = M_i^f \quad \forall i \in \mathcal{N}_f \tag{4.38}$$

$$I_i^f(t_f) = \tilde{M}_i^f \quad \forall i \in \mathcal{N}_f \tag{4.39}$$

where $M_i^f \in \Re_{++}^1$ and $\tilde{M}_i^f \in \Re_{+}^1$ are exogenous. Note that the transportation time for the flow of finished goods is not captured explicitly in the inventory dynamics, however it is accounted for implicitly in the freight rate (tariff) charged per unit of flow. Further, in addition to the terminal time inventory (state) constraints , the model is general enough to handle inventory constraints over the entire planning horizon $[t_0,\ t_f]$. For instance, non-negativity of the inventory (state) variables could be imposed to restrict firms from taking backorder. Naturally

$$\Omega_f = \{(c^f,\ q^f,\ s^f) :\ (4.35) - (4.39)\ hold\}$$

is the set of feasible controls. With the preceding development, we note that firm f's problem is: with the c^{-f} and q^{-f} as exogenous inputs, compute $c^{f,k}, q^f$ and s^f (thereby finding I^f) in order to solve the following extremal problem:

$$\left.\begin{array}{ll} \max & \Phi_f(c^f, q^f, s^f; c^{-f}, q^{-f}) \\ \text{subject to} & \left(c^f, q^f, s^f\right) \in \Omega_f \end{array}\right\} \forall f \in \mathcal{F} \tag{4.40}$$

also for all $f \in \mathcal{F}$. That is, each firm is a Nash agent that knows and employs the current instantaneous values of the decision variables of other firms to make its own non-cooperative decisions. As such, (4.40) is a differential Nash game.

Chapter 5
Mathematical Programming Algorithms of equilibrium problems

There are different algorithms for finite-dimensional mathematical programs, as well as the infinite-dimensional mathematical programs. In this chapter we review three categories of continuous-time algorithms for infinite-dimensional mathematical programming as well as proofs of convergence.

5.1 Steepest descent methods

In general steepest descent is applied to unconstrained optimization wherein we follow direction of the negative gradient when minimizing. Such algorithm can be used to solve infinite-dimensional mathematical programs too. In this section we consider infinite-dimensional mathematical programs of the following form:

$$\begin{aligned} &min\ J(u) \\ &Subject\ to: \quad u \in V \end{aligned} \tag{5.1}$$

Where V is a Hilbert space. This algorithm is also applicable to the optimal control problems:

$$\begin{aligned} &min J(u) = K[x(t_f), t_f] + \int_{t_0}^{t_f} f_0(x, u, t)dt \\ &Subject\ to: \end{aligned}$$

$$\frac{dx}{dt} = f(x,u,t)$$
$$x(t_0) = x_0$$

Also Hamiltonian is introduced as:

$$H(x,u,\lambda,t) = f_0(x,u,t) + \lambda^T f(x,u,t)$$

5.1.1 Steepest Descent Algorithm:

Step 0. Initialization
Pick $u^0(t) \in (L^2[t_0,t_f])^m$. Set K=0
Step 1. Find state trajectory. Use $u^k(t)$ and solve the state dynamics and the initial-value equations

$$\frac{dx}{dt} = f(x,u^0,t)$$
$$x(t_0) = x_0$$

Call this solution $x^k(t)$.
Step 2. Find adjoint trajectory Use $u^k(t)$ as well as $x^k(t)$ solve the following adjoint final-value problem

$$(-1)\frac{d\lambda}{dt} = \frac{\partial H(x^k,u^k,\lambda,t)}{\partial x}$$
$$\lambda(t_f) = \frac{\partial K[x(t_f),t_f]}{\partial x}$$

Let's call this solution $\lambda^k(t_f)$
Step 3. Find gradient Use $u^k(t), x^k(t)$ and $\lambda^k(t)$, and calculate

$$\nabla_u J(u^k) = \Big[\frac{\partial H(x^k,u^k,\lambda,t)}{\partial u}\Big]^T$$
$$\frac{\partial H(x^k,u^k,\lambda,t)}{\partial u} = \frac{\partial f_0(x^k,u^k,t)}{\partial u} + (\lambda^k)^T\frac{\partial f(x^k,u^k,t)}{\partial u}$$

Step 4. Update and apply stopping test For a sufficiently small step size θ_k, follow these steps and update

$$u^{k+1} = u^k - \theta_k \nabla_u J(u^k)$$

Check the stepping criterion and if it is met then the solution is

$$u^*(t) = u^{k+1}(t)$$

Otherwise put $k = k + 1$ and start begin from step 1

5.1.2 Convergence of the Steepest Descent Algorithm

Suppose the functional $J : V \to \mathcal{R}^1$ and suppose this fictional is convex and weakly bounded from below and has a well-defined gradient and also V is a reflective Banach Space. Take $\nabla J(u)$, as a uniformly continuous function. Let's determine the optimal step size θ_k based on

$$1 > \theta_k > 0 \tag{5.2}$$

$$d^k = -\frac{\nabla J(u^k)}{||\nabla J(u^k)||} \tag{5.3}$$

$$\frac{d}{d\theta} J(u^k + \theta_k d^k) = \langle \nabla J(u^k + \theta_k d^k), d^k \rangle = 0 \tag{5.4}$$

$$J(u^k + \theta_k d^k) \leq J(u^k + \theta d^k) \quad \forall \theta \in [0, \theta_k] \tag{5.5}$$

Then, if the condition

$$\lim_{\|u\| \to \infty} J(u) \to \infty$$

holds , then the steepest descent algorithm converges to a minimum u^* of J on V.
For proof please see [38]

5.2 projected gradient methods

In this section we will review projected gradient algorithm for the constrained infinite-dimensional mathematical program . Consider the infinite-dimensional mathematical program introduced in 5.1. And suppose that $J(u)$ is G-differentiable on U and that $V = (L^2[t_0, t_f])^m$. Also G-derivative of the functional $J(u)$ is well defined and allows the articulation of the first-order necessary for the optimal solution $u^* \in U$:

$$\delta J(u^*, \phi) = \langle \nabla J(u^*), u - u^* \rangle \geq 0, \quad \forall u \in U$$

Therefore it seems reasonable to drive some algorithms based on the gradient of the functional. However, because there is the constraint ($U = V$), we can not directly use the steepest descent algorithm. So we need an algorithm to modify the gradient direction when it points out of U in order to get an alternative feasible direction. Projected gradient methods can be serve as this alternative by using the minimum norm projection.

5.2.1 Structure of the Gradient Projection Algorithm

The update rule for the gradient projection algorithm is as follows:

$$u^{k+1} = P_U\Big[u^k - \theta_k \nabla J(u^k)\Big]$$

Where P_U shows the norm projection, k is the iteration and θ_k is the step size. We are now ready to review the steps of the gradient projection algorithm:

Step 0. Initialization Pick $u^0(t) \in (L^2[t_0, t_f])^m$. Set K=0

Step 1. Find state trajectory Use $u^k(t)$ and solve the state dynamics and the initial-value equations

$$\frac{dx}{dt} = f(x, u^0, t)$$
$$x(t_0) = x_0$$

Call this solution $x^k(t)$.

Step 2. Find adjoint trajectory Use $u^k(t)$ as well as $x^k(t)$ solve the following adjoint final-value problem

$$(-1)\frac{d\lambda}{dt} = \frac{\partial H(x^k, u^k, \lambda, t)}{\partial x}$$
$$\lambda t_f = \frac{\partial K[x(t_f), t_f}{\partial x}$$

Let's call this solution $\lambda^k(t_f)$

Step 3. Find gradient Use $u^k(t), x^k(t)$ and $\lambda^k(t)$, and calculate

$$\nabla_u J(u^k) = \frac{\partial H(x^k, u^k, \lambda, t)}{\partial u} = \frac{\partial f_0(x^k, u^k, t)}{\partial u} + (\lambda^k)^T \frac{\partial f(x^k, u^k, t)}{\partial u}$$

Step 4. Update and apply stopping test For a sufficiently small step sizeθ_k, follow these steps and update

$$u^{k+1} = P_U\Big[u^k - \theta_k \nabla J(u^k)\Big]$$

Check the stepping criterion and if it is met then the solution is

$$u^*(t) = u^{k+1}(t)$$

Otherwise put $k = k+1$ and start begin from step 1 To show the convergence of the algorithm, we need to define two other concepts.

Definition 5.2.1. *We suppose $J : V \to \mathcal{R}^1$ is a functional on V and V is a normed vector space. J is said to be coercive (or α-convex) if there is a real scalar $\alpha > 0$ such that*

$$J[(1-\theta)u + \theta v] \leq (1-\theta)J(u) + \theta J(v) - \frac{\alpha}{2}\theta(1-\theta)\|(u-v)\|^2$$

for all $u, v \in V$ and $\theta \in (0,1)$. The difference between these concept and convexity is that here α is nor zero.

Theorem 5.2.1. *If J is G-differentiable at every point of V, then J is coercive (α-convex) if and only if*

$$\delta J(u, u-v) - \delta J(v, u-v) \geq \alpha\|(u-v)\|^2 \quad \forall u, v \in V$$

Also, when the gradient of J is defined on V, then the right-hand side of the equation will be:

$$\langle \nabla J(u) - \nabla J(v), u - v\rangle \geq \alpha\|(u-v)\|^2$$

5.2.2 Convergence of the Gradient Projection Algorithm

Theorem 5.2.2. *Suppose the functional* $J : U \subset V \to \mathcal{R}^1$ *is coercive (*α*-convex) with* $\alpha > 0$ *and* $\nabla J(u)$ *is defined and met the Lipschitz condition.*

$$\|\nabla J(u) - \nabla J(v)\| \leq \beta \|u - v\| \quad \forall u, v \in U$$

Then the projection gradient algorithm converges to the minimum u^* *of* J *on* U *for the following fixed step size*

$$\theta \in \Big(0, \frac{2\alpha}{\beta^2}\Big)$$

For proof please see [38]

5.3 penalty function methods

Sometimes infinite-dimensional mathematical programs contains constraints which violates the special structure of the introduced algorithms. In these cases, the base approach is to penalize those constraints in the objective function. Therefore it is possible to convert a constrained optimization problem into an unconstrained problem whose solution eventually converges to the solution of the original problem.

5.3.1 Structure of the Penalty Function Algorithm

In the penalty function algorithm we follow the following step

$$u^k = arg\Big\{minJ_{\rho k}(u) = J(u) + \rho_k P(u)\Big\} \tag{5.6}$$

Where k is the iteration index and ρ_k is the penalty function multiplier which should be large enough and increasing. In the case that the stopping test is met then we can declare the following solution at iteration k:

$$u^*(t) = u^k(t)$$

Otherwise select $\rho_{k+1} > \rho_k$, set $k = k + 1$ and go to the step 5.6.

5.3.2 Convergence of the Penalty Function Method

Theorem 5.3.1. *Let's assume the following properties for J*

- *Weakly lower semicontinous*
- *bounded from below*
- $J(u) \to \infty$ *as* $\|u\| \to \infty$

Also assume that U is weakly closed. Then we can conclude that the penalty function method converges and every weak cluster point of sequence u_p^* *is an optimal solution of the the Penalty Function Method J.*

For proof please see [38]

Chapter 6

Dynamic optimization and Differential Multi-leader-Follower Games Applied to Linear Freight Service Pricing

6.1 Introduction and Literature Review

In the shipper-carrier problems, the upper level is a non-cooperative game among freight service providers as leaders, also subsequently referred to as carriers, who try to set the optimal transportation price while simultaneously seeking the optimal truck routing decisions in a dynamic urban network. On the lower level, several retailers, referred to as shippers, react to the transportation prices set by the upper-level leaders by finding the best freight service in order to maximize their own utilities. Since shipper's demand depends on the service introduced by the carriers and carriers decisions are impacted by the reaction of shippers, it is clear that the model should capture the behaviors of both types of decision makers simultaneously [149].

Static shipper-carrier problems have been widely used in the literature; however, there are comparatively fewer studies of the dynamic shipper-carrier model. Friesz et al. [150] were among the first researchers who developed a model considering both shippers and carriers. This model was extended by Friesz et al. [6] to include the case of simultaneous loading by shippers and carriers. In addition, Harker and Friesz (1986b) [151] introduced the spatial price equilibrium model. Xiao and Yang have done a research on a three agents-three layer problem which emphasizes on a nonlinear pricing. In this research the carriers and infrastructure companies are assumed to behave cooperatively while making coalitions and it is shown that in the case of nonlinear pricing, equilibrium flows can maximize the total profit [50]. Also, Fernandez et al. presented another model on multi-level shippers in which they make decision based on their expected level of service, existence and uniqueness of the model.

In a recent research by Mutlu and Cetinkaya [152], two channels are introduced to model the shipper-carrier problem. They introduce the centralized channel as the layer in which the system is seeking to maximize the total profit and the decentralized channel as the layer in which shippers and carriers are selfish to optimize their own profit. In addition, Lawphongpanich and Yin established a model on tolling, where the toll price follows a piecewise linear function to determine tolls user equilibrium distribution [51]. In a later study, Shah and Brueckner [153] presented a model in which the carriers compete by deciding about the price and amount of shipments.

Besides static shipper-carrier problems, there exist few studies on the dynamic shipper-carrier model too. Agrawal et al. [154] were among the first ones who presented a model on equilibrium of dynamic case of shipper-carriers problem, In their research an iterative variational inequality takes market equilibrium as the input from the carriers to the shippers. Also, Unnikrishnan et al. [155] presented a model on dynamic shippers-carriers problem with stochastic demand. The model decides about the optimal inventory capacity at transit nodes. There are three approaches introduced by Harker and Friesz (1986a) [151] to predict freight flows, the econometric model, the spatial price equilibrium model, and the freight network equilibrium model. The later approach is going to be disused and used in this dissertation.

In this section, we present the application of differential games to a class of predictive intercity freight models: freight network equilibrium models. We are interested in computing urban freight service prices when an oligopoly of freight service providers compete with one another to carry cargo for large and complex, dynamic network markets. We also consider multiple producers of a single abstract homogeneous commodity that is brought to market by the aforementioned freight service providers. Such produces, acting as shippers, also constitute an oligopoly in the market for their output of the single homogeneous good. For such an environment, we study dynamic freight service pricing from the point of view of multi-leader-follower games.

The problem we have described is an important one because it corresponds to circumstances through which most modern economies have passed some time during their evolution. Moreover, by making the competitive pricing for freight services computable, it is possible to quantify the benefits gained through regulated pricing. The resulting organizational behavior can be called an equilibrium problem with equilibrium constraints (EPEC). In that the decision environment we study is dynamic, we refer to the model we present below as a differential-EPEC, which we abbreviate as D-EPEC.

Notice that our model for the producers competition is neither Cournot nor Bertrand; see the note by Farahat and Perakis [156] for comparisons between the Cournot and Bertrand Nash games. Cournot equilibrium has been introduced by Sherali et al. [157]. This equilibrium considers $N + 1$ players when N of them are followers and Nash agents who selfishly optimize their objective function assuming that other agents' strategies are predetermined.

The other player is the leader who can predict followers' reaction and take optimal strategies based on that. Inspired by the general model by Judd [158], we introduced both price and production quantity as the control variable for a producer/shipper in order to better capture the firms capability of dynamically controlling their inventories. This type of modeling has been widely accepted in the literature, see the work by Adida and Perakis [159] for example

After proper time-discretization, a finite dimensional D-EPEC will be formulated, allowing us to conduct numerical experiments with diagonalization method and solve the MPEC sub-problem with off-the-shelf software package such as GAMS. Our numerical experiments will be conducted (1) to compare the benefits/disadvantages of the producing companies' decision in outsourcing the freight service; (2) to model the effects that regulations could bring to the freight service competition; and (3) to incorporate, more realistic, nonlinear pricing schedules, such as two-part or three part tariffs and premiums charged for priority services.

6.2 Model Description

The setting will be one for which several freight carriers, forming an oligopoly, provide freight services to producers of a single homogeneous good, when those firms constitute a different oligopoly in the market for their output. The freight service firms and the producers of the homogeneous good under consideration operate in a hierarchical decision environment that may be called a generalized Stackelberg game(GSG). The specific GSG we consider herein is one for which the transportation firms anticipate the reactions of the manufacturers to their pricing policies but the manufacturers do not anticipate the prices of freight services. As such the model studied below is an example of a so called equilibrium problem with equilibrium constraints (EPEC). Because the decision environment is explicitly dynamic the EPEC of interest is a differential EPEC, which we denote by writing D-EPEC. We illustrate the structure of our model with Figure 6.1.

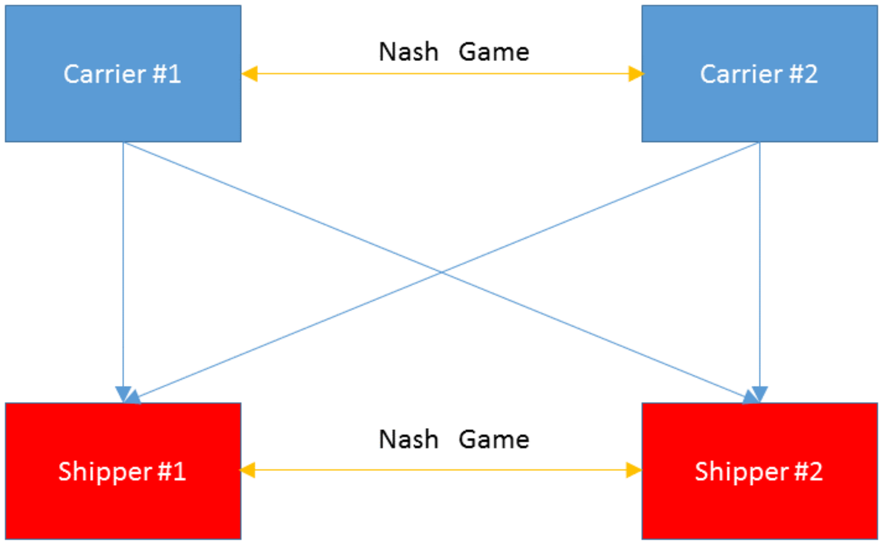

Figure 6.1. Illustration of the multi-leader-follower game by a 2-shipper 2-carrier example.

Furthermore the firms forming the product oligopoly of interest are embedded in a network economy and the trajectories of their inventories and back-orders are based on dynamics that correspond to flow conservation for each firm at each node of the network of interest. Within the network economy the networks nodes constitute local markets; points of inventory and transshipment; and producers manufacturing sites. The flows of goods between nodes are accomplished by freight service providers. The time scale we consider is neither short nor long, but rather of sufficient length to allow output and shipping pattern adjustments but not long enough for manufacturing firms to re-locate or enter or leave the network economy.

6.3 The Upper Level, Freight Service Oligopoly

6.3.1 Notations

The time period is a continuous time $t \in [t_0,\ t_f] \subset \Re^1_+, t_0 \in \Re^1_{++}$ denotes the initial time and $t_f \in \Re^1_{++}$ denotes the terminal time. There are $k \in \mathcal{K}$ number of carriers provide transportation service to $f \in \mathcal{F}$ shippers and decide on their price to gain the maximum benefit out of the market. The network consists of $\mathcal{A}$ arcs, $\mathcal{N}$ nodes and $\mathcal{W}$ (OD) pairs. (OD) pairs will be denoted by $(i,\ j)$ too. Demand function will be determined based on the shipper's response of the product price plus optimal selection of offered transportation prices. Therefor the objective function for each carrier $k \in \mathcal{K}$ is maximizing the revenue as the difference be- tween benefit and cost. The necessary notations for this section's formulation will be as follows:

- $\gamma_{i,j}^{k}(t)$ Dynamic freight service price
- h_{p}^{k} Routing decisions
- $p \in P$ Paths on the network
- ρ Constant nominal interest rate
- $c_{p}^{k}(t)$ Unit transportation cost function on path p for carrier k
- s_{ij}^{kf*} Shipment rate of the homogeneous commodity of interest between origin- destination pair $(i,\ j) \in \mathcal{W}$ carried by carrier k for shipper f

6.3.2 A Service Provider's Objective and Constraints

The carriers are freight service providers whose decision variables are the dynamic freight service price $\gamma_{ij}^{k}(t)$ and a series of routing decisions h_{p}^{k}, which denotes the flow of commodity on path $p \in P$. We do not need a freight service demand function; rather, freight demand is considered a derived demand derived from the shippers' best response that we subsequently describe. Each service provider has the objective of maximizing their profit, which is the different between the carrier's revenue and transportation cost. Taking $s_{ij}^{kf*}(t)$ as given, for each freight service providers $k \in \mathcal{K}$ the objective is

$$max J_k(\gamma^k,\ h^k; \gamma^{-k}) = \sum_{f\in\mathcal{F}} \sum_{(i,j)\in\mathcal{W}} \int_{t_0}^{t_f} e^{-\rho t}\gamma_{ij}^{k}(t) s_{ij}^{kf*}(t)dt - \sum_{(i,j)\in\mathcal{W}p} \sum_{\in P_{ij}} \int_{t_0}^{t_f} c_{p}^{k}(t)h_{p}^{k}(t)]dt \quad (6.1)$$

where ρ is the constant nominal interest rate, $L_{+}^{2}[t_0,\ t_f]$ is the space of nonnegative square-integrable functions, $\mathcal{W}$ is the set of origin-destination nodes for which freight service, $c_{p}^{k}(t)$ is the unit transportation cost function on path p for carrier k. Also s_{ij}^{kf*} is the shipment rate of the homogeneous commodity of interest between origin-destination pair $(i,\ j) \in \mathcal{W}$ carried by carrier k for shipper f. Naturally there should be a flow balance constraint for each OD pair:

$$\sum_{p \in P_{ij}} h_p^k(t) = \sum_{f \in \mathcal{F}} s_{ij}^{kf*}(t) \forall (i,\ j) \in \mathcal{W} \tag{6.2}$$

We also assume, based on market regulations on the carriers, that:

$$\underline{M}_{ij} \leq \gamma_{ij}^k \leq \overline{M}_{ij},\ \forall (i,\ j) \in \mathcal{W} \tag{6.3}$$

Thus, we define the feasible region for each carrier's policy

$$\Gamma_k = \{(\gamma^k,\ h^k)\ s.t.\ (6.2)\ and\ (6.3)\ hold\} \tag{6.4}$$

The carrier's best response problem is:

$$max J_k(\gamma^k,\ h^k; \gamma^{-k}) s.t.\ (\gamma^k,\ h^k) \in \Gamma_k \tag{6.5}$$

6.4 The Lower Level, Oligopoly of Producers

6.4.1 Notations

Notations for the shippers problem are as follows:

- $f \in \mathcal{F}$ Specific shipper
- q^f Production output rates for firm f
- $c^{f,k}$ Allocation of output produced by firm f, carried by firm k

- s^f Shipping pattern for firm f
- I^f Inventories for firm f as the state variables
- p_i^f Price set on node i by firm f,
- ψ_i^f Firm f's inventory cost at node i
- I_i^f Inventory/backorder of firm f at node i.
- $\mathcal{D}_i(p_i^f + \gamma_{ji}^{f,k}; \mathrm{p}^{-f},\ \gamma)$ Demand on node i for given price p_i^f and transportation cost γ
- $Z_f[I^f(t_f)\ ,\ t_f]$ Liquidation value of inventory remaining at the terminal time.

6.4.2 A Producing Firm's Objective and Constraints

In addition to the notation introduced in Section 6.1.3 relative the freight service oligopoly, we need to define a number of important decision variables and parameters critical to the mathematical articulation of the producers' oligopoly. In particular, each firm $f \in \mathcal{F}$ controls production output rates q^f, allocation of output to meet demand $c^{f,k}$ and shipping pattern s^f. Inventories I^f are state variables determined by the controls.

$$I(p,\ q,\ s) : (L^2[t_0,\ t_f])^{|\mathcal{N}|\times|\mathcal{F}|} \times (L_+^2[t_0,\ t_f])^{|\mathcal{N}|\times|\mathcal{F}|} \times (L_+^2[t_0,\ t_f])^{|\mathcal{W}|\times|\mathcal{F}|} \to (\mathcal{H}^1[t_0,\ t_f])^{|\mathcal{N}|\times|\mathcal{F}|}$$

where $L_+^2[t_0,\ t_f]$ is the space of nonnegative square-integrable functions and $\mathcal{H}^1[t_0,\ t_f]$ is a Sobolev space for the real interval $[t_0,\ t_f] \in \Re_+^1$

Notice that our model for the producers' competition is neither Cournot nor Bertrand, see the note by Farahat and Perakis [156] for comparisons between the Cournot and Bertrand Nash games. Inspired by the general model by Judd [158], we introduced both price and production quantity as the control variable for a producer/shipper in order to better capture the firms capability of dynamically controlling their inventories. This type of modeling has been widely accepted in the literature, see the work by Adida and Perakis [159] for example. Following the spirit of Bertrand competition, we are choosing price as the control variable for each producer in order to capture a pricing structure similar to the so called mill pricing (see for example [160]), where the delivered price equal to the sum of commodity pricing plus the price charged for shipping.

Each producer has the objective of maximizing net profit expressed as revenue less cost and taking the form of an operator acting on allocations of output to meet demands, production rates and shipment patterns. Taking the freight service rate offered by different carriers as exogenously given,for each firm $f \in \mathcal{F}$, the net profit is:

$$\Phi_f(p^f,\ q^f,\ s^f; p^{-f},\ q^{-f}) = e^{-\rho t_f} Z_f[I(t_f),\ t_f] \\ + \int_{t_0}^{t_f} e^{-\rho t}\{\sum_{k\in\mathcal{K}}\sum_{(j,i)\in\mathcal{W}} \mathcal{D}_i(p_i^f + \gamma_{ji}^{f,k}; \mathrm{p}^{-f},\ \gamma)\cdot p_i^f - \sum_{i\in\mathcal{N}} V_i^f(q^f,\ t) - \sum_{i\in\mathcal{N}} \psi_i^f(I_i^f,\ t)\}dt \quad (6.6)$$

Where p_i^f is the price set on node i by firm f, ψ_i^f is firm f's inventory cost at node i, and I_i^f is the inventory/backorder of firm f at node i. In addition, $\mathcal{D}_i(p_i^f + \gamma_{ji}^{f,k}; \mathrm{p}^{-f},\ \gamma)$ is the demand on node i for given price p_i^f and transportation cost γ (charged by carrier k for products by producer f to transport to node i). Also $Z_f[I^f(t_f)\ ,\ t_f]$ is the liquidation value of inventory remaining at the terminal time. We also introduce a short-hand notation that

$$\Psi_f(p^f,\ q^f,\ s^f; p^{-f},\ q^{-f}) \\ = e^{-\rho t}\{\sum_{k\in\mathcal{K}}\sum_{(j,i)\in\mathcal{W}} \mathcal{D}_i(p_i^f + \gamma_{ji}^{f,k}; \mathrm{p}^{-f},\ \gamma)\cdot p_i^f - \sum_{i\in\mathcal{N}} V_i^f(q^f,\ t) - \sum_{i\in\mathcal{N}} \psi_i^f(I_i^f,\ t)\}$$

Also $V_i^f(q,\ t)$ is the variable cost of production for firm $f \in \mathcal{F}$ at node $i \in \mathcal{N}$. Note that $\Phi_f(p^f,\ q^f,\ s^f; p^{-f},\ q^{-f})$ is a functional that is completely determined by the controls p^f, q^f and s^f when p^{-f} and q^{-f} are non-own price and non-own production rates and taken as exogenous data by firm f. The first term of the functional $\Phi_f(p^f,\ q^f,\ s^f; p^{-f},\ q^{-f})$ in expression (6.6) is the firm's revenue; the second term is the firm's cost of production; the third term is the firm's shipping costs; and the last term is the firm's inventory or holding cost.

All consumption, production and shipping variables are non-negative and bounded from above; that is

$$P^f \geq p^f \geq 0 \quad (6.7)$$

$$Q^f \geq q^f \geq 0 \quad (6.8)$$

$$S^f \geq s^f \geq 0 \quad (6.9)$$

with $P^f \in \Re_{++}^{|\mathcal{F}|}, Q^f \in \Re_{++}^{|\mathcal{F}|}$ and $S^f \in \Re_{++}^{|\mathcal{W}_f|}$ Constraints (6.7), (6.8) and (6.9) are recognized as pure control constraints.

Firm f solves an optimal control problem to determine its production q^f, allocation of production to meet demand $c^{f,k}$, and shipping pattern s^f –thereby also determining inventory I^f via dynamics we articulate momentarily–by maximizing its profit functional $\Phi_f(p^f,\ q^f,\ s^f; p^{-f},\ q^{-f})$ subject to inventory dynamics expressed as flow balance equations and pertinent production and inventory constraints. The inventory dynamics for firm $f \in \mathcal{F}$, expressing simple flow conservation, obey

$$\frac{dI_i^f}{dt} = q_i^f + \sum_{k\in\mathcal{K}} \sum_{(j,i)\in\mathcal{W}} s_{ji}^{kf} - \sum_{k\in\mathcal{K}} \sum_{(i,j)\in\mathcal{W}} s_{ij}^{kf}$$

$$\sum_{(j,i)\in\mathcal{W}} \sum_{k\in\mathcal{K}} \mathcal{D}_i(p_i^f + \gamma_{ji}^{f,k}; \mathrm{p}^{-f},\ \gamma) \forall i \in \mathcal{N}_f \tag{6.10}$$

$$I_i^f(t_0) = M_i^f \ \forall i \in \mathcal{N}_f \tag{6.11}$$

where $M_i^f \in \Re_{++}^1$ and $\tilde{M}_i^f \in \Re_{+}^1$ are exogenous. Note that the transportation time for the flow of finished goods is not captured explicitly in the inventory dynamics, however it is accounted for implicitly in the freight rate (tariff) charged per unit of flow. Further, in addition to the terminal time inventory (state) constraints (6.11), the model is general enough to handle inventory constraints over the entire planning horizon $[t_0,\ t_f]$. For instance, non-negativity of the inventory (state) variables could be imposed to restrict firms from taking backorder. Naturally

$$\Omega_f = \{(p^f,\ q^f,\ s^f):\ (6.7),\ (6.8),\ (6.9),\ (6.10)\ \mathit{and}\ (6.11)\ \mathit{hold}\} \tag{6.12}$$

is the set of feasible controls. With the preceding development, we note that firm f's problem is: with the c^{-f} and q^{-f} as exogenous inputs, compute $c^{f,k}, q^f$ and s^f (thereby finding I^f) in order to solve the following extremal problem:

$$\left.\begin{array}{ll}\max & \Phi_f(p^f, q^f, s^f; p^{-f}, q^{-f}) \\ \text{subject to} & \left(p^f, q^f, s^f\right) \in \Omega_f\end{array}\right\} \forall f \in \mathcal{F} \tag{6.13}$$

also for all $f \in \mathcal{F}$. That is, each firm is a Nash agent that knows and employs the current instantaneous values of the decision variables of other firms to make its own non-cooperative decisions. As such, (6.13) is a differential Nash game.

6.4.3 Necessary and Sufficient conditions of the Producers' Oligopoly

Following the spirit of (Friesz et al., 2006)and to prepare us for further (numerical) analysis, we here list the qualitative analysis on the lower level oligopoly by the producers. Note that in this section the freight service pricing schemes posted by the carriers are taken as exogenously given. Denote by H_f the Hamiltonian of (6.13) for any $f \in \mathcal{F}$:

$$\begin{aligned}\mathcal{H}_f &= -\Psi_f(p^f,\ q^f,\ s^f; c^{-f},\ q^{-f}) \\ &+ \sum_{i\in\mathcal{N}_f} \lambda_i (q_i^f + \sum_{k\in\mathcal{K}} \sum_{(j,i)\in\mathcal{W}} s_{ji}^{kf} - \sum_{k\in\mathcal{K}} \sum_{(i,j)\in\mathcal{W}} s_{ij}^{kf} - \sum_{(j,i)\in\mathcal{W}} \sum_{k\in\mathcal{K}} \mathcal{D}_i(p_i^f + \gamma_{ji}^{f,k}; \mathrm{p}^{-f},\ \gamma))\end{aligned} \tag{6.14}$$

We can quickly write out the necessary conditions for (6.13) for any $f \in \mathcal{F}$:

(i) Minimum principle:

$$(p^f,\ q^f,\ s^f) := \arg \min_{(p^f,q^f,s^f)\in\Omega_f} \mathcal{H}_f \tag{6.15}$$

(ii) Adjoint equations:

$$\frac{d\lambda_i^f}{dt} = -\frac{\partial \mathcal{H}_f}{\partial I_i^f} = -\frac{\partial \psi_i^f(I_i^f, t)}{\partial I_i^f} \tag{6.16}$$

(iii) Transversality conditions:

$$\lambda_i^f(t_f) = \frac{-e^{-\rho t_f} \partial Z_f[I(t_f), t_f]}{\partial I_i^f(t_f)} \tag{6.17}$$

(iv) Dynamics:

$$\frac{dI_i^f}{dt} = q_i^f + \sum_{k\in\mathcal{K}} \sum_{(j,i)\in\mathcal{W}} s_{ji}^{kf} - \sum_{k\in\mathcal{K}} \sum_{(i,j)\in\mathcal{W}} s_{ij}^{kf} \tag{6.18}$$
$$- \sum_{(j,i)\in\mathcal{W}} \sum_{k\in\mathcal{K}} \mathcal{D}_i(p_i^f + \gamma_{ji}^{f,k}; \mathrm{p}^{-f},\ \gamma)\ \forall i \in \mathcal{N}_f$$

In summary, the necessary conditions for the producers' competition consists of (i) – (iv) above. In addition, we assume this game is regular in the sense of the following:

Assumption 1
The dynamic oligopolistic network competition problem introduced above will be considered regular if:· (1) the state operator $I(c,\ q,\ s)$ exists and is unique, while each of its components is continuous and G-differentiable; (2) the demand, production cost and inventory cost functions are continuously differentiable with respect to controls and states,· and (3) for each $f \in \mathcal{F}$, the terminal cost function $Z_f[I^f(t_f)\ ,\ t_f]$ is continuously differentiable with respect to $I_i^f(t_f)$ for all $i \in \mathcal{N}_f$.

We introduce a short-hand notation

$$\Lambda := \{\lambda : \lambda_i^f(t_f) = \frac{-e^{-\rho t_f}\partial Z_f[I(t_f), t_f]}{\partial I_i^f(t_f)}, \tag{6.19}$$
$$\frac{d\lambda_i^f}{dt} = -\frac{\partial \psi_i^f(I_i^f, t)}{\partial I_i^f},\ \forall i \in \mathcal{N}_f,\ \forall f \in \mathcal{F}\} \tag{6.20}$$

Familiarity with variational inequalities suggests that the following variational inequality has solutions that are differential Nash equilibria for a non-cooperative game in which individual firms maximize net profits in light of current information about their competitors:

$$Find\ (p^{f*},\ q^{f*},\ s^{f*}) \in \Omega, \lambda^* \in \Lambda \tag{6.21}$$
$$Such\ that$$
$$0 \geq \sum_{f\in\mathcal{F}} \int_{t_0}^{t_f} [\sum_{i\in\mathcal{N}_f} \frac{\partial \mathcal{H}_f^*}{\partial p_i^f}(p_i^f - p_i^{f*}) + \sum_{i\in\mathcal{N}_f} \frac{\partial \mathcal{H}_f^*}{\partial q_i^f}(q_i^f - q_i^{f*})$$
$$+ \sum_{k\in\mathcal{K}} \sum_{(i,j)\in\mathcal{W}_f} \frac{\partial \mathcal{H}_f^*}{\partial s_{ij}^{kf}}(s_{ji}^{kf} - s_{ji}^{kf*})]dt$$
$$For\ all (p,\ q,\ s) \in \Omega, \forall \lambda \in \Lambda \tag{6.22}$$

where $\mathcal{H}_f^* = \mathcal{H}_f(p^{f*},\ q^{f*},\ s^{f*},\ \lambda^{f*}; p^{-f},\ q^{-f}; t)$, $\Omega = \Pi_{f\in\mathcal{F}}\Omega_f$.

We note that (6.21) and (6.22) is a differential variational inequality expressing the differential Nash game that is our present interest. The issue of immediate concern is to formally demonstrate that solutions of (6.21) and (6.22) are differential Nash equilibria.

In fact we state the following result:

1. Any solution of (6.21) and (6.22) is a solution of the dynamic oligopolistic network competition problem when regularity in the sense of Definition 5.1.1 holds.

2. When this variational inequality is regular in the sense of Definition 5.1.1, there exists a solution of the dynamic oligopolistic network competition problem.

6.5 Freight Pricing Oligopoly as a Multi-leader-follower Game

6.5.1 Mathematical Formulation of the EPEC

The proposed bi-level model of the freight pricing assumes that the carriers have knowledge on the optimal strategy of producers involved in the oligopolistic competition. Therefore, each of these carriers $k \in \mathcal{K}$ is solving the following mathematical programming with equilibrium constraints (MPEC) problem:

$$\begin{aligned} & max J_k(\gamma^k,\ h^k; \gamma^{-k}) \\ & s.t. (\gamma^k,\ h^k) \in \Gamma_k \cap \{(p^{f*},\ q^{f*},\ s^{f*})\ solves\ (6.21)\ and\ (6.22),\ \forall f\} \end{aligned} \tag{6.23}$$

For simplicity, we denote by $MPEC_k(\gamma^{-k*})$ the set of solutions of the k-th carrier, bearing in mind that h^k is internal to each carrier's decision. Assuming that the service carriers competes with each other in a Nash-like manner, the complete formulation constitutes an equilibrium problem with equilibrium constraints (EPEC), which is given as

$$\gamma^k \in MPEC_k(\gamma^{-k*}) \ , \ \forall k \in \mathcal{K} \tag{6.24}$$

Thus we have the following definition of freight price equilibrium:

Definition 6.5.1. *(The Freight Pricing Equilibrium)*
A tuple of pricing schemes $(\gamma^{1*}, \ \cdots, \ \gamma^{k*}, \ \gamma^{|\mathcal{K}|*})$ *is said to be a multi-leader-follower freight pricing equilibrium if (6.24) is solved for every leader* k *together with the corresponding lower level equilibrium* $(p^*, \ q^*, \ s^*)$ *. And we realized that the freight service equilibrium problem among the MPECs described above is an Equilibrium Problem with Equilibrium Constraints (EPEC).*

6.6 Reformulation and Algorithms

6.6.1 Reformulation of the lower level equilibrium

First of all, we notice that the lower-level problem can be represented by the following formulation involving differential complementarity constraints:

$$0 \leq -\frac{\partial \Psi_f}{\partial p_i^f} - \lambda_i^f \frac{\partial D_i(p_i^f + \gamma_{ji}^{f,k}; \mathrm{p}^{-f}, \gamma)}{\partial p_i^f} + \rho_{p_i}^f \perp p_i^f \geq 0, \ \forall i \in \mathcal{N}_f, \forall f \in \mathcal{F}, \ \forall k \in \mathcal{K} \tag{6.25}$$

$$0 \leq -\frac{\partial \Psi_f}{\partial q_i^f} + \lambda_i^f + \rho_{q_i}^f \perp q_i^f \geq 0, \ \forall i \in \mathcal{N}_f, \ \forall f \in \mathcal{F} \tag{6.26}$$

$$0 \leq -\frac{\partial \Psi_f}{\partial s_{ij}^{kf}} + \lambda_j^f - \lambda_i^f + \rho_{s_{ij}}^{kf} \perp s_{ij}^{kf} \geq 0, \ \forall (i, \ j) \in \mathcal{W}_f^k, \forall f \in \mathcal{F}, \ \forall k \in \mathcal{K} \tag{6.27}$$

$$0 \leq C^f - c_i^f \perp \rho_{c_i}^f \geq 0, \ \forall i \in \mathcal{N}_f, \ \forall f \in \mathcal{F} \tag{6.28}$$

$$0 \leq Q^f - q_i^f \perp \rho_{q_i}^f \geq 0, \ \forall i \in \mathcal{N}_f, \ \forall f \in \mathcal{F} \tag{6.29}$$

$$0 \leq S^{kf} - s_{ij}^{kf} \perp \rho_{s_{ij}}^{kf} \geq 0,\ \forall (i,\ j) \in \mathcal{W}_f^k,\ \forall f \in \mathcal{F},\ \forall k \in \mathcal{K} \tag{6.30}$$

$$\frac{d\lambda_i^f}{dt} = -\frac{\partial \psi_i^f(I_i^f, t)}{\partial I_i^f},\ \forall i \in \mathcal{N}_f,\ \forall f \in \mathcal{F} \tag{6.31}$$

$$\lambda_i^f(t_f) = \frac{-e^{-\rho t_f} \partial Z_f[I(t_f), t_f]}{\partial I_i^f(t_f)},\ \forall i \in \mathcal{N}_f,\ \forall f \in \mathcal{F} \tag{6.32}$$

$$\frac{dI_i^f}{dt} = q_i^f + \sum_{k \in \mathcal{K}} \sum_{(j,i) \in \mathcal{W}} s_{ji}^{kf} - \sum_{k \in \mathcal{K}} \sum_{(i,j) \in \mathcal{W}} s_{ij}^{kf} \tag{6.33}$$

$$\sum_{(j,i) \in \mathcal{W}} \sum_{k \in \mathcal{K}} \mathcal{D}_i(p_i^f + \gamma_{ji}^{f,k}; \mathrm{p}^{-f},\ \gamma) I_i^f(t_0) = M_i^f,\ \forall i \in \mathcal{N}_f,\ \forall f \in \mathcal{F},\ \forall i \in \mathcal{N}_f,\ \forall f \in \mathcal{F} \tag{6.34}$$

With the complementarity formulation (6.25) to (6.34), the MPEC problem for each carrier is reformulated as a dynamic mathematical program with complementarity constraint (MPCC). Our scheme of time discretization is as follows: define the discrete instant of time $t_m = t_0 + m\triangle t$, with $m = 1, \cdots M$ where $\triangle t$ is the time step. At terminal time we have $t_M = t_f$ thus $M = \dfrac{t_f - t_0}{\triangle t}$. Hence, a finite dimensional Nonlinear Program with Equilibrium Constraints (NLPEC) will be formulated, allowing us to conduct numerical experiments by GAMS with the NLPEC solver. We need to mention that although DVI is infinite dimension, maximum principle is finite dimension as it is a characteristic of optimal control problem (OCT) [91].

6.7 Numerical set up

Our numerical experiments will be conducted (1) to compare the benefits/disadvantages of the producing companies' decision in outsourcing the freight service; (2) to model the effects that regulations could bring to the freight service competition; and (3) to incorporate, for carriers, more realistic, nonlinear pricing schedules, such as two-part or three part tariffs and premiums charged for priority service.

6.7.1 A Small Numerical Implementation

Let us consider a small network with 6 arcs and 4 nodes. 2 shippers located at node 1 and node 2 offers identical commodity to the market on each node and 2 carriers will offer the freight service. Figure 6.2 illustrates this network. Note that for simplicity we consider that the two shippers only have production facility and inventory in one node, which is their headquarter location: node 1 for shipper 1 and node 2 for shipper 2. Under this settings we do not have any shipment between inventory facilities and thus all s_{ij} variables are omitted. Carriers are deciding about the sipping cost and the best routes to choose to ship the demand. The time interval of interest is $[0,\ 20\ ]$, with $t_0 = 0, t_f = 20$; the sets of OD pairs are, respectively $\mathcal{W}_1 = \{(1,1)\ ,\ (1,\ 2)\ ,\ (1,\ 3)\ ,\ (1,\ 4)\}$ for shipper 1 and $\mathcal{W}_2 = \{(2,1)\ ,\ (2,\ 2)\ ,\ (2,\ 3)\ ,\ (2,\ 4)\}$ for shipper 2. The OD $(1,\ 1)$ and $(2,\ 2)$ corresponds to local delivery.

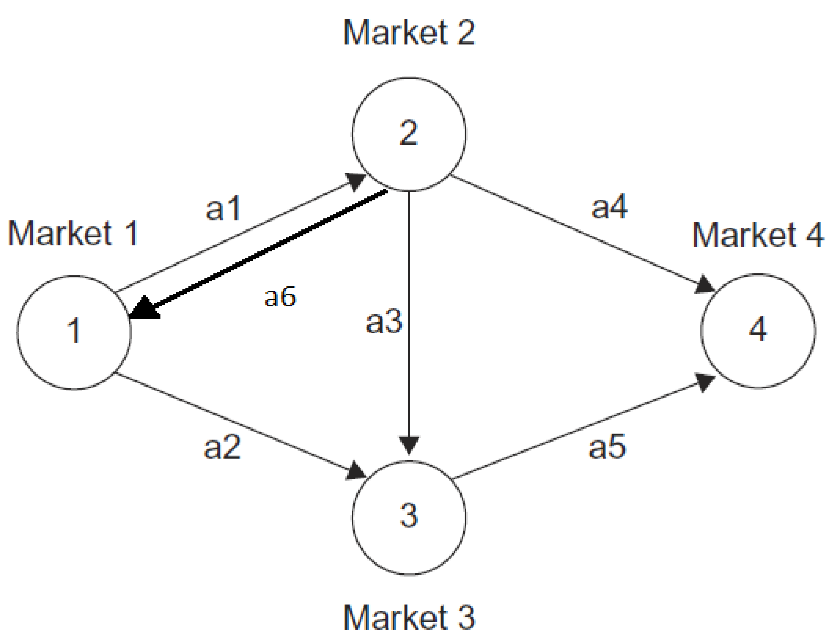

Figure 6.2. Network with 6 Arcs and 4 Nodes

The initial inventory for each shipper is:

$$I_1^1(0) = 350$$
$$I_2^2(0) = 350$$

These serves as the initial condition for the following inventory dynamics:

$$\frac{dI_1^1}{dt} = q_1^1 - \sum_{k=1,2} (D_{11}^{1k} + D_{12}^{1k} + D_{13}^{1k} + D_{14}^{1k}) \tag{6.35}$$

$$\frac{dI_2^2}{dt} = q_2^2 - \sum_{k=1,2} (D_{21}^{2k} + D_{22}^{2k} + D_{23}^{2k} + D_{24}^{2k}) \tag{6.36}$$

The demand function on each node j for firm f and carrier k take the following linear form:

$$\begin{aligned} \mathcal{D}_j^{fk}(p_{i_f}^f + \gamma_{i_f j}^k; \mathrm{p}^{-f},\ \gamma^{-k}) &= \alpha_j - \beta_j^1(p_{i_f}^f + \gamma_{i_f j}^k) + \sum_{-k} \beta_j^2(p_{i_f}^{-f} + \gamma_{ij}^{-k}) \\ &+ \sum_{-f} \beta_j^3(p_{i-f}^{-f} + \gamma_{i_{-f}j}^k) + \sum_{-k}\sum_{-f} \beta_j^4(p_{i-f}^{-f} + \gamma_{i_{-f}j}^{-k}) \end{aligned} \tag{6.37}$$

where $\alpha_j, \beta_j^1, \beta_j^2, \beta_j^3$ and β_j^4 are constants. The total demand of firm f is

$$\mathcal{D}^f = \sum_{k\in\mathcal{K}} \sum_{j\in\mathcal{N}} \mathcal{D}_j^{fk} \tag{6.38}$$

We assume the holding/inventory costs and the production costs are quadratic and of the form:

$$\psi_i^f = \frac{1}{2} A_{\psi_i}^f (I_i^f)^2$$
$$V_i^f = \frac{1}{2} A_{V_i}^f (q_i^f)^2$$

Please see Table (6.1) for details of the fixed parameters

List of fixed parameters			
Parameter Name	value	Parameter Name	value
α_i	40	$A_{\psi_1}^{f1}$	0.3
$\beta_i^1, i = 1, 2, 3, 4$	0.37,0.38,0.36,0.37	$A_{\psi_2}^{f2}$	0.1
$M(i)$	350	$A_{V_1}^{f1}$	2
ρ	0.01	$A_{V_2}^{f2}$	0.5

Table 6.1. Parameters for the Small Network

Also we assume the cost of each arc is monotonically increasing over time. Arcs are separable and independent of each other.

6.7.2 Numerical Results

The solution algorithm for each carrier's MPEC is implemented in GAMS in conjunction with the MPEC solver [161]. Matlab is used to interact with GAMS and apply the diagonalizaton algorithm to solve EPEC for the final number of carriers. The optimal solution is reported after 2952 iteration using a laptop with Intel(R) Core(TM) processor and 8.00 GB RAM. We achieved an optimal solution using nonlinear complementarity formulation for an oligopolistic market. Computational time for the example is 4888.71 seconds. 544 major iterations were run implementing the diagonalization algorithm to take one carrier's price decision and calculate the other carrier's decision. The numerical results for the time-varying carriers' optimal prices on different paths are displayed in figure (6.3) and (6.4).

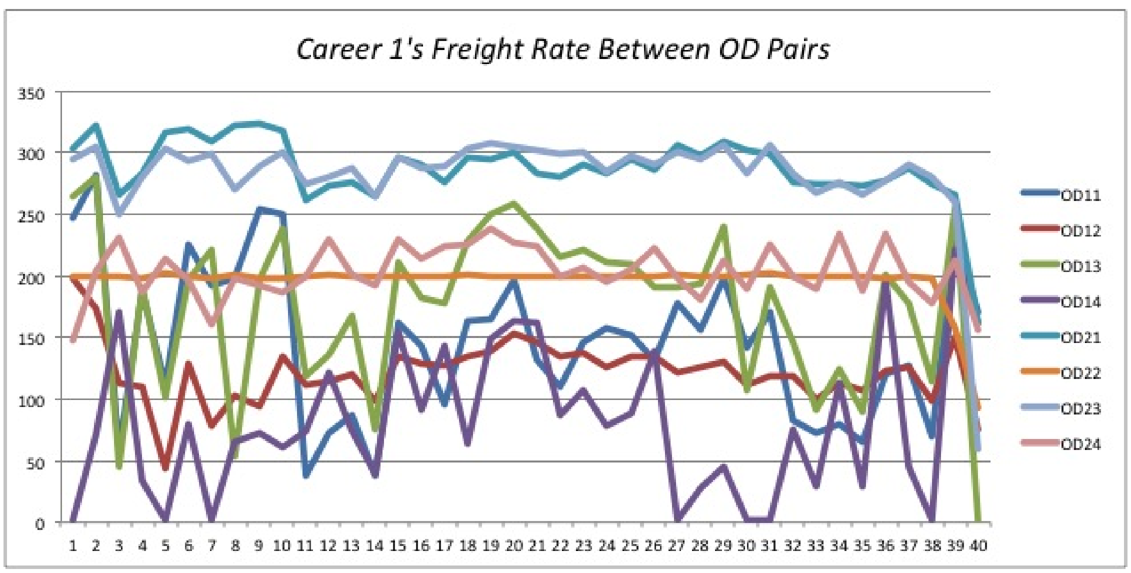

Figure 6.3. Carrier 1's freight price

And

As shown in figures (6.3) and (6.4), carriers are following almost the same pricing strategy overtime. Equilibrium takes place where the marginal cost equals marginal benefit and since carriers' marginal cost is almost the same on each path, equilibrium solution of carriers is

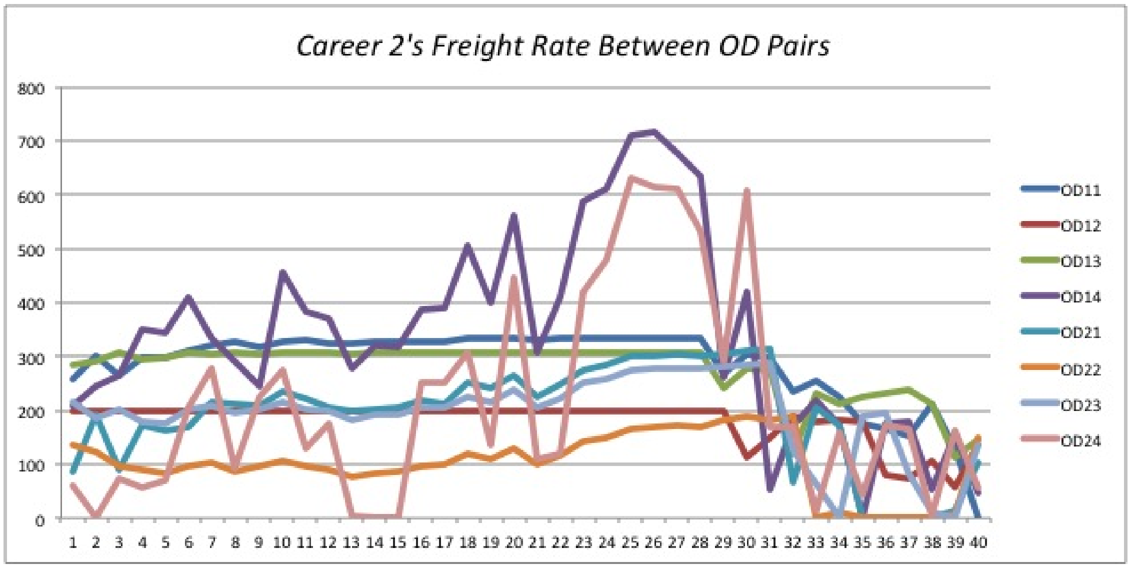

Figure 6.4. Carrier 2's freight price

almost the same on a same path. Also price is changing over time due to changes in the dynamic demand and the dynamic travel cost for each carrier at each time.

We also depicted the production rate and mill price of each shippers. For better comparison, both shippers' production rates are plotted in figure (6.5) and both shippers' mill prices are displayed in figure (6.6).

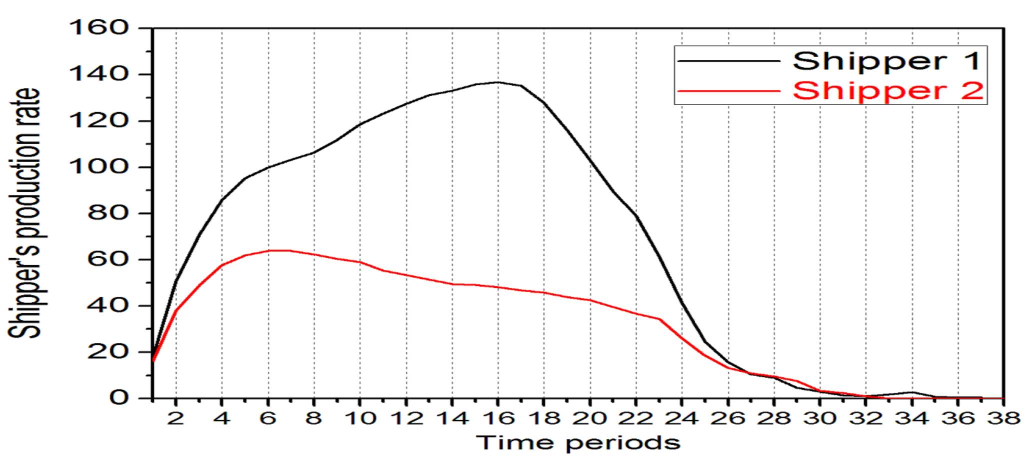

Figure 6.5. Shippers' production rate

And

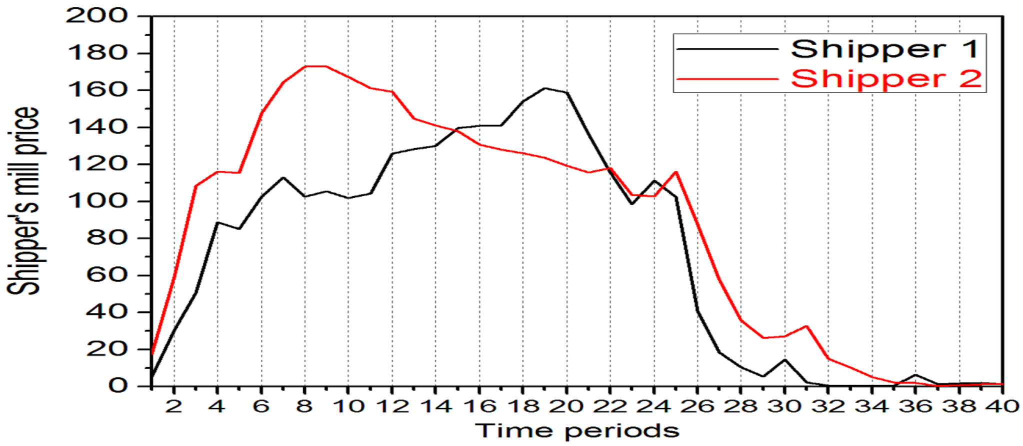

Figure 6.6. Shippers' mill price

Figure (6.5) shows that the firms operate to satisfy the increasing demand until time t = 20; at the end of horizon, this rate decreases to avoid the terminal inventory. Firm 1 has a higher production cost than firm 2 and therefore experiences a lower production rate and a higher mill pricing. As the results, It has been shown that dynamic oligopolistic network competition can be naturally converted to mathematical problem with complementarity constraints. We have provided an example to solve stackelberg dynamic carriers-shippers games formulated as MPCC after a proper time-discretization.

Even though we are considering two carriers-two shippers on a network of 6 arcs and 4 nodes, the numerical results show complicated temporal behavior. On the other hand, due to the non-convexity in each agent's problem, EPEC has been mentioned as a complicated problem to solve in the literature. Because of drawbacks associated with the diagonalization algorithm, the convergence rate and computational time might become more efficient by using another solution methodologies rather than diagonalization.

6.8 Conclusion

In this section, we proposed a model in supply chain management describing the equilibrium problems with equilibrium constraints between two layers of carriers and shippers. The so called generalized stackelburg problem models the linear pricing decision of the carriers as leaders given the reaction of followers to their price decisions. The goal of this chapter is to study a dynamic pricing problem and its connection to revenue management in freight models.

Revenue management (RM) decides about the production availability and optimal price to maximize revenue growth. Revenue management has been studied in many fields started from airline industry with a fixed price and continued in other industries such as car rental agencies and hotels. It recently attracted more attentions in more industries such as retailers, broadcasting, health care, railways, electric power and many more. There are various models in these fields considering different perspective of RM, however they all consider the basic aspects of RM. We also considered those aspects in our model including finite horizon, price-based demand, capacity and etc.

There are two strategies level in revenue management: price and duration control [162]. Price can be discussed as two questions: what the optimal price is (demand-based price) and what price each group of customers should pay (price discrimination). From the RM perspective we focused on the first question in his chapter and will continue working on the second question in the next chapter. In fact, In this chapter we designed a revenue management freight model for the shippers (lower level) who are selling their product to a price sensitive group of customers. They optimized production level and price to maximize their revenue. Also, the focus of the upper level problem is on the strategic role of the price in optimizing carriers' revenue. In addition, our model controls the inventory and production level regulations and helps revenue management to act more efficiently.

Next chapter will focus on nonlinear pricing setting of oligopolistic networks. We will focus on a bi-level Stackelberg game formulation for a combined nonlinear pricing-routing game among urban freight services with an explicit and dynamic network traffic flow component. Since one drawback of the diagonalization algorithm is that it may fail finding the optimal solution, we intend to apply double adjoint algorithms to guarantee finding optimal solution and reducing the computational time. A more complicate example will be established and solved by the aforementioned algorithm.

Chapter 7 Dynamic Optimization and Differential Multi-leader-follower Games Applied to Nonlinear Freight service pricing

7.1 Introduction and Literature Review

The dynamic shipper-carrier problems were introduced in chapter 6. We are interested to extend the problem to the case that companies apply non-linear pricing to their transportation services and try to optimize their routing strategy.

This chapter presents a bi-level Stackelberg game formulation for a combined pricing-routing game among urban freight services with an explicit and dynamic network traffic flow component. In particular, on the upper level of the problem we consider a non-cooperative game among freight service providers as leaders, also subsequently referred to as carriers, who try to set the optimal transportation price while simultaneously seeking the optimal truck routing decisions in a dynamic urban network.

On the lower level, several retailers, referred to as shipper, react to the transportation prices set by the upper-level leaders by finding the best freight service in order to maximize their own utilities. The carriers are considered as leaders and shippers are considered followers of this problem and this problem is formulated as a Stackelberg game, in which the leaders make optimal decisions by predicting the reactions of the followers and the resulting equilibrium states at the lower level.

For each leader, such an optimization problem is a mathematical program with equilibrium constraints (MPEC). On top of this, we aim to find a Nash equilibrium among the leaders, thereby coupling multiple MPECs into a single equilibrium problem with equilibrium constraints (EPEC). We show the computability of this EPEC model by proposing novel yet practical algorithms based on computational intelligence and high performance computing.

We presented the major part of the literature review related to shipper-carrier problems in chapter 4. In addition, there are significant researches related to nonlinear pricing and routing game-theoretical problems in transportation systems. With respect to this project, relevant literature can be found in some areas such as vehicle routing problems, dynamic and static game theoretic problems, network user equilibriums problems and supply chain games. Developed freight service nowadays need to be treated in a more complicated way.

There are many studies in which the companies compete given a linear tariff rate. However, many of the price schemes in economical systems are not linear. In general, the term Nonlinear Pricing is used to refer to the pricing scheme when the total price of the contract/product is not proportional to the quantity of the commodity or service [37]. Classic examples of nonlinear pricing are public utilities, e.g. electricity supply. A good daily example of this price scheme is city's taxi rides, while the cost of the taxi does not grow linearly and mostly depends on other criteria such as distance and the existing traffic congestion on the roads. In the freight service industry, such pricing is more often observed and more complicated, not only counting the weight of the item to be shipped, but also volume, express service, etc.

Examples of recent studies include: Hurley and Peterson's paper on carrier competitions which allows carriers to choose two-part tariffs depending on volume [49]. Xiao and Yang have done a research on a three agents-three layer problem which emphasizes on a linear pricing. The carriers and infrastructure companies are assumed to behave cooperatively while making coalitions [50]. In addition Lawphongpanich and Yin established a model on tolling in which the toll price follows a piecewise linear functions to determine tolls user equilibrium distribution [51].

The contribution of this part of research is to assume nonlinear pricing in its fully general setting as in [37]. This is a more realistic model in contrast to the limited pricing schemes which only consider two-part or three-part pricing, etc. This problem is not simple since adding this feature requires a careful modeling of the demand profile which introduces one more dimension to the problem. In addition, due to non-convexity associated with MPECs, solving EPECs are generally complicated and time consuming and the popular diagonalization algorithms might not be able to find the optimal solution [25]. To overcome this difficulty we will introduce dual adjoint algorithms to guarantee finding optimal solution and reducing the computational time.

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7.1.1 Model Description

The carrier are assumed to provide transportation services for a single homogeneous product, and charge the shippers certain prices. A price discrimination model with nonlinear pricing is presented in a multi-leaders-followers form. On the transportation side, the shippers make optimal routing decisions. This model is new and non-trivial for the following reasons. First, we assume a nonlinear pricing in its fully general form to represent a realistic pricing scheme, which, in the case of this research, is the so-called quantity discount pricing and able to capture a more general form of price discrimination.. The shippers are *followers* of this problem. The decision-making hierarchy is illustrated in Figure(7.1).

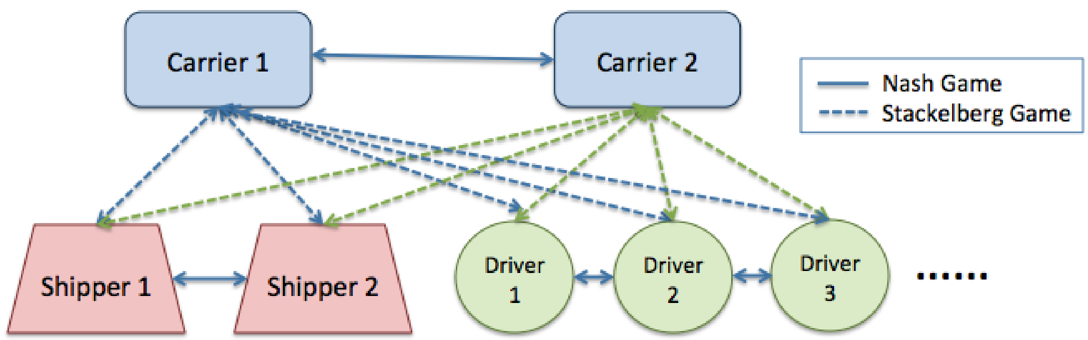

Figure 7.1. Competition and decision hierarchy of the proposed model.

This problem is formulated as a Stackelberg game, in which the leaders make optimal decisions by predicting the reactions of the followers and the resulting equilibrium states at the lower level. For each leader, such an optimization problem is a *mathematical program with equilibrium constraints* (MPEC). On top of this, we aim to find a Nash equilibrium among the leaders, thereby coupling multiple MPECs into a single *equilibrium problem with equilibrium constraints* (EPEC). Overall, this model encounters two types of entities (shipper, carrier) and numerous players in a multi- level, distributed decision environment, and thus is highly non-trivial and challenging to solve. We show the computability of this model by proposing novel yet practical algorithms based on computational intelligence and high performance computing.

7.2 Mathematical formulation

The model is conceived in a within-day time scale, we denote by $[t_0,\ t_f]$ the time horizon. The price is a real time price and varies over the defined interval. We denote the continuous time by scalar $t \in \mathcal{R}_+^\infty$ while the initial time is denoted by $t_0 \in \mathcal{R}_+^\infty$ and the final time by

$t_f \in \mathcal{R}_+^\infty$ ands $t_0 < t_f$ We let C and S be the sets of carriers and shippers respectively. In addition, we denote by $\mathcal{A}$ directed arcs, $\mathcal{N}$ for nodes and $\mathcal{W}$ for origin-destination (OD) pairs. $\mathcal{N}$ is considered as the set of locations in the network with shipping demand.

7.2.1 Freight Service Problem on the Upper Level

Under the nonlinear pricing scheme, a profile demand, denoted $N(q,\ p)$, shows the number of customers (shippers) wiling to purchase q units at the marginal price p [37].. The decision variable for carriers is the transportation price with quantity discount and set of routing decisions. The net profit maximization for each carrier $c \in C$ is expressed as:

$$max_{\gamma_{ij}^c,(i,j)\in W} J^c(\gamma_J^c x,\ h^c \gamma_{i,j}^{-c}) = \int_{t_0}^{t_f} e^{-\rho t} \Bigg[\sum_{s\in\mathcal{S}} \sum_{(i,j)\in\mathcal{W}} \int_0^{q_{ij}^{c,s}} \gamma_{ij}^c(q,\ t) N_{ij}(\gamma_{ij}^c(q,\ t),\ q,\ t) dq - \sum_{(i,j)\in W} \sum_{p\in\mathcal{P}_{ij}} \Psi_p^c(t, h_p^c) h_p^c(t) \Bigg] dt \tag{7.1}$$

where $\mathcal{P}^c$ is the set of paths followed by trucks, which starts and ends at the same depot. $h^c(t) = (h_p^c(t) : p \in \mathcal{P}^c)$ denotes the vector of path flows of the trucks from carrier $c1.h^*(t) = (h_p^*(t) : p \in \mathcal{P})$ is the vector of equilibrium path flows corresponding to the travelers, and $\mathcal{P}$ is the set of paths utilized by travelers. $\Psi_p(t,\ h^*; h)$ represents the path travel cost (associated with travel time, fuel, personnel, etc.) for the trucks, which obviously depends on t, ph^*, and h. Intuitively the transportation cost on each path depends on the traffic background on the path. this delay operator function for each carrier is summation of travel cost and penalty cost.

Also naturally there should be a flow balance constraint for each OD pair:

$$\sum_{p\in P_{ij}} h_p^c(t) = \sum_{s\in\mathcal{S}} q_{ij}^{cs*}(t) \forall (i,\ j) \in \mathcal{W} \tag{7.2}$$

where q_{ij}^{cs} is the shipment rate of the homogeneous commodity of interest between origin-destination pair $(i,\ j) \in \mathcal{W}$ ordered by shipper $s \in S$ and carried by carrier $c \in C$. We also have the assumption that there are regulations on the price range which is:

$$\underline{M}_{c,i} \leq \gamma_{ij}^{c} \leq \overline{M}_{c,i}\ \forall c \in C,\ (i,\ j) \in W \tag{7.3}$$

Here, for the sake of simplicity we assume that the truck flows are continuous and separable, which is easy to relax to incorporate more realistic truck flows.

All price and flow variables are non-negative:

$$\begin{aligned} \gamma_{ij}^{c} &\geq 0\ \forall c \in C, (i,j) \in \mathcal{W} \\ h_{p}^{c} &\geq 0\ \forall c \in C, p \in \mathcal{P}_{ij} \end{aligned} \tag{7.4}$$

Thus the feasible region will be defined as:

$$\Gamma^{c} = \{(\gamma_{ij}^{c}, h_{p}^{c}),\quad s.t.\ (7.2),\ (7.3),\ (7.4) hold\} \tag{7.5}$$

The career optimal response problem is of the form:

$$\left.\begin{array}{ll} \max & J_c(\gamma^c, h^c; \gamma^{-c}) \\ \text{subject to} & ((\gamma^c, h^c) \in \Gamma_c \end{array}\right\} \forall c \in \mathcal{C} \tag{7.6}$$

7.2.2 Shippers' Problem on the Lower Level

Each shipper tries to maximize the surplus by minimizing its utility function. This function which is denoted by $U_{c,l}^{s}(\gamma_{ij}^{c}(q_{ij}^{c,s}))$, defines the benefit that shipper $s \in S$ receives for selecting carrier $c \in C$ for the customer location $l \in L$ for the service size q. Also $\gamma_{ij}^{c}(q_{ij}^{c,s})$ shows the quantity based marginal transportation price. We assume that shippers might use services from different carriers for the same location.

Assuming rational shippers, the second group of followers will face a following dynamic (convex) optimization problem.

$$max\ U^s(q^s,\ q^{-s},\ \alpha^s,\ \alpha^{-s}) = \sum_{c\in C}\sum_{(i,j)\in W} U^s_{c,l}(\gamma^c_{ij}(q^{c,s}_{ij}))$$

Shippers try to maximize their benefit taking into account the number of customers they will gain or lose due to selecting a specific carrier. for this purpose they have to make a decision about q^c_{ij} which shows the quantity being sent by the selected carrier $c \in C$ from node i to node j and $\alpha^s_i(t)$ as the product price at location i. We assume that shippers can pick more than one carrier for each location (due to the capacity constraint). Therefor shippers' surplus is as follows:

$$U^s_{c,l}(\gamma^c_{ij}(q^{c,s}_{ij})) = e^{-\rho t_f} Z_f[I(t_f),\ t_f] + \int_{t_0}^{t_f} e^{-\rho t_f}\Big\{D^s_i(\gamma^c_{ji} + \alpha^s_i, \boldsymbol{\alpha}^{-s}, \gamma).\alpha^s_i(t) - V^s(q^{c,s}_{ij}(t), t)\Big\}dt \tag{7.7}$$

Considering α^s_i as the uniform marginal unit price by shipper $s \in S$ in location $(i,\ j) \in W, D^s_i(\gamma^c_{ji} + \alpha^s_i,\ \alpha^{-s},\ \gamma)$ shows the number of final customers at location i who are willing to pay the final price for the product. Also $V^s(q^{c,s}_{ij}(t),\ t)$ is the production cost for the company. The state variable would be defined as the inventory/backorder.

We define the state variable as the difference of the demand quantity, D^s_i and number of shipment to each location, $q^{c,s}_{ij}$ inspired by the general model by Friesz [38], we introduce inventory as flow balance between the demand and the quantity which is going to be shipped by carriers. In addition, the cost corresponded to inventory/backorder should be added to the objective function. Finally, the utility function and the dynamic will be as follows:

$$U^s(q^s,\ q^{-s},\ \alpha^s,\ \alpha^{-s}) = e^{-\rho t_f} Z_f[I(t_f),\ t_f] + \int_{t_0}^{t_f} e^{-\rho t}\{\sum_{c\in C}\sum_{(j,i)\in\mathcal{W}} D^s_i(\gamma^c_{ji} + \alpha^s_i,\ \alpha^{-s},\ \gamma).\alpha^s_i(t) - \sum_{c\in C}\sum_{(i,j)\in\mathcal{W}} V^s(q^{c,s}_{ij}(t),\ t) - \sum_{i\in\mathcal{N}} \psi^s(I^s_i(t),\ t)\}dt \tag{7.8}$$

where $\psi^s(I^s_i(t),\ t)$ is the inventory/backorder penalty function, and:

$$\frac{dI_i^s}{dt} = \sum_{c \in C} \sum_{(j,i) \in \mathcal{W}} [q_{ji}^{c,s} - q_{ij}^{c,s} - D_i^s(\gamma_{ji}^c + \alpha_i^s,\ \alpha^{-s},\ \gamma)] \forall i \in \mathcal{N} \tag{7.9}$$

All production and product price variables are non-negative and bounded from above;

$$\begin{aligned} 0 \le \alpha_i^s \le \boldsymbol{\alpha}^s \\ 0 \le q_{ij}^{c,s} \le Q^s \end{aligned} \tag{7.10}$$

And the terminal inventory should obey:

$$I^s(t_f) = M^f \tag{7.11}$$

In general:

$$\Omega^s = \{(\zeta y^s,\ q_{ij}^{c,s}) : ((7.9), (7.10) and (7.11) hold\} \tag{7.12}$$

is the set of feasible controls. Shippers' problem is to take ζy^{-s} and γ_{ij}^c in order to solve

$$\left. \begin{array}{l} \max\ U^s .\ \forall s \in S \\ Subject\ to \\ (\alpha^s,\ q_{ij}^{c,s}) \in \Omega^s \end{array} \right\} \forall s \in S \tag{7.13}$$

Shippers consider the decision of other shippers to make their own non-cooperative decision. This makes (7.13) a differential Nash game.

7.2.3 Necessary and Sufficient conditions of the Producers' Oligopoly

Let's introduce a short-hand notation that

$$\Phi^s(q^s,\ q^{-s},\ cx^s,\ \alpha^{-s}) \ = \ e^{-\rho t}\{\sum_{c\in C}\sum_{(j,i)\in\mathcal{W}} D_i^s(\gamma_{ji}^c+\alpha_i^s,\ \alpha^{-s},\ \gamma).\alpha_i^s(t) - \sum_{c\in C}\sum_{(i,j)\in\mathcal{W}} V^s(q_{ij}^{c,s}(t),\ t) - \sum_{\dot{x}\in\mathcal{N}}\psi^s(I_{\dot{x}}^s(t),\ t)\} \tag{7.14}$$

$$\Psi^s(q^s,\ \alpha^s,\ ,\ I^s,\ \lambda^s) \ = \ \sum_{i\in\mathcal{N}}\lambda_i^s\{\sum_{c\in C}\sum_{(j,i)\in\mathcal{W}}[q_{ji}^{c,s}-q_{ij}^{c,s}-D_i^s(\gamma_{ji}^c+\alpha_{x'}^s,\ \alpha^{-s},\ \gamma)]\} \tag{7.15}$$

Denote by H^s the Hamiltonian for any $s \in S$:

$$\mathcal{H}^s = -\Phi^s(q^s,\ q^{-s},\ \alpha^s,\ \alpha^{-s}) + \Psi^s(q^s,\ \alpha^s,\ ,\ I^s,\ \lambda^s) \tag{7.16}$$

And the necessary conditions will be as follows:

(i) Maximum principle:

$$(q^s,\ \alpha^s) := \arg\max_{(q^s,\alpha^s)\in\Omega^s}\mathcal{H}^s \tag{7.17}$$

(ii) Adjoint equations:

$$\frac{d\lambda_i^s}{dt} = -\frac{\partial\mathcal{H}^s}{\partial I_{x'}^s} = -\frac{\partial\psi^s(I_l^s(t),t)}{\partial I_i^s} \tag{7.18}$$

(iii) Transversality conditions:

$$\lambda_i^s(t_f) = \frac{-e^{-\rho t_f}\partial Z_f[I(t_f),t_f]}{\partial I_i^s(t_f)} \tag{7.19}$$

(iv) Dynamics:

$$\frac{dI_i^s}{dt} = \sum_{c \in C} \sum_{(j,i) \in \mathcal{W}} [q_{ji}^{c,s} - q_{ij}^{c,s} - D_i^s(\gamma_{ji}^c + \alpha_i^s,\ \alpha^{-s},\ \gamma)] \forall i \in \mathcal{N} \tag{7.20}$$

In summary, the necessary conditions for the producers' competition consists of (7.17-7.20). We introduce a short-hand notation:

$$\Lambda^s := \{\lambda : \lambda_i^s(t_f) = \frac{-e^{-\rho t_f} \partial Z_f[I(t_f), t_f]}{\partial I_i^s(t_f)}, \tag{7.21}$$

$$\frac{d\lambda_i^s}{dt} = -\frac{\partial \psi^s(I_i^s(t), t)}{\partial I_i^s},\ \forall i \in \mathcal{N},\ \forall s \in S\} \tag{7.22}$$

Familiarity with variational inequalities suggests that the following variational inequality has solutions that are differential Nash equilibria for a non-cooperative game in which individual firms maximize net profits in light of current information about their competitors:

$$Find\ (q^{s*},\ \alpha^{s*}) \in \Omega^s, \lambda^* \in \Lambda^s such\ that \tag{7.23}$$

$$0 \geq \sum_{s \in \mathcal{S}} \int_{t_0}^{t_f} [\sum_{c \in C} \sum_{(i,j) \in \mathcal{W}} \frac{\partial \mathcal{H}^{s*}}{\partial q_{ij}^{cs}} (q_{ij}^{c,s} - q_{ij}^{c,s*}) + \sum_{i \in \mathcal{N}} \frac{\partial \mathcal{H}^{s*}}{\partial \alpha_i^s} (\alpha_i^s - \alpha_i^{s*})] dt$$

$$\forall (q,\ \alpha) \in \Omega^s, \forall \lambda \in \Lambda^s \tag{7.24}$$

Where $\mathcal{H}^{s*} = \mathcal{H}^s(q^{s*},\ \alpha^{s*},\ \lambda^{s*}; q^{-s},\ \alpha^{-s}; t)$, $\Omega = \Pi_{s \in \mathcal{S}} \Omega^s$

We note that (7.23) and (7.24) is a differential variational inequality expressing the differential Nash game that is our present interest.

The issue of immediate concern is to formally demonstrate that solutions of (7.23) and (7.24) are differential Nash equilibria. In fact we state the following result:

Theorem 7.2.1. *(solution to the dynamic oligopolistic network competition problem)*
Any solution of (7.23) and (7.24) is a solution of the dynamic oligopolistic network competition problem when regularity in the sense of Definition 7.1.1 holds.

Proof. (solution to the dynamic oligopolistic network competition problem) *following Friesz [27], we note that* (7.24) *is equivalent to the following optimal control problem*:

$$Max\ L(q,\ \alpha) = \sum_{s\in\mathcal{S}} \int_{t_0}^{t_f} [\sum_{c\in C} \sum_{(i,j)\in\mathcal{W}} \frac{\partial \phi^{s*}}{\partial q_{ij}^{cs}}(q_{ij}^{c,s}) + \sum_{i\in\mathcal{N}} \frac{\partial \phi^{s*}}{\partial \alpha_i^s}(\alpha_i^s)]dt \tag{7.25}$$

$$(q,\ \alpha) \in \Omega^s, \forall \lambda \in \Lambda^s \quad s.t.(7.9), (7.10) and (7.11)$$

where $L(q,\ \alpha)$ *is a linear functional. Since this functional assume knowledge of the solution to our game,* $L(q,\ cy)$ *is a mathematical construct to be used in the analysis and is not used for the computational purposes. The following shows the augmented Hamiltonion for the optimal control problem*:

$$H_0 = \sum_{s\in\mathcal{S}} [\sum_{c\in C} \sum_{(i,j)\in\mathcal{W}} \frac{\partial \phi^{s*}}{\partial q_{ij}^{cs}}(q_{ij}^{c,s*}) + \sum_{i\in\mathcal{N}} \frac{\partial \phi^{s*}}{\partial \alpha_{0'}^s}(\alpha_i^{s*})] + \sum_{s\in S} \Psi_s$$

And when maximum principle is applied:

$$max\ H_0,\ s.t.0 \leq (q^s,\ \alpha^s) \leq (Q^s,\ \alpha^s)$$

And the necessary condition for this mathematical program are as follows:

$$\frac{\partial H_0^*}{\partial q_{ij}^{c,s}} = \frac{\phi^{s*}}{q_{ij}^{c,s}} + \frac{\Psi^{s*}}{q_{i'_J}^{c,s}} = \frac{\partial H_s^*}{q_{ij}^{c,s}}$$

$$\frac{\partial H_0^*}{\partial cy_i^s} = \frac{\phi^{s*}}{\partial \alpha_i^s} + \frac{\Psi^{s*}}{\partial \alpha_i^s} = \frac{\partial H_s^*}{\partial cy_i^s}$$

While

$$H_0^* = \sum_{s\in\mathcal{S}} [\sum_{c\in C} \sum_{(i,j)\in\mathcal{W}} \frac{\partial \phi^{s*}}{\partial q_{ij}^{cs}}(q_{ij}^{c,s*}) + \sum_{i\in\mathcal{N}} \frac{\partial \phi^{s*}}{\partial cx_i^s}(\alpha_i^{s*})] + \sum_{s\in S} \Psi^{s*}$$

And

$$\Psi^{s*} = \Psi^s(q^s,\ \alpha^s,\ I^s,\ \lambda^s)$$

□

7.2.4 Existence and Uniqueness for DVI of the Producers' Oligopoly

7.2.4.1 Existence

Even though there are relatively simple multi-leader multi-follower games which admit no equilibrium, the proof of existence of global/local equilibrium of the multi-leader, multi-follower games has not been widely disused in the literature. In general, most of the theorems are still restrictive or model specific.

The existence of Stackelberg games are mostly known for the situation that follower-level equilibrium is unique for each strategy profile of the leaders. These results can be achieved by explicitly replacement of the equilibrium and then analysis of the leader-level equilibrium ([163], [164]). It is clear that such results are only applicable to the problem with simple objective function and constraints associated with players, which demands a weaker notion of equilibrium ([165], [166]).

Kulkarni and Shanbhag. have introduced the first general existence result for equilibrium of Stackelberg games [167]. They introduced the concept of the quasipotential function and expressed the existence result of the global and local minimizers of a suitably defined optimization problem. We will briefly introduce this theorem as well as the Browder's fixed-point theorem for DVI problems. Browder's fixed-point theorem will be used later to prove the existence of DVI of the producers' Oligopoly.

Let $\mathcal{K} = 1, 2, ..., K$ shows the set of leaders where leader i solves the optimization problem. The multi-leader multi-follower game η is denoted by following:

$$min_{x_i,y_i} \alpha_i(x_i, y_i; x^{-i}) \;\; s.t \;\; x_i \in \mathcal{X}, \; y_i \in E(x),$$

Where α is the leader i's objective. Also

$$x^{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_K)$$
$$(\bar{x}_i, x^{-i}) = (x_1, ..., x_{i-1}, \bar{x}_i, x_{i+1}, ..., x_K)$$

Also $E(x)$ is the set of follower equilibrium and y_i is the strategy set of all followers. We next define the new concepts of Implicit Potential multi-leader multi-follower games as well as Quasi-potential multi-leader multi-follower games introduced in [167].

Definition(Implicit Potential multi-leader multi-follower games)

An implicit potential multi-leader multi-follower game, $\tilde{\eta}$ is a multi-leader multi-follower game where there is a unique follower equilibrium. Therefore in this case the ith leader solves the following problem:

$$min_{x_i}\ \alpha_i(x_i, E(x); x^{-i})\ s.t.\ x_i \in \mathcal{X}_i)$$

And there exist a function π such that:

$$\alpha_i(x_i, E(x); x^{-i}) - \alpha_i(\acute{x}_i, E(\acute{x}_i, x^{-i}); x^{-i}) = \pi(x_i, E(x); x^{-i}) - \pi(\acute{x}_i, E(\acute{x}_i, x^{-i}); x^{-i})$$

In this case, any global minimizer of the introduced implicit potential multi-leader multi-follower game, denoted by p^{imp} is a global Nash equilibrium of $\tilde{\eta}$

$$p^{imp} \qquad min_x\ \ \pi(x)\ \ s.t.\ \ x \in \mathcal{X}$$

However, the challenges for the above problem arises while $E(.)$ is not single-valued and , even if it is, determining the potentiality of this implicit game is difficult since it requires a closed-form expression for $E(.)$. The new concept is then introduced in [167] to account for these challenges.

Definition(Quasi-potential multi-leader multi-follower games)

η is called a quasi-potential game if the followings hold:

- There exist a function $\Phi_1(x), ..., \Phi_K(x)$, and function $h(x, y_i)$, while $\alpha_i(x_i, y_i; x^{-i}) = \Phi_i(x) + h(x, y_i)$
- There exists a function $\pi(.)$, while $\Phi_i(x_i; x^{-i}) - \Phi_i(\acute{x}_i; x^{-i}) = \pi(x_i; x^{-i}) - \pi(\acute{x}_i; x^{-i})\ \ \forall i = 1, ..., \mathcal{K},\ \ \forall\ x \in \mathcal{X}$

Also any global minimizer of the introduced quasi-potential multi-leader multi-follower game, denoted by p^{quasi} is a global Nash equilibrium of η and follows:

$$p^{quasi} \qquad min_{x,w}\ \ \pi(x) + h(x, w)\ \ s.t.\ \ (x, w) \in \mathcal{J}^{quasi}$$

Where

$$\mathcal{J}^{quasi} = \{(x,w) \mid x_i \in \mathcal{X}_i,\ i \in \mathcal{K},\ w \in\ E(x)\}$$

Now we review two basic theorems for optimistic and pessimistic leader, while optimistic leader minimize over y_i and a pessimistic leader would maximize over y_i while minimizing over x_i.

Theorem(Existence of global equilibrium of η):

Let's assume $\mathcal{J}^{quasi}$ as a nonempty set and continuous objective function for a quasi-potential multi-leader multi-follower game η. Based on the introduced definitions, if a minimizer J^{quasi} exists, (i.e. if either π is a coercive function on $\mathcal{J}^{quasi}$ or if $\mathcal{J}^{quasi}$ is compact); then there exists an equilibrium for η.

Also to introduce the existence theorem for pessimistic leader we need one more definition:

Definition(pessimistic leader formulation)

In the pessimistic formulation, the i^{th} leader solves the following formulation

$$min_{x_i}\ max_{y_i}\ \ \alpha_i(x_i, y_i; x^{-i})\ s.t.\ x_i \in \mathcal{X}_\rangle,\ y_i \in E(x))$$

While $\hat{\eta}$ is the Nash equilibrium of this problem. Also consider the problem

$$\mathcal{J}^{quasi} = min_{x_i}, max_w \pi(x) + h(x,w)\ s.t\ , x \in \mathcal{X},\ w \in\ E(x)$$

Theorem(Existence of pessimistic formulation equilibrium of η)

Consider a game with quasi-potential objective functions. Then if (x,w) is a solution of J^{quasi} then (x,y) is an equilibrium of $\hat{\eta}$ where $y_i = w, \forall i \in K$ and vice versa.

Kulkarni and Shanbhag [167] also proposed a modified formulation of Stackelberg in which every leader is recognized of the equilibrium constraints of all leaders. They proved that the equilibrium of this modified problem is the equilibrium of the original game. The modified formulation has a shared constraint that restricts each player's optimization problem. They proved that if the leader objective function is a potential function, then the global minimizers of the potential function over the shared constraint are equilibrium of the modified formulation. Their approach is based on Shared-constraint games in where there exists a set C in the product space of strategies such that for any tuple of strategies of other players, Φ^{-i}, the feasible strategies Φ_i for player i are those that satisfy $(\Phi_i, \Phi^{-i}) \in C$.

In this section we use Browder's fixed-point theorem to prove the existence of DVI of the producers' Oligopoly. To use the theorem, we first need to define the regularity condition.

Definition 7.2.1. *(Regularity conditions of dynamic oligopolistic network competition problem)*
The regularity conditions we use here are borrowed from the conditions in [27]. Therefore the dynamic oligopolistic network competition problem introduced above is assumed regular because:

- *The demand, production cost and inventory cost functions are continuously differentiable with respect to the controls and states so* $\Phi^s(q^s,\ \alpha^s; q^{-s},\ \alpha^{-s})$ *is continuously differentiable with respect to controls and states.*
- *The state operator* $I(q,\ \alpha)$ *exists and is unique, while each of its components is continuous and G-differentiable*
- *For each* $f \in \mathcal{F}$*, the terminal cost function* $Z_f[I^f(t_f)\ ,\ t_f]$ *is continuously differentiable with respect to* $I_i^f(t_f)$ *for all* $i \in \mathcal{N}_f$*.*

We next discus the Browder's fixed-point theorem and prove that the existence result holds for our optimal control problem:

Theorem 7.2.2. *(Browder's fixed-point theorem for infinite-dimensional variational inequalities)*

Let U *be a compact convex subset of the locally convex topological vector space* V *and* F *a continuous (single-valued) mapping of* U *into* V^* *(the dual space of* V*). Then there exists* $u^* \in U$ *such that*

$$\langle F(u^*)\ ,\ u^* - u\rangle \geq 0 \forall u \in U \tag{7.26}$$

Proof. ([168])
We prove the theorem through the contradiction. Suppose there exist a $u \in U$ *such that:*

$$\langle F(u^*)\ ,\ u^* - u\rangle < 0,\ \forall u \in U \tag{7.27}$$

For each $u^* \in U$, *we let* $G(u^*)$ *to be*

$$G(u^*) = \{u \ : \ u \in U, \ \langle F(u^*), \ u^* - u \rangle < 0\} \tag{7.28}$$

Since we assumed existence of a solution for (7.27), $G(u^*)$ *is not empty. Also* $G(u^*)$ *is convex for* $\forall u^*$ *We also define function* f *as:*

$$f(u, \ v) = \langle F(u), u - v \rangle \tag{7.29}$$

Since F *is a continuous mapping of set* U *which is compact (and hence bounded), then* $f(u, \ v)$ *is a continuous function of* v *on* U *for each fixed* $u \in U$*. Therefore, we can conclude that* $G^{-1}(u)$ *is open in* $U, \forall u \in U$*. Also based on Browder's elementary fixed-point theorem, an element* $\overline{u} \in U$ *exist such that* $\overline{u} \in G(\overline{u})$ *. For this* $\overline{u}$ *we have:*

$$G(\overline{u}) = \{u \ : \ u \in U, \ \langle F(\overline{u}), \ u^* - u \rangle < 0\} \tag{7.30}$$

$\overline{u} \in G(\overline{u})$ *, therefore*

$$0 \geq \langle F(\overline{u}, \overline{u} - u \rangle = 0 \tag{7.31}$$

which is a contraction. Therefore the theorem is proved. □

Now we are ready to prove that the existence result holds for $DVI(\Phi, \ f, \ \Psi, \ \Omega, \ I(\mathrm{O}), \ t_0, \ t_f)$ in (7.24):

Theorem 7.2.3. (*Existence of a solution to* $DVI(\Phi, \ f, \ \Psi, \ \Omega, \ I(\mathrm{O}), t_0, t_f)$ *)*

When regularity in the sense of Definition 7.2.1 holds and Ω *is compact,* $DVI(\Phi, \ f, \ \psi, \ \Omega, \ I(\mathrm{O}), \ t_0, \ t_f)$ *has a solution.*

Proof. By the assumption of regularity $DVI(\Phi, \ f, \ \psi, \ \Omega, \ I(\mathrm{O}), \ t_0, \ t_f)$ *is well defined and continuous. So* $F(x(u, \ t), \ u, \ t)$ *is continuous in* u*.*

Also, by regularity, we know Ω *is convex and compact. Consequently, by Theorem 7.2.2, the dynamic oligopolistic network competition problem introduced in this dissertation has a solution.* □

Next we discuss the uniqueness of solutions to DVI problems.

7.2.4.2 Uniqueness

To discuss the uniqueness of DVI solutions, we introduce the notion of monotonicity of a vector function:

Definition 7.2.2. *(Monotonically increasing function)*

We say function $F(y) : \mathbb{R}^n \to \mathbb{R}^n$ monotonically increasing on Λ if

$$[F(y^1) - F(y^2)]^T (y^1 - y^2) \geq 0 \forall y^1,\ y^2 \in \Lambda \tag{7.32}$$

Definition 7.2.3. *(Strictly monotonically increasing function)*

We define function $F(y)$ as strictly monotonically increasing on Λ if

$$[F(y^1) - F(y^2)]^T (y^1 - y^2) > 0,\ \forall y^1,\ y^2 \in \Lambda,\ suchthat\ y^1 \neq y^2 \tag{7.33}$$

Also by reversing the directions of the inequalities, we can get the monotone decreasing definition of the above definitions.

We now are ready to define the following uniqueness result:

Theorem 7.2.4. ($Uniqueness\ of\ VI(F,\ \Lambda)$ *)*

If $y \in \Lambda \subseteq \mathbb{R}^n$ is a solution of $VI(F,\ \Lambda)$ and $F(x)$ is strictly monotonically increasing then y is unique.

Proof. We prove the theorem through contradiction. Suppose there are two solutions $y^1 \in \Lambda$ and $y^2 \in \Lambda$ while $y^1 \neq y^2$ Therefore

$$F(y^1)(y^2 - y^1) \geq 0 \tag{7.34}$$

$$F(y^2)(y^1 - y^2) \geq 0 \tag{7.35}$$

By adding inequalities (7.34) and (7.35) we will have:

$$[F(y^1) - F(y^2)]^T (y^1 - y^2) \leq 0 \tag{7.36}$$

Which is contradicting the strict monotonicity definition in (7.2.2). Hence $y^1 = y^2$ which means any solution is unique.

Theorem 7.2.5. *(Uniqueness of dynamic oligopolistic network competition problem)*
If demand, production and inventory cost functions are strictly convex,for an instance their Hessians are positive semidefinite in the entire domain, then the dynamic oligopolistic network competition problem introduced above will have a unique solution.

Before proving theorem (7.2.5) on dynamic oligopolistic problem of shippers we need to define one more theorem. □

Theorem 7.2.6. *(Relationship of convexity and monotonicity)*
If the differential function $E(x) : \Lambda \subseteq \mathbb{R}^n \to \mathbb{R}$ is (strictly) convex for $\forall x \in \Lambda$, then its gradient, $\nabla E(x)$ is (strictly) monotonically increasing for $\forall x \in \Lambda$.

Proof. $(Uniqueness\ of\ VI(F,\ \Lambda)\ )$
Based on convexity and differentiability we have

$$E(y^1) \geq E(y^2) + [\nabla E(y^2)]^T (y^1 - y^2) \tag{7.37}$$
$$E(y^2) \geq E(y^1) + [\nabla E(y^1)]^T (y^2 - y^1) \tag{7.38}$$

We will add inequalities (7.37) and (7.38) $\forall y^1, y \in \Lambda$ which leads to:

$$0 \geq [\nabla E(y^2)]^T (y^1 - y^2) + [\nabla E(y^1)]^T (y^2 - y^1) \tag{7.39}$$

And therefore:

$$-[\nabla E(y^2)]^T (y^1 - y^2) + [\nabla E(y^1)]^T (y^1 - y^2)] \geq 0 \tag{7.40}$$

Which gives us:

$$\{[\nabla E(y^1)] - [\nabla E(y^2)]^T\}(y^1 - y^2) \geq 0 \quad (7.41)$$

Which is the condition for monotonically increasing function. □

Proof of theorem (7.2.5)

When theorem (7.2.5) holds, H^{s*} is strictly convex. Therefore, based on theorem (7.2.6) $\frac{\partial H^{s*}}{\partial q_{ij}^{cs}}$ and $\frac{\partial H^{s*}}{\partial \alpha_i^s}$ are strictly monotonically increasing.

Since these two functions are strictly monotonically increasing based on theorem (7.2.4) the solutions to variational inequality defined in (7.23) are unique.

7.2.5 Freight Pricing Oligopoly as a Multi-leader-follower Game

7.2.5.1 Mathematical Formulation of the EPEC

The proposed bi-level model of the freight pricing assumes that the carriers have knowledge on the optimal strategy of producers involved in the oligopolistic competition. Therefore, each of these carriers $c \in C$ is solving the following mathematical programming with equilibrium constraints (MPEC) problem:

$$\max J_c(\gamma^c,\ h^c; \gamma^{-c}) s.t. (\gamma^c,\ h^c) \in \Omega_c \cap \{(q^{s*},\ (y^{s*})\ solves\ (7.23)\ and\ (7.24), \forall s\} \quad (7.42)$$

For simplicity, we denote by $MPEC_c(\gamma^{-c*})$ the set of solutions of the c-th carrier, bearing in mind that h^c is internal to each carrier's decision. Assuming that the service carriers competes with each other in a Nash-like manner, the complete formulation constitutes an equilibrium problem with equilibrium constraints (EPEC), which is given as

$$\gamma^c \in MPEC_c(\gamma^{-c*})\ ,\ \forall c \in C. \quad (7.43)$$

Thus we have the following definition of freight price equilibrium:

Definition 7.2.4. *(The Freight Pricing Equilibrium)*
A triple of pricing schemes $(\gamma^{1*},\ \cdots,\ \gamma^{c*},\ \gamma^{|C|*})$ *is said to be a multi-leader-follower freight pricing equilibrium if (7.43) is solved for every leader c together with the corresponding lower level equilibrium* $(q^*,\ \alpha^*)$ *.*

And we realized that the freight service equilibrium problem among the MPECs described above is an Equilibrium Problem with Equilibrium Constraints (EPEC).

7.2.6 Solution Methodology

7.2.6.1 EPECs and MPECs Algorithms Review

The variational inequality formulation does not have a known solution schemes. One of the known methodologies to solve VI problems, is to reformulate them to complementarity constraints as a mathematical program with complementarity constraints (MPCC). In this section we review some methods for solving MPECs and EPECs problem.

Let f be a function defined on $\mathcal{F} : \Re^n \to \Re^n$ Then in general, mathematical problem with complementarity constraints is considered as following [25]:

$$\begin{aligned} &minimize\ f^k(x^k,\ y; x^{-k}) \\ &st:\ \ g^k(x^k,\ y; x^{-k}) \leq 0,\ h^k(x^k,\ y; x^{-k}) = 0, \\ &0 \leq G(x^k,\ y; x^{-k}) \perp H(x^k,\ y; x^{-k}) \geq 0 \end{aligned} \tag{7.44}$$

There are multiple ways to solve MPECs. Some researchers have showed that MPECs can be reformulated as the following nonlinear problem ([26], [58], [25]).

$$\begin{aligned} &minimize f^k(x^k,\ y; x^{-k}) \\ &st : g^k(x^k,\ y; x^{-k}) \leq 0,\ h^k(x^k,\ y; x^{-k}) = 0, \\ &G(x^k,\ y; x^{-k}) \geq 0,\ H(x^k,\ y; x^{-k}) \geq 0, \\ &G(x^k,\ y; x^{-k}) \circ H(x^k,\ y; x^{-k}) \leq 0, \end{aligned} \tag{7.45}$$

This NLP can be solved by any of the nonlinear optimization algorithms and the solution will be the solution of the original MPEC. As it is obvious NLP (7.45) does not have a feasible solution to satisfy the inequalities strictly. Therefore the Mangasarian-Fromovitz constraints qualification can not be held at any feasible point. However, there are other types of MPEC constraints qualification such as linear independence (MPEC-LICQ) which guarantee the existence of the Lagrangian multipliers at the local optimal ([26], [25]). For the definition of MPEC-LICQ please see [58].

Theorem 7.2.7. *([56], [57], [25])*
If a local optimal point x^* of the MPEC (7.4) satisfies the MPEC-LICQ, then x^* is a strongly stationary point. i.e. There exist unique MPEC Lagrangian multipliers.

Theorem 7.2.8 (*[26], [58], [25]*)**.** .
A vector x^* is the strong stationary of the MEPC (7.44) if and only if it is equivalent to the KKT point of NLP(7.45) . Therefore, there exist unique multipliers as $\lambda = (\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,\ \lambda_5)$ that satisfies the follows KKT equations:

$$
\begin{aligned}
&\nabla f(x^*) + \lambda_1^T \nabla g(x^*) + \lambda_2^T \nabla h(x^*) - [\lambda_3 - H(x^*) \circ \lambda_5]^T \nabla G(x^*) - [\lambda_4 - G(x^*) \circ \lambda_5]^T \nabla H(x^*) = 0,\\
&G(x^*) \geq 0,\ H(x^*) \geq 0,\ G(X^*) \circ H(x^*) \leq 0,\ h(x^*) = 0,\ g(x^*) \leq 0,\\
&\lambda_1^T g(x^*) \geq 0,\ \lambda_3^T G(x^*) \geq 0,\ \lambda_4^T G(x^*) \geq 0,\ \lambda_5^T [G(x^*) \circ H(x^*)] \geq 0\\
&\lambda_1 \geq 0,\ \lambda_2 \geq 0,\ \lambda_3 \geq 0,\ \lambda_4 \geq 0,\ \lambda_5 \geq 0
\end{aligned}
\tag{7.46}
$$

For the proof please refer to [26].

Facchinei et al. and Fukushima et al. applied a so called perturbed Fischer-Burmeister function to design a sequence of smooth and regular NLPs which approximate the MPEC ([28], [29]). Also, Scholtes's investigated a series of NLPs which can be solved to attain the optimal point of MPECs [169]. This approach finds a sequence of stationary points of a parametric NLP which regularizes a MPEC in the form of complementarity conditions. Let's assume NLP(t) as the following form:

$$
\begin{aligned}
\textit{minimize}\ & f^k(x^k,\ y; x^{-k})\\
& st:\ g^k(x^k,\ y; x^{-k}) \leq 0,\ h^k(x^k,\ y; x^{-k}) = 0\\
& G(x^k,\ y; x^{-k}) \geq 0,\ H(x^k,\ y; x^{-k}) \geq 0\\
& G(x^k,\ y; x^{-k}) \circ H(x^k,\ y; x^{-k}) \leq t
\end{aligned}
\tag{7.47}
$$

According to NLP (7.47), the goal is to find the optimal solution of NLP(0) which is the solution to the MPEC problem in the complementarity form.

There are also exact penalization approaches which move the complementarity constraints to the objective function and solve the resulted problem instead on the original MPEC ([30], [31]). On the other hand, due to non-convexity of EPEC, its solution is considered as a difficult problem. One of the solution strategies for EPECs is the (Jaccobi/Gauss-Seidel) diagonalization method ([32], [10]), where the underlying MPECs are solved in turns until an equilibrium point could be obtained. Also to solve an EPEC, Hu suggested to reformulate each leaders MPEC to NLP and find the first order KKT condition.Then the system of all first order conditions can be solved to find the equilibrium solution [33].

Other solution strategies for EPECs include sequential nonlinear complementarity algorithm as in [25]. For example, Su extended the Scholtes's Regularization scheme by relaxing the complementarity constraints and perturbing the coefficient of objective function along with any sequence t tending to zero [25].The problem then becomes solving a sequence of perturbed NLPs while any sequence tends to zero. Other EPEC algorithms include mixed complementarity formulation using big-M method by Ehrenmann [34] and parametric smoothing approach by Bouza Allende with the focus on existence results and more efficient

rate of the convergence [35].More comprehensive studies about solution methodologies for MPECs and EPECs can be found in ([36], [34]). Also, Computationally, solving EPECs are generally complicated and time consuming due to the non-convexity associated with MPECs. Also popular diagonalization algorithms might not be able to find the optimal solution [25]. To overcome this difficulty we will introduce dual adjoint algorithms to guarantee finding optimal solution and reducing the computational time.

7.2.6.2 Double Adjoint Method

Consider the following dynamic MPEC problem.

$$min \int_{t_0}^{t_f} g(x(v,\ t),\ v,\ t)dt \tag{7.48}$$

$$s.t \int_{t_0}^{t_f} [F(x(v,\ t),\ y,\ v,\ t)]^T(u-v)dt \geq 0,\ \forall u \in U; \tag{7.49}$$

$$x(u,\ t) \in arg\{\frac{dx}{dt} = f(x,\ u,\ t); x(t_0) = x_0; \Gamma[x,\ x(t_f),\ t_f] = 0\}; \tag{7.50}$$

$$y \in \mathcal{Y} \tag{7.51}$$

where $u \in U \subseteq (\mathcal{L}^2[t_0,\ t_f])^m, U$ is closed and bounded. Notice that the underlying differential variational inequality can be formulated as:

$$\min \gamma^T = \Gamma[x(t_f),\ t_f] + \int_{t_0}^{t^f} [F(\chi,\ y,\ u,\ t)]^T u dt \tag{7.52}$$

$$s.t.\ \frac{dx}{dt} = f(x,\ u,\ t) \tag{7.53}$$

$$x(t_0) = x_0; \tag{7.54}$$

$$u \in U \tag{7.55}$$

The Hamiltonian for the above problem yields:

$$H(y,\ x,\ \chi,\ u,\ \lambda,\ t) = [F(\chi,\ y,\ v,\ t)]^T u + \lambda^T f(x,\ u,\ t) \tag{7.56}$$

The necessary conditions for the optimality is then given as

- State dynamics

$$\frac{dx}{dt} = f(x,\ u,\ t) \tag{7.57}$$

- initial time conditions

$$x(t_0) = x_0 \tag{7.58}$$

- adjoint equations

$$\frac{d\lambda_l^s}{dt} = -H_x = -u^T F_x - \lambda^T f_x \tag{7.59}$$

- transversality conditions

$$\lambda(t_f) = \frac{\partial \text{I}_x^{\neg T}[x_(t_f), t_f]}{\partial x(t_f)} \tag{7.60}$$

- minimum principle

$$u \in arg \min_{u \in U} H(y,\ u) \tag{7.61}$$

In a special case when there is no constraints to u, we have that the minimum principle can be represented by

$$\frac{\partial H(y,u)}{du} = -F + \lambda^T f_u = 0 \tag{7.62}$$

Another special case is when $U = u : A^T u(t) = b$, then the minimum principle can be written as

$$\frac{\partial H(y,u)}{du} = -F + \lambda^T f_u + A\gamma = 0 \tag{7.63}$$

$$A^T u(t) = b \tag{7.64}$$

We will introduce a creative algorithm as double adjoint to change MPEC and consequently EPEC problems to a system of differential equations and linear constraints. After proper time-discretization, this system can be solved through existing NLP solvers.

The steps of double adjoint approach is as following:

- Write necessary conditions for the lower level
- Write the best response as DVI and restate it as an optimal control problem. We call this problem sub optimal control problem.
- Write necessary conditions for the sub optimal control problem.
- Compute the control law from the best response in step 3. the control law along with the state dynamic, the two adjoint equations derived in step 1 and 3 and all other constraints will be constraints for the upper level problem so we will end up with a single level oligopolistic game in this step. Since we will have two adjoint variables in this approach we call it double adjoint algorithm.
- repeat step 1 to 4 for the leaders. Write necessary conditions, DVI and its related optimal control problem along with other constraints. the problem is now a system of differential equations and linear constraints which can be solved through proper time-discretization methods.

We will solve a simple example to show the applicability of double adjoint algorithm. In this section we solve an illustrative differential game with two leaders, while $\{l_1,\ l_2\} \in \mathcal{L}$, shows set of leaders and two followers while, $\{f_1,\ f_2\} \in \mathcal{F}$, shows set of followers. This example (with a small change) was originally proposed and solved by Dockner and Jorgensen [102] with only one leader and one follower.

Consider two leaders trying to maximize the following objective function:

$$J(v,\ x) = \int_0^1 [v(t) - \frac{1}{2}(v^2(t) + x^2(t))]dt \tag{7.65}$$

And followers are maximizing the following optimal control problem:

$$K(u,\ x) = \int_0^1 [u(t) - \frac{u^2(t)}{2} - \frac{x^2(t)}{2}]dt \tag{7.66}$$

$$\frac{dx}{dt} = x(t) + u(t) \tag{7.67}$$

$$x(0) = 0 \tag{7.68}$$

Step 1 Necessary conditions for the lower level.

The Hamiltonian for the followers' problem is:

$$H^{lo} = \phi(u,\ x) + \Psi(u,\ x,\ \lambda) \tag{7.69}$$

Where

$$\phi(u,\ x) = u(t) - \frac{u^2(t)}{2} - \frac{x^2(t)}{2} \tag{7.70}$$

$$\Psi(u,\ x,\ \lambda) = \lambda(t)(x(t) + u(t)) \tag{7.71}$$

When the adjoint variable, $\lambda(t)$, obeys:

$$\frac{d\lambda}{dt} = -\nabla_x H^{lo} = x(t) - \lambda(t) \tag{7.72}$$

$$\lambda(1) = 0 \tag{7.73}$$

Then, the maximum principle implies:

$$(x,\ u) := arg\ max\ H^{lo} \quad s.t.(7.67) \tag{7.74}$$

Step 2 Writing the DVI and its related optimal control problem

The finite dimensional variational inequality principle from the necessary conditions requires any optimal solution of (7.74) to satisfy:

$$\int_0^1 \frac{\partial(u(t) - \frac{u^2(t)}{2} - \frac{x^2(t)}{2})}{\partial u}(u - u^*) \leq 0,\ \forall l \in \mathcal{L} \tag{7.75}$$

Note that based on proof 7.2.1, introduced DVI in (7.75) is equivalent to the following optimal control problem:

$$Max \quad \int_0^1 \frac{\partial(u(t) - \frac{u^2(t)}{2} - \frac{x^2(t)}{2})}{\partial u} u = \int_0^1 (1 - u(t))u(t) \tag{7.76}$$

$$s.t. \frac{dx}{dt} = x(t) + u(t) \tag{7.77}$$

Step 3 Writing necessary conditions for the sub optimal control problem of (7.76 & 7.77) Let's introduce the new Hamiltonian as H^{lo2}

$$H^{lo2} = u(t) - u(t)^2 + \lambda(t)u(t) + \mu(t)(x(t) + u(t)) \tag{7.78}$$

The new adjoint variable, $\mu(t)$, should obey:

$$\frac{\mu}{dt} = -\nabla_x H^{lo2} = -\mu(t) \tag{7.79}$$

$$\mu(1) = 0 \tag{7.80}$$

And maximum principle implies:

$$\nabla_u H^{lo2} = 0 \tag{7.81}$$

Step 4 Make the EPEC problem a single level oligopolistic game The control law will be computed from the maximum principle

$$\nabla_u H^{lo2} = 0$$
$$1 - 2u(t) + \lambda = 0 \tag{7.82}$$

Using control law (7.82) to replace the control variable u, the EPEC problem can be converted into the following single level problem:

$$Max J(v,\ x) = \int_0^1 [v(t) - \frac{1}{2}(v^2(t) + x^2(t))]dt \tag{7.83}$$

$$
\begin{aligned}
& Subjectto: \\
& \frac{dx}{dt} = x(t) + \frac{1+\lambda}{2} && (7.84) \\
& \frac{d\lambda}{dt} = -\nabla_x H^{lo} = x(t) - \lambda(t) && (7.85) \\
& \frac{d\mu}{dt} - \nabla_x H^{lo2} = -\mu(t) && (7.86) \\
& x(0) = 0 && (7.87) \\
& \lambda(1) = 0 && (7.88) \\
& \mu(1) = 0 && (7.89)
\end{aligned}
$$

Where μ and λ are the double adjoint variables.

Step 5 Repeat all the steps for the problem (7.83-7.89) in step 4

One should note that the state variable for the problem (7.83-7.89) is the set of $(x(t),\ \lambda(t),\ \mu(t))$. Hence the necessary conditions is: Introducing the Hamiltonian as H^{up}:

$$
H^{up} = v(t) - \frac{1}{2}(v^2(t) + x^2(t)) + \alpha^1(t)(x(t) + \frac{1+\lambda(t)}{2}) + \alpha^2(t)(x(t) - \lambda(t)) - \alpha^3(t)\mu(t) \tag{7.90}
$$

While the minimum principle implies:

$$
(x,\ u) := arg\ max\ H^{up} \tag{7.91}
$$

And the adjoint variables, α^1, α^2 and α^3 should satisfy:

$$
\begin{aligned}
& \frac{d\alpha^1}{dt} = -\nabla_x H^{up} = x(t) - \alpha^1(t) - \alpha^2(t) && (7.92) \\
& \frac{d_{Cy^2}}{dt} = -\nabla_\lambda H^{up} = -\frac{\alpha(t)^1}{2} + \alpha^2(t) && (7.93) \\
& \frac{d\alpha^3}{dt} = -\nabla_\mu H^{up} = \alpha^3(t) && (7.94) \\
& \alpha^1(1) = 0 && (7.95) \\
& \alpha^2(1) = 0 && (7.96) \\
& \alpha^3(1) = 0 && (7.97)
\end{aligned}
$$

Following DVI has a solution which is optimal solution to the problem (7.91)

$$\int_0^1 \frac{\partial H^{up}}{\partial v}(v - v^*) \leq 0 \forall l f \in \mathcal{F} \tag{7.98}$$

(7.98) is equivalent to the following optimal control problem:

$$Max \int_0^1 \frac{\partial H^{up}}{\partial v} v = \int_0^1 (1 - v(t))v(t) \tag{7.99}$$

$$s.t.$$

$$\frac{dx}{dt} = x(t) + \frac{1 + \lambda(t)}{2} \tag{7.100}$$

$$\frac{d\lambda}{dt} = -\nabla_x H^{lo} = x(t) - \lambda(t) \tag{7.101}$$

$$\frac{d\mu}{dt} = -\nabla_x H^{lo2} = -\mu(t) \tag{7.102}$$

We will compute the necessary conditions for this sub optimal control problem to solve maximum principle of of (7.91)

Let's introduce new Hamiltonian as H^{up2}

$$H^{up2} = v(t) - (v(t))^2 + \gamma^1(t)(x(t) + \frac{1 + \lambda(t)}{2}) + \gamma^2(t)(x(t) - \lambda(t)) - \gamma^3(t)\mu(t) \tag{7.103}$$

(7.103) Where the new adjoint variables, $\gamma^1(t), \gamma^2(t)$ and $\gamma^3(t)$, obeys:

$$\frac{d\gamma^1(t)}{dt} = -\nabla_x H^{up2} = -\gamma^1(t) - \gamma^2(t) \tag{7.104}$$

$$\gamma^1(1) = 0 \tag{7.105}$$

$$\frac{d\gamma^2(t)}{dt} = -\nabla_\lambda H^{up2} = -\frac{\gamma^1}{2}(t) + \gamma^2(t) \tag{7.106}$$

$$\gamma^2(1) = 0 \tag{7.107}$$

$$\frac{d\gamma^3(t)}{dt} = -\nabla_\mu H^{up2} = \gamma^3 \tag{7.108}$$

$$\gamma^3(1) = 0 \tag{7.109}$$

therefore the control law will be derived for maximum principle (7.94) as:

$$\nabla_v H^{up2} = 0 \tag{7.110}$$

$$1 - 2v(t) = 0 \tag{7.111}$$

Now, final formulation of the EPEC problem can be represented as system of differential equations and linear constraints:

$$\frac{dx}{dt} = \frac{1}{2} + \frac{1+\lambda}{2} \tag{7.112}$$

$$\frac{d\lambda}{dt} = -\nabla_x H^{lo} = x(t) - \lambda(t) \tag{7.113}$$

$$\frac{d\mu}{dt} = -\nabla_x H^{lo2} = -\mu(t) \tag{7.114}$$

$$\frac{d\alpha^1}{dt} = x(t) - \alpha^1(t) - \alpha^2(t) \tag{7.115}$$

$$\frac{d\alpha^2}{dt} = -\frac{\alpha(t)^1}{2} + \alpha^2(t) \tag{7.116}$$

$$\frac{d\alpha^3}{dt} = \alpha^3(t) \tag{7.117}$$

$$\frac{d\gamma^1(t)}{dt} = -\gamma^1(t) - \gamma^2(t) \tag{7.118}$$

$$\frac{d\gamma^2(t)}{dt} = -\frac{\gamma^1}{2}(t) + \gamma^2(t) \tag{7.119}$$

$$\frac{d\gamma^3(t)}{dt} = \gamma^3 \tag{7.120}$$

$$x(0) = 0 \tag{7.121}$$

$$\mu(1) = 0 \tag{7.122}$$

$$\alpha^1(1) = 0 \tag{7.123}$$

$$\alpha^2(1) = 0 \tag{7.124}$$

$$\alpha^3(1) = 0 \tag{7.125}$$

$$\lambda(1) = 0 \tag{7.126}$$

$$\gamma^1(1) = 0 \tag{7.127}$$

$$\gamma^2(1) = 0 \tag{7.128}$$

$$\gamma^3(1) = 0 \tag{7.129}$$

Problem (7.112–7.129) can be solved through discretization or other existing methods.

7.2.6.3 Double Adjoint on Nonlinear Pricing of Shipper-Carrier Problem

Based on proof 7.2.1 the DVI in (7.24) is equivalent to the following optimal control problem

$$Max \sum_{s\in\mathcal{S}} \int_{t_0}^{t_f} [\sum_{c\in C} \sum_{(i,j)\in\mathcal{W}} \frac{\partial \Phi^{s*}}{\partial q_{ij}^{cs}} q_{ij}^{c,s} + \sum_{i\in\mathcal{N}} \frac{\partial \Phi^{s*}}{\partial \alpha_i^s} \alpha_i^s t] dt \tag{7.130}$$

$$s.t$$

$$\frac{dI_i^s}{dt} = \sum_{c\in C} \sum_{(j,i)\in\mathcal{W}} [q_{ji}^{c,s} - q_{ij}^{c,s} - D_i(\gamma_{ji}^c + \alpha_i^s,\ \alpha^{-s},\ \gamma)],\ \forall i \in \mathcal{N} \tag{7.131}$$

$$0 \le \alpha_i^s \le \alpha^s \tag{7.132}$$

$$0 \le q_{ij}^{c,s} \le Q^s \tag{7.133}$$

$$\tag{7.134}$$

Then the necessary and sufficient conditions for sub-optimal problem of the lower level is as follows: Let's Denote by H^2 the Hamiltonian:

$$\mathcal{H}^2(q^s,\ \alpha^s,\ q^{-s},\ \alpha^{-s}) = \sum_{s\in\mathcal{S}} \sum_{c\in C} \sum_{(i,j)\in\mathcal{W}} \frac{\partial \Phi^{s*}}{\partial q_{ij}^{cs}} q_{ij}^{c,s} + \sum_{s\in\mathcal{S}} \sum_{c\in C} \sum_{i\in\mathcal{N}} + \frac{\partial \Phi^{s*}}{\partial \alpha_i^s} \alpha_i^s \tag{7.135}$$

$$+\sum_{i\in\mathcal{N}}\mu_i^s\sum_{c\in C}\sum_{(i,j)\in\mathcal{W}}[q_{ji}^{c,s}-q_{ij}^{c,s}-D_i^s(\gamma_{ji}^c+\alpha_i^s,\ \alpha^{-s},\ \gamma)] \tag{7.136}$$

Which is equivalent to:

$$\begin{aligned}
&\mathcal{H}^2(q^s,\alpha^s,\ q^{-s},\alpha^{-s}) = \\
&\qquad\sum_{s\in\mathcal{S}}\sum_{c\in C}\sum_{(i,j)\in\mathcal{W}}(\frac{\partial V^{s*}(q_{i'_J}^{c,s},t)}{\partial q_{ij}^{c,s}})q_{ij}^{cs}+ \\
&\qquad\sum_{s\in\mathcal{S}}\sum_{c\in C}\sum_{i\in\mathcal{N}}(-\frac{\partial D_i^{s*}(\gamma_{ji}^c+\alpha_i^s,\alpha^{-s},\gamma)}{\partial_{Cy_{x'}^s}}\alpha_i^s-D_i^{s*}(\gamma_{ji}^c+\alpha_i^s,\ \alpha^{-s},\ \gamma)) \\
&\qquad+\sum\mu_i^s\sum\sum[q_{ji}^{c,s}-q_{ij}^{c,s}-D_i^s(\gamma_{ji}^c+\alpha_i^s,\ \alpha^{-s},\ \gamma)]
\end{aligned} \tag{7.137}$$

The new adjoint variable, μ should satisfy:

$$\frac{d\mu_i^s}{dt}=-\frac{\partial\mathcal{H}^2}{\partial I_i^s} \tag{7.138}$$

$$\mu_i^s(t_f)=0 \tag{7.139}$$

And the maximum principle turns to a simple optimization problem which can be reformulated to Kuhn-Tucker conditions

$$\begin{aligned}
&Max\ \mathcal{H}^2(q^s,\ \alpha^s,\ q^{-s},\ \alpha^{-s}) \\
&s.t. \\
&0\le\alpha_{x'}^s\le\alpha^s \\
&0\le q_{ij}^{c,s}\le Q^s
\end{aligned} \tag{7.140}$$

Solving problem (7.140) gives us the closed form of the control law.

Let's denote the control law derived from (7.140) as followings:

$$q_{ij}^{c,s} = g_{ij}^{c,s}(\lambda^s,\ \gamma^c,\ \mu^s)\forall c \in C,\ \forall s \in S,\ \forall (i,\ j) \in \mathcal{W} \tag{7.141}$$

$$\alpha_i^s = k_i^s(\lambda^s,\ \gamma^c,\ \mu^s)\forall s \in S,\ \forall i \in \mathcal{N} \tag{7.142}$$

Where g and k are the results for control variables $q_{ij}^{c,s}$ and α_i^s in optimization problem (7.140).

Therefore, the original EPEC problem has been converted into the single dynamic oligopolistic network competition problem:

$$max_{\gamma_{ij}^c,(i,j)\in W} J^c(\gamma_J^c x,\ h^c \gamma_{i,j}^{-c}) = \int_{t_0}^{t_f} e^{-\rho t}\Bigg[\sum_{s\in\mathcal{S}}\sum_{(i,j)\in\mathcal{W}}\int_0^{q_{ij}^{c,s}} \gamma_{ij}^c(q,\ t)N_{ij}(\gamma_{ij}^c(q,\ t),\ q,\ t)dq - \sum_{(i,j)\in W}\sum_{p\in\mathcal{P}_{ij}} \Psi_p^c(t, h_p^c)h_p^c(t)\Bigg]dt \tag{7.143}$$

Subject to

$$q_{ij}^{c,s} = g_{ij}^{c,s}(\lambda^s,\ \gamma^c,\ \mu^s)\forall c \in C,\ \forall s \in S,\ \forall (i,\ j) \in \mathcal{W} \tag{7.144}$$

$$\alpha_i^s = k_i^s(\lambda^s,\ \gamma^c,\ \mu^s)\forall s \in S, \forall i \in \mathcal{N} \tag{7.145}$$

$$\frac{d\mu_i^s}{dt} = -\frac{\partial \mathcal{H}^{s2}}{\partial I_i^s},\ \forall s \in S,\ \forall i \in \mathcal{N} \tag{7.146}$$

$$\frac{d\lambda_i^s}{dt} = -\frac{\partial \mathcal{H}^s}{\partial I_i^s} = -\frac{\partial \psi^s(I_i^s(t), t)}{\partial I_i^s},\ \forall s \in S, \forall i \in \mathcal{N} \tag{7.147}$$

$$\frac{dI_i^s}{dt} = \sum_{c\in C}\sum_{(j,i)\in\mathcal{W}} q_{ji}^{c,s} - \sum_{c\in C}\sum_{(i,j)\in\mathcal{W}} q_{ij}^{c,s} - \sum_{c\in C}\sum_{(j,i)\in\mathcal{W}} D_i^s(\gamma_{ji}^c + \alpha_i^s, \alpha^{-s}, \gamma),$$

$$\forall s \in S,\ \forall i \in \mathcal{N} \tag{7.148}$$

$$\lambda_i^s(t_f) = \frac{-e^{-\rho t_f}\partial Z_f[I(t_f), t_f]}{\partial I_i^s(t_f)},\ \forall s \in S,\ \forall i \in \mathcal{N} \tag{7.149}$$

$$\mu_i^s(t_f) = 0,\ \forall s \in S,\ \forall i \in \mathcal{N} \tag{7.150}$$

$$I^s(t^f) = M^f \tag{7.151}$$

$$\underline{M}_{c,i} \leq \gamma_{ij}^c \leq \overline{M}_{c,i},\ \forall c \in C,\ (i,\ j) \in W \tag{7.152}$$

In general:

$$\Omega^{c2} = \{(\gamma_{ij}^c,\ h^c) : ((7.144) - (7.152))\ hold\} \tag{7.153}$$

is the set of feasible controls. Carriers' problem solves the following extremal problem:

$$\left.\begin{aligned} &\max && J^c \\ &Subject\ to\ (\gamma_{ij}^c,\ h^c) \in \Omega^{c2} \end{aligned}\right\}\forall c \in C \tag{7.154}$$

Oligopoly of carriers consider the decision of other shippers to make their own non-cooperative decision. This makes (7.154) a differential Nash game.

7.2.7 Necessary and Sufficient Conditions of The Carriers' Oligopoly

Following the spirit of (Friesz et al., [68]) and to be prepared for further numerical analysis, we here list the qualitative analysis on the upper level oligopoly to write the necessary and sufficient conditions of the carriers' Oligopoly problem.

Let us first introduce the short hand notation η for the objective functional (7.143):

$$\eta^c(\gamma_{ij}^c,\ h^c) = e^{-\rho t}[\sum_{s\in\mathcal{S}}, \sum_{(ij)\in\mathcal{W}} \int_0^{q_{ij}^{c,s}} \gamma_{ij}^c(q,\ t)N_{ij}(\gamma_{ij}^c(q,\ t),\ q,\ t)dq - \sum_{(ij)\in W}\sum_{p\in\mathcal{P}_{ij}} \Psi_p^c(t)h_p^c(t)] \tag{7.155}$$

Denote by H^c the Hamiltonian of (7.155) for any $\forall c \in C$

$$
\begin{aligned}
\mathcal{H}^c = -\eta^c(\gamma_{ij}^c,\ h^c) + \sum_{i\in\mathcal{N}}\sum_{s\in\mathcal{S}}\beta1_i^s\sum_{c\in C}\sum_{(j,i)\in\mathcal{W}}[q_{ji}^{c,s} - q_{ij}^{c,s} - D_i^s(\gamma_{ji}^c + \alpha_i^s, \alpha^{-s}, \gamma)]\\
+\sum_{x\in\mathcal{N}}\sum_{s\in\mathcal{S}}\beta2_i^s\{-\frac{\partial\psi^s(I_i^s(t),t)}{\partial I_i^s}\}\\
+\sum_{i\in\mathcal{N}}\sum_{s\in\mathcal{S}}\beta3_i^s\{-\frac{\partial\mathcal{H}^{s2}}{\partial I_i^s}\}
\end{aligned}
\tag{7.156}
$$

We can quickly write out the necessary conditions for the upper level $\forall c \in C$

- Maximum principle:

$$
(\gamma_{ij}^c,\ h^c)\ := \arg\max_{(\gamma_{ij}^c, h^c)\in\Omega^c}\mathcal{H}^c \tag{7.157}
$$

- Adjoint equations:

$$
\begin{aligned}
\frac{d\beta1_i^s}{dt} = -\frac{\partial\mathcal{H}^c}{\partial I_i^s}\\
\frac{d\beta2_i^s}{dt} = -\frac{\partial\mathcal{H}^c}{\partial \mu_i^s}\\
\frac{d\beta3_i^s}{dt} = -\frac{\partial\mathcal{H}^c}{\partial \lambda_i^s}
\end{aligned}
\tag{7.158}
$$

- Transversality conditions:

$$
\beta1_i^s(t_f)\ ,\ \beta2_i^s(t_f)\ ,\ \beta3_i^s(t_f) = 0 \tag{7.159}
$$

- Dynamics:

$$
\begin{aligned}
\frac{dI_i^s}{dt} &= \sum_{c\in C}\sum_{j,i)\in\mathcal{W}} q_{ji}^{c,s} - \sum_{c\in C}\sum_{i,j)\in\mathcal{W}} q_{ij}^{c,s} - \sum_{c(\in C}, \sum_{\prime} D_i^s(\gamma_{ji}^c + (y_{\dot{x}}^s,\ \alpha^{-s},\ \gamma)\} \\
\frac{d\mu_i^s}{dt} &= -\frac{\partial\mathcal{H}^{s2}}{\partial I_i^s},\ \forall s\in S,\ \forall i\in\mathcal{N} \qquad (7.160) \\
\frac{d\lambda_i^s}{dt} &= -\frac{\partial\mathcal{H}^s}{\partial I_i^s} = -\frac{\partial\psi^s(I_i^s(t),t)}{\partial I_i^s},\ \forall s\in S,\ \forall i\in\mathcal{N} \\
\lambda_i^s(t_f) &= \frac{-e^{-\rho t_f}\partial Z_f[I(t_f),t_f]}{\partial I_i^s(t_f)},\ \forall s\in S,\ \forall i\in\mathcal{N} \\
\mu_i^s(t_f) &= 0,\ \forall s\in S,\ \forall i\in\mathcal{N} \\
I^s(t^f) &= M^f \\
\underline{M}_{c,i} &\leq \gamma_{ij}^c \leq \overline{M}_{c,i},\ \forall c\in C,\ (i,\ j)\in W
\end{aligned}
$$

In summary, the necessary conditions for the carriers" competition consists of (7.157-7.160) above

Familiarity with variational inequalities suggests that the following variational inequality has solutions that are differential Nash equilibria for a non-cooperative game in which individual firms maximize net profits in light of current information about their competitors:

$$
\begin{aligned}
&0 \geq \sum_{c\in C}\int_{t_0}^{t_f}[\sum_{c\in C}\sum_{(i,j)\in\mathcal{W}} \frac{\partial\mathcal{H}^{c*}}{\partial\gamma_{ij}^c}(\gamma_{ij}^c - \gamma_{ij}^{c*}) + \sum_{i\in\mathcal{N}}\frac{\partial\mathcal{H}_{c*}}{\partial h^c}(h^c - h^{c*})]dt \\
&\forall(\gamma,\ h)\in((7.158)-(7.160)) \qquad (7.161)
\end{aligned}
$$

The issue of immediate concern is to formally demonstrate that solutions of (7.161)are differential Nash equilibria. In fact we state the following result:

Theorem 7.2.9. *(Solution of DVI (7.161))*
Any solution of (7.161) is a solution of the dynamic oligopolistic network competition problem when regularity in the sense of Definition 7.2.1 holds.
Based on the first part of 7.2.1 the maximum principle in (7.161) can be reformulated as the following optimal control problem

$$
\begin{aligned}
&Max\ \sum_{c\in C}\int_{t_0}^{t_f}\left[\sum_{c\in C}\sum_{(i,j)\in\mathcal{W}}\frac{\partial\eta^{c*}}{\partial\gamma_{ij}^c}\gamma_{ij}^c + \sum_{i\in\mathcal{N}}\frac{\partial\eta^{c*}}{\partial h^c}h^c\right]dt \\
&\qquad s.t.\ \ \underline{M}_{ij} \leq \gamma_{ij}^k \leq \overline{M}_{ij},\ \forall(i,\ j)\in\mathcal{W} \qquad (7.162)
\end{aligned}
$$

Therefore by converting the maximum principle to (7.162) we are going to solve a following single layer optimal control problem:

$$Find\ (7.162) \tag{7.163}$$

$$Such\ that\ (\gamma,\ h) \in ((7.158) - (7.160)) \tag{7.164}$$

In the final formulation (7.163), the EPEC problem for each carrier is reformulated as a single level optimal control problem which can be solved by time discretization. Our scheme of time discretization is as follows: define the discrete instant of time $t_m = t_0 + m\triangle t$, with $m = 1, \cdots M$ where $\triangle t$ is the time step. At terminal time we have $t_M = t_f$ thus $M = \dfrac{t_f - t_0}{\triangle t}$. Hence, a finite dimensional nonlinear program will be formulated, allowing us to conduct numerical experiments by GAMS with the conopt solver.

Chapter 8
Numerical Example

8.1 Example of a Small Network

Let us consider a small network with 6 arcs and 4 nodes. 2 shippers offer identical commodity at every node to and 2 carriers will offer the freight service. Figure 8.2 illustrates this network.

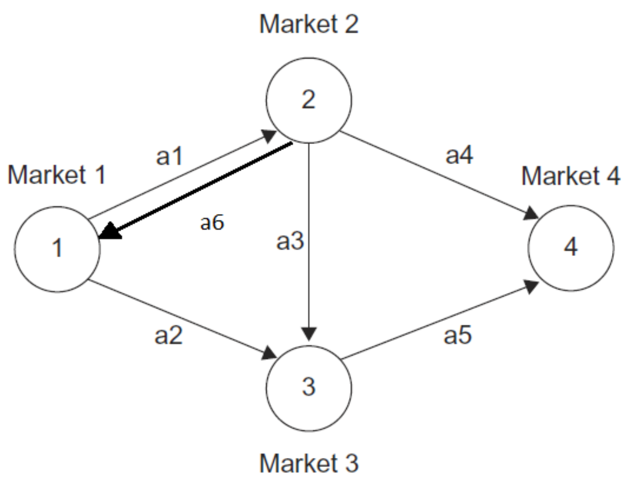

Figure 8.1. Network with 6 Arcs and 4 Nodes

The time interval of interest is [0; 15], with $t_0 = 0$, $t_f = 15$; Two shippers offer identical commodity at every node; they are interested in transporting their commodity between OD pairs within sets $W1$ and $W2$; respectively, where W1 $W_1 = (1,2),(1,3),(2,4),(3,4)$ for shipper 1 and $W2 = (1,2),(2,3),(2,4),(3,4)$ for shipper 2. The OD (1, 1) and (2, 2) corresponds to local delivery. Also carriers are trying to decide about the non-linear price for the first 10 units of the product.

The initial inventory for each shipper is:

$$I_1(0) = 40 \tag{8.1}$$
$$I_1(0) = 40$$

We impose the condition for the inventory allowed at terminal time by any shipper at any node:

$$I_i^s(15) = 0, \ \forall s \in S, \ i \in \mathcal{N} \tag{8.2}$$

These serves as the initial condition for the following inventory dynamics:

$$\begin{aligned}
\frac{dI_1^1}{dt} &= q_{11}^1 - q_{12}^1 - q_{13}^1 - q_{14}^1 - D_1^1 && (8.3)\\
\frac{dI_2^1}{dt} &= q_{12}^1 - D_2^1\\
\frac{dI_3^1}{dt} &= q_{13}^1 - D_3^1\\
\frac{dI_4^1}{dt} &= q_{14}^1 - D_4^1\\
\frac{dI_1^2}{dt} &= q_{21}^2 - D_1^2 && (8.4)\\
\frac{dI_2^2}{dt} &= q_{22}^2 - q_{21}^2 - q_{23}^2 - q_{24}^2 - D_2^2\\
\frac{dI_3^2}{dt} &= q_{23}^2 - D_3^2\\
\frac{dI_4^2}{dt} &= q_{24}^2 - D_4^2
\end{aligned}$$

We assume the linear average of the prices as P_{ij}^s which takes the following mathematical form:

$$P_{ij}^c = \frac{\Sigma_{s\in\mathcal{S}}\Sigma_{q=0}^{q_{ij}^{c,s}}\gamma_{ij}^c(q)}{\Sigma_{s\in\mathcal{S}}q_{i'_J}^{c,s}} \tag{8.5}$$

Also, we assume that the demand function on each node $i \in \mathcal{N}$ for firm s take the following linear form:

$$D_i^s(\gamma_{ji}^c + cy_i^{\mathcal{S}},\ \alpha^{-s},\ \gamma) = \theta_i^0 - \beta_i^1(P_{\dot{x}j}^c + Cy_{i_s}^s) + \sum_{-c} \beta_i^2(P_{ij}^{-c} + \alpha_{\dot{x}_f}^s)$$
$$+ \sum_{-s} \beta_i^3(P_{i-sj}^c + \alpha_{i-s}^{-s}) + \sum_{-c}\sum_{-s} \beta_i^4(P_{i-sj}^{-c} + \alpha_{-i-s}^{-s}) \qquad (8.6)$$

where $\theta_{x'}^0, \beta_i^1, \beta_i^2$ and β_i^3 are constants. In addition, the total demand of firm s is

$$\mathcal{D}^s = \sum_{(i,j)\in\mathcal{W}} \mathcal{D}_i^s,\ \forall s \in S \qquad (8.7)$$

We assume the holding/inventory costs are quadratic and the production costs are linear and of the form:

$$\psi_i^s = \frac{1}{2} A_{\psi_i}^s (I_i^s)^2 \qquad (8.8)$$
$$V_i^s = \frac{1}{2} A_{V_i}^s (\Sigma_{c\in C}\Sigma_{i'_J\in\mathcal{W}} q_{ij}^{cs})^2 \qquad (8.9)$$

Please see Table 8.1 for details of the fixed parameters s

Parameters for the Small Network			
Parameter Name	value	Parameter Name	value
$\theta_i^0,\ i = 1,2,3,4$	40,40,40,40	$A_{V_1}^1$	0.3
$\beta_i^1,\ i = 1,2,3,4$	0.37,0.38,0.36,0.35	$A_{V_2}^2$	0.5
$\beta_i^2,\ i = 1,2,3,4$	0.15,0.14,0.13,0.15	$A_{\psi_1}^1$	4
$\beta_i^3,\ i = 1,2,3,4$	0.15,0.14,0.13,0.15	$A_{\psi_2}^2$	2
$\beta_i^4,\ i = 1,2,3,4$	0.05,0.01,0.05,0.07	ρ	0.01

Table 8.1. Parameters for the Small Network

We assume that the quadratic holding cost function is reasonable since opportunity costs usually increase more than proportionally to the amount of capital left inactive in cash and the inventory cost for backorder should also be positive. Also we assume the cost of each arc is monotonically increasing over time. The cost of arcs are separable and independent of each other.

Moreover, the following bounds on will be imposed on the control variables:

$$
\begin{aligned}
&Q_s = 40,\ \forall i \in \mathcal{N},\ \forall s \in S \\
&\alpha_s = 500,\ \forall s \in S \\
&\underline{M}_{c,x'} = 1 \forall c \in C,\ \forall i \in \mathcal{N} \\
&\overline{M}_{c,i} = 2000 \forall c \in C,\ \forall i \in \mathcal{N}
\end{aligned}
$$

Also, as mentioned before, the demand profile, denoted $N(\gamma,\ p)$, shows the number of customers (shippers) wiling to purchase q^{th} units at the marginal price γ [37]. We assume that profile demand on each node $i \in N$ for firm s take the following linear form:

$$N^c_{ij}(\gamma^c_{ij}(q,\ t),\ q,\ t) = \theta^1_{c,ij} - \beta^5_{c,ij} * (\gamma^c_{ij}(q,\ t)) - \beta^6_{c,ij} * q \tag{8.10}$$

Please see table 8.2 for details of the fixed parameters for the demand profile.

Parameters for The Demand Profile			
c, w_{ij}	$\theta^1_{c,ij}$	$\beta^5_{c,ij}$	$beta^6_{c,ij}$
$c1, W11$	50	0.18	1.60
$c1, W12$	60	0.12	1.62
$c1, W13$	55	0.14	1.44
$c1, W14$	45	0.13	1.43
$c1, W21$	50	0.22	1.42
$c1, W22$	55	0.15	1.15
$c1, W23$	65	0.17	1.52
$c1, W24$	50	0.19	1.34
$c2, W11$	60	0.20	1.13
$c2, W12$	60	0.22	1.13
$c2, W13$	55	0.24	1.60
$c2, W14$	65	0.23	1.62
$c2, W21$	60	0.22	1.44
$c2, W22$	45	0.15	1.43
$c2, W23$	45	0.17	1.42
$c2, W24$	60	0.19	1.50

Table 8.2. Parameters for The Demand Profile

Path-Arc relationship		
Path	Arc Sequence	OD Pair
p_1	a_1	(1,2)
p_2	a_2	(1,3)
p_3	a_1, a_2	(1,3)
p_4	a_1, a_4	(1,4)
p_5	a_1, a_2, a_5	(1,4)
p_6	a_6	(1,4)
p_7	a_2	(2,1)
p_8	a_6, a_2	(2,3)
p_9	a_4	(2,3)
p_{10}	a_3, a_5	(2,4)
p_{11}	a_5	(3,4)
p_{12}	a_2, a_5, a_6	(2,4)

Table 8.3. Path-Arc relationship

We believe that it is natural to consider the demand profile function to decrease over time because popularity of the product will decrease.

To find the travel cost, we first need to find the arc flow. The flow of commodity on path $p \in \mathcal{P}$ traversing arc a at time t is denoted by $h_p(t)$. so, the arc flow is given by

$$f_a(t) = \sum_{p \in P_i j} \sigma_{ap} h_p(t) \forall a \in A, \ \forall (i, \ j) \in \mathcal{W} \tag{8.11}$$

While $f_a(t)$ shows the arc flow on arc a and A shows the sets on arcs on the network of this numerical example. Table 8.3 gives the relationships between arc and path variables.

Therefore, arc-path incidence matrix will be as follows:

$$\Delta_p = (\delta_a p) \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Now, we are ready to define the travel time (path delay function) for each arc W_{ij}:

$$D_{c,ij} = A_{c,ij} + B_{c,ij}(f_{ij})^n, \ i = 1, \ 2, \ 3, \ 4, \ 5, \ 6 \tag{8.12}$$

Where A_a and $B_a \in \mathbb{R}^1_{++}$ are known constants.

Please see table 8.4 for details of the fixed parameters in path delay function

Path Delay Parameters		
c, w_{ij}	$A_{c,ij}$	$B_{c,ij}$
$c1, W11$	1	0.1
$c1, W12$	1	0.2
$c1, W13$	1.2	0.4
$c1, W14$	1.2	0.4
$c1, W21$	1	0.3
$c1, W22$	1.2	0.2
$c1, W23$	1	0.2
$c1, W24$	1.2	0.3
$c1, W11$	1.5	0.5
$c1, W12$	1.5	0.5
$c1, W13$	1.5	0.2
$c1, W14$	1.5	0.5
$c1, W21$	1.6	0.3
$c1, W22$	1.6	0.4
$c1, W23$	1.4	0.5
$c1, W24$	1.5	0.3

Table 8.4. Parameters for The Path Delay

8.1.1 Computational Results of the Small Network

As mentioned earlier, the finite dimensional time discretization is used to solve the problem. We defined the discrete instant of time $t_m = t_0 + m\triangle t$, with $m = 1, \cdots M$ where $\triangle t$ is the time step. At terminal time we have $t_M = t_f$ thus $M = \dfrac{t_f - t_0}{\triangle t}$. This finite dimensional Nonlinear Program will be solved by GAMS in conjunction with the conopt solver.

The optimal solution is reported after 2952 iteration using a laptop with Intel(R) Core(TM) processor and 8.00 GB RAM. The computational time is 00:03:06 which shows the remarkable improvement using double adjoint algorithm compared to the diagnolization algorithm used in chapter 3. The numerical results are reported in this section in graphical forms.

Figures 8.2-8.17 show the marginal transportation price set by carriers for different OD pairs for 15 time periods and 10 units of product. Each figure shows how the price changes over time, for different quantities on different arcs. Since the profile demand has negative correlation with the transportation price and quantity, carriers adopt a decreasing-price policy over time and quantity. The x axis shows time, y axis shows the transportation marginal price at each quantity while colors distinguish between the different units of the product.

The pattern clearly shows the quantity discount at different units. For example for carrier 1 on ODII the first unit should be charged as 77.34 at time 1 and the second one as 75.14 and so on. so for somebody who is willing to buy 10 units, the final price would be summation of all the marginal prices from 1 to 10. Therefore the final price would be $77.34 + 75.14 + 72.93 + 70.73 + 68.52 + 66.32 + 64.12 + 61.93 + 59.73 + 57.54 = 674.33$. Since in linear pricing setting, buyers should buy 10 units of this product at the price $77.34 * 10 = 773.4$, we could say nonlinear pricing setting, gives the buyers a discount equal to 99.15 compared to linear pricing situation.

Also, figures 8.18 and 8.19 show the transportation price trajectories for the first unit of the product for the different paths and the two carriers. This graph can be used to compare the different policies that carriers take on different paths.

From the figures 8.2-8.19 we can observe two facts. Since shippers must satisfy their terminal inventory condition, the carriers take advantage of this fact and ship more products from the shipper with higher initial inventory. In our example, shipper 2 has a higher initial inventory and as a result we see that the transportation prices on OD pairs belong to shipper 2 are generally higher than the ones belong to shipper l. In addition, since the parameters of the problem are the same for different units of the product, the transportation price pattern at different units is almost the same over time. Secondly, the prices for OD pair *OD14* is higher compared to other OD pairs of shipper 1, which can be the result of the high transportation cost that carriers might face in this maps.

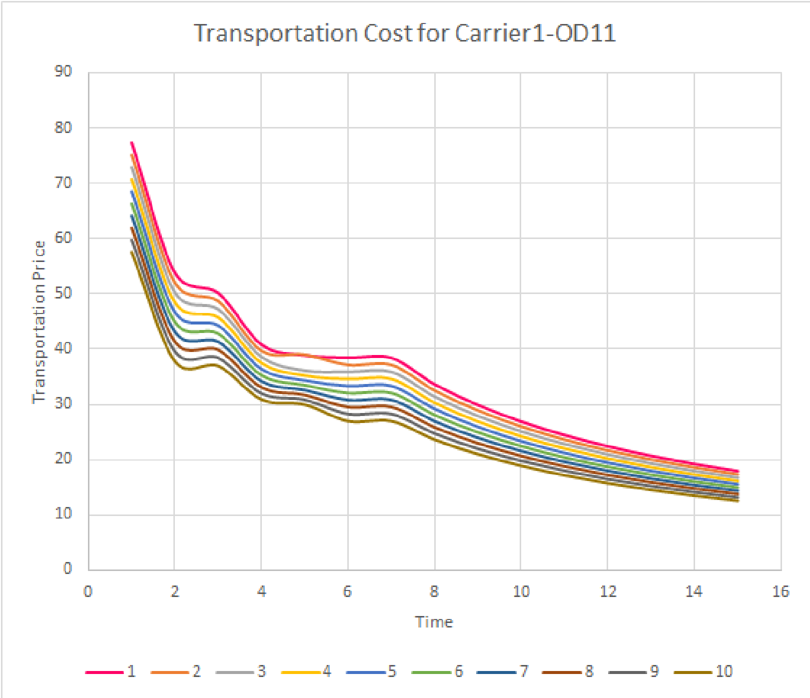

Figure 8.2. Transportation Price on ODII for carrier 1

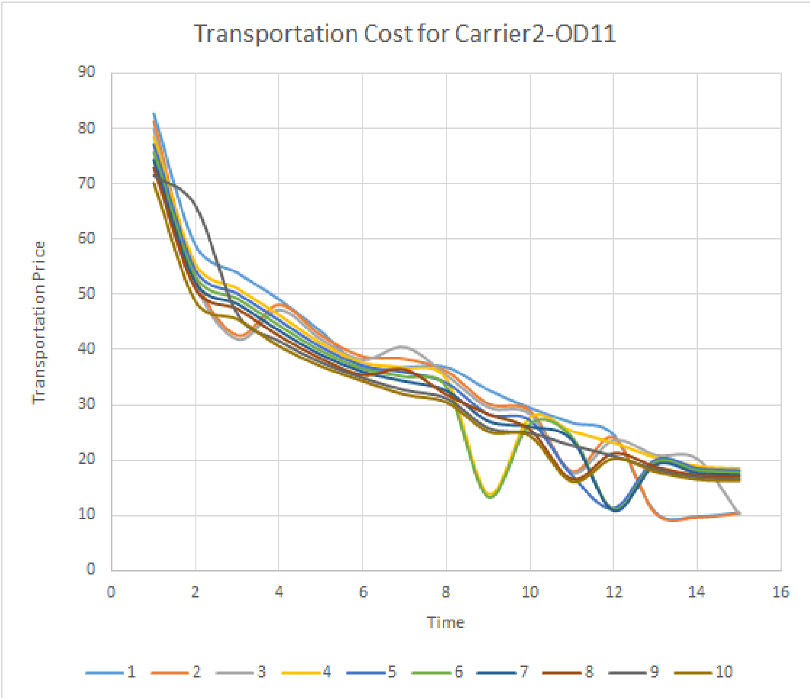

Figure 8.3. Transportation Price on ODII for carrier 2

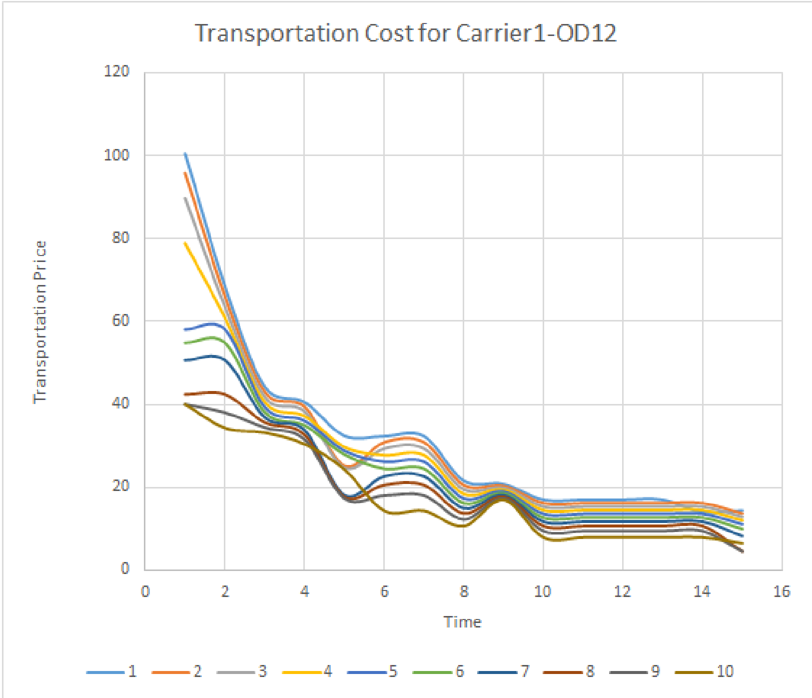

Figure 8.4. Transportation Price on OD12 for carrier 1

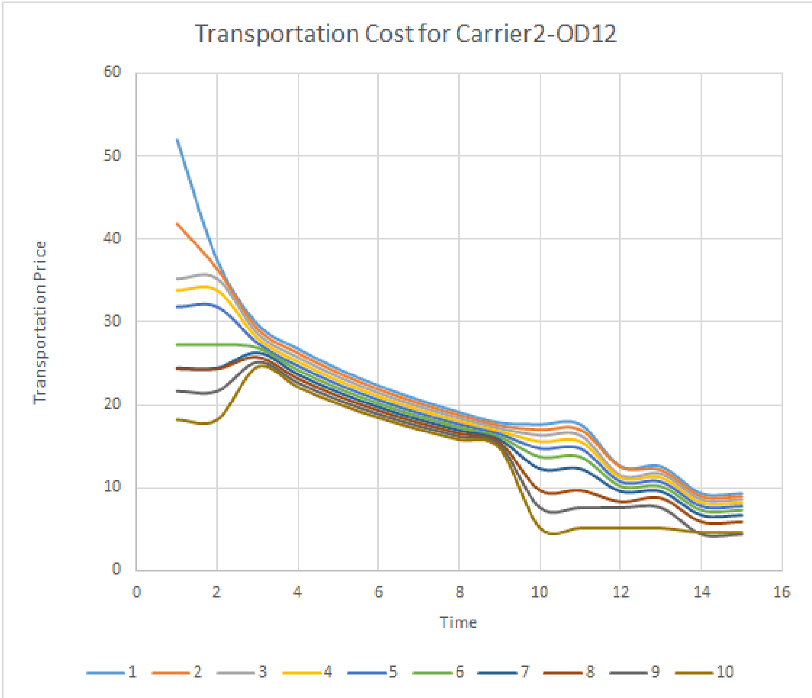

Figure 8.5. Transportation Price on OD12 for carrier 2

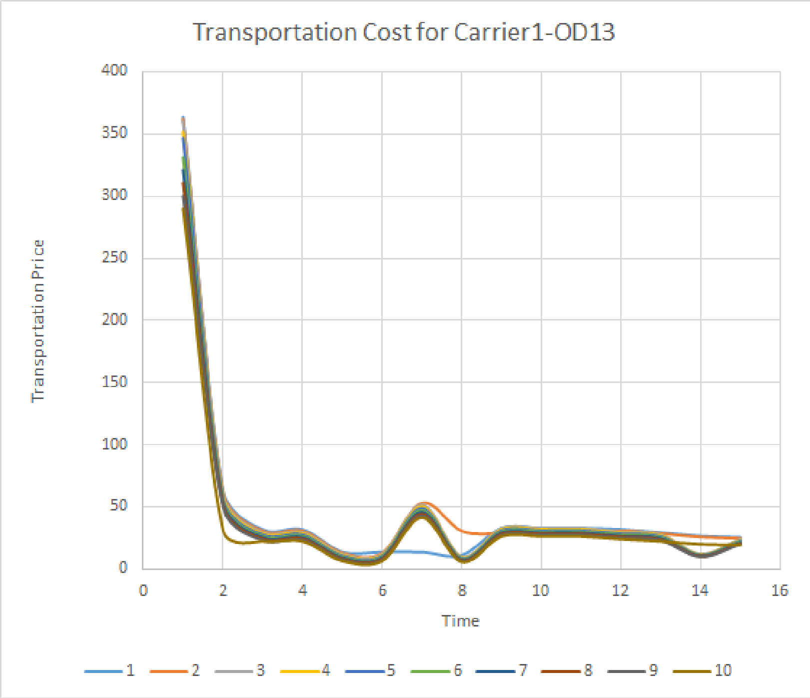

Figure 8.6. Transportation Price on OD13 for carrier 1

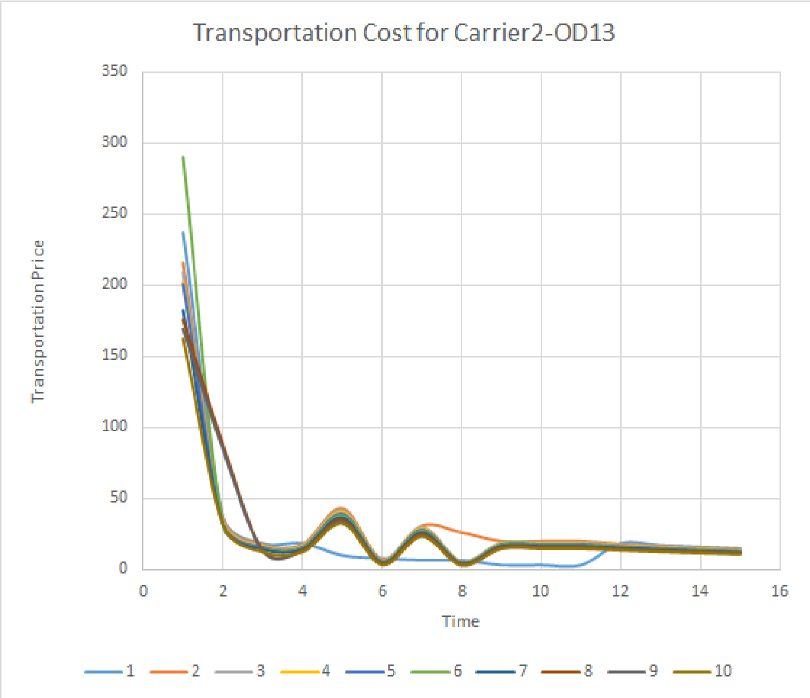

Figure 8.7. Transportation Price on OD13 for carrier 2

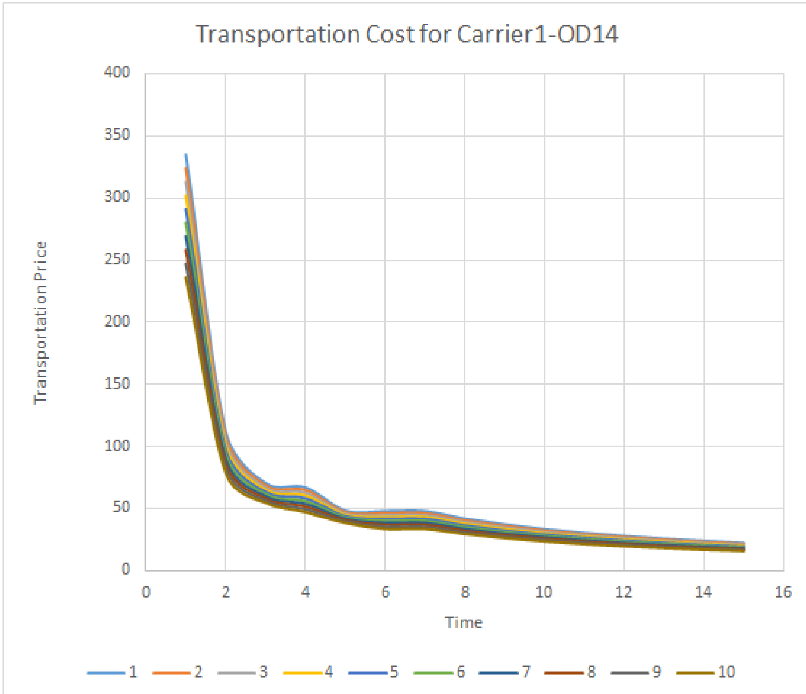

Figure 8.8. Transportation Price on OD14 for carrier 1

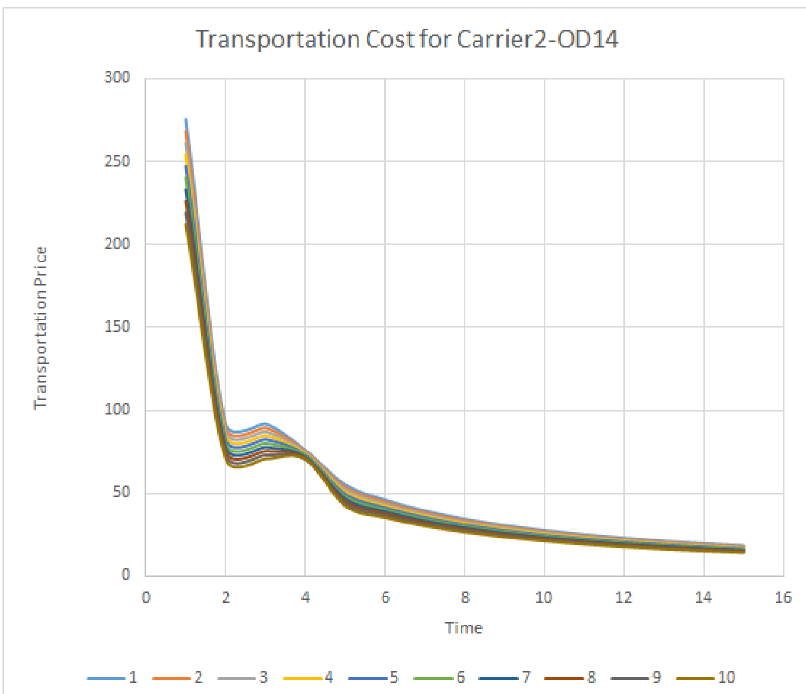

Figure 8.9. Transportation Price on OD14 for carrier 2

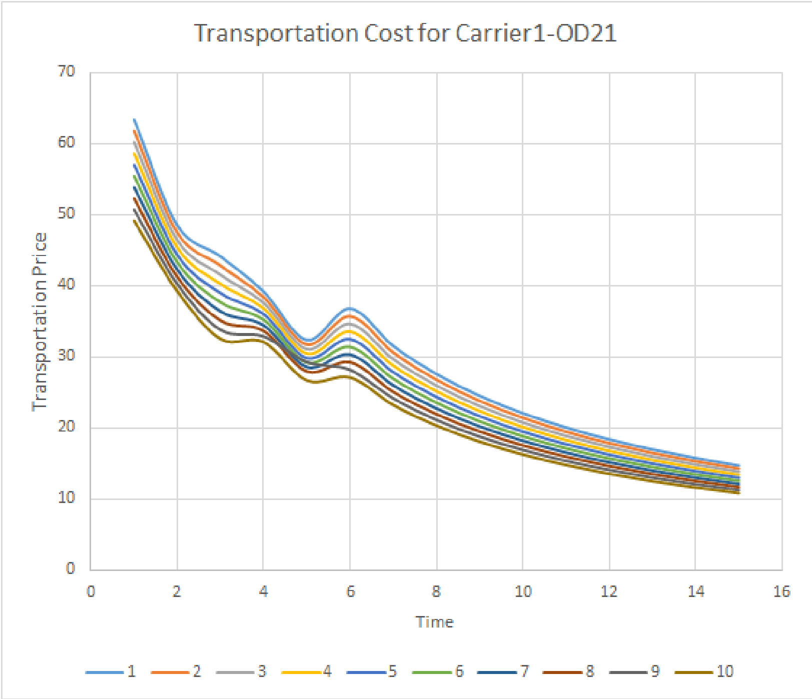

Figure 8.10. Transportation Price on OD21 for carrier 1

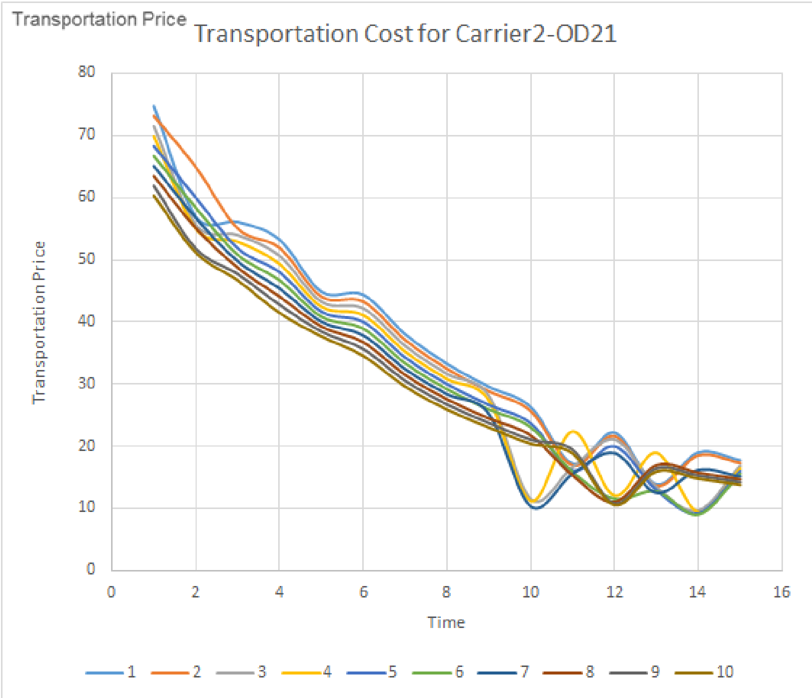

Figure 8.11. Transportation Price on OD21 for carrier 2

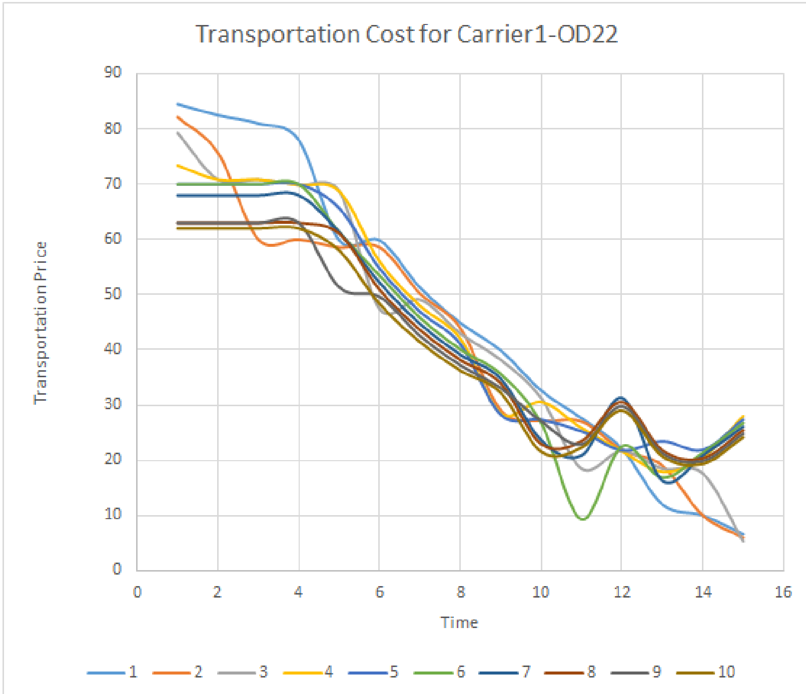

Figure 8.12. Transportation Price on OD22 for carrier 1

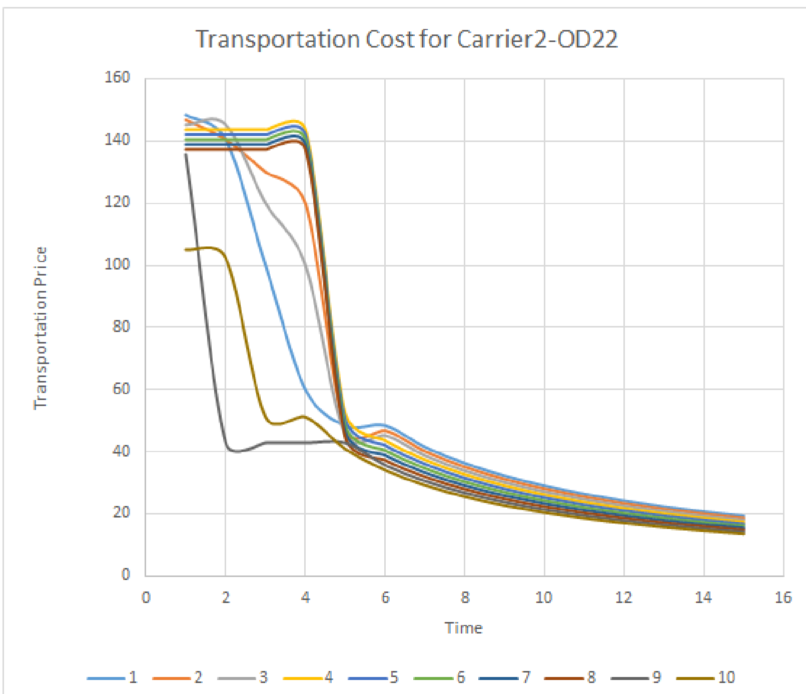

Figure 8.13. Transportation Price on OD22 for carrier 2

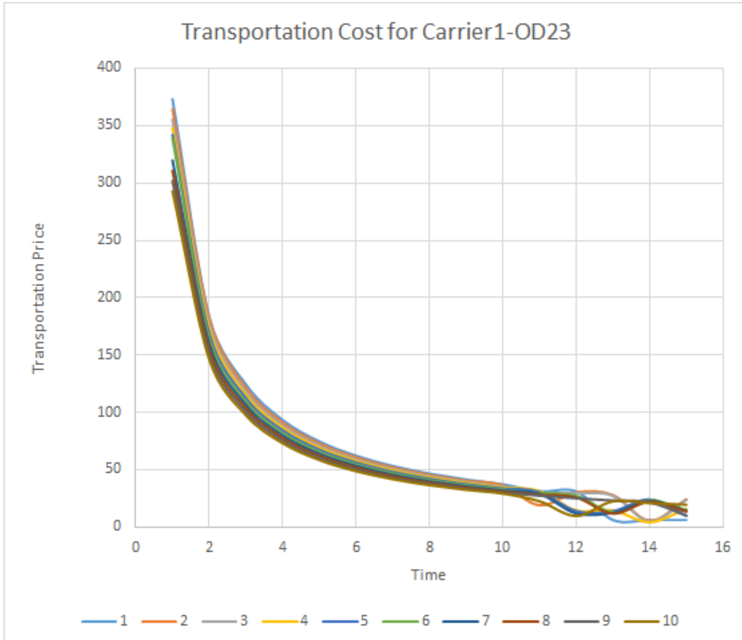

Figure 8.14. Transportation Price on OD23 for carrier 1

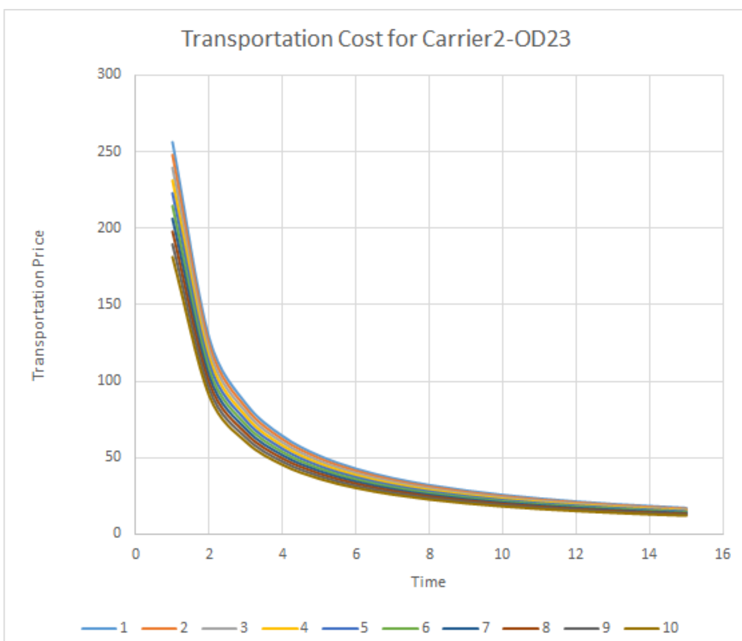

Figure 8.15. Transportation Price on OD23 for carrier 2

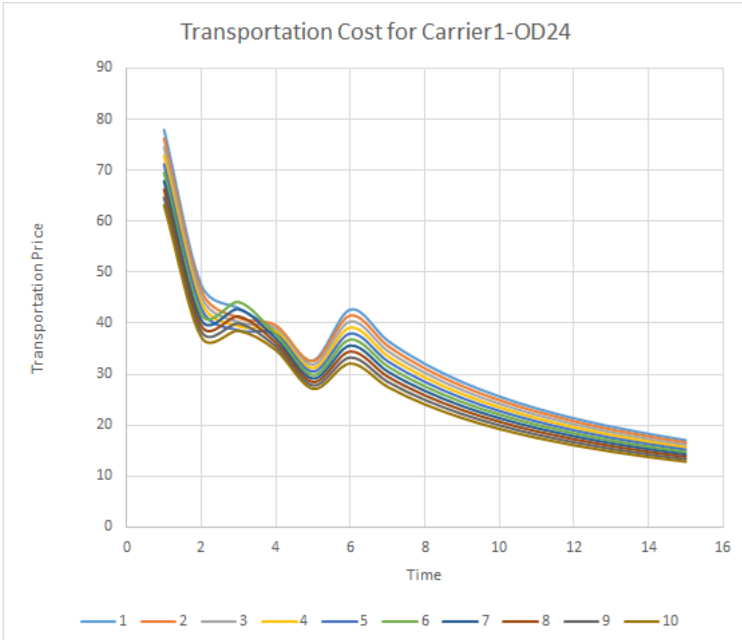

Figure 8.16. Transportation Price on OD24 for carrier 1

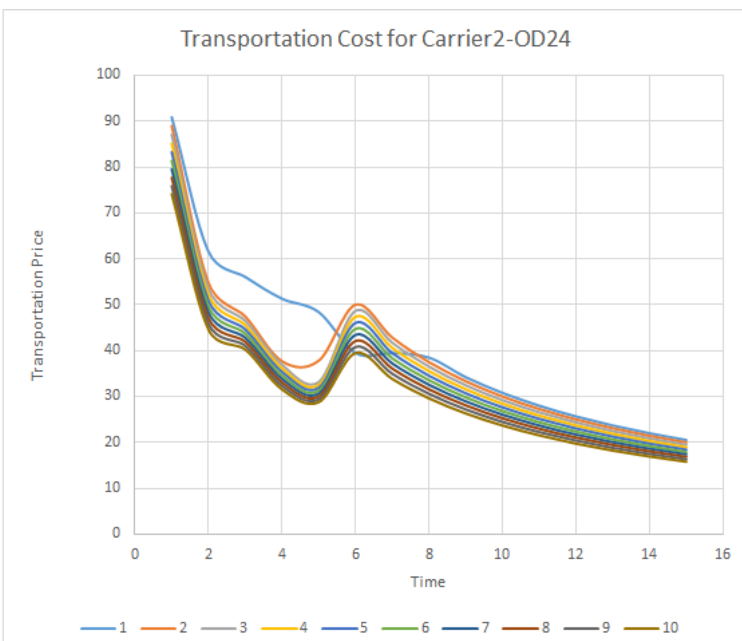

Figure 8.17. Transportation Price on OD24 for carrier 2

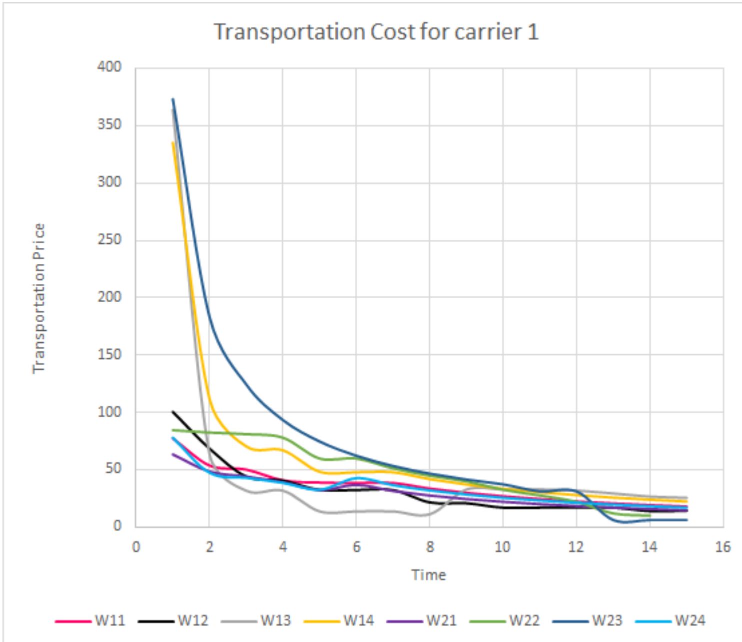

Figure 8.18. Transportation Price on different OD pairs for the first unit for carrier 1

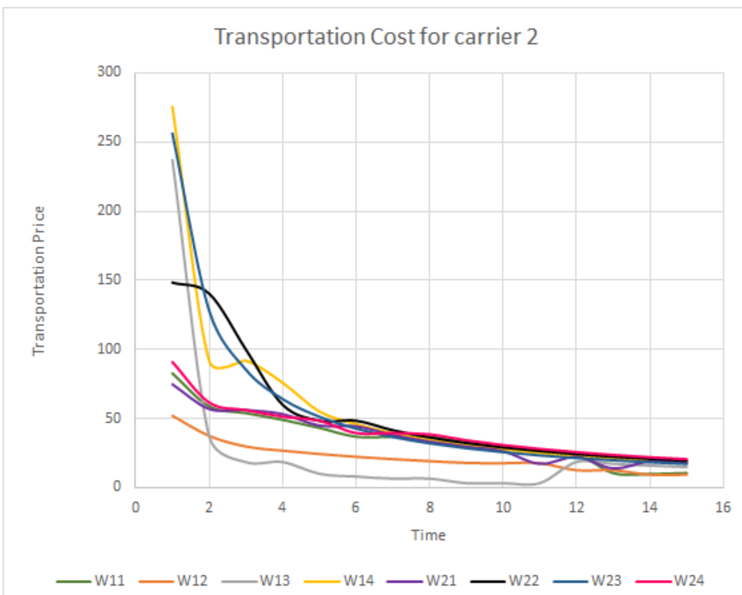

Figure 8.19. Transportation Price on different OD pairs for the first unit for carrier 2

Also Figure (8.20) and (8.21) depict the commodity price for shippers. We may observe that the carriers adopt an increasing-price policy over time. Also, since the inventory cost for shipper 1 is higher, shipper 1 sets lower commodity price compared to shipper 2 in order to keep less inventory. Figures (8.22) and (8.23) confirm this conclusion too.

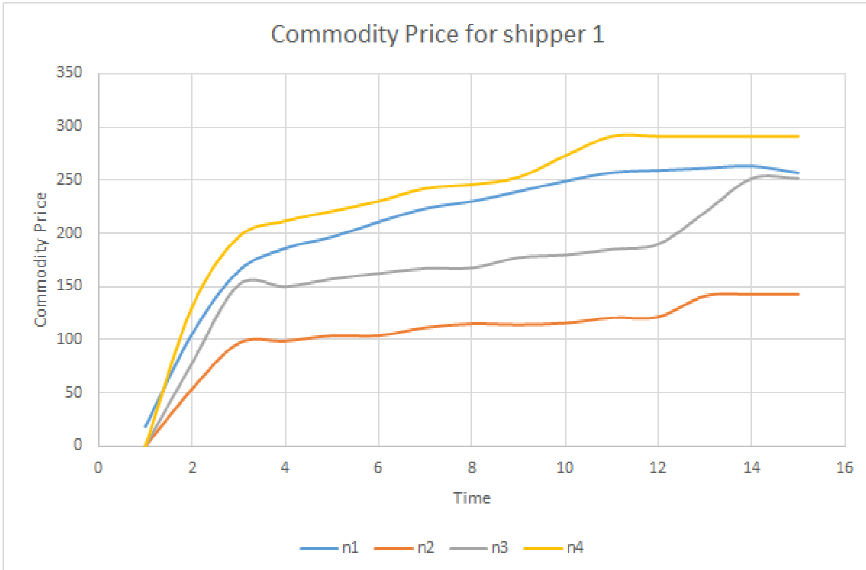

Figure 8.20. Commodity price for shippers1

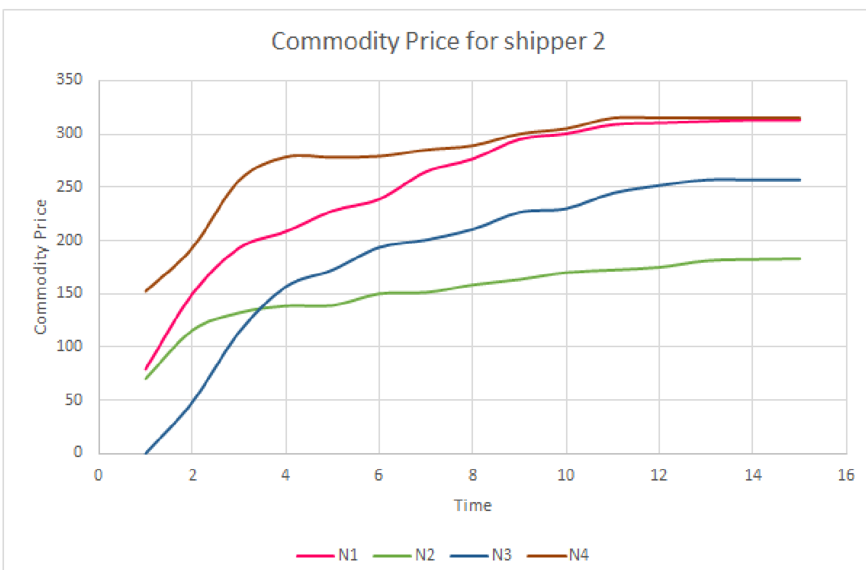

Figure 8.21. Commodity price for shippers2

In this numerical example, due to the lowest and highest demand sensitivity to price at node 4 and node 2 respectively, the commodity price at N4 is the highest and at N2 is the lowest. Also, due to the high commodity price at node 1 in comparison to node 2 and low unit transportation cost for shipment on pair OD21, shipper 1 prefer to transport the commodity from node 2 to 1 rather than keep it at node 2. These observations can be clearly found in 8.20, 8.21 while, 8.24 confirms them too.

Figures 8.22 and 8.23 show the inventory trajectories of shippers. Shippers try to balance the inventory by sometimes adopting a policy of back-ordering which is represented by a negative inventory level. At the end, both shippers' terminal inventory is equaled to zero.

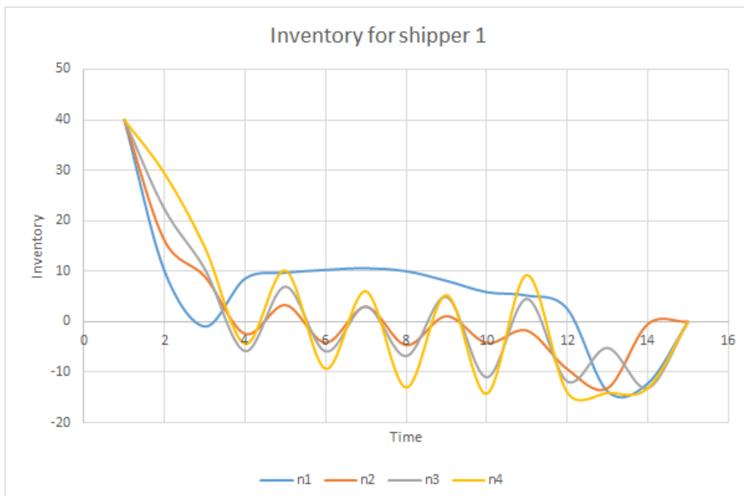

Figure 8.22. Inventory trajectory for shippers1

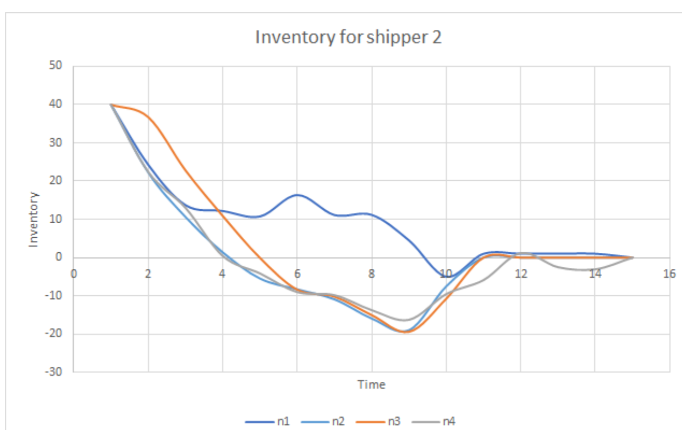

Figure 8.23. Inventory trajectory for shippers2

Figure 8.24 and 8.25 presents commodity shipment trajectories for different markets, on different paths over time. Different shippers follow different sales patterns on different nodes. There is relatively variable transport of goods until the terminal time arrives; then goods are moved between nodes to satisfy the terminal inventory.

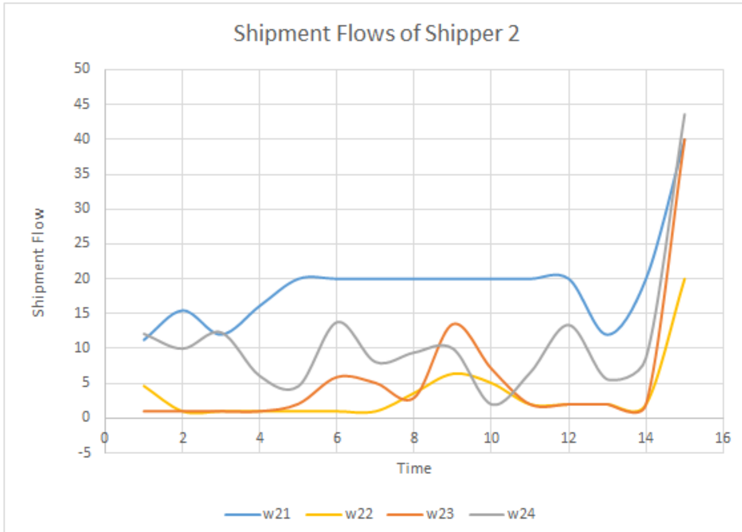

Figure 8.24. Commodity shipment trajectories for shipperl

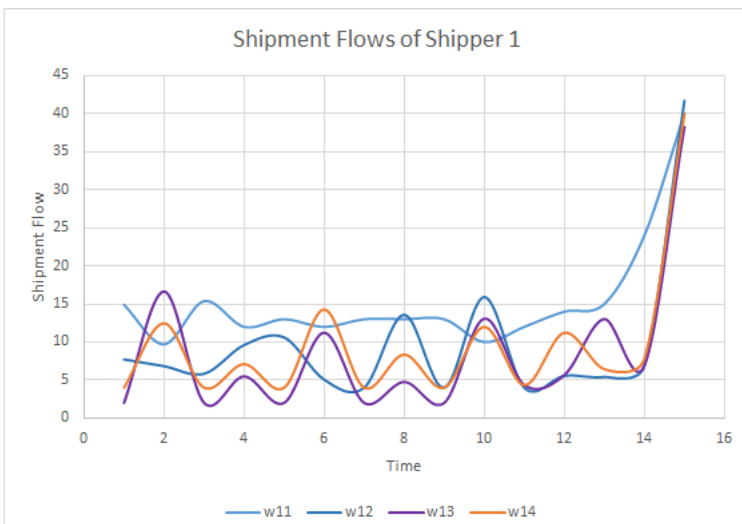

Figure 8.25. Commodity shipment trajectories for shipper2

8.2 Example of a Medium Network

Let us consider a medium network with 12 arcs and 9 nodes while 2 shippers offer identical commodity at every node. Figure 8.2 illustrates this network.

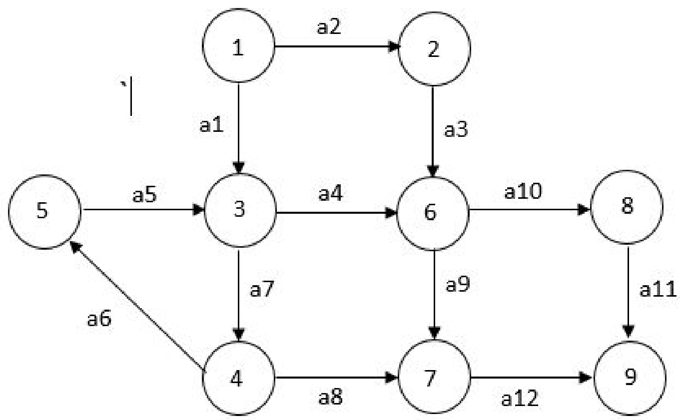

Figure 8.26. Network with 12 Arcs and 9 Nodes

The time interval of interest is [0; 15], with $t_0 = 0$, $t_f = 15$; Two shippers offer identical commodity at every node; they are interested in transporting their commodity between OD pairs within sets $W1$ and $W2$; respectively, where $W1 = W2 =${(1,1), (1, 2), (2,6), (6,7), (1,3), (4,7),(3,4), (3,6), (5,3), (4,5), (6,8), (7,9)}.

The initial inventory for each shipper is:

$$I_1(0) = 40 \tag{8.13}$$
$$I_1(0) = 60$$

We impose the condition for the inventory allowed at terminal time by any shipper at any node:

$$I_i^s(15) = 0, \ \forall s \in S, \ i \in \mathcal{N} \tag{8.14}$$

These serves as the initial condition for $i = 1, ..., 9$ for the following inventory dynamics:

$$\frac{dI_1^i}{dt} = q_{11}^i - q_{12}^i - q_{13}^i - D_1^i \tag{8.15}$$

$$\frac{dI_2^i}{dt} = q_{12}^i - q_{26}^i - D_2^i$$
$$\frac{dI_3^i}{dt} = q_{13}^i - q_{34}^i - q_{36}^i - D_3^i$$
$$\frac{dI_4^i}{dt} = q_{34}^i - q_{45}^i - D_4^i$$
$$\frac{dI_5^i}{dt} = q_{45}^i - q_{53}^i - D_5^i$$
$$\frac{dI_6^i}{dt} = q_{26}^i + q_{36}^i - q_{68}^i - D_6^i$$
$$\frac{dI_7^i}{dt} = q_{67}^i + q_{47}^i - q_{79}^i - D_7^i$$
$$\frac{dI_8^i}{dt} = q_{68}^i - D_8^i$$
$$\frac{dI_9^i}{dt} = q_{79}^i - D_9^i$$

Holding/inventory costs and demand function on each node $i \in \mathcal{N}$ are assumed same as the previous numerical example.Also, please see Table 8.5 for details of the fixed parameters

Parameters for the Small Network			
Name	value	Name	value
$\theta_i^0,\ i = 1,2,3,4,5,6,7,8,9$	40,40,40,40,40,40,40,40,40	$A_{V_i}^1$	0.3
$\beta_i^1,\ i = 1,2,3,4,5,6,7,8,9$	0.37,0.38,0.36,0.37,0.30,0.4,0.25,0.28,0.38	$A_{V_i}^2$	0.5
$\beta_i^2,\ i = 1,2,3,4,5,6,7,8,9$	0.15,0.14,0.13,0.15,0.14,0.14,0.16,0.13,0.14	$A_{\psi_i}^1$	4
$\beta_i^3,\ i = 1,2,3,4,5,6,7,8,9$	/0.1,0.1,0.1,0.1,0.1,0.1,0.1,0.1,0	$A_{\psi_i}^2$	2
$\beta_i^4,\ i = 1,2,3,4,5,6,7,8,9$	0.01,0.01,0.01,0.01,0.01,0.01,0.01,0.01,0.01	ρ	0.01

Table 8.5. Parameters for the Small Network

Same bounds and same shape of the demand profile from previous numerical exam is assumed for this network too. Also, please see table 8.6 for details of the fixed parameters for the demand profile.

8.2.1 Computational Results of the Medium Network

As mentioned earlier, the finite dimensional time discretization is used to solve the problem. We defined the discrete instant of time $t_m = t_0 + m\triangle t$, with $m = 1, \cdots M$ where $\triangle t$ is the time step. At terminal time we have $t_M = t_f$ thus $M = \dfrac{t_f - t_0}{\triangle t}$. This finite dimensional Nonlinear Program will be solved by GAMS in conjunction with the conopt solver.

Parameters for The Demand Profile			
c, w_{ij}	$\theta^1_{c,ij}$	$\beta^5_{c,ij}$	$beta^6_{c,ij}$
$c1, W11$	70	0.13	0.23
$c1, W12$	80	0.11	0.4
$c1, W26$	75	0.14	0.54
$c1, W67$	70	0.14	0.64
$c1, W13$	73	0.13	0.14
$c1, W47$	82	0.12	0.52
$c1, W34$	72	0.12	0.22
$c1, W36$	63	0.13	0.03
$c1, W53$	75	0.15	0.20
$c1, W45$	65	0.10	0.50
$c1, W68$	85	0.15	0.45
$c1, W79$	73	0.13	0.50
$c2, W11$	80	0.11	0.60
$c2, W12$	80	0.11	0.41
$c2, W26$	76	0.16	0.16
$c2, W67$	62	0.12	0.40
$c2, W13$	65	0.13	0.53
$c2, W47$	65	0.15	0.35
$c2, W34$	80	0.11	0.61
$c2, W36$	75	0.17	0.57
$c2, W53$	85	0.11	0.65
$c2, W45$	60	0.13	0.43
$c2, W68$	77	0.12	0.32
$c2, W79$	85	0.15	0.15

Table 8.6. Parameters for The Demand Profile

The optimal solution is reported using a laptop with Intel(R) Core(TM) processor and 8.00 GB RAM. The computational time is 00:06:33 which shows the remarkable improvement compared to the diagnolization algorithm.

Figures 8.27-8.38 show the marginal transportation price set by carriers. Each figure shows how the price changes over time, for different quantities on different arcs. Similar to the previous network carriers adopt a decreasing-price policy over time and quantity.

Also, figures 8.39 shows the transportation price trajectories for the first unit of the product for the different paths and the two carriers. This graph can be used to compare the different policies that carriers take on different paths.

Since the parameters of the problem are the same for different units of the product, the transportation price pattern and values at different units in figures 8.39 are almost the same

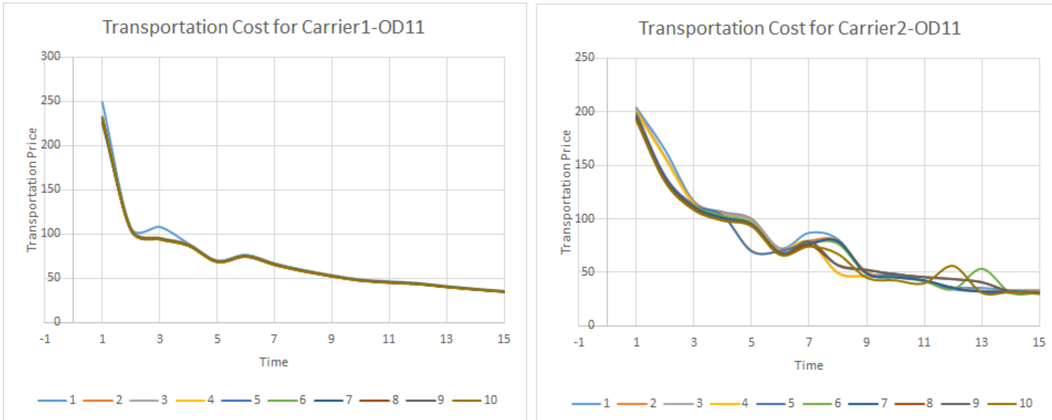

Figure 8.27. Transportation Price on OD11

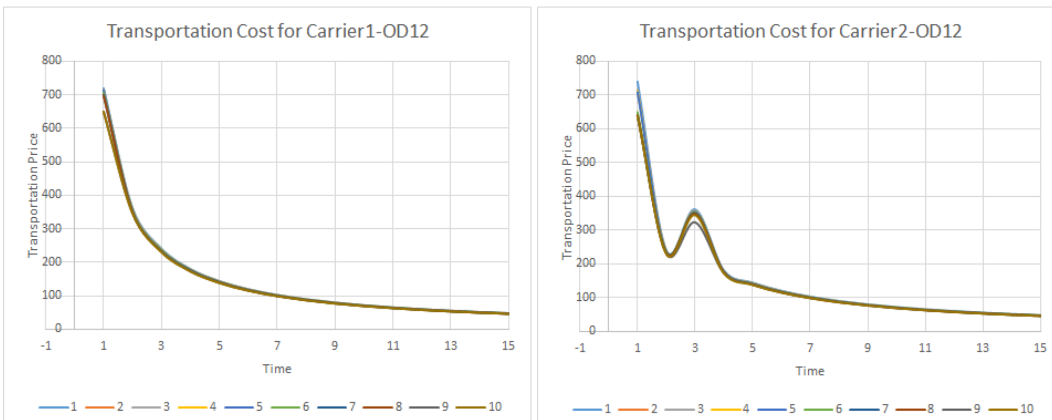

Figure 8.28. Transportation Price on OD12

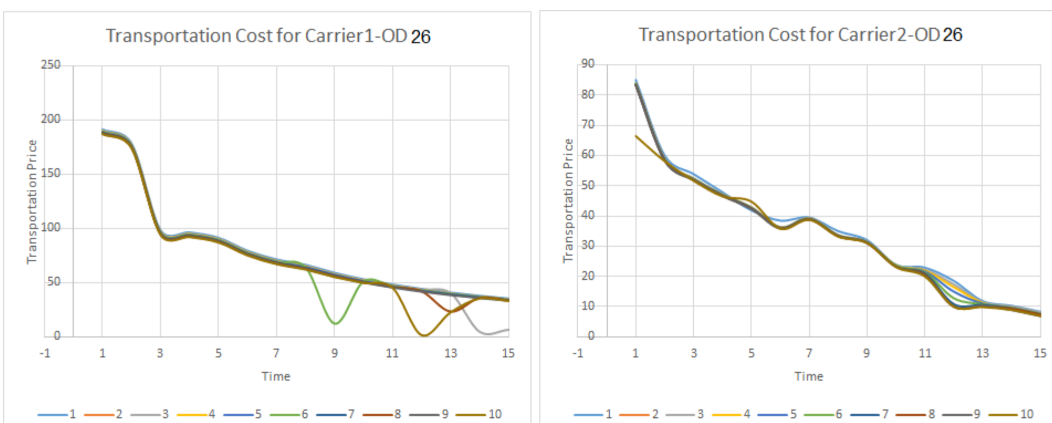

Figure 8.29. Transportation Price on OD26

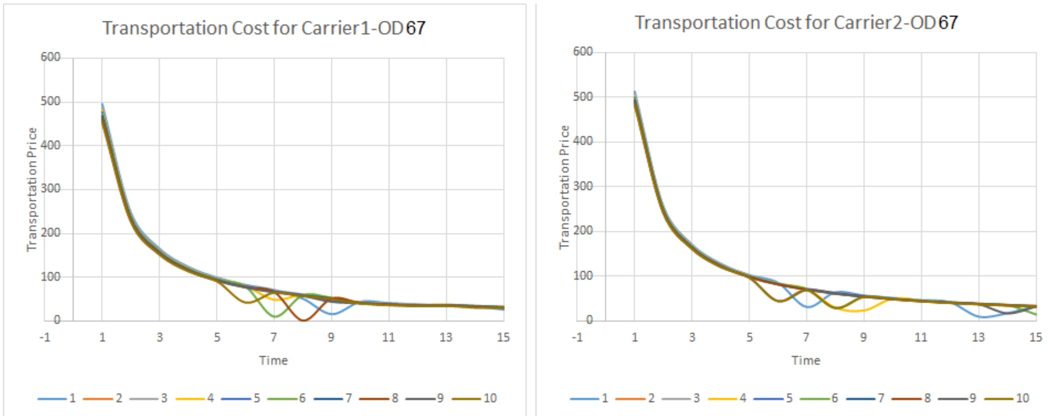

Figure 8.30. Transportation Price on OD67

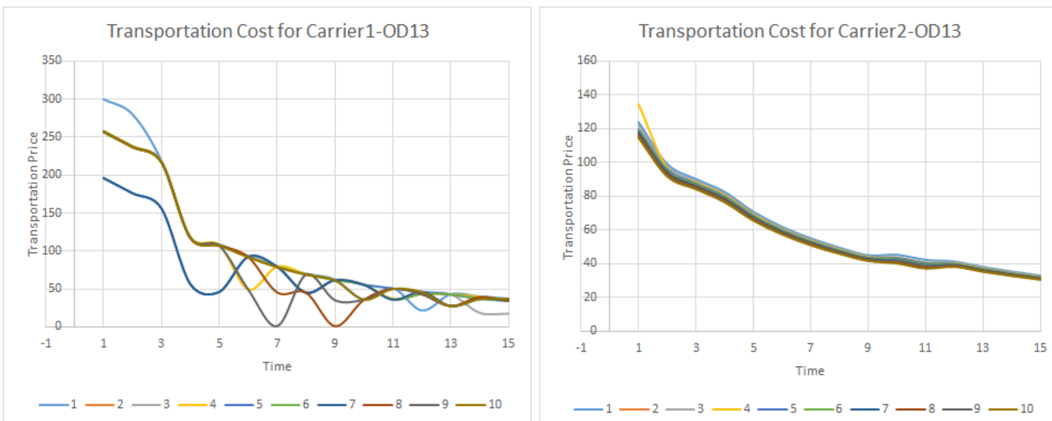

Figure 8.31. Transportation Price on OD13

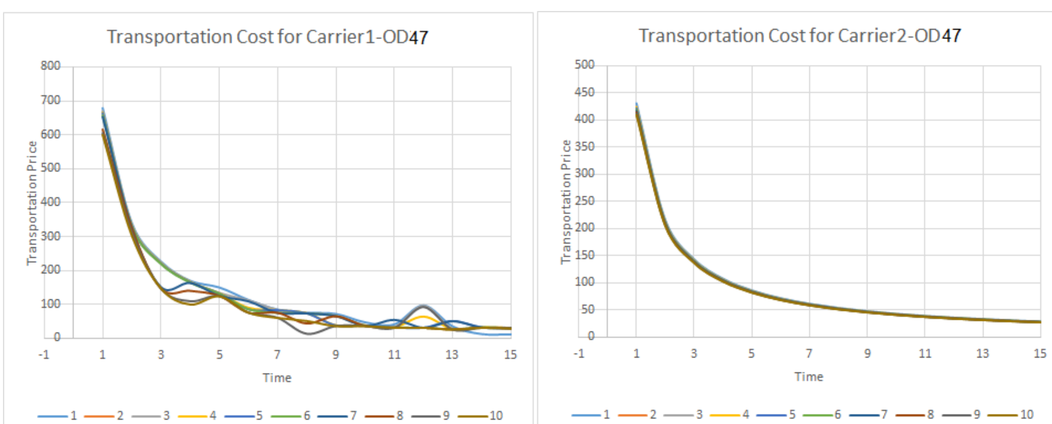

Figure 8.32. Transportation Price on OD47

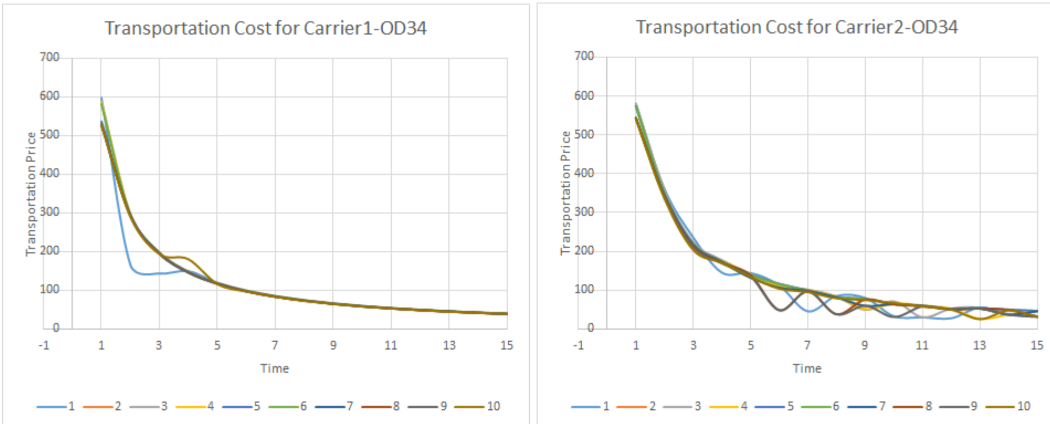

Figure 8.33. Transportation Price on OD34

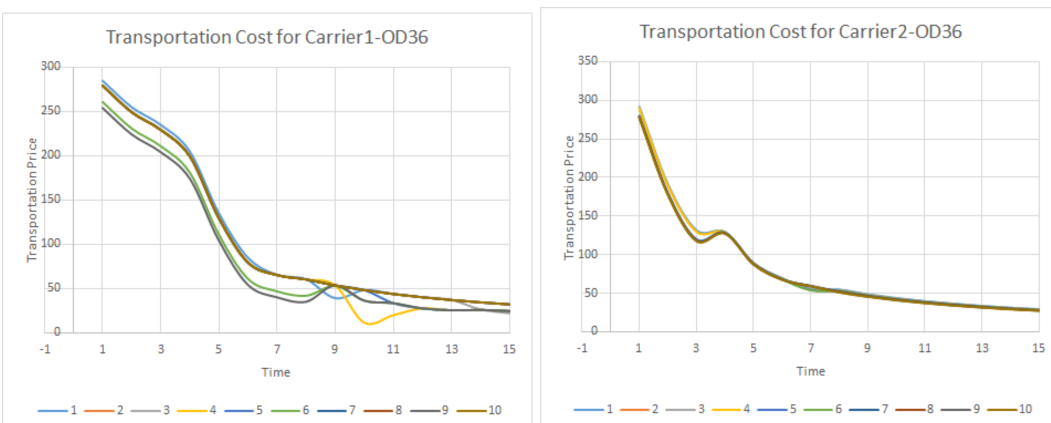

Figure 8.34. Transportation Price on O36

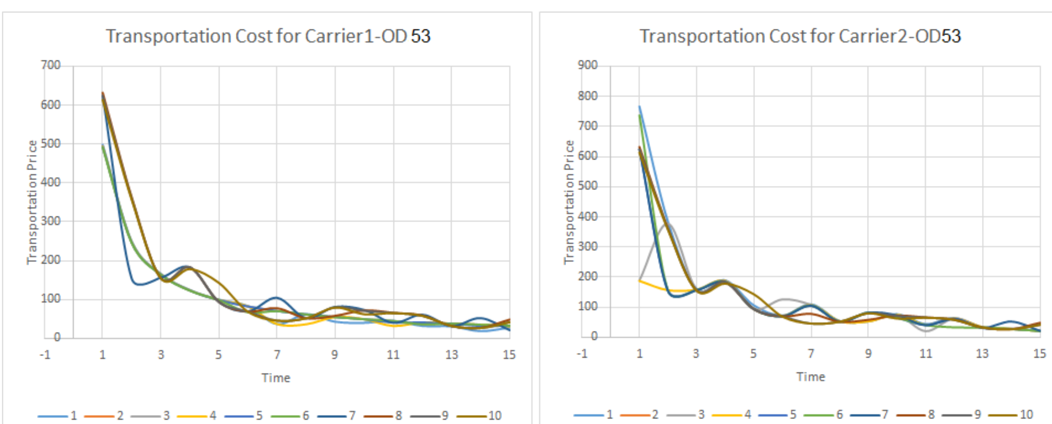

Figure 8.35. Transportation Price on OD53

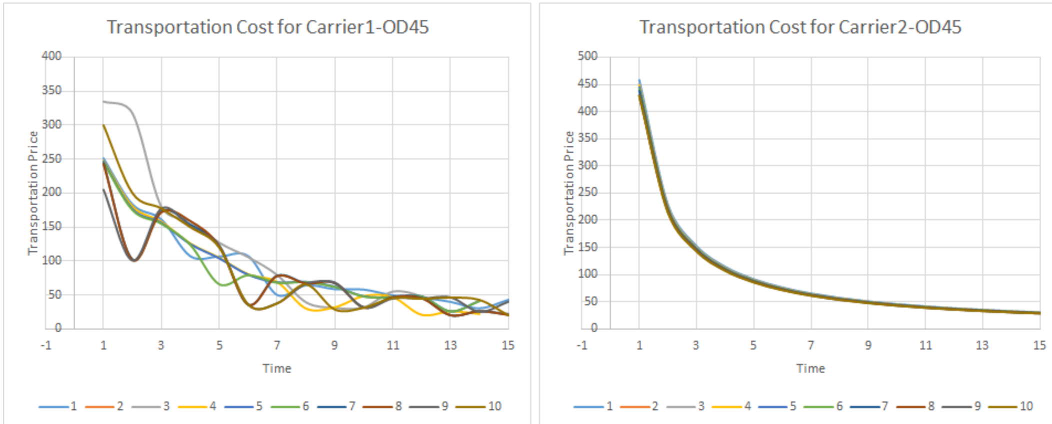

Figure 8.36. Transportation Price on OD45

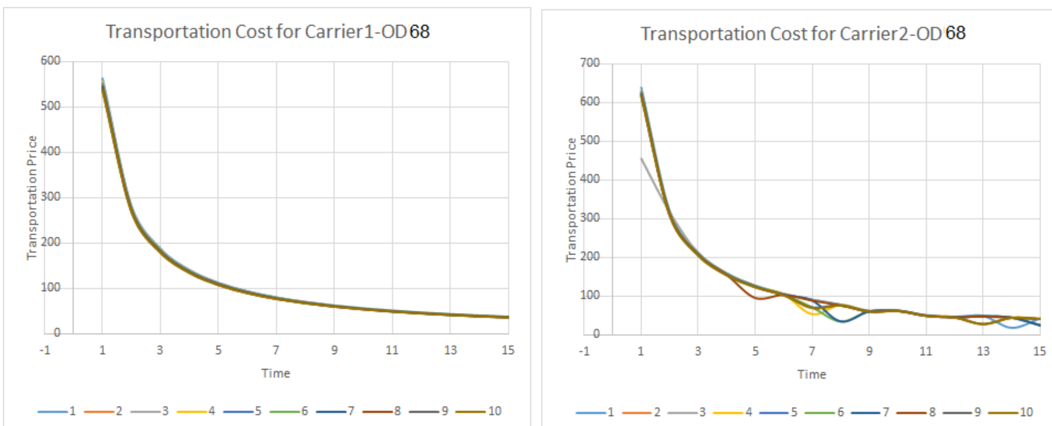

Figure 8.37. Transportation Price on OD68

over time. In addition, the transportation price on OD pairs *OD12, OD34, OD37* are higher compared to other OD pairs, which can be the result of the high transportation cost that carriers might face in this maps.

Figure 8.40 depicts the commodity price for shippers. We may observe that the carriers adopt an increasing-price policy. Also, since the inventory cost for shipper1 is higher, shipper 1 sets lower commodity price compared to shipper 2 in order to keep less inventory.

In this numerical example, due to the lowest and highest demand sensitivity to price at node 7 and node 9 respectively, the commodity price at N7 is the highest and at N9 is the lowest. More insights and conclusions can be found in the conclusion section of this thesis.

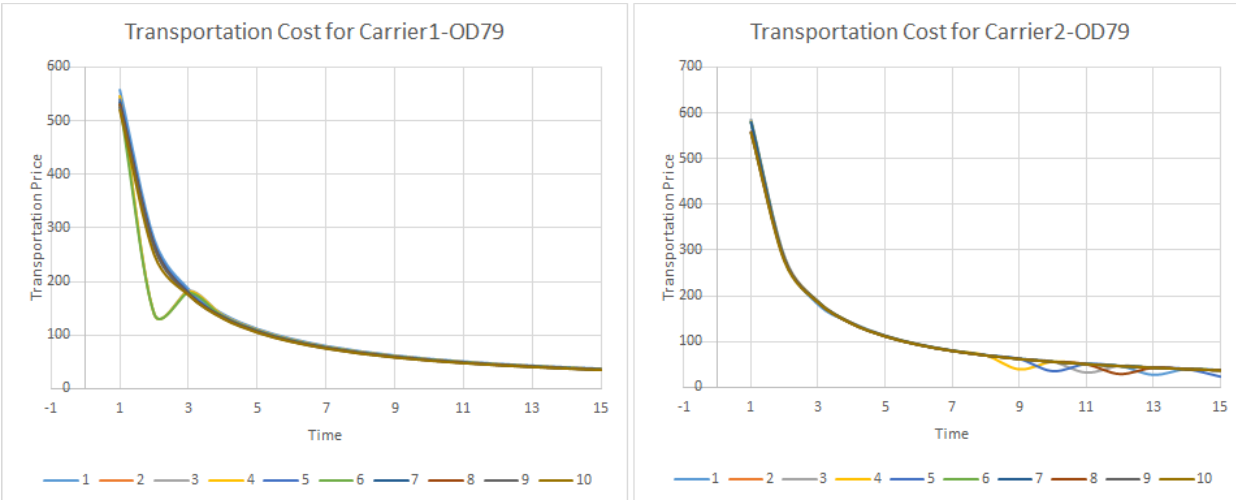

Figure 8.38. Transportation Price on OD79

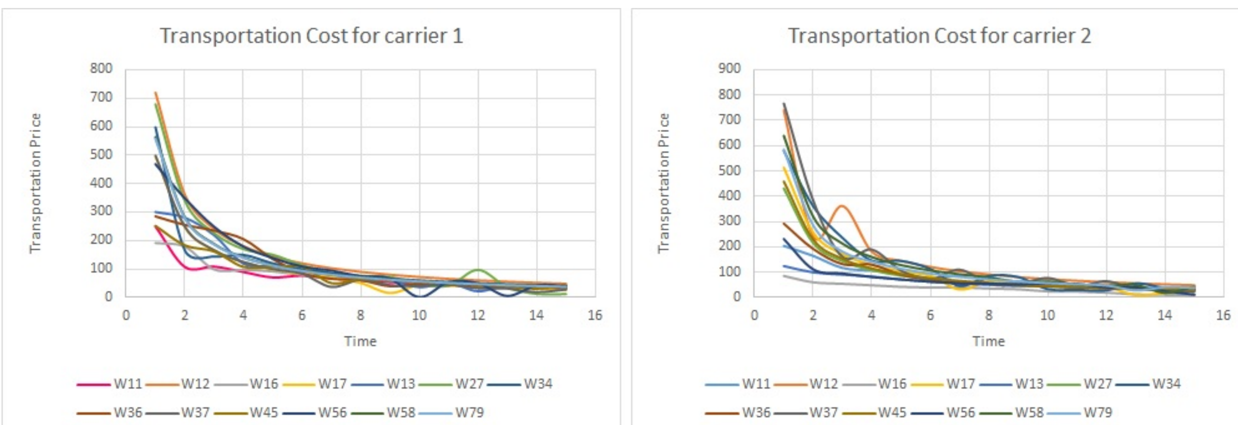

Figure 8.39. Transportation Price on different OD pairs for the first unit

8.3 Conclusion

In this chapter, we discuses and developed a nonlinear pricing model for a non-cooperative game with carriers on the upper level and shippers on the lower level. Carriers analyze the demand profile, denoted by $N(\gamma, p)$, which shows the number of customers (shippers) wiling to purchase q^{th} units at the marginal price γ. Then they try to take the best price discrimination strategy to satisfy the demand for customers and optimize their own objective function.

We formulated the dynamic freight service problem as a Stackelberg-Cournot game. In this game, the leaders make optimal decisions by predicting the reactions of the followers

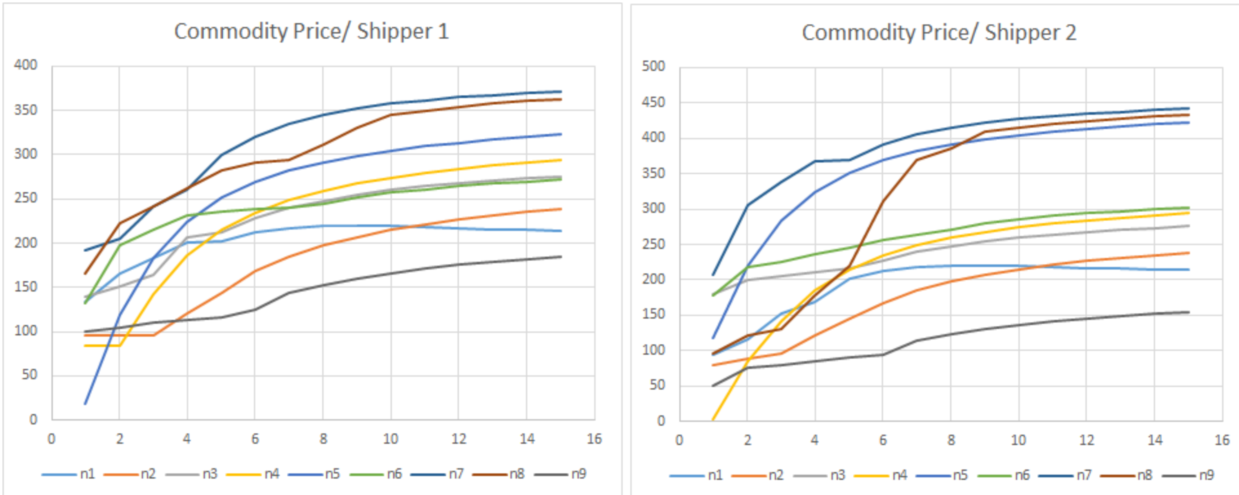

Figure 8.40. Commodity price

and the resulting equilibrium states at the lower level. For each leader, we formulated the problem as a mathematical program with equilibrium constraints (MPEC). On top of this, we had to find a Nash equilibrium among the leaders, thereby coupling multiple MPECs into a single equilibrium problem with equilibrium constraints (EPEC). Also, as mentioned before, due to the non-convexity of EPEC problem, these problems are very challenging to solve. We introduced an algorithm denoted as double adjoint algorithm to change the MPEC and consequently EPEC problem to the single level optimization problem with a system of differential equations. At the end, a discrete-time approximation method has been employed to solve this single level optimal.

The advantage of the double adjoint algorithm is that in spite of the diagonalization algorithm, this method can guarantee to find an optimal solution with quicker convergence rate and shorter computational time. In addition, after reformulation of the problem with such algorithm, one may employ a standard nonlinear mathematical programming package to obtain a solution to the dynamic shipper-carrier freight assignment model.

We have solved a numerical example that suggests the double adjoint algorithm can be applicable to solve dynamic Stackelberg games formulated as MPEC and EPEC structures. It successfully captured the dynamic nature of the problem with a very efficient computational time

Chapter 9 Conclusion

This dissertation tried to present two projects in the multi-leader-follower freight service games. We aimed to contribute to the existing literature in transportation pricing optimization and games by introducing the first, linear pricing structure in a bi-level problem with game-theoretic consideration in chapter 3/; and a general quantity discount pricing in the same bi-level problem for a large and complex, dynamic network in chapter 4/.

From a theoretical point of view, we have shown that the lower level problems of differential Stackelberg-Cournot-Nash games can be expressed as its necessary and sufficient conditions (control constraints) including minimum principle, adjoint equations and transversality conditions. In the linear pricing part of our research, we have shown that DVIs can be reformulated as a set of a finite number of equations known as a nonlinear complementarity problem (NCP). NCP formulation can be employed to convert the bi-level Stackelberg-Cournot-Nash problem into a single-level problem. Then diagonalization was used to solve the EPEC problem.

Then, in the nonlinear pricing part of our research, we showed that the minimum principle can be reformulated to a differential variational inequality which can be expressed as a mathematical optimal control problem. Rewriting the necessary conditions for this optimal control problem gives us a nonlinear optimization problem along with the two adjoint equations and transversality conditions. This method is denoted as double adjoint algorithm. By repeating the same steps of the double adjoint algorithm, we proved that the EPEC problem can finally be expressed as a single level optimization problem with a system of differential equations and other constraints. At the end, a discrete-time approximation method has been employed to solve this single level optimal.

We solved an example in this dissertation to show the efficiency of the EPEC formulation and the computational methods we used. Even though the example is not derived from real data, we tested the computational applicability in this dissertation.

We believe that this dissertation is the first one introducing computationally applicable formulation and approach to differential multi leader-follower Stackelberg games. This type of game leads to a competition in pricing among freight companies as well as the carriers which makes it possible to quantify the benefits gained through regulated pricing. Also, this paper is the first which assumes nonlinear pricing in its fully general form in a differential multi leader-follower Stackelberg game.

Although, the example we solved in this dissertation is hypothetical, this application to pricing can be tested if real data is collected and employed.

To solve the problem with real data the following data is needed:

- Topology of the network; To define the set of nodes, arcs and paths of the network
- Shape of demand curve; To define form of the demand function on each node for shippers
- Form of holding/inventory cost function and production cost function
- The initial inventory for shippers
- The inventory allowed at terminal time for shippers
- Market regulations such as the bounds on the market price of products and their shipments
- Shape of the the demand profile, denoted by $N(\gamma, p)$, which shows the number of customers (shippers) wiling to purchase q^{th} units at the marginal price γ
- travel time (path delay function) for each arc in the network

This data is needed for the entire time period considered in the problem and for each firm in the oligopoly of carriers and shippers.

In summary, we have accomplished the followings in this dissertation:

- Provided a model for a hierarchical decision environment that may be called a generalized Stackelberg game (GSG)

- Provided a theoretical framework for converting a bi-level generalized differential Stackelberg game into a single-level problem by converting the DVI in the lower level Nash game to a NCP formulation (in the linear pricing project for carrier-shipper network)

- Provided a theoretical framework for converting a bi-level generalized differential Stackelberg game, into a single level problem with a system of differential equations by introducing double adjoint algorithm (in the nonlinear pricing project for carrier-shipper network)

- incorporating the price discrimination and demand profile into the dynamic transportation pricing model

- Solving EPEC using double adjoint algorithm to guarantee finding optimal solution and reducing the computational time. As we observed the computational time is reduced from 4888.71 seconds in diagonalization algorithm to 186 seconds by using double adjoint algorithm. Therefore we conclude that the double adjoint algorithm has an effective role in reducing the complexity of the problem and consequently the computational time.

- Discussed the regularity, existence and stability theorems and showed that the carrier-shipper problem we introduced hold those theorems' conditions.

An extension and potential future researches are recommended as follows:

- **FW1. Stability** Developing a results for the stability of the generalized differential Stackelberg game of the carrier-shipper problem can be considered for the future work of multi leader-followers games. Stability theory is a very old subject and gives this opportunity to conclude about the behavior of a system with no need to compute the direct solution trajectories. stability in the modern sense was first studied by Lagrange [119]. He used Lagrangian mechanics to analyze the mechanical systems and concluded that in the absence of external forces, the equilibrium of the mechanical system is stable. Today following definition is being used for stability. Even though stabilizing nonlinear system is challenging, Lyapunov developed a theory on stability which has been the most popular and successful method so far [120]. Lyapunov functions introduced first by Aleksandr Lyapunov to prove stability of equilibrium nonlinear. Since there is not a standard approach for Lyapunov functions, it is still difficult to obtain it for general nonlinear systems. However, the idea is still being applied extensively in the literature of control for nonlinear systems.

- **FW2. Price of Anarchy**
  Developing analysis for the price of anarchy to check the inefficiency associated with the Nash game and differential Stackelberg game

- **FW3. Real Data Employment**
  Developing an example with the real data employment. Even though the computational applicability of the algorithm is shown by a hypothetical example, collecting and employing real data can provide a better observation on efficiency and computability of the presented approach.

- **FW4. Stochastic Stackelberg Game**
  Developing the stochastic case of the generalized Stackelberg game (GSG) for dynamic transportation pricing

- **FW5. Other Types of Nonlinear Pricing Settings** Extending the nonlinear pricing model from the quantity discount to other types of nonlinear pricing settings

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