TRANSMISSION PROBLEMS FOR PARABOLIC EQUATIONS
AND APPLICATIONS TO THE FINITE ELEMENT METHOD

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Abstract

We study theoretical and practical issues of the second-order parabolic equation $u_t + Lu = f$, where $L = -\text{div}(A\nabla)$ is a second-order operator with piecewise smooth coefficient matrix $A$, with possibly jump discontinuities across a finite number of curves, called the interface. First we concentrate on the problems with certain homogeneous or non-homogeneous boundary and interface conditions on smooth domain $\Omega$ with smooth interface $\Gamma$. Afterwards we analyze the problem on polygonal domains. Under some additional conditions we establish well-posedness in weighted Sobolev spaces. When Neumann boundary conditions are imposed on adjacent sides of the polygonal domain, or when the interfaces are not smooth, we fail to acquire well-posedness on weighted Sobolev space but we are able to obtain the decomposition $u = u_{\text{reg}} + w_s$, into a function $u_{\text{reg}}$ with better decay at the vertices and a function $w_s$ that is locally constant near the vertices, thus proving well-posedness in an augmented space. Based on the theoretical analysis we are able to implement a certain Finite Element scheme with improved graded meshes, which can recover the rate of convergence for piecewise polynomials of degree $m \geq 1$. Three numerical tests are included in the last.

Key Words: Well-posedness, Neumann-Neumann vertex, non-smooth interface, transmission problem, broken weighted Sobolev space, semigroup theory, finite element method with graded mesh,
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0.1 Introduction

My dissertation is mainly about the parabolic problem on bounded domains, in the form of \((L + \frac{\partial}{\partial t})u = f\), where \(L = -\text{div}(A\nabla)\) is a second-order operator, and coefficient matrix \(A\) is piecewise smooth on the whole domain, with possibly jump discontinuities across a finite number of curves, called the interface. Apart from certain boundary conditions, some additional conditions are imposed on the interface, called transmission conditions. Problems with such operator \(L\) are called transmission problems or interface problems.

The parabolic transmission problem can be applied in many fields, among which a well-known application is the heat transfer among composite materials. We know the heat transfer follows the equation \(u_t - \alpha \Delta u = f\), with \(f\) be the heat source function, and \(\alpha\) be the thermal diffusivity. Since the value of \(\alpha\) depends only on the type of material and varies among different materials, a heat equation with jump discontinuities can be set up to analyze the heat flow among the whole composites. We can assume there is no heat loss across the interface of different materials, or sometimes there is a heat source on the interface (solar panels, for instance). In addition, we always assume the temperature to be the same on double sides of the interface. These two assumptions yield to transmission conditions on the interface.

In spite of the large quantity of practical applications, the parabolic problem is not studied for many times as expected. Instead, there are hundreds of literatures on the corresponding elliptic transmission problem \(Lu = f\) on polygonal or polyhedral domains. In 1967, Professor V.A.Kondratev first established the theory of elliptic problem on domains that contain conical or angular points in [17]. Short after that, there are some famous work done by Professors M. Costabel [9] [10] [12], M. Dauge [11] [13], S. Nicaise [24] [27] [28] [29], and V.G.Mazya [21] [22]. Also, there are quite a few recent work, let us mention here the papers by Z.D. Milovanovic [25], P.E. Druet [14], J. Xiong and J. Bao [32]. However, the theory of parabolic transmission problem on polygonal domains are not yet well established, especially in the case where Neumann boundary conditions are imposed on both sides of some vertex of the polygon. In my dissertation, I will rely on the previous results for the elliptic problem to exploit a theoretical analysis on the corresponding parabolic problem, especially the well-posedness of solution which yield to the construction of numerical solution using the Finite Element Method with graded mesh.

The outline of the dissertation is as follows. In the first chapter, we concentrate on the behavior of the strong solution to the parabolic transmission problem with smooth boundary and smooth interface. In section 1.1, we assign the homogeneous transmission conditions so that the strong solution satisfies the weak formulation. In section 1.2, we study the existence
of strong solution with such homogeneous interface and boundary conditions, and present the
well-posedness of solution in higher-order broken Sobolev space, provided the initial data are
sufficiently regular, with higher-order compatibility conditions satisfied on the boundary and
interface. In section 1.3, we let the interface and boundary conditions to be non-homogeneous,
and make the solution have fixed jump discontinuities across the interface. Under this con-
dition, we can prove the strong solution still exists and satisfies certain regularity results. In
section 1.4, we introduce a finite difference scheme to obtain a full discrete numerical solution
to our problem in 1 dimensional case.

In the second chapter, we start to analyze the parabolic transmission problem on 2d
polygonal domains. In section 2.1, we state the mathematical formulation of the problem, and
the structure of domain. In section 2.2, we mention several literatures about the elliptic problem
on polygons. Most importantly, we state a lemma from the paper [18] that we can rely on to
establish our theory on the parabolic problem. In section 2.3, we consider a special case that
no Neumann boundary conditions are imposed on both sides of any vertex, and the interface
are smooth. Under such condition, we are able to obtain the well-posedness of solution on the
broken weighted Sobolev space. Moreover, we deduce that the solution will lie in higher-order
broken weighted Sobolev space when the initial data are sufficiently regular, with higher-order
compatibility conditions are met. In section 2.4, we calculate how singular the solution is near
each vertex. In section 2.5, we study the case when Neumann boundary conditions are imposed
on both sides of one vertex. In this case, we introduce the semigroup theory to help analyze
the behavior of solution. The result implies that such operator $\text{div}(A\nabla)$ does not generate a
strongly continuous semigroup in certain weighted Sobolev space, hence the well-posedness of
solution fails.

In the third chapter, based on the theory in chapter 2, we introduce a finite element
scheme with graded mesh to obtain a numerical solution to our parabolic problem. The struc-
ture of the finite element space is the same as in [18] to the corresponding elliptic problem. In
section 3.1, we list a few papers written by others that introduce numerical methods on similar
problems. In section 3.2, we state the construction of numerical solution in the paper [18] to
the elliptic problem on polygons, in which the crucial part is the construction of finite element
space. In section 3.3, we provide how to construct the numerical solution on such finite element
space with Backward Euler method. Moreover, we give an estimate on the convergence rate of
such scheme. In section 3.4, we implement such scheme in three concrete examples, and see
how the solutions behave across the interfaces, or near each vertices. We also study how fast
the numerical solution converges to the exact solution.
0.2 Future Work

Several problems still remain open in our dissertation.

1. *Generation of Semigroup in Weighted Sobolev Space*. In section 2.5 we reveal that, when \( \Omega \) is a polygonal domain and \( 0 < \alpha < 1 \), the generation of a \( C_0 \) semigroup for \( L \) in the space \( X_\alpha^0(\Omega) \) fails if Neumann boundary conditions are imposed on adjacent sides of some vertex in \( \Omega \). In the future, we are interested in finding a new weighted space where the second-order operator \( L \) can generate a \( C_0 \) semigroup. So far few literature appears on this topic.

2. *Well-Posedness of Parabolic Problem on Polyhedrons*. In the past decade numerous results have been published on the elliptic problem on polyhedral domains, such as [1] [6] [7] [8]. However, few regularity results comes out on the parabolic transmission problem. The main difficulties are, how to estimate the singularity of solution near the vertices and the edges, and how to construct a finite element space to obtain a numerical solution with quasi-optimal convergence rate. In the meantime, we are trying to extend our results from the polygonal domains into the polyhedral domains.

3. *A General Inverse Function Theorem for nonlinear Elasticity on Polygonal Domain*. This is an extension of the elliptic problem on polygonal domains. The paper [23] presents a regularity result for the anisotropic linear elasticity equation with mixed boundary on a polyhedron domain. Also, the book [20] reveals in chapter 6 that the nonlinear elasticity equation with homogenous boundary condition is well-posed on smooth domains. However, there are several challenges on studying a nonlinear elasticity equation on polygonal domains. First, the inverse function theorem only applies to continuously differentiable functions, but it is not the case when there are corners in the domain. Second, the Korn’s inequality does not always hold on polygonal domains. In the future we will start from a simpler nonlinear elasticity model and apply the same tool as in the parabolic problem. It is strongly possible that the model remains static under small perturbation.
Chapter 1

Parabolic Problems with Interface on Smooth Domains

1.1 Preliminaries

1.1.1 Weak Solution of the Initial-Boundary Value Problem

Let us study the parabolic problem

$$
\begin{aligned}
&u_t + Lu = f \quad \text{in } \Omega_T; \\
&u = 0 \quad \text{on } \partial \Omega \times [0, T]; \\
&u = g \quad \text{on } \Omega \times \{t = 0\}.
\end{aligned}
$$

(1.1.1)

Where the domain $\Omega$ is assumed to be an open, connected, bounded subset of $\mathbb{R}^n$, $\Omega_T = \Omega \times (0, T]$ for some fixed time $T > 0$. Also, the operator $L$ is a uniformly elliptic, second order operator, defined as

$$
Lu = - \sum_{i,j=1}^{n} (a^{ij}(x, t)u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i(x, t)u_{x_i} + c(x, t)u
$$

for given coefficients $a^{ij}, b^i, c \in L^\infty(\Omega_T)$, $i, j = 1, 2, \ldots, n$. We will also always suppose $a^{ij} = a^{ji}$ $(i, j = 1, \ldots, n)$. Then if the initial data $f$ and $g$ are not regular enough, we are still able to define a solution of the IBVP (1.1.1) in the variational form.

1.1.1 Definition. Assume $f \in L^2(0, T; H^{-1}(\Omega))$, $g \in L^2(\Omega)$, then we say a function $u$ is a weak/variational solution of the IVP (1.1.1) if

1. $u \in L^2(0, T; H^1_0(\Omega))$, with $u' \in L^2(0, T; H^{-1}(\Omega));$

2. $\langle u', \nu \rangle + B[u, \nu] = \langle f, \nu \rangle$ for each $\nu \in H^1_0(\Omega)$ and a.e. time $0 \leq t \leq T$. where $\langle , \rangle$ represents the pairing of $H^{-1}(\Omega)$ and $H^1_0(\Omega)$, and

$$
B[u, v] = \int_{\Omega} \sum_{i,j=1}^{n} a_{ij} u_{x_i} v_{x_j} + \sum_{i=1}^{n} b_i u_{x_i} v + cu v\,d\mathbf{x};
$$
Then it is well-known that, given $f$ and $g$ in the above space, there exist a unique weak solution to the problem (1.1.1). Based on this result, we are able to set up a new initial/boundary value problem called ”transmission problem”, after giving some improved regularity to the initial data. We follow Evans [15] here.

1.1.2 Strong Solution of the Initial-Boundary Value Problem

Assume the boundary of $\Omega$ is smooth. Now we divide the domain $\Omega$ into $K$ smooth, open and disjoint subdomains $\Omega_k, k = 1,2,\ldots,K$, such that $\overline{\Omega} = \bigcup_{k=1}^{K} \overline{\Omega}_k$. We denote $\Gamma = (\bigcup_{k=1}^{K} \partial \Omega_k) \setminus \partial \Omega$ to be the interface. It is not hard to observe that $\Gamma$ is smooth as well. We assume $\partial \Omega_k$ and $\Gamma$ do not touch each other.

In addition, we assume that the operator $L$ has no lower order terms, and do not depend on time:

$$Lu = -\sum_{i,j=1}^{n} (a^{ij}(x)u_{x_i})_{x_j},$$

with coefficients

$$a_{ij} \in \cap_{k=1}^{K} C^\infty(\Omega_k), \quad i,j = 1,2,\ldots,n;$$

where $m > 0$ is a fixed integer. Observe that $L$ is now self adjoint, and is elliptic on each subdomain $\Omega_k$ as well. However, the coefficients of $L$ may jump across $\Gamma$. Now based on this piecewise smooth operator $L$, we are able to define the strong solution to the parabolic problem (1.1.1).

1.1.2 Definition. If $L$ and $\Omega$ are defined as above, then we say a function $u$ is a strong solution of (1.1.1) if

1. $u \in L^2(0,T;H^2(\Omega_k))$ and $u' \in L^2(0,T;L^2(\Omega)), \ (k = 1,2,\ldots,K)$;
2. $u_t + Lu = f$ a.e. on each $\Omega_k, \ (k = 1,2,\ldots,K)$;
3. $u = 0$ on $\partial \Omega \times [0,T], u = g$ on $\Omega \times \{t = 0\}$, a.e.

According to the definition, the strong solution is defined almost everywhere on each subdomain $\Omega_k, k = 1,2,\ldots,K$. Different from the weak solution, there is no uniqueness to the strong solution, as we did not restrict the value of $u$ on the interface $\Gamma = \Omega \setminus \bigcup_{k=1}^{K} \Omega_k$. $u$ does not have to be in $H^1_0(\Omega)$, thus a jump of value across the interface is allowed.
Another observation is, if there exists a strong solution \( u \), then by definition, the function \( f = u_t + Lu \) must lie in \( L^2(0, T; L^2(\Omega)) \). Also, by the theorem 5.9.4 in [15], \( u \in C([0, T]; H^1(\Omega_k)) \), thus by continuity \( g = u(0) \) must lie in \( H^1(\Omega_k) \), \( (k = 1, 2, ..., K) \).

### 1.1.3 Transmission Condition on the Interface

Now we are looking for some function \( u \in L^2(0, T; H^2(\Omega_j)) \cap L^2(0, T; H^1_0(\Omega)), \ j = 1, 2, ..., K \), such that \( u \) is both the weak solution and a strong solution to the IVP (1.1.1). In other words, \( u \) is a strong solution which satisfies the weak formulation. Since the weak solution is unique, we can expect at most one function \( u \) having this property. Now let us suppose this function \( u \) exists, and then we are going to study the behavior of \( u \) on the interface \( \Gamma \).

First of all, as a weak solution, \( u \) lies in \( H^1_0(\Omega) \) for a.e. \( t \in (0, T] \). Since jump is not allowed for any \( H^1 \) function, we therefore obtain one restriction of \( u \) on the interface

\[
\int_{\Omega_k} (u_t + Lu)v\,dx = \int_{\Omega_k} fv\,dx
\]

for each \( v = v(t) \in H^1_0(\Omega) \). We do integration by part to the left hand side, on each sub domain \( \Omega_k, \ k = 1, 2, ..., K \). This gives:

\[
\sum_{k=1}^K \left[ (u_t, v)_{\Omega_k} + B[u, v]_{\Omega_k} + \int_{\partial \Omega_k} \nabla_v^A u(x) v(x)\,ds \right] = \int_{\cup_{k=1}^K \Omega_k} fv\,dx, \tag{1.1.3}
\]

where the notation \( \nabla_v^A u(x) = \sum_{i,j=1}^n a_{ij}^A(x) u_{x_j}(x) n_j(x) \) is called the conormal derivative of \( u \) across the interface, and \( n_j(x) \) is the \( j \)th component of the unit outward normal vector. Recall the fact that all subdomains \( \Omega_k \) are disjoint, so each part of \( \Gamma \) belongs to two adjacent subdomains, therefore we have \( \sum_{k=1}^K \partial \Omega_k = \Gamma^+ + \Gamma^- + \partial \Omega \). Since \( v = 0 \) on \( \partial \Omega \), we obtain

\[
\int_{\partial \Omega} \nabla_v^A u(x) v(x)\,ds = 0,
\]

then we sum up the equation (1.1.3) and it becomes

\[
(u_t, v) + B[u, v] + \int_{\Gamma} (\nabla_v^A u^+ - \nabla_v^A u^-)(x) v(x)\,ds = (f, v). \tag{1.1.4}
\]
with the notation $\nabla^A u^\pm(x) = \sum_{i,j=1}^n a_{ij}(x)\pm u^\pm(x) n_j(x)$. Recall $u$ satisfies the weak formulation $\langle u_t, v \rangle + B[u, v] = \langle f, v \rangle$, and in our case $u_t$ and $f$ both lie in $L^2(\Omega)$ for a.e. $t > 0$, which results in $\langle u_t, v \rangle + B[u, v] = \langle f, v \rangle$.

We subtract this by (1.1.4) to get

$$
\int_{\Gamma} (\nabla^A u^+(x) - \nabla^A u^-(x)) v(x) \, ds = 0.
$$

As $v \in H^1_0(\Omega)$ is arbitrary, to make this equality hold, $u$ needs to satisfy

$$
\nabla^A u^+(x) - \nabla^A u^-(x) = 0 \quad \text{a.e. } x \in \Gamma. \tag{1.1.5}
$$

The equations (1.1.2) and (1.1.5) together are called transmission conditions on the interface. According to the calculation above, these conditions are necessary in making the strong solution satisfy the weak formulation.

1.1.3 Remark. 1. If $u$ is the strong solution of (1.1.1) and $u$ satisfy the transmission conditions, then $u$ will satisfy the weak formulation. This result can be shown by reversing the above calculation.

2. From the above subsection, we know if a strong solution $u$ to the problem (1.1.1) exists, then $f \in L^2([0, T]; L^2(\Omega))$ and $u \in C([0, T]; H^1(\Omega_k))$ for all $k = 1, 2, \ldots, K$. If in addition, the solution $u$ satisfy the transmission conditions, then $u$ does not jump across the interface for a.e. $t > 0$. By continuity of $u$ in time we obtain $g = u(0)$ does not jump across $\Gamma$ as well. Therefore $g \in H^1_0(\Omega)$.

1.1.4 Parabolic Transmission Problem with Homogeneous Boundary Condition

Adding the transmission conditions to the definition of the strong solution of (1.1.1), we obtain the following problem:

$$
\begin{align*}
\begin{cases}
  u_t + Lu = f, & \text{in } \bigcup_{k=1}^K \Omega_k \times (0, T), \\
  u = 0, & \text{on } \partial\Omega \times [0, T], \\
  u = g, & \text{on } \Omega \times \{t = 0\}, \\
  u^+ - u^- = 0, & \text{on } \Gamma \text{ and a.e. } 0 < t < T, \\
  \nabla^A u^+ - \nabla^A u^- = 0, & \text{on } \Gamma \text{ and a.e. } 0 < t < T,
\end{cases}
\end{align*}
$$

(1.1.6)
This initial-boundary-interface value problem is well defined. From the subsection above, we already know if \( u \) is a strong solution to the problem (1.1.6), then \( u \) must satisfy the weak formulation. This implies we can expect at most one strong solution to this problem. In addition, we have presented one necessary condition for the existence of strong solution is \( f \in L^2(0, T; L^2(\Omega)) \) and \( g \in H^1_0(\Omega) \). However, we can still interpret our problem in a weak sense if the initial data \( f \) and \( g \) is not regular enough.

Now given the function \( f \) and \( g \) lie in some proper spaces, we will study the existence and the regularity of solution.

### 1.2 Existence and Regularity

In this section, we first state the existence of solution to the elliptic problem in a weak sense. Secondly, we mimic the proof in section 7.1 of Evans [15] to describe the well-posedness of solution to the parabolic problem (1.1.6) in certain broken Sobolev spaces.

#### 1.2.1 A Study of the Elliptic Problem

**1.2.1 Theorem** (Hengguang Li, Yu Qiao, Victor Nistor, [19]). Assume \( L \) and \( \Omega \) are defined as above, with distributions \( f \in H^{-1}(\Omega) \), \( h \in H^{-1/2}(\Gamma) \), then if a function \( u \in H^1_0(\Omega) \) solves the following boundary value elliptic problem

\[
\begin{align*}
Lu &= f, \quad \text{on } \cup_{k=1}^{K} \Omega_k, \\
u &= 0, \quad \text{on } \partial \Omega, \\
u^+ - u^- &= 0, \quad \text{on } \Gamma, \\
\nabla^A u^+ - \nabla^A u^- &= h, \quad \text{on } \Gamma.
\end{align*}
\]  

(1.2.1)

Then \( u \) will satisfy the usual weak formulation

\[
B[u, v] = (f, v)_{\Omega} + (h, v)_{\Gamma}, \quad \forall v \in H^1_0(\Omega). 
\]  

(1.2.2)

where the first \( \langle , \rangle_{\Omega} \) represents the \( H^{-1}(\Omega) \) and \( H^1_0(\Omega) \) dual, while the second \( \langle , \rangle_{\Gamma} \) represents the \( H^{-1/2}(\Gamma) \) and \( H^{1/2}(\Gamma) \) dual. Moreover, We have the estimate

\[
||u||_{H^1_0(\Omega)} \leq \beta^{-1} ||f||_{H^{-1}(\Omega)} + \frac{C_0}{\beta} ||h||_{H^{-1/2}(\Gamma)},
\]  

(1.2.3)
where $C_0$ is a constant depend only on $\Omega$, and $\beta$ is a constant depend only on $L$ such that
\[
\beta \|u\|_{H^1_0(\Omega)}^2 \leq B[u,u] \text{ for all } u \in H^1_0(\Omega).
\]

**Proof.** We recall a proof for completeness of the paper [19]. Now let us give a detailed proof. Assume this solution $u \in H^1_0(\Omega)$ exists, with the conormal derivative $\nabla^A u^\pm \in H^{-1/2}(\Gamma)$ well defined on each side of $\Gamma$. As the operator $L$ is smooth on each subdomain $\Omega_k$, we can study separately the behavior of $u$ on each $\Omega_k$. Since $f \in H^{-1}(\Omega)$ and $u \in H^{1/2}(\Gamma)$, we can construct sequences $(f_m)_{m=1}^\infty \in C^\infty(\Omega)$ and $(p_m)_{m=1}^\infty \in C^\infty(\Gamma)$ such that $f_m$ converges to $f$ in $H^{-1}(\Omega)$, with $p_m$ converges to $u$ in $H^{-1/2}(\Gamma)$. For each integer $m$ and $k \in [1, K]$, we consider the following Dirichlet boundary value problem on $\Omega_k$:
\[
\begin{cases}
L u_m = f_m, & \text{on } \Omega_k, \\
u_m = 0, & \text{on } \partial \Omega_k \cap \partial \Omega, \\
u_m = p_m, & \text{on } \partial \Omega_k \cap \Gamma.
\end{cases}
\]

(1.2.4)

The classical results reveals that, there exists a unique solution $u_m$ to the problem (1.2.4), with $u_m \in C^\infty(\Omega_k)$.

On one hand, let us study the relation between $u_m$ and $u$ on the domain $\Omega_k$. Let $e_m = u - u_m$. Comparing the equations (1.2.1) and (1.2.4), we can observe that $e_m \in H^1(\Omega_k)$ and solves the following boundary problem:
\[
\begin{cases}
L e_m = f - f_m, & \text{on } \Omega_k, \\
e_m = 0, & \text{on } \partial \Omega_k \cap \partial \Omega, \\
e_m = u - p_m, & \text{on } \partial \Omega_k \cap \Gamma.
\end{cases}
\]

(1.2.5)

According to the theory of the non-homogeneous boundary value elliptic problems, we can obtain the regularity estimate of $e_m$:
\[
\|e_m\|_{H^1(\Omega_k)} \leq C(\|f - f_m\|_{H^{-1}(\Omega_k)} + \|u - p_m\|_{H^{1/2}(\partial \Omega_k \cap \Gamma)}).
\]

(1.2.6)

Therefore, by our assumption it is clear to see $\|e_m\|_{H^1(\Omega_k)} \to 0$ as $m \to \infty$. This reveals $u_m \to u$ strongly in $H^1(\Omega_k)$. Moreover, as $u_m \in C^\infty(\Omega_k)$ and $\nabla^A u \in H^{-1/2}(\partial \Omega_k)$ is defined, we can
assert that $\nabla^A u_m \to \nabla^A u$ in $H^{-1/2}(\partial \Omega_k \cap \Gamma)$ by the trace theorem.

On the other hand, we apply the above definition of $u_m$ on each subdomain $\Omega_k$, $k = 1, 2, \ldots, K$, to make $u_m$ be defined on the whole domain $\Omega$.

It is not hard to derive $u_m^2 H^{1,0}(\Omega)$ from the equation (1.2.4), since the value of $u_m$ does not jump across $\Gamma$. Now we do integration by parts. For any $v \in H^1(\Omega)$ and each $k = 1, 2, \ldots, K$, we have

$$\int_{\Omega_k} \nabla^A u_m \cdot \nabla v \, dV = B[u_m,v]_{\Omega_k} + \int_{\partial \Omega_k \cap \Gamma} \nabla^A u_m \cdot v \, ds. \quad (1.2.7)$$

Then we sum up the equation (1.2.7) on each $\Omega_k$, this results in

$$\int_{\Omega} \nabla^A u_m \cdot \nabla v \, dV = B[u_m,v]_{\Omega} + \int_{\partial \Omega \cap \Gamma} (\nabla^A u_m^+ - \nabla^A u_m^-) \cdot v \, ds. \quad (1.2.8)$$

for all $v \in H^1(\Omega)$. Recall $Lu_m = f_m$ on each $\Omega_k$, we obtain

$$\int_{\Omega} f_m \cdot v \, dV = \int_{\bigcup_{k=1}^K \Omega_k} Lu_m \cdot v \, dV = \int_{\bigcup_{k=1}^K \Omega_k} \nabla^A u_m \cdot \nabla v \, dV = B[u_m,v]_{\Omega} + \int_{\partial \Omega} (\nabla^A u_m^+ - \nabla^A u_m^-) \cdot v \, ds. \quad (1.2.9)$$

Passing to the limit $m \to \infty$, we obtain

$$\langle f, v \rangle_{\Omega} = B[u,v] + \int_{\Gamma} (\nabla^A u^+ - \nabla^A u^-) \cdot v \, ds \quad \forall \ v \in H^1(\Omega), \quad (1.2.10)$$

this is exactly

$$B[u,v] = \langle f, v \rangle_{\Omega} + \langle h, v \rangle_{\Gamma} \quad \forall \ v \in H^1(\Omega). \quad (1.2.11)$$

What remains to be shown is the regularity result. Indeed, since $L$ is elliptic, the bilinear form $B[u,v]$ will satisfy the Lax-Milgram condition, thus

$$\beta \|u\|_{H^1(\Omega)}^2 \leq B[u,u] = \langle f, u \rangle_{\Omega} + \langle h, u \rangle_{\Gamma} \leq \|f\|_{H^{-1}(\Omega)} \|u\|_{H^1(\Omega)} + \|h\|_{H^{-1/2}(\Gamma)} \|u\|_{H^{1/2}(\Gamma)}. \quad (1.2.12)$$

By the trace theorem, there exists a constant $C_0 > 0$ depend only on $\Omega$ such that $\|u\|_{H^{1/2}(\Gamma)} \leq C_0 \|u\|_{H^1(\Omega)}$. Thus the inequality above becomes

$$\beta \|u\|_{H^1(\Omega)}^2 \leq \|f\|_{H^{-1}(\Omega)} + C_0 \|h\|_{H^{-1/2}(\Gamma)}. \quad (1.2.13)$$
Hence we finished the proof.

1.2.2 Remark. 1. We can consider \( F(v) = (f, u)_{\Omega} + (h, u)_{\Gamma} \) as a functional on \( H^1_0(\Omega) \), so that by the Lax-Milgram theorem there exists a unique weak solution to the equation \((1.2.2)\) in \( H^1_0(\Omega) \). Since the existence of strong solution is not guaranteed to the elliptic transmission problem \((1.2.1)\), we define \( u \in H^1_0(\Omega) \) to be the weak solution of \((1.2.1)\) if \( u \) satisfies the weak formulation \((1.2.2)\).

2. If \( h = 0 \), then the weak solution of \((1.2.1)\) is the weak solution to the regular elliptic problem \( B[u, v] = (f, v) \). Denote \( u = L^{-1} f \), apply the estimate \((1.2.3)\) we obtain \( \| L^{-1} \|_{(H^{-1}, H^1_0)} \leq 1/\beta \).

1.2.3 Definition. To explain our results more clearly, we define the broken Sobolev space using the following notations:

1. \( \hat{H}^m(\Omega) = \{ v : \Omega \to \mathbb{R}, v \in H^m(\Omega_k), \forall 1 \leq k \leq K \} \);

2. \( \hat{W}^{m,\infty}(\Omega) = \{ v : \Omega \to \mathbb{R}, \partial^\alpha v \in L^{\infty}(\Omega_k), \forall 1 \leq k \leq K, |\alpha| \leq m \} \);

In addition, we define the two spaces

\[ D_m = \hat{H}^{m+1}(\Omega) \cap \{ u = 0 \text{ on } \partial \Omega \} \cap \{ u^+ - u^- = 0 \text{ on } \Gamma \}; \]

and

\[ R_m = \hat{H}^{m-1}(\Omega) \oplus H^{m-1/2}(\Gamma). \]

In particular, it is clear that \( D_m = \hat{H}^{m+1}(\Omega) \cap H^1_0(\Omega) \). Now let us define a new operator \( \hat{L}_m^A \) on the space \( D_m \) such that:

\[ \hat{L}_m^A u = (L u|_\Omega, (\nabla^A_v u^+ - \nabla^A_v u^-)|_\Gamma). \]

It is not hard to observe that \( \hat{L}_m^A : u \to (f, h) \) maps \( D_m \) to the space \( R_m \), since \( L \) maps \( H^{m+1}(\Omega_k) \) to \( H^{m-1}(\Omega_k) \) for each \( k = 1, 2, ..., K \), and \( \nabla^A_v \) maps \( H^{m+1}(\Omega_k) \) to \( H^{m-1/2}(\partial \Omega_k \cup \Gamma) \) by the trace theorem.

1.2.4 Theorem (Hengguang Li, Yu Qiao, Victor Nistor, [19]). Assume the operator \( L \) and the domain \( \Omega \) is defined as before. then
1. by the Theorem 1.2.1, The elliptic problem

\[
\begin{cases}
    Lu = f, & \text{in } \bigcup_{k=1}^{K} \Omega_k, \\
    u = 0, & \text{on } \partial \Omega, \\
    u^+ - u^- = 0, & \text{on } \Gamma, \\
    \nabla^A u^+ - \nabla^A u^- = h, & \text{on } \Gamma.
\end{cases}
\]

(1.2.14)

is well defined and has a unique weak solution \( u \in H^1_0(\Omega) \) when \( f \in H^{-1}(\Omega) \) and \( h \in H^{-1/2}(\Gamma) \).

2. The operator \( \tilde{L}_m^A \) is well-defined from \( \mathcal{D}_m \) onto \( \mathcal{R}_m \) and is invertible.

Moreover, let \( \|L^{-1}\| \) denote norm of the inverse map \( L: \mathcal{D}_m \to \mathcal{R}_0 \) as in the Theorem 1.2.1.

Then, there exists a constant \( \tilde{C}_1 = \tilde{C}_1(m, \|L^{-1}\|, \|A\|_{\tilde{W}^m,\infty(\Omega)}) \) such that

\[
\|u\|_{\tilde{H}^{m+1}(\Omega)} + \|u\|_{H^1_0(\Omega)} \leq \tilde{C}_1 (\|f\|_{\tilde{H}^{m-1}(\Omega)} + \|h\|_{H^{m-1/2}(\Gamma)}),
\]

(1.2.15)

for all \( u \in \mathcal{D}_m \). Where \( A = (a^{ij}) \) denotes the coefficients of operator \( L \).

The proof of this theorem is an induction on \( m \). Here is a brief description. The case \( m = 0 \) is already presented, when \( u \) is defined in a weak sense. For \( m > 0 \), the weak solution \( u \) solves the problem (1.2.14) strongly. Besides that, assume the estimates (1.2.15) is true for some integer \( m \), that is \( u \in \tilde{H}^{m+1}(\Omega) \) when \( (f, h) \in \mathcal{R}_m \). After flatten the interface and do some calculation nearby, it reveals all the tangential derivatives of \( u \) lie in \( \tilde{H}^{m+1}(\Omega) \) if \( (f, h) \in \mathcal{R}_{m+1} \). The regularity of the conormal derivative then follows from the fact \( Lu = f \). Hence the proof is done.

1.2.5 Corollary. Let \( h(x) \equiv 0 \), the equation (1.2.15) becomes

\[
\|u\|_{\tilde{H}^{m+1}(\Omega)} + \|u\|_{H^1_0(\Omega)} \leq \tilde{C}_1 \|f\|_{\tilde{H}^{m-1}(\Omega)}.
\]

(1.2.16)

From the theorem we see (1.2.14) presents a bijection between the space \( \tilde{H}^{m+1}(\Omega) \cap H^1_0(\Omega) \cap \{u \mid \nabla^A u^+ = \nabla^A u^-\} \) and \( \tilde{H}^{m-1}(\Omega) \). Under this condition, we use the letter \( P \) to denote the operator from \( u \) to \( f \), that is, \( Pu = f \) and \( u = P^{-1} f \). Notice \( Pu \) is defined only when \( u \) lies in the space mentioned above. In this section, we will only study the case when \( h(x) \equiv 0 \).
1.2.6 Remark. It is not hard to check that $\tilde{H}^1(\Omega)$ is compactly embedded in $L^2(\Omega)$.

The Theorem 1.2.1 and 1.2.4 in this subsection present the existence and regularity of solution to the elliptic transmission problem (1.2.1). Based on these results, we can move on to study the behavior of solution to the parabolic problem (1.1.6).

1.2.2 Analysis of the Parabolic Transmission Problem

1.2.7 Lemma. According to our Theorem 1.2.4 and its corollary, the problem (1.2.14) describes a bijection $P$ between $u$ and $f$ in the case of $h = 0$. We denote as $P\bar{u} = f$. Then there exists a countable orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions of $P$, say $(w_k)_{k=1}^{\infty}$. Moreover, $(w_k)_{k=1}^{\infty}$ is also a complete $B$-orthogonal basis of $H^1_0(\Omega)$, that is, $B[w_k, w_j] = 0$ for each $k \neq j$.

Proof. From the estimate (1.2.16), we obtain that for any $f \in L^2(\Omega)$,

$$
\|u\|_{H^1_0(\Omega)} \leq \tilde{C}_1 \|f\|_{L^2(\Omega)},
$$

for some constant $\tilde{C}_1$. That is $\|P^{-1}f\|_{H^1_0(\Omega)} \leq \tilde{C}_1 \|f\|_{L^2(\Omega)}$. By the Rellich-Kondrachov Theorem, $H^1_0(\Omega)$ is compactly embedded on $L^2(\Omega)$. This reveals $P^{-1}$ is a compact operator on $L^2(\Omega)$.

At the same time, for any $f_1$, $f_2 \in L^2(\Omega)$ and $u_i = P^{-1}f_i$, $i = 1, 2$, we have $(P\bar{u}_1, u_2) = B[u_1, u_2] = (u_1, P\bar{u}_2)$. That is, $\langle f_1, P^{-1}f_2 \rangle = \langle P^{-1}f_1, f_2 \rangle$. Thus $P^{-1}$ is symmetric on $L^2(\Omega)$.

Apply the Spectral Theorem of compact self-adjoint operators (See the Appendix D of [15]), there exists a countable orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions of $P^{-1}$. Since $P$ and $P^{-1}$ have exactly the same eigenfunctions, we finished the first part of lemma.

Now let us analyze the sequence of functions $(w_k)_{k=1}^{\infty}$. Since they are eigenfunctions of $P^{-1}$, we have $P^{-1}w_k = \lambda_k w_k$. We observe from (1.2.14) that $w_k \in H^1_0(\Omega)$ and $\nabla_y w_k^+ = \nabla_y w_k^-$. Also, we have $B[w_k, w_j] = (Pw_k, w_j) = \lambda_k \langle w_k, w_j \rangle = 0$ for $k \neq j$, this means $(w_k)_{k=1}^{\infty}$ are $B$-orthogonal. Now let us prove $\text{span}(w_k)_{k=1}^{\infty}$ is complete in $H^1_0(\Omega)$. Assume there exists a function $g \in H^1_0(\Omega)$ such that $g \in \text{span}(w_k)_{k=1}^{\infty}$ with respect to $B$, then by definition $B[g, w_k] = 0$ for all $k \in \mathbb{N}^+$. Since $B[g, w_k] = (g, Pw_k) = (g, \nabla_y^+ w_k - \nabla_y^- w_k) = (g, Pw_k) = \lambda_k (g, w_k)$ and $\lambda_k \neq 0$ (otherwise $w_k \equiv 0$), we obtain $(g, w_k) = 0$ as well. This result reveals $g = 0$ as $\text{span}(w_k)_{k=1}^{\infty}$ is complete in $L^2(\Omega)$. Hence we complete the proof.

$\Box$
1.2.8 Remark. To make the notations clear, we always write $P u = f$ or $u = P^{-1} f$ when $u$ solves the boundary value problem

\[
\begin{aligned}
\begin{cases}
L u &= f, \\
u &= 0, \\
u^+ - u^- &= 0,
\end{cases}
\text{in } \bigcup_{k=1}^{K} \Omega_k,
\end{aligned}
\]

on $\partial \Omega$,
(1.2.18)

And we use the notation $\mathcal{T}$ to represent the following space:

\[
\mathcal{T} = \{ u \mid u \in H^1_0(\Omega) \cap \dot{H}^2(\Omega) \text{ and } \nabla^A u^+ - \nabla^A u^- = 0 \}.
\]

(1.2.19)

Observe that the expression $Pu$ make sense if and only if $u \in \mathcal{T}$.

1.2.9 Theorem. Assume $L$ and $\Omega$ are defined as above, with $f \in L^2(0, T; H^{-1}(\Omega))$, $g \in L^2(\Omega)$, then the weak solution of the parabolic transmission problem (1.1.6) is defined as $B[u, v] + \langle u_t, v \rangle = (f, v)$ for all $v \in H^1_0(\Omega)$ and a.e. $t \in (0, T]$, with the initial condition $u(0) = g$. Therefore, by the classical results, there exists a unique weak solution $u \in L^2(0, T; H^1(\Omega))$, with $u' \in L^2(0, T; H^{-1}(\Omega))$.

Proof. Since we can consider the problem as

\[
\begin{aligned}
\begin{cases}
L u &= f - u_t, \\
u &= 0, \\
u^+ - u^- &= 0,
\end{cases}
\text{on } \bigcup_{k=1}^{K} \Omega_k,
\end{aligned}
\]

on $\partial \Omega$,
(1.2.20)

for a.e. $t > 0$. Apply the Theorem 1.2.1 in the case of $h = 0$, we obtain the following weak formulation

\[
B[u, v] = \langle f - u_t, v \rangle \quad \forall v \in H^1_0(\Omega).
\]

(1.2.21)

According to the standard results on the linear parabolic problem, there exists a unique function $u \in L^2(0, T; H^1(\Omega)) \cap H^1(0, T; H^{-1}(\Omega))$ satisfying the weak formulation (1.2.21) with the initial condition $u(0) = g$, with the estimate

\[
\|u\|_{L^2(0, T; H^1(\Omega))} + \|u'\|_{L^2(0, T; H^{-1}(\Omega))} \leq C(\|f\|_{L^2(0, T; H^{-1}(\Omega))} + \|g\|_{L^2(\Omega)}).
\]

(1.2.22)
1.2.10 Remark. In the Theorem 7.1.2 of Evans [15], the weak solution \( u \) to the problem (1.2.21) is constructed by a certain sequence \( \{u_m\}_{m=1}^{\infty} \) and then passing to limits. Here is the sketch of his proof: First, we choose a set of functions \( \{w_k\}_{k=1}^{\infty} \) such that \( \{w_k\}_{k=1}^{\infty} \) is an orthonormal basis of \( L^2(\Omega) \), as well as a complete basis of \( H^1_0(\Omega) \). Now fix a positive integer \( m \), we look for a function \( u_m : [0, T] \to H^1_0(\Omega) \) of the form

\[
u_m(t) = \sum_{k=1}^{m} d_m^k(t) w_k, \quad (1.2.23)\]

where the coefficients \( d_m^k(t) \) \( 0 \leq t \leq T, \ k = 1, 2, ..., m \) is selected to satisfy the equations

\[
d_m^k(0) = (g, w_k) \quad (k = 1, ..., m); \quad (1.2.24)\]

and

\[
(u_m', w_k) + B[u_m, w_k] = \langle f, w_k \rangle \quad (0 \leq t \leq T, \ k = 1, ..., m). \quad (1.2.25)\]

According to standard existence theory for ODEs, there exists a unique absolutely continuous function \( d_m(t) = (d_m^1(t), ..., d_m^{m}(t)) \) satisfying the condition (1.2.24) and (1.2.25) for a.e. \( 0 \leq t \leq T \). Therefore, \( u_m \) is uniquely determined. By a standard energy estimate, the sequence \( \{u_m\}_{m=1}^{\infty} \) is uniformly bounded in \( L^2(0, T; H^1_0(\Omega)) \cap H^1(0, T; H^{-1}(\Omega)) \) with the estimate

\[
\|u_m\|_{L^2(0, T; H^1_0(\Omega))} + \|u_m'\|_{L^2(0, T; H^{-1}(\Omega))} \leq C(\|f\|_{L^2(0, T; H^{-1}(\Omega))} + \|g\|_{L^2(\Omega)}). \quad (1.2.26)\]

Thus there exists a subsequence \( \{u_{m_L}\}_{L=1}^{\infty} \subset \{u_m\}_{m=1}^{\infty} \) which converges weakly to a function \( u \) in both norms. By passing to the weak limit, \( u \) satisfy the weak formulation (1.2.21) as well as the initial condition \( u(0) = g \). Moreover, we check \( u \equiv 0 \) when \( f \equiv g \equiv 0 \), which suffices to show such solution \( u \) is unique.

In our transmission problem, let us choose \( \{w_k\}_{k=1}^{\infty} \) as mentioned above to be the complete set of normalized eigenfunctions of \( P \) in \( H^1_0(\Omega) \), where \( P \) is the operator defined in our Lemma 1.2.7. As a consequence of Lemma 1.2.7, the set of functions \( \{w_k\}_{k=1}^{\infty} \) meet the requirement, after a normalization in \( L^2(\Omega) \). More specifically, \( \|w_k\|_{L^2(\Omega)} = 1 \) and \( B[w_k, w_j] = 0 \) when \( k \neq j \). We will need the B-orthogonality of \( \{w_k\}_{k=1}^{\infty} \) to obtain some regularity results.
In the next theorem, we will study the existence of strong solution to the problem (1.1.6), after giving some higher regularity to the initial data.

**1.2.11 Theorem.**

1. Assume \( g \in H^1_0(\Omega) \), \( f \in L^2(0,T;L^2(\Omega)) \), \( L \) and \( \Omega \) are defined as above. Suppose also \( u \) is the weak solution of the parabolic problem (1.1.6), then in fact

\[
 u \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1_0(\Omega)), u' \in L^2(0,T;L^2(\Omega)),
\]

and we have the estimate

\[
 \text{ess sup}_{0 \leq t \leq T} \| u(t) \|_{H^1_0(\Omega)} + \| u \|_{L^2(0,T;H^2(\Omega))} + \| u' \|_{L^2(0,T;L^2(\Omega))} \leq C(\| f \|_{L^2(0,T;L^2(\Omega))} + \| g \|_{H^1_0(\Omega)}).
\]

The constant \( C \) depend only on \( \Omega \), \( T \) and the coefficient of \( L \).

2. If in addition, \( f \in L^2(0,T;H^1(\Omega)) \), \( f' \in L^2(0,T;H^{-1}(\Omega)) \) and \( g \in \mathcal{I} \), where \( \mathcal{I} \) is a function space defined in the Remark 1.2.8 that

\[
 \mathcal{I} = \{ g \mid g \in H^1_0(\Omega) \cap H^2(\Omega) \text{ and } \nabla^A g^+ - \nabla^A g^- = 0 \}. \tag{1.2.27}
\]

Then

\[
 u \in L^\infty(0,T;H^2(\Omega)), u' \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega)), u'' \in L^2(0,T;H^{-1}(\Omega)),
\]

with the estimate

\[
 \text{ess sup}_{0 \leq t \leq T} \left( \| u'(t) \|_{L^2(\Omega)} + \| u(t) \|_{H^2(\Omega)} + \| u' \|_{L^2(0,T;H^2_0(\Omega))} + \| u'' \|_{L^2(0,T;H^{-1}(\Omega))} \right)
\]

\[
 \leq C(\| f \|_{H^1(0,T;H^{-1}(\Omega))} + \| f \|_{L^2(0,T;H^1_0(\Omega))} + \| g \|_{H^2(\Omega)}).
\]

**Proof.**

1. We rely on the proof of theorem 7.1.5 in Evans [15]. Recall our Remark 1.2.10, the sequence of functions \( \{ u_m \}_{m=1}^\infty \) defined by (1.2.23) and (1.2.25) contains a subsequence \( \{ u_{m'} \}_{m'=1}^\infty \subset \{ u_m \}_{m=1}^\infty \) which converges weakly to \( u \). Our method is to analyze the regularity of \( u_m \) and then pass to a weak limit.

Let us fix \( m \geq 1 \), we multiply equation (1.2.25) by \( d_m^k(t) \) and sum \( k = 1, 2, \ldots, m \) to discover

\[
 (u_m', u_m') + B[u_m, u_m'] = \langle f, u_m' \rangle
\]
for a.e. $0 \leq t \leq T$. Now

$$B[u_m, u'_m] = \int_{\Omega} \sum_{i,j=1}^{n} a_{ij} u_{m,x_i} u'_{m,x_j} \, dx.$$ 

Since $a_{ij} = a_{ji}$ (i, j = 1, 2, ..., n) and these coefficients do not depend on t, we see $B[u_m, u'_m] = \frac{d}{dt} (\frac{1}{2} B[u_m, u_m])$. Moreover, we have

$$|\langle f, u'_m \rangle| \leq \frac{1}{2} \|f\|^2_{L^2(\Omega)} + \frac{1}{2} \|u'_m\|^2_{L^2(\Omega)}.$$ 

Therefore

$$\left\| u'_m \right\|^2_{L^2(\Omega)} + \frac{d}{dt} \frac{1}{2} B[u_m, u_m] \leq \frac{1}{2} \|f\|^2_{L^2(\Omega)} + \frac{1}{2} \|u'_m\|^2_{L^2(\Omega)}.$$ 

Integrating t from 0 to T, we find,

$$\int_0^T \left\| u'_m \right\|^2_{L^2(\Omega)} \, dt + \sup_{0 \leq t \leq T} B[u_m(t), u_m(t)] \leq C(B[u_m(0), u_m(0)] + \int_0^T \|f\|^2_{L^2(\Omega)} \, dt).$$

Recall the equation (1.2.24), since $\{w_k\}_{k=1}^{\infty}$ is an $B$-orthogonal basis of $H^1_0(\Omega)$, and $u_m(0)$ is a partial sum of $g$, we have the estimate

$$\left\| u_m(0) \right\|^2_{H^1_0(\Omega)} \leq C \sup_{0 \leq t \leq T} B[u_m(t), u_m(t)] \leq C \sup_{0 \leq t \leq T} B[u_m(t), u_m(t)] \leq C \left\| g \right\|^2_{H^1_0(\Omega)}.$$ 

Combine the above two inequalities, we find that

$$\left\| u'_m \right\|^2_{L^2(0,T;L^2(\Omega))} \leq C(\|g\|^2_{H^1_0(\Omega)} + \|f\|^2_{L^2(0,T;L^2(\Omega))}),$$

and

$$\sup_{0 \leq t \leq T} \left\| u_m(t) \right\|^2_{H^1_0(\Omega)} \leq \sup_{0 \leq t \leq T} B[u_m(t), u_m(t)] \leq C(\|g\|^2_{H^1_0(\Omega)} + \|f\|^2_{L^2(0,T;L^2(\Omega))}).$$

Passing to limits as $m = m_t \to \infty$ we deduce $u \in L^\infty(0, T; H^1_0(\Omega)), u' \in L^2(0, T; L^2(\Omega))$, with the stated bounds.
Now we recall the weak formulation of $u$:

$$B[u, v] = (f - u, v) \quad \forall v \in H^1_0(\Omega).$$

As $f - u \in L^2(0, T; L^2(\Omega))$, Resulting from Theorem 1.2.4 directly with $m = 1$, we deduce $Lu = f - u$ a.e. on each $\Omega_j$ and have the following inequality

$$\|u\|_{L^2(\Omega)} \leq C \|Lu\|_{L^2(\Omega \setminus \Gamma)}.$$  \hspace{1cm} (1.2.28)

Integrating from $0$ to $T$, we obtain

$$\|u\|_{L^2(0, T; L^2(\Omega))} \leq C \|f - u\|_{L^2(0, T; L^2(\Omega))} \leq C(\|f\|_{L^2(0, T; L^2(\Omega))} + \|g\|_{H^1(\Gamma)}).$$  \hspace{1cm} (1.2.29)

Here we complete the proof of (1).

1.2.12 Corollary. Now we see $Lu + u = f$ a.e. on each subdomain $\Omega_j$. After integration by parts, we obtain the transmission condition $\nabla^A v^+ u^- - \nabla^A u^- = 0$ a.e. on $x \in \Gamma$. Therefore, $u$ solves the parabolic transmission problem (1.1.6) strongly.

2. Now suppose $g \in \mathcal{T}, f \in H^1(\Omega_j; H^{-1}(\Omega)) \cap L^2(0, T; H^1(\Omega))$. Fix $m \geq 1$ and differentiate equation (1.2.25) with respect to $t$. Since $L$ does not depend on $t$, we find

$$(\tilde{u}^{'}, w_k) + B[\tilde{u}_m, w_k] = \langle f', w_k \rangle \quad (k = 1, \ldots, m),$$

Where $\tilde{u}_m = u^{'}. Multiply this equation by $d^{'m}(t)$ and sum $k=1,2,\ldots,m$:

$$(\tilde{u}', \tilde{u}_m) + B[\tilde{u}_m, \tilde{u}_m] = \langle f', \tilde{u}_m \rangle.$$  \hspace{1cm} (1.2.30)

By our previous results, $(\tilde{u}', \tilde{u}_m) = \frac{1}{2} \frac{d}{dt} \|\tilde{u}_m\|_{L^2(\Omega)}^2, B[\tilde{u}_m, \tilde{u}_m] \geq \beta \|\tilde{u}_m\|_{H^1(\Omega)}^2$ and $\langle f', \tilde{u}_m \rangle \leq \frac{\beta}{2} \|\tilde{u}_m\|_{H^1(\Omega)}^2 + \frac{2}{p} \|f'\|_{H^{-1}(\Omega)}^2$ for some constant $\beta > 0$. Apply these inequalities to the equation (1.2.30), we find

$$\frac{d}{dt} \|\tilde{u}_m\|_{L^2(\Omega)}^2 + \|\tilde{u}_m\|_{H^1(\Omega)}^2 \leq C(\|\tilde{u}_m\|_{L^2(\Omega)}^2 + \|f'\|_{H^{-1}(\Omega)}^2).$$  \hspace{1cm} (1.2.31)
Using Gronwall’s Inequality, we deduce

\[
\sup_{0 \leq t \leq T} \| \hat{u}_m(t) \|_{L^2[\Omega]}^2 \leq C(\| \hat{u}_m(0) \|_{L^2[\Omega]}^2 + \| f' \|_{L^2[0,T;H^{-1}(\Omega)]}^2). \tag{1.2.32}
\]

Since \( \hat{u}_m(0) = u'_m(0) \), we recall the equation (1.2.25) to find \((u'_m(0), w_k) + B[u_m(0), w_k] = (f(0), w_k)\). Here \( f(0) \in L^2(\Omega) \) according to the interpolation theorem of Sobolev Space. As \( \{w_k\}_{k=1}^\infty \) is orthonormal in \( L^2(\Omega) \) and B-orthogonal in \( H_0^1(\Omega) \), we have the equality

\[
B[u_m(0), w_k] = B[g, w_k] = (Lg, w_k)_{\Omega \setminus \Gamma} + (\nabla^A v^+ - \nabla^A v^-, w_k)_{\Gamma} = (Pg, w_k)_{\Omega}.
\]

Here \( P \) is an operator defined in the Remark 1.2.8. This equation reveals \((u'_m(0), w_k) = (f(0) - Pg, w_k)\). By orthogonality of \( w_k \), we obtain the following estimate

\[
\| u'_m(0) \|_{L^2[\Omega]}^2 \leq \| f(0) - Pg \|_{L^2[\Omega]}^2 \leq C(\| f \|_{H^1(0,T;H^{-1}(\Omega))}^2 + \| f' \|_{L^2[0,T;H^1(\Omega)]}^2 + \| g \|_{H^2(\Omega)}^2).
\]

Now let us look back to (1.2.32). Apply the last inequality, we deduce

\[
\sup_{0 \leq t \leq T} \| u'_m(t) \|_{L^2[\Omega]}^2 \leq C(\| f \|_{H^1(0,T;H^{-1}(\Omega))}^2 + \| f' \|_{L^2[0,T;H^1(\Omega)]}^2 + \| g \|_{H^2(\Omega)}^2). \tag{1.2.33}
\]

Integrating the equation (1.2.31) from 0 to \( T \), we obtain the estimate

\[
\| u'_m \|_{L^2[0,T;H^1_0(\Omega)]}^2 = \int_0^T \| \hat{u}_m \|_{H^1_0(\Omega)}^2 \, dt \leq C(\| f \|_{H^1(0,T;H^{-1}(\Omega))}^2 + \| f' \|_{L^2[0,T;H^1(\Omega)]}^2 + \| g \|_{H^2(\Omega)}^2).
\tag{1.2.34}
\]

Passing to limits as \( m = m_l \to \infty \) we deduce \( u' \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega)) \), with the stated bounds.

At the same time, recall the estimate (1.2.28). Since \( f \) and \( u' \) both lies in \( L^\infty(0,T;L^2(\Omega)) \), we have

\[
\sup_{0 \leq t \leq T} \| u(t) \|_{H^2(\Omega)}^2 \leq \sup_{0 \leq t \leq T} \| Lu(t) \|_{H^2(\Omega)}^2 \leq \sup_{0 \leq t \leq T} \| f(t) \|_{L^2(\Omega)}^2 \leq \sup_{0 \leq t \leq T} \| u'(t) \|_{L^2(\Omega)}^2 \leq C(\| f \|_{H^1(0,T;H^{-1}(\Omega))}^2 + \| f' \|_{L^2[0,T;H^1(\Omega)]}^2 + \| g \|_{H^2(\Omega)}^2).
\tag{1.2.35}
\]
Hence $u \in L^\infty(0,T;\dot{H}^2(\Omega))$, with the above estimate.

What remains to be shown is $u'' \in L^2(0,T;H^{-1}(\Omega))$. Now fix any $v \in H^1_0(\Omega)$, with $\|v\|_{H^1_0(\Omega)} \leq 1$, and write $v = v^1 + v^2$, where $v^1 \in \text{span}\{w_k\}_{k=1}^m$ and $B[v_2,w_k] = 0$ ($k = 1, ..., m$). Since the functions $\{w_k\}_{k=1}^\infty$ are B-orthogonal in $H^1_0(\Omega)$, we have $\|v^1\|_{H^1_0(\Omega)} \leq C \|v\|_{H^1_0(\Omega)} \leq C$ for some constant $C$. Utilizing (1.2.25), we deduce for a.e. $0 \leq t \leq T$ that

$$ \langle \tilde{u}'_m, v^1 \rangle + B[\tilde{u}'_m, v^1] = \langle f', v^1 \rangle. \quad (1.2.36) $$

As $\{w_k\}_{k=1}^\infty$ is orthogonal in $L^2(\Omega)$, this implies

$$ \langle \tilde{u}'_m, v \rangle = \langle \tilde{u}'_m, v^1 \rangle = \langle \tilde{u}'_m, v^1 \rangle. $$

Consequently, by the equation (1.2.36)

$$ |\langle \tilde{u}'_m, v \rangle| \leq C(\|f\|_{H^{-1}_0(\Omega)} + \|\tilde{u}'_m\|_{H^1_0(\Omega)}), $$

since $\|v^1\|_{H^1_0(\Omega)} \leq C$. Thus

$$ \|\tilde{u}'_m\|_{H^{-1}_0(\Omega)} \leq C(\|f\|_{H^{-1}_0(\Omega)} + \|\tilde{u}'_m\|_{H^1_0(\Omega)}), $$

and therefore

$$ \int_0^T \|\tilde{u}'_m\|^2_{H^{-1}_0(\Omega)} \, dt \leq C \int_0^T \|f'\|^2_{L^2(\Omega)} + \|\tilde{u}'_m\|^2_{H^1_0(\Omega)} \, dt \leq C(\|g\|^2_{\dot{H}^1(\Omega)} + \|f'\|^2_{L^2(0,T;L^2(\Omega))}). $$

Passing to limits as $m = m_1 \to \infty$, we deduce $u'' = \tilde{u}' \in L^2(0,T;H^{-1}(\Omega))$, with the above estimate.

\[\blacksquare\]

1.2.3 Higher Regularity of Solution with Compatible Initial Data

The second part of the theorem 1.2.11 reveals that $u$ lies in a better space when a better initial data is given. Now we are going to show that, for the parabolic transmission problem
(1.1.6), if the initial data is smooth enough and satisfy some compatibility condition, the solution will be smooth enough on each subdomain as well.

1.2.13 Lemma. Assume \( g \in \mathcal{T}, f \in L^2(0, T; \hat{H}^1(\Omega)), \tilde{f} = f' \in L^2(0, T; H^{-1}(\Omega)), \tilde{g} = f(0) - Pg \in L^2(\Omega) \).

Suppose also \( w \in L^2(0, T; H^1_0(\Omega)), \) with \( w_0 \in L^2(0, T; H^{-1}_0(\Omega)) \), is the weak solution of

\[
\begin{align*}
\left\{ \begin{array}{ll}
w_t + Lw = f, & \text{in } \bigcup_{k=1}^K \Omega_k \times [0, T], \\
w = 0, & \text{on } \partial \Omega \times [0, T], \\
w = \tilde{g}, & \text{on } \Omega \times \{t = 0\}, \\
w^+ - w^- = 0, & \text{on } \Gamma \text{ and a.e. } 0 \leq t \leq T, \\
\nabla_x^+ w^+ - \nabla_x^- w^- = 0, & \text{on } \Gamma \text{ and a.e. } 0 \leq t \leq T.
\end{array} \right.
\end{align*}
\]

(1.2.37)

And \( u \in L^2(0, T; H^1_0(\Omega)), \) with \( u_0 \in L^2(0, T; H^{-1}_0(\Omega)) \) is the weak solution of (1.1.6).

Then in fact \( u' = w, \) a.e.

Proof. We already proved that, both of the initial value problems (1.2.37) and (1.1.6) have a unique weak solution. Also, \( u \) satisfy the weak formulation and the initial condition

\[
B[u, v] + \langle u_t, v \rangle = \langle f, v \rangle, \ u(0) = g, \ \forall \ v \in H^1_0(\Omega);
\]

while \( w \) satisfy

\[
B[w, v] + \langle u_t, v \rangle = \langle \tilde{f}, v \rangle, \ w(0) = \tilde{g}, \ \forall \ v \in H^1_0(\Omega).
\]

(1.2.38)

Now we define a new function \( \hat{u} \) related to \( w \) by:

\[
\hat{u}(t) = \int_0^t w(s) ds + g, \quad \text{for a.e. } x \in \Omega, \ t \geq 0.
\]

(1.2.39)

Notice \( w \in L^2(0, T; H^1_0(\Omega)) \cap H^1(0, T; H^{-1}(\Omega)) \), this results in \( w \in C([0, T]; L^2(\Omega)) \). Therefore, the function \( \hat{u} \) is well defined and is continuous in space \( L^2 \) norm. Obviously, \( \langle \hat{u}_t, v \rangle = \langle w, v \rangle \) for all \( v \in H^1_0(\Omega) \) and a.e. \( t \). Now we integrate the equation (1.2.38) through \( 0 \) to \( t \):

\[
\int_0^t B[w(s), v] ds + \int_0^t \langle w_t(s), v \rangle ds = \int_0^t \langle \tilde{f}'(s), v \rangle ds.
\]

(1.2.40)
Here we set \( v \) to be constant in time. By the fact that \( w(s) \in L^2(0, T; H^1_0(\Omega)) \), \( w_t(s) \in L^2(0, T; H^{-1}(\Omega)) \), and \( t'(s) \in L^2(0, T; H^{-1}(\Omega)) \), we can apply Theorem 4 in 5.9.2 in Evans [15] to obtain

\[
B[\tilde{u}(t) - \tilde{u}(0), v] + \langle w(t) - w(0), v \rangle = (f(t) - f(0), v); \tag{1.2.41}
\]

That is

\[
B[\tilde{u}(t), v] + \langle \tilde{u}_t(t), v \rangle = (f(t), v) + B[\tilde{u}(0), v] + \langle w(0), v \rangle - (f(0), v). \tag{1.2.42}
\]

As we can compute

\[
B[\tilde{u}(0), v] + \langle w(0), v \rangle - (f(0), v) = B[g, v] + (f(0) - Pg, v) - (f(0), v) = (Pg + f(0) - Pg - f(0), v) = 0, \tag{1.2.43}
\]

Therefore we obtain

\[
B[\tilde{u}(t), v] + \langle \tilde{u}_t(t), v \rangle = (f(t), v). \tag{1.2.44}
\]

Recall the fact (1.2.39) that \( \tilde{u}(0) = g \) and \( \tilde{u} \in H^1_0(\Omega) \) for a.e. \( t \), we can observe \( \tilde{u} = u \) and is the unique weak solution to the IVP (1.1.6). As \( w = \tilde{u}_t \) solves the IVP (1.2.37), we complete the proof.

1.2.14 Theorem. (i) Assume

\[
\frac{d^k f}{dt^k} \in L^2(0, T; H^{2m-2k}(\Omega)), \quad k = 0, 1, \ldots, m.
\]

suppose also that the following \( m^{th} \) order compatibility conditions hold:

\[
g_0 = g \in \mathcal{T}, \; g_1 = f(0) - Pg_0 \in \mathcal{T}, \; \ldots, \; g_{m-1} = \frac{d^{m-2} f}{dt^{m-2}}(0) - Pg_{m-2} \in \mathcal{T}, \; g_m = \frac{d^{m-1} f}{dt^{m-1}}(0) - Pg_{m-1} \in H^1_0(\Omega).
\]

Then

\[
\frac{d^k u}{dt^k} \in L^2(0, T; H^{2m+2-2k}(\Omega)), \quad k = 0, 1, \ldots, m + 1;
\]

and we have the estimate

\[
\sum_{k=0}^{m+1} \left\| \frac{d^k u}{dt^k} \right\|_{L^2(0, T; H^{2m+2-2k}(\Omega))} \leq C \left( \sum_{k=0}^{m} \left\| \frac{d^k f}{dt^k} \right\|_{L^2(0, T; H^{2m-2k}(\Omega))} + \| g \|_{H^{2m+1}(\Omega)} \right),
\]
where the constant $C$ only depend on $m$, $\Omega$, $\Gamma$, $T$ and the coefficients of $L$.

(ii) If in addition, $g_m \in \mathcal{F}$, and

\[
\frac{d^k f}{dt^k} \in L^2(0, T; \dot{H}^{2m-2(2k+1)}(\Omega)), \quad k = 0, 1, \ldots, m, m+1.
\]

Then

\[
\frac{d^k u}{dt^k} \in L^\infty(0, T; \dot{H}^{2m+2-2k}(\Omega)), \quad k = 0, 1, \ldots, m+1;
\]

and we have the estimate

\[
\sum_{k=0}^{m+1} \left\| \frac{d^k u}{dt^k} \right\|_{L^\infty(0, T; \dot{H}^{2m+2-2k}(\Omega))} \leq C \sum_{k=0}^{m+1} \left\| \frac{d^k f}{dt^k} \right\|_{L^2(0, T; \dot{H}^{2m-2(2k+1)}(\Omega))} + \| g \|_{\dot{H}^{2m+2}(\Omega)},
\]

where the constant $C$ only depend on $m$, $\Omega$, $\Gamma$, $T$ and the coefficients of $L$.

1.2.15 Remark. Taking into account Theorem 4 in 5.9.2 in Evans, we see that

\[
f(0) \in \dot{H}^{2m-1}(\Omega), \quad f'(0) \in \dot{H}^{2m-3}(\Omega), \quad \ldots, \quad f^{(m-1)}(0) \in \dot{H}^{1}(\Omega),
\]

and consequently,

\[
g_0 \in \dot{H}^{2m+1}(\Omega), \quad g_1 \in \dot{H}^{2m-1}(\Omega), \quad \ldots, \quad g_m \in \dot{H}^{1}(\Omega).
\]

The compatibility conditions are consequently the requirements that in addition each of these functions satisfy the boundary and interface conditions, in the trace sense.

Proof. 1. The proof is an induction on $m$, the case $m = 0$ being Theorem 1.2.11 above.

Assume now the theorem is valid for some nonnegative integer $m$, and suppose then

\[
g \in \dot{H}^{2m+3}(\Omega), \quad \frac{d^k f}{dt^k} \in L^2(0, T; \dot{H}^{2m+2-2k}(\Omega)), \quad k = 0, \ldots, m+1,
\]
and the \((m + 1)^{th}\) order compatibility conditions hold. Now set \(\bar{u} = u'\). Resulting from lemma 1.2.13, we check that \(\bar{u}\) is the unique weak solution of

\[
\begin{align*}
\bar{u}_t + L\bar{u} &= f, & \text{in } \Omega_T; \\
\bar{u} &= 0, & \text{on } \partial\Omega \times [0, T]; \\
\bar{u} &= \bar{g}, & \text{on } \Omega \times \{t = 0\}; \\
\bar{u}^+ - \bar{u}^- &= 0, & \text{on } \Gamma \text{ and a.e. } 0 \leq t \leq T; \\
\nabla_y^A \bar{u}^+ - \nabla_y^A \bar{u}^- &= 0, & \text{on } \Gamma \text{ and a.e. } 0 \leq t \leq T.
\end{align*}
\]

for \(\bar{f} = f_t\), \(\bar{g} = f(0) - Lg\). In particular, for \(m = 0\) we rely upon Theorem 1.2.11 to be sure that \(\bar{u} \in L^2(0, T; H^1_0(\Omega)), \bar{u}' \in L^2(0, T; H^{-1}(\Omega))\).

Since \(f\) and \(g\) satisfy the \((m + 1)^{th}\) order compatibility conditions, it follows that \(\bar{f}\) and \(\bar{g}\) satisfy the \(m^{th}\) order compatibility condition. Thus applying the induction assumption, we deduce

\[
\frac{d^k u}{dt^k} \in L^2(0, T; \dot{H}^{2m+2-2k}(\Omega)), \quad k = 0, \ldots, m + 1,
\]

and

\[
\sum_{k=0}^{m+1} \left\| \frac{d^k u}{dt^k} \right\|_{L^2(0, T; \dot{H}^{2m+2-2k}(\Omega))} \leq C \left( \sum_{k=0}^{m} \left\| \frac{d^k f}{dt^k} \right\|_{L^2(0, T; \dot{H}^{2m-2k}(\Omega))} + \| g \|_{\dot{H}^{2m+1}(\Omega)} \right)
\]

for \(\bar{f} = f'\). Since \(\bar{u} = u'\), we can rewrite the foregoing:

\[
\frac{d^k u}{dt^k} \in L^2(0, T; \dot{H}^{2m+4-2k}(\Omega)), \quad k = 1, \ldots, m + 2;
\]

\[
\sum_{k=1}^{m+2} \left\| \frac{d^k u}{dt^k} \right\|_{L^2(0, T; \dot{H}^{2m+4-2k}(\Omega))} \leq C \left( \sum_{k=1}^{m+1} \left\| \frac{d^k f}{dt^k} \right\|_{L^2(0, T; \dot{H}^{2m+2-2k}(\Omega))} + \| Lg \|_{\dot{H}^{2m+1}(\Omega)} + \| f(0) \|_{\dot{H}^{2m+1}(\Omega)} \right)
\]

\[
\| f(0) \|_{\dot{H}^{2m+1}(\Omega)} \leq C \left( \sum_{k=0}^{m+1} \left\| \frac{d^k f}{dt^k} \right\|_{L^2(0, T; \dot{H}^{2m+2-2k}(\Omega))} + \| g \|_{\dot{H}^{2m+3}(\Omega)} \right).
\]

(1.2.46)
Here we use the estimate
\[ \| f(0) \|_{\tilde{H}^{2m+1}(\Omega)} \leq C(\| f \|_{L^2(0,T;\tilde{H}^{2m+2}(\Omega))} + \| f' \|_{L^2(0,T;\tilde{H}^{2m}(\Omega))}) , \]
which follows from Theorem 4 in 5.9.2 in Evans.

Now write for a.e. \( 0 \leq t \leq T \): \( Lu = f - u' = h \) in \( \Omega \setminus \Gamma \). According to theorem 1.2.4, we have
\[ \| u \|_{\tilde{H}^{2m+4}(\Omega)} \leq C(\| h \|_{\tilde{H}^{2m+2}(\Omega)} + \| u \|_{L^2(\Omega)}) \]
\[ \leq C(\| f \|_{\tilde{H}^{2m+2}(\Omega)} + \| u' \|_{\tilde{H}^{2m}(\Omega)} + \| u \|_{L^2(\Omega)}). \]

Integrating with respect to \( t \) from \( 0 \) to \( T \) and add the resulting expression to the equation (1.2.46), we deduce,
\[ \sum_{k=0}^{m+2} \left\| \frac{d^k u}{dt^k} \right\|_{L^2(0,T;\tilde{H}^{2m+4-2k}(\Omega))} \leq C(\sum_{k=0}^{m+1} \left\| \frac{d^k f}{dt^k} \right\|_{L^2(0,T;\tilde{H}^{2m+2-2k}(\Omega))} + \| g \|_{\tilde{H}^{2m+3}(\Omega)} + \| u \|_{L^2(0,T;L^2(\Omega))}). \]

Since
\[ \| u \|_{L^2(0,T;L^2(\Omega))} \leq C(\| f \|_{L^2(0,T;L^2(\Omega))} + \| g \|_{L^2(\Omega)}), \]
We thereby obtain the assertion of the theorem for \( m + 1 \). Hence we complete part (1).

2. Apply induction on \( m \) again. The case \( m = 0 \) being the part (2) of Theorem 1.2.11 above. Assume now the theorem is valid for some nonnegative integer \( m \), and suppose then
\[ g \in \tilde{H}^{2m+3}(\Omega), \frac{d^k f}{dt^k} \in L^2(0,T;\tilde{H}^{2m+3-2k}(\Omega)), \quad k = 0, \ldots, m + 2; \]
and the \( (m + 1)^{th} \) order compatibility conditions hold. Set \( \tilde{u} = u' \) again, still, \( \tilde{u} \) solves the equation (1.2.45). Since \( f \) and \( g \) satisfy the \( (m + 1)^{th} \) order compatibility conditions, it follows that \( \tilde{f} \) and \( \tilde{g} \) satisfy the \( m^{th} \) order compatibility condition. Thus applying the induction assumption, we deduce
\[ \frac{d^k \tilde{u}}{dt^k} \in L^\infty(0,T;\tilde{H}^{2m+2-2k}(\Omega)), \quad k = 0, \ldots, m + 1, \]
and
\[
\sum_{k=0}^{m+1} \left\| \frac{d^k u}{dt^k} \right\|_{L^\infty(0,T; \widetilde{H}^{2m+2-2k}(\Omega))} \leq C \left( \sum_{k=0}^{m} \left\| \frac{d^k f}{dt^k} \right\|_{L^\infty(0,T; \widetilde{H}^{2m-2k}(\Omega))} + \|g\|_{\widetilde{H}^{2m+1}(\Omega)} \right),
\]
we can rewrite the foregoing:
\[
\frac{d^k u}{dt^k} \in L^\infty(0,T; \widetilde{H}^{2m+4-2k}(\Omega)), \quad k = 1, \ldots, m+2;
\]
\[
\sum_{k=1}^{m+2} \left\| \frac{d^k u}{dt^k} \right\|_{L^\infty(0,T; \widetilde{H}^{2m+4-2k}(\Omega))} \leq C \left( \sum_{k=1}^{m+1} \left\| \frac{d^k f}{dt^k} \right\|_{L^\infty(0,T; \widetilde{H}^{2m+2-2k}(\Omega))} + \|Lg\|_{\widetilde{H}^{2m+1}(\Omega)} + \|f(0)\|_{\widetilde{H}^{2m+2}(\Omega)} \right)
\]
\[
\|f(0)\|_{\widetilde{H}^{2m+2}(\Omega)} \leq C \left( \sum_{k=0}^{m+1} \left\| \frac{d^k f}{dt^k} \right\|_{L^\infty(0,T; \widetilde{H}^{2m+2-2k}(\Omega))} + \|g\|_{\widetilde{H}^{2m+3}(\Omega)} \right).
\]
(1.2.47)
Here we use the estimate
\[
\|f(0)\|_{\widetilde{H}^{2m+2}(\Omega)} \leq C (\|f\|_{L^2(0,T; \widetilde{H}^{2m+3}(\Omega)}) + \|f'\|_{L^2(0,T; \widetilde{H}^{2m+1}(\Omega))}),
\]
which follows from Theorem 4 in 5.9.2 in Evans.
Now write for a.e. \(0 \leq t \leq T\): \(Lu = f - u' = h\) in \(\Omega \setminus \Gamma\). According to theorem 1.2.4, we have
\[
\|u\|_{\widetilde{H}^{2m+4}(\Omega)} \leq C (\|h\|_{\widetilde{H}^{2m+2}(\Omega)} + \|u\|_{L^2(\Omega)})
\]
\[
\leq C (\|f\|_{\widetilde{H}^{2m+2}(\Omega)} + \|u'\|_{\widetilde{H}^{2m+2}(\Omega)} + \|u\|_{L^2(\Omega)}).
\]
Taking the \(L^\infty\) norm in time and add the resulting expression to the equation (1.2.47), we deduce,
\[
\sum_{k=0}^{m+2} \left\| \frac{d^k u}{dt^k} \right\|_{L^\infty(0,T; \widetilde{H}^{2m+4-2k}(\Omega))} \leq C \left( \sum_{k=0}^{m+1} \left\| \frac{d^k f}{dt^k} \right\|_{L^\infty(0,T; \widetilde{H}^{2m+2-2k}(\Omega))} + \|g\|_{\widetilde{H}^{2m+3}(\Omega)} + \|u\|_{L^\infty(0,T; L^2(\Omega))} \right),
\]
Since
\[ \|u\|_{L^\infty(0,T;L^2(\Omega))} \leq C(\|f\|_{L^2(0,T;L^2(\Omega))} + \|g\|_{L^2(\Omega)}), \]

We thereby obtain the assertion of the theorem for \( m + 1 \).

\( \square \)

1.2.16 Remark. There are two ways to construct the functions \( f \) and \( g \) meeting all the compatibility conditions. One is to pick any function \( g \) in \( \tilde{H}^{2m+2}(\Omega) \cap \mathcal{T} \) and match this \( g \) with an appropriate function \( f \). We define \( f(t) \) to be a function-valued polynomial with respect to \( t \):

\[ f(t) = f(0) + f'(0)t + f''(0)\frac{t^2}{2!} + \ldots + \frac{d^{m-1}f}{dt^{m-1}}(0)\frac{t^{m-1}}{(m-1)!}. \]

Now \( g_0 = g \) is given. For each given \( g \in \mathcal{T}, k = 0, 1, \ldots, m - 1 \), we construct a function \( \frac{d^k}{dt^k}(0) \) that lies in \( \tilde{H}^{2m-2k}(\Omega) \) and satisfy \( \frac{d^k}{dt^k}(0) - Pg_k \in \mathcal{T} \). After \( m \) steps we will eventually obtain the function \( f \).

Another way to meet the compatibility conditions is to apply the set of eigenfunctions \( \{w_j\}_{j=1}^\infty \) in the lemma 1.2.7, where \( Pw_j = \lambda_j w_j, j = 1, 2, \ldots \). Recall this lemma, \( w_j \in H^1_0(\Omega) \) and solves the following problem strongly:

\[ \begin{align*}
Lu &= \lambda_j w_j, & \text{in } \bigcup_{k=1}^K \Omega_k, \\
u &= 0, & \text{on } \partial \Omega, \\
u^+ - \nu^- &= 0, & \text{on } \Gamma, \\
\nabla\nu^+ - \nabla\nu^- &= 0, & \text{on } \Gamma.
\end{align*} \tag{1.2.48} \]

The theorem 1.2.4 guarantees \( w_j = u = P^{-1}(\lambda_j w_j) \in \tilde{H}^3(\Omega) \cap \mathcal{T}, j = 1, 2, \ldots \). By induction we can deduce \( w_j = P^{-m}(\lambda_j^{-m} w_j) \in \mathcal{T} \cap \tilde{H}^{2m+1}(\Omega) \) for all integer \( m \geq 1 \). This reveals that, if we define \( f \) and \( g \) to be the finite sum of some eigenfunctions of \( P \):

\[ g = c_1 w_1 + c_2 w_2 + \ldots + c_N w_N, \]

and

\[ f = d_1(t) w_1 + d_2(t) w_2 + \ldots + d_N(t) w_N, \]

where \( \{c_i\}_{i=1}^N \in \mathbb{R}, \{d_i(t)\}_{i=1}^N \in C^m[0,T] \), then the compatibility conditions will automatically hold.
1.3 Problem with Generalized Transmission Condition

From the previous section, we studied that the solution to the parabolic transmission problem (1.1.6) can be sufficiently smooth on each subdomain given proper initial data. In this section, we intend to generalize the transmission condition (1.1.5) and (1.1.2) on the interface \( \Gamma \), by allowing \( u \) and \( \nabla^A \gamma u \) to be discontinuous across the interface.

1.3.1 Generalized Initial-Boundary Value Problem

1.3.1 Theorem. Consider the following non-homogenous initial value problem:

\[
\begin{aligned}
& u_t + Lu = f \quad \text{in } \bigcup_{k=1}^{K} \Omega_k \times [0, T], \\
& u = 0 \quad \text{on } \partial \Omega \times [0, T], \\
& u = g \quad \text{on } \bigcup_{k=1}^{K} \Omega_k \times \{t = 0\}, \\
& u^+ - u^- = p \quad \text{on } \Gamma \text{ and a.e. } 0 \leq t \leq T, \\
& \nabla^A_\gamma u^+ - \nabla^A_\gamma u^- = h \quad \text{on } \Gamma \text{ and a.e. } 0 \leq t \leq T.
\end{aligned}
\]  

(1.3.1)

Assume the functions \( f \in L^2(0, T; L^2(\Omega)), g \in H^1_0(\Omega), p \in L^2(0, T; H^{3/2}(\Gamma)) \cap H^1(0, T; H^{-1/2}(\Gamma)), h \in L^2(0, T; H^{1/2}(\Gamma)), \) with the compatibility condition \( g^+ - g^- = p(0) \) on \( \Gamma \).

Then there exist a unique strong solution \( u \in L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; \dot{H}^1(\Omega)) \) to IVP (1.3.1), with \( u' \in L^2(0, T; L^2(\Omega)) \), with the estimate:

\[
\begin{aligned}
\text{ess sup}_{0 \leq t \leq T} \|u(t)\|_{H^1_0(\Omega)} + \|u\|_{L^2(0, T; H^2(\Omega))} + \|u'\|_{L^2(0, T; L^2(\Omega))} \\
\leq C(\|f\|_{L^2(0, T; L^2(\Omega))} + \|g\|_{H^1_0(\Omega)} + \|p\|_{L^2(0, T; H^{3/2}(\Gamma))} + \|p\|_{H^1(0, T; H^{-1/2}(\Gamma))} + \|h\|_{L^2(0, T; H^{1/2}(\Gamma))}).
\end{aligned}
\]  

(1.3.2)

The constant \( C \) depend only on \( \Omega, T \), and the coefficient of \( L \).
This is our main theorem in this section. Before going to the general problem, we need to concentrate on a special case when \( p \equiv 0 \). That is, to study the following initial value problem:

\[
\begin{aligned}
&u_t + Lu = f \quad \text{in } \bigcup_{k=1}^{K} \Omega_k \times [0, T], \\
&u = 0 \quad \text{on } \partial \Omega \times [0, T], \\
&u = g \quad \text{on } \bigcup_{k=1}^{K} \Omega_k \times \{ t = 0 \}, \\
&u^+ - u^- = 0 \quad \text{on } \Gamma \text{ and a.e. } 0 \leq t \leq T, \\
&\nabla^A_v u^+ - \nabla^A_v u^- = h \quad \text{on } \Gamma \text{ and a.e. } 0 \leq t \leq T.
\end{aligned}
\]

(1.3.3)

Where \( f \in L^2(0, T; L^2(\Omega)) \), \( g \in H^1_0(\Omega) \), and \( h \in L^2(0, T; H^{1/2}(\Gamma)) \).

### 1.3.2 Problems with no Jump Discontinuities across the Interface

Now we intend to build a weak solution to the IVP (1.3.3). To get started, we have to derive the weak formulation. As for any \( t > 0 \), we can write (1.3.3) as

\[
\begin{aligned}
&Lu = f - u_t \quad \text{in } \bigcup_{k=1}^{K} \Omega_k, \\
&u = 0 \quad \text{on } \partial \Omega, \\
&u^+ - u^- = 0 \quad \text{on } \Gamma, \\
&\nabla^A_v u^+ - \nabla^A_v u^- = h \quad \text{on } \Gamma.
\end{aligned}
\]

(1.3.4)

Apply the theorem 1.2.1 in our previous section, the weak formulation to this problem is:

\[
B[u, v] = (f - u_t, v)_{\Omega} + (h, v)_{\Gamma} \quad \text{for all } v \in H^1_0(\Omega);
\]

(1.3.5)

where the first \( \langle , \rangle_{\Omega} \) represents the \( H^{-1}(\Omega) \) and \( H^1(\Omega) \) dual, while the second \( \langle , \rangle_{\Gamma} \) represents the \( H^{-1/2}(\Gamma) \) and \( H^{1/2}(\Gamma) \) dual. Considering \( f \) and \( h \) has higher regularity here, we can rewrite the above weak formulation as

\[
B[u, v] + (u_t, v) = (f, v) + (h, v)_{\Gamma} \quad \text{for all } v \in H^1_0(\Omega).
\]

(1.3.6)

Therefore, our goal is to seek for a function \( u \in L^2(0, T; H^1_0(\Omega)) \) satisfying the equation (1.3.6) for all \( t > 0 \), with the initial condition \( u(0) = g \). Notice, the right hand side of (1.3.6) can be
written as a functional $F$ on $v$:

$$F(v) = (f, v) + (h, v)_Γ$$

(1.3.7)

when $v \in H₀^1(Ω)$, with the estimate

$$\|F\|_{H⁻¹(Ω)} = \sup_v \frac{(F, v)}{\|v\|_{H₀^1(Ω)}} \leq \|f\|_{H⁻¹(Ω)} \|v\|_{H₀^1(Ω)} + \|h\|_{H⁻¹/₂(Γ)} \|v\|_{H¹/₂(Γ)} \leq C(\|f\|_{H⁻¹(Ω)} + \|h\|_{H⁻¹/₂(Γ)})$$

(1.3.8)

Similarly,

$$\|F\|_{L²(Ω)} \leq C(\|f\|_{L²(Ω)} + \|h\|_{H¹/₂(Γ)})$$

(1.3.9)

for $v \in L²(Ω)$. Now the equation (1.3.6) becomes

$$B[u, v] + (u_t, v) = F(v) \quad \text{for all } v \in H₀^1(Ω).$$

(1.3.10)

Based on our results in the theorem 1.2.9, there exists a unique weak solution $u \in L²(0, T; H₀¹(Ω))$, with $u' \in L²(0, T; H⁻¹(Ω))$, to the weak formulation above, subject to the initial condition $u(0) = g$. Moreover, by the first part of theorem 1.2.11, the solution $u$ solves the IVP (1.3.3) strongly on each subdomain $Ω_k$, $k = 1, 2, ..., K$, with the estimate

$$\text{ess sup}_{0 \leq t \leq T} \|u(t)\|_{H₀¹(Ω)} + \|u\|_{L²(0, T; H²(Ω))} + \|u'\|_{L²(0, T; L²(Ω))} \leq C(\|f\|_{L²(0, T; L²(Ω))} + \|g\|_{H₀¹(Ω)})$$

$$\leq C(\|f\|_{L²(0, T; L²(Ω))} + \|h\|_{L²(0, T; H¹/₂(Γ))} + \|g\|_{H₀¹(Ω)}).$$

(1.3.11)

The constant $C$ depend only on $Ω$, $T$ and the coefficients of $L$.

1.3.2 Remark. By a similar argument as in the theorem 1.2.14, we can obtain the following higher regularity results: Assume

$$\frac{d^kf}{dt^k} \in L²(0, T; H²m−2k(Ω)), \quad \frac{d^kh}{dt^k} \in L²(0, T; H²m−2k+1/2(Γ)), \quad (k = 0, 1, ..., m),$$

and the following $m^{th}$-order compatibility conditions hold:

$$g_0 = g \in Δ_1, \quad g_1 = f(0) − Lg_0 \in Δ_1, \quad ..., \quad g_m = \frac{d^{m−1}f}{dt^{m−1}}(0) − Lg_{m−1} \in Δ_1;$$
\[ \nabla^A g^+ - \nabla^A g^- = \frac{d^k h}{dt^k}(0) \text{ on } \Gamma, \quad k = 0, 1, \ldots, m. \]

Where the space \( \mathcal{D}_1 \) is defined as

\[ \mathcal{D}_1 = \mathcal{H}^2(\Omega) \cap \{ u = 0 \text{ on } \partial\Omega \} \cap \{ u^+ - u^- = 0 \text{ on } \Gamma \}. \]

Then

\[ \frac{d^k u}{dt^k} \in L^2(0, T; \mathcal{H}^{2m+2-2k}(\Omega)) \quad k = 0, 1, \ldots, m + 1; \]

and we have the estimate

\[ \sum_{k=0}^{m+1} \left\| \frac{d^k u}{dt^k} \right\|_{L^2(0, T; \mathcal{H}^{2m+2-2k}(\Omega))} \leq C \sum_{k=0}^{m} \left\| \frac{d^k f}{dt^k} \right\|_{L^2(0, T; \mathcal{H}^{2m+2-2k}(\Omega))} + \sum_{k=0}^{m} \left\| \frac{d^k h}{dt^k} \right\|_{L^2(0, T; \mathcal{H}^{2m+2-2k+1/2}(\Gamma))} + \| g \|_{\mathcal{H}^{2m+1}(\Omega)} \]

(1.3.12)

the constant \( C \) only depending on \( m, \Omega, \Gamma, T \) and the coefficients of \( L \).

This result can be proved by an induction on \( m \) as well. However, the compatibility conditions are hectic, we have to choose proper initial data \( f, g \) and \( h \) so that they can "match" the compatibility conditions at both the interface and the boundary.

### 1.3.3 Problems with Jump Discontinuities across the Interface

Now we are going to study the general IVP (1.3.1). This time the solution \( u \) will no longer lie in \( \mathcal{H}^1(\Omega) \), as it is discontinuous across the interface. Therefore, we cannot derive any weak formulation and find a weak solution using that formulation. However, since the jump of \( u \) across the interface is a fixed function \( p \), we can seek for a proper function \( v \) independent of \( u \), in which \( v \) is supported only on some disconnected subspace \( \Omega_{k_j} \subset \{ \Omega_{k_j} \}_{j=1}^K \) \( j = 1, 2, \ldots \), and the jump function of \( v \) across the interface \( \Gamma \) is also \( p \). Once we find this \( v \), we can subtract it by \( u \) in order to “kill” the discontinuity across the interface and get an \( \mathcal{H}^1 \) function on \( \Omega \). Thus we can obtain the regularity of \( u \) by studying the behavior of function \( v \) and the \( \mathcal{H}^1 \) function \( u - v \).

Notice that we already assumed \( \Gamma \) and \( \partial\Omega \) are disjoint. Consider first the special case \( K = 2 \), that is, \( \overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2 \), with \( \Gamma = \partial\Omega_1 \cap \partial\Omega_2 \). Then, let us take a look at the following initial
value problem restricted in $\Omega_1 \times [0, T]$:

\[
\begin{aligned}
Lw + w_t &= 0 & \text{on } \Omega_1 \times [0, T], \\
w &= \begin{cases} 
p & \text{on } \Gamma \times [0, T], \\
0 & \text{on } \partial \Omega_1 \setminus \Gamma \times [0, T],
\end{cases} \\
w &= g_1 & \text{on } \Omega_1 \times \{t = 0\},
\end{aligned}
\] (1.3.13)

where $g_1 \in H^1(\Omega_1)$ is an extension function from $\partial \Omega_1$ to $\Omega_1$, with the form

\[
g_1 = \begin{cases} 
p(0) & \text{on } \Gamma, \\
0 & \text{on } \partial \Omega_1 \setminus \Gamma 
\end{cases}
\] (1.3.14)

along with the following $H^1$ bound:

\[
\|g_1\|_{H^1(\Omega_1)} \leq C \|g_1\|_{H^{1/2}(\partial \Omega_1)} = C \|p(0)\|_{H^{1/2}(\Gamma)}.
\] (1.3.15)

Trace theorem guarantees the existence of the extension $g_1$. Since $p \in L^2(0, T; H^{3/2}(\Gamma)) \cap H^1(0, T; H^{-1/2}(\Gamma))$, we apply theorem 5.9.4 in Evans Book and discover $p \in C([0, T], H^{1/2}(\Gamma))$. Therefore $p(0) \in H^{1/2}(\Gamma)$, with the estimate

\[
\|p(0)\|_{H^{1/2}(\Gamma)} \leq \|p\|_{C([0, T]; H^{1/2}(\Gamma))} \leq C(\|p\|_{L^2(0, T; H^{3/2}(\Gamma))} + \|p'\|_{L^2(0, T; H^{-1/2}(\Gamma))}).
\]

Now we look back to the IVP (1.3.13). As $L$ is a smooth operator in $\Omega_1$, by the elliptic regularity theorem on the smooth domain, we can expect a unique weak solution $w \in L^2(0, T; H^2(\Omega_1))$ to this Dirichlet problem, with the following estimates

\[
\begin{aligned}
\|w\|_{L^2(0, T; H^2(\Omega_1))} &\leq C(\|p\|_{L^2(0, T; H^{3/2}(\Gamma))} + \|g_1\|_{H^1(\Omega_1)}), \\
\|w\|_{L^\infty(0, T; H^1(\Omega_1))} &\leq C(\|p\|_{C([0, T]; H^{1/2}(\Gamma))} + \|g_1\|_{H^1(\Omega_1)}), \\
\|w'\|_{L^2(0, T; L^2(\Omega_1))} &\leq C(\|p'\|_{L^2(0, T; H^{-1/2}(\Gamma))} + \|g_1\|_{H^1(\Omega_1)}).
\end{aligned}
\] (1.3.16)
Hence we obtained a regular function $w$ defined on $\overline{\Omega_1} \times [0, T]$. Now, let us extend the function $w$ from $\overline{\Omega_1} \times [0, T]$ into $\overline{\Omega} \times [0, T]$ such that

$$w_1 = \begin{cases} w & \text{if } x \in \overline{\Omega_1}, \\ 0 & \text{if } x \in \overline{\Omega \setminus \Omega_1}. \end{cases}$$  

(1.3.17)

Observe that $w_1 = p$ on one side of $\Gamma$, and $w_1 = 0$ on the other side, so $w_1$ is not continuous across $\Gamma$. More precisely, $w_1^+ - w_1^- = p$ on $\Gamma$. Looking back into the IVP (1.3.1), the solution $u$ need to satisfy $u^+ - u^- = p$ on $\Gamma$. Therefore, if we define a new function

$$v = u - w_1$$

on the whole domain $\Omega \times [0, T]$, then the difference across the interface $\Gamma$ can be eliminated, i.e. $v^+ - v^- = 0$ on $\Gamma$. Also, comparing the functions $u$ and $w_1$, we deduce that $v$ satisfies the following initial value problem:

$$\begin{align*}
Lv + v_t &= f & & \text{on } \Omega_k \times [0, T], & k = 1, 2, \\
v &= 0 & & \text{on } \partial \Omega \times [0, T], \\
v(0) &= \begin{cases} g + g_1 & \text{on } \Omega_1, \\ g & \text{on } \Omega_2, \end{cases} \\
v^+ - v^- &= 0 & & \text{on } \Gamma \times [0, T], \\
\nabla_v^+ v^+ - \nabla_v^- v^- &= h + \nabla_v^+ w_1^- & & \text{on } \Gamma \times [0, T],
\end{align*}$$

(1.3.18)

Notice we already assume the compatibility condition $g^+ - g^- = p(0)$ hold, that is, $(g + g_1)^+ = g^-$. This condition guarantees the continuity of $v(0)$ across $\Gamma$. In addition, we have

$$\left\| \nabla_v^+ w_1^- \right\|_{L^2(0, T; H^{1/2}(\Gamma))} = \left\| \nabla_v^- w \right\|_{L^2(0, T; H^{1/2}(\Gamma))} \leq C \left\| w \right\|_{L^2(0, T; H^{1/2}(\Omega_1))} \leq C \left\| w \right\|_{L^2(0, T; \dot{H}^2(\Omega))}$$

$$\leq C \left( \| p \|_{L^2(0, T; H^{1/2}(\Gamma))} + \| g_1 \|_{H^1(\Omega_1)} \right).$$

(1.3.19)

The equation above shows $(h + \nabla_v^+ w_1^-) \in L^2(0, T; H^{1/2}(\Gamma))$. Based on our study of the IVP (1.3.4) and the inequality (1.3.12), there exists a unique weak solution $v \in L^2(0, T; \dot{H}^2(\Omega)) \cap$
\( L^2(0, T; H^1_0(\Omega)) \), with the following regularity estimates:

\[
\|\nu\|_{L^2(0, T; H^2(\Omega))} + \|\nu\|_{L^\infty(0, T; H^1(\Omega))} + \|\nu'\|_{L^2(0, T; L^2(\Omega))}
\leq C(\|f\|_{L^2(0, T; L^2(\Omega))} + \|h + \nabla^A w^+\|_{L^2(0, T; H^{1/2}(\Gamma))} + \|g + g_1\|_{H^1(\Omega)} + \|g\|_{H^1(\Omega_2)} )
\leq C(\|f\|_{L^2(0, T; L^2(\Omega))} + \|h\|_{L^2(0, T; H^{1/2}(\Gamma))} + \|\nabla^A w^+\|_{L^2(0, T; H^{1/2}(\Gamma))} + \|g\|_{H^{3/2}(\Omega)} + \|g_1\|_{H^1(\Omega_1)})
\leq C(\|f\|_{L^2(0, T; L^2(\Omega))} + \|h\|_{L^2(0, T; H^{1/2}(\Gamma))} + \|p\|_{H^1(\Omega_1)} + \|g\|_{H^{3/2}(\Omega)} + \|g_1\|_{H^1(\Omega_1)})
\leq C(\|f\|_{L^2(0, T; L^2(\Omega))} + \|h\|_{L^2(0, T; H^{1/2}(\Gamma))} + \|p\|_{C([0, T]; H^{1/2}(\Gamma))} + \|g\|_{H^{3/2}(\Omega)} + \|g_1\|_{H^1(\Omega_1)})
\leq C(\|f\|_{L^2(0, T; L^2(\Omega))} + \|h\|_{L^2(0, T; H^{1/2}(\Gamma))} + \|p\|_{C([0, T]; H^{1/2}(\Gamma))} + \|g\|_{H^{3/2}(\Omega)} + \|p\|_{L^2(0, T; H^{3/2}(\Gamma))} + \|p'\|_{L^2(0, T; H^{1/2}(\Gamma))} + \|p'\|_{L^2(0, T; H^{-1/2}(\Gamma))}).
\tag{1.3.20}
\]

Recall the equation \( u = v - w_1 \), we can expect a unique weak solution of \( u \) from the uniqueness of \( w_1 \) and \( v \). Combine the estimate (1.3.16) for \( w \) and the estimate (1.3.20) for \( v \), we deduce the following bound of \( u \):

\[
\text{ess sup}_{0 \leq t \leq T} \|u\|_{H^2_t(\Omega)} + \|u\|_{L^2(0, T; H^2(\Omega))} + \|u'\|_{L^2(0, T; L^2(\Omega))}
\leq C(\|f\|_{L^2(0, T; L^2(\Omega))} + \|h\|_{L^2(0, T; H^{1/2}(\Gamma))} + \|p\|_{L^2(0, T; H^{1/2}(\Gamma))} + \|g\|_{H^{3/2}(\Omega)} + \|p\|_{C([0, T]; H^{1/2}(\Gamma))} + \|g_1\|_{H^1(\Omega_1)})
\leq C(\|f\|_{L^2(0, T; L^2(\Omega))} + \|h\|_{L^2(0, T; H^{1/2}(\Gamma))} + \|g\|_{H^{3/2}(\Omega)} + \|p\|_{L^2(0, T; H^{3/2}(\Gamma))} + \|p'\|_{L^2(0, T; H^{1/2}(\Gamma))} + \|p'\|_{L^2(0, T; H^{-1/2}(\Gamma))}).
\tag{1.3.21}
\]

Therefore we finish the proof of the estimate (1.3.2) in the case \( K = 2 \), that is, the case when \( \Omega = \Omega_1 \cup \Omega_2 \).

For the case \( K > 2 \), we can still follow the same procedure to obtain a set of functions \( w_1, w_2, \ldots, w_J \) such that each function \( w_j \) is supported only on a subdomain \( \Omega_{k_j} \) for \( j = 1, 2, \ldots, J \), where \( \{\Omega_{k_j}\}_{j=1}^J \) are disconnected to each other. Then we choose proper \( w_j \) to make sure the jump function of \( \sum_{j=1}^J w_j \) across \( \Gamma \) is \( p \), in order to make the function \( v = u - \sum_{j=1}^J w_j \) continuous across \( \Gamma \). After analyzing the behavior of the function \( v \) on \( \Omega \) and each function \( w_j \) on \( \Omega_{k_j} \), we will obtain the existence of \( u \) and the regularity result (1.3.2).
1.4 Convergence Analysis on the Finite Difference Method for 1d Parabolic Transmission Problem

Let us return to the parabolic transmission problem (1.1.6). In this section we intend to build a numerical solution $u_h$ to this transmission problem by a certain Finite Difference scheme in one dimensional case.

Now we set the domain $\Omega$ to be the unit open interval $(0, 1)$, with the time $t$ between 0 and 1. That is, $\Omega = (0, 1) \times [0, 1]$. We then let $x = 1/2$ to be the interface, therefore, the IVP (1.1.6) can be written as

$$
\begin{align*}
\frac{\partial u}{\partial t} - (au_x)_x &= f \quad \text{in } ((0, 1/2) \cup (1/2, 1)) \times [0, 1]; \\
u &= 0 \quad \text{on } x = 0, 1; \\
u &= g \quad \text{in } (0, 1) \times \{t = 0\}; \\
u^+ &= u^- \quad \text{on } x = 1/2; \\
a^+ u_x^+ &= a^- u_x^- \quad \text{on } x = 1/2.
\end{align*}
$$

(1.4.1)

By our previous assumption, the coefficient $a(x)$ is independent of $t$ and is sufficiently smooth on both subintervals, with $\beta_2 > a(x) > \beta_1 > 0$ for some constant $\beta_1$ and $\beta_2$. The first part of the Theorem 1.2.11 reveals that, we can expect a unique strong solution $u \in L^\infty([0, 1]; H^1_0(0, 1))$ if the initial data $g$ lies in $H^1_0(0, 1)$ and $f$ lies in $L^2([0, 1]; L^2(0, 1))$.

To build a proper numerical solution $u_h$ to this problem, we want the exact solution $u$ to be at least in $L^1([0, 1], W^{4, 1}(0, 1))$, so that the fourth order derivative $u_{xxxx}$ will be bounded on each subdomain. Resulting from the Theorem 1.2.14, if we assume for now that

$$
\begin{align*}
f(x, t) &\in H^2([0, 1]; L^2(0, 1)) \cap H^1([0, 1]; \hat{H}^2(0, 1)) \cap L^2([0, 1]; \hat{H}^1(0, 1)), \\
g(x) &\in \hat{H}^5(0, 1) \cap \mathcal{T},
\end{align*}
$$

$$
g_1(x) = f(x, 0) - Pg(x) \in \mathcal{T}, \quad g_2(x) = f_t(x, 0) - Pg_t(x) \in H^1_0(0, 1),
$$

Then it will be sufficient to make $u$ lie in the space $L^\infty([0, 1], \hat{H}^5(0, 1))$, a subspace of $L^\infty([0, 1], \hat{W}^{4, \infty}(0, 1))$ by Sobolev Embedding Theorem. Now let us start to build a discretized solution $u_h$. 

1.4.1 Construction of ODEs from Space Discretization

We divide the space interval \((0, 1)\) into \(2n\) subintervals of equal length \(h = \frac{1}{2n}\). For any \(t \in [0, 1]\) and \(i = 0, 1, ..., 2n\), we denote \(u_i(t) = u(\frac{i}{2n}, t)\) as the value of the exact solution at the point \((\frac{i}{2n}, t)\), with \(u_0(0) = g_i = g(\frac{i}{2n})\) as the initial value. Similarly, for \(i \neq n\), we use the notation that \(a_i = a(\frac{i}{2n})\) and \(f_i(t) = f(\frac{i}{2n}, t)\). Since \(a(x)\) may be discontinuous at \(x = 1/2\), we denote \(a^+ = \lim_{x \to \frac{1}{2}^+} a(x)\) as the values of \(a\) on either side of interface. The boundary condition implies \(u_0(t) = u_{2n}(t) = 0\).

Now our first step is to write the equation (1.4.1) into a discrete version. We let \(t > 0\) to be fixed. Since \(u\) is regular on each subdomain, we apply Taylor expansion near the point \(x = \frac{i}{2n}\) to discover that

\[
\frac{d^2u_i}{dt^2} = \frac{1}{h^2}(u_{i-1} - 2u_i + u_{i+1}) + O(h^2),
\]

for \(i = 1, 2, ..., n-1, n+1, ..., 2n-1\). As \((au)\) \(x = a\cdot u + au_{xx} = a'\cdot u + au''\), we rewrite the first equation of (1.4.1) into \(u_i - au'' = a'\cdot u' + f\), that is

\[
\frac{du_i}{dt} - \frac{1}{h^2}(a_i u_{i-1} - 2a_i u_i + a_i u_{i+1}) - a_i' u_i' - f_i = O(h^2) \quad i = 1, 2, ..., n-1, n+1, ..., 2n-1.
\]

(1.4.2)

This is a system of \(2n - 2\) equations. The first line of (1.4.1) does not apply to the case \(i = n\), where \(u_n\) represents the value of \(u\) on the interface. However, we can study the 5th line of (1.4.1) to obtain one equation for \(u_n\). Still, let us do the Taylor expansion near \(x = 1/2\):

\[
u_n - u_{n-1} = hu_n' - \frac{h^2}{2} u_n'' + \frac{h^3}{6} u_n''' + O(h^4)
\]

\[
= hu_n' - \frac{h^2}{2} (u_{n-1}' + hu_{n-1}''') + \frac{h^3}{6} u_{n-1}''' + O(h^4) = hu_n' - \frac{h^2}{2} u_{n-1}'' - \frac{h^3}{3} u_{n-1}''' + O(h^4).
\]

(1.4.3)

Similarly,

\[
u_{n+1} - u_n = hu_n' + \frac{h^2}{2} u_{n+1}'' - \frac{h^3}{3} u_{n+1}''' + O(h^4).
\]

(1.4.4)
As the transmission condition $a_n^+ u_n^+ = a_n^- u_n^-$ holds, we apply it to the RHS of (1.4.3) and (1.4.4) to discover

$$a_n^- (u_n - u_{n-1}) - a_n^+ (u_{n+1} - u_n) = a_n^- \left( -\frac{h^2}{2} u_{n-1}''' - \frac{h^3}{6} u_{n-1}'' + \frac{h^4}{3} u_{n+1}'' - \frac{h^5}{5} u_{n+1}'' \right) + O(h^6)$$

$$= -\frac{h^2}{2} (a_n^- u_{n-1}''' + a_n^+ u_{n+1}''') - \frac{h^3}{3} (a_n^- u_{n-1}'' + a_n^+ u_{n+1}'') + O(h^4)$$

$$= \frac{h^2}{3} a_n^- u_{n-2}''' - 5h^2 a_n^- u_{n-1}''' - 5h^2 a_n^+ u_{n+1}''' + \frac{h^2}{3} a_n^+ u_{n+2}'' + O(h^4).$$

(1.4.5)

Thus we obtain an equation for $u_n$:

$$u_n = \frac{a_n^-}{a_n^- + a_n^+} u_{n-1} + \frac{a_n^+}{a_n^- + a_n^+} u_{n+1} + h^2 \left( \frac{a_n^-}{a_n^- + a_n^+} \frac{1}{3} u_{n-2}'' \right)$$

(1.4.6)

$$- \frac{a_n^-}{a_n^- + a_n^+} 5 \frac{1}{6} u_{n-1}''' - \frac{a_n^+}{a_n^- + a_n^+} 5 \frac{1}{6} u_{n+1}''' + \frac{a_n^+}{a_n^- + a_n^+} \frac{1}{3} u_{n+2}'' + O(h^4).$$

We then differentiate (1.4.6) with respect to $t$:

$$\frac{du_n}{dt} = \frac{a_n^-}{a_n^- + a_n^+} \frac{du_{n-1}}{dt} + \frac{a_n^+}{a_n^- + a_n^+} \frac{du_{n+1}}{dt} + O(h^2).$$

(1.4.7)

Combine these two equations above, we obtain

$$\frac{du_n}{dt} - \frac{1}{h^2} (a_n^- u_{n-1} - (a_n^- + a_n^+) u_n + a_n^+ u_{n+1}) = \frac{a_n^-}{a_n^- + a_n^+} \frac{du_{n-1}}{dt} + \frac{a_n^+}{a_n^- + a_n^+} \frac{du_{n+1}}{dt}$$

$$+ \frac{a_n^-}{a_n^- + a_n^+} \frac{1}{3} u_{n-2}''' - \frac{a_n^-}{a_n^- + a_n^+} \frac{5}{6} u_{n-1}''' - \frac{a_n^+}{a_n^- + a_n^+} \frac{5}{6} u_{n+1}''' + \frac{a_n^+}{a_n^- + a_n^+} \frac{1}{3} u_{n+2}'' + O(h^2).$$

(1.4.8)
We then apply (1.4.2) to estimate the terms \(\frac{du_n}{dt}\) and \(\frac{du_{n+1}}{dt}\), the above equation (1.4.8) therefore becomes:

\[
\frac{du_n}{dt} - \frac{1}{h^2} (a_n^- u_{n-1} - (a_n^- + a_n^+) u_n + a_n^+ u_{n+1})
- \frac{1}{3} a_n^- \frac{u_n''}{u_n-2} - \left(\frac{a_n^-}{a_n^+ + a_n^-} a_{n-1} - \frac{5}{6} a_n^-\right) u_n'' - \left(\frac{a_n^+}{a_n^+ + a_n^-} a_n^{+1} - \frac{5}{6} a_n^+\right) u_{n+1}'' - \frac{1}{3} a_n^+ u_n'' \tag{1.4.9}
\]

\[
-\left(\frac{a_n^- a_{n-1}^-}{a_n^- + a_n^+} u_{n-1}' + \frac{a_n^+ a_{n+1}^+}{a_n^- + a_n^+} u_{n+1}'\right) + \frac{f_{n-1} a_n^- + f_{n+1} a_n^+}{a_n^- + a_n^+} = O(h^2)
\]

Therefore, the equations (1.4.2) and (1.4.9) construct a system of \((2n - 1)\) equations. We regard (1.4.9) as the \(n\)th equation in this system, for it is an estimate of \(\frac{du_n}{dt}\).

As \(u_i'' = \frac{1}{h^2} (u_{i-1} - 2u_i + u_{i+1}) + O(h^2)\) and \(u_i' = \frac{1}{2h} (u_{i+1} - u_{i-1}) + O(h^2)\) for all \(i = 1, 2, ..., n - 1, n + 1, ... 2n - 1\), we can replace all the first and second derivative terms of \(u_i\) by the above expressions, the system of equations (1.4.2) and (1.4.9) will thereby become

\[
\frac{dU}{dt} - AU - F = h^2 V. \tag{1.4.10}
\]

where \(U = (u_1 u_2 ... u_{2n-1})^T\), \(F = (f_1 f_2 ... f_{n-1} \frac{f_{n-1}}{a_n^- + a_n^+} a_n^- + f_{n+1} a_n^+ f_{n+1} ... f_{2n-1})^T\), \(V\) is the error vector which comes from the RHS of (1.4.2) and (1.4.9), and has the bound

\[
\|V\|_{L^\infty} \leq C \|u\|_{W^{4,\infty}[0,1]}, \tag{1.4.11}
\]

for some constant \(C > 0\). And finally, \(A\) is a \((2n - 1) \times (2n - 1)\) matrix of the form

\[
A = -\frac{1}{h^2} J + \frac{1}{h} M, \tag{1.4.12}
\]
1.4.2 Semi-discretized Finite Difference Scheme

Here the notations \( \beta = \frac{a^-_n - a^-_{n-1}}{a^-_n + a^-_{n+1}} - \frac{5}{6} a^-_n \), \( \alpha = \frac{a^+_n - a^-_{n-1}}{a^-_n + a^-_{n+1}} - \frac{5}{6} a^+_n \). And then

\[
J = \begin{bmatrix}
2a_1 & -a_1 & 0 \\
-a_2 & 2a_2 & -a_2 \\
& & \ddots \\
-a_{n-1} & 2a_{n-1} & -a_{n-1} \\
-\frac{1}{2} a^-_n & \frac{2}{3} a^-_n - \beta & -\frac{4}{3} a^-_n + 2\beta & a^-_n + a^+_n - \beta - \alpha & 2\alpha - \frac{4}{3} a^+_n - \alpha + \frac{2}{3} a^+_n & -\frac{1}{3} a^+_n \\
& & & & & \\
& & & & & \\
0 & 2a_{n-1} & -a_{n-1} & \ddots & \ddots & -a_{2n-2} & 2a_{2n-2} & -a_{2n-2} \\
0 & -a_{2n-1} & 2a_{2n-1} & & & -a_{2n-1} & 2a_{2n-1}
\end{bmatrix}
\]

(1.4.13)

\[
M = \begin{bmatrix}
0 & a'_1 & 0 \\
-a'_2 & 0 & a'_2 \\
& & \ddots \\
-a'_{n-1} & 0 & a'_{n-1} \\
-\frac{a'_n a'_{n-1}}{a'_n + a'_{n-1}} & 0 & a'_n + a'_{n-1} & 0 & a'_n & a'_{n+1} & 0 \\
& & & & & & \ddots \\
& & & & & & \ddots \\
0 & 2a'_{2n-2} & -a'_{2n-2} & 0 & a'_{2n-2} & 0 & 0
\end{bmatrix}
\]

(1.4.14)

The equation (1.4.10) holds for all time \( t > 0 \), with the initial condition \( U(0) = G = (g_1, g_2, \ldots, g_{2n-1}) \).

1.4.2 Semi-discretized Finite Difference Scheme

In this subsection, we remain the space discretized and let time \( t \) to be continuous. We want to build up a numerical solution \( U_h(t) = (u_{h1}(t) \ u_{h2}(t) \ \ldots \ u_{h2n-1}(t))^T \) to approximate the exact solution of the problem (1.4.1), such that \( U_h \rightarrow U \) in \( L^\infty([0, 1]; l^2) \) as \( h \rightarrow 0 \). Now based
on (1.4.10), let us construct \( U_h(t) \) by solving the following system of ODEs:

\[
\begin{align*}
U_h(0) &= G; \\
\frac{dU_h}{dt} - AU_h - F &= 0 \quad \text{for all } t \geq 0.
\end{align*}
\]

(1.4.15)

Where \( G = (g_1, g_2, \ldots, g_{2n-1}) \).

Notice that \( n \) is a finite number, \( F \) is continuous in time, and \( A \) is a matrix depend only on \( a(x) \). According to standard existence theory for Ordinary Differential Equations, there exists a unique absolutely continuous function \( U(t) \) satisfying the equation (1.4.15) for \( 0 \leq t \leq 1 \). Now let us take denote this vector function \( U_h \) to be a numerical solution to the problem (1.4.1).

1.4.1 Theorem. Let \( U \) and \( U_h \) to be the functions defined above. Then, there exists a constant \( C \) dependent only on \( a, f, g \), and a constant \( \varepsilon > 0 \) depend only on \( a(x) \), such that

\[
\|U_h - U\|_{L^\infty([0,1];t^2)} \leq Ch^{3/2}
\]

(1.4.16)

when \( 0 < h < \varepsilon \). That is, the convergence rate on this scheme is \( O(h^{3/2}) \).

Proof. Let \( e_i = u_i - u_{hi}, i = 1, 2, \ldots, 2n - 1 \). We compare the equations (1.4.10) with (1.4.15) to discover

\[
\begin{align*}
E(0) &= 0; \\
\frac{dE}{dt} - AE &= h^2V \quad \text{for all } t \geq 0.
\end{align*}
\]

(1.4.17)

where \( E = (e_1, e_2, \ldots, e_{2n-1}) \). Now let us concentrate on the behavior of the matrix \( A \). Recall (1.4.12), \( A \) has the form

\[
A = -\frac{1}{h^2}J + \frac{1}{h}M.
\]

(1.4.18)

Therefore we can write as:

\[
\frac{dE}{dt} + \frac{1}{h^2}JE - \frac{1}{h}ME = h^2V.
\]

(1.4.19)

Since \( J \) is not positive definite, we are going to seek for some proper matrix \( P \), in order to make the matrix \( D = PJ \) be better constructed. Let \( P \) to be the following \((2n-1) \times (2n-1)\)
matrix:

\[
P = \begin{bmatrix}
\frac{a_n}{a_1} & \frac{a_n}{a_2} & \cdots & \frac{a_n}{a_{n-1}} & 0 \\
\frac{a_n}{3a_{n-2}} & -\frac{a_n}{a_{n-2}} & \cdots & \frac{a_n}{a_{n-1}} & 1 & \cdots & -\frac{a_n}{3a_{n+2}} \\
& \frac{a_n}{a_{n+1}} & \cdots & \frac{a_n}{a_{n+1}} & \cdots & \frac{a_n}{a_{2n-2}} & \frac{a_n}{a_{2n-1}} \\
& & & & & & \\
0 & 0 & \cdots & 0 & \frac{a_n}{a_{n+1}} & \frac{a_n}{a_{n+1}} & \frac{a_n}{a_{2n-2}} & \frac{a_n}{a_{2n-1}} \\
& & & & & & \end{bmatrix}
\]

Then as a result of calculation, we get the following \((2n - 1) \times (2n - 1)\) matrix \(D\):}

\[
D = PJ = \begin{bmatrix}
2a_n & -a_n & \cdots & -a_n \\
-a_n & 2a_n & -a_n & \cdots & -a_n \\
& & & & \ddots & \cdots & -a_n \\
& & & & \cdots & 2a_n & -a_n & \cdots & -a_n \\
& & & & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots & \cdots & -a_n \\
& & & & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & -a_n \\
0 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots & \cdots & -a_n \\
& & & & & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -a_n \\
& & & & & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -a_n \\
\end{bmatrix}
\]

It is not hard to observe that \(D\) is a symmetric positive-definite matrix. We then left-multiply both sides of (1.4.19) by this matrix \(P\) to obtain:

\[
P \frac{dE}{dt} + \frac{1}{h^2} \frac{dE}{dt} \cdot D - \frac{1}{h} \frac{dE}{dt} \cdot \frac{dE}{dt} \cdot PME = h^2 PV. \tag{1.4.22}
\]

In addition, let us left-multiply (1.4.22) by \(\left(\frac{dE}{dt}\right)^T\), the transpose of \(\frac{dE}{dt}\). Then the equation becomes:

\[
\left(\frac{dE}{dt}\right)^T P \frac{dE}{dt} + \frac{1}{h^2} \left(\frac{dE}{dt}\right)^T \cdot D - \frac{1}{h} \left(\frac{dE}{dt}\right)^T \cdot \frac{dE}{dt} \cdot PME = h^2 \left(\frac{dE}{dt}\right)^T PV. \tag{1.4.23}
\]
Now let us study each term in (1.4.23) carefully. First, we want the estimate
\[
\left(\frac{dE}{dt}\right)^T P \frac{dE}{dt} \geq \beta_3 \left\| \frac{dE}{dt} \right\|_2^2. \tag{1.4.24}
\]
For some constant $\beta_3 > 0$. This estimate will hold when the matrix $P$ is positive-definite. That is to say, the following $5 \times 5$ matrix:

\[
\begin{bmatrix}
\frac{a_n}{a_{n-2}} & 0 & 0 & 0 & 0 \\
0 & \frac{a_n}{a_{n-1}} & 0 & 0 & 0 \\
-\frac{a_n}{3a_{n-2}} & -\frac{\beta}{a_{n-1}} & 1 & -\frac{a_n}{a_{n+1}} & -\frac{a_n}{3a_{n+2}} \\
0 & 0 & 0 & \frac{a_n}{a_{n+1}} & 0 \\
0 & 0 & 0 & 0 & \frac{a_n}{a_{n+2}}
\end{bmatrix} \tag{1.4.25}
\]

is positive-definite. From calculation, this is equivalent to the following inequality:

\[
\frac{4a_n}{a_{n-2}} + \frac{4a_n}{a_{n+2}} + \frac{25a_n}{a_{n-1}} + \frac{25a_n}{a_{n+1}} + \frac{36(a_n a_{n-1} + a_n a_{n+1})}{(a_n^2 + a_n^2)^2} < 208. \tag{1.4.26}
\]

In fact, since $a(x)$ is smooth enough on each subinterval, this condition can be easily satisfied if we pick the value $h$ on a small interval $(0, \varepsilon)$ for some $\varepsilon > 0$. Under this condition, (1.4.24) will hold and the constant $\beta_3$ depends only on the function $a(x)$.

And thereafter, on the second term of (1.4.23), we have $\frac{1}{h^2} \left(\frac{dE}{dt}\right)^T D E = \frac{d}{dt} \frac{1}{h^2} (E^T D E)$ by the symmetry of $D$, where

\[
\frac{1}{h^2} E^T D E = a_n \sum_{i=0}^{n-1} \left(\frac{e_i - e_{i+1}}{h}\right)^2 + a_n^+ \sum_{i=n}^{2n-1} \left(\frac{e_i - e_{i+1}}{h}\right)^2 = (\nabla E)^T Q (\nabla E). \tag{1.4.27}
\]

Here $e_0 = e_{2n} = 0$, $Q =$ diag $(a_n^- \ldots a_n^- a_n^+ \ldots a_n^+)$ is a $2n \times 2n$ diagonal matrix, and

\[
\nabla E = (\frac{e_1 - e_0}{h}, \frac{e_2 - e_1}{h}, \ldots, \frac{e_{2n-1} - e_{2n-2}}{h}, \frac{e_{2n} - e_{2n-1}}{h}) \text{ is a vector of } 2n \text{ components.}
\]

Then we come to the next term. We rewrite it as

\[
\frac{1}{h} \left(\frac{dE}{dt}\right)^T P M E = \frac{dE}{dt}^T N (\nabla E) \tag{1.4.28}
\]
where $N$ is a $(2n - 1) \times 2n$ matrix, with the components

\[
N = \begin{bmatrix}
\frac{a_1' a_n^-}{a_1} & \frac{a_2' a_n^-}{a_2} & \ldots & \frac{a_{n-1}' a_n^-}{a_{n-1}} & \frac{\alpha_1}{a_n+1} \\
\frac{a_1' a_n^-}{a_2} & \frac{a_2' a_n^-}{a_2} & \ldots & \frac{a_{n-1}' a_n^-}{a_{n-1}} & \frac{\alpha_2}{a_n+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{a_1' a_n^-}{a_{n-1}} & \frac{a_2' a_n^-}{a_{n-1}} & \ldots & \frac{a_{n-1}' a_n^-}{a_{n-1}} & \frac{\alpha_{n-1}}{a_n+1} \\
\frac{\alpha_n}{a_{n+1}} & \frac{\alpha_n}{a_{n+1}} & \ldots & \frac{\alpha_n}{a_{n+1}} & \frac{\alpha_n}{a_{n+1}} \\
0 & 0 & \ldots & 0 & 0
\end{bmatrix}
\]  

(1.4.29)

Here $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) = \left( -\frac{a_{n-2} a_n^-}{3a_{n-2}}, \frac{a_{n-2} a_{n-1}}{a_n+1} - \frac{\beta a_{n-1}}{a_{n-1}}, \frac{a_{n-2} a_{n-3}}{3a_{n-2}}, \frac{a_{n-2} a_{n-3}}{a_n+1} - \frac{\beta a_{n-3}}{a_{n-1}}, \frac{a_{n-3} a_{n-2} a_n^+}{a_{n-1}} - \frac{\beta a_{n-3} a_n^+}{3a_{n-2}}, \frac{a_{n-3} a_{n-2} a_n^+}{a_n+1} - \frac{\beta a_{n-3} a_n^+}{a_{n-1}} \right)$.

Since $N$ is bounded, there exists a constant $\beta_4 > 0$ dependent only on $a(x)$ such that

\[
\left\| \frac{dE}{dt} \right\| N \nabla E \leq \beta_4 \left\| \frac{dE}{dt} \right\| \left\| \nabla E \right\|_2 \leq \frac{\beta_4^2}{\beta_3} \left\| \nabla E \right\|_2^2 + \frac{\beta_3}{4} \left\| \frac{dE}{dt} \right\|_2^2 \leq \beta_5 \left( \nabla E \right)^T Q(\nabla E) + \frac{\beta_3}{4} \left\| \frac{dE}{dt} \right\|_2^2.
\]  

(1.4.30)

Here $\beta_5 = \frac{\beta_4^2}{\beta_3 \min(a_n, a_n^+)} > 0$.

Finally, as $P$ is a bounded, 5-diagonal matrix, while $V$ has an $L^\infty$ bound by (1.4.11), we deduce $\|PV\|_{L^\infty}$ is bounded. Thus we can estimate the last term of (1.4.23):

\[
h^2 \left\| \frac{dE}{dt} \right\|^T PV \leq h^2 \left\| \frac{dE}{dt} \right\| \left\| PV \right\|_{L^\infty} \leq \beta_6 h^{3/2} \left\| \frac{dE}{dt} \right\|_2 \leq \frac{\beta_4^2 h^3}{\beta_3} + \frac{\beta_3}{4} \left\| \frac{dE}{dt} \right\|_2^2
\]  

(1.4.31)

for some constant $\beta_6 > 0$. Combining all the estimates (1.4.24), (1.4.27), (1.4.30) and (1.4.31), the equation (1.4.23) results in

\[
\frac{\beta_3}{2} \left\| \frac{dE}{dt} \right\|_2^2 + \frac{d}{dt} \left( \nabla E \right)^T Q(\nabla E) \leq \frac{\beta_4^2 h^3}{\beta_3} + \beta_5 \left( \nabla E \right)^T Q(\nabla E).
\]  

(1.4.32)

This inequality holds for each $t \in [0, 1]$. Now let us denote $\eta(t) = (\nabla E(t))^T Q(\nabla E(t))$. Then $\eta(t)$ is a continuous, positive-valued function of $t$. It is not hard to observe $\eta(0) = 0$ from the fact
\[ E(0) = 0. \] Also, from (1.4.32) we deduce

\[ \eta'(t) \leq \frac{\beta_2^2}{\beta_3^2} h^3 + \beta_2 \eta(t) \]  

(1.4.33)

when \( 0 \leq t \leq 1 \). Considering \( \beta_i \) are all positive, the differential form of Gronwall’s inequality yields to the estimate

\[ \eta(t) \leq e^{\beta_2 t} \eta(0) + \frac{\beta_2^2}{\beta_3^2} \int_0^t h^3 ds \leq \beta_\gamma h^3 \]  

(1.4.34)

for some constant \( \beta_\gamma > 0 \). Thereafter, we integrate (1.4.32) from 0 to \( t \) to discover

\[ \frac{\beta_2}{2} \left\| \frac{dE}{dt} \right\|_{L^2(0,t; l^2_t)}^2 + \eta(t) \leq \frac{\beta_2^2}{\beta_3^2} h^3 + \beta_5 \int_0^t \eta(s) ds. \]  

(1.4.35)

Apply the results in (1.4.34) and the fact \( \eta(t) \geq 0 \), we deduce

\[ \left\| \frac{dE}{dt} \right\|_{L^2(0,t; l^2_t)}^2 \leq \frac{2\beta_2^2}{\beta_3^2} h^3 + \frac{2\beta_5}{\beta_3} \beta_\gamma h^3. \]  

(1.4.36)

Therefore,

\[ \left\| E(t) \right\|_{L^2(0,t; l^2_t)}^2 = \left( \int_0^t E^t(s) ds \right)^2 \leq t \int_0^t \left( E^t(s) \right)^2 ds = t \left\| E^t \right\|_{L^2(0,t; l^2)}^2 \leq \beta_\delta t^2 h^3 \]  

(1.4.37)

where \( \beta_\delta = \frac{2\beta_2^2}{\beta_3^2} + \frac{2\beta_5}{\beta_3} \beta_\gamma > 0 \) is a constant. Hence \( \max_{t \in [0,1]} \| E(t) \|_{l^2_t} \leq \sqrt{\beta_\delta h^{3/2}} = O(h^{3/2}) \). This result implies that \( U_h \) converges to \( U \) in \( L^\infty([0,1]; l^2_t) \) with a rate of \( O(h^{3/2}) \).

1.4.2 Remark. The classical results of the finite difference scheme implies, if the domain \( \Omega \) contains no interface, the convergence rate \( U_h \to U \) will be \( O(h^2) \) in \( L^\infty(l^\infty) \), and \( O(h^{3/2}) \) in \( L^\infty(t^2) \). However, in our problem with interface, yet we have no idea on showing the \( L^\infty(l^\infty) \) convergence.

1.4.3 Fully Discretized Finite Difference Scheme in 1d: Backward Euler

In the last subsection, we denoted \( U(t) = (u_1(t) \ u_2(t) \ldots u_{2n-1}(t)) \) as the semi-discrete exact solution to the problem (1.4.1), and we have built a numerical solution \( U_h \) to approximate
Here \( U \) and \( U_h \) are both continuous in time. Now, let us divide the time interval \( [0, 1] \) into \( m \) parts with equal length \( k = 1/m \). And for each integer \( 0 \leq j \leq m \), we denote

\[
U_j = U(jk) = (u_1(jk) \ u_2(jk) \ldots \ u_{2n-1}(jk)) = (u(h, jk) \ u(2h, jk) \ldots \ u(1-h, jk)).
\]

Thus \( \tilde{U} = (U_1 \ U_2 \ldots \ U_m) \) constructs a \( (2n-1) \times m \) matrix, in which the \( i^{th} \) row and \( j^{th} \) column has the element \( u(ih, jk) \). We consider this \( \tilde{U} \) as the fully discretized exact solution to the problem (1.4.1). Now our goal is to build a fully discretized numerical solution \( \tilde{U}_h \) such that \( \tilde{U}_h \to \tilde{U} \) in \( l^2 \).

Resulting from our assumption, the exact solution \( u \) lies in \( L^\infty ([0, 1]; \hat{W}^{4, \infty}(0, 1)) \), also \( u_t \in L^\infty ([0, 1]; \hat{W}^{2, \infty}(0, 1)) \), and \( u_{tt} \in L^\infty ([0, 1]; L^1(0, 1)) \). Thus for each \( j = 0, 1, \ldots , m-1 \), we have the estimate

\[
U_{j+1} = U_j + kU_{j+1}^t + k^2 W_j,
\]

where \( W_j \) is the error term that has a bounded \( l^\infty \) norm. Recall the equation (1.4.10), at time \( t = jk \) we have:

\[
U_j^t - AU_j - F_j = h^2 V_j,
\]

where \( F_j = F(jk) \) and \( V_j = V(jk) \). Here \( V(jk) \) has a bounded \( l^\infty \) norm as well. Therefore (1.4.38) can be written as

\[
(I - kA)U_{j+1} - U_j - kF_j = kh^2 V_j + k^2 W_j.
\]

Still, we have the following initial condition:

\[
U_0 = G.
\]

Now based on the equations (1.4.40) and (1.4.41), let us construct a numerical solution \( \tilde{U}_h = (U_{h1} \ U_{h2} \ldots \ U_{hm}) \) by following the iteration below

\[
\begin{cases}
U_{h0} = G; \\
U_{hj+1} = (I - kA)^{-1}(U_{hj} + kF_j) \quad j = 0, 1, \ldots , m-1.
\end{cases}
\]

Similar to the semi-discretized scheme, we need to show that \( \tilde{U}_h \) is a good approximation of the exact solution \( \tilde{U} \), under some conditions.

**1.4.3 Theorem.** Assume \( \tilde{U}_h \) and \( \tilde{U} \) are two matrices defined as above, then there exists a constant \( C \) dependent only on \( a, f, g \), and a constant \( \epsilon > 0 \) depend only on \( a(x) \), such that when \( 0 < h < \epsilon \), we
have
\[ \|U_{hj} - U_j\|_{L^2} \leq C(h^{3/2} + kh^{-1/2}) \] (1.4.43)

for each j = 1, ..., m. That is, the convergence rate of this scheme is \(O(h^{3/2} + kh^{-1/2})\) on \(l^\infty(l^2)\).

**Proof.** Still, we denote the error vector \(E_j = U_j - U_{hj}, j = 0, 1, ..., m\). Resulting from the equations above, we have

\[ \begin{cases} E_0 = 0; \\ (I - kA)E_{j+1} - E_j = k^2W_j + kh^2V_j & j = 0, 1, ..., m - 1. \end{cases} \] (1.4.44)

Here the second equation can be written as

\[ \frac{E_{j+1} - E_j}{k} - AE_{j+1} = kW_j + h^2V_j. \] (1.4.45)

Apply our results in the previous subsection, \(A\) has the form \(-\frac{1}{h^2}J + \frac{1}{h}M\). Similarly, we left-multiply (1.4.45) by \(P\) to obtain

\[ P\frac{E_{j+1} - E_j}{k} + \frac{1}{h^2}DE_{j+1} - \frac{1}{h}NE_{j+1} = kW_j + h^2V_j. \] (1.4.46)

Here \(D\) and \(N\) is the same as before. Now we left-multiply this equation again by \((\frac{E_{j+1} - E_j}{k})^T\), the transpose of \(\frac{E_{j+1} - E_j}{k}\), to get

\[ (\frac{E_{j+1} - E_j}{k})^TP\frac{E_{j+1} - E_j}{k} + \frac{1}{h^2}(\frac{E_{j+1} - E_j}{k})^TDE_{j+1} - \frac{1}{h}(\frac{E_{j+1} - E_j}{k})^TNE_{j+1} = (\frac{E_{j+1} - E_j}{k})^T(kW_j + h^2V_j). \] (1.4.47)

By the same argument as on the equation (1.4.23) in the theorem 1.4.1, we can find a small number \(\varepsilon > 0\) such that \(P\) is positive definite when \(0 < h < \varepsilon\). Under this condition, we mimic the process from (1.4.23) to (1.4.32). The equation (1.4.47) will then result in the following estimate

\[ C_0 \|\frac{E_{j+1} - E_j}{k}\|_{L^2}^2 + \frac{1}{h^2}(\frac{E_{j+1} - E_j}{k})^TDE_{j+1} \leq C_1 \frac{1}{h^2}E_{j+1}^TDE_{j+1} + C_2 \frac{(h^2 + k)^2}{h} \] (1.4.48)
for some constant $C_0, C_1, C_2 > 0$ independent of $E_j, h$ and $k$. That is

$$
(1 - kC_1) \frac{1}{h^2} E_j^T D E_{j+1} \leq \frac{1}{h^2} E_j^T D E_j + C_2 \frac{k(h^2 + k)^2}{h}.
$$

(1.4.49)

As the matrix $D$ is SPD, by Cauchy’s inequality we have $E_j^T D E_{j+1} \leq \frac{1}{2} E_j^T D E_j + \frac{1}{2} E_{j+1}^T D E_{j+1}$. Thus (1.4.49) becomes

$$
(1 - 2kC_1) \frac{1}{h^2} E_j^T D E_{j+1} \leq \frac{1}{h^2} E_j^T D E_j + 2C_2 \frac{k(h^2 + k)^2}{h}.
$$

(1.4.50)

Since this estimate holds for each $j = 0, 1, \ldots, m$, the induction of the above inequality gives us:

$$
\frac{1}{h^2} E_j^T D E_j \leq (1 - 2kC_1)^{-j} \frac{1}{h^2} E_0^T D E_0 + \sum_{s=0}^{j-1} (1 - 2kC_1)^{-s} (2C_2 \frac{k(h^2 + k)^2}{h}) \leq \frac{C_2(h^2 + k)^2}{C_1 h}.
$$

(1.4.51)

Now let us go back to the equation (1.4.48). We sum up (1.4.48) for $j = 0, 1, \ldots, m - 1$ and apply Cauchy’s inequality $E_j^T D E_{j+1} \leq \frac{1}{2} E_j^T D E_j + \frac{1}{2} E_{j+1}^T D E_{j+1}$ again to the left hand side, it becomes

$$
C_0 \sum_{j=0}^{m-1} \frac{\|E_{j+1} - E_j\|_2^2}{k} \leq C_1 \sum_{j=1}^{m} \frac{1}{h^2} E_j^T D E_j + C_2 \frac{m(h^2 + k)^2}{h}.
$$

(1.4.52)

Apply (1.4.51), we deduce

$$
\sum_{j=0}^{m-1} \frac{\|E_{j+1} - E_j\|_2^2}{k} \leq \frac{C_1}{C_0} \sum_{j=1}^{m} \frac{C_2(h^2 + k)^2}{C_1 h} \frac{C_2 m(h^2 + k)^2}{C_0 h} \leq \frac{2C_2(h^2 + k)^2}{C_0 kh}.
$$

(1.4.53)

Hence

$$
\|E_j\|_2^2 = k^2 \sum_{j=0}^{m-1} \left( \frac{E_{j+1} - E_j}{k} \right)^2 \leq k^2 \sum_{j=0}^{m-1} \left( \frac{E_{j+1} - E_j}{k} \right)^2 \leq \frac{2C_2(h^2 + k)^2}{C_0 h} \frac{2C_2(h^2 + k)^2}{C_0 h} \leq \frac{2C_2(h^2 + k)^2}{C_0 h}.
$$

(1.4.54)

That is, $\|E_j\|_2 \leq \sqrt{\frac{2C_2}{C_0}(h^{3/2} + k h^{-1/2})}$ for each $j = 0, 1, \ldots, m$. This result implies that $U_{h_j}$ converges to $U_j$ in $l^2$ for each $j$, with a rate of $O(h^{3/2} + k h^{-1/2})$. □
Chapter 2

Parabolic Transmission Problem on 2d Polygonal Domain

2.1 Preliminaries

In the first chapter, we discussed the parabolic transmission problem on a smooth, bounded domain $\Omega$. Now in this chapter, we study the case that $\Omega$ is a two dimensional, bounded polygonal domain. That is, a 2D domain bounded by a finite chain of straight line segments. This means the boundary of $\Omega$ is smooth except at the vertices. We do not restrict $\Omega$ to be convex. Still, we divide the domain $\Omega$ into $K$ open, disjoint subdomains $\Omega_k$, $k = 1, 2, ..., K$, so that $\Omega = \bigcup_{k=1}^{K} \Omega_k$. Let $\Gamma$ denote the interface among these subdomains, and we assume $\Gamma$ consist of straight line segments as well, so that all the subdomains are polygons, with $\bigcup_{k=1}^{K} \partial \Omega_k = \Gamma + \partial \Omega$. Different from Chapter 1, we remove the assumption that $\partial \Omega$ and $\Gamma$ do not touch each other. In other words, they may have common points. We use $V$ to denote the set of vertices of all the polygons $\bigcup_{k=1}^{K} \Omega_k$.

![Polygonal Domain with Interface](image-url)

Fig. 2.1: Polygonal Domain with Interface
Now, similar to the smooth case, let us set up the parabolic transmission problem on this polygonal domain $\Omega$ with homogeneous boundary and interface condition:

\[
\begin{aligned}
\begin{cases}
L u + u_t = f & \text{in } \bigcup_{k=1}^{K} \Omega_k \times T; \\
u = 0 & \text{on } \partial_D \Omega; \\
\nabla^A_v u = 0 & \text{on } \partial_N \Omega; \\
\nabla^A_v u^+ = \nabla^A_v u^- & \text{on } \Gamma \times T.
\end{cases}
\end{aligned}
\tag{2.1.1}
\]

Here $\partial_D \Omega$ represents the Dirichlet part of the boundary $\partial \Omega$, $\partial_N \Omega$ is the Neumann part of boundary, and $\partial \Omega = \partial_D \Omega + \partial_N \Omega$. We assume $\partial_D \Omega \neq \emptyset$, and $\partial_N \Omega = \partial \Omega \setminus \partial_D \Omega$ to be open. Also, the points where the boundary condition changes can only appear at the vertices in $V$. If one such point is not at a vertex, then we consider this point as an “artificial vertex” and regard it as an element of $V$, with the interior angle $\pi$. Also, same as the last chapter, the operator $L$ is in the form of:

\[
Lu := -\text{div} (A \nabla u) = - \sum_{i,j=1}^{2} \partial_j (a_{ij} \partial_i u).
\]

Here $a_{ij}$ are sufficiently smooth on each subdomains, with $a_{12} = a_{21}$. When $\partial_D \Omega \neq \emptyset$, we can apply Poincare inequality to discover that $L$ is uniformly elliptic. We assume $f$ and $g$ lie in proper spaces.

By a similar argument as theorem 1.2.1 in the smooth case, the weak formulation of this transmission problem is

\[
B[u, v] + \langle u_t, v \rangle = \langle f, v \rangle, \quad \forall \, v \in H^1_D(\Omega).
\tag{2.1.2}
\]

Where

\[
H^1_D(\Omega) = \{ u \mid u \in H^1(\Omega), \, u = 0 \text{ on } \partial_D \Omega; \, \nabla^A_v u = 0 \text{ on } \partial_N \Omega \}.
\]

According to standard results, if $f$ lies in $L^2(0, T; (H^1_D(\Omega))')$ and $g$ lies in $L^2(\Omega)$, we will obtain a unique weak solution $u \in L^2(0, T; H^1_D(\Omega))$, with $u_t \in L^2(0, T; (H^1_D(\Omega))')$. However, as the elliptic regularity theorem does not apply under non-smooth domain, we can hardly raise the space regularity of $u$ into $\hat{H}^2(\Omega)$ as before. But still, we may be able to seek for a weaker regularity
results on u. We will need to study the previous results about the elliptic transmission problem
on the polygonal domain first.

2.2 Previous results on elliptic problem

We study the following boundary value problem first:

\[
\begin{align*}
L_u &= f, \quad \text{on } \bigcup_{k=1}^{K} \Omega_k, \\
u &= 0, \quad \text{on } \partial_D \Omega, \\
\nabla_v^A u &= 0 \quad \text{on } \partial_N \Omega, \\
u^+ - u^- &= 0, \quad \text{on } \Gamma, \\
\nabla_v^A u^+ - \nabla_v^A u^- &= 0, \quad \text{on } \Gamma.
\end{align*}
\] (2.2.1)

The elliptic transmission problem on the polygonal domain has been widely studied, around 20 papers of them are listed in the introduction part of my dissertation. In this chapter, we mainly rely on the paper [18] written by my advisor Anna Mazzucato and her former student Hengguang Li. To better explain their results, let us define the following space, named weighted Sobolev Spaces:

\[
\mathcal{K}_m^a(\Omega) = \{ u \in L^2_{\text{loc}}(\Omega), \rho^{\lfloor \alpha \rfloor - a} \partial^\alpha u \in L^2(\Omega), |\alpha| \leq m \}. \tag{2.2.2}
\]

Here m is a non-negative integer, a is a real number. \( \rho = \rho(x) \) is defined as

\[
\rho(x) = \prod_{Q \in V} d(x, Q). \tag{2.2.3}
\]

where V is the set of all vertices on the boundary and on the interface of \( \Omega \), and \( d(x, Q) \) represents the distance between the point \( x \in \Omega \) and the vertex Q. We see, when x approaches one of the vertices, the value of \( \rho(x) \) vanishes linearly. We call the function \( \rho \) as the weight function.

The \( \mathcal{K}_m^a \) norm is defined as

\[
\| u \|^2_{\mathcal{K}_m^a(\Omega)} = \sum_{|\alpha| \leq m} \left\| \rho^{\lfloor \alpha \rfloor - a} \partial^\alpha u \right\|^2_{L^2(\Omega)}. \tag{2.2.4}
\]

By definition, we easily check that \( \mathcal{K}_0^0(\Omega) = L^2(\Omega), \mathcal{K}_m^m(\Omega) \subset \mathcal{K}_m^m(\Omega), \text{ and } \mathcal{K}_m^{m+1}(\Omega) \subset \mathcal{K}_m^m(\Omega) \) for any \( b > 0 \) and any integer \( m \geq 0 \).
In our assumption, each subdomain $\Omega_k$ is polygon, so we can define the spaces $\mathcal{X}_a^m(\Omega_k)$ as well. Similar as the broken Hilbert space, we define the broken weighted Sobolev space on $\Omega$ as the following:

\[
\hat{\mathcal{X}}_a^m(\Omega) = \{ u \in L^2_{\text{loc}}(\Omega_k), \rho^{m-a} u \in L^2(\Omega_k), |a| \leq m, k = 1, 2, ..., K \}. \tag{2.2.5}
\]

That is, $\hat{\mathcal{X}}_a^m(\Omega) = \bigcap_{K=1}^{K} \mathcal{X}_a^m(\Omega_k)$.

Observe that, if a continuous function $u \in \hat{\mathcal{X}}_1^1(\Omega)$, then $u$ must equal 0 on all the vertices in $\mathcal{V}$, otherwise we check $\|u\|_{\hat{\mathcal{X}}_1^1(\Omega)} \geq \|u/\rho\|_{L^2(\Omega)} \to \infty$ as $1/\rho \notin L^2_{\text{loc}}(\Omega)$, this gives rise to a contradiction. Even in the case $u$ is not continuous, $u$ has to behave sufficiently small near a neighborhood of $\mathcal{V}$ to sit inside $\hat{\mathcal{X}}_1^1(\Omega)$.

Recall that we use letter $\mathcal{V}$ to denote the set of all vertices on the boundary and on the interface of $\Omega$. Specifically, there are 3 types of vertices:

1. vertices lies on $\partial \Omega$, and is the vertex of only one subdomain;
2. vertices lies on $\partial \Omega$, and is the vertex of more than one subdomains;
3. vertices lies on $\Gamma$ but not on $\partial \Omega$.

There are two boundary edges for type 1 vertices, two boundary edges and a few interface edges for type 2 vertices, and a few interface edges for type 3 vertexes. Now let us separate the vertex set $\mathcal{V}$ into two subsets $\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2$, where $\mathcal{V}_1$ consists of all the vertices in type 1 and 2 such that at least one boundary edge is Dirichlet, and $\mathcal{V}_2$ contains all the remaining vertices. That is, the set $\mathcal{V}_2$ consists of all the type 3 vertices, and type 1 and 2 vertices that have Neumann-Neumann boundary edges.
Recall the definition, if a function $u \in H^1_D(\Omega)$, then $u = 0$ on $\partial_D \Omega$. This means $u = 0$ on all $Q \in V_1$, but $u$ does not necessarily vanish on any vertices in $V_2$.

For each vertex $Q \in V_2$, we choose a function $\chi_Q \in C^\infty(\Omega)$ that is constant equal to 1 in a neighborhood of $Q$. We can choose these functions to have disjoint supports. We denote $W_s$ to be the linear span of the functions $\chi_Q$. Notice for any $w_s \in W_s$, we have $w_s \notin K^0_1(\Omega)$.

2.2.1 Lemma (Anna Mazzucato, Hengguang Li, Victor Nistor, [18]). Assume $f \in \hat{K}^{m-1}_{a-1}(\Omega)$, where $m \geq 1$ is an integer and $a \in (0, \eta)$ for some fixed $\eta > 0$, then the elliptic problem (2.2.1) has a unique weak solution $u \in H^1_D(\Omega)$ such that $u = u_{\text{reg}} + w_s$, where $u_{\text{reg}} \in \hat{K}^{m+1}_{a+1}(\Omega) \cap K^1_{a+1}(\Omega)$, and $w_s \in W_s$.

The constant $\eta$ is called the singularity constant. It depends on the operator $L$ and the shape of the polygon $\Omega$.

In addition, the linear map $u \rightarrow f$ in the equation (2.2.1) describe an isomorphism between the space $V^{m+1}_{a+1}(\Omega)$ and $\hat{K}^{m-1}_{a-1}(\Omega)$, where

$$V^{m+1}_{a+1}(\Omega) = \{ u | u = u_{\text{reg}} + w_s, \ u_{\text{reg}} \in \hat{K}^{m+1}_{a+1}(\Omega) \cap K^1_{a+1}(\Omega), \ w_s \in W_s \}$$

$$\cap \{ u | u = 0 \text{ on } \partial_D \Omega, \ \nabla^A u = 0 \text{ on } \partial_N \Omega, \ \nabla^A u = \nabla^A u^- \text{ on } \Gamma \}.$$  

(2.2.6)
We use the letter $P$ to denote the isomorphism, that is $Pu = f$, $P : V^{m+1}_{a+1}(\Omega) \rightarrow \mathcal{K}^{m-1}_{a-1}(\Omega)$. Also, we have the following estimate:

$$
\|u_{\text{reg}}\|_{\mathcal{K}^{1}_{a+1}(\Omega)} + \|u_{\text{reg}}\|_{\mathcal{K}^{m+1}_{a+1}(\Omega)} + \|w_s\|_{L^2(\Omega)} \leq C \|f\|_{\mathcal{K}^{m-1}_{a-1}(\Omega)}.
$$

(2.2.7)

Here $C$ is a constant that depend on $L$ and $\Omega$.

This lemma was proved by using the Babuška-Lax-Milgram theorem and the Fredholm index theory. In the case $m = 0$, the isomorphism $P : V^1_{a+1}(\Omega) \rightarrow (V^1_{a+1}(\Omega))'$ will also hold but has to be interpreted in a weak sense. As the weighted space $\mathcal{K}^m_{a}(\Omega)$ does not define for any negative $m$, we always use $\mathcal{K}^{-1}_{a-1}(\Omega)$ to denote the dual space of $V^1_{a+1}(\Omega)$ for $a \in (0, \eta)$. Based on this lemma, we are ready to work on the parabolic transmission equation (2.1.1) on the polygonal domain.

### 2.3 Well-posedness Results for Parabolic Transmission Problem

In this section, we make the assumption that $V_2 = \emptyset$. This means all the vertices in $V$ must have at least one Dirichlet side, and there is no interior vertex on the interface $\Gamma$. Under this assumption, we have $W_s = \{0\}$, hence $u \equiv u_{\text{reg}}$ in the lemma 2.2.1.

#### 2.3.1 Standard Regularity Results

First of all, Let us introduce the following weighted Poincaré inequality:

**2.3.1 Lemma (See [23]).** Let $\mathcal{C} = \mathcal{C}_{r, \alpha} := \{(r \cos \theta, r \sin \theta) \in \mathbb{R}^2, 0 < r < R, 0 < \theta < \alpha\}$, $0 < \alpha \leq 2\pi$. Then

$$
\int_{\mathcal{C}} \frac{|u|^2}{r^2} \, dx \leq \left(\frac{2\alpha}{\pi}\right)^2 \int_{\mathcal{C}} \frac{\partial_\theta u|^2}{r^2} \, dx \leq \left(\frac{2\alpha}{\pi}\right)^2 \int_{\mathcal{C}} |\nabla u|^2 \, dx
$$

(2.3.1)

for any $u$, $\nabla u \in L^2(\mathcal{C})$ satisfying $u(r, \theta) = 0$ if $\theta = 0$, in the trace sense.

This lemma is well-known and is very useful to our problem. It shows the relation between the spaces $H^1_D(\Omega)$ and $\mathcal{K}^1(\Omega)$. Under our assumption, we directly deduce $H^1_D(\Omega) \subset \mathcal{K}^1(\Omega)$ by the inequality. If both sides of the sector $\mathcal{C}$ is Neumann, such lemma will fail. Now we are ready to prove our main theorem in this section.

**2.3.2 Theorem.** Let $\eta$ be the constant in the lemma 2.2.1. Then there exists a positive constant $\delta \leq \eta$, such that, if $f \in L^2(0, T; (V^1_{a+1}(\Omega))')$, and $g \in \mathcal{K}^0_{a}(\Omega)$ for some $a \in (0, \delta)$, then the weak
solution \( u \) to the parabolic transmission problem (2.1.1) will lie in the space \( L^2(0, T; \mathcal{X}^{-1}_{a+1}(\Omega)) \cap H^1(0, T; (\mathcal{V}^1_{a+1}(\Omega))^\prime) \), with the following bound:

\[
\|u\|_{L^2(0, T; \mathcal{X}^{-1}_{a+1}(\Omega))}^2 + \|u_t\|_{L^2(0, T; (\mathcal{X}^1_{a+1}(\Omega))^\prime)}^2 \leq \|f\|_{L^2(0, T; (\mathcal{X}^1_{a+1}(\Omega))^\prime)}^2 + \|g\|_{X^0_a(\Omega)}^2.
\]

A more precise estimate of \( \delta \) will be presented later.

**Proof.** According to the results in the section 7.1 of Evans book, there exists a unique weak solution \( u \in L^2(0, T; H^1(\Omega)) \) to the problem (2.1.1), since \( f \in L^2(0, T; (H^1(\Omega))^\prime) \), \( g \in L^2(\Omega) \) and the weak formulation of this problem is

\[
B[u, v] + \langle u_t, v \rangle = \langle f, v \rangle, \quad v \in H^1_D(\Omega)
\]

with \( u(0) = g \). Here \( B[u, v] = \int_{\Omega} \sum_{i,j=1}^{2} a_{ij} u_{x_i} v_{x_j} \ dx \). However, we want the regularity of \( u \) to be raised into \( L^2(0, T; \mathcal{X}^{-1}_{a+1}(\Omega)) \), this means we have to do some analyze near each corner of the subdomains.

By our assumption, we have \( f \in L^2(0, T; \mathcal{X}^{-1}_{a+1}(\Omega)) \), \( g \in \mathcal{X}^0(\Omega) \). As the existence of weak solution is known, we are going to define an adapted weight function \( \vartheta(x) > 0 \) on \( \Omega \), such that

1. the value of \( \vartheta \) behaves like \( \rho \) pointwise. That is, there exist two constants \( 0 < C_1 < C_2 \) depend only on \( L \), such that \( C_1 \rho < \vartheta < C_2 \rho \) pointwise.
2. \( \vartheta \) is sufficiently smooth (at least \( C^1 \)) inside each subdomain, except near the corners;
3. \( \nabla^A_v(\vartheta^a) = 0 \) on \( \partial_N \Omega \);
4. \( \vartheta \) is continuous across the interface. that is, \( \vartheta^+ - \vartheta^- = 0 \) on \( \Gamma \);
5. \( \nabla^A_v(\vartheta^a)^+ = \nabla^A_v(\vartheta^a)^- = 0 \) on \( \Gamma \).
If we can find one function \( \vartheta \) satisfying the conditions above, then we denote \( \bar{u} = u/\vartheta^a \).

After plugging into the origin problem (2.1.1), we notice \( \bar{u} \) solves the following problem:

\[
\begin{align*}
\begin{cases}
\partial^{-a} L \partial^{a} \bar{u} + \bar{u}_t = \bar{f} & \text{in } \bigcup_{k=1}^{K} \Omega_k \times \mathcal{T}; \\
\bar{u} = 0 & \text{on } \partial_D \Omega; \\
\nabla^A_v (\partial^{a} \bar{u}) = 0 & \text{on } \partial_N \Omega; \\
\bar{u} = \bar{g} & \text{on } \bigcup_{k=1}^{K} \Omega_k \times \{ t = 0 \}; \\
(\partial^{a} \bar{u})^+ = (\partial^{a} \bar{u})^- & \text{on } \Gamma \times \mathcal{T}; \\
\nabla^A_v (\partial^{a} \bar{u})^+ = \nabla^A_v (\partial^{a} \bar{u})^- & \text{on } \Gamma \times \mathcal{T}.
\end{cases}
\end{align*}
\]

(2.3.3)

Here \( \bar{f} = f/\vartheta^a \in L^2(0, T; \mathbb{R}^{-1}(\Omega)) = L^2(0, T; (H^1_D(\Omega))^\prime) \), \( \bar{g} = g/\vartheta^a \in L^2(\Omega) \) by the 1st condition.

Taking all the other conditions into account, we have

\[
\begin{align*}
\begin{cases}
\partial^{-a} L \partial^{a} \tilde{u} + \tilde{u}_t = \bar{f} & \text{in } \bigcup_{k=1}^{K} \Omega_k \times \mathcal{T}; \\
\tilde{u} = 0 & \text{on } \partial_D \Omega; \\
\nabla^A_v \tilde{u} = 0 & \text{on } \partial_N \Omega; \\
\tilde{u} = \bar{g} & \text{on } \bigcup_{k=1}^{K} \Omega_k \times \{ t = 0 \}; \\
\tilde{u}^+ = \tilde{u}^- & \text{on } \Gamma \times \mathcal{T}; \\
\nabla^A_v \tilde{u}^+ = \nabla^A_v \tilde{u}^- & \text{on } \Gamma \times \mathcal{T}.
\end{cases}
\end{align*}
\]

(2.3.4)

Now let us try to construct the function \( \vartheta \). As the domain \( \Omega \) is a 2D polygon with straight sides, we have \( \vartheta = \vartheta(x, y) \). First we consider the case when the coefficient of \( L \), \( a_{ij} \), \( 1 \leq i, j \leq 2 \), are constants on each subdomain. For each vertex \( Q_i = (x_i, y_i) \in \mathcal{V} \) and each subdomain \( \Omega_{ij}, \ 1 \leq i, j \leq L \) that contains the vertex \( Q_i \), we define the function

\[
\vartheta_i(x, y) = \frac{C_{ij}}{\sqrt{a_{11}a_{22} - a_{12}^2}} \left( a_{22}(x - x_i)^2 - 2a_{12}(x - x_i)(y - y_i) + a_{11}(y - y_i)^2 \right)^{1/2}
\]

\[
= C_{ij} \left( (x - x_i, y - y_i) \left( \begin{array}{cc}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{array} \right)^{-1} (x - x_i, y - y_i)^T \right)^{1/2}
\]

(2.3.5)

\[
= C_{ij} \left( (x - x_i, y - y_i) A^{-1} (x - x_i, y - y_i)^T \right)^{1/2}.
\]
Here the constant \( C_{ii} \) is chosen to make \( \theta_i \) continuous across \( \Gamma \). We can let \( C_{ii} = 1 \), all the \( C_{ii} \) will then be uniquely determined by the above equation. The value of \( \theta_i \) can be defined on the interfaces as well. Now for each \( i \), let us sketch the level curves of the function \( \theta_i \) in a neighborhood of \( Q_i \). We choose a proper small constant \( C_0 > 0 \) so that the level sets

\[
\mathcal{B}_{Q_i}(C_0) = \{(x, y) \mid \theta_i(x, y) \leq C_0\}
\]

for each \( Q_i \in \mathcal{V} \) are disjoint in \( \Omega \).

At the same time, we define the following constant function

\[
\theta_0(x, y) \equiv 1
\]

throughout \( \Omega \). Now we are able to construct the function \( \theta \) by applying partition of unity. For each point \((x, y) \in \Omega\), we let

\[
\theta(x, y) = \theta_0(x, y) = 1
\]

if \((x, y) \in \Omega \setminus \bigcup_{Q_i \in \mathcal{V}} \mathcal{B}_{Q_i}(C_0)\); and

\[
\theta(x, y) = \theta_i(x, y)
\]

if \((x, y) \in \mathcal{B}_{Q_i}(C_0/2)\); finally

\[
\theta(x, y) = (1 - \zeta_i(x, y))\theta_0(x, y) + \zeta_i(x, y)\theta_i(x, y)
\]

if \((x, y) \in \mathcal{B}_{Q_i}(C_0)\) but \((x, y) \notin \mathcal{B}_{Q_i}(C_0/2)\). Here \( \zeta_i \) is a continuous function defined on the level set \( \{(x, y) \mid C_0/2 \leq \theta_i(x, y) \leq C_0\} \), and have the following properties:

1. \( \zeta_i \) is constant along any level curve of \( \theta_i \). That is, \( \zeta_i = \kappa(\theta_i) \) for some continuous function \( \kappa \) defined on the closed interval \([C_0/2, C_0]\);

2. \( \kappa = 1 \) on \([C_0/2, 2C_0/3]\), \( \kappa = 0 \) on \([5C_0/6, C_0]\), and \( \kappa \) is smooth\( (C^1 \text{ will do}) \) and non-decreasing on \([C_0/2, C_0]\).

We pick a function \( \kappa \) that satisfy the two conditions above (for example, a polynomial with degree 3 will do), then each partition function \( \zeta_i \) is well defined on its own domain. Hence we have constructed the function \( \theta \) on the whole domain, and are ready to check if it satisfy all the conditions above. Condition 1 is clear by noticing that the matrix \( A = (a_{ij}) \) is SPD on each subdomain. Condition 2 is fine as the functions \( \theta_0, \theta_i \) and \( \zeta_i \) are all smooth on each subdomain,
except near the vertices. Moreover, it is not hard to check the functions $\vartheta_i$ satisfy the conditions 3,4,5 on the set $B_{Q_i}(C_0)$, the functions $\zeta_i$ and $\vartheta_0$ also satisfy the conditions 3,4,5 on their own domain. After a calculation we can derive $\vartheta$ satisfy these 3 conditions as well. Hence the function $\vartheta$ we construct meets all our requirements.

Now let us try to construct $\vartheta$ when $L$ does not have constant coefficients on each subdomain. The idea is similar. In our construction above, the level curves of $\vartheta_i$ near the vertex $Q_i : (x_i, y_i)$ is a set of adapted sectors. For each point $(x, y)$ on one of the curves, the tangent line of the level curve at this point is in the direction of

$$A \cdot n(x, y) = (A(x, y) \cdot (\frac{y - y_i}{-x + x_i}),$$

We then do the same thing in the case when $A$ is smooth on each subdomain. Fix one Dirichlet side of $Q_i$, let $\vartheta_i(x, y) = d((x, y), Q_i)$ when $(x, y)$ is on this side and $d((x, y), Q_i) \leq C_0$. Then for each of these points, we sketch a curve starting from this point and satisfies the following condition: if one point $(x, y)$ inside $\Omega$ lies on this curve, then the curve will have the tangent vector

$$A \cdot n(x, y) = (A(x, y) \cdot (\frac{y - y_i}{-x + x_i})$$

at the point $(x, y)$. As $A$ is smooth on each subdomain, we see this curve is uniquely determined by the initial point on the fixed Dirichlet side. Also, two curves will not intersect if they are generated by two different initial points. Now let us construct the function $\vartheta_i(x, y)$ in a neighborhood of $Q_i$, such that the set of curves we sketched are the level curves of $\vartheta_i(x, y)$. Still, we choose proper $C_0 > 0$ to make the support of each $\vartheta_i$ be disjoint. By a similar argument as the above case, we check $\vartheta_i(x, y)$ behaves as $d((x, y), Q_i)$ and is continuous across the interfaces. Moreover, we discover from the theory of level curves that

$$\nabla^A \vartheta_i = (\nabla u)^T A \vartheta_i = 0$$

on the boundary and interface of $\Omega$ where $\vartheta_i$ is defined.

Let $\vartheta_0(x, y) \equiv 1$ on $\Omega$, we do the same partition of unity as above to construct the function

$$\vartheta(x, y) = (1 - \zeta_i(x, y))\vartheta_0(x, y) + \zeta_i(x, y)\vartheta_i(x, y)$$
for some smooth cut-off function $\zeta_i(x, y)$. By a similar argument, we see $\vartheta(x, y)$ is defined on the whole domain and satisfy all the five conditions.

Now we return to the parabolic problem (2.3.4). Our goal now is to prove $\bar{u} \in L^2(0, T; H^1_D(\Omega))$. In fact, the most typical way is to build a weak solution by the Galerkin approximation method. That is, to construct solutions of certain finite-dimensional approximations and then pass to limits. First of all, we need to construct an orthonormal basis of $L^2(\Omega)$.

According to the lemma 2.2.1 and the fact $H^1_D(\Omega) \subset H^1(\Omega)$, the elliptic problem (2.2.1):

$$
\begin{align*}
L u &= f, & \text{on } \bigcup_{k=1}^{K} \Omega_k; \\
u &= 0, & \text{on } \partial_D \Omega; \\
\nabla^A_{\nu} u &= 0, & \text{on } \partial_N \Omega; \\
u^+ - u^- &= 0, & \text{on } \Gamma; \\
\nabla^A_{\nu} u^+ - \nabla^A_{\nu} u^- &= 0, & \text{on } \Gamma.
\end{align*}
$$

(2.3.11)

describes an isomorphism between $u \in H^1_D(\Omega)$ and $f \in (H^1_D(\Omega))^\prime$, with the following variational formulation:

$$B[u, v] = \langle f, v \rangle, \quad v \in H^1_D(\Omega).$$

Here $B[u, v] = \int_{\Omega} \sum_{i,j=1}^{2} a_{ij} u_i v_j \, dx$. We denote it as $Pu = f$ and $u = P^{-1}f$. From the fact $B[u, v] = B[v, u]$ we see $P$ is symmetric in $L^2(\Omega)$. Moreover, we see $P^{-1}$ maps $L^2(\Omega)$ function into $H^1_D(\Omega)$, and $H^1_D(\Omega)$ is compactly embedded in $L^2(\Omega)$. Based on the Fredholm theory of compact and symmetric operators on Hilbert space, we get a conclusion that there exists a countable orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions of $P$, say $(w_i)_{i=1}^{\infty}$. Moreover, $(w_i)_{i=1}^{\infty}$ is also a complete B-orthogonal basis of $H^1_D(\Omega)$, that is, $B[w_i, w_j] = 0$ for each $i \neq j$.

Now we have $Pw_i = \lambda_i w_i$ with $\lambda_i > 0$ for each $i$. As $w_i \in H^1_D(\Omega)$, we deduce $Lw_i = \lambda_i w_i$ pointwise on each subdomain $\Omega_k$. Also, each $w_i$ satisfies the boundary and transmission conditions in the equations (2.2.1). From the lemma 2.2.1, we see $w_i \in \mathcal{K}^{1+\alpha}_{1+\alpha}(\Omega) \cap \mathcal{K}^{2}_{1+\alpha}(\Omega)$ for any $0 < \alpha < \eta$. Notice that, if we compare the elliptic problem (2.2.1) with the parabolic problem (2.3.4), we will see each $w_i$ satisfies the same boundary and interface condition as $\bar{u}$ do. Therefore, in the remaining part of the proof, we will take $(w_i)_{i=1}^{\infty}$ as the basis of the Galerkin approximation in the problem (2.3.4).
First, let us derive the weak formulation of the problem: assume \( \bar{u} \) and \( v \) are as regular as the eigenfunctions, then we calculate

\[
\langle f - \bar{u}_t, v \rangle = \langle \tilde{\theta}^{-a} L \tilde{\theta}^a \bar{u}, v \rangle = \langle \tilde{\theta}^{-a} \sum_{i,j=1}^{2} (-a_{ij} (\tilde{\theta}^a \bar{u}))_{x_i} x_j, v \rangle = \int_{\Omega} \sum_{i,j=1}^{2} a_{ij} (\tilde{\theta}^a \bar{u})_{x_i} (\tilde{\theta}^{-a} v)_{x_j} \, dx \\
+ \int_{\Gamma} (\nabla^A (\tilde{\theta}^a \bar{u})^+ - \nabla^A (\tilde{\theta}^a \bar{u})^-) \tilde{\theta}^{-a} v \, ds + \int_{\partial \Omega} \nabla^A (\tilde{\theta}^a \bar{u}) \tilde{\theta}^{-a} v \, ds = B[\tilde{\theta}^a \bar{u}, \tilde{\theta}^{-a} v] \\
= (\tilde{\theta}^a \nabla \bar{u} + \alpha a^{-1} \tilde{\theta} \bar{u}, \tilde{\theta}^{-a} \nabla v - \alpha a^{-1} \tilde{\theta} \phi v)_A \\
= (\nabla \bar{u}, \nabla v)_A - a^2 (\frac{\tilde{u}^2}{\tilde{\theta}^2} \nabla \bar{u}, \nabla \phi)_A + a^2 (\frac{\bar{u}^2}{\tilde{\theta}^2} \nabla \bar{u}, \nabla \phi)_A - a^2 (\frac{\bar{u}^2}{\tilde{\theta}^2} \nabla \bar{u}, \nabla \phi)_A.
\]

(2.3.12)

Here the notation \( (X, Y)_A = \int_{\Omega} X^T A Y \, dx \). We denote the above expression as \( B_a[\bar{u}, v] \).

Now let us check \( B_a[\bar{u}, v] \) is bounded. In fact, by (2.3.12) we have

\[
B_a[\bar{u}, v] \leq \alpha_1 \| \nabla \bar{u} \|_{L^2(\Omega)} \| \nabla v \|_{L^2(\Omega)} + \alpha_2 \| \bar{u} / \tilde{\theta} \|_{L^2(\Omega)} \| v / \tilde{\theta} \|_{L^2(\Omega)} + \alpha_3 \| \bar{u} / \tilde{\theta} \|_{L^2(\Omega)} \| \nabla v \|_{L^2(\Omega)} + \alpha_4 \| \nabla \bar{u} \|_{L^2(\Omega)} \| v / \tilde{\theta} \|_{L^2(\Omega)} \leq \alpha \| \nabla \bar{u} \|_{L^2(\Omega)} \| \nabla v \|_{L^2(\Omega)}
\]

(2.3.13)

for some positive constant \( \alpha = \alpha(a) > 0 \). In addition, we want the following bound

\[
B_a[\bar{u}, \bar{u}] \geq \beta \| \nabla \bar{u} \|_{L^2(\Omega)}^2 - \gamma \| \bar{u} \|_{L^2(\Omega)}^2 , \quad (2.3.14)
\]

for all \( \bar{u} \in H^1_D(\Omega) \) and some constant \( \beta, \gamma > 0 \). This requires some conditions on \( a \). Now we are going to show there exist a constant \( \delta > 0 \) such that, the inequality (2.3.14) hold when \( 0 < a < \delta \).

We return to the equation (2.3.12) and plug in the function \( \tilde{\theta} \) to discover

\[
B_a[\bar{u}, \bar{u}] = (\nabla \bar{u}, \nabla \bar{u})_A - a^2 (\frac{\bar{u}^2}{\tilde{\theta}^2} \nabla \bar{u}, \nabla \tilde{\theta})_A \geq \bigcup_{Q_i(\tilde{\theta}/2)} (\nabla \bar{u})^T A(\nabla \bar{u}) \, dS \\
- a^2 \int_{B_{Q_i}(\tilde{\theta}/2)} \frac{\bar{u}^2}{(x - x_i, y - y_i)A^{-1}(x - x_i, y - y_i)} \, dS + \beta_0 \| \nabla \bar{u} \|_{L^2(\Omega \setminus \bigcup_{Q_i(\tilde{\theta}/2)} B_{Q_i}(\tilde{\theta}/2))}^2 \\
- \gamma_0 a^2 \| \bar{u} \|_{L^2(\Omega \setminus \bigcup_{Q_i(\tilde{\theta}/2)} B_{Q_i}(\tilde{\theta}/2))}^2
\]

(2.3.15)
for some $\beta_0, \gamma_0 > 0$ that does not depend on $\bar{u}$. Let us study the first term of (2.3.15) then. On each vertex $Q_i$ we define $r_i = \inf_{\theta(X) = \langle \bar{u}, Q_i \rangle} d(X, Q_i)$ so that the whole sector centered at $Q_i$ with radius $r_i$ lies inside $B_{Q_i}(C_0/2)$. We use $D_i$ to denote the sector, and $\omega_i$ as the angle of the sector, that is, the angle of $\Omega$ at the vertex $Q_i$. As $A$ is symmetric positive definite, by lemma 2.3.1 we have

$$
\int_{B_{Q_i}(C_0/2)} (\bar{u}_x, \bar{u}_y) A(\bar{u}_x, \bar{u}_y)^T dS \geq \int_{D_i} (\bar{u}_x, \bar{u}_y) A(\bar{u}_x, \bar{u}_y)^T dS \geq \beta_1 \|\nabla \bar{u}\|_{L^2(D_i)}^2
$$

$$
\geq \beta_1 \left( \frac{\pi}{2\omega_i} \right)^2 \int_{D_i} \frac{\bar{u}_x^2}{d((x, y), Q_i)} dS \geq \beta_2 \frac{\beta_2}{\omega_i^2} \int_{D_i} \frac{\bar{u}_x^2}{d((x, y), Q_i)} dS \geq \beta_3 \frac{\beta_3}{\omega_i^2} \|\bar{u}\|_{L^2(D_i)}^2
$$

(2.3.16)

for some constant $\beta_2, \beta_3 > 0$ depend only on $A$. We then let $\omega = \max_{Q_i \in V} (\omega_i)$ and choose $\delta = \min(\sqrt{\beta_2 / \omega^2}, \eta)$. Combining the inequalities (2.3.15) and (2.3.16) together we can conclude, if $a \in (0, \delta)$, then $a^2 < \beta_2 / \omega_i^2$ for each $i$, and this is sufficient to make (2.3.14) hold.

Now let us pick this $\delta$ and return to the parabolic problem (2.3.4). The weak formulation of the problem (2.3.4) becomes:

$$
B_a[\bar{u}, v] + \langle \bar{u}_t, v \rangle = \langle \bar{f}, v \rangle, \quad \forall v \in H^1_D(\Omega);
$$

(2.3.17)

with the initial condition $\bar{u}(0) = \bar{g}$. As the bilinear form $B_a[\ , \ ]$ satisfy the property (2.3.13) and (2.3.14), we can mimic the proof of theorem 7.1.2 in the book of Evans [15]: We take $\{w_i\}_{i=1}^\infty$ to be the basis for the Garlekin approximation. For each integer $m \geq 1$, we define

$$
\bar{u}_m(t) = \sum_{i=1}^m d^i_m(t) w_i, \quad i = 1, 2, \ldots, m.
$$

(2.3.18)

Here $\{d^i_m(t)\}_{i=1}^m$ is a set of continuous functions of $t$ with the initial condition

$$
d^i_m(0) = (\bar{g}, w_i), \quad i = 1, 2, \ldots, m.
$$

(2.3.19)
Also, for each $t > 0$, $\bar{u}_m(t)$ satisfies the following system of equations

$$B_a [\bar{u}_m(t), w_i] + (\bar{u}_m'(t), w_i) = (\bar{f}_i, w_i), \quad i = 1, 2, ..., m. \quad (2.3.20)$$

From the classical results we know the function $\bar{u}_m(t)$ exists and is unique for each integer $m \geq 1$. In addition, the energy estimate tells us

$$\|\bar{u}_m\|^2_{L^2(0,T;H^{1}_0(\Omega))} + \|\bar{u}_m'\|^2_{L^2(0,T;H^{-1}(\Omega))} + \|\bar{u}_m\|^2_{L^\infty(0,T;L^2(\Omega))} \leq C(\|\bar{f}\|^2_{L^2(0,T;H^1_u(\Omega))} + \|\bar{g}\|^2_{L^2(\Omega)}) \quad (2.3.21)$$

with the constant $C$ depend only on $\Omega$ and $A = (a_{ij})_{i,j=1,2}$. Now we denote $u_m = \partial^a \bar{u}_m$. For each $t > 0$, by lemma 2.3.1 we can deduce

$$\left\| \frac{u_m}{\partial^{a+1}} \right\|_{L^2(\Omega)} = \left\| \frac{\bar{u}_m}{\partial} \right\|_{L^2(\Omega)} \leq C \left\| \nabla \bar{u}_m \right\|_{L^2(\Omega)}; \quad (2.3.22)$$

and

$$\left\| \frac{\nabla u_m}{\partial^a} \right\|_{L^2(\Omega)} \leq C \left( \left\| \frac{u_m}{\partial^a} \right\|_{L^2(\Omega)} + \left\| \frac{u_m}{\partial^{a+1}} \right\|_{L^2(\Omega)} \right) \leq C \left\| \nabla \bar{u}_m \right\|_{L^2(\Omega)} \quad (2.3.23).$$

These 2 inequalities and (2.3.21) reveal that $u_m$ has a uniform bound in $L^2(0, T; K^{1}_{a+1}(\Omega))$. We choose a weakly convergent subsequence $\{u_{mk}\}_{k \geq 1} \subset \{u_m\}_{m \geq 1}$ and pass to limits. It is not hard to check $\partial^a \bar{u}_m = u_m \rightharpoonup u = \partial^a \bar{u}$, and $u$ has the following estimate

$$\|u\|^2_{L^2(0,T;K^{1}_{a+1}(\Omega))} \leq \|f\|^2_{L^2(0,T;V^{1}_{a+1}(\Omega))} + \|g\|^2_{X_0^a(\Omega)} \quad (2.3.24)$$

Similarly, we apply (2.3.21) and pass to limits again to discover

$$\|u\|^2_{L^\infty(0,T;X_0^a(\Omega))} + \|u_t\|^2_{L^2(0,T;V^{1}_{a+1}(\Omega))'} \leq \|f\|^2_{L^2(0,T;V^{1}_{a+1}(\Omega))'} + \|g\|^2_{X_0^a(\Omega)} \quad (2.3.25).$$

### 2.3.2 Higher Regularity of Solution

In this subsection, we will use one theorem to describe how regular the solution behaves when given more regular data $f$ and $g$. 
2.3.3 Theorem (Advanced Regularity). In the parabolic problem (2.1.1), if \( f \in L^2(0,T;\mathcal{X}^0_a(\Omega)) \), and \( g \in \mathcal{X}^{1}_{a+1}(\Omega) \cap H^1_D(\Omega) \) for some \( a \in (0,\delta) \), then we will have \( u \in L^\infty(0,T;\mathcal{X}^{1}_{a+1}(\Omega)) \cap H^1(0,T;\mathcal{X}^0_a(\Omega)) \).

**Proof.** We continue from the proof of theorem 2.3.2. Still, \( \bar{u}_m \) solves the problem (2.3.4), and we have the estimate (2.3.21). In fact, we can mimic the proof of theorem 1.2.11(1) in the previous chapter. Let \( \bar{f} = f/\theta^a, \bar{g} = g/\theta^a \) as well, then \( \bar{f} \in L^2(0,T;L^2(\Omega)), \) and \( \bar{g} \in L^2(0,T;H^1_D(\Omega)) \). We notice

\[
\|\bar{u}_m(0)\|_{H^1_D(\Omega)}^2 \leq \beta_4 B[\bar{u}_m, \bar{u}_m] \leq \beta_4 B[\bar{g}, \bar{g}] \leq \beta_5 \|\bar{g}\|_{H^1_D(\Omega)}^2 \leq C \|g\|_{\mathcal{X}^1_{a+1}(\Omega)}^2.
\]

Applying (2.3.20), for each \( 0 < t < T \) we can deduce

\[
\|\bar{u}_m(t)\|_{H^1_D(\Omega)}^2 \leq \beta_4 B[\bar{u}_m(t), \bar{u}_m(t)] \leq \beta_4 (B[\bar{u}_m(0), \bar{u}_m(0)] + \int_0^t B[\bar{u}_m(\tau), \bar{u}_m'(\tau)] \ d\tau).
\]

\[
\leq \beta_4 B[\bar{u}_m(0), \bar{u}_m(0)] + \beta_4 \int_0^t -(\bar{u}_m'(\tau), \bar{u}_m'(\tau)) + (\bar{u}_m'(\tau), \bar{f}(\tau)) \ d\tau \leq \beta_5 \|\bar{g}\|_{H^1_D(\Omega)}^2 + \beta_4 \|\bar{g}\|_{H^1_D(\Omega)}^2 + \frac{\beta_4}{2} \|\bar{u}_m\|_{L^2(0,T;L^2(\Omega))}^2.
\]

That is

\[
\|\bar{u}_m\|_{L^\infty(0,T;H^1_D(\Omega))}^2 + \|\bar{u}_m'\|_{L^2(0,T;L^2(\Omega))}^2 \leq C(\|\bar{g}\|_{H^1_D(\Omega)}^2 + \|\bar{f}\|_{L^2(0,T;L^2(\Omega))}^2).
\]

Recall \( u_\ast = \theta^a \bar{u}_m \), by a similar argument as (2.3.23), we deduce

\[
\|u_m\|_{L^\infty(0,T;\mathcal{X}^{1}_{a+1}(\Omega))}^2 + \|u_m'\|_{L^2(0,T;\mathcal{X}^0_a(\Omega))}^2 \leq C(\|g\|_{\mathcal{X}^1_{a+1}(\Omega)}^2 + \|f\|_{L^2(0,T;\mathcal{X}^0_a(\Omega))}^2).
\]

Passing to limits \( u_m \to u \) and we can see \( u \in L^\infty(0,T;\mathcal{X}^{1}_{a+1}(\Omega)) \cap L^\infty(0,T;H^1_D(\Omega)) \), with \( u_\ast \in L^2(0,T;\mathcal{X}^0_a(\Omega)) \), with the given bound. \( \Box \)

2.3.4 Remark. Under this condition, \( u \) solves the parabolic problem (2.1.1) strongly, that is, \( Lu = f - u_\ast \) almost everywhere in \( \cup_{k=1}^K \Omega_k \times T \), with \( u \) satisfying the boundary and transmission conditions. These reveals \( Pu = f - u_\ast \) for every \( t > 0 \). Since \( f - u_\ast \in L^2(0,T;\mathcal{X}^0_a(\Omega)) \), from the lemma 2.2.1 we deduce \( u = P^{-1}(f - u_\ast) \in L^2(0,T;V^2_{b+1}(\Omega)) \) for any \( 0 < b < \min\{ \eta, a + 1 \} \), where \( \eta \) is the constant in the lemma 2.2.1.
2.3.5 Lemma. Assume for some \( a \in (0, \delta) \) we have \( g \in V^2_{a+1}(\Omega) \), \( f' \in L^2(0, T; (V^1_{a+1}(\Omega))') \), \( f(0) - Pg \in K_0^0(\Omega) \), and \( u \) is the weak solution of IVP (2.1.1). Then, the function \( u_t \) lies in \( L^2(0, T; \mathcal{X}^1_{a+1}(\Omega)) \cap L^2(0, T; H^1_D(\Omega)) \) and is the unique weak solution of the IBVP

\[
\begin{aligned}
Lw + w_t &= f', \quad \text{in } \Omega_k \times T; \\
w &= 0, \quad \text{on } \partial_D \Omega; \\
\nabla_y w &= 0, \quad \text{on } \partial_N \Omega; \\
w(0) &= f(0) - Pg, \quad \text{on } \Omega_k \times \{t = 0\}.
\end{aligned}
\] (2.3.30)

Proof. This lemma is based on lemma 1.2.13 in the previous chapter. We study the IBVP (2.3.30) first. By the classical results, \( w \) is well posed and satisfy the weak formulation

\[
B[w, v] + \langle w'_t, v \rangle = \langle f', v \rangle \quad \forall \, v \in H^1_D(\Omega) \text{ and a.e. } 0 \leq t \leq T;
\] (2.3.31)

along with the initial condition \( w(0) = f(0) - Pg \in K_0^0(\Omega) \). We apply the theorem 2.3.2 and discover \( w \in L^2(0, T; \mathcal{X}^1_{a+1}(\Omega)) \cap L^2(0, T; H^1_D(\Omega)) \cap L^\infty(0, T; K_0^0(\Omega)) \).

Now we define a new function \( \tilde{u} \) related to \( w \):

\[
\tilde{u}(t) = \int_0^t w(\tau) d\tau + g
\] (2.3.32)

for each \( t > 0 \). Notice \( w \in C([0, T]; K_0^0(\Omega)) \), hence the function \( \tilde{u}(t) \) is well defined and is Lipschitz continuous in \( K_0^0(\Omega) \) norm. Obviously, \( \langle \tilde{u}_t, v \rangle = \langle w, v \rangle \) for all \( v \in H^1_D(\Omega) \) and a.e. \( t > 0 \). Now we integrate the equation (2.3.31) through \( 0 \) to \( t \):

\[
\int_0^t B[w(\tau), v] d\tau + \int_0^t \langle w'_t(\tau), v \rangle d\tau = \int_0^t \langle f'(\tau), v \rangle d\tau.
\] (2.3.33)

Here we set \( v \) to be constant in time, and rewrite the equation as

\[
B\left[\int_0^t w(\tau) d\tau, v\right] + \left(\int_0^t w'_t(\tau) d\tau, v\right) = \left(\int_0^t f'(\tau) d\tau, v\right).
\] (2.3.34)
By the fact that \( w(\tau) \in L^2(0,T;\mathcal{C}^1_{a+1}(\Omega)), w'(\tau) \in L^2(0,T; (V^1_{a+1}(\Omega))'), f'(\tau) \in L^2(0,T; (V^1_{a+1}(\Omega))') \), we can apply the theorem 2 in the section 5.9.2 of Evans book to discover:

\[
B[\tilde{u}(t) - \tilde{u}(0), v] + \langle w(t) - w(0), v \rangle = (f(t) - f(0), v); \tag{2.3.35}
\]

That is

\[
B[\tilde{u}(t), v] + \langle \tilde{u}'(t), v \rangle = (f(t), v) + B[\tilde{u}(0), v] + \langle w(0), v \rangle - (f(0), v). \tag{2.3.36}
\]

As \( g \in V^2_{a+1}(\Omega) \), we can compute

\[
B[\tilde{u}(0), v] + \langle w(0), v \rangle - (f(0), v) = B[g, v] + (f(0) - Lg, v) - (f(0), v) = (Lg + f(0) - Lg - f(0), v) = 0, \tag{2.3.37}
\]

Therefore we obtain

\[
B[\tilde{u}(t), v] + \langle \tilde{u}'(t), v \rangle = (f(t), v). \tag{2.3.38}
\]

Recall the fact (2.3.32) that \( \tilde{u}(0) = g \) and \( \tilde{u}(t) \in \mathcal{C}^1_{a+1}(\Omega) \cap H_0^1(\Omega) \) for a.e. \( t > 0 \), we can observe \( \tilde{u} \) is the unique weak solution of the IVP(2.1.1). As \( w = \tilde{u}_t \) solves the IVP(2.3.30), we complete the proof.

2.3.3 Higher Regularity of Solution under Compatibility Conditions

Now let us study how regular the solution will behave when \( f \) and \( g \) lies on sufficiently regular spaces, with highly compatible conditions.

2.3.6 Theorem (Higher regularity). Let \( \delta > 0 \) be the constant in the theorem 2.3.2, \( m \geq 0 \) be an integer, \( a \in (0, \delta) \), \( g \in \mathcal{K}^{2m+1}_{a+1}(\Omega) \), and

\[
\frac{d^i f}{dt^i} \in C([0,T]; \mathcal{K}^{2m-1-2i}_{a}(\Omega)), \quad i = 0, 1, ..., m - 1.
\]

\[
\frac{d^i f}{dt^i} \in L^2(0,T; \mathcal{K}^{2m-2i}_{a}(\Omega)), \quad i = 0, 1, ..., m.
\]

Suppose also the following \( m^{th} \) order compatibility conditions hold:

\[
g_0 = g \in V^2_{a+1}(\Omega) \cap \mathcal{K}^{2m+1}_{a+1}(\Omega),
\]
\[g_1 = f(0) - Pg_0 \in \mathcal{V}_{a+1}^2(\Omega) \cap \mathcal{X}_{a+1}^{2m-1}(\Omega),\]

...  

\[g_{m-1} = \frac{d^{m-2}f}{dt^{m-2}}(0) - Pg_{m-2} \in \mathcal{V}_{a+1}^2(\Omega) \cap \mathcal{X}_{a+1}^{3}(\Omega),\]

\[g_m = \frac{d^{m-1}f}{dt^{m-1}}(0) - Pg_{m-1} \in \mathcal{X}_{a+1}^1(\Omega) \cap H_D^1(\Omega).\]

Assume \(u\) solves the parabolic problem (2.1.1), then,

\[\frac{d^i u}{dt^i} \in L^\infty(0, T; \mathcal{X}_{a+1}^{2m+1-2i}(\Omega)) \cap L^\infty(0, T; \mathcal{X}_{a+1}^i(\Omega)) \quad i = 0, 1, ..., m;\]

and we have the estimate

\[\sum_{i=0}^{m} \left\| \frac{d^i u}{dt^i} \right\|_{L^\infty(0, T; \mathcal{X}_{a+1}^{2m+1-2i}(\Omega))} \leq C \left( \sum_{i=0}^{m} \left\| \frac{d^i f}{dt^i} \right\|_{L^2(0, T; \mathcal{X}_{a}^{2m-2i}(\Omega))} + \sum_{i=0}^{m-1} \left\| \frac{d^i f}{dt^i} \right\|_{L^\infty(0, T; \mathcal{X}_{a}^{2m-2i-2}(\Omega))} + \sum_{i=0}^{m} \left\| g_i \right\|_{\mathcal{X}_{a+1}^{2m+1-i}}.\]

Here the constant \(C\) only depend on \(m, \Omega, T\) and the coefficients of \(L\).

Proof. The proof is an induction on \(m\), the case \(m = 0\) being theorem 2.3.3 above.

Assume now the theorem is valid for some nonnegative integer \(m\), and suppose then

\[g \in \mathcal{X}_{a+1}^{2m+3}(\Omega), \quad \frac{d^{m+1}f}{dt^{m+1}} \in L^2(0, T; \mathcal{X}_{a}^1(\Omega)),\]

\[\frac{d^i f}{dt^i} \in L^2(0, T; \mathcal{X}_{a}^{2m+2-2i}(\Omega)) \cap C([0, T]; \mathcal{X}_{a}^{2m+1-2i}(\Omega)), \quad i = 0, 1, ..., m;\]

and the \((m + 1)^{th}\) order compatibility conditions hold. Now set \(\tilde{u} = u' = u_t\). By the result of lemma 2.3.5, we check that \(\tilde{u}\) is the unique weak solution of

\[
\begin{align*}
\tilde{u}_t + Lu &= \tilde{f} \quad \text{in} \ \Omega_T, \\
\tilde{u} &= 0 \quad \text{on} \ \partial_D \Omega \times [0, T], \\
\nabla^n v \tilde{u} &= 0 \quad \text{on} \ \partial_N \Omega \times [0, T], \\
\tilde{u} &= \tilde{g} \quad \text{on} \ \Omega \times \{t = 0\}, \\
\tilde{u}^+ - \tilde{u}^- &= 0 \quad \text{on} \ \Gamma \times [0, T], \\
\nabla^n v \tilde{u}^+ - \nabla^n v \tilde{u}^- &= 0 \quad \text{on} \ \Gamma \times [0, T].
\end{align*}
\]
for $f = f'$, $\bar{g} = g_1 = f(0) - Pg$. In particular, for $m = 0$ we have $\bar{f} \in L^2(0, T; \mathcal{K}^0(\Omega))$ and $\bar{g} \in \mathcal{K}^1_{a+1}(\Omega) \cap H^1_D(\Omega)$. We rely upon Theorem 2.3.2 to be sure that $\bar{u} \in L^\infty(0, T; H^1_D(\Omega)) \cap L^\infty(0, T; \mathcal{K}^1_{a+1}(\Omega))$.

Since $f$ and $g$ satisfy the $(m + 1)^{th}$ order compatibility conditions, it follows that $\bar{f}$ and $\bar{g}$ satisfy the $m^{th}$ order compatibility condition. Thus applying the induction assumption, we deduce

$$
\frac{d^i\bar{u}}{dt^i} \in L^\infty(0, T; \mathcal{K}^{2m+1-2i}_{a+1}(\Omega)) \quad i = 0, \ldots, m;
$$

and

$$
\sum_{i=0}^{m} \left\| \frac{d^i\bar{u}}{dt^i} \right\|_{L^\infty(0, T; \mathcal{K}^{2m+1-2i}_{a+1}(\Omega))} \leq C \left\{ \sum_{i=0}^{m} \left\| \frac{d^i\bar{f}}{dt^i} \right\|_{L^2(0, T; \mathcal{K}^{2m-2i}_{a}(\Omega))} + \sum_{i=0}^{m-1} \left\| \frac{d^i\bar{f}}{dt^i} \right\|_{L^2(0, T; \mathcal{K}^{2m-2i}_{a}(\Omega))} + \sum_{i=0}^{m} \left\| \frac{\bar{g}_i}{dt^i} \right\|_{\mathcal{K}^{2m+1}_{a+1}(\Omega)} \right\}.
$$

for $\bar{f} = f'$. Since $\bar{u} = u'$ and $\bar{g}_i = g_{i+1}$ for each $i = 0, 1, \ldots, m$, we can rewrite the foregoing:

$$
\frac{d^{i+1}\bar{u}}{dt^{i+1}} \in L^\infty(0, T; \mathcal{K}^{2m+3-2i}_{a+1}(\Omega)) \quad i = 1, \ldots, m + 1;
$$

\begin{align}
\sum_{i=1}^{m+1} \left\| \frac{d^{i+1}\bar{u}}{dt^{i+1}} \right\|_{L^\infty(0, T; \mathcal{K}^{2m+3-2i}_{a+1}(\Omega))} &\leq C \left\{ \sum_{i=1}^{m} \left\| \frac{d^{i+1}\bar{f}}{dt^{i+1}} \right\|_{L^\infty(0, T; \mathcal{K}^{2m+1-2i}_{a+1}(\Omega))} + \sum_{i=1}^{m+1} \left\| \frac{d^{i+1}\bar{f}}{dt^{i+1}} \right\|_{L^\infty(0, T; \mathcal{K}^{2m+1-2i}_{a+1}(\Omega))} + \sum_{i=1}^{m+1} \left\| \frac{\bar{g}_i}{dt^i} \right\|_{\mathcal{K}^{2m+3-2i}_{a+1}(\Omega)} \right\},
\end{align}

(2.3.40)

Now write for a.e. $0 \leq t \leq T$: $Pu = f - u' = h$ in $\Omega$. According to lemma 2.2.1, as $a < \delta < \eta$, we have

$$
\left\| u \right\|_{\mathcal{K}^{2m+3}_{a+1}(\Omega)} \leq C \left\| h \right\|_{\mathcal{K}^{2m+1}_{a-1}(\Omega)} \leq C \left( \left\| f \right\|_{\mathcal{K}^{2m+1}_{a-1}(\Omega)} + \left\| u' \right\|_{\mathcal{K}^{2m+1}_{a-1}(\Omega)} \right) \leq C \left( \left\| f \right\|_{\mathcal{K}^{2m+1}_{a-1}(\Omega)} + \left\| u' \right\|_{\mathcal{K}^{2m+1}_{a-1}(\Omega)} \right).
$$

This reveals

$$
\left\| u \right\|_{L^\infty(0, T; \mathcal{K}^{2m+3}_{a+1}(\Omega))} \leq C \left( \left\| f \right\|_{L^\infty(0, T; \mathcal{K}^{2m+1}_{a+1}(\Omega))} + \left\| u' \right\|_{L^\infty(0, T; \mathcal{K}^{2m+1}_{a+1}(\Omega))} \right).
$$
Adding the resulting expression to the equation (2.3.40), we deduce,

\[
\sum_{i=0}^{m+1} \left\| \frac{d^i u}{dt^i} \right\|_{L^\infty(0,T; \mathbb{R}^{2^{m+1-2i}(\Omega)})} \leq C \left( \sum_{i=0}^{m} \left\| \frac{d^i f}{dt^i} \right\|_{L^\infty(0,T; \mathbb{R}^{2^{m+1-2i}(\Omega)})} + \sum_{i=0}^{m+1} \left\| \frac{d^i f}{dt^i} \right\|_{L^2(0,T; \mathbb{R}^{2^{m+1-2i}(\Omega)})} + \sum_{i=0}^{m+1} \left\| g_i \right\|_{\mathbb{R}^2} \right).
\]

We thereby obtain the assertion of the theorem for \( m + 1 \).

\[\square\]

2.3.7 Remark. There are two ways to construct the functions \( f \) and \( g \) meeting all the compatibility conditions. One is to pick any function \( g \) in \( \mathbb{R}^{2^{m+1}(\Omega)} \cap \mathbb{M}^{2^m}_{a+1} \) and match this \( g \) with an appropriate function \( f \). We define \( f(t) \) to be a function-valued polynomial with respect to \( t \):

\[
f(t) = f(0) + f'(0)t + f''(0)\frac{t^2}{2!} + \ldots + f^{m-1}(0)\frac{t^{m-1}}{(m-1)!}.
\]

Now \( g_0 = g \) is given. For each given \( g_i \in \mathbb{V}^{2^m}_{a+1} \cap \mathbb{M}^{2^m-2i}_{a+1} \), \( i = 0, 1, \ldots, m-1 \), we construct a function \( \frac{d^i f}{dt^i} \) that lies in \( \mathbb{R}^{2^{m-2i}}(\Omega) \) and satisfy \( \frac{d^i f}{dt^i}(0) - P g_i \in \mathbb{V}^{2^m-2i}_{a+1} \cap \mathbb{M}^{2^m-2i-1}_{a+1} \). After \( m \) steps we will eventually obtain the function \( f \).

Another way to meet the compatibility conditions is to apply the set of eigenfunctions \( \{w_j\}_{j=1}^\infty \) in the theorem 2.3.2, where \( P w_j = \lambda_j w_j \), \( j = 1, 2, \ldots \). Recall this theorem, \( w_j \in \mathbb{H}^1_D(\Omega) \) solves the following elliptic transmission problem strongly:

\[
\begin{cases}
Lu = \lambda_j w_j & \text{in } \bigcup_{k=1}^{K} \Omega_k, \\
u = 0 & \text{on } \partial_D \Omega, \\
\nabla^A u = 0 & \text{on } \partial_N \Omega, \\
u^+ - u^- = 0 & \text{on } \Gamma, \\
\nabla^A u^+ - \nabla^A u^- = 0 & \text{on } \Gamma.
\end{cases}
\]

(2.3.41)

The lemma 2.2.1 guarantees \( w_j = u = P^{-1}(\lambda_j w_j) \in \mathbb{R}^{2^m}_{a+1}(\Omega) \cap \mathbb{M}^{2^m}_{a+1} \), \( j = 1, 2, \ldots \). By induction we can deduce \( w_j = P^{-m\lambda_j^-m} w_j \in \mathbb{M}^{2^m-2i}_{a+1} \cap \mathbb{M}^{2^m-2i-1}_{a+1} \) for all integer \( m \geq 1 \). This reveals that, if we define \( f \) and \( g \) to be the finite sum of some eigenfunctions of \( P \):

\[
g = c_1 w_1 + c_2 w_2 + \ldots + c_N w_N,
\]
and
\[ f = d_1(t)w_1 + d_2(t)w_2 + ... + d_N(t)w_N, \]
where \( \{c_i\}_{i=1}^N \in \mathbb{R}, \{d_i(t)\}_{i=1}^N \in C^m[0,T], \) then the compatibility conditions will automatically hold.

### 2.4 Vertex Singularities

#### 2.4.1 Theorem. Let \( \eta > 0 \) be the constant in the lemma 2.2.1, and \( \delta > 0 \) be the constant in the theorem 2.3.2. Still, we only consider the case when \( \mathcal{V}_2 = \emptyset. \) Also, we assume that the coefficients of operator \( L \) are constant on each subdomain \( \Omega_k, \ k = 1, 2, \ldots, K. \) Recall the proof of theorem 2.3.2 and the equation (2.3.15), we have the following bound for \( \eta \) and \( \delta: \)

\[
\max \delta = \max \eta = \inf_{Q_i \in \mathcal{V}, \bar{u} \in H^1_0(\Omega)} \left( \frac{\int_{B_{Q_i}(C_\delta/2)} (\nabla \bar{u})^T A(\nabla \bar{u}) dS}{\int_{B_{Q_i}(C_\delta/2)} \bar{u}^2 \frac{1}{(x-x_i,y-y_i) \lambda^{-1}(x-x_i,y-y_i)^T} dS} \right)^{1/2}. \tag{2.4.1}
\]

In particular, if \( L = -\Delta \) on each subdomain \( \Omega_k, \) then we have

\[
\max \delta = \max \eta = \min_{Q_i \in \mathcal{V}} \left( \frac{\pi}{\sigma_i \omega_i} \right), \tag{2.4.2}
\]

where \( \omega_i \) is the interior angle of \( \Omega \) in the vertex \( Q_i, \) \( \sigma_i = 1 \) if \( Q_i \) has two Dirichlet edges, \( \sigma_i = 2 \) if \( Q_i \) has one Neumann edge and one Dirichlet edge.

#### 2.4.2 Remark. The value \( \max \eta \) describes how singular the solution is near the vertices of the domain, which plays a very important role on both elliptic and parabolic problems with vertices. It depend on the operator \( L \) and the shape of the domain \( \Omega. \) In our paper, we denote \( \max \eta \) as the domain singularity. For each vertex \( Q_i, \) we call the infimal of the above expression (2.4.1) as the singularity of \( Q_i. \) Since finding the infimal of such expression is essentially an eigenvalue problem on polar coordinates, so the minimal exists and equals to the infimal. The domain singularity for \( L = -\Delta \) is well known, which is initially mentioned in the book of Professor V.A.Kondratev in 1967 (See [17]). While in 1990, Professor S. Nicaise dealt with the case that \( L \) equals to a scalar multiple of Laplacian on each subdomain (See [28] [29]).

In the part (a) of the proof, we will derive that, the domain singularity \( \delta \) of the parabolic problem is the same as the corresponding elliptic problem \( \eta, \) when given the same \( L \) and same \( \Omega_k. \) Also, we will show that, the value \( \delta \) can be chosen up to the above infimal.
In the part (b), we will show that, in the case of \( L = -\Delta \), the above infimal equals to a value, in which the value is well known, and \( \delta \) cannot exceed this value.

In the part (c), we are going to present that, in the case of \( L = -C_k \Delta \), the above infimal equals to a value, in which \( \delta \) cannot exceed this value. Such value can be obtained by solving a 1 dimensional eigenvalue problem.

In the part (d), we will move on to the case when \( L \) is a general second order operator with constant coefficients. Under such case, we can proof the above infimal equals to a value, in which \( \delta \) cannot exceed this value. Such value can be obtained by doing dilation and rotation near each vertices and solving a 1 dimensional eigenvalue problem afterwards.

**Proof.** In the following content, we denote

\[
F_i(\bar{u}) = \left( \int_{B_{Q_i}(C_0/2)} (\nabla \bar{u})^\top A(\nabla \bar{u}) \, dS \right)^{1/2} - \int_{B_{Q_i}(C_0/2)} \frac{\bar{u}^2}{(x-x_i, y-y_i)A^{-1}(x-x_i, y-y_i)^\top} dS
\]

for \( \bar{u} \in H^1_D(\Omega) \).

(a). We look back to the equation (2.3.15). Recall the definition of the function \( \theta \) and the region \( B_{Q_i}(C_0/2) \), if \( 0 < a < \inf_{\bar{u} \in H^1_D(\Omega)} F_i(\bar{u}) \) for each \( i \), then

\[
\int_{B_{Q_i}(C_0/2)} (\nabla \bar{u})^\top A(\nabla \bar{u}) \, dS - a^2 \int_{B_{Q_i}(C_0/2)} \frac{\bar{u}^2}{(x-x_i, y-y_i)A^{-1}(x-x_i, y-y_i)^\top} dS
\]

\[
= (1 - \inf_{\bar{u} \in H^1_D(\Omega)} F^2_i(\bar{u})) \int_{B_{Q_i}(C_0/2)} (\nabla \bar{u})^\top A(\nabla \bar{u}) \, dS
\]

\[
+ a^2 \left( \inf_{\bar{u} \in H^1_D(\Omega)} F^2_i(\bar{u}) \right) - \int_{B_{Q_i}(C_0/2)} \frac{\bar{u}^2}{(x-x_i, y-y_i)A^{-1}(x-x_i, y-y_i)^\top} dS
\]

\[
\geq (1 - \inf_{\bar{u} \in H^1_D(\Omega)} F^2_i(\bar{u})) \int_{B_{Q_i}(C_0/2)} (\nabla \bar{u})^\top A(\nabla \bar{u}) \, dS \geq C \|\nabla \bar{u}\|_{L^2(B_{Q_i}(C_0/2))}^2
\]

for some constant \( C > 0 \). When apply this inequality to (2.3.15), we see

\[
B_a[\bar{u}, \bar{u}] \geq \beta \|\nabla \bar{u}\|_{L^2(\Omega)}^2 - \gamma \|\bar{u}\|_{L^2(\Omega)}^2
\]

(2.4.4)
for some $\beta, \gamma > 0$.

Now let us keep $0 < a < \inf_{Q, \in \nu, \in H_{1,}^l(\Omega)} \mathbb{F}_1(\bar{u})$ so that the bounded bilinear form $B_a[\cdot, \cdot]$ satisfies (2.4.4), and then look back to the lemma 2.2.1 in the case $\mathcal{V}_2 = \emptyset$. The weak formulation of the elliptic problem (2.2.1) is

$$B_a[\bar{u}, \nu] = \langle \bar{f}, \nu \rangle \quad \forall \nu \in H_{D}^1(\Omega) \quad (2.4.5)$$

for $\bar{u} = u/\theta^a$ and $\bar{f} = f/\theta^a$. The existence and uniqueness of weak solution is known by the fact that $u = P^{-1}f \in H_{D}^1(\Omega)$.

Next we are going to show $\|\bar{u}\|_{L^2(\Omega)} \leq C \|\bar{f}\|_{(H_{D}^1(\Omega))^\prime}$, for some constant $C > 0$. If this is not true, there would exist sequences $\{\bar{f}_k\}_{k=1}^\infty \subset (H_{D}^1(\Omega))^\prime$ and $\{\bar{u}_k\}_{k=1}^\infty \subset H_{D}^1(\Omega)$ such that

$$B_a[\bar{u}_k, \nu] = \langle \bar{f}_k, \nu \rangle \quad \forall \nu \in H_{D}^1(\Omega),$$

but

$$\|\bar{u}_k\|_{L^2(\Omega)} > k \|\bar{f}_k\|_{(H_{D}^1(\Omega))^\prime}.$$

As we may with no loss suppose $\|\bar{u}_k\|_{L^2(\Omega)} = 1$, we see $\bar{f}_k \to 0$ in $(H_{D}^1(\Omega))^\prime$. Now let us do the usual energy estimates

$$\beta \|\nabla \bar{u}_k\|_{L^2(\Omega)}^2 - \gamma \|\bar{u}_k\|_{L^2(\Omega)}^2 \leq B_a[\bar{u}_k, \bar{u}_k] = \langle \bar{f}_k, \bar{u}_k \rangle \leq \frac{\beta}{2} \|\bar{u}_k\|_{H_{D}^1(\Omega)}^2 + \frac{1}{2\beta} \|\bar{f}_k\|_{(H_{D}^1(\Omega))^\prime}^2.$$

That is

$$\|\bar{u}_k\|_{H_{D}^1(\Omega)}^2 \leq C(\|\bar{u}_k\|_{L^2(\Omega)}^2 + \|\bar{f}_k\|_{(H_{D}^1(\Omega))^\prime}^2). \quad (2.4.6)$$

This shows the sequence $\{\bar{u}_k\}_{k=1}^\infty$ is bounded in $H_{D}^1(\Omega)$. Thus there exists a subsequence $\{\bar{u}_{k_j}\}_{j=1}^\infty \subset \{\bar{u}_k\}_{k=1}^\infty$ such that

$$\begin{cases}
\bar{u}_{k_j} \rightharpoonup \bar{u}_0 & \text{in } H_{D}^1(\Omega), \\
\bar{u}_{k_j} \to \bar{u}_0 & \text{in } L^2(\Omega).
\end{cases} \quad (2.4.8)$$

Here “$\rightharpoonup$” represents weak convergence. Since $\bar{f}_{k_j} \to 0$ in $(H_{D}^1(\Omega))^\prime$, we deduce $\bar{u}_0$ satisfies the weak formulation

$$B_a[\bar{u}_0, \nu] = 0 \quad \forall \nu \in H_{D}^1(\Omega). \quad (2.4.9)$$
This is
\[ B[\delta^a \bar{u}_0, \bar{v}] = 0 \quad \forall \bar{v} \in H^1_D(\Omega). \]  
(2.4.10)

Since \( \delta^a \bar{u}_0 \in H^1_D(\Omega) \), and \( B[\ , \ ] \) is positive definite, thus \( \bar{u}_0 \equiv 0 \). However, \( \bar{u}_{\kappa_i} \to \bar{u}_0 \) in \( L^2(\Omega) \) implies that \( \|\bar{u}_0\|^2_{L^2(\Omega)} = 1 \), this results in a contradiction.

Hence we have shown \( \|\bar{u}\|_{L^2(\Omega)} \leq C \|\bar{f}\|_{(H^1_D(\Omega))'} \) for some constant \( C > 0 \). We apply the energy estimate (2.4.7) once again to obtain \( \|\bar{u}\|_{H^1_D(\Omega)} \leq C \|\bar{f}\|_{(H^1_D(\Omega))'} \) as well. Since \( \bar{u} = u/\delta^a \) and \( \bar{f} = f/\delta^a \), by the lemma 2.3.1 and a similar argument as (2.3.23) we can deduce

\[ \|u\|_{X^{m+1}_{\kappa_i}(\Omega)} \leq \|f\|_{X^{m+1}_{\kappa_i}(\Omega)}' . \]  
(2.4.11)

Recall the theorem 4 in the paper [23] on the two dimensional case, if \( u \) solves the elliptic problem (2.2.1), then for each \( m \geq 1 \),

\[ \|u\|_{X^{m+1}_{\kappa_i}(\Omega)} \leq C(\|f\|_{X^{m-1}_{\kappa_i}(\Omega)} + \|u\|_{X^{m}_{\kappa_i}(\Omega)}). \]  
(2.4.12)

That is

\[ \|u\|_{X^{m+1}_{\kappa_i}(\Omega)} \leq C(\|f\|_{X^{m-1}_{\kappa_i}(\Omega)} + \|u\|_{X^1_{\kappa_i}(\Omega)}) \]  
(2.4.13)

by induction. Apply the inequality (2.4.11) we see

\[ \|u\|_{X^{m+1}_{\kappa_i}(\Omega)} \leq C \|f\|_{X^{m-1}_{\kappa_i}(\Omega)} \]

hold for any \( m \geq 1 \). This reveals that, if \( a \) is under such condition \( 0 < a < \inf_{\bar{u} \in H^1_D(\Omega)} \bar{F}_i(\bar{u}) \), the lemma 2.2.1 will hold when \( \Omega \) does not contain Neumann-Neumann vertices or non-smooth interface.

Now let us move on to the parabolic problem (2.1.1). We review the proof of theorem 2.3.2 to discover, if the bilinear form \( B_a[\ , \ ] \) satisfies

\[ B_a[\bar{u}, \bar{v}] \geq \beta \|\nabla \bar{u}\|^2_{L^2(\Omega)} - \gamma \|\bar{u}\|^2_{L^2(\Omega)} \]  
(2.4.14)

for some \( \beta, \gamma > 0 \) and all \( \bar{u} \in H^1_D(\Omega) \), then the theorem 2.3.2 will hold, as well as theorem 2.3.3 and theorem 2.3.6. From our calculation in the equation (2.4.3) we deduce, the condition

\( 0 < a < \inf_{Q_i \in V, \bar{u} \in H^1_D(\Omega)} \bar{F}_i(\bar{u}) \) is sufficient for making this inequality hold.
Hence, we are able to choose

\[
\eta = \delta = \inf_{Q_i \in V, \bar{u} \in H^1_D(\Omega)} F_i(\bar{u}) = \inf_{Q_i \in V, \bar{u} \in H^1_D(\Omega)} \left( \frac{\int_{B_{Q_i}(C_0/2)} (\nabla \bar{u})^T A (\nabla \bar{u}) \, dS}{\int_{B_{Q_i}(C_0/2)} \frac{u^2}{(x-x_i, y-y_i)^2} \, dS} \right)^{1/2}
\]

in the lemma 2.2.1 for the elliptic problem, and in the theorem 2.3.2 for the parabolic problem. Next, we are going to show this value is the best possible for \( \eta \) and \( \delta \). We will need to find the exact value of \( \inf_{Q_i \in V, \bar{u} \in H^1_D(\Omega)} F_i(\bar{u}) \).

(b). Let us start from the easiest case when \( L = -\Delta \) on each subdomain \( \Omega_i \), under which condition \( A \) is the identity matrix. Recall the definition of function \( \delta \) and the related region \( B_{Q_i}(C_0/2) \), for each vertex \( Q_i \in V \), we let \( \delta = \delta_i = d(x, y), Q_i) = ((x-x_i)^2 + (y-y_i)^2)^{1/2} \) near \( Q_i \), so that the region \( B_{Q_i}(C_0/2) \) becomes a sector of radius \( C_0/2 \) and angle \( \omega_i \). Hence by calculation

\[
F_i(\bar{u}) = \frac{\|\nabla \bar{u}\|_{L^2(B_{Q_i}(C_0/2))}}{\|\bar{u}/\delta\|_{L^2(B_{Q_i}(C_0/2))}}.
\]

As \( Q_i \) can be either a Neumann-Dirichlet vertex or a Dirichlet-Dirichlet vertex, we will consider these two cases differently.

1. Assume \( Q_i \) has one Neumann edge and one Dirichlet edge, then we apply the lemma 2.3.1 directly to discover \( F_i(\bar{u}) \geq \frac{\pi}{2 \omega_i} \). This is

\[
\int_{B_{Q_i}(C_0/2)} (\nabla \bar{u})^T A (\nabla \bar{u}) \, dS \geq \left( \frac{\pi}{2 \omega_i} \right)^2 \int_{B_{Q_i}(C_0/2)} \frac{u^2}{(x-x_i, y-y_i)^2} \, dS
\]

One the other hand, if for each \( \varepsilon > 0 \) we let

\[
\bar{u}_\varepsilon(\tau, \phi) = \chi_i(\tau) \varepsilon \sin\left( \frac{\pi \phi}{2 \omega_i} \right),
\]

where \((\tau, \phi)\) represents the polar coordinate with pole \( Q_i \) and polar axis in the direction of the Dirichlet edge, while \( \chi_i(\tau) \) is a smooth cut-off function. It equals 1 on the interval \([0, C_0/2]\) and vanishes on \([C_0, \infty)\).

then it is clear that \( \bar{u}_\varepsilon \in H^1_D(\Omega) \) and has compact support in \( B_{Q_i}(C_0) \). By the definition of \( F_i \), we can calculate \( F_i(\bar{u}_\varepsilon) = \frac{(1+\varepsilon)\pi}{2 \omega_i} \) for this particular \( \bar{u}_\varepsilon \).
As \( \epsilon > 0 \) is arbitrary, we deduce \( \inf_{u \in H^1_{b,0}(\Omega)} F_i(\bar{u}) = \frac{\pi}{2\omega_i} \) for each \( Q_i \in \mathcal{V} \).

Now we are going to present that, the lemma 2.2.1 and the theorem 2.3.2 does not hold when \( a = \frac{\pi}{2\omega_i} \). In the lemma 2.2.1, if we let

\[
\begin{align*}
  u(r, \phi) &= \chi_i(r)r^{\frac{\pi}{2\omega_i}} \sin\left(\frac{\pi \phi}{2\omega_i}\right), \\
  (r, \phi) &\text{ represents the polar coordinate defined as above, then as a result of calculation,} \\
  -\Delta u &= 0 \quad \text{in } B_{Q_i}(C_0/2); \\
  | -\Delta u | &< C \quad \text{in } B_{Q_i}(C_0) \setminus B_{Q_i}(C_0/2); \\
  -\Delta u &= u = 0 \quad \text{in } \Omega \setminus B_{Q_i}(C_0).
\end{align*}
\]

Here \( C \) is a constant depend only on \( C_0, \chi_i \) and \( \omega_i \). This estimate results in \(-\Delta u \in \mathcal{K}^0_{b}(\Omega)\) for any positive \( b \). However, it is clear that \( u \notin \mathcal{K}^0_{1 + \frac{\pi}{2\omega_i}}(\Omega) \) since \( 1/r \notin L^2(\mathcal{B}_{Q_i}(C_0/2)) \).

Therefore the lemma 2.2.1 fails when \( a = \frac{\pi}{2\omega_i} \). In other words, \( \eta \leq \frac{\pi}{2\omega_i} \).

The idea of constructing a counterexample in the theorem 2.3.2 on \( a = \frac{\pi}{2\omega_i} \) is similar. We let

\[
  u(r, \phi, t) = \chi_i(r)r^{\frac{\pi}{2\omega_i}} \sin\left(\frac{\pi \phi}{2\omega_i}\right)t.
\]

First, the initial condition \( g = u(0) \equiv 0 \) at \( t = 0 \). When \( t > 0 \), same as above, we can deduce \(-\Delta u \in \mathcal{K}^0_{b}(\Omega)\) for any positive \( b \). Also, \( u_t = \chi_i(r)r^{\frac{\pi}{2\omega_i}} \sin\left(\frac{\pi \phi}{2\omega_i}\right) \in \mathcal{K}^0_{1 + \frac{\pi}{2\omega_i}}(\Omega) \).

This implies \( f = -\Delta u + u_t \in \mathcal{K}^0_{1 + \frac{\pi}{2\omega_i}}(\Omega) \) for each \( t > 0 \). However, it is clear that \( u(t) \notin \mathcal{K}^0_{1 + \frac{\pi}{2\omega_i}}(\Omega) \) for any \( t > 0 \), and hence \( u \notin L^2(0,T;\mathcal{K}^0_{1 + \frac{\pi}{2\omega_i}}(\Omega)) \). Thus the theorem 2.3.2 fails at \( a = \frac{\pi}{2\omega_i} \). That is, \( \delta \leq \frac{\pi}{2\omega_i} \).

Since \( Q_i \in \mathcal{V} \) is arbitrary, we therefore obtain \( \eta \leq \min_{Q_i \in \mathcal{V}} \frac{\pi}{2\omega_i} = \inf_{u \in H^1_{b,0}(\Omega), Q_i \in \mathcal{V}} F_i(\bar{u}) \), and

\[
  \delta \leq \min_{Q_i \in \mathcal{V}} \frac{\pi}{2\omega_i} = \inf_{u \in H^1_{b,0}(\Omega), Q_i \in \mathcal{V}} F_i(\bar{u}).
\]

2. Assume both edges of \( Q_i \) are Dirichlet, then let us divide the sector \( B_{Q_i}(C_0/2) \) into two equal sectors \( e_{i1} \) and \( e_{i2} \), each of which has an angle of \( \omega_i/2 \). Since both sectors have one
Dirichlet side, we apply the lemma 2.3.1 on each of them to see

\[
\int_{c_{i1}} (\nabla \tilde{u})^T A(\nabla \tilde{u}) \, dS \geq \left( \frac{\pi}{2(\omega_i/2)} \right)^2 \int_{c_{i1}} \frac{\tilde{u}^2}{(x-x_i, y-y_i)^T A^{-1}(x-x_i, y-y_i)^T} dS,
\]

\[
\int_{c_{i2}} (\nabla \tilde{u})^T A(\nabla \tilde{u}) \, dS \geq \left( \frac{\pi}{2(\omega_i/2)} \right)^2 \int_{c_{i2}} \frac{\tilde{u}^2}{(x-x_i, y-y_i)^T A^{-1}(x-x_i, y-y_i)^T} dS.
\]

We combine these two inequalities to see

\[
\int_{B_{Q_i}(C_0/2)} (\nabla \tilde{u})^T A(\nabla \tilde{u}) \, dS \geq \left( \frac{\pi}{\omega_i} \right)^2 \int_{B_{Q_i}(C_0/2)} \frac{\tilde{u}^2}{(x-x_i, y-y_i)^T A^{-1}(x-x_i, y-y_i)^T} dS,
\]

That is, \( F_i(\tilde{u}) \geq \frac{\pi}{\omega_i} \). Still, the function

\[
\tilde{u}_\epsilon(r, \phi) = \chi_\epsilon(r) \frac{r}{2} \sin\left( \frac{\pi \phi}{2\omega_i} \right), \quad \epsilon > 0
\]

lies in \( H^1_D(\Omega) \) and makes \( F_i(\tilde{u}) = \frac{(1+\epsilon)\pi}{\omega_i} \). As \( \epsilon > 0 \) is arbitrary, we deduce \( \inf_{\tilde{u} \in H^1_D(\Omega)} F_i(\tilde{u}) = \frac{\pi}{\omega_i} \) for each \( Q_i \in \mathcal{V} \). Still, we need to show the lemma 2.2.1 and theorem 2.3.2 will no longer work when \( a = \frac{\pi}{\omega_i} \). Similarly, the counterexamples are

\[
u(r, \phi, t) = \chi_t(r) \frac{r}{\omega_i} \sin\left( \frac{\pi \phi}{\omega_i} \right) t.
\]

for lemma 2.2.1, and

\[
u(r, \phi, t) = \chi_t(r) \frac{r}{\omega_i} \sin\left( \frac{\pi \phi}{\omega_i} \right) t. \tag{2.4.17}
\]

for theorem 2.3.2. Here \((r, \phi)\) represents the polar coordinate with pole \( Q_i \) and polar axis in the direction of one Dirichlet side, while \( \chi_t(r) \) is a smooth cut-off function. It equals 1 on the interval \([0, C_0/2]\) and vanishes on \([C_0, \infty)\). These reveal \( \eta \leq \min_{Q_i \in \mathcal{V}} \frac{\pi}{\omega_i} = \inf_{\tilde{u} \in H^1_D(\Omega), Q_i \in \mathcal{V}} F_i(\tilde{u}) \), and \( \delta \leq \min_{Q_i \in \mathcal{V}} \frac{\pi}{\omega_i} = \inf_{\tilde{u} \in H^1_D(\Omega), Q_i \in \mathcal{V}} F_i(\tilde{u}) \).

(c). Now let us start the case when \( L = -C_k \Delta \) on each subdomain \( \Omega_k \). Here \( C_k > 0 \), and therefore \( A = C_k I \) on \( \Omega_k \) for each \( k = 1, 2, ..., K \). Still, fix a vertex \( Q_i \in \mathcal{V} \), and we are going to
study the exact value of $\inf_{\tilde{u} \in H^1_D(\Omega)} F_i(\tilde{u})$. Let the subdomains that contain $Q_l$ be $\Omega_{1l}$, ..., $\Omega_{ll}$, $l = 1, 2, \ldots, L$, with the matrix $A = C_{ll} I$ on each $\Omega_{ll}$, while their interfaces be in the directions of $\omega_{il}$, $l = 1, 2, \ldots, L-1$, $0 = \omega_{i0} < \omega_{i1} < \ldots < \omega_{ii} = \omega_i < 2\pi$, under the polar coordinates $(r, \phi)$, in which the pole is $Q_i$ and the polar axis is in the direction of one Dirichlet edge of $Q_i$. Recall the definition of $\partial_i$, we can let $\partial_i = d(x, y)$, $Q_i = ((x-x_i)^2 + (y-y_i)^2)^{1/2}$ near $Q_i$, and the corresponding region $B_{Q_i}(C_0/2)$ becomes a sector of radius $C_0/2$ and angle $\omega_i$. Now we can calculate

$$F_i(\tilde{u}) = \sum_{l=1}^{L} C_{ll} \frac{||\nabla \tilde{u}||^2_{L^2(B_{Q_i}(C_0/2) \cap \Omega_{ll})}}{||\tilde{u}/r||^2_{L^2(B_{Q_i}(C_0/2) \cap \Omega_{ll})}}^{1/2}.$$ (2.4.18)

This is

$$F_i^2(\tilde{u}) = \frac{\sum_{l=1}^{L} C_{ll} \left(||\tilde{u}/r||^2_{L^2(B_{Q_i}(C_0/2) \cap \Omega_{ll})} + ||\tilde{u}/r||^2_{L^2(B_{Q_i}(C_0/2) \cap \Omega_{ll})}\right)}{\sum_{l=1}^{L} C_{ll} \frac{||\nabla \tilde{u}||^2_{L^2(B_{Q_i}(C_0/2) \cap \Omega_{ll})}}{||\tilde{u}/r||^2_{L^2(B_{Q_i}(C_0/2) \cap \Omega_{ll})}}}.$$ (2.4.19)

And hence

$$F_i^2(\tilde{u}) \geq \frac{\sum_{l=1}^{L} C_{ll} \left(||\tilde{u}/r||^2_{L^2(B_{Q_i}(C_0/2) \cap \Omega_{ll})} + ||\tilde{u}/r||^2_{L^2(B_{Q_i}(C_0/2) \cap \Omega_{ll})}\right)}{\sum_{l=1}^{L} C_{ll} \frac{||\nabla \tilde{u}||^2_{L^2(B_{Q_i}(C_0/2) \cap \Omega_{ll})}}{||\tilde{u}/r||^2_{L^2(B_{Q_i}(C_0/2) \cap \Omega_{ll})}}}.$$ (2.4.20)

To minimize $F_i(\tilde{u})$, let us reduce it into a one dimensional minimizing problem: take only the arc of the sector $B_{Q_i}(r)$ for some $0 < r < C_0/2$, denoted as $C_{Q_i}(r)$. We see $C_{Q_i}(r)$ is divided into pieces by the interfaces that cross $Q_i$. Corresponding to the equation (2.4.20), we are going to study the following problem on the arc $C_{Q_i}(r)$:

$$\min_{\tilde{u} \in H^1([0, \omega_i]; \tilde{u}(0) = 0)} \sum_{l=1}^{L} \frac{1}{\omega_{il}} \int_{\omega_{il}}^{\omega_{i(l-1)}} C_{ll} \tilde{u}^2 \, d\phi$$ (2.4.21)

when $Q_i$ is a Newmann-Dirichlet vertex, or study

$$\min_{\tilde{u} \in H^1([0, \omega_i]; \tilde{u}(0) = \bar{u}(\omega) = 0)} \sum_{l=1}^{L} \frac{1}{\omega_{il}} \int_{\omega_{il}}^{\omega_{i(l-1)}} C_{ll} \tilde{u}^2 \, d\phi$$ (2.4.22)
when $Q_i$ is a Dirichlet-Dirichlet vertex.

Now let us start from the case when $Q_i$ is a Newmann-Dirichlet vertex. We obtain from the 1d Poincare inequality that the above fraction has a lower bound. Assume a function $\tilde{u} \in H^1[0, \omega]$ minimize the above fraction, with the minimal value $\lambda_{i0}^2$, for some $\lambda_{i0} > 0$, then for any $v \in H^1[0, \omega]$ with $v(0) = 0$, we have

$$\frac{d}{dt} \left( \frac{\sum_{l=1}^{L} \int_{\omega_{l-1}}^{\omega_l} C_{i\ell} \left( \tilde{u}_\phi + tv_{\phi} \right)^2 \, d\phi}{\sum_{l=1}^{L} \int_{\omega_{l-1}}^{\omega_l} C_{i\ell} \left( \tilde{u} + tv \right)^2 \, d\phi} \right)_{t=0} = 0.$$  

We calculate this expression to discover

$$\frac{\sum_{l=1}^{L} \int_{\omega_{l-1}}^{\omega_l} C_{i\ell} \tilde{u}_\phi^2 \, d\phi}{\sum_{l=1}^{L} \int_{\omega_{l-1}}^{\omega_l} C_{i\ell} \tilde{u}^2 \, d\phi} = \frac{\sum_{l=1}^{L} \int_{\omega_{l-1}}^{\omega_l} C_{i\ell} \tilde{u}_\phi v_{\phi} \, d\phi}{\sum_{l=1}^{L} \int_{\omega_{l-1}}^{\omega_l} C_{i\ell} \tilde{u} v \, d\phi} = \lambda_{i0}^2$$

for all $v \in H^1[0, \omega]$ with $v(0) = 0$. Here the ratio is positive since the LHS is always positive and is invariant of $v$.

Now we see

$$\sum_{l=1}^{L} \int_{\omega_{l-1}}^{\omega_l} C_{i\ell} \tilde{u}_\phi v_{\phi} \, d\phi = \lambda_{i0}^2 \sum_{l=1}^{L} \int_{\omega_{l-1}}^{\omega_l} C_{i\ell} \tilde{u} v \, d\phi.$$  

Apply integration by parts, we can rewrite the LHS as

$$\sum_{l=1}^{L} \int_{\omega_{l-1}}^{\omega_l} C_{i\ell} \tilde{u}_\phi v_{\phi} \, d\phi = - \sum_{l=1}^{L} \int_{\omega_{l-1}}^{\omega_l} C_{i\ell} \tilde{u}_\phi v \, d\phi + \sum_{l=1}^{L-1} \left( C_{i l+1} \tilde{u}_\phi^+(\omega_l) - C_{i l} \tilde{u}_\phi^-(\omega_l) \right) v(\omega_l) + C_{iL} \tilde{u}_\phi(\omega_L) v(\omega_L)$$
As \( v \) is arbitrary, it implies that \( \tilde{u} \) is the unique weak solution to the following eigenfunction problem

\[
\begin{align*}
-\tilde{u}_{\phi\phi} &= \lambda^2_{i_0} \tilde{u} \quad \text{in } (\omega_{i_{l-1}}, \omega_{i_l}), \ l = 1, 2, \ldots, L, \\
\tilde{u}(0) &= 0, \\
\tilde{u}_\phi(\omega_i) &= 0, \\
\tilde{u}^+(\omega_{i_l}) - \tilde{u}^-(\omega_{i_l}) &= 0 \quad l = 1, 2, \ldots, L - 1, \\
C_{i_{l+1}} \tilde{u}^+_\phi(\omega_{i_l}) - C_{i_l} \tilde{u}^-_\phi(\omega_{i_l}) &= 0 \quad l = 1, 2, \ldots, L - 1. 
\end{align*}
\] (2.4.24)

While \( \lambda^2_{i_0} \) is the corresponding eigenvalue. Since our goal is to minimize \( \lambda^2_{i_0} \), the problem therefore becomes, to find the smallest positive eigenvalue of the above 1 dimensional elliptic problem.

By the standard theory of ODE, the solution \( \tilde{u} \) can be expressed as

\[
\tilde{u}(\phi) = b_{i_l} \sin(\lambda_{i_0} \phi + d_{i_l}), \quad \phi \in (\omega_{i_{l-1}}, \omega_{i_l})
\]

for some number \( b_{i_l}, d_{i_l} \) that depend on \( c_{i_l} \) and \( \omega_{i_l}, l = 1, 2, \ldots, L \). Specifically, \( \lambda_{i_0}, b_{i_l} \) and \( d_{i_l} \) solves the following system of equations:

\[
\begin{align*}
d_{i_1} &= 0; \\
b_{i_l} \sin(\lambda_{i_0} \omega_{i_l} + d_{i_l}) &= b_{i_{l+1}} \sin(\lambda_{i_0} \omega_{i_{l+1}} + d_{i_{l+1}}), \quad l = 1, 2, \ldots, L - 1; \\
c_{i_l} b_{i_l} \cos(\lambda_{i_0} \omega_{i_l} + d_{i_l}) &= c_{i_{l+1}} b_{i_{l+1}} \cos(\lambda_{i_0} \omega_{i_{l+1}} + d_{i_{l+1}}), \quad l = 1, 2, \ldots, L - 1; \\
\cos(\lambda_{i_0} \omega_{i_l} + d_{i_l}) &= 0. 
\end{align*}
\] (2.4.25)

And \( \lambda_0 \) is the smallest positive number that solves the system of equations.

Similarly, for a Dirichlet-Dirichlet vertex \( Q_{i'}, \) the minimizer \( \tilde{u} \) and the minimal value \( \lambda^2_{i_0} \) of the expression (2.4.22) satisfy the following problem

\[
\begin{align*}
-\tilde{u}_{\phi\phi} &= \lambda^2_{i_0} \tilde{u} \quad \text{in } (\omega_{i_{l-1}}, \omega_{i_l}), \ l = 1, 2, \ldots, L, \\
\tilde{u}(0) &= \tilde{u}(\phi) = 0, \\
\tilde{u}^+(\omega_{i_l}) - \tilde{u}^-(\omega_{i_l}) &= 0 \quad l = 1, 2, \ldots, L - 1, \\
C_{i_{l+1}} \tilde{u}^+_\phi(\omega_{i_l}) - C_{i_l} \tilde{u}^-_\phi(\omega_{i_l}) &= 0 \quad l = 1, 2, \ldots, L - 1. 
\end{align*}
\] (2.4.26)
And still, $\tilde{u}$ is in the form of

$$
\tilde{u}(\phi) = b_{t1}\sin(\lambda_{t0}\phi + d_{t1}), \quad \phi \in \{\omega_{i1}, \omega_{il}\}
$$

where $\lambda_{t0}$, $b_{t1}$ and $d_{t1}$ solves the following system of equations:

$$
\begin{align*}
&\begin{cases}
  d_{t1} = 0; \\
  b_{t1}\sin(\lambda_{t0}\omega_{il} + d_{t1}) = b_{t1+1}\sin(\lambda_{t0}\omega_{il+1} + d_{t1+1}), & l = 1, 2, \ldots, L-1; \\
  c_{t1}b_{t1}\cos(\lambda_{t0}\omega_{il} + d_{t1}) = c_{t1+1}b_{t1+1}\cos(\lambda_{t0}\omega_{il+1} + d_{t1+1}), & l = 1, 2, \ldots, L-1; \\
  \sin(\lambda_{t0}\omega_{il} + d_{t1}) = 0. 
\end{cases}
\end{align*}
$$

(2.4.27)

Now for either case of boundary conditions, we return to the equation (2.4.20), for each vertex $Q_i \in \mathcal{V}$, we discover,

$$
\begin{align*}
&\sum_{l=1}^{L} C_{il} \left\| \tilde{u}_\phi \right\|_{B_{Q_i}(C_\phi/2) \cap \Omega_{il}}^2 - \lambda_{t0}^2 \sum_{l=1}^{L} C_{il} \left\| \tilde{u} \right\|_{B_{Q_i}(C_\phi/2) \cap \Omega_{il}}^2 \\
&= \int_0^{C_\phi/2} \sum_{l=1}^{L} \int_{\omega_{il}}^{\omega_{il-1}} C_{il} \tilde{u}_\phi^2 (r, \phi) r d\phi dr - \lambda_{t0}^2 \sum_{l=1}^{L} \int_{\omega_{il}}^{\omega_{il-1}} C_{il} \tilde{u}^2 (r, \phi) r d\phi dr \\
&\int_0^{C_\phi/2} \sum_{l=1}^{L} \int_{\omega_{il}}^{\omega_{il-1}} C_{il} \tilde{u}_\phi^2 (r, \phi) d\phi - \lambda_{t0}^2 \sum_{l=1}^{L} \int_{\omega_{il}}^{\omega_{il-1}} C_{il} \tilde{u}^2 (r, \phi) d\phi) r dr \geq 0.
\end{align*}
$$

(2.4.28)

And hence

$$
F^2_t(\tilde{u}) \geq \sum_{l=1}^{L} C_{il} \left\| \tilde{u}_\phi \right\|_{B_{Q_i}(C_\phi/2) \cap \Omega_{il}}^2 \geq \lambda_{t0}^2.
$$

On the other hand, we choose a function

$$
\tilde{u}(r, \phi) = \chi(r)r^\epsilon \tilde{u}(\phi) \in H^1_D(\Omega),
$$
where χ(r) is a smooth cut-off function that it equals 1 on [0, C_0/2], and equals 0 on [C_0, ∞). We then let $ε \to 0$ to find the following estimate

$$\inf_{u \in H_0^1(\Omega)} F_i^2(\bar{u}) = \inf_{u \in H_0^1(\Omega)} \sum_{l=1}^L C_{il} \left( \left\| u_r \right\|_{L^2(B_{Q_i}(C_0/2) \cap \Omega_{il})}^2 + \left\| \bar{u}_r / r \right\|_{L^2(B_{Q_i}(C_0/2) \cap \Omega_{il})}^2 \right) \leq \lim_{ε \to 0} \sum_{l=1}^L C_{il} \left( \left\| r^{ε-1} \bar{u}_r (\phi) \right\|_{L^2(B_{Q_i}(C_0/2) \cap \Omega_{il})}^2 + \left\| r^{ε-1} \bar{u}_r (\phi) \right\|_{L^2(B_{Q_i}(C_0/2) \cap \Omega_{il})}^2 \right) \leq \lim_{ε \to 0} \sum_{l=1}^L \int_{\omega_{i1}^{w_{i1}}} C_{il} \left( \bar{u}_r (\phi) \right) \frac{d\phi}{r} = \sum_{l=1}^L \int_{\omega_{i1}^{w_{i1}}} C_{il} \bar{u}_r^2 \frac{d\phi}{r} = \lambda_{i0}^2.$$

(2.4.29)

The above two equations implies that

$$\inf_{u \in H_0^1(\Omega)} F_i^2(\bar{u}) = \lambda_{i0}^2,$$

(2.4.30)

or

$$\inf_{u \in H_0^1(\Omega)} F_i(\bar{u}) = \lambda_{i0}.$$

What remains to be shown is $η \leq \lambda_{i0}$ and $δ \leq \lambda_{i0}$. In fact, in the elliptic problem 2.2.1, if we let

$$\bar{u}(r, \phi) = χ(r) r^{λ_{i0}} \bar{u}_r (\phi),$$

where $χ_i(r)$ is a smooth cut-off function that equals 1 on the interval $[0, C_0/2]$ and vanishes on $[C_0, ∞)$, then $\bar{u}$ is supported on the sector $B_{Q_i}(C_0)$, and

1. $\bar{u}$ satisfies the boundary conditions and the transmission conditions on the interface.
2. $Δ\bar{u} = 0$ on the sectors $B_{Q_i}(C_0/2) \cap \Omega_{il}$, which implies $Δ\bar{u} \in H^2_{λ_{i0}}(Ω)$.
3. $\bar{u} \notin H^{λ_{i0}+1}_{λ_{i0}}(Ω)$ since $1/r \notin L^2(Ω)$.

This example is sufficient to show $η \leq \lambda_{i0}$.

By a similar argument, for the parabolic problem 2.3.2, if we let

$$\bar{u}(r, \phi, t) = χ(r) r^{λ_{i0}} \bar{u}_r (\phi) t,$$
then we will have:

1. \( \bar{u} \) satisfy the boundary conditions and the transmission conditions on the interface.

2. \( \Delta \bar{u} = 0 \) on the sectors \( \mathcal{B}_{Q_i}(C_0/2) \cap \Omega_{1'i} \), which implies \( \Delta \bar{u} \in L^2(0, T; \mathcal{K}_{\lambda_{i0}}^0(\Omega)) \). Also, it is not hard to check \( \bar{u}_t \in L^2(0, T; \mathcal{K}_{\lambda_{i0}}^0(\Omega)) \).

3. \( \bar{u} \notin L^2(0, T; \mathcal{K}_{\lambda_{i0}+1}^1(\Omega)) \).

This implies \( \delta \leq \lambda_{i0} \).

As a result, we can eventually obtain that

\[
\max \eta = \max \delta = \min_{Q_i \in \mathcal{V}} \lambda_{i0} = \inf_{Q_i \in \mathcal{V}, \bar{u} \in H^1_D(\Omega)} F_i(\bar{u}).
\]

(d). Finally, let us move on to the case where the operator \( L = -\text{div}(A \nabla \cdot) \) is a general second order operator with constant coefficients on each subdomain. In fact, we can reduce this case into the case that \( L \) is a multiple of Laplacian by scaling.

Fix a vertex \( Q_i \in \mathcal{V} \), our purpose is to minimize

\[
F_i(\bar{u}) = \left( \frac{\int_{\mathcal{B}_{Q_i}(C_0/2)} (\nabla \bar{u})^\top A(\nabla \bar{u}) \, dS}{\int_{\mathcal{B}_{Q_i}(C_0/2)} (x-x_i,y-y_i)^\top A^{-1}(x-x_i,y-y_i)^\top \, dS} \right)^{1/2},
\]

for \( \bar{u} \in H^1_D(\Omega) \).

Still, let the subdomains that contain \( Q_i \) be \( \Omega_{i1}, ..., \Omega_{iL} \), with the matrix \( A = A_{i1} \) on \( \Omega_{i1} \), while their interfaces be in the directions of \( \omega_{i1}, \, l = 1, 2, ..., L \), under the polar coordinates with pole \( Q_i \) and polar axis in the direction of a Dirichlet side. And \( 0 = \omega_{i0} < \omega_{i1} < ... < \omega_{iL} = \omega_i \). Recall the definition of \( \mathcal{B}_{Q_i}(C_0/2) \):

\[
\mathcal{B}_{Q_i}(C_0/2) = \{ (x, y) | \vartheta_i(x, y) = C_i \left\langle (x-x_i, y-y_i) A^{-1} (x-x_i, y-y_i)^\top \right\rangle^{1/2} \leq C_0/2 \}.
\]

It is an adapted sector centered at \( Q_i \) with adapted radius \( C_0/2 \), and the arc of the sector is continuous across the interfaces.

Now we begin to set up a new polar coordinate system \( (r', \phi') \) by scaling the above system: still, we let \( Q_i \) be the pole, and take the same polar axis as the old one, in the direction
of a Dirichlet side. For each point inside the sector \( B_{Q_1}(C_0) \) with coordinate \((r, \phi)\), we let

\[
(r' \cos(\phi'), r' \sin(\phi'))^T = P(\phi) \ (r \cos \phi, r \sin \phi)^T, \quad l = 1, 2, ..., L.
\]

Here \( P(\phi) \) is a \( 2 \times 2 \) matrix defined on \([0, \omega]\), and is constant on each interval \((\omega_{l-1}, \omega_l)\), say \( P_{il} \), \( l = 1, 2, ..., L \). We let \( P_{il} \) satisfy the following conditions:

1. \( \det(P_{il}) > 0; \)
2. \( P_{i1}(r, 0)^T = (r, 0)^T; \)
3. \( P_{il}(r \cos(\omega_{il}), r \sin(\omega_{il}))^T = P_{i l+1}(r \cos(\omega_{il}), r \sin(\omega_{il}))^T, \) \( l = 1, 2, ..., L - 1; \)
4. \( P_{il} A_{il} P_{il}^T = d_{il} I \) for some constant \( d_{il} > 0. \)

Here the condition 3 is to make the map \( P(\phi) \) continuous across the interface, condition 1 is to make \( P(\phi) \) a bijection map, and condition 4 is to make the image of \( B_{Q_1}(C_0) \) be a sector. These four conditions uniquely determine the matrices \( P_{il} \) and the constant \( d_{il} \). Details are as follows.

Now we obtain the new coordinate \((r', \phi')\). Let us denote it as \( P^0 : (r, \phi) \rightarrow (r', \phi'). \)

From the conditions 2 and 3, we can denote as

\[
P_{il}(r \cos(\omega_{il}), r \sin(\omega_{il}))^T = P_{i l+1}(r \cos(\omega_{il}), r \sin(\omega_{il}))^T = (r' \cos(\omega'_{il}), r' \sin(\omega'_{il}))^T
\]

for \( l = 1, 2, ..., L - 1 \), and

\[
P_{i1}(\cos 0, \sin 0)^T = (\cos 0, \sin 0)^T, \quad P_{iL}(\cos \omega_i, \sin \omega_i)^T = (\cos \omega_i, \sin \omega_i)^T.
\]

The new boundary are now in the directions of \( \phi' = \omega'_{i0} = 0, \phi' = \omega'_{i1} = \omega'_i \), and the new interfaces are now in the direction of \( \phi' = \omega'_{il}, \) \( l = 1, 2, ..., L - 1 \), under the new polar coordinates. These interfaces divide the domain into \( L \) subdomains near the vertex \( Q_1 \), denotes as \( \Omega'_{il}, \) \( l = 1, 2, ..., L \). By the condition 1, we obtain

\[
0 = \omega'_{i0} < \omega'_{i1} < \omega'_{i2} < ... < \omega'_{iL} = \omega'_i < 2\pi.
\]

Back to the expression of \( B_{Q_1}(C_0/2) \), we see from the condition 4 that

\[
(x - x_i, y - y_i) A_{il}^{-1} (x - x_i, y - y_i)^T = (r \cos \phi, r \sin \phi) A_{il}^{-1} (r \cos \phi, r \sin \phi)^T
\]
Hence
\[ \mathcal{B}_{Q_i}(C_0/2) \cap \Omega_{il} = \{ (r', \phi') | C_{il} \sqrt{d_{il}} r' \leq C_0/2, \omega_{il-1} \leq \phi' \leq \omega_{il} \}. \]

It implies that \( P(\phi) \) maps each subdomain \( \mathcal{B}_{Q_i}(C_0/2) \cap \Omega_{il} \) into a small sector. As the arc of \( \mathcal{B}_{Q_i}(C_0/2) \) is continuous across the interface, and by condition 3 the map \( P(\phi) \) is continuous across the interface, we can conclude that the image \( P : \mathcal{B}_{Q_i}(C_0/2) \) is a sector with continuous arc across the interface as well. We denote the common radius as \( r'_0/2 \) and the sector as \( \mathcal{B}'_{Q_i}(r'_0/2) \). We see \( r'_0 = C_0\sqrt{d_{il}}/C_{il} \) for each \( l \).

Back to the expression of \( F_i(\bar{u}) \). For each \( u \in H^1_D(\Omega) \), we let \( \bar{u}(r', \phi') = \bar{u}(r, \phi) \). Then \( \bar{u} \) is a scaling of \( \bar{u} \) and is well defined on the sector \( \mathcal{B}'_{Q_i}(r'_0) \), with \( (\nabla \bar{u}) = (P^{-1})^T(\nabla \bar{u}) \).

\[
F_i^2(\bar{u}) = \frac{\int_{\mathcal{B}_{Q_i}(C_0/2)} (\nabla \bar{u})^T A(\nabla \bar{u}) \, dS}{\int_{\mathcal{B}_{Q_i}(C_0/2)} \bar{u}^2 (x-x_i, y-y_i) A^{-1}(x-x_i, y-y_i) \, dS} = \frac{\sum_{l=1}^L \int_{\mathcal{B}'_{Q_i}(r'_0/2) \cap \Omega_{il}} (\nabla \bar{u})^T(d_{il} l) (\nabla \bar{u}) \, \text{Det}(P^{-1}) \, dS'}{\sum_{l=1}^L \int_{\mathcal{B}'_{Q_i}(r'_0/2) \cap \Omega_{il}} d_{il} \bar{u}^2 \, \text{Det}(P^{-1}) \, dS'}
\]

\[
= \frac{\sum_{l=1}^L (\text{Det}(A_{il}))^{1/2} \, (\nabla \bar{u})^T(\nabla \bar{u}) \, \text{dS'}}{\sum_{l=1}^L (\text{Det}(A_{il}))^{1/2} \, \|\bar{u}/r'\|^2_{L^2(\mathcal{B}'_{Q_i}(r'_0/2) \cap \Omega_{il})}}
\]

We denote the above expression as \( F_i^2(\bar{u}) \). Now let us derive, if \( \bar{u}(r, \phi) \in H^1_D(\Omega) \), then what boundary conditions \( \bar{u}(r', \phi') \) need to satisfy on the sector \( \mathcal{B}'_{Q_i}(r'_0) \). First, we have \( \bar{u}(r, 0) = \bar{u}(r) \), which implies \( \phi' = 0 \) is a Dirichlet boundary of \( \bar{u} \). Second, if \( \phi = \omega_i \) is a Dirichlet side of \( \bar{u} \), we will have \( \bar{u}(r', \omega_i') = \bar{u}(r, \omega_i) = 0 \) and hence \( \phi' = \omega_i' \) is a Dirichlet side of \( \bar{u} \) as well; if \( \phi = \omega_i \) is a Neumann side of \( \bar{u} \), then we will have

\[ \nabla_{i} \bar{u}(r', \omega_i') = (\nabla \bar{u}(r', \omega_i')) \cdot n'(r', \omega_i') = (P^{-1})^T(\nabla \bar{u}(r, \omega_i)) \cdot \frac{r}{r'} \text{Det}(P_{il}) (P^{-1})^T n(r, \omega_i) \]
\[ \frac{r}{r'} \text{det}(P_{il}) (\nabla \bar{u}(r, \omega_i))^T \frac{P^{-1}_{il}}{P^{-1}_{il}} (P_{il}^{-1})^T \nabla \bar{u}(r, \omega_i) = \frac{\text{det}(P_{il})}{d_{il} r'} (\nabla \bar{u}(r, \omega_i))^T A_{il} n(r, \omega_i) \]

\[ = \frac{\text{det}(P_{il})}{d_{il} r'} \nabla^A \bar{u}(r, \omega_i) = (\text{det}(A_{il}))^{-1/2} \nabla^A \bar{u}(r, \omega_i) = 0, \quad (2.4.32) \]

This reveals \( \phi' = \omega'_i \) is a Neumann side of \( \bar{u} \) as well, in the form of \( \nabla \bar{u} \cdot n' = 0 \). Here in the first line of the above equation we used the following identity:

\[ r' n'(r', \omega'_i) = \begin{pmatrix} -r' \sin \omega'_i \\ r' \cos \omega'_i \end{pmatrix} = \begin{pmatrix} 0 & r' \cos \omega'_i \\ 1 & r' \sin \omega'_i \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} P_{il} \begin{pmatrix} r \cos \omega_i \\ r \sin \omega_i \end{pmatrix} = \text{det}(P_{il}) (P_{il}^{-1})^T r \ n(r, \omega_i). \quad (2.4.33) \]

Return to the equation (2.4.31), we can observe

\[ \inf_{\bar{u} \in H^1_D(\Omega)} \bar{F}^2_i(\bar{u}) = \inf_{\bar{u} \in H^1(D \setminus \Omega)} \bar{F}^2_i(\bar{u}) = \inf_{\bar{u} \in H^1(D \setminus \Omega)} \bar{F}^2_i(\bar{u}) \quad (2.4.34) \]

when \( Q_i \) is a Dirichlet-Neumann vertex of \( \bar{u} \), and

\[ \inf_{\bar{u} \in H^1_D(\Omega)} \bar{F}^2_i(\bar{u}) = \inf_{\bar{u} \in H^1(D \setminus \Omega)} \bar{F}^2_i(\bar{u}) \quad (2.4.35) \]

when \( Q_i \) is a Dirichlet-Dirichlet vertex of \( \bar{u} \). Here we rewrite the expression of \( \bar{F}^2_i(\bar{u}) \):

\[ \bar{F}^2_i(\bar{u}) = \frac{1}{\lambda_{i0}^2} \left( \text{det}(A_{il}) \right)^{1/2} \left( \left\| \nabla \bar{u} \right\|^2_{L^2(D \setminus \Omega)} + \left\| \nabla \bar{u} \right\|^2_{L^2(D \setminus \Omega)} \right), \quad (2.4.36) \]

Now let us denote \( \inf_{\bar{u} \in H^1_D(\Omega)} \bar{F}^2_i(\bar{u}) = \lambda_{i0}^2 \) for some \( \lambda_{i0} > 0 \). Recall the proof of the theorem in the case that \( \text{L} \) is a multiple of Laplacian on each subdomain, compare the expression of \( \bar{F}^2_i(\bar{u}) \) with the equation (2.4.19), we can directly apply the results of the equations (2.4.29) and (2.4.30) on \( \bar{F}^2_i(\bar{u}) \) to discover, if \( Q_i \) is a Dirichlet-Neumann vertex of \( \bar{u} \), then \( \lambda_{i0}^2 \) is the smallest eigenvalue.
of the following one dimensional elliptic problem on the interval \([0, \omega_i']\):

\[
\begin{aligned}
&-\bar{u}_{\phi', \phi'} = \lambda_{i_0}^2 \bar{u} \\
&\bar{u}(0) = 0, \\
&\bar{u}_{\phi'}(\omega_i') = 0, \\
&\bar{u}^+(\omega_i') - \bar{u}^-(\omega_i') = 0 \\
&(\text{Det}(A_{i_l+1}))^{1/2} \bar{u}_{\phi'}^+(\omega_i') - (\text{Det}(A_{i_l}))^{1/2} \bar{u}_{\phi'}^-(\omega_i') = 0
\end{aligned}
\]

(2.4.37)

And similarly, if \(Q_i\) is a Dirichlet-Dirichlet vertex of \(\bar{u}\), then \(\lambda_{i_0}^2\) is the smallest eigenvalue of the following problem:

\[
\begin{aligned}
&-\bar{u}_{\phi', \phi'} = \lambda_{i_0}^2 \bar{u} \\
&\bar{u}(0) = \bar{u}(\omega_i') = 0, \\
&\bar{u}^+(\omega_i') - \bar{u}^-(\omega_i') = 0 \\
&(\text{Det}(A_{i_l+1}))^{1/2} \bar{u}_{\phi'}^+(\omega_i') - (\text{Det}(A_{i_l}))^{1/2} \bar{u}_{\phi'}^-(\omega_i') = 0
\end{aligned}
\]

(2.4.38)

Hence we are able to get the value of \(\lambda_{i_0}\) by calculation. Still, what remains to be shown is \(\eta \leq \lambda_{i_0}\) and \(\delta \leq \lambda_{i_0}\). In fact, in the elliptic problem 2.2.1, if we let

\[
\hat{u}(r', \phi') = \chi(r') r^{\lambda_{i_0}} \bar{u}_{\phi'}(\phi'),
\]

where \(\chi_i(r)\) is a smooth cut-off function that equals 1 on the interval \([0, r'_0/2]\) and vanishes on \([r'_0, \infty)\), then \(\hat{u}\) is supported on the sector \(B'_{Q_i}(r'_0)\), with \(\Delta \hat{u} = 0\) on the sectors \(B'_{Q_i}(r'_0/2) \cap \Omega'_{i_0}\).

Now let us consider the function \(\bar{u}\) defined by:

\[
\bar{u}(r, \phi) = P^{-1} \hat{u}(r', \phi')
\]

(2.4.39)

on the adapted sector \(B_{Q_i}(C_0)\). As \(P : B_{Q_i}(C_0) \to B'_{Q_i}(r'_0)\) is a bijection map, the function \(\bar{u}\) is well defined on this sector. Moreover, we let \(\bar{u} = 0\) on \(\Omega \setminus B_{Q_i}(C_0)\). Therefore, \(\bar{u}\) is now defined
on the whole domain \( \Omega \). We can check \( \bar{u} \) satisfy the following conditions:

\[
\begin{cases}
\bar{u} \in H^1_D(\Omega); \\
\nabla^A_{\gamma} \bar{u}^+(r, \omega_i) = \nabla^A_{\gamma} \bar{u}^-(r, \omega_i) \quad \text{on } r' \leq r_0'; \\
\mathcal{L} \bar{u} = 0 \quad \text{on } \mathcal{B}_{Q_1}(C_0/2) \cap \Omega_{ii}; \\
\bar{u} \not\in \mathcal{X}_{\lambda_{10}}(\Omega).
\end{cases}
\] (2.4.40)

This example is sufficient to show \( \eta \leq \lambda_{i0} \).

By a similar argument, the example that \( \bar{u} = 0 \) on \( (\Omega \setminus \mathcal{B}_{Q_1}(C_0)) \times T \), and \( \bar{u} = \mathcal{P}^{-1} \bar{u} \) on \( \mathcal{B}_{Q_1}(C_0) \times T \), where

\[\hat{u}(r' , \phi', t) = \chi(r') r'^{\lambda_{i0}} \bar{u}_{\phi'}(\phi') t\]

is sufficient to show \( \delta \leq \lambda_{i0} \) for the parabolic problem 2.3.2.

Therefore, we can make the conclusion that

\[
\max \eta = \max \delta = \min_{Q_i \in \mathcal{V}} \lambda_{i0} = \inf_{Q_i \in \mathcal{V}, \bar{u} \in H^1_D(\Omega)} \mathcal{F}_i(\bar{u}).
\]

Hence we finish the proof. This result only applied for the operators that has constant coefficient on each subdomain. When the operator does not have constant coefficients, it is still an eigenvalue problem, however, other tools are required.

2.4.3 Example. Now we end up with an example: let \( Q_i \in \mathcal{V}_1 \) be the vertex of two subpolygons \( \Omega_{i1} \) and \( \Omega_{i2} \), with \( \omega_{i0} = 0 \) be the Dirichlet side, \( \omega_{i1} = \pi/4 \) be the interface, and \( \omega_i = \omega_{i2} = 3\pi/4 \) be the Neumann side. Also, we assume the coefficient matrices are

\[
A_1 = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 100 & 100 \\ 100 & 200 \end{pmatrix}.
\]

Then by calculation we can check

\[
P_1 = \begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 2 & -1/2 \\ -1 & 3/2 \end{pmatrix},
\]
and the corresponding value $d_1 = 1/2$, $d_2 = 250$. Moreover,

$$
P_1(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad P_1(\begin{pmatrix} 1 \\ 1 \end{pmatrix}) = P_2(\begin{pmatrix} 1 \\ 1 \end{pmatrix}) = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}, \quad P_2(\begin{pmatrix} -1 \\ 1 \end{pmatrix}) = \begin{pmatrix} -5/2 \\ 5/2 \end{pmatrix}.
$$

These reveals $\omega_{i0}' = 0$, $\omega_{i1}' = \arctan(1/3)$ and $\omega_{i2}' = 3\pi/4$. We follow (2.4.37) to set up the following 1d eigenvalue problem

$$
\begin{align*}
-\tilde{u}_{\phi'} &= \lambda_{i0}^2 \tilde{u} \\
\tilde{u}(0) &= 0,
\end{align*}
$$

(2.4.41)

After solving the above system of equations we deduce the smallest eigenvalue $\lambda_{i0} \approx 0.1332$. This is the corner singularity of the given operator at $Q_i$.

### 2.5 Domains with Neumann-Neumann Vertices or Non-smooth Interface: Semigroup Theory

From the theorem 2.3.2, we see if $\mathcal{V}_2 = \emptyset$, and $f \equiv 0$, then the initial-boundary value problem

$$
\begin{align*}
u_t + Lu &= 0 \quad \text{in } \bigcup_{k=1}^{K} \Omega_k \times T; \\
u &= 0 \quad \text{on } \partial_D \Omega; \\
abla v \cdot u &= 0 \quad \text{on } \partial_N \Omega; \\
u &= g \quad \text{on } \bigcup_{k=1}^{K} \Omega_k \times \{t = 0\}; \\
u^+ &= u^- \quad \text{on } \Gamma \times T; \\
abla v u^+ &= \nabla v u^- \quad \text{on } \Gamma \times T.
\end{align*}
$$

(2.5.1)

describes a one to one map from the initial data $g \in K_0^0(\Omega)$ into the weak solution $u \in L^2(0, T; K_1^{1,1}(\Omega))$ if and only if $\alpha \in (0, \eta)$. In this section, we will concentrate on the case when $f \equiv 0$ and $\mathcal{V}_2 \neq \emptyset$, that is, there exists two adjacent sides where Neumann boundary conditions are imposed.
First, let us check if the weak solution \( u \) will lie in \( L^2(0, T; \mathcal{X}^{1}_{a+1}(\Omega)) \) when \( g \in \mathcal{X}^{0}_{a}(\Omega) \) for some positive \( a \). The answer is no. Here is a counterexample: we choose an eigenfunction \( u_0 \) of \( L \) such that \( Lu_0 = \lambda u_0 \) on each subdomain, with the boundary and interface conditions preserved. In addition, \( u_0 \) does not vanish on at least one of the Neumann-Neumann vertices. By a simple calculation, if we set the initial data \( g = u_0 \) then the solution \( u(t) = e^{-t}u_0 \). Since \( u_0 \in H^1_D(\Omega) \), we have \( u_0 \in \mathcal{X}^{0}_{a}(\Omega) \) for all \( a \in (0, 1) \) by Sobolev embedding theorem. On the other hand, \( u_0 \notin \mathcal{X}^{0}_{a+1}(\Omega) \) for any nonnegative \( a \), which implies \( u = e^{-1}u_0 \notin L^2(0, T; \mathcal{X}^{0}_{a+1}(\Omega)) \).

Second, fix a constant \( a \in (0, 1) \), let us see if the weak solution \( u \) lies in \( C((0, T); \mathcal{X}^{0}_{a}(\Omega)) \) when given the initial data \( g \in \mathcal{X}^{0}_{a}(\Omega) \). This is a very challenging problem. We will need to introduce the semigroup theory.

2.5.1 Definition (mild solution). A continuous function \( u \in C((0, T); \mathcal{X}^{0}_{a}(\Omega)) \) is called a mild solution to the initial-boundary value problem (2.5.1) if

1. \( \int_{0}^{t} u(s) ds \in D(A) \),
2. \( A \int_{0}^{t} u(s) ds = u(t) - g \).

Here \( A : u \to h \) is the one-to-one map corresponding to the following elliptic problem:

\[
\begin{cases}
-Lu = h & \text{in } \bigcup_{k=1}^{K} \Omega_k; \\
u = 0 & \text{on } \partial_D \Omega; \\
\nabla^A_v u = 0 & \text{on } \partial_N \Omega; \\
u^+ = u^- & \text{on } \Gamma; \\
\nabla^A_v u^+ = \nabla^A_v u^- & \text{on } \Gamma.
\end{cases}
\] (2.5.2)

And \( D(A) = \{ u \mid \exists h \in \mathcal{X}^{0}_{a}(\Omega) \text{ s.t. } u = A^{-1}h \} \).

Since \( \mathcal{X}^{0}_{a}(\Omega) \subset L^2(\Omega) \), from our previous lemma 2.3.1, we see the map \( A^{-1} : \mathcal{X}^{0}_{a}(\Omega) \to D(A) \) is an isomorphism.

2.5.2 Definition (\( C_0 \) semigroup). (See Pazy [30]) A strongly continuous semigroup(\( C_0 \) semigroup) on the Banach Space \( X = \mathcal{X}^{0}_{a}(\Omega) \) is a map \( T : \mathbb{R}_+ \to L(X) \) such that

1. \( T(0) = I, \) (identity operator on \( X \))
2. \( \forall t, s \geq 0 : T(t+s) = T(t)T(s) \).
3. \( \forall x_0 \in X : \| T(t)x_0 - x_0 \| \to 0, \) as \( t \downarrow 0. \)

The following theorem connects our problem and strongly continuous semigroups.

2.5.3 Lemma (Arendt et al. Theorem 3.1.12 [2]). Let \( A \) be a closed operator on the Banach space \( X = \mathcal{K}_a^\alpha(\Omega). \) The following assertions are equivalent:

- for all \( g \in X \) there exists a unique mild solution to the problem (2.5.1),
- the operator \( A \) generates a strongly continuous semigroup,
- the resolvent set of \( A \) is nonempty and for all \( g \in D(A) \) there exists a unique classical solution to the problem (2.5.1).

When these assertions hold, the solution of the Cauchy problem (2.5.1) is given by \( u(t) = T(t)g \) with \( T \) the strongly continuous semigroup generated by \( A. \)

2.5.4 Lemma. If the mild solution \( u \in C([0, T]; \mathcal{K}_a^\alpha(\Omega)) \) to the problem (2.5.1) exists, then it coincides with the weak solution.

Proof. It is clear that \( \mathcal{K}_a^\alpha(\Omega) \subset L^2(\Omega), \) and \( A^{-1} : L^2(\Omega) \to H^1_D(\Omega) \cap H^2(\Omega), \) so we only need to show, if

1. \( \int_0^t u(s)ds \in H^1_D(\Omega) \cap H^2(\Omega), \)
2. \( A \int_0^t u(s)ds = u(t) - g \)

for any \( t > 0, \) then we will have \( u \in L^2(0, T; H^1_D(\Omega)) \) and \( u_t \in L^2(0, T; (H^1_D(\Omega))'), \) with the following equation hold

\[
\langle u_t, v \rangle + B[u, v] = 0, \quad v \in H^1_D(\Omega). \tag{2.5.3}
\]

On the website Mathematics Stack Exchange, a Russian professor with user id mkl314 gives an instructive proof to the above problem. See [26].

From the above two lemmas, we see that if we can prove \( A \) generates a strongly continuous semigroup on \( \mathcal{K}_a^\alpha(\Omega), \) the mild and weak solution \( u \) will then lie on \( C([0, T]; \mathcal{K}_a^\alpha(\Omega)). \) The following theorem gives a necessary and sufficient condition on the generation of a \( C_0 \) semigroup.
2.5.5 Theorem (Hille-Yosida). (See Pazy [30]) Let $A$ be a closed linear operator defined on a linear subspace $D(A)$ of the Banach space $X$, $\omega$ a real number, and $M > 0$. Then $A$ generates a strongly continuous semigroup $T$ that satisfies $\|T(t)\|_X \leq Me^{\omega t}$ if and only if

1. $D(A)$ is dense in $X$, and

2. every real $\lambda > \lambda_0$ belongs to the resolvent set of $A$ and for such $\lambda$ and for all positive integers $n$

$$\| (\lambda I - A)^{-n} \|_X \leq \frac{M}{(\lambda - \lambda_0)^n}.$$ 

To see if our operator $A$ generate a strongly continuous semigroup, let us first check some pre-requisite conditions of Hille-Yosida Theorem for $A$.

2.5.6 Theorem. Let $A$ be defined by the elliptic problem (2.5.2), then

1. $A$ is closed,

2. $D(A)$ is dense,

3. every positive real number belongs to the resolvent set of $A$.

Proof. 1. We already acquire from the lemma 2.3.1 that the operator $A^{-1}$ describes an isomorphism from $X$ onto $D(A)$. Now let us assume $\{x_n\}_{n=1}^{\infty}$ is a sequence in $D(A)$ converging to $x \in X$ such that $Ax_n = y_n \to y \in X$ as $n \to \infty$. We denote $A^{-1}y = x_0$, then $x_0 \in D(A)$. Since $\| y_n - y \|_{L^2(\Omega)} \to \| y_n - y \|_X \to 0$, we obtain $\| x_n - x_0 \|_{H^1_0(\Omega)} \to 0$ by Lax-Milgram theorem. On the other hand, the fact $\| x_n - x \|_{K^0_k(\Omega)} \to 0$ implies $x_0 = x$, hence $x \in D(A)$ and $Ax = y$. $A$ is therefore a closed operator.

2. Pick any fixed function $u \in X = X^0_k(\Omega)$ with $\| u \|_X = 1$, we have $u \in X^0_k(\Omega_k)$, $k = 1, 2, \ldots, K$. Now let $V_\delta = \{ y : y \in \Omega, \text{ dist}( y, \Gamma \cup \partial\Omega) > \delta \}$, and we define the function

$$\chi_{V_\delta}(x) = \begin{cases} 1 & \text{if } x \in V_\delta; \\ 0 & \text{if } x \in \Omega \setminus V_\delta. \end{cases}$$

(2.5.4)
Then for any small $\varepsilon > 0$, we can find a $\delta > 0$ such that $\|u - u_{V_\delta}\|_X < \varepsilon/2$. Moreover, we define the following two dimensional mollifier

$$
\phi(x) = \begin{cases} 
I_2 e^{-\frac{1}{|x|^2}} & \text{if } |x| < 1; \\
0 & \text{if } |x| \geq 1.
\end{cases}
$$

Here $I_2$ is a constant that makes $\|\phi\|_{L^2(\mathbb{R}^2)} = 1$. We denote $u_{\gamma}(x) = \phi(\gamma x)$. Since $u_{\gamma}$ is supported in $V_\delta$, we can find a small positive $\alpha < \delta^2$ such that $\|u_{\gamma} - u_{\gamma} \ast \phi\|_X < \varepsilon/2$. Hence we obtain $\|u - u_{\gamma} \ast \phi\|_X < \varepsilon$. Since $u_{\gamma} \ast \phi$ is sufficiently smooth and is supported on $V_{\delta + \delta^2}$, which is away from the interfaces and the boundary, we see $u_{\gamma} \ast \phi \in D(A)$. This implies $D(A)$ is dense in $X$.

3. Since for any $\lambda > 0$ and any nonzero function $u \in X$, we have $(Au - \lambda u, u)_{L^2(\Omega)} = -B[u,u] - \lambda(u,u)_{L^2(\Omega)} < 0$, hence $Au - \lambda u \neq 0$. This implies that $\lambda$ belongs to the resolvent set of $A$.

Therefore, in order to prove $A$ generates a $C_0$ semigroup, it is suffice show

$$
\left\| (\lambda I - A)^{-n} \right\|_{X^0_u(\Omega)} \leq \frac{M}{\lambda^n}
$$

for some fixed $M, \lambda_0 > 0$ and all $\lambda > \lambda_0$. In other words, that is

$$
\left\| (1 - \frac{1}{\lambda} A)^{-n} \right\|_{X^0_u(\Omega)} \leq M.
$$

However, our conjecture is, there is no uniform bound $M > 0$ for the above inequality, hence $A$ cannot generate a $C_0$ semigroup. Although we fail to present a counterexample for general operator $A$ and general polygonal domain $\Omega$, we are able to do it in the following special case.

2.5.7 Theorem. Let $\Omega$ be a sector with radius $r_0 > 0$, and let $A = \Delta$ with the Neumann boundary condition imposed on the two radii, and Dirichlet boundary condition imposed on the arc. Then we have

$$
\sup_{\lambda > 0} \left\| (1 - \frac{1}{\lambda} A)^{-1} \right\|_{X^0_u(\Omega)} = +\infty.
$$

(2.5.7)
This implies $A$ does not generates a $C_0$ semigroup on $X = X^0_\alpha (\Omega)$.

Proof. Let $Q$ be center of the sector, and $RS$ be the arc. Then for any $u \in X = X^0_\alpha (\Omega)$,

$$
\|u\|_X = \left\| \frac{u}{(d(x, Q) - d(x, R) - d(x, S))^{\alpha}} \right\|_{L^2(\Omega)}
$$

First of all, we see the operator $(I - \frac{1}{\lambda} A)$ is invertible on $X$, since it is invertible on $L^2(\Omega)$ and $X \subset L^2(\Omega)$. Our purpose is to show, for any small $0 < \varepsilon < 1$, there exists a value $\lambda > 0$ and a corresponding function $u = u(\lambda) \in D(A)$, such that

$$
\frac{\| (I - \frac{1}{\lambda} A) u \|_X}{\| u \|_X} = \varepsilon
$$

In the following content, we try to prove an easier statement: for any $0 < \varepsilon < 1$, we can find $\lambda > 0$ and $u = u(\lambda) \in \tilde{D}(A)$ such that

$$
\frac{\| (I - \frac{1}{\lambda} A) u \|_{d^a(x, Q)}}{\| d^a(x, Q) \|_{L^2(\Omega)}} = \varepsilon
$$

Here $D(A) = \{ u \mid \exists f \in L^2(\Omega) \text{ s.t. } u = A^{-1}(d^a(x, Q) f) \}$.

To begin with, let us denote $r = d(x, Q)$, and $v = u/r^a$, then the above equation can be written as

$$
\left\| (I - \frac{1}{\lambda} r^{-a} A r^a) v \right\|_{L^2(\Omega)} = \varepsilon \left\| v \right\|_{L^2(\Omega)},
$$

this is

$$
( I - \frac{1}{\lambda} r^{-a} A r^a) v, ( I - \frac{1}{\lambda} r^{-a} A r^a) v ) = \varepsilon^2 (v, v).
$$

Fix $\varepsilon > 0$, we need to construct the function $v$ and the corresponding value $\lambda$ to meet (2.5.11), under the restriction $r^a v = u \in \tilde{D}(A)$, or in other words,

1. $r^{-a} \Delta (r^a v) \in L^2(\Omega),$
2. $\partial (r^a v) / \partial n = 0$ on the radii QR and QS;
3. $r^a v = 0$ on the arc RS.
Here (3) implies \( \nu = 0 \) on RS, and (2) implies \( \partial \nu / \partial n = 0 \) on QR and QS, due to the fact that \( \partial r^a / \partial n = 0 \) on the radii. Hence for any \( w \in L^2(\Omega) \) such that \( r^{-a} w \in \bar{D}(A) \), we have

\[
( (I - \frac{1}{\lambda} r^{-a} A r^a) ) \nu, w ) = (\nu, w) - (A(r^a \nu), r^{-a} w) = (\nu, w) - (r^a \nu, A(r^{-a} w)) = (\nu, (I - \frac{1}{\lambda} r^{-a} A r^a) w).
\]

(2.5.12)

Now we start to construct the function \( \nu \) and value \( \lambda \), by solving the following eigenvalue problem:

\[
\begin{align*}
(I - \frac{1}{\lambda} r^a A r^{-a})(I - \frac{1}{\lambda} r^{-a} A r^a)\nu &= \epsilon^2 \nu \quad \text{on } \Omega; \\
r^a \nu &= 0 \quad \text{on } \bar{D}(A); \\
r^{-a}(I - \frac{1}{\lambda} r^{-a} A r^a) \nu &= 0 \quad \text{on } \text{RS}.
\end{align*}
\]

(2.5.13)

If \( \nu \) satisfies the above conditions, then the expression \( (I - \frac{1}{\lambda} r^a A r^{-a})(I - \frac{1}{\lambda} r^{-a} A r^a) \) make sense, and we are able to derive (2.5.11) by plugging \( w = (I - \frac{1}{\lambda} r^{-a} A r^a) \nu \) into the equation (2.5.12). So our goal is the construction of such \( \nu \). To make it easier, let us restrict \( \nu \) to be a radial function, that is, the value of \( \nu \) depends only on \( r \). Then it is clear \( \partial \nu / \partial n = \partial r / \partial n = 0 \) on the radii. The eigenvalue problem (2.5.13) therefore can be simplified into

\[
\begin{align*}
(I - \frac{1}{\lambda} r^a A r^{-a})(I - \frac{1}{\lambda} r^{-a} A r^a)\nu &= \epsilon^2 \nu \quad \text{on } \Omega; \\
r^a \nu &= 0 \quad \text{on } \bar{D}(A); \\
r^{-a}(I - \frac{1}{\lambda} r^{-a} A r^a) \nu &= 0 \quad \text{on } \text{RS}.
\end{align*}
\]

(2.5.14)

Moreover, since the operator \( \Delta \) can be written as \( \frac{1}{r} \partial_r(r \partial_r) + \frac{1}{r^2} \partial_{\theta \theta} \) under the polar coordinates, while \( r \) and \( \nu \) are radial and have no angular component, we can discover

\[
\epsilon^2 \nu = (I - \frac{1}{\lambda} r^a A r^{-a})(I - \frac{1}{\lambda} r^{-a} A r^a)\nu = \nu - \frac{1}{\lambda} \left[ \frac{1}{r^{1+a}} (r(r^a \nu)_r)_r + \frac{1}{r^{1-a}} (r^{1-2a}(r^a \nu)_r)_r \right] + \frac{1}{\lambda^2} \left[ \frac{1}{r^{1-a}} (r(r^{-a} \nu)_r)_r \right].
\]

(2.5.15)

To make the calculation simpler, we plug back \( u = r^a \nu \), so that \( \nu = u/r^a \). After replacing \( \nu \) by \( u \), the above equation become

\[
\epsilon^2 u = r^a \epsilon^2 \nu = r^a (I - \frac{1}{\lambda} r^a A r^{-a})(I - \frac{1}{\lambda} r^{-a} A r^a)\nu = u - \frac{1}{\lambda} \left[ \frac{1}{r} (ru)_r + \frac{1}{r^{1-2a}} (r^{1-2a} u)_r \right] + \frac{1}{\lambda^2} \left[ \frac{1}{r^{1-2a}} (r(r^a u)_r)_r \right].
\]

(2.5.16)
And we update the eigenvalue problem (2.5.14) into

\[
\begin{cases}
    u - \frac{1}{\lambda} \left[ \frac{1}{r} (ru)_r + \frac{1}{r-2a} (r^{1-2a} u)_r \right] + \frac{1}{\lambda^2} \left[ \frac{1}{r-2a} (r^{1-2a} (ru)_r)_r \right] = \epsilon^2 u & \text{on } \Omega; \\
    u = 0 & \text{on } \partial \Omega; \\
    r^{2a} (u - \frac{1}{\lambda} (ru)_r) = 0 & \text{on } \partial \Omega.
\end{cases}
\]

(2.5.17)

Since \( u \) is radial and depend only on \( r \), let us consider \( u = u(r) \) to be a function of 1 variable, and satisfy the equation

\[
\begin{cases}
    u - \frac{1}{\lambda} \left[ \frac{1}{r} (ru)_r + \frac{1}{r-2a} (r^{1-2a} u)_r \right] + \frac{1}{\lambda^2} \left[ \frac{1}{r-2a} (r^{1-2a} (ru)_r)_r \right] = \epsilon^2 u & 0 \leq r < r_0; \\
    \end{cases}
\]

(2.5.18)

with the boundary condition

\[
    u(r_0) = (ru)_r (r_0) = 0.
\]

(2.5.19)

Then by calculation, a general solution for the equation (2.5.18) can be in the form

\[
    u(r) = \beta J_0 (\sqrt{\lambda} r) + \gamma I_0 (\sqrt{\lambda} r),
\]

(2.5.20)

where \( \beta, \gamma \in \mathbb{R} \),

\[
    J_0 (\sqrt{\lambda} r) = c_0 + c_2 r^2 + c_4 r^4 + c_6 r^6 + ...
\]

and

\[
    I_0 (\sqrt{\lambda} r) = d_2 r^{2a+2} + d_4 r^{2a+4} + d_6 r^{2a+6} + ...
\]

Here the coefficients \( c_i, d_i \) are given by the following equalities

\[
    0 = \left[ -(2a)^2 c_0 + 2^2 (2a)^2 c_2 \right] r^{-2} + \sum_{i=1}^{\infty} [(1-\epsilon^2) c_{2i-2} - ((2i)^2 + (2i-2a)^2) c_{2i} + (2i+2)^2 (2i-2a)^2 c_{2i+2}] r^{2i};
\]

(2.5.21)

and

\[
    0 = \left[ -(2^2 + (2a + 2)^2) d_2 + 2^2 (2a + 4)^2 d_4 \right] r^{2a} + \sum_{i=1}^{\infty} [(1-\epsilon^2) d_{2i} - ((2i+2)^2 + (2i + 2a + 2)^2) d_{2i+2} + (2i + 2)^2 (2i + 2a + 4)^2 d_{2i+4}] r^{2i+2a}.
\]

(2.5.22)
The boundary condition (2.5.19) becomes

\[ \beta J_0(\sqrt{\lambda}r_o) + \gamma I_0(\sqrt{\lambda}r_o) = \beta J_1(\sqrt{\lambda}r_o) + \gamma I_1(\sqrt{\lambda}r_o) = 0, \quad (2.5.23) \]

here

\[ J_1(r) = \frac{1}{r}(rJ_0)_r(r) = 2^2 c_2 r^2 + 4^2 c_4 r^4 + 6^2 c_6 r^6 + \ldots \]

and

\[ I_1(r) = \frac{1}{r}(rI_0)_r(r) = (2a + 2)^2 d_2 r^{2a} + (2a + 4)^2 d_4 r^{2a+2} + (2a + 6)^2 d_6 r^{2a+4} + \ldots \]

This boundary condition implies, if we can find a solution \( r_1 > 0 \) to the following equation

\[ \frac{J_1(r)}{J_0(r)} = \frac{I_1(r)}{I_0(r)}, \quad (2.5.24) \]

then the function

\[ u(r) = \frac{J_0(\frac{r}{r_0}r)}{J_0(r_1)} - \frac{I_0(\frac{r}{r_0}r)}{I_0(r_1)} \quad (2.5.25) \]

satisfy the above boundary conditions (2.5.23) at \( r = r_0 \). Comparing \( u(r) \) with the expression (2.5.20), we observe that it solves the eigenvalue problem (2.5.17), with the corresponding value \( \lambda = \left( \frac{r_1}{r_0} \right)^2 \). Therefore, we can make the conclusion that \( u \in \tilde{D}(A) \) and is a solution to the equation (2.5.9).

So, what remains to be shown is, there do exist a solution \( r \) to the equation (2.5.24). This has to be done by studying the behavior of the coefficients \( c_{2i}, d_{2i}, i = 0, 1, 2, \ldots \), of the function \( J_0, I_0 \).

Let us study \( J_0 \) first. As we mentioned, the sequence \( \{c_{2i}\}_{i \in \mathbb{N}^+} \) satisfy:

\[ \begin{align*}
& c_2 = c_0 / 4; \\
& c_{2i+2} = \frac{1}{(2i+2)^2(2i-2a)^2} \left[ ((2i)^2 + (2i - 2a)^2) c_{2i} - (1 - \epsilon^2) c_{2i-2} \right].
\end{align*} \quad (2.5.26) \]

When \( \epsilon = 0 \), we can easily check \( c_{2i+2} = (2i)^2 c_{2i} \) for all \( i \in \mathbb{N}^+ \), so \( \frac{J_0(r)}{I_1(r)} \equiv 1. \)
When $\epsilon > 0$, let us define the sequence $\{e_{2i}\}_{i \in \mathbb{N}^+}$ such that

$$
\begin{align*}
    e_0 &= 0; \\
    e_{2i} &= \frac{(2i)^2}{1 + \epsilon} e_{2i-2} e_{2i-2}.
\end{align*}
$$

(2.5.27)

After we plug it into the equation (2.5.26), we can discover

$$
\begin{align*}
    \begin{cases}
        e_0 = 1, \\
        e_2 = \frac{1}{1 + \epsilon} \\
        (1 + \epsilon) \frac{(2i - 2\alpha)^2}{(2i)^2} e_{2i+2} = (1 + \frac{(2i - 2\alpha)^2}{(2i)^2}) e_{2i} - (1 - \epsilon) e_{2i-2}.
    \end{cases}
\end{align*}
$$

(2.5.28)

This is

$$
(1 + \epsilon) \frac{(2i - 2\alpha)^2}{(2i)^2} [e_{2i+2} - \frac{(2i)^2 + (2i - 2\alpha)^2}{2 (2i - 2\alpha)^2} e_{2i}] = (1 - \epsilon) \frac{(2i)^2 + (2i - 2\alpha)^2}{2 (2i)^2} e_{2i} - e_{2i-2} \\
= (1 - \epsilon) \frac{(2i)^2}{2 (2i)^2} [e_{2i} - \frac{(2i - 2\alpha)^2 + (2i - 2\alpha - 2)^2}{2 (2i - 2\alpha)^2} e_{2i-2}] + (1 - \epsilon) \frac{a(a+1)(2i-a)^2}{4i^2(i-a-1)^2} e_{2i-2} \\
= (1 - \epsilon) \frac{(2i)^2 + (2i - 2\alpha)^2}{2 (2i)^2} [e_{2i} - \frac{(2i - 2\alpha)^2 + (2i - 2\alpha \alpha - 2)^2}{2 (2i - 2\alpha)^2} e_{2i-2}] + O(i^{-2}).
$$

(2.5.29)

At the same time

$$
(1 + \epsilon) \frac{(2i - 2\alpha)^2}{(2i)^2} e_{2i+2} - e_{2i} = (\frac{(2i - 2\alpha)^2}{(2i)^2} - \epsilon) e_{2i} - (1 - \epsilon) e_{2i-2} \\
= (1 - \epsilon) \frac{(2i - 2\alpha)^2}{(2i - 2)^2} [e_{2i} - e_{2i-2}] - \frac{(2i - 2\alpha)(2i - 2\alpha - 2) a}{(i-1)^2 i^2} e_{2i} \\
= (1 - \epsilon) \frac{(2i - 2\alpha)^2}{(2i - 2)^2} [e_{2i} - e_{2i-2}] - O(i^{-1}).
$$

(2.5.30)

The above 2 equality implies, for sufficiently large $i$, we have

$$
\frac{(2i)^2 + (2i - 2\alpha)^2}{2 (2i - 2\alpha)^2} e_{2i} < e_{2i+2} < \frac{(2i)^2}{(2i - 2\alpha)^2} e_{2i}.
$$
and

\[
\lim_{t \to \infty} 2(2i - 2a)^2 \frac{e^{2i+2}}{e^{2t}} - (2i)^2 + (2i - 2a)^2 = C \quad (2.5.31)
\]

for some constant C independent of i. Now back to the equation (2.5.24) and (2.5.27), we can discover

\[
\frac{J_1(r)}{I_0(r)} = \frac{2^2 c_2 + 4^2 c_4 r^2 + 6^2 c_6 r^4 + 8^2 c_8 r^6 + ...}{c_0 + c_2 r^2 + c_4 r^4 + c_6 r^6 + ...} = (1 + \varepsilon) \frac{\frac{\varepsilon_2}{e_2} r^2 + \frac{\varepsilon_4}{e_4} r^4 + \frac{\varepsilon_6}{e_6} r^6 + ...}{c_0 + c_2 r^2 + c_4 r^4 + c_6 r^6 + ...}
\]

\[
= \left(1 + \varepsilon\right) \frac{\sum_{i=0}^{\infty} \frac{e_{2i+2}}{e_{2i}} \left(\prod_{j=1}^{i} \frac{e_{2j}}{e_{2j-2}}\right) \left(\frac{r^2}{1+c}\right)^i}{\sum_{i=0}^{\infty} \left(\prod_{j=1}^{i} \frac{e_{2j}}{e_{2j-2}}\right) \left(\frac{r^2}{1+c}\right)^i}.
\]

(2.5.32)

Secondly, let us study the function \(I_0(r)\). The sequence \(\{d_{2l}\}_{l \in \mathbb{N}^+}\) satisfies

\[
\begin{aligned}
d_4 &= \frac{2^2 + (2a + 2)^2}{2^2(2a + 4)^2} d_2, \\
d_{2l+4} &= \frac{1}{(2l+2)^2(2l+2a+4)^2} \left[ ((2l + 2)^2 + (2l + 2a + 2)^2) d_{2l+2} - (1 - \varepsilon^2) d_{2l} \right].
\end{aligned}
\]

(2.5.33)

Still, for \(\varepsilon > 0\), let us define the sequence \(\{f_{2l}\}_{l \in \mathbb{N}^+}\) such that

\[
\begin{aligned}
f_2 &= d_2, \\
f_{2l+2} &= \frac{(2l+2+2a)^2}{2(2l)^2 (1+c)} d_{2l+2}. \\
\end{aligned}
\]

(2.5.34)

We plug it into the equation (2.5.33) to discover

\[
\begin{aligned}
f_2 = 1, \\
f_4 &= \frac{2^2 + (2a + 2)^2}{2^2(2a + 4)^2}, \\
(1 + \varepsilon) \frac{(2l+2)^2}{(2l+2a)^2} f_{2l+2} &= (1 + \frac{(2l+2)^2}{2(2l)^2}) f_{2l} - (1 - \varepsilon) f_{2l-2}.
\end{aligned}
\]

(2.5.35)

By a similar argument, we can obtain

\[
\frac{(2l + 2a)^2 + (2l)^2}{2(2l)^2} f_{2l} < f_{2l+2} < \frac{(2l + 2a)^2}{(2l)^2} f_{2l}
\]
for sufficiently large \(i\), and

\[
\lim_{i \to \infty} 2 \left(2i\right)^2 \frac{f_{2i+2}}{f_{2i}} - ((2i)^2 + (2i + 2)^2) = C
\]  

(2.5.36)

for some constant \(C\) independent of \(i\). Now back to the equation (2.5.24) and (2.5.34), we discover

\[
\frac{I_1(r)}{I_0(r)} = \frac{(2a + 2)^2 d_4 r^{2a} + (2a + 4)^2 d_4 r^{2a+2} + (2a + 6)^2 d_4 r^{2a+4} + (2a + 8)^2 d_4 r^{2a+6} \ldots}{0 r^{2a} + d_2 r^{2a+2} + d_4 r^{2a+4} + d_6 r^{2a+6} \ldots}
\]

\[
= (1 + \varepsilon) \frac{(2a + 2)^2 d_4 r^{2a} + \sum_{i=1}^{\infty} \frac{f_{2i+2}}{f_{2i}} \left( \prod_{j=2}^{i} \frac{f_{2j}}{(2j+2a)^2 f_{2j-2}} \right) \left( \frac{r^2}{1+\varepsilon} \right)^i}{\sum_{i=1}^{\infty} \left( \prod_{j=2}^{i} \frac{f_{2j}}{(2j+2a)^2 f_{2j-2}} \right) \left( \frac{r^2}{1+\varepsilon} \right)^i}
\]

(2.5.37)

Our purpose is to solve the equation \(I_1(r)/I_0(r) = I_1(r)/I_0(r)\), that is,

\[
\sum_{i=0}^{\infty} \frac{e_{2i+2}}{e_{2i}} \left( \prod_{j=1}^{i} \frac{e_{2j}}{(2j)^2 e_{2j-2}} \right) \left( \frac{r^2}{1+\varepsilon} \right)^i = (2a + 2)^2 + \sum_{i=1}^{\infty} \frac{f_{2i+2}}{f_{2i}} \left( \prod_{j=2}^{i} \frac{f_{2j}}{(2j+2a)^2 f_{2j-2}} \right) \left( \frac{r^2}{1+\varepsilon} \right)^i.
\]  

(2.5.38)

As we have derived, for \(i\) sufficiently large, the ratio \(\frac{e_{2i+2}}{e_{2i}}\) and \(\frac{f_{2i+2}}{f_{2i}}\) will follow the limit (2.5.31) and (2.5.36), which is independent of \(\varepsilon\). Hence

\[
\lim_{r \to \infty} \frac{I_1(r)}{I_0(r)} = \lim_{r \to \infty} \frac{I_1(r)}{I_0(r)} = 1.
\]

Moreover, for sufficiently large \(r\), we check from (2.5.38) that the value of \(I_1(r)/I_0(r)\) and \(I_1(r)/I_0(r)\) is largely based on \(\frac{r^2}{1+\varepsilon}\), which implies, the existence of solution to (2.5.24) is consistent for any \(\varepsilon > 0\). Apparently, the eigenvalue problem (2.5.17) have a solution when \(\varepsilon = 1\), that is to say, there exists at least one solution to the equation (2.5.38) for \(\varepsilon = 1\). Hence (2.5.38) has a solution for any \(\varepsilon > 0\). Lastly, let the solution to be \(r = r_\nu\), then as we mentioned, the
function (2.5.25) meets the conditions (2.5.18) and (2.5.19), hence the equation (2.5.9) holds, i.e. \[
\left\| \frac{(I - \frac{1}{\lambda} A) u}{r} \right\|_{L^2(\Omega)} / \left\| \frac{u}{r} \right\|_{L^2(\Omega)} = \epsilon \text{ for such } u.
\]

Finally, we are ready to show the existence of solution to our main equation (2.5.8) in the beginning. In fact, for any \(0 < \epsilon < 1\), we can use the same idea to construct a function \(u \in D(A) \subset \tilde{D}(A)\) such that, \(u\) is radial and is supported in a small neighborhood of the center \(Q\) (say \(u(r) \equiv 0\) for \(r > r_0/2\)), with the condition (2.5.9) satisfied. Under such situation, we have \[
\left\| (I - \frac{1}{\lambda} A) u \right\|_X < 3 \left\| \frac{1}{\lambda} (I - \frac{1}{\lambda} A) u \right\|_{L^2(\Omega)}' \|u\|_X > \left\| \frac{u}{r} \right\|_{L^2(\Omega)}' \|u\|_X < 3 \left\| \frac{(I - \frac{1}{\lambda} A) u/r}{r} \right\|_{L^2(\Omega)} = 3\epsilon. \]
We are eventually able to derive \[
\sup_{\lambda > 0} \left\| (I - \frac{1}{\lambda} A)^{-1} \right\|_X = +\infty \text{ by choosing arbitrary } \epsilon.
\]

\[\square\]

2.5.8 Definition (quasi-contraction semigroup). A \(C_0\) semigroup \(T(t), \ t \geq 0\), is called a quasi-contraction semigroup if there is a constant \(\lambda_0 \geq 0\) such that \(\|T(t)\| \leq e^{\lambda_0 t}\) for all \(t \geq 0\). \(T(t)\) is called a contraction semigroup if \(\|T(t)\| \leq 1\) for all \(t \geq 0\).

2.5.9 Theorem (Lumer-Phillips). (See Pazy [30]) Let \(A\) be a linear operator defined on a linear subspace \(D(A)\) of the Banach space \(X = K^0_a(\Omega)\). Then \(A\) generates a quasi contraction semigroup if and only if

1. \(D(A)\) is dense in \(X\),
2. \(A\) is closed,
3. \(A\) is quasidissipative, i.e. there exists an \(\lambda_0 \geq 0\) such that \(A - \lambda_0 I\) is dissipative operator, and
4. \(A - \lambda I\) is surjective for some \(\lambda > \lambda_0\), where \(I\) denotes the identity operator.

2.5.10 Theorem. The operator \(A\) does not generate a quasi-contraction semigroup on \(X = K^0_a(\Omega)\).

Proof. We only need to show \(A\) is not quasidissipative on \(X\). Let us first consider the case that \(A\) is Laplacian in a neighborhood of the vertex \(Q_i\), say \(B_{Q_i}(r_0)\), with Neumann boundary condition imposed on both sides of \(Q_i\). Now we define a sequence of functions \(\{u_n\}_{n \geq 1} \in H^2(\Omega)\), such that \(u_n\) is a radial function centered at \(Q_i\) and supported in \(B_{Q_i}(r_0)\), \(n = 1, 2, \ldots\), with the
following conditions hold:

\[
\begin{align*}
 u_n(x) &= \begin{cases} 
 0 & \text{if } r > r_0; \\
P(r) & \text{if } r \in [r_0/2, r_0]; \\
r^a & \text{if } r \in [r_0/2n, r_0/2]; \\
R_n(r) & \text{if } r \in [r_0/4n, r_0/2n]; \\
0 & \text{if } r < r_0/4n.
\end{cases}
\end{align*}
\]  
(2.5.39)

Here \( r = d(x, Q_i) \). We construct the polynomials \( P(r), R_n(r) \) of degree \( \leq 3 \) such that \( \partial u_n / \partial r \) is continuous, that is,

\[
P(r_0) = P'(r_0) = 0, \quad P(r_0/2) = (r_0/2)^a, \quad P'(r_0/2) = a(r_0/2)^{a-1};
\]

and

\[
Q_n(r_0/4n) = Q'_n(r_0/4n) = 0, \quad Q_n(r_0/2n) = (r_0/2n)^a, \quad Q'_n(r_0/2n) = a(r_0/2n)^{a-1}.
\]

Under such condition, \( u_n \) will be second order differentiable and lie on \( \mathcal{K}_0^{2} (\Omega) \subset D(A) \). Now let us calculate

\[
(\Delta u_n, u_n)_X = (\nabla \frac{u_n}{r^\alpha}, \nabla \frac{u_n}{r^\alpha})_{L^2} - \frac{2}{r^{a+1}} (\frac{u_n}{r^a}, \frac{u_n}{r^a})_{L^2},
\]

where

\[
(\nabla \frac{u_n}{r^\alpha}, \nabla \frac{u_n}{r^\alpha})_{L^2} = \omega \int_{r_0/4n}^{r_0/2n} \frac{d R_n}{dr} r^2 dr + \omega \int_{r_0/2}^{r_0} \frac{d P}{dr} r^2 dr = M_1,
\]

\[
\left(\frac{u_n}{r^a}, \frac{u_n}{r^a}\right)_{L^2} \geq \omega \int_{r_0/2n}^{r_0/2} \frac{1}{r} r dr = \omega \ln n.
\]

In addition, we have

\[
(u_n, u_n)_X = \omega \int_{r_0/4n}^{r_0/2n} R_n^2 r^{1-2a} dr + \omega \int_{r_0/2n}^{r_0/2} r^{1-2a} dr + \omega \int_{r_0/2}^{r_0} P^2 r^{1-2a} dr < M_2.
\]

The constants \( M_1, M_2 \) are independent of \( n \). It is clear that \( \lim_{n \to \infty} (\Delta u_n, u_n)_X = -\infty \), while

\[
\lim_{n \to \infty} (u_n, u_n)_X = \omega \int_0^{r_0/2} r^{1-2a} dr + \omega \int_{r_0/2}^{r_0} P^2 r^{1-2a} dr < \infty.
\]

This sequence \( \{u_n\}_{n \geq 1} \) implies that \( A \) is not quasi-dissipative on the domain \( X = \mathcal{K}_0^{2} (\Omega) \).
For the case when $L$ is not a Laplacian, the idea is exactly the same, we can always find a sequence that $(Lu_n, u_n) \to -\infty$, while $\|u_n\|_X$ is uniformly bounded, which implies the operator $L$ cannot be quasi-dissipative on $X$. □

2.5.11 Remark. Our observation in this section is, when there is a vertex that has Neumann boundary condition imposed on both sides, the well-posedness of solution will fail in the weighted Sobolev space $\mathcal{H}^0_a$. It implies that the weighted space is not a good tool to exploit the regularity of solution in such case.
Chapter 3

Numerical Solution: FEM with Graded Mesh

3.1 Preliminaries

Now let us return to the parabolic transmission problem (2.1.1). From our previous results, we see the solution $u$ lies on some weighted Sobolev space when given appropriate data $f$ and $g$. Hence, we are able to construct a numerical solution by applying a certain finite element scheme in space with Backward Euler in time.

3.2 Previous Results on Elliptic Transmission Problem

In the past decades, the numerical implement of elliptic transmission problem has been studied in a very large quantity of papers, especially in Finite Element Method. Let us mention the paper by M. Bourlard [4], C. Bacuta [6] [7], P. Huang [16], Y. Cao [5] and L. Wang [31] for instance.

In this chapter, we will mostly rely on the results from the paper [18] and extend it to the parabolic problem. Recall the lemma 2.2.1 we cite from [18], if $f \in H^{m-1}_{a+1}(\Omega)$ for $0 < a < \eta$ and $m \geq 1$, the elliptic transmission problem (2.2.1):

$$
\begin{cases}
Lu = f, & \text{on } \bigcup_{k=1}^{K} \Omega_k; \\
u = 0, & \text{on } \partial \Omega; \\
\nabla^A u = 0 & \text{on } \partial_N \Omega, \\
\nabla^A u_+ - \nabla^A u_- = 0, & \text{on } \Gamma;
\end{cases}
$$

(3.2.1)

will have a unique strong solution $u$, such that $u$ can be decomposed into $u = u_{\text{reg}} + w_s$, while $u_{\text{reg}} \in H^{m+1}_{a+1}(\Omega) \cap H^1_{a+1}(\Omega)$, $w_s \in W_s$ is $C^\infty$ and is locally constant in a neighborhood of each
vertex in $\mathcal{V}_2$, with the following bound

$$
\left\| u_{\text{reg}} \right\|_{\mathcal{K}_{a+1}^m(\Omega)} + \left\| u_{\text{reg}} \right\|_{\mathcal{K}_{a+1}^{m+1}(\Omega)} + \left\| w_{s} \right\|_{L^2(\Omega)} \leq C \left\| f \right\|_{\mathcal{K}_{a-1}^{m-1}(\Omega)}. 
$$

(3.2.2)

Here $C$ is a constant that depends on $L$ and $\Omega$.

In 2010, Hengguang Li presented in [18] a finite element scheme to this elliptic transmission problem when all the subdomains $\Omega_k$ have straight sides. The main idea is, to construct a sequence of special triangulations $\{T_n\}_{n \geq 0}$ of $\Omega$ by induction, such that $T_n$ has uniform triangular mesh away from the vertices, but has graded triangular mesh near the vertices. Moreover, let $S_n := S(T_n, m) \subset H^1_D(\Omega)$ be the finite element space of continuous functions on $\Omega$ that restrict to a polynomial of degree $m \geq 1$ on each triangle of $T_n$, and let $u_n \in S_n$ be the finite element approximation of $u$, given by the equation

$$
B[u_n, v_n] := \int_{\Omega} A_{ij} \frac{\partial u_n}{\partial j} \frac{\partial v_n}{\partial i} \, dx = (f, v_n) \quad \forall v_n \in S(T_n, m). 
$$

(3.2.3)

Then according to the main results of his paper [18], $u_n$ is a finite element approximation of $u$, with the convergence rate

$$
\left\| u - u_n \right\|_{H^1(\Omega)} \leq C_n h^m \left\| f \right\|_{\mathcal{K}_{a-1}^{m-1}(\Omega)} 
$$

(3.2.4)

and

$$
\left\| u - u_n \right\|_{L^2(\Omega)} \leq C_n h^{m+1} \left\| f \right\|_{\mathcal{K}_{a-1}^{m-1}(\Omega)} 
$$

(3.2.5)

for some constant $C_n > 0$. Here $h$ is the length of the triangle of $T_n$ that is away from the vertices. With the above inequalities hold, we say that the finite element scheme with such triangulation provides quasi-optimal rates of convergence for $f \in \mathcal{K}_{a-1}^{m-1}(\Omega)$.

The crucial part of the above scheme is the construction of the mesh. Here $T_0$ is the initial uniform mesh, $T_n = \kappa(T_{n-1})$ is constructed by induction, where for each vertex $Q$ on $\partial \Omega \cup \Gamma$, we make refinement by a fixed ratio $\kappa_Q$ on the triangular mesh that contains $Q$, and refine regularly on the other triangles (See [18]). This makes the new triangulation $T_n$ contain quadruple numbers of triangles over $T_{n-1}$, and largely reduces the area of the triangles that contain the vertices. The fixed ratio $\kappa_Q$ should not exceed $2^{-m} \eta_Q$, where $\eta_Q$ is the singularity of the vertex $Q$ and can be determined by our theorem 2.4.1.
3.3 Fully Discrete Finite Element Method for Parabolic Problem

In the parabolic transmission problem (2.1.1), we are going to implement finite element method in space, with backward Euler method in time to seek for a discrete numerical solution. Same as in the elliptic problem (see section 5 of [18]), we apply \( \{T_n\}_{n \geq 1} \) as the mesh, and \( S_n := S(T_n, m) \) as the finite element space. Here the space \( S_n := S(T_n, m) \subset H^1_D(\Omega) \) is the space of all polynomial functions with degree \( \leq m \) on each triangle of \( T_n \). Thereafter, let us construct a numerical solution \( u^n_j \in S_n, j = 0, 1, \ldots \) which solves the following system of equations:

\[
\frac{u^{n+1} - u^n}{k} + B[u^{n+1}, v_n] = (f^{n+1}, v_n) \quad \forall v_n \in S_n, j = 0, 1, \ldots
\]

(3.3.1)

with the initial condition

\[
(u^0_n, v_n) = (g, v_n) \quad \forall v_n \in S_n.
\]

(3.3.2)

Here \( k \) is the time step size, and \( f^n_j = f(jk), j = 0, 1, \ldots, M, M = T/k \).

3.3.1 Theorem (Error Estimates). Let \( u \) be the exact solution of the parabolic problem (2.1.1). Also, we fix \( m = 1 \) so that \( S_n := S(T_n, 1) \subset H^1_D(\Omega) \) is the space of linear continuous functions on each triangle of \( T_n \), and \( u^n_1 \in S_n \) is the numerical solution generated by the equation (3.3.1) and (3.3.2). If we assume

\[
0 < a < \min(1, \eta),
\]

and

\[
f'' \in L^2(0, T; (H_D^1(\Omega))^a), f' \in C(0, T; L^2(\Omega)), f \in C(0, T; \mathcal{K}^2_a(\Omega));
g \in V^2_{a+1}(\Omega), g_1 = f(0) - Pg \in V^2_{a+1}(\Omega), g_2 = f'(0) - Pg_1 \in L^2(\Omega);
\]

(3.3.3)
then by bootstrap we can derive that \( u \) can be decomposed into \( u = u_{\text{reg}} + w_s \), where

\[
\begin{align*}
u_{\text{reg}} & \in W^{2,\infty}(0, T; L^2(\Omega)) \cap W^{1,\infty}(0, T; \mathbb{R}^2_{a+1}(\Omega)) \cap L^\infty(0, T; \mathbb{R}^4_{a+1}(\Omega)); \\
\end{align*}
\]

and \( w_s \in W_s \) is \( C^\infty \), finite dimensional, and is locally constant in a neighborhood of each vertex in \( V_s \), with the following bound

\[
\begin{align*}
\|w_s\|_{L^\infty(0, T; \mathbb{R}^2(\Omega))} + \|u_{\text{reg}}\|_{W^{2,\infty}(0, T; L^2(\Omega))} + \|u_{\text{reg}}\|_{W^{1,\infty}(0, T; \mathbb{R}^2_{a+1}(\Omega))} + \|u_{\text{reg}}\|_{L^\infty(0, T; \mathbb{R}^4_{a+1}(\Omega))} \\
\leq C\left(\left\|f''\right\|_{L^2(0, T; (H^1_0(\Omega))^\prime)} + \|f\|_{C(0, T; \mathbb{R}^2(\Omega))} + \|g\|_{V^2_{a+1}(\Omega)} + \|f(0) - Pg\|_{V^2_{a+1}(\Omega)}\right).
\end{align*}
\]

(3.3.4)

Under such condition, \( u^n_j \) is a finite element approximation of \( u \), with the convergence rate

\[
\begin{align*}
\max_{0 \leq j \leq M} \left\| u(jk) - u^n_j \right\|_{L^2(\Omega)} &= O(h^2 + k); \\
\max_{0 \leq j \leq M} \left\| u(jk) - u^n_j \right\|_{H^1_0(\Omega)} &= O(h + \frac{h^2 + k}{\sqrt{k}}).
\end{align*}
\]

(3.3.5)

Here \( k \) is the time step size, \( M = T/k \), and \( h \) is the length of the triangle of \( T_n \) that is away from the vertices.

Proof. 1. The regularity results can be obtained by using bootstrap. Although the well-posedness of solution in weighted space fails when \( V_s \neq \emptyset \), we still have the well-posedness of solution on regular Sobolev spaces. Differentiating (2.1.1) two times, we can check \( u_{tt} \) lies in \( L^\infty(0, T; L^2(\Omega)) \) by classical results. Since \( Pu_t = f - u_{tt} \) for a.e. \( t > 0 \), we obtain from (3.2.2) that \( u_t = P^{-1}(f - u_{tt}) \in L^\infty(0, T; V^2_{a+1}(\Omega)) \), since \( P^{-1} \) describes an isomorphism from \( \mathcal{X}^0_{a-1}(\Omega) \) to \( V^2_{a+1}(\Omega) \) and \( L^2(\Omega) \subset \mathcal{X}^0_{a-1}(\Omega) \). By the same argument, the equation \( u = P^{-1}(f - u_t) \) helps us control the \( L^\infty \mathcal{X}^4_{a+1} \) norm of \( u \).

2. For the numerical part of the theorem, we mimic the proof in the lecture notes written by professor Douglas N. Arnold in 2011 (See [3]). Since \( u \) is the exact solution of (2.1.1), and \( f - u' \in W^{1,\infty}(0, T; L^2(\Omega)) \), there exists a unique function \( w_n \in S_n \times [0, T] \) such that

\[
B[ w_n , v_n ] = (f - u', v_n), \quad \forall v_n \in S_n \text{ and } t \geq 0.
\]


Also, since $B[u, v] = (f - u', v)$ for all $v \in H^1_D(\Omega)$ and $t \geq 0$, we have

$$B[w_n, v_n] = B[u, v_n], \quad \forall v_n \in S_n \text{ and } t \geq 0.$$  

Differentiate the equation in time, we see that $w'_n$ is the elliptic projection of $u'$ onto $S_n$:

$$B[w'_n, v_n] = B[u', v_n] = (f' - u'', v_n), \quad \forall v_n \in S_n \text{ and } t \geq 0.$$  

This means $w'_n \in S_n$ is the numerical solution of the elliptic problem $Pw'_n = f' - u''$, where the exact solution is $u'$. Apply the estimate (3.2.4), (3.2.5) and plug in $m = 1$, we deduce

$$\left\| u' - w'_n \right\|_{L^2(\Omega)} \leq C_n h^2 \left\| f' - u'' \right\|_{\mathcal{K}^{a-1}_{\alpha-1}(\Omega)}; \quad (3.3.6)$$

$$\left\| u' - w'_n \right\|_{H^1_b(\Omega)} \leq C_n h \left\| f' - u'' \right\|_{\mathcal{K}^{a-1}_{\alpha-1}(\Omega)}; \quad (3.3.7)$$

for some constant $C_n > 0$ and all $t \geq 0$. Now for any $t \geq 0$ and $v_n \in S_n$, let us analyze the term

$$\left( -\frac{w_n((j+1)k) - w_n(jk)}{k}, v_n \right) + B[w_n((j+1)k), v_n] = B[u((j+1)k), v_n] +$$

$$\left( -\frac{u((j+1)k) - u(jk)}{k}, v_n \right) + \left( -\frac{w_n((j+1)k) - u((j+1)k) - [w_n(jk) - u(jk)]}{k}, v_n \right). \quad (3.3.8)$$

Since we have

$$\left\| \frac{u((j+1)k) - u(jk)}{k} - u'((j+1)k) \right\|_{L^2(\Omega)} = \frac{k}{2} \left\| u''(jk + \epsilon) \right\|_{L^2(\Omega)} \leq \frac{k}{2} \left\| u'' \right\|_{L^\infty(0,T;L^2(\Omega))},$$

and by (3.3.6),

$$\left\| \frac{w_n((j+1)k) - u((j+1)k) - [w_n(jk) - u(jk)]}{k} \right\|_{L^2(\Omega)} \leq \frac{1}{k} \int_{jk}^{(j+1)k} \left[ w'_n(s) - u'(s) \right] ds \right|_{L^2(\Omega)} \leq C_n h^2 \left\| f' - u'' \right\|_{L^\infty(0,T;\mathcal{K}^{a-1}_{\alpha-1}(\Omega))} \leq C_n h^2 \left\| f' - u'' \right\|_{L^\infty(0,T;L^2(\Omega))}. \quad (3.3.9)$$
combining the above two inequalities and plug into (3.3.8), we can discover

\[
\begin{align*}
&\left(\omega_n((j+1)k) - \omega_n(jk)\right) + B[w_n((j+1)k), v_n] = B[u((j+1)k), v_n] + (u'(j+1)k), v_n) \\
&+ (z^{j+1}_n, v_n) = (f((j+1)k), v_n) + (z^{j+1}_n, v_n) = (f^{j+1}_n, v_n) + (z^{j+1}_n, v_n) \quad \forall v_n \in S_n, j = 0, 1, ...
\end{align*}
\]

(3.3.10)

where

\[
\left\|z^j_n\right\|_{L^2(\Omega)} \leq C_n h^2 \left\|f'\right\|_{L^\infty(0,T;L^2(\Omega))} + (C_n h^2 + k) \left\|u''\right\|_{L^\infty(0,T;L^2(\Omega))} \quad j = 0, 1, ...
\]

(3.3.11)

Now let us analyze \(u^j_n\) generated by the equation (3.3.1) and (3.3.2). Since

\[
\left(\omega_n - \omega_n\right) + B[u^{j+1}_n, v_n] = (f^{j+1}_n, v_n) \quad \forall v_n \in S_n, j = 0, 1, ...
\]

(3.3.12)

Compare it with (3.3.10), we let \(y^j_n = u^j_n - \omega_n\) and get

\[
\left(\omega_n - \omega_n\right) + B[y^{j+1}_n, v_n] = (z^{j+1}_n, v_n) \quad \forall v_n \in S_n, j = 0, 1, ...
\]

(3.3.13)

Now for each \(j \in \mathbb{N}\) we choose \(v_n = y^{j+1}_n \in S_n\), and the equation becomes

\[
(y^{j+1}_n, y^{j+1}_n) + kB[y^{j+1}_n, y^{j+1}_n] = (y^j_n, z^{j+1}_n, y^{j+1}_n)
\]

(3.3.14)

Since the bilinear form \(B[\cdot, \cdot]\) is positive definite, we can deduce

\[
\left\|y^{j+1}_n\right\|_{L^2(\Omega)} \leq \left\|y^j_n + k z^{j+1}_n\right\|_{L^2(\Omega)} \leq \left\|y^j_n\right\|_{L^2(\Omega)} + k \left\|z^{j+1}_n\right\|_{L^2(\Omega)}.
\]

By iteration, it becomes

\[
\max_{0 \leq j \leq M} \left\|y^j_n\right\|_{L^2(\Omega)} \leq \left\|y^0_n\right\|_{L^2(\Omega)} + k \sum_{s=1}^{M} \left\|z^s_n\right\|_{L^2(\Omega)} = k \sum_{s=1}^{M} \left\|z^s_n\right\|_{L^2(\Omega)}.
\]
Here the observation \( y_0^0 = 0 \) comes from that \( B[u_0^n, v_n] = B[u(0), v_n] = B[w_n(0), v_n] \) for all \( v_n \in S_n \). By the estimate (3.3.11) we deduce

\[
\max_{0 \leq j \leq M} \left\| \frac{y_j}{n} \right\|_{L^2(\Omega)} \leq C_n T h^2 \left\| f' \right\|_{L^\infty(0,T;L^2(\Omega))} + (C_n h^2 + k)T \left\| u'' \right\|_{L^\infty(0,T;L^2(\Omega))}
\]

\[
\leq C_n (h^2 + k) \left( \left\| f'(0) - Pg \right\|_{X^{a+1}_2(\Omega)} + \left\| f' \right\|_{L^\infty(0,T;L^2(\Omega))} + \left\| f'' \right\|_{L^2(0,T;H^1_s(\Omega)')} + \left\| g \right\|_{X^{a+1}_2(\Omega)} \right).
\]

(3.3.15)

This implies

\[
\max_{0 \leq j \leq M} \left\| u_j - w_n(kj) \right\|_{L^2(\Omega)} = O(h^2 + k).
\]

Integrating the equation (3.3.6) from 0 to \( kj \), we see

\[
\max_{0 \leq j \leq M} \left\| u_j - u(kj) \right\|_{L^2(\Omega)} = O(h^2 + k).
\]

Next, we recall the equation (3.3.14). Since

\[
(y_j + k z_j + y_{j+1} - y_j, y_{j+1}) = k B[y_j, y_{j+1}] \geq C k \left\| y_j \right\|^2_{H^1_0(\Omega)},
\]

We apply the \( L^2 \) estimate on \( y_j^n \) in the equation (3.3.15) to discover

\[
\max_{0 \leq j \leq M} \left\| y_j \right\|_{H^1_0(\Omega)} \leq C \frac{\max_{0 \leq j \leq M} \left\| y_j \right\|_{L^2(\Omega)}}{\sqrt{k}} + C \max_{0 \leq j \leq M} \left\| y_j \right\|_{L^2(\Omega)}^{1/2} \max_{0 \leq j \leq M} \left\| y_j \right\|_{L^2(\Omega)}^{1/2} = O\left(\frac{h^2 + k}{\sqrt{k}}\right).
\]

Still, we integrate the equation (3.3.7) from 0 to \( kj \) to obtain \( \left\| u - w_n \right\|_{H^1_0(\Omega)} = O(h) \), which implies

\[
\max_{0 \leq j \leq M} \left\| u_j - u(kj) \right\|_{H^1_0(\Omega)} = O(h + \frac{h^2 + k}{\sqrt{k}}).
\]

Hence we obtain the convergence rate for this FEM scheme.
3.3.2 Remark. In fact, for the above scheme, if we apply quadratic elements on each triangular mesh, that is, fix $m = 2$ and use the sequence of finite element space $S_n := S(T_n, 2)$ for $n \in \mathbb{N}^+$, then we will obtain a set of more accurate numerical solution $u_n$. By a similar argument, the convergence rate for $u_n$ will be

$$
\max_{0 \leq j \leq M} \left\| u_j^n - u(kj) \right\|_{L^2(\Omega)} = O(h^3 + k),
$$

(3.3.16)

The proof is technique, we need to re-control the term in (3.3.9). This result shows, if we take $k = 4h^3$, the numerical solution $u_j^n$ will have the convergence rate $O(h^3)$ in $L^\infty L^2$, and $O(h^{3/2})$ in $L^\infty H^1$. Here the estimate for $L^\infty H^1$ convergence rate may not be optimal.

If we apply polynomial elements of degree $\leq 3$ on each triangular mesh: $S_n = S(T_n, 3)$, then we can derive

$$
\max_{0 \leq j \leq M} \left\| u_j^n - u(kj) \right\|_{H^1_0(\Omega)} = O(h^4 - \epsilon + k)
$$

for any small $\epsilon > 0$, which implies $u_j^n$ has the quasi-optimal convergence rate in $h$ to the exact solution $u \in L^\infty(0, T; \hat{H}^d_{a+1}(\Omega) + W^d_s)$.

3.4 Numerical Test

In this section, we will study three specific examples for our problem, and implement the above finite element scheme with graded mesh $\{T_n\}_{n \geq 1}$ to get a sequence of numerical solutions. To be more convenient, let us rewrite the statement of our parabolic problem (2.1.1):

$$
\begin{cases}
Lu + u_t = f & \text{in } \bigcup_{k=1}^{K} \Omega_k \times T; \\
u = 0 & \text{on } \partial_D \Omega; \\
\nabla^A u = 0 & \text{on } \partial_N \Omega; \\
u = g & \text{on } \bigcup_{k=1}^{K} \Omega_k \times \{t = 0\}; \\
u^+ = u^- & \text{on } \Gamma \times T; \\
\nabla^A u^+ = \nabla^A u^- & \text{on } \Gamma \times T.
\end{cases}
$$

(3.4.1)
In the examples, we will choose appropriate data $L$, $f$, $g$ that meet the conditions

\[ f'' \in L^2(0, T; L^2(\Omega)), \quad f' \in L^2(0, T; H^1(\Omega)), \quad g \in V^2_{a+1}(\Omega), \quad f(0) - Pg \in V^2_{a+1}(\Omega), \]

so we can make sure

\[ u \in W^{2,\infty}(0, T; L^2(\Omega)) \cap W^{1,\infty}(0, T; \mathcal{H}^2_{a+1}(\Omega)) \]

by our theorem 2.3.6.

Now we take the mesh to be $\mathcal{T}_n$ mentioned in [18] and our section 3.2: At every step, for each vertex $Q$, we refine by a fixed ratio $\kappa_Q$ on the triangular mesh that contains $Q$, and refine regularly on the other meshes. Also, we let the finite element space be $S_n := S(\mathcal{T}_n, 2)$ so that it consists of quadratic functions as elements on each mesh. Then the fixed ratio $\kappa_Q$ can be $2^{-n} \eta_Q$ on each vertex $Q$. Here we are able to calculate $\eta_Q$ using our theorem 2.4.1.

The convergence rate (3.3.16) from our theoretical results is to be expected. In the following three examples, we are going to test how the numerical solution $u^h_n \in S_n$ converges to the exact solution $u$ under $L^{\infty}(H^1)$ and $L^{\infty}(L^2)$ norm. The exact solution is not solvable for these specific examples, however, since our theoretical results (3.3.16) guarantee that the sequence of solutions $(u^h_n)_{n \geq 1}$ converge to the exact solution $u$, we can analyze the convergence rate by comparing the numerical solution $u^h_n$ and $u^h_{n+1}$ for each nonnegative integer $n$.

For each example below, we let the mesh size of our initial triangulation $\mathcal{T}_0$ be $h = 0.25$, so $\mathcal{T}_n$ will have the mesh size $h = 2^{-n-2}$ for triangles away from vertices. In addition, for each triangulation, we let the time mesh size $k = 4h^3$ so that we can expect $O(h^{\frac{7}{2}})$ error in $L^{\infty}H^1$ at most, which are derived from the estimate (3.3.16).

### 3.4.1 Domain with Nonsmooth Interface

In the first test, we let the domain be a square that is divided into two polygons by the interface. Specifically, let $\Omega = (0, 1) \times (0, 1)$, $T = [0, 1]$, $\partial \Omega = \partial_D \Omega$, with the interface $\Gamma = \{(1 - y, y) \mid y \in \left(\frac{1}{2}, 1\right)\} \cup \{(\frac{1}{2}, y) \mid y \in (0, \frac{1}{2})\}$. Notice that $\Gamma$ has an interior vertex at $(\frac{1}{2}, \frac{1}{2})$. Then the two subdomains

\[ \Omega_1 = \{(x, y) \mid x \in (0, \frac{1}{2}), \ y \in (0, 1 - x)\}, \quad \Omega_2 = \Omega \setminus \Omega_1; \]
Moreover, we let the operator be

\[
L = \begin{cases} 
-\Delta & \text{on } \Omega_1; \\
-10\Delta & \text{on } \Omega_2; 
\end{cases}
\]  

(3.4.2)

with the data \(g(x,y) \equiv 0\) and \(f(x,y,t) = \sin(\pi x) \sin(\pi y) t\) for all \((x,y,t) \in (0,1) \times (0,1) \times [0,1]\).

Thereafter, we are able to construct our initial uniform mesh \(T_0\), with mesh size \(h = 0.25\). Notice that there are 6 vertices on the boundary and interface of \(\Omega\), where the solution \(u\) results in a singularity. We follow our theorem 2.4.1 to discover, at each corner of the unit square, say \((0,0), (0,1), (1,0)\) and \((1,1)\), the singularity \(\eta = 2\); at the vertices \((\frac{1}{2},0)\) and \((\frac{1}{2}, \frac{1}{2})\), the singularity \(\eta\) equals 1 and \(0.839 < 1\). Recall how we do the next mesh refinement from figure 3.2, at each corner of the unit square, the decay ratio of triangles in subsequent refinements \(\kappa = 2^{-m} \eta = \eta/4 = 1/2\), hence we can refine uniformly at those corners. At the vertices \((\frac{1}{2},0)\) and \((\frac{1}{2}, \frac{1}{2})\), we have the ratio \(\kappa = 1/4 = 0.25\) and \(\kappa = 0.839/4 = 0.210\), so we have to use graded triangular mesh near these 2 vertices with the above decay ratio, see figure 3.5.
Fig. 3.5: First 3 mesh refinements $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ after the initial uniform mesh

Now, given the triangulation $\{\mathcal{T}_n\}_{n \geq 1}$ and the corresponding function space $S_n = S(\mathcal{T}_n, 2)$, we follow the equation (3.3.1) and (3.3.2), with the space mesh size $h = 2^{-n-2}$ and time mesh size $k = 4h^2 = 2^{-3n-4}$. After numerical implement, we obtain the numerical solutions $u^h_i \in S_i$ for $i = 0, 1, 2, 3$ as follows.

<table>
<thead>
<tr>
<th></th>
<th>$h - u^h_0$</th>
<th>$h - u^h_1$</th>
<th>$h - u^h_2$</th>
<th>$h - u^h_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t=0.0$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$t=0.25$</td>
<td>1.202e-5</td>
<td>7.166e-7</td>
<td>1.189e-7</td>
<td>7.261e-5</td>
</tr>
<tr>
<td>$t=0.5$</td>
<td>5.248e-5</td>
<td>3.113e-6</td>
<td>5.076e-7</td>
<td>3.137e-4</td>
</tr>
<tr>
<td>$t=1.0$</td>
<td>2.191e-4</td>
<td>1.295e-5</td>
<td>2.037e-6</td>
<td>1.305e-3</td>
</tr>
</tbody>
</table>

Fig. 3.6: Diagram: Domain with Non-smooth Interface

We aim to confirm the $O(h^2)$ convergence of our scheme:

$$
\lim_{i \to \infty} e_i = \lim_{i \to \infty} \frac{\|u^h_i - u^h_{i-1}\|_{H^1_D(\Omega)} + \|u^h_{i+1} - u^h_i\|_{H^1_D(\Omega)}}{2} \geq 2\sqrt{2}.
$$

(3.4.3)
From the diagram, we check that $e_1 \approx 17$, and $e_2 \approx 6$. Different from running the elliptic transmission problems, the complexity of our parabolic problem significantly increases for $i \geq 3$. However, the information in the above diagram do reveal the convergence from $u_h^n$ to $u$.

3.4.2 Domain with Neumann-Neumann Vertices

Now let us study the second example, in which the domain is non-convex and contains one vertex with Neumann boundary conditions imposed on both sides. To be more specific, let the domain $\Omega$ be the unit square $(0, 1) \times (0, 1)$ minus the triangle with vertices $(0, 1)$, $(0, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2})$. Moreover, we let $L = -\Delta$ on $\Omega$ so that there is no interface. On the boundary, we impose Neumann condition on the sides from $(0, 0)$ to $(1, 0)$ and from $(1, 0)$ to $(1, 1)$, and impose Dirichlet condition on the other sides. Lastly, same as in the previous example, we let the time interval be $[0, 1]$, with the functions $g(x, y) \equiv 0$ and $f(x, y, t) = \sin(\pi x) \sin(\pi y) t$ for all $(x, y, t) \in (0, 1) \times (0, 1) \times [0, 1]$. 

Fig. 3.7: Numerical solution $u_h^2 \in S_2 : S(T_2, 2)$ at time $t = 0.5$ and $t = 1$.
Similar to the last example, let us calculate the singularity of each vertex by the theorem 2.4.1. As a result, for each step of mesh refinement, we can refine uniformly at the vertices (0, 1), (1, 0) and (0, $\frac{1}{2}$). At the vertices (0, 0) and (1, 1) where the boundary condition changes from Dirichlet to Neumann, the decay ratio of triangles in subsequent refinements are both $\kappa = 2^{-2} = 0.25$. At the vertex $(\frac{1}{2}, \frac{1}{2})$, the decay ratio $\kappa = 2^{-2} \times \frac{7}{4} = 0.143$. Hence the triangulation $T_i$ at $i = 1, 2, 3$ should be as in the figure 3.10.

Now given the triangulation $\{T_n\}_{n \geq 1}$ and the corresponding function space $S_n = S(T_n, 2)$, we follow the equation (3.3.1) and (3.3.2), with the space mesh size $h = 2^{-n-2}$ and time mesh size $k = 4h^3 = 2^{-3n-4}$. After numerical implement, we obtain the numerical solutions $u^h_i \in S_i$ for $0 \leq i \leq 3$ in the diagram 3.11 below.
Fig. 3.10: First 3 mesh refinements $T_1, T_2, T_3$ after the initial uniform mesh

<table>
<thead>
<tr>
<th></th>
<th>$| u_0^h - u_1^h |_{H^1_D(\Omega)}$</th>
<th>$| u_1^h - u_2^h |_{H^1_D(\Omega)}$</th>
<th>$| u_2^h - u_3^h |_{H^1_D(\Omega)}$</th>
<th>$| u_3^h |_{H^1_D(\Omega)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>t=0.0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>t=0.25</td>
<td>9.012e-5</td>
<td>2.447e-5</td>
<td>7.520e-6</td>
<td>3.028e-4</td>
</tr>
<tr>
<td>t=0.5</td>
<td>5.834e-4</td>
<td>1.448e-4</td>
<td>4.011e-5</td>
<td>1.878e-3</td>
</tr>
<tr>
<td>t=1.0</td>
<td>3.100e-3</td>
<td>7.263e-4</td>
<td>1.883e-4</td>
<td>9.737e-3</td>
</tr>
</tbody>
</table>

Fig. 3.11: Diagram: Domain with Neumann-Neumann Vertices

Still, we are ready to verify the $O(h^3)$ convergence of our scheme:

$$\lim_{i \to \infty} e_i = \lim_{i \to \infty} \frac{\| u_i^h - u_{i-1}^h \|_{H^1_D(\Omega)}}{\| u_i^h - u_{i+1}^h \|_{H^1_D(\Omega)}} \geq 2\sqrt{2}.$$ (3.4.4)

From the diagram, we check that at time $t = 1$, the above ratio $e_1 \approx 4.27$, and $e_2 \approx 3.86$. Since in this example there is no interface, the error of the numerical solutions $u_1^h$ and $u_2^h$ are comparably small. As a consequence $e_i$ should converge very quickly. We can expect that $\{e_i\}_{i \geq 1}$ converges to some constant no less than $2\sqrt{2}$. 
Fig. 3.12: Numerical solution $u^h_2 \in S_2 : S(T_2,2)$ at time $t = 0.5$ and $t = 1$

3.4.3 Problem with High Contrast across Interface

In the last test, we let the domain be a square that is divided equally into two sub-rectangles by the interface. Specifically, let $\Omega = (0,1) \times (0,1)$, $T = [0,1]$, $\partial\Omega = \partial_D \Omega$, with the two subdomains

$$\Omega_1 = (0, \frac{1}{2}) \times (0,1), \quad \Omega_2 = (\frac{1}{2},1) \times (0,1);$$

Moreover, we let the operator be

$$L = \begin{cases} -\Delta & \text{on } \Omega_1; \\ -100\Delta & \text{on } \Omega_2; \end{cases} \quad (3.4.5)$$

which has a high contrast across the interface $\Gamma$. Now, to meet the compatibility conditions, we let the data $f \equiv 0$, and

$$g(x, y) = \begin{cases} y(1-y)(\frac{31300}{808}x - 200x^3 + 100x^4) & \text{on } \Omega_1; \\ y(1-y)(\frac{1303}{808}(1-x) - 2(1-x)^3 + (1-x)^4) & \text{on } \Omega_2. \end{cases} \quad (3.4.6)$$
Here the singularity of the vertices $(\frac{1}{2},0)$ and $(\frac{1}{2},1)$ are both 1, so the decay ratio of triangles in subsequent refinements are $\kappa = 2^{-2} = 0.25$. At the 4 corners of the unit square, we have $\kappa = 0.5$ so that we can apply uniform mesh near them. The triangulation $T_i$ for $i = 1, 2, 3$ are as follows.

After numerical implement, we obtain the numerical solutions $u^h_i \in S_i$, $i = 1, 2, 3$ as in the diagram below.
We can observe from the above diagram that, the ratio \( e_i = \frac{\|u^h_i\|_{H^1_1(\Omega)}}{\|u^h_{i+1}\|_{H^1_1(\Omega)}} \) at \( t = 1 \) for \( i = 0, 1, 2 \) is \( e_0 \approx 3.1 \times 10^{16} \), \( e_1 \approx 3.3 \times 10^6 \) and \( e_2 \approx 1.1 \times 10^2 \). Although \( u^h_i \) is approaching the exact solution \( u \), we have to take many more steps of mesh refinement to obtain a satisfactory numerical solution. The large error happens due to the high contrast of the operator \( L \) across the interface \( \Gamma \). Mostly, for the operator \( L = -100 \Delta \) on \( \Omega \), we usually have to take the mesh size \( h < \frac{1}{200} \) so that the error of the numerical solution can be < 10%.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( h^0 - u^h_1 ) ( H^1_1(\Omega) )</th>
<th>( h^1 - u^h_2 ) ( H^1_1(\Omega) )</th>
<th>( h^2 - u^h_3 ) ( H^1_1(\Omega) )</th>
<th>( h^3 - u^h_4 ) ( H^1_1(\Omega) )</th>
</tr>
</thead>
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<tr>
<td>0.0</td>
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<td>1.250e-2</td>
<td>2.381e-3</td>
<td>20.31</td>
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<td>0.25</td>
<td>3.060e-4</td>
<td>1.491e-8</td>
<td>9.717e-10</td>
<td>1.736e-10</td>
</tr>
<tr>
<td>0.5</td>
<td>4.588e-9</td>
<td>2.410e-17</td>
<td>4.505e-20</td>
<td>1.425e-22</td>
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</tbody>
</table>
Bibliography


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