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EFFECTS OF A FRACTION EQUIVALENCE INTERVENTION COMBINING CRA-I AND NUMBER LINE REPRESENTATIONS

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ABSTRACT

This study examines the effects of an instructional intervention focused on fraction equivalence content using explicit instruction and concrete-representational-abstract-integrated (CRA-I) format lessons. The intervention was designed to help students build fraction magnitude knowledge and make connections between area-model and number line model representations of fractions. The four-week intervention was implemented with fourth grade students at risk for mathematics failure and with identified disabilities in a pretest-intervention-posttest comparison group experimental design study. Results indicated that students in the intervention group \((n = 31)\) significantly outperformed students in the control group \((n = 29)\) on tests of fraction equivalence and fraction number line estimation accuracy, but not on an assessment aligned with students’ core mathematics instruction or on an assessment composed of released Partnership for Assessment of Readiness for College and Careers (PARCC) fractions items. Implications for research and practice are discussed.

Keywords: fractions, magnitude knowledge, CRA, number line, special education, learning disabilities
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INTRODUCTION

Fractions represent a major bottleneck in mathematics education. The many students who do not develop fractions competence in elementary and middle school are missing foundational knowledge that is critical for success in Algebra (Booth & Newton, 2012; Booth, Newton, & Twiss-Garrity, 2014), and failure to pass Algebra I seriously limits access to post-secondary education and career-training opportunities (National Mathematics Advisory Panel Report [NMAP], 2008). The negative consequences associated with fractions failure disproportionately affect students at-risk for and with identified disabilities who have demonstrably poorer fractions performance than their typically achieving peers (Hecht, Vagi, & Torgesson, 2007; Hecht & Vagi; 2010; Mazzocco & Devlin, 2008).

Fraction equivalence concepts are a major focus of early fractions instruction and underpin student understanding of fraction arithmetic procedures and rational number concepts, but are especially difficulty for many students to grasp (e.g., Behr, Lesh, Post, & Silver, 1983; Behr, Wachsmuch Post, & Lesh, 1984; Hunting, 1984; Kamii & Clark, 1995; Larson-Novillis, 1979; Post, Wachsmuth, Lesh, & Behr, 1985). Ni (2001) characterized fraction equivalence as “One of the most important and abstract mathematical ideas that elementary school children ever encounter…” (p. 400). Given its importance, equivalence is heavily emphasized in the Common Core State Standards for mathematics (CCSS-M; Council of State School Officers and National Governors Association Center for Best Practices, 2010).

In third grade, the CCSS mathematics standards state that students should (a) understand that equivalent fractions are the same size and occupy the same point on the number line (3.NF.3.a); (b) generate equivalent fractions and explain why fractions are equivalent using visual representations (3.NF.3.b); and (c) express whole numbers as fractions, and recognize
fractions that are equivalent to whole numbers (3.NF.3.c; CCSS, 2010). In fourth grade, standard 4.NF.1 states that students are expected to apply, explain, and use visual representations to model the principle: A fraction a/b is equivalent to a fraction (n x a)/(n x b).

Improving students’ understanding of fraction equivalence promotes future fractions learning, but few studies have tested the efficacy of interventions designed specifically to improve knowledge of fraction equivalence for students at-risk for mathematics failure and with identified disabilities. This introduction will present the theoretical framework for the intervention used in the present study, review relevant past intervention research, and outline how the present study contributes to the literature.

Theoretical Framework

Research shows there is a bidirectional relation between conceptual and procedural understanding in mathematics, but that growth in conceptual understanding has a stronger influence on growth in procedural learning than vice versa (Hecht & Vagi, 2010; Matthews & Rittle-Johnson, 2009; Rittle-Johnson, Siegler, & Alibali, 2001). Although there are five conceptual interpretations of fractions (i.e., part-whole, measure, operator, quotient, and ratio; Behr, Harel, Post, & Lesh, 1992), the part-whole and measurement interpretations are considered the most fundamental to early fractions instruction (Hecht & Vagi, 2010). The part-whole interpretation conceptualizes a fraction, for example 2/3, as 2 out of 3 equal parts of a whole; and part-whole representations typically use area models (e.g., a pie chart divided into 3 equal size pieces, with 2 pieces shaded). The measurement interpretation of fractions reflects cardinal size (Hecht, 1998; Hecht, Close, & Santisi, 2003), and the most common measurement representations are number line models (e.g., to represent 2/3, the interval between 0 and 1 on the
number line is divided into three equal-length segments, with a tick mark at the second segment marker).

Both conceptual interpretations are clearly reflected in early fractions instructional standards. In third grade, for example, part-whole understanding is reflected in standard 3.NF.1: Understand that a fraction 1/b as the quantity formed by 1 part when a whole is partitioned into b equal parts. Measurement understanding is reflected in standard 3.NF.2: Understand a fraction as a number on the number line (CCSS, 2010). Both interpretations are also emphasized in practice recommendations (NCTM, 2006; NMAP, 2008). Yet, classroom instruction typically prioritizes the part-whole interpretation over the measurement interpretation (e.g., Wu, 1999). Initial instruction on symbolic fraction notation, fraction equivalence, and fraction arithmetic operations generally relies on part-whole reasoning and area model representations, while the measurement interpretation and number line models are generally introduced later and with less emphasis (e.g., Fuchs et al., 2013; Wu, 1999).

Researchers have long suspected that over-reliance on the part-whole interpretation of fractions and over-use of area model representations of fractions may contribute to poor fractions performance by propagating whole number bias, or students’ tendency to overgeneralize whole number concepts and procedures to fractions (Gelman & Williams, 1998; Ni & Zhou, 2005; Vamvakoussi & Vosniadou, 2004, 2010). The results of recent intervention research and theoretical models of fractions learning also indicate that measurement understanding is more central to overall fractions learning than part-whole understanding (e.g., Bailey et al., 2015; Fuchs et al., 2013; Siegler, Thompson, & Schneider, 2011). Notably, results of Fuchs and colleagues’ Fraction Face-Off intervention studies (Fuchs et al., 2013; Fuchs et al., 2014; Fuchs et al., 2015; Fuchs et al., 2016) demonstrate that an instructional program focused primarily on
the measurement interpretation of fractions can result in significant and differential improvements over typical, part-whole focused instruction on assessments of fraction magnitude knowledge, computation, and problem solving for at-risk fourth graders. Further, results from this line of research also show that fraction magnitude knowledge fully mediated improvement in part-whole understanding, while part-whole understanding did not influence measurement understanding (Fuchs et al., 2013; Fuchs et al., 2014).

The integrated theory of numerical development (Siegler et al., 2011; Siegler & Lortie-Forgues, 2014) points to the special relation between fraction magnitude knowledge and overall mathematics achievement as one indication that magnitude knowledge is “the common core of numerical development.” The theory posits that numerical development primarily involves a process of learning to accurately represent magnitude for non-symbolic numbers, then for small whole numbers, larger whole numbers, and finally rational numbers (including fractions). Evidence that fraction magnitude knowledge is strongly correlated with fraction arithmetic (e.g., Siegler et al., 2011; Siegler & Pyke, 2013), and predictive of overall mathematics achievement (e.g., Bailey, Hoard, Nugent, & Geary, 2012) also support the integrated theory and the argument that fraction magnitude knowledge provides a critical foundation for overall fraction learning.

Although theoretical frameworks (Bailey et al., 2015; Siegler et al., 2011) and innovative intervention approaches (e.g., Fuchs et al., 2013) strongly emphasize the primacy of the measurement interpretation of fractions, the part-whole interpretation still represents an important facet of fractions conceptual understanding (Hansen et al., 2015; Hecht & Vagi, 2010). Rather than view the two conceptual interpretations as being at odds with each other, it may be possible to use instruction designed to leverage students’ part-whole knowledge and competence with area-models to scaffold measurement knowledge and competence with number lines.
Fuchs, Schumacher, et al. (2014), Fuchs, Malone, et al. (2015) and Gabriel et al. (2012) reported positive intervention outcomes as a result of requiring students to use area model (i.e., fraction circle) manipulatives to model fraction magnitude and support magnitude comparison and ordering decisions. Using area model manipulatives mapped directly onto the number line (i.e., proportional bar charts or fraction tile manipulatives) is potentially an even more effective approach for connecting the two conceptual interpretations. Bright, Behr, Post, and Wachsmith (1988), for example, found evidence that making connections between bar charts and number line models during instruction resulted in more growth on fraction outcomes than using number line models alone.

If part-whole concepts are easier for students to grasp and students demonstrate greater competence when interpreting and using area models (e.g., Bright et al., 1988; Keijzer & Terwel, 2003); but magnitude knowledge is of greater importance in regard to overall fraction learning and mathematics achievement (e.g., Fuchs et al., 2013; Siegler et al., 2011); then systematically teaching students to make connections between area models and measurement models may represent a successful approach for scaffolding measurement understanding. Further, if teaching students to make connections between area models and number line models within the context of instruction focused on finding equivalent fractions improves students’ conceptual understanding of the procedure for finding an equivalent fraction (i.e., students understanding of why and how multiplication can be used to find an equivalent fraction \( \frac{a}{b} = \frac{a \times n}{b \times n} \)) and leads to subsequent growth in fraction magnitude knowledge, this approach could provide empirical support for Bailey et al.’s (2015) speculative model of fraction learning.
Previous Fractions Intervention Research

A recent review of the fraction intervention literature identified explicit instruction as necessary for supporting the fractions learning of students at-risk for and with identified disabilities (Misquitta, 2011). Several measurement-focused fractions interventions successfully improved student performance on a variety of fraction outcomes, but only Fuchs and colleagues’ *Fraction Face-Off* intervention included participants classified as at-risk for disabilities and was designed in alignment with the principles of explicit instruction (Fuchs et al., 2013; Fuchs et al., 2014; Fuchs et al., 2015; Fuchs et al., 2016). *Fraction Face-Off* participants engage in 3, 30-minute small-group intervention sessions per week for 12 weeks. Each session includes an introduction, group work, a fluency-building activity, and independent work. The first seven weeks of the program focus on identifying and naming fractions, comparing and ordering fractions, and placing fractions on the number line. Addition and subtraction of fractions are introduced in weeks 8 and 9, and the final weeks (10-12) are spent on cumulative review.

The core *Fraction Face-Off* intervention is designed to compensate for at-risk fourth grade students’ lower-than-average working memory capacity, attentive behavior, and language ability by using instructional strategies like chunking material into small units, using a system to monitor and reward attentive behavior, using simple language during instruction, and stopping frequently to check understanding. Results consistently demonstrate that the 36-lesson program leads to very large performance gains on fraction number line estimation, magnitude comparison, addition and subtraction, and problem solving outcomes (Fuchs et al., 2013; Fuchs et al., 2014; Fuchs et al., 2015; Fuchs et al., 2016).

Other intervention studies targeting fraction magnitude knowledge have included typically achieving students and used various instructional approaches. Gabriel et al. (2012)
tested the effect of an intervention for fourth and fifth grade students that involved using fraction cards to play modified versions of five games: war (fraction magnitude comparison), memory and old maid (equivalence), blackjack (addition and estimation), and treasure hunt (comparison, addition, estimation). Students also had access to fraction circle manipulatives to model fraction magnitude during game play. After 20, 30-minute sessions (2 sessions per week for 10 weeks), intervention participants outperformed the business-as-usual (BAU) control group on a fraction concepts test, but control group students outperformed the intervention group on a measure of fraction arithmetic. Interestingly, error analysis of arithmetic test items revealed that intervention group participants made fewer errors associated with whole number bias (e.g., adding across numerators and denominators) than control group participants even though their performance was less accurate overall.

Saxe, Diakow, and Gearhart (2012) and Moss and Case (1999) both implemented experimental curricula designed to help typically achieving fourth and fifth grade students develop fraction magnitude knowledge by using the number line as the central representation for instruction across whole number and rational number topics. Both interventions used an inquiry-based approach to instruction, focused on discussion and group activities, and compared intervention group performance against the performance of BAU control groups. Both interventions also introduced equivalence concepts on the number line. Moss and Case, like Gabriel et al. (2012), found that the intervention group outperformed the control group on assessment items targeting conceptual (magnitude) knowledge, but not on standard arithmetic items. Saxe et al. reported the intervention group outperformed the control group on a researcher-created assessment of intervention content and on an end-of-year assessment based on the BAU mathematics curriculum.
A final study targeting fraction magnitude knowledge for typically achieving students focused primarily on representing and ordering fractions on the number line (Bright et al., 1988). The results of two, small-group teaching experiments and a larger group experiment all showed that students’ scores on a test of fraction knowledge improved from pre to posttest, but students’ responses to interview questions suggested that they struggled to use number lines in fraction problem solving (Bright et al., 1988). Intervention duration was increased from 4 to 8 days after the first, small-group experiment based on evidence that students needed additional instruction on fraction equivalence and more practice translating between area-model and number line representations of fractions. A full review of the descriptive research focused on fraction magnitude knowledge outcomes with attention to connections between the descriptive and intervention literatures is located in Appendix A.

Fraction equivalence was the specific focus of two additional studies including participants at-risk for or with identified disabilities (Butler, Miller, Crehan, Babbitt, & Pierce, 2003; Hunt, 2013). A third study targeted fraction equivalence but also included instruction on fraction identification, addition, subtraction, and comparison for mixed-ability fourth graders (Jordan, Miller, & Mercer, 1999). Hunt’s (2013) fraction equivalence intervention for at-risk 8-10-year-olds used an explicit instructional approach within a tier II intervention format (daily, 30-minute intervention sessions for four weeks). Intervention instruction focused on the ratio interpretation of fractions and emphasized the use of ratio-representations to help solve contextualized fractions problems. Intervention participants outperformed (BAU) control group participants on a researcher-created assessment of fraction equivalence items drawn from the district curriculum materials.
Butler et al. (2003) and Jordan et al. (1999) both implemented fractions equivalence interventions using an explicit instructional approach and graduated instructional sequence routines (i.e., concrete-representational-abstract [CRA] instruction). CRA instruction guides students through lessons using concrete manipulatives to model and solve problems, followed by lessons using representations (e.g., drawings) of objects to model and solve problems, and finally, lessons focused on abstract notation. Results of CRA intervention studies show that the strategy has been used effectively to improve the performance of students with disabilities across a variety of mathematics skill (e.g., Butler et al., 2003; Cass, Cates, Smith, & Jackson, 2003; Maccini & Hughes, 2000; Maccini & Ruhl, 2000; Witzel, Mercer, & Miller, 2003; Witzel, 2005; Witzel, Riccomini, & Schneider, 2008). Participants in Jordan et al.’s (1999) CRA fraction intervention outperformed a BAU control group and maintained higher performance post intervention on the outcome measure.

Butler et al. (2003) tested the effect of traditional CRA instruction against RA instruction (i.e., an instructional sequence without a concrete phase) on middle school students’ performance before and after participating in 10 explicit, scripted lessons focused on fraction equivalence. Instruction focused on the part-whole and ratio interpretations of fractions. Students in the CRA condition used beans, fraction circles, and fraction squares during the concrete phase, and students in both conditions used pictorial representations of beans (dots), circles, and squares during the representational phase of instruction. Both groups of students improved from pre to posttest on measures of basic fraction skills and equivalence concepts, but the CRA group demonstrated greater gains than the RA group.

These studies provide evidence that measurement-focused fractions instruction can significantly improve outcomes for at-risk students (e.g., Fuchs et al., 2013, Fuchs et al., 2014),
but that magnitude knowledge-building instruction alone will not improve procedural outcomes in the absence of direct procedural instruction (e.g., Gabriel et al., 2012; Moss & Case, 1999). In addition, fraction number line representations can initially be difficult for students to interpret (Bright et al., 1988), but systematic instruction using number line representations can support students’ conceptual understanding of fraction equivalence (e.g., Moss & Case, 1999; Saxe et al., 2012). Further, teaching students to make direct connections between area-models and number line models of fractions can support students’ ability to understand and use fraction number lines (e.g., Bright et al., 1988). Finally, explicit instructional methods and graduated sequence instruction routines promote fraction learning for students at-risk for and with identified disabilities (e.g., Butler et al., 2003; Fuchs et al., 2013).

**Contribution to the Literature**

Past interventions targeting fraction equivalence have not focused on the measurement interpretation of fractions, and past interventions focused on magnitude knowledge have devoted relatively little instructional time to fractions procedures. The present study fills these gaps in the literature, and tests the efficacy of an innovative instructional approach. The intervention teaches students to make connections between area model representations and number line representations to promote fraction magnitude knowledge during instruction focused on fraction equivalence. The present study also extends the research base on graduated sequence instruction by using the CRA-I (concrete-representational-abstract integration) instructional routine (e.g., Stickland & Maccini, 2013; Strickland, 2016). The research questions addressed in the present study include:

1. After controlling for pretest scores and disability status, do group (intervention/control) posttest outcomes differ on:
a. a test comprised of released Partnership for Assessment of Readiness for College and Careers (PARCC) fractions items,

b. a curriculum-aligned unit test, and/or

c. a test of fraction equivalence?

2. After controlling for pretest scores and disability status, do group (intervention/control) posttest outcomes differ on tests of 0-1 and 0-2 fraction number line estimation (i.e., fraction magnitude knowledge)?

3. How well do experimental intervention and comparison condition participants maintain performance on the 0-1 NLE test and the fraction equivalence test two months post-intervention?; and

4. Do intervention group students enjoy the intervention lessons and materials and perceive the intervention to be beneficial for their fraction learning?
Method

Participants and Setting

Study participants were members of the fourth grade class in a rural public intermediate school (grades 4-6) in the Mid-Atlantic region of the United States. The school serves approximately 1,200 students, with approximately 40% of students receiving free or reduced lunch, 18% of students receiving special education services, and 10% receiving gifted programming. All participants were classified as at-risk for mathematics failure or had been identified as having a disability and had math-related IEP goals. Students who scored at the basic or below basic level (i.e., below the 35th percentile) on the previous year’s standardized state mathematics assessment were identified as at-risk (Fuchs et al., 2016; Tian & Siegler, 2016). In addition, all participants were identified as having a need for the intervention by scoring below 60% accuracy on a screening measure of intervention content. Other participant characteristics (i.e., gender and disability classification) are summarized in Table 1.

Experimental Design

Researchers used a pretest-intervention-posttest comparison group experimental design to assess the effect of intervention. The 60 fourth graders identified for participation in the study were assigned to receive either intervention instruction or curriculum-aligned supplemental instruction in small, intact classroom groups during their regularly scheduled supplemental mathematics instructional periods. All participants continued to receive their regular, core mathematics instruction, and students in both study conditions received supplemental instruction during 4, 15-minute sessions per week for 4 weeks. Students in the intervention group and the comparison group received the same amount of supplemental instruction (240 minutes) during the study. After students were assigned to study conditions, ANCOVAs were performed on
group PSSA-M scores and screening test scores controlling for math teacher and disability status to assess group equivalence.

ANOVA results on the PSSA-M revealed no significant difference between intervention and control group scores, $F(1, 56) = 1.48, p = .23$, after controlling for the effects of teacher and disability status. Teacher, $F(1, 56) = 1.33, p = .25$, and disability status, $F(1, 56) = 0.87, p = .36$, were not significant covariates in the model. ANCOVA results on the screening test revealed no significant difference between intervention and control group scores, $F(1, 56) = 0.22, p = .64$, after controlling for the effects of teacher and disability status. Teacher, $F(1, 56) = 1.33, p = .25$, and disability status, $F(1, 56) = 1.49, p = .23$, were not significant covariates in the model. Results indicate that groups were equivalent at pretest. Attendance of students assigned to both groups was also similar throughout the duration of the study.

Measures

Measures included an (a) intervention-aligned test of fraction equivalence, (b) a curriculum-aligned unit test, (c) a transfer assessment of released PARCC fourth grade fractions items, (d) a 0-1 fraction number line estimation (NLE) test and a 0-2 fraction NLE test, and (e) a social validity questionnaire. The first author created the intervention-aligned test of fraction equivalence and released PARCC items test. The curriculum-aligned unit test was sourced from students’ core mathematics curriculum materials, the NLE tasks were paper-and-pencil adaptations of the tasks used by Resnick et al. (2016), and the social validity questionnaire was adapted from Butler et al. (2003). All measures can be found in Appendix B.

**Intervention-aligned test of fraction equivalence.** Equivalent forms of a researcher-created test of fraction equivalence were given as a screening and pretest measure, immediately post-intervention, and one month post-intervention to assess maintenance. The assessment was
comprised of 3 items requiring students to write the equal or not equal symbol between a pair of fractions, 3 items requiring students to find a missing numerator or denominator to find an equivalent fraction for a given fraction, and 3 items requiring students to find an equivalent fraction for a given fraction. Students could score 1 point for each correct answer on the 9 items. Test-retest reliability for this measure was .80.

**Curriculum-aligned unit test.** An end-of-unit test from the core curriculum was given pre and post intervention to measure student performance on core instructional objectives. The assessment contained a total of 32 items targeting fraction equivalence, fraction magnitude comparison and ordering, and ability to find a common denominator for two fractions. Items were scored correct or incorrect using a scoring key. Test-retest reliability was .78.

**Released PARCC items test.** An assessment comprised of 7 released PARCC items targeting fraction equivalence, comparison, and ordering from the fourth grade mathematics assessment was given pre and post intervention as a transfer measure of fraction equivalence and comparison. On the assessment, 4 items targeted fraction comparison and ordering, 2 items targeted fraction equivalence, and 1 item targeted both ordering and equivalence. Students could score 24 total points for accurate responses on the assessment (some items contained multiple questions). All items were scored for accuracy using a scoring key. Test-retest reliability was .82.

**Fraction number line estimation tests.** A paper-and-pencil assessment of 0-1 and 0-2 fraction NLE was given pre and post-intervention to assess students’ fraction magnitude knowledge (e.g., Siegler et al., 2011; Resnick et al., 2016). Target fractions were presented above the midpoint of 19 cm number lines, and students were asked to make a tic mark on the number line to represent the magnitude of the target fraction. Before beginning work on the 0-1 number line items (1/5, 13/14, 2/13, 3/7, 5/8, 1/3, ½, 1/19, 5/6), a demonstration was given to show
students how to place the fraction 1/8 on the number line. Before beginning work on the 0-2 number line items (1/3, 7/4, 12/13, 1 11/12, 3/2, 5/6, 5/5, ½, 7/6, 1 2/4, 3/8, 1 5/8 2/3, 1 1/5, 7/9, 1/19, 1 5/6, 4/3), a demonstration was given to show students how to place 1/8 and 1 1/8 on the 0-2 number line. Test-retest reliability was .84 on the 0-1 NLE test and .86 on the 0-2 NLE test.

**Fractions attitudes survey and social validity questionnaire.** A fractions attitude survey was given to both groups at pre and posttest and a social validity questionnaire was given to the intervention group post intervention. These measures were used to obtain feedback from students in both experimental groups about their attitudes towards fractions instruction, and from students in the intervention group about their perceptions about the most and least helpful elements of intervention instruction. Items used 5-point Likert scale.

**Comparison Group Instruction**

Students in both conditions received core mathematics instruction using the fourth grade Go Math! curriculum (Houghton Mifflin Harcourt, 2012). During the study, core mathematics instruction focused on fraction equivalence and comparison with emphasis on the part-whole interpretation of fractions and area-model representations. In addition to core instruction, the comparison group received 16, 15-minute small group instruction sessions led by students’ regularly assigned supplemental mathematics period teachers using activities aligned with core instruction. For example, comparison group students engaged in worksheet practice focused on finding and identifying equivalent fractions with pie chart models and used Plickers* to practice identifying equivalent fractions.

Research team members including the school principal, vice principal, and mathematics coach observed each comparison group at least twice (once in the first half of the study and once in the second half of the study) and completed 21 total observations (20% of comparison group
sessions). Observation times were selected at random and teachers were not notified of observations beforehand. Observers filled out descriptive observation reports. Of the 21 sessions observed, 17 were focused on instruction or remediation of content related to fraction equivalence and 4 were used for other purposes (2 sessions were used for multiplication fact fluency practice and 2 were used for testing). Of the 17 sessions focused on content instruction, all incorporated review or re-teaching worksheets sourced from core curriculum materials, 76% (13 out of 17) sessions incorporated area-model representations, 12% (2 out of 17) incorporated number line representations, and 18% (3 out of 17) sessions focused on multiplication procedures for finding an equivalent fraction using only abstract fractions (no representations).

**Fractions Intervention**

The experimental intervention program was designed to teach fraction equivalence procedures and concepts using representations that systematically connect area-models (fraction strips) and the number line (measurement model). The intervention was comprised of 16, 15-minute sessions led by the first author over 4 weeks to 9 small groups of 3-6 students. All sessions were audio-taped. After introductory sessions focused on key number line and fractions vocabulary terms, lessons used a CRA-I lesson format (e.g., Strickland & Maccini, 2013; Strickland, 2016) and features of explicit instruction (e.g., advance organizer and review of prior knowledge, teacher modeling, meaningful guided and independent practice opportunities; e.g., Archer & Hughes, 2011). Example CRA-I models used during intervention are located in Appendix C.

In traditional CRA sequence instruction, a concept or skill is taught to mastery using manipulatives (concrete objects) before transitioning to lessons using representations (drawings of objects). Students must then achieve mastery at the representational phase of instruction.
before moving on to abstract lessons, or lessons that present the concept or skill using only abstract notation (Strickland & Maccini, 2010). In CRA-I instruction, students work through all phases in each lesson (Strickland & Maccini, 2013; Strickland, 2016). Students often struggle with the transition from the representational phase to the abstract phase (e.g., Hudson & Miller, 2006) and have difficulty generalizing material (e.g., Gagnon & Maccini, 2001). CRA-I format lessons are designed to mitigate these difficulties and to promote better understanding and transfer of a skill or concept by explicitly connecting all three phases of instruction (Pashler et al., 2007; Strickland & Maccini, 2013).

In intervention instruction, the concrete materials were fraction strip manipulatives. These were scaled so that 1 whole strip was the same length as the unit interval (0-1) of fraction number line representations. In CRA-I format lessons, students manipulated fraction strips mapped onto the number line, then represented their work by marking and labeling a proportionate number line, and finally wrote an equation to represent the problem. An example of this process is illustrated in Figure 1.

Intervention lessons also used explicit instruction. Each lesson started with an advance organizer to assess relevant prerequisite skills, inform students of lesson objectives, and to identify a rationale for the lesson. Next, a demonstration of the lesson skill or concept was presented through teacher modeling and think-aloud. Following the model, students engaged in multiple guided practice opportunities while the teacher gradually withdrew support as students’ competence improved. Finally, students engaged in independent practice opportunities (Archer & Hughes, 2011; Gersten, 2009). Interventions sessions were only 15 minutes long, so some lessons stretched over multiple intervention meetings. An outline of intervention session topics is presented in Table 2.
**Fidelity of Intervention Instruction**

All intervention sessions were audio-taped. Researchers sampled 20% of sessions (i.e., 4 audio-taped lessons) from each of the 9 small groups to be assessed for fidelity. The first author and two trained research assistants independently assessed fidelity using an intervention fidelity checklist. The checklists required present/not present responses for each key feature of intervention sessions. Point-by-point inter-rater reliability across lesson elements was 98%. Intervention fidelity was rated as high by both observers, with 93% of lesson elements observed across all sessions rated as “present.”

**Procedure**

First, students who scored at the basic or below basic level on their third grade standardized state mathematics assessment were identified and the screening measure was administered to the entire group. Sixty of the 98 students who had qualifying standardized assessment scores and who scored at or below 60% accuracy on the screening measure were assigned to treatment or comparison instruction in intact classroom groups. In all, 9 intervention groups of 3-6 students each, and 9 comparison groups of 3-6 students each participated in the study. After assignment was complete, group equivalence was assessed and the remaining four pre-test measures were administered to all participants the week prior to the start of intervention. All assessments were group administered by the first author and participating classroom teachers.

All participants continued to receive their regular, core mathematics (Go Math!) instruction during the study and participated in 16, 15-minute small-group intervention or comparison instructional sessions during their supplemental mathematics instruction periods. The first author delivered intervention instruction and participants’ regularly assigned
supplemental mathematics classroom teachers led comparison group instruction. Comparison group instruction involved re-teaching and remediation activities aligned with the Go Math! Core curriculum materials from the unit of instruction on fraction equivalence and ordering. Intervention lessons used a CRA-I lesson format and focused on making connections between area model and number line representations of equivalent fractions. Post-tests were administered following the final intervention and comparison instructional sessions, and again two months post intervention to assess maintenance. A study timeline is located in Appendix D.
Results

Group comparison tests were performed on posttest scores to evaluate the effect of intervention instruction on student outcomes. In order to address the first research question, analyses were performed to evaluate group differences on a) released PARCC item posttest scores, b) curriculum-aligned unit posttest scores, and c) fraction equivalence posttest scores. A second set of analyses were performed to address the second research question and to assess group differences at posttest on the fraction NLE tests. The third research question regarding the maintenance of student performance post-intervention and the fourth research question regarding fractions attitudes survey and social validity questionnaire responses are addressed descriptively.

PARRC Released Items Test, Curriculum-Aligned Unit Test, and Fraction Equivalence Test Group Comparisons

A series of one-way analyses of covariance (ANCOVAs) were conducted to assess group-based differences at posttest on the released PARCC items test, the curriculum-aligned unit test, and the fraction equivalence test. Across analyses, treatment group (control, intervention) was entered as the between-subjects factor, while disability status (no disability, disability) and pretest scores based on each of the outcomes were entered as covariates. A Bonferroni correction was applied to the analyses to control for Type I error (α/4=0.01). Pretest and posttest means and standard errors are reported in Table 3.

On the released PARCC items test, an ANCOVA analysis revealed no significant differences in posttest scores, $F(1, 54)=0.30, p=.59, \eta_p^2=.01$, after controlling for the effects of pretest scores and disability status. Pretest scores, $F(1, 54)=0.30, p=.59, \eta_p^2=0.01$, and disability status, $F(1, 54)=0.84, p=.37, \eta_p^2=0.02$, were not significant covariates in the model. The findings suggested that the intervention had no effect on released PARCC items test scores.
On the curriculum-aligned unit test, an ANCOVA analysis revealed no significant differences in posttest scores, $F(1, 54)=0.21, p=.65, \eta^2_p=0.01$, after controlling for the effects of pretest scores and disability status. Pretest scores, $F(1, 54)=0.96, p=.33, \eta^2_p=0.02$, and disability status, $F(1, 54)=1.21, p=.28, \eta^2_p=.02$, were not significant covariates in the model. The findings suggested that intervention had no effect on curriculum-aligned unit test scores.

In contrast, significant differences in scores on the fraction equivalence test were obtained after controlling for pretest scores and disability status, $F(1, 55)=38.89, p < .001, \eta^2_p=0.41$. Pretest scores, $F(1, 54)=3.03, p=.09, \eta^2_p=0.05$, and disability status, $F(1, 54)=0.16, p=.69, \eta^2_p=.01$, were not significant covariates in the model. Students in the intervention group obtained significantly higher scores on the measure of fraction equivalence at posttest, $M=7.90, SD=0.98$, than students in the control group, $M=5.38, SD=1.64$.

To test for the presence of statistically significant differences in posttest scores on the fraction equivalence test among the four groups of students (i.e., 1) students with disabilities in the intervention group, 2) students without disabilities in the intervention group, 3) students with disabilities in the control group, 4) students without disabilities in the control group), a Kruskal-Wallis analysis was conducted. The Kruskal-Wallis test was selected to examine differences by student group because it has been shown to be robust to unbalanced and small subsample sizes. The test indicated a significant difference in median fraction equivalence scores, $\chi^2(3, N=60)=30.42, p < .001, \eta^2=0.52$. Follow-up Mann-Whitney U tests were next conducted to evaluate specific pairwise differences among the four groups. The Bonferroni correction was again used ($\alpha/6=.008$) to control for Type I error.

Mann-Whitney tests indicated that fraction equivalence posttest scores were similar for 1) students in the intervention group with disabilities ($Mdn=8.00$) and without disabilities
(Mdn=8.00), Mann-Whitney U=73.00, p = .92, and 2) students in the control group with disabilities (Mdn=5.50) and without disabilities (Mdn=5.00), Mann-Whitney U=83.50, p=.98. Fraction equivalence posttest scores were significantly higher for students with disabilities in the intervention group (Mdn=8.00) than for students with disabilities in the control group (Mdn=5.50), Mann-Whitney U=3.00, p=.005. Scores were also higher for students without disabilities in the intervention group (Mdn=8.00) than for students without disabilities in the control group (Mdn=5.00), Mann-Whitney U=56.5, p < .001. Finally, students with disabilities in the intervention group demonstrated higher median scores on the fraction equivalence posttest (Mdn=8.00) than students without disabilities in the control group (Mdn=5.00), Mann-Whitney U=15.50, p = .003.

**Fraction Number Line Estimation Test Group Comparisons**

Fraction NLE accuracy is depicted in Figure 2 and is indexed and reported in Table 4 as percent absolute error. Percent absolute error (PAE) is calculated as:

\[ PAE = \frac{\text{Student’s Estimate – Exact Magnitude}}{\text{Numerical Range of the Number Line}} \]

PAE was computed for each item and was averaged across items on the 0-1 and 0-2 NLE tests to calculate total scores. Smaller PAE reflects more accurate performance on the assessments. Average PAE was approximately 30% for both groups on both pretests (0-1 and 0-2 NLE). At posttest, the control group averaged 23.41% PAE and the intervention group averaged 10.07% PAE on 0-1 NLE. On the 0-2 NLE posttest, the control group averaged 22.71% PAE and the intervention group averaged 20.94% PAE.

An ANCOVA analysis was performed on raw difference scores (|Student’s Estimate in cm – Exact Magnitude in cm|) to test for differences in group estimation accuracy at posttest controlling for pretest accuracy and disability status and using a Bonferroni adjustment.
(α/2=0.03) to control for Type I error. On the 0-1 NLE assessment, ANCOVA revealed a significant difference in raw difference scores by group, $F(1, 47) = 15.27, p < .001, \eta_p^2 = .245$. The intervention groups’ estimates were significantly more accurate ($M=1.89, SD=1.09$) than the control groups’ estimates at posttest ($M=3.87, SD=2.19$). Pretest scores, $F(1, 47)=5.02, p=.05$, $\eta_p^2=0.09$, and disability status, $F(1, 47)=0.33, p =.57, \eta_p^2=0.01$, were not significant covariates in the model. The findings indicated intervention instruction led to significantly more accurate 0-1 NLE accuracy than comparison group instruction.

An ANCOVA analysis on 0-2 NLE outcomes revealed no significant difference in raw scores by group, $F(1, 47) = .002, p =.967, \eta_p^2 = .000$. The intervention groups’ estimates ($M=3.54, SD=1.46$) and the control groups’ estimates ($M=3.86, SD=1.47$) were similarly accurate. Pretest scores, $F(1, 47)=.58, p=.45, \eta_p^2=0.01$, and disability status, $F(1, 47)=1.41, p =.24, \eta_p^2=0.03$, were not significant covariates in the model. The findings indicated that intervention had no effect on students’ 0-2 NLE accuracy.

**Maintenance of Intervention Effects**

The fraction equivalence test and the 0-1 NLE test were administered a final time two months after the last intervention session. A one-way ANOVA showed that the intervention group maintained a significantly higher mean score than the control group at two months post intervention on the fraction equivalence test, $F(1, 45) = 9.87, p = .003$. Both groups’ scores fell between 1 and 2 points from posttest to maintenance test, but the intervention groups’ scores were still significantly higher ($M=6.10, SD=2.19$) than the control groups’ scores ($M=4.19, SD=1.96$).

Maintenance scores on the 0-1 NLE test remained similar to those obtained at posttest. A one-way ANOVA showed a significant difference in raw difference scores by group, $F(1, 47) =$
15.56, \( p < .001 \). The intervention groups’ estimates were significantly more accurate (\( M=1.86, SD=1.06 \)) than the control groups’ estimates at posttest (\( M=4.28, SD=2.05 \)). At two months post-intervention, average PAE was 22.93% for the control group and 10.56% for the intervention group.

**Fractions Attitudes Survey and Social Validity Questionnaire Responses**

Pre and posttest outcomes on the fractions attitude survey and posttest outcomes for the intervention group on the social validity questionnaire are presented in Table 5. Based on their responses to fractions attitude survey items, students in both groups agreed more strongly at posttest (than at pretest) that fractions are easy and that they know how to solve fractions problems. Students in both groups also disagreed more strongly at posttest (than at pretest) that it is hard to understand fractions and that learning fractions is a waste of time. Only students in the intervention group were more likely to agree at posttest (than at pretest) that number lines are helpful for learning about fractions. On the social validity questionnaire, intervention group students indicated that they enjoyed the intervention lessons and materials, that the lessons and materials were helpful for learning about fractions, and that they would recommend the intervention program to other students. Intervention group students were also asked to identify what they liked most and least about intervention instruction. The most frequent positive responses referenced the CRA-I materials. Students indicated that they enjoyed using the velcro fraction strips and white boards. Negative responses indicated that students disliked taking tests associated with the study and thought that the intervention sessions were too brief.
Discussion

The present intervention significantly improved struggling fourth graders’ ability to identify and generate equivalent fractions and increased fraction magnitude estimation accuracy on 0-1 number lines, and both of these effects were maintained two months post-intervention. These results extend Strickland and Maccini’s (2013) work with the CRA-I instructional sequence by providing evidence that CRA-I can be successfully applied to fractions content (in addition to Algebra content). Results also align with previous findings reported by Fuchs and colleagues (Fuchs et al., 2013; Fuchs et al., 2014) by demonstrating that fractions intervention instruction focused on the measurement interpretation and magnitude knowledge can improve struggling students’ fraction outcomes.

In contrast to results reported by Fuchs and colleagues (Fuchs et al., 2013; Fuchs et al., 2014), results of the present intervention study do not indicate that improved fraction magnitude knowledge (i.e., improved 0-1 fraction number line estimation accuracy) leads to improved performance on assessments of conceptual understanding of fractions or complex fraction problem solving. Intervention and control group participants in the present study achieved similar posttest outcomes on an assessment of released PARCC fractions items and on a curriculum-aligned fractions unit test despite intervention group students’ differential improvement on fraction number line estimation accuracy and on a test of fraction equivalence. This discrepancy and results on the various outcome measures are examined in greater detail in the discussion below.

PARRC Released Items Test and Curriculum-Aligned Unit Test Outcomes

Results indicate that intervention did not lead to stronger posttest performance than control group instruction on the PARCC released items test and curriculum-aligned unit test.
Both of these distal assessments included items that were related to intervention content but were presented in uninstructed formats and that required complex problem solving. As a result, improvements on these assessments would have required generalization and extension of intervention skills and concepts. Given the results of past fraction intervention studies (e.g., Fuchs et al., 2013), we hypothesized that intervention group students’ development of stronger fraction magnitude knowledge could both support generalization and help students benefit more from core mathematics instruction, which in turn could lead to greater gains on the curriculum-aligned unit test and/or the released PARCC items test. This hypothesis was incorrect.

One explanation for this outcome is simply that our intervention group students, like many low-performing students, struggled to generalize or transfer their learning (e.g., Troia, 2002) to the novel contexts of the released PARCC items test and curriculum-aligned unit test. Our intervention was also significantly shorter in duration and narrower in scope than Fuchs and colleagues’ Fraction Face-Off intervention. The present intervention provided four hours of instruction focused on a small set of fraction equivalence objectives, while the Fraction Face-Off program provides 18 hours of instruction and focuses on a much broader set of fractions learning objectives. The brevity and narrow instructional scope of the present intervention likely accounts, at least in part, for the failure of intervention to produce differential gains on distal measures similar to those achieved by the Fractions Face-Off intervention.

In addition, participants in the present study were among the lowest performing students in their classes and their mathematics deficits may have contributed to limited intervention effects. Anecdotally, intervention students made many errors that could be attributed to deficits in multiplication fact knowledge and multiplicative reasoning. Research suggests that multiplication fact fluency is strongly related to fraction performance and that multiplicative
reasoning helps students make accurate judgments about the relations between numerators and denominators (Hansen et al., 2015; Resnick et al., 2016). For example, a student with strong multiplicative reasoning skills can compare a given fraction to one half by quickly assessing whether the denominator is greater than or less than roughly twice the value of the numerator. Multiplicative reasoning skills also support students’ ability to simplify fractions and use multiplication to find equivalent fractions. Overall, low mathematics performance and specific deficits in multiplication fact knowledge and multiplicative reasoning create barriers to fraction learning and may have hindered intervention effectiveness for participating students.

It is also possible that intervention effects did not generalize/extend to the distal measures because the intervention was missing an important/necessary component for fostering strong conceptual understanding: situating number line problems in a meaningful context. A recent practice article (Rodrigues, Dyson, Hansen, & Jordan, 2016) recommends presenting fraction number line instruction and problems in the context of a racecourse. Authors propose that anchoring number line instruction using the real-world scenario of a racecourse engages students and provides a meaningful context for students to interpret fractions as magnitudes (Rodrigues et al., 2016). Past research supports that idea that abstract, context-free number lines are difficult for students to interpret (Bright et al., 1988). The present intervention did not provide a meaningful context for fraction number line problems, the absence of which may help explain the similar intervention and control group outcomes on the distal assessments.

**Fraction Equivalence Test Outcomes**

Intervention group students significantly outperformed control group students on the test of fraction equivalence and follow up statistical tests indicated that students with disabilities in the intervention group outperformed students with and without disabilities in the control group.
Items on the equivalence test were similar to the abstract items students learned to solve during CRA-I intervention instruction. Test items required students to write the equal or not equal sign between three pairs of fraction, find the missing numerator or denominator to complete three pairs of equivalent fractions, and to find an equivalent fraction for three given fractions. Outcomes on the equivalence test provide support for the effectiveness of teaching students to find equivalent fractions by mapping fraction strips onto the number line using the CRA-I instructional sequence and may indicate that this method of instruction is appropriate for use during core mathematics instruction.

**Fraction Number Line Estimation Outcomes**

Students in the intervention group also significantly outperformed control group students at posttest on 0-1 number line estimation accuracy but not on 0-2 number line estimation accuracy. Intervention group students learned to place fractions less than or equal to 1 on the 0-1 number line by interpreting the denominator of the fraction as the number of equal-length segments in the unit or the distance between 0 and 1 on the number line. Students learned to interpret the numerator as the number of equal-length segments to count over from 0 to find the point on the number line that represents the magnitude of the fraction. This method provided students with an effective procedure for representing fraction magnitudes on the number line but this procedural knowledge may not have engendered conceptual understanding of fraction magnitude. In past research, fraction number line estimation performance has been used as an indicator of fraction magnitude knowledge and strong correlations have been reported between performance on fraction number line estimation tests and other measures of fraction performance (e.g., Bailey et al., 2012; Siegler et al., 2011; Siegler & Pyke, 2013). However, in the present study, intervention group students’ performance on 0-1 number line estimation at posttest likely
reflects procedural skill rather than conceptual magnitude knowledge and was therefore less likely to affect performance on other measures of fraction knowledge.

Further, similar gains may not have been achieved on the 0-2 number line estimation test because intervention instruction focused exclusively on fractions smaller than or equal to 1 and used only number lines with endpoints 0 and 1. It may be notable that on the 0-2 test, intervention group students’ performance was least accurate on improper fractions (i.e., fractions with a numerator greater than the denominator). We chose to limit intervention examples and problems to proper fractions (i.e., fractions less than 1) on number lines with endpoints 0 and 1 in order not to overwhelm or confuse students with mathematics difficulties. However, this decision may have had the unintended consequence of reinforcing the misconception that all fractions are less than 1 (Mazzocco & Devlin, 2008). If this explanation were true, it could help to explain the failure of intervention effects to generalize to number line estimation accuracy on number lines with endpoints 0 and 2. Relatedly, intervention instruction may not have included sufficient emphasis on the concept that fraction magnitude is determined in consideration of the reference unit (i.e., 2/3 of 1 vs. 2/3 of 2). The same procedure used in the intervention for placing fractions on the 0-1 number line could be easily extended and applied to placing improper fractions and mixed numbers on number lines extending beyond 1, but students may have needed explicit instruction in order to learn to do so.

**Implications for Practice and Research**

Results of the present intervention suggest that the CRA-I instructional sequence (Strickland & Maccini, 2013) can be successfully applied to fractions content. In addition to academic effects, students’ enjoyment of the CRA-I sequence intervention materials and lessons
provide compelling support for using the CRA-I sequence in mathematics instruction. Strickland (2016) explains how to apply CRA-I detail for teachers interested in implementing the strategy.

Future studies might examine the effectiveness of the instructional sequence across other mathematics content. In regard to the present intervention, future iterations might incorporate several changes designed to enhance student outcomes. For example, intervention sessions and overall duration could be extended, the intervention could incorporate instructional components designed to improve students’ multiplication and division fact fluency and multiplicative reasoning skills, and equivalence problems could be situated in a meaningful context to support students’ ability to interpret fraction magnitudes. In addition, intervention instruction should incorporate number lines that extend beyond 1. In future studies, it would also be useful to include additional measures. Including a measure of multiplication fact fluency as a covariate in analyses could help to explain the role of fact fluency in students’ response to intervention. Including an assessment designed to measure students’ conceptual understanding of equivalence could also help to define the relations between magnitude knowledge and fraction equivalence skills and concepts.

Limitations

Several limitations should be considered when interpreting study results. The study was conducted at the beginning of the school year prior to the regularly scheduled fractions unit of instruction, but teachers and administrators decided to rearrange the sequence of units in order to keep core mathematics instruction in alignment with intervention instruction. After the completion of the study, teachers indicated that changing the order of instructional units may have been a mistake and that students might have been better prepared to benefit from fractions instruction if they had followed the curriculum scope and sequence.
In addition, few students with identified disabilities were included as participants in either condition. Small subsample sizes were considered in analyses, but readers should exercise caution in generalizing study results across student populations. Finally, intervention instruction was delivered by the first author (researcher). External validity should be established by training students’ regular classroom teachers to implement intervention instruction.

Conclusion

In conclusion, the results of this study support the use of the CRA-I instructional sequence to help struggling fourth graders students make connections across area models and number line models of fractions, place fractions on the number line accurately, and solve fraction equivalence problems. Results also indicate that improvements in fraction number line estimation accuracy do not necessarily indicate an improvement of fraction magnitude knowledge or coincide with improvements in performance on measures of fraction problem solving or conceptual understanding. Future research can further explore application of the CRA-I instructional strategy to fractions content and the relations between students’ fraction equivalence learning across procedural and conceptual dimensions.
References


abstract instruction in the acquisition and retention of fraction concepts and skills.


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https://doi.org/10.1207/S15327035EX1004_3


Appendix A

Literature Review

Building fraction magnitude knowledge: Implications for teaching and learning

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Abstract

The present review synthesizes results across the body of descriptive research investigating relations between students’ general competencies (e.g., working memory, attentive behavior), mathematics competencies (e.g., whole number arithmetic, fraction arithmetic) and their fraction magnitude knowledge. Results are reported by competency and discussed in consideration of relevant intervention and theoretical research. Working memory, attentive behavior, and language ability influence fraction magnitude knowledge directly and indirectly. Whole number magnitude knowledge, whole number fact fluency, long division skill, and fraction arithmetic procedural fluency are identified as especially relevant mathematics competencies for supporting development of fraction magnitude knowledge.

Keywords: fraction magnitude knowledge
Nick has a whole pizza and says he will eat ½ of the pizza, he will give ¾ of the pizza to Sam, and give ⅜ of the pizza to Joe. Can Nick do what he says? The answer is no, but only 26% of fourth graders were able to show or explain why Nick’s plan would not work in response to this item on the 2013 National Assessment of Educational Progress (NAEP; Kloosterman, Mohr, & Walcott, 2016). Although struggling with fractions seems to be a nearly universal experience, mastering fractions is especially challenging for students with or at-risk for mathematics disabilities (Algozzine, O’Shea, Crews, & Stoddard, 1987; Hecht, Vagi, & Torgesson, 2007; Hecht & Vagi, 2010). Evidence suggests that students who struggle with fractions early continue to struggle in later grades (Hecht & Vagi, 2010; Mazzocco & Devlin, 2008), and because fractions provide a critical foundation for more advanced mathematics and science (e.g., algebra; Booth & Newton, 2012; Booth, Newton, & Twiss-Garrity, 2014), fractions failure in the early grades can effectively exclude students from a wide variety of future academic and professional opportunities (National Mathematics Advisory Panel [NMAP], 2008; Sadler & Tai, 2007).

Mastering fractions skills and concepts is difficult for a variety of reasons. Fractions computation procedures are different and more complex than whole number computation procedures (Lortie-Forgues, Tian, & Siegler, 2015), and the principles and rules that apply to fractions diverge from those that apply to whole numbers (e.g., Ni & Zhou, 2005). For example, whole numbers have unique successors and each whole number refers to a distinct magnitude, while fractions do not have unique successors and many fractions can refer to the same magnitude (e.g., ½ = 2/4 = 3/6 = 100/200). In addition, fractions can be understood conceptually as a part of a whole, a measure on a number line, a ratio, an operator, or a quotient (Behr, Harel, Post, & Lesh, 1992; Behr & Post, 1994). Students understandably become confused when
multiple conceptual interpretations are presented concurrently (Berch, 2016; Wu, 1999). The results of recent empirical and theoretical research indicate that an instructional focus on the measurement interpretation of fractions targeting improvement in fraction magnitude knowledge may be especially important for fractions learning (e.g., Fuchs et al., 2013; Siegler, Thompson, & Schneider, 2011).

**Magnitude Knowledge**

Numerical magnitude knowledge refers to an understanding that numbers have magnitude and can be ordered and compared, and is distinct from other numerical abilities like counting and arithmetic (Fazio, Bailey, Thompson, & Siegler, 2014). Fraction magnitude knowledge (MK) is typically defined as an understanding that fractions are numbers with magnitude and can be represented as points on the number line (Siegler et al., 2011). Although there are five conceptual interpretations of fractions (e.g., Behr et al., 1992), researchers suggest that the measurement interpretation and the part-whole interpretation are most vital to students’ early fraction learning (Hecht & Vagi, 2010). Moderate to strong correlations have been reported between measures of fraction MK and fraction arithmetic performance (e.g., Siegler et al., 2011; Siegler & Pyke, 2013) and between fraction MK and assessments of general fractions knowledge (e.g., Hecht, 1998; Hecht et al., 2003). In addition, the results of a cross-lagged analysis of fraction MK and overall mathematics achievement led researchers to conclude that fraction MK drives gains in general math achievement (i.e., sixth grade fraction MK scores predicted seventh grade math achievement), but that general math achievement does not drive gains in fraction MK (Bailey, Hoard, Nugent, & Geary, 2012).

Geary, Hoard, Nugent, and Byrd-Craven (2008) and Siegler et al. (2011) have put forth theoretical accounts of fractions learning, which argue improvements in fraction MK are critical.
for development of overall fraction competence. Siegler’s (2011) integrated theory of numerical development, argues that an understanding of fraction magnitude is the key conceptual insight underpinning a mature understanding of fractions, and that a mature understanding of fractions is essential for overall numerical development. The integrated theory also suggests that a strong understanding of fraction magnitude may mitigate or combat the negative influence of the ‘whole number bias’ on fractions performance (Siegler et al., 2011).

Further, the results of a series of intervention studies (Fuchs et al., 2013; Fuchs et al., 2014; Fuchs et al., 2015; Fuchs et al., 2016) targeting fraction MK for at-risk fourth graders seem to support theorists’ contention that fraction MK is central to fractions learning. Participants in Fuchs and colleagues’” Fraction Face-Off” intervention program showed significant improvements in fraction arithmetic and problem solving, even though the measurement-focused intervention devoted limited instructional time to these skills. In addition, intervention study results showed that participants’ measurement understanding completely mediated improvement in their part-whole understanding, while improvement in part-whole understanding did not mediate the effect of intervention on measurement outcomes. Fuchs and colleagues suggest results indicate that fraction magnitude knowledge transfers to fraction arithmetic performance, and that MK is more important (than part-whole knowledge) for the development of fractions competence (Fuchs et al., 2013).

**How Does Fraction Magnitude Knowledge Develop?**

Due to research indicating that fraction MK may be the key to overall fractions learning and achievement, a number of recent descriptive studies have investigated how fraction MK develops and which other competencies are related to/and or contribute to fraction MK outcomes. An early model of fractions learning (Hecht, Close, & Santisi, 2003) and related
theories of overall mathematics development (e.g., Geary et al., 2004) identify both domain
general (i.e., global cognitive processes that contribute to learning) and specific mathematics
competencies as contributing to fractions learning. Hecht et al. (2003) hypothesized that attentive
behavior and working memory (i.e., domain-general competencies) would predict fraction
outcomes (computation, word problem solving, and estimation) both directly and indirectly
through their effects on intervening math skills (i.e., whole number arithmetic and fractions
concepts). A test of the model yielded evidence that both domain general competencies were
stronger predictors of the intervening math skills (simple arithmetic knowledge and conceptual
understanding of fraction [part-whole and MK]) than of fraction outcomes directly (Hecht et al.,
2003). In addition, the intervening math skills were not equally predictive of all studied fraction
outcomes: Simple arithmetic knowledge was only related to fractions computation, while
fractions conceptual knowledge was related to all three fractions outcomes (word problem
solving, estimation, and computation). In effect, Hecht et al.’s (2003) study indicated that there
are complex relations between different competencies and fraction outcomes.

In the 16 years since Hecht et al.’s (2003) model was proposed and tested, a number of
additional descriptive studies have contributed to the field’s understanding about how various
competencies relate to fractions outcomes. Given the recent emphasis on fraction MK, a growing
number of recent descriptive studies have focused on investigating relations between various
competencies and fraction MK specifically. The following section of the introduction will present
brief summaries pertaining to each competency with a hypothesized relation to fraction MK.
Summaries include a general definition of the competency, information about how the
competency relates to other fractions outcomes (e.g., arithmetic), and the rationale researchers
have proposed for including the competency in studies predicting fraction MK. Domain general competencies will be presented first, followed by mathematical competencies.

**Domain general competencies.** The domain general competencies with hypothesized relations with fraction MK include working memory, attention or attentive behavior, language (verbal reasoning, listening comprehension), nonverbal reasoning, and specific sub-domains of nonverbal reasoning (visual spatial memory and non-symbolic proportional reasoning).

**Working memory.** Working memory refers to one’s capacity to hold and simultaneously manipulate information in short-term memory (Baddeley, 1986; 1992), and weak working memory capacity is related to overall difficulty in mathematics (e.g., Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007; Hitch & McAuley, 1991). In regard to fractions, studies have demonstrated that working memory contributes unique variance to fraction arithmetic performance while controlling for whole number arithmetic performance (Jordan et al., 2013; Seethaler, Fuchs, Star, & Bryant, 2011). Working memory capacity may influence fraction MK because making magnitude judgments requires holding and processing multiple pieces of information simultaneously (Jordan et al., 2013).

**Attention.** Attention helps students stay focused and on-task during mathematics instruction, influences the quality and quantity of the practice they receive, and is consistently linked to overall mathematics performance (e.g., Cirino, Fletcher, Ewing-Cobbs, Barnes, & Fuchs, 2007; Fuchs, Geary, Compton, Fuchs, Hamlett, & Bryant, 2010; Fuchs, Geary, Compton, Fuchs, Hamlett, Seethaler, et al., 2010; Passolunghi & Pazzaglia, 2004; Zentall, 1990). In regard to fractions, strong attention may contribute to MK development by supporting the ability to attend to multi-step procedures, inhibit whole number bias, and persist through challenging tasks (Namkung & Fuchs, 2016).
Language. Language ability is central to math learning in general (e.g., LeFevre et al., 2010), and fractions vocabulary knowledge influences students’ mental representations of fractions (Miura, Okamoto, Kim, Steere, & Fayol, 1993). Although language ability is not associated with whole number arithmetic (e.g., Fuchs et al., 2005, 2006, 2008, 2013; Fuchs Geary, Compton, Fuchs, Hamlett, & Bryant, 2010; Fuchs, Geary, Compton, Fuchs, Hamlett et al., 2010; Seethaler et al., 2011), language ability is a significant predictor of fraction arithmetic performance (Seethaler et al., 2011). Strong language ability may support of fraction MK due to the complexity of language used to explain magnitude concepts and tasks (Jordan et al., 2013).

Nonverbal reasoning. Nonverbal reasoning refers to the ability to interpret and form accurate representations of quantitative and qualitative relations among numbers (Primi, Ferrao, & Almeida, 2010), to recognize patterns, and to infer and use rules (Nutley et al., 2011). Geary’s (2004) model of mathematical learning purports that the nonverbal cognitive system is important for the ability to represent and manipulate mathematical information. There is mixed evidence regarding the relation between nonverbal reasoning and fraction arithmetic (Fuchs et al., 2013; Jordan et al., 2013; Seethlather et al., 2011), but it is suggested that nonverbal reasoning ability may support fraction MK by contributing to students’ ability to interpret the relation between numerator and denominator (Jordan et al., 2013).

Visual spatial memory. Visual spatial memory refers to the ability to maintain and use visual-spatial information in short-term memory. Visual-spatial memory may contribute to fractions MK development because students must maintain and simultaneously manipulate visual-spatial representations of fractions to make magnitude judgments (LeFevre, 2010; Vukovic et al., 2014).
**Non-symbolic proportional reasoning.** Non-symbolic proportional reasoning refers to the ability to reason about the relative magnitude of quantities represented pictorially (rather than as Arabic numerals). Evidence suggests that non-symbolic proportional reasoning underlies early, informal fraction knowledge (i.e., the ability to identify which of two boxes has a greater proportion of objects; Spinillo & Bryant, 1991), but there is also consistent evidence that non-symbolic tasks share weaker relations with mathematics outcomes than symbolic tasks (Fazio et al., 2014; Holloway & Ansari, 2009; Mazzocco, Feingenson, & Halberda, 2011; Soltesz, Szucs, & Szucs, 2010). Non-symbolic proportional reasoning skill may support fraction MK by contributing to students’ ability to assess multiplicative relationships between fractional quantities (Boyer and Levine, 2012).

**Mathematics competencies.** The mathematics competencies with hypothesized relations with fraction MK include fractions arithmetic, whole number knowledge, whole number arithmetic, and whole number MK.

**Fractions arithmetic.** Fractions arithmetic performance is commonly categorized as an indicator of procedural knowledge, while fraction MK is categorized as conceptual knowledge (Bailey, Siegler, & Geary, 2014; Bailey et al., 2015; Byrnes & Wasik, 1991; Rittle-Johnson & Alibali, 1999; Rittle-Johnson, Siegler, & Alibali, 2001). Both conceptual and procedural knowledge are important for developing fraction competence, and correlational evidence consistently demonstrates that the two are moderately to strongly related (Byrnes & Wasik, 1991; Hallett, Nunes & Bryant, 2010; Hecht et al., 2003; Siegler & Pyke, 2013; Siegler et al., 2011; Torbeyns, Schneider, Xin, & Siegler, 2015). While some earlier theories of fractions learning suggested that procedural and conceptual knowledge develop independently of one
another (e.g., Nesher, 1986; Resnick & Omanson, 1987), more recent theoretical models and empirical studies consider how the two types of knowledge exert influence on one another.

One viewpoint privileges procedural knowledge and suggests that the development of fractions procedural knowledge supports the development of fractions conceptual knowledge (Gagne, 1983; VanLehn, 1990). A second viewpoint privileges conceptual knowledge and suggests that developed fractions conceptual knowledge (e.g., MK) supports the development of procedural knowledge (Byrnes & Wasik, 1991; Hecht, 1998; Siegler et al., 2011; Siegler & Lortie-Forgues, 2014). A third viewpoint suggests that conceptual and procedural fractions knowledge influence each other in a bidirectional fashion, with development in each domain contributing to growth in the other (e.g., Bailey et al., 2015).

**Whole number knowledge.** Number knowledge (i.e., the ability to represent and manipulate small numerals) predicts later mathematics achievement (Bailey et al., 2014; Geary, 2011; Geary et al., 2013). Theories of mathematical development suggest that number knowledge contributes to later fraction learning by providing a foundation for conceptual understanding of magnitude and proportional reasoning (Geary, 2011; LeFevre et al., 2010; Siegler et al., 2011).

**Whole number arithmetic.** Whole number arithmetic performance uniquely predicts fractions arithmetic performance (Bailey et al., 2014; Hecht et al., 2003; Jordan et al., 2013), but the relation between whole number arithmetic and conceptual knowledge of fractions is debated. Some suggest that whole number knowledge can interfere with fraction MK development through the whole number bias, or students’ tendency to overgeneralize whole number conceptual and procedural knowledge to fractions (Gelman & Williams, 1998; Ni & Zhou, 2005; Vamvakoussi & Vosniadou, 2004, 2010). For example, whole number arithmetic knowledge is
cited as interfering with fraction MK when students mistakenly apply whole number arithmetic procedures to fraction addition (e.g., \( \frac{3}{4} + \frac{1}{4} = \frac{4}{8} \) or \( \frac{1}{2} \)) and as a result, receive invalid magnitude feedback (i.e., the sum of two fractions should be greater than either addend, but \( \frac{1}{2} \) is less than \( \frac{3}{4} \)). Proponents of the integrated theory of numerical development argue that very strong whole number arithmetic knowledge may actually protect against whole number bias and contribute to fraction MK (Siegler et al., 2011; Siegler & Lortie-Foruges, 2014). Siegler and Pyke (2013), for example, found that students with high scores on whole number arithmetic tasks were less likely to make fraction arithmetic errors commonly associated with whole number bias.

Over and above generalized whole number arithmetic performance, math fact fluency and division skill may share unique relations with fraction MK. Fact fluency, or the ability to retrieve solutions from long-term memory quickly and accurately, supports performance with more complex procedures (Geary, 1993; Jordan & Hanich, 2003; Locuniak & Jordan, 2008), while disfluent fact performance is characteristic of mathematics learning disabilities (Geary, 2004; Jordan & Hanich, 2000). Multiplication fact fluency in particular may be associated with fraction MK because it could facilitate students’ ability to recognize multiplicative relations between fractions (Hansen et al., 2015). Empson and Levi (2008) called division an “intuitive entry point” for fractions learning, and moderate to strong correlations have been reported between long division performance and fractions outcomes (Siegler et al., 2012; Siegler & Pyke, 2013; Hansen et al., 2015).

**Whole number magnitude knowledge.** Whole number MK predicts overall mathematics achievement (e.g., Booth & Siegler, 2006, 2008; Halberda, Mazzocco, & Feingenson, 2008; Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007; Geary, Nugent, Hoard, & Byrd-Craven, 2007; Holloway & Ansari, 2008; Jordan et al., 2013), and a deficit in whole number magnitude
processing (i.e., in the ability to judge the value of one numeral relative to another) is characteristic of learning disabilities in mathematics (Butterworth, Varma, & Laurillard, 2012; Rouselle & Noelle, 2007). Fraction MK also predicts overall mathematics achievement (Bailey et al., 2012; Siegler & Pyke, 2013; Siegler et al., 2011, 2012), and whole number MK and fraction MK appear related (e.g., Bailey et al., 2012), but there is debate regarding the nature of the relation between whole number and fraction MK.

The integrated theory of numerical development argues that whole number MK provides a foundation for the development of fraction MK, and that coming to see fractions as numbers with magnitude is the key conceptual insight underlying a mature understanding of fractions (Siegler et al., 2011). Alternatively, it is possible that performance on whole number and fraction MK tasks (i.e., number line estimation [NLE] tasks) are related because both require proportional reasoning ability (Barth & Paladino, 2011; Slusser, Santiago, & Barth, 2013). Proponents of this viewpoint explain that placing a numeral (e.g., 5) on a number line with endpoints 0 and 100 involves estimating the magnitude of 5 relative to the whole (100) and that the ability to make this type of proportional judgment with whole numbers underlies students’ later ability to make similar judgments involving fractions (Barth & Paladino, 2011; Slusser, Santiago, & Barth, 2013).

**Purpose of the Review**

Theoretical frameworks suggest that fraction MK is critically important for fraction learning and overall mathematics development (e.g., Bailey et al., 2015; Siegler et al., 2011), and policy documents and evidence from intervention research suggest that fraction MK is malleable (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; Fuchs, et al., 2013). This review is designed to synthesize results of descriptive
studies investigating the relations between various domain general and mathematics competencies and fraction MK, and to consider results in light of the results of intervention research and theoretical accounts of fractions learning. The synthesis of results will be used to generate recommendations for practice targeting fraction MK development.

Method

Inclusion Criteria

Included descriptive studies investigated the effects of any combination of domain general and mathematics competencies on at least one dependent measure of fraction MK. For the purpose of this review, measures of fraction MK include fraction number line estimation tasks, fraction magnitude comparison tasks, and fraction concepts tests containing items targeting both part-whole and fraction MK. Fraction number line estimation tasks require students to represent the magnitude of a given fraction by placing it on a horizontal number line with endpoints 0 and 1, 0 and 2, or 0 and 5 (e.g., Siegler et al., 2011). Fraction magnitude comparison tasks require students to select the greater or lesser of two fractions (e.g., Hecht et al., 1998; Hecht et al., 2003).

Although part-whole and magnitude understanding represent two distinct funds of knowledge (Hecht, 1998), the decision to include studies that used fractions concepts tests comprised of items targeting both types of knowledge is based on evidence that outcomes on assessments of part-whole and MK are highly correlated (e.g., Hecht, 1998) and that correlations between mathematics and domain general competency variables and both assessments of part whole and MK are similar in strength and direction (e.g., Hansen et al., 2015). Nevertheless, type of dependent measure is taken into consideration throughout the presentation and discussion of results.
**Literature Search**

A search of the ProQuest, PsychInfo, Web of Science, and Google Scholar databases was conducted using combinations of the terms: fraction*, MK, magnitude concept*, magnitude understanding, magnitude comparison, and fraction number line. All retrieved studies were screened for adherence to inclusion criteria. Ten descriptive studies were identified for inclusion (Bailey et al., 2014; Bailey et al., 2015; Hansen et al., 2015; Hecht et al., 2003; Hecht & Vagi, 2010; Jordan et al. 2013; Mou et al., 2016; Namkung & Fuchs, 2016; Resnick et al., 2016; Vukovic et al., 2014).

**Results**

Important features of descriptive studies including (a) data analysis methods, (b) dependent measures of fraction MK, (c) independent variable (mathematics and domain general competency) measures, and (d) bivariate correlations between the two (when available) are summarized in Table 1 (domain general competencies) and Table 2 (mathematics competencies). The results of descriptive studies describing relations between independent variables and fraction MK are summarized in text and are diagrammed to present a visual synthesis of study results (Figure 1). The diagram uses Hecht et al.’s (2003) conceptual framework as a model for situating domain general and mathematics competencies. Results and discussion of descriptive studies are organized by independent variable/competency. Results and are briefly summarized in text and discussed in consideration of relevant intervention study results and theoretical accounts of fraction learning and development. A conceptual model of fraction MK learning is depicted in Figure 2, and implications of the review for practice are discussed.
**Working Memory**

Authors of two descriptive studies (Bailey et al., 2014; Hansen et al., 2015) reported that working memory was a relatively weak but statistically significant predictor of fraction MK for eighth graders and sixth graders, respectively. Mou et al. (2016) reported that working memory measured at first grade was included in the best-fitting Bayes Model predicting ninth grade fraction MK, but was not included in the best-fitting model predicting fraction MK at eighth grade or in the model predicting fraction MK growth from eighth to ninth grade. Authors of the remaining four studies investigating working memory reported that working memory was not a significant predictor of fraction MK for typically achieving fourth graders (Jordan et al., 2013; Vukovic et al., 2014), at-risk fourth graders (Namkung & Fuchs, 2016), or fifth graders (Hecht et al., 2003).

Although there is mixed evidence regarding the effect of working memory on fraction MK across included descriptive studies, patterns of results could indicate that (1) working memory (i.e., listening recall) is more strongly related to performance on more complex 0-5 fraction NLE tasks than to performance on simpler, 0-1 NLE tasks; (2) that working memory becomes more strongly predictive of fraction MK as students age and fractions tasks become more demanding; and/or (3) that common data analysis procedures underestimate the predictive power of working memory. In addition, evidence from the intervention literature suggests that working memory moderates the effect of fractions MK intervention and that instruction designed to compensate for weak working memory capacity can improve fraction MK outcomes (Fuchs et al., 2013; Fuchs et al., 2014; Fuchs et al., 2015).

All studies reporting that working memory accounted for unique variance in fraction MK outcomes used the 0-5 NLE task as the dependent variable (or as part of a composite DV). In
effect, it is possible that results reflect a unique relation between working memory capacity and the complex demands of the 0-5 NLE task. Jordan et al. (2013) observed that it was equally common for students to place the fraction \( \frac{1}{2} \) at the midpoint of 0-1 and 0-5 number lines. It may be that working memory capacity contributes to students’ ability to manage the extra steps necessary for making magnitude judgments on a 0-5 number line. Bailey et al. (2014) also suggest the possibility that working memory is more salient for novel (unpracticed) tasks, and note that students are unlikely to have encountered 0-5 NLE tasks during typical classroom instruction.

Further, working memory is not a unitary construct (Baddeley, 1986), and the counting recall subtest of the Working Memory Test Battery for Children (WMTB-C; Pickering & Gathercole, 2001) in particular may have been a poor match for the specific executive processes relevant to fraction NLE tasks. The results of Namkung and Fuchs’ (2016) study suggest that verbal working memory (measured using the listening recall subtest of the WMTB-C; Pickering & Gathercole, 2001) was much more strongly related to fraction NLE than numerical working memory (i.e., counting recall). Findings across the intervention literature corroborate this finding (Fuchs et al., 2013; Fuchs et al., 2014; Fuchs et al., 2015), and suggest that listening recall relates more strongly to fraction NLE, while counting recall may relate more strongly to fraction magnitude comparison. In effect, the impact of working memory on fraction MK could have been underestimated in descriptive studies using the counting recall subtest of the WMTB-C as a predictor of fraction NLE.

Researchers have also suggested that working memory may become more important for fractions outcomes as students get older and fractions content becomes more complex (Jordan et al., 2013; Namkung & Fuchs, 2013). The pattern is not clear-cut in the dataset under review, but
all descriptive studies involving younger students reported non-significant results for working memory (Hecht et al., 2003; Jordan et al., 2013; Namkung & Fuchs, 2016; Vukovic et al., 2014), and studies involving older students reported significant relations between working memory and fraction MK (Bailey et al., 2014; Hansen et al., 2015; Mou et al., 2016).

Finally, common data analysis techniques may underestimate the influence of working memory on fractions MK. Mou et al. (2016) reported that working memory was included as a predictor of ninth grade fraction MK in the best-fitting Bayes model, but not in the matching regression model. In explanation, the authors suggest that regression analyses may underestimate domain general predictor variables (including working memory) due to the effects of collinearity on coefficient estimates (Mou et al., 2016). If this explanation holds true, the effect of working memory could be widely underestimated across included descriptive studies.

Fuchs and colleagues’ studies involving their highly successful Fraction Face-Off intervention have also contributed to a greater understanding of how working memory limitations influence fraction MK (Fuchs et al., 2013; Fuchs et al., 2014; Fuchs et al., 2015). The Fraction Face-Off intervention is designed to compensate for domain general limitations, and consistently resulted in substantial and differential improvements in at-risk fourth graders’ fraction MK, computation, and problem solving performance (Fuchs et al., 2013; Fuchs et al., 2014; Fuchs et al., 2015; Fuchs et al., 2016). In regard to compensating for working memory limitations specifically, the intervention teaches strategies for chunking and segmenting measurement tasks and engages students in fraction magnitude fluency-building practice. Results of the initial intervention study showed that working memory mediated the effect of intervention on 0-1 NLE and fraction magnitude comparison, but that compensatory strategies (segmenting,
chunking, fluency-building) were more effective in relation to outcomes on NLE than on FMC (Fuchs et al., 2013).

A second study investigated the effects of the original intervention plus five additional minutes of fluency-building activities using flashcards (designed to build automaticity with the chunking and segmenting strategies taught during intervention) or MK concept-building activities requiring students to represent and describe fraction magnitudes using manipulatives (Fuchs et al., 2014). Results of this study indicated that students with very weak working memory benefitted more from the conceptual activity, while students with slightly stronger working memory capacity benefited more from fluency-building activities. Results may imply that chunking and segmenting strategies for making magnitude judgments are helpful but still tax working memory, and as a result, are more effective for students who come to the intervention with greater working memory capacity (Fuchs et al., 2014). Students with weaker working memory benefit more from conceptual practice designed to help consolidate their MK through building concrete fraction representations and using those representations to explain and justify magnitude judgments (Fuchs et al., 2014).

A third study extended the investigation of the potential benefit of students’ ability to explain fraction magnitude judgments by testing the effects of the original intervention plus seven additional minutes of instruction and practice with ‘supported self-explanation’ of fraction magnitude comparison or seven additional minutes of word problem instruction and practice (Fuchs et al., 2015). In the supported self-explanation condition, students learned a four-step procedure for analyzing, representing, and explaining magnitude judgments. Instruction in the word problem condition used schema-based instruction to teach students to systematically analyze and solve “splitting” and “grouping” fraction word problems.
Results showed that students in the self-supported explanation condition were more accurate on fraction magnitude comparisons and provided much higher quality comparison explanations than students in the word-problem condition, even though both groups learned the same process for making comparisons in the base intervention program. Results also showed that working memory moderated comparison outcomes, that self-supported explanation instruction was more effective for students with weaker working memory capacity, and that word-problem instruction was more effective for students with stronger verbal reasoning ability (Fuchs et al., 2015). Authors suggest that overall, results indicate that self-supported explanation instruction effectively compensates for working memory limitations and improves comparison performance by decreasing cognitive load. Taken together, results across the descriptive and intervention literature indicate that students’ working memory capacity strongly influences both fraction 0-5 NLE and fraction magnitude comparison accuracy; that fluency-building and chunking and segmenting strategies can help mitigate the negative influence of working memory limitations on fraction NLE; and that teaching students to explain fraction magnitude judgments can help mitigate the negative influence of working memory limitations on fraction magnitude comparison tasks.

Attentive Behavior

Resnick et al. (2016) defined three distinct classes to describe the fraction MK learning trajectories of students as they progressed from third grade to sixth grade. Students in class 1 had high fraction MK at the beginning of the study, and high MK at the end; students in class 2 had a steep-growth trajectory (i.e., they started with low fraction MK and ended with high fraction MK); and students in class 3 had a low-growth trajectory (i.e., they started and ended with low fraction MK; Resnick et al., 2016). Authors reported that students with attention ratings 1
standard deviation above average were 1.6 times more likely to be assigned to class 1 than to class 2; but that higher attention ratings did not significantly increase students’ odds of being assigned to class 2 over class 3 (Resnick et al., 2016). Jordan et al. (2013), Hecht et al., (2003), and Hansen et al. (2015) reported that attention was a unique predictor of fraction MK for fourth, fifth, and sixth graders, respectively.

Vukovic et al. (2014) reported that attention measured at first grade had a direct effect on whole number arithmetic at second grade and mediated the relation between whole number arithmetic and fraction MK at fourth grade. Namkung and Fuchs (2016) also found that attention contributed significantly to whole number and fractions arithmetic, but did not find evidence that attention predicted fraction MK. Finally, Mou et al. (2016) also reported that attention was not a significant variable in models predicting fraction MK at grade 8, grade 9, or in growth from grade 8 to grade 9.

Results across descriptive studies seem to indicate that attentive behavior is important for fractions MK learning, but that attention may contribute more strongly to performance on fraction magnitude comparison tasks and broader fractions concepts tests than to performance on fraction NLE tasks (on which working memory is more influential). For example, Resnick et al. (2016) reported that above average attention ratings did not significantly increase odds of assignment to the fraction NLE steep-growth class over the low-growth class, and Vukovic et al. (2014) reported that attention was linked indirectly to fraction NLE through whole number arithmetic. In addition, Fuchs and colleagues’ (2013) intervention study results also indicate that while working memory moderated the effects of intervention on fraction NLE, attention moderated the effects of intervention on a fractions concepts test and a fraction magnitude comparison task.
Fuchs et al. (2013, 2014, 2015) employed a token exchange system designed to encourage and reward attentive behavior within their larger fractions intervention, but mediation analysis results led authors to conclude that this system did not adequately compensate for the negative influence of inattentive behavior on fractions magnitude comparison and concepts tests. It may be that the influence of chronic, long-term inattentive behavior (that would contribute to inadequate prerequisite skills and lower overall achievement) is so powerful, that good attentive behavior during fractions intervention is not enough to mitigate negative effects.

In addition, a combination of weak working memory capacity and inattentive behavior likely work together to interfere with performance on fraction MK tasks that require strong inhibitory control. Inhibitory control and working memory, for example, would likely both affect a students’ ability to compare two fractions with unlike denominators. In order to make such a comparison, a student has to work through a complex series of steps to determine the magnitude of one fraction, hold that information in memory as they work through the steps again to determine the magnitude of the second fraction, and process all of relevant information to make a comparison (Fuchs et al., 2014). Overall, results corroborate Vukovic et al.’s (2014) contention that poor attention is an early risk factor for poor fractions performance, and that the inhibitory control aspect of inattentive behavior contributes to poor fractions MK outcomes.

Language

Jordan et al. (2013) reported that language ability was a unique predictor of fourth grade fraction MK, and Namkung and Fuchs (2016) reported that language ability was the strongest unique predictor of fourth grade fraction MK. Vukovic et al. (2014) reported that language ability measured at first grade had direct effects on second grade whole number NLE and whole
number arithmetic, and mediated the relations between both second grade outcomes and fourth grade fractions MK.

Results seem to indicate that like other domain general competencies (working memory and attention) language ability exerts an influence on fraction MK both directly and through its impact on important intervening mathematics competencies. Language ability mediated the relations between whole number MK and fraction MK and between whole number arithmetic and fraction MK (Vukovic et al., 2014). No descriptive studies investigated whether the relation between fraction arithmetic and fraction MK was also mediated by language, but Namkung and Fuchs (2016) reported that language is a strong predictor of fraction arithmetic, and Fuchs et al. (2013) reported that listening comprehension was a significant moderator of MK intervention effects on fraction addition and subtraction outcomes.

Language ability may be supportive of the relation between fraction arithmetic and fraction MK because of the complexity of verbal explanations of fractions arithmetic procedures, and because a deep understanding of fraction arithmetic procedures may help strengthen fraction MK. For example, the following language might be used to begin to explain subtraction of fractions with unlike denominators: *In order to subtract one fraction from another, the two fractions must refer to the same unit. So, if two fractions have different denominators, we convert one or both into equivalent fractions with a common denominator.* These two sentences are likely difficult for many adults to follow, let alone students with listening comprehension and/or verbal reasoning limitations. Students may be able to carry out the arithmetic procedure to subtract fractions with unlike denominators without understanding why common denominators are necessary or why the procedure works, but strong language ability likely supports conceptual understanding about the procedure, and as a result, helps students process and make sense of
feedback about the magnitude of the operands and solutions (e.g., \( \text{Subtracting } \frac{1}{3} \text{ from } \frac{5}{6} \) equals \( \frac{3}{6} \). \( \frac{3}{6} \) is equal to \( \frac{1}{2} \), so \( \frac{5}{6} \) must be greater than \( \frac{1}{2} \)).

Bailey et al. (2015) proposed a conceptual model of fractions learning which suggests that fraction MK and basic fraction arithmetic knowledge are mutually supportive of one another and that both feed into students’ deeper conceptual understanding of fractions procedures, which in turn, feeds back into fraction MK development. If Bailey et al.’s (2015) model accurately reflects the learning process, and if language mediates the relation between fraction procedures and MK, then poor language ability could represent a critical weak link in fractions learning. Bailey’s (2015) model must be tested, but it and results across descriptive and intervention research indicate that language supports are an important component of effective fractions instruction. Fuchs and colleagues’ highly successful fraction MK intervention was designed to accommodate language limitations by providing explanations in simple language and requiring students to repeat explanations in their own words (Fuchs et al., 2013; Fuchs et al., 2014; Fuchs et al., 2015). Saxe, Diakow, and Gearhart (2012) successful fraction MK intervention also accounted for language related needs by focusing on helping students to generate and apply definitions for fractions terms and concepts. Overall, results indicate that it is important to provide fractions explanations in simple language, to provide explicit vocabulary instruction, and to frequently check on student understanding (Vukovic et al., 2014).

**Nonverbal Reasoning**

Both Jordan et al. (2013) and Namkung & Fuchs (2016) found that nonverbal reasoning (matrix reasoning subtest of the Wechsler Abbreviated Scale of Intelligence [WASI]; Wechsler, 1999) uniquely predicted fractions MK at fourth grade and was the largest, unique predictor of whole number MK (at third and fourth grades, respectively). Vukovic et al. (2014), in contrast,
found that nonverbal reasoning (matrix reasoning) did not have a direct effect on whole number MK or whole number arithmetic at second grade and did not have any indirect effects on fourth grade fraction MK. Vukovic et al. (2014) also reported that visual-spatial memory (mazes memory and block recall subtests of the WMTB-C; Pickering & Gathercole, 2001) mediated the relation between second grade whole number MK and fourth grade fractions MK. Bailey et al. (2014), in contrast, reported that controlling for visual-spatial memory (mazes memory and block recall) did not affect the relation between first graders’ whole number MK and their fraction MK at eighth grade.

Non-symbolic Proportional Reasoning

Jordan et al. (2013) and Hansen et al. (2015) used different measures of non-symbolic proportional reasoning and reported contradictory results. Jordan et al. (2013) used the ANS task to measure non-symbolic proportional reasoning, but ultimately decided not to include this predictor in regression models after noting that the task did not make significant contributions to any of the models in preliminary analyses. In contrast, Hansen et al. (2015) used a task that represented proportions as area-models (i.e., shaded rectangles) within the context of mixing juice and water and reported that non-symbolic proportional reasoning outcomes uniquely predicted fraction MK at sixth grade (second in strength only to whole number MK).

Taken together, results may indicate that visual-spatial memory is a particularly salient aspect of nonverbal reasoning in regard to its influence on students’ fraction MK, but that it exerts its influence via whole number MK (rather than directly). In addition, the influence of visual-spatial memory may fade as students age and become more competent with fractions. Vukovic et al. (2014) suggest that early visual-spatial memory ability would reflect on both students’ ability to map spatial representations of magnitude onto symbolic representations (i.e.,
to connect their mental number line representation to physical or pictorial number line representations) and to hold spatial and symbolic magnitude representations in memory in order to make proportional judgments (i.e., to place whole numbers on a number line), and that these abilities reflect on fraction NLE in later grades. In effect, practice estimating the magnitude of whole numbers and fractions on number lines may support students in learning to integrate visual-spatial and symbolic representations of magnitude (Vukovic et al., 2014).

The fact that the ANS task did not predict fraction MK in Jordan et al.’s (2013) study aligns with other empirical evidence suggesting that performance on nonsymbolic tasks is not strongly related to performance on more conventional, symbolic mathematics tasks (Fazio et. al., 2014; Holloway & Ansari, 2009; Soltesz, Szucs, & Szucs, 2010). Fazio et al. (2014) for example, found very small, statistically non-significant correlations between fifth graders’ performance on the ANS task and performance on both a fraction magnitude comparison task (-.11) and a fraction NLE task (.03). Contradictory evidence of a strong relation between non-symbolic proportional reasoning and fraction MK seems attributable to the fact that the visual representations used in Hansen et al.’s (2015) non-symbolic proportional reasoning task resemble familiar area-model representations of fractions.

Hansen et al. (2015) suggest that performance on their juice-mixing equivalence task contributed to students’ fraction MK over and above other predictors because it was designed to tap students’ understanding of scale relations and multiplicative reasoning, which are both important components of fractions conceptual understanding. Alternatively, results might simply reflect the strong relation between part-whole understanding and fraction MK (Hecht, 1998). Even if this is the case, part-whole models and representations are much more common than measurement models (e.g., number lines) in typical fractions instructional materials; and as a
result, students are more competent in interpreting part-whole models (e.g., Bright et al., 1988). In effect, results of Hansen et al.’s (2015) study may indicate that a stronger ability to make connections between part-whole (area) models and measurement (number line) models could help support development of fraction MK.

Results of several intervention studies corroborate this interpretation (Bright, Behr, Post, & Lesh; 1988; Fuchs et al., 2013; Gabriel et al., 2012; Yoshida and Sawano, 2002). Fuchs et al. (2013) incorporate part-whole models in the first weeks of their Fraction Face-Off intervention program, and also required students to use area-model drawings and manipulatives to model and explain fraction magnitude comparison problems in supplemental intervention components (Fuchs et al., 2014; Fuchs et al., 2015). Gabriel et al. (2012) provided area-model manipulatives for students to use to model fraction magnitude while playing a variety of card games targeting fraction MK, and Yoshida and Sawano’s (2002) intervention heavily emphasized part-whole models and concepts to support fraction MK learning. In all cases, interventions resulted in significant fraction MK growth and differential achievement over comparison condition instruction. Results of Bright et al.’s (1988) study also indicate that it may be useful to integrate area-model (part-whole) and number line (measurement) representations during instruction. Researchers found that teaching students to make connections between part-whole models and number line models of fractions resulted in greater MK growth than presenting only number line models (Bright et al., 1988).

Results of Fuchs et al.’s (2013) mediation analysis show that improvement in MK completely mediated the intervention effect on part-whole outcomes, but that improvements in part-whole understanding did not mediate the intervention effect on MK outcomes. Authors argue that these results suggest that MK is more fundamental or central to fraction learning than
part-whole knowledge. This interpretation supports the integrated theory of numerical development (Siegler & Lortie-Forgues, 2014; Siegler et al., 2011) and is in alignment with the results of other intervention studies that have compared the effects of a part-whole approach against a measurement approach to instruction (Keijzer & Terwel, 2002; Saxe et al., 2012). Yet, it does appear that making strategic connections between part-whole fraction representations and magnitude representations can support fraction MK development. This approach deserves consideration, given that students and teachers are generally more familiar and comfortable with part-whole models as a result of their overrepresentation in typical instruction.

**Fractions Arithmetic**

Hecht and Vagi (2010) reported that fraction arithmetic was a significant predictor of fraction MK, and that fraction MK mediated group differences (between typically achieving and low achieving students) in fractions arithmetic (before controlling for the effects of fraction word problem solving and estimation). Mou et al. (2016) found that fraction addition performance at seventh grade was included in the best-fitting Bayes Model predicting fraction MK in ninth grade (but not in the model predicting fraction MK at eighth grade or in the model predicting growth in fraction MK from eighth to ninth grade).

Bailey et al. (2014) reported that the relation between first graders’ whole number MK and their eighth grade fraction arithmetic performance was fully mediated by their eighth grade fraction MK. In contrast, the relation between first grade whole number MK and eighth grade fraction MK was not mediated by eighth grade fraction arithmetic knowledge (Bailey et al., 2014). Bailey et al. (2015) investigated outcomes for students in China and the US, and reported that country performance differences in fraction arithmetic were fully mediated by country
performance differences in fraction MK. Conversely, country differences in fraction MK only partially mediated country differences in fractions arithmetic (Bailey et al., 2015).

Overall, results suggest that there is a bidirectional relation between growth in fractions arithmetic and fractions MK, but that the influence of fraction MK on fraction arithmetic is stronger than the effect of fraction arithmetic on fraction MK. This conclusion aligns with past research indicating that conceptual knowledge influences procedural knowledge more strongly than vice versa (e.g., Hecht & Vagi, 2010; Matthews & Rittle-Johnson, 2009; Rittle-Johnson & Alibali, 1999). In Mou et al.’s (2016) study, authors did not find evidence to include fraction arithmetic performance in models explaining fraction MK performance in five out of six analyses. Similarly, results of Bailey et al. (2014)’s mediation analysis show that the relation between students’ early whole number MK and their later fraction MK was not significantly influenced by their fraction arithmetic performance. In both cases, less-than-expert fractions arithmetic knowledge had little effect on fraction MK.

Intervention study results also demonstrate how improved fraction MK can have an influence on fractions arithmetic performance. Results of Fuchs and colleagues’ (2013) study indicated that their intervention led to significant improvements in fraction addition and subtraction, even though instruction focused almost exclusively on fraction MK. Similarly, results of Gabriel et al.’s (2012) study revealed that students committed fewer fraction arithmetic errors commonly associated with ‘whole number bias’ post-intervention, even though the MK intervention did not involve any fractions arithmetic instruction.

The evidence also seems to suggest that fractions arithmetic knowledge does not exert a beneficial influence on fraction MK unless or until students reach a very high level of fraction arithmetic proficiency. Bailey et al. (2015) examined fraction outcomes for students in the US
and China and reported that country differences in fraction arithmetic performance (favoring Chinese students) fully mediated country differences in fraction NLE (favoring Chinese students). Based on their results, Bailey et al. (2015) proposed a model of fractions learning that reflects the primacy of fraction MK, but also emphasizes how fractions MK, arithmetic performance, and conceptual understanding of arithmetic procedures are mutually reinforcing. Hecht and Vagi (2010) suggested that increased instructional focus on fraction arithmetic would likely have little effect on fractions conceptual knowledge; but Bailey et al.’s (2015) model and results related to language ability synthesized here suggest that fractions arithmetic practice to mastery and with instructional focus on teaching how procedures and solutions reflect magnitude could strengthen fraction MK and support overall fractions learning.

**Number Knowledge**

The two descriptive studies that included number sets (Geary, 2009) outcomes as predictors of fraction MK reported contradictory outcomes. Mou et al. (2016) found that first grade number sets outcomes were not included in the best-fitting Bayes models and did not account for unique variance in regression models predicting eighth and ninth graders’ fraction MK or growth in fraction MK. In contrast, Vukovic et al. (2014) reported that first grade number sets outcomes did not predict second grade whole number NLE or whole number arithmetic, but were the only first grade competency that directly predicted fourth grade fraction MK.

Contradictory findings between the two studies may relate primarily to participant age. Vukovic et al. (2014) studied fraction MK in a younger population than Mou et al. (4th graders in contrast to 8th graders), and it may be that students’ facility in representing and adding small whole numbers is important in early development of students’ fraction MK, but fades in importance as students’ conceptual understanding of fractions improves (Mou et al., 2016).
Vukovic et al. (2014) however, suggest that because number sets outcomes were predictive of fraction MK and not of earlier whole number MK or arithmetic, the quantitative ability assessed by the number sets test (i.e., speed and accuracy in representing and adding whole numbers < 10) may be uniquely important for fractions (rather than for mathematics skills more generally). Following this logic, Vukovic et al. (2014) hypothesize that speed and accuracy in representing and adding small whole numbers may facilitate students’ ability to make “the conceptual leap” from whole numbers to fractions. In other words, this competency may protect against whole number bias. As a result, researchers suggest that the number sets test may be useful as a diagnostic assessment for early identification of students who are likely to struggle with fractions in later grades (Vukovic et al., 2014). Relatedly, practice subitizing (i.e., recognizing the numerosity represented by a small set of objects) and adding/combining small quantities in early mathematics instruction could support later fractions development.

**Whole Number Arithmetic**

Jordan et al. (2013) and Bailey et al. (2014) found that students’ first grade whole number arithmetic performance (addition fluency +) accounted for unique variance in fraction MK at fourth grade and eighth grade, respectively. In contrast, Namkung and Fuchs (2016) reported that whole number arithmetic performance (mixed operation accuracy) was not predictive of fraction MK (or whole number MK) for their sample of at-risk fourth graders. Vukovic et al. (2014) found that both attention and language measured at first grade mediated the relation between whole number arithmetic (mixed operation accuracy) at second grade and fourth grade MK.

Hansen et al.’s (2015) identified fifth grade long division skill, but not multiplication fact fluency, as a statistically significant predictor of sixth graders’ fraction MK. Resnick et al. (2016), however, reported that multiplication fact fluency strongly influenced fraction MK
growth class assignment. Students with multiplication fluency scores 1 standard deviation above average were approximately 35% less likely to be assigned to the low-growth group than to the steep-growth group (Resnick et al., 2016).

Overall, the evidence from both descriptive and intervention studies indicate that overall whole number arithmetic achievement is not a strong predictor of fraction MK, but that math fact fluency (addition and multiplication) and competence in solving long division problems support fraction MK development. Results from Namkung and Fuchs (2016) descriptive study and from Fuchs et al.’s (2013) Fraction Face-Off intervention study both indicate that overall whole number arithmetic achievement and fraction MK are not strongly related. Fuchs et al. (2013) reported that whole number arithmetic knowledge did not mediate the effect of intervention on fraction MK outcomes, or in other words, that intervention effects were similar for students with varying levels of incoming whole number calculation skill.

In contrast, evidence from descriptive and intervention studies does suggest that math fact fluency and long division skill support fraction MK development. Addition and multiplication fluency may support fraction MK by strengthening students’ ability to recognize and interpret multiplicative relationships (Resnick et al., 2016), and long division may help students develop pre-requisite skills necessary for making magnitude judgments (Hansen et al., 2015). Bright et al. (1988) reported that intervention participants’ most frequently used strategies for solving MK problems involved skip counting, multiplication, and division.

The fact that multiplication fact fluency did not predict fraction MK in Hansen et al.’s (2015) study may simply indicate that division skill is even more strongly related to fraction MK than multiplication fact fluency. The correlations between multiplication fact fluency and fraction MK (.419**) and long division skill and fraction MK (.519**) show that both skills
were related to fraction MK, but the fact that only division accounted for unique variance in fraction MK outcomes in the regression model may be an artifact of multicollinearity (the correlation between long division and multiplication fluency was .604**).

**Whole Number Magnitude Knowledge**

Results of four studies show that whole number MK was the strongest predictor of fraction MK for students spanning grades 4 through 9 (Bailey et al., 2014; Hansen et al., 2015; Jordan et al., 2013; Mou et al., 2016). Vukovic et al. (2014) reported that second grade whole number MK predicted fourth grade fraction MK, and that the relation between whole number MK and fraction MK was mediated by first grade language ability and visual spatial memory.

Results of two studies also show that whole number MK predicted growth in fraction MK. Resnick et al. (2016) reported that students with whole NLE scores 1 standard deviation above average were 70% more likely to be assigned to the consistently highest achieving group of students (class 1) than to the steep-growth group of students; and, that students with whole number NLE scores 1 standard deviation below average were approximately twice as likely to be assigned to the low-growth group of students than to the steep-growth group. Mou et al. (2016) reported that whole number NLE was the only variable included in the best-fitting Bayes model predicting fraction MK growth from eighth grade to ninth grade.

Descriptive study results indicate that whole number MK is strongly predictive of fraction MK, and provide insight into (1) the relation between whole number NLE and proportional reasoning ability, and (2) differences between whole number MK and fraction MK development. In alignment with the integrated theory of numerical development (Siegler et al., 2011), evidence from intervention studies indicates that systematically using the number line to
help students make connections between their whole number and fraction MK can lead to large improvements in students’ fraction MK (Moss & Case, 1999; Saxe et al., 2012).

Synthesis of descriptive study results regarding relations between whole number NLE, proportional reasoning ability, and fraction MK indicate that both whole number NLE and proportional reasoning are independently related to fraction MK. In effect, students sometimes use part-whole understanding (proportional reasoning) to place fractions on number lines, and sometimes make fraction number line estimates based on reasoning about magnitude. It follows that providing whole number NLE practice and teaching students to use both part-whole/proportional reasoning strategies and magnitude reasoning strategies to make whole number magnitude judgments will support students’ future development of fraction MK.

Resnick et al. (2016) suggest that their results provide evidence that the developmental progression for whole number MK differs from the developmental progression for fraction MK. While previous research indicates that students represent small whole numbers on the number line first and learn to represent larger numbers with greater accuracy over time (Booth & Siegler, 2006; Siegler & Opfer, 2003), Resnick et al.’s results suggest that students learn to represent unit fractions and other proper fractions first, and improper fractions and mixed number later. In effect, students do not simply learn to represent larger and larger fractions, their fraction MK development relies on their understanding of the properties of fractions and their ability to integrate their new fraction MK with existing whole number MK (Resnick et al., 2016; Siegler et al., 2011).

Relatedly, results may support the supposition that an overemphasis on proper fractions in early fractions instruction reinforces the common misconception that all fractions are smaller than one (Vosniadou, Vamvakoussi, & Skopeliti, 2008). Early and adequate experience with
improper fractions and mixed numbers, however, may provide valuable opportunities for students to integrate their whole number and fraction MK. Introducing 0-2 and 0-5 number lines earlier in fractions instruction, explicitly teaching NLE strategies, and representing both whole numbers and fractions simultaneously could all support fraction MK development.

Evidence from the intervention literature supports the integrated theory of numerical development (Siegler et al., 2011) and suggests that using number line representations to help students make connections between whole number and fraction magnitude can dramatically improve students’ fraction MK. Saxe et al. (2012) taught a series of lessons about integers and used the number line to model integer vocabulary terms and concepts. Next, students participated in a series of lessons about fractions and used the number line to model fraction terms and concepts while making connections and noting differences between the two sets of terms and concepts. Moss and Case (1999) taught students to represent percentages, then decimals, and finally fractions on the number line (rather than teach fractions first, as in the traditional sequence for rational number instruction). Researchers argued that beginning instruction with percentages allowed students to make connections between whole and rational numbers and to understand the differences between the properties of whole number and rational numbers. In addition, they argued that emphasizing number line representations throughout instruction on percentages, decimals, and fractions allowed students to make connections between the different rational number representations and to understand the proportional nature of all rational numbers (Moss & Case, 1999). The success of these interventions and evidence from the descriptive literature build a strong rationale for teaching student to connect their whole number and fraction MK on the number line.
General Discussion and Implications for Practice

The results of this review suggest that there are key mathematics competencies that instruction should target before fractions are introduced and after to promote the development of fraction MK. In addition, mathematics instruction at all levels should be designed to accommodate/compensate for weaknesses in domain-general competencies (i.e., attention, language ability, and working memory), as these seem to exert an influence on fraction MK development both directly and indirectly through the impact they have on intervening mathematics competencies (e.g., Hecht et al., 2003; Geary, 2008). Figure 2 depicts a conceptual model of the fraction MK learning that includes both domain general competencies and key mathematics competencies.

In order to prevent future difficulties with and promote development of fraction MK, mathematics instruction prior to the introduction of fractions should target number knowledge, whole number MK and NLE, addition and multiplication fact fluency, and long division competence. Once fractions are introduced, teachers can support the development of fraction MK by teaching students to place fractions on the number line (alongside whole numbers) and by teaching students to make connections between area-model representations of fraction magnitude and number line representations of fraction magnitude. Teachers should also teach fraction arithmetic procedures to mastery, teach how and why procedures work, and help students to make connections between arithmetic problems and fraction magnitude.

Teachers should also promote attentive behavior and use instructional strategies designed to compensate for language and working memory limitations at all levels of mathematics instruction. In order to address language limitations, teachers should provide explicit vocabulary instruction; explain terms, concepts, and procedures in simple language (and teach students to
explain terms, concepts and procedures); and frequently stop to check student understanding during instruction. In order to address working memory limitations, teachers should break up (i.e., chunk or segment) explanations and procedures into small units and provide fluency-building practice for key skills and ideas.

**Limitations**

There are several limitations of the present review that should be noted. Authors’ choice of the variables/competencies to include in models, the assessments used to measure those competencies, and the type of analyses performed all reflect on study results, and because many of the included descriptive studies were conducted by the same researchers, studies and results may reflect author biases. In addition, included descriptive studies generally did not account for or describe the instruction participants received. This omission precludes any discussion about the influence of instruction on study outcomes. Finally, conclusions based on study results should be considered preliminary, given that primary study results reflect ‘snapshots’ of student performance.

**Conclusion**

Overall, results of descriptive studies investigating the relations between various domain general and mathematics competencies and fraction MK highlight the importance of both types of competencies. Domain general competencies have direct and indirect effects on fraction MK, and in effect, fractions instruction and mathematics instruction in general must be designed to accommodate or compensate for domain general weaknesses to the extent possible. In addition, several key mathematical competencies seem to warrant increased instructional focus. In early elementary grade mathematics instruction, number knowledge, whole number MK and math fact fluency appear to be of particular importance. In later elementary grade mathematics, whole
number division, procedural fluency with fraction arithmetic operations, and the ability to extend understanding of number line representations to include fractions appear to be especially important in regard to supporting the development of fraction MK.
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Note. Dependent measure abbreviations: FMC = fraction magnitude comparison; NLE = number line estimation; PW = part-whole knowledge items; MK = magnitude knowledge; Working memory measures: listening recall, counting recall, and backward digit recall subtests are from the Working Memory Test Battery for Children (WMTB-C; Pickering & Gathercole, 2001), counting span test (Hitch & McAuley, 1991); Attentive behavior measures: SWAN = Strengths and Weaknesses of ADHD Symptoms and Normal Behavior Scale (Swanson et al., 2004, 2006, 2008), SSRS = Social Skills Rating System (Gresham & Elliott, 1990); Nonverbal reasoning measures: the matrix reasoning subtest is from the Wechsler Abbreviated Scale of Intelligence (WASI; Wechsler, 1999), the mazes memory and block recall subtests are from theWMTB-C (Pickering & Gathercole, 2001), ANS = Approximate Number System task (Halberda et al., 2008), NVR = nonverbal reasoning, PR = proportional reasoning, VSM = visual spatial memory; Language measures: PPVT = Peabody Picture Vocabulary Test-fourth edition (Dunn & Dunn, 2007), vocab. = vocabulary subtest of the Woodcock Diagnostic Reading Battery (WDRB; Woodcock, 1997), listening comp. = listening comprehension subtest of the WDRB (Woodcock, 1997); all bivariate correlations are statistically significant at p > .05 or higher unless otherwise specified with <sup>NS</sup> = not significant.
Table 2

Descriptive study information and bivariate correlations between measures of fraction MK and mathematics competencies

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<td>Untimed, addition and multiplication</td>
<td>5 .50</td>
<td>Multiplication accuracy</td>
<td>5 .37</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hecht &amp; Vagi (2010)</td>
<td>FMC</td>
<td>4</td>
<td>Regression</td>
<td>Untimed, addition and multiplication</td>
<td>4 .51</td>
<td></td>
<td>4 .35</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>FMC</td>
<td>5</td>
<td></td>
<td>Untimed, addition and multiplication</td>
<td>5 .49</td>
<td></td>
<td>5 .38</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Jordan et al. (2013)</td>
<td>PW + MK concepts</td>
<td>5</td>
<td>Regression</td>
<td>Untimed, addition and subtraction</td>
<td>4 .62</td>
<td>Addition fact fluency (WIAT)</td>
<td>4 .44</td>
<td>0-1000 NLE</td>
<td>4 .69</td>
</tr>
<tr>
<td>Mou et al. (2016)</td>
<td>0-5 NLE</td>
<td>8</td>
<td>Regression, Bayes Models</td>
<td>Timed, addition</td>
<td>7 -.51</td>
<td></td>
<td>0-100 NLE</td>
<td>1 .52 Number Sets</td>
<td>1 -.52</td>
</tr>
<tr>
<td></td>
<td>0-5 NLE</td>
<td>9</td>
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<td>Timed, addition</td>
<td>7 -.50</td>
<td></td>
<td>0-100 NLE</td>
<td>1 .54 Number Sets</td>
<td>1 -.55</td>
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<tr>
<td></td>
<td>FMC</td>
<td>7</td>
<td></td>
<td>Timed, addition</td>
<td>7 .36</td>
<td></td>
<td>0-100 NLE</td>
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<td>1 .45</td>
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<td></td>
<td>0-100 NLE</td>
<td>6</td>
<td></td>
<td>Untimed subtraction</td>
<td>4 .14*</td>
<td>WRAT-4 calc., mixed operation</td>
<td>4 .20</td>
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<tr>
<td></td>
<td>0-1000 NLE</td>
<td>6</td>
<td></td>
<td>Timed, Double-digit subtraction</td>
<td>4 .11*</td>
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<td>Namkung &amp; Fuchs (2010)</td>
<td>0-1 NLE</td>
<td>4</td>
<td>Path Modeling</td>
<td>Untimed subtraction</td>
<td>4 .22</td>
<td></td>
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<tr>
<td>Study</td>
<td>Dependent Measure(s)</td>
<td>Grade</td>
<td>Method(s) of Analysis</td>
<td>Fraction arithmetic</td>
<td>Grade</td>
<td>r</td>
<td>Whole Number Arithmetic</td>
<td>Grade</td>
<td>r</td>
</tr>
<tr>
<td>-------------------------------</td>
<td>----------------------</td>
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<td>-----------------------</td>
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<td>-------</td>
<td>---</td>
<td>------------------------</td>
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<td>---</td>
</tr>
<tr>
<td>Resnick et al. (2016)</td>
<td>0-1 and 0.2 fraction NLE</td>
<td>4-5</td>
<td>Latent growth class analysis; Multinomial logistic regression</td>
<td></td>
<td></td>
<td></td>
<td>Timed, Double-digit addition</td>
<td>4</td>
<td>.14.12</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Multiplication fact fluency (WIAT)</td>
<td>4(1)</td>
<td>.42</td>
<td>.0-1000 NLE</td>
<td>4(1)</td>
<td>.46</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Multiplication fact fluency (WIAT)</td>
<td>4(2)</td>
<td>.43</td>
<td>.0-1000 NLE</td>
<td>4(2)</td>
<td>.47</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Multiplication fact fluency (WIAT)</td>
<td>5(1)</td>
<td>.47</td>
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<td>.45</td>
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<td>PW + MK concepts</td>
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<td>Mediation analysis</td>
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<td></td>
<td>0.100 NLE</td>
<td>1</td>
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</table>

Note. WRAT I = Wide Range Achievement Math Computation Test I (Wilkinson, 2008); WRAT II = Numerical Operations subtest of the Wechsler Individual Achievement Test II Abbreviated (Wechsler, 2001); Addition Fluency subtest of the WIAT (The Psychological Corporation, 1992); Multiplication Fact Fluency subtest of the WIAT (The Psychological Corporation, 1992); Addition and subtraction subtests of the Double-Digit Calculation Test (Fuchs, Hamlett, & Powell, 2003); Number Sets (Gee, 2009). All bivariate correlations are statistically significant at p > .05 or higher unless otherwise specified with NS = not significant.
Figure 1. Diagram of statistically significant relations between competencies and fraction magnitude knowledge reported across results of included studies. Arrows show the direction of reported relations. Connecting lines are labeled with numbers that correspond with studies. Text labels indicate indirect relations reported in mediation analyses.
Figure 2. Conceptual model of relations between domain general and intervening mathematics competencies and fraction magnitude knowledge. Arrows indicate the direction of the relation.
Appendix B

Outcome Measures

I. Intervention-Aligned Test of Fraction Equivalence

Name: ___________________________          Date: __________________

Write = or ≠ in the space between each pair of fractions:

\[
\begin{align*}
\frac{1}{3} & \quad \frac{4}{12} & \quad \frac{3}{5} & \quad \frac{6}{20} & \quad \frac{5}{6} & \quad \frac{15}{12} \\
\frac{1}{2} & \quad \frac{4}{4} & \quad \frac{2}{3} & \quad \frac{6}{9} & \quad \frac{4}{10} & \quad \frac{40}{40} \\
\frac{5}{6} & \quad \frac{3}{8} & \quad \frac{2}{5} & \quad \frac{2}{5} & \quad \frac{2}{5} & \quad \frac{2}{5}
\end{align*}
\]
II. Curriculum-Aligned Unit Test

Tell whether the fractions are equivalent. Write = or ≠

1. \( \frac{5}{10} \) \( \bigcirc \) \( \frac{1}{2} \)
2. \( \frac{2}{3} \) \( \bigcirc \) \( \frac{3}{6} \)
3. \( \frac{6}{8} \) \( \bigcirc \) \( \frac{3}{4} \)
4. \( \frac{7}{12} \) \( \bigcirc \) \( \frac{4}{6} \)

Write two equivalent fractions for each.

5. \( \frac{2}{3} \)
6. \( \frac{5}{10} \)
7. \( \frac{4}{12} \)
8. \( \frac{4}{5} \)

Write the fraction in simplest form.

9. \( \frac{6}{12} \)
10. \( \frac{2}{10} \)
11. \( \frac{4}{6} \)
12. \( \frac{3}{12} \)
13. \( \frac{6}{10} \)

Write the pair of fractions as a pair of fractions with a common denominator.

14. \( \frac{2}{3} \) and \( \frac{5}{6} \)
15. \( \frac{3}{5} \) and \( \frac{1}{2} \)
16. \( \frac{1}{4} \) and \( \frac{5}{12} \)

17. \( \frac{7}{8} \) and \( \frac{3}{4} \)
18. \( \frac{3}{10} \) and \( \frac{1}{5} \)
19. \( \frac{3}{4} \) and \( \frac{1}{3} \)
Problem Solving

20. Peggy completed $\frac{5}{6}$ of the math homework and Al completed $\frac{4}{5}$ of the math homework. Did Peggy or Al complete more of the math homework?

Show all of your work.

21. Nell made a pizza. She cut the pizza into fourths. Then she cut each fourth into four pieces. Nell and her friends ate 6 of the smaller pieces of pizza.

Show all of your work.

What fraction of the pizza did Nell and her friends eat?

_____________________

What fraction of the pizza did Nell and her friends NOT eat?

_____________________

97
Compare. Write <, >, or =

22. \[
\frac{2}{6} \, \bigcirc \, \frac{3}{4}
\]

23. \[
\frac{6}{8} \, \bigcirc \, \frac{1}{4}
\]

24. \[
\frac{5}{6} \, \bigcirc \, \frac{2}{4}
\]

25. \[
\frac{1}{3} \, \bigcirc \, \frac{4}{12}
\]

26. \[
\frac{1}{6} \, \bigcirc \, \frac{1}{8}
\]

27. \[
\frac{2}{3} \, \bigcirc \, \frac{4}{6}
\]

28. \[
\frac{3}{10} \, \bigcirc \, \frac{3}{12}
\]

29. \[
\frac{7}{8} \, \bigcirc \, \frac{4}{4}
\]

Write the fractions in order from least to greatest.

30. \[
\frac{1}{2} \, \bigcirc \, \frac{1}{4} \, \bigcirc \, \frac{5}{8}
\]

31. \[
\frac{2}{3} \, \bigcirc \, \frac{1}{6} \, \bigcirc \, \frac{9}{10}
\]

32. \[
\frac{3}{5} \, \bigcirc \, \frac{3}{4} \, \bigcirc \, \frac{3}{8}
\]
III. PARCC Released Items Test

1. Circle the **three** fractions that are equivalent to $\frac{1}{2}$

\[
\frac{5}{10} \quad \frac{4}{6} \quad \frac{8}{12} \quad \frac{4}{8} \quad \frac{2}{4}
\]

2. Write the correct comparison symbol ($<$, $>$, $=$) on the line to compare each pair of fractions.

\[
\frac{5}{6} \text{ yard} \quad \frac{3}{4} \text{ yard}
\]

\[
\frac{5}{6} \text{ yard} \quad \frac{7}{8} \text{ yard}
\]

\[
\frac{5}{6} \text{ yard} \quad \frac{10}{12} \text{ yard}
\]

\[
\frac{5}{6} \text{ yard} \quad \frac{2}{3} \text{ yard}
\]
3. The table shows the heights of three different plants.

<table>
<thead>
<tr>
<th>Type of Plant</th>
<th>Height (feet)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tomato</td>
<td>1 3</td>
</tr>
<tr>
<td>Pepper</td>
<td>3 6</td>
</tr>
<tr>
<td>Bean</td>
<td>5 12</td>
</tr>
</tbody>
</table>

Which statements about the heights of the plants are true?

Select the **three** correct statements.

a. The bean plant is the tallest plant.
b. The tomato plant is the shortest plant.
c. The pepper plant is taller than the bean plant.
d. The tomato plant is shorter than the bean plant.
e. The pepper plant is shorter than the tomato plant.
4. Of the students in one school, \( \frac{1}{12} \) play soccer, \( \frac{3}{8} \) play basketball, \( \frac{2}{5} \) take music lessons, \( \frac{2}{6} \) take dance lessons.

**Part A**

Which fraction is equivalent to the fraction of students who take music lessons at the school?

a. \( \frac{3}{6} \)

b. \( \frac{5}{8} \)

c. \( \frac{4}{10} \)

d. \( \frac{4}{12} \)

**Part B**

Which list orders the fractions from least to greatest?

a. \( \frac{1}{12} , \frac{2}{5} , \frac{2}{6} , \frac{3}{8} \)

b. \( \frac{2}{5} , \frac{3}{8} , \frac{2}{6} , \frac{1}{12} \)

c. \( \frac{2}{5} , \frac{2}{6} , \frac{3}{8} , \frac{1}{12} \)

d. \( \frac{1}{12} , \frac{2}{6} , \frac{3}{8} , \frac{2}{5} \)
5. Jasmine ate $\frac{1}{4}$ of a pie. She drew a model to represent the fraction of the pie that she ate.

Which fraction is equivalent to the fraction of the pie that Jasmine ate?

a. $\frac{2}{5}$

b. $\frac{3}{6}$

c. $\frac{2}{8}$

d. $\frac{1}{12}$
6. Isabel used $\frac{2}{3}$ cup of strawberries in a fruit salad. She used less than $\frac{2}{3}$ cup of blueberries in the same salad. Which of the following could be the fraction of a cup of blueberries that Isabel used?

Select the **three** fractions that could represent the fraction of a cup of blueberries.

a. $\frac{1}{2}$

b. $\frac{1}{4}$

c. $\frac{4}{6}$

d. $\frac{5}{6}$

e. $\frac{3}{8}$
7. The table shows the lengths of five different animals in a zoo. For each animal, shade an oval in the table to show whether it is less than or greater than \( \frac{5}{10} \) meter in length.

Shade in one oval in each row.

<table>
<thead>
<tr>
<th>Animal</th>
<th>Length (in meters)</th>
<th>Less than ( \frac{5}{10} ) meter</th>
<th>Greater than ( \frac{5}{10} ) meter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blue jay</td>
<td>25/100</td>
<td>◯</td>
<td></td>
</tr>
<tr>
<td>Cottontail rabbit</td>
<td>4/10</td>
<td>◯</td>
<td>◯</td>
</tr>
<tr>
<td>Raccoon</td>
<td>8/10</td>
<td>◯</td>
<td>◯</td>
</tr>
<tr>
<td>Snowy owl</td>
<td>67/100</td>
<td>◯</td>
<td>◯</td>
</tr>
<tr>
<td>Thread snake</td>
<td>11/100</td>
<td>◯</td>
<td>◯</td>
</tr>
</tbody>
</table>

**Part B**

Use the lengths in the table to compare the lengths of the animals.

Write in the correct comparison symbol (<, >, =) to compare each pair of animals.

Blue jay          Cottontail rabbit
Raccoon          Snowy owl
Thread snake     Blue jay
Number Line Estimation
Section 1: 0–1 Number Lines
2 minutes

Name

Teacher

Period
Example

\[
\frac{1}{8}
\]

\[
\frac{1}{5}
\]

\[
\frac{13}{14}
\]

\[
\frac{2}{13}
\]

\[
\frac{3}{7}
\]
Example 1

Example 2

Example 2
V. Fraction Attitudes Survey and Social Validity Questionnaire

**Student Social Validity Survey**

Directions: Read each question carefully. Circle the number that you think best matches your own feelings.

- I totally agree 5 🎉
- I agree a little 4 😊
- Not sure 3 😐
- I disagree a little 2 😞
- I totally disagree 1 😞

<table>
<thead>
<tr>
<th>😊</th>
<th>😊</th>
<th>😐</th>
<th>😞</th>
<th>😞</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

The **most** helpful part of the program was ____________________________

because ____________________________________________________________.

The **least** helpful part of the program was ____________________________

because ____________________________________________________________.
<table>
<thead>
<tr>
<th>I totally agree</th>
<th>I agree a little</th>
<th>Not sure</th>
<th>I disagree a little</th>
<th>I totally disagree</th>
</tr>
</thead>
<tbody>
<tr>
<td>🎉</td>
<td>😊</td>
<td>😐</td>
<td>😞</td>
<td>😞</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>The lessons helped me to learn about fractions.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>I enjoyed the lessons.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>I enjoyed working with the materials.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>The materials helped me to learn about fractions.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>I would recommend this program to other students.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Appendix C
Sample CRA-I Model

Fraction Equivalence Intervention

\[
\frac{1}{2} = \frac{2}{4} = \frac{16}{32} = \frac{50}{100}
\]

Stephanie Morano

Intervention Session Topic Overview

<table>
<thead>
<tr>
<th>Session</th>
<th>Week 1: Introduction to fraction on the number line</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Number line introduction, vocabulary</td>
</tr>
<tr>
<td>2</td>
<td>Fraction vocabulary (on the number line)</td>
</tr>
<tr>
<td>3-4</td>
<td>Placing unit fraction on the number line</td>
</tr>
<tr>
<td></td>
<td>Week 2: Identifying fractions as equivalent or not equivalent</td>
</tr>
<tr>
<td>5-6</td>
<td>Placing non-unit fraction on the number line (i.e., composition)</td>
</tr>
<tr>
<td>7</td>
<td>Fractions equivalent to 1; Identifying fraction as equivalent or not equivalent</td>
</tr>
<tr>
<td>8</td>
<td>Fractions equivalent to ( \frac{1}{2} ); Identifying fractions as equivalent or not equivalent</td>
</tr>
<tr>
<td></td>
<td>Week 3: Finding equivalent fractions on the number line; Connecting models and procedures [( \frac{a}{b} = \frac{(a \times n)}{(b \times n)} )]</td>
</tr>
<tr>
<td>9</td>
<td>Finding equivalent fractions: Missing number problems</td>
</tr>
<tr>
<td>10</td>
<td>Finding equivalent fractions: Missing denominator problems</td>
</tr>
<tr>
<td>11-12</td>
<td>Finding an equivalent fraction for a given fraction</td>
</tr>
<tr>
<td></td>
<td>Week 4: Finding equivalent fractions using multiplication; Modeling on the number line</td>
</tr>
<tr>
<td>13-15</td>
<td>Practice using multiplication to find equivalent fractions; Model on the number line</td>
</tr>
</tbody>
</table>
Week 1

- Number line intro, vocabulary
- Fraction vocabulary (on number line)
- Placing Unit fractions on number line

Number line introduction; Number line and fraction vocabulary
- Number line: unit interval, equal-length segments, right endpoint, left endpoint
- Fraction: number with magnitude; magnitude is defined by relation between numerator and denominator
  - Denominator = number of equal-length segments in the unit interval
  - Numerator = number of equal-length segments represented
Placing unit fractions (one in the numerator) on the number line:
- Denominator = number of equal-length segments in the unit interval
- Numerator = number of equal-length segments represented

Week 2

» Placing non-unit fractions on the number line
» Fractions equal to 1, ½
Placing non-unit fractions (not one in the numerator) on the number line:
- Denominator = number of equal-length segments in the unit interval
- Numerator = number of equal-length segments represented
- All fractions can be composed with unit fractions

Fractions equal to 1:
- Can compose fractions equal to 1 with unit fractions
- $a/a = 1$
More Practice: Fractions equal to 1, Fractions Equal to ½
- $a/a = 1$
- $a/(a \times 2) = ½$
- If time, almost 1; almost 1/2; a little more than 1; a little more than 1/2

Week 3
- Finding equivalent fractions: missing numerator
- Finding equivalent fractions: missing denominator
- Finding Equivalent fractions
Finding Equivalent Fractions (Missing Numerator)
- Match/find using fraction strips; recreate on number line
- Find- how many sub-segments?
- Check/verify with multiplication

Finding Equivalent Fractions (Missing Denominator)
- Match/find using fraction strips; recreate on number line
- Find- how many sub-segments?
- Check/verify with multiplication
Finding Equivalent Fractions
- Match/find using fraction strips; recreate on number line
- Find- how many sub-segments?
- Check/verify with multiplication
- Rule: \( \frac{a}{b} = \frac{(a \times n)}{(b \times n)} \)

Week 4

More practice- finding equivalent fractions
Work Backwards- Use Equation to Create Model for Equivalent Fractions
- Model first fraction, with fraction strips, on number line

Equal/Not Equal
- Model first fraction, model second fraction
Appendix D

Study Timeline

<table>
<thead>
<tr>
<th>September 2016</th>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present study to 4th grade parents</td>
<td>9/13</td>
</tr>
<tr>
<td>Identify all eligible 4th grade students- all those who scored at the Basic or Below Basic level on their 3rd grade PSSA-M</td>
<td>By 9/16</td>
</tr>
<tr>
<td>Obtain consent from group of eligible students</td>
<td>Ongoing</td>
</tr>
<tr>
<td>Give all eligible 4th graders the screening assessment (10-15 minutes)</td>
<td>9/19-9/23</td>
</tr>
<tr>
<td>Meet with 4th grade REM teachers about comparison group instruction (Go Math activities in comparison condition booklet)</td>
<td>By 9/30</td>
</tr>
<tr>
<td>Identify all students who scored below 60% accuracy on the screener- These are the study participants: - Obtain consent - Obtain demographic information - Obtain REM class schedules</td>
<td>By 9/30</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>October 2016</th>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pretest all students participants</td>
<td>10/3-10/7</td>
</tr>
<tr>
<td>Assign study participants to conditions</td>
<td>By 10/14</td>
</tr>
<tr>
<td>Intervention Week 1</td>
<td>10/17-10/21</td>
</tr>
<tr>
<td>Intervention Week 2</td>
<td>10/24-10/28</td>
</tr>
<tr>
<td>Intervention Week 3</td>
<td>10/31-11/4</td>
</tr>
</tbody>
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<table>
<thead>
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<th>November 2016</th>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intervention Week 4</td>
<td>11/14-11/18</td>
</tr>
<tr>
<td>Post-testing</td>
<td>11/21-11/23</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>December 2016</th>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maintenance testing</td>
<td>12/19/12/21</td>
</tr>
</tbody>
</table>
Appendix E

Dissertation Tables and Figures

I. Table 1

Table 1

Participant Characteristics by Treatment Group

<table>
<thead>
<tr>
<th>Gender</th>
<th>Intervention (n = 29)</th>
<th>Comparison (n = 31)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>18</td>
<td>11</td>
</tr>
<tr>
<td>Female</td>
<td>15</td>
<td>16</td>
</tr>
<tr>
<td>Classification</td>
<td></td>
<td></td>
</tr>
<tr>
<td>At-Risk</td>
<td>25</td>
<td>21</td>
</tr>
<tr>
<td>LD</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>SLD</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>OHI</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Note. *t*-tests showed no significant differences between groups on screening measure and PSSA-M scores. Chi-square tests showed no significant differences by classification status. LD = learning disabilities; SLD = speech/language impairment; OHI = Other health impairment
II. Table 2

Table 2

<table>
<thead>
<tr>
<th>Session</th>
<th>Week 1: Introduction to fractions on the number line</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Number line introduction, vocabulary</td>
</tr>
<tr>
<td>2</td>
<td>Fraction vocabulary (on the number line)</td>
</tr>
<tr>
<td>3-4</td>
<td>Placing unit fractions on the number line</td>
</tr>
<tr>
<td></td>
<td>Week 2: Identifying fractions as equivalent or not equivalent</td>
</tr>
<tr>
<td>5-6</td>
<td>Placing non-unit fractions on the number line (i.e., composition)</td>
</tr>
<tr>
<td>7</td>
<td>Fractions equivalent to 1; Identifying fractions as equivalent or not equivalent</td>
</tr>
<tr>
<td>8</td>
<td>Fractions equivalent to (\frac{1}{2}); Identifying fractions as equivalent or not equivalent</td>
</tr>
<tr>
<td></td>
<td>Week 3: Finding equivalent fractions on the number line; Connecting models and procedure (\frac{a}{b} = \frac{(a \times n)}{(b \times n)})</td>
</tr>
<tr>
<td>9</td>
<td>Finding equivalent fractions: Missing numerator problems</td>
</tr>
<tr>
<td>10</td>
<td>Finding equivalent fractions: Missing denominator problems</td>
</tr>
<tr>
<td>11-12</td>
<td>Finding an equivalent fraction for a given fraction</td>
</tr>
<tr>
<td></td>
<td>Week 4: Finding equivalent fractions using multiplication; Modeling on the number line</td>
</tr>
<tr>
<td>13-16</td>
<td>Practice using multiplication to find equivalent fractions; Model on the number line</td>
</tr>
</tbody>
</table>
### III. Table 3

#### Pre and Posttest Group Score Means (and Standard Errors) on the Released PARCC Items Test, Curriculum-aligned Unit Test, and Fraction Equivalence Test

<table>
<thead>
<tr>
<th>Test</th>
<th>Intervention ($n = 31$)</th>
<th>Control ($n = 29$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>With disabilities ($n = 6$)</td>
<td>Without disabilities ($n = 25$)</td>
</tr>
<tr>
<td><strong>Released PARCC items test</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pretest</td>
<td>5.67 (1.26)</td>
<td>7.58 (0.68)</td>
</tr>
<tr>
<td>Posttest</td>
<td>15.00 (1.55)</td>
<td>15.21 (0.65)</td>
</tr>
<tr>
<td><strong>Curriculum-aligned unit test</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pretest</td>
<td>4.67 (0.96)</td>
<td>6.42 (0.67)</td>
</tr>
<tr>
<td>Posttest</td>
<td>20.17 (3.87)</td>
<td>21.08 (1.44)</td>
</tr>
<tr>
<td><strong>Fraction equivalence test</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pretest</td>
<td>1.33 (0.33)</td>
<td>1.38 (0.22)</td>
</tr>
<tr>
<td>Posttest</td>
<td>7.83 (0.48)</td>
<td>7.96 (0.20)</td>
</tr>
</tbody>
</table>
### IV. Table 4

<table>
<thead>
<tr>
<th>Item Level PAE on the 0-1 and 0-2 NLE Tests by Group at Pre and Posttest</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>0-1 NLE Items</strong></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>1. 1/5</td>
</tr>
<tr>
<td>2. 13/14</td>
</tr>
<tr>
<td>3. 2/13</td>
</tr>
<tr>
<td>4. 3/7</td>
</tr>
<tr>
<td>5. 5/8</td>
</tr>
<tr>
<td>7. 1/2</td>
</tr>
<tr>
<td>8. 1/19</td>
</tr>
<tr>
<td>9. 5/6</td>
</tr>
<tr>
<td><strong>0-1 NL Total</strong></td>
</tr>
<tr>
<td><strong>0-2 NLE Items</strong></td>
</tr>
<tr>
<td>1. 1/3</td>
</tr>
<tr>
<td>2. 7/4</td>
</tr>
<tr>
<td>3. 12/13</td>
</tr>
<tr>
<td>4. 11/12</td>
</tr>
<tr>
<td>5. 3/2</td>
</tr>
<tr>
<td>6. 5/6</td>
</tr>
<tr>
<td>7. 5/5</td>
</tr>
<tr>
<td>8. 1/2</td>
</tr>
<tr>
<td>9. 7/6</td>
</tr>
<tr>
<td>10. 12/4</td>
</tr>
<tr>
<td>11. 3/8</td>
</tr>
<tr>
<td>12. 15/8</td>
</tr>
<tr>
<td>13. 2/3</td>
</tr>
<tr>
<td>14. 11/5</td>
</tr>
<tr>
<td>15. 7/9</td>
</tr>
<tr>
<td>16. 1/19</td>
</tr>
<tr>
<td>17. 15/6</td>
</tr>
<tr>
<td>18. 4/3</td>
</tr>
<tr>
<td><strong>0-2 NL Total</strong></td>
</tr>
</tbody>
</table>
Table 5
Fractions Attitudes Survey and Social Validity Questionnaire Item Means (and Standard Deviations) at Pre and Posttest

<table>
<thead>
<tr>
<th>Item</th>
<th>Control Pretest</th>
<th>Control Posttest</th>
<th>Intervention Pretest</th>
<th>Intervention Posttest</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fractions are easy.</td>
<td>3.43 (1.20)</td>
<td>3.81 (1.42)</td>
<td>3.23 (1.57)</td>
<td>4.33 (.99)</td>
</tr>
<tr>
<td>I know how to solve fractions problems.</td>
<td>3.07 (.98)</td>
<td>4.19 (1.17)</td>
<td>2.93 (1.20)</td>
<td>3.97 (1.07)</td>
</tr>
<tr>
<td>It is hard to understand fractions.</td>
<td>3.04 (1.53)</td>
<td>2.00 (1.41)</td>
<td>2.83 (1.53)</td>
<td>1.97 (1.22)</td>
</tr>
<tr>
<td>Number lines are helpful for learning about fractions.</td>
<td>2.64 (1.64)</td>
<td>2.50 (1.83)</td>
<td>3.37 (1.52)</td>
<td>3.81 (1.45)</td>
</tr>
<tr>
<td>Learning about fractions is a waste of time.</td>
<td>2.46 (1.67)</td>
<td>1.44 (.73)</td>
<td>2.67 (1.83)</td>
<td>1.70 (1.32)</td>
</tr>
<tr>
<td>The lessons helped me to learn about fractions.</td>
<td></td>
<td></td>
<td>4.47 (.86)</td>
<td></td>
</tr>
<tr>
<td>I enjoyed the lessons.</td>
<td></td>
<td></td>
<td>4.77 (.68)</td>
<td></td>
</tr>
<tr>
<td>I enjoyed working with the materials.</td>
<td></td>
<td></td>
<td>4.73 (.58)</td>
<td></td>
</tr>
<tr>
<td>The materials helped me to learn about fractions.</td>
<td></td>
<td></td>
<td>4.50 (.90)</td>
<td></td>
</tr>
<tr>
<td>I would recommend this program to other students.</td>
<td></td>
<td></td>
<td>4.70 (.92)</td>
<td></td>
</tr>
</tbody>
</table>

Note. Scale: 1 = I totally disagree; 2 = I disagree a little; 3 = Not sure; 4 = I agree a little; 5 = I totally agree. Standard deviations are in parentheses.
VI.  Figure 1

### Example CRA-I Instructional Sequence

- **C/R**
  - \( \frac{2}{3} = \frac{4}{6} \)
  - **Find the missing numerator to make an equivalent pair of fractions**
  - \( \frac{2}{3} \)
  - \( \frac{1}{3} \)
  - \( \frac{1}{3} \)
  - \( \frac{1/6}{6} \)
  - \( \frac{1/6}{6} \)
  - \( \frac{1/6}{6} \)
  - \( \frac{1/6}{6} \)

- **A**
  - \( \frac{2}{3} \cdot \frac{2}{2} = \frac{4}{6} \)
  - Two one-sixths are equal to \( \frac{1}{3} \), so four one-sixths are equal to \( \frac{2}{3} \)
  - \( \frac{2}{3} = \frac{4}{6} \)

*Figure 1.* Illustration of the process of finding a missing numerator in an equivalent pair of fractions using the CRA-I format.
VII. Figure 2

Figure 2. Representation of student estimates on each item on the 0-1 Number Line Estimation Test. Small black tie marks represent student estimates and large red tie marks represent the exact magnitude for each item.
VITA

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