The Pennsylvania State University
The Graduate School
Eberly College of Science

ON SOME PROBLEMS IN LAGRANGIAN DYNAMICS AND FINSLER GEOMETRY

A Dissertation in
Mathematics
by
Dong Chen

Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Doctor of Philosophy

August 2017

The dissertation of Dong Chen was reviewed and approved* by the following:

Dmitri Burago
Distinguished Professor of Mathematics
Dissertation Advisor
Chair of Committee

Federico Rodriguez Hertz
Professor of Mathematics

Mark Levi
Professor of Mathematics

Runze Li
Verne M. Willaman Professor of Statistics

Yuxi Zheng
Francis R. Pentz and Helen M. Pentz Professor of Science
Department Head

*Signatures are on file in the Graduate School.

Abstract

The purpose of this dissertation is to present several applications of enveloping functions and dual lens maps to geometry and dynamical systems. In Chapter 1 we have a brief review on basic notions and theory we need to understand the main results. In Chapter 2 we prove that given a point on a Finsler surface, one can always find a neighborhood of the point and isometrically embed this neighborhood into a Finsler torus without conjugate points. The major tool is enveloping functions.

In Chapter 3 we introduce the dual lens map technique developed by Burago and Ivanov. It derives from enveloping functions and symplectic geometry. We then show how this technique is used to perturb the geodesic flows of flat Finsler tori.

In Chapter 4 we show how dual lens map can be used in KAM theory. The celebrated KAM Theory says that if one makes a small perturbation of a non-degenerate completely integrable system, we still see a huge measure of invariant tori with quasi-periodic dynamics in the perturbed system. These invariant tori are known as KAM tori. What happens outside KAM tori draws a lot of attention. We show two types of Lagrangian perturbations of the geodesic flow on flat Finsler tori. The perturbations are C^∞ small but the resulting flows has a positive measure of trajectories with positive Lyapunov exponent. The measure of this set is of course extremely small. Still, the flow has positive metric entropy. From this result we get positive metric entropy outside some KAM tori and it gives positive answer to a question asked by Kolmogorov.

Table of Contents

List of Figures

Acknowledgments

I would love to take this opportunity to express my deepest gratitude to Dima Burago, who has been an awesome advisor throughout my six years' graduate studies in Penn State. He is always there to listen, to support, to guide, and to help me succeed in every way. His immense knowledge and patient endure guided my research into the right direction. My entire graduate experience became so meaningful and enjoyable because of Dima.

I would also extend my appreciation to my dissertation committee members: Profs. Federico Rodriguez Hertz, Marc Levi, Runze Li, as well as Anton Petrunin, who served in my committee during my comprehensive exam. I am indebted to you all. I am also thankful to Profs. Vadim Kaloshin, Anatoly Katok, Sergei Ivanov, Leonid Polterovich, Yakov Sinai, Daniel Thompson and Anatoly Vershik, for their invitation, interest in my work and useful comments.

I am grateful to the graduate students, especially those in the dynamical group and on the fourth floor of McAllister, for providing a friendly, stimulating and enthusiastic work environment. I greatly value their friendship and deeply appreciate their faith in me. Many thanks also go to the faculty and staff in Department of Mathematics. Thank you for making this a memorable and fruitful experience.

Finally, special thanks are due to my parents for their unconditional love and continued support on my dreams. I owe you everything and love you.

Chapter 1 Preliminaries

1.1 Hamiltonian flow on a cotangent bundle

Let (Ω^{2n}, ω) be a $2n$-dimensional symplectic manifold. Let H be a smooth function on T^*M. We can define the *Hamiltonian vector field* X_H as the unique solution to the equation

$$\omega(X_H, V) = dH(V)$$

for any smooth vector field V on Ω. X_H is well-defined due to nondegeneracy of ω. The flow Φ_H^t on Ω defined by $\dot{\Phi}_H^t = X_H$ is called *the Hamiltonian flow on* Ω *with Hamiltonian* H. One can easily verify that Φ_H^t preserves ω and hence the volume form ω^n. The *metric entropy of* Φ_H^t is defined to be the measure theoretical entropy with respect to the volume form ω^n.

If ω is exact (i.e. $\omega = d\theta$ for a 1-form θ), θ is preserved on each energy level by the Hamitonian flow Φ_H^t if and only if H is positively homogeneous in coordinates of cotangent spaces (i.e. $H(q, \lambda p) = \Theta(\lambda)H(q,p)$ for some positive function Θ, see [23]). Such Hamiltonians are called *generalized homogeneous.*

Let H be a generalized homogeneous Hamiltonian and h be a noncritical value of H, then θ is a contact form on the level set $H^{-1}(h)$ and Φ_H^t is a contact flow. The measure defined by $\theta \wedge (d\theta)^{n-1}$ is an invariant measure of the flow Φ_H^t. This volume form is called the *Liouville measure.* The Hamiltonian flow Φ_H^t has positive metric entropy if and only if its restriction on $H^{-1}(h)$ has positive measure theoretical entropy with respect to the Liouville measure.

1.2 Geodesic flows on Finsler manifolds and its entropy

A typical example of a Hamiltonian flow with generalized homogeneous Hamiltonian is the geodesic flow on a Finsler manifold. Let M be a smooth manifold. A *Finsler metric* φ on M is a smooth family of quadratically convex norms $\varphi(x, \cdot)$ on each tangent space T_xM. It is *reversible* if $\varphi(x, v) = \varphi(x, -v)$ for all $x \in M, v \in T_xM$. Let (M, φ) be a Finsler manifold with its unit tangent bundle UTM. We define the dual norm on cotangent bundle T^*M by

$$\varphi^*(\alpha) := \sup_{v \in UT_xM} \{\alpha(v)\}, \text{ for } \alpha \in T_x^*M.$$

The cotangent bundle T^*M has an exact natural symplectic form ω. The geodesic flow g_t on (M, φ) is defined to be the Hamiltonian flow on the cotangent bundle T^*M with generalized homogeneous Hamiltonian $(\varphi^*)^2/2$.

For any point x in (M, φ), the unit ball B_x in T_xM is a convex body. By F. John [22], among all ellipsoids contained in B_x, there exists a unique ellipsoid E_x with maximum volume. E_x is the unit sphere of some quadratic form on T_xM. In this way we can define quadratic forms on each tangent spaces and these forms are close to Finsler norms. In this way we can associate with the Finsler metric φ a Riemannian metric g_φ, from which UTM inherits a Riemannian structure (see [33] for details). This metric is called the Sasaki metric. For each vector $\zeta \in T_vUTM$ we define the Lyapunov exponent by

$$\chi^+(v, \zeta) := \limsup_{t \to \infty} \frac{\ln ||Dg_t\zeta||}{t}$$

and the upper Lyapunov exponent by

$$\chi^+(v) := \max_{\zeta \in T_vUTM} \chi^+(v, \zeta).$$

For our purpose, there is no need to recall the precise definition of the metric entropy h_μ for the Liouville measure μ on UTM. Indeed, it is enough to know that Pesin's inequality [32]

$$h_\mu \geq \int_{UTM} \chi^+(v) d\mu(v) \tag{1}$$

provides a lower bound of metric entropy. Indeed, this formula tells us that the metric entropy is no less than the mean of upper Lyapunov exponent.

1.3 Geometry on Finsler manifolds

If $\gamma : [a, b] \to M$ is a smooth curve on a Finsler manifold (M, φ), then one defines the length of γ by

$$L(\gamma) := \int_a^b \varphi(\gamma(t), \gamma'(t))dt.$$

Using this definition of length we define a non-symmetric metric (i.e. a positive definite function on $M \times M$ satisfying the triangle inequality) on M by letting the distance $d(x, y)$ from x to y be the infimum of the lengths of all piecewise smooth curves starting from x and ending at y. It can be non-symmetric since $d(x, y)$ may not be equal to $d(y, x)$. Under this non-symmetric metric we can define geodesics in the following way: a curve $\gamma : [a, b] \to M$ is said to be a *geodesic* of (M, φ) if for every sufficiently small interval $[c, d] \subseteq [a, b]$, $\gamma|_{[c,d]}$ realizes the distance from $\gamma(c)$ to $\gamma(d)$. In this thesis we will always assume that a geodesic is unit-speed, i.e. if γ is a geodesic, then $\varphi(\gamma(s), \gamma'(s)) = 1$, for $s \in [a, b]$. A geodesic $\gamma : [a, b] \to M$ is called *minimal* if for $a \leq t_1 < t_2 \leq b, d(\gamma(t_1), \gamma(t_2)) = t_2 - t_1$. And a C^k Finsler metric φ on M is called *simple* if every pair of points on M is connected by a unique geodesic depending C^k smoothly on the endpoints.

Let γ be a ray with unit speed in a Finsler manifold. Define the *Busemann function* $b_\gamma : M \to \mathbb{R}$ with respect to γ by

$$b_\gamma(x) := \lim_{t\to\infty}(t - d(x, \gamma(t))).$$

1.4 Entropy non-expansive flows

Let Φ^t be a flow on a metric space (X, d). We say Φ^t is *entropy non-expansive* if for any $\epsilon > 0$, there exists an orbit γ such that the set of trajectories which stay forever within distance no more than ϵ from γ contains an open invariant set on which the dynamic has positive metric entropy [4]. Basically it means that positive metric entropy can be generated in an arbitrarily small neighborhood of an orbit of the system. The issue attracted a lot of interest, see for instance... In particular, D. Burago introduced this notion in 1988 being in mathematical isolation in the former Soviet Union, see .... This situation is a bit counter-intuitive since hyperbolic dynamics tends to expand and occupy all space. In our situation, however, it is generated even near a periodic orbit, meaning that hyperbolic dynamics is localized in a small neighborhood not only in the phase space but in the configuration space too. The paper [9] gave a construction of an entropy non-expansive flow however not in the context of the KAM Theory.

Chapter 2
Local Structures of Finsler Tori Without Conjugate Points

2.1 Introduction

In this chapter we study the universality of local structures of 2-dimensional Finsler tori without conjugate points. It is known that 2-dimensional Riemannian tori without conjugate points are flat, which was proved by E.Hopf [20] in 1940s. Hopf's paper is a partial answer to a question asked by Hedlund and Morse [21], that is, whether the same result still holds in all dimensions. The positive answer to this question is now known as Hopf's conjecture. After that many other people studied this problem with various assumptions. In 1994, D.Burago and S.Ivanov [7] proved the Hopf's conjecture. This breakthrough shows the rigidity of Riemannian tori without conjugate points.

Hopf's problem is originally formulated for Riemannian manifolds. On the other hand, if you look into the world of Finsler manifolds, the whole picture of Finsler tori without conjugate points remains veiled. There are examples of non-flat Finsler tori without conjugate points, thus the original Hopf's conjecture does not hold in Finsler case. One can construct such a non-flat Finsler 2-torus by making symplectic (contact) perturbations on the Euclidean torus [25] or constructing some metric of revolution [35].

Before Burago and Ivanov, Croke and Kleiner [15] have shown that in the Riemannian case, if a torus without conjugate points has a smooth (or bi-Lipschitz) Heber foliation [19], then it is flat. Smoothness of the Heber foliation is (more or less) equivalent to the assertion that the geodesic flow is smoothly conjugate to that of some flat Finsler torus. It is still an open question if the Heber foliation of a Finsler manifold without conjugate points is smooth, and whether the geodesic flow of such manifold is smoothly conjugate to that of some flat Finsler torus. We even do not know whether every geodesic with an irrational rotation number is dense in a 2-dimensional Finsler torus without conjugate points.

In this chapter I extend an approach suggested by Burago-Ivanov to show that there are no local restrictions for a metric to be the metric of a Finsler torus without conjugate points. Therefore the world of Finsler tori without conjugate points is much wider than the examples to the best of my knowledge. See Theorem 2.1 for precise formulation.

In order to proof Theorem 2.1 we generalize the concept of Busemann functions on Finsler manifold to an enveloping function. Such extension does not depend on the ray. And we can get back the Finsler metric from the enveloping function. By perturbing the enveloping function we can get a perturbation of the Finsler metric.

2.2 Enveloping functions

We use some notation and techniques from [8]. To make this note more reader-friendly, we copy them here.

Definition 2.1. A function f on a Finsler manifold (M, φ) is called *forward 1-Lipschitz* if for $p, q \in M$, $f(p) - f(q) \leq d(q, p)$.

Let (M, φ) be a Finsler manifold. We have a norm φ^* on the cotangent bundle T^*M given by:

$$\varphi^*(\alpha) := \sup\{\alpha(v) | v \in T_xM, \varphi(v) = 1\},$$

for $x \in M, \alpha \in T_x^*M$. And we denote by UM and U^*M the bundles of unit spheres of φ and φ^*. Since φ is Minkowski on each tangent space, φ^* is also Minkowski on each cotangent space, hence U_x^*M is quadratically convex for all $x \in M$. A C^1 function on M is called *distance-like* if $\varphi^*(d_xf) = 1$ for all $x \in M$.

Notice that a distance-like function is always forward 1-Lipschitz. In fact, for any $x, y \in M$ and any unit-speed curve $c : [a, b] \to M$ starting at x and ending at

y, if f is distance-like, then

$$f(y) - f(x) = \int_a^b df_{c(s)}(c'(s))ds \leq b - a = L(c)$$

By taking the infimum for all c, f is forward 1-Lipschitz.

Let S be a smooth manifold diffeomorphic to S^{n-1} where $n = \dim M$.

Definition 2.2. A continuous function $F : S \times M \to \mathbb{R}$ is called a $\boldsymbol{C^k}$ *enveloping function* for φ if F is C^k smooth outside $S \times \partial M$ and the following conditions are satisfied:

(a) For every $p \in S$, the function $F_p := F(p, \cdot)$ is distance-like.

(b) For every $x \in M$, the map $p \to d_x F_p$ is a diffeomorphism from S to $U_x^* M$.

If M is a manifold with boundary S, φ is a C^k simple Finsler metric and F is given by $F(p, x) := d(p, x)$, then F is a C^k enveloping function. On the other hand, given an enveloping function F we can define a distance function on $M \times M$ by

$$d_F(x, y) := \sup_{p \in S} F(p, y) - F(p, x).$$

By d_F we can define a metric φ_F on TM, and the unit sphere of φ_F^* in $T_x^* M$ is the image of the map $S \to T_x^* M, p \mapsto d_x F_p$.

Lemma 2.1. Let F be an enveloping function for φ. Then there exists a function $\delta : M \to \mathbb{R}$ depending only on φ such that for every $\tilde{F} : S \times M \to \mathbb{R}$ with

$$||d_x F_\cdot - d_x \tilde{F}_\cdot||_{C^2(S, T_x^* M)} < \delta(x)$$

for all $x \in M$, we can find a Finsler metric $\tilde{\varphi}$ on M such that $\tilde{F}$ is an enveloping function for $\tilde{\varphi}$. In particular, if M is compact or φ is flat, $\delta(x)$ can be chosen to be a constant.

Proof. Since φ is a Finsler metric, the image of the map $S \to T_x^* M, p \mapsto d_x F_p$ is quadratically convex. And $p \mapsto d_x \tilde{F}_p$ is a C^2 small perturbation of this map, hence also has quadratically convex image, therefore the image is the unit sphere of some Minkowski norm on $T_x^* M$. And the dual norm $\tilde{\varphi}$ is a Finsler norm at x. $\square$

Definition 2.3. Let f be a distance-like function on a Finsler manifold (M, φ), and $\gamma : [a, b] \to M$ be a geodesic. We say that γ is *calibrated by f* if $f(\gamma(t_2)) - f(\gamma(t_1)) = t_2 - t_1$, for any $a \leq t_1 < t_2 \leq b$.

The *(Finslerian) gradient* of a distance-like function $f : D \to \mathbb{R}$ at $x \in D$, denoted $\mathrm{grad} f(x)$, is defined to be the unit tangent vector $v \in U_x D$ such that $d_x f(v) = 1$. If γ is calibrated by f, then for all points on γ, the tangent vector of γ coincide with the gradient of f.

Lemma 2.2. If we have an enveloping function F on a Finsler manifold (M, φ), then M has no conjugate points.

Proof. If f is a distance-like function on M, then any integral curve of $\mathrm{grad} f$ is a minimal geodesic. In fact, let $\gamma : [a, b] \to M$ be such a unit-speed curve and $a \leq t_1 < t_2 \leq b$. Then for all $s \in (a, b), df_{\gamma(s)}(\gamma'(s)) = 1$ since $\gamma'(s)$ is the gradient of f at $\gamma(s)$. Thus

$$t_2 - t_1 \geq d(\gamma(t_1), \gamma(t_2)) \geq f(\gamma(t_2)) - f(\gamma(t_1)) = \int_{t_1}^{t_2} df_{\gamma(s)}(\gamma'(s))ds = t_2 - t_1.$$

This implies $d(\gamma(t_1), \gamma(t_2)) = t_2 - t_1 = f(\gamma(t_2)) - f(\gamma(t_1))$. Hence γ is a minimal geodesic and it is calibrated by f.

Now let $\sigma : [a, b] \to M$ be a geodesic. Since a geodesic is a local minimizer, we can find $\delta > 0$ such that $d(\sigma(a), \sigma(a + \delta)) = \delta$. Let $p \in S$ be a point such that $d_{\sigma(a)} F_p$ is the dual to $\sigma'(a)$, then the integral curve γ of $\mathrm{grad} F_p$ with $\gamma(a) = \sigma(a)$ is a minimal geodesic calibrated by F_p. Since γ and σ are geodesics with the same starting point and initial direction, $L(\gamma) = b - a = L(\sigma)$, therefore $\gamma = \sigma$. Therefore σ is a minimal geodesic. This implies any geodesic is a minimal one, so M has no conjugate points.

$\square$

Remark 2.1. By the proof of Lemma 2.2, if a geodesic γ is calibrated by a distance-like function, then γ is minimal.

2.3 Total flexibility of local structures of Finsler tori without conjugate points

Theorem 2.1. [12] Suppose (M, φ) is a $C^k (k \geq 3)$ Finsler surface. Then for any $p_0 \in M$, we can find a neighborhood U of p_0, and an isometric embedding $\Psi : (U, \varphi|_{TU}) \to (\mathbb{T}^2, \tilde{\varphi})$, where $(\mathbb{T}^2, \tilde{\varphi})$ is a C^k Finsler torus without conjugate points. If in addition, φ is reversible, then $\tilde{\varphi}$ can be chosen to be reversible too.

Proof. Lei $\psi : U_0 \to \mathbb{R}^2$ be a local chart around p_0 mapping p_0 to origin. Since ψ is a diffeomorphism we can define the metric on $\psi(U_0)$ simply by pushing forward that on U_0 through ψ. Once we get this isometric embedding, we can assume the image of ψ is $U_\epsilon := \{(\alpha, \beta) \in \mathbb{R}^2 | \alpha^2 + \beta^2 < \epsilon^2\}$, and we identify U_0 with U_ϵ. By choosing small ϵ and let D_ϵ be the closure of U_ϵ, we get a simple Finsler metric on D_ϵ. Let φ_0 be a constant Finsler metric on $\mathbb{R}^2$ which is identical to $\varphi|_{T_{p_0}D}$. For each $x \in \mathbb{R}^2$ denote by S_x the unit circle of φ_0 in $T_x\mathbb{R}^2$. For any $q \in S_x$ there exists a unique $q^* \in T_x^*\mathbb{R}^2$ supporting q (i.e. $q^*(q) = 1$ and $q^*(S_x) \leq 1$.) and we denote $S_x^* := \{q^* : q \in S_x\}$. Due to the smoothness of φ, we can choose small ϵ so that

$$||\varphi^*(x, \cdot) - \varphi_0^*(x, \cdot)||_{C^k(S_x^*)} < \delta(x), \tag{*}$$

for all $x \in U_\epsilon$, where δ is the function in Lemma 1.

For $p \in S_{p_0}$, let $\gamma_p^0 : [-a_0, b_0] \to D_\epsilon$ be the geodesic in the Finsler disk (D_ϵ, φ_0) with $\gamma_p^0(0) = p_0$ and $(\gamma_p^0)'(0) = p$. Let $\gamma_p : [-a, b] \to D_\epsilon$ be the geodesic in (D_ϵ, φ) with $\gamma_p(0) = p_0, \gamma_p'(0) = p$. Then we can define a function F on $S_{p_0} \times D_\epsilon$ by the following: if x lies on the left hand side of the direction of γ_p, then $F(p, x) := d(x, \gamma_p)$, otherwise define $F(p, x) := -d(\gamma_p, x)$. Then F is a C^k enveloping function for φ.

Similar as above we get a C^∞ enveloping function F^0 on $S_{p_0} \times \mathbb{R}^2$ for the constant metric φ_0. From (*) we know that

$$||d_x F_\cdot - d_x F^0_\cdot||_{C^k(S_{p_0}, T^* D_\epsilon)} < \delta(x), \tag{**}$$

for all $x \in D_\epsilon$. Extend F to $S_{p_0} \times \mathbb{R}^2$ so that (**) holds for all $x \in \mathbb{R}^2$.

Take a large r and let g be a function on $\mathbb{R}^2$ with value 1 on D_ϵ and value 0 outside D_r. By choosing r large enough we may assume g has very small ith$(1 \leq i \leq k)$ derivatives. Take $l > r$ and define a function $\tilde{F}$ on $S_{p_0} \times [-l, l]^2$ by

$$\tilde{F}(p, x) = F(p, x)g(x) + F^0(p, x)(1 - g(x)).$$

Extend $\tilde{F}$ to $S_{p_0} \times \mathbb{R}^2$ by setting

$$\tilde{F}(p, x + (2lm, 2ln)) = F^0(p, x + (2lm, 2ln)) + \tilde{F}(p, x) - F^0(p, x), \text{ for } (m, n) \in \mathbb{Z}^2.$$

$\tilde{F}$ satisfies (**) if we replace F by $\tilde{F}$. By Lemma 1, $\tilde{F}$ is an enveloping function for some Finsler metric $\tilde{\varphi}$ on $\mathbb{R}^2$. By Lemma 2 we know that $\tilde{\varphi}$ has no conjugate points. Since $\tilde{F}$ is quasiperiodic on x, the metric $\tilde{\varphi}$ is periodic on x, hence it projects to a Finsler metric on $\mathbb{T}^2 := \mathbb{R}^2/(2l\mathbb{Z})^2$. $\tilde{\varphi}$ agrees with φ on TD_ϵ and it agrees with φ_0 on $T(\mathbb{T}^2 \backslash D_r)$.

Suppose φ is symmetric, use the same notations as above, then γ_p and γ_{-p} are the same curve with different directions. Therefore we have

$$F(p, x) = -F(-p, x), \tag{***}$$

for all $x \in D_\epsilon$. Extend F to $S_{p_0} \times \mathbb{R}^2$ so that (***) holds. Repeating the procedures as above we get a function $\tilde{F}$ on $S_{p_0} \times \mathbb{R}^2$. Now define

$$\tilde{d}(x, y) = \max_{p \in S_{p_0}} \tilde{F}(p, x) - \tilde{F}(p, y),$$

then $\tilde{d}$ is symmetric and it is C^k close to d_0, which is the metric on $\mathbb{R}^2$ generated by φ_0. As we get such metric $\tilde{d}$, we can define a Finsler metric on the tangent bundle in the following way: for $x \in \mathbb{R}^2, v \in T_x\mathbb{R}^2$, let $c : (-\epsilon_1, \epsilon_1) \to D_r$ be a curve with $c(0) = x, c'(0) = v$. Define

$$\tilde{\varphi}(x, v) := \lim_{t \to 0} \frac{\tilde{d}(x, c(t))}{t}.$$

Then $\tilde{F}$ is an enveloping function for $\tilde{\varphi}$. By symmetry of $\tilde{d}$ we get symmetry of $\tilde{\varphi}$. $\square$

Chapter 3
Dual Lens Maps and Its Application to Geodesic Flows

3.1 Dual lens map

Here we use the notions and definitions from [9].

Definition 3.1. A Finsler metric φ on an n-dimensional disc D is called *simple* if it satisfies the following three conditions:

(S1) Every pair of points in D is connected by a unique geodesic.

(S2) Geodesics depend smoothly on their endpoints.

(S3) The boundary is strictly convex, that is, geodesics never touch it at their interior points.

Once (D, φ) is simple, denote by U_{in}, U_{out} the set of inward, outward pointing unit tangent vectors with base points in ∂D respectively. With any vector $\nu \in U_{in}$, we can associate a unique vector $\beta(\nu) \in U_{out}$, namely the tangent vector of the (unique) geodesic with initial velocity ν at its next intersection point with ∂D. This defines a map $\beta : U_{in} \to U_{out}$, which is called the lens map of φ. If φ is reversible, then the lens map is reversible in the following sense: $-\beta(-\beta(\nu)) = \nu$ for every $\nu \in U_{in}$.

We denote by UT^*D the unit sphere bundle with respect to the dual norm φ^*. Let $\mathscr{L} : TD \to T^*D$ be the Legendre transform of the Lagrangian $\varphi^2/2$. It maps UTD to UT^*D. For a tangent vector $\nu \in UT_xD$, its Legendre transform $\mathscr{L}(\nu)$ is the unique covector $\chi \in U_x^*D$ such that $\chi(\nu) = 1$.

Then consider subsets $U_{in}^* = \mathscr{L}(U_{in})$ and $U_{out}^* = \mathscr{L}(U_{out})$ of UT^*D. The dual lens map of φ is the map $\sigma : U_{in}^* \to U_{out}^*$ given by $\sigma := \mathscr{L} \circ \beta \circ \mathscr{L}^{-1}$ where β is the lens map of φ. If φ is reversible then σ is symmetric in the sense that $-\sigma(-\sigma(\chi)) = \chi$ for all $\chi \in U_{in}^*$.

Note that U_{in}^* and U_{out}^* are $(2n-2)$-dimensional submanifolds of T^*D. The restriction of the canonical symplectic 2-form of T^*D to U_{in}^* and U_{out}^* determines the symplectic structure. And the dual lens map σ is symplectic. In [9], by using enveloping functions, Burago and Ivanov proved the following theorem:

Theorem 3.1 (Burago-Ivanov [9])**.** Assume that $n \geq 3$. Let φ be a simple metric on $D = D^n$ and σ its dual lens map. Let W be the complement of a compact set in U_{in}^*. Then every sufficiently small symplectic perturbation $\tilde{\sigma}$ of σ such that $\tilde{\sigma}|_W = \sigma|_W$ is realized by the dual lens map of a simple metric $\tilde{\varphi}$ which coincides with φ in some neighborhood of ∂D.

The choice of $\tilde{\varphi}$ can be made in such a way that $\tilde{\varphi}$ converges to φ whenever $\tilde{\sigma}$ converges to σ (in C^∞). In addition, if φ is a reversible Finsler metric and $\tilde{\sigma}$ is symmetric then $\tilde{\varphi}$ can be chosen reversible as well.

In the same paper they proved the above theorem for $n = 2$ with additional requirement:

Proposition 3.1 (Burago-Ivanov [9])**.** Let φ be a simple metric on D^2 and σ its dual lens map. If $\tilde{\sigma}$ satisfies the conditions in Theorem 3.1 and moreover, there is an open subset $O \subseteq S$ such that

$$\tilde{\sigma}|_{O_{in}^*} = \sigma|_{O_{in}^*},$$

here $O_{in}^* := \pi^{-1}(O) \cap U_{in}^*$, then $\tilde{\sigma}$ is a dual lens map of some simple Finsler metric in D^2 which coincides with φ in some neighborhood of ∂D. The convergence and reversibility are the same as in Theorem 3.1.

3.2 Perturbation on flat Finsler tori

Let $(\mathbb{T}^n, \varphi_0)$ be a torus with flat Finsler metric φ_0 and $UT^*\mathbb{T}^n$ be its unit cotangent bundle with standard coordinates $(q_1, ..., q_n, p_1, ..., p_n)$. It is not hard to see that $(q_1, ..., q_n, p_1, ..., p_n)$ are action-angle coordinates of the geodesic flow. We think of $\mathbb{T}^n$ as the cube $[-1/2, 1/2]^n$ with sides identified. Take a submanifold $T_0 := \{q_n = -1/2\}$ and a section $\Gamma_0 := \{(q_1, ..., q_n, p_1, ..., p_n) \in UT^*\mathbb{T}^n : q_n = -1/2, p_n > 0\}$. Γ_0 inherits a natural symplectic form from $T^*\mathbb{T}^n$. We set $R_0 : \Gamma_0 \to \Gamma_0$ to be the Poincaré map to Γ_0 of the geodesic flow on $(\mathbb{T}^n, \varphi_0)$.

Denote by $\mathbf{q} = (q_1, ..., q_{n-1}), \mathbf{p} = (p_1, ..., p_{n-1})$. We can find a neighborhood $O_p \subseteq \mathbb{R}^3$ of $\mathbf{0}$ such that for the covectors in Γ_0 with $\mathbf{p} \in O_p$ we have $p_n = \psi(\mathbf{p})$ for some positive function ψ. Let $\Pi : \Gamma_0 \to T^*T_0$ be the canonical projection defined by

$$\Pi(\mathbf{q}, -1/2, \mathbf{p}, p_n) = (\mathbf{q}, \mathbf{p})$$

It is clear that Π is a symplectic bijection between $\Pi^{-1}(T_0 \times O_p)$ and $T_0 \times O_p$.

Define

$$R_1 := \Pi \circ R_0 \circ \Pi^{-1} : T_0 \times O_p \to T_0 \times O_p.$$

By a simple calculation we know the map $R : \mathbb{R}^{n-1} \times O_p \to \mathbb{R}^{n-1} \times O_p$ defined by

$$R(\mathbf{q}, \mathbf{p}) = (\mathbf{q} + \psi(\mathbf{p})^{-1}\mathbf{p}, \mathbf{p})$$

is a lift of R_1 to the universal cover. We say a compact set $K \subseteq \mathbb{R}^{n-1} \times O_p$ is *penetrating* if there exists $r_0 < 1$ such that

$$\pi(K) \subseteq B_0(r_0) \text{ and } \pi(R(K)) \subseteq B_0(r_0^{-1}),$$

here $\pi : T^*\mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ is the bundle projection and $B_0(r)$ is the Euclidean open ball with radius r and center the origin. Since $r_0 < 1$ we can also regard a penetrating K as a subset of B^*T_0. In particular, if O_p and $\pi(K)$ are both small neighborhoods around $\mathbf{p} = \mathbf{0}$ and $\mathbf{q} = \mathbf{0}$ respectively, then K is penetrating.

Proposition 3.2. Assume that $n \geq 2$ and K is a penetrating compact set in $T_0 \times O_p$. For any sufficiently C^∞-small symplectic perturbation $\tilde{R}_1$ of R_1 coinciding with R_1 outside K, there exists a Finsler metric $\tilde{\varphi}$ on $\mathbb{T}^n$ that agrees with φ_0 on Γ_0 such that the Poincaré map to Γ_0 of the geodesic flow on $(\mathbb{T}^n, \tilde{\varphi})$ is $\Pi^{-1} \circ \tilde{R}_1 \circ \Pi$. The convergence and reversibility are the same as in Theorem 3.1.

Proof. Denote by D^n the n-dimensional ball inscribed in $[-1/2, 1/2]^n$ and $\sigma : U^*_{in} \to U^*_{out}$ be the dual lens map of the Finsler disc (D^n, φ_0). Denote $\Gamma_\pm := \{(q_1, ..., q_n, p_1, ..., p_n) \in UT^*(\mathbb{R}^n) : q_n = \pm 1/2, p_n > 0\}$ and define the projections

$\Pi_{\pm} : \Gamma_{\pm} \to B^*\mathbb{R}^3$ by

$$\Pi_{\pm}(\mathbf{q}, \pm 1/2, \mathbf{p}, p_n) = (\mathbf{q}, \mathbf{p}).$$

It is clear that both $\Pi_{\pm}$ are symplectic bijections.

For any $\alpha_1 \in \Pi_-^{-1}(K)$ (resp. $\alpha_2 \in \Pi_+^{-1}(R(K))$), consider its orbit (resp. backward orbit) under the geodesic flow generated by φ_0. Since K is penetrating, the orbit (resp. backward orbit) will intersect U^*_{in}(resp. U^*_{out}) transversally and we denote by $\phi_1(\alpha_1)$(resp. $\phi_2(\alpha_2)$) the first intersection. This defines a map $\phi_1 : \Pi_-^{-1}(K) \to U^*_{in}$ (resp. $\phi_2 : \Pi_+^{-1}(R(K)) \to U^*_{out}$). It is clear that both ϕ_1 and ϕ_2 are symplectic bijections to their images.

The restriction of R on K can be decomposed as

$$R|_K = \Pi_+ \circ \phi_2^{-1} \circ \sigma \circ \phi_1 \circ \Pi_-^{-1}.$$

Let $\tilde{R}$ be a lift of $\tilde{R}_1$ to the universal cover. Define a dual lens map $\tilde{\sigma} : U^*_{in} \to U^*_{out}$

by

$$\tilde{\sigma}(\alpha) := \begin{cases} \phi_2 \circ \Pi_+^{-1} \circ \tilde{R} \circ \Pi_- \circ \phi_1^{-1}(\alpha), & \text{if } \alpha \in \phi_1(\Pi_-^{-1}(K)); \\ \sigma(\alpha), & \text{otherwise.} \end{cases}$$

By definition, $\tilde{\sigma}$ coincides with σ outside a compact set. Moreover $\tilde{\sigma} \to \sigma$ in C^∞ as $\tilde{R} \to R$ in C^∞.

By Theorem 3.1, there exists a Finsler metric $\tilde{\varphi}$ in D^n agreeing with φ_0 around the boundary ∂D^n and the dual lens map for $(D^n, \tilde{\varphi})$ is $\tilde{\sigma}$. Extend $\tilde{\varphi}$ to $[-1/2, 1/2]^n$ by φ_0. Now extend $\tilde{\varphi}$ to the whole $[-1/2, 1/2]^n$ by setting it equal to φ_0 outside D^n. It is flat in a neighborhood of the boundary $\partial[-1/2, 1/2]^n$ so it projects to a Finsler metric $\tilde{\varphi}$ (we abuse notation again) on $\mathbb{T}^n$. The Poincaré map onto Γ_0 is $\Pi^{-1} \circ \tilde{R}_1 \circ \Pi$. Since $\tilde{R}_1$ has positive metric entropy, so does $\Pi^{-1} \circ \tilde{R}_1 \circ \Pi$. Thus we can make a C^∞ perturbation of φ_0 on any small tubular neighborhood of a closed orbit γ and the resulting metric has positive metric entropy.

If φ_0 is reversible, we define $\tilde{\sigma}$ by:

$$\tilde{\sigma}(\alpha) = \begin{cases} \phi_2 \circ \Pi_+^{-1} \circ \tilde{R} \circ \Pi_- \circ \phi_1^{-1}(\alpha), & \text{if } \alpha \in \phi_1(\Pi_-^{-1}(K)); \\ -\phi_1 \circ \Pi_-^{-1} \circ \tilde{R}^{-1} \circ \Pi_+ \circ \phi_2^{-1}(-\alpha), & \text{if } \alpha \in -\phi_2(\Pi_+^{-1}(R(K))); \\ \sigma(\alpha), & \text{otherwise.} \end{cases}$$

It is clear that $\tilde{\sigma}$ is symmetric. By Theorem 3.1, $\tilde{\varphi}$ can be chosen to be reversible. $\square$

Remark 3.1. We only give the proof for $\mathbb{T}^n$ glued out of a cube. Similar arguments work not only for general tori glued out of parallelepipeds, but also for perturbations in any neighborhood of any closed orbit.

Chapter 4
Positive Metric Entropy in KAM Systems

4.1 Introduction

Already in the early 50's the study of nearly integrable Hamiltonian systems has drawn the attention of many outstanding mathematicians such as Arnol'd, Kolmogorov and Moser. Indeed, for any integrable Hamiltonian system the whole phase space is foliated by invariant Lagrangian submanifolds that are diffeomorphic to tori, generally called *KAM tori*, and on which the dynamics is conjugated to a rigid rotation. Therefore, it is natural to ask what happens to such a foliation and to these stable motions once the system is slightly perturbed. In 1954 Kolmogorov [24] - and later Arnol'd [1] and Moser [26] in different contexts - proved that, for small perturbations of an integrable system it is still possible to find a big measure set of KAM tori. This result, commonly referred to as *KAM theorem*, contributed to raise new interesting questions, for instance about the destiny of the stable motions that are destroyed by effect of the perturbation (in other words, about the dynamics outside KAM tori). In this context, Arnol'd [2] constructed an example of a perturbed integrable system, in which some orbits outside KAM tori have a wide range in action variables (even though the rate of change of action variables is exponentially small [29]). This striking phenomenon, known as *Arnol'd diffusion* and still quite far from being fully understood, shows the presence of some randomness in the dynamics outside KAM tori. The question we address in the present paper is therefore the following: how much random can the motion outside

KAM tori be?

It is well-known that, C^2-generically the Hamiltonian flow has positive topological entropy (cf. [30], see also [14] for an analogous statement for Riemannian geodesic flows). Once we turn our attention to metric entropy, the problem becomes more challenging and one cannot simply derive positive metric entropy from positive topological entropy. In fact, Bolsinov and Taimanov [5] built an example of a Riemannian manifold on which the geodesic flow has positive topological entropy but zero metric entropy.

Recently Burago and Ivanov [9] used dual lens map to construct a reversible Finsler metric C^∞-close to the standard metric on $S^n, n \geq 4$, such that its geodesic flow has positive metric entropy. However the geodesic flow on the sphere is degenerate, hence it does not lie in the realm of KAM theory.

Unlike the case of spheres, the geodesic flow on flat tori are nondegenrate. In this paper we therefore provide examples analogous to Burago-Ivanov's one on torus (Theorem 4.1 and Theorem 4.2). Our theorem shows that in the complement of KAM tori, the behavior of nearly integrable Hamiltonian flows can be quite stochastic.

4.2 Non-ergodic Donnay-Burns-Gerber tori

Definition 4.1. We say that a centrally symmetric cap $\mathscr{C} = \{r \leq r_1\} \subseteq \mathbb{R}^2$ is a **non-ergodic Donnay-Burns-Gerber (DBG) cap** if:

(a) $\mathscr{C}$ has two parallel geodesics C_{r_0} and C_{r_1}, where $C_{r_i} := \{r = r_i\}$ for $i = 0, 1$.

(b) The Gaussian curvature is positive on $\{r \leq r_0\}$, negative at C_{r_1}, and strictly decreasing from center to boundary.

If a torus contains a non-ergodic DBG cap and outside the cap the Gaussian curvature is nonpositive, then we call it a **non-ergodic DBG torus**.

Lemma 4.1. The geodesic flow on a non-ergodic DBG torus has positive metric entropy.

Sketch of proof. The proof is similar to the proof of Theorem 1.1 in [11]. By virtue of Clairaut's integral, any geodesic entering the cap $\mathscr{C}$ will go out of the cap.

Let $c : [-T_1, T_1] \to \mathscr{C}$ be an arc-length parametrized geodesic with endpoints in C_{r_1} such that $c(0)$ is the point of c closest to the origin; suppose furthermore that $c(\pm T_2)$ lie in C_{r_0}, for some $0 < T_2 < T_1$. Let J_S, J_C be two Jacobi fields on c with $J_S(0) = 0, J'_S(0) = 1, J_C(0) = 1, J'_C(0) = 0$. Let $u_S = J'_S/J_S, u_C = J'_C/J_C$ and $K(t)$ be the Gaussian curvature at $c(t)$. Then both u_S and u_C satisfy the Riccati equation:

$$u'(t) + u(t)^2 + K(t) = 0$$

By imitating the proofs of Lemma 2.5 and Lemma 2.6 in [11], we get

(A) $u_S(\pm T_1) = u_S(\pm T_2) = 0$. and $J_S(t)$ vanishes only at $t = 0$.

(B) There is a $\tau \in (0, T_2)$ such that $\lim_{t\to\tau^-} u_C(t) = -\infty$.

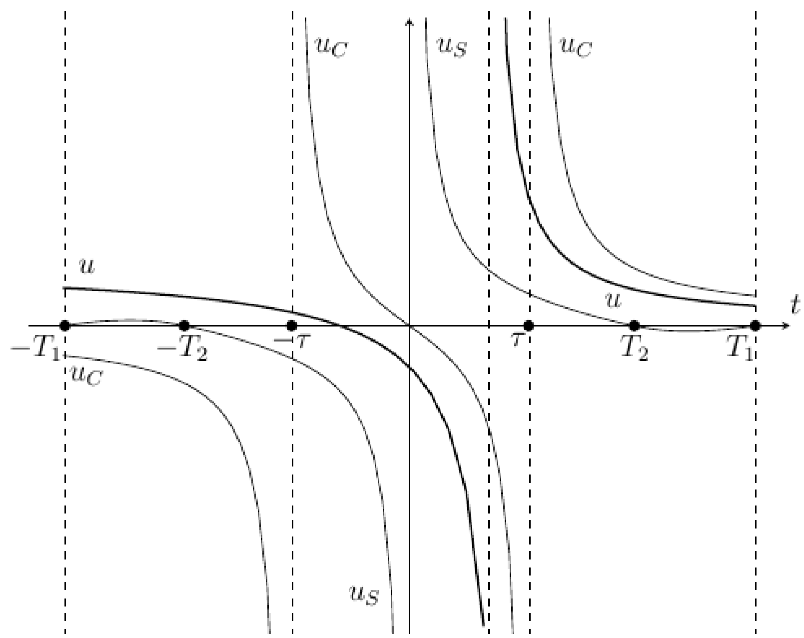

Figure 4.1. Graphs of u_S, u_C and u

If a Jacobi field J on c satisfies $J'(-T_1)J(-T_1) \geq 0$ then $u := J'/J$ satisfies the Riccati equation with $u(-T_1) \geq 0$. This means the graph of u must lie above that of u_S. By (A) and (B) we have $u(T_1) \geq 0$. So the cone $J'J \geq 0$ is preserved by the cap.

By Poincaré recurrence theorem, almost every vector in $UT\mathscr{C}$ will come back infinitely many times. For any geodesic c entering the cap $\mathscr{C}$ at time t_0, when it returns to the cap again, say at time $t_1 > t_0$, the image of the cone $\{J'(t_0)J(t_0) \geq 0\}$ under the translation will lie strictly in the interior of $\{J'(t_1)J(t_1) \geq 0\}$. By Wojkowski's cone field theory [34], the vectors with non-zero Lyapunov exponents form a set with positive Liouville measure. By Pesin's inequality (1) the geodesic flow has positive metric entropy. $\square$

4.3 Construction of a non-ergodic DBG torus

In this section we construct a conformal metric on $[-1/2, 1/2] \times [-1/2, 1/2]$ which is flat outside a disc and centrally symmetric inside the disc. More precisely we want to build a function $g : [0, 1] \to (0, 1]$ such that the torus with conformal metric

$$ds^2 = g(r)^2(dx^2 + dy^2), \text{ where } r := \sqrt{x^2 + y^2}. \tag{2}$$

is a non-ergodic DBG torus.

In order to get such a function g we change our coordinate system to geodesic polar coordinates. However before doing this we need some preliminary.

Definition 4.2. We say a function $\rho : I \to \mathbb{R}$ is *even* (resp. *odd*) at a point $a \in I$ if all odd (resp. even) derivatives of ρ vanish at a.

Lemma 4.2. For any smooth function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ which is odd at 0, $\rho'(0) = 1$ and is positive except at 0, there exist smooth functions $g, l : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that l is odd at 0, $l(0) = 0, l'(r) = g(r), g(0) = 1, \rho(l(r)) = rg(r)$, and g is positive.

Proof. Since

$$\rho = r\frac{dl}{dr},$$

we have

$$\frac{dr}{r} = \frac{dl}{\rho}. \tag{$*$}$$

Both sides of $(*)$ have singularity at 0. Since ρ is odd at 0, $\rho'(0) = 1$, for small l we have

$$\frac{1}{\rho} = \frac{1}{l}\left(\frac{1}{1 + \rho^{(3)}(0)l^2/6 + o(l^3)}\right) = \frac{1}{l}\left(\frac{1}{1 + l^2 O(1)}\right) = \frac{1}{l}(1 + l^2\tilde{\rho}(l)) = \frac{1}{l} + l\tilde{\rho}(l),$$

where $\tilde{\rho}$ is a smooth function that is even at 0. We integrate both sides of $(*)$ regarding r as a function of l with $r(0) = 0$. Then we get

$$\lim_{l\to 0}(\ln r - \ln l) = \lim_{l\to 0}\int_0^l s\tilde{\rho}(s)ds = 0.$$

Therefore $\lim_{l\to 0}\ln(r/l) = 0$ and $\ln(r/l)$ is even at 0. By direct computation, it is now easy to see that r/l is even at 0. This implies that r is odd at 0. From $(*)$ we have

$$\frac{d\ln r}{dl} = \frac{dr}{rdl} = \frac{1}{\rho} > 0.$$

Therefore $\ln r(l)$ is strictly increasing and smooth, so is $r(l)$. By the Inverse Function Theorem there exists a smooth $l : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ which is the inverse function of $r(l)$. Moreover $l(0) = 0$, $l'(0) = 1$ and l is odd at 0. Finally we define $g(r) := l'(r)$. It is clear that g is even at 0 and positive. □

By Lemma 4.2 we have only to find $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with the following properties:

(i) ρ satisfies the conditions in Lemma 4.2;

(ii) $\rho'(l_0) = \rho'(l_1) = 0$ for some $0 < l_0 < l_1$.

(iii) Let $K(l) := -\rho''(l)/\rho(l)$. Then $K(l) > 0$ on $[0, l_0]$, $K(l_1) < 0$.

(iv) $K'(l) < 0$ on $[0, l_1]$.

(v) There exists $l_2 > l_1$ such that $K(l)$ is negative on $[l_1, l_2)$ and $\rho'(l) = 1$ for $l \geq l_2$.

Indeed once we have such a function ρ, by Lemma 4.2 we have smooth functions $g, l : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\rho(l(r)) = rg(r)$ and $l(r) = \int_0^r g(t)dt$. Consider the metric defined by (2). By changing the coordinate system to geodesic polar coordinates, the metric becomes

$$ds^2 = dl^2 + \rho(l)^2 d\theta^2. \tag{3}$$

Note that $\rho'(l) = 0$ iff the parallel at l is a geodesic, and the Gaussian curvature is given by $K(l) = -\rho''(l)/\rho(l)$. Let $r_i := l^{-1}(l_i)$ for $i = 0, 1, 2$. (ii) implies (a) in the definition of a non-ergodic DBG cap, while (b) can be derived from (iii) and (iv). (v) guarantees the metric is negatively curved on the annulus $\{r_1 < r < r_2\}$ and is flat outside $\{r = r_2\}$. So once ρ satisfies (i)-(v), the torus with metric (3) will be a non-ergodic DBG torus.

Here is the construction of $\rho(l)$:

For any $a > 0$, let $\lambda_1 : \mathbb{R}_{\geq 0} \to [0, 1]$ be a C^∞ function with the properties that $\lambda_1 \equiv 1$ on $[0, \frac{1}{\sqrt{5a}}]$ and $\lambda_1 \equiv 0$ on $[\frac{1}{2\sqrt{a}}, +\infty)$. Let $\lambda_2 : \mathbb{R}_{\geq 0} \to [0, 1]$ be another C^∞ function with $\lambda_2 \equiv 0$ on $[0, \frac{1}{\sqrt{5a}}] \cup [\frac{1}{2\sqrt{a}}, +\infty)$ and positive on $(\frac{1}{\sqrt{5a}}, \frac{1}{2\sqrt{a}})$. Define $\rho''(l)$ by

$$\rho''(l) = \lambda_1(l)(-30al + 200a^2l^3) + C(1 - \lambda_1(l))\lambda_2(l),$$

where C is a positive constant such that $\int_0^\infty \rho''(l)dl = 0$. Notice that $\rho''(l) = 0$ on $[\frac{1}{2\sqrt{a}}, +\infty)$. Define ρ by setting $\rho(0) = 0, \rho'(0) = 1$. Then $\rho(l) = l - 5al^3 + 10a^2l^5$ on $[0, \frac{1}{\sqrt{5a}}]$ and $\rho'(l) \equiv 1$ for $l \geq \frac{1}{2\sqrt{a}}$. The graph of ρ is shown in Figure 3.

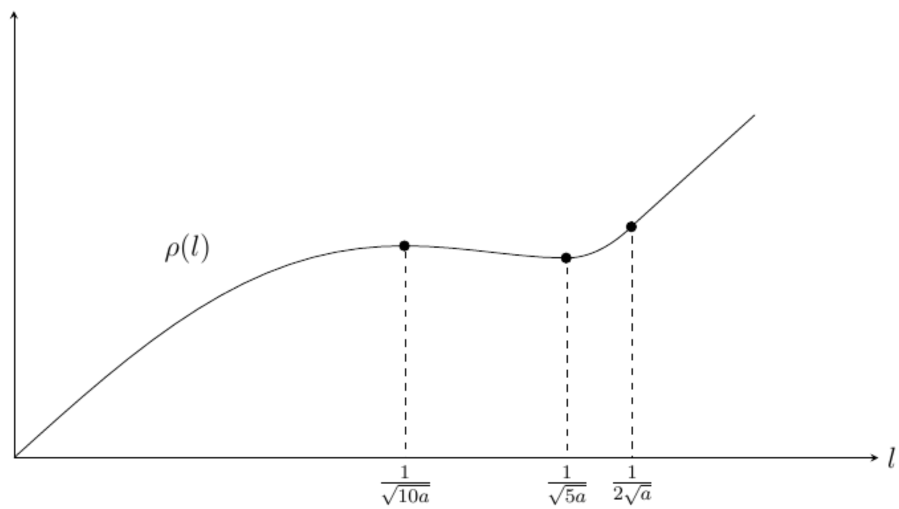

Figure 4.2. Graph of ρ

It is easy to see that ρ satisfies (i) and (ii) for $l_0 = \frac{1}{\sqrt{10a}}$ and $l_1 = \frac{1}{\sqrt{5a}}$. $\rho''(l) = -30al + 200a^2l^3$ on $[0, l_1]$ and it is positive on $[l_1, \frac{1}{2\sqrt{a}})$, so $K(l)$ is positive on $\{l \leq \frac{1}{\sqrt{10a}}\}$ and negative in the annulus between $\{l = l_1\}$ and $\{l = \frac{1}{2\sqrt{a}}\}$. Hence ρ satisfies (iii) and (v) for $l_2 = \frac{1}{2\sqrt{a}}$. The last part to be verified is (iv). Since

$$K'(l) = -\frac{\rho\rho''' - \rho'\rho''}{\rho^2},$$

we only need to verify that $\rho\rho''' - \rho'\rho'' = 100a^2l^3(1 + 12al^2 - 40a^2l^4)$ is positive on $(0, \frac{1}{\sqrt{5a}}]$. This can be done by direct calculation. This finishes the construction.

Remark 4.1. The function g constructed in this way is strictly decreasing on $[0, r_2]$ and constant for $r \geq r_2$ since

$$\frac{d\rho}{dl} = \frac{d\rho}{dr}\frac{dr}{dl} = \frac{g + rg'}{g} = 1 + \frac{rg'}{g}$$

and $\rho'(l) < 1$ on $(0, \frac{1}{2\sqrt{a}})$, $\rho'(l) = 1$ for $l \geq \frac{1}{2\sqrt{a}}$. So the supremum of g is $g(0) = 1$. From Lemma 4.2 we know that the lower bound is positive.

Remark 4.2. If g satisfies the condition that a torus with metric $g(r)^2(dx^2+dy^2)$ is non-ergodic DBG, we can find a constant δ_0 such that for all $\delta \in (-\delta_0, \delta_0)$, a torus with metric $(g(r)^2+\delta)(dx^2+dy^2)$ is also non-ergodic DBG. This follows from the fact that being a non-ergodic DBG torus is an open condition.

Remark 4.3. By choosing a sufficiently large a we can shrink the support of g' to be as small as we want. Indeed notice that $\rho(s) \le s$ and $\rho'(s) > 0$ on $(\frac{1}{\sqrt{5a}}, \frac{1}{2\sqrt{a}})$. Therefore

$$\begin{aligned}
\max_{s\ge 0}\ s-\rho(s) &= \max_{0\le s\le \frac{1}{2\sqrt{a}}} \int_0^s 1-\rho'(t)dt \\
&= \frac{1}{\sqrt{5a}} - \rho(\frac{1}{\sqrt{5a}}) + \max_{0\le s\le \frac{1}{2\sqrt{a}}} \int_{\frac{1}{\sqrt{5a}}}^s 1-\rho'(t)dt \\
&< \frac{1}{\sqrt{5a}} - \rho(\frac{1}{\sqrt{5a}}) + \max_{0\le s\le \frac{1}{2\sqrt{a}}} s - \frac{1}{\sqrt{5a}} < \frac{1}{3\sqrt{a}}.
\end{aligned}$$

From $(*)$ we have

$$\begin{aligned}
\ln r_2 - \ln(\frac{1}{2\sqrt{a}}) &= \int_0^{\frac{1}{2\sqrt{a}}} \left(\frac{1}{\rho(s)} - \frac{1}{s}\right) ds \\
&= \int_0^{\frac{1}{\sqrt{5a}}} \left(\frac{1}{s-5as^3+10a^2s^5} - \frac{1}{s}\right) ds + \int_{\frac{1}{\sqrt{5a}}}^{\frac{1}{2\sqrt{a}}} \left(\frac{1}{\rho(s)} - \frac{1}{s}\right) ds \\
&< \int_0^{\frac{1}{\sqrt{5a}}} \frac{5as-10a^2s^3}{1-5as^2+10a^2s^4} ds + \int_{\frac{1}{\sqrt{5a}}}^{\frac{1}{2\sqrt{a}}} \left(\frac{1}{s-\frac{1}{3\sqrt{a}}} - \frac{1}{s}\right) ds \\
&< 10a \int_0^{\frac{1}{\sqrt{5a}}} 2s - 4as^3 ds + \ln\left(1 - \frac{1}{3s\sqrt{a}}\right)\Bigg|_{\frac{1}{\sqrt{5a}}}^{\frac{1}{2\sqrt{a}}} < 2 - \ln(3-\sqrt{5}).
\end{aligned}$$

Thus $r_2 \to 0$ as $a \to \infty$.

4.4 Perturbation of the Hamiltonian H_0

Suppose the fundamental domain of the deck group on the universal cover of our torus $\mathbb{T}^2 \cong \mathbb{R}^2/\mathbb{Z}^2$ is $\{-1/2 < q_1, q_2 < 1/2\}$. We use p_1, p_2 to denote the coordinates in the cotangent space and denote $B^*\mathbb{T}^2 := \{(q_1,q_2,p_1,p_2) \in T^*\mathbb{T}^2 : p_1^2+p_2^2 < 1\}$.

In this section we want to perturb the kinetic Hamiltonian

$$H_0(q_1, q_2, p_1, p_2) := \frac{p_1^2 + p_2^2}{2}$$

in such a way that the Hamiltonian flow if the resulting Hamiltonian has positive metric entropy. More precisely, we want to prove the following:

Lemma 4.3. There exists a family $\{H_\epsilon\}_{\epsilon>0}$ of smooth perturbations of H_0 such that for all $\epsilon > 0$, there exists an open interval I_ϵ with the property that for any $h \in I_\epsilon$, the Hamiltonian flow $\Phi^t_{H_\epsilon}$ on the level set $\{H_\epsilon = h\}$ has positive metric entropy.

Proof. Let $\xi : \mathbb{R}_{\geq 0} \to [0, 1]$ be a smooth function with $\xi \equiv 1$ on $[0, 1/3]$ and $\xi \equiv 0$ on $[2/3, 1]$. And let g be the function we built in 4.3. We define

$$H_\epsilon := H_0 + \epsilon(1 - g(r)^2)\xi(p_1^2 + p_2^2), \text{ where } r = \sqrt{q_1^2 + q_2^2}.$$

Since g is positive and $0 \leq 1 - g^2 < 1$ (by Remark 4.1), we have

$$\epsilon > \max_{(x,y)\in\mathbb{T}^2} \epsilon(1 - g(r)^2)\xi(p_1^2 + p_2^2).$$

Notice that if $H_\epsilon < 1/6$ then $p_1^2 + p_2^2 < 1/3$, therefore $\xi \equiv 1$ whenever the total energy is small. By the Maupertuis principle, the Hamiltonian flow $\Phi^t_{H_\epsilon}$ on the level set $\{H_\epsilon = \epsilon\}$ is a time change of the geodesic flow on $\mathbb{T}^2$ with metric

$$ds^2 = \epsilon g(r)^2(dq_1^2 + dq_2^2).$$

This metric has positive metric entropy since, by Lemma 4.1, the metric $ds^2 = g(r)^2(dq_1^2 + dq_2^2)$ does.

Let δ_0 be the constant we get from Remark 4.2 and define $I_\epsilon := (\epsilon - \epsilon\delta_0, \epsilon + \epsilon\delta_0)$. By using Maupertuis principle again we prove the lemma. □

4.5 Perturbation of $\tilde{H}_0 = -\sqrt{1 - 2H_0}$

In this section we prove that a smooth perturbation of

$$\tilde{H}_0(q_1, q_2, p_1, p_2) := -\sqrt{1 - p_1^2 - p_2^2}$$

can be derived from a suitable perturbation of H_0. Since this result holds for all degrees of freedom, we use $(\mathbf{q}, \mathbf{p})$ to denote the coordinates instead of (q_1, q_2, p_1, p_2).

Suppose $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ has coordinates $\mathbf{q} = (q_1, ..., q_n)$ and let $\mathbf{p} = (p_1, ..., p_n)$ be the coordinates in the cotangent bundle. Denote $B^*\mathbb{T}^n = \{(\mathbf{q}, \mathbf{p}) : \sum p_i^2 < 1\}$. Define

$$H_0(\mathbf{q}, \mathbf{p}) := \frac{1}{2}\sum_{i=1}^{n} p_i^2, \qquad \tilde{H}_0(\mathbf{q}, \mathbf{p}) := -\sqrt{1 - 2H_0(\mathbf{q}, \mathbf{p})}.$$

Then

$$\Phi^t_{\tilde{H}_0}(\mathbf{q}, \mathbf{p}) = (\mathbf{q} + \frac{t\mathbf{p}}{\sqrt{1 - \sum p_i^2}}, \mathbf{p}).$$

Let $V(\mathbf{q}, \mathbf{p})$ be a C^2-smooth function on $B^*\mathbb{T}^n$. We perturb H_0 and $\tilde{H}_0$ by V in the following way:

$$H_\epsilon(\mathbf{q}, \mathbf{p}) := \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \epsilon V(\mathbf{q}, \mathbf{p}), \qquad \tilde{H}_\epsilon(\mathbf{q}, \mathbf{p}) := -\sqrt{1 - 2H_\epsilon(\mathbf{q}, \mathbf{p})}.$$

Then we have

Lemma 4.4. If supp$V \subseteq \{\sum p_i^2 \le C < 1\}$ for some $C \in \mathbb{R}_+$, then for every $\delta, m, \mathcal{T} > 0$, there exists $\epsilon = \epsilon(V, \delta, m, \mathcal{T}) > 0$ such that for each $0 \le T \le \mathcal{T}$ we have

$$||\Phi^T_{\tilde{H}_\epsilon} - \Phi^T_{\tilde{H}_0}||_{C^m(B^*\mathbb{T}^n)} < \delta.$$

Proof. Denote $\Phi^T_{\tilde{H}_\epsilon}(\mathbf{q}, \mathbf{p}) - \Phi^T_{\tilde{H}_0}(\mathbf{q}, \mathbf{p})$ by $(\Delta\mathbf{q}, \Delta\mathbf{p})$ as they usually do this in calculus books. Put $(\mathbf{q}(t), \mathbf{p}(t)) := \Phi^t_{\tilde{H}_\epsilon}(\mathbf{q}, \mathbf{p})$. Suppose that $H_\epsilon(\mathbf{q}, \mathbf{p}) = E$. Then

$$\dot{\mathbf{q}}(t) = \frac{\partial \tilde{H}_\epsilon}{\partial \mathbf{p}} = \frac{\mathbf{p} + \epsilon V_{\mathbf{p}}}{\sqrt{1 - 2E}}, \qquad \dot{\mathbf{p}}(t) = -\frac{\partial \tilde{H}_\epsilon}{\partial \mathbf{q}} = -\frac{\epsilon V_{\mathbf{q}}}{\sqrt{1 - 2E}}.$$

If $\sum p_i^2 > C$, then $\dot{\mathbf{p}}(t) \equiv 0$, hence $\Delta\mathbf{p} = 0$. Consider the trajectory $(\mathbf{q}(t), \mathbf{p}(t))$, V_p vanishes along it, hence $\Delta\mathbf{q} = 0$. Therefore we only need to consider the case $\sum p_i^2 \le C$. Since V is compactly supported we may assume that ϵ is small enough so that $\sum p_i^2 + 2\epsilon V < (1 + C)/2 < 1$. In this case

$$\Delta\mathbf{p} = \int_0^T \dot{\mathbf{p}}(t)dt = -\int_0^T \frac{\epsilon V_{\mathbf{q}}}{\sqrt{1 - 2E}}dt = -\frac{\epsilon}{\sqrt{1 - \sum p_i^2 - 2\epsilon V}}\int_0^T V_{\mathbf{q}}dt.$$

$$\begin{aligned}
\Delta\mathbf{q} &= \int_0^T \dot{\mathbf{q}}(t)dt - \frac{\mathbf{p}T}{\sqrt{1-\sum p_i^2}} = \int_0^T \left(\dot{\mathbf{q}}(0) + \int_0^t \ddot{\mathbf{q}}(s)ds\right)dt - \frac{\mathbf{p}T}{\sqrt{1-\sum p_i^2}} \\
&= T\left(\sqrt{1-\sum p_i^2} - \sqrt{1-\sum p_i^2 - 2\epsilon V}\right)_{\mathbf{p}} + \frac{\int_0^T\int_0^t \dot{\mathbf{p}}(s) + \epsilon\dot{\mathbf{p}}(s)\cdot V_{\mathbf{pp}} + \epsilon\dot{\mathbf{q}}(s)\cdot V_{\mathbf{qp}}dsdt}{\sqrt{1-2E}} \\
&= T\left(\sqrt{1-\sum p_i^2} - \sqrt{1-\sum p_i^2 - 2\epsilon V}\right)_{\mathbf{p}} + \frac{\int_0^T\int_0^t -\epsilon V_{\mathbf{q}} - \epsilon^2 V_{\mathbf{q}}\cdot V_{\mathbf{pp}} + \epsilon(\mathbf{p}+\epsilon V_{\mathbf{p}})\cdot V_{\mathbf{qp}}dsdt}{1-\sum p_i^2 - 2\epsilon V}.
\end{aligned}$$

We can see from the above calculation that since $\sum p_i^2 + 2\epsilon V < (1+C)/2 < 1$, $(\Delta\mathbf{q}, \Delta\mathbf{p})$ converges to 0 uniformly in C^m as $\epsilon \to 0$. □

4.6 Entropy exapansive cases

Theorem 4.1. [13] For every $\epsilon > 0$ there exists a reversible Finsler metric on $\mathbb{T}^3$ which is ϵ-close to the Euclidean metric in the C^∞-sense and such that the associated geodesic flow has positive metric entropy.

Proof. Suppose $(\mathbb{T}^3, \varphi_0)$ is the Euclidean 3-torus. From 3.2 we can choose $O_p = B^*T_0 = \{(q_1, q_2, p_1, p_2) \in T^*T_0 : p_1^2 + p_2^2 < 1\}$. Then we have

$$\psi(p_1, p_2) = \sqrt{1 - p_1^2 - p_2^2}.$$

Hence the lift of R_1 to universal cover is

$$R(q_1, q_2, p_1, p_2) = \left(q_1 + \frac{p_1}{\sqrt{1-p_1^2-p_2^2}}, q_2 + \frac{p_2}{\sqrt{1-p_1^2-p_2^2}}, p_1, p_2\right)$$

Therefore $R_1 = \Phi^1_{\tilde{H}_0}$. Note that $\Phi^t_{\tilde{H}_0}$ and $\Phi^t_{H_0}$ are the same up to time reparametrization. Let H_ϵ be the perturbation of H_0 as in Lemma 4.3, and define

$$\tilde{H}_\epsilon := -\sqrt{1 - 2H_\epsilon}.$$

$\Phi^t_{\tilde{H}_\epsilon}$ has the same trajectories as $\Phi^t_{H_\epsilon}$, hence $\Phi^t_{\tilde{H}_\epsilon}$ has positive metric entropy since $\Phi^t_{H_\epsilon}$ does. Since the support of perturbation is contained in $\{p_1^2 + p_2^2 < 2/3\}$, $\tilde{H}_\epsilon \to \tilde{H}_0$ in C^∞. From Lemma 4.3 we know that $\Phi^1_{\tilde{H}_\epsilon} \to \Phi^1_{\tilde{H}_0} = R_1$ in C^∞. By Proposition 3.2 we get the desired metric. □

4.7 Entropy non-exapansive cases

In this section we genralizes the methods in [9] and obtains the Burago-Ivanov type result for flat Finsler torus.

Theorem 4.2. [6] The flat Finsler metric φ_0 on $\mathbb{T}^n (n \geq 4)$ can be perturbed in the class of Finsler metrics so that the resulting geodesic flow has positive metric entropy and is entropy non-expansive. Such perturbations can be made C^∞ small. Moreover, if φ_0 is reversible, the resulting metric can be chosen to be reversible.

One primary distinction between our examples and those in Theorem 4.1 is the entropy non-expansiveness. We still do not know if one can do such perturbation in the class of Riemannanian metrics or if one can lower the dimension to $n = 3$.

Proof. We will only give a proof for $n = 4$. Higher dimensional cases are straightforward generalizations.

Recall from Chapter 3.2,

$$R_1 := \Pi \circ R_0 \circ \Pi^{-1} : T_0 \times O_p \to T_0 \times O_p.$$

And $R : \mathbb{R}^3 \times O_p \to \mathbb{R}^3 \times O_p$ defined by

$$R(\mathbf{q}, \mathbf{p}) = (\mathbf{q} + \psi(\mathbf{p})^{-1}\mathbf{p}, \mathbf{p})$$

is a lift of R_1 to the universal cover.

Lemma 4.5. We can find a neighborhood $O_q \subseteq \mathbb{R}^3$ of $\mathbf{q} = \mathbf{0}$ and a symplectic change of coordinates in $O_q \times O_p$ such that in the new coordinates $(\mathbf{Q}, \mathbf{P})$, the map R is the following:

$$R(\mathbf{Q}, \mathbf{P}) = (\mathbf{Q} + \mathbf{P}, \mathbf{P}).$$

Namely, locally ψ can be chosen to be $\psi \equiv 1$ or any positive function.

Proof. Define

$$\mathbf{P} := \psi(\mathbf{p})^{-1}\mathbf{p}.$$

Then we have

$$d\mathbf{P} = d\mathbf{p}\, \Phi(\mathbf{p})$$

for some matrix function $\Phi : O_p \to Mat(3, \mathbb{R})$. Notice that $\Phi(\mathbf{0}) = \psi(\mathbf{0})^{-1}\mathbf{I}_3$. By choosing smaller O_p if necessary we may assume Φ are all invertible. Let O_q be a small neighborhood of $\mathbf{q} = \mathbf{0}$. We make the following coordinate change in $O_q \times O_p$:

$$(\mathbf{Q}, \mathbf{P}) := (\mathbf{q}\Phi(\mathbf{p})^{-1}, \psi(\mathbf{p})^{-1}\mathbf{p}).$$

By direct computation we have

$$d\mathbf{Q} \wedge d\mathbf{P} = d\mathbf{q} \wedge d\mathbf{p}.$$

Under the new coordinates $(\mathbf{Q}, \mathbf{P})$, the map $R|_{O_q \times O_p}$ is the following:

$$R(\mathbf{Q}, \mathbf{P}) = (\mathbf{Q} + \mathbf{P}, \mathbf{P}).$$

$\square$

Denote by $\mathbf{P}^2 := P_1^2 + P_2^2 + P_3^2$ and $\mathbf{Q}^2 := Q_1^2 + Q_2^2 + Q_3^2$. It is not hard to verify $R|_{O_q \times O_p}$ is the time-one map of the Hamiltonian flow on $O_q \times O_p$ with Hamiltonian H_0 defined by:

$$H_0(\mathbf{Q}, \mathbf{P}) := \frac{\mathbf{P}^2}{2}.$$

Define a perturbed Hamiltonian on $O_q \times O_p$ by:

$$H_\epsilon(\mathbf{Q}, \mathbf{P}) := H_0 + \frac{\epsilon\mathbf{Q}^2}{2}\xi(\mathbf{P}^2)\xi(\mathbf{Q}^2)$$

where $\epsilon < 1$ and ξ is a smooth function on $[0, 2]$ with $\xi \equiv 1$ on $[0, \delta]$ and $\xi \equiv 0$ on $[2\delta, 2]$ for some given $\delta > 0$. The change is supported by $\{\mathbf{P}^2 < 2\delta, \mathbf{Q}^2 < 2\delta\}$. Define

$$\Xi := \{(\mathbf{Q}, \mathbf{P}) | \mathbf{P}^2 + \epsilon\mathbf{Q}^2 < \epsilon\delta\}.$$

We have

$$H_\epsilon|_\Xi = H_0 + \frac{\epsilon\mathbf{Q}^2}{2}.$$

Ξ is $\Phi^t_{H_\epsilon}$-invariant and all orbits in Ξ are closed with period $2\pi/\sqrt{\epsilon}$.

We firstly choose small δ so that the support of the perturbation has tiny size in $O_q \times O_p$. Then we choose appropriate ϵ so that $2\pi/\sqrt{\epsilon} = N$ for some positive integer N. As $N \to \infty$, $\Phi^t_{H_\epsilon}$ converges to $\Phi^t_{H_0}$ in C^∞ topology. The time one map

$T := \Phi^1_{H_\epsilon}$ satisfies $T^N = id$ on Ξ.

After the first perturbation, we want to perturb T in order to get positive metric entropy. In [9], Burago and Ivanov proved the following lemma for 6-dimensional disc. In fact, similar arguments also work for general $2n$-dimensional discs with $n \geq 4$.

Lemma 4.6 (Burago-Ivanov [9])**.** There exists a symplectomorphism $\theta : D^6 \to D^6$ which is arbitrarily close to the identity in C^∞, coincides with the identity map near the boundary, and has positive metric entropy.

Let $D \subseteq \Xi$ be a closed set such that $D, T(D), T^2(D), ..., T^{N-1}(D)$ are disjoint. We can choose D to be symplectomorphic to the standard unit disc D^6. Let $\theta : D^6 \to D^6$ be the map in Lemma 4.6. We extend this map by identity to a map from $T_0 \times O_p$ to itself. We abuse notation and still use θ to denote this map. The restriction of $(T \circ \theta)^N$ to D is θ. Therefore $(T \circ \theta)^N$ has positive metric entropy, thus so does $T \circ \theta$.

Since T and θ are C^∞-close to R_1 and id respectively, $\tilde{R}_1 := T \circ \theta$ can be as close to R_1 in C^∞ topology as we want. Moreover, the support of $\tilde{R}_1 - R_1$ can be arbitrarily small given we choose tiny δ.

By Proposition 3.2, we get the desired Finsler metric. $\square$

Appendix
Nondense Irrational Geodesics in Nearly Flat Finsler Tori

1 Twist maps, minimal configurations and Peierls' barrier

1.1 Twist maps and generating functions

Definition .1 ([23]). $f : S^1 \times (a, b) \to S^1 \times (a, b)$ is *an area-preserving twist map* if:

(1) f is area preserving and preserves orientation.

(2) f preserves boundary components in the sense that there exists an $\epsilon > 0$ such that if $(x, y) \in S^1 \times (x, x + \epsilon)$ then $f(x, y) \in S^1 \times (a, \frac{a+b}{2})$.

(3) if $F = (F_1, F_2)$ is a lift of f to the universal cover $\mathbb{R} \times (a, b)$ then $\frac{\partial F_1}{\partial y}(x, y) > 0$. Here (a, b) can be an open interval or the whole real line.

If in addition to (1)-(3) we have

(4) f twists infinitely at either end. Namely, for all $x \in S^1$ we have

$$\lim_{y \to a+} F_1(x, y) = -\infty, \lim_{y \to b-} F_1(x, y) = +\infty,$$

then we say f is *an area-preserving twist map with infinite twist.* The collection of all area-preserving twist maps with infinite twist from $S^1 \times (a, b)$ to itself is denoted $IFT(a, b)$.

Let $F : \mathbb{R} \times (a, b) \to \mathbb{R} \times (a, b)$ be a lift of $f \in IFT(a, b)$ to the universal cover, the *generating function* $h(x,x')$ is uniquely characterized by

$$F(x,y) = (x',y') \Longleftrightarrow y = -\frac{\partial h}{\partial x}(x,x'), y' = \frac{\partial h}{\partial x'}(x,x').$$

Example .1. The map $f_0 : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ defined by $f(x,y) = (x+y,y)$ is an area-preserving twist map with infinite twist. The generating function is given by

$$h_0(x,x') = \frac{(x'-x)^2}{2}.$$

Example .2. Define $f_1 : S^1 \times (-1,1) \to S^1 \times (-1,1)$ by $f(x,y) = (x + \frac{y}{\sqrt{1-y^2}}, y)$. Then $f_1 \in IFT(-1,1)$ and the generating function is given by

$$h_1(x,x') = \sqrt{(x'-x)^2 + 1}.$$

Given a $f \in IFT(a,b)$, if the amount of twisting in (3) has a uniform lower bound β, then its generating function h will satisfy all the following conditions $(H_1) - (H_{6\theta})$ with $\theta = \cot\beta$ [27]:

$$(H_1) \qquad h(x,x') = h(x+1,x'+1)$$

$$(H_2) \qquad \lim_{|\xi|\to\infty} h(x,x+\xi) = +\infty, \text{ uniformly in } x$$

There exists a positive continuous function ρ on $\mathbb{R}^2$ such that for $x < \xi, x' < \xi'$:

$$(H_5) \qquad h(\xi,x') + h(x,\xi') - h(x,x') - h(\xi,\xi') \geq \int_x^\xi \int_{x'}^{\xi'} \rho$$

$$(H_{6\theta}) \qquad \begin{cases} x \to & \theta x^2/2 - h(x,x') \text{ is convex for any } x' \\ x' \to & \theta x'^2/2 - h(x,x') \text{ is convex for any } x \end{cases}$$

Here θ is a positive number. We say h satisfies (H_6) if it satisfies $(H_{6\theta})$ for some $\theta > 0$. There was (H_3) and (H_4) but they can be derived from others. We use $\mathscr{H}_\theta$ to denote the collection of all continuous functions $h : \mathbb{R}^2 \to \mathbb{R}$ satisfying $(H_1) - (H_{6\theta})$.

1.2 Properties of functions in $\mathscr{H}_\theta$

Mather [28] proves that for a given $h \in \mathscr{H}_\theta$ there exists a unique Borel measure μ_h on $\mathbb{R}^2$ such that for any $x < \xi, x' < \xi'$

$$\mu_h([x,\xi] \times [x',\xi']) = h(\xi, x') + h(x, \xi') - h(x, x') - h(\xi, \xi').$$

and two unique Borel measures ν_h^1, ν_h^2 on $\mathbb{R}$ such that

$$\nu_h^1(y, z] = \theta(y - z) + \partial_1 h(y+, y) - \partial_1 h(z+, z),$$

$$\nu_h^2(y, z] = \theta(y - z) + \partial_2 h(y, y+) - \partial_2 h(z, z+).$$

It is clear ν_h^i is invariant under the translation $y \to y + 1$ and $\nu_h^i(y, y+1] = \theta$. For $x \leq \xi$, we have

$$\mu_h([x,\xi]^2) \leq (\xi - x)\nu_h^i(x, \xi), i = 1, 2. \tag{.1}$$

For any sequence $(x_i)_{i\in\mathbb{Z}}$,

$$\sum_{i=j}^{k-1} h(x_i, x_{i+1}) = \sum_{i=j}^{k-1} h(x_i, x_i) + \int_{x_j}^{x_k} \partial_2 h(y, y+) dy + \sum_{i=j}^{k-1} \mu_h(\Delta[x_i, x_{i+1}]), \tag{.2}$$

where Δ_i is the triangle

$$\{(y, z) : x_i \leq y \leq z \leq x_{i+1}\} \text{ or } \{(y, z) : x_{i+1} \leq y \leq z \leq x_i\}$$

according to whether x_i or x_{i+1} is greater. For the proofs of the results listed above, see [28].

If h_1 and h_2 are two real-valued continuous functions on $\mathbb{R}^2$ satisfying (H_2), then the *conjunction of* h_1 *and* h_2 is defined to be

$$h_1 * h_2(x, x') = \min_y h_1(x, y) + h_2(y, x').$$

If $h_1, h_2 \in \mathscr{H}_\theta$, then $h_1 * h_2 \in \mathscr{H}_\theta$ [27].

1.3 Minimal configuration and Rotation symbols

We refer to [4] [18] [27] [28] for the definitions and results we will need in the following.

A *configuration* is a bi-infinite sequence $\mathbf{x} = (..., x_i, ...) \in \mathbb{R}^{\mathbb{Z}}$ (with product topology of $\mathbb{R}^{\mathbb{Z}}$). The *Aubry graph* of $\mathbf{x}$ is the graph of the piecewise linear function $\Phi : \mathbb{R} \to \mathbb{R}$ determined by $\Phi(i) = x_i$ at every $i \in \mathbb{Z}$.

Suppose h is a function on $\mathbb{R}^2$ satisfying $(H_1) - (H_6)$. Define

$$h(x_j, ..., x_k) := \sum_{i=j}^{k} h(x_i, x_{i+1}).$$

A segment $(x_j, ..., x_k)$ is said to be *minimal* (for h) if it is a minimizer for $h(x_j^*, ..., x_k^*)$ with $x_j^* = x_j$ and $x_k^* = x_k$, A configuration is minimal if all its segments are minimal. We use $\mathscr{M} = \mathscr{M}_h$ to denote the set of all minimal configurations. The Aubry graphs of minimal configurations cross at most once (see [4] (3.1)). In the survey [4] Bangert shows how minimal geodesics on torus are related to minimal configurations.

A configuration $\mathbf{x}'$ is a *translate* of $\mathbf{x}$ if there exist integers j, k such that $x_i' = x_{i+j} + k$ for all i. In [4] Bangert use the notation $T_{(a,b)}$ to denote the translation $T_{(a,b)}\mathbf{x} = \mathbf{x}'$ where $x_i' = x_{i-a} + b$.

A translate of minimal configuration is always minimal. A basic result of Aubry says that the set of translates of a minimal configuration is totally ordered with $\mathbf{x} < \mathbf{y}$ being defined to be $x_i < y_i$ for all integers i ([4] (3.13)). Aubry's result has a consequence that if $\mathbf{x}$ is a minimal configuration, then there is a number $\omega = \rho(\mathbf{x})$, called the *rotation number* of $\mathbf{x}$, such that if $x_i' = x_{i+j} + k$ with $j > 0$, then $\mathbf{x}' > \mathbf{x}$ (resp. $\mathbf{x}' < \mathbf{x}$) if $j\omega + k > 0$ (resp. $j\omega + k < 0$).

When $\rho(\mathbf{x})$ is irrational, it is also called *rotation symbols* $\tilde{\rho}(\boldsymbol{x})$ *of* $\boldsymbol{x}$. When $\rho(x) = p/q \in \mathbb{Q}, q > 0$, we investigate $x_i' = x_{i+q} - p$ i.e. $\mathbf{x}' = T_{(-q,-p)}\mathbf{x}$, and we define

$$\tilde{\rho}(\mathbf{x}) = \begin{cases} p/q+ & \text{if } \mathbf{x}' > \mathbf{x} \\ p/q & \text{if } \mathbf{x}' = \mathbf{x} \\ p/q- & \text{if } \mathbf{x}' < \mathbf{x} \end{cases}$$

Since minimal configurations cross at most once, if $\tilde{\rho}(\mathbf{x}) = \tilde{\rho}(\mathbf{x}') = p/q$, then the Aubry graphs of $\mathbf{x}$ and $\mathbf{x}'$ do not cross.

The space $\mathscr{S}$ of *rotation symbols* is the disjoint union $\mathbb{R} \sqcup \mathbb{Q}+ \sqcup \mathbb{Q}-$. Here $\mathbb{Q}\pm$ are copies of $\mathbb{Q}$. For $p/q \in \mathbb{Q}$, let $p/q\pm$ be the corresponding elements in $\mathbb{Q}\pm$. The *underlying number* is defined to be the projection image on $\mathbb{R}$, denoted ω^*. We provide $\mathscr{S}$ with the unique total order for which $p/q- < p/q < p/q+$ and the map $\omega \mapsto \omega^*$ is weakly order preserving [28]. For any $\omega \in \mathscr{S}$, $\mathscr{M}_\omega = \mathscr{M}_{\omega,h}$ denotes the set of all minimal configurations of rotation symbol ω or ω^* (for example, $\mathscr{M}_{p/q+,h}$ contains the minimal configurations with rotational symbol $p/q+$ or p/q). $\mathscr{M}_\omega$ is nonempty for all $\omega \in \mathscr{S}$, see [4]. Define a projection pr_0 by $pr_0(\mathbf{x}) := x_0$. Let $A_\omega := pr_0(\mathscr{M}_\omega)$. Then A_ω is closed and $pr_0 : \mathscr{M}_\omega \to A_\omega$ is a homeomorphism.

Remark .1. (i) If $\mathbf{x} \in \mathscr{M}_{p/q}$, then $\mathbf{x}$ is a minimum of $h_{q,p} : P_{q,p} \to \mathbb{R}, x \mapsto h(x_0, ..., x_q)$, here $P_{q,p}$ denotes the $T_{q,p}$-invariant confirgurations. In particular $H_{q,p}$ is constant on $\mathscr{M}_{p/q}$. See [4].

(ii) The Aubry graph of configurations in $\mathscr{M}_{p/q+,h}$ and $\mathscr{M}_{p/q-,h}$ do not cross. Suppose $\mathbf{x} < \mathbf{x}'$ are two neighborhood configurations in $\mathscr{M}_{p/q}$, then there exists configuration $\mathbf{y}^-$ (resp. $\mathbf{y}^+$) between $\mathbf{x}$ and $\mathbf{x}'$ with rotation symbol $p/q-$ (resp. $p/q+$) such that it is ω-asymptotic (resp. α-asymptotic) to $\mathbf{x}$ and α-asymptotic (resp. ω-asymptotic) to $\mathbf{x}'$. See [4].

(iii) For any $h \in \mathscr{H}_\theta$, let $H(x, x') := h^{*q}(x, x'+p)$, where $h^{*q} = h * \cdots * h$(q times) denotes the q-fold conjunction with itself. Then by Section A.1.2 we know $H \in \mathscr{H}_\theta$. It is clear that $A_{\omega,h} = A_{q\omega-p,H}$ and $P_{\omega,h} = P_{q\omega-p,H}$ as long as the rotational symbol ω is not a rational number or is a rational number whose denominator is divisible by q.

1.4 Peierls' barrier

Peierls' barrier $P_\omega(\xi) = P_{\omega,h}(\xi)$ is defined for every real number ξ. If $\xi \in A_\omega$, then $P_\omega(\xi) = 0$. Otherwise, ξ belongs to some complementary interval (x_0^-, x_0^+) of A_ω in $\mathbb{R}$, where $\mathbf{x}^-, \mathbf{x}^+ \in \mathscr{M}_\omega$. Suppose $\mathbf{x}$ is a configuration with $x_0 = \xi, \mathbf{x}^- \leq \mathbf{x} \leq \mathbf{x}^+$, and if $\omega = p/q$, we require $x_{i+q} = x_i + p$. Then $P_\omega(\xi)$ is defined to be the minimum of the following formula taken over all such $\mathbf{x}$:

$$\sum_i h(x_i, x_{i+1}) - h(x_i^-, x_{i+1}^-).$$

For any $h \in \mathscr{H}_\theta$, $P_{\omega,h}(\xi)$ exists, is non-negative, vanishes only on A_ω and is a

Lipschitz function of ξ with Lipschitz constant 2θ. See [27] for details.

In [27] and [28] Mather shows a modulus of continuity for Peierls' barrier:

Theorem .1. There exists a positive real number such that the following holds. For any $h \in \mathscr{H}_\theta$, $p/q \in \mathbb{Q}$ and ω a rotation symbol, we have

(1) $|P_{p/q}(\xi) - P_\omega(\xi)| \leq C\theta(q^{-1} + |q\omega^* - p|)$;

(2) $|P_{p/q+}(\xi) - P_\omega(\xi)| \leq C\theta|q\omega^* - p|$ for $\omega \geq p/q+$;

(3) $|P_{p/q-}(\xi) - P_\omega(\xi)| \leq C\theta|q\omega^* - p|$ for $\omega \leq p/q-$.

2 An extension of Mather's destruction of invariant circle

Proposition .1. For any $f \in IFT(a,b)$ and any Liouville number ω, we can find a C^∞ small perturbation $\tilde{f} \in IFT(a,b)$ and a compact $K \subseteq S^1 \times (a,b)$ such that $\tilde{f} - f$ has support K and there is no $\tilde{f}$-invariant circle with rotation number ω.

Remark .2. In [28] Mather proved that for any Liouville number ω and twist map in $IFT(-\infty,+\infty)$ there exists a C^∞ small perturbation with no invariant circle admitting rotation number ω. But the perturbation of Mather is not compactly supported. Nevertheless we will imitate Mather's construction to build up our perturbation.

Proof of Proposition .1. We are going to prove that for any $\epsilon > 0$ and $r \geq 1$, we can find $\tilde{f} \in IFT(a,b)$ such that $||\tilde{f} - f||_{C^r} \leq \epsilon$ and there is no $\tilde{f}$-invariant circle with rotation number ω. The general idea is firstly choose a rational number p/q close to ω, and make a C^{r+1} small perturbation h' on the generating function h so that the $P_{p/q,h'}$ is positive at some point. When $p/q < \omega$, we make the second perturbation h'' so that the maximum of $P_{p/q+,h''}$ is bounded from below by a constant depending only on q and r. Once p/q is sufficiently close to ω (here we use the property that ω is Liouville), by Theorem .1 we can see $P_{\omega,h''}$ does not vanish identically. For $p/q > \omega$, we find a lower bound of $P_{p/q-,h''}$ instead of $P_{p/q+,h''}$ and the rest proceeds in a similar way.

Without loss of generality we may assume the twisting amount $\partial F_1/\partial y$ has a lower bound β so that we can use the formulas in Theorem .1. In fact, if (a,b) is finite, $\partial F_1/\partial y$ will have lower bound due to (3) and (4) in the definition of twist map. If $a = -\infty$ or $b = +\infty$, see the first paragraph in the proof of Theorem 2.1 in [28].

Let $F : \mathbb{R} \times (a, b) \to \mathbb{R} \times (a, b)$ be the lift of f to universal cover and h be the generating function. By the above assumption $h \in \mathscr{H}_\theta$. Suppose $\mathbf{x}$ is a minimal configuration in $\mathscr{M}_{p/q}$.

We now explain how to construct the perturbation of h when $\omega > p/q$. Choose an interval J with length$\geq q^{-1}$ in the complement to the set $\{x_i + j\}_{i,j\in\mathbb{Z}}$. Without loss of generality we may assume $J = (x_j, x_k + m)$ for some $j, k, m \in \mathbb{Z}$.

For any $\epsilon > 0$ and any integer $r \geq 1$, we choose a C^∞ nonnegative function u on $\mathbb{R}$ with the following properties:

(a) u has support $\bar{J}$.

(b) $||u||_{C^{r+1}} \leq \epsilon/2$.

(c) $u(\xi) \geq C_1(r)\epsilon/q^{r+1}$, for $\xi \in J'$, here J' is the middle third of J and $C_1(r)$ is a constant depending only on r.

Here is how to construct such a function: Define a function $\Psi : \mathbb{R} \to \mathbb{R}$ by:

$$\Psi(t) = \begin{cases} \exp\left(\dfrac{1}{t^2-1}\right), & \text{for } |t| < 1 \\ 0, & \text{otherwise} \end{cases}$$

Denote $C_0(r) := ||\Psi||_{C^{r+1}}$. Define a function u_0 by

$$u_0(t) = \frac{\epsilon}{2^{r+2}q^{r+1}C_0(r)}\Psi(2qt).$$

and let

$$C_1(r) = \Psi\left(\frac{1}{3}\right)2^{-r-2}C_0(r)^{-1}.$$

It is not hard to check that u_0 satisfies (a)-(c) for $J = (-1/2q, 1/2q)$. For a general J, we have only to move and rescale u_0.

Define a function v on $\mathbb{R}$ by

$$v(t) = \begin{cases} C_2(r)q^{-r-1}\Psi(2q(t - x_{j+1})), & \text{for } t \in [x_{j+1} - 1/2q, x_{j+1}); \\ C_2(r)q^{-r-1}\Psi(0), & \text{for } t \in [x_{j+1}, x_{k+1} + m); \\ C_2(r)q^{-r-1}\Psi(2q(t - x_{k+1} - m)), & \text{for } t \in [x_{k+1} + m, x_{k+1} + m + 1/2q); \\ 0, & \text{otherwise,} \end{cases}$$

where $C_2(r) = 2^{-r-1}C_0(r)^{-1}$. Note that v is nonnegative, C^∞, supported by an interval with length$\leq 3/q$ and $||v||_{C^{r+1}} = 1$.

Now we make a first perturbation on h:

$$h'(x,x') = h(x,x') + \sum_{i\in\mathbb{Z}} u(x+i)v(x'+i).$$

Note that for each point (x,x'), the sum in the right hand side contains at most one nonzero term, hence h' is well-defined. Moreover we have

$$||h'-h||_{C^{r+1}} \leq ||u||_{C^{r+1}}||v||_{C^{r+1}} \leq \epsilon/2.$$

It is clear that h' satisfies $(H_1), (H_2)$ and $(H_{6\theta'})$ for $\theta' = \theta+1$ given $r \geq 1$ and ϵ small. For (H_5), since h is a generating function of $f = (f_1, f_2)$, for $x < \xi, x' < \xi'$,

$$h(\xi,x') + h(x,\xi') - h(x,x') - h(\xi,\xi') = \int_x^\xi \int_{x'}^{\xi'} -\frac{\partial f_1}{\partial y}.$$

The integrand is bounded from below by a constant, therefore it remains positive under small C^r perturbation. Hence h' satisfies (H_5).

Now let us see how does this perturbation affect the Peierls' barrier $P'_{p/q}$ associated to h'. For any $\xi \in J$, suppose (x_0^-, x_0^+) is the complementary interval of $A'_{p/q}$ containing ξ. Then we have $x_j \leq x_0^- \leq x_0^+ \leq x_k + m$. Take any configuration $\mathbf{y}$ with $y_0 = \xi$ and $\mathbf{x}^- \leq \mathbf{y} \leq \mathbf{x}^+$. Since $\mathbf{x}, \mathbf{x}^\pm \in \mathscr{M}'_{p/q}$, their Aubry graphs do not cross(see Remark 5). Hence we have

$$x_{j+1} \leq x_1^- \leq y_1 \leq x_1^+ \leq x_{k+1} + m.$$

Therefore

$$\begin{aligned} P'_{p/q}(\xi) &= \min_{\mathbf{x}^-\leq\mathbf{y}\leq\mathbf{x}^+, y_0=\xi} \sum_{i=0}^{q-1} h'(y_i, y_{i+1}) - h'(x_i^-, x_{i+1}^-) \\ &\geq P_{p/q}(\xi) + \min_{\mathbf{x}^-\leq\mathbf{y}\leq\mathbf{x}^+, y_0=\xi} u(y_0)v(y_1) \\ &= P_{p/q}(\xi) + C_2(r)\Psi(0)u(\xi)/q^{r+1} \end{aligned}$$

Let $H'(x,x') := h'^{*q}(x, x'+p)$. Since $\mathbf{x} \in A_{p/q,h'} = A_{0,H'}$, we add a constant to

H' so that $H'(x_i, x_i) = 0$ for all i. For any $\xi \in J'$,

$$H'(\xi,\xi) = P_{0,H'}(\xi) = P_{p/q,h'}(\xi) \geq C_2(r)\Psi(0)u(\xi)/q^{r+1} \geq C_3(r)\epsilon/q^{2r+2}, \qquad (.3)$$

where $C_3(r) := C_1(r)C_2(r)\Psi(0)$.

To simplify the notation, we denote by $J^- < J^+$ the endpoints of J. Since $P'_{p/q}$ is positive on J, $J^\pm$ are neighborhood elements in $A_{0,H'}$ and H' is positive in J. From [4] Theorem 5.3 there exist a minimal configuration $\mathbf{y} \in \mathscr{M}_{0+,H'}$ such that $y_i \to J^\pm$ as $i \to \pm\infty$.

In order to do further perturbation we need to find a lower bound of $\max_i |y_{i+1} - y_i|$. We consider the point $y_i \in J'$(if no such y_i exists then we take the length of J' as our lower bound). Mather [28] shows that

$$H'(\mathbf{y}) = \sum_{i=-\infty}^{\infty} h'(y_i, y_{i+1})$$

is absolutely convergent. Hence in this case we can extend formula (.2) to infinite sums

$$H'(\mathbf{y}) = \sum_{i=-\infty}^{\infty} h'(y_i, y_i) + \int_{J^-}^{J^+} \partial_2 h'(y, y+) dy + \sum_{i=-\infty}^{\infty} \mu(\Delta[y_i, y_{i+1}]),$$

where $\mu = \mu_{H'}$. Let $\mathbf{y}'$ be the configuration obtained from $\mathbf{y}$ by removing y_i, i.e. $y'_j = y_j$ for $j < i$ and $y'_j = y_{j+1}$ for $j \geq i$. $H'(\mathbf{y}')$ is finite since $H'(\mathbf{y})$ is. Hence we can also use formula (.2) to calculate $H(\mathbf{y}')$. By taking difference we have

$$H'(\mathbf{y}') - H'(\mathbf{y}) = \mu(\Delta[y_{i-1}, y_{i+1}]) - \mu(\Delta[y_{i-1}, y_i]) - \mu(\Delta[y_i, y_{i+1}]) - H'(y_i, y_i).$$

On the other hand the left hand side is nonpositive since $\mathbf{y}$ is minimal. Hence

$$C_3(r)\epsilon/q^{2r+2} \leq H'(y_i, y_i) \leq \mu(\Delta[y_{i-1}, y_{i+1}]) \leq \theta' |y_{i+1} - y_{i-1}|.$$

Here the first inequality comes from formula (.3) and the last inequality comes from formula (.1). This implies

$$\max_i |y_{i+1} - y_i| \geq C_4(r)\epsilon/q^{2r+2},$$

where $C_4(r) = C_3(r)/2\theta'$. Choose i such that $|y_{i+1} - y_i| \geq C_4(r)\epsilon/q^{2r+2}$ and denote $I := [y_i, y_{i+1}]$. Use similar construction as u, we can build up a C^∞ function w with support in I with $||w||_{C^{r+1}} \leq \epsilon/2$ and

$$\max w \geq C_5(r)\epsilon^{r+2}/q^{2(r+1)^2},$$

where $C_5(r) = C_4(r)^{r+1}C_0(r)^{-1}2^{-r-2}$. We set

$$h''(x, x') = h'(x, x') + \sum_{i\in\mathbb{Z}} w(x + i)v(x' + i)$$

If ϵ is small enough, $h'' \in \mathscr{H}_{\theta'}$. Moreover

$$||h'' - h||_{C^{r+2}} \leq ||h' - h||_{C^{r+2}} + ||w||_{C^{r+1}}||v||_{C^{r+1}} \leq \epsilon.$$

and

$$P''_{p/q+}(\xi) \geq P'_{p/q+}(\xi) + w(\xi)C_2(r)\Psi(0)/q^{r+1}.$$

So

$$P''_{p/q+}(\xi_0) \geq C_6(r)\epsilon^{r+2}/q^{2(r+1)(r+2)}, \tag{.4}$$

where ξ_0 is where w reaches its maximum and $C_6(r) := C_2(r)C_5(r)\Psi(0)$. Notice that $C_6(r)$ is independent of p/q and ω is Liouville, hence we can choose p/q so close to ω that

$$C\theta'|\omega q - p| < C_6(r)\epsilon^{r+2}/q^{2(r+1)(r+2)}, \tag{.5}$$

where C is the constant in Theorem .1. When $\omega > p/q$, by (.4)(.5) and Theorem .1,

$$P''_\omega(\xi_0) \geq P''_{p/q+}(\xi_0) - C\theta'|\omega q - p| > 0.$$

When $\omega < p/q$, instead of choosing $\mathbf{y}$ from $\mathscr{M}_{0+,H'}$, we choose $\mathbf{y}$ from $\mathscr{M}_{0-,H'}$ and use similar construction to increase $P''_{p/q-}$.

This proves Proposition .1. $\square$

3 Nondense irrational geodesics

Theorem .2. For any Liouville number ω, one can perturb the flat Finsler metric φ_0 on $\mathbb{T}^2$ in the class of Finsler metric so that the resulting metric has a non-dense geodesic with rotation vector $\left(\frac{\omega}{\sqrt{1+\omega^2}}, \frac{1}{\sqrt{1+\omega^2}}\right) \in S^1$. Such perturbation can be made C^∞ small. If the unperturbed Finsler metric is reversible, the resulting Finsler metric can be chosen to be reversible as well.

Proof. For any Liouville number ω, by taking an action of some matrix in $SL(2, \mathbb{Z})$ on the lattice and then translate in universal cover, we may assume ω is close to 0. Recall in Section 3.2, when $n = 2$, the map R_1 is given by

$$R_1 : S^1 \times (-1, 1) \to S^1 \times (-1, 1), (x, y) \mapsto \left(x + \frac{y}{\psi(y)}, y\right).$$

We have $R_1 \in IFT(-1, 1)$ and its generating function is

$$h(x, x') = \kappa(x - x') := \sqrt{d_{\varphi_0}((x, 0), (x', 1))}.$$

For any $\epsilon > 0, r \geq 1$, choose p/q sufficiently close to ω so that formula (.5) holds. We take $J = (-1/2q, 1/2q)$ and and use .1 to construct $h'' : \mathbb{R}^2 \to \mathbb{R}$ with $||h'' - h||_{C^{r+1}} \leq \epsilon$ and the twist map $R_1'' \in IFT(-1, 1)$ associated to h'' has no invariant circle with rotational number ω. From Aubry-Mather theory the absence of R_1''-invariant circle implies the existence of a minimal R_1''-invariant Cantor set whose projection to S is also Cantor. Let K be the support of $R_1'' - R_1$. It is not hard to see $\pi(K) \subseteq (-1/q, 1/q)$ and $\pi(R(K)) \subseteq (-\omega - \frac{3}{q}, \omega + \frac{3}{q})$. Hence K is penetrating for large q.

From Proposition 3.2 there exists a reversible Finsler metric $\tilde{\varphi}$ on $\mathbb{T}^2$ such that the Poincaré map of the geodesic flow is $\Pi^{-1} \circ R_1'' \circ \Pi$. The R_1''-invariant Cantor set with rotation number ω implies the existence of a nondense geodesic with rotation vector $\left(\frac{\omega}{\sqrt{1+\omega^2}}, \frac{1}{\sqrt{1+\omega^2}}\right)$. This proves Theorem 3.1. $\square$

Bibliography

[1] V. Arnold, *Proof of a Theorem by A. N. Kolmogorov on the invariance of quasi-periodic motions under small perturbations of the Hamiltonian,* Russian Math. Survey 18 (1963), 13-40.

[2] V. Arnold, *Instability of dynamical systems with several degrees of freedom,* Soviet Mathematics 5 (1964), 581-585.

[3] S. Aubry, P. Le Daeron, *The discrete Frenkel-Kontorova model and its extensions.I. Exact results for the ground-states*, Phys. D, 8(3):381–422, 1983.

[4] V. Bangert, *Mather Sets for Twist Maps and Geodesics on Tori.* Dynamics Reported, 1, (1988), 1-56.

[5] A. Bolsinov, I. Taimanov, *Integrable geodesic flows on the suspensions of toric automorphisms,* Proceedings of the Steklov Institute of Math. 231 (2000), 42-58.

[6] D. Burago, D. Chen, S. Ivanov, *An example of entropy non-expansive nearly integrable system*, manuscript in preparation.

[7] D. Burago, S. Ivanov, *Riemannian tori without conjugate points are flat*, Geom. Funct. Anal. 4 (1994), no. 3, 259-269.

[8] D. Burago, S. Ivanov, *Boundary rigidity and filling volume minimality of metrics close to a flat one*, Ann. of Math. 171 (2010), no. 2, 1183-1211.

[9] D. Burago, S. Ivanov. *Boundary distance, lens maps and entropy of geodesic flows of Finsler metrics,* Journal of Geometry and Topology, 20 (2016), 469-490.

[10] D. Burago, S. Ivanov, B. Kleiner. *On the structure of the stable norm of periodic metrics*, Mathematical Research Letters 4 (1997), 791-808.

[11] K. Burns, M. Gerber. *Real analytic Bernoulli geodesic flows on S2,* Ergodic Theory and Dynamical Systems, Volume 9 (1989), 27-45.

[12] D. Chen, *On total flexibility of local structures of Finsler tori without conjugate points*, https://arxiv.org/pdf/1310.7299.pdf

[13] D. Chen, *Positive metric entropy arises in some nondegenerate nearly integrable systems*, Journal of Modern Dynamics, vol. 11, pp. 43-56, 2017.

[14] C. Contreras, *Geodesic flows with positive topological entropy, twist map and hyperbolicity,* Annals of Mathematics, Second Series, Vol. 172, No. 2 (2010), 761-808.

[15] C. Croke, B. Kleiner, *On tori without conjugate points*, Invent. Math. 120 (1995), 241–257.

[16] V. Donnay, *Geodesic flow on the two-sphere, Part I: Positive measure entropy,* Ergodic Theory and Dynamical Systems, 8 (1989), 531-553.

[17] V. Donnay, C. Liverani, *Potentials on the two-torus for which the Hamiltonian flow is ergodic,* Communications in Mathematical Physics, 135 (1991), No. 2, 267-302.

[18] G. Forni. *Analytic destruction of invariant circles*, Ergodic Theory Dynam. Systems, 14(2):267–298, 1994.

[19] J. Heber, On the geodesic flow of tori without conjugate points, Math. Z. 216 (1994), 209-216.

[20] E. Hopf. Closed surfaces without conjugate points. Proceedings of the National Academy of Sciences 34 (1948), 47-51.

[21] G. A. Hedlund, M. Morse, Manifolds Without Conjugate Points, Trans. Am. Math. Soc., 51, (1942), 362-386.

[22] F. John, *Extremum problems with inequalities as subsidiary conditions,* Courant Anniversary Volume, New York, Interscience (1948), 187-204.

[23] A. Katok, B. Hasselblat, *Introduction to the Modern Theory of Dynamical Systems*, Vol. 54. Cambridge university press (1997).

[24] A. Kolmogorov, *On the conservation of conditionally periodic motions under small perturbation of the Hamiltonian,* Dokl. Akad. Nauk SSSR 98 (1954), 525-530.

[25] B. Kleiner. Personal communication.

[26] J. Moser, *On invariant curves of area-preserving mappings of an annulus.* Nach. Akad. Wiss. Göttingen, Math. Phys. Kl. II 1 (1962), 1-20.

[27] J. Mather, *Modulus of continuity for Peierls's barrier. Periodic Solutions of Hamiltonian Systems and Related Topics*, ed. P.H. Rabinowitz et al. NATO ASI Series C 209. Reidel: Dordrecht, 1987, 177-202.

[28] J. Mather, *Destruction of invariant circles*, Ergodic Theory and Dynamical Systems, 8, (1988) 199-214.

[29] N. Nekhoroshev, *Behavior of Hamiltonian systems close to integrable,* Functional Analysis and Its Applications, Volume 5, Issue 4 (1971), 338-339.

[30] S. Newhouse, *Quasi-elliptic periodic points in conservative dynamical systems*, American Journal of Mathematics, Volume 99, No. 5 (1977), 1061–1087.

[31] S. Newhouse, *Continuity Properties of Entropy*, Annals of Mathematics, Second Series, Volume 129, No. 1 (1989), 215–235.

[32] Y. Pesin, *Lyapunov characteristic exponents and smooth ergodic theory,* Uspekhi Mat. Nauk, 32:4(196) (1977), 55–112.

[33] S. Sasaki, *On the differential geometry of tangent bundles of Riemannian manifolds,* Tohoku Math. J. 10 (1958), 338-354.

[34] M. Wojtkowski, *Invariant families of cones and Lyapunov exponents*, Ergodic Theory and Dynamical Systems, **5** (1985), 145–161.

[35] N. Zinov'ev, *Examples of Finsler metrics without conjugate points: Metrics of revolution*, St. Petersburg Math. J. 20 (2009), 361-379.

Vita
Dong Chen

EDUCATION

Ph.D., Mathematics, Graduate Minor in Statistics **Expected: August 2017**
Concentration Area: Differential Geometry and Dynamical Systems
Dissertation Title: "On some problems in Lagrangian Dynamics and Finsler Geometry"
Dissertation Chair: Dmitri Burago

B.S., Mathematics **2011**
Department of Mathematics
Sun Yat-Sen University, China

PUBLICATIONS

Chen, D., *Positive metric entropy arises in some nondegenerate nearly integrable systems,* Journal of Modern Dynamics, vol. 11, pp. 43-56, 2017.

Chen, D., *On total flexibility of local structures of Finsler tori without conjugate points,* submitted.

Burago, D., **Chen, D.**, & Ivanov S., *An example of entropy non-expansive nearly integrable system,* manuscript in preparation.

Chen, D., & Burago, D., *Nondense geodesics with irrational rotational vector in C^∞-small perturbation of flat tori*, manuscript in preparation.

Burago, D., Burago, Y., & **Chen, D.**, *Elementary Introduction to Modern Geometry*, book in preparation. This book is planned to be a text for instructors, graduate students and advanced undergraduate students interested in modern geometry.

TEACHING

Instructor The Pennsylvania State University
- MATH 021 - College Algebra I (FA12, SP13, FA15, SP16)
- MATH 230 - Calculus and Vector Analysis (FA14, SP15, SP17)

Teaching Assistant The Pennsylvania State University
- MATH 497C - Winding Number in Topology and Geometry (and the Rest of Mathematics) (FA13). Mathematics Advanced Study Semesters (MASS) Program
- MATH 536 – Abstract Algebras (SP14, SP17)