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ABSENCE OF MIXING FOR SMOOTH FLOWS ON GENUS TWO SURFACES

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Dmitri Scheglov

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The dissertation of Dmitri Scheglov was reviewed and approved* by the following:

Anatole Katok
Professor of Mathematics
Dissertation Advisor, Chair of Committee

Omri Sarig
Professor of Mathematics

Svetlana Katok
Professor of Mathematics

Martin Bojowald
Professor of Physics

John Roe
Professor of Mathematics
Department Head

*Signatures are on file in the Graduate School.

Abstract

We prove that typical area-preserving flows with linearly isomorphic non-degenerate saddles on the genus two surface are not mixing.

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Chapter 1

Introduction to the problem.

Let M be a compact orientable surface of genus $g = 2$ and $p_1, p_2 \in M$. Let $H(p_1, p_2)$ be the stratum of all holomorphic abelian differentials,which vanish at p_1 and p_2 and do not vanish anywhere else. We also require that for any $\nu \in H(p_1, p_2)$, the vertical foliation has no homologically trivial leaves.

Let ω be any smooth volume form on M and let $\Psi(\omega, p_1, p_2)$ be the space of all smooth ω-preserving vector fields on M such that for any $V \in \Psi(\omega, p_1, p_2)$:

1. V has linearly isomorphic critical saddles of index -1 at p_1, p_2 in the sense that the eigenvalues of V coincide at the saddles and V does not vanish anywhere else;

2. V is tangent to the vertical foliation of some $\nu \in H(p_1, p_2)$.

We define a measurable structure on $\Psi(\omega, p_1, p_2)$ by the pullback of the measurable structure on $H(p_1, p_2)$. The main goal of the paper is to prove the following theorem.

Theorem 1.0.1. *For almost all $V \in \Psi(\omega, p_1, p_2)$ the generated flow U^t is not mixing.*

The study of mixing properties of flows on compact surfaces began in 1953 with the work of Kolmogorov [16]. Using Fourier series expansions, he showed that analytic flows on the 2-dimensional torus which preserve analytic density, do not have fixed points, and have a Diophantine rotation number ρ (see [8],[11] for a precise definition for rotation number of a flow) are analytically conjugate to linear flows, and so are not mixing.

On any transversal to the U^t-trajectories, there is a Poincaré map T and a T-invariant measure μ. The map T is isomorphic to an interval exchange transformation (IET) and the flow can be represented as a special flow constructed using T and a roof function f with a finite number of singularities.

Any type of degeneracy of a fixed point of U^t corresponds to a power singularity of f. Kocergin [13] showed that ergodic flows having only degenerate fixed points are mixing. Mixing in such flows is caused by strong stretching of any small rectangle moving near power singularities of f (a "black hole"-like effect). For C^5-smooth flows without fixed points on the torus and sufficiently well approximated rotation numbers, the absence of mixing was proved by Katok in 1967 [7]. In 1972, this result was improved by Kocergin [12]. Using reparametrizations of irrational flows on the two-torus, Shklover in 1967 [21] proved the existence of analytic flows that are weakly mixing but not mixing.

Non-degenerate fixed points lead to roof functions with symmetric logarithmic singularities (the precise notion will be given shortly). Kocergin considered the case when T is a circle rotation and $f(x) = f_0(x) + \sum_{i=1}^{N}(-A_i \ln\{x - x_i\} - B_i \ln\{x_i - x\})$, where $A_i, B_i > 0$, f_0 is a function of bounded variation, and $\sum_{i=1}^{N} A_i = \sum_{i=1}^{N} B_i$. In [12], he proved the absence of mixing in such special flows for typical circle rotations T. He used an abstract criterion for absence of mixing in special flows, good approximation properties of circle rotations, and a special 'compensation effect' of symmetric logarithmic singularities. We will use Kocergin's abstract criterion and compensation effect. Similar symmetric cancellations were used in a different setting in [23] by Sinai and Ulcigrai. They considered the case of circle rotations in the base. Fraczek and Lemanczyk [6] pointed out that non-mixing takes place for *every* irrational rotation and f with $A_i = B_i$ $\forall i$.

Katok [10] proved that for all IETs and roof functions with bounded variations, the IETs themselves and corresponding special flows are non-mixing. He used special "rigidity times" for interval exchanges, analogous to proper times for irrational circle rotations. The existence of rigidity times will be essential in our proof.

Roof functions with *asymmetric* logarithmic singularities correspond to flows on surfaces with boundaries and arose naturally in Arnold's paper [2], where he asked a question about their mixing properties. This was answered by Sinai and Khanin [22], who proved that for any asymmetric logarithmic roof function, almost any circle rotation gives rise to a mixing flow. The asymmetry of a roof function causes stretching of any small rectangle in the phase space, which leads to mixing.

We can also consider the Hamiltonian flow corresponding to a multivalued Hamiltonian arising from the integration of a cohomologically nontrivial closed form on a surface of finite genus. The orbit foliation splits into a component filled with periodic orbits and a "nontrivial" component, where the flow is isomorphic to a special flow over an interval exchange map [29]. Ulcigrai [24] generalized the result of Sinai and Khanin to the case of almost all interval exchanges and a functions with asymmetric logarithmic singularities, proving mixing for such special flows. Her proof uses deep Diophantine-like properties of Rauzy induction (see [3]).

In order to prove our main theorem, we will show the absence of mixing for symmetric logarithmic roof functions and typical (in the measure-theoretic sense) interval exchanges of 5 intervals of the type (54321). The permutations of the type(54321) naturally appear for IETs for flows on the genus two surface. To apply Kocergin's abstract criterion, we study combinatorial properties of Rauzy induction and prove a special combinatorial lemma, which is of independent interest. After that, we get an estimate on the ergodic sums of the roof function at different points, largely based on the ideas of a recent paper by Kocergin ([15]). To complete the proof, we use an integrability property of the *Zorich* (or *Kontsevich-Zorich*) cocycle ([17, 27, 28]).

Chapter 2

Main definitions and scheme of the proof

In this chapter, we will define all the needed objects and notations.

Definition 2.0.2. Consider an irreducible permutation π on n symbols. This means $\pi\{1,2,..,s\} = \{1,2,..,s\}$ only if $s = n$. Let $v = (v_1, v_2, ..., v_n)^t$ be a vector with positive components and $|v| = v_1 + ... + v_n$. Let $u_i = \sum_{j=1}^{i} v_j$ and $I^i = (u_{i-1}, u_i)$ for i = 1,...,n . The *interval exchange transformation (IET)* $T = (v,\pi)$: $[0,|v|]\to$ $[0,|v|]$ is a piecewise orientation preserving isometry which permutes I^i according to the permutation π.

Clearly, $T = (v,\pi)$ preserves Lebesgue measure on [0, $|v|$]. The set of all IETs of n intervals has natural locally-affine, projective, and Lebesgue measurable structures induced by those on $\mathbb{R}^n_+$.

2.0.1 Rauzy induction

Given an interval exchange T=(v,π) of n intervals such that $v_n \neq v_{\pi^{-1}(n)}$, we have two possibilities:

1. (Rauzy rule A) $v_n < v_{\pi^{-1}(n)}$. In this case, let $I = [0,\ |v| - v_n]$.
2. (Rauzy rule B) $v_n > v_{\pi^{-1}(n)}$. In this case, let $I = [0,\ |v| - v_{\pi^{-1}(n)}]$

The first return map of T on I is again an interval exchange map (v', π') of n intervals. The new permutation π' depends only on A or B (the independence on v is easily seen from the definition) and will be denoted correspondingly $A\pi$ or $B\pi$.

Since $v_n \neq v_{\pi^{-1}(n)}$ for almost all $v \in \mathbb{R}^n$, we have a map R: $(v, \pi) \to (v', \pi')$ defined on the full measure set of IETs. The map R is called *the Rauzy induction*.

2.0.2 Rauzy classes and Rauzy graphs

. Given an irreducible permutation π on n symbols, we may consider all possible permutations which may arise by iterations of operations A or B. The corresponding finite set of permutations $R(\pi)$ is called *the Rauzy class* of π.

Now consider the finite oriented graph $G(\pi)$ whose vertices are elements of $R(\pi)$. Two vertices π_1 and π_2 are connected by a directed labeled edge $\pi_1 \xrightarrow{a} \pi_2$ if $\pi_2 = A\pi_1$ and $\pi_1 \xrightarrow{b} \pi_2$ if $\pi_2 = B\pi_1$. The graph $G(\pi)$ is called the *Rauzy graph* of π.

2.0.3 Zorich induction

For a given n and a permutation σ, consider the set $\Delta_{n-1} = \left\{ v \in \mathbb{R}^n_+ : |v| = 1 \right\}$. For almost all interval exchanges $(v, \pi) \in \Delta_{n-1} \times R(\sigma)$ (in the sense of the Lebesgue measure on a symplex Δ_{n-1}), there is a uniquely defined sequence of Rauzy rules A or B arising from the iterations of R applied to (v, π).

Consider the first moment k when this sequence changes from A to B or from B to A. Let $(v^{'}, \pi^{'}) = R^k(v, \pi)$. Then the normalized vector $w = v'/|v'|$ defines a new interval exchange $(w, \pi') \in \Delta_{n-1} \times R(\sigma)$. This means we have a map $Z : (v, \pi) \to (w, \pi')$ defined almost everywhere on $\Delta_{n-1} \times R(\sigma)$. The map Z is called the *Zorich induction*. Zorich induction preserves a finite absolutely continuous measure μ_Z on $\Delta_{n-1} \times R(\sigma)$ and is ergodic according to this measure [27].

2.0.4 Rauzy and Zorich cocycles

Given a number n and a permutation σ, there is a matrix-valued function $RC(v, \pi)$ on $\mathbb{R}^n_+ \times R(\sigma)$ uniquely defined by the rule that for almost all $(v, \pi) \in \mathbb{R}^n_+ \times R(\sigma)$, if $(v^{'}, \pi^{'}) = R(v, \pi)$, then $v = RC(v, \pi)v^{'}$. For a given typical (v, π), the function $RC(v, \pi)^{-1}$ depends only on whether we have case A or B and is called the *Rauzy cocycle*.

Given an interval exchange $(v, \pi) \in \Delta^{n-1} \times R(\sigma)$ and $(v', \pi') = Z(v, \pi)$, there is a number k such that $v' = w/|w|$, where $(w, \pi') = R^k(v, \pi)$. Let $ZC(v, \pi) = RC(v, \pi) \cdot \ldots \cdot RC(R^{k-1}(v, \pi))$. $ZC(v, \pi)^{-1}$ is a matrix-valued function defined almost everywhere on $\Delta^{n-1} \times R(\sigma)$ and is called the *Zorich cocycle*.

2.0.5 Special flows

Consider a strictly positive integrable function $f(x)$ defined on [0, 1]. Let T be a measure preserving automorphism of [0,1] to itself. There is a natural flow S_t on the set $X = \{(x, p)|x \in [0, 1), 0 \leq p < f(x)\}$ (the subgraph of $f(x)$). Namely, every point (x, p) on the subgraph moves vertically with unit speed until it reaches the level $f(x)$ and then jumps down to the point $(T(x), 0)$ to continue its vertical move.

The flow S_t preserves Lebesgue measure on the subgraph and is called a *special flow*. $f(x)$ is the *roof function* and T is the *base transformation*.

In our case the base transformation T will be an interval exchange map of 5 intervals of the type (54321). The roof function will be a linear combination of functions of the type $f(x) = -\ln\{x - a\} - \ln\{b - x\}, x \in [0, 1]$, where $[a, b]$ is one of the exchanged intervals. Here $\{\} : \mathbb{R} \to [0, 1)$ is the function with period 1 such that $\{x\} = x, x \in [0, 1)$ In chapter five we will explain the detailed relation between the special flows we consider and flows on the genus two surface.

For general reference on interval exchange maps and Rauzy induction one may see [19, 20]. Rauzy and Zorich cocycles and their ergodic properties are studied in [19, 26, 27, 28].

2.0.6 The scheme of the proof

Let $I = [a, b], a, b \in [0, 1]$. We consider the function $f_I : [0, 1] \to \mathbb{R}_+$,

$$f_I(x) = -\ln\{x - a\} - \ln\{b - x\}.$$

Let $T{=}(v, \pi)$ be an interval exchange of 5 intervals, $|v| = 1$, $\pi = (54321)$ and let $I^1, \ldots, I^5$ be the rearranged intervals.

Definition 2.0.3. A *symmetric logarithmic function* for T is a function $f : [0,1] \to \mathbb{R}_+$ defined by

$$f(x) = \sum_{p=1}^{5} a_p \cdot f_{I^p}(x) + b \cdot f_{[0,1]}(x) + g(x),$$

where $a_p \in \mathbb{R}_+, b \in \mathbb{R}$, and g has bounded variation.

The most important part in the proof of the main result is the following:

Theorem 2.0.4. *For almost all $T \in \Delta_4 \times R(54321)$, the special flow constructed using the base transformation T and any symmetric logarithmic function for T is not mixing.*

The proof will consist of four basic parts. In chapter 2, we describe special partitions and substitutions associated with the iterated Rauzy induction maps. After that, we describe the relation of these substitutions with the Rauzy graph and prove an abstract combinatorial lemma about the Rauzy graph. In chapter 3, we get estimates on the ergodic sums of the roof function $f(x)$ along the trajectory of T. The estimates will essentially use the combinatorial lemma. In chapter 4, we use integrability of the Zorich cocycle, ergodic estimates from chapter 3, and an abstract Kocergin's criterion to complete the proof of Theorem 2.0.4. In chapter 5, we deduce the main result from Theorem 2.0.4 .

Chapter 3

The combinatorial part

3.0.7 Labeled partitions and substitutions

Suppose (v, π) is an ergodic interval exchange and $(v_1, \pi_1) \dots (v_n, \pi_n)$ is a sequence of its images under the action of the Rauzy induction map. For this sequence, we construct a corresponding sequence Θ_n of labeled partitions of $[0, 1]$ on subintervals.

From the definition of Rauzy induction, it follows that there is a subinterval $I_n \subset [0, 1]$ such that (v_n, π_n) is an induced map of (v, π) on I_n. Let $I_n^1, \dots, I_n^5$ be the induced subintervals taken from the left to the right. Consider all the images of $I_n^1, \dots, I_n^5$ under the action of (v, π) up to first return to I_n. By ergodicity of (v, π) and by the construction of Rauzy induction, they form a finite partition of $[0, 1]$.

Now let all the images up to first return time of I_n^k have some label k, $k \in \{1, \dots, 5\}$. What we get is a finite labeled partition Θ_n of $[0, 1]$ to subintervals. From the definition of Rauzy induction, it follows that Θ_n refines Θ_{n+1}, and moreover, for almost all IETs, it converges to ϵ (the partition on points). Since Θ_n is a finite labeled partition of $[0, 1]$, we can 'read' it from the left to the right to get a word Ω_n in the alphabet $\{1, \dots, 5\}$.

Now consider two interval exchanges (v_n, π_n) and (v_{n+1}, π_{n+1}). The interval I_n has a labeled partition $\Theta_n|_{I_n} = (I_n^1, \dots, I_n^5)$ and also a labeled subpartition $\Theta_{n+1}|_{I_n}$. That means that every interval I_n^k, $k = 1, \dots, 5$ consists of some elements of the partition Θ_{n+1}. Reading these elements from the left to the right for every I_n^k, $k = 1, \dots, 5$, we get some word η_k in the alphabet $\{1, \dots, 5\}$ for each I_n^k. In other words, we get a substitution on 5 symbols: $k \to \eta_k$, $k = 1, \ldots, 5$. It easily follows from the definition of Rauzy induction that this substitution depends only on π_n, π_{n+1}, and which Rauzy rule (A or B) is applied to π_n to get π_{n+1}. In other words, it depends only on the corresponding edge of the Rauzy graph. Since the elements of the partitions Θ_n and Θ_{n+1} are images of elements of I_n and I_{n+1} correspondingly, a simple observation shows that the whole word Ω_{n+1} is obtained from Ω_n by the same substitution described above.

There are several objects associated to every edge of the Rauzy graph:

1. The Rauzy rule (A or B) and corresponding Rauzy map on the space of IETs.
2. The matrix $R(v, \pi)$, as in the definition of Rauzy cocycle, which depends only on the edge.
3. The substitution described above, which also depends only on the edge.

Given any path Γ in the Rauzy graph, natural compositions of objects above give rise to the following:

1. The word in the alphabet $\{A, B\}$, and the map on the space of all IETs.
2. The matrix, corresponding to the path, as a product of the edge matrices.
3. The substitution, as a composition of the edge substituitions.

More formally, we consider the category whose objects are the vertices of the Rauzy graph and whose morphisms are the connecting oriented paths. Then the associated objects above are the covariant functors to corresponding categories. In this paper, we associate any path in the Rauzy graph with its image under one of these functors when it is clear from context and does not lead to ambiguity.

Given a typical $T = (v, \pi)$, the n-th iteration of the Rauzy induction map defines an n-path γ in the Rauzy graph and a subinterval I on which the induction is done. In this paper, the interval I will be associated with γ since they are both uniquely defined for a given typical T. By a Rauzy cycle, we mean any closed loop in the Rauzy graph. The Rauzy graph for the Rauzy class of (54321) with substitutions and Rauzy rules on edges is given in the Appendix.

We have described substitutions naturally arising from iterations of the Rauzy induction map and the corresponding labeled partitions of $[0, 1]$, and now we are ready to formulate and prove a combinatorial lemma related to the Rauzy graph, which will be used in the next sections. The author wants to emphasize that the proof of this lemma is purely combinatorial and does not clarify the reasons why the lemma is true. The author believes that a possible geometric explanation might arise from a reformulation of the lemma in terms of the Teichmüller geodesic flow.

3.0.8 The combinatorial lemma

Let A be the alphabet $\{1,2,3,4,5\}$ and U a fixed countable alphabet. A *symmetric word* is any word in the alphabet U which is symmetric in the usual sense. We may consider any symmetric word as a 'word function' of its symbols. For example, the symmetric word 'aba' can be considered as a function on the variables a and b: $p(a,b) = aba$. Here a and b are variables, taking values in the space of words and ab is a natural operation of concatenation of words. Another example is $w(a,b,c) = cabbac$. Depending on the context, any symmetric word will be considered as a word or as a word function of its symbols. Substitutions have a natural right action on the words, thought of as word functions, by composition. For example, if ζ is a substitution, $\zeta : a \to 12345, b \to 4$, then $p \circ \zeta(a,b) = p(12345,4) = 12345412345$.

Let S be a set of words in the alphabet A. We take any symmetric word $w = w(x_1,\ldots,x_n), x_1,\ldots,x_n \in U$, and any substitution $\zeta : x_1,\ldots,x_n \to S$.

Definition 3.0.5. $Symm\,(S) = \{w \circ \zeta \;\mid\; \text{any } \; w, \zeta \; \text{ described above}\}$.

So $Symm\,(S)$ is a subset of words in the alphabet A. If w is a word in A, then $w^{\mathbb{Z}_+} = \{\varnothing, w, ww, www, \ldots\}$ and $C = Symm(2,3,4,1\,(2345)^{\mathbb{Z}_+}, 5\,(4321)^{\mathbb{Z}_+})$. According to the natural correspondences described above, paths in the Rauzy graph act on words in A. Now we are ready to prove the combinatorial lemma.

This lemma is a key argument for the rest of the proof. Informally speaking, it says that almost-symmetric words transform to almost-symmetric words when one moves along the cycles in the Rauzy graph. We will apply this lemma to the Rauzy induction iterated process to produce balanced (almost-symmetric) partitions (the exact definition of a balanced partition will be given later). The almost-symmetry of the partitions will be combined with the symmetry of the logarithmic roof function to get symmetric cancellations in the ergodic sum in order to finally apply Kocergin's criterion.

Theorem 3.0.6. *Any cycle of the Rauzy graph which starts and ends at* (54321) *maps the class* C *to itself.*

Proof. We start our proof with the left part of the graph.

Lemma 3.0.7. *Any cycle in the graph X starting and ending at (25143) maps the class* $X = Symm(1, 4, 31, 3(453)^{\mathbb{Z}_+}, 5(42)^{\mathbb{Z}_+})$ *into itself.*

Proof. Obviously, it's enough to prove the lemma for cycles which start and return to (25143) only once. The substitution τ for such a cycle is: $1 \to 1; 2 \to 2; 3 \to 3; 4 \to 4(54)^n; 5 \to 542$, where n is the number of one-step subcycles at (24153).

We have

$$\begin{aligned} \tau(1) &= 1 \in X, \\ \tau(4) &= 4(54)^n \in X, \\ \tau(31) &= 31 \in X, \\ \tau(23(453)^k) &= 23(4(54)^n5423)^k \in X, \text{ and} \\ \tau(5(42)^s) &= 542(4(54)^n2)^s \in X. \end{aligned}$$

Thus $\tau(X) \in X$. Here and further we use the simple observation that the action of a symmetric substitution (i.e. the images of all symbols are symmetric words) for a symmetric word results in a symmetric word. □

Lemma 3.0.8. *Any cycle in the graph Z starting and ending at (25431) maps the class* $Z = Symm(1, 3, 4, 2(345)^{\mathbb{Z}_+}, 51(4321)^{\mathbb{Z}_+})$ *to itself.*

Proof. Any cycle Γ which starts and returns to (25431) only once can be uniquely decomposed: $\Gamma = (25431) \xrightarrow{\alpha} (25143) \xrightarrow{\beta} (25143) \xrightarrow{\chi} (25314) \xrightarrow{\eta} (25314) \xrightarrow{\nu} (25431)$. So we have:

$$\alpha : Symm(1, 3, 4, 2(345)^{\mathbb{Z}_+}, 51(4321)^{\mathbb{Z}_+}) \to$$
$$Symm(1, 4, 5, 23(453)^{\mathbb{Z}_+}, 31(54231)^{\mathbb{Z}_+}) \subset X,$$

$\beta : X \to X$ according to Lemma 3.0.7,

$\chi : X \to Symm(1, 5, 41, 234(534)^{\mathbb{Z}_+}, 3(523)^{\mathbb{Z}_+})$,

$\eta : Symm(1, 5, 41, 234(534)^{\mathbb{Z}_+}, 3(523)^{\mathbb{Z}_+}) \to$

$$Symm(1, 52^n, 41, 234(52^n34)^{\mathbb{Z}_+}, 3(52^n3)^{\mathbb{Z}_+}),$$

where n is a number of one-step subcycles at (25314), and

$$\nu : Symm(1, 52^n, 41, 234(52^n34)^{\mathbb{Z}_+}, 3(52^n3)^{Z_+}) \to$$
$$Symm(1, 3(23)^n, 51, 2345(3(23)^n45)^{Z_+}, 4(3(23)^n4)^{\mathbb{Z}_+})$$

The set in the last inclusion is a subset of Z. □

Lemma 3.0.9. *Any cycle in the graph Y starting and ending at (32541) maps the class $Y = Symm(1, 2, 4, 3(45)^{\mathbb{Z}_+}, 521(4321)^{\mathbb{Z}_+})$ to itself.*

Proof. It is enough to prove the lemma for the cycles which start and return to (32541) only once. The substitution τ for any such cycle is $1 \to 1; 2 \to 2; 3 \to 345; 4 \to 4(34)^n; 5 \to 5$, where n is the number of one-step subcycles at (32514). We have $\tau(1) = 1 \in Y$, $\tau(2) = 2 \in Y$, $\tau(4) = 4 \in Y$, $\tau(3(45)^k) = 345(4(34)^n5)^k \in Y$, and $\tau(521(4321)^s) = 521(4(34)^n34521)^s \in Y$. From this, it follows that $\tau(Y) \in Y$. □

Now we can prove Theorem 2.0.4 for the left part of the Rauzy graph. Any cycle Γ which starts and returns at (54321) only once can be uniquely represented as $\Gamma = (54321) \xrightarrow{\alpha} (25431) \xrightarrow{\beta} (25431) \xrightarrow{\chi} (32541) \xrightarrow{\eta} (32541) \xrightarrow{\kappa} (43251) \xrightarrow{\sigma} (43251) \xrightarrow{\xi} (54321)$, so we have:

$\alpha : C \to Z$,

$\beta : Z \to Z$ according to Lemma 3.0.8,

$\chi : Z \to Y$,

$\eta : Y \to Y$ according to Lemma 3.0.9,

$\kappa : Y \to Symm(1, 2, 4, 3(453)^{\mathbb{Z}_+}, 5321(4321)^{\mathbb{Z}_+})$,

$$\sigma : Symm(1, 2, 4, 3(453)^{\mathbb{Z}_+}, 5321(4321)^{\mathbb{Z}_+}) \to$$
$$Symm(1, 2, 45^n, 3(45^{n+1}3)^{\mathbb{Z}_+}, 5321(45^n321)^{\mathbb{Z}_+}),$$

where n is a number of one-step subcycles at (43251), and

$$\xi : Symm(1, 2, 45^n, 3(45^{n+1}3)^{\mathbb{Z}_+}, 5321(45^n321)^{\mathbb{Z}_+}) \to C.$$

This finishes the proof of the combinatorial lemma for the left part of the Rauzy graph. The proof for the right part of the Rauzy graph is completely analogous and we skip it here. □

The following combinatorial technical lemma will be needed later.

Lemma 3.0.10. *There is a Rauzy cycle L, starting and ending at (54321), such that:*

1. *The corresponding matrix has only positive entries.*
2. *If S is any other Rauzy cycle starting and ending at (54321) than the substitution τ for the cycle SL has the following structure:*

$$\begin{aligned}
\tau(1) &= 12345\omega_1 12345,\\
\tau(2) &= 2\omega_2 2,\\
\tau(3) &= 3\omega_3 3,\\
\tau(4) &= 4\omega_4 4,\\
\tau(5) &= 5\omega_5 5,\\
\tau(12345) &= 12345\omega 12345.
\end{aligned}$$

3. *The labels for the last two edges of L are BA*

Proof. The path can be taken to be $L = QL'$. Here Q is any path, starting and ending at (54321), such that the corresponding matrix has only positive entries. Its existence is a well-known fact in the theory of IETs (see, for example, [26, 28]). Now consider the path given by L'=AAAABBBBAAABA.

To finish the proof, we 'diagram chase' along the path L. Move along the path and observe what happens with the first and last symbols at every step. This is sensible because on every step the first and last symbols of the word depend only on the first and last symbols of the previous word. We skip the details here. □

Chapter 4

The ergodic sum estimate

Let $f(x)$ be a symmetric logarithmic function and $\varphi(x) = f'(x)$. In this section, we prove several technical theorems and get a uniform estimate on $|f_n(x) - f_n(y)|$, where x and y are two points from a set of 'large' measure, and f_n is n-th ergodic sum for f. The estimate will be obtained under some 'good' assumptions on the interval exchange. In the next chapter, we will prove that the needed assumptions take place for almost all IETs of the type (54321), and using the ergodic sum estimate and an abstract criterion by Kocergin, we get the final result.

4.0.9 The uniform balance estimate

Definition 4.0.11. Consider a word C in the alphabet $\{1, 2, 3, 4, 5\}$. The word C is called K-*balanced*, with *balance constant* K, if:

1. Every subword of length K contains all symbols 1, 2, 3, 4, 5.

2. For any symbols $a, b \in \{1, 2, 3, 4, 5\}$, if L_i is the left subword of C ending at the i-th letter a from the left and R_i is the right subword of C starting at the i-th letter a from the right, then for any i, the number of letters b in L_i should differ from the number of letters b in R_i by no more than K .

Definition 4.0.12. Let (v, π) be an IET, $\pi = (54321)$, and $I_0^1, \dots, I_0^5$ exchanged intervals. An infinite sequence of subintervals [0,1] $\supseteq I_1 \supseteq I_2 \supseteq \dots \supseteq I_n \supseteq$ ... is called *good* if it satisfies the following properties:

1. On any subinterval I_n, the induced IET on I_n $T = (v, \pi)$ has five intervals and $\pi = (54321)$.

2. $I_n = [0, |I_n|]$, and $|I_{n+1}| < |I_n|/2$

3. There is a universal constant K such that for any n, the word Ω_n corresponding to the labeled partition Θ_n, defined by I_n has the following properties:

   (a) $\Omega_n|_{I_0^1} = 12345\omega_1 12345$ and is K-balanced.

   (b) $\Omega_n|_{I_0^2} = 2\omega_2 2$ and is K-balanced.

   (c) $\Omega_n|_{I_0^3} = 3\omega_3 3$ and is K-balanced.

   (d) $\Omega_n|_{I_0^4} = 4\omega_1 4$ and is K-balanced.

   (e) $\Omega_n|_{I_0^5} = 5\omega_5 5$ and is K-balanced.

   (f) $\Omega_n = 12345\omega 12345$ and is K-balanced.

   Here ω_i are words and the motivation for this rather technical propery is that we will need a local symmetry of the partition near singularities, which this property represents.

4. For any n, the interval I_n consists of subintervals I_n^1, ..., I_n^5 with lengths $|I_n^1|$, ..., $|I_n^5|$. There exists d_n such that for any $i \in \{1, \dots, 5\}$, $1 \leq |I_n^i|/d_n \leq 1.001$.

The intervals I_n and the partitions Θ_n will be also called good.

Definition 4.0.13. Let (v, π) be an IET, $\pi = (54321)$, and $I_0^1, \dots, I_0^5$ be exchanged intervals. Let I_n be a good interval, and let $\omega_1, \dots, \omega_5$ be as in Definition 4.0.12. The restriction of Θ_n to the disjoint union of intervals related to the words ω_1, ..., ω_5 is called *the reduced partition* Θ'_n.

We want to emphasize that all $\omega_1, \dots, \omega_5$ depend on n.

Now we are going to get some ergodic sum estimates for an interval exchange which admits a good sequence of subintervals, but first we set some notation:

- $T = (v, \pi)$ is an interval exchange which admits a good sequence, and the exchanged intervals are $I_0^1, \dots, I_0^5$;

- I_n is n-th good interval, and $I_n^1, \ldots, I_n^5$ are the exchanged subintervals;
- Θ_n is a labeled partition of [0, 1] induced by I_n, and Ω_n is a corresponding word;
- d_n is a number for I_n as given in Definition 4.0.12;
- $\Theta_n^{'}$ is a reduced partition for I_n;
- $\lambda_n = |I_n|/|I_{n-1}|$ is a relative length of I_n; and
- $f_n(x) = f(x) + f(Tx) + \ldots + f(T^{n-1}x)$ is the n-th ergodic sum for $f(x)$.

The next two theorems (4.0.14 and 4.0.15) will be formulated for any symmetric logarithmic functions, but proved only for the model case $f(x) = \ln(x) + \ln(1-x)$ and $\varphi(x) = \frac{1}{x} - \frac{1}{1-x}$. All the arguments work in a general case provided that:

1. The theorems have 'additive' property, namely, it is enough to prove them for any f_{I^i}, independently;
2. From the definition of a good sequence, it follows that the induced partitions are K-balanced on any interval I^i and any f_{I^i} is symmetric on I^i;
3. Any function f_{I^i} has a bounded variation outside I^i, and it will be clear from the proofs that we can ignore any terms with bounded variation.

The informal reason why we prove the theorems only for the model case is that even for that case, the proofs are quite technically difficult, and for the sake of clarity we do not want to overload the proof with additional indices and notations.

All of the changes which have to be made in order to prove Theorems 4.0.14 and 4.0.15 for a different function f_{I^i} consist only of a change of notation.

Theorem 4.0.14. *Let $f(x)$ be a symmetric logarithmic function for (v, π), $\varphi(x) = f'(x)$ and S is a finite set satisfying:*

1. *S has points only in the elements of Θ_n' and*
2. *The number of points of S in every element of $\Theta_n^{'}$ is at most p and depends only on the element label.*

Under these assumptions, the following inequality holds:

$$\left|\sum_{x\in S}\varphi(x)\right| \le C\cdot\frac{p}{|I_n|},$$

where C is a universal constant.

Proof. Let $f(x)=\ln(x)+\ln(1-x)$ and $\varphi(x)=\frac{1}{x}-\frac{1}{1-x}$. First, we assume that $p=1$. Represent S as a union of 5 sets $S_1,\dots,S_5$, corresponding to the given label $1,\dots,5$. Without loss of generality, say S_1 is not empty. We consider the i-th point of the set S_1 from the left and the i-th point of S_1 from the right. Let those points be denoted by x_i and $\tilde{x}_i$. Then

$$\begin{aligned}\varphi(x_i)+\varphi(\tilde{x}_i) &= (1/x_i-1/(1-x_i))+(1/\tilde{x}_i-1/(1-\tilde{x}_i))\\ &= (1/x_i-1/(1-\tilde{x}_i))+(1/\tilde{x}_i-1/(1-x_i)).\end{aligned}$$

Let us estimate only the first term; the second term is estimated in the same way:

$$|1/x_i-1/(1-\tilde{x}_i)| = \left|\frac{(1-\tilde{x}_i)-x_i}{x_i\cdot(1-\tilde{x}_i)}\right|.$$

Since the word is balanced it is easy to see that $|x_i-(1-\tilde{x}_i)|\le K\cdot C\cdot d_n$, where C is some universal constant(it follows from the fact that the legths of all partition elements are uniformly close to d_n). Also, $x_i\ge\ i\cdot d_n$ and $1-\tilde{x}_i\ge\ i\cdot d_n$. So, finally, we get

$$\left|\frac{(1-\tilde{x}_i)-x_i}{x_i\cdot(1-\tilde{x}_i)}\right| \le \frac{C\dot{K}\cdot d_n}{(i\cdot d_n)\cdot(i\cdot d_n)} = \frac{C\dot{K}}{i^2\cdot d_n}.$$

By summing up the terms for all i, summing up these estimates for $S_1,\dots,S_5$, and taking into account that $d_n>|I_n|/6$, we complete the proof when $p=1$.

For $p>1$ we split S into no more than p sets, satisfying the previous case, which is possible because of property 2, and finish the proof. □

4.0.10 The ergodic estimate

Before obtaining the estimate, we prove the following theorem.

Theorem 4.0.15. *Let $f(x)$ be a symmetric logarithmic function, $\varphi(x)=f'(x)$, and $x \in I_n^1 \cup I_n^5$ be a point located on the distance of at least $d_n/10$ from the ends of I_n^1 and I_n^5. Let t be any positive integer less than the first return time of x into I_n and S be a piece of x-trajectory, $S = \{x, Tx, \ldots, T^{t-1}x\}$. Then the following inequality on the derivative holds:*

$$\left|\sum_{y \in S} \varphi(y)\right| \leq -C \cdot \frac{\ln \lambda_n}{|I_n|} - C \cdot \frac{\ln \lambda_{n-1}}{|I_{n-1}|} - \ldots - C \cdot \frac{\ln \lambda_1}{|I_1|} + C,$$

where C is a constant depending only on $f(x)$.

Proof. Let $f(x) = -\ln(x) - \ln(1-x)$ and $\varphi(x) = -\frac{1}{x} + \frac{1}{1-x}$, let t^{n-1} be the last return time of x to I_{n-1}, and set $x_{n-1} = T^{t^{n-1}}x$. In general, we let t^i be the last return time of x_{i+1} to I_i and set $x_i = T^{t^i}x_{i+1}$, for $i = 0, \ldots, n-1$.

Notice that any t^k *can*, in principle, be equal to 0. For example, $t^{n-1} = 0$ means that $S \bigcap I_{n-1} = \{x\}$.

The main idea of the proof is based on the following simple observation. Let S_m ($y \in [0,1]$, $1 \leq m \leq n$) be a piece of y-trajectory such that $S_m \bigcap I_m = \varnothing$. Then S_m has *at most* one element in every interval of the partition Θ_m.

For any I_m, let $I_m^{sym} = [1-|I_m|, 1]$ be a symmetric interval. From the definition of a good sequence (property 3), it follows that I_m^{sym} consists of images of $I_m^1, \ldots, I_m^5$ taken in the same order from the left to the right for any m.

Now, we are going to divide S into the proper pieces. The author wants to emphasize that he came to this idea while reading Kocergin's paper [14], where an analogous procedure was applied for irrational circle rotations. First, we define our proper pieces. Let $P_n = \{x, Tx, \ldots, x_{n-1}\}$ and $W_n = \{x, Tx, \ldots, x_0\}$, and for $i = 1, \ldots, n-1$, let $P_i = \{x_i, Tx_i, \ldots, x_{i-1}\}$ and $W_i = \{x_i, Tx_i, \ldots, x_0\}$. Then, let $S_n = P_n \bigcap([|I_{n-1}, 1-|I_{n-1}|])$ and $K_n = W_n \bigcap \left(I_{n-1} \cup I_{n-1}^{sym}\right)$, and for $n = 2, \ldots, n-1$, let $S_i = P_i \bigcap([I_{i-1}, 1-I_{i-1}])$ and $K_i = W_i \bigcap \left(I_{i-1} \cup I_{i-1}^{sym}\right) \setminus K_{i+1}$. Finally, set $S_1 = \emptyset$ and $K_1 = W_1 \setminus K_2$.

The informal meaning of the sets defined above is as follows. P_k is the piece of S between the last visit to I_k and the last visit to I_{k-1}, W_k is the tail of S since the last visit to I_k, S_k is the part of P_k lying between I_{k-1} and I_{k-1}^{sym}, and K_k is the subset of W_k lying in the union of I_{k-1} and I_{k-1}^{sym}. From the definitions above,

$S = S_n \cup S_{n-1} \ldots \cup S_1 \bigcup K_n \cup K_{n-1} \ldots \cup K_1$, and the union is disjoint.

We are going to estimate $\sum_S \varphi$ by estimating $\sum_{S_k} \varphi$ and $\sum_{K_k} \varphi$. After that, using the triangle inequality, we will complete the proof. Each S_k is a K-balanced piece and $\sum_{S_k} \varphi$ is estimated with the help of Theorem 4.0.14. Notice that each K_k is not K-balanced, but has controlled cardinality, and $\sum_{K_k} \varphi$ is estimated by a simple calculation.

Let us consider the set $S_n = P_n \bigcap([|I_{n-1}|, 1 - |I_{n-1}|])$. From the definition of P_n, it follows that the number p of points of S_n on any element of the reduced partition Θ'_{n-1} depends only on the label of the element. Since any element of Θ_n contains at most one point of S, it follows that p is less than the maximal possible number of elements of Θ_n in any element of Θ_{n-1} which gives $p < C \cdot \frac{|I_{n-1}|}{|I_n|}$ for some universal C. Now we can apply Theorem 4.0.14 to get $\sum_{S_n} \varphi < \frac{C}{|I_n|}$.

Now, let us estimate $\sum_{K_n} \varphi$. For clarity's sake, let us first estimate $\sum_{W_n \cap I_{n-1}} \frac{1}{x}$. We enumerate the points of $W_n \bigcap I_{n-1}$ from the left to the right as $x_1, x_2, \ldots, x_s$. The point $x = x_1$ is estimated separately using $x_1 > d_n/10$, by the theorem's assumption. The reason we estimate it separately is that the properties of good sequences allow us to control the distribution of partition elements, and hence trajectory points inside [0,1]. But the definition of a good sequence does not give any information about the point location *inside* the element, and it could happen that the trajectory point stays too close to the singularity. This is why we need the additional assumption on a distance from the cutting points.

Also, $x_i > d_n \cdot (i-1)$ if $i \geq 2$, because every element of Θ_n contains at most one point of W_n. For the same reason, we have $s < C \cdot \frac{|I_{n-1}|}{|I_n|}$, and finally we get

$$\sum_{i=1}^{s} \frac{1}{x_i} < \frac{10}{d_n} + \sum_{i=2}^{s} \frac{1}{d_n \cdot (i-1)} < -C \cdot \frac{\ln \lambda_n}{|I_n|}.$$

In the second inequality above, we use the fact that $|I_n|$ and d_n are uniformly proportional, and a well-known analysis estimate of partial sums of the harmonic series by a logarithm. Since $\frac{1}{x} < 2$ on I_{n-1}^{sym}, we have

$$\sum_{K_n \cap I_{n-1}^{sym}} \frac{1}{x} < 2 \cdot \frac{|I_{n-1}|}{|I_n|}.$$

Summing up the estimates for I_{n-1} and I_{n-1}^{sym}, we have

$$\sum_{K_n} \frac{1}{x} < -C \cdot \frac{\ln \lambda_n}{|I_n|}. \tag{4.0.1}$$

Due to the symmetry argument we have

$$\sum_{K_n} \frac{1}{1-x} < -C \cdot \frac{\ln \lambda_n}{|I_n|}. \tag{4.0.2}$$

In the inequality (4.0.2), the point y of K_n closest to 1 is also estimated separately and satisfies $|1-y| > d_n/10$ because of the theorem's assumption. All the other estimates near 0 in obtaining (4.0.1) take place symmetrically near 1, and we skip the details here.

Summing up (4.0.1) and (4.0.2), we have

$$\sum_{K_n} \varphi < -const \cdot \frac{\ln \lambda_n}{|I_n|}.$$

Now, we consider the tail W_{n-1} starting at x_{n-1}, and note that from the definition of x_{n-1}, it follows that any element of Θ_{n-1} contains at most one point of W_{n-1}. Also, note that from the definition of K_{n-1} and K_n, it follows that for any $y \in K_{n-1}, |y| > |I_{n-1}|$ and $|1-y| > |I_{n-1}|$.

After these remarks, we may shift the index $n \longrightarrow n-1$ and repeat the previous argument word by word, except that now we do not need any separate estimates, provided that in this case we don't have any trajectory points which can be arbitrarily close to singularities. So we have

$$\sum_{S_{n-1}} \varphi < \frac{C}{|I_{n-1}|}$$

and

$$\sum_{K_{n-1}} \varphi < -C \cdot \frac{\ln \lambda_{n-1}}{|I_{n-1}|}.$$

We may then remark that from the definition of K_{m-1} and K_m, it directly follows that for any $y \in K_{m-1}, |y| > |I_{m-1}|$ and $|1-y| > |I_{m-1}|$. Let us also remind the reader that for any m, at most one point of P_m lies in any element of Θ_m.

These two remarks allow us to continue shifting indices, and we obtain for any $m \in \{2, \ldots, n\}$,

$$\sum_{S_{m-1}} \varphi < \frac{const}{|I_{m-1}|}$$

and

$$\sum_{K_{m-1}} \varphi < -const \cdot \frac{\ln \lambda_{m-1}}{|I_{m-1}|}.$$

Now, summing up the estimates for all m we have,

$$\left| \sum_{y \in S} \varphi(y) \right| \leq -C \cdot \frac{\ln \lambda_n}{|I_n|} - C \cdot \frac{\ln \lambda_{n-1}}{|I_{n-1}|} - \ldots - C \cdot \frac{\ln \lambda_1}{|I_1|} + C.$$

□

Lemma 4.0.16. *Let Θ_n be the n-th good partition, $f(x)$ a symmetric logarithmic function, and $\varphi(x) = f'(x)$. Suppose we have a finite set S of pairs $\{x_i, y_i\}$ such that different pairs of S lie entirely on different elements of Θ_n and are a distance of at least $d_n/10$ from the cutting points. Then there is a constant C depending only on $f(x)$ such that*

$$\left| \sum_S (f(x_i) - f(y_i) - \varphi(x_i) \cdot (x_i - y_i)) \right| \leq C.$$

The proof is a simple calculation using the Taylor expansion of the logarithmic function and convergence of $\sum \frac{1}{n^2}$. We skip the details here.

The following theorem is the main ergodic estimate.

Theorem 4.0.17. *Let $f(x)$ be a symmetric logarithmic function and Θ_n a good K-balanced partition. There exist constants $\beta > 0$ and C depending only on $f(x)$ and K and a set $G_n \in [0,1]$ with $\mu(G_n) > \beta$ such that for any $x, y \in G_n$,*

$$|f_{r_n}(x) - f_{r_n}(y)| < C \cdot \left(1 - \ln \lambda_n - \frac{\ln \lambda_{n-1}}{2} - \ldots - \frac{\ln \lambda_1}{2^{n-1}}\right),$$

where r_n is the total number of Θ_n-elements, labeled 1 or 5.

Proof. Let Γ_n be the union of all Θ_n-elements labeled 1 or 5. Let G_n be the union of all $x \in \Gamma_n$ lying a distance of at least $d_n/5$ from the Θ_n-cutting points. The distance $d_n/5$ from the Θ_n-cutting points, combined with property 4 of Definition 4.0.12, guarantees that the assumption of Theorem 4.0.15 will be satisfied for any $x \in \Gamma_n$ along the x-orbit of the length r_n.

Since Θ_n is a K-balanced partition, every subword of length K contains the symbols 1 and 5. The lengths of all Θ_n-elements are uniformly proportional since Θ_n is good. From this, the existence of $\beta > 0$ such that $\mu(G_n) > \beta$ follows. Let us pick a point $x \in G_n$, and let S be the x-trajectory of length r_n. Let $x' \in I_n^1$, $T^t x' = x$, $t < r_n$. Let $\hat{x} \in I_n^1$; then $\hat{x} = T^{r_n} x' = T^{r_n - t} x$. The existence of such $x', \hat{x}$, and t clearly follows from the following observations:

1. Properties (1) and (4) in Definition 4.0.12 guarantee that I_n^1, when it returns to I_n, almost (up to a relative error 0.001), coincides with I_n^5. Correspondingly, I_n^5, when it returns to I_n, up to a relative error 0.001, coincides with I_n^1. From this, it follows that there exists a subinterval $\hat{I}_n^1 \subset I_n^1$ such that $T^{r_n} \hat{I}_n^1 \subset I_n^1$, $|\hat{I}_n^1| \geq 0.998 \cdot |I_n^1|$, and the orbit of $\hat{I}_n^1$ of length r_n consists of intervals completely belonging to the elements of Θ_n labeled 1 or 5.

2. From the definition of G_n, it follows that any point of the x'-trajectory or $\hat{x}$-trajectory of length r_n lies a distance of at least $d_n/10$ from the cutting points of Θ_n, which allows us to apply the Theorem 4.0.15 and Lemma 4.0.16.

Clearly, $f_{r_n}(x) - f_{r_n}(x') = f_t(\hat{x}) - f_t(x')$. Suppose t is less than the first return time of $\hat{x}$ into I_n. According to Lemma 4.0.16:

$$|f_t(\hat{x}) - f_t(x^{'})| < C \cdot (1 + |\varphi_t(\hat{x})| \cdot |\hat{x} - x^{'}|) < C \cdot (1 + |\varphi_t(\hat{x})| \cdot |I_n|).$$

Now we apply Theorem 4.0.15 to $|\varphi_t(z)|$ and use the fact that $|I_k| < |I_{k-1}|/2$ for any k to get

$$|f_t(\hat{x}) - f_t(x^{'})| < C \cdot \left(1 - \ln \lambda_n - \frac{\ln \lambda_{n-1}}{2} - \ldots - \frac{\ln \lambda_1}{2^{n-1}}\right).$$

If t is greater than the first return time of $\hat{x}$ to I_n, then let $w = T^r \hat{x}$ be the first return image of $\hat{x}$ in I_n and $u = T^r x'$ be the first return image of x' in I_n.

Now, we may consider the points w and u and the time $t-r$ and use exactly the same arguments which we used for the points $\hat{x}$ and x' and the time r to get

$$|f_{t-r}(w) - f_{t-r}(u)| < C \cdot \left(1 - \ln\lambda_n - \frac{\ln\lambda_{n-1}}{2} - \ldots - \frac{\ln\lambda_1}{2^{n-1}}\right).$$

The use of the triangle inequality gives

$$|f_t(\hat{x}) - f_t(x^{'})| \leq |f_r(\hat{x}) - f_r(x^{'})| + |f_{t-r}(w) - f_{t-r}(u)|,$$

and thus proves the inequality for any $t \leq r_n$.

Finally, we may pick any $x, y \in G_n$, find corresponding $x', y' \in I_n^1$, and use the triangle inequality

$$|f_{r_n}(x) - f_{r_n}(y)| \leq |f_{r_n}(x) - f_{r_n}(x^{'})| + |f_{r_n}(x^{'}) - f_{r_n}(y^{'})| + |f_{r_n}(y^{'}) - f_{r_n}(y)|$$

to complete the proof. $\square$

The theorem in this chapter was proved under assumption that the roof function is ' model'. In the next chapter we will briefly explain that the 'extra'-function of bounded variation causes no essential problems.

Chapter 5

Absence of mixing for special flows and flows on surfaces

Here we will need the following abstract criterion for the absence of mixing by Kocergin.

Theorem 5.0.18. *Let X be a Lebesgue space with probability measure μ, positive $f(x) \in L^1(X, \mu)$, $\int f(x)d\mu = 1$, and $T : X \to X$ an automorphism. Let there exist an increasing sequence of natural numbers q_n, a sequence of finite partitions ξ_n of the space X converging to the partition on points, and a sequence of sets $A_n \subset X$ and numbers $M, c > 0$ so that for arbitrary numbers n:*

1. *$\mu(A_n) > c$,*

2. *If $x, y \in A_n$ then $|f_{q_n}(x) - f_{q_n}(y)| < M$, and*

3. *$T^{q_n}(C \bigcap A_n) \subset C$ for arbitrary elements $C \in \xi_n$.*

Then the special flow U^t constructed by the automorphism T and the function f is not mixing.

In this chapter, we combine Lemma 3.0.6 and the main ergodic estimate to prove Theorem 1.0.1.

Proof. Let L be a Rauzy cycle from Lemma 3.0.10 and W be a corresponding matrix. Let Γ be any Rauzy cycle starting at the vertex (54321) and ending with the cycle L ,and let $\tau(\Gamma)$ be a corresponding substitution on 5 symbols. From Theorem 3.0.6 and positivity of all components of W, it follows that there is a constant K depending only on W so that $\tau(\Gamma)(1), \ldots, \tau(\Gamma)(5), \tau(\Gamma)(12345)$ are K-balanced words.

Let $X = \{T = (v, \pi) : \|v\| = 1, \pi = (54321), \exists D \forall v_i : 1 < |v_i|/D < 1.001\}$. It is easy to see that $X \subset \Delta_4 \times R(54321)$ is a positive measure set of IETs. The set $Y = W^{-1}(X)$, $Y \subset \Delta_4 \times R(54321)$, is also a positive Lebesgue measure set of IETs, which means that $\mu_Z(Y) > 0$ since μ_Z is equivalent to the Lebesgue measure (here W is considered as a projective transformation, which is justified because every $x \in \mathbb{R}^5_+$ can be uniquely identified with a one-dimensional subspace coming through x and, correspondingly, with some element $y \in \Delta_4$).

Since $Z : \Delta_4 \times R(54321) \to \Delta_4 \times R(54321)$ is ergodic, for almost every $T \in \Delta_4 \times (54321)$, there is a uniquely defined sequence $n_k : Z^{n_k}(T) \in Y$. Let q be the total number of changes from A to B or from B to A in L. Since the last two edges of L change labels from B to A, it follows that $Z^{n_k+q}(T) \in X$.

For any given $T \in \Delta_4 \times (54321)$, there is a unique sequence Ξ_k of Rauzy cycles and a sequence I_k of inducing intervals corresponding to Z^{n_k+q}. From the definition of W, definition of X, Theorem 3.0.6, applied to Ξ_k, and Lemma 3.0.10, it follows that I_k is a good sequence.

There is a function $H : Y \to \mathbb{R}_+$ defined for almost all $T \in Y$. Namely, let $Z_Y : Y \to Y$ be the induced map for Z on Y. If $r_Y : Y \to \mathbb{Z}_+$ is the first return time function for Z on Y, then clearly $Z_Y = Z^{r_Y}$. Let L_1 be the induced interval corresponding to the Rauzy cycle $T \to Z^q \circ Z_Y(T)$, and let L_0 be the induced interval corresponding to the Rauzy cycle $T \to Z^q(T)$. Then $H(p) = -\ln \frac{|L_1|}{|L_0|}$

Informally speaking, $Z^q(T)$ is a good interval exchange and $Z^q \circ Z_Y(T)$ is the first good induced interval exchange. Then $\frac{|L_1|}{|L_0|}$ is the relative length of L_1 according to L_0.

Lemma 5.0.19. *H is integrable on Y with respect to μ_Z.*

Proof. We define a function $\lambda : \Delta_4 \times R(54321) \to \mathbb{R}_+$ as follows. If T is an interval exchange , then $\lambda(T)$ is the length of the inducing interval corresponding to $Z(T)$.

Then

$$H(T) = -\sum_{k=0}^{r_Y(T)-1} \ln \lambda(Z^k(T)) - \sum_{k=0}^{q-1} \ln \lambda(Z^k(Z_Y(T))) + \sum_{k=0}^{q-1} \ln \lambda(Z^k(T)).$$

Let $\Psi : Y \to \mathbb{R}_+$,

$$\Psi(T) = -\sum_{k=0}^{r_Y(T)-1} \ln \lambda(Z^k(T)).$$

Since $\Psi(T)$ is the ergodic sum of $-\ln \lambda(\cdot)$ along the trajectory of T until the first return of T to Y and Z is ergodic and preserves μ_Z, it follows that

$$\int_Y \Psi \, d\mu_Z = \int_{\Delta_4 \times R(54321)} -\ln \lambda d\mu_Z.$$

Let $\Lambda = \int_{\Delta_4 \times R(54321)} -\ln \lambda d\mu_Z$. If we show that $\Lambda < \infty$, then the integrability of Ψ and the other terms in the expression for H clearly follow. If $T = (v, \pi) \in \Delta_4 \times R(54321)$ is an interval exchange and u is a length vector for an induced interval corresponding to $Z(T)$, then $v = ZC(T)u$, where ZC^{-1} is the Zorich cocycle. From this it follows that $1 = |v| \leq \|ZC(T)\| \cdot |u| = \|ZC(T)\| \cdot \lambda$ (the matrix norm here is the sum of the absolute values of all entries).

Now, we have $\Lambda = \int -\ln \lambda d\mu_Z \leq \int \ln \|ZC\| d\mu_Z < \infty$. The last inequality was proved by Zorich ([27]) and completes the proof of the lemma. $\square$

We pick an interval exchange $T \in \Delta_4 \times R(54321)$ and consider the good sequence I_k described above. We denote $\lambda_1 = |I_1|$ and $\lambda_k = \frac{|I_k|}{|I_{k-1}|}, k \geq 2$.

From the definition of n_k, I_k and $H(\cdot)$, it clearly follows that if $p = Z^{n_1}(T)$, then $\lambda_k(p) = H(Z_Y^k p), k \geq 2$, because n_k is a sequence of return times of T to Y under the action of Z.Let $f(x)$ be a symmetric logarithmic function for T and let $F_k(T) = \sup_{x,y \in G_k} |f_{r_k}(x) - f_{r_k}(y)|$, where r_k and G_k are defined in Theorem 4.0.17 for the sequence I_k. According to Theorem 3.4, we have

$$F_k(T) < C \cdot \left(1 - \ln \lambda_k - \frac{\ln \lambda_{k-1}}{2} - \ldots - \frac{\ln \lambda_1}{2^{k-1}}\right),$$

which means

$$F_k(T) < C \cdot \left(1 + H(Z_Y^k(p)) + \frac{H(Z_Y^{k-1}(p))}{2} + \ldots + \frac{H(p)}{2^{k-2}} - \frac{\ln \lambda_1(T)}{2^{k-1}}\right).$$

For any $p \in Y$, let

$$E_k(p) = H(Z_Y^k(p)) + \frac{H(Z_Y^{k-1}(p))}{2} + \ldots + \frac{H(p)}{2^{k-2}}.$$

Let $D(p) = \liminf E_k(p)$. Since $H = \int_Y H(p) d\mu_Z < \infty$ and Z_Y preserves $\mu_Z|_Y$, according to Fatou's Lemma $\int_Y D d\mu_Z \leq 2 \cdot H$. The last inequality shows that $D(p)$ is finite a.e. on Y, which immediately implies that for almost all $T \in \Delta_4 \times (54321), \liminf F_k(T) < \infty$.

Let $g(x)$ be a function of bounded variation and

$$\Omega_k = \sup_{x,y \in G_k} |g_{r_k}(x) - g_{r_k}(y)|.$$

Directly applying Katok's argument ([10]), we have that Ω_k is uniformly bounded for all k. Informally, the argument uses the fact that if the images of the small interval $[x, y]$ do not intersect under m iterations of T, then $|g_m(x) - g_m(y)| \leq Var(g)$. This implies that, for almost all T, there is a uniformly bounded subsequence of $F_k(T) + \Omega_k(T)$. Kocergin's criterion is now applied and completes the proof. □

Now we are ready to prove the main result of the dissertation. Let us briefly remind the reader of the construction of the measurable structure on $H(p_1, p_2)$. We pick up an abelian differential $\nu \in H(p_1, p_2)$ and in a small enough neighborhood Ω of ν (in a sense of the moduli space topology) we define the *period map*. Namely, for any $\tau \in \Omega$ we integrate τ by any smooth path connecting p_1 and p_2. This defines a linear functional on the relative homology group $H_1(M, p_1, p_2, \mathbb{C})$ or an element of the relative cohomology group $H^1(M, p_1, p_2, \mathbb{C})$. This way, the period map gives a local homeomorphism to an open subset of a vector space, and the pullback of the Lebesgue measurable structure gives a measurable structure on Ω. Thus, we get a measurable structure on the whole $H(p_1, p_2)$. The details may be found in [17].

For any measurable set $X \subset H(p_1, p_2)$, there is a uniquely defined set $Y \subset \Psi(\omega, p_1, p_2)$ such that any vector field $V \in Y$ is tangent to the vertical foliation of some abelian differential from X. This gives an induced measurable structure on $\Psi(\omega, p_1, p_2)$.

Now we start the proof of Theorem 1.0.1.

Proof. Let $\Upsilon \subset H(p_1, p_2)$ be a set of abelian differentials such that for all $\nu \in \Upsilon$, there exists $V \in \Psi(\omega, p_1, p_2)$ such that V is tangent to the vertical foliation of ν and the generated flow U^t is mixing.

There is no canonical choice of a horizontal segment for all differentials from $H(p_1, p_2)$, but there exists a locally canonical choice (here, 'locally' is taken in the sense of the moduli space topology). Moreover, the choice can be taken in such a way that the horizontal holonomy map defines an IET $T = (v, \pi)$, $\pi = (54321)$, and the preimage of any zero measure set $\Gamma \subset \Delta_4 \times (54321)$ under this locally canonical map has measure zero (this construction is well known as a space of 'zippered rectangles,' and has been widely used in the study of IETs and Teichmüller geodesic flow, see for details [4, 17, 18, 19, 26]).

Now, we take an abelian differential ν and the corresponding horizontal segment I. Clearly, any vector field V tangent to the vertical foliation of ν defines the same first return map to I as a holonomy map to the horizontal foliation, which is $T = (v, \pi)$, $\pi = (54321)$. The cutting points of T correspond to separatrices of fixed points of the generated flow U^t. The flow U^t can be clearly represented as a special flow over T and the cutting points correspond to the singularities of the roof function $f(x)$. Because the solutions of ODEs depend smoothly on the initial conditions, f is a piecewise smooth function on I with discontinuities in cutting points.

In order to understand the behaviour of f near cutting points, we may directly apply the corresponding calculations by Kocergin [12]. Since the fixed points are locally linearly isomorphic, due to the calculations by Kocergin one may see that the roof function can be represented as a symmetric logarithmic function over T. We skip the details here.

Now we may apply Theorem 2.0.4 to see that for any local choice of a vertical segment, the set of IETs allowing a corresponding mixing flow has measure zero, which means that the corresponding subset in $H(p_1, p_2)$ has measure zero. The observation that $H(p_1, p_2)$ has a countable base of open sets implies that the measure of Υ equals to zero and completes the proof. $\square$

Chapter 6

The related result by Ulcigrai.

After the main result of the dissertation was accepted for publication in paper form in The Journal of Modern Dynamics, C. Ulcigrai put a preprint of a paper, containing a more general result [25].

Definition 6.0.20. The function f has symmetric logarithmic singularities if

$$f(x) = C_0^+|\ln(x)| + \sum C_i^+|\ln(x-a_i)| + \sum C_i^-|\ln(a_i - x)| + C_0^-|\ln(1-x)|$$

and $\sum C_i^+ = \sum C_i^-$.

Theorem 6.0.21. *For any irredusible permutation π and almost any vector $v = (v_1, \ldots, v_n$ the special flow, constructed by the interval exchange $T = (v, \pi)$ and any symmetric logarithmic function f is not mixing.*

As a corollary she is able to treat flows on the surfaces of higher genus.

The main idea of the proof is to use Kochergin criterion and symmetric cancellation of logarithmic singularities. But the main technical step is much easier and more elegant: namely C. Ulcigrai notices that instead of strong form of symmetric cancellations, presented in the combinatorial lemma it's enough to use a weaker form to cancel logarithmic singularities.

Namely, instead of strong balance of induced partition it is enough to require that the sum of deviations uniformly converges, which, as C.Ulcigrai has shown, quite easily follows from integrability of Zorich cocycle and ergodicity of the Rauzy induction map.

6.0.11 Relative comparison of Ulcigrai's result and the result of the dissertation.

The result of C. Ulcigrai provides the proof of more general statement and the proof is more clear technically, since it does not use any complicated combinatorics, but uses intuitively clear arguments related to integrability of Zorich cocycle and summable deviations of induced interval trajectories from arithmetic progressions.

However it has the following disadvantage: the result strongly relies on the integrability of Zorich cocycle, and ergodicity of the Rauzy induction, since it requires that a trajectory of IET under the Rauzy induction returns infinitely many times to some very specific set of positive measure. Which means that in the possible attempt to generalize the proof to the case of all (rather than almost all) interval exchanges one would have to describe which interval exchanges visit some specific set K of positive measure infinitely many times.

Moreover the construction of the set is non-explicit and relies on the use of Egorov's theorem from measure theory and integrability of Zorich cocycle.

Having this said we come to the relative advantage of our proof. The main tool of the proof, the combinatorial lemma, requires that the trajectory of a given interval exchange T under the Rauzy induction map visits the set $K = \{(v,\pi)|\pi = (54321)\}$ infinitely many times. One may easily see that this assumption must be satisfied for any ergodic T.

Namely, suppose the opposite is true and the trajectory of some T visits K only finitely many times. But that implies that after some number of iterations the trajectory stays inside the left or inside the right part of the Rauzy graph. Which immediately implies (from the combinatorial structure of the graph) that the interval I_1 or the interval I_5 are not divided under the action of the Rauzy induction map. And that in turn implies that there exists a nontrivial invariant set of positive measure for T, consisting of a finite union of intervals.

So we have just shown that any ergodic interval exchange must return to the set K infinitely many times.

In order to get the proof for all interval exchanges of the type (54321) we still have to substitute the use of integrability of Zorich cocycle. In the other words we need estimates on the lengths of the induced intervals once the iteration of T returns to the set K under the Rauzy induction map.

In our opinion it is easier to do using our proof than the proof by Ulcigrai, because our set K has an easier description, than the corresponding set in the proof of Ulcigrai, because of the following reasons, already mentioned:

1) The definition of the set K does not use the integrability of Zorich cocycle. (The set K of C. Ulcigrai does use it essentially)

2) Any ergodic interval exchange must return to the set K infinitely many times. (It is not so for the set K of C. Ulcigrai)

3) Our set K has more elementary description: $K = \{(v, \pi) | \pi = (54321)\}$ and the set K in the proof of C. Ulcigrai is highly non-explicit.

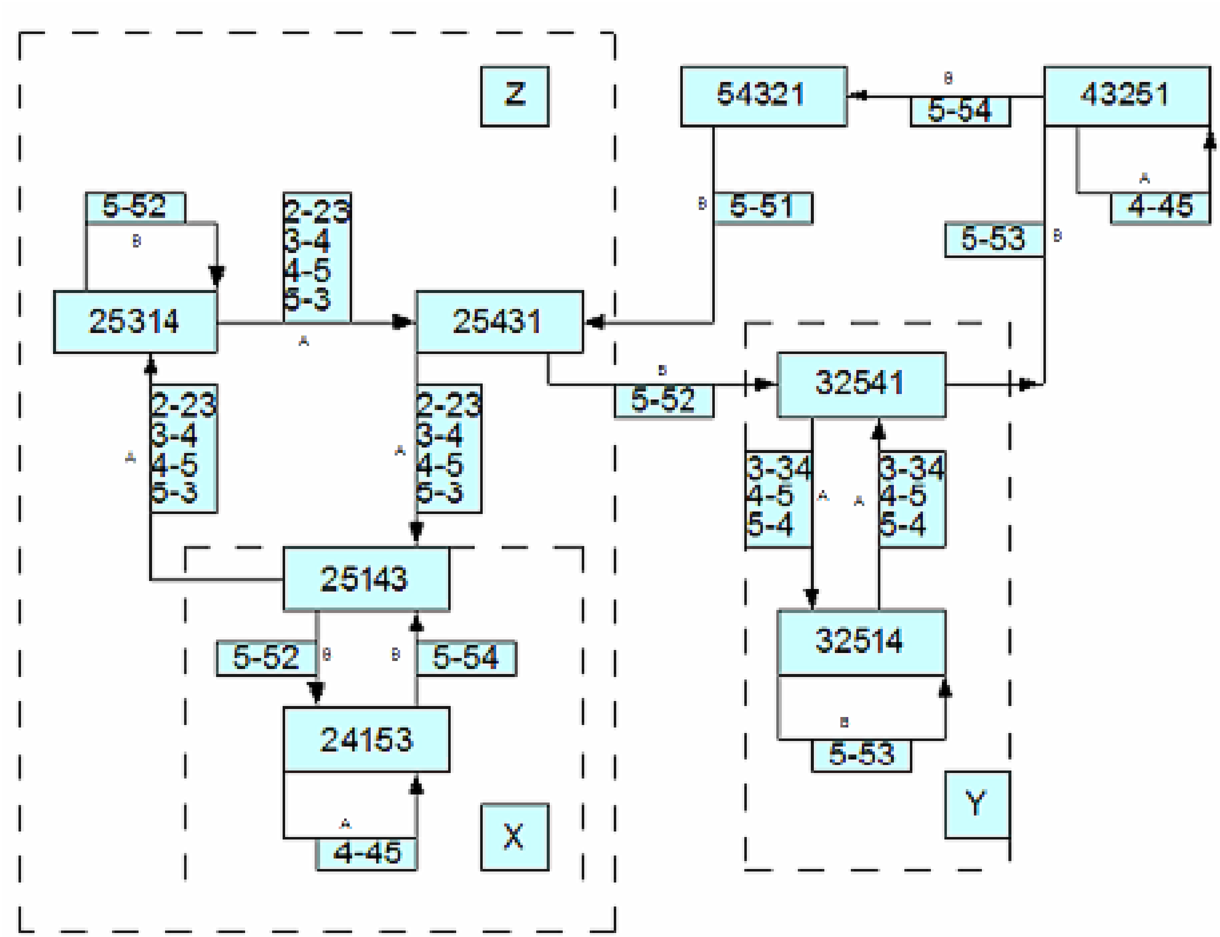
Summarizing all, what was said before we get to the following conclusion:

The proof of C. Ulcigrai works better and easier for almost-all interval exchanges and more elementary in the combinatorial sence. And it also works for much wider class of permutations.

The main element of our proof is more constructive and must take place for all ergodic interval exchanges of the type (54321), which could possibly lead to the proof of the result for all interval exchanges of a given type, while the proof of C. Ulcigrai essentially uses typical properties of interval exchanges so can not be directly applied for all interval exchanges.

Appendix **A**

Rauzy graph with substitutions and labels on edges



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Vita

Dmitri Scheglov

Personal information:

Name: Dmitri Vadimovich Scheglov
Date of Birth: March 18, 1980
Place of Birth: Chelyabinsk, Russia
Citizenship: Russia
Address: Department of Mathematics, 018 McAllister Building, The Pennsylvania State University, University Park, PA 16802
E-mail: dvs117@psu.edu
Phone: (814) 865-42-91

Education:
B.A. Moscow State University, Moscow, Russia, 2002.
Thesis advisor: V. I. Oseledets.
Thesis title: *Adic coding of ergodic automorphisms of the Lebesgue space.*

Ph.D. Pennsylvania State University, August, 2009.
Thesis advisor: A. B. Katok.
Thesis title: *Absence of mixing for smooth flows on the genus two surface.*

Positions:
August 2005 - May 2009: Graduate Teaching Assistant, Department of Mathematics, Pennsylvania State University.

Publications:
Absence of mixing for smooth flows on the genus two surface, Journal of Modern Dynamics, Vol 3(1), 2009.

Talks:
Semi-annual Workshop in Dynamical Systems and Related Topics, State College, PA, October 18-21, 2007.
Title: Absence of mixing for special flows over interval exchange maps.

Honors and awards:
Soros undergraduate student, Moscow State University, 1998.
University Graduate Fellowship, Pennsylvania State University, 2005.