INSIGHTS INTO THE BEHAVIOR OF HETEROGENEOUS THERMOSTATICALLY CONTROLLED LOADS AND BATTERY PACKS

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by
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ABSTRACT

This dissertation analyzes and determines fundamental insights into the behavior of heterogeneous populations of two types of energy storage devices: (i) thermostatically controlled loads (TCLs) harnessed for demand response and (ii) battery cells in a battery pack. Determined through reduced-order models, the insights are intended to facilitate control of these devices. The selection of these devices comes from two similarities. First, populations of these devices are utilized for filtering differences between power demanded and power supplied. Control of power demanded by TCLs is a proposed method to integrate renewable resources into the grid without the need to shed excess supplied energy or disrupt grid frequency. Likewise, stationary battery packs are implemented for renewable resource integration, and packs are used in partially or fully electric vehicles to match the power requirements.

Second, beyond the application connection there exists a fundamental connection: the aggregate behavior of both systems is inherently different from the average unit behavior. As the literature states, despite the fact that individual TCL power dynamics are oscillatory, heterogeneous populations of TCLs exhibit damped aggregate power dynamics. This damping phenomenon is beneficial to control methods attempting to match power demand to supply. As this damping introduced by heterogeneity changes the nature of the aggregate dynamics, there is a focus on developing low-order aggregate power dynamic models that incorporate heterogeneity. Similarly, studies and experiments in the battery literature show that pack capacity degradation is not captured by the average battery cell. The battery pack capacity and other pack state of health (SOH) parameters are unfavorably influenced by the unhealthiest cells within the pack, impacting pack control through the SOH parameter-determined current and voltage limits. Furthermore, studies show that heterogeneity found within the cells influences the degradation of other cells within the pack, with and without cell charge balancing strategies.

Though the literature discusses these aggregate behavior dynamics, there are still fundamental insights missing for both systems. Studies do exist that attempt to determine a relationship between TCL heterogeneity and the damped aggregated dynamics, but there is a lack of a concrete study relating how heterogeneity of multiple parameters impacts the aggregate dynamics. There are studies that attempt to determine how heterogeneity of battery cells impact aggregate pack SOH, but there is a lack of insights determined from analyzing a model specifically developed to capture heterogeneity dynamics. This dissertation proposes the solution
to both of these problems through the development of reduced-order models that are built with the explicit purpose of determining the insights missing from the literature.

For the TCL problem, a model of an individual TCL is reformulated and a stochastic parameter integral is applied to develop a reduced-order aggregate power demand model. A set of second-order linear time-varying (LTV) differential equations characterize this model, with the time-varying damping ratios describing the strength of the damping observed in the aggregate power demand. This model, shown to be accurate for low-order approximations, furnishes several insights, including: (i) Only heterogeneity in the TCL power demand’s characteristic frequency creates damping in the aggregate response. Other heterogeneities, such as in the demand’s duty cycle, do not. (ii) Low levels of heterogeneity in the characteristic frequencies also create a beating phenomenon in the aggregate power dynamics. (iii) The damping ratios depend on the dominant characteristic frequencies and decay over time. These insights lead to understanding useful for system design through intelligent selection of TCL populations to meet operator needs. The damping ratios lend themselves to better-informed controller design, potentially allowing more aggressive demand response controllers without inducing instability.

For the battery pack problem, a framework is developed to determine insights into the behavior of heterogeneity within a pack for different balancing strategies. An electrochemical cell model is reduced to describe heterogeneity between cells through an LTV model for two different studies. In the first study, charge and capacity heterogeneity are considered, and insights lead to the knowledge that voltage balancing algorithms do not remove capacity differences within the lifespan of the pack. A novel control algorithm is developed from these insights, which balances not voltage, but rather charge and capacity, and is shown to increase pack lifespan by up to 9.2%. In the second study, charge and temperature heterogeneity are considered, and the tradeoffs between balancing these are formulated. It is discovered that temperature heterogeneity removal can be assisted by control of an average current through two cells using entropic and ohmic effects, and a model predictive control (MPC) algorithm is developed to show this.

Through this dissertation, an understanding is gained regarding how the aggregate response of heterogeneous groups of energy storage devices is inherently different than individual device response. For both the TCL demand response problem and the battery pack problem, the insights coupled with reduced-order models are intended to facilitate the development of accurate and efficient control methods.
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Nomenclature

\( A \)\( _\text{TCL w ave height} \) [kW] (Chapter 2)

Electrode area [cm\(^2\)] (Chapters 3-5)

\( A_{\text{min}}, A_{\text{max}} \) Minimum/maximum TCL wave height [kW] (Chapter 2)

\( A \) State matrix (Chapter 4)

\( A_{\text{diag}} \) Diagonal state matrix (Chapter 4)

\( a \) Specific interfacial area [cm\(^{-1}\)] (Chapters 3-5)

\( a_1 \) Time-varying parameter \( \frac{\partial h(X, Q, U)}{\partial x} \) (Chapter 4)

Time-varying parameter \( \frac{\partial h(T, X, U)}{\partial X} \) (Chapter 5)

\( a_2 \) Time-varying parameter \( \frac{\partial h(T, X, U)}{\partial t} \) (Chapter 5)

\( b_1 \) Time-varying parameter \( 2 \frac{\partial h(X, Q, U)}{\partial u} \) (Chapter 4)

Constant parameter (Chapter 5)

\( b_2 \) Time-varying parameter \( 2 \frac{\partial h(T, X, U)}{\partial u} \) (Chapter 5)

\( C \) Thermal capacitance \( \left[ \frac{\text{kWh}}{\degree C} \right] \) (Chapter 2)

\( C_{\text{avg}} \) Average thermal capacitance \( \left[ \frac{\text{kWh}}{\degree C} \right] \) (Chapter 2)

\( C_{\text{min}}, C_{\text{max}} \) Minimum/maximum thermal capacitance \( \left[ \frac{\text{kWh}}{\degree C} \right] \) (Chapter 2)

\( c \) Concentration of lithium ions in an electrode \( \left[ \frac{\text{mol}}{\text{cm}^3} \right] \) (Chapters 3-5)

\( c_1 \) Constant for thermal capacitance pdf \( \left[ \frac{\degree C}{\text{kWh}} \right] \) (Chapter 2)

Time-varying parameter \( \frac{\partial g(X, Q, U)}{\partial x} \) (Chapter 4)

\( c_2 \) Time-varying parameter \( \frac{\partial g(X, Q, U)}{\partial q} \) (Chapter 4)

\( c_e \) Average electrolyte concentration \( \left[ \frac{\text{mol}}{\text{cm}^3} \right] \) (Chapters 3-5)

\( c_{\text{max}} \) Maximum solid phase concentration \( \left[ \frac{\text{mol}}{\text{cm}^3} \right] \) (Chapters 3-5)

\( c_s \) Surface concentration \( \left[ \frac{\text{mol}}{\text{cm}^3} \right] \) (Chapters 3-5)

\( \bar{c} \) Average concentration \( \left[ \frac{\text{mol}}{\text{cm}^3} \right] \) (Chapters 3-5)
\( D \)  
TCL duty cycle (Chapter 2)

\[ D \]  
Solid phase Li diffusion coefficient \( \frac{\text{cm}^2}{\text{s}} \) (Chapters 3-5)

\( D_{\text{min}}, D_{\text{max}} \)  
Minimum/maximum TCL duty cycle (Chapter 2)

\( d_1 \)  
Time-varying parameter \( 2 \frac{\partial g(X, Q, U)}{\partial U} \) (Chapter 4)

\( E_{\text{act}} \)  
Reaction rate activation energy \( \frac{1}{\text{mol}} \) (Chapters 3-5)

\( F \)  
Faraday’s constant \( \frac{\text{C}}{\text{mol}} \) (Chapters 3-5)

\( g \)  
Output voltage function (Chapters 3-5)

\( h \)  
Capacity loss dynamics function (Chapters 3-5)

\( h_T \)  
Temperature dynamics function (Chapters 3-5)

\( I \)  
Current [A] (Chapters 3-5)

\( i_{o,s} \)  
Side reaction exchange current density \( \frac{\text{A}}{\text{cm}^2} \) (Chapters 3-5)

\( J \)  
Main current density \( \frac{\text{A}}{\text{cm}^3} \) (Chapters 3-5)

\( J_s \)  
Side current density \( \frac{\text{A}}{\text{cm}^3} \) (Chapters 3-5)

\( J_{\text{tot}} \)  
Total current density \( \frac{\text{A}}{\text{cm}^3} \) (Chapters 3-5)

\( K \)  
Voltage balancing gain [A/V] (Chapter 4)

\( K_1, K_2 \)  
Multivariable feedback balancing gains [s\(^{-1}\)] (Chapter 4)

\( K_z \)  
Charge balancing feedback gain [s\(^{-1}\)] (Chapter 5)

\( K_t \)  
Temperature balancing feedback gain [A/K] (Chapter 5)

\( k \)  
Rate constant \( \frac{\text{cm}^{2.5}}{\text{mol}^{0.5} \text{s}} \) (Chapters 3-5)

\( k_{\text{ref}} \)  
Reference rate constant \( \frac{\text{cm}^{2.5}}{\text{mol}^{0.5} \text{s}} \) (Chapters 3-5)

\( L \)  
Electrode thickness [cm] (Chapters 3-5)

\( M_A \)  
Number of TCL wave height pieces [kW] (Chapter 2)

\( M_D \)  
Number of TCL duty cycle pieces (Chapter 2)

\( M_p \)  
SEI layer Molar mass \( \frac{\text{kg}}{\text{mol}} \) (Chapters 3-5)

\( M_\xi \)  
Number of TCL initial fraction pieces (Chapter 2)

\( M_\omega \)  
Number of TCL frequency pieces \( \text{[rad/s]} \) (Chapter 2)

\( m C_p \)  
Thermal capacitance [J/K] (Chapters 3-5)
\( N \) Number of Fourier series terms (Chapter 2)

\( N_L \) Number of mass-spring or TCL units (Chapter 2)

\( n_{Li} \) Number of moles of lithium \([\text{mol}]\) ( Chapters 3-5)

\( P \) Air conditioning unit power \([\text{kW}]\) (Chapter 2)

\( P \) Coordinate transformation matrix (Chapter 4)

\( P_{1,i,n}, P_{2,i,n} \) Variables for TCL population second-order differential equations (Chapter 2)

\( P_{\text{TCL}} \) Unit power demand \([\text{kW}]\) (Chapter 2)

\( P_{\text{TCL}}^{\text{peak}} \) Expected power demand \([\text{kW}]\) (Chapter 2)

\( \bar{P}_{\text{TCL}, \text{peak}} \) Approximate beat peak for TCL power demand \([\text{kW}]\) (Chapter 2)

\( pdf \) Probability density function (Chapter 2)

\( Q_{\text{loss}} \) Capacity loss \([\text{Ah}]\) (Chapters 3-5)

\( Q_{\text{nom}} \) Healthy cell capacity \([\text{Ah}]\) (Chapters 3-5)

\( q_1 \) Sum of cell capacities \([\text{Ah}]\) (Chapters 3-5)

\( q_2 \) Difference between cell capacities \([\text{Ah}]\) (Chapters 3-5)

\( R \) Thermal resistance \(\frac{\text{[°C]}}{\text{[kW]}}\) (Chapter 2)

\( R_c \) Particle radius \([\text{cm}]\) (Chapters 3-5)

\( R_film \) Contact and unmodeled resistance \([\Omega]\) (Chapters 3-5)

\( R_{\text{SEI}} \) Film layer resistance \([\Omega \text{ m}^2]\) (Chapters 3-5)

\( R_u \) Initial SEI layer resistance \([\Omega \text{ m}^2]\) (Chapters 3-5)

\( r \) Universal gas constant \(\frac{[\text{J}]}{[\text{mol K}]}\) (Chapters 3-5)

\( r \) Electrode particle radial coordinate \([\text{cm}]\) (Chapters 3-5)

\( SOC_{\text{min}}, SOC_{\text{max}} \) SOC limits (Chapter 5)

\( s \) Switching control variable (Chapter 2)

\( s_0 \) Initial switching variable value (Chapter 2)

\( T \) Temperature of the TCL \(\text{[°C]}\) (Chapter 2)

\( T_{\text{min}}, T_{\text{max}} \) Temperature of the battery cell \(\text{[K]}\) (Chapters 3-5)

\( T_0 \) Initial temperature \(\text{[°C]}\) (Chapter 2)

\( t_f \) Final time of window \([\text{s}]\) (Chapter 5)

\( T_{\text{min}}, T_{\text{max}} \) TCL temperature boundaries \(\text{[°C]}\) (Chapter 2)

\( x_{\text{iii}} \) Temperature limits \([\text{K}]\) (Chapter 5)
\( T_{ref} \) Reference cell temperature \([K]\) (Chapters 3-5)

\( T_{sp} \) Set point temperature \( ^[\circ C] \) (Chapter 2)

\( T_{co} \) Ambient TCL temperature \( ^[\circ C] \) (Chapter 2)

\( T_{\infty} \) Ambient air temperature \([K]\) (Chapters 3-5)

\( t \) Time \([s]\)

\( U_{ref} \) Electrode reference potential \([V]\) (Chapters 3-5)

\( U_{ref,s} \) Side reaction equilibrium potential \([V]\) (Chapters 3-5)

\( u \) Unit step function (Chapter 2)

\( u_1 \) Main current \([A]\) (Chapters 3-5)

\( u_2 \) Balancing current \([A]\) (Chapters 3-5)

\( u_{min},u_{max} \) Current limits (Chapter 5)

\( V \) Battery voltage \([V]\) (Chapters 3-5)

\( v_1 \) Sum of cell voltages \([V]\) (Chapters 3-5)

\( v_2 \) Difference between cell voltages \([V]\) (Chapters 3-5)

\( w_T \) Temperature difference weighting term \([K^{-2}]\) (Chapter 5)

\( w_X \) Charge difference weighting term \([C^{-2}]\) (Chapter 5)

\( x \) Charge level \([Ah]\) (Chapters 3-5)

\( x_{1,i} \) Mass \( i \)'s position \([m]\) (Chapter 2)

\( x_{2,i} \) Mass \( i \)'s velocity \([m/s]\) (Chapter 2)

\( x_{10} \) Initial mass position \([m]\) (Chapter 2)

\( y \) Expected position of mass-spring population \([m]\) (Chapter 2)

\( y_1,y_2 \) Variables for mass-spring population second-order differential equation (Chapter 2)

\( y_{peak} \) Mass-spring system peak of the beat \([m]\) (Chapter 2)

\( z_1 \) Sum of cell charges \([Ah]\) (Chapters 3-5)

\( z_2 \) Difference between cell charges \([Ah]\) (Chapters 3-5)

\( z_{2,e} \) Equilibrium charge imbalance \([Ah]\) (Chapter 5)

\( z_{2,free} \) Charge difference free response term \([Ah]\) (Chapters 3-5)

\( z_{2,forced} \) Charge difference forced response term \([Ah]\) (Chapters 3-5)

\( \alpha_a, \alpha_c \) Charge transfer coefficient (Chapters 3-5)
\( \gamma_{i,j,k,l} \) Value of piecewise uniform pdf for piece \( i, j, k, l \) (Chapter 2)
\( \gamma_j \) Cell thermal coupling term \([\text{K/s}]\) (Chapters 3-5)
\( \Delta_{db} \) Temperature deadband width \([\text{°C}]\) (Chapter 2)
\( \Delta t \) Time step \([\text{s}]\) (Chapter 5)
\( \Delta \phi \) Solid and electrolyte potential difference \([\text{V}]\) (Chapters 3-5)
\( \delta_{film,n} \) Negative electrode film thickness \([\text{cm}]\) (Chapters 3-5)
\( \varepsilon \) Active material faction (Chapters 3-5)
\( \varepsilon_t \) Infinitesimal time delay \([\text{s}]\) (Chapter 2)
\( \zeta_{1,2} \) Mass-spring system damping ratios (Chapter 2)
\( \zeta_{1,i,n}, \zeta_{2,i,n} \) TCL damping ratios (Chapter 2)
\( \eta \) Coefficient of performance (Chapter 2)
\( \eta_{n}, \eta_{p} \) Overpotential \([\text{V}]\) (Chapters 3-5)
\( \eta_s \) Side reaction overpotential \([\text{V}]\) (Chapters 3-5)
\( \theta_{0\%} \) Stoichiometry at 0\% SOC (Chapters 3-5)
\( \theta_{100\%} \) Stoichiometry at 100\% SOC (Chapters 3-5)
\( \kappa_{p} \) SEI ionic conductivity \([\frac{\text{S}}{\text{cm}}]\) (Chapters 3-5)
\( \lambda \) Thermal eigenvalue \([\text{s}^{-1}]\) (Chapters 3-5)
\( \lambda_z \) Desired charge imbalance eigenvalue \([\text{s}^{-1}]\) (Chapters 3-5)
\( \lambda_{\tau} \) Desired temperature imbalance eigenvalue \([\text{s}^{-1}]\) (Chapters 3-5)
\( \mu \) Entropy coefficient \([\text{V/K}]\) (Chapters 3-5)
\( \xi \) TCL initial fraction (Chapter 2)
\( \xi_{min}, \xi_{max} \) Minimum/maximum TCL initial fraction (Chapter 2)
\( \rho_{p} \) SEI layer density \([\frac{\text{kg}}{\text{cm}^3}]\) (Chapters 3-5)
\( \sigma \) Standard deviation for thermal capacitance pdf \([\frac{\text{kWh}}{\text{°C}}}]\) (Chapter 2)
\( \tau_1 \) Sum of cell temperatures \([\text{K}]\) (Chapters 3-5)
\( \tau_2 \) Difference between cell temperatures \([\text{K}]\) (Chapters 3-5)
\( \tau_{2,e} \) Equilibrium temperature difference \([\text{K}]\) (Chapter 5)
\( \tau_{2,free} \) Temperature difference free response term \([\text{K}]\) (Chapters 3-5)
\( \tau_{2,forced} \) Temperature difference forced response term \([\text{K}]\) (Chapters 3-5)
\( \tau_{\text{off}} \) Period of time the pulse is off [s] (Chapter 2)
\( \tau_{\text{on}} \) Period of time the pulse is on [s] (Chapter 2)
\( \tau^* \) Period of time before the pulse train starts [s] (Chapter 2)
\( \phi \) Solid potential [V] (Chapters 3-5)
\( \phi_e \) Electrolyte potential [V] (Chapters 3-5)
\( \omega \) TCL characteristic frequency [rad/s] (Chapter 2)
\( \omega_{\text{hl},i} \) Process noise [\(^\circ\text{C}/\text{s}\)] (Chapter 2)
\( \omega_{\text{min}}, \omega_{\text{max}} \) Minimum/maximum natural frequency of TCL units [rad/s] (Chapter 2)
\( \omega_{n,i} \) Mass \( i \)'s natural frequency [rad/s] (Chapter 2)
\( \omega_{n,\text{min}}, \omega_{n,\text{max}} \) Minimum/maximum natural frequency of mass-spring units [rad/s] (Chapter 2)
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Chapter 1

Introduction

1.1 Motivation

This dissertation determines fundamental insights into the behavior of heterogeneous populations of energy storage devices through analysis of reduced-order controls-oriented models. These insights are discovered for the purpose of improved design and control of these populations. Specifically, the energy storage devices explored are (i) thermostatically controlled loads (TCLs) used for demand response and (ii) battery cells used in a battery pack. These devices are chosen because of two similarities they share:

Application: Populations of both devices are implementable for the regulation of differences between power demanded and power supplied. The power demanded by TCLs can be regulated to match a power supplied for the purpose of integrating intermittent renewable supplies into the electric grid [1,2]. Stationary battery packs can be implemented for the same purpose by charging and discharging the battery [3–5]. Battery packs in transportation applications are suitable for manipulating the power split between multiple power sources, such as the internal combustion engine and electric motors within a hybrid electric vehicle (HEV) [6].

Fundamental Aggregate Behavior: The aggregate behavior of populations of both devices exhibit dynamic responses substantially different than those represented by an average unit. Individual TCLs have power demands that are oscillatory in nature, but the aggregate power demand of a heterogeneous population is damped [1]. Battery cells degrade due to a variety of factors, but heterogeneities in cell parameters and conditions create a pack degradation trend that is severely different than what an average cell could represent [7–12].

These two similarities are linked to each other. With insights related to collective dynamics, control methods can be improved. This means that in order to improve the application of these devices for power regulation, the aggregate behavior needs to be understood at a fundamental level. This is of great importance, as the motivation behind controlling these devices for filtering power differences is linked to paramount challenges in the current research field.
There is a push to increase renewable resource penetration into the electric grid [2,4] and a need for more efficient and environmentally-friendly transportation vehicles [6]. To address these challenges, this dissertation takes individual TCL and battery cell models from the literature, and reduces these models into a form suitable for analysis. By analyzing heterogeneous populations through these models, fundamental insights are obtained which describe the impact parameter and state heterogeneity have on system behavior. These insights, in turn, are useful for improving controller development of these devices and contributing to the solution of these overarching challenges. The next two sections provide an explanation of how the two types of energy storage devices are used and a survey of the literature. The final section of this chapter lists the contributions of this dissertation to address gaps in each body of literature.

### 1.2 Thermostatically Controlled Loads

There is a growing interest in exploiting TCLs for demand response. Demand response, also known as load management, is the practice of controlling the power demanded by consumers to match the power supplied by the electric grid. This can be accomplished through control of TCLs such as refrigerators, air conditioners, or heating systems within buildings [2], or even the manipulation of electricity prices during the day [13]. Demand response technologies are becoming increasingly valuable with the growing investment in renewable generation from resources such as solar and wind. The intermittencies associated with such renewables can burden the grid’s frequency regulation services, and demand response is a potential means for lessening this burden.

Typically, a TCL receives a desired set point temperature as an input from the homeowner. As the actual temperature of the TCL cannot be maintained exactly, the TCL temperature oscillates between an upper and lower temperature bound determined by the set point, as Figure 1-1 shows. Therefore, the device controlling temperature is turned on and off in a cyclic manner, resulting in the oscillatory power demand given in Figure 1-2. Chapter 2 outlines the dynamic equations and corresponding parameter set that produce these figures. Manipulation of the power demand cycle is a direct control method to implement demand response. For example, one form of demand response involves the manipulation of the set point temperature by the grid operator (instead of the homeowner) to change the TCL power demand as desired [1]. A second form is performed by directly turning certain TCLs in a population on or off [14].

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Callaway’s work is the first to develop a framework of using TCL set point temperature control for demand response. The proposed topology involves the use of a centralized controller to introduce small variations in set point temperatures with respect to time [1]. This makes it possible to change aggregate TCL power demand dynamics, thereby compensating for intermittencies from resources such as wind. Callaway likens the manipulation of set point
temperature to battery charging and discharging, but utilizing the thermal energy storage capacity of TCLs instead of the chemical energy storage capacity of a battery. For buildings with AC systems, an increase in set point temperature increases the number of loads that are off, which lowers the aggregate power demand. The opposite occurs during a decrease in set point temperature. Callaway notes that this approach can cause aggregate TCL power to be oscillatory, especially when TCL dynamics are homogenous. This can be challenging for control purposes. Two factors are known to damp these oscillatory dynamics: process noise used to represent unmodeled heat terms [15], and TCL parameter heterogeneity [1]. The damping effect induced by heterogeneity in TCL parameters (e.g., thermal capacitance) is valuable for control design. Figure 1-3 shows an example of the damping inherent to a heterogeneous TCL population’s power dynamics. This means that an average TCL unit, which is oscillatory by nature, cannot capture the damped aggregate power demand. Again, Chapter 2 details the TCL dynamics that can produce such a power demand.

![Graph showing oscillatory power demand](image)

**Figure 1-3.** Example of the aggregate power demand of a heterogeneous population of TCLs.

The insights from this work indicate that models of homogeneous TCLs are not accurate enough to implement control over heterogeneous TCLs. As a result, there is a growing body of literature on the low-order modeling of heterogeneous populations of TCLs [14–21]. For example, Koch et al. neglect process noise and present a linear model that incorporates capacitance heterogeneity by sampling heterogeneous TCL dynamics into discrete “bins.” They tune the number of bins to match the damped oscillations created by heterogeneity [14]. Zhang et
al. propose a clustering-based approach to capture heterogeneity, along with a second-order temperature model and a solar heat gain term to capture internal TCL dynamics more accurately [16]. Moura et al. use Malhamé and Chong’s homogeneous population model of coupled Fokker Planck equations [15] and add a second diffusion term to supplement the one from process noise. Identification of these diffusion coefficients allows their model to capture the damping effects created by parameter heterogeneity [17]. Ghaffari et al. present analytical partial differential equation (PDE) dynamics for a heterogeneous population of TCLs under the approximation of no process noise. Their model is developed by using an auxiliary parameter to describe heterogeneity in thermal capacitance, resistance and power parameters [18].

In order to improve heterogeneous model development, there is a focus on characterizing the effects of heterogeneity on a population of TCLs. Perfumo et al. develop an analytic expression for the power demanded by TCLs under step changes in set point temperature, assuming TCL capacitance is log-normally distributed. An approximate solution is used to determine the oscillation period as well as the decaying amplitudes [22,23]. In a separate body of work, a sensitivity analysis is performed to describe the nature of the underdamped oscillations under step changes in set point temperature. It is found that increased heterogeneity leads to stronger damping effects. Moreover, the oscillatory nature of the response appears to be robust with respect to the shape of the heterogeneous parameter distribution [24]. Ghaffari et al. break a heterogeneous population into subsets of homogeneous populations, and a stability analysis is performed [25]. Hao et al. represent a heterogeneous population of TCLs through two battery models, and use this to analytically characterize the flexibility of TCLs [26].

The insights determined by the above studies show that TCL parameter heterogeneity and random heat generation within the TCLs can both cause aggregate power demand dynamics to be damped. The literature attempts to study each of these two effects by either modeling process noise or incorporating parameter heterogeneity in its analyses of aggregate TCL dynamics. However, there is a noticeable gap in this literature: the lack of a model that quantifies the structure of the damped solution and its relationship to underlying heterogeneities for a broad range of TCL parameters. Additionally, the fact that the aggregate dynamics can also exhibit beating remains relatively less studied.
1.3 Battery Packs

Battery packs are currently being implemented for both the smart grid and within the transportation sector. The reasons for using stationary battery packs within the smart grid are similar to using TCLs for demand response: this helps integrate intermittent renewable resources into the electric grid. If, at any point, a renewable resource provides an excess of power, it is stored in the battery pack. When the renewable source provides too little power, the battery pack is discharged to provide the extra power required [3,4]. In transportation applications, battery packs are incorporated into vehicles in an attempt to increase vehicle efficiency and reduce the cost of fuel [27]. Furthermore, in the case of plug-in hybrid electric vehicles (PHEVs) and fully electric vehicles (EVs), these battery packs also have the ability to be used for renewable resource integration [28].

Widespread implementation of both stationary and transportation battery packs is impeded by pack degradation [5]. A battery pack is expensive, and for one to be cost-effective, it needs to be designed to last a number of years before being declared “dead” [29]. As a result, it is generally necessary for control algorithms to place an emphasis on minimizing degradation, typically by placing limitations on the pack’s input current, output voltage or power, or even the states of the pack. For example, it is desirable to maintain a cell’s state of charge (SOC) within safe limits, as charging outside the limits increases degradation of the cell [5]. These limitations are, in turn, impacted by the current temperature and state of health (SOH) of the battery cell. Operation limits are placed to avoid high cell temperatures which lead to more rapid degradation, though the reduced resistance in these scenarios is noted for being beneficial for cell performance [30]. A cell with a worse SOH, i.e. a lower capacity or higher resistance, has a more limited safe operating range than a healthier cell. This ultimately leads to cells with lower SOH values negatively impacting the battery pack SOH [9,12,31].

This work focuses on the lithium-ion (Li-ion) battery chemistry. The literature contains numerous models and experiments that provide insights into Li-ion cell degradation dynamics [32–40]. In contrast, most Li-ion pack studies emphasize the mitigation of charge heterogeneity, which is known to limit a pack’s performance and longevity [12,41]. Numerous voltage balancing [42,43] and charge balancing [5,31,44,45] algorithms and topologies exist to remove charge imbalance. There is comparatively little work on pack degradation dynamics, and how state and parameter heterogeneity impact end-of-life (EOL) of a battery pack. While research into pack degradation is just starting to blossom, multiple bodies of work report a common insight: a pack...
of heterogeneous battery cells has a degradation trend that is more severe than the homogeneous case [7–11]. This leads to more severe limitations on control methods to avoid increased degradation. Three bodies of preliminary work have formed in an attempt to understand this phenomenon. The first body focuses on experimental results from cycling heterogeneous battery cells [11,46–52]. The second body develops battery pack models that are useful for determining insights through simulation and analysis [7–10,48,49,53–57]. The third body develops control algorithms that remove imbalance or prevent its impact on pack performance and lifespan [12,41,57–62].

Most experiments performed focus on running the same current cycle through all cells tested. These cells have parameter heterogeneities due to either manufacturing differences or environmental conditions. Baumhöfer et al. test 48 cells to determine how capacity of the cells spread due to cycling. They find that all cells degrade, but the distribution spreads greatly over time [46]. Rothgang and Baumhöfer, as well as Paul et al., confirm this spread through other sets of experiments, but find that the initial capacity does not necessarily determine which cells degrade faster, i.e. the initially unhealthiest cell does not necessarily remain the unhealthiest [47,48]. Chiu et al. test two cells in series to examine the impact of temperature heterogeneity on degradation, and find that pack capacity is limited by the highest temperature cell [49]. Several studies consider experimental testing of cells in parallel. Gogoana et al. find that heterogeneity in internal resistances of the cells leads to a negative impact on the cycle life of the pack [11], and Gong et al. reach a similar conclusion for heterogeneous cell capacities [50]. Shi et al. find that cells in parallel have increasingly large imbalance currents, which is the primary factor for capacity degradation in the pack [51]. Contrary to some studies, Pastor-Fernández et al. cycle four cells in parallel and find that capacity and resistance heterogeneity decrease over time due to healthier cells receiving larger currents than the weaker cells [52].

In lieu of pure experimentation, several researchers focus on developing models to capture the degradation behavior of a heterogeneous battery pack. A common practice is to assign a model, typically an equivalent-circuit model, to each cell in the pack [53,54]. With each cell model having its own set of parameters, this modeling style employs Monte Carlo methods for pack representation. Recent publications use these Monte Carlo methods to understand how the degradation of the battery pack is impacted by individual, heterogeneous cells. Several go a step further and examine how charge balancing strategies affect the pack SOH. In particular:
• Paul et al. and Chiu et al. supplement their experimental work with Monte Carlo simulations to explore how individual cell capacities spread due to heterogeneous initial parameter values [48] and heterogeneous temperature spreads [49].

• Onori et al. consider two heterogeneous battery cells in series using equivalent-circuit models. By examining the system equations and running simulations, they explain the impact an “upstream” cell has on a “downstream” cell under cycling. Through this study, it is found that if the upstream cell ages faster, current increases to satisfy the power demanded of the string. This current increase magnifies degradation of all cells, causing the downstream cell to age faster [7].

• The work of [7] is expanded to simulate more battery pack topologies, such as parallel configurations [8]. The studies show that aged cells impact other cells less for series strings than parallel configurations, as current increases dramatically for healthy cells parallel to an unhealthy cell. Retroactivity is used to examine the thermal coupling existing between cells in a pack, and it is shown that this thermal coupling is not as dominant in parallel configurations as compared to series strings.

• Cordoba-Arenas attempts to understand heterogeneity’s importance for a variety of pack topologies using Monte Carlo methods with equivalent-circuit models for each cell. It is shown that having thermally coupled cells in a pack degrades cells faster within the pack as compared to thermally insulated cells. Through simulation, it is found that placing series strings of cells in parallel is more robust to aging propagation than sets of parallel cells placed in series. This work also shows the impact passive and active charge balancing strategies have on degradation within the pack for the scenarios examined [9].

• Zheng et al. present a graphical method to relate pack capacity to the cells with the minimum chargeable and dischargeable electric quantities. A graphical method is used to derive a simple equation relating pack degradation to the degradation of cells with the worst electric quantity properties. The cell degradation model chosen leads to the insight that loss of lithium in the anode impacts pack degradation significantly [10].

• LeBel et al. develop a model of cells in parallel that accounts for parameter and temperature heterogeneity, and determine through simulation that open-circuit voltage (OCV) differences lead to some cells aging prematurely [55].
Yang et al. simulate cells in parallel for a variety of temperature differences. Temperature heterogeneity leads to higher temperature cells receiving larger current portions, and is found to create more severe imbalance currents and thus accelerated aging per cycle [56].

Ganesan et al. develop an electro-thermal battery pack model, and are able to determine individual cell characteristics from the sensitivity of the voltage with respect to the parameter in question. A reconfigurable battery pack is considered, and current deviation between modules with different capacities is related to capacity losses over time [57].

In light of the above analyses on heterogeneous pack degradation, some studies develop control algorithms to mitigate the impact of state and parameter differences. Rehman et al. develop a balancing method to operate cells with lower internal resistances over a wider SOC range. This balancing method allows pack lifespan to be defined by the average cell capacity instead of the worst cell capacity [12]. This is used as a basis to develop another control strategy which switches between meeting performance criteria and eliminating capacity mismatch over time. By assigning each cell a different reference SOC based on its capacity, weaker cells are intuitively set to lower SOCs to improve capacity homogeneity and extend pack lifespan up to 40% [58]. Ganesan et al. supplement their studies with a temperature control algorithm designed to manipulate the heat transfer coefficient in order to reach a reference temperature [57]. Altaf et al. examine temperature imbalance created through resistance heterogeneity and coolant temperature gradients. They find that charge balancing hardware is suitable for temperature heterogeneity reduction. Within a cell, an irreversible heating term exists related to the current squared. By using balancing hardware to manipulate the current through each cell, cooler cells can be heated faster than others, leading to temperature equalization within the battery pack [41]. Other studies show that control of multi-level converters (MLCs) is suitable for balancing both charge and temperature within a battery pack. The control decisions for the MLCs can be made using convex optimization with full future input knowledge [59] or using linear quadratic model predictive control (MPC) based methods [60], for both unipolar and bipolar control [61]. As these studies note that there are some tradeoffs between charge and temperature balancing [41], algorithms are designed to maintain low charge imbalances instead of outright elimination [59]. Barreras et al. shows that additional objectives of homogenizing internal resistances and fault detection can also be considered when making balancing hardware control decisions [62].
The insights developed in these studies show that SOH and temperature heterogeneity in a battery pack can limit performance and lifespan. However, these studies tend to lack a formal framework to develop concrete insights by analyzing a model of the imbalance dynamics using linear system and control theory. This prevents new discoveries that can be used to develop improved control algorithms with the specific intention of removing heterogeneity from the pack. There is a lack of published understanding of the impact of no balancing, voltage balancing, and charge balancing on the growth or decay of capacity, resistance, and temperature heterogeneity within a pack. As a result, only a minimal number of algorithms exist which are specifically built to remove these heterogeneities using balancing hardware and state feedback-based algorithms.

1.4 Contributions

A survey of the literature indicates the need of insight-favorable, reduced-order models for both heterogeneous populations of TCLs and heterogeneous battery cells within a pack. For TCLs, there is a need to concretely determine how a variety of parameter heterogeneities impact aggregate power dynamics. For batteries, there is a need to provide insights into how heterogeneous cells impact pack SOH through understanding of the heterogeneity dynamics. This dissertation addresses these issues, recognizing that complex unit (single TCL or cell) models cannot be used easily for insight development. Model reduction and reformulation tools are proposed for increasing mathematical tractability of the individual unit models, which simplifies aggregate or heterogeneity model development while still maintaining core dynamic system information. With the models and insights determined, novel control algorithms and system design selection can be used to improve performance of the populations for their select purpose. This leads to two groups of contributions to the literature. For the TCL work, the primary contributions of this dissertation are:

1a. A low-order model representative of the aggregate behavior of a heterogeneous population of TCLs.
1b. The insight that only heterogeneity in a population’s characteristic frequencies leads to damped aggregate power demand profiles.
1c. The insight that a beating phenomenon is triggered by low levels of heterogeneity in the population’s characteristic frequencies.
1d. The parameter-dependent, time-varying damping ratios of the low-order model describe the strength of the damping observed in the aggregate power demand.

For the battery pack work, the primary contributions of this dissertation to the literature are:

2a. A framework for the development and analysis of a model describing the heterogeneity dynamics within a battery pack.

2b. An analysis into impact voltage balancing has on capacity and resistance heterogeneity.

2c. A control algorithm which balances charge and capacity heterogeneity and increases pack lifespan by up to 9.2%.

2d. An analysis of the tradeoffs between balancing charge and temperature heterogeneity within a battery pack.

2e. The insight that control of an average current passing through two cells in series is able to improve the rate of temperature heterogeneity dissipation.

Chapter 2 of this dissertation covers development of the TCL problem contributions 1a-1d. Chapter 3 presents the framework of contribution 2a and applies the first two steps of the framework to an electrochemical battery model with thermal and degradation dynamics. Chapter 4 applies the remaining steps of the framework for a battery pack with charge and SOH heterogeneity, and furnishes contributions 2b and 2c. Chapter 5 analyzes a pack with charge and temperature heterogeneity, and develops contributions 2d and 2e. Chapter 6 provides a summary of the contributions and concluding remarks.
Chapter 2

Demand Response Using Heterogeneous Thermostatically Controlled Loads: Characterization of Aggregate Power Dynamics

2.1 Introduction

This chapter examines the aggregate dynamics of large, heterogeneous populations of TCLs connected to the electric grid. There is a growing interest in the use of these systems for demand response, especially since the aggregate power demand of heterogeneous TCLs is known to be damped. This damping is useful for control of these systems, and a variety of models are proposed in the literature that capture heterogeneous power dynamics [14,16–21]. Further work focuses on the understanding of heterogeneity's impact on the aggregate power dynamics through approximation solutions [22,23], Monte Carlo studies [24], stability analysis of homogeneous subsets [25], or using a battery model to characterize flexibility of TCLs [26].

The above work shows that TCL parameter heterogeneity can cause aggregate power demand dynamics to be damped. The literature attempts to study this effect by incorporating parameter heterogeneity in its analyses of aggregate TCL dynamics. However, there is a noticeable gap in this literature: the lack of a model that quantifies the structure of the damped solution and its relationship to underlying heterogeneities for a broad range of TCL parameters. Additionally, the fact that the aggregate dynamics can also exhibit beating remains relatively less studied. This chapter addresses these gaps by contributing a novel reduced-order model of the aggregate power dynamics for heterogeneous TCL populations to the literature. In order to emphasize the impact of parameter heterogeneity on the damping effect, this work does not consider process noise. The chapter's LTV aggregate model fosters three additional contributions to the TCL literature. First, the development of the model furnishes an insight that damping is created only by heterogeneity in the characteristic frequency of the TCL dynamics. Second, the time-varying damping ratios of the model analytically quantify the damping in the system. Third, the potential beating phenomenon is described and related to the decay present in the aggregate response. The reduced-order model, insights, and damping ratios together are suitable for
designing improved demand response controllers, as well as state and parameter estimators. The work of this chapter is published in a conference paper [63] and journal article [64].

The remainder of this chapter is organized as follows. The next section examines a population of oscillators more simplistic than the TCL population in order to develop simple insights (Section 2.2). By solving for the oscillatory power demand for a single TCL, parallels are drawn between the TCL problem and the simple oscillator example (Section 2.3). The aggregate power demand for a population of TCLs is explored using univariate parameter distributions, leading to discoveries that explain which TCL parameters, when distributed, create damping and beating (Section 2.4). A reduced-order model of the aggregate power demand is developed for the case of multivariate parameter distributions. The damping ratios of this approximate model characterize the damping in the dynamics (Section 2.5). A realistic TCL population is simulated for a triggered demand response event, and the accuracy of the approximate model is validated against a Monte Carlo model (Section 2.6). This chapter ends by summarizing the main discoveries found from the analysis, and the benefits they provide for the TCL demand response problem.

2.2 Mass-Spring Population

This section examines a population of mass-spring oscillators with heterogeneous natural frequencies. In this example, each device (a mass-spring) is described by oscillatory dynamics (mass displacement), while the output of the population contains damped dynamics (expected displacement of all masses). This is analogous to the TCL problem: each device (a TCL) is described by oscillatory dynamics (power demand), while the output of the population contains damped dynamics (expected power demand of all TCLs). Analyzing this example provides insights into the expected dynamics, and these insights are translatable to the TCL demand response problem.

Consider a population of \( i = 1, \ldots, N_L \) mass-spring devices, each with natural frequency \( \omega_{ni} \). Each unit is described by the states \( \tilde{x}_i(t) = [x_{1,i}(t), x_{2,i}(t)]^T \), with \( x_{1,i}(t) \) as the mass's position and \( x_{2,i}(t) \) as the mass's velocity. Equation (2-1) presents the state space equations for each unit under free response, with initial conditions \( \tilde{x}_i(0) = [x_{10,i}, 0]^T \). Equation (2-2) presents the analytic solution for \( x_{1,i}(t) \), and Figure 2-1 presents the displacement for example oscillators.
with varying natural frequencies. The output of the state space system, $y(t)$, is defined to be the expected position (in a statistical sense) of the units, as defined by Equation (2-3).

$$\hat{x}_i(t) = \begin{bmatrix} 0 & 1 \\ -\omega_{n,i}^2 & 0 \end{bmatrix} \hat{x}_i(t)$$  

(2-1)

$$x_{1,i}(t) = x_{10} \cos(\omega_{n,i} t)$$  

(2-2)

$$y(t) = \frac{1}{N_L} \sum_{i=1}^{N_L} x_{1,i}(t)$$  

(2-3)

![Figure 2-1. Oscillatory displacement of a population of mass-springs with heterogeneous natural frequencies.](image)

Figure 2-1. Oscillatory displacement of a population of mass-springs with heterogeneous natural frequencies.

Suppose the natural frequencies are distributed uniformly between $\omega_{n,\text{min}}$ and $\omega_{n,\text{max}}$, i.e. the probability density function (pdf) is $pdf_{\omega_n}(\omega_n) = \frac{1}{\omega_{n,\text{max}} - \omega_{n,\text{min}}}$. Equation (2-4) presents the solution for $y(t)$ for the case of $N_L \to \infty$.

$$y(t) = \int_{\omega_{n,\text{min}}}^{\omega_{n,\text{max}}} x_1(t) pdf_{\omega_n}(\omega_n) d\omega_n = \frac{x_{10}}{t(\omega_{n,\text{max}} - \omega_{n,\text{min}})} \left( \sin(\omega_{n,\text{max}} t) - \sin(\omega_{n,\text{min}} t) \right)$$  

(2-4)

The result of Equation (2-4) allows the development of the first insight for this example: *for a uniform distribution of frequency, the expected displacement is damped by a $1/t$ term.* Despite that an individual unit's displacement is described by oscillatory dynamics, the expected dynamics are damped. The decay of the expected dynamics indicates that the oscillators are desynchronizing over time, i.e. the behavior is nonrecurrent [65]. Figure 2-2 presents $y(t)$, with $x_{10} = 1$ m, $\omega_{n,\text{min}} = 0.9 \text{ rad/s}$ and $\omega_{n,\text{max}} = 1.1 \text{ rad/s}$, to visualize the damping in the expected
displacement. To quantify the damping in the system, damping ratios are determined from differential equations representing the output. Rewriting Equation (2-4):

\[ y(t) = \frac{x_{10}}{\omega_{n,max} - \omega_{n,min}} (y_1(t) - y_2(t)) \quad (2-5a) \]

\[ y_1(t) = \frac{1}{t} \sin(\omega_{n,max} t) \quad (2-5b) \]

\[ y_2(t) = \frac{1}{t} \sin(\omega_{n,min} t) \quad (2-5c) \]

![Expected Response](image)

Figure 2-2. Expected displacement of uniformly distributed mass-springs with \( \omega_{n,min} = 0.9 \text{ rad/s} \) and \( \omega_{n,max} = 1.1 \text{ rad/s} \).

Two second-order differential equations, one for \( y_1(t) \) and one for \( y_2(t) \), describe the system. For convenience, and without impacting the final result, both sides of Equation (2-6) are multiplied by \( t \), as Equation (2-6) presents. Equation (2-7) presents the result of applying a second-order derivative to Equation (2-6). By combining Equations (2-5b) and (2-7), a second-order differential equation is obtained for \( y_1(t) \) (Equation (2-8)). Similar steps are repeated for \( y_2(t) \) to produce Equation (2-9). These differential equations are LTV, due to the \( 2/t \) coefficient of \( y_1(t) \) and \( y_2(t) \).

\[ ty_1(t) = \sin(\omega_{n,max} t) \quad (2-6) \]

\[ t\ddot{y}_1(t) + 2\dot{y}_1(t) = -\omega_{n,max}^2 \sin(\omega_{n,max} t) \quad (2-7) \]

\[ \ddot{y}_1(t) + \frac{2}{t} \dot{y}_1(t) + \omega_{n,max}^2 y_1(t) = 0 \quad (2-8) \]

\[ \ddot{y}_2(t) + \frac{2}{t} \dot{y}_2(t) + \omega_{n,min}^2 y_2(t) = 0 \quad (2-9) \]
Using the standard formula for a second-order system \((\ddot{z}(t) + 2\zeta \omega_n \dot{z}(t) + \omega_n^2 z(t) = 0)\), damping ratios for this LTV model are determined. The two time-varying damping ratios, \(\zeta_1(t) = \frac{1}{\omega_{n,\text{max}} t}\) and \(\zeta_2(t) = \frac{1}{\omega_{n,\text{min}} t}\), provide explicit quantification of the damping in the expected mass displacement for times \(t > 0\). Equivalently, these ratios are able to describe the rate at which oscillators in the population desynchronize.

This example highlights a second insight: a uniform distribution of frequency can potentially lead to a beating effect in the expected mass displacement. Equation (2-10) presents an alternative expression for the output \(y(t)\). For small differences between \(\omega_{n,\text{min}}\) and \(\omega_{n,\text{max}}\), a beating effect can exist in the system: a slowly varying amplitude along with a more quickly varying oscillatory motion [66]. Figure 2-2 shows these two oscillation periods for the example case. The existence of a beat period, \(\frac{2\pi}{\omega_{n,\text{max}} - \omega_{n,\text{min}}}\), indicates that there is some level of repeated synchronization and desynchronization between the individual oscillators.

\[
y(t) = \frac{2x_{10}}{m\pi} \cos\left(\frac{\omega_{n,\text{max}} + \omega_{n,\text{min}}}{2} t\right) \sin\left(\frac{\omega_{n,\text{max}} - \omega_{n,\text{min}}}{2} t\right)
\]

(2-10)

In order to determine the impact of the potential beating effect, the peaks of the beats are compared against the initial amplitude of the system. The peak of each beat is approximated to occur halfway through the beat, i.e. at \(t = \frac{m\pi}{\omega_{n,\text{max}} - \omega_{n,\text{min}}}\) for \(m = 3, 5, \ldots\), and it is approximated that \(\cos\left(\frac{\omega_{n,\text{max}} + \omega_{n,\text{min}}}{2} t\right) \sin\left(\frac{\omega_{n,\text{max}} - \omega_{n,\text{min}}}{2} t\right) \approx 1\) at these times. Equation (2-11) expresses the peak of the beat, \(y_{\text{peak}}(m)\), where \(\frac{m-1}{2}\) is the number of the beat analyzed. Equation (2-12) presents the ratio of the peak of the beat to the initial amplitude \(y(0)\), obtained by using L'Hopital's rule.

\[
y_{\text{peak}}(m) = \frac{2x_{10}}{m\pi}
\]

(2-11)

\[
\frac{y_{\text{peak}}(m)}{y(0)} = \frac{2}{m\pi}
\]

(2-12)

Equation (2-12) gives an approximation of the decay of the beating effect over time. Figure 2-3 presents a plot of Equation (2-12) against the beat number. This shows that the amplitude decays to less than 5% of the initial expected displacement after the sixth beat. After this point, the beating effect is not very noticeable.
This section reviews the modeling of a population of TCLs. The state equation for an individual TCL is reviewed, as well as the equation for the expected power demand of a population of TCLs. Though TCLs can include a variety of systems, such as those with water heaters or refrigerators, this work focuses on air conditioning (AC) systems for simplicity. An analytic solution for the power demanded by a single TCL in free response is presented in the form of a rectangular wave using the Fourier series. This expression allows for the introduction of parameters more suitable for representing the expected power demand of a TCL.

**TCL Monte Carlo Model**

A TCL is defined to be a building with thermal capacitance $C$ and a thermal resistance $R$. The lumped capacitance approximation is used, such that the temperature of the building is represented by an average value $T(t)$. The environment outside the house has an ambient temperature $T_\infty$. For this work, the sign convention assumes $T_\infty$ is greater than the desired house temperature. The TCL air conditioning unit, of constant power $P$, is set to maintain temperature
within a deadband surrounding the set point using switching control. Equation (2-13) gives the state equation for the temperature of the building [67].

\[
\dot{T}_i(t) = \frac{1}{c_i R_i} \left( T_{\infty,i} - T_i(t) - R_i P_i s_i(t) \right) + \omega_{hi,i} \tag{2-13}
\]

Note that \( i = 1, ..., N_L \) denotes individual TCLs, with \( N_L \) as the number of TCLs. The variable \( s(t) \) represents the discrete switching control of the TCL used to maintain temperature within the desired bounds \([T_{min}, T_{max}]\). Equation (2-14) gives the conditions that determine \( s(t) \), and Equations (2-15)-(2-16) provide expressions for \( T_{min,i} \) and \( T_{max,i} \) with respect to \( T_{sp} \) (set point temperature) and \( \Delta_{db} \) (temperature deadband width). The parameter \( \varepsilon_t \) is an infinitesimal time delay [68]. As this work does not require a forced change in set point temperature to analyze TCL dynamics, \( T_{sp} \) remains a constant for each TCL. The process noise term, \( \omega_{hi,i} \), represents unmodeled heat loads, and is neglected in this work.

\[
s_i(t) = \begin{cases} 
0 & \text{if } s_i(t - \varepsilon_t) = 1 \text{ and } T_i(t) \leq T_{min,i} \\
1 & \text{if } s_i(t - \varepsilon_t) = 0 \text{ and } T_i(t) \geq T_{max,i} \\
s_i(t - \varepsilon_t) & \text{else}
\end{cases} \tag{2-14}
\]

\[
T_{min,i} = T_{sp,i} - \frac{\Delta_{db,i}}{2} \tag{2-15}
\]

\[
T_{max,i} = T_{sp,i} + \frac{\Delta_{db,i}}{2} \tag{2-16}
\]

Equation (2-17) presents the expression for the expected power demanded (in a statistical sense) by all TCLs, \( \bar{P}_{TCL} \). In this expression, \( \eta \) is defined as the coefficient of performance of the air conditioning unit, and \( P_{TCL}(t) \) is the individual unit power demand. The literature typically refers to the expected power demand as the aggregate power demand per TCL. The insights determined in this chapter, as well as the model developed, for the expected power demand are applicable to the aggregate power demand.

\[
\bar{P}_{TCL}(t) = \frac{1}{N_L} \sum_{i=1}^{N_L} \frac{P_i}{\eta_i} s_i(t) = \frac{1}{N_L} \sum_{i=1}^{N_L} P_{TCL,i}(t) \tag{2-17}
\]

**Power Demand as a Rectangular Wave**

In order to develop an analytic expression for a heterogeneous system, a single TCL is first examined. From Equation (2-17), it is easily seen that for a single TCL in continuous time, the power demand follows the form of a rectangular wave (i.e., pulse train) [69]. Table 2-1 presents a nominal set of TCL parameters from the literature [1]. While it is not the intention of the author to state that these parameters represent a real TCL exactly, the parameters (such as the
thermal capacitance) are within the same order of magnitude as presented in other works [70,71]. Using these nominal values and initial conditions $T(0) = T_{sp}$ and $s(0) = s_0$, Figure 1-1 shows an example of temperature variation for the lone TCL case, and Figure 1-2 plots the corresponding power demand of the AC unit. Equation (2-18) presents the analytic expression for a rectangular wave, with $u$ as the unit step function. There are four parameters that describe the rectangular wave:

1. The height of the wave, $A = \frac{p}{\eta}$.
2. The period of time the pulse is on, $\tau_{on}$.
3. The period of time the pulse is off, $\tau_{off}$.
4. The period of time before the pulse train starts, $\tau^*$.

$$
P_{TCL}(t) = A \sum_{k=0}^{\infty} \left( u \left( t - \left( \tau^* + k\tau_{on} + k\tau_{off} \right) \right) \right) - A \sum_{j=1}^{\infty} \left( u \left( t - \left( \tau^* + j\tau_{on} + (j-1)\tau_{off} \right) \right) \right)$$

(2-18)

Table 2-1. TCL parameter values (from [1]).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C$</td>
<td>10 kWh/°C</td>
</tr>
<tr>
<td>$R$</td>
<td>2 °C/kW</td>
</tr>
<tr>
<td>$P$</td>
<td>14 kW</td>
</tr>
<tr>
<td>$\eta$</td>
<td>2.5</td>
</tr>
<tr>
<td>$T_{sp}$</td>
<td>20°C</td>
</tr>
<tr>
<td>$\Delta_{db}$</td>
<td>0.5°C</td>
</tr>
<tr>
<td>$T_\infty$</td>
<td>32°C</td>
</tr>
</tbody>
</table>

Equation (2-19) presents the solution to Equation (2-13), with $T_0$ as the initial temperature. This allows for the determination of the time periods. Equation (2-20) solves for $t = \tau_{on}$ using $T(t) = T_{min}$, $T_0 = T_{max}$, and $s(t) = 1$. Equation (2-21) solves for $t = \tau_{off}$ using $T(t) = T_{max}$, $T_0 = T_{min}$, and $s(t) = 0$.

$$
T(t) = T_0 e^{-\frac{1}{CR}t} + \int_t^\infty e^{-\frac{1}{CR}(t-v)} \left[ \frac{1}{CR} \frac{p}{\eta} \right] \left\{ \frac{1}{T_\infty} s(v) \right\} dv
$$

(2-19)

$$
\tau_{on} = -CRC ln \left( \frac{T_{min} - T_\infty + PR}{T_{max} - T_\infty + PR} \right)
$$

(2-20)
\[ \tau_{off} = -CR\ln\left( \frac{T_{max} - T_\infty}{T_{min} - T_\infty} \right) \tag{2-21} \]

As the wave can begin partially through an on or off cycle, \( \tau^* \) depends on the initial conditions \( T_0 \) and \( s(0) = s_0 \). Equation (2-22) gives a generic solution for \( \tau^* \) based on the initial conditions, under the restriction that \( T_{\text{min}} \leq T_0 \leq T_{\text{max}} \). Equation (2-18) can be formatted in a way to avoid this restriction, but doing so does not fundamentally alter the results or conclusions of this work.

\[ \tau^* = -CR\ln\left( \frac{T_{max} - T_\infty + PRs_0}{T_0 - T_\infty + PRs_0} \right) \tag{2-22} \]

**Fourier Series Representation**

The rectangular wave solution, while exact, is piecewise continuous. The Fourier series allows the representation of this solution using smooth functions. The infinite Fourier series for this rectangular wave is:

\[
P_{TCL}(t) = A \frac{\tau_{on}}{\tau_{on} + \tau_{off}} + \frac{2A}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left( \frac{n\pi\tau_{on}}{\tau_{on} + \tau_{off}} \right) \cos\left( \frac{2\pi n t - 2\pi n \tau^* - \pi n \tau_{on}}{\tau_{on} + \tau_{off}} \right) \tag{2-23} \]

Analyzing Equation (2-23), it is found that there are more appropriate parameters to define the Fourier series expression for power dynamics rather than \( \tau_{on}, \tau_{off}, \) and \( \tau^* \). The characteristic frequency of this oscillatory system, \( \omega \), is defined in Equation (2-24). The ratio of time that power is demanded to total time period (i.e., the duty cycle), is defined as \( D \) in Equation (2-25).

In addition, there is a ratio of the initial delay time to the total time period, as Equation (2-26) presents. For lack of a better term, this is referred to as the initial fraction, \( \xi \). Equation (2-27) presents the Fourier series using these new parameters. Equations (2-27) and (2-17) are the TCL system equivalent to Equations (2-2) and (2-3) of the mass-spring system.

\[
\omega = \frac{2\pi}{\tau_{on} + \tau_{off}} \tag{2-24}
\]

\[
D = \frac{\tau_{on}}{\tau_{on} + \tau_{off}} \tag{2-25}
\]

\[
\xi = \frac{\tau^*}{\tau_{on} + \tau_{off}} \tag{2-26}
\]

\[
P_{TCL}(t) = AD + \frac{2A}{\pi} \sum_{n=1}^{\infty} \frac{\sin(nD\pi)}{n} \cos(n\omega t - 2\pi n \xi - \pi nD) \tag{2-27}
\]
2.4 Heterogeneous Populations with Univariate Distributions

This section analyzes the impact different types of parameter heterogeneity have on the expected power demand of a population of TCLs. Four univariate parameter distributions are explored, each one associated with a different Fourier series parameter. Analytic solutions are presented for each of these cases through the use of a stochastic integral. By drawing parallels between the TCL system and the mass-spring system, the insights of the simple example are translated to this more complex case. A sensitivity analysis is performed to determine which underlying TCL parameters impact the aggregate response the most.

Analytic Expressions for Univariate Distributions

There are several similarities that exist between a population of mass-springs and a population of TCLs. By comparing Equations (2-27) and (2-17) with Equations (2-2) and (2-3), it can be seen that $P_{TCL}(t)$ and $x_1(t)$ are equivalent, and $\bar{P}_{TCL}(t)$ and $y(t)$ are equivalent. Both systems contain frequency parameters ($\omega$ and $\omega_n$), but the TCL oscillatory dynamics are described by a summation of cosine terms, rather than a single cosine term. Additionally, there is a constant term ($AD$) and a phase ($-2\pi n \xi - \pi n D$) within the TCL system. Despite these differences, the same steps used for the mass-spring system are applicable to the TCL system.

Suppose one of the Fourier series parameters, $\theta$, is uniformly distributed between a minimum value $\theta_{min}$ and a maximum value $\theta_{max}$, i.e. $pdf_\theta(\theta) = \frac{1}{\theta_{max} - \theta_{min}}$. While profiles more complex than uniform distributions can be used, desirable insights do not require any additional complexity. Equation (2-28) presents the solution to $\bar{P}_{TCL}(t)$ for a large population of TCLs. Table 2-2 presents the analytic solutions for the four different cases of $\theta = A, \omega, D, \xi$.

$$\bar{P}_{TCL}(t) = \int_{\theta_{min}}^{\theta_{max}} P_{TCL}(t)pdf_\theta(\theta)d\theta$$  \hspace{1cm} (2-28)
Table 2-2. Exact expressions for different types of parameter heterogeneity.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>pdf$_\theta(\theta)$</th>
<th>$P_{TCL}(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega$</td>
<td>$\frac{1}{\omega_{\text{max}} - \omega_{\text{min}}} \frac{2A}{\pi(\omega_{\text{max}} - \omega_{\text{min}})} t \sum_{n=1}^{\infty} \left( \frac{\sin(\pi nD)}{n^2} \left( \sin(n\omega_{\text{max}}t - 2\pi n\xi - \pi nD) - \sin(n\omega_{\text{min}}t - 2\pi n\xi - \pi nD) \right) \right)$</td>
<td>$AD + \frac{4A}{\pi(\omega_{\text{max}} - \omega_{\text{min}})} t \sum_{n=1}^{\infty} \left( \frac{\sin(\pi nD)}{n^2} \cos\left(\frac{n(\omega_{\text{max}} + \omega_{\text{min}})t - 4\pi n\xi - 2\pi nD}{2}\right) \right)$</td>
</tr>
<tr>
<td>$D$</td>
<td>$\frac{1}{D_{\text{max}} - D_{\text{min}}} \frac{A(D_{\text{max}}^2 - D_{\text{min}}^2)}{2(D_{\text{max}} - D_{\text{min}})} + \frac{-A}{2\pi^2(D_{\text{max}} - D_{\text{min}})} \sum_{n=1}^{\infty} \left( \frac{1}{n^2} \cos(n\omega t - 2\pi n\xi - 2\pi nD_{\text{max}}) \right)$</td>
<td>$\frac{1}{A_{\text{max}} - A_{\text{min}}} \frac{(A_{\text{max}}^2 - A_{\text{min}}^2)D}{2(A_{\text{max}} - A_{\text{min}})} + \frac{(A_{\text{max}}^2 - A_{\text{min}}^2)}{\pi(A_{\text{max}} - A_{\text{min}})} \sum_{n=1}^{\infty} \left( \frac{\sin(\pi nD)}{n} \cos(n\omega t - 2\pi n\xi - \pi nD) \right)$</td>
</tr>
</tbody>
</table>

From the expressions of Table 2-2, several insights are determined:

**Insight:** Heterogeneity in the characteristic frequency is the only type of parameter heterogeneity that creates a damped expected power demand. As noted in previous works in the literature, damping is expected in heterogeneous systems [24]. Of the four analytic solutions presented in Table 2-2, damping only exists in the case of a distributed frequency parameter. This is through the $1/t$ term in front of the summation. For all other cases of univariate parameter heterogeneity, the dynamics remain purely oscillatory.

**Insight:** The insights developed from the mass-spring example are translatable to the TCL problem for the case of TCL frequency heterogeneity. The expression of Table 2-2 for TCL frequency heterogeneity parallels Equation (2-4). The insights from the mass-spring example are translated to the TCL problem:

- Heterogeneity in the TCL characteristic frequency creates damping in the expected power demand relating to a $1/t$ term.
- Two second-order LTV differential equations are able to characterize each Fourier series term $n$ for the case of frequency heterogeneity. Repeating the same steps as with the mass-spring system, two damping ratios are determined for each term:
\[ \zeta_{1,n}(t) = \frac{1}{n\omega_{\text{max}}t} \text{ and } \zeta_{2,n}(t) = \frac{1}{n\omega_{\text{min}}t}, \text{ applicable for } t > 0. \]

Truncating the Fourier series to \( N \) terms leads to \( 2N \) damping ratios describing the system.

- Table 2-2 presents two equivalent expressions for \( \bar{P}_{\text{TCL}}(t) \) in the case of frequency heterogeneity. The second of these expressions shows the potential for beating in the system, through the multiplication of the cosine term by a sine term.

Figure 2-4 presents a simulation of \( N_L = 10000 \) TCLs using Equation (2-18), with Fourier series parameters calculated using Table 2-1, \( T_0 = T_{sp}, s_0 = 1, \) and \([\omega_{\text{min}}, \omega_{\text{max}}] = [3.88 \text{ rad/h}, 4.74 \text{ rad/h}] \) (a \( \pm 10\% \) variation of the characteristic frequency). To determine the approximate peak of the beat, it is again approximated that the peak occurs at \( t = \frac{m\pi}{\omega_{\text{max}} - \omega_{\text{min}}} \) for \( m = 3, 5, \ldots \). At these times, the full summation term of the second expression for \( \bar{P}_{\text{TCL}}(t) \) in Table 2-2 is approximated as equal to 1. Subtracting out the average amplitude \( AD \), a comparison can be made between the approximate beat peak \( \bar{P}_{\text{TCL,peak}}(m) \) and the initial amplitude \( \bar{P}_{\text{TCL}}(0) \), as Equation (2-29) presents.

Note that this expression for beat peak amplitude is applicable only for the case of a uniform distribution of frequency.

\[ \frac{\bar{P}_{\text{TCL,peak}}(m) - AD}{\bar{P}_{\text{TCL}}(0) - AD} = \frac{2/(m\pi)}{\sum_{n=1}^{\infty} \frac{\sin(n\pi D)}{n} \cos(2\pi n(\zeta - \pi D))} \quad (2-29) \]

**Insight:** Only a few Fourier series terms are necessary to approximate the expected power demand for the case of frequency heterogeneity. For a distributed TCL frequency, each Fourier series term \( n \) parallels the \( y(t) \) expression of Equation (2-10). The amplitude of the Fourier series term is related to \( \frac{1}{n^2} \), indicating that as \( n \) increases, the magnitude of that term decreases. Additionally, it is known from the mass-spring system analysis that the decay rate from the initial amplitude to beat peak is \( \frac{2}{m\pi} \). The first term in the Fourier series, with a beat period of \( \frac{2\pi}{\omega_{\text{max}} - \omega_{\text{min}}} \), decays to \( \frac{2}{3\pi} \) of its initial amplitude at time \( t = \frac{3\pi}{\omega_{\text{max}} - \omega_{\text{min}}} \). By contrast, the third term in the Fourier series, with a shorter beat period of \( \frac{2\pi}{3(\omega_{\text{max}} - \omega_{\text{min}})} \), decays to \( \frac{2}{9\pi} \) of its initial amplitude at the same time. The larger \( n \) is, the faster the Fourier series term’s response decays. These two facts together indicate that after a few Fourier series terms, the approximate power demand approaches the true power demand.
Figure 2.4. Expected power demand for a uniform distribution of frequency with $\omega_{\text{min}} = 3.88 \text{ rad/h}$ and $\omega_{\text{max}} = 4.74 \text{ rad/h}$.

**Insight:** Increased levels of frequency heterogeneity lead to stronger damping and less apparent beating effects. Other studies note that increasing the heterogeneity level for a distribution of capacitance leads to stronger damping effects [24]. This insight applies for this system, as the TCL characteristic frequency is a function of the capacitance. Thus, increasing frequency heterogeneity is equivalent to increasing capacitance heterogeneity, which leads to the stronger damping effects. The approximate beat period, $\frac{2\pi}{\omega_{\text{max}} - \omega_{\text{min}}}$, decreases as frequency heterogeneity increases in magnitude. Relating to the mass-spring system, this leads to less apparent beating, as the beating phenomenon dies out faster.

A reasonable question to ask is: what is the benefit of the above insights in the context of demand response system design and control? On a basic level, these insights allow the increased understanding of the core behavior of the aggregate TCL power dynamics. This understanding is useful for a variety of applications. For example, suppose a grid operator desires to select TCLs from a population to implement demand response. The most desirable population is one that is able to respond to commanded trajectories as closely as possible. In a realistic scenario, the grid operator commands a step change in set point temperature to adjust power demand by a set
amount. This induces oscillations in the aggregate power demand before steady state is eventually reached. These oscillations, created through the synchronization and desynchronization of the TCLs, are not desirable to have in the aggregate power demand [22]. To eliminate these oscillations and reach steady state power demand quickly, a more damped system is desired. The operator is able to use the insights of this work, and select TCLs from the population that create the largest distribution of the frequency parameter. A more damped system is obtained, more suitable to the operator's needs. If desired, the TCLs can be selected to minimize or eliminate potential beating, an effect which represents increased TCL synchronization and desynchronization. In addition to TCL population selection, the damping ratios allow for better-informed controller and estimator design. By quantifying the damping strength in the system, the developer of the control algorithm is able to determine how aggressive the controller can be before introducing instability into the system.

**TCL Frequency Sensitivity**

The TCL frequency parameter is dependent on six of the underlying TCL model parameters: $C$, $R$, $P$, $T_{\infty}$, $\Delta_{db}$, and $T_{sp}$. Therefore, heterogeneity in any of these parameters leads to heterogeneity in the frequency parameter of the population. To understand the impact of each of these parameters on frequency, a sensitivity analysis is performed. Table 2-3 presents the sensitivity of the frequency parameter to each of these underlying parameters, evaluated at the nominal parameter values. The sensitivity of frequency to these parameters directly relates to the sensitivity of the damping ratio, and thus quantifies the impact variations of these parameters have on the aggregate power demand. Table 2-3 shows that a small change in $\Delta_{db}$ impacts the frequency substantially as compared to the other parameters. This makes sense – even a small change in the deadband substantially impacts the time it takes for a building to heat up or cool down. Frequency is moderately dependent on $C$, $R$, and $P$, but is not greatly impacted by $T_{\infty}$ and $T_{sp}$. This indicates that variations in these last two parameters do not substantially impact the damping characteristics of the system.
Table 2.3. Sensitivity of the TCL frequency parameter with respect to underlying TCL parameters, evaluated at nominal values.

<table>
<thead>
<tr>
<th>Sensitivity term</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\partial \omega}{\partial C}$</td>
<td>$-0.43 \frac{\text{rad}}{\text{C}^\circ}$</td>
</tr>
<tr>
<td>$\frac{\partial \omega}{\partial R}$</td>
<td>$-0.5 \frac{\text{rad}}{\text{kW} h^2}$</td>
</tr>
<tr>
<td>$\frac{\partial \omega}{\partial P}$</td>
<td>$0.23 \frac{\text{rad}}{\text{kWh}}$</td>
</tr>
<tr>
<td>$\frac{\partial \omega}{\partial T_{sp}}$</td>
<td>$0.09 \frac{\text{rad}}{\text{h}^\circ}$</td>
</tr>
<tr>
<td>$\frac{\partial \omega}{\partial \Delta_{db}}$</td>
<td>$-9.62 \frac{\text{rad}}{\text{h}^\circ}$</td>
</tr>
<tr>
<td>$\frac{\partial \omega}{\partial T_{\infty}}$</td>
<td>$-0.09 \frac{\text{rad}}{\text{h}^\circ}$</td>
</tr>
</tbody>
</table>

### 2.5 Heterogeneous TCL Population with Multivariate Distributions

This section presents the steps to determine the expected power demand for a population of TCLs with multivariate parameter distributions. By approximating the parameter distributions as piecewise uniform, an analytic expression for the power demand is determined for any type of parameter heterogeneity. Truncating the Fourier series to $N$ terms allows for a low-order model of the power demand. As with the previous section, this model allows for the representation of the system using LTV differential equations, which contain analytic formulas for the damping ratios that characterize damping in the power demand.

### Reduced-Order Model for Aggregate Power Demanded per TCL

While the parameter distributions selected in the previous section are useful for insight development, none are necessarily realistic. In order to model the expected power demand for more general cases, multiple parameters must be considered heterogeneous. Suppose one or more
of the Monte Carlo model parameters, \( \vec{\varphi} = [C, R, T_w, P, \ldots] \), are heterogeneous. Each parameter is distributed in a way such that \( \int_{\varphi_{\min, i}}^{\varphi_{\max, i}} p\varphi_i(q_i) d\varphi_i = 1 \). In addition to any parameter distributions, there are pdfs that describe the initial conditions: \( pdf_{on}(T_0) \) is associated with the initial conditions for \( s_0 = 1 \), and \( pdf_{off}(T_0) \) is associated with the initial conditions for \( s_0 = 0 \). Together, \( \int_{T_{min}}^{T_{max}} (pdf_{on}(T_0) + pdf_{off}(T_0)) dT_0 = 1 \). By knowing the parameter and initial condition distributions, it is possible to obtain distributions for the Fourier series parameters, \( pdf_A(A), pdf_\omega(\omega), pdf_D(D), pdf_\xi(\xi) \), and an overall multivariate distribution \( pdf(A, \omega, D, \xi) \). Each Fourier series parameter varies between a minimum and maximum value. In order to find the power dynamics for the heterogeneous case, the product of the pdf and Equation (2-27) are integrated over the parameter range, as done in Equation (2-30). Obviously, if a parameter is not heterogeneous, it is not necessary to integrate over that parameter.

\[
\bar{P}_{TCL}(t) = \int_{\omega_{\min}}^{\omega_{\max}} \int_{\xi_{\min}}^{\xi_{\max}} \int_{D_{\min}}^{D_{\max}} \int_{A_{\min}}^{A_{\max}} P_{TCL}(t) pdf(A, \omega, D, \xi) d\omega dD d\xi dA
\]

Equation (2-30) presents a general method to determine the aggregate power demand per TCL, \( \bar{P}_{TCL}(t) \). However, without a known form for the pdf, it is impossible to determine if this method always yields a closed-form solution. In order to guarantee this, the pdf is approximated as piecewise uniform. While other shapes can be used to approximate the pdf, the literature notes that different distribution shapes are not impactful on the oscillatory and damped behavior of the aggregate power demand [24]. To determine the piecewise uniform approximation of \( pdf(A, \omega, D, \xi) \), a two step process is outlined:

1. Discretize the ranges of each Fourier series parameter. For example, \( \omega \) is split into \( i = 1, \ldots, M_\omega \) pieces. Each piece has a minimum value \( \omega_{\min, i} \) and a maximum value \( \omega_{\max, i} \). The length of the piece is \( \Delta \omega = \omega_{\max, i} - \omega_{\min, i} \).

2. Calculate the average value for this section. This can be accomplished by using a variety of methods, and Equation (2-31a) presents one option. Equation (2-31b) presents the expression for the piecewise uniform pdf for \( \omega_{\min} < \omega < \omega_{\max}, D_{\min} < D < D_{\max}, \xi_{\min} < \xi < \xi_{\max}, \) and \( A_{\min} < A < A_{\max} \).

\[
\gamma_{i,j,k,l} = \frac{\int_{A_{\min}}^{A_{\max}} \int_{\omega_{\min}}^{\omega_{\max}} \int_{D_{\min}}^{D_{\max}} \int_{\xi_{\min}}^{\xi_{\max}} pdf(A, \omega, D, \xi) d\omega dD d\xi dA}{\Delta \omega \Delta D \Delta \xi \Delta A}
\]

\[
pdf(A, \omega, D, \xi) = \gamma_{i,j,k,l}
\]
Using Equations (2-30) and (2-31) together, the expression for the expected power dynamics of heterogeneous TCLs is determined under the approximation of a piecewise constant pdf. Equation (2-32) gives the exact expression for this.

\[
\bar{P}_{TCL}(t) = \bar{P}_{TCL,avg} + \sum_{i=1}^{M_o} \sum_{n=1}^{N_i} \left[ \frac{1}{t} (\beta_1 e^{i\omega_{max,1}t} + \beta_2 e^{-i\omega_{max,1}t} - \beta_1 e^{i\omega_{min,1}t} - \beta_2 e^{-i\omega_{min,1}t}) \right]
\]

(2-32a)

\[
\bar{P}_{TCL,avg} = \sum_{i=1}^{M_o} \sum_{j=1}^{N_D} \sum_{k=1}^{M_D} \sum_{l=1}^{M_A} \frac{\gamma_{i,j,k,l}}{4} (A_{max,1}^2 - A_{min,1}^2) (D_{max,1}^2 - D_{min,1}^2) \cdot (\omega_{max,1} - \omega_{min,1})(\xi_{max,k} - \xi_{min,k})
\]

(2-32b)

\[
\beta_{1/2} = \frac{1}{16\pi^3 n^4} \sum_{j=1}^{M_D} \sum_{k=1}^{M_D} \sum_{l=1}^{M_A} [\gamma_{i,j,k,l}(A_{max,1}^2 - A_{min,1}^2)\rho_{1/2}]
\]

(2-32c)

\[
\rho_1 = \left( e^{-2\pi n D_{max,1}} - e^{-2\pi n D_{min,1}} + 2\pi n (D_{max,1} - D_{min,1}) \right)
\]

(2-32d)

\[
\rho_2 = \left( e^{2\pi n D_{max,1}} - e^{2\pi n D_{min,1}} - 2\pi n (D_{max,1} - D_{min,1}) \right)
\]

(2-32e)

Equation (2-32) presents a novel reduced-order model of the expected power demand of a population of heterogeneous TCLs. Note that while the Monte Carlo model is time-invariant, this low-order representation is LTV. This analytic model is ideal for both demand response controller design and state/parameter estimator design. High-order models present in the literature, such as the Monte Carlo model, are not suited for tractable development of controller or estimator algorithms. The low-order nature of the approximate model developed in this work allows for simplified controller or estimator design, and is a key feature of the model.

**Damping Ratios of the Reduced-Order Model**

The model provided for the piecewise uniform pdf case is ideal for simulation of the system. However, the differential equation representation of the system provides the damping ratios necessary to characterize the expected power demand's damping. To obtain the damping ratios, the same steps applied to the mass-spring system are followed. Equation (2-32a) is rewritten into the form of Equation (2-33):

\[
\bar{P}_{TCL}(t) = \bar{P}_{TCL,avg} + \sum_{i=1}^{M_o} \sum_{n=1}^{N_i} \left[ P_{1,i,n}(t) - P_{2,i,n}(t) \right]
\]

(2-33a)

\[
P_{1,i,n}(t) = \frac{1}{t} (\beta_1 e^{i\omega_{max,1}t} + \beta_2 e^{-i\omega_{max,1}t})
\]

(2-33b)
\[ P_{2,i,n}(t) = \frac{1}{t} \left( \beta_1 e^{i n \omega_{\min} t} + \beta_2 e^{-i n \omega_{\min} t} \right) \]  \quad (2-33c)

Two second-order, LTV differential equations are associated with each Fourier series term \( n \) for each piece \( i \). In total, \( 2M_\omega N \) LTV differential equations exist to describe the system. The same steps are repeated as with the mass-spring case: multiply \( P_{1,i,n}(t) \) and \( P_{2,i,n}(t) \) by \( t \), and take a second-order derivative with respect to time. Equation (2-34) presents the differential equations for these two terms. The damping ratios from these differential equations are

\[ \zeta_{1,i,n}(t) = \frac{1}{n \omega_{\max} t} \]  and \[ \zeta_{2,i,n}(t) = \frac{1}{n \omega_{\min} t} \].

\[ \ddot{P}_{1,i,n}(t) + \frac{2}{t} \dot{P}_{1,i,n}(t) + \left( n \omega_{\max} t \right)^2 P_{1,i,n}(t) = 0 \]  \quad (2-34a)

\[ \ddot{P}_{2,i,n}(t) + \frac{2}{t} \dot{P}_{2,i,n}(t) + \left( n \omega_{\min} t \right)^2 P_{2,i,n}(t) = 0 \]  \quad (2-34b)

### 2.6 Example Population of TCLs

This section validates the reduced-order power demand model by comparing simulated results against the Monte Carlo model. Realistic parameter distributions are used to represent the population of heterogeneous TCLs, including initial conditions representative of a demand response event. The Fourier series is truncated to different levels in order to determine the necessary number of terms to accurately represent the system. The damping ratios are plotted for this example case to show the change in the damping effect over time.

#### Population Distribution

In order to provide a realistic example, an appropriate distribution for the capacitance parameter must be utilized. In this example, capacitance is distributed normally, but limited between \( C_{\min} \) and \( C_{\max} \). Equation (2-35) presents \( pdf_C(C) \), with \( C_{avg} \) as the average capacitance, \( C_{\min} = 0.9 C_{avg} \), \( C_{\max} = 1.1 C_{avg} \), \( \sigma = 0.5 \frac{\text{kWh}}{^\circ \text{C}} \) as the standard deviation, and \( c_1 = 0.023 \frac{^\circ \text{C}}{\text{kWh}} \) as the value to make \( \int_{C_{\min}}^{C_{\max}} pdf_C(C) = 1 \). For the initial conditions, it is desired to match initial conditions created by a demand response event. In this scenario, the population’s power demand is at steady state, indicating the TCLs are distributed uniformly from \( T_{\min} \) to \( T_{\max} \), with 43% of TCLs on [68]. The grid owner requires less power to be demanded, and forces
23% of all TCLs from on to off uniformly. After this demand response event occurs, \( pdf_{on}(T_0) = \frac{0.2}{T_{max} - T_{min}} \) and \( pdf_{off}(T_0) = \frac{0.8}{T_{max} - T_{min}} \). All nominal parameters for this simulation are those presented in Table 2-1.

\[
\begin{aligned}
    pdf_c(c) &= \begin{cases} 
        0 & \text{for } c < C_{min} \\
        \frac{1}{\sigma \sqrt{2\pi}} \exp\left(-\frac{(c - c_{avg})^2}{2\sigma^2}\right) + c_1 & \text{for } C_{min} < c < C_{max} \\
        0 & \text{for } c > C_{max}
    \end{cases} \\
\end{aligned}
\]  

(2-35)

With the parameters defined, it is possible to determine \( pdf(A, \omega, D, \xi) \), as Equation (2-36) expresses.

\[
\begin{aligned}
    pdf(A, \omega, D, \xi) &= pdf_\omega(\omega)pdf_\xi(\xi) \\
    pdf_\omega(\omega) &= \left(\frac{1}{\sigma \sqrt{2\pi}} \exp\left(-\frac{(\omega - \omega_{avg})^2}{2\sigma^2}\right) + c_1\right) \\
    &\cdot \left[\frac{2\pi}{\omega_{avg}^2}\right] \text{ for } \omega_{min} < \omega < \omega_{max} \\
    pdf_\xi(\xi) &= \begin{cases} 
        \frac{1}{T_{max} - T_{min}} |a_1(T_{max} - T_\infty + PR) \exp(a_1 \xi)|, & \xi_{min} < \xi < 0 \\
        \frac{1}{T_{max} - T_{min}} |a_1(T_{max} - T_\infty) \exp(a_1 \xi)|, & 0 < \xi < \xi_{max}
    \end{cases}
\end{aligned}
\]  

(2-36a)

\[
\begin{aligned}
    a_1 &= - \frac{\ln(T_{min} - T_{min} + PR)}{T_{max} - T_{min} + PR} + \frac{\ln(T_{max} - T_{min})}{T_{max} - T_{min} + PR}
\end{aligned}
\]  

(2-36b)

The steps of Section 2.5 are repeated to determine the piecewise uniform approximation of \( pdf(A, \omega, D, \xi) \). To develop the piecewise uniform approximation, \( M_\omega = 10, M_\xi = 20, \Delta \omega = 0.08\frac{rad}{s}, \Delta \xi = 0.0429 \) for \( \xi_{min} < \xi < 0 \) and \( \Delta \xi = 0.0571 \) for \( 0 < \xi < \xi_{max} \).

Reduction-Order Model Accuracy

With the piecewise uniform pdf determined, it is now possible to develop the exact expression for the approximate power demand. Equation (2-32a) holds true, and Equation (2-37) presents the expressions of the variables for this specific example. Figure 2-5 plots two of the dominant damping ratios of this example population, \( \zeta_{1, M_\omega 1}(t) = \frac{1}{\omega_{max} t} \) and \( \zeta_{2, M_\xi 1}(t) = \frac{1}{\omega_{min} t} \), both associated with Fourier series term \( n = 1 \). As the plot indicates, damping is initially very strong in the system at low \( t \), but the damping strength decreases substantially over time.

\[
\begin{aligned}
    P_{TCL,avg} &= \sum_{i=1}^{M_\omega} \sum_{k=1}^{M_\xi} Y_{i,k} AD(\omega_{max,i} - \omega_{min,i})(\xi_{max,k} - \xi_{min,k})
\end{aligned}
\]  

(2-37a)
As stated in Section 2.4, it is expected that only a few Fourier series terms are necessary to accurately capture the expected power demand. The approximate model, truncated to \( N = 1, 5, \) and 10, is compared against the Monte Carlo model with \( N_L = 10000 \) TCLs and a time step of 0.0025 h. Figure 2-6 plots the aggregate power demand (per TCL) for the Monte Carlo model and the \( N = 1 \) reduced-order model, and Figure 2-7 plots the error between the reduced-order models and the Monte Carlo model. From these plots, it is shown that the low-order approximate models are less accurate at the beginning of time, but the initial error decreases as \( N \) increases. A histogram of the error (Figure 2-8) shows that for all order approximate models, over 75\% of data points are within \( \pm 0.025 \frac{\text{kW}}{\text{TCL}} \) of the Monte Carlo model.
Figure 2-6. Expected power demand from the Monte Carlo model and the reduced-order model ($N = 1$).

Figure 2-7. Difference in expected power demand between the Monte Carlo model and the reduced-order models over time.
2.7 Summary of Insights and Contributions

This chapter presents an analysis of the damping found within the aggregate power demand of a population of parameter-heterogeneous TCLs. By applying the results of a simplistic population of oscillators to the more complex population of TCLs, the relationship relating TCL parameter heterogeneity and damping is expanded upon. A linear, time-varying, reduced-order model is developed to represent the aggregate demand per TCL for a wide range of parameter heterogeneities through the use of stochastic integrals. This model is ideal for the improved development of control or estimation algorithms utilized for demand response. The case studies and reduced-order model allow for the following contributed insights to the TCL demand response literature:

- Only heterogeneity in the characteristic frequencies of the TCLs introduces damping into the aggregate power demand.
- The time-varying damping ratios of the reduced-order model characterize the damping in the aggregate power demand.
- A potential beating effect exists in the aggregate power demand. The damping in the dynamics creates a decay in each beat amplitude, directly correlating with the beat period.
Chapter 3

Framework for the Analysis of Heterogeneity Dynamics in Battery Packs

3.1 Introduction

This chapter outlines the battery heterogeneity problem and the framework used to analyze pack heterogeneity growth and decay. There is a growing body of literature which indicates that capacity, resistance or temperature heterogeneity [7–11,30] increases the rate at which battery packs degrade and limit pack performance. Several studies take action on this problem, and develop control algorithms that can remove the impact from capacitance and resistance heterogeneity [12,58] and temperature heterogeneity [59–62,72]. These studies indicate the value of analyzing heterogeneity dynamics within a battery pack. While several studies provide such analyses [7–10,55,56], there is a noticeable gap in the literature: the absence of studies determining core insights through a model representative of the heterogeneity dynamics. This chapter addresses this gap by presenting a framework for examining the coevolution of charge, maximum capacity, and temperature imbalance within a battery pack. Within this chapter, the first two steps of this framework are applied in order to reduce an electrochemical battery model into a form suitable for analysis. The remaining steps of the framework are applied in Chapters 4 and 5 for two different cases. The first case analyzes charge and capacity heterogeneity and develops a novel control algorithm to balance capacity heterogeneity. The second case analyzes charge and temperature heterogeneity to determine the impact of balancing one on the other.

The sections of this chapter are organized as follows. The framework to analyze cell-to-cell heterogeneity is presented (Section 3.2). Following the first step of the framework, an electrochemical battery model with the desired degradation and temperature dynamics is selected (Section 3.3). The second step of the framework is applied, and the model is reduced into a form suitable for analysis (Section 3.4). A brief summary ends the chapter (Section 3.5).
3.2 Framework for Analysis of Imbalance Behavior

This section provides the framework used to determine insights into the growth or decline of heterogeneity within a pack of battery cells. The framework is appropriate for a variety of cell models with or without balancing hardware. The five main steps of the framework are as follows:

1. **Select the electrochemical cell model with appropriate dynamics.** The model is selected to be representative of the individual battery cells within the pack, and is the basis for analysis.

2. **Reduce the cell model complexity.** Most electrochemical models contain many algebraic constraints and dynamics that intensify the complexity of subsequent mathematical analyses. Appropriate approximations are made to allow for straightforward application of control and system theory techniques, while retaining a model with physical meaning.

3. **Separate pack dynamics into two subspaces.** The states describing charge, capacity loss, resistance growth and temperature are separable into (i) a subspace of the sum of the values for all cells and (ii) a subspace that describes the differences between the values of the cells.

4. **Apply a Taylor series to both subspaces.** This technique linearizes the dynamics of the differences subspace. An LTV model of the heterogeneity dynamics is produced which is suitable for determining the growth or decay of imbalance in the pack.

5. **Analyze the stability of the differences subspace model using linear system theory.** Linear system theory makes it possible to analyze the stability of both linear time-invariant and LTV systems. Applying this, conclusions are drawn about the stability of the subspace containing discrepancies in state variables among imbalanced cells, and thus whether heterogeneity grows or decays with time. This is applicable for packs without balancing or with voltage balancing, or for packs with multivariable state feedback balancing.

The first two steps of this framework are applied in the remainder of Chapter 3. Chapters 4 and 5 apply the remaining steps of the framework.
3.3 Battery Cell Model

This section presents the basics of battery components and operation, and outlines the electrochemical model used in the remainder of this dissertation. Lithium-ion batteries contain five major components: the positive and negative electrodes, the electrolyte, the separator, and the current collectors (Figure 3-1). During charging, lithium ions deintercalate from the positive electrode, which is a mass of solid particles typically made of a lithium metal oxide. The ions move through the liquid or gel electrolyte and past the separator, intercalating into the solid particles of the negative electrode, typically a lithiated carbon. Electrons cannot move through the electrolyte or separator, and instead travel to the negative electrode using the current collectors. The energy state of the negative electrode is higher than that of the positive electrode, allowing the battery to store energy. During discharge, the opposite process occurs, with electrons flowing through the external circuit connected to the current collectors to generate a current, and with the potential difference of the two electrodes generating a voltage.

For step #1 of the framework, the single-particle model with thermal effects (SPM-T) is used to represent a single battery cell. This model approximates electrolyte concentration as constant, and the particles of each electrode are represented as uniformly-sized spheres [73]. The SPM and SPM-T are known to provide accurate representations of the physics behind Li-ion battery dynamics at currents up to 1C [74–76]. Li-ion battery cells degrade through several mechanisms, including lithium plating and fracturing [32,77,78]. For cells with graphite anodes,
solid-electrolyte interphase (SEI) film layer growth substantially impacts the cycle-life of the cell [77], and consideration of this degradation mechanism is important when attempting to improve cycle-life. This work uses the model presented by Ramadass et al. [40] to represent SEI layer growth, though the framework allows for expansion to include different growth models, such as the one proposed by Safari et al. [79]. Parameters for the SPM-T are defined in Table 3-1, which also provides the parameter values used in this work [80–82]. Table 3-2 defines the degradation-related parameters along with their values [79,81]. The negative electrode is composed of lithium graphite, and its reference potential, \( U_{\text{ref}, n} \), is defined from Randall et al. [83]. The positive electrode is composed of LiFePO\(_4\), and its reference potential, \( U_{\text{ref}, p} \), is determined by adding \( U_{\text{ref}, n} \) to the open-circuit voltage curve of Docimo et al. [84].

Equation (3-1) defines the relationship between the input current \( I \) (positive for discharge) and the current densities \( J_{\text{tot}, i} \), for \( i = n, p \) (negative and positive electrode respectively). In the negative electrode, there is a side current density \( J_s \) that relates to the loss of active material to the SEI layer. The intercalation process in the solid particles of each electrode is represented by spherical diffusion, provided in Equation (3-2), with \( c_i \) as the concentration of lithium ions in that electrode.

\[
J_{\text{tot}, n} = J_n + J_s = \frac{I}{A_{n}\bar{n}} \\
J_{\text{tot}, p} = J_p = -\frac{I}{A_p\bar{p}} \\
\frac{\partial c_i}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 D_i \frac{\partial c_i}{\partial r} \right) \\
\left. \frac{\partial c_i}{\partial r} \right|_{r=R_i} = 0 \\
D_i \left. \frac{\partial c_i}{\partial r} \right|_{r=R_i} = -\frac{I_i}{a_i F} \\
\]

Equation (3-3a) gives the Butler-Volmer equation, which relates the main current density \( J_i \) of each electrode to the overpotential \( \eta_i \), surface concentration \( c_{s,i} \) and temperature. Equation (3-3b) shows the rate constant \( k_i \)'s dependence on temperature through Arrenhius’ correlation [76]. Equation (3-4) provides the relationship between \( J_s \) and the side reaction overpotential \( \eta_s \).

\[
J_i = a_i k_i (c_e)^{\alpha_{a,i}}(c_{\text{max},i} - c_{s,i})^{\alpha_{a,i}}(c_{s,i})^{\alpha_{c,i}} \left( \exp \left( \frac{a_{a,i} F}{R_a T} \eta_i \right) - \exp \left( -\frac{a_{c,i} F}{R_a T} \eta_i \right) \right) \\
k_i = k_{\text{ref}, i} \exp \left( \frac{E_{\text{a,i}}}{R_a \left( \frac{1}{T} - \frac{1}{T_{\text{ref}}} \right)} \right) \\
J_s = -i_{o,s} a_n \exp \left( -\frac{a_{c,n} F}{R_a T} \eta_s \right) \\
\]

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Table 3-1. Battery cell electrochemical and thermal parameters [80–82].

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Neg. Electrode</th>
<th>Pos. Electrode</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thickness, $L$ (cm)</td>
<td>$3.4 \cdot 10^{-3}$</td>
<td>$7.0 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>Particle radius, $R$ (cm)</td>
<td>$3.5 \cdot 10^{-4}$</td>
<td>$3.65 \cdot 10^{-6}$</td>
</tr>
<tr>
<td>Active material volume fraction, $\varepsilon$</td>
<td>0.55</td>
<td>0.41</td>
</tr>
<tr>
<td>Maximum solid phase concentration, $c_{\text{max}}$</td>
<td>$31.07 \cdot 10^{-3}$</td>
<td>$22.806 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>Stoichiometry at 0% SOC, $\theta_{0%}$</td>
<td>0</td>
<td>0.76</td>
</tr>
<tr>
<td>Stoichiometry at 100% SOC, $\theta_{100%}$</td>
<td>0.80</td>
<td>0.03</td>
</tr>
<tr>
<td>Average electrolyte concentration, $c_e$</td>
<td>$1.2 \cdot 10^{-3}$</td>
<td></td>
</tr>
<tr>
<td>Charge transfer coefficient, $\alpha_a, \alpha_c$</td>
<td>0.5, 0.5</td>
<td>0.5, 0.5</td>
</tr>
<tr>
<td>Solid phase Li diffusion coefficient, $D$</td>
<td>$5.29 \cdot 10^{-11}$</td>
<td>$1.18 \cdot 10^{-14}$</td>
</tr>
<tr>
<td>Reference rate constant, $k_{\text{ref}} \left( \frac{\text{cm}^2\text{s}}{\text{mol}^2\text{s}} \right)$</td>
<td>0.0790</td>
<td>1.0614 $\cdot 10^{-4}$</td>
</tr>
<tr>
<td>Reaction rate activation energy, $E_a_r \left( \frac{1}{\text{mol}} \right)$</td>
<td>$40 \cdot 10^{3}$</td>
<td>$25 \cdot 10^{3}$</td>
</tr>
<tr>
<td>Area, $A$ (cm$^2$)</td>
<td>3580</td>
<td>3487</td>
</tr>
<tr>
<td>$\alpha$ (cm$^{-1}$)</td>
<td>$3\varepsilon_n/R_n$</td>
<td>$3\varepsilon_p/R_p$</td>
</tr>
<tr>
<td>Faraday’s constant $F \left( \frac{\text{C}}{\text{mol}} \right)$</td>
<td>96487</td>
<td></td>
</tr>
<tr>
<td>Contact and unmodeled resistance, $R_c$ (\Omega)</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Thermal capacitance, $mC_p$ (J/K)</td>
<td>48.52</td>
<td></td>
</tr>
<tr>
<td>Reference cell temperature, $T_{\text{ref}}$ (K)</td>
<td>298.15</td>
<td></td>
</tr>
<tr>
<td>Ambient air temperature, $T_{\infty}$ (K)</td>
<td>298.15</td>
<td></td>
</tr>
<tr>
<td>Universal gas constant, $R_u \left( \frac{1}{\text{mol} K} \right)$</td>
<td>8.314</td>
<td></td>
</tr>
<tr>
<td>Thermal eigenvalue, $\lambda$ (s$^{-1}$)</td>
<td>$1.7 \cdot 10^{-3}$</td>
<td></td>
</tr>
</tbody>
</table>
Table 3-2. Degradation parameters for a lithium iron phosphate battery cell [79,81].

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Side reaction equilibrium potential, $U_{ref,s}$ (V)</td>
<td>0.4</td>
</tr>
<tr>
<td>Side reaction exchange current density, $i_{o,s}$ (A cm$^{-2}$)</td>
<td>$3.9 \cdot 10^{-12}$</td>
</tr>
<tr>
<td>SEI layer Molar mass, $M_p$ (kg mol$^{-1}$)</td>
<td>0.162</td>
</tr>
<tr>
<td>SEI layer density, $\rho_p$ (kg cm$^{-3}$)</td>
<td>$1690 \cdot 10^{-6}$</td>
</tr>
<tr>
<td>SEI ionic conductivity, $\kappa_p$ (S cm$^{-1}$)</td>
<td>0.0575</td>
</tr>
<tr>
<td>Initial SEI layer resistance, $R_{SEI}$ (Ω m$^2$)</td>
<td>0.001</td>
</tr>
</tbody>
</table>

The overpotentials further relate to the difference in potential between the solid and electrolyte of each electrode, $\Delta\phi_i = \phi_i - \phi_e$, as Equation (3-5a) presents. Equation (3-5b) presents a similar expression for $\eta_s$.

\[
\eta_i = \Delta\phi_i - U_{ref,s}(c_{s,i}) - \frac{J_{tot,i}}{\alpha_i}R_{film,i}
\]

\[
\eta_s = \Delta\phi_n - U_{ref,s} - \frac{J_{tot,n}}{\alpha_n}R_{film,n}
\]  

(3-5a) 

(3-5b)

The output voltage $V$ of the battery cell relates to the difference in solid potentials $\phi_i$ and any extra resistance $R_c$ due to connections and unmodeled effects:

\[
V = \phi_p - \phi_n - R_c I
\]  

(3-6)

Equation (3-7) presents the temperature dynamics of cell $j$, approximated by a first-order lumped capacitance model [76]. The first term, $\lambda(T_\infty - T)$, represents convective heat transfer between the cell and the surrounding fluid. The second term represents reversible heat generation, relating to the surface concentration-dependent entropy coefficient $\mu = \frac{\partial U_{ref,p}}{\partial T} - \frac{\partial U_{ref,n}}{\partial T}$. This term is not typically considered in the charge/temperature balancing literature, but it has a specific importance— for low discharge rates and a positive entropy coefficient, it is possible to cool a cell electrically. The third term corresponds to irreversible heat generation relating to the resistance terms, and the fourth term $\gamma_j$ is an additional coupling term that models heat transfer between cells within the battery pack. This fourth term is neglected hereafter, but its inclusion does not substantially alter the derivations or conclusions presented in this dissertation.

\[
\dot{T}_j = \lambda(T_\infty - T) - \frac{\pi}{mC_p}\mu - \frac{1}{mC_p}(\eta_p - \eta_n - \left(\frac{R_{film,p}}{A_pL_p\alpha_p} + \frac{R_{film,n}}{A_nL_n\alpha_n} + R_c\right)I) + \gamma_j
\]  

(3-7)
For degradation, Equation (3-8a) expresses how $J_s$ relates to the rate of change of capacity loss $Q_{\text{loss}}$. Equation (3-8b) shows how $J_s$ also relates to growth of the film thickness $\delta_{\text{film},n}$ in the negative electrode. The film thickness growth then directly relates to a change in resistance of the film layer, as Equation (3-9) shows. For the remainder of this dissertation, the positive electrode’s film layer resistance is $R_{\text{film},p} = 0$. 

$$Q_{\text{loss}} = -L_n A_n J_s$$ \hspace{1cm} (3-8a)

$$\delta_{\text{film},n} = -\frac{M_p}{\alpha_n \rho_p F} J_s$$ \hspace{1cm} (3-8b)

$$R_{\text{film},n} = R_{\text{SEI}} + \frac{\delta_{\text{film},n}}{\kappa_p}$$ \hspace{1cm} (3-9)

### 3.4 Reduced-Order Cell Model

Applying step #2 of the framework, the complexity of the cell model is reduced to simplify later analysis. Five tasks are set for this: (1) Algebraic relationships are solved. (2) Resistance growth is represented exactly as a function of capacity loss. (3) Surface concentration is approximated by the average concentration and negative electrode concentration is represented using positive electrode concentration. (4) Positive electrode concentration is related to the charge of the battery. (5) The entropy coefficient is related to the charge of the battery cell.

There are several algebraic relationships between the parameters, and complexity of the model’s representation is substantially reduced by solving these expressions. Equation (3-10) presents the solution for $\eta_p$ by combining Equations (3-1b) and (3-3a) for the case of $\alpha_{a,p} = \alpha_{c,p} = 0.5$.

$$\eta_p = -\frac{2R_u}{F} \sinh^{-1}\left(\frac{1}{2A_p L_p a_p k_p (c_e c_s, c_{max, p} - c_{s,p})^{\frac{1}{2}}}ight)$$ \hspace{1cm} (3-10)

Under the approximations of the SPM and $\alpha_{a,n} = \alpha_{c,n} = 0.5$, it is possible to determine a simplified expression for $J_s$ that can be solved iteratively [83] or algebraically [85]. Equation (3-11) presents an equivalent solution to the algebraic expression for $J_s$ presented by Plett [85]. As this solution shows, $J_s$ is a function of both $c_{s,n}$ and $I$. The solution to $J_s$ allows the determination of $\eta_n$ and $\eta_s$, and with Equations (3-10) and (3-11), $\phi_p - \phi_n$ is determinable, and thus so is the voltage output.
\[ J_s = \left( 1 - \frac{2f_2(c_{s,n}) + i_{o,s}a_n}{2(f_2(c_{s,n}) + i_{o,s}a_n)} \right) \frac{I}{\Lambda_n L_n} + \frac{-i_{o,s}a_n}{2(f_2(c_{s,n}) + i_{o,s}a_n)} \sqrt{\left( \frac{I}{\Lambda_n L_n} \right)^2 + 4f_1(c_{s,n})(f_2(c_{s,n}) + i_{o,s}a_n)} \]  

(3-11a)

\[ f_1(c_{s,n}) = a_n k_n \left( c_e c_{s,n}(c_{\text{max},n} - c_{s,n}) \right)^{0.5} \exp \left( -\frac{F}{2R_u T} (U_{\text{ref},s} - U_{\text{ref},n}(c_{s,n})) \right) \]  

(3-11b)

\[ f_2(c_{s,n}) = a_n k_n \left( c_e c_{s,n}(c_{\text{max},n} - c_{s,n}) \right)^{0.5} \exp \left( -\frac{F}{2R_u T} (U_{\text{ref},s} - U_{\text{ref},n}(c_{s,n})) \right) \]  

(3-11c)

The relationship between capacity loss and resistance growth is exploited to remove redundant expressions. Equation (3-8a) and (3-8b) show that \( Q_{\text{loss}} \) and \( \delta_{\text{film,n}} \) both relate to \( J_s \). By integrating these equations from 0 to \( t \) and equating, \( \delta_{\text{film,n}} \) can be calculated as a function of initial film layer thickness \( \delta_{\text{film,n}}(0) \), initial capacity loss \( Q_{\text{loss}}(0) \), and the capacity loss at time \( t \):

\[ \delta_{\text{film,n}}(t) = \delta_{\text{film,n}}(0) + \frac{M_p}{\Lambda_n A_n \rho_p F} (Q_{\text{loss}}(t) - Q_{\text{loss}}(0)) \]  

(3-12)

Equation (3-12) explicitly relates capacity degradation to resistance growth in this model, so by knowing one, the other can be calculated.

Though the framework is designed to include a variety of electrochemical dynamics, a volume average of the concentration within each electrode is used. This substantially reduces mathematical complexity in the analysis without impacting the final results and messages of this work significantly. A corroborate of this, as well as a method to include diffusion dynamics within an analysis, is presented in the appendix of this dissertation. For this chapter, the surface concentration is approximated as equal to the average concentration, i.e. \( c_{s,i} = \bar{c}_i \). Equation (3-13) presents the governing equation for the positive electrode’s average concentration using a first-order Padé approximation [86].

\[ \dot{\bar{c}}_p = \frac{1}{F \varepsilon_p A_p \rho_p} I \]  

(3-13)

As lithium is conserved within the battery, the total number of moles of lithium \( n_{Li} \) can be related to the concentrations of the electrodes [87]. Equation (3-14a) presents this relationship, augmenting it to include the moles of lithium lost to the SEI layer, \( \frac{Q_{\text{loss}}}{F} \). The value of \( n_{Li} \) is easily calculated using the electrode stoichiometries of a healthy cell at 0% SOC. Equation (3-14b) rewrites this expression to remove the need for a dynamic expression for \( \bar{c}_n \).

\[ n_{Li} = \varepsilon_p L_p A_p \bar{c}_p + \varepsilon_n L_n A_n \bar{c}_n + \frac{Q_{\text{loss}}}{F} \]  

(3-14a)

\[ \bar{c}_n = \frac{1}{\varepsilon_n L_n A_n} \left( n_{Li} - \varepsilon_p L_p A_p \bar{c}_p - \frac{Q_{\text{loss}}}{F} \right) \]  

(3-14b)
As this work analyzes the charge level of batteries, it is ideal to relate the concentrations of the electrodes to a charge level $x$. Equation (3-15) presents an affine relationship outlining how $\bar{c}_p$ translates to this charge level. Equation (3-16) presents the dynamic equation for $x$, with the initial charge $x(0)$ relating to initial concentration $\bar{c}_p(0)$. This choice of representation for charge level meets two important criteria: for a healthy cell ($Q_{\text{loss}} = 0$), $x = 0$ for a cell at 0% SOC and $x = Q_{\text{nom}}$ for a cell at 100% SOC. The variable $Q_{\text{nom}}$, the capacity of a healthy cell, is calculated using Equation (3-17).

$$\bar{c}_p = -\frac{1}{F\varepsilon_p A_p L_p} x + \theta_{0\%p} c_{\text{max},p}$$ \hspace{1cm} (3-15)

$$\dot{x} = -I, \quad x(0) = F\varepsilon_p A_p L_p \left( \theta_{0\%p} c_{\text{max},p} - \bar{c}_p(0) \right)$$ \hspace{1cm} (3-16)

$$Q_{\text{nom}} = F\varepsilon_p A_p L_p \left( \theta_{0\%p} - \theta_{100\%p} \right) c_{\text{max},p}$$ \hspace{1cm} (3-17)

By using charge dynamics, the entropy coefficient becomes a function of the charge of the battery. The entropy coefficient is approximated by a polynomial with parameters defined in Table 3-3 [80]:

$$\mu = \mu_0 + \mu_1 \left( \frac{x}{Q_{\text{nom}}} \right) + \mu_2 \left( \frac{x}{Q_{\text{nom}}} \right)^2 + \mu_3 \left( \frac{x}{Q_{\text{nom}}} \right)^3 + \mu_4 \left( \frac{x}{Q_{\text{nom}}} \right)^4$$ \hspace{1cm} (3-18)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_0$ (V/K)</td>
<td>$-3.23 \cdot 10^{-4}$</td>
</tr>
<tr>
<td>$\mu_1$ (V/K)</td>
<td>$5.89 \cdot 10^{-4}$</td>
</tr>
<tr>
<td>$\mu_2$ (V/K)</td>
<td>$3.03 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>$\mu_3$ (V/K)</td>
<td>$-7.11 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>$\mu_4$ (V/K)</td>
<td>$3.93 \cdot 10^{-4}$</td>
</tr>
</tbody>
</table>

Table 3-4 presents the final, simplified model for a single battery cell. The function $h(x, T, Q_{\text{loss}}, I)$ represents the capacity loss dynamics, $h_T(x_j, T_j, Q_{\text{loss},j}, I_j)$ represents the temperature dynamics excluding $\gamma_j$ for cell $j$, and $g(x, T, Q_{\text{loss}}, I)$ represents the voltage output.
3.5 Summary

This chapter describes a framework to determine the influence balancing strategies have on the charge, capacity, resistance and temperature imbalances between cells in a lithium-ion battery pack. The framework provides a step-by-step process to develop a model representative of heterogeneity dynamics within a battery pack which can be used for stability analysis. Within this chapter, the first two steps of the framework are applied. First, an electrochemical model with temperature and degradation dynamics is selected to be representative of a battery cell. Second, model reduction is applied to reformulate the model in order to simplify complexities for analysis of heterogeneity behavior. This sets up for the application of the remainder of the framework to two different studies contained in Chapters 4 and 5.
Table 3-4. Summary of the reduced-order battery model.

| Charge dynamics: | $\dot{x} = -l$ |
| Temperature dynamics ($j = 1, 2, \ldots$ for each cell): | $T_j = \frac{h(T_j, T, Q_{loss, j}, I_j)}{mC_p} + \gamma_j$ |
| Capacity loss dynamics: | $\dot{Q}_{loss} = h(x, T, Q_{loss, j}) = -L_nA_n I_s$ |
| Concentration relationships: | $c_p = \frac{1}{p_f \rho F} - \frac{\theta \rho_c \rho_{c, p}}{x} + \frac{1}{\varepsilon_L A_n L_n}$ |
| Side reaction current density: | $J_s = \left(1 - \frac{2f_2 + i_o \alpha d_n}{f_2 + i_o \alpha d_n}\right) \frac{I}{A_n L_n}$ |
| Film layer resistance: | $\delta_{film,n}(t) = \delta_{film,n}(0) + \frac{M_p}{L_n A_n a_n \rho_f F} (Q_{loss}(t) - Q_{loss}(0))$ |
| Overpotential ($i = p$): | $\eta_p = \frac{2 R_u T}{F} \sinh^{-1} \left( \frac{-I}{A_p L_p} \right)$ |
| Overpotential ($i = n$): | $\eta_n = U_{ref, n} - U_{ref} \left(\varepsilon_n\right) = \frac{2 R_u T}{F} \ln \left( \frac{J_s}{i_o d_n} \right)$ |
| Entropy coefficient: | $\mu = \mu_0 + \mu_1 \left(\frac{x}{Q_{nom}}\right) + \mu_2 \left(\frac{x}{Q_{nom}}\right)^2 + \mu_3 \left(\frac{x}{Q_{nom}}\right)^3 + \mu_4 \left(\frac{x}{Q_{nom}}\right)^4$ |
| Rate constant ($i = n, p$): | $k_i = k_{ref} e^{\frac{E_{a_{ref}}}{R_u} \left(\frac{1}{T} - \frac{1}{T_{ref}}\right)}$ |
| Cell voltage: | $V = g(x, T, Q_{loss, j}) = \left(\eta_p + U_{ref, p}(\varepsilon_p) - \frac{I}{a_p A_p L_p} R_{film, p}\right)$ |

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Chapter 4
Multivariable State Feedback Control as a Foundation for Lithium-Ion Battery Pack Charge and Capacity Balancing

4.1 Introduction

This chapter applies step #3-5 of the framework from Chapter 3 to a battery pack with charge and capacity heterogeneity. In this study, cell temperatures are considered balanced and at a steady state temperature $T = 298.15$ K. This allows the framework to highlight an important insight: balancing cell voltage is severely limited in improving long-term pack performance. This insight motivates the development of a novel balancing strategy, one which balances cell capacities in addition to charge. As the degradation model presented in Chapter 3 directly relates capacity and resistance degradation, the balancing of capacities leads to simultaneous balancing of resistances as well. By removing heterogeneity in these quantities, the balancing algorithm is able to extend the lifespan of the battery pack. To the best of the author’s knowledge, no existing publication makes this assertion or explores the fundamental benefits of multivariable state feedback for the balancing of battery charge, capacity and resistance. The contributions of this chapter are archived [88]. The rest of this chapter presents the development of an LTV model to represent charge and capacity heterogeneity within the battery pack (Section 4.2), and analyzes the model for no balancing and voltage balancing strategies (Section 4.3), as well as multivariable balancing strategies (Section 4.4). The final section of this chapter ends with a summary of the insights and benefits of using state feedback to simultaneously balance charge, capacity and heterogeneity (Section 4.5).

4.2 A Mathematical Representation of Cell Imbalance

This section presents the batteries examined in this work, and covers step #3 and step #4 of the framework from Chapter 3. In order to increase the clarity of the mathematical analysis, the battery pack consists of two cells in series. The two cells have identical parameters, with the only
difference being the initial conditions for the two state variables. Because the two state variables represent an amount of charge and charge capacity, respectively, the heterogeneity in initial conditions reflects a multivariable imbalance scenario. The first cell, cell #1, is described by $x_1, Q_{loss,1}$, and $V_1$, while cell #2 is described by $x_2, Q_{loss,2}$, and $V_2$. The current $I$ is split into two signals: $u_1$ and $u_2$. The main current, $u_1$, is the current demanded by the pack’s application, and goes through both cells. The balancing current $u_2$ is the current between the two cells used to remove heterogeneity. With $u_2$, charge is shuttled from cell #2 into cell #1, making $I_1 = u_1 + u_2$ for cell #1 and $I_2 = u_1 - u_2$ for cell #2. Figure 4-1 presents a diagram of the battery string, with cells #1 and #2 identified.

![Diagram of battery string with cells #1 and #2 identified](image)

Figure 4-1. Cells #1 and #2, with application current $u_1$ and balancing current $u_2$.

This work is concerned with analyzing the differences between the charges and capacities (and thus resistances) between the cells. The representation of these cells is split into two subspaces, as per step #3 in the framework of Chapter 3:

1. The subspace of the sums. In this, $z_1 = x_1 + x_2$ and $q_1 = Q_{loss,1} + Q_{loss,2}$. These variables relate to the average behavior of the cells.
2. The subspace of the differences. In this, $z_2 = x_1 - x_2$ and $q_2 = Q_{loss,1} - Q_{loss,2}$. These variables relate to the differences, or imbalance, between the states of the cells.

With $z_1, z_2, q_1$ and $q_2$, the two battery cells are described in a manner that allows the direct representation of the heterogeneity to be analyzed. These four variables are referred to as states, as they represent the dynamics of the system. The dynamic equations, $\dot{x}_1, \dot{x}_2$, $\dot{Q}_{loss,1}$ and $\dot{Q}_{loss,2}$, are replaced with the equivalent representation of Equation (4-1), with function $h$ defined in Table 3-4. Note that $h$ no longer depends on $T$, which is considered constant in this chapter.

\[
\begin{align*}
\dot{z}_1 &= -2u_1 \quad (4-1a) \\
\dot{q}_1 &= h\left(\frac{x_1 + x_2}{2}, \frac{q_1 + q_2}{2}, u_1 + u_2\right) + h\left(\frac{x_1 - x_2}{2}, \frac{q_1 - q_2}{2}, u_1 - u_2\right) \quad (4-1b)
\end{align*}
\]
\[ \dot{z}_2 = -2u_2 \quad (4-1c) \]
\[ \dot{q}_2 = \dot{h} \left( \frac{z_1 + z_2}{2}, \frac{q_1 + q_2}{2}, u_1 + u_2 \right) - h \left( \frac{z_1 - z_2}{2}, \frac{q_1 - q_2}{2}, u_1 - u_2 \right) \quad (4-1d) \]

The voltages of the cells are broken up in a similar manner, with \( v_1 = V_1 + V_2 \), \( v_2 = V_1 - V_2 \), and function \( g \) defined in Table 3-4. Once again, \( g \) is no longer written as a function of \( T \), as it is considered constant. The expressions for \( v_1 \) and \( v_2 \) are:

\[ v_1 = g \left( \frac{z_1 + z_2}{2}, \frac{q_1 + q_2}{2}, u_1 + u_2 \right) + g \left( \frac{z_1 - z_2}{2}, \frac{q_1 - q_2}{2}, u_1 - u_2 \right) \quad (4-2a) \]
\[ v_2 = g \left( \frac{z_1 + z_2}{2}, \frac{q_1 + q_2}{2}, u_1 + u_2 \right) - g \left( \frac{z_1 - z_2}{2}, \frac{q_1 - q_2}{2}, u_1 - u_2 \right) \quad (4-2b) \]

Equations (4-1b), (4-1d), (4-2a) and (4-2b) are nonlinear expressions. If \( z_2 \), \( q_2 \) and \( u_2 \) are relatively small, they can be viewed as perturbations about \( z_1 \), \( q_1 \) and \( u_1 \), and a Taylor series is justifiable to approximate these expressions. Applying step #4 of the framework, Equations (4-3a) and (4-3b) present the first-order Taylor series for \( \dot{q}_1 \) and \( \dot{q}_2 \), and Equations (4-4a) and (4-4b) present the same for \( v_1 \) and \( v_2 \). In these expressions, the derivative terms are evaluated at \( X = \frac{z_1}{2} \), \( Q = \frac{q_1}{2} \), and \( U = u_1 \).

\[ \dot{q}_1 = 2\dot{h} \left( \frac{z_1}{2}, \frac{q_1}{2}, u_1 \right) \quad (4-3a) \]
\[ \dot{q}_2 = \frac{\partial h(X,Q,U)}{\partial x} z_2 + \frac{\partial h(X,Q,U)}{\partial q} q_2 + 2 \frac{\partial h(X,Q,U)}{\partial u} u_2 \quad (4-3b) \]
\[ v_1 = 2g \left( \frac{z_1}{2}, \frac{q_1}{2}, u_1 \right) \quad (4-4a) \]
\[ v_2 = \frac{\partial g(X,Q,U)}{\partial x} z_2 + \frac{\partial g(X,Q,U)}{\partial q} q_2 + 2 \frac{\partial g(X,Q,U)}{\partial u} u_2 \quad (4-4b) \]

Equations (4-1a), (4-3a), and (4-4a) show that \( z_1 \), \( q_1 \) and \( v_1 \) are independent of \( z_2 \) and \( q_2 \). To determine \( z_1 \) and \( q_1 \) as functions of time, only knowledge of the input current trajectory \( u_1 \) is necessary. However, Equations (4-3b) and (4-4b) show the dependence of \( q_2 \) and \( v_2 \) on \( z_1 \), \( q_1 \) and \( u_1 \). As \( z_1 \), \( q_1 \) and \( u_1 \) change over time, Equations (4-3b) and (4-4b) are LTV equations. This indicates two things: (i) The differences between cells do not impact the sums of the charges, capacity losses, or voltages. (ii) The imbalance between the cells is dependent on the trajectory of the sum (or average) of the charges and capacities.

Moving forward, the impact of the subspace of the sums on the subspace of differences is examined using the following time-varying parameters: \( a_1 = \frac{\partial h(X,Q,U)}{\partial x} = -\frac{\partial h(X,Q,U)}{\partial q} \), \( b_1 = 2 \frac{\partial h(X,Q,U)}{\partial u} \), \( c_1 = \frac{\partial g(X,Q,U)}{\partial x} \), \( c_2 = \frac{\partial g(X,Q,U)}{\partial q} \), and \( d_1 = 2 \frac{\partial g(X,Q,U)}{\partial u} \). Given these definitions,
Equations (4-5) and (4-6) present the dynamics of the imbalance of the system, with matrix \( A = \begin{bmatrix} 0 & 0 \\ a_1 & -a_1 \end{bmatrix} \).

\[
\begin{align*}
\{z_2\} &= \begin{bmatrix} 0 & 0 \\ a_1 & -a_1 \end{bmatrix} \{q_2\} + \begin{bmatrix} -2 \\ b_1 \end{bmatrix} u_2 \\
\{z_2\} &= \begin{bmatrix} c_1 & c_2 \end{bmatrix} \{z_2\} + d_1 u_2
\end{align*}
\] (4-5) (4-6)

4.3 Analysis of No Balancing and Voltage Balancing

This section presents an analysis of the charge and capacity imbalances of the two cells in the battery string, following step #5 of the framework. A case study for a battery pack without balancing hardware is analyzed to determine the stability of the charge and capacity imbalances. A second case study with analysis is provided for voltage balancing strategies. Simulated results are provided to fortify the insights determined in each case study.

No Balancing

The model of heterogeneity between cells in the battery pack, Equations (4-5) and (4-6), is propitious toward determining the growth or decay of \( z_2 \) and \( q_2 \). If these states approach equilibrium, such that \( \dot{z}_2 = 0 \) and \( \dot{q}_2 = 0 \), growth is not unbounded and the subspace is stable. If these states do not approach an equilibrium, the subspace is unstable and heterogeneity increases as time increases. In this first case study, an analysis is preformed to determine stability for a battery pack without any balancing, i.e. \( u_2 = 0 \). As Equation (4-5) is LTV, stability is determined using LTV theory. The traditional method to determine stability for LTV systems is to determine the state transition matrix [89]. However, because of the nature of matrix \( A \) in Equation (4-5), an alternative is to determine the eigenvalues of \( A \) by making it diagonal. The transformation matrix required for this diagonalization is constant, finite, and invertible: a fact that makes it possible to use this diagonalization as a foundation for LTV stability analysis. Equation (4-7) presents the diagonal matrix \( A_{diag} \), using coordinate transformation matrix \( P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \) [89].

\[
A_{diag} = PAP^{-1} + \dot{PP}^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & -a_1 \end{bmatrix}
\] (4-7)
From Equation (4-7), the eigenvalues of the subspace are 0 and $-a_1$. The first eigenvalue, 0, indicates that $z_2$ is marginally stable - without balancing, charge imbalance remains constant. The second eigenvalue, $-a_1$, indicates different results depending on its value over time. If $-a_1$ is a positive number on average over time, $q_2$ is unstable, and grows unbounded. If $-a_1$ is a negative number on average over time, $q_2$ is stable. The difference between capacity losses decays to an equilibrium value over time, with the decay rate relating to the magnitude of $-a_1$. Figure 4-2 plots $a_1$ as a function of $\bar{c}_n$ and $u_1$, showing the entire possible range of values for $a_1$. This plot shows that $-a_1 < 0$ for all combinations of $\bar{c}_n$ and $u_1$, but the magnitude is very small. This indicates that $q_2$ is stable, and does decay to an equilibrium value, but at a very slow rate.

![Figure 4-2. Range of values of $a_1$.](image)

While $q_2$ approaches its equilibrium, this does not necessarily imply that capacities balance over time. Solving for equilibrium using Equation (4-5) with $\dot{z}_2 = 0$ and $\dot{q}_2 = 0$, it is found that $z_2$ is a free variable in equilibrium (equal to $z_2(0)$) and $q_2 = z_2$ in equilibrium. Therefore, $q_2$ approaches a constant value in equilibrium that is not zero, so capacities do not balance. The main insights for the case of no balancing are described as follows:
**Insight:** In the case of no balancing, the difference between the charges of the two cells remains constant. The difference between the capacity losses slowly approaches a value equal to the difference between the charges, at a rate relating to $-a_1$.

Table 4-1 describes the different simulation tests used in this chapter for validation. Figure 4-3 presents the results of applying test #1 to two cells in a string and no balancing implemented, using the summarized model of Table 3-4 for each cell. For the initial conditions listed, this yields $z_2(0) = -0.04Q_{nom}$ and $q_2(0) = -0.01Q_{nom}$. Film thicknesses are chosen to be $\delta_{film,n,1}(0) = 0$ cm and $\delta_{film,n,2}(0) = 2.78 \cdot 10^{-6}$ cm to maintain that the voltage output equations are consistent between the two cells. The simulation is run with a time step size of 60 seconds. Figure 4-3a shows that charge imbalance remains a constant over time, while capacity imbalance approaches the same value as charge imbalance. As the magnitude of $a_1$ is very small, this rate is so slow that 20 years pass before $q_2$ becomes sufficiently close to its equilibrium. Figure 4-3b plots a voltage imbalance that wavers over time. This is due to the impact of resistance heterogeneity in Equation (4-6), specifically the variation of the $c_2q_2$ term. Figure 4-3c shows that in this time, for the model and parameters chosen, the average cell capacity loss reaches nearly 4 Ah, close to the initial capacity of the cell, $Q_{nom} = 4.4$ Ah. The plot of $\frac{Q_{loss}}{2}$ shows that after only 5 years, 50% of the initial capacity is lost. This is due in part to the high negative electrode concentrations at which the cells are maintained. A high concentration reduces surface potential, leading to an increase in side reaction current density, which induces more rapid SEI layer growth.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Test 1</th>
<th>Test 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_1$</td>
<td>0 A</td>
<td>$\frac{Q_{nom}}{3600}$ sgn $\left(\sin\left(\frac{\pi t}{1860}\right)\right)$</td>
</tr>
<tr>
<td>$x_1(0)$</td>
<td>0.95$Q_{nom}$</td>
<td>0.95$Q_{nom}$</td>
</tr>
<tr>
<td>$x_2(0)$</td>
<td>0.99$Q_{nom}$</td>
<td>0.99$Q_{nom}$</td>
</tr>
<tr>
<td>$Q_{loss,1}(0)$</td>
<td>0 As</td>
<td>0 As</td>
</tr>
<tr>
<td>$Q_{loss,2}(0)$</td>
<td>0.01$Q_{nom}$</td>
<td>0.05$Q_{nom}$</td>
</tr>
</tbody>
</table>
Figure 4-3. No balancing case study (a) charge and capacity imbalances, (b) voltage imbalance, and (c) average capacity of the cells.

**Voltage Balancing**

The second case study involves the use of voltage balancing between the two battery cells. In voltage balancing, the balancing current is related to the voltage difference between
imbalanced cells. One example of this is a proportional voltage feedback policy, where \( u_2 = K \nu_2 \). The gain \( K \) relates to the speed at which voltage is balanced, and is determined by the balancing hardware characteristics. Combining this expression with Equation (4-6), \( u_2 \) is solved for in terms of \( z_2 \) and \( q_2 \):

\[
u_2 = \frac{K}{1-Kd_1} [c_1 \ c_2] \{z_2 \ q_2\} \tag{4-8}
\]

Equation (4-9) presents the closed-loop dynamics for the case of voltage feedback, found by combining Equations (4-5) and (4-8).

\[
\begin{pmatrix}
\dot{z}_2 \\
\dot{q}_2
\end{pmatrix} =
\begin{pmatrix}
\frac{-2Kc_1}{1-Kd_1} & \frac{-2Kc_2}{1-Kd_1} \\
\frac{b_1Kc_1}{1-Kd_1} & -a_1 + \frac{b_1Kc_2}{1-Kd_1}
\end{pmatrix} \begin{pmatrix}
z_2 \\
q_2
\end{pmatrix}\tag{4-9}
\]

In order to analyze the stability of \( z_2 \) and \( q_2 \), knowledge of the physical system is exploited. The degradation process for a battery is much slower than its other dynamics [7]. From this, it is reasonable to assume the difference between capacities changes at a much slower rate than the difference between charges for cells with balancing. Under this condition, singular perturbation theory is applicable, allowing the separation of the fast dynamics and the slow dynamics. The main concepts behind singular perturbation theory are [90]:

1. The fast dynamics (\( \dot{z}_2 \)) view the slow state (\( q_2 \)) as a nearly constant input. The fast dynamics reach equilibrium (\( \dot{z}_2 = 0 \)) quickly. Equation (4-10) presents the equilibrium value of \( z_2 \) for the voltage balancing case study.

\[
z_2 = -\frac{c_2}{c_1} q_2 \tag{4-10}
\]

\[
\dot{q}_2 = -a_1 \left( 1 + \frac{c_2}{c_1} \right) q_2 \tag{4-11}
\]

Equation (4-10) shows that voltage balancing leads to a charge imbalance proportional to the capacity loss imbalance. The eigenvalue of Equation (4-11), \( -a_1 \left( 1 + \frac{c_2}{c_1} \right) \), is once again useful for determining the stability of the capacity difference state. Figure 4-4 plots \( \frac{c_2}{c_1} \) for the case of \( q_1 = 0 \), and varying values of \( z_1 \) and \( u_1 \). This shows \( \frac{c_2}{c_1} < 0 \), with \( \left| \frac{c_2}{c_1} \right| < 1 \) for the situation examined. This shows that \( q_2 \) remains stable, but approaches its equilibrium value over time at a rate slower than the case without any balancing, as \( \left| -a_1 \right| > \left| -a_1 \left( 1 + \frac{c_2}{c_1} \right) \right| \). Noteworthy, the gain \( K \) does not appear in the eigenvalue, indicating that the choice of voltage balancing strategy does
not impact the rate at which capacity imbalance approaches equilibrium. In contrast to the no balancing case study, solving Equation (4-11) for equilibrium yields \( q_2 = 0 \). The main insights from the voltage balancing case study are as follows:

**Insight:** *In the case of voltage balancing, the difference between the charges of the two cells is proportional to the difference between the capacities. The difference between the capacities (and resistances) decays to zero at a rate relating to \(-a_1 \left(1 + \frac{c_2}{c_1}\right)\). This rate is independent of the gain selected for voltage balancing.*

![Graph showing the range of values for \( \frac{c_2}{c_1} \) for no average capacity loss.](image)

Figure 4-4. Range of values for \( \frac{c_2}{c_1} \) for no average capacity loss.

Figure 4-5 presents a simulation of voltage balancing circuits with different \( K \) values, for test #1 in Table 4-1. The nominal gain, \( K = 0.104 \), is determined from the average balancing current of a flyback capacitor [42], with parameters from Speltino et al. [91]. Gain values one order of magnitude above and below are also used for comparison. Figure 4-5a shows that charge imbalance approaches equilibrium after a very brief period of time, confirming that the application of singular perturbation theory is justified. The value of \( z_2 \) is not constant, as \( \frac{c_2}{c_1} \) varies with time, but as \( 0 < \frac{c_2}{c_1} < 1 \), it is a small value. The capacity imbalance approaches zero as time increases, but because the rate is so slow, it takes over twenty years for this to decrease to a negligible value. Figure 4-5b shows that voltage imbalance disappears very quickly, as expected.
for a voltage balancing circuit. Figure 4-5c shows the average capacity loss of the cells to be near 4 Ah again. This shows that for capacity to become homogeneous within the pack, the cells are degraded beyond their useful life (i.e., capacity loss exceeds 20%).

Figure 4-5. Voltage balancing case study (a) charge and capacity imbalances, (b) voltage imbalance, and (c) average capacity of the cells.
4.4 Multivariable State Feedback Balancing

This section develops a new control algorithm that balances charge, capacity, resistance, and voltage within the lifespan of the battery pack. The new balancing strategy is analyzed, following step #5 of the framework for this new scenario. The results of the voltage balancing case study reveal that voltage balancing slowly reduces capacity and resistance heterogeneity.

Typically, a lithium-ion battery cell is declared dead after 20% of its initial capacity is lost [29]. In the voltage balancing case study, capacities balance after 91% of the cells’ initial capacities are lost. To eliminate the differences in capacity/resistance in a period of time that is meaningful for improving pack performance, a novel cell balancing method is proposed based on the concept of state feedback. In state feedback, the balancing current is designed as a function of the two states $z_2$ and $q_2$, expressed in Equation (4-12).

$$u_2 = K_1 z_2 + K_2 q_2 \quad (4-12)$$

The gains, $K_1$ and $K_2$, are chosen in a manner to reach a desired performance from the battery pack’s management system. This type of balancing requires (i) more freedom over the balancing current and (ii) knowledge of $z_2$ and $q_2$ over time. The first requirement is met by utilizing more advanced balancing hardware already used by other balancing circuits, such as DC-DC converters [12]. The second requirement is realizable through the use of estimation algorithms, which exist for both charge/SOC [91–93] and health parameters directly relating to capacity [94–96]. Equation (4-12) is combined with Equation (4-5) to yield the closed-loop system with state feedback control:

$$\begin{bmatrix} \dot{z}_2 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -2K_1 & -2K_2 \\ a_1 + b_1 K_1 & -a_1 + b_1 K_2 \end{bmatrix} \begin{bmatrix} z_2 \\ q_2 \end{bmatrix} \quad (4-13)$$

As with voltage balancing, the dynamics of Equation (4-13) are split apart using singular perturbation theory. For $\dot{z}_2$, $-2K_2 q_2$ is viewed as a constant input. With this, the eigenvalue relating to $z_2$ is $-2K_1$. The gain $K_1$ is designed in a manner to determine how fast $z_2$ approaches its equilibrium value. Equation (4-14) presents the equilibrium value for $z_2$. The equilibrium value for $z_2$ is again placed into the $\dot{q}_2$ expression of Equation (4-13), yielding Equation (4-15).

$$z_2 = -\frac{K_2}{K_1} q_2 \quad (4-14)$$

$$\dot{q}_2 = -a_1 \left( 1 + \frac{K_2}{K_1} \right) q_2 \quad (4-15)$$
The eigenvalue relating to $q_2$, as determined from Equation (4-15), is $-a_1 \left(1 + \frac{K_2}{K_1}\right)$. To control the rate at which $q_2$ approaches equilibrium, $\frac{K_2}{K_1}$ must be designed to match the desired rate. As with voltage balancing, solving Equation (4-15) for equilibrium yields $q_2 = 0$.

The selection of $\frac{K_2}{K_1}$ conveys an important trade-off that exists when implementing state feedback. Increasing this value increases the rate at which capacity and resistance imbalance fades, but also increase the equilibrium value for $z_2$. This is an important insight: in order to quickly eliminate capacity or resistance imbalance, a charge imbalance must be temporarily induced. The weaker cell’s charge level is lowered to impede degradation, while the healthier cell’s charge is increased, leading to increased degradation. To exhibit the trade-offs associated with choosing $K_1$ and $K_2$, the choices for these values based on different objectives are listed:

1. If the priority is to reduce voltage imbalance, select $\frac{K_2}{K_1} = \frac{c_2}{c_1}$. This voltage balancing case study falls into this category of controller.
2. If the priority is to eliminate charge imbalance, a larger $K_1$ leads to faster charge balancing, and $K_2 = 0$. Capacity imbalance decays at a rate of $-a_1$, and so capacity imbalance takes does not disappear by the end of the battery pack’s life.
3. If the priority is to eliminate capacity and resistance imbalance, $K_1$ is designed to be just large enough to have $z_2$ approach equilibrium quickly. The gain $K_2$ is designed to increase $\frac{K_2}{K_1}$, and thus increase the rate of capacity heterogeneity decay. However, $K_2$ is also designed to maintain a realistic value for charge imbalance. This trade-off is important, as charge imbalance limits the effective capacity of the battery pack. This method eliminates charge and capacity heterogeneity within the lifespan of the battery pack.

For all of the listed objectives, it is possible to make the gains time-varying if extra flexibility is desired. In this, there is potential to balance multiple objectives: during battery pack operation, charge or voltage balancing should be the focus. During rest or minimal pack usage, capacity balancing should be the focus. This potentially offsets a challenge of making the priority to eliminate capacity imbalance, as removal of charge imbalance during operation eliminates the possibility of lowered effective pack capacity.

Figure 4-6 presents a simulation study for the state feedback balancing algorithm for test #1. The initial conditions and parameters are maintained from the previous case studies. The first gain value, $K_1 = 1.045 \cdot 10^{-5}$, is selected to be the equivalent for the voltage balancing case. The
second gain value, $K_2 = 3K_1$, is chosen to emphasize a focus on balancing capacity between the cells. The state feedback balancer is compared against the voltage balancing circuit with $K = 0.104$. Figure 4-6a shows that $z_2$ reaches its equilibrium value quickly, but the equilibrium value is a larger amount of charge heterogeneity as compared to the voltage balancing circuit. This drawback is subsided by the benefit of capacity balancing, as $q_2$ decays to 28% of its initial value in less than one year. This is a substantial improvement over the voltage balancer, as $q_2$ only decays to 75% of its initial value during the same time period. Figure 4-6b plots the voltage imbalance, and shows that the state feedback balancer leads to a larger voltage imbalance over the standard algorithm. This is rectified over time, as the charge and capacity imbalances decrease. Figure 4-6c plots the average capacity loss of the two cells over the simulation time period. As this shows, the state feedback balancing algorithm does not increase the average amount of degradation. The weakest cell in the pack, cell #2, is the limiting factor that determines pack lifespan. With multivariable state feedback control implemented, cell #2 reaches 80% of $Q_{nom}$ after 1.28 years. Again, this short amount of time is due in part to the high negative electrode concentrations at which the cells are maintained. However, this is still 2.2% longer than cell #2’s lifespan when implementing voltage balancing. As the two cells’ capacities converge while using multivariable control, this means that the lifespan of the battery pack is now limited by the average cell capacity, and not the weakest cell.

Figure 4-7 presents a simulation study for a pack under operation, using test #2 of Table 4-1. The main current, $u_1$, is a 1C square wave, and from this the cells charge up and down 50% of $Q_{nom}$, switching between discharging and charging every 30 minutes. With the initial conditions provided, this yields $z_2(0) = -0.04Q_{nom}$ and $q_2(0) = -0.05Q_{nom}$. The gains $K_1$ and $K_2$ remain the same as the previous study. Figure 4-7a plots the values for $z_2$ and $q_2$ after the completion of several cycles. This shows the state balancing algorithm still reduces capacity differences at a similar rate as before, and that the charge difference is still larger than that with voltage balancing. Figure 4-7b shows that the voltage imbalance for the state feedback controller is still greater than that in the voltage balancing case, with large fluctuations due to the oscillating current. Figure 4-7c shows the negligible difference in the average capacity loss between the two algorithms. Using the same method to calculate lifespan as with the previous study, there is a 9.2% increase in the lifespan of the battery pack using state feedback over the case using voltage feedback. This number is dependent on the selected current cycle, gain values, and underlying models, but provides an encouraging example as to the benefits of multivariable state feedback for balancing.
Figure 4-6. State feedback balancing case study ($u_1 = 0$) (a) charge and capacity imbalances, (b) voltage imbalance, and (c) average capacity of the cells.
Figure 4-7. State feedback balancing case study with cycled batteries ($u_1 \neq 0$) (a) charge and capacity imbalances, (b) voltage imbalance, and (c) average capacity of the cells.

4.5 Summary of Insights and Contributions

This chapter applies the last three steps of the framework of Chapter 3 to determine the influence balancing strategies have on the charge, capacity and resistance imbalances between
cells in a lithium-ion battery pack. By using control theory techniques, this procedure allows for
the determination of the rates at which these imbalances grow or decay. Applying the framework
to a single-particle model with SEI layer growth to represent degradation, it is determined that
voltage balancing strategies are ineffective in improving the removal of capacity and resistance
imbalance. A novel multivariable state feedback controller is developed using control theory, and
the framework is applied to this balancing algorithm to obtain several important insights:

- The proposed controller is able to remove charge, capacity and resistance imbalances
  within the lifespan of the battery pack. This is shown to increase the useful life of the
  pack by as much as 9.2% for the gain values and underlying model selected.
- Capacity imbalance decay rate is increased by inducing charge imbalances between
  the cells. This creates two competing objectives, charge and capacity balancing, that
  must be weighed when designing the controller.
- The proposed controller is capable of focusing on voltage, charge, or capacity
  balancing as determined by the needs of the battery pack.
Chapter 5

Analysis and Control of Charge and Temperature Imbalance within a Lithium-Ion Battery Pack

5.1 Introduction

This chapter applies steps #3-5 of the framework of Chapter 3 to a battery pack with charge and temperature heterogeneity. In this study, degradation is considered negligible (i.e., $Q_{loss} = 0$, $\delta_{fim,n} = 0$ and $J_s = 0$), and the negative electrode overpotential simplifies to the negative of the expression for $\eta_p$ in Table 3-4 with $i = n$ instead. Recent literature presents a potential option for temperature heterogeneity reduction through the use of existing hardware intended for charge balancing. Within a cell, an irreversible heating term exists related to the current squared. By using balancing hardware to manipulate the current through each cell, cooler cells can be heated faster than others, leading to temperature equalization within the battery pack [41,59–62]. The literature notes that there are some tradeoffs between charge and temperature balancing [41], and algorithms are designed to maintain low charge imbalances instead of outright elimination [59].

In order to show the potential tradeoffs between charge and temperature balancing, a formal analysis of how charge balancing impacts temperature heterogeneity, and vice-versa, is necessary. The primary goal of this chapter is address this gap in the literature, and answer the following primary questions: Is control of a common current through two cells able to assist in removing temperature heterogeneity? What are some of the conditions under which charge balancing helps remove or increases temperature heterogeneity? What are some of the conditions under which temperature balancing helps remove or increases charge heterogeneity? To the best of the author's knowledge, the literature has yet to examine these questions using a formal framework capable of providing broad insights. Applying the framework to this system allows the development of insights that answer these questions. In addition, while existing studies in this area only consider using ohmic effects for temperature balancing, this chapter additionally considers entropic effects. For small currents, entropic effects generate a reversible heating term.
greater than or equal to the irreversible heating term from ohmic effects. The inclusion of this term allows temperature balancing to be accomplished without deliberate energy dissipation, and impacts the insights found in this chapter. It should be noted that the focus in this chapter is on the use of power electronics, rather than fluidic means, for mitigating battery pack imbalance. One advantage of this approach is that it can potentially enable faster balancing compared to fluidic balancing, or perhaps work in conjunction with it. The contributions of this work are currently under review for archival [97].

The chapter is structured as follows. An LTV state space model is developed to represent charge and temperature heterogeneity within the battery pack (Section 5.2). Three different cases are examined to determine the core insights: no balancing, charge balancing, and temperature balancing (Section 5.3). An MPC algorithm is presented determine the optimal current, and simulation results are presented for a variety of cases in order to validate the insights (Section 5.4). The chapter ends with a brief summary of the knowledge obtained through the analysis (Section 5.5).

5.2 Model of Cell Imbalance

This section reviews the application of step #3 and step #4 from the framework of Chapter 3. Two cells are considered in series, with the corresponding state variables being the charges \( x_{1/2} \) and the temperatures \( T_{1/2} \) of cell \#1, 2, respectively. There is a main current \( u_1 \) which passes through both cells, and a balancing current \( u_2 \) which shuttles charge from cell \#2 into cell \#1, i.e. \( I_1 = u_1 + u_2 \) and \( I_2 = u_1 - u_2 \). Applying step #3, the two subspaces of the system are defined as follows:

1. The subspace of the sums: \( z_1 = x_1 + x_2 \) and \( \tau_1 = T_1 + T_2 \).
2. The subspace of the differences: \( z_2 = x_1 - x_2 \) and \( \tau_2 = T_1 - T_2 \).

Using these new states, the battery string dynamics are expressed using Equations (5-1)-(5-4). Note that \( h_T \) is defined in Table 3-4, and it is no longer written as a function of the constant \( \dot{Q}_{loss} \).

\[
\dot{z}_1 = -2u_1 \tag{5-1}
\]

\[
\dot{\tau}_1 = h_T \left( \frac{x_1 + x_2}{2}, \frac{\tau_1 + \tau_2}{2}, u_1 + u_2 \right) + h_T \left( \frac{x_1 - x_2}{2}, \frac{\tau_1 - \tau_2}{2}, u_1 - u_2 \right) \tag{5-2}
\]

\[
\dot{z}_2 = -2u_2 \tag{5-3}
\]

\[
\dot{\tau}_2 = h_T \left( \frac{x_1 + x_2}{2}, \frac{\tau_1 + \tau_2}{2}, u_1 + u_2 \right) - h_T \left( \frac{x_1 - x_2}{2}, \frac{\tau_1 - \tau_2}{2}, u_1 - u_2 \right) \tag{5-4}
\]
Following step #4, a first-order Taylor series is applied to Equations (5-2) and (5-4) about the average cell trajectories \( X = \frac{x_1}{2}, T = \frac{\tau_1}{2}, \) and \( U = u_1 \), yielding:

\[
\begin{align*}
\dot{\tau}_1 & \approx 2h_T \left( \frac{x_1}{2}, \frac{\tau_1}{2}, u_1 \right) \\
\dot{\tau}_2 & \approx \frac{\partial h_T(X,T,U)}{\partial x} z_2 + \frac{\partial h_T(X,T,U)}{\partial T} \tau_2 + 2 \frac{\partial h_T(X,T,U)}{\partial u} u_2
\end{align*}
\]

From (5-1) and (5-5), it can be seen that \( z_1 \) and \( \tau_1 \) do not depend on the differences subspace states, and only \( u_1 \) is necessary to determine these. However, because the derivatives found in (5-6) are evaluated about these average trajectories, \( z_2 \) and \( \tau_2 \) depend on the sums subspace states and inputs. Defining time-varying parameters \( a_1 = \frac{\partial h_T(X,T,U)}{\partial x}, a_2 = \frac{\partial h_T(X,T,U)}{\partial T}, b_1 = -2 \) and \( b_2 = 2 \frac{\partial h_T(X,T,U)}{\partial u} \), the differences subspace is represented as an LTV system:

\[
\begin{bmatrix}
\dot{z}_2 \\
\dot{\tau}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 0 \\
a_1 & a_2
\end{bmatrix}
\begin{bmatrix}
z_2 \\
\tau_2
\end{bmatrix} +
\begin{bmatrix}
b_1 \\
b_2
\end{bmatrix} u_2
\]

Equation (5-7) presents a model suitable for the analysis of charge and temperature heterogeneity between the cells.

### 5.3 State Feedback Analysis

This section follows step #5 of the framework and analyzes the evolution of charge and temperature differences over time using the LTV model of Section 5.2. Three cases are explored. The first is for a battery pack that lacks any balancing hardware. The second is for a battery pack that uses a balancing current to remove charge imbalance. The third is for a battery pack that uses the balancing current to more quickly remove temperature heterogeneity between the cells. In order to obtain core insights into the heterogeneous dynamics, state feedback is used to explore these cases. However, the insights hold true for different control methods, and these insights are validated in Section 5.4 using a more advanced MPC algorithm to determine balancing current.

#### No Balancing Current

The first case examines the behavior of charge and temperature imbalance in the absence of balancing current, i.e. \( u_2 = 0 \). Equation (5-7) shows that for this case the differences subspace eigenvalues are along the diagonal of the state matrix. Moreover, because the off-diagonal term in
this matrix is always finite, these eigenvalues suffice for stability analysis for this LTV system. The first eigenvalue, $0$, shows that charge heterogeneity is constant without any balancing, regardless of whether the main current flowing into the battery string is zero or nonzero. The second eigenvalue, $a_2$, is time-varying, and its value determines the growth/decay rate of the temperature difference between the cells. Figure 5-1 plots an example of $a_2$’s possible values for a range of average charge $x = z_1/2$ and current $u_1$ values, with average temperature $\tau_1/2 = T_\infty$. For this broad range of values, $a_2 < 0$, indicating that temperature heterogeneity decays to a steady state value. Equation 5-8 presents the equilibrium values, denoted by the subscript $e$, for the differences states assuming $a_2 \neq 0$.

\[
\begin{align*}
\begin{bmatrix} z_{2,e} \\ \tau_{2,e} \end{bmatrix} &= \begin{bmatrix} z_2(0) \\ -\frac{a_1}{a_2} z_{2,e} \end{bmatrix} \\
(5-8)
\end{align*}
\]

Equation (5-8) reveals that it is possible for battery cells in a string to exhibit a steady state temperature imbalance, driven directly by a steady state charge imbalance. This is an interesting detail that can be explained as follows: steady state differences in battery charge can correspond to steady state differences in entropy coefficients among cells, since entropy coefficients are charge-dependent. If a nonzero main current $u_1$ is applied to a battery pack in the presence of differences in entropy coefficients, this will directly result in a difference in reversible heat generation rates among the cells. Indeed, the term $a_1$ in Equation (5-8) depends on the current $u_1$: $a_1 = 0$ when $u_1 = 0$, and is typically nonzero when $u_1 \neq 0$. Thus, a battery pack with charge imbalance can experience differences in heat generation among different cells simply
as a result of the application of a nonzero main current, \( u_1 \). The fact that both \( \tau_{2,e} \) and \( a_2 \) depend on \( u_1 \) leads to the main insight for this first case:

**Insight:** *In the case of no balancing hardware, it is possible to control the same current through two cells to assist in removing temperature heterogeneity.*

In order to manipulate the temperature heterogeneity dynamics, the common current through both cells, \( u_1 \), can be manipulated to change the magnitude of \( a_2 \) and potentially increase the rate at which \( \tau_2 \) converges to equilibrium. The impact of this is strongly dependent on the cell parameters. At the same time, manipulation of \( u_1 \) changes \( \tau_{2,e} \) for nonzero charge differences. It is therefore possible for a current profile to induce temperature imbalance if charge imbalance exists. However, it is also possible to manipulate \( u_1 \) in a manner favorable to removing temperature heterogeneity. This is an important fact – it shows that the quick removal of temperature heterogeneity can be accomplished without any balancing hardware or cooling systems.

**Charge Balancing**

The second case examines the behavior of charge and temperature imbalance under the condition of a state feedback-based balancing current for charge balancing. In this case the current \( u_2 = K_z z_2 \) is applied to Equation (5-7):

\[
\begin{bmatrix}
\dot{z}_2 \\
\dot{\tau}_2
\end{bmatrix} =
\begin{bmatrix}
b_1 K_z & 0 \\
a_1 + b_2 K_z & a_2
\end{bmatrix}
\begin{bmatrix}
z_2 \\
\tau_2
\end{bmatrix}
\]  

Equation (5-9)

In Equation (5-9), the eigenvalues are along the diagonal of the matrix, and solving \( \lambda_z = b_1 K_z \) allows the determination of the gain \( K_z \) to stabilize the charge imbalance dynamics with eigenvalue \( \lambda_z \). The temperature imbalance state remains governed by eigenvalue \( \lambda_z \). Solving Equation (5-9) for equilibrium shows that \( z_{2,e} = 0 \) and \( \tau_{2,e} = 0 \), which indicates that all heterogeneity is removed if charge balancing is utilized. Equation (5-10) presents the temperature imbalance dynamics, knowing that \( z_2 \) easily determinable in Equation (5-9):

\[
\dot{\tau}_2 = a_2 \tau_2 + \left( a_1 + \frac{b_2 \lambda_z}{b_1} \right) z_2
\]  

Equation (5-10)

Equation (5-10) is a first-order ordinary differential equation with a time-varying coefficient, and its solution is determinable using an integrating factor:

\[
\tau_2(t) = e^{\int a_2 dt} \left( \int e^{-\int a_2 dt} \left( a_1 + \frac{b_2 \lambda_z}{b_1} \right) z_2(t) \right) dt + c_0
\]  

Equation (5-11)
Note that $c_0$ is a constant determined using the initial conditions. Equations (5-10) and (5-11) show that while $\tau_2 = 0$ in equilibrium, there is still an impact on $\tau_2$ from $z_2$ as charge is balanced. This fact leads to the second major insight of this work:

**Insight:** It is possible that a balancing current used to remove charge heterogeneity between two cells can simultaneously help remove or increase temperature heterogeneity.

In order to exemplify the potential impact charge balancing has on temperature heterogeneity, Equations (5-1), (5-5), (5-9) and (5-11) are solved for $u_1 = 0$ and constant $\lambda_z$:

\[
\begin{align*}
  z_2 &= z_2(0)e^{\lambda_z t} \\
  \tau_2 &= \tau_{2,\text{free}} + \tau_{2,\text{forced}} \\
  \tau_{2,\text{free}} &= \tau_2(0)e^{-\lambda t} \\
  \tau_{2,\text{forced}} &= \{c_1(e^{\lambda_z t} - e^{-\lambda t}) + c_2(e^{(\lambda - \lambda) t} - e^{-\lambda t})\}
\end{align*}
\]

\[
\begin{align*}
  c_1 &= \left(\frac{z_2(0)\lambda_x}{mC_p \mu(z_{2,\text{nom}})}\right)\frac{2T_\infty}{\lambda_x + \lambda} \\
  c_2 &= \left(\frac{z_2(0)\lambda_x}{mC_p \mu(z_{2,\text{nom}})}\right)\frac{\tau_1(0) - 2T_\infty}{\lambda_x + \lambda}
\end{align*}
\]

In Equation (5-13), the free response term ($\tau_{2,\text{free}}$) represents the inherent change in temperature heterogeneity over time, while the forced response term ($\tau_{2,\text{forced}}$) relates to the impact of charge balancing on the temperature heterogeneity. The comparison of these two terms allows one to determine the significance of charge balancing’s impact on temperature heterogeneity. For example, consider three different scenarios:

- Both $\tau_{2,\text{free}}$ and $\tau_{2,\text{forced}}$ have the same sign over time (e.g., $u_1 = 0$, $z_2(0) < 0$, $\mu\left(\frac{z_1}{2Q_{\text{nom}}}\right) > 0$, $\tau_2(0) > 0$). This scenario leads to a slower rate to approach equilibrium, as the forced term opposes the reduction of temperature heterogeneity. It is possible for this scenario to lead to a temporary increase in temperature imbalance if the forced term’s magnitude is large enough.

- $\tau_{2,\text{free}}$ and $\tau_{2,\text{forced}}$ have opposite signs and $|\tau_{2,\text{forced}}| \gg |\tau_{2,\text{free}}|$ for a substantial amount of time (e.g., $u_1 = 0$, $z_2(0) < 0$ and large, $\mu\left(\frac{z_1}{2Q_{\text{nom}}}\right) > 0$, $\tau_2(0) < 0$). In this scenario, the forced term negates the free response term, but causes $\tau_2$ to switch signs but have a similar or potentially larger magnitude.
• $\tau_{2,\text{free}}$ and $\tau_{2,\text{forced}}$ have opposite signs and $|\tau_{2,\text{forced}}| \leq |\tau_{2,\text{free}}|$ over time (e.g., $u_1 = 0$, $z_2(0) < 0$ and small, $\mu \left(\frac{z_1}{2q_{\text{nom}}}\right) > 0$, $\tau_2(0) < 0$). In this ideal scenario, the forced response term cancels out the natural temperature heterogeneity, leading to a faster reduction in imbalance.

These different scenarios show that it is possible for charge balancing to help or hurt temperature balancing. Similar conclusions apply for Equation (5-11), but knowledge of $u_1$ is required to perform the analysis. Ideally, charge balancing algorithms can be designed to avoid the first two scenarios and take advantage of the third.

**Temperature Balancing**

The third case examines the behavior of charge and temperature imbalance when using balancing current $u_2$ to remove the temperature difference. In order to simplify the stability analysis for this case it is assumed that $a_1 = 0$. This indicates that either $u_1 = 0$ or both the overpotential terms and the entropy coefficient together are not strong functions of concentration.

Applying $u_2 = K \tau_2$ to Equation (5-7) yields:

$$
\begin{bmatrix}
\dot{z}_2 \\
\dot{\tau}_2
\end{bmatrix} =
\begin{bmatrix}
0 & b_1 K \\
0 & a_2 + b_2 K
\end{bmatrix}
\begin{bmatrix}
z_2 \\
\tau_2
\end{bmatrix}
$$

(5-16)

In Equation (5-16), the eigenvalues are again along the diagonal of the matrix. As with the first case, the charge difference is marginally stable. However, the temperature difference state has an equilibrium of zero, and the rate at which this decays to equilibrium can be designed by solving $\lambda_\tau = a_2 + b_2 K$. Note that in order to control temperature imbalance, $b_2$ needs to be nonzero. Figure 5-2 plots $b_2$ for a constant $\tau_1 = 2T_{\text{co}}$, and shows that it is possible for $b_2 = 0$. In these scenarios, state feedback cannot directly impact temperature imbalance and $\lambda_\tau = a_2$. When this is not the case, the solution for $\tau_2$ is found by using separation of variables, yielding Equation (5-17). With $\tau_2$’s trajectory known, an analytic solution for $z_2$ is determined, as presented in Equations (5-18)-(5-20).
Figure 5-2. Range of values for parameter $b_2$.

\[
\tau_2 = \tau_2(0) \exp \left( \int_0^t \lambda_\tau(\sigma) d\sigma \right) \tag{5-17}
\]

\[
z_2 = z_{2,\text{free}} + z_{2,\text{forced}} \tag{5-18}
\]

\[
z_{2,\text{free}} = z_2(0) \tag{5-19}
\]

\[
z_{2,\text{forced}} = \tau_2(0) b_1 \int_0^t K_r(\xi) \exp \left( \int_0^\xi \lambda_\tau(\sigma) d\sigma \right) d\xi \tag{5-20}
\]

Equations (5-17)-(5-20) show parallels between charge balancing and temperature balancing, and leads to the third major insight of this chapter:

**Insight:** It is possible that a balancing current used to remove temperature heterogeneity between two cells can simultaneously help remove or increase charge heterogeneity.

This third insight mirrors the second insight, and can be exemplified by again comparing the free response and forced response terms ($z_{2,\text{free}}$ and $z_{2,\text{forced}}$, respectively) for three different scenarios:

- Both $z_{2,\text{free}}$ and $z_{2,\text{forced}}$ have the same sign over time (e.g., $u_1 = 0$, $z_2(0) < 0$, $\mu \left( \frac{z_1}{2Q_{\text{nom}}} \right) > 0$, $\tau_2(0) > 0$). In this, temperature balancing increases the charge imbalance within the cell.

- $z_{2,\text{free}}$ and $z_{2,\text{forced}}$ have opposite signs and $|z_{2,\text{forced}}| > |z_{2,\text{free}}|$ in steady state (e.g., $u_1 = 0$, $z_2(0) < 0$ and small, $\mu \left( \frac{z_1}{2Q_{\text{nom}}} \right) > 0$, $\tau_2(0) < 0$). In this scenario, charge imbalance switches its sign but is not removed. If $|z_{2,\text{forced}}| > 2|z_{2,\text{free}}|$, the magnitude of charge imbalance increases.
• $z_{2,\text{free}}$ and $z_{2,\text{forced}}$ have opposite signs and $|z_{2,\text{forced}}| \leq |z_{2,\text{free}}|$ in steady state (e.g., $u_1 = 0$, $z_2(0) < 0$ and large, $\mu \left( \frac{z_1}{2Q_{\text{nom}}} \right) > 0$, $\tau_2(0) < 0$). In this scenario, charge imbalance does not switch signs and is reduced. It is possible for imbalance to be completely removed if the two terms have equal magnitude.

These three scenarios show that it is possible for temperature balancing to impact charge imbalance. If the most ideal scenario, $z_{2,\text{free}} = -z_{2,\text{forced}}$ in steady state, is not achievable, it still may be possible to design a temperature balancing algorithm that avoids any non-beneficial scenarios.

5.4 Model Predictive Control for Charge and Temperature Imbalance Removal

This section validates the three major insights of Section 5.3. An MPC problem is set up to optimize the current input into the cell in order to remove charge heterogeneity, temperature heterogeneity, or both. The MPC problem is first solved to determine a profile for $u_1$ and validate the first insight, and then is solved for $u_2$ to validate the second and third insights. A final test explores the tradeoffs found when simultaneously balancing both charge and temperature between the cells.

Optimization Problem Formulation

In order to determine a current to minimize charge and temperature heterogeneity, the optimization problem of Equation (5-21) is solved over a $t_f = 60$ s window. The window time is selected after extensive testing to most effectively minimize heterogeneity in the system.

$$\min \int_0^{t_f} (w_x(x_1 - x_2)^2 + w_T(T_1 - T_2)^2)dt$$

s.t. Table III (for each cell $j = 1,2$)

$SOC_{\text{min}} \leq \frac{x_j}{Q_{\text{nom}}} \leq SOC_{\text{max}}$

$T_{\text{min}} \leq T_j \leq T_{\text{max}}$

$u_{\text{min}} \leq U \leq u_{\text{max}}$

In this problem, the polynomial $\bar{\theta}_0 + \bar{\theta}_1 t + \bar{\theta}_2 t^2$ describes the optimized current $U$, with $\bar{\theta} = [\theta_0, \theta_1, \theta_2]$ as the optimization variables. Either $U = u_1$ is optimized and $u_2 = 0$ (for
optimizing the same current through both cells) or $u_1$ is a known profile and $U = u_2$ is optimized (for optimizing the balancing current). The optimized current $U$ is limited between minimum and maximum values $u_{min}$ and $u_{max}$. Similarly, the $SOC = \frac{x}{Q_{nom}}$ is limited between $SOC_{min} = 0$ and $SOC_{max} = 1$ to avoid under or overcharging the battery, and the temperature is limited between minimum and maximum values $T_{min} = 283.15$ K and $T_{max} = 310.15$ K as well. The ratio of the weights $w_X$ and $w_T$ determines the relative emphasis among charge and temperature balancing. Once the optimal current is determined, the initial value is input into the plant (the full SPM-T). It is assumed that cell charges and temperature are known, and the optimization process is rerun at every time step $\Delta t = 1$ s. The optimization problem is solved using a nonlinear optimizer, initialized with multiple initial parameter guesses in order to avoid poor local minima.

### Using a Common Current for Temperature Balancing

The purpose of this subsection is to validate the first insight of Section 5.3 and show that a common current is able to assist in the reduction of temperature imbalance. To do this, $u_1$ is optimized with $u_{min} = -1$ A and $u_{max} = 1$ A, $w_X = 0$, and $w_T = 1$. The initial conditions of the states are $x_1(0) = 0.45Q_{nom}$, $x_2(0) = 0.5Q_{nom}$, $T_1(0) = 296.15$ K, and $T_2(0) = 298.15$ K, meaning there is initial heterogeneity $z_2(0) = -0.05Q_{nom}$ and $\tau(0) = -2$ K.

Figure 5-3 presents the optimized current profile $u_1$ over time. With this profile, the cells are discharged by 18.3%, while the charge imbalance remains constant. Figure 5-4 and Figure 5-5 plot $\tau_1$ and $\tau_2$, respectively, over time for cases with and without $u_1$ control. Without $u_1$ control, the magnitude of temperature heterogeneity reduces to less than 2% of its initial value after 2300 seconds. With $u_1$ control, it only takes 1490 seconds, showing that temperature heterogeneity is removed in 35% less time. Figure 5-5 additionally shows the change in the equilibrium value $\tau_2$ when using $u_1$ control, while Figure 5-6 shows the change in $a_2$. These figures show that the eigenvalue $a_2$ does not substantially change over time, and thus is not the main factor in the more rapid reduction of temperature heterogeneity. Instead, temperature imbalance is removed faster due to the manipulation in the equilibrium value. To explain why this is, consider two scenarios: in the first, $\tau_2$ is designed to be zero, and $\tau_2(0)$ is slightly negative. In the second scenario, $\tau_2$ is designed to be briefly positive with a large magnitude and $\tau_2(0)$ is still slightly negative. For a constant eigenvalue $a_2$, the $\tau_2$ of the second scenario reaches equilibrium in a shorter amount of
time, as there is a larger difference between $\tau_{2,e}$ and $\tau_2$ initially. This is exactly what occurs in this test. Note that if temperature heterogeneity is initially positive instead of negative, $u_1$ would become negative in this test, in order to flip the sign of $\tau_{2,e}$ initially.

![Figure 5-3. Optimized current for the case of controlling $u_1$.](image)

![Figure 5-4. Average cell temperature $\tau_1/2$ for the case of controlling $u_1$. Black for $u_1 = 0$, blue for $u_1$ control.](image)
Single Variable and Multivariable Balancing

In order to validate the second insight, that removing charge heterogeneity has the potential to help or slow the removal of temperature heterogeneity, \( u_2 \) is optimized with weights \( w_X = 1 \) and \( w_T = 0 \). The balancing current is limited to \( u_{\text{min}} = -0.5 \) A and \( u_{\text{max}} = 0.5 \) A, and
for these tests $u_1 = 0$. Figure 5-7 presents the optimized $u_2$ profile for the same initial conditions as with the previous subsection. Figure 5-8 plots $z_2$ for both the optimized $u_2$ case and for the no balancing case ($u_2 = 0$). Likewise, Figure 5-9 plots $\tau_2$ for both of these cases. In this scenario, there is a faster reduction in temperature heterogeneity when balancing charge. By plotting $\tau_{2, free}$ and $\tau_{2, forced} = \tau_2 - \tau_{2, free}$ (Figure 5-10), it can be seen that $\tau_{2, free}$ and $\tau_{2, forced}$ have opposite signs and $|\tau_{2, forced}| \leq |\tau_{2, free}|$. This is an ideal case described earlier, in which temperature heterogeneity removal is assisted by charge balancing. A second test is run with initial conditions $x_1(0) = 0.45Q_{nom}$, $x_2(0) = 0.5Q_{nom}$, $T_1(0) = 298.15$ K, and $T_2(0) = 296.15$ K. As the charge imbalance is the same, the optimized current and $z_2$ trajectory remain the same as with Figure 5-7 and Figure 5-8. Figure 5-11 and Figure 5-12 plot the equivalent of Figure 5-9 and Figure 5-10 for this new scenario. From these results, it can be seen that charge balancing hurts temperature balancing, as both $\tau_{2, free}$ and $\tau_{2, forced}$ have the same sign over time, due to $\tau_2$ having its sign switched.

Figure 5-7. Optimized balancing current $u_2$ for the first charge balancing test.
Figure 5-8. Charge difference $z_2$ between cells for the first charge balancing test.

Figure 5-9. Temperature difference $\tau_2$ for the first charge balancing test.
Figure 5-10. Free and forced temperature difference terms $\tau_{2,\text{free}}$ and $\tau_{2,\text{forced}}$ when controlling $u_2$ for the first charge balancing test.

Figure 5-11. Temperature difference $\tau_2$ for the second charge balancing test.
In order to validate the third insight, that the faster removal of temperature heterogeneity may increase or reduce charge heterogeneity, \( u_2 \) is optimized with the same limits, but \( w_X = 0 \) and \( w_T = 1 \). Figure 5-13 plots the optimized \( u_2 \), Figure 5-14 plots \( z_2 \), and Figure 5-15 plots \( \tau_2 \), with and without balancing. As these figures show, temperature imbalance is removed at a faster rate, and \( z_2 \) increases in magnitude but switches signs. Figure 5-16 plots \( z_{2,\text{free}} \) and \( z_{2,\text{forced}} = z_2 - z_{2,\text{free}} \) over time, and shows that this example matches with the second temperature balancing scenario of Section 5.3. Note that the improved removal of temperature heterogeneity is strongly dependent on factors such as the balancing current limits, the initial temperature difference, and the entropy coefficient magnitude. An increase in any of these would lead to more substantial differences between the balancing and no balancing cases.
Figure 5-13. Optimized balancing current $u_2$ for the temperature balancing test.

Figure 5-14. Charge difference $z_2$ between cells for the temperature balancing test.
The ratio of the weights, $w_X/w_T$, determines how much to focus on the $z_2$ or $\tau_2$ when applying multivariable balancing. The ratio approaching zero places a greater emphasis on removing $\tau_2$, and it approaching infinity prioritizes removing $z_2$. For $x_1(0) = 0.45Q_{nom}$, $x_2(0) = 0.5Q_{nom}$, $T_1(0) = 298.15$ K, and $T_2(0) = 296.15$ K, Figure 5-17, Figure 5-18 and Figure 5-19 plot the change in $u_2$, $z_2$ and $\tau_2$ over time for $w_X/w_T = \left[0, \frac{50}{Q_{nom}}, \frac{100}{Q_{nom}}, \infty \right]$. As these figures indicate, if temperature imbalance removal is prioritized, a temporary charge imbalance is
induced to increase the rate of temperature heterogeneity dissipation. The charge imbalance induced increases in magnitude and is dissipated at a slower rate as this weighting increasingly favors temperature imbalance. Once the temperature difference becomes small enough in magnitude, charge balancing becomes the focus.

Figure 5-17. Optimized balancing current $u_2$ for the multivariable balancing test with ratios $w_x = \left[ 0, \frac{50}{Q_{\text{nom}}}, \frac{100}{Q_{\text{nom}}}, \infty \right]$.

Figure 5-18. Charge difference $z_2$ between cells for the multivariable balancing test with ratios $w_x = \left[ 0, \frac{50}{Q_{\text{nom}}}, \frac{100}{Q_{\text{nom}}}, \infty \right]$. 
Figure 5-19. Temperature difference $\tau_2$ for the multivariable balancing test with ratios $\frac{w_x}{w_T} = \left[0, \frac{50}{Q_{\text{nom}}^2}, \frac{100}{Q_{\text{nom}}^2}, \infty\right]$.

5.5 Summary of Insights and Contributions

This chapter applies the last three steps of the framework of Chapter 3 to determine the influence current-controlled balancing strategies have on the charge and temperature imbalances between cells in a lithium-ion battery pack. By splitting pack dynamics into separate subspaces, a linear time-varying model is developed that represents charge and temperature imbalance between cells. State feedback is used to analyze the stability and equilibrium values of the imbalance states and determine key insights into the behavior of charge and temperature heterogeneity for different cases. First, an average current passing through two cells in series is able to reduce the amount of time it takes to remove temperature heterogeneity by up to 35% in the example study presented. Second, charge balancing can have a positive or negative impact on temperature heterogeneity, dependent on the signs and magnitudes of the free and forced responses of the temperature heterogeneity dynamics relative to one another. Third, temperature balancing can have a positive or negative impact on charge heterogeneity, dependent on the signs and magnitudes of charge heterogeneity’s free and forced dynamics. Simulated results using realistic battery parameters and a model predictive controller to determine the optimized current input confirm the validity of the insights.
Chapter 6

Conclusions

This dissertation provides fundamental insights into the behavior of heterogeneous populations of thermostatically controlled loads (TCLs) and battery cells within a battery pack. Both of these devices are implementable for regulating power demand and power supply differences: TCL control is a suggested option for implementing demand response for renewable resource integration, and battery packs within transportation applications regulate power between different power sources. In addition, the aggregate behavior of populations of these devices cannot be represented accurately using an average unit model. By developing models representative of either the aggregate dynamics or heterogeneity dynamics, analysis of the population dynamics is possible. Analysis leads to insight development, in order to better understand the behavior of these populations. With this knowledge, it is possible to improve system design or develop new control algorithms that take advantage of these insights. These improvements and algorithms lend themselves to improved power regulation and help address major challenges within the energy sector.

It is known that the aggregate power demand of a population of TCLs is damped, and that this is beneficial for demand response purposes. This dissertation develops a reduced-order model that describes the aggregate power demand of a heterogeneous TCL population for a variety of parameter distributions. An individual unit’s power demand is modeled using a Fourier series with four parameters that relate to underlying TCL parameters such as thermal capacitance and resistance: amplitude, characteristic frequency, duty cycle and initial fraction. Aggregating this representation using a piecewise uniform approximation of the joint probability density function (pdf) leads to the reduced-order model, which furnishes several key insights. First, by studying uniform, univariate distributions of the Fourier series parameters, it is determined that only heterogeneity in the characteristic frequencies of the TCLs creates a damped aggregate power demand. Second, low levels of characteristic frequency heterogeneity lead to an apparent beating phenomenon which can be quantified by the model for uniform distributions. Third, the parameter-dependent, time-varying damping ratios of the reduced-order model quantify the prevalence of damping effects within the aggregate power demand for times after zero. The low-
order model and insights have the ability to improve demand response algorithm development and understanding of a population’s ability to implement demand response.

Heterogeneity of charge, capacity, resistance and temperature between battery cells within a battery pack is known to reduce the lifespan and performance of the pack. This dissertation presents a framework to develop a linear time-varying (LTV) model for, and analyze the evolution of, heterogeneity within the pack. The steps of the framework reduce an electrochemical battery model, separate the sums and differences subspace dynamics, and analyze the LTV heterogeneity model using linear system theory. In this work, the base battery model is the single-particle model (SPM) with thermal dynamics and solid-electrolyte interphase (SEI) layer growth used to represent degradation. While insights are drawn from analysis of this model, the framework is adaptable to other electrochemical models with different degradation effects.

The framework is applied to two different studies. In the first study, charge and capacity heterogeneity are considered with constant and uniform cell temperatures, and SEI layer growth is related to capacity loss and resistance growth. Analysis of two cells in series leads to the insight that voltage balancing is not able to remove capacity heterogeneity within the lifespan of the battery pack. With this knowledge, a novel state feedback controller is developed which simultaneously balances charge and capacity, and is shown to potentially increase pack lifespan by up to 9.2% through extending the lifespan of the weakest cell in the pack, though this number varies based on charging and initial conditions. In the second study, charge and temperature heterogeneity are considered using an electrothermal battery model as a basis. Entropic and ohmic effects are utilized to balance temperature between cells using current instead of control of coolant flow. For cells with non-negligible entropy coefficients, analysis indicates that charge (temperature) balancing can assist in the removing of, or in the growth of, temperature (charge) imbalance, and this depends on the magnitudes and signs of free and forced response terms. An insight is developed that indicates an average current can be controlled to remove temperature heterogeneity between two cells, provided entropy coefficients vary for different charge levels and a charge heterogeneity exists between the two cells. An MPC algorithm is designed and implemented to show the validity of these insights.

The models and insights developed in this dissertation increase understanding of the behavior of heterogeneous populations of energy devices. The suggested control approaches developed from these insights assist in addressing the major energy-related challenges faced today.
References


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Appendix

This appendix provides reasoning for the simplification of the battery diffusion dynamics, and indicates potential methods for inclusion within the battery framework of Chapters 3 and 4 if desired. Suppose diffusion is not neglected, and an N\textsuperscript{th}-order Padé approximation is used to determine the surface concentration of the positive electrode. The equations describing the surface concentration of the positive electrode are [84,86]:

\[
\begin{align*}
\dot{x} &= -I \\
\dot{x}_2 &= -\lambda_2 x_2 + I \\
&\vdots \\
\dot{x}_N &= -\lambda_N x_N + I \\
c_{sp} &= \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_N x_N 
\end{align*}
\]

where \( i = 2, \ldots, N \), \( x_i \) are the additional states relating to the diffusion dynamics, \( \lambda_i \) relate to the eigenvalues associated with diffusion, and \( \beta_i \) weight the contribution of each state to the surface concentration calculation. Note that the first state in this expression is the same as Equation (3-16), and relates to the average concentration. Step #3 of the framework of Chapter 3 can now be applied with the additional states considered. To do this, the following terms are defined: \( x_{i,j} \) is \( x_i \) for cell \( j = 1,2 \), \( z_i = x_{i,1} + x_{i,2} \) represents the sum states, and \( Z_i = x_{i,1} - x_{i,2} \) represents the difference states. With \( I = u_1 + u_2 \) for the first cell and \( I = u_1 - u_2 \) for the second cell, the subspace dynamics for the new states are:

\[
\begin{align*}
\dot{z}_i &= -\lambda_i z_i + 2u_1 \\
\dot{z}_i &= -\lambda_i Z_i + 2u_2 
\end{align*}
\]

Equation (A-2) shows that both the sum and difference between each additional spherical diffusion state is stable, as all \( -\lambda_i < 0 \). This is important – for a small balancing current \( u_2 \), the differences \( Z_i \) decay to small values quickly as compared to the rate of degradation. In other words, the difference between the surface concentrations of the two cells’ positive electrodes can be approximated as the difference between the average concentrations. Though it is possible to include these addition difference states \( Z_i \) in the analysis, meaningful impact on the results of the analysis only occurs if balancing current is maintained at a high value for long periods of time.

By including diffusion dynamics, the only change is the difference in the linearization point: instead of linearizing around the sum of the average concentration (or charge), linearization is performed around the sum of the surface concentrations, which includes the new \( Z_i \) terms. The only impact on the differences subspace is seen through the time varying coefficients.
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