TRACTABLE RADAR WAVEFORM DESIGN
UNDER PRACTICAL CONSTRAINTS

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by
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Abstract

Waveform design is one of the central aspects of radar systems. It can determine many of the radar properties. A well-designed waveform can improve the signal-to-interference-plus-noise ratio (SINR), enable suitable delay (range) resolution, and utilize the spectrum efficiently. Moreover, for multiple array radar system, waveform diversity can be employed to enhance the flexibility of the transmit beampattern design and enable efficient management of radar radiation power in directions of interest.

While unconstrained waveform design is straightforward, a key open challenge is to enforce some of the significant practical constraints of constant modulus, waveform similarity and spectral interference constraints. Incorporating these constraints in an analytically tractable manner is a longstanding open challenge. This is due to the fact that the optimization problem subject to these constraints is a hard non-convex problem. Decades of the past work have shown a stiff trade-off between analytical tractability (achieved by relaxations to manageable constraints) and realistic design that exactly obeys these practical constraints but is computationally troublesome. In this dissertation, we propose a new framework that breaks this classical trade-off.

In the first part, we address the problem of a joint transmit waveform and receive filter design for radar systems under the important practical constraints of constant modulus and waveform similarity. We develop a new analytical approach that involves solving a sequence of convex Quadratically Constrained Quadratic Programming (QCQP) problems, which we prove converges to a sub-optimal solution. Because an improvement in SINR results via solving each problem in the sequence, we call the method Successive QCQP Refinement (SQR). We evaluate SQR against other candidate techniques with respect to SINR performance, beampattern and pulse compression properties in a variety of scenarios. Results show that SQR outperforms state of the art methods that also employ constant modulus and/or similarity constraints while being computationally less burdensome.

In the second part, we address the problem of designing a beampattern for Multiple-Input-Multiple-Output (MIMO) radar, which in turn is determined by the transmit waveform. A new approach is proposed in our work, which involves solving the hard non-
convex problem of beampattern design using a sequence of convex equality constrained Quadratic Programs (QP), each of which has a closed form solution. The converged solution achieves constant modulus and satisfies the Karush-Kuhn-Tucker (KKT) optimality conditions, which we prove formally is possible under realistic assumptions. We evaluate the proposed successive closed forms (SCF) algorithm against state-of-the-art MIMO beampattern design techniques and show that SCF breaks the trade-off between desirable performance and the associated computation cost. Furthermore, we address the problem of designing a beampattern for MIMO radar under a spectral interference constraint, which in turn is determined by the transmitted waveform, as well as the constant modulus constraint. One key challenge when jointly enforcing the spectral interference constraint and the constant modulus constraint is to ensure feasibility of the optimization problem. A new approach is proposed in our work, which also involves solving a sequence of constrained quadratic programs, each of which results in a closed form solution. We formally prove that feasible set of each QP problem in the proposed beampattern with interference control (BIC) algorithm is always non-empty. We evaluate BIC algorithm against the state-of-the-art MIMO beampattern design techniques under the constant modulus constraint and show that BIC achieves a higher performance while maintaining a low spectral interference level in the desired bands.

In the final part of this dissertation, we address the problem of radar ambiguity function shaping. Ambiguity function shaping continues to be one of the most challenging open problems in cognitive radar. Analytically, a complex quartic function should be optimized as a function of the radar waveform code. We develop a new approach called Adaptive Sequential Refinement (ASR) to suppress the clutter returns for a desired range-Doppler, i.e. ambiguity function response. ASR solves the aforementioned optimization problem in a unique iterative manner such that the formulation is updated depending on the iteration index. We establish formally that: 1.) the problem in each step of the iteration has a closed form solution, and 2.) monotonic decrease of the cost function until convergence is guaranteed. Experimental validation shows that ASR perform better than state of the art alternatives even as its computational burden is orders of magnitude lower.

In summary, this dissertation develop a new sequence of convex problems approaches towards solving the hard non-convex problems that appear in practical radar waveform design. Finally, we present some ideas for future research which address waveform design for cooperative radar-communication systems optimization, MIMO ambiguity function shaping and waveform design for autonomous vehicle.
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Dedication

I dedicate this thesis to

my parents, Sa’ad and Wafiqah, my wife, Norah

and my beloved son and daughter, Yosef and Aseel.
Chapter 1

Introduction

Radar systems transmit an electromagnetic signal in the space to detect a desirable object called target. Then, the radar receives returns from the target, undesired objects in the space called clutter, signals from other radiators called interference in addition to noise. A successful radar system should be able to separate the target returns from clutter, interference and noise [1]. During the past five decades, many adaptive techniques have been developed to suppress the disturbance at the receiver side such as false alarm rate (CFAR) detectors and space-time adaptive processing (STAP). Since the target and clutter returns are essentially due to the radar transmitted signal (signal-dependent), the shape of the transmitted waveform is very crucial for target detection and clutter rejection. In 1965, H. Van Trees [1–3] noted that selecting an appropriate transmit waveform is more effective than optimum receiver design. Since then, transmit waveform design has been an ongoing topic of research for transmit adaptive radar systems. The basic idea of transmit adaptive radars is to utilize a priori knowledge about the environment that affects the system performance such as target, clutters, locations of electromagnetic interferences. Then, this environmental knowledge is used to design the transmitted waveform to improve the radar performance. This process is illustrated in Fig 1.1.

Some of the waveform design approaches rely on maximizing the detection performance. Therefore, in the situations where the disturbance is not white, finding the waveform that maximize the detection performance is equivalent to maximizing the radar signal-to-interference-plus-noise ratio (SINR) [4]. For example, let \( x \in \mathbb{C}^N \) be the transmit waveform vector with \( N \) time samples the target detection problem for fixed target can be formulated in the framework of the following hypothesis testing:

\[
H_0 : \quad r = n \\
H_1 : \quad r = \alpha x + n
\]

where \( r \) is the received signal, \( \alpha \) is the propagation coefficient and \( n \) is is the \( N \)-dimensional complex column vector containing the filtered disturbance with known positive definite covariance matrix \( E[n^Hn] = R \). In this case, the generalized likelihood ratio test can be given as:

\[
|r^H R^{-1} x| \begin{array}{c} H_1 \approx t \\ \geq \end{array} H_0
\]
where $t$ is a threshold which is determined by the probability of false alarm. The probability of detection ($P_d$) can be given as a function of the false alarm ($P_f$) using the Marcum Q function:

$$P_d = Q(\sqrt{2|\alpha|^2 x^H R^{-1} x}, \sqrt{-2 \ln P_f})$$

$$= Q(\sqrt{2|\alpha|^2 \text{SINR}}, \sqrt{-2 \ln P_f})$$

where $\text{SINR} = x^H R^{-1} x$. In this case, $P_d$ is an increasing function of the SINR and, hence, the maximization of $P_d$ can be obtained by maximizing the SINR. Research pertaining waveform design using the SINR as an objective function can be divided into two main lines regarding the disturbance type. The first one, considers the signal-independent disturbance caused by interference plus noise only [4–7] where developed algorithms find the transmit waveform that maximizes the SINR.

The second one, considers both the signal-independent disturbance caused by clutter and the signal-independent disturbance [8–12] where some knowledge is assumed of a priori electromagnetic reflectivity and spectral clutter models. Therein, the developed algorithms find the joint transmitter waveform and receiver filter that maximize the SINR.

However, under intense disturbance caused by clutter and/or interference, the SINR may not be the most important factor [3]. Multiple-input multiple-output (MIMO) radar, which allows each antenna element to transmit a different waveform is likely to play a significant role in clutter intense applications [3, 13, 14]. This property can be used to electronically concentrate the radar energy in the expected target direction. In this case, the objective is to find the transmit waveform to achieve the desired beampattern. Beampattern design has been a well-studied subject and there is a sizable literature.
available offering numerical optimization algorithms [15–22].

1.1 Motivation and Challenges

Whether the objective is to maximize the SINR or to achieve the desired beampattern, waveform design under practical constraints is a long standing open challenge. Unlike the radar receive filter design, the transmit waveform has many practical limitations. The first practical limitation is the requirement of a Constant Modulus Constraint (CMC). The importance of the CMC has been well documented [1, 23] and analyzed in terms of SINR performance loss. Most radar systems utilize non-linear power amplifiers which can not be efficiently utilized without CMC. More specifically, the output of the amplifier will be a clipped version of the optimized waveform as shown in Fig. 1.2, therefore, the system performance will degrade because the transmit waveform is different from the desired one. In fact, [1, 24] demonstrate that the SINR of some algorithms that use energy constraint is often lower than the SINR of the unoptimized reference waveform.

On the other hand, the Similarity Constraint (SC), uses a reference signal as a benchmark to produce an optimized waveform that shares some of the desirable autocorrelation properties of the reference waveform. As noted in [23, 25], the resulting waveforms from algorithms that do not enforce SC suffer from undesirable range resolution and/or low autocorrelation sidelobe levels, as illustrated in Fig. 1.3.
Another fundamental problem of interest in designing radar waveforms is to ensure co-existence of radar and communication systems [7, 26–33]. In this case, interference constraints must be incorporated additionally. The joint incorporation of CMC and SC with interference constraints remains an open problem.

Decades of the past work have shown a stiff trade-off between analytical tractability (achieved by relaxations to manageable constraints) and realistic design that exactly obeys these practical constraints but is computationally troublesome. In this dissertation, I present new theoretical results and experimental validation to support the following: The enhancement of a radar system performance via waveform design under the joint incorporation of CMC, SC and/or spectral interference constraint can be achieved by a realistic and computationally efficient analytical framework.

1.2 Literature Review: Waveform Design Under Practical Constraint

To overcome some of the practical requirements listed in section 1.1, recent work pertaining to SINR maximization and beampattern design have developed numerical optimization algorithms. In this section, we discuss representative algorithms, the Phase Coding Algorithm with Similarity Constraint (PCA-SC), the Sequential Optimization Algorithm (SOA), the Wideband Beampattern Formation via Iterative Techniques (WBFIT) algorithm.

1.2.1 SINR Maximization Methods

1.2.1.1 Phase Coding Algorithm with Similarity Constraint (PCA-SC)

Let \( x \in \mathbb{C}^N \) be the transmit waveform vector with \( N \) time samples and \( R \in \mathbb{C}^{N \times N} \) be the interference covariance matrix. The SINR is given by:

\[
SINR = x^H \Phi x
\]

where \( \Phi = R^{-1} \). The problem of maximizing SINR under joint CMC and SC can be formulated as:

\[
(P_1) \quad \max_x x^H \Phi x
\]

subject to: \( |x| = 1 \)

\[
||x - x_0||_\infty \leq \epsilon
\]

where \( x_0 \) is the reference waveform signal. Note that, problem \((P_1)\) involves maximization of a non-concave function since \( \Phi \) is a positive-definite matrix which is equivalent to minimizing a non-convex function [34]. Moreover, the CMC requires the absolute value of each element to be 1 which is also non-convex, therefore, both the objective function and the feasible set of problem \((P_1)\) are non-convex. De Maio et al. [4] proposed the Phase Coding Algorithm with Similarity Constraint (PCA-SC) to solve problem \((P_1)\). The algorithm focuses on signal-independent interference only and for Single-Input-Single-Output
(SISO) radar. The main steps in the PCA-SC algorithm rely on the Semi-Definite Relaxation (SDR) and randomization [35]. Due to the non-convexity of the problem, PCA-SC does not guarantee fast convergence or a local minimum solution.

1.2.1.2 Sequential Optimization Algorithm (SOA)

The framework of PCA-PC has been extended recently by Cui et al. [36] for MIMO radar under signal-dependent clutter where $SINR = x^H \Phi(x)x$. Therein, two sequential algorithms have been developed, namely, the Sequential Optimization Algorithm 1 (SOA1) and 2 (SOA2). In both algorithms, SDR with randomization is used to tackle the non-convexity of the problem accompanied with an iterative method found in [10,23] to resolve the dependence of the disturbance covariance matrix on the waveform vector $x$ as summarized in the following:

1. Start from an initial point $x_0$. Repeat the following until convergence:

2. Set $\Phi_m = \Phi(x_{m-1})$.

3. Solve the following using SDR with randomization:

$$\begin{align*}
(P_2) \max_{x} & \quad x^H \Phi_m x \\
\text{subject to:} & \quad |x| = 1 \\
& \quad ||x - x_0||_\infty \leq \epsilon
\end{align*}$$

4. Set $x_m = x^*(P_2)$ and $m = m + 1$. Go to step (2)

However, both SOA1 and SOA2 need a large number of randomization trials and do not guarantee convergence.

1.2.2 Beampattern Design

1.2.2.1 Wideband Beampattern Formation via Iterative Techniques (WB-FIT)

Consider a Uniform Linear Array (ULA) MIMO radar system with $M$ transmit antennas. The signal transmitted by the $m$-th element is denoted by $x_m(n)$, $n = 0, ..., N - 1$ where $N$ is the number of time samples. The discrete Fourier Transform (DFT) of $x_m(n)$ is denoted by $y_m(p)$ and is given by:

$$y_m(p) = \sum_{n=1}^{N} x_m(n)e^{-j2\pi \frac{np}{N}}, \quad p = \frac{N}{2}, ..., 0, ..., \frac{N}{2} - 1 \quad (1.1)$$

For the steering vector of a wideband ULA can be written as:

$$a(\theta, f) = [1 \quad e^{j2\pi(f + f_c) \frac{d \cos \theta}{c}} \quad ... \quad e^{j2\pi(f + f_c) \frac{(M-1)d \cos \theta}{c}}] \quad (1.2)$$
Note that $a(\theta, f)$ is continuous in both phase and frequency. Therefore, it can be expressed as a discrete frequency and a discrete angle for the interval $[0^\circ, 180^\circ]$ with $K$ angles:

$$a_{kp} = a\left(\theta_k, \frac{p}{NT_s}\right), \quad k = 1, 2, ..., K$$

(1.3)

where $T_s$ is the sampling rate. The beampattern can be expressed as the following discrete angle-frequency grid as:

$$P_{kp} = |a_{kp}^H y_p|^2$$

(1.4)

where $y_p = [y_0(p) \; y_1(p) \; ... \; y_{M-1}(p)]^T$. The wideband beampattern matching problem with no SC can be formulated as:

$$\left\{ \begin{array}{l} \min_x \sum_{k=1}^{K} \sum_{p=-N/2}^{N/2-1} \left[ d_{kp} - |a_{kp}^H y_p|^2 \right] \\ \text{s.t.:} |x| = 1 \end{array} \right.$$  

(1.5)

where $d_{kp} \in \mathbb{R}$ is the desired beampattern. He et al. in [37] proposed the Wideband Beampattern Formation via Iterative Techniques (WBFIT) to solve the problem indirectly in two stages. In the first stage, the algorithm solves the unconstrained minimization problems:

$$\min_{y_p} \sum_{k=1}^{K} \sum_{p=-N/2}^{N/2-1} \left[ d_{kp} - |a_{kp}^H y_p|^2 \right]$$

(1.6)

for $p = -N/2, ..., 0, ..., N/2 - 1$ by using the least-square estimate and the Sussman-Gerchberg-Saxton algorithm found in [38] also called the cyclic algorithm. In the second stage, in the second stage they aim to find the constant modulus approximation of the solution.

### 1.2.3 Ambiguity Function Shaping

In radar signal processing, a key role is played by the range-Doppler response of the waveform used to probe the environment. The aforementioned response is usually referred to as the classic waveform *ambiguity function* [39] and the problem of designing signals sharing a desired ambiguity has attracted the interest of many radar researchers since the early days of radar [38, 40–42].

The explicit shaping of the ambiguity function by an optimal selection of the transmit waveform is known to be notoriously difficult because the cost function involves a non-convex complex quartic [39]. It has in fact been shown recently that in the presence of the constant modulus constraint, the resulting optimization is NP-hard [39].

### 1.3 Goals and Contributions

Following this brief overview of the key ideas that will constitute this dissertation, it is now appropriate to state the goals of this research as follows:
• A new tractable analytical framework for SINR maximization that jointly enforces both CMC and SC. In contrast to existing work, which relies on SDR with randomization and its extensions [11,35,43,44], our approach involves solving a sequence of convex problems (each a QCQP). In each iteration of the sequence, the designed waveform satisfies the similarity constraint while constant modulus is successively achieved at convergence; hence, the method is called – Successive QCQP Refinement (SQR). We formally prove that the SINR resulting from the proposed successive QCQP solution is non-decreasing in each step. Moreover, the proposed SQR can easily incorporate the practical interference constraints jointly with CMC and SC.

• A new cost-efficient framework for beampattern design that jointly enforces both CMC and spectral interference constraint. We develop a new algorithm for MIMO beampattern design that involves solving the hard non-convex problem of beampattern design using a sequence of convex Equality Constrained Quadratic Programs, each of which has a closed form solution, such that constant modulus is achieved at convergence. Because each QP in the sequence has a closed form, the proposed successive closed forms (SCF) algorithm has significantly lower complexity than competing methods that incorporate CMC. We formally prove that the sequence of cost functions representing the deviation from the desired beampattern, that occurs in the proposed SCF algorithm is non-increasing. We further show that the sequence of waveform solutions (via solving each QP) converges to a Karush-Kuhn-Tucker (KKT) point under mild and realistic assumptions. Furthermore, a new algorithmic solution for beampattern design under both the spectral interference constraint and the constant modulus constraint. We formally prove that feasible set of each QP problem in the proposed beampattern with interference control (BIC) algorithm is always non-empty. The proposed BIC algorithm has significantly lower complexity than SDR with relaxation [36,43].

Adaptive Sequential Refinement (ASR): A new tractable analytical framework for ambiguity function shaping under waveform CMC. We develop a new algorithm for ambiguity function shaping that involves solving the non-convex quartic problem using a sequence of convex Quadratic Programs (QP) where the cost function as well as constraints are updated in each iteration of the sequence. We show further that each of these QPs has a closed form solution leading to significantly reduced complexity.

Remark: Although the antenna configuration for the radar system in this dissertation is mainly assumed to be a Uniform Linear Array (ULA) with half-wavelength spacing, all the proposed algorithms are applicable to any array set-up as long as the steering characteristic of the array antenna is known. The Algorithms does not use any assumptions on the antenna configurations. Moreover, some of the algorithms has been applied to a ULA antenna with 2.5-wavelength spacing, as in sec 3.4.2 (Case III).

The remainder of this document is organized as follows. Chapters 2, 3 and 4 respectively present the three main proposed contributions of this dissertation. The conclusion of the dissertation is outlined in Chapter 5.
Chapter 2

Successive QCQP Refinement for MIMO Radar Waveform Design Under Practical Constraints
2.1 Introduction

The resolution and target detection performance of a radar are highly dependent on its signal waveform shape. A well designed (optimized) waveform can significantly improve the signal-to-interference-plus-noise ratio (SINR) \([12,45]\) and probability of detection \([46,47]\). Other design criterion include desirable autocorrelation \([48]\), suitable ambiguity function shaping \([3,39,40]\), mutual information \([47,49–51]\) and beam pattern \([21,22,52]\). Further, with the advances in Multiple-Input Multiple-Output (MIMO) radar sensing, waveform design becomes more challenging since the design must allow for exploiting spatial diversity. Despite significant advances in radar waveform design, many practical challenges remain. One such challenge is the hardware requirement of a constant modulus waveform signal. Another challenge is to produce a design that does not compromise the autocorrelation properties by enforcing a strong similarity constraint. A vast majority of existing work however either ignores or relaxes \([1]\) these constraints in favor of tractable analytical solutions.

2.1.1 Motivation and Challenges

Recent work pertaining to radar waveform optimization can be divided into two categories. The first category uses an energy constraint instead of the Constant Modulus Constraint (CMC) such as \([53]\). This is typically combined with a \(L_2\) norm similarity constraint or a uniform elemental power constraint \([54–56]\). The advantage of these relaxations is an analytically tractable solution. However, the importance of the CMC has been well documented \([1,23]\) and analyzed in terms of SINR performance loss. Most radar systems utilize non-linear power amplifiers which can not be efficiently utilized without CMC. More specifically, the output of the amplifier will be a clipped version of the optimized waveform, therefore, the system performance will degrade because the transmit waveform is different from the desired one. In fact, \([1,24]\) demonstrate that the SINR of some algorithms that use energy constraint is often lower than the SINR of the unoptimized reference waveform.

On the other hand, the Similarity Constraint (SC), uses a reference signal as a benchmark to produce an optimized waveform that shares some of the desirable autocorrelation properties of the reference waveform. As noted in \([23,25]\), the resulting waveforms from algorithms that do not enforce SC suffer from undesirable artifacts in pulse compression and ambiguity function properties.

The second category includes the CMC without SC such as \([21,22,52,57,58]\) where the objective is to match desired beam pattern or with the SC such as \([4,44]\). However, it is well known that radar waveform design jointly with both CMC and SC is a hard non-convex problem. Existing approaches to address this issue invariably involve semi-definite relaxation (SDR) and randomization as in \([11,35,36,43]\). In this approach, a Semidefinite Programing (SDP) problem is first solved to find a waveform distribution and based on this a large number of trials are generated, followed by exhaustive search to find the closest waveform that has the maximum SINR. Although SDR with randomization gives a good approximate solution for problems with CMC \([35,44]\) there is no guarantee of a reasonable solution when introducing a SC to the problem. In fact, as shown in \([11]\)
the objective SINR drops significantly when SC is introduced (at least four dB under strong SC and one dB for a very weak SC). Further, randomization based methods can sometimes have prohibitively high computational complexity [11].

Another key problem of interest in designing radar waveforms is to ensure co-existence of radar and communication systems [7, 26–33]. In this case, interference constraints must be incorporated additionally. The joint incorporation of CMC and SC with interference constraints remains an open problem. Fortunately, the extension of our proposed SQR method to incorporate the spectral coexistence under CMC and SC is straightforward and tractable, as is shown in this chapter.

Set Up: We consider waveform design of a narrow band colocated MIMO radar involving point like stationary targets for two cases: in presence of white Gaussian noise, and in the presence of noise as well as signal-dependent clutter (modeled as a collection of stationary points). Optimization of both the transmit waveform as well as receive filter is pursued in a manner akin to [10, 11] where the receive filter admits a closed form in terms of the transmit waveform.

### 2.1.2 Our contributions

To overcome the challenges mentioned above, we develop a new algorithm for MIMO waveform design which jointly enforces CMC and SC. Specifically, this work makes the following contributions:

- **SQR: A new tractable analytical framework for waveform design that jointly enforces both CMC and SC.** As argued above, the radar waveform design jointly with CMC and SC involves solving a hard non-convex problem. In contrast to existing work, which relies on SDR with randomization and its extensions [11, 35, 36, 43, 44], our approach involves solving a sequence of convex problems (each a QCQP) such that in each iteration of the sequence the designed waveform satisfies the similarity constraint. Constant modulus is successively achieved at convergence, hence the method is called – Successive QCQP Refinement (SQR). No randomization is needed in our algorithm, which in turn also leads to significantly reduced computational complexity.

- **Analysis of the SQR based algorithm.** We formally prove that the SINR resulting from the proposed successive QCQP solution is non-decreasing in each step. We further prove that the sequence of solutions converges.

- **Extensions of SQR to incorporate spectral interference constraint.** The proposed SQR can easily incorporate the practical interference constraints jointly with CMC and SC since the spectral interference constraint is modeled as a convex quadratic constraint [7, 26] and the SQR already relies on solving a QCQP.

- **Experimental insights and validation.** Experimental validation is performed via Monte Carlo simulations. The SQR algorithm is shown to outperform SDR with randomization [35], the sequential optimization algorithm (SOA) [11] which also enforces CMC and SC; this is true especially in the medium (realistic) range of similarity. Further, upon the addition of interference constraints for spectral
co-existence, the proposed SQR is competitive with the recent cognitive radar code optimization (CRCO) [7] which in fact does not enforce CMC.

2.1.3 Notation

We denote vectors by boldface letters, e.g. \( \mathbf{a} \) (lowercase) and denote matrices by boldface letters, e.g. \( \mathbf{A} \) (uppercase). The \( l \)-th element of \( \mathbf{a} \) is denoted by \( a_l \) and denoted by \( A(m,l) \) the element located in the \( m \)-th row and \( l \)-th column of the matrix \( \mathbf{A} \). The Hermitian, conjugate and transpose operators are denoted by \((.)^H\), \((.)^*\) and \((.)^T\), respectively. We denote the Kronecker product by \( \otimes \). We denote by \( \lambda_{\text{max}}(\mathbf{A}) \), \( \lambda_{\text{min}}(\mathbf{A}) \) and \( \text{tr}(\mathbf{A}) \) the maximum eigenvalue, the minimum eigenvalue and the trace of \( \mathbf{A} \), respectively; \( \text{diag}(\mathbf{a}) \) represents a diagonal matrix such that the \( i \)-th element of the diagonal is equal to \( a(i) \). The curled inequality \( \succeq \) when used for matrices denotes the generalized matrix inequality and when used for vectors \( \mathbf{a} \succeq \mathbf{b} \) means \( a_i \geq b_i \) for all \( i \). For a complex number \( a \), we denote \( \text{Re}(a) \) and \( \text{im}(a) \) to the real and imaginary part \( a \), respectively; also we denote \( |a| \) and \( \arg a \) to the amplitude and phase of \( a \), respectively. We use \( j = \sqrt{-1} \) as the imaginary unit number. Finally, we use \( \mathbf{x}^* \) to denote the optimal solution, \( \mathbf{x}^*(P) \) to denote an optimal solution \( \mathbf{x}^* \) of the the optimization problem \( P \) and \( v(P) \) to denote its optimal value.
2.2 MIMO Signal Model

Consider a colocated narrow band MIMO radar system with $N_T$ transmit antennas and $N_R$ receive antennas. Each transmit element emits a different waveform $x_m(n)$, $m = 1, ..., N_T$, $n = 1, ..., N$ where $N$ is the number of samples. Let $x(n)$ be an $N_T \times 1$ vector denoting the $n$-th sample of the $N_T$ antennas. In addition, let $\mathbf{x}$ be the concatenated and complete $N_T N \times 1$ vector of the transmit waveform, $\mathbf{x} = [x^T(1), ..., x^T(N)]^T$. Then we have the following signal model [11]:

$$
\mathbf{r} = \alpha_0 \mathbf{U}(\theta_0) \mathbf{x} + \sum_{k=1}^{K} \alpha_k \mathbf{U}(\theta_k) \mathbf{x} + \mathbf{n} \tag{2.1}
$$

where $\mathbf{r}$ is an $N_R N \times 1$ receive waveform, $\mathbf{n}$ is an $N_R N \times 1$ circular complex Gaussian noise vector with zero mean and covariance matrix $\sigma^2_n \mathbf{I}$, $\alpha_0$ and $\alpha_k$ denote respectively to the complex amplitudes of the target and the $k$-th clutter source, $\theta_0$ and $\theta_k$ are the angle of the target and the angle of the $k$-th clutter source, respectively and $\mathbf{U}(\theta)$ is the steering matrix of a Uniform Linear Array (ULA) antenna with half-wavelength separation between the antennas given as:

$$
\mathbf{U}(\theta) = \mathbf{I}_N \otimes [\mathbf{a}_r(\theta) \mathbf{a}_t(\theta)^T]
$$

where $\mathbf{I}_N$ is the $N \times N$ identity matrix, $\mathbf{a}_t$ and $\mathbf{a}_r$ are the transmit and and receive steering vector, respectively, as defined in [11].

The most common criterion in waveform design involves SINR maximization, which involves a joint optimization of the transmit waveform and the receive filter. In particular, the receive filter is assumed to be a linear Finite Impulse Response (FIR) filter $\mathbf{w} \in \mathbb{C}^{N_R N}$. In this case, the output of filter $r_f$ can be given by:

$$
r_f = \mathbf{w}^H \mathbf{r} = \alpha_0 \mathbf{w}^H \mathbf{U}(\theta_0) \mathbf{x} + \sum_{k=1}^{K} \alpha_k \mathbf{w}^H \mathbf{U}(\theta_k) \mathbf{x} + \mathbf{w}^H \mathbf{n} \tag{2.2}
$$

Therefore, the SINR can be expressed as:

$$
\text{SINR} = \frac{\sigma^2 |\mathbf{w}^H \mathbf{U}(\theta_0) \mathbf{x}|^2}{\mathbf{w}^H \Sigma(\mathbf{x}) \mathbf{w} + \mathbf{w}^H \mathbf{w}} \tag{2.3}
$$

where $\sigma = E[|\alpha_0|^2]/\sigma^2_n$ and

$$
\Sigma(\mathbf{x}) = \sum_{k=1}^{K} I_k \mathbf{U}(\theta_k) \mathbf{x} \mathbf{x}^H \mathbf{U}^H(\theta_k) \tag{2.4}
$$

where $I_k = E[|\alpha_k|^2]/\sigma^2_n$.
2.2.1 Problem Formulation

Our objective is to optimize the SINR in eq. (2.3) subject to the CMC and SC, i.e, solve the following optimization problem:

\[
\begin{align*}
\max_{w, x} & \quad \frac{\sigma^2 w^H U(\theta_0)x^2}{w^H \Sigma(x)w + w^H w} \\
\text{s.t.} & \quad ||x - x_0||_\infty \leq \epsilon \\
& \quad |x| = 1/\sqrt{N_T N}
\end{align*}
\]

(2.5)

The constant modulus constraints ($|x| = 1/\sqrt{N_T N}$) implies that $|x_m(n)| = 1/\sqrt{N_T N}$ for $m = 1, \ldots, N_T$ and $n = 0, \ldots, N-1$. It has been shown in [11] that the joint optimization problem of eq. (2.5) is equivalent to the following optimization problem:

\[
\begin{align*}
(P) \quad \max_{x} & \quad x^H \Phi(x) x \\
\text{s.t.} & \quad ||x - x_0||_\infty \leq \epsilon \\
& \quad |x| = 1/\sqrt{N_T N}
\end{align*}
\]

(2.6)

where $\Phi(x)$ is the SINR matrix. As shown in [11], the positive-semidefinite SINR matrix can be given as:

\[
\Phi(x) = U(\theta_0)^H [\Sigma(x) + I]^{-1} U(\theta_0)
\]

(2.7)

Note that the similarity constraint can be rewritten as:

\[
\arg x_l \in [\gamma_l, \gamma_l + \delta] \quad l = 1, 2, \ldots, N_T N
\]

(2.8)

where $x_l$ is the $l$-th element of $x$, $\gamma_l$ and $\delta$ are given by:

\[
\gamma_l = \arg x_{0l} - \arccos(1 - \epsilon^2/2)
\]

\[
\delta = 2 \arccos(1 - \epsilon^2/2)
\]

and $0 \leq \epsilon \leq 2$. If $\epsilon = 0$, the waveform $x$ will be identical to the reference waveform $x_0$. On the other hand, if $\epsilon = 2$, there will be no SC and the problem will have only a CMC.

In the existing literature [11, 23], the dependence of $\Phi(x)$ on the waveform $x$ has been resolved iteratively assuming $\Phi(x) = \Phi$ for a fixed $x$ and repeatedly optimizing $x$ with a new $\Phi$ till convergence. A key recent example is the sequential optimization algorithm (SOA) of Cui et al [11]. Nevertheless, even for a fixed $\Phi$, the optimization of $x$ presents a hard non-convex problem, most popular solutions of which involve SDR with randomization [11, 35, 36, 43].
2.3 Proposed Method: Successive QCQP Refinement

The optimization problem in (2.6) for signal independent clutter (i.e. \( \Phi(x) = \Phi \)) is equivalent to the following non-convex problem:

\[
\begin{align*}
(NC) \quad & \max_x \quad x^H(\Phi - \lambda I)x \\
& \text{s.t.:} \quad \arg x_l \in [\gamma_l, \gamma_l + \delta] \quad l = 1, 2, \ldots, N_T N \quad |x| = 1 / \sqrt{N_T N}
\end{align*}
\]

where \( \lambda \geq \lambda_{\text{max}}(\Phi) \) and \( \lambda_{\text{max}}(\Phi) \) is the largest eigenvalue of \( \Phi \) so that \( \Phi - \lambda I \) is negative-semidefinite. Since \( x^H(\Phi - \lambda I)x = x^H\Phi x - \lambda x^H x = x^H\Phi x - \lambda \). This implies that \( x^*(P) = x^*(NC) \).

The optimization problem in (2.9) can be relaxed to the following convex optimization problem \((CP)\):

\[
\begin{align*}
(CP) \quad & \max_x \quad x^H Q x \\
& \text{s.t.:} \quad |x|^2 \leq 1 / (N_T N) \quad l = 1, 2, \ldots, N_T N \\
a_l \text{Re}(x_l) + b_l \text{Im}(x_l) \geq c_l \quad l = 1, 2, \ldots, N_T N
\end{align*}
\]

where \( Q = (\Phi - \lambda I) \) while the parameters \( a_l, b_l \) and \( c_l \) represents the line that intersects with the constant modulus at the interval \( [\gamma_l, \gamma_l + \delta] \), as shown in Fig. 2.1 (a). This relaxation becomes closer to \((NC)\) as the value of \( \delta \) becomes smaller. For instance, if \( \delta = \frac{\pi}{2} \), then the feasible value of \( |x| \) lies between \( \frac{1}{\sqrt{2N_T N}} \) and \( \frac{1}{\sqrt{N_T N}} \), as shown in Fig. 2.1 (a). Therefore, this property can be used to make \( |x| \) approach \( \frac{1}{\sqrt{N_T N}} \) by iteratively reducing \( \delta \), as will be shown later in this section.

Furthermore, the problem \((CP)\) can be converted to the following problem with real variables [59]:

\[
\begin{align*}
\max_v \quad & v^T S v \\
\text{s.t.:} \quad v^T E_l v \leq 1 / (N_T N), \quad l = 1, 2, \ldots, N_T N \\
& A v \geq c
\end{align*}
\]

where:

\[
S = \begin{bmatrix}
\text{Re}\{Q\} & - \text{Im}\{Q\} \\
\text{Im}\{Q\} & \text{Re}\{Q\}
\end{bmatrix}
\]
\[ v = [\Re\{x^T\} \Im\{x^T\}]^T, \]

\[ A(i, j) = \begin{cases} a_l & \text{if } i = j = l, \\ b_l & \text{if } i = l, \text{ and } j = l + N_T N, \\ 0 & \text{Otherwise.} \end{cases} \]

\[ c = \frac{1}{\sqrt{N_T N}} [1 \ 1 \ ... \ 1]^T, \]

and

\[ E_l(i, j) = \begin{cases} 1 & \text{if } i = j = l, \\ 1 & \text{if } i = l + N_T N, \text{ and } j = l + N_T N, \\ 0 & \text{Otherwise.} \end{cases} \]

Problem (2.11) is a real convex Quadratically Constrained Quadratic Program (QCQP), which can be easily converted to Second Order Cone Program (SOCP) [60] [34] and solved efficiently [34, 61].

Next, we will present two algorithms to achieve a feasible solution of \( NC \) via successive refinements of (2.11). The first one –Successive QCQP Refinement- Binary Search (SQR-BS)– converges very fast while the second one –Successive QCQP Refinement- Non-Decreasing (SQR-ND)– converges to a relatively higher SINR and produces a non-decreasing sequence in each iteration if combined with the sequential algorithm in [11].


2.3.1 Successive QCQP Refinement- Binary Search (SQR-BS) Algorithm

Consider the following problem:

\[
(RC^{(n)}) \begin{cases} 
\max_v & v^T Sv \\
\text{s.t.:} & v^TE_l v \leq 1/(N_T N), \\
& l = 1, 2, \ldots, N_T N \\
& A_n v \preceq c \\
& v^{(n-1)T} P v \geq v^{(n-1)T} P v^{(n-1)}
\end{cases}
\]

where \(v^{(n-1)}\) is the optimal solution of \(RC^{(n-1)}\), \(S\) as defined in eq. (2.12) is negative-semidefinite, while \(P = S + \lambda I\) is positive definite. Note that, the SINR value of the \(n\)-th refinement is given as \(SINR^n = v^{(n)T} P v^{(n)}\).

At \(n = 0\), \(A_0\) is chosen such that \(\arg x_l \in [\gamma_l, \gamma_l + \delta]\) as follows:

\[
A_0(i, j) = \begin{cases} 
\frac{\cos(\arg x_{il})}{\sin(\arg x_{il})} & \text{if } i = j = l, l = 1, 2, \ldots, N_T N \\
\frac{\cos(\arg x_{il})}{\sin(\arg x_{il})} & \text{if } i = l, j = l + N_T N, \\
0 & \text{Otherwise.}
\end{cases}
\]

which represents the straight line in Fig. 2.2 (a). Denote the solution of \(RC^{(0)}\) by \(v^{(0)}\) and denote the complex solution of \(CP\) by \(x^{(0)}\). In this case, there will be two possibilities:

1. If \(\arg x_l^{(0)} \geq \gamma_l + \delta/2\), we set the new SC as \([\gamma_l + \delta/2, \gamma_l + \delta]\), i.e, the new constraint angles \(\gamma_l^{(1)} = \gamma_l + \delta/2\) and \(\delta^{(1)} = \delta/2\).

2. If \(\arg x_l^{(0)} < \gamma_l + \delta/2\), then we set the new SC as \([\gamma_l, \gamma_l + \delta/2]\), i.e, the new constraint angles \(\gamma_l^{(1)} = \gamma_l\) and \(\delta^{(1)} = \delta/2\).

In other words, the feasible SC interval is reduced to half according to the location of \(\arg x_l^{(0)}\). In the next refinement we solve \(RC^{(1)}\) same as problem \(RC^{(0)}\) but with the new SC \(([\gamma_l^{(1)}, \gamma_l^{(1)} + \delta^{(1)}])\). Continuing in the same fashion for \(RC^{(n)}, n = 2, 3, \ldots, F\), the interval \(([\gamma_l^{(n)}, \gamma_l^{(n)} + \delta^{(n)})]\) will get smaller and smaller \((\delta^{(n)} = \delta^{(n-1)}/2)\) and eventually the modulus of \(x_l^{(n)}\) will converge to one, as illustrated in Fig. 2.2. This is similar to a binary search for \(x_l^{(n)}\). Note that the following affine constraint:

\[
\cos(\gamma + \frac{\delta}{2}) \text{Re}(x_l) + \sin(\gamma + \frac{\delta}{2}) \text{Im}(x_l) \geq \frac{\cos(\delta/2)}{\sqrt{N_T N}}
\]

combined with the quadratic constraint: \(|x_l|^2 \leq 1/(N_T N)|\) enforces the angle of \(x_l\) to be in the interval \([\gamma_l, \gamma_l + \delta]\). Therefore, the angle \(\gamma_l + \delta/2\) defines the line rotation while \(\delta\) defines the width of the feasible interval. In the next refinement, the width of the feasible interval is reduced by half (i.e. \(\delta^{(n)} = \delta^{(n-1)}/2\)), hence, we have the following affine constraint at the \(n^{th}\) refinement:

\[
L_n(x) = \cos(\gamma_l^{(n)} + \delta/2^n) \text{Re}(x_l) + \]

16
Figure 2.2: Illustration of the successive approximation of problem (2.9): (a) The convex hull of problem (2.9) is the blue area. (b) The solution point of the convex problem (2.10) in red. Now, we consider only the upper half of the similarity constraint and solve again. (c) Sconced refinement (d) Third refinement, here solution in the third refinement is very close to unity.

\[
\sin(\gamma_l(n) + \delta/2^n) \Im(x_l) \geq \frac{\cos(\delta/2^n)}{\sqrt{N_T N}}
\]

where \(\gamma_l(n)\) is selected according to the two possibilities described earlier. Dividing \(L_n(x)\) by \(\cos(\delta/2^n)\) for \(\delta < \pi\), the general form of \(A_n\) is given by:

\[
A_n(i, j) = \begin{cases} 
\frac{\cos(\gamma_l(n) + \delta/2^n)}{\cos(\delta/2^n)} & \text{if } i = j = l, \\
\frac{\sin(\gamma_l(n) + \delta/2^n)}{\cos(\delta/2^n)} & \text{if } i = l, \text{ and } j = l + N_T N, \\
0 & \text{Otherwise.}
\end{cases}
\]  

(2.14)

The SQR algorithm for a fixed \(\Phi\) (under signal independent clutter) is given in Algorithm 1.

**Computational Complexity:** Based on the complexity of QCQP [60] in each refinement, the overall computational complexity of SQR is \(O(FN_T^{3.5} N_3^{3.5})\) where \(F\) is the total number of refinements. In comparison, SDR with randomization has a computational complexity of \(O(N_T^{3.5} N_3^{3.5}) + O(LN_T^2 N^2)\) [44]. The SQR algorithm hence typically has much lower complexity. As will be shown later, the number of required refinements \(F\) is independent of \(N_T N\) and in fact \(F << N_T N\). On the other hand, SDR with randomization invariably needs a large number of randomization trials \(L\) [11] which makes the term \(O(LN_T^{-2} N^2)\) much larger.
Algorithm 1: Successive QCQP Refinement- Binary Search (SQR-BS)

**Inputs:** $x_0$, $\Phi$, $\epsilon$, $F$ and $\zeta$ (the desired threshold value).

**Output:** A solution $x^*$ for (NC)

1. Set $n = 0$, $\gamma_k^{(0)} = \gamma_l$, and $\delta^{(0)} = \delta$.
2. Solve $RC(n)$ and get $v(n)$ (the complex version of $v(n)$).

(The computational complexity of this step is $O(N_T^3N^{3.5})$)

3. Compute $SINR^n = x^{(n)H}_n \Phi x^{(n)}$

if $SINR^n - SINR^{n-1} \leq \zeta$ or $n \geq F$ then

STOP

end if

(The computational complexity of this step is $O(N_T^2N^2)$ [62])

4. Do the following:

   for $k = 1, 2, ..., N_T N$ do

   change the similarity constraint:

   if $\arg x_l^{(n)} < \gamma_l^{(n)} + \delta^{(n)}/2$ then

   $\gamma_k^{(n+1)} = \gamma_l^{(n)}$

   else

   set $\gamma_k^{(n+1)} = \gamma_l^{(n)} + \delta^{(n)}/2$

   end if

   set $\delta_k^{(n+1)} = \delta^{(n)}/2$;

end for

5. Generate $A_{n+1}$ in eq. (2.14) base on the new values $\gamma^{(n+1)}$ and $\delta^{(n+1)}$.

(The computational complexity of this step is $O(N_T N)$)

6. Set $n = n + 1$ GOTO step (2)

**Output:** $x^* = 1/\sqrt{N_T N} \exp(j \arg x^{(n)})$

For $\epsilon \geq \sqrt{2}$ (or equivalent for $\delta \geq \pi$), the linear constraint that defines $A_0$ is given as:

$$\cos(\arg x_{00}) \Re(x_l) + \sin(\arg x_{00}) \Im(x_l) \geq \frac{\cos(\delta/2)}{\sqrt{N_T N}}$$

In this case, $\cos(\delta/2)/\sqrt{N_T N} \leq 0$. Therefore, $x = 0$ is a feasible point, as shown in Fig. 2.1 (b). Equivalently, $v = 0$ is also a feasible point where $v$ is the real version of $x$. Since $S$ in negative definite, the optimal value of $\max_v v^T S v = 0$ and the optimal solution is always $v^* = 0$. To avoid this, the affine constraints in the problem as well as Algorithms 1 and 2 must be slightly modified, while still retaining the sequence of QCQP structure, for the case of $\delta \geq \pi$ or $\epsilon \geq \sqrt{2}$. For more details see [63].

### 2.3.2 Convergence Analysis of SQR-BS

The SINR of the SQR algorithm is non-decreasing with each refinement, unlike the SOA1 algorithm found in [11,44]. To this end, we can prove that:

**Lemma 2.3.1.** Let $v^{(n-1)}$ and $v^{(n)}$ be the optimal solutions of $RC^{(n-1)}$ and $RC^{(n)}$, respectively. then: $SINR^{n-1} = v^{(n-1)T} P v^{(n-1)} \leq v^{(n)T} P v^{(n)} = SINR^n$. In other words,
the sequence \( \{\text{SINR}^n\}_{n=0}^\infty \) is non-decreasing.

Proof. See Section 1.1 of the Appendix.

Based on Lemma 2.3.1, the SQR algorithm improves the SINR after each refinement. Moreover, the sequence SINR\(^n\) converges as shown in the following lemma:

**Lemma 2.3.2.** The sequence SINR\(^n\) converges to a finite value SINR\(^*\).

Proof. See Section 1.2 of the Appendix.

Since SINR of the SQR algorithm is non-decreasing and converges to a fixed value, it is important to have some insight about the rate of convergence. In the following, the maximum improvement from one refinement to another is evaluated. This in turn can be used to set a predefined stopping criterion or equivalently determine the number of refinements in advance. Define the maximum improvement \( i(n) \) of SINR\(^n\) over SINR\(^{n-1}\) as:

\[
i(n) = \sup \{\text{SINR}^n - \text{SINR}^{n-1}\}
\]

Let \( x^{(n-1)} \) and \( x^{(n)} \) be the complex versions of \( v^{(n-1)} \) and \( v^{(n)} \), respectively, i.e. \( v^{(n)} = [\text{Re}(x^{(n)}), \text{Im}(x^{(n)})]^T \). Let \( \bar{x} = x^{(n)} - x^{(n-1)} \). Then:

\[
\text{SINR}^n - \text{SINR}^{n-1} = x^{(n)} H \Phi x^{(n)} - x^{(n-1)} H \Phi x^{(n-1)} \\
= x^{(n-1)} H \Phi \bar{x} + \bar{x} H \Phi x^{(n-1)} + \bar{x} H \Phi \bar{x} \\
\leq 2 \sqrt{x^{(n-1)} H \Phi x^{(n-1)}} ||\bar{x}||_2 + ||\bar{x}||_2 \\
\Rightarrow i(n) = \lambda_{\text{max}}(2||x^{(n-1)}||_2 ||\bar{x}||_2 + ||\bar{x}||_2^2)
\]

where the first inequality holds from the Cauchy-Schwarz theorem and \( \lambda_{\text{max}} \) is the largest eigenvalue of \( \Phi \). The first equality holds if and only if \( x^{(n-1)} = a \bar{x} \) where \( a \) is a scalar.

Therefore, \( i(n) \) can be rewritten as:

\[
i(n) = \lambda_{\text{max}}(2a + 1)||\bar{x}||_2^2
\]

For \( \delta \leq \pi/2 \), \( ||\bar{x}_l|| \) can achieve its maximum value if \( |x_l^{(n)}| \) is at its maximum feasible value and \( |x_l^{(n-1)}| \) is at its minimum feasible value, as illustrated in Fig. 2.3. In this case:

\[
\max_{x^{(n-1)} = a \bar{x}} ||\bar{x}_l|| = \max\{|x_l^{(n)}| - |x_l^{(n-1)}|\} \\
\leq \max\{|x_l^{(n)}|\} - \min\{|x_l^{(n-1)}|\} \\
= \frac{1}{\sqrt{NTN}}(1 - \beta(n))
\]

where \( \beta(n) = \cos \left( \frac{\pi}{2(n+1)} \right) \) is the minimum value of \( |x_l^{(n-1)}| \) at the \( n \)-th refinement.

Therefore,

\[
\max ||\bar{x}||_2^2 = (1 - \beta(n))^2
\]
In this case $a = \max|\bar{x}_l|/\min|x_l^{(n-1)}| = \beta(n)/(1 - \beta(n))$ which implies that:

$$i(n) = \lambda_{max}(1 - \beta^2(n))$$

To get an insight towards the maximum increase in each refinement, for the case of $\lambda_{max} = 1$, Fig. 2.4 plots $i(n)$. This shows that after five refinements, the maximum increment is less than 0.24% of the maximum achievable value. The algorithm can hence be programmed to stop after a fixed number of refinements. Therefore, the maximum improvement shows how fast SQR-BS converges and it can be used to calculate the maximum number of refinements needed prior to SQR-BS execution. For example, if the stopping criterion is such that $SINR^n - SINR^{n-1} \leq \zeta$ where $\zeta$ is a threshold value, then the maximum number of refinements is $n'$ such that $i(n') \leq \zeta$.

### 2.3.3 SQR-BS Algorithm Under Signal Dependent Clutter

Under signal dependent clutter sources, the SINR matrix is a function of $x$ as given in (2.7). This problem can be solved using a sequential algorithm developed by [10, 11] where in the first step the SINR matrix is computed using an initial waveform $x_0$. In the second step, a problem similar to (2.9) is formulated and solved using SDR with randomization then the solution $x^*$ is used to update SINR matrix for the next optimization procedure until the solution converges. The same sequential algorithm procedure yields our extension of the SQR-BS algorithm to the case of signal dependent clutter, as summarized in Algorithm 2. Note that, we use $x_n$ in Algorithm 2 to denote to the solution of the $n$-th iteration.
Algorithm 2 Sequential SQR-BS

Inputs: $x_0$, $\theta_i$ and $\alpha_i$ for $i = 0, 1, 2, \ldots K$
Output: A solution $x^*$ for $(P)$

1. Set $n = 1$, $\gamma_k^{(0)} = \gamma_l$, and $\delta^{(0)} = \delta$.
2. Compute:
   \[
   \Phi(x_{n-1}) = U(\theta_0)^H \left[ \Sigma(x_{n-1}) + I \right]^{-1} U(\theta_0)
   \]
   \[
   \Sigma(x_{n-1}) = \sum_{k=1}^{K} I_k U(\theta_k)x_{n-1}x_{n-1}^H U^H(\theta_k)
   \]
   (The computational complexity of this step is $O(N^{-3}N^3)$ [11])
3. Use Alg. 1 with $\Phi = \Phi(x_{n-1})$ as the SINR matrix and $x_0(Alg.1) = x_0$ set $x_n = x^*(Alg.1)$.
   (The computational complexity of this step is $O(N^{-3.5}N^3.5)$)
4. Compute $SINR^n = x_n^H \Phi(x_n)x_n$
   if $SINR^n - SINR^{n-1} \leq \epsilon$ or $n \geq N$ then
     STOP
   end if
   (The computational complexity of this step is $O(N^{-2}N^2)$)
5. Set $n = n + 1$ GOTO step (2)
Output: $x^* = 1/\sqrt{NTN} \exp(j \arg x_n)$

2.3.4 Successive QCQP Refinement- Non-Decreasing (SQR-ND) Algorithm

Although Algorithm 2 is a fast and efficient algorithm, the combination of the sequential method with Algorithm 1 does not guarantee anymore a non-decreasing SINR. Therefore, we propose an adjustment to the refinement part of SQR-BS which guarantees a non-decreasing SINR even if it is combined with the sequential method. Furthermore, another advantage of this new modification is an easier calculation for the feasible region when introducing an interference constraint as will be shown later in sec. 2.3.6. The new algorithm is called SQR Non-Decreasing (SQR-ND) and is given in Algorithm 3.

The affine constraint in the $n^{th}$ refinement is given by:

\[
A_n(i, j) = \begin{cases} 
\frac{\cos(\arg x_l^{(n-1)})}{\cos(\delta/2)} & \text{if } i = j = l, \\
\frac{\sin(\arg x_l^{(n-1)})}{\cos(\delta/2)} & \text{if } i = l, \text{ and } j = l + NTN, \\
0 & \text{Otherwise.}
\end{cases}
\]

with $x^{(0)}$ being any feasible point and $x^{(n-1)}$ is the complex version of $v^{(n-1)}$. In this algorithm, the set of affine constraints $A_n v$ rotate according to $v^{(n-1)}$, as shown in Fig. 5 (a) and (b), which may violate the SC. Therefore, two sets of affine constraints has been introduced to ensure SC is satisfied. The matrices for these additional constraints are given as:
Finally, the refinement optimization problem is rewritten as:

\[
\max_v \quad v^T S v \\
\text{s.t.:} \quad \begin{align*}
v^T E_l v &\leq 1/(N_T N), \\
A_n v &\succeq c \\
B_+ v &\preceq 0 \\
B_- v &\preceq 0
\end{align*}
\]  \tag{2.15}

In particular, let \(x^{(n-1)}\) be the complex version of the optimal solution to \(RCND^{(n-1)}\), i.e. \(v^{(n-1)} = [\text{Re}(x^{(n-1)T}) \text{Im}(x^{(n-1)T})]^T\), the SQR-ND Algorithm 3 adjusts the affine constraints so that the feasible set of the next refinement will include what we call as the constant modulus version of \(x^{(n-1)}\) given by \(x^{(n-1)} = \exp(j \arg(x^{(n-1)})))/\sqrt{N_T N}\), as illustrated in Fig. 2.5. In fact, the feasible set also includes \(\cos(\delta/2)x^{(n-1)}\) which is the constant modulus version of \(x^{(n-1)}\) with the smallest magnitude in the feasible set. If \(x^{(n)} = \cos(\delta/2)x^{(n-1)}\), then the constraints of the next problem \(RCND^{(n+1)}\) are the same as problem \(RCND^{(n)}\) which means \(x^{(n+1)} = x^{(n)}\) and, hence, the algorithm converges. Otherwise, the feasible set is adapted to include the constant modulus versions of \(x^{(n)}\). Convergence is then guaranteed by Lemma 2.3.3 which establishes that the SINR sequence that results by using the solution of each refinement, is in fact non-decreasing and converges.

Under signal dependent clutter sources, the same sequential algorithm procedure yields our extension of the SQR-ND, as summarized in Algorithm 4.

**Remark:** Note that, in Algorithm 3, the variable \(n\) denotes the \(n\)-th refinement and \(x^{(n)}\) is the optimal solution of \(RCND^{(n)}\). Whereas, in Algorithm 4, \(n\) denotes the \(n\)-th iteration and \(x_n\) is final result of Algorithm 3, i.e. the optimized waveform at the \(n\)-th iteration.

### 2.3.5 Convergence Analysis of SQR-ND

**Lemma 2.3.3.** Let \(x^{(n-1)}\) and \(x^{(n)}\) be the complex version of the optimal solutions \(v^{(n-1)}\) and \(v^{(n)}\), respectively, where \(n\) is the refinement index. Define \(\text{SINR}^n = x^{(n)H}Qx^{(n)}\) then: \(\text{SINR}^{n-1} = x^{(n-1)H}Qx^{(n-1)} \leq x^{(n)H}Qx^{(n)} = \text{SINR}^n\). In other words, the sequence \(\{\text{SINR}^n\}_{n=0}^{\infty}\) is non-decreasing over refinements.

**Proof.** See Section 1.3 of the Appendix. \(
\)

**Lemma 2.3.4.** The sequence \(\text{SINR}^n\) defined in Algorithm 3 converges to a finite value \(\text{SINR}^*\).

**Proof.** See Section 1.4 of the Appendix. \(
\)
Figure 2.5: Illustration of the successive approximation of problem (2.15): (a) The convex hull of problem $RCND^{(0)}$ is the blue area and the solution is marked in red. (b) Now, we rotate the line to make the new solution at the closest point. (c) the solution of the second refinement (d) Again we rotate the line, if the solution is found at the minimum distance from origin for all the point, then the algorithm converge.

Lemma 2.3.5. If the solution in Algorithm 3 converges to constant modulus, then the sequence $\text{SINR}^n$ defined in Algorithm 4 is non-decreasing over iterations and converges.

Proof. See Section 1.5 of the Appendix.

Remark: Contrasting SQR-BS with SQR-ND, while SQR-ND offers convergence via non-decreasing SINR even under the sequential updates of $\mathbf{x}$, a practical benefit of SQR-BS is faster convergence over the refinements. This is verified in Section 2.3.2.

2.3.6 Waveform design with Spectral Interference Constraint

2.3.6.1 Interference Constraint Formulation

The problem of spectral co-existence has been of great interest recently [7,26–32], and involves minimization of interference caused by radar transmission at victim communication receivers in the same frequency band. For the MIMO radar spectral coexistence, we assume that the $k$-th licensed radiator operates on a frequency band $B_k = [f_{1k}, f_{2k}]$, where $f_{1k}$ and $f_{2k}$ are the lower and upper normalized frequency. The amount of interfering
Algorithm 3 Successive QCQP Refinement - Non-Decreasing (SQR-ND)

**Inputs:** $x_0, x^{(0)}$ (optional), $\Phi$, $\epsilon$, $F$ and $\zeta$.
**Output:** A solution $x^*$ for ($NC$)

1. Set $n = 0$ and generate $B_\pm$. If $x^{(0)}$ is not specified, set $x^{(0)} = x_0$.
2. Solve $R$CND$^n$ and get $x^{(n)}$.
   (The computational complexity of this step is $O(N_T^{3.5} N^{3.5})$)
3. Compute $SINR^n = x^{(n)H}Qx^{(n)}$, where $Q = \Phi - \lambda I$ as in (2.10)
   if $SINR^n - SINR^{n-1} \leq \zeta$ or $n \geq F$ then
     STOP
   end if
   (The computational complexity of this step is $O(N_T^2 N^2)$)
5. Generate $A_{n+1}$ based on $x^{(n)}$.
   (The computational complexity of this step is $O(N_T N^2)$)
6. Set $n = n + 1$ GOTO step (2)

Output: $x^* = 1/\sqrt{N_T N} \exp(j \arg x^{(n)})$

Algorithm 4 Sequential SQR-ND

**Inputs:** $x_0$, $\epsilon$, $\zeta$, $\theta_i$ and $\alpha_i$ for $i = 0, 1, 2, \ldots K$
**Output:** A solution $x^*$ for ($P$)

1. Set $n = 1$, $\gamma^{(0)} = \gamma_l$, and $\delta^{(0)} = \delta$.
2. Compute:
   $$
   \Phi(x_{n-1}) = U(\theta_0)^H[\Sigma(x_{n-1}) + I]^{-1}U(\theta_0)
   $$
   $$
   \Sigma(x_{n-1}) = \sum_{k=1}^{K} I_k U(\theta_k) x_{n-1} x_{n-1}^H U^H(\theta_k)
   $$
   (The computational complexity of this step is $O(N_T^3 N^3)$ [11])
3. Use Alg. 3 with $\Phi = \Phi(x_{n-1})$ as the SINR matrix, $x_0 (Alg.3) = x_0$, $x^{(0)} (Alg.3) = x_{n-1}$ and set $x_n = x^*(Alg.3)$.
   (The computational complexity of this step is $O(N_T^{3.5} N^{3.5})$)
4. Compute $SINR^n = x_n^H\Phi(x_n)x_n$
   if $SINR^n - SINR^{n-1} \leq \zeta$ or $n \geq T$ (largest number of iterations) then
     STOP
   else
     Set $n = n + 1$ GOTO step (2)
   end if
   (The computational complexity of this step is $O(N_T^2 N^2)$)

Output: $x^* = 1/\sqrt{N_T N} \exp(j \arg x_n)$
energy due to the \( k \)-th licensed radiator \( E_l \) is given by:

\[
E_l = \int_{f_1^k}^{f_2^k} \sum_{i=1}^{N} a_i(\psi_k)^T x \{i\} e^{-j2\pi f_i} \, df
= x^H U_t(\psi_k)^H R_I^k U_t(\psi_k) x
\]  

(2.16)

where \( a_i(\psi_k) \) is the \( N_T \times 1 \) steering vector \([11]\), \( U_t(\psi_k) = I_N \otimes a_i(\psi_k)^T \), \( \psi_k \) is the angle of the \( k \)-th radiator and \( R_I^k \) is the \( N \times N \) co-existence matrix found in \([26]\) to MIMO systems and is given by:

\[
R_I^k(m,l) = \begin{cases} 
    f_2^k - f_1^k & \text{if } m = l \\
    \frac{e^{j2\pi f_2^k (m-l)} - e^{j2\pi f_1^k (m-l)}}{j2\pi(m-l)} & \text{if } m \neq l
\end{cases}
\]  

(2.17)

For \( K \) radiators, the total amount of energy is basically the sum of the coexistence matrices. However, since some of the radiators might be more important than the others or located at a closer distance, the final amount of energy \( R_I \) can be computed as a weighted sum:

\[
R_I = \sum_{i=1}^{K} w_k U_t(\psi_k)^H R_I^k U_t(\psi_k)
\]  

(2.18)

where the non-negative weights \( w_k \), \( k = 1, \ldots, K \) capture the relative importance of radiators. In this case, the SINR matrix is given by:

\[
\Phi(x) = U(\theta_0)^H [M + \Sigma(x) + I]^{-1} U(\theta_0)
\]  

(2.19)

where \( M \) is a function of \( R_I \).

### 2.3.6.2 SQR-ND with Interference Constraint

A major advantage of our proposed convex formulation in (2.13) and (2.15), is the simplicity of adding a new convex interference constraint. However, since the algorithm must be solved in several refinements, we need to ensure that the feasible set is not empty before each refinement. Nevertheless, in the case of interference constraint, we have a quadratic constraint that is shaped by \( R_I \) as defined in (2.18). Fortunately, using (2.15), we can compute the feasibility of the problem once and ensure that the problem will be feasible in each refinement. First, the lowest interference energy value achievable by the SQR-ND method can be evaluated by solving the following optimization problem:

\[
(RCF^{(n)}) \begin{cases} 
    \min_v \quad v^T C_I v \\
    \text{s.t.:} \quad v^T E_l v \leq 1/(N_T N), \\
    l = 1, 2, \ldots, N_T N \\
    \quad A_n v \geq c \\
    \quad B_v^+ v \geq 0 \\
    \quad B_v^- v \leq 0
\end{cases}
\]  

(2.20)
where:
\[
C_I = \begin{bmatrix}
\text{Re}\{R_I\} & -\text{Im}\{R_I\} \\
\text{Im}\{R_I\} & \text{Re}\{R_I\}
\end{bmatrix}.
\]

Problem (2.20) is a minimization problem and since \(C_I\) is positive semi-definite, therefore, it is a convex problem and it can be solved using an approach similar to Algorithm 3. At convergence, the solution will have a constant modulus and the minimum achievable value is \(m_{E_I} = v^* C_I v^* = x^T R_I x^*\) where \(x^*\) is the complex version of \(v^*\). After this, our final problem can be formulated as:

\[
(\text{RCI}^{(n)}) \left\{ \begin{array}{l}
\text{max}_v v^T S v \\
\text{s.t.:} \\
v^T E_l v \leq 1/(N_T N), \quad l = 1, 2, ..., N_T N \\
A_n v \succeq c \\
B_+ v \preceq 0 \\
B_- v \preceq 0 \\
v^T C_I v \leq E_I \cos(\delta/2)
\end{array} \right. \tag{2.21}
\]

where \(E_I\) is the desired maximum spectral interference energy. Problem \(\text{RCI}^{(n)}\) is feasible for any value of \(E_I \geq m_{E_I}\) and \(\delta < \pi\). The cosine scaling is introduced due to the fact that SQR-ND converges to a scaled constant modulus (\(|x_l| = \cos(\delta/2)/\sqrt{N_T N}\) for all \(k\), as shown in 2.5 (d). Therefore, the final solution is \(x^* = x/\cos(\delta/2)\). As a result, the true spectral interference constraint is given by:

\[
(x^T / \cos(\delta/2)) R_I (x/ \cos(\delta/2)) \leq E_I
\]

or \(x^T R_I x \leq E_I \cos^2(\delta/2)\) as in eq. (2.21). To ensure that the problem in (2.21) is always feasible, we should adjust \(A_0\) to include the point \(\cos(\delta/2)x^*(RCF)\) by making \(x^{(0)} = x^*(RCF)\). In this case, if \(E_I = m_I\) (the minimum achievable interference energy), then the feasible point \(\cos(\delta/2)x^*(RCF)\) is guaranteed to satisfy the interference constraint. Finally, the SQR algorithm for the spectral interference constraint is similar to Algorithm 3 and Algorithm 4 with the replacement of problem \(\text{RCND}^{(n)}\) by \(\text{RCI}^{(n)}\) and we call it SQR-ND with Interference constraint (SQR-ND-Int).
2.4 Numerical Results

2.4.1 Experimental Setup

In this section, we examine the performance of the proposed waveform design and compare it to competing methods using numerical simulations. The number of transmit and receive antennas are $N_T = 4$ and $N_R = 8$ elements, respectively. For the reference signal $x_0$, we considered the orthogonal linear frequency modulation (LFM) waveform. It can be defined by the space-time waveform matrix:

$$X_0(k,n) = \exp\left\{j2\pi k(n-1)/N\right\} \exp\left\{j\pi(n-1)^2/N\right\} \sqrt{N_T}$$

where $k = 1, ..., N_T$ and $n = 1, ..., N$. The reference waveform vector $x_0$ is the generated by stacking the column of $X_0$. In section 2.4.2, we compare our algorithms (both SQR-BS and SQR-ND) to SDR with randomization method for a fixed SINR matrix $\Phi$, i.e. absence of signal dependent clutter. Then we compare our algorithm to the Sequential Optimization Algorithm 1 and 2 (SOA1 and SOA2), i.e. the sequential SDR and randomization approach of [11]. In both these scenarios, the number of randomization trials used was 20000, as in [11] and our proposed SQR algorithms involved four refinement steps, i.e. $F = 4$. The noise variance is $\sigma_n = 0$ dB. In section 2.4.4, we introduce a spectral interference constraint and compare our method to the recently developed CRCO in [7].

2.4.2 Waveform design with Constant Modulus and Similarity Constraint

In the absence of signal dependent clutter, the SINR matrix $\Phi$ is constant and independent of the waveform signal. Our set up involves a target located at an angle $\theta_0 = 15^\circ$ with a reflecting power of $|\alpha_0|^2 = 10$ dB. Note Algorithm 1 and Algorithm 3 as detailed in Sections 2.3.1 and 2.3.4 are used for SQR-BS and SQR-ND results respectively.

Figure 2.6 shows the SINR behavior of the output waveform versus $\epsilon$ the similarity constraint parameter – see (2.6). Clearly, the proposed SQR-BS and SQR-ND outperform the SDR with randomization and the performance gap increases as $\epsilon$ increases. The SQR-BS has higher SINR values particularly when the similarity constraint is strong, i.e. lower values of $\epsilon$, even with one or two refinements. This is verified in Fig. 2.7. In particular, Fig. 2.7 further reveals that SQR-BS with two refinements achieves better SINR than SDR with randomization for $\epsilon \leq 0.88$ and for $\epsilon \leq 0.43$ in case of one refinement. SQR-ND is always better than SDR with randomization for $F = 2$ refinements. Examining the performance of SQR-BS and SQR-ND for varying $F$ is insightful from a complexity standpoint. Recall, the complexity of SQR-BS and SQR-ND is $O(FN_T^{3.5}N^{3.5})$, note that SDR with randomization requires $O(LN^2N_T^2)$ more computations than the one refinement of SQR-BS. Moreover, often in practice, the number of randomization trials $L$ is much greater than $N^{1.5}N_T^{1.5}$, the computational...
Figure 2.6: The SINR of the optimal waveform design versus the similarity parameter $\epsilon$ for constant SINR matrix (signal independent clutter). SQR-BS and SQR-ND are compared against SDR with randomization [4,11].

Figure 2.7: The SINR of the optimal waveform design versus the similarity parameter $\epsilon$ for constant SINR matrix (no signal dependent clutter). SQR-BS and SQR-ND are compared against the SDR with randomization.

complexity of the SDR with randomization could in fact exceed that of SQR-ND and SQR-BS even with $F = 4$ refinements.

2.4.3 Waveform design with Constant Modulus and Similarity Constraint Under Signal Dependent Clutter

We now compare the performance of the proposed SQR algorithms against an extension of SDR with randomization called the SOA1 and SOA2 algorithms [11,44], in the presence of clutter. For consistency with the results reported in [11], all parameter choices are as in [11]. The target is located at an angle $\theta_0 = 15^\circ$ with a reflecting power of $|\alpha_0|^2 = 20$
dB and three fixed interferences located at $\theta_1 = -50^\circ$, $\theta_2 = -10^\circ$ and $\theta_3 = 40^\circ$ reflecting a power of $|\alpha_1|^2 = |\alpha_2|^2 = |\alpha_3|^2 = 30$ dB. Note Algorithm 2 and Algorithm 4 as detailed in Sections 2.3.3 and 2.3.5 are used for SQR-BS and SQR-ND results respectively. Note also, that henceforth the term iteration (unless specified otherwise explicitly), refers to a sequential iteration as in SOA1 [11] and Algorithm(s) 2 and 4.

Fig. 2.8 (a) shows the SINR improvement in each iteration for the SOA1, SOA2 and the sequential SQR-BS/ND with $\epsilon$ equals to 0.5. It is evident from Fig. 2.8 (a), both the SQR-BS and SQR-ND achieve an SINR 1.25 dB higher than SOA1 for the same CMC and SC. Note also that Fig. 2.8 also shows the SINR performance of SQR-BS with only CMC and no SC, i.e. $\epsilon = 2$. As expected, this forms a practical upper bound for the SINR achieved in the presence of a similarity constraint, i.e. $\epsilon < 2$. To show the suppression capability of the resulting waveform, the beam pattern is shown in Fig. 2.8 (b). The SQR optimized beam pattern resulting from the proposed algorithms exhibits much better suppression performance at $\theta = -50^\circ$ and $\theta = -10^\circ$ when compared to SOA1.

Table 2.1 shows the computational complexity and the simulation time of the different methods.

<table>
<thead>
<tr>
<th>Method</th>
<th>Order of Complexity</th>
<th>Sim. Time (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SOA1 [11]</td>
<td>$O(N_T^{3.5}N^{3.5}) + O(LN_T^2N^2)$</td>
<td>11.2 s</td>
</tr>
<tr>
<td>SOA2 [11]</td>
<td>$O(N_T^{3.5}N^{3.5}) + O(LN_T^2N^2)$</td>
<td>10 s</td>
</tr>
<tr>
<td>SQR-BS</td>
<td>$O(FN_T^{3.5}N^{3.5})$</td>
<td>6.3 s</td>
</tr>
<tr>
<td>SQR-ND</td>
<td>$O(FN_T^{3.5}N^{3.5})$</td>
<td>7.4 s</td>
</tr>
</tbody>
</table>

Results of a similar experiment are shown in Fig. 2.9 and Fig. 2.10 but with $\epsilon = 0.7$ and $\epsilon = 1.2$, respectively. As expected overall SINR values in Fig. 2.9 are higher than the
Figure 2.9: (a) The SINR of the optimal waveform design in each iteration and in (b) the beampattern $P(\theta)$ for SOA1, SOA2, SQR-BS and SQR-ND algorithms with $\epsilon = 0.7$ for all of them.

Figure 2.10: (a) The SINR of the optimal waveform design in each iteration and in (b) the beampattern $P(\theta)$ for SOA1, SOA2, SQR-BS and SQR-ND algorithms with $\epsilon = 1.2$ for all of them.

case of $\epsilon = 0.5$ owing to a softer similarity constraint. Note however, that SQR-BS and SQR-ND achieve a gain of 1.59 dB and 1.71 dB over the SOA1 algorithm, respectively.

Fig. 2.11 shows a demanding scenario where the target is located very close to one of the clutter sources. Namely, we set $\theta_0 = 35^\circ$ close to the third clutter source $\theta_3 = 40^\circ$. In this case, the SINR as expected drops dramatically as shown in Fig. 2.11. However, the two proposed SQR algorithms still perform better with about 0.9 dB gain for SQR-ND and 0.5 dB for SQR-BS. Note that in Fig. 2.11, the SINR value for SQR-ND is non-decreasing with each iteration unlike SQR-BS and SOA1, which do not offer such a guarantee. Note that, since the clutter scatters are not widely separated in space in
Figure 2.11: (a) The SINR of the optimal waveform design in each iteration and in (b) the beampattern $P(\theta)$ for SOA1, SOA2, SQR-BS and SQR-ND algorithms with $\epsilon = 0.7$ for all of them.

Figure 2.12: The converged SINR values of SQR-BS and SOA1 vs. $\epsilon$

In this case, the rejection performance of MIMO radar through waveform design is limited compared with regular SIMO/SISO radar.

A plot of the final SINR value (at convergence) vs. the similarity constraint parameter $\epsilon$ for sequential SQR-BS and SOA1 is shown in Fig. 2.12. Remarkably, the SOA1 increases approximately linearly with $\epsilon$ while the SQR-BS exhibits a superlinear increase. Note that for $\epsilon > 0.9$ the SQR-BS is used with the modifications described earlier in section 2.3.3.

Finally, the output waveform pulse compression for each algorithm is compared with the LFM reference signal as shown in Fig. 2.13. This is the waveform signal of the first antenna. The pulse compression was computed using the same procedure as in [11,64]. As shown in Fig. 2.13, the side lobes increases with $\epsilon$ and, therefore, there is a trade-off between better SINR and low side lobes. It should be noted also that the SINR values of
SQR-ND algorithm are greater than SOA1 by 0.7 dB, 1.26 dB and 1.85 dB for $\epsilon = 0.2$, 0.5, and 1.2, respectively. Likewise, SQR-BS algorithm exceeds SINR achieved via SOA1 by 0.5 dB, 1.15 dB and 1.22 dB for $\epsilon = 0.2$, 0.5, and 1.2, respectively.

### 2.4.4 Waveform design with SC, CMC and Spectral Interference Constraint

First, we will compare our method with only CMC to a state of the art recent method that uses an energy constraint: Algorithm 1 for Cognitive Radar Code Optimization (CRCO1) found in [7] for SISO case $N_T = N_R = 1$ and no signal dependent interference to obtain some insight about how much loss is incurred by incorporating the CMC. In this case, we use the following SINR matrix:

$$\Phi = \sigma_0 I + \sum_{k=1}^{K_J} \frac{\sigma_{J,k}}{\Delta f^k_J} R^k_J$$

and

$$R_I = \sum_{k=1}^{K} w_k R^k_I$$

where $\sigma_0 = 0$ dB is the noise level, $K_J = 2$ is the number of active unlicensed radiators or jammers, $\sigma_{J,k} = 50$ dB for $k = 1, 2$ is the energy of the $k$-th active unlicensed radiator using the normalized frequency band $B^k_J = [f^k_{J,1}, f^k_{J,2}]$ and $\Delta f^k_J = f^k_{J,2} - f^k_{J,1}$ is the bandwidth used by the $k$-th coexisting radiator. Furthermore, the normalized frequency band of the first and second unlicensed radiator are $B^1_J = [0.2, 0.22]$ and $B^2_J = [0.6, 0.635]$, respectively. For $R_I$ we used $B^1 = [f^1_1, f^1_2] = [0.05, 0.08]$, $B^2 = [f^2_1, f^2_2] = [0.4, 0.435]$, $w_1 = w_2 = 1$ and $E_I = 0.005$. Fig. 2.14 shows the the Energy Spectral Density (ESD) of the proposed algorithm SQR-ND-Int with CMC against CRCO1 as well as the reference LFM waveform. Since CRCO1 does not enforce CMC, it manages to reduce the interference level in the unlicensed band (shown between the gray vertical lines) by a larger amount although both SQR-ND-Int and CRCO1 achieve the same interference levels in the licensed band. The SINR values corresponding to different waveforms are -0.0007 dB, -0.0262 dB and -0.5122 dB for the CRCO1, SQR-ND-Int and the LFM waveforms, respectively. In summary, a relatively small loss (against CRCO1) in SINR value is seen via the proposed SQR-ND-Int even as CMC is captured simultaneously with interference constraint and in an analytically tractable manner.

We now present results for the MIMO scenario. Moreover, we also include the signal dependent clutter, therefore, the signal waveform design should reduce both the spectral coexistence interference and reject clutter. In this simulation, we assumed a LFM pulse with $N = 120$ and MIMO antennas with $N_T = 2$ and $N_R = 4$. We assume three clutter interferences with the same power and amplitudes as in Section 2.4.3. The covariance matrix $M$ of the signal-independent interference has been modeled as the MIMO generalization of [26] as:

$$M = \sum_{k=1}^{K} \frac{\sigma_{I,k}}{\Delta f^k_I} U(\psi_k) R^k_I U^H(\psi_k)$$  \hspace{1cm} (2.23)
Figure 2.13: The pulse compression profile of the output waveform form SOA1, SQR-BS, and SQR-ND compared to the LFM reference signal. (a) For $\epsilon = 0.2$ (b) $\epsilon = 0.5$ (c) $\epsilon = 1.2$
Figure 2.14: The plot ESD in dB for: 1) reference waveform $x_0$ in black dashed line, 2) the output waveform $x^*$ using CRCO1 in blue dotted line, 3) the output waveform using SQR-ND-Int in red.

where $\psi_k$ is the angle of the $k$-th radiator, $K = 2$ is the number of licensed radiators, $\psi_1 = 30^\circ$ while $\psi_1 = -20^\circ$, $\sigma_{1,k} = 10$ dB for $k = 1, 2$ is the energy of the $k$-th coexisting radiator using the normalized frequency band $B^k = [f^k_1, f^k_2]$ and $\Delta f_k = f^k_2 - f^k_1$ is the bandwidth used by the $k$-th coexisting radiator. Furthermore, the normalized frequency band of the first and second coexisting radiator are $B^1 = [0.05, 0.08]$ and $B^2 = [0.4, 0.435]$, respectively.

In Fig. 2.15, the ESD of the waveform optimized via SQR-ND-Int (corresponding to the first antenna) is shown for different values of $\epsilon$. Also plotted in Fig. 2.15 for comparison, is the unoptimized reference waveform. As expected, the optimized waveforms from the proposed algorithm reduce the interference in the specified frequency bands. Moreover, the stop-band reduction can be controlled by increasing the value of $\epsilon$ as it gives the algorithm more degree of freedom, however, this will trade-off the similarity of the output waveform to the reference signal.
Figure 2.15: The plot ESD in dB for: 1) reference waveform $x_0$ in black dashed line, 2) the output waveform $x^*$ for $\epsilon = 0.5$ in blue dotted line, 3) the output waveform $x^*$ for $\epsilon = 0.6$ in green, 4) the output waveform $x^*$ for $\epsilon = 0.7$ in red.
2.5 Conclusion

Our work achieves tractable waveform design for MIMO radar in the presence of constant modulus and similarity constraints. The central idea of our analytical contribution is to successively refine and achieve constant modulus (at convergence), while solving a sequence of quadratically constrained quadratic programs, such that each optimization in the sequence satisfies a similarity constraint. We show this approach called SQR can achieve superior SINR and beam pattern with desirable supression results against state of the art, remarkably at a lower computational cost.
Chapter 3

Tractable Transmit MIMO Beampattern Design Under Constant Modulus and Spectral Interference Constraint
3.1 Introduction

The management of the radar radiation power by shaping the beampattern has become crucial to efficiently utilize consumed energy, reduce interference, and increase the probability of detection [3,65].

Optimization of MIMO waveform to achieve the desired beampattern design has been a topic of much recent interest [15–22,37,52,58,66–74]. Some of these works focus on receive beampattern design [69,70] while others focus on the transmit beampattern [37,71–74]. In practice, the transmit beampattern is more difficult to design due to the requirement of a constant modulus constraint (CMC) on the radar transmit waveform, i.e. a constant envelop transmit signal [24].

The importance of the waveform CMC has been well documented and analyzed in terms of performance loss [1,23,24]. Most radar systems utilize non-linear power amplifiers which cannot be efficiently utilized without CMC. Specifically, the output of the amplifier will be a clipped version of the optimized waveform, which often leads to a significant degradation in the system performance.

Both narrowband and wideband MIMO transmit beampattern design under waveform CMC have been studied in [20–22,37,75]. It is well-known that the problem of minimizing deviation of the designed beampattern vs. an idealized one subject to the constant modulus constraint (CMC) constitutes a hard non-convex problem. To ensure tractability, some existing approaches pursue relaxations of or approximations to the CMC. An exemplary approach in this category is [37,71], where an approximation to constant modulus was pursued using the peak-to-average ratio (PAR) waveform constraint. While the CMC is not explicitly represented in the optimization process, the resulting solution is converted to the nearest constant modulus solution. This indirect approximation makes the problem more tractable, however, it degrades the design accuracy. Some recent efforts directly enforce CMC and hence lead to better performance. However, they involve computationally expensive procedures such as the gradient-based methods in [21] or Semidefinite Relaxation (SDR) with randomization [35,36,43]. Moreover, the design criterion of some of the recent work does not allow full control of power allocation such as [28] where the objective is to minimize radiation power in a few selected directions.

An interesting recent advance that forces CMC for (narrowband) beampattern design has been proposed in [20] which sets up waveform design as a phase optimization problem and solves it using a typical iterative numerical method but with no known analytical guarantees of the resulting solution.

On the other hand, to ensure co-existence of radar and communication systems, the design criterion of some recent work does not allow full control of power allocation where the objective is to minimize radiation power in a few selected directions. We address the problem of designing a beampattern for multiple-input multiple-output (MIMO) radar under a spectral constraint, which in turn is determined by the transmitted waveform, as well as the constant modulus constraint. While unconstrained beampattern design is straightforward, a key open challenge is jointly enforcing the spectral constraint in addition to the constant modulus constraint on the radar waveform. To our knowledge, no existing method addresses this problem in the literature.
3.1.1 Our contributions

Our principal aim is to develop an algorithmic approach that can design constant modulus MIMO waveforms in a tractable manner while retaining high levels of performance in the sense of measures such as closeness to an idealized beampattern. As argued above, existing work invariably trades off performance vs. computation in a rigid manner. Two algorithms have been developed for: 1) Beampattern design under waveform CMC. 2) Beampattern design under joint CMC and spectral constraint. Specifically, this chapter includes the following contributions:

3.1.1.1 Beampattern Design under Waveform CMC.

- **Sequence of Closed Forms**: A new algorithmic solution for both narrowband and wideband beampattern design under waveform CMC. To overcome the challenges mentioned above, we develop a new algorithm for MIMO beampattern design that involves solving the hard non-convex problem of beampattern design using a sequence of convex equality constrained Quadratic Programs (QP), each of which has a closed form solution, such that constant modulus is achieved at convergence. Because each QP in the sequence has a closed form, the proposed successive closed forms (SCF) algorithm has significantly lower complexity than competing methods that incorporate CMC. Moreover, the SCF algorithm can be used to minimize power in selected directions or to allow full control of the power allocation as in [37].

- **Convergence of the SCF Algorithm.** We formally prove that the sequence of cost functions representing deviation from the desired beampattern, that occurs in the proposed SCF algorithm is non-increasing, i.e. an improvement is obtained by solving each problem in the sequence. We further prove that the sequence of waveform solutions (via solving each QP) converges to constant modulus.

- **Properties of the SCF solution.** We prove that the SCF solution satisfies the Karush-Kuhn-Tucker (KKT) conditions of the non-convex optimization problem, which are necessary conditions for optimality.

- **Experimental insights and validation.** Experimental validation is performed via numerical simulations. We considered two scenarios: 1) Narrowband null forming where the SCF algorithm shows significant power suppression in the desired directions. 2) Wideband beampattern design where the proposed SCF is shown to achieve a beampattern much closer to the ground truth unconstrained design against state of the art alternatives. In addition, the proposed SCF is robust to the presence of noise in the designed waveform, making it even more appealing from a practical standpoint.

3.1.1.2 Beampattern Design under Joint CMC and Spectral Constraint

- **A new algorithmic solution for beampattern design under both the spectral constraint and the constant modulus constraint.** We develop a new
algorithm for MIMO beampattern design that involves solving the hard non-convex problem of beampattern design using a sequence of convex equality and inequality constrained quadratic programs (QP), each of which has a closed form solution, such that constant modulus is achieved at convergence. The proposed beampattern with interference control (BIC) algorithm has significantly lower complexity than SDR with relaxation [35,36] which is a representative algorithm for the constant modulus constraint.

- **Feasibility of the sequence of QP.** One key challenge when jointly enforcing the spectral constraint and the constant modulus constraint is to ensure feasibility of the optimization problem. Since the feasible set of each QP problem consists of equality and inequality constraints, the intersection of these constraints might be empty. We formally prove that feasible set of each QP problem in the proposed BIC algorithm is always feasible.

- **Convergence of the BIC Algorithm.** We formally prove that the sequence of cost functions representing a deviation from the desired beampattern, that occurs in the proposed BIC algorithm, is non-increasing, i.e. an improvement is always obtained by solving each problem in the sequence.

- **Experimental insights and validation.** Experimental validation is performed via numerical simulations. We considered two scenarios: 1) null forming where the BIC algorithm shows significant power suppression in the desired directions even with the presence of the spectral constraint, and 2) full beampattern design where the proposed BIC is shown to achieve a beampattern much closer to the ground truth against state of the art alternatives that have no spectral constraint.

### 3.2 System Model

Consider a wideband MIMO radar with a uniform linear array (ULA) of $N_T$ antennas and equal spacing distance of $d$ as shown in Fig. 3.1. The signal transmitted from the $m$-th element is denoted by $z_m(t)$. Let $z_m(t) = x_m(t)e^{j2\pi f_c t}$ where $x_m(t)$ is the baseband signal and $f_c$ is the carrier frequency. We assume that the spectral support of $x_m(t)$ is within the interval $[-B/2, B/2]$ where $B$ is the bandwidth in Hz. The sampled baseband signal transmitted by the $m$-th element is denoted by $x_m(n) \triangleq x_m(t = nT_s)$, $n = 0, ..., N - 1$ with $N$ being the number of time samples and $T_s = 1/B$ is the sampling rate.
discrete Fourier transform (DFT) of $x_m(n)$ is denoted by $y_m(p)$ and it is given by

$$y_m(p) = \sum_{n=0}^{N-1} x_m(n)e^{-j2\pi \frac{np}{N}}, \quad p = -\frac{N}{2}, \ldots, 0, \ldots, \frac{N}{2} - 1$$ \hspace{1cm} (3.1)

### 3.2.1 Far-Field Beampattern

According to [37], the discrete frequency beampattern at the angle $\theta$ in the frequency band $p$ in the far-field is given by

$$P(\theta, p) = |a^H(\theta, p)y_p|^2$$ \hspace{1cm} (3.2)

where

$$a(\theta, p) = [1 \quad e^{j2\pi(\frac{p}{N_T}+f_c) \frac{d\cos\theta}{c}} \ldots \quad e^{j2\pi(\frac{(N_T-1)p}{N_T}+f_c) \frac{d\cos\theta}{c}}]^T$$ \hspace{1cm} (3.3)

and

$$y_p = [y_0(p) \quad y_1(p) \ldots \quad y_{N_T-1}(p)]^T$$ \hspace{1cm} (3.4)

where $c$ is the speed of wave propagation. Note that $a(\theta, p)$ is continuous in phase. It can be expressed as a discrete angle vector by dividing the interval $[0^\circ, 180^\circ]$ into $K$ angle bins. Using the same simplified notation found in [37], it can be written as

$$a_{kp} = a(\theta_k, p), \quad k = 1, 2, \ldots, K$$ \hspace{1cm} (3.5)

In this case, the beampattern can be given by the following discrete angle-frequency grid

$$P_{kp} = |a_{kp}^H y_p|^2 = |a_{kp}^H \mathbf{W}_p \mathbf{x}|^2$$ \hspace{1cm} (3.6)

where $\mathbf{x} \in \mathbb{C}^{N_TN}$ is the concatenated vector i.e. $\mathbf{x} = [x_0^T \quad x_1^T \ldots \quad x_{N_T-1}^T]^T$ where $x_m = [x_m(0) \quad x_m(1) \ldots \quad x_m(N-1)]^T \in \mathbb{C}^N$ and $\mathbf{W}_p \in \mathbb{C}^{N_T \times N_TN}$ is given by

$$\mathbf{W}_p = \mathbf{I}_{N_T} \otimes \mathbf{e}_p^H$$ \hspace{1cm} (3.7)

where $\otimes$ is a Kronecker product operator, $\mathbf{e}_p^H = [1 \quad e^{-j2\pi \frac{p}{N}} \ldots \quad e^{-j2\pi \frac{(N-1)p}{N}}] \in \mathbb{C}^N$ and $\mathbf{I}_{N_T}$ is an $N_T \times N_T$ identity matrix.

### 3.2.2 Spectral Constraint

The problem of spectral co-existence has been of great interest recently [7, 26–32] and involves minimization of interference caused by radar transmission at victim communication receivers operating in the same frequency band. In this case, the beampattern of the transmit waveform is required to have nulls in these bands to prevent interference. For $J$ communication receivers, we suppose that the $j$-th communication receiver operating on a frequency band $B_j = [p^j_l, p^j_u]$, where $p^j_l$ and $p^j_u$ are the lower and upper normalized frequency, respectively. We denote the desired (discrete) spectrum shape by

$$\hat{\mathbf{y}} = [\hat{\gamma}_{N^{-\frac{N}{2}}}, \hat{\gamma}_{N^{-\frac{N}{2}}+1}, \ldots, \hat{\gamma}_{N^{-\frac{N}{2}}-1}] \in \mathbb{C}^{N \times 1}$$ defined as

$$\hat{\gamma}_p = \begin{cases} 
0 & \text{for } p \in B_j = [p^j_l, p^j_u], \\
\gamma & \text{otherwise.}
\end{cases} \quad j = 1, 2, \ldots, J$$
where $\gamma$ is a scalar such that $\hat{y}^H F F^H \hat{y} = N$ and $F$ is the DFT matrix. In SHAPE algorithm proposed by Rowe et al. [58], a least-squares fitting approach for the spectral shaping problem for SISO has been formulated by minimizing the following cost function

$$\| F^H x - \hat{y} \odot e^{j\beta}\|_2^2$$

(3.8)

where the phase vector $\beta$ is an auxiliary vector and $\odot$ represents the element-wise product operation. We extend (3.28) for MIMO radar and employ it as a constraint in the optimization problem as follows

$$\|(I_{N_T} \otimes \hat{y} \odot e^{j\beta}) - x\|_2^2 = \| \bar{F}^H \bar{y} - x\|_2^2 \leq E_R$$

(3.9)

where $1_{N_T} = [1, 1, \ldots, 1] \in \mathbb{R}^{M \times 1}$, $\bar{F} = I_{N_T} \otimes F^H$, and $\bar{y} = 1_{N_T} \otimes \hat{y} \odot e^{j\beta}$, and $E_R$ is the maximum tolerable spectral error.

### 3.2.3 Problem Formulation

The optimization problem can be formulated as the following matching problem:

$$\left\{ \begin{array}{l}
\min_x \quad \sum_{k=1}^{K} \sum_{p=-N}^{N-1} \left[ d_{kp} - |a_{kp}^H W_p x| \right]^2 \\
\text{s.t.:} \quad |x| = 1 \\
\quad \| \bar{F}^H \bar{y} - x\|_2^2 \leq E_R
\end{array} \right.$$  

(3.10)

where $d_{kp} \in \mathbb{R}$ is the desired beampattern. The constant modulus constraints ($|x| = 1$) implies that $|x_m(n)| = 1$ for $m = 1, \ldots, N_T$ and $n = 0, \ldots, N - 1$. These constraints are neither convex nor linear and it is well known in the literature that (3.10) is a hard non-convex problem even without the spectral constraint. He et al. [37] proposed a solution to problem (3.10) without the interference constraint by employing a peak-to-average ratio constraint as a relaxation of the constant modulus constraint. However, they used the cyclic algorithm [38,76] to solve the unconstrained problem $\min_{y_p} \sum_{k=1}^{K} \sum_{p=-N}^{N-1} \left[ d_{kp} - |a_{kp}^H y_p| \right]^2$ in the first stage and then in the second stage they aim to find the constant modulus approximation of the solution. The algorithm does not directly minimize the cost function under constant modulus constraint or any relaxed version thereof. In this chapter, we propose a new solution that minimizes the cost function of interest subject to the constant modulus constraint and the interference constraint by solving a sequence of problems under a relaxed convex constraint such that constant modulus is still achieved at convergence. The proposed new solution has the ability to break the computational cost-solution quality trade-off that has been demonstrated in past work such as SDR with randomization [11,35,36,43] or the simulated annealing approach [37].
3.3 Proposed Method for Beampattern under CMC: Sequence of Closed Forms Solutions (SCF)

As shown in [37], it is more convenient to rewrite the objective function of eq. (3.10) as:

$$\sum_{k=1}^{K} \sum_{p=-\frac{N}{2}}^{\frac{N}{2}} |d_{kp} e^{j\phi_{kp}} - a_{kp}^{H} W_{p} x|^2$$  (3.11)

where $\phi_{kp} = \arg\{a_{kp}^{H} W_{p} x\}$. Since $x$ is unknown, $\phi_{kp}$ is also unknown for all values of $k$ and $p$. In the existing literature [37,38,76], this problem has been resolved by an iterative method. This method minimizes eq. (3.11) by fixing the values of $\{\phi_{kp}\}$ and minimizing w.r.t. $x$ and then fixing $x$ and minimizing w.r.t. $\{\phi_{kp}\}$. It has been shown that such an iterative method, ensures that the cost function is monotonically decreasing and converges to a finite value [37,38,76]. Therefore, we focus on solving the following constrained problem for fixed values of $\{\phi_{kp}\}$:

$$\min_{x} \sum_{k=1}^{K} \sum_{p=-\frac{N}{2}}^{\frac{N}{2}} |d_{kp} e^{j\phi_{kp}} - a_{kp}^{H} W_{p} x|^2 \quad \text{s.t.:} \quad |x| = 1$$  (3.12)

Now, let us define the following:

$$A_p = \begin{bmatrix} a_1^H \\ \vdots \\ a_K^H \end{bmatrix}, \quad d_p = \begin{bmatrix} d_{1p} e^{j\phi_{1p}} \\ \vdots \\ d_{Kp} e^{j\phi_{Kp}} \end{bmatrix},$$  (3.13)

The objective function of problem (3.12) can be rewritten in terms of $A_p$ and $d_p$ as:

$$f(x) = \sum_{p} ||d_p - A_p W_{p} x||^2_2$$
$$= \sum_{p} x^{H} F_p^H A_p^H A_p W_{p} x - d_p^{H} A_p W_{p} x - x^{H} F_p^H A_p^H d_p$$
$$+ \sum_{p} d_p^{H} d_p$$
$$= x^{H} \left( \sum_{p} F_p^H A_p^H A_p W_{p} \right) x - \left( \sum_{p} d_p^H A_p W_p \right) x$$
$$- x^{H} \left( \sum_{p} F_p^H A_p^H d_p \right) + \sum_{p} d_p^H d_p$$
$$= x^{H} P x - q^{H} x - x^{H} q + r$$  (3.14)

where $P = \sum_{p} F_p^H A_p^H A_p W_{p}$, $q = \sum_{p} F_p^H A_p^H d_p$ and $r = \sum_{p} d_p^H d_p$. Problem $P'$ is equivalent to the following problem:

$$\min_{x} \quad x^{H} P x - q^{H} x - x^{H} q + r \quad \text{s.t.:} \quad |x| = 1$$  (3.15)
which can be converted to the following problem with real (as opposed to complex) variables:

\[
\begin{align*}
\min_{\mathbf{u}} \quad & \mathbf{u}^T \mathbf{G} \mathbf{u} - \mathbf{t}^T \mathbf{u} - \mathbf{u}^T \mathbf{t} + r \\
\text{s.t.:} \quad & u_l^2 + u_{l+L}^2 = 1, \quad l = 1, 2, \ldots, L
\end{align*}
\] (3.16)

where \(\mathbf{u} = [\text{Re}\{\mathbf{x}\}^T \text{Im}\{\mathbf{x}\}^T]^T\), \(u_l\) is the \(l\)-th element of \(\mathbf{u}\), \(L = N_T N\),

\[\mathbf{G} = \begin{bmatrix} \text{Re}\{\mathbf{P}\} & -\text{Im}\{\mathbf{P}\} \\ \text{Im}\{\mathbf{P}\} & \text{Re}\{\mathbf{P}\} \end{bmatrix} \quad \text{and} \quad \mathbf{t} = \begin{bmatrix} \text{Re}\{\mathbf{q}\} \\ \text{Im}\{\mathbf{q}\} \end{bmatrix}\]

Problem (3.16) can be rewritten as:

\[
\begin{align*}
(RP) \quad \min_{\mathbf{s}} \quad & \mathbf{s}^T (\mathbf{R} + \lambda \mathbf{I}) \mathbf{s} \\
\text{s.t.:} \quad & \mathbf{s}^T \mathbf{E}_l \mathbf{s} = 1, \quad l = 1, 2, \ldots, L + 1
\end{align*}
\] (3.17)

where \(\lambda\) is a positive number,

\[\mathbf{R} = \begin{bmatrix} \mathbf{G} & -\mathbf{t} \\ -\mathbf{t}^T & r \end{bmatrix}, \mathbf{s} = \begin{bmatrix} \mathbf{u} \\ 1 \end{bmatrix} = \begin{bmatrix} \text{Re}\{\mathbf{x}\} \\ \text{Im}\{\mathbf{x}\} \end{bmatrix}, \quad \lambda \geq 0\]

and \(\mathbf{E}_l\) is a \((2L + 1) \times (2L + 1)\) matrix given by:

\[\mathbf{E}_l(i, j) = \begin{cases} 
1 & \text{if } i = j = l, \text{ and } l \leq L, \\
1 & \text{if } i = l + L, j = l + L, \text{ and } l \leq L, \\
1 & \text{if } i = j = 2L + 1, \text{ and } l = L + 1, \\
0 & \text{Otherwise.}
\end{cases}\]

Note that, since \(\sum_p \|d_p - A_p W_p x\|_2^2 = x^H P x - q^H x - x^H q + r \geq 0\), then, according to page 530 of [77], \(\mathbf{R}\) is positive semidefinite. Further, because the problem \(RP\) enforces \(\mathbf{s}^T \mathbf{E}_l \mathbf{s} = 1, \quad l = 1, 2, \ldots, L\) then \(\lambda \mathbf{s}^T \mathbf{s}\) is a constant value (i.e. \(\lambda \mathbf{s}^T \mathbf{s} = \lambda (L + 1)\)). As a result, the optimal solution of \(P'\) and (the complex version of) the optimal solution of \(RP\) are identical for any \(\lambda \geq 0\). Now, consider the following sequence of equality constrained QPs:

\[
(CP^{(n)}) \quad \min_{\mathbf{s}} \quad \mathbf{s}^T (\mathbf{R} + \lambda \mathbf{I}) \mathbf{s} \\
\text{s.t.:} \quad \mathbf{B}^{(n)} \mathbf{s} = \mathbf{1}
\] (3.18)

where \(\mathbf{B}^{(n)} = [b_1^{(n)}, b_2^{(n)}, \ldots, b_{L+1}^{(n)}]^T \in \mathbb{R}^{(L+1)\times(2L+1)}\) such that the line defined by \(b_l^{(n)} \mathbf{s} = 1\) is a tangent to the circle \(\mathbf{s}^T \mathbf{E}_l \mathbf{s} = 1\) for \(l = 1, 2, \ldots, L\) and \(b_{L+1}^{(n)} = [0, \ldots, 0, 1]^T\). In particular, let \(\mathbf{s}^{(n)} \in \mathbb{R}^{(2L+1)\times 1}\) be the optimal solution of \(CP^{(n)}\) and \(\mathbf{x}^{(n)} \in \mathbb{C}^{L\times 1}\) be the complex version defined as:

\[x_l^{(n)} = s_l^{(n)} + js_{l+L}^{(n)}, \quad l = 1, 2, \ldots, L\] (3.19)
where \( x_l^{(n)} \) and \( s_l^{(n)} \) are the \( l \)-th element of \( x^{(n)} \) and \( s^{(n)} \), respectively. Thus, \( s^{(n)} = [\{\text{Re}(x^{(n)})\}^T \text{Im}(x^{(n)})]^T \). In this case, we define the matrix \( B^{(n)} \) as:

\[
B^{(n)}(i,j) = \begin{cases} 
\cos(\arg x_i^{(n-1)}) & \text{if } i = j = l, \ l \leq L, \\
\sin(\arg x_i^{(n-1)}) & \text{if } i = l, j = l + L, \ l \leq L, \\
1 & \text{if } i = L + 1, j = 2L + 1, \\
0 & \text{Otherwise.}
\end{cases}
\]

Note that, problem \( CP^{(n)} \) is a convex quadratic minimization with linear equality constraints. Using the optimality conditions for problem \( CP^{(n)} \) \([34]\), we have:

\[
\begin{bmatrix} \bar{R} & B^{(n)}^T \\ B^{(n)} & 0 \end{bmatrix} \begin{bmatrix} s^{(n)} \\ v^{(n)} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

(3.20)

where \( \bar{R} = 2(R + \lambda I) \) and \( v^{(n)} \in \mathbb{R}^{(L+1) \times 1} \) is the Lagrange multiplier associated with the equality constraints. Solving (3.20) by block elimination gives:

\[
s^{(n)} = \bar{R}^{-1} B^{(n)}^T \left( B^{(n)} \bar{R}^{-1} B^{(n)}^T \right)^{-1} 1
\]

(3.21)

Since \( b_1^{(n)}, b_2^{(n)}, ..., b_{L+1}^{(n)} \) by construction are linearly independent for all \( n \) and \( \bar{R} \) is positive definite for any \( \lambda > 0 \), then according to theorem 2.1 of \([78]\) all the eigenvalues of \( K \) in (3.20) are nonzero i.e. \( K \) is nonsingular. As a consequence, equation (3.21) always has a unique solution \( s^{(n)} \).

Although the problem in (3.18) does not result in a constant modulus solution, a sequence of such problems (in the index \( n \)) can ensure a non-increasing sequence of cost function values, such that the corresponding solution converges to constant modulus. To recognize this, let \( x^{(n-1)} \) be the complex version defined in eq. (3.19), i.e. \( s^{(n-1)} = [\{\text{Re}(x^{(n-1)})\}^T \text{Im}(x^{(n-1)})]^T \). The affine constraints of \( CP^{(n)} \) are adjusted so that the feasible set of \( CP^{(n)} \) includes what we call as the constant modulus version of \( x^{(n-1)} \) given by \( x^{(n-1)} = \exp(j \arg(x^{(n-1)})) \). If \( x^{(n)} = x^{(n-1)} \), then the constraints of the next problem \( CP^{(n+1)} \) are the same as problem \( CP^{(n)} \) which means \( x^{(n+1)} = x^{(n)} \) and, hence, the algorithm converges. Otherwise, the feasible set of \( CP^{(n)} \) is adapted to include the constant modulus version of \( x^{(n-1)} \). Convergence is then guaranteed by Lemma 3.3.1 which establishes that the cost function sequence that results by using the constant modulus version of the solution at each iteration, is in fact non-increasing and converges. This procedure is visually illustrated in Fig. 3.2 and formally described in Algorithm 5.

**Computational Complexity:** Based on the computational cost of solving (3.20) in each iteration, the overall computational complexity of SCF is \( O(FL^2) \) if the matrix inversion method in \([79]\) is used to compute \( s^{(n)} \) in Eq (3.21), where \( F \) is the total number of iterations. In comparison, SDR with randomization has a computational complexity of \( O(L^{1.5}) + O(TL^2) \) \([44]\) where \( T \) is the number of randomization trails. It invariably needs a large number of randomization trials \( T \gg L \) \([11]\) which makes the term \( O(TL^2) \) very significant. Therefore, the SCF algorithm has much lower complexity.
3.3.1 Convergence Analysis of SCF

The value of the objective function of problem (P) as a function of the constant modulus version \( x_{(n)} \) is non-increasing in \( n \). This can be proved via the following Lemma.

**Lemma 3.3.1.** Let \( x^{(n)} \) be the complex version defined in (3.19). Denote by \( x_{(n)} \) the constant modulus version of \( x^{(n)} \), i.e. \( x_{(n)} = \exp(j \arg(x^{(n)})) \). Define \( f(x) = \sum_p \|d_p - A_p W_p x\|_2^2 \) and \( \lambda_P \) as the maximum eigenvalue of \( P \). If \( \lambda \geq \frac{1}{8} \lambda_P + ||q||_2 \) then:

\[
f(x_{(n-1)}) \geq f(x_{(n)})
\]

(3.22)

In other words, the sequence \( \{f(x_{(n)})\}_{n=0}^{\infty} \) is non-increasing. Moreover, the sequence \( \{f(x_{(n)})\}_{n=0}^{\infty} \) converges to a finite value \( f^* \).

**Proof.** See subsection 2.1 of the Appendix.

It is well known in constrained optimization [34], that the first order necessary conditions for optimality are the so-called Karush-Kuhn-Tucker (KKT) conditions. So far, we have shown that our solution is feasible, i.e. a constant modulus is guaranteed at convergence. Now, we prove that the final converged solution of the proposed SCF algorithm is in fact KKT optimal.

**Lemma 3.3.2.** Let \( C \) be the smallest number of iterations needed for convergence, i.e. \( f(x_{(n-1)}) = f(x_{(n)}) \) for \( n \geq C \). If \( \lambda \geq \frac{1}{8} \lambda_P + ||q||_2 \), then for \( n \geq C \):
Algorithm 5 Successive Closed Form (SCF)

Inputs: $d_p$, $W_p$, $a_{kp}$ for $p = -\frac{N}{2}, ..., 0, ..., \frac{N}{2} - 1$, $k = 1, 2, ..., K$, an initial value $x^{(0)}$ and a threshold value $\zeta$.

Output: A solution $x^*$ for $(P)$.

(1) Set $n = 1$ and compute $\bar{R}$.
(2) Compute $B^{(n)}$.
(3) Compute $s^{(n)} = \bar{R}^{-1}B^{(n)^T}(B^{(n)}\bar{R}^{-1}B^{(n)^T})^{-1}$. 
(4) Set $x_l^{(n)} = \exp\{j\arg(s_l^{(n)} + j\delta_l^{(n)})\}, l = 1, 2, ..., L$.
(5) Check the following:
if $f(x^{(n)}) - f(x^{(n-1)}) < \zeta$ then
  STOP.
else
  set $n = n + 1$ GOTO step (2).
end if

Output: $x^* = x^{(n)}$

Proof. See subsection 2.2 of the Appendix.

Note that SCF as described in Algorithm 5 obtains the optimal $x$ for a fixed $\phi_{kp}$. The complete SCF Algorithm to solve $P'$ is given in Algorithm 6.

Algorithm 6 Successive Closed Form (SCF) for problem $P'$

Inputs: $d_{kp}$, $W_p$, $a_{kp}$, for $p = -\frac{N}{2}, ..., 0, ..., \frac{N}{2} - 1$, $k = 1, 2, ..., K$ and $\zeta'$ (the desired threshold value).

Output: A solution $x^*$ for $(P')$.

(1) Set $m = 1$.
(2) Set $\phi_{kp}^{(m)} = \arg\{a_{kp}^H W_p x^{(m-1)}\}$ for all $k$ and $p$.
(3) Set $d_p^{(m)} = [d_1 e^{j\phi_{1p}^{(m)}}, ..., d_K e^{j\phi_{Kp}^{(m)}}]^T$.
(4) Use Algorithm 5 to compute $x^{(m)}$ with $d_p = d_p^{(m)}$ and $x^{(0)} = x^{(m-1)}$ as inputs.
(5) Check the following:
if $f'(x^{(m)}) - f'(x^{(m-1)}) < \zeta'$ where $f'(x) = \sum_p ||d_p|-|A_p W_p x||^2_2$ then
  STOP.
else
  set $m = m + 1$ GOTO step (2).
end if

Output: $x^* = x^{(m)}$
3.3.2 Narrowband null forming beampattern design

Narrowband null forming beampattern design can be seen as a special case of our wideband beampattern design. In this case the beampattern design is conducted in spatial domain only i.e. $N = 1$. However, unlike the problem formulation in (3.10) the goal of null forming beampattern design is to form a beampattern with nulls in desired directions denoted by $\{\theta_k\}_{k=1}^K$, therefore, we consider only few angular directions unlike the full design for the beampattern across all angular directions made earlier in eq. (3.11). Let $x = [x_0 ... x_{N_T-1}] \in \mathbb{C}^{N_T \times 1}$ where $x_m$ is the coding waveform transmitted from the element $m$. The objective function can be defined as [28]:

$$f(x) = x^H V^H \Sigma V x$$

where $V$ and $\Sigma$ are expressed as:

$$V = \begin{bmatrix} v_1(\theta_1) & v_1(\theta_2) & ... & v_1(\theta_K) \\ v_2(\theta_1) & v_2(\theta_2) & ... & v_2(\theta_K) \\ \vdots & \vdots & \ddots & \vdots \\ v_M(\theta_1) & v_M(\theta_2) & ... & v_M(\theta_K) \end{bmatrix}$$

and,

$$\Sigma = \text{diag}\{\sigma_1,...,\sigma_K\}$$

where $v_m(\theta_k) = e^{j2\pi(f+f_c)md\cos\theta_k}$ and $\sigma_k$ is the weight factor of the radiation power in the $k$-th direction. Therefore, the problem can be formulated as:

$$\begin{cases} 
\min_x & x^H V^H \Sigma V x \\
\text{s.t.:} & |x| = 1
\end{cases}$$

In this case, the optimization problem reduces to problem $CP^{(n)}$ in (3.18) with:

$$R = \begin{bmatrix} \text{Re}\{V^H \Sigma V\} & -\text{Im}\{V^H \Sigma V\} \\ \text{Im}\{V^H \Sigma V\} & \text{Re}\{V^H \Sigma V\} \end{bmatrix}, s = \begin{bmatrix} \text{Re}\{x\} \\ \text{Im}\{x\} \end{bmatrix},$$

and $B^{(n)} = [b_1^{(n)}, b_2^{(n)}, ..., b_L^{(n)}]^T \in \mathbb{R}^{L \times 2L}$ given by:

$$B^{(n)}(i,j) = \begin{cases} 
\cos(\arg x_{l}^{(n-1)}) & \text{if } i = j = l, \\
\sin(\arg x_{l}^{(n-1)}) & \text{if } i = l, j = l + L, \\
0 & \text{Otherwise.}
\end{cases}$$

Since $V^H \Sigma V$ is positive semi-definite and there are no linear terms in the objective function (i.e. $q = 0$), then both Lemma 3.3.1 and 3.3.2 hold for $\lambda \geq \frac{L}{8} \lambda_P$. Note that, in this case, we use only Algorithm 5 with $R$ and $B^{(n)}$ as mentioned above.

3.4 Numerical Results

In this section, we examine the performance of the proposed SCF based beampattern design and compare it to competing methods using numerical simulations.
3.4.1 Simple Example: SCE performance compared to global optimal value

In this section, we compare the optimal solution of the proposed SCE algorithm to the actual global optimal. The actual global optimal value is found using some types of the branch-and-bound algorithms via BARON software [80,81] under fairly general assumptions. However, this method require an extensive search and a very large number of iterations which make it practically unfeasible. Therefore, we used this method for a toy example where $M = 2$ and $5$.

Experimental set up: A linear MIMO radar antenna array with half-wavelength spacing and $\lambda = 0.1$. Case I: Two antenna elements ($M = 2$), $\sigma_k = 1/2$ for $k = 1, 2$, $\theta = [-20^\circ, -50^\circ]$. Table 3.1 shows the optimal value of the cost function of SCF compared to the global optimal value obtained by BARON. In this case, the global optimal value is identical to the optimal value obtained by the SCF algorithm.

Table 3.1: Cost function in dB of the proposed method vs. Global (Case I)

<table>
<thead>
<tr>
<th>Method</th>
<th>Cost Function (dB)</th>
<th>No. Iterations</th>
<th>Run Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>BARON [80,81]</td>
<td>−5.01453</td>
<td>2068101</td>
<td>3 days</td>
</tr>
<tr>
<td>SCF</td>
<td>−5.01453</td>
<td>15</td>
<td>0.0066902 sec</td>
</tr>
</tbody>
</table>

Case II: Five antenna elements ($M = 5$), $K = 3$, $\theta = [-10^\circ, 20^\circ, 45^\circ]$ and the wave factors $\sigma_k = 1/3$ for $k = 1, 2, 3$. Table 3.2 shows the optimal value of the cost function of SCF compared to the global optimal value obtained by BARON. In this case, the global optimal value is very close to the optimal value of the SCF algorithm. The optimal value of SCF is only 0.01001 dB higher than the global optimal value.

Table 3.2: Cost function in dB of the proposed method vs. Global (Case II)

<table>
<thead>
<tr>
<th>Method</th>
<th>Cost Function (dB)</th>
<th>No. Iterations</th>
<th>Run Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>BARON [80,81]</td>
<td>−12.14509</td>
<td>3630721</td>
<td>&gt; 8 days</td>
</tr>
<tr>
<td>SCF</td>
<td>−12.13507</td>
<td>15</td>
<td>0.008484 sec</td>
</tr>
</tbody>
</table>

3.4.2 Narrowband null forming beampattern

For narrowband beampattern design, we compare the proposed method to the state-of-the-art narrowband phase-only variable metric method (POVMM) method [20], Alternating Direction Method of Multipliers (ADMM) [82] and SDR with randomization [36]. Experimental set up: A linear MIMO radar antenna array of $M = 16$ elements with half-wavelength spacing and $\lambda = 0.1$. Three beamforming cases are considered Case I: $K = 1$, $\theta_1 = -20^\circ$ and the wave factor $\sigma_1 = 1$; Case II: $K = 6$, $\theta = [-60^\circ, -20^\circ, 20^\circ, 45^\circ, 60^\circ, 80^\circ]$ and the wave factors $\sigma_k = 1/6$ for $k = 1, 2, ..., 6$ and, finally, Case III: 2.5-wavelength spacing with $K = 6$, $\theta = [-60^\circ, -20^\circ, 20^\circ, 45^\circ, 60^\circ, 80^\circ]$ and the wave factors $\sigma_k = 1/6$ for $k = 1, 2, ..., 6$.

Figures 3.3, 3.4 and 3.5 show the resulting beampattern for Case I, Case II and Case III, respectively. Clearly, SCF outperforms both SDR with randomization, ADMM and
POVMM by a significant amount. For example, in Fig. 3.3 at $\theta_1 = -20^\circ$, SCF has a null that is more than 100 dB lower than competing methods. The cost function defined by $f(x_{(n)}) = x_{(n)}^H V^H \Sigma V x_{(n)}$ for the proposed SCF is non-increasing in each iteration as shown in Figure 3.6. Table 3.3 shows the cost function value as optimized via SCF as well as POVMM and their corresponding computational run times as observed in practice. Although POVMM has lower complexity per iteration, it needs orders of magnitude more iterations to achieve the same performance as SCF. Hence, for a given cost function (designed beam pattern deviation from idealized) specification, SCF can achieve it much sooner. For a fair comparison, SCF as well as competing methods are initialized with the same waveform, which is a pseudo-random vector of unit magnitude complex entries.

The effect of the number of antennas on the cost function is shown in Figure 3.7. In this case we used $K = 2$, $\theta = [-60^\circ, -63^\circ]$ and the wave factors $\sigma_k = 1/2$ for $k = 1, 2$. 
Interestingly, the performance of the proposed SCF method with $M = 6$ is comparable with the performance of POVMM with larger number of antennas $M = 10$.

Next, we consider the effect of random phase errors on the designed waveforms. The phase errors are incurred during waveform transmission and can be modeled as a random noise added to the designed waveform phase [28]. In the presence of phase errors hence, the actual transmitted waveforms are:

$$\tilde{x} = [\exp(\arg(x_1) + e_1) \ldots \exp(\arg(x_M) + e_M)]^T$$  \hspace{1cm} (3.27)

where $e_m$ is the random phase error for the waveform transmitted from the $m$-th antenna element. The random phase errors are modeled as statistically independent and Gaussian-distributed with zero mean and a standard deviation of $\sigma_e$.

Figure 3.8 shows a comparison between the ideal and the actual radiation beampattern of the SCF method with additive phase errors having a standard deviation of $\sigma_e = 0.25$ for Case II. Fortunately, the average null depth of the actual radiation beampattern is
Table 3.3: Cost function in dB of SCF vs. POVMM, Case II

<table>
<thead>
<tr>
<th>Method</th>
<th>Cost Func. (dB)</th>
<th>Sim. Time (s)</th>
<th>iter.</th>
</tr>
</thead>
<tbody>
<tr>
<td>POVMM</td>
<td>-26.70</td>
<td>0.0122</td>
<td>150</td>
</tr>
<tr>
<td>POVMM</td>
<td>-35.33</td>
<td>0.0178</td>
<td>250</td>
</tr>
<tr>
<td>POVMM</td>
<td>-53.10</td>
<td>0.0240</td>
<td>350</td>
</tr>
<tr>
<td>SCF ($\zeta = 10^{-7}$)</td>
<td>-121.5</td>
<td>0.0117</td>
<td>9</td>
</tr>
</tbody>
</table>

Figure 3.7: The cost function versus number of antennas for the proposed SCF method and POVMM [20].

about -61.34 dB which is quite acceptable.

The average cost function over 1000 independent noise realizations of Case II ($M = 16$) is listed in Table 3.4 for varying $\sigma_e = 0.25$. The SCF algorithm with 7 iterations outperforms the POVMM method for realistic values of $\sigma_e$.

Table 3.4: Effect of random phase errors on the performance (Cost Function in dB)

<table>
<thead>
<tr>
<th>$\sigma_e$</th>
<th>0</th>
<th>0.1</th>
<th>0.25</th>
<th>0.5</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>POVMM</td>
<td>-35.33</td>
<td>-32.8</td>
<td>-31.11</td>
<td>-27.73</td>
<td>-22.77</td>
</tr>
<tr>
<td>SCF</td>
<td>-72.51</td>
<td>-43.2</td>
<td>-35.15</td>
<td>-29.23</td>
<td>-23.07</td>
</tr>
</tbody>
</table>

Figure 3.9 shows the actual radiation beampattern for waveform design for Case II with phase errors of standard deviation $\sigma_e = 0.25$ of SCF compared to POVMM. The average null depth of the actual POVMM radiation beampattern is about -55.83 dB which is about 5.51 dB higher than the average null depth of the actual SCF radiation beampattern.

3.4.3 Wideband beampattern

For wideband beampattern design, we compare SCF to the state-of-the-art Wideband Beampattern Formation via Iterative Techniques (WBFIT) method [37]. The following set-up is used in this numerical experiment: The number of transmit antennas $M = 10$, 

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Figure 3.8: Effect of random phase error on the performance of SCF (Proposed) for Case II with phase deviation \( \sigma_e = 0.25 \).

Figure 3.9: Effect of random phase error on the performance of POVMM and SCF (Proposed) for Case II with phase deviation \( \sigma_e = 0.25 \).

the number of time samples \( N = 32 \), the carrier frequency of the transmit signal \( f_c = 1 \) GHz and the bandwidth \( B = 200 \) MHz. The spatial angle is divided into \( K = 180 \) grid points and we set \( \lambda = 10 \).

To provide an upper bound on achievable performance, we also obtain the beampattern as obtained by an unconstrained optimization of the transmit waveform, i.e. minimize (3.11) w.r.t \( x \) but without any constraints on \( x \).

Example 1: We consider the following desired transmit beampattern:

\[
d(\theta, f) = \begin{cases} 
1 & \theta = [120^\circ] \\
0 & \text{Otherwise}
\end{cases}
\]

Fig. 3.10 shows the angle-frequency plot of the beampattern for (a) the unconstrained beampattern design (No CMC), (b) WBFIT method, (c) SDR with randomization and
Example 2: We consider the following desired transmit beampattern:

\[
d(\theta, f) = \begin{cases} 
1 & \theta = [95^\circ, 145^\circ] \\
0 & \text{Otherwise.}
\end{cases}
\]

Fig. 3.11 shows the angle-frequency plot of the beampattern. Clearly, SCF is closer to the beampattern achieved by an unconstrained waveform and has higher suppression at the undesired angles compared to WBFIT and SDR with randomization with 10,000 randomization trials. Table 3.5 shows the minimum cost function \( \sum_p \|d_p - |A_p W_p x^*|\|_2^2 \) of the proposed method compared to WBFIT method and SDR with randomization. SCF achieves a cost function value that is 5.7 dB and 2 dB lower than WBFIT and SDR with randomization, respectively, which indicates that the proposed method is much closer to the desired beampattern.

Example 3: We consider another example where we have different beampattern shape in different frequency bands. In particular, we consider the following desired beampattern:

(d) SCF. In this somewhat favorable case, all three optimization methods produced a comparable radiation beampattern with SCF slightly outperforming WBFIT.
Figure 3.11: Plot of the beampattern: (a) The unconstrained beampattern (b) WBFIT method (c) SDR with randomization (d) Proposed method

Table 3.5: Cost function in dB of the proposed method vs. WBFIT (Case I)

<table>
<thead>
<tr>
<th>Method</th>
<th>Cost Function (dB)</th>
<th>Simu. Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unconstrained</td>
<td>19.93</td>
<td>-</td>
</tr>
<tr>
<td>WBFIT [37]</td>
<td>29.24</td>
<td>1.15</td>
</tr>
<tr>
<td>SDR with rand. [36]</td>
<td>25.5</td>
<td>3489.2</td>
</tr>
<tr>
<td>SCF</td>
<td><strong>23.54</strong></td>
<td><strong>6.02</strong></td>
</tr>
</tbody>
</table>

\[
d(\theta, f) = \begin{cases} 
1 & \theta = [30^\circ, 50^\circ], -B/2 + f_c \leq f \leq f_c \\
1 & \theta = [130^\circ, 150^\circ], f_c < f \leq B/2 + f_c \\
0 & \text{Otherwise}
\end{cases}
\]

Fig. 3.12 shows the angle-frequency plot of the beampattern. Clearly, SCF is closer to the beampattern achieved by an unconstrained waveform and has higher suppression at the undesired angles compared to WBFIT. Table 3.6 shows the the minimum cost function \(\sum_p \|d_p\| - \|A_p W_p x^*\|_2^2\) of the proposed method compared to WBFIT method. In this case, the difference between SCF and both WBFIT and SDR with randomization is higher, as SCF achieves a cost function value that is 7.5 dB and 5 dB lower than
Figure 3.12: Plot of the beampattern: (a) The unconstrained beampattern (b) WBFIT method (c) Proposed method (d) SCF (Proposed)

WBFIT and SDR with randomization, respectively.

Table 3.6: Cost function in dB of the proposed method vs. WBFIT (Case II)

<table>
<thead>
<tr>
<th>Method</th>
<th>Cost Function (dB)</th>
<th>Simu. Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unconstrained</td>
<td>19.38</td>
<td>-</td>
</tr>
<tr>
<td>WBFIT [37]</td>
<td>32.38</td>
<td>1.54</td>
</tr>
<tr>
<td>SDR with rand. [36]</td>
<td>29.2</td>
<td>4014.6</td>
</tr>
<tr>
<td>SCF</td>
<td>24.1</td>
<td>6.42</td>
</tr>
</tbody>
</table>
3.5 Beampattern Design under Constant Modulus and Spectral Constraints

3.5.1 Formulation of the Spectral Constraint

The problem of spectral co-existence involves minimization of interference caused by radar transmission at victim communication receivers operating in the same frequency band. In this case, the beampattern of the transmit waveform is required to have nulls in these bands to prevent interference. For \( J \) communication receivers, we suppose that the \( j \)-th communication receiver operating on a frequency band \( B_j = [p_j^l, p_j^u] \), where \( p_j^l \) and \( p_j^u \) are the lower and upper normalized frequency, respectively. We denote the desired (discrete) spectrum shape by \( \hat{y} = [\hat{y}_{N^2}, \hat{y}_{N^2}+1, \ldots, \hat{y}_{N^2-1}] \in \mathbb{C}^{N \times 1} \) defined as

\[
\hat{y}_p = \begin{cases} 
0 & \text{for } p \in B_j = [p_j^l, p_j^u], \quad j = 1, 2, \ldots, J \\
\gamma & \text{otherwise}
\end{cases}
\]

where \( \gamma \) is a scalar such that \( \hat{y}^H \hat{F^H} \hat{y} = N \) and \( \hat{F} \) is the DFT matrix. In SHAPE algorithm proposed by Rowe et al. [58], a least-squares fitting approach for the spectral shaping problem for SISO has been formulated by minimizing the following cost function

\[
\| \hat{F^H} \bar{y} - \hat{y} \|^2_2
\]

where the phase vector \( \beta \) is an auxiliary vector and \( \odot \) represents the element-wise product operation. We extend (3.28) for MIMO radar and employ it as a constraint in the optimization problem as follows

\[
\| (I_M \otimes \hat{F}^H)(I_M \otimes \hat{y}) - x \|_2^2 = \| \hat{F}^H \hat{y} - x \|^2_2 \leq E_R
\]

where \( I_M = [1, 1, \ldots, 1] \in \mathbb{R}^{M \times 1} \), \( \hat{F} = I_M \otimes \hat{F}^H \), and \( \bar{y} = I_M \otimes \hat{y} \), and \( E_R \) is the maximum tolerable spectral error.

3.5.2 Non-convex Optimization Problem

Starting with problem:

\[
\min_x \sum_{k=1}^{K} \sum_{p=-N}^{N-1} |d_{kp} - |a_{kp}^H W_p x||^2 \\
\text{s.t.:} \\
| x | = 1 \\
\| \hat{F}^H \bar{y} - x \|^2_2 \leq E_R
\]

Here, we focus on solving the following constrained beampattern design problem under CMC and spectral constraint:

\[
\min_x \quad x^H Px - q^H x - x^H q + r \\
\text{s.t.:} \\
| x | = 1 \\
\| \hat{F}^H \hat{y} - x \|^2_2 \leq E_R
\]

The spectral constraint can also be simplified as:
\[ \left\| \bar{F}^H \bar{y} - x \right\|^2_2 = (\bar{F}^H \bar{y} - x)^H (\bar{F}^H \bar{y} - x) = x^H x - 2 \text{Re}\{\bar{y}^H \bar{F} x\} + \bar{y}^H \bar{F} \bar{F}^H \bar{y} = 2L - 2 \text{Re}\{\bar{y}^H \bar{F} x\} \]

where \( L = N_T N \). Hence, the interference constraint can be rewritten as
\[ \text{Re}\{\bar{y}^H \bar{F} x\} \geq (1 - E_R/2) L \]

The optimization problem (3.31) is equivalent to the following problem.
\[
\begin{cases}
\min \ x^H P x - q^H x - x^H q + r \\
\text{s.t.:} \quad |x| = 1 \\
\quad \text{Re}\{\bar{y}^H \bar{F} x\} \geq (1 - E_R/2) L
\end{cases}
\] (3.32)

Moreover, \( f(x) \) can be converted to the following function with real (as opposed to complex) variables.
\[ f(u) = u^T G u - t^T u - u^T t + r \] (3.33)

where
\[ u = [\text{Re}\{x\}^T \text{Im}\{x\}]^T \] (3.34)
\[ G = \begin{bmatrix} \text{Re}\{P\} - \text{Im}\{P\} \\ \text{Im}\{P\} & \text{Re}\{P\} \end{bmatrix} \] (3.35)
\[ t = \begin{bmatrix} \text{Re}\{q\} \\ \text{Im}\{q\} \end{bmatrix} \] (3.36)

The problem (3.32) can be rewritten as
\[
\begin{cases}
\min \ s^T (R + \lambda I) s \\
\text{s.t.:} \quad s^T E_l s = 1, \quad l = 1, 2, \ldots, L \\
\quad \bar{s}^T s \geq (1 - E_R/2) L
\end{cases}
\] (3.37)

where \( \lambda \) is an arbitrary positive number,
\[ \bar{s} = [\text{Re}\{\bar{F}^H \bar{y}\}^T \text{Im}\{\bar{F}^H \bar{y}\}^T 0]^T \] (3.38)
\[ R = \begin{bmatrix} G & -t \\ -t^T & r \end{bmatrix}, \] (3.39)
\[ s = \begin{bmatrix} \text{Re}\{x\} \\ \text{Im}\{x\} \\ 1 \end{bmatrix}, \] (3.40)

and \( E_l \) is a \( 2L + 1 \times 2L + 1 \) matrix given by
\[ E_l(i, j) = \begin{cases} 1 & \text{if } i = j = l \\ 1 & \text{if } i = l + L, j = l + L \\ 0 & \text{otherwise.} \end{cases} \] (3.41)
Note that, since
\[ s^T R s = x^H P x - q^H x - x^H q + r \]  
\[ = \sum_p \|d_p - A_p W_p x\|^2_2 \]  
\[ \geq 0 \]

, \( R \) is positive semi-definite. Further, because the problem (3.37) enforces constant modulus, i.e., \( s^T E_l s = 1 \) for \( l = 1, 2, \ldots, L \), \( \lambda s^T s \) is a constant value \( (\lambda s^T s = \lambda(L + 1)) \).

As a result, (3.10) and (3.37) are the identical optimization problems and the optimal solution of (3.10) and (the complex version of) the optimal solution of (3.37) are also identical for any \( \lambda \geq 0 \).

### 3.5.3 Sequence of Closed Form Solutions for joint CMC and spectral interfrerence constraint

Now we focus on solving (3.37). Though it is minimization of a convex objective function, it is still non-convex because of the constant modulus constraint. We propose a new sequential approach to solve (3.37) which involves solving a sequence of convex problems. Let us consider the following sequence of constrained QPs where the \( n \)-th QP is given by

\[
(CP)^{(n)} \begin{cases} 
\min_s & s^T (R + \lambda I) s \\
\text{s.t.:} & B^{(n)} s = 1 \\
& \bar{s}^{(n)} T s \geq (1 - E_R/2) L
\end{cases}
\]

where \( \bar{s}^{(n)} \) is given by:

\[
\bar{s}^{(n)} = \begin{bmatrix}
\Re \{ (\mathbf{F}^H \mathbf{y}) \odot e^{j \arg(x^{(n-1)}) - \arg(\mathbf{F}^H \mathbf{y})} \} \\
\Im \{ (\mathbf{F}^H \mathbf{y}) \odot e^{j \arg(x^{(n-1)}) - \arg(\mathbf{F}^H \mathbf{y})} \} \\
0
\end{bmatrix}
\]

and \( B^{(n)} = [b_1^{(n)}, b_2^{(n)}, \ldots, b_{L+1}^{(n)}]^T \in \mathbb{R}^{(L+1) \times (2L+1)} \) such that the line defined by \( b_l^{(n)} T s = 1 \) is a tangent to the circle \( s^T E_l s = 1 \) for \( l = 1, 2, \ldots, L \). Specifically, \( b_l \) is given by

\[
b_l^{(n)}(i) = \begin{cases} 
\cos(\gamma_l^{(n)}) & \text{if } i = l \\
\sin(\gamma_l^{(n)}) & \text{if } i = l + L \\
0 & \text{otherwise.}
\end{cases}
\]

for \( l = 1, \ldots, L \) and \( b_{L+1}^{(n)} = [0, \ldots, 0, 1]^T \) where \( \gamma_l^{(n)} = 2 \arg(x_l^{(n-1)}) - \gamma_l^{(n-1)} \) and \( x_l^{(n)} \) is the \( l \)-th elements of \( x^{(n)} \) which is the complex version of the optimal solution of (3.45), \( s^{(n)} \), that is, \( x_l^{(n)} = s_l^{(n)} + j s_{l+L}^{(n)} \) and conversely \( s^{(n)} = [\Re\{x^{(n)}\}]^T \Im\{x^{(n)}\}^T \).

Although the problem (3.45) does not result in a constant modulus solution, a sequence of such problems (in the index \( n \)) ensures a non-increasing sequence of cost function values, such that the sequence of the corresponding optimal solutions converges to constant modulus under mild condition. To recognize this, we first show that the constraints of \( CP^{(n)} \) in (3.45) are adjusted so that the feasible set of \( CP^{(n)} \) includes \( x^{(n-1)} \).

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Lemma 3.5.1. The feasible set of problem $CP^{(n)}$ contains the optimal solution of problem $CP^{(n-1)}$.

Proof. See subsection 2.3 of the Appendix.

Lemma 3.5.1 proves that the feasible set of each iteration is updated such that it contains the optimal solution of the optimization problem at the previous iteration step. If $|x^{(n)}| = 1$, then the constraints of the next problem $CP^{(n+1)}$ are the same as problem $CP^{(n)}$, which means $x^{(n+1)} = x^{(n)}$ and, hence, the algorithm converges. Convergence of the algorithm is then guaranteed by Lemma 3.5.3 which establishes that the cost function sequence is in fact non-increasing and converges. This procedure is visually illustrated in Fig. 3.13.

Now we focus on how to solve the optimization problem (3.45) at each iteration step. Note that the problem (3.45) is a convex quadratic minimization with linear equality constraints. Using the optimality conditions for problem (3.45), the sufficient and necessary Karush-Kuhn-Tucker (KKT) conditions [34] of (3.45) give the following.

$$2(R + \lambda I)s^{(n)} + B^{(n)}Tv^{(n)} - \mu^{(n)}\bar{s} = 0$$

$$B^{(n)}s^{(n)} = 1$$

Figure 3.13: Illustration of the successive solutions of eq. (3.18).
\[
\begin{align*}
\mu^{(n)}(\bar{s}^{(n)^T}s^{(n)} - (1 - E_R/2)L) &= 0 \quad (3.50) \\
\bar{s}^{(n)^T}s^{(n)} - (1 - E_R/2)L &\geq 0 \quad (3.51) \\
\mu^{(n)} &\geq 0 \quad (3.52)
\end{align*}
\]

We can directly solve these equations to find \( s^{(n)}, v^{(n)} \) and \( \mu^{(n)} \). The complementary slackness condition (3.50) implies that either \( \mu^{(n)} = 0 \) or \( \bar{s}^{(n)^T}s^{(n)} - (1 - E_R/2)L = 0 \) must be satisfied. In the case of \( \mu^{(n)} = 0 \), from Eqs. (3.48) and (3.49), we have

\[
\begin{bmatrix}
\bar{R} & B^{(n)^T} \\
B^{(n)} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
s^{(n)} \\
v^{(n)}
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
[1]
\end{bmatrix}
\quad (3.53)
\]

where \( \bar{R} = 2(R + \lambda I) \) and \( v^{(n)} \in \mathbb{R}^{(L+1) \times 1} \) is the Lagrange multiplier associated with the equality constraints. Solving (3.53) by block elimination gives

\[
\hat{s}^{(n)} = \bar{R}^{-1}B^{(n)^T}(B^{(n)}\bar{R}^{-1}B^{(n)^T})^{-1}L
\quad (3.54)
\]

If \( \hat{s}^{(n)} \) satisfies \( \hat{s}^{(n)^T}\hat{s}^{(n)} - (1 - E_R/2)L \geq 0 \), then \( s^{(n)} = \hat{s}^{(n)} \) is the optimal solution of problem \((CP^{(n)})\). However, if \( \hat{s}^{(n)^T}\hat{s}^{(n)} - (1 - E_R/2)L < 0 \), then \( \hat{s}^{(n)} \) is not the solution since it violates (3.51). Thus, \( \mu^{(n)} = 0 \) can not be valid and, therefore, it is the case that \( \bar{s}^{(n)^T}s^{(n)} - (1 - E_R/2)L = 0 \) must holds. In this case, the KKT conditions (3.48) through (3.50) are given in the matrix form by

\[
\begin{bmatrix}
\bar{R} & B^{(n)^T} & -\hat{s}^{(n)} \\
B^{(n)} & 0 & 0 \\
-\hat{s}^{(n)^T} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
s^{(n)} \\
v^{(n)} \\
\mu^{(n)}
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
[1] \\
-(1 - E_R/2)L
\end{bmatrix}
\quad (3.55)
\]

Using block elimination to solve (3.55) gives

\[
s^{(n)} = \mu^{(n)}R^{-1}(I - B^{(n)^T}\hat{R}B^{(n)}\hat{R}^{-1})\hat{s}^{(n)} + \hat{s}^{(n)}
\quad (3.56)
\]

where

\[
\hat{R} = (B^{(n)}\hat{R}^{-1}B^{(n)^T})^{-1}
\quad (3.57)
\]

\[
\mu^{(n)} = \frac{1}{\alpha^{(n)}}(\hat{s}^{(n)^T}\hat{s}^{(n)} - (1 - E_R/2)L)
\quad (3.58)
\]

\[
\alpha^{(n)} = -\left[\hat{s}^{(n)^T}\begin{bmatrix}\bar{R} & B^{(n)^T} \\
B^{(n)} & 0 \end{bmatrix}^{-1}\hat{s}^{(n)}\right]
\quad (3.59)
\]

Note that (3.51) always holds since \( \bar{s}^T\bar{s}^{(n)} - (1 - E_R/2)L = 0 \) in this case. To confirm all KKT conditions are satisfied, we have to show the dual feasibility condition (3.52) holds. The following lemma proves this.
Algorithm 7 Successive algorithm to solve (3.32)

**Inputs:** \( d_p, W_p, a_{kp} \) for \( p = -\frac{N}{2}, ..., 0, ..., \frac{N}{2} - 1, \) \( k = 1, 2, ..., K \) and \( \zeta \) (the stopping threshold).

**Output:** A solution \( x^* \) for problem (3.32).

1. Set \( n = 1 \) and an initial value for \( x^{(0)} \).
2. Compute \( B^{(n)} = [b_1^{(n)}, b_2^{(n)}, ..., b_{L+1}^{(n)}]^T \) as in (3.47).
3. Compute \( \hat{s}^{(n)} \) via eq. (3.21) and \( \bar{s}^{(n)} \) via eq. (3.46).
4. Check the following:
   - if \( \bar{s}^{(n)}^T \hat{s}^{(n)} - (1 - \frac{E_R}{2})L \geq 0 \) then
     - \( s^{(n)} = \hat{s}^{(n)} \).
   - else
     - \( s^{(n)} = \mu^{(n)} \bar{s}^{(n)} - (I - B^{(n)} R B^{(n)} R^{-1}) \bar{s}^{(n)} + \hat{s}^{(n)} \)
     - where \( \mu^{(n)} \) is defined in (3.58).
   - end if
5. Construct \( x^{(n)} \) where \( x_l^{(n)} = s_l^{(n)} + j s_{l+L}^{(n)} \) for \( l = 1, ..., L \). Check the following:
   - if \( \sum_p \| d_p - A_p W_p x^{(n)} \|_2^2 - \sum_p \| d_p - A_p W_p x^{(n-1)} \|_2^2 < \zeta \) then
     - STOP.
   - else
     - set \( n = n + 1 \) GOTO step (2).
   - end if

**Output:** \( x^* = \exp\{j \arg(x^{(n)})\} \).

---

**Lemma 3.5.2.** If \( \bar{s}^T \hat{s}^{(n)} - (1 - \frac{E_R}{2})L < 0 \) then \( \mu^{(n)} > 0 \).

**Proof.** See subsection 2.4 of the Appendix. \( \square \)

The process of solving (3.32) for fixed \( \{\phi_{kp}\} \) is given in Algorithm 7. Note that both cases lead to the closed form solutions. The complete BIC algorithm to solve (3.31) (including iteration of \( x \) and \( \{\phi_{kp}\} \)) is given in Algorithm 8. The structure of Algorithm 7 bears a high level conceptual similarity to SCF Algorithm since the solution of SCF Algorithm also employs a sequence of convex problems approach. However, the way to update the feasible set is different so that the feasible set of the next step should include the optimal solution of the previous step. Furthermore, the BIC can deal with the spectral constraint as well while SCF Algorithm does not.

**Computational Complexity:** Based on the computational cost of solving (3.55) in each iteration, the overall computational complexity of BIC is \( \mathcal{O}(FL^2.373) - \mathcal{O}(FL^3) \) [79] where \( F \) is the total number of iterations.

**Convergence Analysis:** The value of the objective function of the problem (3.32) as a function of the constant modulus version \( x^{(n)} \) is non-increasing in \( n \). This is proved in the following lemma.

**Lemma 3.5.3.** Define \( g(s) = s^T (R + \lambda I)s \). Then

\[
g(s^{(n-1)}) \geq g(s^{(n)})
\]  

(3.60)
In other words, the sequence \( \{g(s^{(n)})\}_{n=0}^{\infty} \) is non-increasing. Moreover, the sequence \( \{g(s^{(n)})\}_{n=0}^{\infty} \) converges to a finite value \( f^* \).

Proof. Denote the feasible sets of \( CP^{(n-1)} \) and \( CP^{(n)} \) by \( F_{n-1} \) and \( F_n \), respectively. From Lemma 3.5.1, \( s^{(n-1)} \in F_{n-1} \). Since \( CP^{(n)} \) is a convex problem and \( s^{(n)} \) is the optimal solution of \( CP^{(n)} \),

\[
s^{(n-1)^T}(R + \lambda I)s^{(n-1)} \geq s^{(n)^T}(R + \lambda I)s^{(n)} \quad (3.61)
\]

Therefore, the sequence \( \{f(x^{(n)})\}_{n=0}^{\infty} \) is non-increasing. Since \( f(x) \geq 0 \) for all values of \( x \), it is bounded below. Hence, it converges to a finite value \( f^* \) according to the monotone convergence theorem [83].

Fig. 3.14 verifies the cost function is non-increasing and converges. We plot the cost function in dB (blue line) and actual values (red line). The blue and red lines clearly show the non-increasing property and convergence of the proposed algorithm, respectively.

### 3.5.4 Special Case: Nullforming Beampattern Design

Null forming beampattern design can be seen as a special case of our full beampattern design. However, unlike the problem formulation in (3.10), the goal of null forming beampattern design is to form a beampattern with nulls in desired directions denoted by \( \{\theta_k\}_{k=1}^{K} \). Here, the objective function can be defined by

\[
f(x) = \sum_{p=\frac{N}{2}}^{N-1} \left\| A_p W_p x \right\|^2_2 = x^H V x \quad (3.63)
\]
Algorithm 8 Beampattern with Interference Control (BIC)

**Inputs:** \( d_{kp}, W_p, a_{kp}, \) for \( p = -\frac{N}{2}, ..., 0, ..., \frac{N}{2} - 1, \) \( k = 1, 2, ..., K \) and \( \zeta \) (the desired threshold value).

**Output:** A solution \( x^* \) for problem (3.30).

1. Set \( m = 1 \).
2. Set \( \phi_{kp}^{(m)} = \arg\{a_{kp}^H W_p x^{(m-1)}\} \) for all \( k \) and \( p \).
3. Set \( d_{kp}^{(m)} = [d_1 e^{j\phi_{kp}^{(m)}} , ..., d_K e^{j\phi_{kp}^{(m)}}]^T \).
4. Use Algorithm 7 to compute \( x^{(m)} \) with \( d_p = d_{kp}^{(m)} \) and \( x^{(0)} = x^{(m-1)} \) as inputs.
5. Check the following:
   - if \( f'(x^{(m)}) - f'(x^{(m-1)}) < \zeta \) where \( f'(x) = \sum_p \|d_p - |A_p W_p x|\|_2^2 \) then STOP.
   - else set \( m = m + 1 \) GOTO step (2).

**Output:** \( x^* = x^{(m)} \)

where \( V \) is expressed as

\[
V = \sum_{p=-\frac{N}{2}}^{\frac{N}{2}-1} W_p^H A_p^H A_p W_p \tag{3.64}
\]

Therefore, the minimization problem can be formulated as

\[
\begin{align*}
\min_x & \quad x^H V x \\
\text{s.t.:} & \quad |x| = 1 \\
& \quad \|F^H \tilde{y} - x\|_2^2 \leq ER
\end{align*}
\tag{3.65}
\]

In this case, the optimization problem reduces to problem \( CP^{(n)} \) in (3.45) with

\[
R = \begin{bmatrix}
\text{Re}\{V\} & -\text{Im}\{V\} \\
\text{Im}\{V\} & \text{Re}\{V\}
\end{bmatrix} \tag{3.66}
\]

\[
s = \begin{bmatrix}
\text{Re}\{x\} \\
\text{Im}\{x\}
\end{bmatrix} \tag{3.67}
\]

Since \( V \) is positive semi-definite and there are no linear terms in the objective function (i.e. \( q = 0 \) and \( r = 0 \)), then all the lemmas in Section 3.5.3 hold. Note that, in this case, we use only Algorithm 7 with \( R \) as mentioned above.

### 3.6 Numerical Results

In this section, we examine the performance of the proposed BIC based beampattern design and compare it to competing methods using numerical simulations. We evaluate and compare the state-of-the-art algorithms for waveform design and beampattern optimization. Note that none of the following algorithms do not exploit the spectral constraint.
For null forming beampattern design, we compare the proposed BIC to the state-of-the-art phase-only variable metric method (POVMM) method [20] and the SHAPE algorithm [58].

### 3.6.1 Nullforming Beampattern Design

For null forming beampattern design, we compare the proposed BIC to the state-of-the-art phase-only variable metric method (POVMM) method [20] and the SHAPE algorithm [58].

- **Phase-only variable metric method (POVMM) [20]:** POVMM performs null forming beampattern design by optimizing phases of the waveform under the constant modulus constraint but no spectral interference constraint. It solves a phase optimization problem using a typical iterative numerical method.

- **SHAPE [58]:** The SHAPE algorithm is a computationally efficient method of designing sequences with desired spectrum shapes. In particular, the spectral shape is optimized as a cost function subject to the constant modulus constraint but the resulting beampattern is an outcome (not explicitly controlled).

- **Wideband beampattern formation via iterative techniques (WBFIT) [37]:** The WBFIT synthesize wideband MIMO beampattern under the constant modulus or low PAR. They first find the Fourier transformed waveform in the frequency domain and then fit the DFT of the waveform to the result of the first step subject to the enforced PAR constraint.
POVMM performs null forming beampattern design by optimizing phases of the waveform under the constant modulus constraint but no spectral interference constraint. The SHAPE minimizes only the interference constraint (3.28) under the constant modulus constraint. Experimental set up is following. We suppose a linear MIMO radar antenna array of $M = 16$ elements with half-wavelength spacing and number of time samples $N = 32$. We consider $K = 3$, $\theta = \{10^\circ, 40^\circ, 120^\circ\}$. We assume a carrier frequency of $f_c = 300$ MHz and we have access to the 225-328.6 MHz and 335-400.15 MHz bands allocated for the U.S. Federal Government. We then place a notch in the band 328.6-335 MHz.

Fig. 3.15 shows the results for nullforming beampattern of BIC versus POVMM and SHAPE. Fig. 3.15a, we plot the resulting beampattern versus the angle. Note that the BIC and POVMM have nulls in the desired angles while SHAPE captures the spectral constraint without beampattern control. On the other hand, Fig. 3.15b plots the spectrum versus the frequency. Here, BIC and SHAPE effectively suppress the energy in the frequency bands where the transmission should be mitigated. However, POVMM is not working for suppression in the frequency bands. These two figures show that only the BIC can not only make nulls in beampattern but also control the spectral shape of the waveform.

In Fig. 3.15c, we introduce more practical scenario. We assume we have access to licensed television broadcasts (UHF) that occur from 470 to 698 MHz as well as the 225-328.6 MHz and 335-400.15 MHz bands as in Fig. 3.15a. Each television station is allocated 6 MHz of bandwidth and we assume there are 7 stations are licensed for operation (Ch. 21-23, 512-536 MHz and Ch. 36-39, 602-626 MHz). We plot the spectrum with different threshold values in Fig. 3.15c and it is shown that the stronger threshold supresses more in spectrum. It is also shown that we can design the spectrum that can be adjusted to the distance to the stations. For example, we assume that the stations of Ch. 36-39 is closer to the radar than Ch. 21-23. Therefore, the spectrum is more suppressed in those bands. In Fig. 3.15d, we show the converged cost function value of POVMM and the proposed BIC method since these two methods use the same cost function. The BIC method achieves similar cost function values or lower for $E_R \geq 0.03$ even after introducing the spectral constraint.

### 3.6.2 Full Beampattern Design

For wideband beampattern design, we compare BIC to the state-of-the-art WBFIT method [37]. The experimental set-up used in Fig. 3.16 and Fig. 3.17 is following. The number of transmit antennas $M = 10$, the number of time samples $N = 32$, the carrier frequency of the transmit signal $f_c = 1$ GHz and the bandwidth $B = 200$ MHz and the spatial angle is divided into $K = 180$ grid points.

In Fig. 3.16, we place a notch in the band 910-932 MHz and consider the following desired transmit beampattern

$$d(\theta, f) = \begin{cases} 
1 & \theta = [95^\circ, 120^\circ] \\
0 & \text{Otherwise.} 
\end{cases} \quad (3.68)$$

Fig. 3.16 shows the angle-frequency plot of the beampattern for WBFIT method (no spectral constraint) and BIC with the spectral constraint ($E_R = 0.01$). The BIC method
is able to keep the energy of the waveform in particular frequency band low enough thanks to the spectral interference constraint as well as shows higher suppression at the undesired angles compared to WBFIT that has no spectral constraint.

In Fig. 3.17, we simulate more challenging but more practical scenario. We assume that the beampattern should be suppressed at the angles of $40^\circ$ through $80^\circ$ in the frequency band $[943.75 \text{ MHz}, 981.25 \text{ MHz}]$ and at $120^\circ$ through $160^\circ$ in $[962.5 \text{ MHz}, 1,000 \text{ MHz}]$, that is,

\[
\begin{align*}
  d(\theta, f) = \begin{cases} 
    0 & \theta = [40^\circ, 80^\circ] \text{ and } f = [943.75, 981.25] \\
    0 & \theta = [120^\circ, 160^\circ] \text{ and } f = [962.5, 1000] \\
    1 & \text{Otherwise}
  \end{cases}
\end{align*}
\] (3.69)

This ideally appears as black boxes in the angle-frequency beampattern plots. We also assume that transmission should be restricted at all directions in the frequency band $[1.025 \text{ GHz}, 1.0625 \text{ GHz}]$. This restriction can be performed by the spectral constraint. First, as shown in Fig. 3.17b, since WBFIT does not have the spectral constraint, the notch of frequency band $[1.025 \text{ GHz}, 1.0625 \text{ GHz}]$ does not appear. Second, the black boxes are not seen so clearly in Fig. 3.17b. Lastly, WBFIT suppresses the energy of the waveform unnecessarily in the frequency band where we do not have any restriction (e.g. $[1.0625 \text{ GHz}, 1.1 \text{ GHz}]$). On the other hand, the proposed BIC effectively suppresses and restrict the transmitted energy in the desired frequency bands and angles and generate enough power anywhere else.

Table 3.7: Converged cost function values in dB

<table>
<thead>
<tr>
<th>Method</th>
<th>Fig. 3.16</th>
<th>Fig. 3.17</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unconstrained</td>
<td>19.1277</td>
<td>15.4681</td>
</tr>
<tr>
<td>WBFIT</td>
<td>26.4294</td>
<td>34.7744</td>
</tr>
<tr>
<td>BIC</td>
<td>28.5934</td>
<td>31.3286</td>
</tr>
</tbody>
</table>

Finally we compare values of the converged cost function of each algorithm for both scenario in Figs. 3.16 and 3.17 in Table 3.7. Note that the proposed BIC shows comparable or less value of the cost function though it enforces the spectral constraint.

### 3.7 Conclusion

Our work achieves tractable beampattern design by waveform synthesis for MIMO radar in the presence of constant modulus. The central idea of our analytical contribution is to successively achieve constant modulus (at convergence), while solving an equality constrained quadratic program in each step of the sequence. Because each such problem in the sequence has a closed form SCF and BIC, this makes our method computationally attractive. We establish new analytical properties of the SCF algorithm as well as the converged solution. Further, we show experimentally that the proposed SCF can achieve superior beampattern accuracy compared to state-of-the-art and incorporates
more constraints on the beampattern design (in the case of BIC), namely the CMC and the spectral constraint.
Figure 3.16: Plot of the beampattern. (a) unconstrained (b) WBFIT method (c) BIC (Proposed method)
Figure 3.17: Plot of the beampattern. (a) unconstrained (b) WBFIT method (c) BIC (Proposed method)
Chapter 4

Tractable Approach for Ambiguity Function Shaping in Cognitive Radar
4.1 Introduction

The waveform of the radar signal has a major influence on the range-Doppler response of the radar system, which is usually referred as the ambiguity function [3, 84, 85]. A well designed waveform can increase the Signal-to-Interference-Ratio (SIR) [45], controls the Doppler/range resolutions of the system and suppresses the unwanted returns. The design criteria can be extended to achieve a suitable ambiguity function [46], a desired beampattern [40] or a suitable autocorrelation function with mutual information depending on different application domains [51]. Tractable radar waveform design remains an open challenge given that performance metrics need to be balanced with demanding practical constraints [86].

Waveform design methods for the radar system ambiguity function have been studied intensively and can be categorized into the fast-time or the slow-time ambiguity function. The fast-coding ambiguity function, also known as the classic waveform ambiguity function, tackles the problem of designing a desired ambiguity function [38, 40, 41] or a given zero-Doppler ambiguity function cut [42, 54, 87].

On the other hand, the slow-time ambiguity function tackles the problem of minimizing the unwanted returns by shaping the range-Doppler response of the matched filter. In addition, the problem becomes much harder if we take the hardware constraint into consideration by satisfying the constant modulus constraint (CMC). The importance of the waveform CMC has been well documented and analyzed in terms of performance loss [23]. Most radar systems utilize non-linear power amplifiers which cannot be efficiently utilized without CMC. More specifically, the output of the amplifier will be a clipped version of the optimized waveform, which often leads to a significant degradation in the system performance.

Our work focuses on (slow-time) ambiguity function shaping in a cognitive radar scenario where the radar can exploit the ideal range-doppler interference map of the environment. In this setting, prior investigations are limited, particularly in the presence of CMC. A key effort includes the work of Aubry et. al. [39] which also assumes that the radar has knowledge of the scattering environment. Analytically, they approach the quartic cost function via a polynomial time waveform optimization procedure based on Maximum Block Improvement (MBI) [39]. Additionally, constant modulus is handled in their work via (a large number of) random sample generation, and the combination of the same with MBI leads to an effective but computationally burdensome approach. Our goal is to break the trade-off between CMC constrained optimization of the quartic cost function that provides the desired ambiguity function shaping vs. the associated computational cost.

Specifically, this chapter makes the following contributions:

- **ASR: A new tractable analytical framework for ambiguity function shaping under waveform CMC.** To overcome the challenges mentioned above, we develop a new algorithm for ambiguity function shaping that involves solving the non-convex quartic problem using a sequence of convex Quadratic Programs (QP) where the cost function as well as constraints are updated in each iteration of the sequence. We show further that each of these QPs has a closed form solution leading to significantly reduced complexity.
• **Analysis of the ASR algorithm.** We formally prove that the sequence of cost functions representing the energy of the unwanted returns of the range-Doppler response, that occurs in the proposed ASR algorithm is non-increasing. A new strategy is developed that adapts the definition of each problem in the sequence depending on whether the iteration index is **even** or **odd** and we formally show this leads to guaranteed convergence.

• **Experimental insights and validation.** Experimental validation is performed via simulations. The proposed ASR is shown to achieve a much more desirable ambiguity function against state of the art alternatives.

### 4.2 System Model

Consider a monostatic radar system transmitting $N$-coherent burst of slow-time coded pulses, denoted as $x = [x(1), x(2), \ldots, x(N)]^T \in \mathbb{C}^N$ and the receiving pulses as $v = [v(1), v(2), \ldots, v(N)]^T \in \mathbb{C}^N$. The observed codes $v$ can be expressed according to a range-azimuth cell [39]:

$$ v = \alpha_T x \odot p(\nu_{dt}) + d + n, \quad (4.1) $$

where $\alpha_T$ is a complex parameter accounting for channel propagation and backscattering effects from the target within the range-azimuth bin of interest, $s$ is the radar code send $x = [x(1), \ldots, x(N)]^T \in \mathbb{C}^N$. $p(\nu_{dt}) = [1, e^{j2\pi \nu_{dt}}, \ldots, e^{j2\pi(N-1)\nu_{dt}}]^T$ where $\nu_{dt}$ is the normalized target Doppler frequency.

**Interfering vector:** $d$ is the $N$-dimensional column vector containing the interfering echo samples which can be related to cluster returns from different rang-azimuth bins or non-interest targets echoes which include non-threatening targets. Thus the interfering samples $d$ can be expressed as:

$$ d = \sum_{k=1}^{N_t} \rho_k J^r_k (x \odot p(\nu_{dk})) \quad (4.2) $$

where, the range position is denoted as $r_k \in \{0, 1, \ldots, N-1\}$. It denotes $\rho_k$ as the echo complex amplitude, $\sigma^2_k = \mathbb{E}(|\rho_k|^2)$ as the echo mean power from $k$-th interference scatterer.

Also the shift matrix $J^r$ is denoted as:

$$ J^r(l_1, l_2) = \begin{cases} 1, & \text{if } l_1 - l_2 = r \\ 0, & \text{otherwise} \end{cases} \quad (l_1, l_2) \in \{1, \ldots, N\}^2 \quad (4.3) $$

**Noise samples:** $n$ is the $N$-dimensional column vector of the noise samples which is assumingly uncorrelated from $d$. Thus, $\mathbb{E}[n] = 0$ and $\mathbb{E}[nn^H] = \delta^2_n \mathbf{I}$.

Combining (4.1) and (4.2), the matched filter is $MF = x \odot p(\nu_{dt})$, and the output of
filtered received codes is $MF \ast \mathbf{v}$:

$$
(x \odot p(\nu_{d_T}))^H \mathbf{v} = \alpha T \|x\|^2 + \left( x \odot p(\nu_{d_T}) \right)^H \mathbf{n} + \sum_{k=1}^{N_t} g_k (x \odot p(\nu_{d_T}))^H J_{rk} (x \odot p(\nu_{d_k}))
$$

(4.4)

Since $\mathbf{n}$ is uncorrelated with $\mathbf{d}$, the energy of the disturbances in the match filter output can be expressed as:

$$
E[|\text{disturbances in the output}|^2] = E\left[ \left| (x \odot p(\nu_{d_T}))^H \mathbf{n} \right|^2 \right] + \sum_{k=1}^{N_t} \left| g_k (x \odot p(\nu_{d_T}))^H J_{rk} (x \odot p(\nu_{d_k})) \right|^2
$$

(4.5)

$$
= \sigma_n^2 \|x\|^2 + \sum_{k=1}^{N_t} \sigma_k^2 \|x\|^2 E[g_x(r_k, \nu_{d_k} - \nu_{d_T})]
$$

where $g_x(r, \nu)$ is the Ambiguity Function (AF) of transmitted code $x$ which is defined as:

$$
g_x(r, \nu) = \frac{1}{\|x\|^2} |x^H J^{*} (x \odot p(\nu))|^2
$$

(4.6)

In the AF, $r$ represents the time-lag which takes value in $\{0, 1, ..., N - 1\}$ and $\nu \in [-\frac{1}{2}, \frac{1}{2}]$ is the normalized Doppler frequency. Given a $(r, \nu)$ pair, the AF $g_x(r, \nu)$ gives the Doppler response from an interfering patch corresponding with a Doppler frequency of $\nu$ located $r$ time-lag away.

**Optimization Problem Formulation:** Without loss of the generality, all the Doppler frequencies can be expressed in terms of the difference with respect to the $\nu_{d_T}$ by centering them to target frequency. The normalized Doppler interval $[-\frac{1}{2}, \frac{1}{2}]$ is divided into $N_\nu$ bins, where $N_\nu$ is decided in experiments. Thus, the normalized frequency can be represented in discrete manner: $\nu_h = -\frac{1}{2} + \frac{h}{N_\nu}$, $h = 0, ..., N_\nu - 1$. For each $h$, there is a normalized Doppler frequency $\nu_h$. For a given $\nu_{d_k}$ of $k$-th interference, which $\nu_{d_k} \sim \mathcal{U}(\bar{\nu}_{d_k} - \frac{\epsilon_k}{2}, \bar{\nu}_{d_k} + \frac{\epsilon_k}{2})$, $I_k := [\bar{\nu}_{d_k} - \frac{\epsilon_k}{2}, \bar{\nu}_{d_k} + \frac{\epsilon_k}{2}]$. Then the demo $\nu_{d_k}$ can be approximated by discretized normalized Doppler frequency $\nu_h$ of $h \in B_k := \{h : h = 2, ..., 7\}$. Thus, each statistical expectation $E[g_x(r_k, \nu_{d_k})]$ in the disturbance power can be approximated with the sample mean over the Doppler bins indexed in $B_k$:

$$
E[g_x(r_k, \nu_{d_k})] \approx \frac{1}{\text{Card}(B_k)} \sum_{h \in B_k} g_x(r_k, \nu_h)
$$

(4.7)
Therefore, the total disturbance power at the output of the matched filter can be expressed as:

\[
N_t \sum_{k=1}^{N_t} \sigma_k^2 \|x\|^2 \left( \frac{1}{\text{Card}(B_k)} \sum_{h \in B_k} g_x(r_k, \nu_h) \right) + \sigma_n^2 \|x\|^2
\]

\[
= \sum_{r=0}^{N-1} \sum_{h=0}^{N_{\nu} - 1} \|x\|^2 g_x(r, \nu_h) \times \left( \sum_{k=1}^{N_t} \delta(r - r_k) \mathbf{1}_{B_k}(h) \frac{\sigma_k^2}{\text{Card}(B_k)} \right)
+ \sigma_n^2 \|x\|^2 \tag{4.8}
\]

\[
= \sum_{r=0}^{N-1} \sum_{h=0}^{N_{\nu} - 1} p(r, h) \|x\|^2 g_x(r, \nu_h) + \sigma_n^2 \|x\|^2
\]

where, \( \mathbf{1}_{B_k}(h) \) denotes the indicator function of the set \( B_k \) and \( p(r, h) \) is the range-Doppler interference map:

\[
p(r, h) = \sum_{k=1}^{N_t} \delta(r - r_k) \mathbf{1}_{B_k}(h) \frac{\sigma_k^2}{\text{Card}(B_k)} \tag{4.9}
\]

Given a range-Doppler bin \((r, \nu_h)\), \( p(r, h) \) indicates the interference power in that bin. Altering the \( p(r, h) \) gives different testing scenarios. Therefore, this gives us the objective function \( \phi(x) \):

\[
\phi(x) = \sum_{r=0}^{N-1} \sum_{h=0}^{N_{\nu} - 1} p(r, h) \|x\|^2 g_x(r, \nu_h) \tag{4.10}
\]

which is the interference power at the output of the matched filter.

The primary optimization problem will be:

\[
P = \left\{ \begin{array}{l}
\min \quad \phi(x) \\
\text{s.t.} \quad \|x\| = 1
\end{array} \right. \tag{4.11}
\]

The objective function can be simplified as:

\[
\phi(x) = \sum_{r=0}^{N-1} \sum_{h=0}^{N_{\nu} - 1} p(r, h) \|x\|^2 \mathbf{J}^r \text{diag}(\mathbf{p}(\nu_h)) \mathbf{x}^2
\]

\[
= \sum_{i=1}^{N \times N_{\nu}} \left| \mathbf{x}^H \sqrt{p(r, h)} \mathbf{J}^r \text{diag}(\mathbf{p}(\nu_h)) \mathbf{x} \right|^2 =: f(x) \tag{4.12}
\]

where \( i \) is a ono-to-one mapping index that:

\[
i \in \{1, ..., N \times N_{\nu} \} \rightarrow (r, h) \in \{0, ..., N - 1\} \times \{0, ..., N_{\nu} - 1\}
\]

Then the optimization problem \( P \) becomes complex quartic minimization problem.
4.3 Proposed method: Adaptive Sequential Refinement (ASR)

The summation term in (4.12) can be written as:
\[ |\mathbf{x}^H \mathbf{C}_{(r,h)} \mathbf{x}|^2 \]
where \( \mathbf{C}_{(r,h)} = \sqrt{p(r,h)} \mathbf{J} \text{diag}(\mathbf{p}(\nu_h)) \), note that \( \nu_h := -\frac{1}{2} + \frac{h}{N} \), and each \( i \) maps to a \((r,h)\) pair, \( M := N \times N_v \). Then the objective function can be expressed as:
\[
f(x) = \sum_{i=1}^{M} |\mathbf{x}^H \mathbf{C}_i \mathbf{x}|^2 = \sum_{i=1}^{M} (\mathbf{x}^H \mathbf{C}_i \mathbf{x}) (\mathbf{x}^H \mathbf{C}_i^H \mathbf{x})^H
\]
\[
= \sum_{i=1}^{M} \mathbf{x}^H \mathbf{C}_i \mathbf{x}^H \mathbf{C}_i^H \mathbf{x} = \mathbf{x}^H \left[ \sum_{i=1}^{M} \mathbf{T}_i(x) \right] \mathbf{x}. \tag{4.14}
\]

Let \( \mathbf{T}(x) = \sum_{i=1}^{M} \mathbf{T}_i(x) \).

The optimization problem is hence re-written as
\[
Q = \begin{cases} 
\min_x & \mathbf{x}^H \mathbf{T}(x) \mathbf{x} \\
\text{s.t.} & |\mathbf{x}| = 1
\end{cases} \tag{4.15}
\]

Note that if \( \mathbf{x} \) is fixed, the cost function of Eq (4.15) reduces to a quadratic. This motivates us to develop an iterative approach wherein a sequence of quadratic problems is solved such that cost function converges. Further, to handle constant modulus, we work with relaxations of the CMC to a parametrized equality constraint while updating the parameters in each iteration so they eventually converge to constant modulus.

In particular, when written with real-valued variables each problem in the sequence takes the following form:
\[
CQ_{(n)} = \begin{cases} 
\min_{\mathbf{s}} & \mathbf{s}^T \tilde{\mathbf{R}}_{(n)} \mathbf{s} \\
\text{s.t.} & \mathbf{G}_{(n)} \mathbf{s} = 1
\end{cases} \tag{4.16}
\]

where \( \tilde{\mathbf{R}}_{(n)} = \mathbf{R}_{(n)} + \lambda \mathbf{I} \),
\[
\mathbf{R}_{(n)} = \begin{bmatrix} \text{Re}\{\mathbf{T}_{(n)}\} & -\text{Im}\{\mathbf{T}_{(n)}\} \\ \text{Im}\{\mathbf{T}_{(n)}\} & \text{Re}\{\mathbf{T}_{(n)}\} \end{bmatrix} \text{ and } \mathbf{s} = \begin{bmatrix} \text{Re}\{\mathbf{x}\} \\ \text{Im}\{\mathbf{x}\} \end{bmatrix} \tag{4.17}
\]

Note that \( \mathbf{R}_{(n)} \) is derived from \( \mathbf{T}_{(n)} \), where \( \mathbf{T}_{(n)} \) is defined as follows:
\[
\mathbf{T}_{(n)} = \begin{cases} 
\sum_{i=1}^{M} \mathbf{C}_i \mathbf{x}_{(n-1)}^H \mathbf{x}_{(n-1)}^H \mathbf{C}_i^H, & n = 2k + 1 \\
\sum_{i=1}^{M} \mathbf{C}_i^H \mathbf{x}_{(n-1)}^H \mathbf{x}_{(n-1)}^H \mathbf{C}_i, & n = 2k
\end{cases} \tag{4.18}
\]

Note that we define \( \mathbf{s}_{(n)} \) as the solution of \( CQ_{(n)} \) and \( \mathbf{x}_{(n)} \) is the complex version of \( \mathbf{s}_{(n)} \).
Remark: Note that the definition of $T(n)$ is different depending on whether the iteration index is odd or even. We prove that such a choice enables monotonic decrease of the cost function and guaranteed convergence (see Lemma 1).

Note further that $\lambda \mathbf{I}$ is added to $R(n)$ to ensure that the resulting matrix $\bar{R}(n)$ is positive definite, which ensures that $CQ(n)$ is a strictly convex problem. The addition of this term crucially does not alter the cost function so long as $x$ is constant modulus and hence $s^T s = N$.

$$G(n)(i,j) = \begin{cases} 
\cos(2 \arg x_{l(n-1)} - \delta x_{l(n-1)}) , & \text{if } i = j = l \\
\sin(2 \arg x_{l(n-1)} - \delta x_{l(n-1)}) , & \text{if } i = l, j = n + N \\
0, & \text{otherwise}
\end{cases} \quad (4.19)$$

where $x_{l(n-1)}$ is the $l$-th element of the solution in problem $CQ(n-1)$, $\delta x_{l(n-1)} = 2 \arg x_{l(n-2)} - \delta x_{l(n-2)}$. Using the KKT condition [88] and block elimination shows that:

$$s(n) = \bar{R}^{-1}(n) G^T(n) \left( G(n) \bar{R}^{-1}(n) G^T(n) \right)^{-1} 1 \quad (4.20)$$

and the stage solution of the $CQ(n)$ is denoted as $x(n)$.

After certain iterations or the solution $x(n)$ of $CQ(n)$ satisfying the constant modulus (the offset error $\|x(n)\|_N - N < \epsilon$), the algorithm gives the final solution $x$.

**Lemma 4.3.1.** Let $s(n)$ and $s(n-1)$ be the solutions of $CQ(n)$ and $CQ(n-1)$, respectively. Define the function $g(s(n))$ as:

$$g(s(n)) = \sum_{i=1}^M s_{(n)}^T A_i s(n) s_{(n)}^T A_i s(n) + \lambda s_{(n)}^T s(n)$$

$$= f(x(n)) + \lambda x^H(n) x(n)$$

where

$$A_i = \begin{bmatrix} \text{Re}\{C_i\} & -\text{Im}\{C_i\} \\
\text{Im}\{C_i\} & \text{Re}\{C_i\} \end{bmatrix} \quad \text{and} \quad s(n) = \begin{bmatrix} \text{Re}\{x(n)\} \\
\text{Im}\{x(n)\} \end{bmatrix}$$

Then there exits a finite $\lambda > 0$ such that:

$$g(s(n-1)) \geq g(s(n))$$

i.e. the sequence $\{g(s(n))\}_{n=0}^\infty$ is non-increasing and converges to a finite value $g^*$.

**Proof.** See Section 2.5 of the Appendix. \qed

A step by step description of the waveform design using ASR Algorithm is showing in Algorithm 1.
Algorithm 9 ASR

1: Initial setup of \( N, N_\nu, \epsilon, \{ (r, h) \} \) of interferences
2: Randomly initialize \( x_0 \)
3: \( i \leftarrow 1 \)
4: \textbf{for} \((r, h) \) in \( \{ (r, h) \} \) \textbf{do}
5: \hspace{1em} Compute \( J^r \) as (4.3)
6: \hspace{1em} Compute \( \nu_h \) and \( p(\nu_h) \)
7: \hspace{1em} \( C_i \leftarrow \sqrt{p(r, h) J^r \text{diag}(p(\nu_h))} \), \( i \leftarrow i + 1 \)
8: \textbf{end for}
9: \( n \leftarrow 1 \)
10: \textbf{while} \( \| x(n) \|_1 > \epsilon \) \textbf{do}
11: \hspace{1em} if \( n \% 2 == 1 \) then
12: \hspace{2em} \( T(n) \leftarrow \sum_{i=1}^{M} C_i x(n-1) x_H^{(n-1)} C_i^H \)
13: \hspace{1em} else
14: \hspace{2em} \( T(n) \leftarrow \sum_{i=1}^{M} C_i^H x(n-1) x_H^{(n-1)} C_i \)
15: \hspace{1em} end if
16: \hspace{1em} Compute \( \tilde{R}_n \) and \( G_n \) as (4.3.1) and (4.19)
17: \hspace{1em} \( s_n \leftarrow \tilde{R}_n^{-1} G_n^T (G_n \tilde{R}_n^{-1} G_n^T)^{-1} 1 \)
18: \hspace{1em} Compute \( x_n \) by inverse (4.3.1)
19: \hspace{1em} \( n \leftarrow n + 1 \)
20: \textbf{end while}
21: \( x \leftarrow x(n) \)

4.4 Experiments and analyses

In this section, we will compare our proposed ASR Algorithm to the state of the art approach by [39] which involves an MBI type method with a quadratic subroutine and hence called MBIQ.

Specifically in Figure. 4.6, the red blocks represent the regions that range-Doppler returns are unfavorable and need to be suppressed. To display these regions, the Doppler frequency axis is discretized into \( N_\nu = 50 \) bins which gives the \( \nu_h = -\frac{1}{2} + \frac{h}{N_\nu} \), \( h = 0, \ldots, 50 \). For the discretization, \( h = 0 \) is corresponding to \( [-\frac{1}{2}, -\frac{1}{2} + \frac{2}{N_\nu}] \cup [\frac{1}{2} - \frac{2}{N_\nu}, \frac{1}{2}] \). Moreover, in each block, the interference power is assigned to be uniformly distributed, as following:

\[
p(r, h) = \begin{cases} 
1, & (r, h) \in \{2,3,4\} \times \{35,36,37,38\} \\
1, & (r, h) \in \{3,4\} \times \{18,19,20\} \\
0, & \text{otherwise}
\end{cases}
\]

The Signal-to-Interference-Ratio (SIR) provides the numerical assessment of the results and is defined as:

\[
\text{SIR} = \frac{N^2}{\sum_{r=1}^{N} \sum_{h=1}^{N_\nu} p(r, h) \|x\|^2 g_x(r, \nu_h)}
\]
The cost function and SIR performance are examined as a function of the iteration index. The $|f(x)|$ is decreasing and SIR is monotonically increasing as expected using ASR. At iteration 200, the $|f(x)|$ drops below 2.5 and the SIR reaches 28 dB which is an ideal value to produce a practical code $x$. The code provided by the algorithm is indeed constant modulus as designed. Figure 4.1 shows the SIR of ASR compared to MBIQ averaged over 50 trials for different values of $N$. ASR Algorithm achieves a higher average SIR compared to MBIQ.

The contours of the ambiguity function are plotted in Figure 4.6, where the red box shows the blocks as in the experiment setup interferences. From the Figure, the AF indeed provides a lower return within these blocks and the average return is $-48.3$ dB. These “punch holes” at the red blocked locations in the AF indicates the desired suppression in the range-Doppler returns is achieved.

Figure 4.2 plots $g_x(r, \nu_h)$ cuts at different $r$ and $\nu$ values. In particular, the response within the red lines is the desired range, where the response should be suppressed. From the Figure, ASR gives lower returns than MBIQ within the unwanted return regions at cuts $g_x(2, \nu_h)$, $g_x(3, \nu_h)$ and $g_x(4, \nu_h)$. Figure 4.2 (d) also shows the autocorrelation function of $x$ for $N = 20$. Because we did not pose any constraints on the zero-Doppler cut of the ambiguity function, some side-lobe peaks can be observed in the plotted function. Our future research will seek to improve upon this result.

The run times of ASR and MBIQ [39] are reported in Table 4.1. For $N = 20$, ASR takes less than 1s to finish 1000 iterations while MBIQ takes 39.56s for 100 iterations. The reduced complexity makes ASR attractive for practical deployment.
Figure 4.2: AF cuts and autocorrelation function

Table 4.1: Testing $N = 20$, the execution speeds

<table>
<thead>
<tr>
<th>Avg. over 50 trails</th>
<th>Time / 100 iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>MBIQ</td>
<td>39.56s</td>
</tr>
<tr>
<td>ASR</td>
<td>0.071s</td>
</tr>
</tbody>
</table>

4.5 Conclusion

This chapter proposes a Adaptive Sequential Refinement (ASR) algorithm to suppress the unwanted radar signal returns over given range-Doppler bins with the constant modulus constraint. The proposed algorithm has a closed form at each iteration and a unique definition of subproblems depending on the iteration process which guarantees the monotonicity and convergence. Moreover, ASR provides a magnificent improvement of the execution time. Overall evaluation shows the real world usage and the superior performance of ASR comparing to state-of-the-art methods.
Chapter 5

Conclusion

This dissertation developed a tractable and cost-efficient framework for radar waveform design under practical constraints. Here, I briefly summarize the contributions of this dissertation.

5.1 Summary of Contributions

- **Successive QCQP Refinement for MIMO Radar Waveform Design Under Practical Constraints** We developed a new tractable analytical framework for SINR maximization that jointly enforces both CMC and SC. In contrast to existing work, which relies on SDR with randomization and its extensions [11,35,36,43,44], our approach involves solving a sequence of convex problems (each a QCQP). In each iteration of the sequence, the designed waveform satisfies the similarity constraint while constant modulus is successively achieved at convergence; hence, the method is called – Successive QCQP Refinement (SQR). SQR achieves tractable waveform design for MIMO radar in the presence of constant modulus and similarity constraints. The central idea of our analytical contribution is to successively refine and achieve constant modulus (at convergence), while solving a sequence of quadratically constrained quadratic programs, such that each optimization in the sequence satisfies a similarity constraint. We show this approach called SQR can achieve superior SINR and beam pattern with desirable suppression results against state of the art, remarkably at a lower computational cost.
• **Tractable Transmit MIMO Beampattern Design Under Constant Modulus and Spectral Interference Constraint** We developed a new cost-efficient framework for beampattern design that jointly enforces both CMC and spectral interference constraint. It involves solving the hard non-convex problem of beampattern design using a sequence of convex Equality Constrained Quadratic Programs, each of which has a closed form solution, such that constant modulus is achieved at convergence. Because each QP in the sequence has a closed form, the proposed successive closed forms (SCF) and beampattern with interference control (BIC) algorithms have significantly lower complexity than competing methods that incorporate CMC. We formally prove that the sequence of cost functions representing the deviation from the desired beampattern, that occurs in the proposed SCF and BIC algorithm are non-increasing. Further, we show experimentally that the proposed SCF can achieve superior beampattern accuracy compared to state-of-the-art and incorporates more constraints on the beampattern design (in the case of BIC), namely the CMC and the spectral interference constraint.

• **Tractable Approach for Ambiguity Function Shaping in Cognitive Radar** we developed a new tractable analytical framework for ambiguity function shaping under waveform CMC. The new approach called Adaptive Sequential Refinement (ASR). It involves solving the non-convex quartic problem using a sequence of convex Quadratic Programs (QP) where the cost function as well as constraints are updated in each iteration of the sequence. We show further that each of these QPs has a closed form solution leading to significantly reduced complexity. ASR provides a magnificent improvement of the execution time. Overall evaluation shows the real world usage and the superior performance of ASR comparing to state-of-the-art methods.
5.2 Future Research

5.2.1 Cooperative Radar-Communication Systems Optimization

In the research proposed in Chapters 2 and 3, the communication and radar systems do not co-operate and minimal knowledge is assumed by one of the other. Here, we assume ‘light’ information sharing regime (i.e. requiring only low rate side channel) between communication and radar systems that enables new cooperative optimizations that could significantly improve performance over prior (one-sided) approaches; however the achievable gains depend on a) what information is shared and b) how often.

The literature on cooperative optimizations is sparse, and the few existing approaches adopt different perspectives from ours. Petropulu et al. have proposed a cooperative design of MIMO radar and communication systems \[89–92\] based on optimizing a sampling matrix for MIMO radars using matrix completion (MIMO-MC) \[89,90\] or a precoding matrix of MIMO radar \[91,92\] jointly with a communication transmit signal covariance matrix. They minimize the effective interference power (EIP) or maximize SINR at radar subject to the communication transmit power constraint and the communication capacity constraint. A single optimization problem which requires strong cooperation of radar and communication systems is solved that makes its transition to practical systems infeasible. Further their approaches employ orthogonal waveform codes while crucially excluding the constant modulus property. Therefore, a future approach that is based on double optimization problems, one at the communication side and the other at the radar system, is practical relevant under CMC, SC and/or spectral interference constraint.

5.2.2 Waveform Design for Autonomous Vehicle

Waveform design for autonomous vehicle has become an emerging research topic lately \[93\]. Unlike convectional stationary radars, autonomous vehicle radars require multi-target detections and accurate speed estimation, see Figure 5.1. This requirements add more constraints on the shape of the ambiguity function as well as the practical constraints mentioned in this dissertation. Therefore, a new optimization problems is needed to tackle such waveform design for autonomous vehicles.
5.2.3 MIMO Radar Ambiguity Function Shaping

Ambiguity function shaping for MIMO radar system under constant modulus constraints has received little attention in the existing literature. Due to the complexity of the SISO ambiguity function (as it turned out to be complex quartic function as described in Chapter 4) MIMO ambiguity function shaping has been avoided. However, as mentioned in section 5.2.2, MIMO ambiguity function shaping is relevant for autonomous vehicles as waveform diversity at the transmitter enables accurate target direction estimation [93], as shown in Figure 5.1.
1 Proofs of Chapter 2

1.1 Proof of Lemma 2.3.1

Proof. Let $v^{(n)} = v^{(n-1)} + y$ or $y = v^{(n)} - v^{(n-1)}$. Then:

\[
\begin{align*}
&v^{(n)} T P v^{(n)} - v^{(n-1)} T P v^{(n-1)} \\
&= (v^{(n-1)} + y)^T P (v^{(n-1)} + y) - v^{(n-1)} T P v^{(n-1)} \\
&= v^{(n-1)} T P y + y^T P v^{(n-1)} + y^T P y \\
&= 2 v^{(n-1)} T P y + y^T P y \\
&= 2 v^{(n-1)} T P (v^{(n)} - v^{(n-1)}) + y^T P y \\
&= 2 v^{(n-1)} T P v^{(n)} - 2 v^{(n-1)} T P v^{(n-1)} + y^T P y \geq 0
\end{align*}
\]

where the inequality holds since we enforced $v^{(n-1)} T P v^{(n)} \geq v^{(n-1)} T P v^{(n-1)}$ as a constraint in the problem $RC^{(n)}$ and because $P$ is positive semi-definite.

The constraint $A_n v \succeq c$, which successively reduces the feasible set depends on the solution $v^{(n-1)}$ of $RC^{(n-1)}$.

\[ \square \]

1.2 Proof of Lemma 2.3.2

Proof. Let $\lambda_{max}$ be the largest eigenvalue of $P$. Using Rayleigh-Ritz theorem for the Hermitian matrix $P$ we have:

\[
\begin{align*}
\frac{v^{(n)} T P v^{(n)}}{v^{(n)} T v^{(n)}} &\leq \lambda_{max} \\
\Rightarrow v^{(n)} T P v^{(n)} &\leq \lambda_{max} v^{(n)} T v^{(n)} \leq \lambda_{max} \\
\Rightarrow v^{(n)} T P v^{(n)} &\leq \lambda_{max} \\
\Rightarrow 0 &\leq \text{SINR}^{n} \leq \lambda_{max}
\end{align*}
\]


⇒SINR\textsuperscript{n} is bounded

Since SINR\textsuperscript{n} is a monotone non-decreasing (from Lemma 2.3.1) and bounded, then from the Monotone Convergence Criterion [83] the sequence SINR\textsuperscript{n} converges to a finite value SINR\textsuperscript{⋆}.

1.3 Proof of Lemma 2.3.3

Proof. Denote the feasible set of RCND\textsuperscript{(n−1)} and RCND\textsuperscript{(n)} by \( \mathcal{F}_{n−1} \) and \( \mathcal{F}_n \), respectively. Since RCND\textsuperscript{(n)} is a convex problem then \( v^{(n)} \) is the global optimal solution of RCND\textsuperscript{(n)}. Therefore, for every \( v \in \mathcal{F}_n \) we have [34]:

\[
v^T S v \leq v^{(n)}^T S v^{(n)}
\]

Our design ensures that the constraints of problem RCND\textsuperscript{(n)} are adjusted to include the optimal solution of the previous problem \( v^{(n−1)} \), as shown in Fig. 2.5, then \( v^{(n−1)} \in \mathcal{F}_n \). To show this, we need to prove that \( A_n v^{(n−1)} \succeq c \) since all other constraints are the same throughout all refinements. The \( k \)-th row of \( A_n v^{(n−1)} \) in problem (2.15) is given by:

\[
\begin{align*}
\cos(\arg x_i^{(n−1)}) \frac{v^{(n−1)}(k)}{\cos(\delta/2)} + \\
\sin(\arg x_i^{(n−1)}) \frac{v^{(n−1)}(k + N_T N)}{\cos(\delta/2)} \\
= \cos(\arg x_i^{(n−1)}) \text{Re}(x_i^{(n−1)}) + \\
\sin(\arg x_i^{(n−1)}) \text{im}(x_i^{(n−1)}) \\
= \cos^2(\arg x_i^{(n−1)}) |x_i^{(n−1)}|^2 + \\
\sin^2(\arg x_i^{(n−1)}) |x_i^{(n−1)}|^2 \\
= \frac{|x_i^{(n−1)}|}{\cos(\delta/2)}
\end{align*}
\]

since \( \cos(\delta/2)/\sqrt{N_T N} \leq |x_i^{(n−1)}| \leq 1/\sqrt{N_T N} \) then:

\[
\frac{|x_i^{(n−1)}|}{\cos(\delta/2)} \geq 1/\sqrt{N_T N} \text{ for all } k
\]

As a result, \( A_n v^{(n−1)} \succeq c \) and \( v^{(n−1)} \in \mathcal{F}_n \). Therefore:

\[
v^{(n−1)}^T S v^{(n−1)} \leq v^{(n)}^T S v^{(n)}
\]

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or equivalently,
\[
x^{(n-1)H}Qx^{(n-1)} \leq x^{(n)H}Qx^{(n)}
\]
\[\Rightarrow \quad \text{SINR}^{n-1} \leq \text{SINR}^n\]

This implies that the SINR\(^n\) value is non-decreasing in each successive refinement or equivalently SINR\(^n\) is a monotone increasing sequence.

1.4 Proof of Lemma 2.3.4

Proof.

\[
x^{(n)}^TQx^{(n)} = x^{(n)}^T(\Phi - \lambda I)x^{(n)}
= x^{(n)}^T\Phi x^{(n)} - \lambda x^{(n)}^Tx^{(n)}
\]

Since \(0 \leq x^{(n)}^T\Phi x^{(n)} \leq \lambda_{\max}\) and \(\cos^2(\delta/2) \leq x^{(n)}^T x^{(n)} \leq 1\) then \(x^{(n)}^TQx^{(n)}\) is bounded.

Since SINR\(^n\) is a monotone non-decreasing (from Lemma 2.3.3) and bounded, then from the Monotone Convergence Criterion [83] the sequence SINR\(^n\) converges to a finite value SINR\(^*\). 

1.5 Proof of Lemma 2.3.5

Proof. Here \(n\) denotes the iteration index. Then, if Algorithm 3 converges to a constant modulus, let \(x_{n-1}\) and \(x_n\) denote the optimal solution at iterations \(n-1\) and \(n\), respectively. Since \(x_n\) is optimized using \(Q = \Phi(x_{n-1}) - \lambda I\), therefore, it follows that:

\[
x_{n-1}^H(\Phi(x_{n-1}) - \lambda I)x_{n-1} \leq x_n^H(\Phi(x_{n-1}) - \lambda I)x_n
\]

Since \(x_{n-1}\) and \(x_n\) are constant modulus, then \(x_{n-1}^Hx_{n-1} = x_n^Hx_n = 1\). Hence, it follows:

\[
x_{n-1}^H\Phi(x_{n-1})x_{n-1} \leq x_n^H\Phi(x_{n-1})x_n
\]

(.1)

From [11] and Proposition 1.1 of [10]:

\[
x_n^H\Phi(x_n)x_n = \sigma|w_{n-1}^H\Sigma(x)w_n|^2
\]

\[
= \frac{w_{n-1}^H\Sigma(x)w_{n-1} + w_{n-1}^Hw_{n-1}}{w_{n-1}^H\Sigma(x)w_{n-1} + w_{n-1}^Hw_{n-1}}
\]

\[
\leq \max_w \sigma|w^H\Sigma(x)w + w^Hw|
\]

\[
= \frac{\sigma|w_n^H\Sigma(x)w_n|^2}{w_n^H\Sigma(x)w_n + w_n^Hw_n}
\]

\[
= x_n^H\Phi(x_n)x_n
\]

where

\[
w_n = \frac{|\Sigma(x) + I|^{-1}U(\theta_0)x_n}{x_n^H\Sigma(x)U(\theta_0)|\Sigma(x) + I|^{-1}U(\theta_0)x_n}
\]

(.2)
Therefore,
\[ \mathbf{x}_n^H \Phi \mathbf{x}_{n-1} \mathbf{x}_n \leq \mathbf{x}_n^H \Phi \mathbf{x}_n \mathbf{x}_n \]  
(.3)

Combining equations (.1) and (.3) we have:
\[ \mathbf{x}_{n-1}^H \Phi \mathbf{x}_{n-1} \mathbf{x}_n \leq \mathbf{x}_n^H \Phi \mathbf{x}_n \mathbf{x}_n \]  
(.4)

which implies:
\[ \text{SINR}^{n-1} \leq \text{SINR}^n \]  
(.5)

Since SINR\(^n\) is bounded, Proposition 1 of [10], this implies that SINR\(^n\) converges to a finite value.

2 Proofs of Chapter 3

2.1 Proof of Lemma 3.3.1

Proof. Denote the feasible set of CP\(^{(n-1)}\) and CP\(^{(n)}\) by \(\mathcal{F}_{n-1}\) and \(\mathcal{F}_n\), respectively. Let \(s_{(n-1)} = [\text{Re}\{\mathbf{x}^T_{(n-1)}\} \text{Im}\{\mathbf{x}^T_{(n-1)}\} 1]^T\). Clearly, \(s_{(n-1)} \in \mathcal{F}_n\). Since CP\(^{(n)}\) is a convex problem then:
\[ s_{(n-1)}^T (\mathbf{R} + \lambda \mathbf{I}) s_{(n-1)} \geq s_{(n)}^T (\mathbf{R} + \lambda \mathbf{I}) s_{(n)} \]  
(.6)

Now define
\[ g(\mathbf{x}) = \mathbf{x}^H (\mathbf{P} + \lambda \mathbf{I}) \mathbf{x} - \mathbf{q}^H \mathbf{x} - \mathbf{x}^H \mathbf{q} + r + \lambda \]

Given the fact that \(|x_l(n-1)| = 1\) and \(|x_l^{(n)}| \geq 1\) for all \(l\) where \(x_l(n-1)\) is the \(l\)-th element of \(\mathbf{x}_{(n-1)}\) and using (.6), we have
\[ g(\mathbf{x}_{(n-1)}) \geq g(\mathbf{x}^{(n)}) \]  
(.7)

Note further that \(|x_l(n)| = 1\) and because \(|x_l^{(n)}| \geq 1\), then \(\mathbf{x}^{(n)}\) can be written as:
\[ \mathbf{x}^{(n)} = \mathbf{x}^{(n)} + \mathbf{Dx}^{(n)} \]

where \(\mathbf{D} = \text{diag}(\delta_1, \delta_2, ..., \delta_L)\) is a non-negative diagonal matrix, then:
\[ g(\mathbf{x}^{(n)}) - g(\mathbf{x}^{(n)}) \]
\[ = 2 \lambda \mathbf{x}_{(n)}^H \mathbf{Dx}^{(n)} + \mathbf{x}_{(n)}^H (\mathbf{PD} + \mathbf{DP}) \mathbf{x}^{(n)} + \]
\[ \mathbf{x}_{(n)}^H \mathbf{D}(\mathbf{P} + \lambda \mathbf{I}) \mathbf{Dx}^{(n)} - \mathbf{q}^H \mathbf{Dx}^{(n)} - \mathbf{x}_{(n)}^H \mathbf{Dq} \geq 2 \lambda \mathbf{x}_{(n)}^H \mathbf{Dx}^{(n)} + \mathbf{x}_{(n)}^H (\mathbf{PD} + \mathbf{DP}) \mathbf{x}^{(n)} \]
\[ - \mathbf{q}^H \mathbf{Dx}^{(n)} - \mathbf{x}_{(n)}^H \mathbf{Dq} \]
\[ = 2 \lambda  ||\mathbf{Dx}^{(n)}||_1 + \mathbf{x}_{(n)}^H (\mathbf{PD} + \mathbf{DP}) \mathbf{x}^{(n)} \]
\[ - \mathbf{q}^H \mathbf{Dx}^{(n)} - \mathbf{x}_{(n)}^H \mathbf{Dq} \]  
(.8)
where the first inequality holds since $P + \lambda I$ is positive definite and the last equality holds since $\|Dx(n)\|_1 = \sum_{l=1}^{L} |\delta_l x_l(n)| = \sum_{l=1}^{L} |\delta_l x_l(n)| = \sum_{l=1}^{L} \delta_l x_l(n) = x_H^H Dx(n)$.

Using Theorem 7.5 of [94] it can be shown that:

$$x_H^H (PD + DP)y(n) \geq - \frac{L}{4} \lambda_P \lambda_D$$

(9)

where $\lambda_D$ is the maximum eigenvalue of $D$. Note that, $\|Dx(n)\|_\infty = \max_l |\delta_l x_l(n)| = \lambda_D$.

Substituting eq. (9) in eq. (8) gives:

$$g(x^{(n)}) - g(x^{(n)})$$

$$\geq 2 \lambda \|Dx(n)\|_1 - \frac{L}{4} \lambda_P \|Dx(n)\|_\infty$$

$$- q^H Dx(n) - x_M^H Dx(n)$$

$$\geq 2 \lambda \|Dx(n)\|_1 - \frac{L}{4} \lambda_P \|Dx(n)\|_1$$

$$- 2\|q\|_2 \|Dx(n)\|_2$$

$$\geq 2 \lambda \|Dx(n)\|_1 - \left( \frac{L}{4} \lambda_P + 2\|q\|_2 \right) \|Dx(n)\|_1$$

$$\geq 0$$

where the last four inequalities are due to the facts that $\|Dx(n)\|_1 \geq \|Dx(n)\|_\infty$, $\|q\|_2 \|Dx(n)\|_2 \geq x_M^H Dx(n)$, $\|Dx(n)\|_1 \geq \|Dx(n)\|_2$ and $\lambda \geq \frac{L}{8} \lambda_P + \|q\|_2$, respectively. This implies that:

$$g(x^{(n)}) \geq g(x^{(n)})$$

(10)

Combining (7) and (10) leads to:

$$g(x^{(n-1)}) \geq g(x^{(n)})$$

Since $x_H^H x(n-1) = x_H^H x(n) = L$ then:

$$f(x^{(n-1)}) - f(x^{(n)}) = g(x^{(n-1)}) - g(x^{(n)}) \geq 0$$

Therefore, the sequence $\{f(x^{(n)})\}_{n=0}^\infty$ is non-increasing. Since $f(x) \geq 0$ for all values of $x$, then it is bounded below and, hence, it converges to a finite value $f^*$ (from the Monotone Convergence Criterion [95]).

\[\square\]

2.2 Proof of Lemma 3.3.2

Proof. First we prove that $|x^{(n)}| = 1$ for $n \geq C$. At convergence, the following holds:

$$f(x^{(n-1)}) - f(x^{(n)}) = g(x^{(n-1)}) - g(x^{(n)}) = 0$$
In Lemma 3.3.1 it has been shown that:

\[
2 \lambda x^H_{(n)} D x_{(n)} + x^H_{(n)} (PD + DP)x_{(n)} - q^H D x_{(n)} - x^H_{(n)} D q \geq 0
\]

for \( \lambda \geq \frac{L}{2} \lambda P + ||q||_2 \) and \( g(x_{(n-1)}) \geq g(x^{(n)}) \geq g(x_{(n)}) \). This implies:

\[
0 = g(x^{(n)}) - g(x_{(n)}) = 2 \lambda x^H_{(n)} D x_{(n)} + x^H_{(n)} (PD + DP)x_{(n)} + x^H_{(n)} D(P + \lambda I) D x_{(n)} - q^H D x_{(n)} - x^H_{(n)} D q \\
\geq x^H_{(n)} D(P + \lambda I) D x_{(n)} \\
\geq \lambda x^H_{(n)} D^2 x_{(n)}
\]

\[
= \lambda \sum_{i=1}^{L} \delta_i^2
\]

where \( \delta_i \) is the \( i \)-th diagonal element of \( D \). Since \( \lambda > 0 \) then \( g(x^{(n)}) - g(x_{(n)}) = 0 \) holds if and only if \( D = 0 \). Therefore, \( x^{(n)} = x_{(n)} + D x_{(n)} = x_{(n)} \) for \( n \geq C \) and the proof of the first part is complete.

For the second part, since problem \((RP)\) is the real form of \((P)\), then it is sufficient to prove that \( s^{(n)} \) for \( n \geq C \) satisfies the KKT conditions of problem \((RP)\).

Since problem \((RP)\) is non-convex, the KKT conditions may not be applicable unless the problem satisfies a constraint qualification \([96,97]\). Therefore, we will first prove that problem \((RP)\) satisfies the Linear Independent Constraint Qualification (LICQ) i.e. the gradient vectors of the constraints of problem \((RP)\) are linearly independent at \( s^{(n)} \) for \( n \geq C \). Note that:

\[
\nabla h_i(s^{(n)}) = 2 E_i s^{(n)}
\]

where \( h_i(s) = s^T E_i s - 1 \) is the \( i \)-th equality constraint of \((RP)\). Therefore:

\[
\nabla h_i(s^{(n)})^T \nabla h_j(s^{(n)}) = 4 s^{(n)} E_i E_j s^{(n)}
\]

\[
= \begin{cases} 
4 (s_i^{(n)} + s_{i+L}^{(n)}) & \text{if } i = j, \\
0 & \text{Otherwise}
\end{cases} \tag{1.11}
\]

since \( s_i^{(n)} + s_{i+L}^{(n)} = |x_i^{(n)}|^2 = 1 \) for \( n \geq C \), this implies that:

\[
\text{rank} \{ \nabla h_1(s^{(n)}), ..., \nabla h_{L+1}(s^{(n)}) \} = L + 1, \text{ for } n \geq C
\]

and, therefore, problem \((RP)\) satisfies LICQ. The KKT conditions of problem \((RP)\):

\[
2(R + \lambda I)s^{*} + \sum_{i=1}^{L+1} 2 w_i^* E_i s^{*} = 0 \tag{1.12}
\]

\[
s^{*T} E_l s^{*} = 1, \quad l = 1, 2, ..., L + 1 \tag{1.13}
\]

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where \( w^* \in \mathbb{R}^{(L+1)\times 1} \) is the Lagrange multiplier associated with the equality constraints of \((RP)\). We compare this with the KKT conditions of problem \((CP^{(n)})\) given by:

\[
2(R + \lambda I)s^{(n)} + B^{(n)}T v^{(n)} = 0 \\
B^{(n)}s^{(n)} = 1
\]

The \( l \)-th row of \( B^{(n)}s^{(n)} = 1 \) can be written as:

\[
\cos(\arg x_l^{(n)}) \Re\{x_l^{(n)}\} + \sin(\arg x_l^{(n)}) \Im\{x_l^{(n)}\} = 1
\]  \hspace{1cm} (.14)

Since \(|x^{(n)}| = 1\) for \( n \geq C \) then:

\[
\Re\{x_l^{(n)}\}^2 + \Im\{x_l^{(n)}\}^2 = 1
\]  \hspace{1cm} (.15)

Solving equations (.14) and (.15) gives \( \Re\{x_l^{(n)}\} = \cos(\arg x_l^{(n-1)}) \) and \( \Im\{x_l^{(n)}\} = \sin(\arg x_l^{(n-1)}) \). This implies that:

\[
b_l^{(n)} = E_l s^{(n)} \quad \text{for} \quad l = 1, 2, \ldots, L + 1
\]  \hspace{1cm} (.16)

Using eq. (.16), we can rewrite the KKT conditions of problem \((CP^{(n)})\) for \( n \geq C \) as:

\[
2(R + \lambda I)s^{(n)} + \sum_{l=1}^{L+1} v_l^{(n)} E_l s^{(n)} = 0 \\
s^{(n)T} E_l s^{(n)} = 1, \quad l = 1, 2, \ldots, L + 1
\]

where \( v_l^{(n)} \) is the \( l \)-th element of \( v^{(n)} \). Comparing this to the KKT conditions of problem \((RP)\) in (.12)-(.13) implies that \( s^* = s^{(n)} \) and \( w^* = \frac{1}{2} v^{(n)} \) solve the KKT conditions of problem \((RP)\) for \( n \geq C \) and, hence, \( x^{(n)} \) for \( n \geq C \) is a KKT point of \((P)\). \( \Box \)

2.3 Proof of Lemma 3.5.1

Proof. Let \( s^{(n-1)} \) be the optimal solution of \((CP^{(n-1)})\). Then \( B^{(n-1)}s^{(n-1)} = 1 \) and \( s^{(n-1)T} s^{(n-1)} \geq (1 - E_R/2)L \). Let \( x_l^{(n-1)} = \rho_l e^{i\psi_l} \), then \((B^{(n-1)}s^{(n-1)})_l\), the \( l \)-th element of \( B^{(n-1)}s^{(n-1)} \), should be equal to 1. That is,

\[
(B^{(n-1)}s^{(n-1)})_l = \Re\{x_l^{(n-1)}\} \cos(\gamma_l^{(n-1)}) + \\
\Im\{x_l^{(n-1)}\} \sin(\gamma_l^{(n-1)}) \\
= \rho_l \cos(\gamma_l^{(n-1)}) + \\
\rho_l \sin(\gamma_l^{(n-1)}) \\
= 1
\]  \hspace{1cm} (.17)

where \( \gamma_l^{(n)} = 2 \arg(x_l^{(n-1)}) - \delta x_l^{(n-1)} \). This implies

\[
\rho_l = \frac{1}{\cos(\psi_l) \cos(\gamma_l^{(n-1)}) + \sin(\psi_l) \sin(\gamma_l^{(n-1)})}
\]  \hspace{1cm} (.18)
Note that $s^{(n-1)}$ belongs to the feasible set of $CP(n)$ if and only if $\mathbf{B}^{(n)}s^{(n-1)} = 1$ and $\bar{s}^{(n)}T s^{(n-1)} \geq (1 - E_R/2)L$. We have

\[(\mathbf{B}^{(n)}s^{(n-1)})_l = \rho_l \cos(\psi_l) \cos(\gamma_l^{(n)})\]
\[+ \rho_l \sin(\psi_l) \sin(\gamma_l^{(n)})\]
\[= \rho_l \cos(\psi_l - \gamma_l^{(n)})\]  \hspace{1cm} (21)
\[= \rho_l \cos(\psi_l - 2\psi_l + \gamma_l^{(n-1)})\]  \hspace{1cm} (22)
\[= \rho_l \cos(\psi_l - \gamma_l^{(n-1)})\]  \hspace{1cm} (23)
\[= \rho_l \cos(\psi_l) \cos(\gamma_l^{(n-1)})\]
\[+ \rho_l \sin(\psi_l) \sin(\gamma_l^{(n-1)})\]
\[= 1\]  \hspace{1cm} (24)

To show $\bar{s}^{(n-1)}T s^{(n-1)} \geq (1 - E_R/2)L$, let $\bar{x}$ denote the complex version of $\bar{s}$, that is, $\bar{s} = [\text{Re}\{\bar{x}\} \text{Im}\{\bar{x}\}]^T$. Then we have

\[(1 - E_R/2)L \leq \bar{s}^{(n-1)}T s^{(n-1)}\]  \hspace{1cm} (25)
\[= \text{Re}\{\bar{x}^{(n-1)}H^x(n-1)\}\]  \hspace{1cm} (26)
\[= \text{Re}\{\sum_l \bar{x}_l^{(n-1)} \rho_l e^{j\psi_l}\}\]  \hspace{1cm} (27)
\[\leq \left| \sum_l \bar{x}_l^{(n-1)} \rho_l e^{j\psi_l} \right|\]  \hspace{1cm} (28)
\[\leq \sum_l \left| \bar{x}_l^{(n-1)} \rho_l e^{j\psi_l} \right|\]  \hspace{1cm} (29)
\[\leq \sum_l \left| \bar{x}_l^{(n-1)} \right| \rho_l\]  \hspace{1cm} (30)
\[= \left| \sum_l \bar{x}_l^{(n-1)} e^{-j\psi_l} \rho_l e^{j\psi_l} \right|\]  \hspace{1cm} (31)
\[= \left| \sum_l \bar{x}_l^{(n)} \rho_l e^{j\psi_l} \right|\]  \hspace{1cm} (32)
\[= \text{Re}\{\bar{x}^{(n)}H^x(n-1)\}\]  \hspace{1cm} (33)
\[= \bar{s}^{(n)}T s^{(n-1)}\]  \hspace{1cm} (34)

Note that the equality between (33) and (34) holds because we define $\bar{s}^{(n)}$ such that $\text{arg}(\bar{F}^H \bar{y}) = \text{arg}(x^{(n-1)})$. Eqs. (26) and (36) confirm $\mathbf{B}^{(n)}s^{(n-1)} = 1$ and $\bar{s}^{(n)}T s^{(n-1)} \geq (1 - E_R/2)L$. \hfill \qed
2.4 Proof of Lemma 3.5.2

Proof. First, let

\[ K = \begin{pmatrix} \bar{R} & B^{(n)}T & -s^{(n)} \\ B^{(n)} & 0 & 0 \\ -s^{(n)}T & 0 & 0 \end{pmatrix} \]  

(3.37)

\[ K_{11} = \begin{pmatrix} \bar{R} & B^{(n)}T \\ B^{(n)} & 0 \end{pmatrix} \]  

(3.38)

If \( \bar{s}^{(n)} \) is linearly dependent on \( b_1^{(n)}, b_2^{(n)}, \ldots, b_{L+1}^{(n)} \) and \( s^T \bar{s}^{(n)} - (1 - E_R/2)L < 0 \), then there will be no solution to \( CP^{(n)} \) which contradicts Lemma 3.5.1. Therefore, \( b_1^{(n)}, b_2^{(n)}, \ldots, b_{L+1}^{(n)} \), and \( \bar{s} \) must be linearly independent. Moreover, since \( \bar{R} \) is positive definite, all the eigenvalues of \( K \) are nonzero according to Theorem 2.1 in [78], which means \( K \) is nonsingular. Since \( K \) is nonsingular, the Schur complement of the block \( K_{11} \) in \( K \) is also nonsingular (nonzero in our case) according to Section C.4 in [34] and equals to \( \alpha^{(n)} \). This implies

\[ \alpha^{(n)} \neq 0 \]  

(3.39)

Using the block inverse to the matrix \( K_{11} \), Eq. (3.59) can be rewritten as

\[ \alpha^{(n)} = -\bar{s}^{(n)}T (\bar{R}^{-1} - \bar{R}^{-1} B^{(n)}T \hat{R} B^{(n)} \bar{R}^{-1}) \bar{s}^{(n)} \]  

(3.40)

\[ = -\bar{s}^{(n)}T \bar{R}^{-\frac{1}{2}} (I - \bar{R}^{-\frac{1}{2}} B^{(n)}T \hat{R} B^{(n)} \bar{R}^{-\frac{1}{2}}) \bar{R}^{-\frac{1}{2}} \bar{s}^{(n)} \]  

(3.41)

\[ = -y^T (I - \bar{C} \bar{C}^T)^{-1} \bar{C}^T y \]  

(3.42)

\[ = -y^T (I - C (C^T C)^{-1} C^T) y \]  

(3.43)

where \( y = \bar{R}^{-\frac{1}{2}} \bar{s}^{(n)} \) and \( \bar{C} = \bar{R}^{-\frac{1}{2}} B^{(n)} \). Note that \( \bar{C} \bar{C} = C (C^T C)^{-1} C^T \) is an idempotent matrix with eigenvalues of either 0 or 1 [98]. This implies that \( (I - \bar{C}) \) is positive semidefinite. Therefore,

\[ \alpha^{(n)} \leq 0 \]  

(3.44)

Combining (3.39) and (3.44) implies that \( \alpha^{(n)} < 0 \) and, hence, \( \mu^{(n)} > 0 \). \( \square \)

2.5 Proof of Lemma 4.3.1

Proof. Denote \( x^{(n)} \) and \( x^{(n-1)} \) as the solutions of \( Q^{(n)} \) and \( Q^{(n-1)} \), respectively. Then within each \( Q^{(n)} \)’s real version \( CQ^{(n)} \), we have the KKT conditions for the odd iteration:

\[ 2 \sum_{i=1}^{M} A_i s^{(n-1)} s^{T (n-1)} A_i^T s^{(n)} + \lambda s^{(n)} + G^{(n-1)} v = 0 \]  

(1)

\[ G^{(n-1)} s^{(n)} = 1 \]  

(2)

where \( v \) is the Lagrangian multiplier of the equality constraints. Let \( y = s^{(n-1)}, z = s^{(n)} \) and \( y = z + d \) where \( d \neq 0 \). Since \( B^{(n)} y = B^{(n)} z = 1 \), so \( B^{(n)} y = B^{(n)} z + B^{(n)} d = 1 \),
then, $B_{(n)}d = 0$. Multiple $d^T$ with (1), we have
\[
2 \sum_{i=1}^{M} d^T A_i y y^T A_i^T z + \lambda d^T z + d^T B_{(n)}v = 0
\]

Combining with $B_{(n)}d = 0$, we have
\[
\sum_{i=1}^{M} d^T A_i y y^T A_i^T z + \frac{\lambda}{2} d^T z = 0 \quad (3)
\]

Now,
\[
g(s_{(n-1)})
= \sum_{i=1}^{M} y A_i y y^T A_i^T y + \frac{\lambda}{2} y^T y
\]
\[
= y \sum_{i=1}^{M} A_i y y^T A_i^T (z + d) + \frac{\lambda}{2} y^T (z + d)
\]
\[
= y \sum_{i=1}^{M} A_i y y^T A_i^T z + y \sum_{i=1}^{M} A_i y y^T A_i^T d + \frac{\lambda}{2} y^T z + \frac{\lambda}{2} y^T d
\]
\[
= (z + d)^T \sum_{i=1}^{M} A_i y y^T A_i^T z + \frac{\lambda}{2} (z + d)^T z + y \sum_{i=1}^{M} A_i y y^T A_i^T d + \frac{\lambda}{2} y^T d
\]
\[
= g_1 \text{ (corresponds to the even iteration)}
\]
\[
+ y \sum_{i=1}^{M} A_i y y^T A_i^T d + \frac{\lambda}{2} y^T d
\]
\[
= g_1 + (z + d)^T \sum_{i=1}^{M} A_i y y^T A_i^T d + \frac{\lambda}{2} (z + d)^T d
\]
\[
= g_1 + z \sum_{i=1}^{M} A_i y y^T A_i^T d + \frac{\lambda}{2} z^T d + d^T \sum_{i=1}^{M} A_i y y^T A_i^T d + \frac{\lambda}{2} d^T d
\]
\[
= g_1 + \frac{\lambda}{2} z^T + \sum_{i=1}^{M} A_i^T z z^T A_i y + \frac{\lambda}{2} z^T z + g(d, y)
\]
\[
= \frac{y^T}{z^T + d} \sum_{i=1}^{M} A_i^T z z^T A_i y + \frac{\lambda}{2} z^T z + g(d, y)
\]
\[
= z + d = g_1
\]
\[
\begin{align*}
&= z^T \sum_{i=1}^{M} A_i^T z z^T A_i z + \frac{\lambda}{2} z^T z + d^T \sum_{i=1}^{M} A_i^T z z^T A_i z \\
&+ z^T \sum_{i=1}^{M} A_i^T z z^T A_i d + d^T \sum_{i=1}^{M} A_i^T z z^T A_i d + g(d, y) \\
&= z^T \sum_{i=1}^{M} A_i^T z z^T A_i z + \frac{\lambda}{2} z^T z \\
&\quad \quad \quad + d^T \left( \sum_{i=1}^{M} A_i^T A_i y y^T A_i + \sum_{i=1}^{M} A_i^T z z^T A_i d \right) d \\
&\quad \quad \quad + d^T \sum_{i=1}^{M} A_i^T z z^T A_i z + z^T \sum_{i=1}^{M} A_i^T z z^T A_i d + \frac{\lambda}{2} d^T d \\
&= \underbrace{W}_{g(s(n))} z + \underbrace{W}_{g(s(n))} d \\
&\geq 0 \\
&\geq 0.
\end{align*}
\]

If we set
\[
\lambda = \frac{-d^T W z - z^T W d}{d^T d}
\]
then \(\frac{\lambda}{2} d^T d + d^T W z + z^T W d \geq 0\). Since \(d^T d > 0\) and \(|z|\) is bounded, then:
\[
\frac{-d^T W z - z^T W d}{d^T d}
\]
is also bounded and \(\lambda\) is a finite value. Hence:
\[
g(s(n-1)) \geq g(s(n))
\]
Since \(g(s(n)) \geq 0\) for all \(n\) then \(g(s(n))\) is bounded and converges to a finite value \(g^*\). \(\square\)
Bibliography


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Vita

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Mr. Aldayel received his B. S. degree from King Saud University, Riyadh, Saudi Arabia in July 2007 and his M. S. degree from KTH– the Royal Institute of Technology in September 2011. He worked at King Saud University as a lecturer, Riyadh, Saudi Arabia, from 2011 to 2013. His research interests include statistical signal processing, detection and estimation, and convex optimization.

Journal Publications:

