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Abstract

The goal of halo orbit design is to find the initial conditions that lead to a periodic orbit with the desired characteristics. Conventionally, this is done by applying an analytical solution to get an approximation and refining that solution using differential correction to obtain a periodic orbit. Although this method has been used successfully in practice, it is highly dependent on the accuracy of the analytical solution and can lead to significant errors between the desired and achieved orbits. To improve this process, Nath and Ramanan developed a differential evolution scheme to compute halo orbits with a specified Z-amplitude. Differential evolution was shown to be more accurate than the conventional design method and does not require a good initial guess. In this work differential evolution is used to compute halo orbits in the Jupiter-Europa system. In addition to targeting orbits by Z-amplitude, a novel approach is presented to target orbits by X-amplitude and orbital period. It is found, in agreement with Nath et al., that differential evolution can target halo orbits by Z-amplitude more accurately than traditional methods. It is also shown that differential evolution can target halo orbits by X-amplitude and orbital period, but without additional constraints only planar orbits are computed. Finally, it is found that the use of a stochastic mutation factor in differential evolution reduces the chance of convergence to a non-optimal solution.
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CR3BP The Circular Restricted Three-Body Problem

$M_1, M_2, M_3$ Mass of primary body, secondary body, and third body respectively

$\mu$ Normalized mass ratio

$T$ Orbital period

TU Time unit

LU Length unit

$R_{12}$ Distance between primary and secondary

$A_x, A_y, A_z$ X, Y, and Z amplitudes of halo orbit in CR3BP coordinate system

$n$ Mean motion of secondary body

$\phi$, STM State transition matrix

$M$ Monodromy matrix

$\lambda_{max}$ Max eigenvalue of monodromy matrix $M$

$\vec{X}(t)$ Spacecraft state containing the position and velocity

$DE$ Differential evolution

$G$ Generation in differential evolution

$N$ Population size for differential evolution

$P$ Member of population for differential evolution

$F$ Mutation factor for differential evolution

$CR$ Crossover frequency for differential evolution

$V$ Mutant vector for differential evolution
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Chapter 1

Introduction

1.1 The Circular Restricted Three-Body Problem (CR3BP)

Johannes Kepler published his first two laws of planetary motion in 1609 and his third law in 1619, laying the foundation for modern orbital mechanics. In 1687 Newton published *Philosophiae Naturalis Principia Mathematica*, which contained formulations for his laws of motion, universal gravitation, and the three-body problem [1]. In 1767 Leonard Euler proposed a restricted version of the three-body problem with the third mass being infinitesimally small. Euler showed that if the three masses were placed on a straight line, the bodies would exhibit periodic motion about ellipses [2]. Euler was also the first to formulate the circular restricted three-body problem (CR3BP) in a rotating coordinate system. Lagrange studied the CR3BP proposed by Euler and proved the existence of five equilibrium points at which the gravitational forces in the system are balanced [2]. These equilibrium points are also called Lagrange points and are commonly referred to as L1, L2, L3, L4, and L5. L1, L2, and L3 are called the collinear points while L4 and L5 are
referred to as the triangular points because they form an equilateral triangle with the two large bodies. The L1, L2 and L3 points are inherently unstable while the L4 and L5 points are stable for some systems [2].

Figure 1.1. Lagrange point locations

There are 18 first-order ordinary differential equations that describe the three-body problem which require 18 integrals of motion for a closed-form solution. Only 10 integrals of motion exist but if some simplifications are made, a solution is possible [2]. In the CR3BP it is assumed that the mass of the third body is infinitesimally smaller than the two large bodies, which are defined as the primary and secondary, that the motion of the two large bodies is Keplerian, and that the large bodies move in circular orbits about a common barycenter.
1.2 Halo Orbits

Halo orbits are periodic orbits about equilibrium points in the circular restricted three-body problem. They are typically characterized by an out-of-plane or Z-amplitude which is the halo orbit’s excursion from the orbital plane of the primary and secondary. Halo orbits can also be characterized by their in-plane amplitudes in the X and Y directions. These parameters will be referred to as the X-amplitude, Y-amplitude, and Z-amplitude or \( A_x \), \( A_y \) and \( A_z \).

![Figure 1.2. Out-of-plane amplitude \( A_z \)](image)
The first mission to utilize a halo orbit was the International Sun-Earth Explorer-3 (ISEE-3). ISEE-3 proved that halo orbits developed in the CR3BP model can be used in practice. ISEE-3 orbited about the L1 point in between the Sun and the Earth which allowed the spacecraft to study the interaction between Earth’s magnetic field and solar winds [3]. The Genesis Mission also utilized a halo orbit about the Sun-Earth L1 point in order to collect solar wind samples [4]. The Hubble Space Telescope’s replacement, the James Web Space Telescope (JWST), will utilize a halo orbit about the Sun-Earth L2 point in order to get an unobstructed view into the observable universe [5].

Along with their utility in scientific missions, halo orbits have great potential for use as communications satellites. Farquhar was the first to develop halo orbits
and theorized their use as a communications link between Earth and the far side of the moon for future Apollo missions [6]. Furthermore, Pernicka et al. suggest placing two satellites in halo orbits about the L1 and L2 points in the Sun-Mars system to act as communication relays for future inner solar system missions [7].

1.3 Differential Evolution (DE)

Stochastic optimization refers to any method that seeks to minimize an objective function by utilizing randomness [8]. Evolutionary methods are a subset of stochastic optimization and use a population of individuals, each containing a possible solution, to solve an optimization problem [9]. Differential Evolution (DE), a stochastic evolutionary method, was first introduced in 1997 by Storn et al. [10]. In DE each element of the population is initialized to a random value between some bounds. Next, a mutant vector is created that is based on the combination of three random elements of the population. A trial element is created from a combination of the mutant vector and the original element. This trial element is then compared to the original element. If the trial element is a better solution then it replaces the original element in the next generation [10]. DE has proven to be very effective in solving a wide variety of optimization problems and is simple to implement.
1.4 Missions To Europa

Europa is one of the four icy Galilean moons orbiting Jupiter. The Pioneer missions were the first to reach Jupiter in the early 1970’s. The Voyager mission became the first to image Europa, albeit with a very low resolution in 1979. In the late 1990’s the Galileo mission completed 12 flybys of Europa. One of the most important findings was the measurement of Europa’s magnetic field. This measurement implied the existence of conductive fluid beneath Europa’s icy surface. Scientists believe the most likely material to create this magnetic signature is a saltwater ocean [11]. This revelation has sparked intense scientific interest in Europa as one of the leading candidates to find extraterrestrial life. Several missions to Europa are currently in development in order to further study the moon and determine the likelihood of organic life existing beneath the surface. NASA’s Europa Clipper mission intends to address this question and is in the early stages of development. The goal of the mission is to study Jupiter’s moons and conduct multiple flybys of Europa to assess its habitability [12]. ESA’s Jupiter Icy Moons Explorer (JUICE) mission also plans to conduct flybys of Europa to search for organic life. The mission will be the first to attempt subsurface characterization of Europa [13]. In these upcoming missions, halo orbits in the Jupiter-Europa system could be useful for meeting mission requirements. They could provide a good vantage point for imaging Europa or for gathering navigation data.
Chapter 2

Theory

In this chapter the theory of the circular restricted three body problem (CR3BP) is developed and all relevant coordinate systems and units are defined. Additionally, a review of halo orbit design methods is presented and all methods used in this work are described in detail.

2.1 Literature Review

Halo orbit design consists of finding the initial conditions that lead to a periodic orbit of the desired size. This is traditionally done using a two-step process given by Richardson [3]:

1. Obtain the approximate initial conditions using Richardson’s 3rd order analytical solutions.

2. Refine the initial conditions using a differential correction procedure.

Differential correction is an iterative process that repeatedly updates the initial conditions and propagates until a periodic orbit is found. It is highly dependent
on the quality of the approximation given by the analytical solution and can lead to errors between the specified and achieved characteristics of the halo orbit. To reduce this error, additional refinement using pseudo arc-length continuation can be utilized [14].

Grid search methods are often used to find a large number of potential initial conditions for a halo orbit. In these methods the search space for the initial conditions is discretized and the trajectories for each grid point are propagated. If a trajectory is close to periodic, it is refined using a differential correction scheme. Russell used a grid search method to find over 600,000 periodic halo orbit solutions in the Jupiter-Europa system [15]. Grid search methods are computationally intensive and don’t allow for the precise design of orbit parameters, but they give a large number of solutions.

Another popular group of methods for finding halo orbits in the CR3BP utilize Poincaré maps [16]. Poincaré mapping is a mathematical technique used to study periodic solutions in dynamical systems. To create a Poincaré map, a 2-D Poincaré section is chosen from the 3-D state space and the points, each representing a set of initial conditions, are discretized. The Poincaré section is usually chosen to ensure that the velocity of the orbit is transverse at intersection [17]. This increases the likelihood of finding periodic orbits. On the Poincaré map, periodic trajectories are shown as a single point and non-periodic trajectories will appear as multiple points. If a trajectory is close to periodic, the trajectory is refined using differential
correction. One of the main benefits of using Poincaré maps is the ability to reduce the dimensions of the search space from three to two. This significantly lowers the computation time.

Nath and Ramanan utilized a differential evolution (DE) scheme to compute halo orbits about the Sun-Earth L1 point [18]. The differential evolution method conducts a global search for initial conditions that lead to a periodic orbit of a specified size. This method is more computationally intensive than conventional methods, but it does not require a good initial guess. It was shown that DE can compute halo orbits of a specified Z-amplitude to a higher degree of accuracy than traditional methods.

The performance of DE can be fine-tuned using two parameters, the mutation factor $F$ and the crossover frequency $CR$. The mutation factor is usually chosen in the range 0.5 to 1. Using a higher value for the mutation factor reduces the chance of convergence to local minima but can increase computation time [19]. Das et al. suggest using a random mutation factor in the range 0.5 to 1 [20]. The crossover frequency $CR$ is usually chosen in the range 0.3 to 0.9. A higher CR value will speed up convergence but may lead to a sub-optimal solution. It is important to tailor these parameters to the optimization problem.
2.2 Mathematical Models

2.2.1 CR3BP Assumptions

The CR3BP is based off a set of assumptions that allow for a solution to the
differential equations describing the system. The set of assumptions are as follows:

1. There are three bodies considered with masses $M_1$, $M_2$, and $M_3$

2. $M_1 > M_2 >> M_3$

3. The bodies act as point masses

4. $M_1$ and $M_2$ are in circular orbits about the barycenter of the two masses

5. Perturbations are ignored

$M_1$ will be referred to as the primary, $M_2$ the secondary, and $M_3$ the spacecraft.

2.2.2 CR3BP Synodic Coordinate System

The CR3BP synodic coordinate system has an origin at the barycenter of the
primary and secondary body and rotates with respect to an inertial barycentric
coordinate system. The X-axis of the CR3BP synodic frame is in the direction of
the secondary body, the Z-axis is in the direction of the angular momentum of the
secondary body, and the Y-axis completes the coordinate system. The inertial and
synodic frames share the same Z-axis.
2.2.3 Normalized Units

In order to make analysis in the CR3BP easier it is useful to normalize the units of the system. A parameter \( \mu \) is defined:

\[
\mu = \frac{M_2}{M_1 + M_2} \quad (2.1)
\]

After \( \mu \) is introduced, the normalization of the system is as follows:

1. Mass of System \( M_1 + M_2 = 1 \)
2. Mass of Primary Body \( M_1 = 1 - \mu \)
3. Mass of Secondary Body \( M_2 = \mu \)
4. Distance Between $M_1$ and $M_2 = 1$

5. Location of Primary $M_1 : X = -\mu, \ Y = 0, \ Z = 0$

6. Location of Secondary $M_2 : X = 1 - \mu, \ Y = 0, \ Z = 0$

7. Gravitational Parameter $G = 1$

8. Orbital Period of $M_1$ and $M_2$ About Barycenter $T = 2\pi$

The value of the length unit (LU) and time unit (TU) of the system in kilometers and seconds respectively is:

\[
LU = R_{12} \quad km
\]
\[
TU = \frac{T_{\text{secondary}}}{2\pi} = \left(\frac{LU^3}{\mu}\right)^{1/2} \quad \text{Seconds}
\]

(2.2)

where $R_{12}$ is the distance between the primary and secondary bodies and $T_{\text{Secondary}}$ is the orbital period of the secondary body about the primary.
2.2.4 Equations of Motion Derivation

The inertial barycentric acceleration of the spacecraft is given by [21]:

\[
\ddot{\vec{r}}_{s/c}^B = -GM_1 \frac{\vec{r}_{s/c,1}}{r_{s/c,1}^3} - GM_2 \frac{\vec{r}_{s/c,2}}{r_{s/c,2}^3}
\] (2.3)

where \(\vec{r}_{s/c,1}\) denotes the relative position of the spacecraft with respect to \(M_1\). Because of the rotation of the coordinate system, additional terms must be considered.

Using \(\vec{r}_S\), \(\vec{v}_S\), and \(\vec{a}_S\) as the position, velocity, and acceleration of the spacecraft in the synodic frame and \(\vec{\omega}_S\) as the angular velocity of the synodic frame gives:

\[
\ddot{\vec{r}}_{s/c}^B = \ddot{\vec{r}}_S + \dot{\vec{\omega}}_S \times \vec{r}_S + \vec{\omega}_S \times (\vec{\omega}_S \times \vec{r}_S) + 2\vec{\omega}_S \times \vec{v}_S + \vec{a}_{org}
\] (2.4)

where \(\vec{a}_{org}\) is the acceleration of the synodic system with respect to the inertial origin. The synodic origin is not rotating with respect to the inertial origin so \(\vec{a}_{org} = \vec{0}\). Also \(\dot{\vec{\omega}}_S = \vec{0}\) because of the assumed circular motion of the primary and secondary. Simplifying the expression and substituting Cartesian coordinates gives:

\[
\ddot{\vec{r}}_{s/c}^B = \ddot{\vec{r}}_S + \vec{w}_S^2(x\hat{x}_S + y\hat{y}_S) + 2\vec{w}_S(\dot{y}\hat{x}_S - \dot{x}\hat{y}_S)
\] (2.5)

Converting (2.4) into gradient form gives:

\[
\ddot{\vec{r}}_{s/c}^B = \nabla \left( \frac{GM_1}{r_{s/c,1}} + \frac{GM_2}{r_{s/c,2}} \right)
\] (2.6)
Breaking the vector relations into Cartesian components where \( r_{B1} \) and \( r_{B2} \) are the distances of the primary and secondary body from the barycenter respectively gives:

\[
\ddot{x} - 2w_S\dot{y} - w_S^2 x = \frac{\partial}{\partial x} \left( \frac{GM_1}{r_1} + \frac{GM_2}{r_2} \right)
\]

\[
= -\frac{GM_1(x + r_{B1})}{r_1^3} - \frac{GM_2(x - r_{B2})}{r_2^3}
\]

\[
\ddot{y} + 2w_S\dot{x} - w_S^2 y = \frac{\partial}{\partial y} \left( \frac{GM_1}{r_1} + \frac{GM_2}{r_2} \right) = -\frac{GM_1y}{r_1^3} - \frac{GM_2y}{r_2^3}
\]

\[
\ddot{z} = \frac{\partial}{\partial z} \left( \frac{GM_1}{r_1} + \frac{GM_2}{r_2} \right) = -\frac{GM_1z}{r_1^3} - \frac{GM_2z}{r_2^3}
\]

\[
r_1 = \sqrt{(x + r_{B1})^2 + y^2 + z^2} \quad r_2 = \sqrt{(x - r_{B2})^2 + y^2 + z^2}
\]

Now substituting normalized units gives:

\[
M_1 = 1 - \mu = r_{B2}
\]

\[
M_2 = \mu = r_{B1}
\]

\[
\omega_S = 1
\]

\[
G = 1
\]
which allows us to write \( r_1 \) and \( r_2 \) as:

\[
\begin{align*}
    r_1 &= \sqrt{(x + \mu)^2 + y^2 + z^2} \\
    r_2 &= \sqrt{(x - (1 - \mu))^2 + y^2 + z^2}
\end{align*}
\] (2.10)

Finally the non-dimensional rotating equations of motion in the CR3BP are given by:

\[
\begin{align*}
    \ddot{x} - 2\dot{y} - x &= -\frac{(1 - \mu)(x + \mu)}{r_1^3} - \frac{\mu(x - (1 - \mu))}{r_2^3} \\
    \ddot{y} + 2\dot{x} - y &= -\frac{(1 - \mu)y}{r_1^3} - \frac{\mu y}{r_2^3} \\
    \ddot{z} &= -\frac{(1 - \mu)z}{r_1^3} - \frac{\mu z}{r_2^3}
\end{align*}
\] (2.11)

### 2.2.5 Collinear Lagrange Point Locations

The Lagrange points are equilibrium positions in the CR3BP. In equilibrium:

\[
\begin{align*}
    \dot{x} &= \dot{y} = \dot{z} = 0 \\
    \ddot{x} &= \ddot{y} = \ddot{z} = 0
\end{align*}
\] (2.12)

The 3 collinear Lagrange points are all on the x-axis, thus \( y = 0 \) and \( z = 0 \). The x-component of Eq. (2.11) at equilibrium becomes [21]:

\[
\begin{align*}
    x - \frac{(1 - \mu)(x - \mu)}{r_1^3} - \frac{\mu(x + 1 - \mu)}{r_2^3} = 0
\end{align*}
\] (2.13)
This leads to three quintic equations to find the three collinear points L1, L2 and L3 respectively:

\begin{align}
  x^5 + (3 - \mu)x^4 + (3 - 2\mu)x^3 - \mu x^2 - 2\mu x - \mu &= 0 \\
  x^5 - (3 - \mu)x^4 + (3 - 2\mu)x^3 - \mu x^2 + 2\mu x - \mu &= 0 \\
  x^5 + (2 + \mu)x^4 + (1 + 2\mu)x^3 - (1 - \mu)x^2 - 2(1 - \mu)x - (1 - \mu) &= 0
\end{align}

The real part of the solution gives the location of the collinear Lagrange points.

### 2.2.6 Halo Orbit Design

Halo orbit design methods utilize several characteristics of periodic halo orbits in order to find initial conditions [18].

1. Halo Orbits always pass through the X-Z plane orthogonally. At an X-Z plane crossing \( y = \dot{x} = \dot{z} = 0 \)

2. After half of the orbital period halo orbits must pass through the X-Z plane.

To compute halo orbits, the initial conditions \( x_0, z_0, \) and \( \dot{y}_0 \) are found such that the spacecraft state \( \vec{X}(t = \frac{T}{2}) \), which contains the position and velocity of the spacecraft at half of the orbital period is:

\[
\vec{X}(t = \frac{T}{2}) = [x(\frac{T}{2}), 0, z(\frac{T}{2}), 0, \dot{y}(\frac{T}{2}), 0]
\]
Halo orbits can be characterized by their X, Y, and Z amplitudes denoted as $A_x$, $A_y$, and $A_z$. Traditionally when computing halo orbits, only the $A_z$ is specified but this thesis will also look for halo orbits of a specified $A_x$.

### 2.2.6.1 Richardson’s Analytical Solution

Richardson was the first to develop analytical solutions that can be used to compute initial conditions for halo orbits [3]. It is a good approximation that can be refined through differential correction. The third order periodic solutions are as follows:

\[
x = a_{21}A_x^2 + a_{22}A_z^2 - A_x\cos(\tau_1) + (a_{23}A_x^2 - a_{24}A_x^2)\cos(2\tau_1) \\
\quad + (a_{31}A_x^3 - a_{32}A_xA_z^2)\cos(3\tau_1) \\
y = kA_x\sin(\tau_1) + (b_{21}A_x^2 - b_{22}A_z^2)\sin(2\tau_1) \\
\quad + (b_{31}A_x^3 - b_{32}A_xA_z^2)\sin(3\tau_1) \\
z = A_x\cos(\tau_1) + d_{21}A_xA_z(\cos(2\tau_1) - 3) \\
\quad + (d_{32}A_zA_x^2 - d_{31}A_z^3)\cos(3\tau_1)
\]

where:

\[
\tau_1 = \lambda \omega nt + \theta
\]
and $a_{21}, a_{22}, a_{23}, a_{24}, a_{31}, a_{32}, b_{21}, b_{22}, b_{31}, b_{32}, d_{21}, d_{32}, d_{31}, A_x, \lambda, \omega$ are constants that can be calculated using the equations in Appendix B [22]. Additionally $n$ is the mean motion of the secondary body, and $\theta$ determines the family of orbits. For all calculations in this work $\theta = 0$.

To find the initial conditions the derivative of the equation for $y$ is computed and the time $t$ is set to zero. This gives the 3rd order analytical solution for the initial conditions:

$$x_0 = a_{21}A_x^2 + a_{22}A_z^2 - A_x \cos(\theta) + (a_{23}A_x^2 - a_{24}A_x^2)\cos(2\theta)$$

$$+ (a_{31}A_x^3 - a_{32}A_xA_z^2 \cos(3\theta))$$

$$z_0 = A_x \cos(\theta) + d_{21}A_xA_z(\cos(2\theta) - 3)) + (d_{32}A_zA_x^2 - d_{31}A_x^3)\cos(3\theta) \quad (2.18)$$

$$\dot{y}_0 = kA_x(\lambda \omega \cos(\theta)) + (b_{21}A_x^2 - b_{22}A_z^2)(2\lambda \omega \cos(2\theta))$$

$$+(b_{31}A_x^3 - b_{32}A_xA_z^2)(3\lambda \omega \cos(3\theta))$$
2.2.6.2 Differential Correction

Differential correction is used to refine the initial conditions given by Richardson’s analytical solution. The differential correction process is as follows:

1. Choose one parameter, $x_0$, $z_0$, or $\dot{y}_0$ to keep fixed. The values that will be corrected are two of the initial state parameters and the value for half of the orbital period $\frac{T}{2}$. The vector containing these parameters is $\vec{Y}_0$.

2. Numerically integrate the initial conditions from $t_0$ to $t_f = \frac{T}{2}$ to get a reference trajectory $\vec{Y}_{ref}$.

3. Calculate the state transition matrix $\phi$ using the initial conditions. The state transition matrix maps the initial state forward to some time $t$. The procedure for generating the state transition matrix can be found in Appendix A.

4. Evaluate the derivative of the reference trajectory state at $t_f$ to get $\ddot{x}_{ref}(t_f)$, $\ddot{z}_{ref}(t_f)$, and $\dot{y}_{ref}(t_f)$.

5. Construct a differential matrix $DF$:

$$DF = \begin{bmatrix} \phi_{41} & \phi_{45} & \ddot{x}_{ref}(t_f) \\ \phi_{61} & \phi_{65} & \ddot{z}_{ref}(t_f) \\ \phi_{21} & \phi_{25} & \dot{y}_{ref}(t_f) \end{bmatrix}$$

(2.19)
6. Take the inverse of $DF$ to get the matrix $D$

$$D = DF^{-1} \quad (2.20)$$

7. Compute the correction $C$ to the two initial state parameters and half period:

$$C = -D \begin{bmatrix} \dot{x}_{ref}(t_f) \\ \dot{z}_{ref}(t_f) \\ y_{ref}(t_f) \end{bmatrix} \quad (2.21)$$

8. Apply the correction to $\vec{Y}_0$ then set $\vec{Y}_0$ to $\vec{Y}_{corrected}$

$$\vec{Y}_{corrected} = \vec{Y}_0 + \vec{C} \quad (2.22)$$

$$\vec{Y}_0 = \vec{Y}_{corrected} \quad (2.23)$$

9. Repeat steps 1-8 until convergence
2.2.6.3 Differential Evolution

The goal of differential evolution (DE) is to minimize some objective function $J$. In this work three different objective functions are presented for different design goals:

1. Achieve $A_z$: $J = |\dot{x}(\frac{T_2}{2})| + |\dot{z}(\frac{T_2}{2})| + |A_{z, \text{achieved}} - A_{z, \text{desired}}|$

2. Achieve $A_x$: $J = |\dot{x}(\frac{T_2}{2})| + |\dot{z}(\frac{T_2}{2})| + |A_{x, \text{achieved}} - A_{x, \text{desired}}|$

3. Achieve orbital period (days) $\frac{T_2}{2}$: $J = |\dot{x}(\frac{T_2}{2})| + |\dot{z}(\frac{T_2}{2})| + 0.01|\frac{T_2}{2, \text{achieved}} - \frac{T_2}{2, \text{desired}}|$

The value of $|\dot{x}(\frac{T_2}{2})| + |\dot{z}(\frac{T_2}{2})|$ is zero for a perfectly periodic orbit. Given one of these objective functions, the steps of DE for computing halo orbits are as follows [18]:

1. Initialize a population $G$ of size $N$ where each element $P$ of the population contains the three unknowns $[x_0, z_0, \dot{y}_0]$ initialized between some bounds.

2. Propagate each element $P$ and terminate when the trajectory crosses the X-Z plane, then evaluate the objective function $J$. Matlab’s Runge-Kutta 4th order integrator ode45 was used for the integration with 1000 steps and a relative and absolute tolerance of $1 \times 10^{-8}$.

3. Test each element $P$ of the current population $G$ for whether it will be carried on to the next generation $G+1$ by generating a trial element. The indexes $i$ and $j$ represent the element of the population and one of the three components of each element respectively. The testing process is as follows:
(a) A mutant vector $V$ is generated using $V_{i,G+1} = P_{r_1,G} + F(P_{r_2,G} - P_{r_3,G})$

where $r_1$, $r_2$ and $r_3$ are three distinct random integers in the interval $[1, N]$ and $F$ is the mutation factor. $r_1$, $r_2$ and $r_3$ can not be equal to the index of the current element.

(b) A trial element is formed using the current population element $P_i$ and the mutant vector $V$. For each component $j$ of $P$ a random number $rand(j)$ is generated between 0 and 1. Using the crossover frequency $CR$, if $rand(j) > CR$ the $j$ element of $P_i$ is retained in the trial element and if $rand(j) \leq CR$ the $j$ component of the mutant vector $V$ becomes the $j$ component of the trial element.

(c) The objective function is evaluated for the trial element. If the function value of the trial element is lower than the current function value, the trial element replaces the original element in the next generation of the population. If the function value is higher for the trial element then the original element is retained in the next generation. Also if any component of the trial element is outside the specified bounds then the trial element is rejected.

4. Carry out this process for each element of the population to create a new generation.
5. Repeat steps 2-4 until the objective function is lower than some threshold value. In this work the threshold is set to $1 \times 10^{-15}$.

2.2.7 Halo Orbit Stability

The monodromy matrix $M$ is a special case of the state transition matrix that maps the initial state forward by one orbital period. The stability of a halo orbit can be assessed by analyzing the eigenvalues of the monodromy matrix. Broucke developed a stability index based on these eigenvalues that indicates the stability of a halo orbit [23]. Folta et al. utilized a slight modification to Broucke’s stability index that will also be used in this work [24]. The stability index $\nu$ is given by equation 2.24

$$\nu = \frac{1}{2} \left( |\lambda_{\text{max}}| + \frac{1}{|\lambda_{\text{max}}|} \right)$$  \hspace{1cm} (2.24)

where $\lambda_{\text{max}}$ is the value of the largest eigenvalue of $M$. A stable periodic orbit has a stability index $\nu = 1$ and the larger the stability index becomes, the greater the orbit instability. The stability of the orbit is directly related to station-keeping cost so it is important to analyze the stability of each computed halo orbit.
Chapter 3

Halo Orbit Design in the Jupiter-Europa CR3BP

In this chapter the results from using DE to target halo orbits by Z-amplitude, X-amplitude, and orbital period in the Jupiter-Europa CR3BP are presented. All computed orbits are near the Jupiter-Europa L1 or L2 point.

3.1 Problem Formulation

3.1.1 Normalized Units and Lagrange Point Locations

The relevant parameters for the Jupiter-Europa system are:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jupiter-Europa Distance</td>
<td>670,900 km</td>
</tr>
<tr>
<td>Europa Period</td>
<td>3.5511 Days</td>
</tr>
<tr>
<td>Jupiter GM</td>
<td>$1.2668654 \times 10^8 \text{ km}^3 \text{ s}^{-2}$</td>
</tr>
<tr>
<td>Europa GM</td>
<td>$3202.72 \text{ km}^3 \text{ s}^{-2}$</td>
</tr>
</tbody>
</table>

Table 3.1. Jupiter-Europa parameters
Given these parameters the normalized units according to equation 2.2 are:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 LU</td>
<td>670,900 km</td>
</tr>
<tr>
<td>1 TU</td>
<td>1.769322 Days</td>
</tr>
<tr>
<td>$\mu$</td>
<td>$2.52800 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

Table 3.2. Jupiter-Europa normalized units

To find the location of the L1 and L2 points in the Jupiter-Europa System, the quintic equations given in equation 2.14 are solved. This gives:

$L1 = [0.979513, 0, 0] \text{ LU}$ \hspace{1cm} (3.1)

$L2 = [1.0204613, 0, 0] \text{ LU}$

### 3.1.2 Search Space and Additional Constraints

Given the location of the L1 and L2 points, the lower and upper bounds (LB and UB) of the initial conditions $[x_0, z_0, \dot{y}_0]$ for orbits about L1 are:

$LB = [0.95, -0.1, -0.1] \text{ LU}$ \hspace{1cm} (3.2)

$UB = [1.0, 0.1, 0.1] \text{ LU}$
and for orbits about L2:

\[ LB = [1.0, -0.1, -0.1] LU \]

\[ UB = [1.05, 0.1, 0.1] LU \]

(3.3)

To speed up the convergence process some additional constraints were added. If the trajectory passed through the position of Europa on the x-axis \( x = (1 - \mu) LU \) then the initial conditions were rejected. In addition if a trajectory failed to cross the X-Z plane then the initial conditions were rejected.

### 3.1.3 Differential Evolution Parameters

The population size N was set to 40 for each computation. Two different sets of values were used for the mutation factor F and the crossover frequency CR. For the first set of values:

\[ F = 0.5 \]  
\[ CR = 0.8 \]  

(3.4)

as given in [18]. For the second set of values:

\[ F = \text{rand}(0.5, 1) \]  
\[ CR = 0.9 \]  

(3.5)

where \( \text{rand}(0.5,1) \) denotes a random number in the interval \([0.5, 1]\). The
computation time and iterations from using both sets of parameters are presented. All computations were done using Matlab R2017a on a Macbook computer with a 2.6 GHz Intel Core i7.

3.2 Targeting Halo Orbits by Z-Amplitude

3.2.1 Richardson’s Analytical Solutions with Differential Correction

The traditional design method of applying a 3rd order analytical solution to get an approximation for the initial conditions and refining using differential correction was used to compute halo orbits for several values of $A_z$. The tolerance for the differential correction was set to $1 \times 10^{-15}$. The results are given in table 3.3

<table>
<thead>
<tr>
<th>$A_z,_{\text{desired}}$</th>
<th>$A_z,_{\text{achieved}}$</th>
<th>$y_0$</th>
<th>$z_0$</th>
<th>$\dot{y}_0$</th>
<th>$T$ (Days)</th>
</tr>
</thead>
<tbody>
<tr>
<td>200 km</td>
<td>199.19 km</td>
<td>0.977472176</td>
<td>0.000327795</td>
<td>0.017783889</td>
<td>5.375 Days</td>
</tr>
<tr>
<td>2,000 km</td>
<td>1,990.40 km</td>
<td>0.977520259</td>
<td>0.003296133</td>
<td>0.018634493</td>
<td>5.372 Days</td>
</tr>
<tr>
<td>10,000 km</td>
<td>9,466.94 km</td>
<td>0.980221604</td>
<td>0.018007599</td>
<td>0.029756963</td>
<td>5.180 Days</td>
</tr>
</tbody>
</table>

*Table 3.3. Richardson’s analytical solution results*
3.2.2 Differential Evolution

The objective function $J$ to be minimized as given in section 2.2.6.3 is:

$$ J = |\dot{x}(\frac{T}{2})| + |\dot{z}(\frac{T}{2})| + |A_{z,\text{achieved}} - A_{z,\text{desired}}| $$

(3.6)

The results from using DE to target orbits by $Z$-amplitude are given in table 3.4.

<table>
<thead>
<tr>
<th>$A_{z,\text{desired}}$</th>
<th>$A_{z,\text{achieved}}$</th>
<th>$x_0$</th>
<th>$z_0$</th>
<th>$\dot{y}_0$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>200 km</td>
<td>200.07 km</td>
<td>0.977461575</td>
<td>-0.000329294</td>
<td>0.017878528</td>
<td>5.375 Days</td>
</tr>
<tr>
<td>2,000 km</td>
<td>2,000.65 km</td>
<td>0.977511453</td>
<td>0.003313818</td>
<td>0.018727314</td>
<td>5.372 Days</td>
</tr>
<tr>
<td>10,000 km</td>
<td>10,002.40 km</td>
<td>0.980863549</td>
<td>0.019525948</td>
<td>0.030563844</td>
<td>5.102 Days</td>
</tr>
</tbody>
</table>

Table 3.4. DE targeting $Z$-amplitude results

![Targeting Az = 200 km](image)

**Figure 3.1.** Targeting $A_z = 200$ km
Figure 3.2. Targeting $A_z = 2,000$ km

Figure 3.3. Targeting $A_z = 10,000$ km
3.3 Targeting Halo Orbits by X-Amplitude

The Objective Function \( J \) to be minimized is:

\[
J = |\dot{x}(\frac{T}{2})| + |\dot{z}(\frac{T}{2})| + |A_{x,\text{achieved}} - A_{x,\text{desired}}| \tag{3.7}
\]

The results from targeting orbits by X-amplitude are:

<table>
<thead>
<tr>
<th>(A_{x,\text{desired}})</th>
<th>(A_{x,\text{achieved}})</th>
<th>(x_0)</th>
<th>(z_0)</th>
<th>(\dot{y}_0)</th>
<th>(T)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,000 km</td>
<td>999.99 km</td>
<td>1.018845918</td>
<td>0</td>
<td>0.009868772</td>
<td>5.461 Days</td>
</tr>
<tr>
<td>5,000 km</td>
<td>5,000.01 km</td>
<td>1.011521583</td>
<td>0</td>
<td>0.049010103</td>
<td>5.992 Days</td>
</tr>
</tbody>
</table>

Table 3.5. DE targeting X-amplitude results

Figure 3.4. Targeting \(A_x = 1,000 \text{ km}\)
3.4 Targeting Halo Orbits by Orbital Period

The Objective Function $J$ to be minimized is:

$$J = |\dot{x}(\frac{T}{2})| + |\dot{z}(\frac{T}{2})| + 0.01\left|\frac{T}{T_{\text{achieved}}} - \frac{T}{T_{\text{desired}}\}}\right|$$

(3.8)

The results from using DE to target halo orbits by orbital period are given in Table 3.6.

<table>
<thead>
<tr>
<th>$T_{\text{desired}}$</th>
<th>$T_{\text{achieved}}$</th>
<th>$x_0$</th>
<th>$z_0$</th>
<th>$\dot{y}_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6 Days</td>
<td>5.999999 Days</td>
<td>1.011459778</td>
<td>0</td>
<td>0.049347962</td>
</tr>
<tr>
<td>7 Days</td>
<td>7.000000 Days</td>
<td>1.006960155</td>
<td>0</td>
<td>0.078146636</td>
</tr>
</tbody>
</table>

Table 3.6. DE targeting orbital period results
Figure 3.6. Targeting $T = 6$ Days

Figure 3.7. Targeting $T = 7$ Days
3.5 Computation Time and Iterations

The computation time and iterations for the differential correction procedure are given by table 3.7.

<table>
<thead>
<tr>
<th>Target Parameter</th>
<th>Computation Time (seconds)</th>
<th>Iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_z = 200$ km</td>
<td>5.21</td>
<td>8</td>
</tr>
<tr>
<td>$A_z = 2,000$ km</td>
<td>5.48</td>
<td>8</td>
</tr>
<tr>
<td>$A_z = 10,000$ km</td>
<td>10.43</td>
<td>14</td>
</tr>
</tbody>
</table>

Table 3.7. Computation time and iterations for differential correction

The computation time and iterations for differential evolution using $F = 0.5$ and $CR = 0.8$ are:

<table>
<thead>
<tr>
<th>Target Parameter</th>
<th>Computation Time (seconds)</th>
<th>Iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_z = 200$ km</td>
<td>126.27</td>
<td>256</td>
</tr>
<tr>
<td>$A_z = 2,000$ km</td>
<td>180.39</td>
<td>326</td>
</tr>
<tr>
<td>$A_z = 10,000$ km</td>
<td>169.06</td>
<td>282</td>
</tr>
<tr>
<td>$T = 6$ Days</td>
<td>138.26</td>
<td>310</td>
</tr>
<tr>
<td>$T = 7$ Days</td>
<td>622.70</td>
<td>1287</td>
</tr>
<tr>
<td>$A_x = 1,000$ km</td>
<td>146.05</td>
<td>279</td>
</tr>
<tr>
<td>$A_x = 5,000$ km</td>
<td>362.27</td>
<td>680</td>
</tr>
</tbody>
</table>

Table 3.8. Computation time and iterations for $F = 0.5$, $CR = 0.8$
and the results from using $F = \text{rand}(0.5, 1)$ and $CR = 0.9$ are:

<table>
<thead>
<tr>
<th>Target Parameter</th>
<th>Computation Time (seconds)</th>
<th>Iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_z = 200$ km</td>
<td>249.26</td>
<td>576</td>
</tr>
<tr>
<td>$A_z = 2,000$ km</td>
<td>179.69</td>
<td>345</td>
</tr>
<tr>
<td>$A_z = 10,000$ km</td>
<td>167.69</td>
<td>271</td>
</tr>
<tr>
<td>$T = 6$ Days</td>
<td>187.43</td>
<td>342</td>
</tr>
<tr>
<td>$T = 7$ Days</td>
<td>294.50</td>
<td>525</td>
</tr>
<tr>
<td>$A_z = 1,000$ km</td>
<td>190.42</td>
<td>338</td>
</tr>
<tr>
<td>$A_z = 5,000$ km</td>
<td>194.01</td>
<td>381</td>
</tr>
</tbody>
</table>

Table 3.9. Computation time and iterations for $F = \text{rand}(0.5, 1)$, $CR = 0.9$

The use of a random mutation factor $F$ resulted in much more consistent trials and slightly faster computation times when compared to a constant $F$. Also trials using $F = 0.5$ frequently became stuck in local minima and had to be restarted. Using $F = \text{rand}(0.5, 1)$ was effective in avoiding these local minima.
3.6 Stability of Computed Orbits

The stability index is used to determine the stability of the computed orbits. The closer the stability index is to 1, the more stable the orbit. Using Equation 2.24 to calculate the stability index for halo orbits, where DC denotes orbits computed by differential correction, gives:

<table>
<thead>
<tr>
<th>Target Parameter</th>
<th>Stability Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_z = 200$ km (DC)</td>
<td>901</td>
</tr>
<tr>
<td>$A_z = 2,000$ km (DC)</td>
<td>838</td>
</tr>
<tr>
<td>$A_z = 10,000$ km (DC)</td>
<td>102</td>
</tr>
<tr>
<td>$A_z = 200$ km</td>
<td>893</td>
</tr>
<tr>
<td>$A_z = 2,000$ km</td>
<td>834</td>
</tr>
<tr>
<td>$A_z = 10,000$ km</td>
<td>66</td>
</tr>
<tr>
<td>$T = 6$ Days</td>
<td>456</td>
</tr>
<tr>
<td>$T = 7$ Days</td>
<td>264</td>
</tr>
<tr>
<td>$A_x = 1,000$ km</td>
<td>936</td>
</tr>
<tr>
<td>$A_x = 5,000$ km</td>
<td>460</td>
</tr>
</tbody>
</table>

Table 3.10. Stability index for each trial

The most stable orbit is for the target parameter $A_z = 10,000$ km and the least stable orbit is for $A_z = 1,000$ km. The two orbits are plotted below. The unstable orbit in figure 3.8 drifts far away from the nominal halo orbit after two orbital periods and actually comes close to colliding with Europa. The most stable orbit, shown in figure 3.9, stays within the vicinity of the nominal trajectory during its 2nd orbital period but escapes the nominal trajectory during its 3rd orbital period as shown in figure 3.10.
Figure 3.8. Least stable orbit, targeting $A_x = 1,000$ km, two orbital periods

Figure 3.9. Most stable orbit, targeting $A_z = 10,000$ km, two orbital periods
Figure 3.10. Most stable orbit, targeting $A_z = 10,000$ km, three orbital periods
Chapter 4

Discussion

4.1 Targeting Orbits With Differential Evolution

Differential evolution was successfully used to compute halo orbits with a specified $Z$-amplitude. $DE$ was also more accurate than the traditional design method in achieving the specified $Z$-amplitude, but required much longer computation times. Applying the differential correction scheme for a desired $A_z = 10,000$ km resulted in an orbit with $A_z = 9,466$ km while differential evolution computed an orbit with $A_z = 10,002$ km. $DE$ also computed more stable orbits than the differential correction scheme for all three cases.

$DE$ accurately targeted orbits by $X$-amplitude with an error of only 0.01 km for both trials. Additionally both computed orbits are planar. It is easier for $DE$ to find planar orbits because there is no $Z$-component, so the initial $Z$-position is zero. This effectively reduces the dimensions of the search space from three to two. Some additional constraints may be necessary to find non-planar orbits, but it is important not to over-constrain the problem.
DE successfully targeted halo orbits by orbital period to a high degree of accuracy. Again without specifying the Z-amplitude all calculated orbits are planar. The ability to target halo orbits by orbital period could be a useful tool for mission design.

4.2 Stability Analysis

None of the generated orbits appear stable enough to have practical use. The orbit generated from targeting $A_z = 10,000$ km is the most stable orbit with a stability index of 66. A stability index of one indicates a stable orbit, so the orbit is still fairly unstable. The orbit stays near the nominal trajectory for two orbital periods as shown in figure 3.9 but escapes the nominal trajectory after three orbital periods as shown in figure 3.10. The behavior of the most unstable orbit is shown in figure 3.8. The stability of halo orbits is strongly associated with station-keeping costs, so the unstable orbit would require large amounts of propellant to stay in orbit. Adding a constraint for orbital stability was attempted but unsuccessful.

4.3 Differential Evolution Parameters

Comparing the two different sets of DE parameters, $F = 0.5$, $CR = 0.8$ and $F = rand(0.5, 1)$, $CR = 0.9$, the use of a random mutation factor F and a higher crossover frequency CR resulted in slightly faster computation times. The range
of the random mutation factor, 0.5 - 1, was chosen based on the work of Das et al. [20]. Additionally Storn et al., the creators of differential evolution, state that the effective range for F is between 0.4 and 1 [10]. If F was chosen in the range [0 0.5] the DE scheme would be more susceptible to convergence to local minima.

Trials with a constant mutation factor frequently became stuck in a local minima and had to be restarted. Using a random mutation factor worked well to avoid local minima, and resulted in much more consistent trials. Assigning a high value to the crossover frequency and a random value to the mutation factor seems to be an effective combination. The high crossover frequency speeds up convergence while the random mutation factor helps to avoid local minima.
Chapter 5

Summary and Conclusions

5.1 Summary

The goal of this thesis is to demonstrate the power and flexibility of DE in computing halo orbits, to compare the results from DE to the conventional design method, and to compare the effectiveness of different crossover frequencies and mutation factors in DE. The Jupiter-Europa system was selected because of the intense scientific interest in Europa and the limited literature on halo orbits in the system.

There are several main results from this thesis. It was shown, in agreement with Nath et al., that differential evolution can compute halo orbits with a specified Z-amplitude to a higher degree of accuracy than the traditional method of differential correction. It was also shown that differential evolution can accurately target halo orbits by X-amplitude and orbital period.
Finally it was shown that using a random mutation factor $F$ in the range $[0.5, 1]$ paired with a high crossover frequency reduced the chance of convergence to a local minimum and resulted in much more consistent results when compared to a constant mutation factor.

### 5.2 Recommendations For Future Work

The ability to constrain differential evolution to target stable orbits would be incredibly useful for halo orbit design. It was attempted in this work, but adding the additional constraint seemed to over-constrain the problem. The DE scheme should also be used to design halo orbits in a more realistic full ephemeris model. This will require much more computational power than using the idealized circular-restricted three body problem. Additionally DE should be used to design orbits about Lagrange points like Lissajous and Lyapunov orbits. These orbits are quasi-periodic and have been used in a variety of missions. Finally researchers should apply variants of DE to compute halo orbits. A different variation may be better suited for this optimization problem.
Appendix A

Calculating the STM

The calculation of the state transition matrix (STM) requires the integration of the STM differential equation along with the state vector. The differential equation for the STM is [18]:

\[ \dot{\phi} = A\phi \]  \hspace{1cm} (A.1)

where the initial condition \( \phi_0 \) is a 6x6 identity matrix and A is a 6x6 matrix defined as:

\[
A = \begin{bmatrix}
\text{zeros}(3x3) & I(3x3) \\
C(3x3) & K(3x3)
\end{bmatrix}
\]  \hspace{1cm} (A.2)

where zeros(3,3) is a 3x3 matrix of zeros, I(3x3) is a 3x3 identity matrix and K is given by:

\[
K = \begin{bmatrix}
0 & 2 & 0 \\
-2 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]  \hspace{1cm} (A.3)
The 3x3 C matrix, where $\mu$ is the normalized mass ratio, $r_1$ is the s/c distance from the primary, $r_2$ is the s/c distance from the secondary, and $x$, $y$, and $z$ denote the position of the s/c in the CR3BP frame is:

\[
\begin{align*}
C_{11} &= \frac{\partial \ddot{x}}{\partial x} = 1 - \frac{1 - \mu}{r_1^3} \left(1 - \frac{3(x + \mu)^2}{r_1^2} - \frac{\mu}{r_2^3} (1 - 3 \frac{(x - (1 - \mu))^2}{r_2^2}) \right) \\
C_{22} &= \frac{\partial \ddot{y}}{\partial y} = 1 - \frac{1 - \mu}{r_1^3} \left(1 - \frac{3y^2}{r_1^2} - \frac{\mu}{r_2^3} (1 - \frac{3y^2}{r_2^2}) \right) \\
C_{33} &= \frac{\partial \ddot{z}}{\partial z} = -\frac{1 - \mu}{r_1^3} \left(1 - \frac{3z^2}{r_1^2} - \frac{\mu}{r_2^3} (1 - \frac{3z^2}{r_2^2}) \right) \\
C_{12} &= \frac{\partial \ddot{x}}{\partial y} = 3y(1 - \mu) \frac{x + \mu}{(r_1^5)} + 3\mu y \frac{(x - (1 - \mu))}{(r_2^5)} = C_{21} = \frac{\partial \ddot{y}}{\partial x} \\
C_{13} &= \frac{\partial \ddot{x}}{\partial z} = 3z(1 - \mu) \frac{x + \mu}{(r_1^5)} + 3\mu z \frac{(x - (1 - \mu))}{(r_2^5)} = C_{31} = \frac{\partial \ddot{z}}{\partial x} \\
C_{23} &= \frac{\partial \ddot{y}}{\partial z} = 3z(1 - \mu) \frac{y}{(r_1^5)} + 3\mu z \frac{y}{(r_2^5)} = C_{32} = \frac{\partial \ddot{z}}{\partial y}
\end{align*}
\] (A.4)

with:

\[
\begin{align*}
\begin{align*}
&\quad r_1 = \sqrt{(x + \mu)^2 + y^2 + z^2} \\
&\quad r_2 = \sqrt{(x - (1 - \mu))^2 + y^2 + z^2}
\end{align*}
\] (A.5)
Appendix B

Richardson’s Analytical Solution Constants

The steps for computing the constants in Richardson’s analytical solution, where \( \mu \) is the normalized mass ratio, are as follows:

1. Calculate the quantity \( \gamma_L \):

\[
\gamma_L = \frac{r_L}{R_{12}} \tag{B.1}
\]

where \( r_L \) is the distance of the secondary body from the Lagrange point and \( R_{12} \) is the distance between the primary and secondary bodies.

2. Find the constants \( c_2 \), \( c_3 \), and \( c_4 \):

\[
c_2 = \frac{1}{\gamma_L^3} \left[ \mu + \frac{(1 - \mu)\gamma_L^3}{(1 - \gamma_L)^3} \right]
\]

\[
c_3 = \frac{1}{\gamma_L^3} \left[ \mu - \frac{(1 - \mu)\gamma_L^4}{(1 - \gamma_L)^4} \right] \tag{B.2}
\]

\[
c_4 = \frac{1}{\gamma_L^3} \left[ \mu + \frac{(1 - \mu)\gamma_L^5}{(1 - \gamma_L)^5} \right]
\]
3. Compute the linearized frequency $\lambda$ by solving the following equation and selecting the positive real root:

$$\lambda^4 + (c_2 - 2)\lambda^2 - (c_2 - 1)(1 + 2c_2) = 0 \quad (B.3)$$

4. Compute the constant $k$:

$$k = \frac{2\lambda}{\lambda^2 + 1 - c_2} \quad (B.4)$$

5. Compute the constants $d_1$ and $d_2$

$$d_1 = \frac{3\lambda^2}{k} [k(6\lambda^2 - 1) - 2\lambda] \quad (B.5)$$
$$d_2 = \frac{8\lambda^2}{k} [k(11\lambda^2 - 1) - 2\lambda]$$

6. Calculate $a_{21}$, $a_{22}$, $a_{23}$ and $a_{24}$

$$a_{21} = \frac{3c_3(k^2 - 2)}{4(1 + 2c_2)}$$

$$a_{22} = \frac{3c_3}{4(1 + 2c_2)}$$

$$a_{23} = -\frac{3c_3\lambda}{4kd_1} [3k^3\lambda - 6k(k - \lambda) + 4]$$

$$a_{24} = -\frac{3c_3\lambda}{4kd_1} (2 + 3k\lambda)$$
7. Calculate $d_{21}$, $d_{31}$ and $d_{32}$

\[
d_{21} = -\frac{c_3}{2\lambda^2}
\]
\[
d_{31} = \frac{3}{64\lambda^2}(4c_3a_{24} + c_4)
\]
\[
d_{32} = \frac{3}{64\lambda^2}[4c_3(a_{23} - d_{21}) + c_4(4 + k^2)]
\]

(B.7)

8. Compute $b_{21}$, $b_{22}$, $b_{31}$ and $b_{32}$

\[
b_{21} = -\frac{3c_3\lambda}{2d_1}(3k\lambda - 4)
\]
\[
b_{22} = \frac{3c_3\lambda}{d_1}
\]
\[
b_{31} = \frac{3}{8d_2} \left\{8\lambda[3c_3(kb_{21} - 2a_{23}) - c_4(2 + 3k^2)]
\right.
\]
\[
+ (9\lambda^2 + 1 + 2c_2)[4c_3(ka_{23} - b_{21}) + kc_4(4 + k^2)]\right\}
\]
\[
b_{32} = \frac{1}{d_2} \left\{9\lambda[c_3(kb_{22} + d_{21} - 2a_{24}) - c_4]
\right.
\]
\[
+ \frac{3}{8}(9\lambda^2 + 1 + 2c_2)[4c_3(ka_{24} - b_{22}) + kc_4]\right\}
\]

(B.8)

9. Compute $a_{31}$ and $a_{32}$

\[
a_{31} = -\frac{9\lambda}{4d_2} \left\{4c_3(ka_{23} - b_{21}) + kc_4(4 + k^2)\right\}
\]
\[
+ \left\{\frac{9\lambda^2 + 1 - c_2}{2d_2}\right\}[3c_3(2a_{23} - kb_{21}) + c_4(2 + 3k^2)]\right\}
\]
\[
a_{32} = -\frac{1}{d_2} \left\{\frac{9\lambda}{4}[4c_3(ka_{24} - b_{22}) + kc_4]
\right.
\]
\[
+ \frac{3}{2}(9\lambda^2 + 1 - c_2)[c_3(kb_{22} + d_{21} - 2a_{24}) - c_4]\right\}
\]

(B.9)
10. Calculate the frequency corrections $s_1$ and $s_2$

\[
s_1 = \frac{1}{2\lambda[\lambda(1 + k^2) - 2k]} \left\{ \frac{3}{2} c_3 [2a_{21}(k^2 - 2) - a_{23}(k^2 + 2) - 2kb_{21}] ight. \\
\left. - \frac{3}{8} c_4 (3k^4 - 8k^2 + 8) \right\}
\]

\[
s_2 = \frac{1}{2\lambda[\lambda(1 + k^2) - 2k]} \left\{ \frac{3}{2} c_3 [2a_{22}(k^2 - 2) + a_{24}(k^2 + 2) + 2kb_{22} + 5d_{21}] ight. \\
\left. + \frac{3}{8} c_4 (12 - k^2) \right\}
\]  

(B.10)

11. Calculate the constants $a_1$ and $a_2$:

\[
a_1 = -\frac{3}{2} c_3 (2a_{21} + a_{23} + 5d_{21}) - \frac{3}{8} c_4 (12 - k^2)
\]

\[
a_2 = \frac{3}{2} c_3 (a_{24} - 2a_{22}) + \frac{9}{8} c_4
\]  

(B.11)

12. Calculate $l_1$ and $l_2$

\[
l_1 = a_1 + 2\lambda^2 s_1
\]

\[
l_2 = a_2 + 2\lambda^2 s_2
\]  

(B.12)

13. Find the constant $\Delta$:

\[
\Delta = \lambda^2 - c_2
\]  

(B.13)
14. Compute the in-plane amplitudes $A_x$ and $A_y$:

$$A_x = \sqrt{-\frac{l_2 A_z^2 - \Delta}{l_1}}$$

$$A_y = kA_x$$  \hspace{1cm} (B.14)

15. Find the constant $\omega$:

$$\omega = 1 + s_1 A_x^2 + s_2 A_z^2$$  \hspace{1cm} (B.15)

The values of these constants in the Jupiter-Europa CR3BP for $A_z = 2,000$ Km, $\mu = 2.52800 \times 10^{-5}$ and $r_L = 1.3744 \times 10^4$ km are given in table B.1.

The results of Richardson’s analytical solution, given by Equation 2.16, are the initial conditions for a halo orbit in an L1 centered coordinate frame normalized by the Lagrange point-secondary distance $r_L$. To convert these initial conditions to the CR3BP synodic coordinate frame the steps are as follows, where $LU$ and $TU$ are the length and time units in km and seconds respectively:

1. Multiply each initial condition $[x_0, z_0, \dot{y}_0]$ by the constant $\gamma_L$. The resulting coordinates will be denoted as $[x_0^{\gamma_L}, z_0^{\gamma_L}, \dot{y}_0^{\gamma_L}]$

2. Convert the initial conditions to the CR3BP frame:

$$x_0 = \frac{LU - r_L + x_0^{\gamma_L}}{LU} \quad z_0 = \frac{z_0^{\gamma_L}}{LU} \quad \dot{y}_0 = \frac{\dot{y}_0^{\gamma_L} \times TU}{LU}$$  \hspace{1cm} (B.16)
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Table B.1. Jupiter-Europa analytical solution constants for $A_z = 2,000$ Km
Bibliography


