MECHANISTIC MODEL FOR LEAD RUBBER BEARINGS

A Thesis in

Civil Engineering

by

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ABSTRACT

Seismic isolation is a technique used to shift the fundamental period of a structure to a long range period which reduces the forces a structure attracts during a seismic event. Two, widely used bearings in seismic base isolation of structures are elastomeric and lead rubber bearings. A typical elastomeric bearing consists of a number of layers of rubber alternated with steel shims bonded between two rubber layers. The addition of a lead core inserted in a central mandrel hole results in a lead rubber bearing (LRB). The lead enhances the bearings energy dissipating in an earthquake event. When elastomeric or LRBs are simultaneously subjected to vertical compressive load and increasing lateral displacement, the shear force equilibrium path can exhibit a critical point, beyond which the bearing exhibits negative stiffness. Semi-empirical models to simulate this behavior for elastomeric and LRB have been developed in the past. These models rely on experimentally calibrated parameters, making them impractical for design. Recently, a particular mechanics based model, developed for an elastomeric bearing only, approaches the modelling using vertical springs and a simple bi-linear constitutive relationship to represent the rotational behavior of an elastomeric bearing. Overarching goal of the present study is to build on the mechanics based elastomeric model to develop a LRB model. The elastomeric model is modified to include hysteretic behavior of LRB and also uses a Newton type numerical solution technique to solve for response of the bearing. The model, proposed in this study, utilizing vertical springs approach has shown to be capable of simulating the strength, stiffness and hysteretic behavior of LRB well.
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Chapter 1
INTRODUCTION

1.1 General

Base isolation is used in seismic resistance of structures by introducing flexibility to a structural system and, hence, shifting the structure’s fundamental natural period to a long period range, e.g., 3 to 5 seconds. This is typically achieved by placing isolators at the base of a structure. During an earthquake event this period shift translates into reduced floor acceleration and reduced inter-story drift demands on the superstructure compared to a conventional (non-isolated) structural system. For a design level earthquake event this means that the structure effectively remains elastic. The elastic behavior of the structure also means no ductile deformations or large deformation/displacements which in turn reduces possibility of structural and non-structural damage. This reduced demand in displacements, however, comes at the condition that the high displacement demand at the structure isolator interface can be taken care of. The advantage of the structure remaining elastic for a design level event makes base isolation one of the most efficient seismic resistance techniques.

Base isolation is done by means of bearings. Two most commonly used base isolation bearings are: a) friction pendulum and b) lead rubber. The focus of this research is on lead rubber bearing (LRB). An elastomeric bearing is low damping rubber bearing which consists of alternating layers of rubber and steel shims bonded together. Fig. 1-1 shows a photograph of a section cut, longitudinally, through the height of the bearing exposing the steel shims and rubber layers. The photograph also shows a hole in the middle of the bearing. If a lead core is plugs this hole it is a LRB it is an elastomeric bearing.
Numerous mathematical models (refer to Chapter 2 for details) have been proposed to simulate behavior of elastomeric and LRB subjected to simultaneous lateral displacement and vertical compressive load, which is a typical state of the bearing under a structure in an earthquake event. LRB models, so developed, are semi-empirical in nature and also lump the plasticity or the hysteretic element in an element that both translates and rotates, however, this does not accurately reflect the kinematics of lead rubber bearings in base isolated structures. Hence, due to these limitations, a mechanistic model development for lead rubber bearings is undertaken in the present study.

1.2 Problem Statement

The typical condition of an elastomeric bearing being subjected to vertical compressive load and increasing lateral displacement simultaneously leads to a state where the shear force can pass through a critical point (Fig. 1-2). The bearing, beyond this critical displacement and critical lateral force exhibits, negative tangential horizontal stiffness and
a condition of unstable equilibrium. This behavior can have important implications on the behavior of a building as well the stability of a bearing. The current study focuses on simulating this critical response of a LRB using a mechanics based model.

![Critical lateral force response of a bearing](image)

**Fig.1-2** Critical lateral force response of a bearing

Experiments have demonstrated this bearing response (Sanchez et al. 2014). Semi-empirical bearing models (Nagarajaiah and Ferrell 1999; Izuka 2000; Yamamoto et al. 2009; Kikuchi et al. 2010) have been shown to simulate the influence of compressive vertical load and lateral displacement on the lateral force response with reasonable accuracy. This approach gives good agreement with the experiment but is not of much use to a design engineer as it involves performing experiments on the bearing in order to calibrate empirical parameters that will enable design and analysis of a bearing.

A recent model proposed by Han and Warn (2014) to simulate the behavior of elastomeric bearings, which utilizes a set of parallel vertical springs and a bilinear constitutive relationship to represent the moment rotational relationship of elastomeric bearings, is adopted and further extended to simulate the behavior of lead-rubber bearings.
1.3 Research Objective

The primary objective of this study is to develop a model that will be able to simulate the lateral force-displacement behavior of LRBs. The approach adopted to achieve this objective is to utilize a mechanics based model proposed by Han and Warn (2014).

1.4 Tasks

1. The model proposed by Han and Warn (2014) employed with some modifications is re-derived and solved using a Newton type numerical scheme to obtain the equilibrium path solution.

2. A Bouc-wen hysteretic element is added to the Han and Warn bearing model to simulate the energy dissipated by the lead-core.

3. Simulations are performed with the re-derived elastomeric bearing model and new lead-rubber bearing model to generate data that is compared with experimental data from Sanchez et al. (2013) to assess the capabilities of the two models for replicating the monotonic force-displacement response under different vertical force levels.

4. Simulations and results are presented for cyclic loading on LRB.

1.5 Scope of Research

The scope of present research is limited to the bearing models being subjected to shear force, lateral displacement and moments. Boundary condition at the top of bearing is free to translate horizontally but restrained against rotation. Simulations of bearing behavior for cyclic and monotonic lateral displacement conditions on one type of elastomeric and LRB each (refer to section 5.1 for details of bearings) are performed.
1.6 Organization of thesis

This report contains six Chapters. A review and description of past models developed for bearings is presented in Chapter 2. Following which the development of Han and Warn (2014) elastomeric bearing model with some modifications is presented in Chapter 3. Chapter 4 contains the addition of Bouc-Wen element to the elastomeric bearing model and details the complete development of the LRB model. Results of lateral force response simulations of the bearing models (elastomeric and LRB) are evaluated against experimental response in Chapter 5. Chapter 6 presents the main conclusions from this study and also provides recommendations for future work. A list of references and an appendix is also included.
Chapter 2
BACKGROUND

2.1 Introduction

Fig. 2-1 (a) illustrates the behavior of the bearing under lateral and vertical compressive force, which is the subject of this research. When increasing lateral displacement is applied on the bearing while the axial load is constant, increasing moment demand is imposed on the bearing. The stability (critical) point in the bearing is reached when the bearing cannot further resist this additional moment demand imposed on it as it has reached its moment capacity at a particular critical displacement. There are important consequences of this behavior on the earthquake response of elastomeric and lead rubber bearings and, hence, on the response of base isolated structures. For response history analysis, it is desired that the bearing model be capable of simulating the reduction in stiffness with increasing displacement and increasing lateral displacement. From a design perspective, the displacement at which the stiffness reaches zero, the critical point, is considered the stability limit of the bearing and, therefore, estimation of $u_{cr}$, critical lateral displacement (Fig. 2-1) is important. For example, for each critical axial load level a corresponding critical displacement is identified forming a curve (Fig. 2-1b) representing critical load as the ordinate and critical displacement as the abscissa. However, a number of semi-empirical models exist that attempt to simulate the force-displacement behavior of these bearings and capture the critical points that are described in the following section.
2.2 Two-spring model

The two-spring model (Fig. 2-2) of Koh and Kelly (1988) was based on linear elastic force deformation relationships for shear and rotational springs. This model was shown to simulate the behavior well, provided the force and displacement did not produce any material non-linearity. The Koh and Kelly (1988) model aptly simulated a decrease in bearing height with increasing lateral displacement with reasonable accuracy when compared to experimental tests, but, was unable to capture the critical behavior of reduction in horizontal stiffness (Warn and Weisman 2010) shown in Fig. 2-1a.
2.3 Semi-empirical models

One way to model the behavior of a bearing is to estimate the critical point \((u_{cr}, P_{cr})\) and another is to estimate the lateral force displacement response. Buckle and Liu (1994) proposed a method known as the reduced area method to estimate the critical load capacity, however, Han et al. (2013) have shown the method to be overly conservative and also that it lacks a rigorous theoretical basis. Other models that estimate the lateral force response of elastomeric and lead rubber bearings are proposed by Nagarajaiah and Ferrell (1999), Iizuka (2000), Yamamoto et al. (2009) and Kikuchi et al. (2010).

Nagarajaiah and Ferrell (1999) built upon the Koh and Kelly model by introducing nonlinearity in the springs, but, the formulations for the nonlinear relationship were semi-empirical and importantly utilized rotational springs. Although Nagarajaiah and Ferrell model was able to capture a reduction in critical load with increase in critical displacement, comparison of the exact response of model against experimental results showed that the model was unable to simulate the results to a high degree of accuracy.

Iizuka (2000) also proposed a semi-empirical model that included the interaction between normal and bending stresses through an interaction equation accounting for the nonlinear behavior of the moment-rotational relationship. The Izuka model was able to circumvent the problem of accurate response.

Yamamoto et al. (2009) introduced a mechanics based bearing model with vertical springs to simulate the moment rotation relationship. The lead in the Yamamoto model was lumped into a shear spring that both translated and rotated with lateral displacement. However the present study models the hysteretic lead element outside the rotating shear
spring and in parallel to the elastomeric. Kikuchi et al. (2010) extended the Yamamoto model to bi-directional for 3D analysis.

All semi-empirical models described here involve experimental calibration of parameters to simulate accurate non-linear behavior. Therefore are of not much use to a design engineer.

2.4 Elastomeric Mechanistic Model

A global sensitivity analysis of the Iizuka model (2000) was performed by Han et al. (2013) with the intention to identify the parameters to which the critical displacement was most sensitive to inform a revised model.

Fig. 2-3. Illustration of elastomeric bearing model (Han and Warn, 2014)
It was found that the model was highly sensitive to an empirical parameter $r$ which governs the non-linear moment-rotation relationship of the bearing. This result of sensitivity analysis agreed with the expectation that the lateral stiffness is highly dependent on rotational stiffness of the bearing. Based on these results, Han and Warn (2014) proposed a mechanics based model that accounted for the rotational stiffness of the bearing and interaction between normal and bending stresses using a distributed system of parallel vertical springs as illustrated in Fig.2-3. The Han and Warn model is a mechanistic model relying only on the geometric and material properties of the bearing while simulating the force displacement of the bearing with reasonable accuracy, including capturing the reduction in critical displacement with increasing vertical force.

2.5 Experimental response

Sanchez et al. (2013) tested bearings using three methods. Method 1 and Method 2 are quasi-static while Method 3 is a dynamic experimental method. Method 2 and its results are utilized in the present study to evaluate accuracy of proposed models. In Method 2, after a preset axial load was applied to the bearing, a monotonically increasing lateral displacement was applied until the bearing reaches the stability point $(u_{cr}, F_{cr})$. This was conducted for a range of axial load levels. Elastomeric as well as LRB specimens were tested using Method 2. Bearing ID 11795 & 11808 are designated as type 1 (elastomeric) and Bearing ID 11772, 11783 & 11792 as type 2 (LRB) by Sanchez et al. (2013). Details of bearings are presented in Table 3-1 (Elastomeric-Type 1) and Table 4-1 (LRB-Type 2). Further, the experimental lateral force response showing equilibrium paths for elastomeric and LRB are presented in the Chapter 6, which includes a comparison of the model simulated response with the experiments.
2.6 Summary

The dependence of semi-empirical models on parameters that must be experimentally calibrated to analyze the bearing renders the model impractical for design of seismic isolation systems. As these empirical parameters might vary from bearing to bearing, experiments would need to be conducted on the bearing or on its scaled model to determine these parameters and ultimately to design the bearing and the structure above it.

The simulation capability of the mechanistic model proposed by Han and Warn (2014) was reasonably accurate, however, no mechanistic model for LRB exists. The Han and Warn model is adopted as the basis of the present study. Incorporating lead in this mechanistic elastomeric model to simulate the behavior of a LRB is the next step and the subject of this study. Further the experimental tests performed by Sanchez et al. (2013) are utilized in this study to evaluate the accuracy of simulated response from the mechanistic models.
Chapter 3
MECHANISTIC MODEL FOR ELASTOMERIC BEARING

3.1 Introduction

The mechanistic model for elastomeric bearing of Han and Warn model (2014) is revisited here, including relevant modifications, derivations and a new solution procedure, for the benefit of discussing its extension to lead-rubber bearings. In order to better represent the deformed shape of the bearing, as illustrated in Fig. 3-1, Han and Warn modified the original Koh and Kelly (1988) two spring model by replacing the rotational spring with a set of parallel vertical springs. This model is further modified, here, to consist of two sets of parallel vertical springs, rather than one set of parallel vertical springs, to simulate interaction of normal and bending stress and the nonlinear moment rotational behavior of the bearing as illustrated in Fig. 3-2 (c).

Fig. 3-1 Un-deformed and deformed shape of elastomeric bearing
To simulate the force-displacement response of elastomeric, and later LRB, the equilibrium and compatibility equations of the bearing in the deformed shape are derived. This results in a system of non-linear equations to be solved in order to obtain the state of the bearing \((F, s, v, \theta)\) for a given set of input \((P, u)\). A detailed derivation of the equilibrium and compatibility equations can be found in the following section. Section 3.2 provides a description of the solution technique used to solve the system of non-linear equations.

![Elastomeric Bearing models](image)

(a) Koh and Kelly (1988) (b) Han and Warn (2014) (c) Modified Han and Warn model

One of the motivations for the distributed springs in the Han and Warn (2014) model was to systematically capture the interaction of normal and bending stresses in the individual rubber layers that could not be directly captured with a concentrated rotational spring. Of course, finer discretization of the bearing into vertical distributed springs would
mean better accuracy in the response of the bearing which is based on the state of
stress/force in a particular spring and therefore a convergence analysis is recommended to
determine the minimum number of spring required.

3.2 Equilibrium and Compatibility equations

From the free body diagram of the bearing in the deformed configuration of the
(Fig. 3-3c), also accounting for geometric non-linearity, the equilibrium and compatibility

\[ F \cos \theta + P \sin \theta - F_s = 0 \]  \hspace{1cm} (3.1)

\[ F (h' - v) + P u - 2M = 0 \]  \hspace{1cm} (3.2)
\[ s \cos \theta + h' \sin \theta - u = 0 \quad (3.3) \]
\[ h'(\cos \theta - 1) - s \sin \theta + v = 0 \quad (3.4) \]

Where \( F \) is the total lateral shear force in the bearing, \( u \) is the total lateral deformation of the bearing, \( P \) is the vertical force on the bearing, \( \theta \) is the inclination/rotation of the bearing model, \( v \) is the reduction in height of the bearing due to rotation and shear deformation, \( F_s \) is the internal shear spring force, \( M \) is the moment in the bearing (or moment of resistance of one set of parallel springs), \( s \) is the shear deformation of the shear spring and \( h' \) is the height of the bearing after reducing the vertical deformation due to axial load is given as:

\[ h' = h - v_p \quad (3.5) \]

Where \( h \) is the initial height of bearing, \( v_p \) is the vertical deformation due to vertical force which is given as:

\[ v_p = \frac{Pl}{A_b E_c} \quad (3.6) \]

Where \( A_b \) is the area of the bearing cross-section, \( E_c \) is the compression modulus of the bearing (Constantinou et al. 1992) give as:

\[ E_c = 6G_oS^2F \quad (3.7) \]

Where \( G_o \) is the initial shear modulus of the bearing, \( S \) is the shape factor the bearing and \( F \) is a modification for annular shaped bearings. (Constantinou et al. 1992)

Certain parameters are characteristic of the physical bearing properties and presented in Table 3-1.
Table 3-1 Details of annular shaped elastomeric bearing used in this study

<table>
<thead>
<tr>
<th>Description</th>
<th>Symbol</th>
<th>Unit</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Shear Modulus</td>
<td>$G_o$</td>
<td>Mpa</td>
<td>1.08</td>
</tr>
<tr>
<td>Outer Diameter</td>
<td>$D_o$</td>
<td>mm</td>
<td>152</td>
</tr>
<tr>
<td>Inner Diameter</td>
<td>$D_i$</td>
<td>mm</td>
<td>30</td>
</tr>
<tr>
<td>Single rubber layer thickness</td>
<td>$t_r$</td>
<td>mm</td>
<td>3</td>
</tr>
<tr>
<td>Single shim layer thickness</td>
<td>$t_s$</td>
<td>mm</td>
<td>3</td>
</tr>
<tr>
<td>Number of rubber layers</td>
<td>$N$</td>
<td>-</td>
<td>20</td>
</tr>
<tr>
<td>Shape factor</td>
<td>$S$</td>
<td>-</td>
<td>10.2</td>
</tr>
<tr>
<td>Geometry coefficient</td>
<td>$F$</td>
<td>-</td>
<td>0.69</td>
</tr>
</tbody>
</table>

Equations (3.1 through 3.4) developed are non-linear in nature and will be referred to as the system of non-linear equations henceforth. The system of non-linear equations is solved for unknowns $F, s, \theta, and v$ when $P and u$ are given using the Newton-Raphson method which is described in the following section.

3.3 Newton-Raphson Method

The Newton-Raphson method is a root finding method where the procedure is to iteratively determine roots of a system of non-linear equations by updating the estimate of the roots (unknowns). To derive the Newton Raphson formula (Eq. 3.9) consider $Q$, a non-linear function of $x$:

$$Q(x) = 0$$ (3.8)

The aim of the problem is to determine roots of Eq. (3.8). As shown in Fig. 3-4, starting with an initial estimate of the root ($x_i$), a tangent is drawn to the function curve at the point [$x_i, Q(x_i)$]. The point where the tangent intersects the $x$-axis represents an improved estimate of the root.
Fig. 3–4 Geometric illustration of Newton Raphson method

Based on geometry in Fig. 3–4:

\[ x_{i+1} = x_i - \frac{Q(x_i)}{Q'(x_i)} \]  \hspace{1cm} (3.9)

Eq. 3.9 is known as the Newton Raphson equation that can be extended to a system of non-linear equations. Equations 3.1 through 3.4 are the system of non-linear equations arranged in a vector as \( \{Q_1, Q_2, Q_3, Q_4\} \). Recognizing that \( F, s, \theta, \& v \) are unknowns and \( P \& u \) are known for an analysis step, by analogy to Eq. (3.9) the following can be written for the system of non-linear equations:

\[ 0 = Q_{1,i} + \frac{\partial Q_{1,i}}{\partial F}(F_{i+1} - F_i) + \frac{\partial Q_{1,i}}{\partial s}(s_{i+1} - s_i) + \frac{\partial Q_{1,i}}{\partial \theta}(\theta_{i+1} - \theta_i) + \frac{\partial Q_{1,i}}{\partial v}(v_{i+1} - v_i) \]  \hspace{1cm} (3.10)
\[0 = Q_{2,i} + \frac{\partial Q_{2,i}}{\partial F} (F_{i+1} - F_i) + \frac{\partial Q_{2,i}}{\partial s} (s_{i+1} - s_i) + \frac{\partial Q_{2,i}}{\partial \theta} (\theta_{i+1} - \theta_i) + \frac{\partial Q_{2,i}}{\partial v} (v_{i+1} - v_i)\]  
(3.11)

\[0 = Q_{3,i} + \frac{\partial Q_{3,i}}{\partial F} (F_{i+1} - F_i) + \frac{\partial Q_{3,i}}{\partial s} (s_{i+1} - s_i) + \frac{\partial Q_{3,i}}{\partial \theta} (\theta_{i+1} - \theta_i) + \frac{\partial Q_{3,i}}{\partial v} (v_{i+1} - v_i)\]  
(3.12)

\[0 = Q_{4,i} + \frac{\partial Q_{4,i}}{\partial F} (F_{i+1} - F_i) + \frac{\partial Q_{4,i}}{\partial s} (s_{i+1} - s_i) + \frac{\partial Q_{4,i}}{\partial \theta} (\theta_{i+1} - \theta_i) + \frac{\partial Q_{4,i}}{\partial v} (v_{i+1} - v_i)\]  
(3.13)

By rearranging we can compactly write the following:

\[
\{X_{i+1}\} = -[Z_i]^{-1}\{Q_i\} + \{X_i\} \quad (3.14)
\]

Where, \(\{X_{i+1}\} = \begin{bmatrix} F_{i+1} \\ s_{i+1} \\ \theta_{i+1} \\ v_{i+1} \end{bmatrix}\), is the updated estimate of roots, \(\{X_i\} = \begin{bmatrix} F_i \\ s_i \\ \theta_i \\ v_i \end{bmatrix}\), is the previous estimate of unknowns, \(Q_i = \begin{bmatrix} Q_{1,i} \\ Q_{2,i} \\ Q_{3,i} \\ Q_{4,i} \end{bmatrix}\), where \(Q_{ni}\) is the value of nth equation (3.1 through 3.4) evaluated at \(i\)th iteration and \(Z_i = \begin{bmatrix} \frac{\partial Q_{1,i}}{\partial F} & \frac{\partial Q_{1,i}}{\partial s} & \frac{\partial Q_{1,i}}{\partial \theta} & \frac{\partial Q_{1,i}}{\partial v} \\ \frac{\partial Q_{2,i}}{\partial F} & \frac{\partial Q_{2,i}}{\partial s} & \frac{\partial Q_{2,i}}{\partial \theta} & \frac{\partial Q_{2,i}}{\partial v} \\ \frac{\partial Q_{3,i}}{\partial F} & \frac{\partial Q_{3,i}}{\partial s} & \frac{\partial Q_{3,i}}{\partial \theta} & \frac{\partial Q_{3,i}}{\partial v} \\ \frac{\partial Q_{4,i}}{\partial F} & \frac{\partial Q_{4,i}}{\partial s} & \frac{\partial Q_{4,i}}{\partial \theta} & \frac{\partial Q_{4,i}}{\partial v} \end{bmatrix}\) is the Jacobian evaluated at \(i\)th iteration.

The Jacobian of equations (3.1 through 3.4) after evaluation, results in the following:
\[
Z_i = \begin{bmatrix}
\cos \theta_i & \left(-\frac{dF_s}{ds}\right)_i & -F_i \sin \theta_i + P \cos \theta_i & 0 \\
h' - v_i & \left(-2 \frac{dM}{d\theta}\right)_i & -F_i \\
0 & \cos \theta_i & -s_i \sin \theta_i + h' \cos \theta_i & 0 \\
0 & -\sin \theta_i & -s_i \cos \theta_i - h' \sin \theta_i & 1 \\
\end{bmatrix}
\tag{3.15}
\]

To calculate an updated estimate \(X_{i+1}\) from Eq. (3.14), \(Z_i\) (Jacobian Matrix Eq. 3.15) and vector \(Q_i\) must be known. As values of \(X_i\) are known from the previous iteration, all elements of matrix \(Z_i\) and \(Q_i\) can be calculated except for \(\left(\frac{dF_s}{ds}\right)_i\), \((F_s)_i\), \(\left(\frac{dM}{d\theta}\right)_i\) and \(M_i\). Evaluation of \(M_i\) and \(\left(\frac{dM}{d\theta}\right)_i\) are dependent on vertical force, \(P\) and rotation, \(\theta_i\). This evaluation requires for characterizing the moment-rotation relationship of a bearing; a subroutine is developed to determine \(M_i\) and \(\left(\frac{dM}{d\theta}\right)_i\) for details see section 3.3. The remaining terms, \(\left(\frac{dF_s}{ds}\right)_i\) and \((F_s)_i\), are evaluated from the following equations:

\[
(F_s)_i = \frac{dF_s}{ds}.s_i
\tag{3.16}
\]

where, \(s_i\) is an element of \(\{X_i\}\), which is the vector of previous estimate and

\[
\frac{dF_s}{ds} = \frac{G(u)A_b}{T_r}
\tag{3.17}
\]

Eq. (3.17) is the shear stiffness of the bearing and is independent of previous estimate, \(\{X_i\}\) where, \(G(u) = G_o \left[ 1 - 0.325 \tanh \frac{u}{r_p} \right]\), is the non-linear relationship proposed by Nagarajaiah and Ferrell (1999) and \(T_r\) is the total thickness of the rubber layers.

After each iteration, that is, obtaining an updated estimate of unknowns \(X_{i+1}\) using Eq. (3.14), a percentage error is calculated based on the difference between each element of vector \(X_i\) and \(X_{i+1}\) to ascertain if the converged solution is achieved. If the
maximum of all calculated percentage errors meets a user specified tolerance criteria, the estimate of unknowns, \( X_{t+1} \), can be termed as the converged solution for this analysis step. Each set of \( P \) and \( u \) is an analysis step and requires iterations to reach a converged solution. Newton-Raphson method relies on a good initial estimate of unknowns to start iterations in order to get the converged solution at a particular analysis step. At a constant axial load level the first analysis step always starts with \( u = 0 \text{mm} \). The initial estimate (1\(^{st}\) iteration) for this analysis step is input by the user. \( \{X_i\} = 0 \), is used as the initial guess (estimate) for the first analysis step, which in our case is also the converged estimate as this is practically observed. For next analysis step and steps subsequently (with \( u > 0 \text{mm} \)) converged solution from previous step are used as the initial estimate to iterate to a converged solution. This procedure is then repeated for various axial load levels to get the complete response of an elastomeric bearing. Following is an algorithm used for the Newton Raphson method:

**Output**
- Shear force in bearing, \( F \)
- Shear displacement, \( s \)
- Rotation, \( \theta \)
- Vertical displacement, \( v \)

**Input**
- Height of bearing, \( H \)
- Total thickness of rubber, \( Tr \)
- Cross-section area of bearing, \( Ab \)
Elastic stiffness of lead, $K_l$
Ratio of inelastic to elastic stiffness, $\alpha$
Yield displacement of lead, $u_y$
Bouc wen hysteretic parameters $(n, \beta, \gamma)$
Permissible error for Newton Raphson method, $e_a$
Maximum permitted iterations for Newton Raphson, $it_{max}$
No of vertical springs in one set of parallel springs, $n_{o}$

**Procedure**

\[ j = 1 \]
\[ \text{for } P = 0, 89, 178, 267 \text{ & } 356 \]
\[ v_p = (P \cdot h)/(Ab \cdot Ec) \]
\[ h' = h - v_p \]
\[ i = 1 \]
\[ \text{for } u = 0 \text{ to } 150 \text{ in increments of } 1 \]
\[ \text{if } u=0; \]
\[ \Delta u = 0 \]
\[ z_0 = 0 \]
\[ \{Xc\}_i = \{0 0 0 0\}' \]
\[ \text{else} \]
\[ \Delta u = u_i - u_{(i-1)}; \]
\[ z_0 = z_{(i-1)}; \]
\[ \{Xc\}_i = \{Xc\}_i-1 \]
\[ \text{endif} \]
\[ ea = 100 \]
\[ it = 2 \]
\[ \{dX\}_1 = \{0 0 0 0\}' \]
\[ \{X\} = \{Xc\}_1 \]
\[ \text{sum} = 0 \]
\[ \text{While } it < it_{max}-1 \text{ & } e_a > ea \]
\[ \text{for } m = 1 \text{ to } it-1 \text{ in increments of } 1 \]
\[ \text{sum} = \text{sum} + dX_m; \]
\[ \text{end} \]
\[ X = X + \text{sum} , \text{ estimate of unknowns (F, s, \theta, v)} \]
\[ [M, dM/d\theta ] = \text{function(\theta, P, u, no.)} \]
\[ [Fs, dFs/ds ] = \text{function(s, u, Tr, Ab)} \]
Evaluate the Jacobian Matrix for estimated unknowns, $[Z]$
Evaluate the system of non-linear equations at estimated unknowns in a vector, $\{Q\}$
\[ dX_{it} = -[Z]^{-1}\{Q\} \]
\[ \{X\} = \{X\} + \{dX_{it}\} \]
\[ eaF = |dX_{it}(1)/(X(1))|*100 \]
\[ eas = |dX_{it}(2)/(X(2))|*100 \]
\[ eaa = |dX_{it}(3)/(X(3))|*100 \]
\[ eav = |dX_{it}(4)/(X(4))|*100 \]
\[ ea = \text{maximum}(eaF, eas, eaa, eav) \]
\[ \{X\} = \{Xc\}_i \]
\[ it = it + 1 \]
\[ \text{sum} = \{0 0 0 0\}' \]
\[ \text{end While} \]
\[ \{Xc\}_i = \{Xc\}_i + \text{sum}(dX_1, dX_2, ..., dX_{it}) \]
clear dX
lateraldisp(i,j) = u
Return Force(i,j) = $Xc_i(1)$
sheardisp(i,j) = $Xc_i(2)$
theta(i,j) = $Xc_i(3)$
vertdisp(i,j) = $Xc_i(4)$
The same Newton Raphson algorithm is used to analyze a LRB including apt modifications, discussed in Chapter 4, pertaining to the introduction of lead.

3.4 Moment-rotation relationship

The requirement to know apriori the moment of resistance, $M$, and the rotational stiffness, $dM/d\theta$, exhibited by the bearing for a particular inclination, $\theta$, and axial load, $P$, at the $i^{th}$ iteration necessitates development of moment-rotation relationship. To develop this relationship, analysis at a bearing crosssection, which is modeled as two sets of parallel vertical springs, must be performed. The inputs for this evaluation are the rotation ($\theta$), axial load ($P$) and no. of springs in one set of distributed vertical springs ($n$) and the output is moment of resistance, $M$, and rotational stiffness, $dM/d\theta$.

![Constitutive relationship for vertical springs](image)

Fig. 3-5 Constitutive relationship for vertical springs (Han and Warn 2014)

The bearing crosssection fibers are represented by vertical springs as presented in Fig. 2-3, and carries force accordingly. Each vertical spring has a bi-linear constitutive
relationship in tension and a linear relationship in compression as presented in Fig. 3-5. When the bearing is subjected to axial load and rotation, stress variation over a fiber is neglected, this is a reasonable assumption if the model is well decritized. Discretizing the bearing into 20 vertical springs in one parallel spring set is used in the current study. Basic bending assumption of plane sections remaining plane before and after bending is valid. This results in a linear distribution of strains as shown in Fig. 3-6:

![Strain Distribution Diagram](image)

**Fig. 3-6 Strain distribution and sign convention**

\[ \varepsilon_j = \frac{(x - y_j)}{l_s} \theta \]  

(3.18)

where \( \varepsilon_j \) is strain in \( j^{th} \) spring (tension is regarded as positive), \( x \) is the distance of the neutral axis from the center of the bearing, \( y_j \) is the distance of the \( j^{th} \) spring from the center of the bearing, \( l_s \) is the initial length of the springs and \( \theta \) is angle of inclination.

To calculate strains in springs from Eq. (3.18) at the crosssection for a particular \( \theta \) and \( P \), the intial length of springs, \( l_s \), and depth of neutral aixs, \( x \), are required. The length
of springs, \( l_s \), is calculated by setting the elastic rotational stiffness of the bearing equal to
the rotation stiffness offered by the two sets of parallel vertical springs in the model. From
eq (3.19):

\[
k_{e\theta} = \frac{\pi^2 E_b l}{T_r} = 2 \frac{E_c \sum A_j y_j^2}{l_s}
\]

(3.19)

where \( k_{e\theta} \) is the elastic rotational stiffness of the bearing, \( T_r \) is the total thickness of rubber
bearing, \( E_b \) is the bending modulus of the bearing and is generally equal to \( E_c/3 \) (Kelly
1997), \( A_j \) is the area of the \( j \)th spring and \( I \) is the second moment of area of the bearing
cross-section.

The elastic neutral axis depth from the bearing center line can be calculated by
equilibrium of forces over the cross-section, which results in the following formula:

\[
x = \frac{P l_s}{A_b E_c \theta}
\]

(3.20)

Eq. (3.20) is only valid when none of the springs have yielded. However, when the rotation
is high enough to yield one or more springs, the depth to the neutral axis will need to be
ascertained by iterating and using the essential condition of equilibrium of forces at the
cross-section.

After strains are obtained, the stresses in the individual spring elements can be
evaluated as follows using the bi-linear stress-strain relationship presented in Fig. 3-6.

\[
\sigma_j = E_c \varepsilon_j, \quad \text{if } \varepsilon_j < \frac{\sigma_y}{E_c}
\]

(3.21a)

\[
\sigma_j = \sigma_y, \quad \text{if } \varepsilon_j \geq \frac{\sigma_y}{E_c}
\]

(3.21b)
where, \( \sigma_j \) is stress in \( j^{th} \) spring, \( E_c \) is the compression modulus of the bearing (see section 3.1 for definition) and \( \sigma_y \) is the yield stress, assumed equal to \( 3G_o \) (Gent 2001).

Having obtained the stresses in each spring for a particular inclination and axial load (Eq. 3.21) the moment of resistance offered by the cross-section, discretized into \( n \) no. of springs, can be calculated from the following:

\[
M = \sum_{j=1}^{n} \sigma_j A_j y_j = \sum_{j=1}^{n} k_j (x - y_j) \theta y_j
\]  

(3.22)

where \( k_j \) is the individual spring stiffness:

\[
k_j = \frac{E_c A_j}{l_s}, \quad \text{if} \ \varepsilon_j < \frac{\sigma_y}{E_c} \hspace{1cm} (3.23a)
\]

\[
k_j = 0, \quad \text{if} \ \varepsilon_j \geq \frac{\sigma_y}{E_c} \hspace{1cm} (3.23b)
\]

Moreover, Eq. (3.22) is differentiated with respect to \( \theta \) to obtain rotational stiffness of the vertical spring model.

\[
\frac{dM}{d\theta} = \sum_{j=1}^{n} k_j (x - y_j) y_j
\]  

(3.24)

Where \( \frac{dM}{d\theta} \) is the rotational stiffness of the bearing.

The Moment-rotation relationship evaluation is a sub-model to the global bearing model. In the global scheme (Newton-Raphson) a function is called upon at the \( i^{th} \) iteration to determine moment of resistance, \( M_i \), and rotational stiffness, \( \frac{dM}{d\theta} \), with inputs as \( \theta_i \), \( P \) and \( n \). Following is an algorithm used to model moment-rotation relationship described above:
Output
Moment of resistance offered by the bearing, M
Rotational stiffness offered by the bearing, K

Input
Rotation of bearing, θ (rad)
Axial load on bearing, P (kN)
No. of springs in one set of parallel vertical springs, n
Outer Diameter of bearing, Do
Inner Diameter of bearing, Di
Single layer rubber thickness, tr
Number of rubber layers, nos
Initial Shear Modulus of bearing, Go
Yield stress of bearing, σy
Elastic compression modulus, Ec

Procedure
Calculate discretized areas of bearing cross-section represented by vertical springs (A1, A2, A3 . . . An)
Calculate distance of each spring from bearing center line (y1, y2...yn)
Ab = A1 +A2 + ...An
Tr = nos x tr
I = Π*(Do^4 - Di^4)/32
Eb = Ec /3
Ke = Π^2*Eb*I/Tr
sy = Ec*σy
ls = (2*Ec*sum(An*y_n^2))/ke
if θ = 0
M = 0
Return M
K = Ke;
Return K
else
x = (P*ls)/(Ab*Ec* θ), depth of Elastic N.A.
Calculate strains (ε1, ε2.. εn) in individual springs
Calculate stresses (σ1, σ2.. σn) individual springs
if ε1 > sy,
Calculate total axial force, F, in the springs from stresses.
while absolute(F-P) > 0.05 kN
x = x + 0.001 (increment the depth of NA)
Calculate strains in springs (ε1, ε2..εn) based on new x
Calculate stresses in springs (σ1, σ2.. σn) based on new x
Reevaluating total new spring force, F based on new x
end while
end if
m=0
j=1
for A = A1 to An
m = m + σj*.A*yj
j=j+1
end for
The moment-rotation evaluation of the bearing can be performed separately from the global Newton-Raphson scheme for arbitrary input values of $\theta$ and $P$ while using $n = 20$ to get the non-linear moment rotation relationship as presented in Fig. 3-7. As expected, elastic rotational stiffness, $k_{e\theta}$, in the elastic range and beyond the elastic range when some springs have yielded, decrease in rotational stiffness is observed in Fig. 3-7.

![Fig. 3-7 Moment rotation relationship](image-url)
Chapter 4
MECHANISTIC MODEL FOR LEAD RUBBER BEARING

4.1 Introduction
The model development for elastomeric bearing described in Chapter 3 is extended to LRB. The presence of lead at the central core of the bearing will significantly enhance the energy dissipation in comparison to an elastomeric bearing and, as such, the model presented in Chapter 3 must be modified to include an appropriate energy dissipation mechanism.

As observed from the deformed shape in Fig. 4-1b, the lead is assumed to undergo only shear deformation. The shear deformation in the lead is equal to the total lateral displacement \( u \) of the bearing. LRB models from the past (Yamamoto et al. 2008) have modeled the lead by incorporating a hysteretic element in the bearing model shear spring.
where the shear spring (see Fig. 4-2) both translates and rotates with lateral displacement. However, as suggested by the illustration in Fig. 4-1, the lead core undergoes a shear deformation that is without rotation. Hence, it is logical to introduce a hysteretic element that is consistent with the deformation of the system, \( u \), but does not undergo rotation as observed from Fig. 4-2. For the present study the hysteretic element is modeled as a Bouc-Wen formulation.

![Fig. 4-2 LRB mechanistic model](image)

(a) Undeformed (b) Deformed under \( P \) and \( u \)

It is important to characterize the behavior of the lead core in an annular shaped bearing to be able to incorporate the constitutive relationship of lead, modelled here as a hysteretic element. The Bouc-Wen formulation for the hysteretic element is described in section 4.1.
4.2 Lead Model

The lead core is under compression and shear. The behavior of lead in the core is also dependent on the thermodynamic behavior of lead, this complexity is beyond the scope of this study. Without a rigorous thermodynamic description of the constitutive relationship for lead, the widely used Bouc-Wen hysteretic formulation is adopted for modeling the energy dissipated by the lead. Past research efforts (Nagarajaiah et al 1991) have modeled seismic isolation bearings, both lead rubber and friction bearings, using a Bouc-Wen hysteretic formulation. However, in these past studies the Bouc-Wen formulation was used to model the global behavior of the bearing not just the lead and as such these models could not capture the critical path and dependency of the critical point on vertical compressive load as experimentally observed.

The Bouc-Wen formulation, used in the present study, was introduced by Bouc (1967) and modified by Wen (1976). The following equations represent the Bouc-Wen hysteretic formulation as relevant to the hysteretic element in the proposed LRB model:

\[ F_L = a ku + (1 - \alpha) k z \]  

where, \( F_L \) is the shear force in lead, \( \alpha \) is the ratio of inelastic to elastic stiffness, \( k \) is the elastic lateral stiffness of lead, \( z \) is a parameter that controls the inelastic force contribution and is calculated from the solution of the following differential equation:

\[ \dot{z} = \dot{u} \left[ 1 - \frac{z}{u_y}^n \times (\beta + \gamma \cdot sgn(\dot{z})) \right] \]  

(4.2a)

Where, \( \dot{z} = \frac{dz}{dt} \) and \( \dot{u} = \frac{du}{dt} \), \( u_y \) is the yield displacement of lead in shear, \( \beta \) & \( \gamma \) are parameters that govern the shape of the hysteretic behavior, \( n \) is a parameter that governs the transition from elastic to inelastic shear deformation.
Eq. (4.2a) involves shear displacement of lead, $u$, as a time varying input but a time varying input is not necessarily needed as the present scope is limited to monotonic increase of lateral displacement. Eliminating $u$ as a function of time in Eq. (4.2a) by writing a Taylor series expansion of Eq. (4.2a) the following algebraic equation is obtained:

$$z = z_0 + \Delta u \left[ 1 - \left( \frac{z}{u_y} \right)^n \times (\beta + \gamma \cdot sgn(\Delta u)) \right]$$

(4.2b)

Eq. (4.2b) is a non-linear equation that can be integrated with the Newton-Raphson solver (from the elastomeric bearing model presented in Chapter 3). At this juncture a simulation is performed for the lead hysteretic element (Fig. 4-3), without the bearing model, for increasing lateral displacement, $u$, using Newton-Raphson method at each analysis step to determine $z$ from Eq. (4.2b).

Fig. 4-3 Illustration of the hysteretic relationship for the Bouc-Wen element
Each analysis step treats $\Delta u$ and $z_0$ (the value of $z$ from previous step) as constants. Furthermore, Eq. (4.1) can be employed at each analysis step to determine lead shear force, $F_L$. $\beta = 0.1$, $\gamma = 0.9$ and $n = 2$, as suggested by Nagarajiah et al. (1991), are used as hysteretic parameters. $u_y = 12.5 \text{ mm}$, $\alpha = 1\%$, and $k = 0.4 \text{kN/mm}$ are adopted as the parameters of lead core. For justification of values of these property parameters see section 5.3.1.

As the objective of this study is to develop a LRB model, the hysteretic element is then modeled in conjunction with the elastomeric bearing to produce a mechanistic model for LRB. The equilibrium and compatibility equations for the LRB model derived from the mechanics of the bearing with the hysteretic element are detailed in the following section.

4.3 Equilibrium and Compatibility equations

The introduction of a lead core requires the inclusion of an additional element to account for the energy dissipation of the LRB by comparison to the elastomeric bearing. From Fig. 4-2 and Eq. (4.1) and (4.2b), equilibrium and compatibility equations for the LRB model can be written as follows:

$$F \cos \theta + P \sin \theta - F_s - F_L \cos \theta = 0 \quad (4.3)$$

$$F(h' - v) + P u - 2M = 0 \quad (4.4)$$

$$s \cos \theta + h' \sin \theta - u = 0 \quad (4.5)$$

$$h'(\cos \theta - 1) - s \sin \theta + v = 0 \quad (4.6)$$

$$\alpha k u + (1 - \alpha)kz - F_L = 0 \quad (4.6)$$

$$z_0 + \Delta u \left[1 - \frac{z}{u_y} \right]^n \times (\beta + \gamma \text{sgn}(\Delta u)) - z = 0 \quad (4.7)$$
Equations (4.3) through (4.7) are similar to those obtained for the elastomeric model (section 3.2). For details of notations in Eq. (4.3) through (4.7) refer to section 3.2 and section 4.1. The LRB used in this study is the same as the elastomeric bearing (Table 3-1) with the exception of a lead plug in the central core.

Table 4-1 Details of annular shaped LRB

<table>
<thead>
<tr>
<th>Description</th>
<th>Symbol</th>
<th>Units</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Shear Modulus</td>
<td>$G_o$</td>
<td>Mpa</td>
<td>1.11</td>
</tr>
<tr>
<td>Outer Diameter</td>
<td>$D_o$</td>
<td>mm</td>
<td>152</td>
</tr>
<tr>
<td>Inner Diameter</td>
<td>$D_i$</td>
<td>mm</td>
<td>30</td>
</tr>
<tr>
<td>Single rubber layer thickness</td>
<td>$t_r$</td>
<td>mm</td>
<td>3</td>
</tr>
<tr>
<td>Single shim layer thickness</td>
<td>$t_s$</td>
<td>mm</td>
<td>3</td>
</tr>
<tr>
<td>Number of rubber layers</td>
<td>$N$</td>
<td>-</td>
<td>20</td>
</tr>
<tr>
<td>Shape factor</td>
<td>$S$</td>
<td>-</td>
<td>12.17</td>
</tr>
<tr>
<td>Geometry coefficient</td>
<td>$F$</td>
<td>-</td>
<td>0.69</td>
</tr>
</tbody>
</table>
Note: Compression modulus, $E_c$, is not the same as that of the elastomeric bearing. However, this change influences some fundamental properties of the bearing that are listed in Table 4-1.

The six equations (4.3 through 4.7) can be solved simultaneously using the Newton-Raphson solver from the elastomeric bearing model developed in MATLAB. All procedures remain the same, however, instead of solving the four non-linear equations earlier, six non-linear simultaneous equations are now solved for unknowns $F, s, \theta, v, z, F_L$ with $P$ and $u$ as inputs for an analysis step.
4.4 Newton-Raphson method

Extending the system of non-linear equations to LRB model:

\[ F \cos \theta + P \sin \theta - F_s - F_L \cos \theta = Q_1 \] (4.8)

\[ F(h' - v) + Pu - 2M = Q_2 \] (4.9)

\[ s \cos \theta + h' \sin \theta - u = Q_3 \] (4.10)

\[ h'(\cos \theta - 1) - s \sin \theta + v = Q_4 \] (4.11)

\[ F_L - aku - (1 - \alpha)kz = Q_5 \] (4.12)

\[ z - z_0 - \Delta u \left[ 1 - \frac{z^n}{u_y} \right] \times (\beta + \gamma sgn(\Delta u)) = Q_6 \] (4.13)

The Newton-Raphson formula (Eq. 3.14) is rewritten as follows:

\[ \{X_{i+1}\} = -[Z_i]\{Q_i\} + \{X_i\} \] (4.14)

where \( \{X_{i+1}\} = \{F_{i+1}, s_{i+1}, \theta_{i+1}, v_{i+1}, F_{L,i+1}, z_{i+1}\} \), is the updated estimate of unknowns, \( \{X_i\} = \{F_i, s_i, \theta_i, v_i, F_{L,i}, z_i\} \), is the previous estimate of unknowns, \( Q_i = \{Q_{1,i}, Q_{2,i}, Q_{3,i}, Q_{4,i}, Q_{5,i}, Q_{6,i}\} \), where \( Q_{ni} \) is the value of nth equation (Eq. 4.8 through 4.13) evaluated at \( i^{th} \) iteration and \( Z_i \) is the Jacobian evaluated at \( i^{th} \) iteration, which when evaluated for the system of non-linear Eq. (4.8) to (4.13) results in the following:
\[
Z = \begin{bmatrix}
\cos \theta & -\frac{dF_z}{ds} & -F \sin \theta + P \cos \theta + F_L \sin \theta & 0 & -\cos \theta & 0 \\
0 & h - v & 0 & -2 \frac{dM}{d\theta} & -F & 0 \\
0 & \cos \theta & -\sin \theta + h \cos \theta & 0 & 0 & 0 \\
0 & -\sin \theta & -s \cos \theta - h \sin \theta & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -K_L \\
0 & 0 & 0 & 0 & 0 & 1 + \Delta u \left[ \frac{z}{H_y} \right]^{n-1} (\beta + \gamma \text{sgn}(\Delta u))
\end{bmatrix}
\]

(4.15)

Other details that are a function of the bearing characteristics, such as evaluation of \((dF_z/ds)_i\), \((F_z)_i\), \((dM/d\theta)_i\) and \(M_i\), are the same as described in Chapter 3. The methodology to solve equations (4.8) through (4.13) is the same as for the elastomeric bearing mechanistic model.
Chapter 5
MODEL RESULTS AND EVALUATION

5.1 General
Simulations were performed with the elastomeric and LRB numerical models by discretizing the bearings into 20 vertical springs both at top and bottom. The simulated response was compared to data collected from an experimental testing program for monotonic tests (Sanchez et al. 2013) to evaluate the accuracy of the models, especially the LRB, for replicating the monotonic force-displacement response. Results for cyclic loading, to demonstrate hysteretic behavior, on LRB are also presented.

5.2 Elastomeric bearing

5.2.1 Experimental response
The experimental data from monotonic tests on an elastomeric bearing (Bearing ID

![Graph](image)

Fig. 5-1 Test result by Sanchez et al (2013) on elastomeric bearing
11795) conducted by Sanchez et al. (2013) is plotted in Fig. 5-1. Fig. 5-1 plots shear force versus lateral displacement equilibrium paths for five vertical compressive force levels including the absence of compressive load. The data demonstrates the influence of vertical compressive force on the shear force response of the elastomeric bearing.

5.2.2 Model Simulation

The elastomeric model is analyzed for a range of constant axial load levels with monotonic increase in lateral displacement while utilizing linear shear spring properties to produce the results in Fig. 5-2.

![Elastomeric model simulation using linear shear spring](image)

After comparing the model simulation in Fig. 5-2 to the experimental response (Fig. 5-1) it was observed that the model exhibits a response that is stronger than the experimental bearing response. For instance considering axial load level $P = 178\text{kN}$, the
experimental response critical point in Fig. 5-1 is observed at $F_{cr} = 14\text{kN}$ and $u_{cr} = 100\text{mm}$ whereas the model numerically simulates $F_{cr} = 20\text{kN}$ and $u_{cr} = 100\text{mm}$ for the same axial load level. A greater strength from the model, as observed, was expected because linear shear spring properties were used in this simulation. However, the model was able simulate the decrease in critical response ($F_{cr}$ and $u_{cr}$) with increasing axial load. Moreover, the model was re-analyzed using a modified version of the non-linear shear modulus equation proposed by Nagarajaiah and Ferrell (1999) (Eq. 5.1) to produce results of Fig. 5-3.

$$G(u) = G_o \left[ 1 - 0.25 \tanh \frac{u}{T_r} \right]$$

Fig. 5-3. Simulation results from elastomeric bearing model with nonlinear shear spring.
5.3 LRB

5.3.1 Experimental response

Plots of experimental test data for monotonic quasi static test, similar to that of an elastomeric bearing (Fig. 5-1), from Sanchez et al. (2013) for lateral force response of LRB (Bearing ID 11772) are shown in Fig. 5-4.

Fig. 5-4 Test result by Sanchez et al. (2013) on Lead Rubber Bearing

Bouc-Wen parameter: yield displacement of lead, \( u_y \), used to model the characteristic strength of lead, describe in section 4.2, is adopted after observing the experimental response in Fig. 5-4. A change in stiffness at all axial load levels (except for \( P = 0 \) kN) can be observed in Fig. 5-4 at \( u = 12.5 \) mm, this signifies the displacement at which the lead yields. Hence, \( u_y = 12.5 \) mm is used in the numerical model simulation presented in Fig. 5-5.
5.3.2 Model simulation for monotonic loading

The elastic stiffness, $k$, ratio of inelastic to elastic stiffness of lead, $\alpha$, needed to be ascertained for the Bouc-Wen formulation to represent the lead characteristics that are used to analyze the LRB model. A range of numerical model simulations using different values of $k$ and $\alpha$ were performed in order to arrive at the simulated response as presented in Fig. 5-5. The LRB response agrees with the experimental response reasonably well (refer to section 5.4 for comparison) for $k = 0.4 \text{kN/mm}$ and $\alpha = 1\%$. Also by observing the initial slope (lateral stiffness) of the experimental plot for $P = 0 \text{kN}$ in Fig. 5-4, $k = 0.4 \text{ kN/mm}$ is considered reasonable and $\alpha = 1\%$ is estimated in solving for the model response in Fig. 5-5.

Fig. 5-5 Monotonic simulation results from lead rubber bearing model with Bouc-Wen element
5.3.3 Model simulation for cyclic loading

The LRB model was subjected to cyclic loading to produce the results in Fig. 5-6. Analysis of the bearing was performed at three axial load levels, 178 kN, 268 kN and 356 kN with maximum displacement as 150mm, 125mm and 100mm respectively. The model simulates elastic, inelastic and critical point lateral response of bearing for positive as well negative displacement values. The hysteretic behavior, in general shape, agrees with the trends found by dynamic experimental test conducted by Sanchez et al. (2013).

![Fig. 5-6 Cyclic simulations results from LRB model](image)
5.4 Comparison of results

A comparison of the bearing simulated lateral force response with the measured experimental results of the bearings is presented in Fig. 5-7 and Fig. 5-8. Additionally, critical point \((u_{cr}, F_{cr})\) from the plots and percentage difference between experimental and simulated response of critical points are tabulated in Table 5-1 and Table 5-2.

The large difference, at \(P = 356\text{kN}\) for \(u_{cr}\), between LRB experimental test result and the LRB simulated response in Table 5-2 can be regarded as an anomaly for the following reasons. First, from the experimental equilibrium path for axial load level \(P = 356\text{kN}\) in Fig. 5-4, \(u_{cr} = 35\text{mm}\) can be observed. For the same axial load level the experimental response of the elastomeric bearing exhibits \(u_{cr} = 50\text{mm}\) in Fig. 5-1. As the introduction of lead is not expected to reduce the critical displacement this hints at the possibility of inaccuracy in test data. Second, the fact that the bearings tested by Sanchez et al. (2013) were not virgin bearings and may have been damaged in this test or previous tests that were conducted on the same bearings, further supports the claim that this critical displacement from the LRB experimental response could be seen as an inaccuracy.

Moreover, comparing the monotonic simulated bearing response for elastomeric with LRB as presented in Fig. 5-3 and Fig. 5-5 respectively, it can be observed that the introduction of lead in the elastomeric doesn’t significantly change the critical displacement however provides additional strength to the bearing which is signified by increase in critical shear force. For instance considering axial load level \(P = 178\text{kN}\), \(F_{cr} = 14\text{kN} \& u_{cr} = 98\text{mm}\) for elastomeric and \(F_{cr} = 21.7\text{kN} \& u_{cr} = 102\text{mm}\) for LRB can be observed from Fig. 5-3 and Fig. 5-5 respectively.
Fig. 5-7 Comparison of elastomeric bearing model simulation against experimental tests for each axial load.
Fig. 5-8 Comparison of LRB model monotonic simulation against experimental tests for each axial load
## Table 5-1 Summary of critical point response for Elastomeric bearing

<table>
<thead>
<tr>
<th>Axial Load, P (kN)</th>
<th>Experimental</th>
<th>Model</th>
<th>% Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>F&lt;sub&gt;cr&lt;/sub&gt; (kN)</td>
<td>u&lt;sub&gt;cr&lt;/sub&gt; (mm)</td>
<td>F&lt;sub&gt;cr&lt;/sub&gt; (kN)</td>
</tr>
<tr>
<td>0</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>89</td>
<td>-</td>
<td>-</td>
<td>25</td>
</tr>
<tr>
<td>178</td>
<td>14.1</td>
<td>90</td>
<td>14.4</td>
</tr>
<tr>
<td>268</td>
<td>7.0</td>
<td>74</td>
<td>7.7</td>
</tr>
<tr>
<td>356</td>
<td>2.5</td>
<td>50</td>
<td>2.4</td>
</tr>
</tbody>
</table>

## Table 5-2 Summary of critical point response for monotonic loading on LRB

<table>
<thead>
<tr>
<th>Axial Load, P (kN)</th>
<th>Experimental</th>
<th>Model</th>
<th>% Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>F&lt;sub&gt;cr&lt;/sub&gt; (kN)</td>
<td>u&lt;sub&gt;cr&lt;/sub&gt; (mm)</td>
<td>F&lt;sub&gt;cr&lt;/sub&gt; (kN)</td>
</tr>
<tr>
<td>0</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>89</td>
<td>-</td>
<td>-</td>
<td>31.2</td>
</tr>
<tr>
<td>178</td>
<td>21</td>
<td>106</td>
<td>21.7</td>
</tr>
<tr>
<td>268</td>
<td>15.3</td>
<td>70</td>
<td>15.6</td>
</tr>
<tr>
<td>356</td>
<td>12.0</td>
<td>35</td>
<td>10.8</td>
</tr>
</tbody>
</table>
5.5 Discussion

The elastomeric model is able to capture the saturation of moment capacity of bearing at the critical point beyond which any increase in shear force or axial load imposes a moment demand which cannot be further resisted and hence, renders the bearing unstable. The elastomeric bearing mechanistic model that employs modified shear non-linearity equation is capable of simulating the critical point or the stability limit \((u_{cr}, F_{cr})\) with reasonable accuracy. The overall behavior which also includes the elastic range in the shear force versus lateral displacement plot of the simulated response agrees well with the experimental response plots.

Decrease in critical point values of experimental as well as simulated response for LRB and elastomeric with increasing axial load can be observed. This is due to the reason that increasing axial load reduces the elastic lateral stiffness (also shown by Koh and Kelly model) of the elastomeric part, following this reduced stiffness equilibrium path, the critical displacement which saturates the moment of resistance of the bearing must be at a reduced value than that at a lower axial load level in order maintain a moment value which is equal to the capacity of the bearing. However, it should be recognized that behavior is coupled in non-linear equations (refer to Chapter 3 and 4).

When comparing the LRB mechanistic model monotonic simulation with the lateral force experimental response, it is observed that the model is capable of simulating the critical stability limit \((u_{cr}, F_{cr})\) of the bearing with reasonable accuracy. \(u_{cr}\) simulation for the axial load level \(P = 356\) kN, does not agree well with the experimental response, as the experimental response is believed to be inaccurate (for details refer to section 5.4).
An increase in initial lateral stiffness with an increase in axial load level can be observed from the experimental test response for LRB in Fig. 5-2. Initial lateral stiffness in the LRB is dependent on the characteristic strength of the lead. Further it can be inferred that the experimental data indicate a dependency of apparent lead strength on the level of vertical compressive force, which also agrees with findings by Ryan et al. (2006). This dependence could be due to the bulging of the rubber layers which provide confining stress to the lead core and thus the apparent strength. The proposed model does not account for this, therefore this behavior is not expected and observed in the numerically simulated response. While simulated and experimental results agree well in general, including this coupling between the vertical compressive force and the strength of the lead core could improve the agreement.

The simulations from elastomeric and LRB model encompass the condition of varying axial load in an earthquake scenario. The bearing response will shift between equilibrium paths presented in the model simulation plots when axial load and lateral displacement change in time.
Chapter 6
SUMMARY AND CONCLUSIONS

6.1 Summary

This study proposes a mechanistic model for simulating the force-displacement response of LRBs. The key contribution is an extension of a modified version of mechanistic model for elastomeric bearings proposed by Han and Warn (2014) to lead rubber bearings. This extension is completed by adding a hysteretic element to the elastomeric model in order to model the energy dissipated by the lead-core in LRBs. Furthermore, a new solution procedure, Newton-Raphson iteration scheme, was introduced to solve the system of nonlinear equations and to obtain the simulated equilibrium paths for both the elastomeric and LRB models. Finally, the bearing models were used to simulate the monotonic force-displacement response of an elastomeric and lead rubber bearing. The simulated response was compared to the experimental data from an experimental testing program by Sanchez et al. (2013) to evaluate the capability of the models for replicating the force-displacement response.
6.2 Conclusions

The present study focused on development of a mechanistic model for Lead Rubber bearings. The mechanistic model was developed utilizing the vertical spring elastomeric model proposed by Han and Warn (2014). From the model development, analysis and evaluation in Chapters 3 through Chapter 5, the following conclusions are drawn:

1. The modified Han and Warn mechanistic model for elastomeric bearings was able to replicate the general trend of decreasing critical force and displacement values with increasing axial load. The model was able to simulate the critical displacement within 6% accuracy and critical shear force within 10% accuracy. Moreover, availability of exact shear strength properties from characterization tests on bearings to obtain shear strength variation will reduce dependency on the change proposed in Nagarajaiah and Ferell (1999) shear modulus equation.

2. Comparing the critical displacement, \( u_{cr} \), from either simulated or experimental response for elastomeric and lead rubber bearings suggests that the introduction of lead does not alter the stability of the bearing in terms of the displacement at which the critical point is reached. Rather, addition of lead increases the critical shear force significantly.

3. Parallelization of the lead hysteretic element with the elastomeric model seems to be a reasonable approach in modelling of LRBs. In contrast to modelling the hysteretic element with the rotating shear spring as done by Yamamoto et al. (2008).
6.3 **Recommendations for future research**

Further advancements can be made that could improve the simulation of proposed LRB model and help in moving towards design of bearings for earthquake resistance.

1. A robust experimental data set is needed to compare the response from monotonic and cyclic loading simulations from analysis of models. Furthermore, characterization tests on shear strength variation needs to be performed to reduce dependency on empirical equation for non-linear shear strength.

2. The coupling of vertical compressive load and the apparent lead core shear strength needs to be studied and incorporated in the LRB model.

3. A dynamic response history analysis by subjecting the bearing model to various earthquake ground motions needs to be performed to study its hysteretic behavior under earthquake conditions with simultaneous loading in both lateral directions.
References


The function developed in MATLAB for elastomeric and lead rubber bearing models is as follows:

**A.1 Elastomeric bearing**

```matlab
function [lateraldisp, Force, sheardisp, theta, vertdisp] = Bearing_nonlin6(es, max_it, no)

H = 117;
j = 1;
Tr=60;
Ab=17439;
Ec = 0.468;
for P = [0 89 178 268 358];
h = H – P*h/Ab/Ec;

Xc = zeros(4,1);
i = 1;
for u = (0:5:150)
eax_max = 100;
it = 2;
if i>1;
    Xc(1,i) = Xc(1,i-1);
    Xc(2,i) = Xc(2,i-1);
    Xc(3,i) = Xc(3,i-1);
    Xc(4,i) = Xc(4,i-1);
end

dXi = zeros(4,1);
Xi = [Xc(1,i);Xc(2,i);Xc(3,i);Xc(4,i)];
su = zeros(4,1);
while it<=max_it-2 && ea_max>es
    for m = 1:it-1
        su = su +dXi(:,m);
    end
    Xi = Xi + su;
    Fi = Xi(1); si = Xi(2); ai = Xi(3); vi = Xi(4);
    [M, k1] = springsn(ai,P,no,0);
    k2 = shear(u,Tr)*Ab/Tr;
    Zi = [cos(ai) -k2 (-Fi*sin(ai))+(P*cos(ai)) 0;...
        h-vi 0 -k1 -Fi;...
        0 -sin(ai) (-si*cos(ai))-(h*sin(ai)) 1];
    Qi = [Fi*cos(ai)-(si*k2)+(P*sin(ai));...
        (Fi*h)-M-(vi*Fi)+(P*u);...
        (si*cos(ai))+(h*sin(ai))-u;...
        -(si*sin(ai))+(h*cos(ai)+vi-h];
    dXi(:,it) = -Zi\Qi;
    if det(Zi)~=0
        eaF = abs(dXi(1,1)/it)/(Xi(1)+dXi(1,1))\100;
        eas = abs(dXi(2,1)/it)/(Xi(2)+dXi(2,1))\100;
        eaa = abs(dXi(3,1)/it)/(Xi(3)+dXi(3,1))\100;
```
eav = abs(dXi(4,it)/(Xi(4)+dXi(4,it))*100);
ea = [eaF;eas;eaa;eav];
Xi_new = Xi + dXi(:,it);
Q_i = [Xi_new(1)*cos(Xi_new(3))-
Xi_new(2)*k2+P*sin(Xi_new(3))-]
Xi_new(2)*k2+P*sin(Xi_new(3)))*100;

if max(abs(Q_i))>10^-3
    ea_max = 1;
else
    ea_max = max(ea);
end
else
    ea_max = 100;
end
it = it + 1;
Xi = [Xc(1,i);Xc(2,i);Xc(3,i);Xc(4,i)];
su = zeros(4,1);
end

%--------------------------------------------------------------
%--------------------------------------------------------------
%--------------------------------------------------------------
%--------------------------------------------------------------
%--------------------------------------------------------------

function [M,K] = springsn(b,P,no,alpha)
Do = 152;
Di = 30;
tr = 3;
S = (pi*(Do^2-Di^2)/4)/(pi*(Do+Di)*tr);
F = ((Do/Di)^2+1)/((Do/Di)-1)^2 + (1+(Do/Di))/(1-(Do/Di))*log(Do/Di));
Tr = tr*20;
I = pi*(Do^4-Di^4)/64;
G = 1.08*10^-3;
E = 6*G*S^2*F;
sy = 3.24*10^-3;
ey = sy/E;
[Area, ~] = Calc_Area(no,Do,Di);
lS = calc_ls(Do,Area,no,I,Tr)*2;

for n = 1:length(b)
ai = b(n,1);

if ai==0
    M(n,1) = 0;
    K(n,1) = pi^2*E*I/Tr/3;
else
    x = -P*ls/sum(Area)/E/ai;
    [ve,vs,y] = Calc_ss(x,ai,ls,E,ey,Area,Do,no,alpha);
    if ve(1,1)>=ey
        F = Calc_F(vs,Area);
        while abs(-F-P)>0.05
            if min(vs)==sy;
                break
            else
                x = x + 0.001;
                [ve,vs] = Calc_ss(x,ai,ls,E,ey,Area,Do,no,alpha);
                F = Calc_F(vs,Area);
        end
    end
else
    ...
end
F = Calc_F(vs,Area);
er(n,1)= abs(-F-P)/P*100;
[ve,vs] = Calc_ss(x,ai,ls,E,ey,Area,Do,no,alpha);
m=0;
j = 1;
for A = Area
    m = m - vs(j,1)*A*y(j,1);
    j=j+1;
end
M(n,1) = m*2;
[~,s1] = Calc_ss1(x,ai,ls,E,ey,Area,Do,ve,no,alpha);
m1=0;
j = 1;
for A = Area
    m1 = m1 - s1(j,1)*A*y(j,1);
    j=j+1;
end
K(n,1) = m1*2;
end
end
end
end

%---------------------------------------------------------------
---

function [ve,vs,y] = Calc_ss(x,ai,ls,E,ey,Area,D,no,alpha)
j = 1;
for A = Area
    y(j,1) = D*(2*j-1)/2/no-0.5*D;
    ve(j,1) = ai*(x-y(j,1))/ls;
    if ve(j,1)>=ey
        vs(j,1) = E*ey+(alpha*E*(ve(j,1)-ey)/100);
else
    vs(j,1) = E*ve(j,1);
end
j = j+1;
end
end

function [e,s] = Calc_ss1(x,a,ls,E,ey,Area,D,ve,no,alpha)
    j = 1;
    for A = Area
        y(j,1) = D*(2*j-1)/2/no-0.5*D;
        e(j,1)=a*(x-y(j,1))/ls;
        if ve(j,1)>ey
            s(j,1)= (0+E*alpha*ve(j,1)/100)*e(j,1)/ve(j,1);
        else
            s(j,1) = E*e(j,1);
        end
        j = j+1;
    end
end

function [F] = Calc_F(vs,Area)
    f = 0;
    j = 1;
    for A = Area
        f = f + vs(j,1)*A;
        j=j+1;
    end
    F = f;
end

function a = calc_ls(D,Area,no,I,Tr)
    i =1;
    sum = 0;
    for A = Area
        y(i,1) = 0.5*D-D*(2*i-1)/2/no;
        sum = sum + A*y(i,1)^2;
        i = i+1;
    end
    a = (3*Tr*sum)/(pi^2*I);
end

function [Area,error] = Calc_Area(no,Do,Di)
    j=1;
    for i = 1:no/2
        D = Do/2-i*Do/no;
\( a = \cos(2D/Do); \)
\[
\text{if } D>Di/2 \\
\quad \text{if } i==1 \\
\quad \quad \text{A}(1,i) = Do^2a/4-\sin(a)*Do/2*D; \\
\quad \text{else} \\
\quad \quad \text{A}(1,i) = Do^2a/4-\sin(a)*Do/2*D-\text{sum(A}(1,1:i-1)); \\
\text{end} \\
\text{elseif } D<Di/2 && Di>0 \\
\quad ah = \cos(D*2/Di); \\
\quad \text{if } j==1 \\
\quad \quad \text{Ah}(1,j) = Di^2ah/4-\sin(ah)*Di/2*D; \\
\quad \quad \text{A}(1,i) = Do^2a/4-\sin(a)*Do/2*D-\text{sum(A}(1,1:i-1))-Ah(1,j); \\
\quad \text{else} \\
\quad \quad \text{Ah}(1,j) = Di^2ah/4-\sin(ah)*Di/2*D-\text{sum(Ah}(1,1:j-1)); \\
\quad \quad \text{A}(1,i) = Do^2a/4-\sin(a)*Do/2*D-\text{sum(A}(1,1:i-1))-Ah(1,j); \\
\quad \text{end} \\
\quad j = j+1; \\
\text{end} \\
\text{end} \\
\text{Area}(1,1:no/2) = A; \\
\text{Area}(1,no/2+1:no) = A(1,no/2:-1:1); \\
\text{error} = (\pi*(Do^2-Di^2)/4 - \text{sum(Area))}/(\pi*(Do^2-Di^2)/4)*100; \\
\text{end} \\

function G = shear(u,Tr) \\
G = 1.08*10^{-3}*(1-0.25*tanh(u/Tr)); \\
\text{end}
A.2 Lead Rubber Bearing

function [lateraldisp, Force, sheardisp, theta, vertdisp, leadforce, z] = Bearing_nonlin11(es, max_it, no)
H = 117;
j = 1;
Tr=60;
Ab=17439;
a_l = 0.01;
Kl = 0.4;
uy = 12.5;
n = 2;
B = 0.1;
G = 0.9;
for P = [0 89 178 268 356];
i = 1;
for u = (0:1:150)
h = H – P*h/Ab/Ec;
if u==0;
    du = 0;
    zp = 0;
else
    du = u-lateraldisp(i-1,j);
    zp = z(i-1,j);
end
ea_max = 100;
it = 2;
if i>1;
    Xc(:,i) = Xc(:,i-1);
else
    Xc = zeros(6,1);
end
dXi = zeros(6,1);
Xi = [Xc(1,i);Xc(2,i);Xc(3,i);Xc(4,i);Xc(5,i);Xc(6,i)];
su = zeros(6,1);
while it<=max_it-2 && ea_max>es
    for m = 1:it-1
        su = su +dXi(:,m);
    end
    Xi = Xi + su;
    Fi = Xi(1); si = Xi(2); ai = Xi(3); vi = Xi(4); FLi = Xi(5); zi = Xi(6);
    [M, k1] = springsn(ai,P,no,0);
k2 = shear(u,Tr)*Ab/Tr;
Zi = [cos(ai) -k2 (-Fi*sin(ai)+P*cos(ai)+FLi*sin(ai)) 0 -cos(ai) 0;...
          h-vi 0 -k1 -Fi 0 0;...
          0 cos(ai) (-si*sin(ai)+h*cos(ai)) 0 0 0;...
          0 -sin(ai) (-si*cos(ai))-(h*sin(ai)) 1 0 0;...
          0 0 0 1 -(1-a_l)*Kl;...
          0 0 0 0 1+du*n*z1*(n-1)/uy^n*(B+G*sign(z1*du))];
Qi = [Fi*cos(ai)-(si*k2)+(P*sin(ai)-FLi*cos(ai));...
      (Fi*h)-M-(vi*Fi)+(P*u);...
      (si*cos(ai))+(h*sin(ai))-u;...
      -(si*sin(ai))+(h*cos(ai))+vi-h;...
FLi-a_l*Kl*u-(1-a_l)*Kl*zi;...
zi-zp-du*(1-abs(z{i}^n/uy^n)*(B+G*sign(z{i}*du)));
dXi(:,it) = -Zi\Qi;
if det(Zi)~=0
  eaF = abs(dXi(1,it)/(Xi(1)+dXi(1,it))*100);
eas = abs(dXi(2,it)/(Xi(2)+dXi(2,it))*100);
eaa = abs(dXi(3,it)/(Xi(3)+dXi(3,it))*100);
eav = abs(dXi(4,it)/(Xi(4)+dXi(4,it))*100);
eaf = abs(dXi(5,it)/(Xi(5)+dXi(5,it))*100);
eaz = abs(dXi(6,it)/(Xi(6)+dXi(6,it))*100);
ea = [eaF;eas;eaa;eav;eaf;eaz];
X_i_new = Xi + dXi(:,it);
Qi = [Xi_new(1)*cos(Xi_new(3))-
      Xi_new(2)*sin(Xi_new(3))-
      Xi_new(5)*cos(Xi_new(3));
      ...
      X_i_new(1)*h-M-Xi_new(4)*Xi_new(1)+P*u;...
      X_i_new(2)*cos(Xi_new(3))+sin(Xi_new(3))*h-u;...
      X_i_new(5)-a_l*Kl*u-(1-a_l)*Kl*Xi_new(6);...
      X_i_new(6)-zp-du*(1-
      abs(Xi_new(6)^n/uy^n)*(B+G*sign(Xi_new(6)*du)))];
if max(abs(Qi))>10^-3
  ea_max = 1;
else
  ea_max = max(ea);
end
else
  break
end
it = it + 1;
Xi = Xc(:,i);
su = zeros(6,1);
end
Xc(1,i) = Xc(1,i)+sum(dXi(1,:));
Xc(2,i) = Xc(2,i)+sum(dXi(2,:));
Xc(3,i) = Xc(3,i)+sum(dXi(3,:));
Xc(4,i) = Xc(4,i)+sum(dXi(4,:));
Xc(5,i) = Xc(5,i)+sum(dXi(5,:));
Xc(6,i) = Xc(6,i)+sum(dXi(6,:));
clear dXi;
lateraldisp(i,j) = u;
Force(i,j) = Xc(1,i);
sheardisp(i,j) = Xc(2,i);
theta(i,j) = Xc(3,i);
vertdisp(i,j) = Xc(4,i);
leadforce(i,j) = Xc(5,i);
z(i,j) = Xc(6,i);
i = i+1;
end
j=j+1;
end
function [M,K] = springsn(b,P,no,alpha)

Do = 152;
Di = 30;
tr = 3;

S = 12.17;
F = ((Do/Di)^2+1)/((Do/Di)-1)^2 + (1+(Do/Di))/((1-(Do/Di))*log(Do/Di));
Tr = tr^20;
I = pi*(Do^4-Di^4)/64;
G = 1.08*10^-3;
E = 6*G*S^2*F;
sy = 3.24*10^-3;
ey = sy/E;

[Area, ~] = Calc_Area(no,Do,Di);
ls = calc_ls(Do,Area,no,I,Tr)*2;

for n = 1:length(b)
    ai = b(n,1);

    if ai==0
        M(n,1) = 0;
        K(n,1) = pi^2*E*I/Tr/3;
    else
        x = -P*ls/sum(Area)/E/ai;
        [ve,vs,y] = Calc_ss(x,ai,ls,E,ey,Area,Do,no,alpha);
        if ve(1,1)>=ey;
            F = Calc_F(vs,Area);
            while abs(-F-P)>0.05
                if min(vs)==sy;
                    break
                else
                    x = x + 0.001;
            end
        end
    end

    F = Calc_F(vs,Area);
    er(n,1)= abs(-F-P)/P*100;
    [ve,vs] = Calc_ss(x,ai,ls,E,ey,Area,Do,no,alpha);
    m=0;
    j = 1;
    for A = Area
        m = m - vs(j,1)*A*y(j,1);
        j=j+1;
    end
    M(n,1) = m*2;
    [~,s1] = Calc_sss(x,1,ls,E,ey,Area,Do,ve,no,alpha);
    m=0;
    j = 1;
for A = Area
    m1 = m1 - s1(j,1)*A*y(j,1);
    j=j+1;
end
K(n,1) = m1*2;
end
end
%
%---------------------------------------------------------------------
---
%
function [ve,vs,y] = Calc_ss(x,ai,ls,E,ey,Area,D,no,alpha)
j = 1;
for A = Area
    y(j,1) = D*(2*j-1)/2/no-0.5*D;
    ve(j,1)=ai*(x-y(j,1))/ls;
    if ve(j,1)>=ey
        vs(j,1) = E*ey+(alpha*E*(ve(j,1)-ey)/100);
    else
        vs(j,1) = E*ve(j,1);
    end
    j = j+1;
end
end
%
%---------------------------------------------------------------------
---
%
function [e,s] = Calc_ss1(x,a,ls,E,ey,Area,D,ve,no,alpha)
j = 1;
for A = Area
    y(j,1) = D*(2*j-1)/2/no-0.5*D;
    e(j,1)=a*(x-y(j,1))/ls;
    if ve(j,1)>ey
        s(j,1)= (0+E*alpha*ve(j,1)/100)*e(j,1)/ve(j,1);
    else
        s(j,1) = E*e(j,1);
    end
    j = j+1;
end
end
%
%---------------------------------------------------------------------
---
%
function [F] = Calc_F(vs,Area)
f = 0;
j = 1;
for A = Area
    f = f + vs(j,1)*A;
    j=j+1;
end
F = f;
end
%--
%---
9
function a = calc_ls(D,Area,no,I,Tr)
i =1;
sum = 0;
for A = Area
    y(i,1) = 0.5*D-D*(2*i-1)/2/no;
    sum = sum + A*y(i,1)^2;
i = i+1;
end
a = (3*Tr*sum)/(pi^2*I);
end

function [Area,error] = Calc_Area(no,Do,Di)
j=1;
for i = 1:no/2
    D = Do/2-i*Do/no;
    a = acos(2*D/Do);
    if D>Di/2
        if i==1
            A(1,i) = Do^2*a/4-sin(a)*Do/2*D;
        else
            A(1,i) = Do^2*a/4-sin(a)*Do/2*D-sum(A(1,1:i-1));
        end
    elseif D<Di/2 && Di>0
        ah = acos(D*2/Di);
        if j==1
            Ah(1,j) = Di^2*ah/4-sin(ah)*Di/2*D;
            A(1,i) = Do^2*a/4-sin(a)*Do/2*D-sum(A(1,1:i-1))-Ah(1,j);
        else
            Ah(1,j) = Di^2*ah/4-sin(ah)*Di/2*D-sum(Ah(1,1:j-1));
            A(1,i) = Do^2*a/4-sin(a)*Do/2*D-sum(A(1,1:i-1))-Ah(1,j);
        end
        j = j+1;
    end
    Area(1,1:no/2) = A;
    Area(1,no/2+1:no) = A(1,no/2:-1:1);
end
error = (pi*(Do^2-Di^2)/4 - sum(Area))/(pi*(Do^2-Di^2)/4)*100;
end

function G = shear(u,Tr)
G = 1.08*10^-3*(1-0.25*tanh(u/Tr));
end