SHEAR DRIVEN GRAVITY WAVES ON A SLOPING FRONT

A Thesis in
Meteorology

by

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Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Master of Science

May 2009
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ABSTRACT

In Synthetic Aperture Radar derived surface wind speed imagery (SDWS), wave-like patterns are seen frequently in association with gently sloping fronts (i.e. warm and occluded fronts), but are not observed away from the fronts or on steeper fronts (i.e. cold fronts). It is hypothesized that these surface wind patterns are caused by internal gravity waves propagating on the frontal inversion, driven by vertical shear between the two air masses, and extending down to the surface for some or all of their extent. This hypothesis is tested by examining how well the classic linear model of this phenomenon matches observations. The classic Kelvin-Helmholtz velocity profile with a rigid lower boundary and a sloping interface is used to approximate a warm frontal region. It is shown that the distance that these wave-like patterns, observed by SAR, extend from the surface warm front into the cool air mass is consistent with a bifurcation along the warm frontal inversion from unstable to neutral solutions. It also is be shown that the maximum wave growth rate occurs near this bifurcation point and hence can explain the SAR-observed pattern of wind perturbation intensity. In addition, a wave crest tracing procedure is developed to show that the waves refract into the direction of the warm front similar to ocean swell refracting on an approaching beach.
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Acknowledgements

I would like to thank George Young, Hampton Shirer, and Todd Sikora for their valuable advice and guidance. I would also like to thank Amanda Walsh for supporting me through the ups and downs of my research over the past few years. Also I would like to thank Andrew Annunzio, Aviva Braun, Luna Rodriguez and all of the other past and present residents of 402 Walker for their support.
Chapter 1: Introduction

Wavelike patterns, hereafter referred to as waves, are observed in the surface wind fields deduced from the sea surface backscatter images obtained by Synthetic Aperture Radar (SAR) (Young et al., 2005). Hereafter these images will be referred to as Synthetic Aperture Radar derived wind speed (SDWS) images. While many of these waves are clearly the surface signature of orographically generated gravity waves (i.e. mountain lee waves, Beal et al, 2004; Li et al. 1998; Li et al 2004) others are observed in locations and synoptic weather regimes that are not consistent with that phenomenon. The focus of this study is on the waves observed by SDWS imagery on the cold side of warm, occluded, and secluded fronts, in the northern Pacific Ocean. These waves are hypothesized to be the SDWS surface signature of shear-driven gravity waves on a warm frontal inversion propagating down to the sea surface (Young et al., 2005). Such SDWS signatures extend $10^4$-$10^5$ m away from the surface warm front and often are not oriented directly perpendicular to the front. The observed wave perturbation amplitude drops off at some finite distance from the surface warm front, with the maximum value typically being approximately 7 ms$^{-1}$. Typical wavelengths are on the order of several kilometers (Figure 1).

Examining the SDWS image in figure 1 we immediately notice a dramatic increase in the analyzed surface wind speed ahead of the surface warm front. This increase in the analyzed wind speed ahead of the front is because the CMOD-4 algorithm requires an accurate wind direction estimate to obtain the surface wind speed and since the wind speed is interpolated from a global model, the SDWS imagery in the region of a synoptic-scale front is often biased (Young et al. 2008). While the mesoscale structures
of the wavelike patterns observed by the SDWS imagery is correct, there amplitude and mean background flow can be biased either high (as in figure 1) or low.

Figure 1. SDWS surface signature of warm frontal region from January 28, 2004 over the Northern Pacific Ocean. Observed wavelength is $3.67 \pm 0.6$ km. The arrows represent the surface wind direction.

The drastic decrease in the wave signature at a finite distance from the front, and hence height of the frontal inversion, is similar to that observed in topographically forced gravity waves during a cold air outbreak over the Great Lakes (Winstead et al., 2002). The difference between the SDWS surface signature observed in frontal waves and the topographically forced waves is that in the frontal case the wave source is shear directly over the location where we see the SDWS surface signature, where as in the cold air outbreak case the wave source is topography upstream of the SDWS surface signature and the wave is propagating along an inversion that is increasing with height as a function away from the wave source, resulting in a SDWS surface signature that
decreases gradually. Likewise, the wavelength and general appearance of these SDWS surface signatures in frontal regions are the same as those for previous observations of atmospheric gravity waves (Beal et al., 2004). Topographical forcing is absent over the open sea however, hence the hypothesis is that these observed waves are driven by shear across the frontal inversion (Young et al. 2005). These waves are observed predominately in warm frontal cases, but also are observed in occluded and secluded frontal cases. In some cases the waves occur in multiple spatially separated packets of waves (Figure 2), but in others the waves are spaced evenly along an extended section of the front (Figure 1). In this paper, we examine the simplest situation, waves in a warm frontal region where the front is straight and the waves are evenly spaced along an extended section of the front, rather than the more complex situation where the waves are distributed in discrete wave packets.

Figure 2. SDWS surface signature of warm frontal region from March 8, 2005 over the Northern Pacific Ocean. Observed wavelength is 4.28 ± 0.6 km. The arrows represent the surface wind direction.
A warm frontal region can be modeled using a two-layer system with different velocities in each layer. Even if the fluid is stably stratified, shear-driven instability can arise at the interface between two layers of fluid (Helmholtz, 1868). If the velocity gradient is discontinuous at the interface, the shear is infinite, causing the system to be unstable for any velocity ‘jump’ across the interface (Chandrasekhar, 1961). This instability is commonly referred to as parallel shear instability or the Kelvin-Helmholtz instability. In the atmosphere, Kelvin-Helmholtz instability is observed in regions of shear, including frontal zones (Nielsen, 1992). These unstable waves grow by extracting energy from the background flow and in turn act to smooth the shear zone (Drazin and Howard, 1966). Neutral solutions (wave growth rate = 0) exist for the Kelvin-Helmholtz instability, at small wavenumbers, in the form of internal gravity waves that propagate energy away from the shear zone (Lindzen, 1974).

The amplitude of these neutral internal gravity waves decays with vertical distance from the shear zone. In the presence of a rigid lower boundary, however, these neutral internal gravity waves can be reflected back towards the shear zone and so act to enhance the Kelvin-Helmholtz instability further (Lindzen, 1974). This reflection leads to unstable solutions (wave growth rate > 0) for the internal gravity wave modes. Therefore, in the presence of a lower boundary, such as the sea surface, wavenumbers that were neutral for internal gravity waves in an infinite domain become unstable (Lindzen, 1976). A sudden jump in the Brunt-Vaisala frequency, at some level away from the region of shear, can cause similar results (Pellacani, 1978). Such a region of higher stability inhibits the vertical propagation of gravity waves in a way similar to that seen with a rigid

Although the Kelvin-Helmholtz problem has been studied extensively since the mid-20th century, observations of the resulting atmospheric turbulence have been limited to in situ observations of clear air turbulence and photographic analysis of billow cloud formations (Reiss et al., 1977). The primary impact of these observations has been to provide a better understanding of clear air turbulence. Clear air turbulence has long been known to be associated with internal gravity waves propagating upward from regions of shear, such as upper tropospheric fronts (Reed et al., 1972). Little has been done, however, to study the phenomenon in other settings. In particular, the occurrence and dynamics of internal gravity waves within and below lower tropospheric frontal zones largely remains unexplored.

Section 4a shows using SDWS imagery and the corresponding Mesoscale Model V (MM5) analysis that the SDWS-observed waves, mentioned above, occur in warm frontal zones. Section 4b shows that the observed distance that these waves extend from the surface warm front is related to a bifurcation from unstable to neutral wave solutions with increasing heights of the warm frontal inversion. This bifurcation is due to the reflection of the waves by the rigid lower boundary back to the shear zone near the front (where height of shear zone is small) and infinite overreflection far from the front (where height of the shear zone is large). Also in section 4b it is shown that the maximum wave growth rate occurs close to the bifurcation point and explains the SDWS-observed wave perturbation amplitude distribution. In addition, a simple wave crest tracing procedure is
used in section 4c to explain the angle that the waves make with respect to the surface warm front.
Chapter 2. Data

The meteorological data used in this study consists of SDWS imagery and MM5 analyses for the Northern Pacific Ocean off the coast of Alaska. RADARSAT-1 SDWS imagery was obtained from the Alaskan SAR Demonstration Project (http://fermi.jhuapl.edu/sar/stormwatch/web_wind/). The RADARSAT-1 satellite is C-band (5.6 cm) and right-looking with horizontal-horizontal polarization (Sikora et al., 2006). These SDWS images are derived from SAR backscatter images using the CMOD-4 algorithm and assuming a priori knowledge of the near-surface wind direction (Young et al. 2008). The near surface wind direction is interpolated to SAR resolution from NOGAPS near-surface wind direction field. The SDWS image resolution has been smoothed from 100 to 600 m to reduce to presence of small-scale oceanic signatures, which can be confused with small-scale atmospheric signatures (Young et al., 2005). The MM5 analyses were obtained from of the University of Washington (http://www.atmos.washington.edu/mm5rt/).

A total of 21 candidate cases were selected by examining the SDWS imagery from 2003 to 2007. These wave signature cases were selected because they occurred along synoptic-scale fronts, over the open sea, and exhibit no signs of orographic origin. Likewise, care was taken to ensure that the fronts themselves were not interacting with the coast as the resulting barrier jets (which can exhibit similar wave signatures, quite possibly from shear-driven gravity waves), but are dynamically and structurally quite different from open sea fronts.

Of these candidate 21 cases, 17 are warm fronts and the remaining 4 cases are either occluded or secluded fronts. Of the 17 warm frontal cases, 6 are straight fronts (not
wrapping around the low). Of these 6 straight warm fronts, 3 have a continuous wave field throughout the SDWS-observed section of the front. In the remaining 3 cases the waves cluster in a number of discrete wave packets along the front. In this study, we only are focusing on the simplest situation, a continuous wave field along an entire section of a straight warm front.

These SDWS images were analyzed to extract the physical and geometrical properties of the waves. The wavelength and span, or distance the waves extend from surface front into the cold air mass, were obtained by direct measurement from the SDWS images. The wavelength is of interest because the vertical decay in wave amplitude away from the shear zone is dependant on wavelength (Lindzen, 1974). The span is of interest because we hypothesize that there is a strong relationship between the observed span and the critical shear layer height at which reflection by the surface no longer occurs. Additional geometrical quantities were calculated using data extracted from the SDWS images and are explained in detail in the next section. The MM5 model output solely was used to determine frontal type and to obtain a general idea of the shear and stability profiles. This restriction applies because the vertical spacing of the available MM5 maps is too coarse to accurately resolve the frontal zones shear and stability.
Chapter 3. Procedures

a) Data Analysis

In this subsection the data analysis procedures are outlined for the SDWS and MM5 data.

i) SDWS

SDWS images were examined for wave signatures occurring adjacent to a near zero-order jump in wind speed, which is the signature of a front in the wind field (Young et al. 2005). For cases in which waves are observed on a SDWS image, we must confirm that the synoptic setting in which they were observed is of dynamical interest, in particular that the waves are indeed on the cold side of a synoptic-scale warm front. The primary surface feature of these synoptic-scale fronts are the quasi-discontinuity of temperature and humidity where the frontal surface intersects the ground. Warm fronts typically exhibit a near zero-order jump in wind speed followed by a gradual increase in wind speed, and cold fronts exhibit a near zero-order jump in wind speed followed by a drastic increase in wind speed. This SDWS-based subjective analysis is confirmed via the examination of MM5 surface analysis as described in the next subsection.

Once a case has been confirmed as being of interest, geometrical properties of the waves are computed. Picking several points out along the wave crests (or wind speed maxima) for the SDWS images, we can calculate, by photogramery, important parameter values such as wavelength, $\lambda$, angle of the wave with respect to warm front, $\theta$, and wave span or distance the waves extend away from the front into the cold air mass (Figure 3). Picking a point along the crest of a wave and the closest on the crest of the adjacent wave allows us to compute the wavelength. Picking a point along the surface front and drawing
a line perpendicular to the front through that point and then finding the first point on that line at which the wave pattern becomes indiscernible allows us to compute the wave span. These calculations are repeated for multiple points along the front and averaged.

Figure 3: Illustration of wave patterns observed by SDWS. Here $\theta$ is the angle between the wave and the front, wavelength is measured from the crest of wave A to the crest of wave B, and the span is measured perpendicular to the surface warm front.

**ii) MM5**

Manual surface analysis was performed on MM5 output corresponding to each of the 3 analyzed images. The primary purpose of this manual analysis is to confirm the frontal type observed by the SDWS images. This step is performed because examining a SDWS image alone does not provide definitive proof of frontal type or the vertical structure associated with the observed zero-order jump in wind speed. We must examine the vertical structure in the region to determine if we are observing a warm front or an
occluded/secluded front. Unfortunately, the wide vertical spacing of the available weather maps (MM5) over the North Pacific Ocean prevents us from extracting information other than the general synoptic situation. A general idea of shear and stability can be obtained, but given the vertical spacing of the MM5 maps, we cannot gather any quantitative data about the profiles.

**b) Modeling**

In this subsection, a two-layer model is developed to explain the observed wavelike pattern in the surface wind-field as observed by SDWS imagery.

**i) Linear Theory**

In a warm frontal region, the warm air mass rides up and over the cool air mass. Because of the shear between the two air masses, thermal wind plus any ageostrophic wind shear, internal gravity waves can develop on the interface and propagate vertically and horizontally along the interface. This propagation can be represented by using a two-layer semi-infinite Boussinesq fluid with a Kelvin-Helmholtz velocity profile, with a sloping interface, and a rigid lower boundary (Figure 4c). A perturbation is introduced on the interfacial surface and the stability of the system is determined as a function of fluid depth. The stability is determined by the value of the imaginary part of the complex phase speed. To determine the complex phase speed of these internal gravity waves as a function of the shear layer elevation, or equivalently, the distance from the surface warm front, we must solve the dispersion relationship for the system, numerically.

Our equations are identical to Lindzen’s and so only a brief review will be given; refer to Lindzen (1974 and 1976) for a thorough review. Our basic model is a two-layer
semi-infinite Boussinesq fluid with a constant Brunt-Vaisala frequency, N, throughout (Figure 4):

\[ N = \sqrt{\frac{g}{\theta}} \frac{\partial \theta}{\partial z} \]  

(1)

Decomposing the Boussinesq equation set into a base state and a perturbation about that base state, in the x-direction, and a perturbation about the vertical direction,

\[ U = U_o + u' \]

\[ w = w' \]

where \( U_o \) represents the base state horizontal wind, \( u' \) represents the perturbation about this base state, and \( w' \) represents the vertical velocity perturbation. This allows us to arrive at a set of equations that describe the behavior of a perturbation at the interface between the two fluids moving at different velocities:

\[ \frac{\partial u'}{\partial t} + U_o \frac{\partial u'}{\partial x} = -\frac{1}{\rho_o} \frac{\partial p'}{\partial x} \]  

(2)

\[ \frac{\partial w'}{\partial t} + U_o \frac{\partial w'}{\partial x} = -\frac{1}{\rho_o} \frac{\partial p'}{\partial z} - g \frac{p'}{\rho_o} \]  

(3)

\[ \frac{\partial \rho'}{\partial t} + U_o \frac{\partial \rho'}{\partial x} = \frac{w' \rho_o}{g} N^2 \]  

(4)

\[ \frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 \]  

(5)

Where \( p' \) is the pressure perturbation and \( \rho' \) is the density perturbation. We can combine these equations (four equations and four unknowns) to form a single equation for the vertical velocity field, \( w' \), in the x-z plane.

\[ \left( \frac{\partial}{\partial t} + U_o \frac{\partial}{\partial x} \right)^2 \left( \frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial z^2} \right) + N^2 \frac{\partial^2 w'}{\partial x^2} = 0 \]  

(6)
Refer to Appendix A for a formal derivation of (6). Assuming an appropriate plane parallel wave solution,

\[ w = \text{Re} \left[ w(z) e^{i(kx-\omega t)} \right] \quad (7) \]

where \( k \) represents zonal wavenumber and \( \omega \) is the frequency of oscillation, we obtain

\[
\frac{d^2 w}{dz^2} + \left[ \frac{N^2}{(U_o-c)^2} - k^2 \right] w = 0
\]

\[
(8)
\]

where \( c \) is the complex phase speed. Equation (8) describes vertical velocity perturbations away from the interfacial region. In the standard two-layer Kelvin-Helmholtz model (Figure 4a), we have a velocity profile of the form,

\[
U_o = \begin{cases} +U & z > 0 \\ -U & z < 0 \end{cases}
\]

We choose to use this simple velocity profile because it is easier to work with analytically than are more complicated models, such as: a three-layer model (Chandrasekhar, 1961) or a model with a continuous shear profile of the form, \( U = U_o \tanh(z/d) \), where \( d \) is the depth of the shear zone (Drazin and Howard, 1966). These models are more complex to solve analytically, but are expected to reveal qualitatively consistent results, which is that the presence of a rigid lower boundary has a destabilizing effect on these interfacial perturbations at low wavenumbers (Lindzen et al. 1976).

Figure 4: a) Standard Kelvin-Helmholtz model in an infinite fluid. b) Standard Kelvin-Helmholtz velocity profile in a semi-infinite fluid with a rigid lower boundary. c) Kelvin-Helmholtz velocity profile in a semi-infinite fluid with a rigid lower boundary and a sloping interfacial region.
Substituting the Kelvin-Helmholtz velocity profile into (8), we obtain two second-order linear homogeneous differential equations,

\[ \frac{d^2 w_1}{dz^2} + l_1 w_1 = 0, \quad z > 0 \]  
\[ \frac{d^2 w_2}{dz^2} + l_2 w_2 = 0, \quad z < 0 \]  

where

\[ l_1^2 = \left[ \frac{N^2}{(U_o - c)^2} - k^2 \right] \]
\[ l_2^2 = \left[ \frac{N^2}{(U_o + c)^2} - k^2 \right] \]

in which the subscripts 1 and 2 represent the upper and lower layer, respectively. Equations (9) and (10) are known as the Taylor-Goldstein equations, which govern perturbations in stratified and parallel flows (Kundu, 2004).

Equations (9) and (10) can be solved simultaneously given appropriate boundary conditions (Lindzen, 1974). For a unique solution, the system requires four boundary conditions. At the interface between the two layers of fluid, the vertical displacement must match.

\[ \frac{w_1(z = 0)}{(U_o - c)} = \frac{w_2(z = 0)}{(U_o + c)} \]

In addition to vertical displacement, the total pressure must be continuous across the interface,

\[ (U_o - c) \frac{dw_1}{dz} \bigg|_{z=0} = (U_o + c) \frac{dw_2}{dz} \bigg|_{z=0} \]
To avoid a trivial solution the vertical velocity must vanish, at the lower rigid boundary,
\[ w(-H) = 0 \]
In addition, since wave amplitude decays with distance away from the shear zone, we will invoke the radiation condition in the upper region. This condition allows us to discard solutions in which the vertical velocity grows exponentially with vertical distance from the shear zone.

We wish to examine solutions in which the amplitude decays in the vertical direction away from the shear zone. For solutions that decay away from the shear zone, \( l_1^2 < 0 \) and \( l_2^2 < 0 \), thus our solutions for the Kelvin-Helmholtz model with a rigid lower boundary (Figure 4b) are,
\[
\begin{align*}
  w_1 &= A_1 e^{-n_1 z} \\
  w_2 &= -A_2 e^{(2n_1 H + n_2 z)} + A_2 e^{-n_2 z}
\end{align*}
\]
where \( n_1 \) and \( n_2 \) are the complex conjugates of \( l_1 \) and \( l_2 \), respectively and \( A_1 \) and \( A_2 \) are arbitrary constants. In the lower layer, the solution is unstable when the imaginary part of the complex phase speed is greater than zero. Continuity of the interfacial boundary conditions, at \( z = 0 \), yields the dispersion relationship (refer to appendix B for derivation),
\[
(U - c_r - ic_i)^2 n_1 (1 - e^{2n_1 H}) = (U + c_r + ic_i)^2 n_2 (1 + e^{2n_2 H})
\]
(12)
The stability of the wave is determined by the sign of the imaginary part of the complex phase speed (\( c_i > 0 \) growth, \( c_i < 0 \) decay). Neutral solutions (\( c_i = 0 \)) also exist and are associated with infinite overreflection (Lindzen, 1976). Infinite overreflection is associated with waves propagating energy away from the shear zone without any energy
returning to the shear zone. Because neutral modes do not interact through reflection with the rigid lower boundary and but act to smooth the shear zone, we will not discuss them in any detail here.

Up to this point, our model is identical to that described Lindzen (1976), but in order to model the evolution of waves on a warm frontal surface, we diverge from his approach and incorporate the sloping interface (Figure 4c). We do this because we wish to determine the point at which $c_i$ goes to zero as a function of fluid depth, or equivalently, distance from the surface warm front into the cool air mass. Applying the sloping interface as a horizontally heterogeneous boundary condition to the two-layer model results in a system that cannot be treated analytically. Therefore, the dynamical equivalent of the independent column approximation is used to replicate the sloping interface. Thus, the dispersion relationship is solved independently at each location along the frontal surface, an approximation that is valid as long as the frontal slope is small, as in warm fronts.

For this analysis we wish to determine the roots of (12) for a given wavenumber, Brunt-Vaisala frequency, and shear value for each fluid depth, $H$. An iterative algorithm known as the two-dimensional false position method (Acton, 1970) is used to find the roots of (12) numerically (refer to appendix C for details). This algorithm finds the zeros of the intersection of the real and imaginary parts of (12). Performing this scheme over a range of $U$, $N$, and $k$ yields the transect of the complex phase speed as a function of fluid depth. We then can relate these fluid depths to a distance from a surface warm front by assuming a constant frontal slope, as is typical of warm fronts. This yields phase speed as a function of position.
The distance from the surface front at which the imaginary part of the phase speed is equal to zero is associated with a change in the wave stability and is known as a bifurcation point (Kundu et al., 2004). Beyond this point, the growth rate due to reflection by the rigid lower boundary back to the shear zone is zero, and therefore beyond this point, the rigid lower boundary has no interaction with the shear zone. Therefore, neutral wave solutions exist beyond this point. The motivation behind this approach is that we want to ascertain if the distance from the front to the bifurcation point is in agreement with the observed wave span. That is, do the observed waves occur only for the shear zone heights predicted by linear theory.

ii) Crest Tracing

The crests of the waves observed by SDWS imagery often are not perpendicular to the surface warm front, but rather are oriented at some angle to the front. It is hypothesized that this behavior is due to the refraction experienced by a wave traveling through an inhomogeneous medium (Howe, 2007). To explore this hypothesis, a wave crest tracing procedure is developed using the transect of the complex phase speed as a function of fluid depth, or equivalently, distance from the surface warm front.

The wave tracing algorithm proceeds as follows. We choose a discrete number of points along an initial wave crest oriented perpendicular to the surface warm front (Figure 5). Then for some small time interval, \( dt \), where \( dt \ll \) period of oscillation, we allow the wave to propagate in the horizontal plane. The increasing magnitude of the complex phase speed with distance away from the warm front acts to refract the wave,
and therefore at later times the crest is no longer perpendicular to the front, but rather it is bent at some angle $\theta$ that itself can vary with distance from the surface front (Figure 5).

![Figure 5: Evolution of wave whose crest is initially perpendicular to the surface warm front. Here $\theta$ is not represented in the above figure, but is defined as the angle between the line tangent to the crest at $t+dt$ and the initial wave crest. In this figure $\theta$ increases with distance from the front.](image)

A piecewise quadratic fit is performed on the new points to determine the slope of the tangent line at each discretization point along the new crest. From this slope, we can determine the angle of refraction at each discretization point (i.e. the angle of the wave crest relative to the front). Using the angle of refraction, we can compute the new wavenumber at each discretization point

$$k = \sqrt{k_x^2 + k_y^2}$$  \hspace{1cm} (13)

where

$$k_y = k_x \tan \theta$$
In (13), the x-component of the wavenumber remains constant, while the y-component increases as a function of the angle of refraction \( \theta \). The argument behind the fixed x-component of wavenumber is as follows. Starting with two wave crests, initially parallel to one another and oriented perpendicular to the surface warm front (Figure 6a), we allow the waves to propagate for a small amount of time, \( \Delta t \) (Figure 6b). Points A and B (Figure 6a) are separated by a distance, \( \lambda \), and in time will propagate to A’ and B’ (Figure 6b), respectively. The distance between points A’ and B’ is the same as the distance between points A and B, because A and B both move in the same direction at the same speed for the same amount of time. Thus, the x-component of the wavelength is conserved, and with it the x-component of the wavenumber, \( k_x \).

![Figure 6: Illustration of wave crests, initially oriented perpendicular to the surface front, and propagation in time. Figure 6a (left) is the initial wave field, and Figure 6b (right) is wave field after time interval, \( \Delta t \).](image)

To compute the phase speed for the next time step, we must first compute the component of wind speed difference relative to the new wave crest. This step is necessary because the propagation of the wave is dependent only on that component wind speed. The wind speed perpendicular to the crest is
Using the new wind speed difference and wavenumber we can compute the complex phase speed at each discrete point along the new wave crest using the two-dimensional false point method (Acton, 1970). The x and y components of the new phase speed are used to determine the new position of the wave crest. The x and y components of the new phase speed are

\[ c_x = c_o \cos \theta \]
\[ c_y = c_o \sin \theta \]

where \( c_o \) is the phase speed at that particular discretization point for the particular iteration. The new position of the wave crest discretization point is

\[ x = x_o + c_x \Delta t \]
\[ y = y_o + c_y \Delta t \]

This process is iterated in time until a steady state is reached (i.e. when no further refraction occurs), thus may be expressed explicitly as

\[ \theta_{t-\Delta t} - \theta_t = 0. \]  \hspace{2cm} (14)

As the wavenumber and wave-crest-relative wind difference across the interface at each discretization point change, a stability boundary can be crossed as the wave propagates in the x-y plane. It is shown in the next chapter that the distance away from the surface warm front which the unstable solution space extends is highly dependent on the wavenumber. Far away from the surface front, near the bifurcation point, the wavenumber is changing the most and hence the stability of the wave is changing the most. It is shown that the larger the wavenumber, the smaller the unstable solution space extends. Hence a wave initially unstable far away from the surface warm front, near the
bifurcation point, will cross a stability boundary into neutral or stable solutions because of the increase in wavenumber due to refraction.
Chapter 4. RESULTS

a.) Observations

We examine three cases of waves observed by RADARSAT-1 SDWS imagery off the coast of Alaska on January 28th, 2004, March 16th, 2004, and March 21st, 2007 (Figure 7a-c), hereafter referred to as cases I, II, III, respectively. In all three of these cases, the waves occur ahead of a near-zero order jump in wind speed and the wave field is continuous rather than organized in a number of discrete wave packets. Spatial parameter values obtained from these three SDWS images are listed in table 1.

<table>
<thead>
<tr>
<th>Case</th>
<th>Wave Span (km)</th>
<th>Wavelength (km)</th>
<th>Angle with Front (deg)</th>
<th>Wave Amplitude (ms⁻¹)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/28/2004</td>
<td>28±1</td>
<td>3.7±1</td>
<td>68±5</td>
<td>8±1</td>
</tr>
<tr>
<td>3/16/2004</td>
<td>36±1</td>
<td>4.2±1</td>
<td>57±5</td>
<td>8±1</td>
</tr>
<tr>
<td>3/21/2007</td>
<td>51±1</td>
<td>4.3±1</td>
<td>32±5</td>
<td>6±1</td>
</tr>
</tbody>
</table>

Table I. Parameters obtained by hand measurement of SDWS images for January 28th, 2004, March 16th, 2004, and March 21st, 2007. The wave span is the distance that the observed wave extends into the cool air mass measured perpendicular to the surface warm front, the wavelength is measured from the crest of one wave perpendicularly to the adjacent wave crest, the angle that the wave creates with the front is calculated using the wave span and a line that runs perpendicular from the surface warm front to the wave, and the wave amplitude is measured by taking the difference between the surface wind speeds in a wave crest and a trough.

Figure 7d-f shows the MM5 surface weather maps corresponding to the three SDWS cases mentioned above. These maps are analyzed manually to determine the synoptic setting in which the waves are observed. For all three MM5 images, we can see that the SDWS-observed waves are occurring ahead of a surface warm front. In order to confirm that these waves are synoptically driven, we look at the extent of the waves along the front. The waves are continuous throughout the SDWS-observed length of the front for all three cases, suggesting that the waves are a feature of the front itself, rather than of one or more mesoscale entities scattered along the front.
Examining the parameters in table 1, we can see that for the three cases the observed wavelength, wave span, and wave orientation lie in a relatively narrow range. This consistency between cases suggests that the dynamic origins of the phenomena are consistent from case to case. In addition, the maximum wave perturbation amplitude is large enough to be operationally important to the maritime community.

Figure 7: SDWS images (a-c) and the corresponding MM5 surface weather maps (d-f). In the SDWS images, the reds correspond to high surface wind speeds while the blues correspond to relatively slow surface wind speeds. In the MM5 maps the lines are isobars and also the wind barbs are shown to illustrate directionality.
b.) Linear Modeling

We can assume that these SDWS-observed waves are unstable because surface interaction in our linear model is associated with waves made unstable by the reflection from the surface. Observing these wavelike perturbations in the SDWS imagery suggests strong interaction with, and thus reflection by the surface. In the atmosphere these waves can reach a nonlinear steady state however, and hence our linear model can explain only the general spatial characteristics of the SDWS observed waves. In addition, a warm frontal zone has an inversion layer which lies between the cool and warm air masses, which our simple two-layer model does not take into account.

i.) Theoretical Wave Span

It is hypothesized that the distance which these SDWS-observed waves extend away from the surface warm front into the cool air mass is associated with a bifurcation from unstable to neutral solutions as the altitude of the shear zone increases. The argument behind this hypothesis is that at some critical fluid depth, \( H_{c_i=0} \), reflection by the sea surface back to the shear zone would no longer be sufficient to cause growth because the amplitude of the vertically traveling wave would decay below a critical threshold during its trip to the surface and back. Therefore, for fluid depths greater than this critical fluid depth waves would not interact with the boundary and we would have neutral solutions. The distance from the surface warm front that these waves extend is related to the depth of the fluid at the bifurcation point by assuming a constant warm frontal slope, \( m \), and hereafter will be referred to as the theoretical span, \( S \):

\[
S = mH_{c_i=0}
\]  

(15)
Because a frontal slope of $1/200$ is representative of a typical warm front, it is used for the remainder of this study. Figure 8a shows the theoretical wave span for a wavelength of 4 km in $(U, N)$ space. Examining figure 8a we immediately can notice linearity in the contours. Also, but not shown, plots for a range of other wavelengths exhibit the same linear trends in the contours.

![Contour plots of several parameters in (U, N) space for a wavelength of 4 km.](image)

Figure 8: Contour plots of several parameters in $(U, N)$ space for a wavelength of 4 km. a.) Theoretical wave span (km) b.) Distance from surface warm front where maximum growth rate occurs (km) c.) Maximum growth rate (s$^{-1}$)

This linearity$^1$ of the contours and the similarity between the contour plots of different wavelengths suggests the existence of a dynamic similarity relationship between the theoretical wave span and the parameters $U$, $N$, and $k$. To find this relationship, we

---

$^1$ The kinks in the contours are due to the fact that solutions are numerical and hence we do not have perfect convergence for every case.
first nondimensionalize the quantities forming the axes in figure 8a in the following manner

\[ \hat{k} \equiv k \frac{U}{N} \quad \hat{S} = \frac{SN}{U} \]

where \( \hat{k} \) is the dimensionless wavenumber and \( \hat{S} \) is dimensionless theoretical wave span (Lindzen, 1976). Plotting these dimensionless variables over a range of values yields graphical confirmation and depiction of the relationship between the dimensionless wavenumber and the dimensionless theoretical wave span (Figure 9). A power law fit is performed on the curve to obtain a numerical approximation for the dimensionless theoretical wave span as a function of the dimensionless wavenumber,

\[ \hat{S} = 0.23 \left( \hat{k} \right)^{-\frac{5}{3}} \] (16)

The \( r^2 \) value for this fit is 0.995, indicating that the relationship is indeed a power law. Dimensionalizing (16) yields a direct relationship between the theoretical wave span and the parameters \( U, N, \) and \( k \).

\[ S = 0.23 (m) \left( \frac{U}{N} \right)^{\frac{2}{3}} \left( k \right)^{-\frac{5}{3}} \] (17)

SDWS-observed wave spans fall in the upper-left quadrant of \( (U, N) \) space in figure 8a. Unfortunately, we do not have stability and shear data with which to compare these results, but we can argue that the values for shear and Brunt-Vaisala frequency in the upper-left quadrant of figure 8a are consistent with observed values across a frontal zone. Examining figure 9 we can see that at low wavenumbers we have a larger theoretical wave span. This can easily be explained, since the amplitude of the wave decays with distance from the wave source, with the e-folding distance being
proportional to wavelength, thus longer waves are able to penetrate through a greater fluid depth before becoming too weak to be seen by SAR or to cause instability through reflection. Thus longer waves have longer wave spans.

Figure 9: Dimensionless wavenumber and dimensionless theoretical wave span for a range of wavelengths between 2 km and 6 km. The red circles represent the dimensionless SDWS-observed wavenumbers for a range of Brunt-Vaisala frequencies and wind differences across the interface.

**ii) Maximum Growth Rate**

Examination of the contour plot for maximum growth rate in \((U, N)\) space (figure 8b) and corresponding plots for a range of wavelengths reveals a dynamic similarity between the maximum growth rate and the parameters \(U\), \(N\), \(k\), and \(H\). We will nondimensionalize the quantities plotted in figure 8b in the following way,
\[
\hat{S} = \frac{SN}{U} \quad \quad \hat{\omega} \equiv \frac{\omega}{N}
\]

where \(\hat{\omega}\) is the dimensionless growth rate (Lindzen, 1976). Again all of the points fall on a single curve (Figure 10) and \(\hat{\omega}\) appears to have a logarithmic dependence on \(\hat{S}\). However, an equation relating span and growth rate could not be obtained. From figure 10 we notice a logarithmic linear dependence of span on wavenumber for intermediate values of growth rate, but at low growth rates the curve diverges from this logarithmic-linear dependence. Examining figure 10 we can see that the larger maximum growth rates are associated with a larger span and this is consistent with figure 9 where larger spans are associated with smaller wavenumbers. This implies that larger growth rates are associated with larger spans, which have a larger region of reflection.
iii.) Location of Maximum Growth Rate

In addition, the fluid depth, or equivalently, distance from surface warm front, where the maximum growth rate occurs can be related to the parameters $U$, $N$, and $k$. We nondimensionalize the quantities plotted in figure 8c in the following manner,

\[
\hat{k} \equiv k \frac{U}{N} \quad \hat{H}_{\omega_{\text{max}}} \equiv \frac{H_{\omega_{\text{max}}}}{U} N
\]
where $\hat{H}_{\omega_{\text{max}}}$ is the dimensionless fluid depth at which the maximum growth rate occurs.

Plotting $\hat{k}$ vs. $\hat{H}_{\omega_{\text{max}}}$ (Figure 11), we see that the points appear to be obeying a power law similar to that obtained for the dimensionless theoretical wave span. Performing a power law fit we obtain an expression relating the dimensionless location of the maximum growth rate and dimensionless wavenumber:

$$\hat{H}_{\omega_{\text{max}}} = 0.013 \left( \hat{k} \right)^{-\frac{5}{3}} \quad (18)$$

The $r^2$ value for this fit is 0.997, demonstrating that the relationship is indeed a power law. In order to express (18) in terms of distance from the surface warm front into the cool air mass we multiply the left side of the equation by a constant frontal slope, $m$. Dimensionalizing yields an expression relating the fluid depth at the maximum growth rate, $H_{\omega_{\text{max}}}$, to distance from the surface front at which the maximum growth rate occurs, $L_{\omega_{\text{max}}}$, in terms of the parameters $U$, $N$, and $k$.

$$L_{\omega_{\text{max}}} = 0.19m \left( \frac{U}{N} \right)^{\frac{2}{3}} \left( \hat{k} \right)^{-\frac{5}{3}} \quad (19)$$

Because both the location of the maximum growth rate and the theoretical wave span obey power laws of the same quantity, there is a linear dependence between the two

$$L_{\omega_{\text{max}}} = \frac{19}{22}S \quad (20)$$

This relationship implies that the maximum growth rate occurs far from the front and near the bifurcation point. This relationship can explain why the SDWS-observed wave perturbation amplitude drops off rapidly at some point near the theoretical wave span, rather than gradually across the entire wave span.
Figure 11: Dimensionless wavenumber vs. dimensionless fluid depth at which the maximum growth rate occurs. The red circles represent the dimensionless SDWS-observed wavenumbers for a range of Brunt-Vaisala frequencies and wind differences across the interface.

c) Crest Tracing

As mentioned above, the SDWS-observed waves lie on the cool side of surface warm fronts and are not oriented perpendicular to the front. Using the transect of the complex phase speed as a function of fluid depth or function of distance from the front, we can follow a wave crest in time. The wave speed transect evolves as the wave refracts and so changes its local wavenumber. We do so because this orientation produces the maximum wave growth rate. We choose to follow the evolution of a wave that is initially perpendicular to the front and is propagating in the direction of the shear vector or equivalently, parallel to the surface warm front. We choose to use the initial transect of
the complex phase speed from a point in the upper-left quadrant of (U, N) space in figure 8. This point is chosen because SDWS-observed wave spans lie in this region of (U, N) space. The real part of this initial transect varies with distance from the surface warm front, as illustrated in figure 12.

Given this initial transect, the wave will refract towards the surface warm front, much as ocean swell refracts on an approaching beach.

Figure 13 shows the propagation of this initial wave crest with time. From this figure, we can see that the wave span decreases as the wave refracts. This decrease occurs because the end of the wave crest furthest from the front experiences the largest change.
in orientation and thus wavenumber and cross-wave component of shear. As a result it crosses a stability boundary from unstable to neutral solutions. The neutral solutions beyond the stability boundary are not explored here. Comparing figure 13 to the SDWS imagery we can see that in both cases the wave crest creates an acute angle with respect to the surface warm front.

Figure 13: Propagation of wave, that which is initially perpendicular to the front (thick black line). $dt = 20$ s. The lines plotted are for every 5 time steps (100s).
Chapter 5. Conclusions

A two-layer semi-infinite Boussinesq fluid with a standard Kelvin-Helmholtz velocity profile with a sloping interface was used to model shear-driven gravity waves in a warm frontal region. It was shown that the wave span predicted through this classical linear theory is consistent with the SDWS-observed wave span. More specifically it was shown that the SDWS-observed wave span corresponds to the location along the frontal shear zone of a bifurcation from unstable to neutral solutions. This change in stability was caused by the decrease in reflection by the rigid lower boundary back to the shear zone as a function of fluid depth or equivalently, distance from the surface warm front. The variation of the SDWS-observed surface wind speed perturbation amplitude with distance from the front was explained by the location of the maximum wave growth rate relative to the front. The location of this maximum wave growth rate occurs near the bifurcation point and hence explains why the wave SDWS-observed surface wind speed perturbation amplitude is relatively uniform for most of the span and then falls off rapidly at some finite distance from the surface warm front.

It was shown that the angle that these waves make with respect to the surface warm front can be explained using a simple wave crest tracing procedure. It was found that an individual wave, its crest initially perpendicular to the surface warm front, refracts in the direction of the shear vector and crosses a stability boundary into neutrally propagating wave solutions at some critical angle of refraction.

Additional modifications can be made to the two-layer model to represent more accurately this stability profile. Observations of warm frontal zones show a transition
layer, or inversion, between the cold and warm air masses. A three-layer model with a transition layer between the upper and lower layers of the two-layer model would represent more accurately a warm frontal region. Work has been done with this three-layer model in the past (Chandrasekhar, 1961; Lindzen and Rosenthal, 1983), but the primary focus of these works has been to better understand clear air turbulence rather than the dynamics of vertically propagating gravity waves in the lowermost layer.

In addition, atmospheric data to support our analytical results was sparse. Numerical weather prediction model output with greater horizontal and vertical resolution would allow us to gather information on the shear and stability profiles of the region in which these gravity waves are being observed. In addition, greater vertical resolution would allow us to gather information of the transitional zone, or inversion, between the cold and warm air masses. The depth of the inversion layer would be a critical parameter in the accurate implementation of a three-layer semi-infinite model. Similar information could be obtained in situ or with dropsondes using research aircraft.

In this study, we tried to explain waves observed ahead of surface warm fronts in which the warm front was straight and the wave field was homogeneous throughout the length of the front. More often than not waves are observed on fronts that are not straight, but rather are curved and wrapped around the low. Also, in many of these cases, waves are observed propagating in a number of discrete wave packets along the front. A more complex model with an inhomogeneous frontal surface would be required to model these observed phenomena.
APPENDIX A:

Wave Equation Derivation

Here we derive the governing wave equation for wavelike perturbations in parallel stratified flow. Decomposing the Boussinesq equation set into a base state and a perturbation about that base state:

\[
U = U_o + u', \\
v = v', \\
w = w' ,
\]

where \( U_o \) represents the base state zonal wind and \( u' \) represents the perturbation about this base state, and \( v' \) and \( w' \), are the perturbation velocities, in the y and z-directions, respectively, we arrive at a set of perturbation equations (a1) – (a5) that describe the behavior of a perturbation at the interface between two fluids moving at different velocities:

\[
\frac{\partial u'}{\partial t} + U_o \frac{\partial u'}{\partial x} = -\frac{1}{\rho_o} \frac{\partial p'}{\partial x} \tag{a1}
\]

\[
\frac{\partial v'}{\partial t} + U_o \frac{\partial v'}{\partial x} = -\frac{1}{\rho_o} \frac{\partial p'}{\partial y} \tag{a2}
\]

\[
\frac{\partial w'}{\partial t} + U_o \frac{\partial w'}{\partial x} = -\frac{1}{\rho_o} \frac{\partial p'}{\partial z} - g \frac{\rho'}{\rho_o} \tag{a3}
\]

\[
\frac{\partial \rho'}{\partial t} + U_o \frac{\partial \rho'}{\partial x} = w \frac{\rho_o}{g} N^2 \tag{a4}
\]

\[
\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \tag{a5}
\]
Here $N^2$ is the Brunt-Visalia frequency, $p'$ is the pressure perturbation, and $\rho'$ is the density perturbation. We can combine these equations (5 equations and 5 unknowns) to form a single equation for the vertical velocity field $w'$. To do this, we first cross-differentiate (a1) with (a3) to eliminate the pressure gradient terms

$$\frac{\partial}{\partial z} (a1) = \frac{\partial^2 u'}{\partial z^2} + U_o \frac{\partial^2 u'}{\partial x^2} = -\frac{1}{\rho_o} \frac{\partial^2 p'}{\partial z \partial x}$$

$$\frac{\partial}{\partial x} (a3) = \frac{\partial^2 w'}{\partial x^2} + U_o \frac{\partial^2 w'}{\partial x^2} = -\frac{1}{\rho_o} \frac{\partial^2 p'}{\partial z \partial x} - \frac{g}{\rho_o} \frac{\partial \rho'}{\partial x}$$

which yields,

$$\left( \frac{\partial}{\partial t} + U_o \frac{\partial}{\partial x} \right) \left( \frac{\partial w'}{\partial x} - \frac{\partial u'}{\partial z} \right) + \frac{\partial b}{\partial x} = 0 \tag{a6}$$

where the buoyancy is defined as,

$$b = \frac{g \rho'}{\rho_o}$$

Next cross-differentiate (a2) and (a3) to eliminate the pressure gradient terms

$$\frac{\partial}{\partial z} (a2) = \frac{\partial^2 v'}{\partial z^2} + U_o \frac{\partial^2 v'}{\partial z \partial y} = -\frac{1}{\rho_o} \frac{\partial^2 p'}{\partial z \partial y}$$

$$\frac{\partial}{\partial y} (a3) = \frac{\partial^2 w'}{\partial y \partial t} + U_o \frac{\partial^2 w'}{\partial y \partial z} = -\frac{1}{\rho_o} \frac{\partial^2 p'}{\partial z \partial y} - \frac{g}{\rho_o} \frac{\partial \rho'}{\partial y}$$

which yields,

$$\left( \frac{\partial}{\partial t} + U_o \frac{\partial}{\partial x} \right) \left( \frac{\partial w'}{\partial y} - \frac{\partial v'}{\partial z} \right) + \frac{\partial b}{\partial y} = 0 \tag{a7}$$

Now in order to eliminate the $u'$ and $v'$ terms in (a6) and (a7) we use the equation of continuity (a5) and

$$\frac{\partial}{\partial x} (a6) + \frac{\partial}{\partial y} (a7) = 0$$
to get,

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial y^2} + \frac{\partial^2 w'}{\partial z^2}\right) + U_o \frac{\partial}{\partial x}\left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial y^2} + \frac{\partial^2 w'}{\partial z^2}\right) + \nabla^2 b = 0 \quad (a8)$$

where $\nabla^2$ represents the horizontal Laplacian. Next, to eliminate the buoyancy term we use (a4). That may also be written as,

$$\frac{\partial b}{\partial t} + U_o \frac{\partial b}{\partial x} = -N^2 w' \quad (a9)$$

Then by taking the horizontal Laplacian of (a9) and solving for the $\nabla^2 b$, we get

$$\left(\frac{\partial}{\partial t} + U_o \frac{\partial}{\partial x}\right)\nabla^2 b = -N^2 \nabla^2 w' \quad (a10)$$

Multiplying (a8) by $\left(\frac{\partial}{\partial t} + U_o \frac{\partial}{\partial x}\right)$ and substituting (a10) into (a8), we get

$$\left(\frac{\partial}{\partial t} + U_o \frac{\partial}{\partial x}\right)^2\left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial y^2} + \frac{\partial^2 w'}{\partial z^2}\right) + N^2 \left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial y^2}\right) = 0$$

In our two-layer model the vertical velocity perturbation $w'$ is independent of the y-direction, and so we get,

$$\left(\frac{\partial}{\partial t} + U_o \frac{\partial}{\partial x}\right)^2\left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial z^2}\right) + N^2 \left(\frac{\partial^2 w'}{\partial x^2}\right) = 0$$

Looking for wavelike solutions to this wave equation will yield the Taylor-Goldstein equation. The Taylor-Goldstein equation governs the behavior of perturbations in parallel stratified flow.
APPENDIX B:

Dispersion Relationship Derivation

In order to derive the unstable dispersion relationship, we must start at the boundary conditions and eliminate the terms $C_1, C_2, C_3,$ and $C_4$.

Continuity of displacement at the interfacial region ($z = 0$) requires that,

$$\frac{w_1(0)}{(U - c_r - ic_i)} = \frac{w_2(0)}{(U + c_r + ic_i)}$$

$$\frac{C_2}{(U - c_r - ic_i)} = \frac{C_3 + C_4}{(U + c_r + ic_i)} \quad (b1)$$

Continuity of pressure at the interfacial region ($z = 0$) requires that,

$$\left( U - c_r - ic_i \right) \frac{dw_1(0)}{dz} = \left( U + c_r + ic_i \right) \frac{dw_2(0)}{dz}$$

$$-\left( U - c_r - ic_i \right) \left( il_1 C_2 \right) = \left( U + c_r + ic_i \right) \left( il_2 \left( C_3 - C_4 \right) \right) \quad (b2)$$

Substituting (b1) into (b2) to eliminate $C_2$, we obtain

$$-\left( U - c_r - ic_i \right) \left( il_1 \frac{(U + c_r + ic_i)}{(U - c_r - ic_i)} \left( C_3 + C_4 \right) \right) = \left( U + c_r + ic_i \right) \left( il_2 \left( C_3 - C_4 \right) \right)$$

$$-\left( U - c_r - ic_i \right)^2 \left( il_1 \left( C_3 + C_4 \right) \right) = \left( U + c_r + ic_i \right)^2 \left( il_2 \left( C_3 - C_4 \right) \right) \quad (b3)$$

Vertical velocity must vanish at the lower rigid boundary,

$$w_2(-H) = 0$$

so that

$$C_3 e^{-il_2 H} + C_4 e^{-il_2 H} = 0$$

and

$$C_3 = -C_4 e^{2il_2 H} \quad (b4)$$
Substituting (b4) into (b3) to eliminate the $C_3$ term,

$$-(U - c_r - ic_i)^2 \left( il_1 \left(-C_4 e^{2i\lambda H} + C_4\right) \right) = \left(U + c_r + ic_i\right)^2 \left( il_2 \left(-C_4 e^{2i\lambda H} - C_4\right) \right)$$

and

$$-(U - c_r - ic_i)^2 \left( il_1 \left(1 - e^{2i\lambda H}\right) \right) = \left(U + c_r + ic_i\right)^2 \left( il_2 \left(1 + e^{2i\lambda H}\right) \right)$$

because, $il_1 \equiv n_1$ and $il_2 \equiv n_2$, we obtain

$$(U - c_r - ic_i)^2 n_1 (1 - e^{2n H}) = (U + c_r + ic_i)^2 n_2 (1 + e^{2n H})$$

This equation gives the dispersion relationship for unstable solutions in a two-layer Kelvin-Helmholtz velocity profile with a rigid lower boundary present.
APPENDIX C:

Two-Dimensional Secant Method

To solve the dispersion relationship numerically for unstable solutions, we use the numerical procedure developed by Acton (1970) and used by Lindzen and Rosenthal (1976) and Pellacini et al. (1978), the two-dimensional false position method.

Start with dispersion relationship (12) for the unstable solutions:

\[
(U + c_R \pm ic_i) \left( k^2 - \frac{N^2}{(U \pm c_R \mp ic_i)} \right) \left( 1 \pm e^{ \frac{2\hat{H} k^2 - \frac{N^2}{(U \pm c_R \mp ic_i)}}{2\hat{H} k^2 - \frac{N^2}{(U \pm c_R \mp ic_i)}}} \right) = 0
\]

(12)

Nondimensionalation is performed on the variables in (12) to make them dependent only on the following dimensionless variables (Lindzen, 1976).

\[
\hat{k} = \frac{k}{N}
\]

\[
\hat{c} = \frac{c}{U}
\]

\[
\hat{H} = \frac{H N}{U}
\]

\[
(1 + \hat{c}_R \pm i\hat{c}_i) \left( \hat{k}^2 - \frac{1}{(1 \pm \hat{c}_R \pm i\hat{c}_i)} \right) \left( 1 \pm e^{ \frac{2\hat{H} \hat{k}^2 - \frac{1}{(1 \pm \hat{c}_R \pm i\hat{c}_i)}}{2\hat{H} \hat{k}^2 - \frac{1}{(1 \pm \hat{c}_R \pm i\hat{c}_i)}}} \right) = 0
\]

(c12)
We want to find the intersection of the zeros of the real and imaginary parts of (c12) denoted as $R(c_r, c_i)$ and $I(c_r, c_i)$, respectively. According to the secant method (Acton, 1970), our next estimate for the root is given by,

$$
\begin{vmatrix}
\hat{c}_{r,4} - \hat{c}_{r,1} & \hat{c}_{r,4} - \hat{c}_{r,2} & \hat{c}_{r,4} - \hat{c}_{r,3} \\
R_1 & R_2 & R_3 \\
I_1 & I_2 & I_3 \\
\end{vmatrix} = 0
$$

and

$$
\begin{vmatrix}
\hat{c}_{i,4} - \hat{c}_{i,1} & \hat{c}_{i,4} - \hat{c}_{i,2} & \hat{c}_{i,4} - \hat{c}_{i,3} \\
R_1 & R_2 & R_3 \\
I_1 & I_2 & I_3 \\
\end{vmatrix} = 0
$$

where $c_{i,j}$ and $c_{r,j}$ ($j=1,2,3$) are three initial estimates for the root. We then solve for new estimate $(c_{r,4}, c_{i,4})$,

$$
\hat{c}_{r,4} = \frac{(R_2 I_3 - R_3 I_2)\hat{c}_{r,1} + (R_3 I_1 - R_1 I_3)\hat{c}_{r,2} + (R_1 I_2 - R_2 I_1)\hat{c}_{r,3}}{R_2 I_3 - R_3 I_2 + R_3 I_1 - R_1 I_3 + R_1 I_2 - R_2 I_1}
$$

and

$$
\hat{c}_{i,4} = \frac{(R_2 I_3 - R_3 I_2)\hat{c}_{i,1} + (R_3 I_1 - R_1 I_3)\hat{c}_{i,2} + (R_1 I_2 - R_2 I_1)\hat{c}_{i,3}}{R_2 I_3 - R_3 I_2 + R_3 I_1 - R_1 I_3 + R_1 I_2 - R_2 I_1}
$$

We discard the first estimate and keep $j = 2, 3, 4$ as our new estimates and repeat until the error satisfies the convergence criterion. The convergence criterion is,

$$
\left\| \left( c_{r,n+3}, c_{i,n+3} \right) - \left( c_{r,n}, c_{i,n} \right) \right\| / \left\| \left( c_{r,n}, c_{i,n} \right) \right\| \leq 10^{-6}
$$

where $n$ represents the particular iterate. This process is performed over a range of $H$ for a given $U$, $N$, and $k$ in order to obtain the transect of the complex phase speed as a function of fluid depth, or equivalently, distance from the surface warm front.
REFERENCES:


Howe, M. S., 2007: Hydrodynamics and Sound. *Cambridge University Press*


Teschl, G. 2007: Ordinary differential equations and dynamical systems.

