THE COHOMOLOGICAL EQUATION AND REPRESENTATION THEORY

A Thesis in Mathematics by David J. Mieczkowski

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Abstract

The purpose of this thesis is to explore the use of representation theory to prove some new results concerning the solvability of the cohomological equation for certain homogeneous actions on $G/\Gamma$, where $\Gamma$ is always a lattice. In the rank one case, there are obstructions to the solvability of the cohomological equation, whereas for higher rank situations those obstructions automatically vanish, and one can always solve the cohomological equation.

We start by characterizing the obstructions to the solutions of the single generator equation $Xf = g$ on $PSL(2,\mathbb{R})/\Gamma$, where $X$ is the generator of the geodesic flow. Then we show how those obstructions automatically vanish in higher rank cases. We show that the first cohomology of actions of certain higher rank abelian groups $A$, acting by left translation on $PSL(2,\mathbb{R}) \times G/\Gamma$, where $G = PSL(2,\mathbb{R}), SL(2,\mathbb{C})$ or $SO(N,1)$, are reduced to constant coefficients. The groups are generated by various mixtures of the diagonal and/or unipotent generators on each factor. In addition, we show that the first cohomology of the action of the upper-triangular group $N$, of $SL(2,\mathbb{C})$, on $SL(2,\mathbb{C})/\Gamma$ is also trivial, where all groups are viewed as real Lie groups. This provides some examples of higher rank abelian actions by parabolic operators.
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Preface

The research in this thesis was motivated by the study of the paper of Livio Flaminio and Giovanni Forni [4]. In that paper the cohomological equation $Uf = g$ was studied in detail using the theory of group representations. Motivation is also drawn from the paper of Anatole Katok and Ralf Spatzier [16], where the first smooth cohomology group of Anosov actions of higher rank abelian groups is shown to be trivial. The present thesis uses the theory of group representations to study cohomological equations on homogeneous spaces $G/\Gamma$, which arise from parabolic systems, as well as some systems that have at least some hyperbolic characteristics.
Chapter 1

Introduction

The rigidity properties of hyperbolic actions of $\mathbb{Z}^k$ or $\mathbb{R}^k$ for $k \geq 2$ have been the object of much study over the last decade or so. A. Katok and R. Spatzier showed that the first cohomology of smooth cocycles over Anosov actions of higher rank abelian groups reduces to constant coefficients [16]. The lack of obstructions to the solution of the cohomological equation is in sharp contrast to the obstructions characterized by Livshitz [16] for a hyperbolic $\mathbb{Z}$ or $\mathbb{R}$ action. There the sum (or integral) of the values of the cocycle along every periodic orbit for the $\mathbb{Z}$ (or $\mathbb{R}$) action must be equal to 0, and represents an infinite-dimensional space of obstructions.

In their paper, L. Flaminio and G. Forni [4] studied the obstructions to the solution of the cohomological equation for the horocycle flow on (compact) homogeneous spaces of the form $PSL(2, \mathbb{R})/\Gamma$. They showed that the space of obstructions to the equation $Uf = g$ (where $f, g \in L^2(PSL(2, \mathbb{R})/\Gamma)$) is of infinite countable dimension. In particular, they showed that the obstructions are in fact distributions of strictly positive order, which are invariant under the horocycle flow. This shows that even for $\mathbb{R}$ actions best described as parabolic, there are still plenty of obstructions to the existence of solutions to the cohomological equation.

In this thesis, we give several examples of higher rank parabolic abelian actions for which all obstructions to the cohomological equation vanish, and the first smooth (or $C^r$) cohomology trivializes. One such example is the action of the upper triangular group $N$ on $SL(2, \mathbb{C})/\Gamma$, where we view $N$ and $SL(2, \mathbb{C})$ as real Lie groups, and $\Gamma$ is an arithmetic lattice. This is a direct generalization of the horocycle flow acting on the unit tangent bundle of a Riemann surface of constant negative curvature. In fact, it is related to the frame flow on the frame bundle over a 3 dimensional hyperbolic space, in much the same way as the horocycle flow is related to the geodesic flow on $PSL(2, \mathbb{R})/\Gamma$.

Now the classical horocycle flow is the flow on $PSL(2, \mathbb{R})/\Gamma$ given by left multiplication by the matrix group $\{U_t = (1 t \ 0 1)\}$. We show that the first smooth (or Sobolev) cohomology group over any higher rank abelian action of $A$ on $(PSL(2, \mathbb{R}) \times G)/\Gamma$,
where $U_t \in A$ and $\Gamma$ is an irreducible lattice (not necessarily cocompact) in $PSL(2, \mathbb{R}) \times G$ and $G$ is a semisimple Lie group with finite center, trivializes. If $A$ contains $\{X_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \}$ then we get the same result.

The method used in [4] is representation theory. Namely, by considering the cohomological equation $U f = g$ for $f, g \in L^2(PSL(2, \mathbb{R})/\Gamma)$, they were able to reduce the problem to a problem on irreducible unitary representations of $PSL(2, \mathbb{R})$, and then obtain some uniform estimates in order to 'glue' the solutions together to get a bona fide solution in $L^2(PSL(2, \mathbb{R})/\Gamma)$. In this thesis, the same method is applied to obtain a characterization of the obstructions, and solution to the cohomological equation $X f = g$ ($f, g \in L^2(PSL(2, \mathbb{R})/\Gamma)$), where $X$ is the generator of the geodesic flow. This provides an alternate description of the results of Livshitz (for this particular hyperbolic action) in the compact case, and extends it to cover non-cocompact lattices.

We then apply representation theory to show that the first smooth cohomology group is trivial for examples of the type $A$ acting on $(PSL(2, \mathbb{R}) \times G)/\Gamma$. Because irreducible unitary representations of $G \times H$ split into tensor products of irreducible unitary representations of $G$ and $H$ separately, we are able to rely heavily on the representation theory of $PSL(2, \mathbb{R})$, and not very much on the representation theory of the second factor. We also make repeated use of the ergodicity of the $A$ action (Moore’s Ergodicity Theorem [24]) in showing that it is sufficient to control the size of only one generator of $A$, in this case the one which comes from the $PSL(2, \mathbb{R})$ factor, in each irreducible unitary representation.

Finally, we show that the first smooth cohomology of the $N$ action on $SL(2, \mathbb{C})/\Gamma$ is trivial using the more complicated representation theory of $SL(2, \mathbb{C})$, viewed as a real Lie group. Here we are forced to use the analytic description, rather than the algebraic description, of irreducible unitary representations. Although the representation theory of $SL(2, \mathbb{C})$ is in many ways the next easiest for a semisimple Lie group (after $SL(2, \mathbb{R})$), already this shows how application of representation theory to the solution of the cohomological equation is both powerful and delicate.

Also, insomuch as this thesis produces some interesting new results concerning higher rank abelian actions which have some or all generators which are parabolic, it also provides some insight into how representation theory may, and may not, effectively be used to study problems in smooth dynamical systems.
1.1 Statement of Results

There is a close connection between the spaces $PSL(2, \mathbb{R})/\Gamma$ and Riemann surfaces, as $PSL(2, \mathbb{R})/\Gamma$ can be interpreted as a fiber space whose base is some Riemann surface and whose fiber is a circle ($PSO(2)$). In fact, the space $PSL(2, \mathbb{R})/\Gamma$ can be identified with the unit tangent bundle of the surface $M = PSO(2) \backslash PSL(2, \mathbb{R})/\Gamma$.

The thesis begins with the treatment of the cohomological equation $Xf = g$ in $PSL(2, \mathbb{R})$. Here $X = \left( \begin{array}{cc} 1/2 & 0 \\ 0 & -1/2 \end{array} \right) \in sl(2, \mathbb{R})$ is the generator of the geodesic flow on $PSL(2, \mathbb{R})/\Gamma$, and $f, g \in L^2(PSL(2, \mathbb{R})/\Gamma)$. Note that $X$ is a densely defined operator on $L^2(PSL(2, \mathbb{R})/\Gamma)$.

First, the $X$-invariant distributions are classified. Let $E'(L^2(PSL(2, \mathbb{R})/\Gamma))$ be the dual space of the space $C^\infty(L^2(PSL(2, \mathbb{R})/\Gamma))$ of smooth vectors in $L^2(PSL(2, \mathbb{R})/\Gamma)$. Then the subspace

$$I(PSL(2, \mathbb{R})/\Gamma) = \{ D \in E'(L^2(PSL(2, \mathbb{R})/\Gamma)) | L_X D = 0 \}$$

of $X$-invariant distributions is determined by the spectrum of the Laplacian $\triangle$.

The spectrum $\sigma(\triangle_M)$ of the Laplacian on $M$ has pure point discrete component of finite multiplicity and an absolutely continuous component on the interval $[1/4, \infty)$ with finite multiplicity equal to the number of cusps of $M$ [8]. The Laplacian $\triangle_M$ has a smallest nonzero eigenvalue, $\mu_0 > 0$. Selberg’s spectral gap conjecture [21] asserts that there are no eigenvalues in $(0, 1/4)$ for any arithmetic lattice. Let $\sigma_{pp}$ denote the pure point spectrum of $\triangle_M$ and $C$ the (finite) set of cusps of $M$. And of course, if $M$ is compact, the spectrum of the Laplacian $\triangle_M$ is supported on $\sigma_{pp}$.

THEOREM 1.1. The space $I(PSL(2, \mathbb{R})/\Gamma)$ has infinite countable dimension. Furthermore, there is a decomposition

$$I(PSL(2, \mathbb{R})/\Gamma) = \bigoplus_{\mu \in \sigma_{pp}} m_\mu \cdot I_\mu \oplus \bigoplus_{n \in \mathbb{Z}^+} m_n \cdot (I_n \oplus I_{-n}) \oplus \bigoplus_{c \in C} I_c \quad (1.1)$$

where, $(s \in \mathbb{R})$

1. for $\mu = 0$, the space $I_0$ is spanned by the $PSL(2, \mathbb{R})$-invariant volume,

2. for $0 < \mu < 1/4$, the space $I_\mu \subset W^{-s}(PSL(2, \mathbb{R})/\Gamma)$, provided $s > 1$, and it has dimension equal to 2.
3. for $\mu \geq 1/4$, the space $\mathcal{I}_\mu \subset W^{-s}(PSL(2, \mathbb{R})/\Gamma)$, provided $s > 1/2$, and it has dimension equal to 2.

4. for $n \in \mathbb{Z}^+$, the space $\mathcal{I}_n \subset W^{-s}(PSL(2, \mathbb{R})/\Gamma)$, provided $s > 0$, and it has dimension equal to 2. Similarly for $\mathcal{I}_{-n}$.

5. for $c \in C$, the space $\mathcal{I}_c \subset W^{-s}(PSL(2, \mathbb{R})/\Gamma)$, provided $s > 1/2$, and it has infinite countable dimension.

and where each $m_\mu$ represents a finite multiplicity.

The $X$-invariant distributions of order $s > 1$ form a complete set of obstructions to the existence of smooth solutions of the cohomological equation $Xf = g$ for functions $g \in W^s(PSL(2, \mathbb{R})/\Gamma)$. Let

$$\mathcal{I}^s(PSL(2, \mathbb{R})/\Gamma) = \{ D \in W^{-s}(PSL(2, \mathbb{R})/\Gamma) | \mathcal{L}_X D = 0 \}$$

Then $\mathcal{I}^s(PSL(2, \mathbb{R})/\Gamma)$ is determined by Theorem 4.1.

**Theorem 1.2.** Let $s > 1$, then there exists a constant $C_{s,t}$ such that $\forall g \in W^s(PSL(2, \mathbb{R})/\Gamma)$,

- if $t < -1$, and $g$ has no component on the trivial sub-representation of $L^2(PSL(2, \mathbb{R})/\Gamma)$, or
- if $t < s - 1$ and $D(g) = 0$ for all $D \in \mathcal{I}^s(PSL(2, \mathbb{R})/\Gamma)$,

then the equation $Xf = g$ has a solution $f \in W^t(PSL(2, \mathbb{R})/\Gamma)$, which satisfies the Sobolev estimate $\|f\|_t \leq C_{s,t}\|g\|_s$. Solutions are unique modulo the trivial sub-representation if $t > 0$.

**Proof.** This will follow from Theorem 4.3 by setting $\mathcal{H} = L^2(PSL(2, \mathbb{R})/\Gamma)$.

Now let $A = A_1 \times A_2 < (PSL(2, \mathbb{R}) \times G)$ be a higher rank abelian subgroup, such that $A_1$ is the one parameter subgroup $\{ \exp tX \}$ or $\{ \exp tU \}$, where $X = (1/2 \ 0 \ 0 \ 0)$, $U = (0 \ 1 \ 0 \ 0) \in sl(2, \mathbb{R})$, and $A_2$ is a non-compact abelian subgroups of $G$. The next results concern the first smooth cohomology for cocycles over the $A$ action on $(PSL(2, \mathbb{R}) \times G)/\Gamma$, where $G$ is either $PSL(2, \mathbb{R})$, $SL(2, \mathbb{C})$, or $SO(N, 1)$ for $N \geq 4$, and $\Gamma$ is an irreducible lattice in $(PSL(2, \mathbb{R}) \times G)$ for which $L^2((PSL(2, \mathbb{R}) \times G)/\Gamma)$ contains no irreducible unitary sub-representations which are arbitrarily close to the trivial representation. That is, we consider smooth cocycles of the form $\beta : A \times (PSL(2, \mathbb{R}) \times G)/\Gamma \rightarrow \mathbb{R}^k$. By smooth cocycle we mean that for all $a \in A$, we must have $\beta(a, -) \in C^\infty(L^2((PSL(2, \mathbb{R}) \times G)/\Gamma))$ and $\beta$ is a smooth function in the usual sense. The first smooth almost cohomology group is the group of cohomolgy classes of cocycles which are almost cohomologous (i.e up to a constant) via a smooth function $P : (PSL(2, \mathbb{R}) \times G)/\Gamma \rightarrow \mathbb{R}^k$. 
**Theorem 1.3.** The first smooth almost cohomology group over the $A$ action on $(\text{PSL}(2, \mathbb{R}) \times G)/\Gamma$ is trivial.

In fact, we can take $P \in C^\infty(L^2((\text{PSL}(2, \mathbb{R}) \times G)/\Gamma))$.

Finally, let $N = \{ \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} | c \in \mathbb{C} \}$ be the upper triangular subgroup of $SL(2, \mathbb{C})$, where we view both $N$ and $SL(2, \mathbb{C})$ as real Lie groups. Then $N$ is a rank 2 abelian subgroup which acts on $SL(2, \mathbb{C})/\Gamma$ by parabolic operators. Let $\beta : N \times SL(2, \mathbb{C})/\Gamma \to \mathbb{R}^k$ be a smooth $\mathbb{R}^k$-cocycle over the $N$ action on $SL(2, \mathbb{C})/\Gamma$, where we again call $\beta$ smooth if $\beta(n, -) \in C^\infty(L^2(SL(2, \mathbb{C})/\Gamma))$ for all $n \in N$, and $\beta$ is smooth in the usual sense. Again, the first smooth almost cohomology group is the group of cohomology classes of cocycles which are almost cohomologous (i.e up to a constant) via a smooth function $P : SL(2, \mathbb{C})/\Gamma \to \mathbb{R}^k$.

Now let $\Gamma$ be any lattice in $SL(2, \mathbb{C})$ such that the direct (sum) integral decomposition of $L^2(SL(2, \mathbb{C})/\Gamma)$ has no component of the complementary series of $SL(2, \mathbb{C})$, then

**Theorem 1.4.** The first smooth almost cohomology group over the $N$ action on $SL(2, \mathbb{C})/\Gamma$ is trivial.

As an artifact of the proof, if $\Gamma$ is not cocompact, we will not be able to take $P \in C^\infty(L^2(SL(2, \mathbb{C})/\Gamma))$ in general. We do not know if this can be overcome.

We note that the Generalized Ramanujan-Selberg conjecture states that if $\Gamma$ is an arithmetic lattice in $SL(2, \mathbb{C})$, then the direct (sum) integral decomposition of $L^2(SL(2, \mathbb{C})/\Gamma)$ has no component of the complementary series of $SL(2, \mathbb{C})$. So the above theorem is true for all arithmetic lattices provided the conjecture is true.

Indeed, the method of proof is to decompose the Hilbert space $L^2(SL(2, \mathbb{C})/\Gamma)$ into a direct integral of irreducible unitary representations, and then study the cohomological equation on each irreducible unitary representation separately. Uniform estimates on the size of the Sobolev norms of the solution, across all irreducible unitary representations which appear in $L^2(SL(2, \mathbb{C})/\Gamma)$, are then needed to obtain a bona fide global solution. At present we cannot prove such uniform estimates across members of the complementary series of $SL(2, \mathbb{C})$, and so must exclude cases in which they appear.
Chapter 2

Overview of Previous Research

The research presented in this thesis is motivated primarily by the paper of A. Katok and R. Spatzier [16] and the paper of L. Flaminio and G. Forni [4]. We now present an overview of those articles.

In their paper, First Cohomology of Anosov Actions of Higher Rank Abelian Groups and Applications to Rigidity [16], Katok and Spatzier showed that the smooth almost cohomology of smooth cocycles over the standard hyperbolic actions of $\mathbb{Z}^k$ and $\mathbb{R}^k$ for $k \geq 2$ is trivial. They also obtain similar results for Hölder and $C^1$ cocycles, although the original proof for the Hölder and $C^1$ results is incorrect; a correct proof is based upon decay estimates obtained by Kleinbock and Margulis [17].

The standard Anosov $\mathbb{Z}^k$ and $\mathbb{R}^k$ actions are split into two essential categories.

**Example 1.** (Automorphisms of tori and nilmanifolds and their suspensions).

Consider a $\mathbb{Z}^k$-action by toral automorphisms, $\mathbb{T}^n \to \mathbb{T}^n$ where $n > k$. Call an action irreducible if no finite cover splits as a product, and call an action by toral automorphisms standard if it is Anosov and if it contains a $\mathbb{Z}^2$-action such that every non-trivial element of $\mathbb{Z}^2$ acts ergodically with respect to Haar measure. Any standard action is irreducible.

These examples can be generalized to Anosov $\mathbb{Z}^k$-actions by automorphisms of nilmanifolds. We can construct $\mathbb{R}^k$-actions from these examples using the construction of a suspension of an action over a manifold.

The second class of examples comes from symmetric spaces,

**Example 2.** (Symmetric space examples).

Let $G$ be a semisimple connected real Lie group of noncompact type and of real rank at least 2. Given any Iwasawa decomposition of $G = KAN$, consider the left action of $A$ on $G/\Gamma$, where $\Gamma$ is an irreducible torsion-free cocompact lattice in $G$. Also, if $M = Z(A)$ is the centralizer of $A$ in $G$, then this action descends to an action on $N = M\setminus G/\Gamma$. This action is called the Weyl chamber flow of $A$, and is an Anosov action. Notice also that every element of $A$ acts ergodically on $G/\Gamma$, by Moore’s ergodicity theorem. These actions are called standard.
From these examples we can also construct the twisted symmetric space examples by forming the semi-direct product $M \rtimes G \rtimes T$, given by a representation $\rho : \Gamma \to SL(n, \mathbb{Z})$ irreducible over $\mathbb{Q}$.

One of the main results of the paper is now recalled.

**Theorem 2.1 (Katok-Spatzier).**

1. Consider a standard Anosov $A$-action on a manifold $M$ where $A$ is isomorphic to $\mathbb{R}^k$ or $\mathbb{Z}^k$ with $k \geq 2$. Then any $C^\infty$ cocycle $\beta : A \times M \to \mathbb{R}$ is $C^\infty$-cohomologous to a constant cocycle.

2. Any Hölder cocycle into $\mathbb{R}^l$ is Hölder cohomologous to a constant cocycle.

The main elements of the proof are as follows. First, by considering coordinate functions, one can assume that $l = 1$. Second, the constant must be $c(b) = \int_M \beta(b, x) dx$. That is, $\beta - c$ is shown to be cohomologous to $0$. If we fix an $a \in A$, and define $f(x) = \beta(a, x)$, then the transfer function is defined (formally) by $P = \sum_{k=0}^\infty a^k f$. One could also define $P = \sum_{k=-\infty}^{-1} a^k f$, but as it turns out, the obstruction $\sum_{k=-\infty}^{\infty} a^k f$ vanishes, regardless of how we choose $a \in A$.

To show that the (formal) solutions coincide, and are $C^\infty$ functions, decay rates on the Fourier coefficients in the toral case, and estimates on the rate of decay of the matrix coefficients of $L^2(G/\Gamma)$ are used for the Weyl chamber flow. The estimates on the rate of decay of the matrix coefficients of $L^2(G/\Gamma)$ for the Weyl chamber flow are used to obtain differentiability in the stable and unstable directions only. Then a generalization of results of Hörmander, which they obtain in [15], is used to show differentiability in all directions. Since we use the same result in section 4.5.7, we state it now.

**Theorem 2.2 (Katok-Spatzier).** Let $D_1, \ldots, D_k$ be $C^\infty$ plane fields on a manifold $M$ such that their sum $\sum_{i=1}^k D_i$ is totally non-integrable and satisfies the following condition,

(*) For each $j$, the dimension of the space spanned by the commutators of length at most $j$ at each point is constant in a neighborhood.

Let $P$ be a distribution on $M$. Assume that for any positive integer $p$ and $C^\infty$ vectorfield $X$ tangent to any $D_j$, the $p$'th partial derivative $X^p(P)$ exists as a continuous or local $L^2$ function. Then $P$ is $C^\infty$ on $M$.

In the present thesis, a similar outline is used, albeit using an infinitesimal formulation. The difference is that the estimates needed to show that formal solutions
are bona fide smooth or $C'$ solutions come from a deeper application of representation theory to the group $G$. In particular, the estimates are obtained in each irreducible unitary representation. The estimates are uniform across all irreducible unitary representations that appear as sub-representations of $L^2(G/\Gamma)$. Of course, our work only concerns groups, and their products, within the $SO(N,1)$ series, where the representation theory is completely known and, when applied to some problems, tractable.

The second paper, *Invariant Distributions and Time Averages for Horocycle Flows*, by Flaminio and Forni [4], provides the motivation to use representation theory to study the cohomology equation for groups that contain parabolic elements. Following a suggestion of A. Katok in [12] (see also [14]), the authors use the full force of the representation theory of $PSL(2,R)$ to study the cohomological equation $Uf = g$, where $U$ is the generator of the classical horocycle flow on the unit tangent bundle of a (compact) Riemann surface, which can be identified with $PSL(2,R)/\Gamma$. They obtain results for the rank one action of $R$ through the horocycle flow. Like the Livshitz theorem for an Anosov flow, in the horocycle case the space of obstructions has infinite countable dimension. Further, if the obstructions vanish for a smooth vector $g \in C^\infty(L^2(PSL(2,R)/\Gamma))$, then there is a smooth vector solution $f \in C^\infty(L^2(PSL(2,R)/\Gamma))$.

In order to study the equation $Uf = g$, the Hilbert space $L^2(PSL(2,R)/\Gamma)$, which is a unitary representation of $PSL(2,R)$ under the left regular representation, is decomposed as a direct integral of irreducible unitary representations for $PSL(2,R)$. The decomposition is,

$$L^2(PSL(2,R)/\Gamma) = \bigoplus_{\mu \in \sigma_{pp}} m_\mu \cdot \mathcal{H}_\mu \bigoplus \bigoplus_{n \in \mathbb{Z}^+} m_n \cdot (\mathcal{H}_n \oplus \mathcal{H}_{-n}) \bigoplus \bigoplus_{c \in \mathcal{C}} \mathcal{H}_c$$

Here $m_\mu$ is the finite multiplicity of an eigenvalue $\mu \in \sigma_{pp}$, and $m_n$ is the dimension of the space of holomorphic sections of the $n$-th power of the canonical line bundle, and is computable via the Riemann-Roch Theorem [Taylor]; therefore

$$m_\mu \cdot \mathcal{H}_\mu = \bigoplus_{i=1}^{m_\mu} \mathcal{H}_\mu, \quad m_n \cdot (\mathcal{H}_n \oplus \mathcal{H}_{-n}) = \bigoplus_{i=1}^{m_n} (\mathcal{H}_n \oplus \mathcal{H}_{-n}), \quad \mathcal{H}_c = \int_\mathcal{C} \mathcal{H}_c(\lambda) ds_c(\lambda)$$

Note that the trivial representation $\mathcal{H}_0$ corresponding to $\mu = 0$ appears with multiplicity one, and is realized as the space of constant functions, and the Stieltjes measures $ds_c$ are
supported on $[1/4, \infty)$ and are absolutely continuous. All representations $\mathcal{H}_\mu$ for any $\mu \geq 0$ are irreducible unitary representations for $PSL(2, \mathbb{R})$.

First, the $U$-invariant distributions are classified. Let $\mathcal{E}'(L^2(PSL(2, \mathbb{R})/\Gamma))$ be the dual space of the space $C^\infty(L^2(PSL(2, \mathbb{R})/\Gamma))$ of smooth vectors in $L^2(PSL(2, \mathbb{R})/\Gamma)$. Then the subspace

$$\mathcal{I}(PSL(2, \mathbb{R})/\Gamma) = \{ D \in \mathcal{E}'(L^2(PSL(2, \mathbb{R})/\Gamma)) | \mathcal{L}_U D = 0 \}$$

of $U$-invariant distributions is determined by the spectrum of the Laplacian $\triangle$.

**Theorem 2.3 (Flaminio-Forni).** The space $\mathcal{I}(PSL(2, \mathbb{R})/\Gamma)$ has infinite countable dimension. Furthermore, there is a decomposition

$$\mathcal{I}(PSL(2, \mathbb{R})/\Gamma) = \bigoplus_{\mu \in \sigma_{pp}} m_\mu \cdot \mathcal{I}_\mu \oplus \bigoplus_{n \in \mathbb{Z}^+} m_n \cdot (\mathcal{I}_n \oplus \mathcal{I}_{-n}) \oplus \bigoplus_{c \in \mathcal{C}} \mathcal{I}_c \quad (2.2)$$

where,

1. for $\mu = 0$, the space $\mathcal{I}_0$ is spanned by the $PSL(2, \mathbb{R})$-invariant volume,

2. for $0 < \mu < 1/4$, there is a splitting $\mathcal{I}_\mu = \mathcal{I}_\mu^+ \oplus \mathcal{I}_\mu^-$, where $\mathcal{I}_\mu^\pm \subseteq W^{-s}(PSL(2, \mathbb{R})/\Gamma)$, iff $s > \frac{1 \pm \sqrt{1-4\mu}}{2}$, and each subspace has dimension equal to 1.

3. for $\mu \geq 1/4$, the space $\mathcal{I}_\mu \subseteq W^{-s}(PSL(2, \mathbb{R})/\Gamma)$, iff $s > 1/2$, and it has dimension equal to 2.

4. for $n \in \mathbb{Z}^+$, the space $\mathcal{I}_n \subseteq W^{-s}(PSL(2, \mathbb{R})/\Gamma)$, iff $s > n$, and it has dimension equal to 1.

5. for $c \in \mathcal{C}$, the space $\mathcal{I}_c \subseteq W^{-s}(PSL(2, \mathbb{R})/\Gamma)$, iff $s > 1/2$, and it has infinite countable dimension.

Our Theorem 1.1 provides a parallel statement. Since we have not striven to obtain sharp optimal regularity results, the most noticeable difference between the two theorems is in statement 4, which arises from the discrete series.

The $U$-invariant distributions of order $s > 1$ form a complete set of obstructions to the existence of smooth solutions of the cohomological equation $Uf = g$ for functions $g \in W^s(PSL(2, \mathbb{R})/\Gamma)$. Let

$$\mathcal{I}^s(PSL(2, \mathbb{R})/\Gamma) = \{ D \in W^{-s}(PSL(2, \mathbb{R})/\Gamma) | \mathcal{L}_U D = 0 \}$$

Then $\mathcal{I}^s(PSL(2, \mathbb{R})/\Gamma)$ is determined.
THEOREM 2.4 (FLAMINIO-FORNI). Let $s > 1$, then there exists a constant $C_{s,t}$ such that, for all $g \in W^s(\mathcal{H})$,

- if $t < -1$, and $g$ has no component on the trivial sub-representation of $L^2(PSL(2, \mathbb{R})/\Gamma)$, or
- if $t < s - 1$ and $D(g) = 0$ for all $D \in T^s(PSL(2, \mathbb{R})/\Gamma)$,

then the equation $U f = g$ has a solution $f \in W^t(PSL(2, \mathbb{R})/\Gamma)$, which satisfies the Sobolev estimate $\|f\|_t \leq C_{s,t}\|g\|_s$. Solutions are unique modulo the trivial sub-representation iff $t > 0$.

The method of proof of both of these results is to use the detailed description of the action of the operator $U$ on the space of $K$-finite vectors in each irreducible unitary representation, and then solve a second order finite difference equation (in one parameter). The present thesis utilizes the same approach to describe the action of the operator $X$, which is the generator of the geodesic flow, on the $K$-finite vectors. We then solve a second order difference equation (in one parameter). The limitations of this approach to study the solution of equations of the type $X f = g$ for a general element $X \in \mathfrak{g}$ in an arbitrary semisimple Lie group are discussed in the concluding remarks.

As a final remark, we note that the results of [4] apply to $PSL(2, \mathbb{R})$ and not to $SL(2, \mathbb{R})$ because they fail to treat the second principal series of representations for $SL(2, \mathbb{R})$ (see section 3.2.1). That is, the second principal series are not irreducible unitary representations for $PSL(2, \mathbb{R})$. However, an analysis of the technical results on each series type actually shows that all results that are obtained for the first principal series actually apply to the second principal series. We will use this extension of their results implicitly for the analysis of $SL(2, \mathbb{C})/\Gamma$. We also note that all of our own results concerning the diagonal element $X$ on $PSL(2, \mathbb{R})$ also hold for the second principal series, and so everywhere that $PSL(2, \mathbb{R})$ appears, we could replace it with $SL(2, \mathbb{R})$ and the same results would hold.
Chapter 3

Mathematical Preliminaries

This chapter is devoted to preliminaries, both notational and mathematical. The collected results are provided for the purposes of a brief introduction to the major elements that will be used in the main results. The results summarized in this chapter in no way represent original results of the author of this thesis. References to sources are given where appropriate.

3.1 Notational Preliminaries

In this section we provide a quick reference to some commonly used notation. More detailed explanations can be found in the following sections. We now define,

\[ SL(2, \mathbb{R}) : \]

\[ X = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \theta = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

\[ U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -Y + \theta, \quad V = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = -Y - \theta \]

\[ \square = -X^2 - Y^2 + \theta^2, \quad \triangle = -(X^2 + Y^2 + \theta^2) \]

\[ \Pi_{\nu,k} = \prod_{i=1}^k \begin{pmatrix} 2i-1 & -\nu \\ 2i+1 & \nu \end{pmatrix}, \quad \nu^2 = 1 - 4\mu \]

\[ b^+(k) = \frac{2k+1+\nu}{4} \quad \text{and} \quad b^-(k) = \frac{2k-1-\nu}{4} \]

\[ \|f\|_{W^s(H_\mu)} = \left( \sum_{s=-\infty}^{\infty} (1 + \mu + 2k^2)^s |\Pi_{\nu,k}| |f_k|^2 \right)^{1/2} \]

\[ SL(2, \mathbb{C}) : \]

\[ \theta_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \theta_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \theta_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \]

\[ X = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U' = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad V' = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix} \]

\[ \mathfrak{f} = \mathfrak{su}(2), \quad \mathfrak{a} = \mathbb{R}X, \quad \mathfrak{n} = \mathfrak{g}_\alpha \]

where \( \mathfrak{g}_\alpha = \{ (c, 0) \mid c \in \mathbb{C} \} = \text{span} \{ U, U' \}, \quad A = \exp \mathfrak{a}, \quad N = \exp \mathfrak{n} \]

\[ D_\lambda = \{ F : \mathbb{C}^2 \to \mathbb{C} \mid F \text{ is homogeneous of degree } (\lambda, \mu), \text{ and } C^\infty \text{ in } z_1, z_2, \bar{z}_1, \bar{z}_2 \in \mathbb{C}^2 \setminus 0 \} \]
3.2 Representation Theory of Semisimple Lie Groups

We provide a brief summary of some basic concepts from the theory of Group representations in this section. Some basic references are [5], [6], [18], [20], [23]. Let $G$ be a Lie group throughout.

**Definition 1.**

1. A representation of $G$ on a complex Hilbert space $\mathcal{H}$ is a homomorphism $\Phi : G \to \text{GL}(\mathcal{H})$, where $\text{GL}(\mathcal{H})$ is the group of bounded linear operators on $\mathcal{H}$ with bounded inverses s.t. $G \times \mathcal{H} \to \mathcal{H}$ is continuous.

2. $K \subset \mathcal{H}$ is an invariant subspace for $\Phi$ if $\Phi(g)K \subset K$ for $\forall g$.

3. $\Phi$ is irreducible if it has no closed invariant subspaces other than $0$ and $\mathcal{H}$.

4. $\Phi$ is unitary if $\Phi(g)$ is unitary operator for $\forall g$, i.e. $\|\Phi(g) \cdot w\| = \|w\|$, $\forall w \in \mathcal{H}$.

5. if $\Phi : G \to \text{GL}(\mathcal{H})$ and $\Phi' : G \to \text{GL}(\mathcal{H}')$ are two representations of $G$, then they are equivalent if $\exists$ a bounded linear $E : \mathcal{H} \to \mathcal{H}'$, with bounded inverse, s.t. $\Phi'(g)E = E\Phi(g)$, $\forall g$.

6. if $\Phi, \Phi'$ are unitary, and $E$ can be chosen to be unitary, then $\Phi, \Phi'$ are unitarily equivalent.

The first basic, but extremely useful, result in representation theory is Schur’s Lemma.

**Lemma 3.1. (Schur’s Lemma)** A unitary rep. $\Phi : G \to \text{GL}(\mathcal{H})$ is irreducible iff the only bounded linear operators on $\mathcal{H}$ commuting with all $\Phi(g)$, are the scalar operators.

For any representation $\mathcal{H}$, an important subspace is the space of smooth vectors, $C^\infty(\mathcal{H}) = \{ v \in \mathcal{H} \mid \text{the map } g \mapsto \pi(g)v \text{ is smooth} \}$. It is a well known fact, originally proved by Garding, that $C^\infty(\mathcal{H})$ is dense in $\mathcal{H}$ [20]. We can similarly define the space of analytic vectors to be $C^\omega(\mathcal{H}) = \{ v \in \mathcal{H} \mid \text{the map } g \mapsto \pi(g)v \text{ is real analytic} \}$. We give some examples.

**Example 3. (C^\infty vectors).**

- In a finite-dimensional representation, every vector is $C^\infty$, and is in fact analytic.

- We can express the principal series representation, $\text{Ind}_{MAN}^G(\sigma \otimes \nu \otimes 1)$, of a group $G$ as the left regular representation on the space of functions $\{ F \in L^2(K, V) \mid F(km) = \sigma(m)^{-1}F(k) \}$. The $C^\infty$ vectors are then the $C^\infty$ functions on $K$ within the space, and the analytic vectors are the real analytic functions.
For compact Lie groups, the representation theory is fairly simple. We summarize as follows.

Let $K$ be a compact Lie group. It is well known that $K$ admits a Haar measure, (i.e. a nonzero regular Borel measure that is invariant under left and right translation). Given any finite dimensional representation $\pi$ of $K$, we can endow $V$ with a Hermitian inner product that makes $\pi$ unitary. We simply let $(\cdot, \cdot)$ be any hermitian inner product on $V$, and then define,

$$< v, w >= \int_K (\pi(k)v, \pi(k)w) \, dk \quad \text{with } dk \text{ Haar measure}$$

**Definition 2.** If $\pi$ is a unitary representation of $G$, then the functions on $G$ defined by $\pi_{v,w}(g) = (\pi(g)v, w)$ are called the matrix coefficients of $\pi$.

If $V$ is finite dimensional, then the character of $\pi$ is the function $\chi_{\pi}(g) = \text{trace } \pi(g) = \sum_i (\pi(g)u_i, u_i)$, where $\{u_i\}$ is an orthonormal basis.

**Theorem 3.1.** (Peter-Weyl Theorem). Let $K$ be a compact Lie group.

1. if $\{\pi^\alpha\}$ is a maximal set of mutually inequivalent finite dimensional irreducible unitary representations of $K$, then the corresponding set of matrix coefficients $\{(d^{\alpha})^{1/2}\pi_{i,j}^\alpha\}_{i,j,\alpha}$ is an orthonormal basis of $L^2(K)$.

2. Every irreducible unitary representation of $K$ is finite dimensional.

3. Every unitary representation of $K$ is the orthogonal sum of finite dimensional irreducible invariant subspaces (i.e. is completely reducible into irreducibles).

**Example 4.** (Fourier Series of the Circle). Let $K = S^1$.

- All irreducible unitary reps. are one dimensional because $S^1$ is abelian (Schur Lemma).
- Haar measure is the usual Lebesgue measure.
- The irreducible unitary representations are $\pi_n(e^{i\theta})z = e^{in\theta}z$.
- The matrix coefficients are just the functions $e^{i\lambda}$.
- They are also the characters.
- Peter-Weyl theorem then just says that $\{e^{i\lambda}\}$ form an o.n. basis for $L^2(S^1)$.
- Similar for $K = T^n$. 
We recall that every semisimple Lie group has an Iwasawa decomposition, $G = KAN$, where $K$ is a maximal compact subgroup, $A$ is abelian, and $N$ is nilpotent. Using this decomposition, and the Peter-Weyl theorem, we can decompose every irreducible representation of a Lie group into a direct sum of irreducible invariant subspaces for the restriction of the representation to the maximal compact subgroup $K$. That is, $\mathcal{H}|_K = \bigoplus_{\tau \in \hat{K}} V_\tau$. The space $\mathcal{H}_K = \bigoplus_{\tau \in \hat{K}} V_\tau$ is often called the space of $K$-finite vectors [20].

**Theorem 3.2.** If $(\pi, \mathcal{H})$ is an irreducible unitary representation of $G$, then every $\tau \in \hat{K}$ has finite multiplicity in $\mathcal{H}$.

This then leads to the definition of admissible representation. A representation $(\pi, \mathcal{H})$ is called *admissible* if every $\tau \in \hat{K}$ has finite multiplicity in $\mathcal{H}$.

We can also define a representation of the Lie algebra $\mathfrak{g}$ to be a homomorphism of $\mathfrak{g} \rightarrow \mathfrak{gl}(\mathcal{H})$. A representation of a Lie group $G$ on $\mathcal{H}$ induces a representation of the Lie algebra $\mathfrak{g}$ on $\mathcal{H}$. Namely, $\pi(X) = d\Pi_e(X)$. Similarly, if $\pi$ is a representation of a Lie algebra $\mathfrak{g}$, then there exists a simply connected Lie group $G$, with $\text{Lie}(G) = \mathfrak{g}$, and a representation $\Pi$ of $G$ such that $\pi = d\Pi_e$. We will make frequent use of the infinitesimal description of representations of a group $G$ through its Lie algebra $\mathfrak{g}$. The KAN decomposition then allows us to define a $(\mathfrak{g}, K)$-module.

**Definition 3.** A (Harish-Chandra) $(\mathfrak{g}, K)$-module is:

1. a vector space $V$ (with no topology).

2. an algebra homomorphism $\rho : \mathcal{U}(\mathfrak{g}) \rightarrow \text{End}(V)$ s.t.

   (a) $\forall v \in V$, we have $\dim (\rho(\mathcal{U}(t))v) < \infty$

   (b) $\exists \rho_K : K \rightarrow \text{Aut}(V)$ s.t. $\text{lie}(\rho_K) = \rho|_K$ (which is unnecessary if $\pi_1(G) = e$)

A primary feature of the definition is that there is no analysis (i.e. Hilbert or Banach space structure) to worry about. Every admissible representation has an associated $(\mathfrak{g}, K)$-module. It is a fundamental fact that representations of $G$ can be characterized by their associated $(\mathfrak{g}, K)$-modules.

**Lemma 3.2.** If $\pi, \pi'$ be unitary representations with equivalent $(\mathfrak{g}, K)$-modules, then $V \simeq V'$. 
3.2.1  \( SL(2, \mathbb{R}) \)

We provide a basic summary of the representation theory of \( SL(2, \mathbb{R}) \). A word of caution at the outset is needed, however. The relative simplicity of the representation theory for \( SL(2, \mathbb{R}) \) allows many facts to be reduced to reasonably tractable brute force calculations, which would not constitute a realistically practical approach for more general \( G \).

First define a basis for \( \mathfrak{sl}(2, \mathbb{R}) \) by,
\[
\left\{ X = \frac{1}{2}(1 \ 0 \ 0 \ -1), \ Y = \frac{1}{2}(0 \ -1 \ -1 \ 0), \ \theta = \frac{1}{2}(0 \ 1 \ -1 \ 0) \right\}.
\]

Now let \( \pi \) be an irreducible unitary representation of \( SL(2, \mathbb{R}) \) on \( \mathcal{H} \). Then, \( \pi|_{SO(2)} \) is also a unitary representation of \( SO(2) \), a maximal compact subgroup. We can then decompose \( \mathcal{H} \) into a direct sum of irreducible representations for \( SO(2) \). We then have
\[
\mathcal{H} = \bigoplus_{k \in \mathbb{Z}} m_k \mathcal{H}_k.
\]

We start by summarizing the classification of the irreducible unitary representations of \( SL(2, \mathbb{R}) \). Let \( \pi \) be one such irreducible unitary representation on \( \mathcal{H} \). Then \( \pi|_{SO(2)} \) is a unitary representation of \( SO(2) \simeq S^1 \) on \( \mathcal{H} \), and so by the Peter-Weyl theorem, it is completely reducible as,
\[
\mathcal{H} = \bigoplus_{k \in \mathbb{Z}} m_k \mathcal{H}_k, \quad (3.1)
\]
where \( \mathcal{H}_k = \mathbb{C}\{e^{ikt}\} \), and \( m_k \) = multiplicity of \( \mathcal{H}_k \) in \( \mathcal{H} \). The action of \( SO(2) \) on \( \mathcal{H}_k \) is then given by,
\[
\pi(\exp s2\theta) = e^{iks},
\]
\[
\pi(2\theta) = ik
\]
We note that \( P - k = (2\pi)^{-1} \int_0^{2\pi} e^{-iks} \pi(\exp s2\theta)ds \) is orthogonal projection onto \( \mathcal{H}_k \).

The Casimir element for \( SL(2, \mathbb{R}) \) is defined by \( \Box = -X^2 - Y^2 + \theta^2 \). It has the property that \( [\Box, \mathfrak{sl}(2, \mathbb{R})] = 0 \), so that \( \pi(\Box) \) commutes with each \( \pi(X) \) for \( X \in \mathfrak{sl}(2, \mathbb{R}) \). By Schur’s Lemma, \( \pi(\Box) \) acts on \( C^\infty(\mathcal{H}) \) by a scalar \( \mu \). Since \( \Box \) is symmetric, \( \mu \in \mathbb{R} \).

**Theorem 3.3.** Let \( \pi, \pi' \) be irreducible unitary representations of \( SL(2, \mathbb{R}) \). Then \( \pi \simeq \pi' \) iff \( m_k = m'_k \) for all \( k \in \mathbb{Z} \), and \( \mu = \mu' \).

**Sketch of Proof.** Define \( R_+ = \pi(X) - i\pi(Y), R_- = \pi(X) + i\pi(Y) \), and \( E = 2\pi(\theta) \). Then the operators \( R_+, R_-, E \) uniquely determine the representation. Now \( [E, R_\pm] = \)
$$\pm 2iR_\pm [R_+, R_-] = -iE,$$

and with respect to the inner product, $R_+^* = R_-$ on $C^\infty(\mathcal{H})$.

Notice also that,

$$4R_+ R_- = E^2 - 2iE - 4\pi(\square) \quad (3.2)$$

$$4R_- R_+ = E^2 + 2iE - 4\pi(\square) \quad (3.3)$$

$$ER_\pm = R_\pm E \pm 2iR_\pm \quad (3.4)$$

Hence if $v_k \in \mathcal{H}_k$, then $R_\pm$ raise and lower eigenvalues,

$$ER_\pm v_k = R_\pm (Ev_k) \pm 2iR_\pm v_k = (k \pm 2)iR_\pm v_k$$

So that $R_+ v_k \in m_{k+2} \cdot \mathcal{H}_{k+2}$, and similarly $R_- v_k \in m_{k-2} \cdot \mathcal{H}_{k-2}$. Now if $v_k \in m_k \mathcal{H}_k$, we notice that $\mathcal{U} = \bigoplus_{n=1}^\infty C\{R^n v_k\} \oplus C\{v_k\} \oplus \bigoplus_{n=1}^\infty C\{R^n v_k\}$ is an invariant subspace for the irreducible representation $\pi$, hence either $\mathcal{U} = 0$ or $\overline{\mathcal{U}} = \mathcal{H}$. Since this is true for every $v_k \in m_k \mathcal{H}_k$, we see that $m_k \leq \dim \mathcal{H}_k = 1$, for all $k \in \mathbb{Z}$.

In addition, since $R_+, R_-$ raise and lower the eigenvalues of $E$ by 2, we must have that $m_k = 0$ for $k$ even or $m_k = 0$ for $k$ odd. In fact, we find that there are only four possibilities for the spectrum of $-iE$.

1. $\text{spec}(-iE) = \{n, n+2, n+4, \ldots\}$

2. $\text{spec}(-iE) = \{\ldots, n-4, n-2, n\}$

3. $\text{spec}(-iE) = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$

4. $\text{spec}(-iE) = \{\ldots, -3, -1, 1, 3, \ldots\}$

(We note that since $\mathcal{H}$ cannot be finite dimensional, as $SL(2, \mathbb{R})$ has no finite dimensional unitary representations, $\text{spec}(-iE)$ cannot be finite).

We now have $m_k = \begin{cases} 1 & \text{if } k \in \text{spec}(-iE) \\ 0 & \text{otherwise} \end{cases}$.

For each $k \in \text{spec}(-iE)$, choose a unit vector $v_k$ in each $\mathcal{H}_k$. Then $\{v_k\}_{k \in \text{spec}(-iE)}$ is an orthonormal basis of $\mathcal{H}$. The action of the operators $R_+, R_-, E$, can then be written as,

$$R_+ v_k = \alpha_k v_{k+2}, \quad R_- v_{k+2} = \beta_k v_k, \quad Ev_k = ikv_k, \quad \pi(\square) = \mu v_k \quad (3.5)$$
And so,

\[-4\alpha_k \beta_k \nu_{k+2} = -4R_+ R_- \nu_{k+2}\]

\[= (-E^2 + 2iE + 4\pi(\Box))\nu_{k+2}\]

\[= ((k + 2)^2 - 2(k_2 + 4\mu)\nu_{k+2}\]

\[= ((k + 1)^2 + 4\mu - 1)\nu_{k+2}\]

and since \(R_+^* = -R_\), we have that \(\beta_k = -\tilde{\alpha}_k\), so that \(|\alpha_k| = \frac{1}{2}((k + 1)^2 + 4\mu - 1)^{1/2}\).

Finally, let \(\pi, \pi'\) be two irreducible representations s.t. \(\pi(\Box) = \pi'(\Box)\) and spec\((-iE) = \text{spec}(-iE')\). Then if we define \(B : \mathcal{H} \to \mathcal{H}'\) by \(B\nu_k = e^{i\phi_k} \nu'_k\), we see that \(R_+ 'B\nu_k = e^{i\phi_k} \alpha_k \nu'_{k+2}\) and \(BR_+ = e^{i\phi_{k+2}} R_+ \nu_{k+2}\). Thus, \(R_+ 'B = BR_+ \iff e^{i(\phi_{k+2} - \phi_k)} = \alpha_k' / \alpha_k\), which is always possible, so long as \(|\alpha_k'| = |\alpha_k|\). This happens whenever \(\text{spec}(-iE) = \text{spec}(-iE')\) and \(\mu = \mu'\), so that \(\pi \simeq \pi'\).

The proof in the opposite direction is straightforward.

The above discussion allows us to see that there are exactly five classes of irreducible unitary representations for \(SL(2, \mathbb{R})\).

First, let \(\text{spec}(-iE) = 2 \cdot \mathbb{Z}\). Then since we must have \(|\alpha_k| = 1/2((k + 1)^2 + 4\mu - 1)^{1/2} > 0\), we must have \(\mu > 0\).

- (First Principal Series) \(\pi_v^0 \simeq \pi_{-v}^0\).
  if \(4\mu = 1 - v^2 \geq 1\) for \(v \in i \cdot \mathbb{R}\), then we can set \(\alpha_k = 1/2(k + 1 + v)\) to get \(|\alpha_k| = 1/2((k + 1)^2 - v^2)^{1/2} = 1/2((k + 1)^2 + 4\mu - 1)^{1/2}\).

- (Complementary Series) \(\pi_v^\circ\).
  if \(0 < \mu < 1/4\), then we still get a representation with \(|\alpha_k| = 1/2((k + 1)^2 - v^2)^{1/2}\)
  where \(4\mu = 1 - v^2\), and with \(v \in (-1, 1) \setminus \{0\}\). (we note that \(\pi_v^\circ \simeq \pi_{-v}^\circ\)).

Now suppose \(\text{spec}(-iE) = 2 \cdot \mathbb{Z} + 1\). Since \(|\alpha_k| = 1/2((k + 1)^2 + 4\nu - 1)^{1/2}\), we get \(\mu > 1/4\).

- (Second Principal Series) \(\pi_v^0 \simeq \pi_{-v}^0\).
  Again take \(\alpha_k = 1/2(k + 1 + \nu)\) with \(1 - \nu^2 = 4\mu\), and \(\nu \in \mathbb{R} \setminus \{0\}\).

For \(\text{spec}(-iE) = \{n, n + 2, n + 4, \ldots\}\), we have \(R_- = 0\) on \(\mathcal{H}_n\), which implies that \(|\beta_n| = 1/2((n - 1)^2 + 4\mu - 1)^{1/2} = 0\). Hence \(4\mu = 1 - (n - 1)^2\).
• (Holomorphic Discrete Series) $\pi^+_n$.

We must have $n \geq 1$. Then $|a_k| = 1/2((k + 1)^2 - (n - 1)^2)^{1/2} > 0$ for $k = n + 2j$ ($j = 0, 1, 2, \ldots$).

And for $\text{spec}(-iE) = \{\ldots, n-4, n-2, n\}$, we must have $R_+ = 0$ on $\mathcal{H}_n$. So $4\mu = 1 - (n + 1)^2$.

• (Anti-Holomorphic Discrete Series) $\pi^-_n$. We must have $n \leq -1$, and so $|a_k| = 1/2((k + 1)^2 - (n + 1)^2)^{1/2} > 0$ for $k < n$.

As a final note, we mention that $\lim_{\nu \to 0} \pi^0_\nu = \pi^+_1 \oplus \pi^-_1$. For this reason, $\pi^+_1$ and $\pi^-_1$ are called “mock discrete series”.

Beyond the algebraic construction above, we can also describe some useful realizations for each irreducible unitary representation.

• (Discrete Series). $\pi^+_n$. The Hilbert space is $\mathcal{H} = \{f \text{ holomorphic for } \Im(z) > 0 \mid \|f\|^2 = \int_{\Im(z)>0} |f(z)|^2(\Im(z))^{n-2}dz\bar{dz} < \infty\}$

The action is given by $\pi^+_n(g)f(z) = (-bz + d)^{-n}f(-bx + d)$ for $a\neq 0$. All matrix coefficients satisfy $(\pi^+_n(g)v, w) \in L^2(\text{SL}(2, \mathbb{R})).$

• (Principal Series). $\pi^e_\nu$ and $\pi^o_\nu$.

The Hilbert space is $L^2(\mathbb{R})$, equipped with the usual norm. The action is given by,

$$\pi^e,o_\nu f(x) = \begin{cases} | - bx + d|^{-1-\nu} f(\frac{ax-c}{bx+d}) & \text{if } e \\ \text{sgn}(-bx + d)| - bx + d|^{-1-\nu} f(\frac{ax-c}{bx+d}) & \text{if } o \end{cases}$$

• (Complementary Series) $\pi^e_\nu$. The Hilbert space is,

$$\mathcal{H} = \{f: \mathbb{R} \to \mathbb{C} \mid \int f(x)f(y)\frac{dx\,dy}{|x-y|^{1-\nu}} < \infty\}$$

And the action is $\pi^e_\nu(g)f(x) = | - bx + d|^{-1-\nu} f(\frac{ax-c}{bx+d})$.
3.2.2 \( SL(2, \mathbb{C}) \)

Here we will discuss the relevant facts about the representation theory of \( SL(2, \mathbb{C}) \). 

KAN decomposition. We define \( t = \mathfrak{su}(2), a = RX, \ n = g_{\alpha}, \) where \( X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) and \( g_{\alpha} = \{ \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \mid c \in \mathbb{C} \}. \)

Then \( 2 \mathfrak{sl}(2, \mathbb{C}) = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}. \) And if \( K = SU(2), \ A = \exp a, \ N = \exp n, \) then \( SL(2, \mathbb{C}) = KAN. \)

We settle upon the following notation.

\( \theta_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ \theta_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \theta_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \) which form a system of generators for \( \mathfrak{k}. \)

\( X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ U = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ U' = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \) which form a system of generators for \( \mathfrak{a} \) and \( g_{\alpha} \) respectively. We also write \( V = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ V' = \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}. \)

There are several ways to realize irreducible unitary representations of \( SL(2, \mathbb{C}) \). We will start by following the now classic treatment given in [5]. There are two types of representations of \( SL(2, \mathbb{C}) \), the so called Principal Series and the Complementary Series. Notice that while \( SL(2, \mathbb{R}) \) has a Discrete Series, \( SL(2, \mathbb{C}) \) does not. We will begin by giving a description of a subset of the representation space, then describe a norm which will give the full representation as a completion.

First, let \( \lambda, \mu \in \mathbb{C} \) be such that \( \lambda - \mu \in \mathbb{Z} \) and write \( \chi = (\lambda, \mu). \)

We define \( D_{\chi} \) to be the space of functions \( F(z_1, z_2) \) on two complex variables which satisfy the following requirements.

- \( F(z_1, z_2) \) is homogeneous of degree \( (\lambda, \mu). \)
- \( F(z_1, z_2) \) is infinitely differentiable in \( z_1, z_2, \) and their complex conjugates, in \( \mathbb{C} \times \mathbb{C} - \{(0,0)\}. \)

To define a topology on \( D_{\chi} \), we will say that a sequence \( \{F_m(z_1, z_2)\} \) of functions in \( D_{\chi} \) converges to zero if, on every closed bounded set not containing \( (0,0) \), these functions converge uniformly to zero together with all their derivatives. \( D_{\chi} \) is complete with respect to this topology. Eventually we will put a family of norms on \( D_{\chi} \) which generate an equivalent topology (see section 4.5.2).

The action of \( SL(2, \mathbb{C}) \) on \( D_{\chi} \) can then be defined by,

\[
\pi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \cdot F(z_1, z_2) = F(dz_1 - bz_2, -cz_1 + az_2)
\]

There is another useful realization of \( D_{\chi} \). We start by considering the complex line \( z_2 = 1 \) (or alternatively the line \( z_1 = 1 \)) in \( \mathbb{C} \times \mathbb{C} - \{(0,0)\}. \) This line intersects each
complex line passing through the origin, except for the line $z_2 = 0$. We get that each $f(z_1, z_2) \in D_\chi$ is then determined uniquely by its values on this line. To each $F(z_1, z_2) \in D_\chi$ we can associate a function $f$ of a single complex variable by defining $f(z) = F(z, 1)$. And we can then recover $F$ from $f$ by setting $F(z_1, z_2) = z_2^n z_2^\mu f(\frac{z_1}{z_2})$.

The space of functions we get from $D_\chi$ in this way has a natural one to one correspondence with $D_\chi$, and so we will also call it $D_\chi$.

The action of $SL(2, \mathbb{C})$ on $D_\chi$ can then be obtained by restricting the action defined above,

$$\pi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \cdot f(z) = \pi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \cdot F(z, 1)$$

$$= F(az + c, bz + d)$$

$$= (-cz + a)^{n_1-1}(cz + a)^{n_2-1} F\left(\frac{dz - b}{cz + a}, 1\right)$$

$$= (-cz + a)^{n_1-1}(cz + a)^{n_2-1} f\left(\frac{dz - b}{cz + a}\right)$$

We wish to translate the conditions on $D_\chi$ given in the first realization to this new realization. Notice that homogeneity was used to make the restriction from two complex variables to one, so we only want an analogue of the differentiability condition. Now, if $F(z_1, z_2)$ is infinitely differentiable, we naturally get that $f$ must be infinitely differentiable in both $z, \bar{z}$. But it satisfies a further condition in the neighborhood of $z = \infty$. Start by writing $F(1, z) = z^{\lambda} z^\mu f(z^{-1})$, so that the function $\tilde{f}(z) = z^{\lambda} z^\mu f(-z^{-1})$ is also infinitely differentiable in both $z$ and $\bar{z}$. We then call $\tilde{f}$ the inversion of $f$.

So we could alternatively define $D_\chi$ to be the space of functions of a complex variable which, together with their inversions, are infinitely differentiable in both $z$ and $\bar{z}$.

The topology which is defined for the functions $F(z_1, z_2)$ then uniquely induces a topology for the functions $f(z)$. Namely, a sequence of functions $\{\varphi_m(z)\}$ is said to converge to zero if in every finite region, every sequence of derivatives to any given order both of the $\varphi_m(z)$ and of the $\tilde{\varphi}_m(z)$ converge uniformly to zero.

We will make extensive use of the following expansion of $f$ into an asymptotic series of the form,

$$f(z) \sim z^\lambda z^\mu \sum_{j,k=0}^{\infty} a_{j,k} z^{-j} \bar{z}^{-k} \quad \text{as } |z| \to \infty$$
In fact, since \( \tilde{f}(z) \) is infinitely differentiable at \( z = 0 \), we get an asymptotic Taylor’s series,

\[
\tilde{f}(z) \sim \sum_{j,k=0}^{\infty} b_{j,k} z^j \bar{z}^k \quad \text{as } z \to 0
\]

Then the expansion for \( f \) follows by setting \( a_{j,k} = (-1)^{-\lambda-\mu+j+k} b_{j,k} \).

The asymptotic Taylor’s series above can in fact be differentiated term by term, which leads immediately to an expression for the derivatives of \( f(z) \) in terms of asymptotic Taylor series.

It is a fact that \( D_\chi = C^\infty(\pi_\rho,\eta) \).

We will make use of both descriptions, alternating back and forth. To avoid confusion, capital letters shall be homogeneous functions, and their lower case counterparts will be their restrictions to the plane \( z_2 = 1 \).

### 3.3 Direct Integral Decompositions

The representation theory of the preceding sections becomes relevant to proving the main results in view of direct integral decompositions. That is, through the theorem of Kolmogorov-Mautner [22] (Theorem 11.4) below, we can reduce the main results to results on irreducible unitary representations for the Lie group \( G \), and then use uniform estimates to obtain the results on our original unitary representation. We make this precise as follows,

**Theorem 3.4. (Kolmogorov-Mautner)** Given any unitary representation \( \pi \) of a locally compact second countable group \( G \) in a separable Hilbert space \( \mathcal{H} \), there exists a Lebesgue-Stieltjes measure \( d\mu \) on \( \mathbb{R} \) such that \( \mathcal{H} \) is the direct integral \( \mathcal{H} = \int_R \mathcal{H}_\mu d\mu \) of Hilbert spaces \( \mathcal{H}_\mu \) with unitary representations \( \pi_\mu \) of the group \( G \) on \( \mathcal{H}_\mu \), where \( \pi(g)f = \int_R \pi_\mu(g)f_\mu d\mu \). For \( d\mu \)-almost all \( \mu \in \mathbb{R} \), the representation \( \pi_\mu \) is irreducible.

Now let \( \mu \) be a left Haar measure on \( G \) (if \( G \) is unimodular, then this is just Haar measure), and let \( \mu_\Gamma \) be the induced measure on \( G/\Gamma \), which is naturally \( G \) left invariant. Then the left regular representation of \( G \) on \( L^2(G/\Gamma, \mu_\Gamma) \), defined by \( g \cdot f(h\Gamma) = f(g^{-1}h\Gamma) \), is a unitary representation of \( G \). Since \( L^2(G/\Gamma) \) is separable, the above theorem applies, and asserts the existence of a direct integral decomposition of \( L^2(G/\Gamma) \). If \( G/\Gamma \) is compact, then the direct integral decomposition of \( L^2(G/\Gamma) \) into irreducible representations, in fact, degenerates into an infinite discrete sum [6]. The best known example of these facts is the classic Fourier series decomposition of \( L^2(\mathbb{R}^n) \) and \( L^2(\mathbb{T}^n) \).
Example 5. (Fourier Series).

Consider the separable Hilbert space \( L^2(\mathbb{R}^n) = \int_{\mathbb{R}^n} \hat{f}(\xi) d\xi \). That is, we can represent any function \( f \in L^2(\mathbb{R}^n) \) as \( f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi \). And for \( \mathbb{R}^n/\mathbb{Z}^n \), we get the classic representation of any \( f \in L^2(T^n) \) as \( f(x) = \sum_{\kappa \in \mathbb{Z}^n} \hat{f}(\kappa) e^{2\pi i \kappa \cdot x} \).

For the groups \( SL(2, \mathbb{R}) \) and \( SL(2, \mathbb{C}) \), whose representation theory we treated above, we get the following results. First, let \( \mathcal{H} \) be the separable Hilbert space of any non-trivial unitary representation of \( SL(2, \mathbb{R}) \). Since the Casimir operator \( \Box \) is in the center of the universal enveloping algebra of \( sl(2, \mathbb{R}) \), and acts on \( \mathcal{H} \) as an essentially self-adjoint operator, there exists an \( SL(2, \mathbb{R}) \) invariant direct integral decomposition \( \mathcal{H} = \int_{\sigma(\Box)} \oplus m(\mu) \cdot \mathcal{H}_\mu ds(\mu) \) with respect to a positive Stieltjes measure \( ds(\mu) \) over the spectrum \( \sigma(\Box) \). Here the Casimir operator acts as the constant \( \mu \in \sigma(\Box) \) on every \( \mathcal{H}_\mu \), and \( m(\mu) \cdot \mathcal{H}_\mu = \bigoplus_{i=0}^{m(\mu)} \mathcal{H}_\mu \) is the direct integral of \( m(\mu) \) copies of \( \mathcal{H}_\mu \), where \( m(\mu) \) is at most countable.

The left regular representation of \( SL(2, \mathbb{C}) \) on \( L^2(SL(2, \mathbb{C})/\Gamma) \) has a direct integral decomposition \( L^2(SL(2, \mathbb{C})/\Gamma) = \int_{\rho} m(\rho, n) \cdot H_{\rho, n} \) into irreducible unitary representations, where again, \( m(\rho, n) \) is again the at most countable multiplicity.

3.4 Sobolev Spaces

We now provide some basic results concerning Sobolev spaces. For a general introduction see [1].

For any semisimple Lie group \( G \), there are two essentially self-adjoint operators that will be of particular importance. They are the Casimir operator \( \Box_g \), and the Laplacian \( \triangle_g \). In order to define them, we first let \( X_i \) be a basis of \( \mathfrak{g} \), and let \( B \) be the Killing form on \( \mathfrak{g} \). Let \( g_{i,j} = B(X_i, X_j) \), and since \( B \) is non-degenerate, \( (g^{i,j}) = (g_{i,j})^{-1} \) be the inverse matrix. Then put \( X^i = \sum_i g^{i,j} X_j \). Then define,

\[
\Box_g = \sum_{i,j} g_{i,j} X^i X^j
\]  

(3.6)

Since the Killing form can be extended to a left invariant, pseudo-Riemannian metric on \( G \), the Casimir can be extended to the (left invariant) Laplace-Beltrami operator on \( G \) for the metric defined by the Killing form.
If \( \theta \) is any Cartan involution, then using the Killing form, we can form an associated positive definite bilinear form \( B_\theta(X,Y) = -B(\theta X,Y) \). This form gives rise to a bona fide Riemannian metric on \( G \). Under \( \theta \), \( g \) has a decomposition into the \(+1\) eigenspace \( k \), and the \(-1\) eigenspace \( p \). If we let \( K \) be the maximal compact subgroup of \( G \) with Lie algebra \( \mathfrak{t} \), then we get the Cartan decomposition of \( G = K \exp \mathfrak{p} \). If we now take an orthonormal basis \( X_i \) of \( \mathfrak{k} \) and \( Y_i \) of \( \mathfrak{p} \) with respect to the positive definite bilinear form \( B_\theta \), then we have a particularly nice expression for the Casimir operator of \( G \), and of \( K \), and can also define the Laplacian for this metric in this basis as,

\[
\Box_g = - \sum X_i^2 + \sum Y_i^2 \\
\Box_K = - \sum X_i^2 \\
\triangle_g = \sum X_i^2 + \sum Y_i^2
\]

(3.7)  
(3.8)  
(3.9)

So that the Casimir and the Laplacian are related as \( \triangle_g = \Box_g - 2 \Box_K \).

We can now define the Sobolev space of order \( s \in \mathbb{Z}_+ \) for any Hilbert space \( \mathcal{H} \) of a unitary representation of \( G \). We define \( \mathcal{W}^s(\mathcal{H}) \) to be the maximal domain of the operator \((I - \triangle_g)^{s/2}\) endowed with the inner product

\[
< f, g >_s = < (I - \triangle_g)^s f, g >_{\mathcal{H}}
\]

(3.10)

If the operator \((I - \triangle_g)\) is positive, then it makes sense to extend the definition to all \( s \in \mathbb{R}_+ \).

In fact, the spaces \( \mathcal{W}^s(\mathcal{H}) \) are Hilbert spaces that coincide with the completion of \( \mathcal{C}^\infty(\mathcal{H}) \), with respect to the norm \( \|f\|_{\mathcal{W}^s(\mathcal{H})} = \|(I - \triangle_g)^{s/2} f\|_\mathcal{H} \), induced from the inner product (3.10). The space \( \mathcal{C}^\infty(\mathcal{H}) \) coincides with the intersection of the spaces \( \mathcal{W}^s(\mathcal{H}) \) for all \( s \geq 0 \).

The Sobolev spaces with negative exponent \( \mathcal{W}^{-s}(\mathcal{H}) \), which are defined as the Hilbert space duals of the spaces \( \mathcal{W}^s(\mathcal{H}) \), are subspaces of the space \( \mathcal{E}'(\mathcal{H}) \) defined as the dual space of \( \mathcal{C}^\infty(\mathcal{H}) \).

We could also define the Sobolev spaces of integer orders as defined above using an equivalent norm. That is, define \( \mathcal{H}^k(\mathcal{H}) \) for \( k \geq 0 \) to be the completion of \( \mathcal{C}^\infty(\mathcal{H}) \) with respect to the norm,

\[
\|f\|^2_{\mathcal{H}^k(\mathcal{H})} = \sum \|X_1 X_2 \ldots X_l f\|^2_{\mathcal{H}}
\]

(3.11)

The sum being taken over all possible choices of \( 0 \leq l \leq k \) and all possible choices of not necessarily distinct elements \( X_1, X_2, \ldots, X_l \in \mathfrak{g} \).
LEMMA 3.3. The norms $\|f\|_{H^k(H)}^2$ and $\|f\|_{W^k(H)}^2$ are equivalent for all $k \geq 0$.

We will be concerned mostly with the representation spaces $L^2(G/\Gamma)$, and the irreducible unitary representations $\mathcal{H}_g$ of $G$ for semisimple $G$. By the Subrepresentation Theorem [20], every irreducible unitary representation $\mathcal{H}$ of $G$ can be realized as a subrepresentation of a member of the non-unitary principal series, and hence is a subrepresentation of a space equivalent to $L^2(K, V)$, where $K$ is a maximal compact subgroup. Therefore, the $C^\infty$ vectors of irreducible unitary components are just the smooth functions $f : K \rightarrow V$, which satisfy $f(km) = \sigma(m)^{-1}f(k)$. For the space $L^2(G/\Gamma)$, the $C^\infty$ vectors are the smooth functions $f \in L^2(G/\Gamma)$, which have $X^a f \in L^2(G/\Gamma)$ for all $X^a \in \mathcal{U}(g)$ – the universal enveloping algebra of $g$. If $G/\Gamma$ is compact, then this is the same as the space of smooth functions $f : G/\Gamma \rightarrow \mathbb{C}$.

Since all of the operators in the universal enveloping algebra are decomposable with respect to the direct integral decomposition, we can also decompose each Sobolev space $W^s(\mathcal{H}) = \int_\mu W^s(\mathcal{H}_\mu) d\mu$, with respect to the measure $d\mu$.

Because of the equivalence of norms on integer orders, we can also define norms which measure regularity with respect to preferred directions. Namely, let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. Then we can define the Sobolev norm of order $k$ with respect to the subalgebra $\mathfrak{h}$ as follows,

$$\|f\|_{H^k(\mathcal{H}, \mathfrak{h})}^2 = \sum \|X_1 X_2 \ldots X_l f\|_{\mathcal{H}}^2$$

(3.12)

where the sum is now taken over all possible choices of $0 \leq l \leq k$, and all possible choices of not necessarily distinct elements $X_1, X_2, \ldots, X_l \in \mathfrak{h}$. This norm is obviously dominated by $\|f\|_{H^k(\mathcal{H})}^2$.

Provided $\mathfrak{h}$ is semisimple, this norm is equivalent to $\|f\|_{W^k(\mathcal{H}, \mathfrak{h})} = \|(I - \Delta_\mathfrak{h})^{k/2} f\|_{\mathcal{H}}$. And since all operators in $\mathcal{U}(\mathfrak{h}) \subset \mathcal{U}(g)$ are decomposable with respect to the direct integral decomposition, we again have decomposability of the spaces $H^s(\mathcal{H}, \mathfrak{h}) = \int_\mu W^s(\mathcal{H}_\mu, \mathfrak{h}) d\mu$, with respect to the measure $d\mu$.

3.5 Cocycles

The main results are concerned with showing that the first $C^\infty$ cohomology group of $C^\infty$ cocycles over an abelian action on the homogeneous space $G/\Gamma$ reduces to constant coefficients. Therefore, let $G$ be a semisimple Lie group with finite center, and $A < G$ be an abelian subgroup. A good survey of the uses of cocycles in dynamics is [14]. We recall the definition of cocycle.
**Definition 4.** Let \((X, \mu)\) be a Lebesgue space, \(A\) and \(H\) be locally compact second countable groups. Suppose that \(A\) acts via a measurable left action on \(X\). Then a \(G\) (one) cocycle over the \(A\) action is a measurable function \(\beta : A \times X \to H\) such that

\[
\beta(a_1 + a_2, x) = \beta(a_1, a_2 x)\beta(a_2, x) \quad (3.13)
\]

Here we consider the cocycle \(\beta : A \times G/\Gamma \to H\), where \(H = \mathbb{R}^k\) or \(\mathbb{C}^k\), and \(A\) acts via left multiplication on \(G/\Gamma\). Or, alternatively, by writing \(f(x) \cdot a = f(ax)\), we will consider one-cocycles to be measurable functions \(f : G/\Gamma \to H\) which satisfy \(f(a_1 + a_2) = f(a_2) + f(a_1) \cdot a_2\). Here \(F(G/\Gamma, H)\) is the space of all measurable functions \(f : G/\Gamma \to H\). The main results are concerned with restricting to subspaces of \(F(SL(2, \mathbb{C})/\Gamma)\) such as \(L^2(G/\Gamma), C^\infty(G/\Gamma), C^\infty(L^2(G/\Gamma))\), or \(W^s(G/\Gamma)\).

The fundamental question that arises when studying cocycles over an action, is that of equivalence.

**Definition 5.** Two \(H\) cocycles \(\alpha, \beta\) over a \(A\) action are called cohomologous if there exists a measurable map \(P : X \to H\) such that,

\[
\beta(a, x) = P(ax)^{-1} \alpha(a, x) P(x) \quad (3.14)
\]

It is important to note that if \(\alpha\) is a cocycle, and \(P : X \to H\) is a measurable function, then the function \(\beta\) defined by (3.14) is also a cocycle.

Once we have a notion of equivalence, it is natural to ask what the proper notion of trivial is.

**Definition 6.** A cocycle \(\alpha\) is called a coboundary if there exists a measurable map \(P : X \to H\) such that,

\[
\alpha(a, x) = P(ax)^{-1} P(x) \quad (3.15)
\]

Now let \(\phi : A \to H\) be a homomorphism. The the map \(\tilde{\phi} : A \times X \to H\) defined by \(\tilde{\phi}(a, x) = \phi(a)\) is a cocycle. It is then also natural then to define,

**Definition 7.** A cocycle \(\alpha\) is called an almost coboundary if it is cohomologous to a homomorphism \(\phi : A \to H\).

Now if \(H\) is abelian, and for our purposes it will always be either \(\mathbb{R}^k\) or \(\mathbb{C}^k\), then the set \(Z^1(A, H)\) of all (one) cocycles becomes an abelian group by defining the product of
two cocycles to be the pointwise product (i.e. \((a + \beta)(a, x) = a(a, x) + \beta(a, x))\). Then if we define \(B^1(A, H)\) to be the subgroup of all (one) coboundaries, we can define the (first) cohomology group as \(H^1(A, H) = Z^1(A, H) / B^1(A, H)\). If we define \(B^1_\alpha(A, H)\) to be the subgroup of all (one) almost coboundaries, then we can define the (first) almost cohomology group as \(H^1_\alpha(A, H) = Z^1(A, H) / B^1_\alpha(A, H)\). If all of the cocycles are \(C^\infty\) or \(W^s\), and all of the cohomology classes are defined using only \(C^\infty\) or \(W^s\) transfer functions \(P\), then we call the corresponding cohomology group \(C^\infty\) or \(W^s\).

If, as usual, we set \(H = \mathbb{R}^k\) or \(\mathbb{C}^k\), and let \(A < G\) be an abelian subgroup of the semisimple Lie group \(G\). Let \(\omega(v) = \frac{d}{dt}\beta(\exp tv)|_{t=0}\) be the infinitesimal generator for \(\beta\) (provided it exists). Then the cocycle equation implies that \(\omega\) is a closed form on the orbit foliation. That is,

\[
\omega(X + Y) = \frac{d}{dt}\beta(\exp t(X + Y))|_{t=0} \\
= \frac{d}{dt}(\beta(\exp tX) \cdot \exp tY + \beta(\exp tY))|_{t=0} \\
= \omega(X) + \beta(e) \cdot Y + \omega(Y) \\
= \omega(X) + \omega(Y)
\]

Hence, \(\omega\) is a linear (one) form on \(a\).

We can then take the exterior derivative, and noting that \(A\) is abelian, we get

\[
d\omega(X, Y) = \omega(Y) \cdot X - \omega(X) \cdot Y - \omega([X, Y]) \\
= \frac{d}{dt}\frac{d}{ds}\left(\beta(\exp sY) \cdot \exp tX - \beta(\exp tX) \cdot \exp sY\right)|_{s=t=0} \\
= \frac{d}{dt}\frac{d}{ds}\left(\beta(\exp sY + tX) - \beta(\exp tX) - \beta(\exp tX + sY) + \beta(\exp sY)\right)|_{s=t=0} \\
= 0
\]

We can also recover \(\beta\) from \(\omega\) by \(\beta(\exp X) = \int_0^1 \omega(X) \cdot \exp tX dt\). Therefore we can restrict our attention to the infinitesimal situation. In particular, the infinitesimal version of the cohomology equation is \(\omega = \eta - dP\). Thus the \(C^\infty\) or \(W^s\) first cohomology group is trivial if for every \(C^\infty\) or \(W^s\) one form \(\omega\) is exact via a function \(P \in C^\infty\) or \(W^s\).

**Example 6.** Let \(M\) be an \(n\) dimensional manifold, and let \(G\) left act on \(M\) by diffeomorphisms. Then for every \(x \in M\), and \(g \in G\), define \(\alpha(g, x) = (dg)_{g^x} : T_x M \to T_x M\), where \(TM\) is the tangent bundle. Then the chain rule for differentiation is just the cocycle identity. Further, under any trivialization of \(TM\), \(\alpha\) will be identified with a cocycle \(\alpha : G \times M \to \text{GL}(n, \mathbb{R})\), and different trivializations yield cohomologous cocycles.
An even more basic example of cocycles, and how determining the cohomology classes provides useful information in dynamics is the case of time change of a flow.

**Definition 8.** A flow \( \psi^t \) on a manifold \( M \) is a time change of the flow \( \phi^t \), if for each \( x \in M \), 
\[ O_\phi(x) = \{ \phi^t x \}_{t \in \mathbb{R}} \text{ coincides with } O_\psi(x) = \{ \psi^t x \}_{t \in \mathbb{R}} \text{ and the orientations given by } t > 0 \text{ are the same.} \]

Each time change of a flow determines an \( \mathbb{R} \)-cocycle over an \( \mathbb{R} \)-action.

Since orbits coincide, for each \( x \in M \) we must have \( \phi^t x = \psi^{\alpha(t,x)} x \). The additive property of flows, \( \phi^{s+t} = \phi^s \circ \phi^t \), implies the cocycle equation for \( \alpha \), 
\[ \alpha(t+s,x) = \alpha(t,x) + \alpha(s,\phi^t x) \]

The implicit function theorem gives \( \alpha \) the same regularity as the flows.

\( C^r \) flows can be differentiated, to get vector fields \( \xi = \frac{d\phi^t}{dt} \big|_{t=0} \) and \( \eta = \frac{d\psi^t}{dt} \big|_{t=0} \).

\( \phi^t x = \psi^{\alpha(t,x)} x \) then becomes \( \xi(x) = a(x)\eta(x) \), where \( a(x) \geq 0 \) is a real valued scalar function which vanishes only at the fixed point of \( \phi^t \). That is, \( a(x) = \frac{d\alpha(t,x)}{dt} \big|_{t=0} \).

Now two \( C^r \) flows \( \phi^t : M \to M \) and \( \psi^t : N \to N \) are \( C^m \)-flow equivalent \( (m \leq r) \) if there exists a \( C^m \) diffeomorphism \( h : M \to N \) s.t. \( \phi^t = h^{-1} \circ \psi^t \circ h \). So let \( h(x) = \phi^\beta(x)(x) \), where \( \beta \) is differentiable, and \( \left. \frac{d\beta(\phi^t x)}{dt} \right|_{t=0} > 0 \). We get \( h(\phi^t x) = \phi^{\beta(\phi^t x)}(\phi^t x) = \phi^\beta(\phi^{t+1} x)(x) \) implies \( h^{-1}(x) = \phi^{-\beta(\phi^t x)}(x) \). Then the flow \( \psi^t(x) = h \circ \phi^t \circ h^{-1}(x) = \phi^{\alpha(t,x)} x \) is both flow equivalent to and a time change of \( \phi^t \). Therefore \( \alpha(t,x) = \beta(x) + t - \beta(\phi^t x) \). That is, \( \alpha \) is an almost coboundary. If we differentiate with respect to \( t \), and set \( t = 0 \), then \( \alpha(t,x) = \beta(x) + t - \beta(\phi^t x) \) becomes \( \xi\beta = a(x) - 1 \). Therefore we seek solutions to the cohomological equation \( \xi f = g \).
Chapter 4

The Main Results

In this chapter we present our main results.

4.1 The equation $Xf = g$ in $PSL(2, \mathbb{R})$

This section begins with the treatment of the cohomological equation $Xf = g$ in $PSL(2, \mathbb{R})$. Here $X = \left( \frac{1}{2} 0 \atop 0 -\frac{1}{2} \right) \in \mathfrak{sl}(2, \mathbb{R})$ is the generator of the geodesic flow on $PSL(2, \mathbb{R})/\Gamma$, and $f, g \in L^2(PSL(2, \mathbb{R})/\Gamma)$. Note that $X$ is a densely defined operator on $L^2(PSL(2, \mathbb{R})/\Gamma)$. Recall that we defined a basis, which is orthonormal with respect to the Killing form, for $\mathfrak{sl}(2, \mathbb{R})$ by,

$$\{ X = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \theta = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \} \quad (4.1)$$

The cohomological equation $Xf = g$ is studied, following the general outline used in [4], by using the full force of the representation theory of $PSL(2, \mathbb{R})$. Yet, our results concerning the cohomological equation over the geodesic flow are, by and large, not new. One notable new contribution that we have made is that we obtain useful Sobolev estimates on the size of the solution, which are put to good use in sections 4.2-4.4.

The behavior of cohomological equation over a $\mathbb{Z}$ or $\mathbb{R}$ action in the primary hyperbolic situation is already given by the Livshitz theory [14]. There the obstructions are given by periodic data. Another geometric description of the obstructions to the solution of the cohomological equation is also obtained for partially hyperbolic $\mathbb{R}$ and $\mathbb{Z}$ actions by Katok and Kononenko [13], where the obstructions are given by periodic cycle functionals.

The primary contribution of this section is, therefore, to provide an alternative description of the obstructions to the solution of the cohomological equation $Xf = g$, in terms of representation theory. Given that the results concerning the cohomological
equation over the horocycle flow, established in [4], are obtained through a detailed analysis via representation theory, and that a representation theoretic approach is the only one known to be fruitful, our results concerning the geodesic flow show how robust the representation theory approach is. Indeed, there is a high degree of similarity between the geodesic and horocycle flow on this level, and we essentially follow the same approach used in [4]. The reader is then encouraged to compare theorem 1.1 concerning the geodesic flow, and theorem 2.3 concerning the horocycle flow. The most obvious difference concerns statement 4 of each, which arises from the discrete series.

Indeed, for the discrete series, in the case of the horocycle flow, the invariant distributions have growing Sobolev order (in the integral parameter $n$), while in the case of the geodesic flow, the invariant distributions have a fixed order which is independent of $n$.

### 4.1.1 Statement of Results and Initial Reductions

There is a close connection between the spaces $PSL(2, \mathbb{R})/\Gamma$ and Riemann surfaces, as $PSL(2, \mathbb{R})$ can be interpreted as a fiber space whose base is some Riemann surface and whose fiber is a circle ($PSO(2)$). In fact, the space $PSL(2, \mathbb{R})/\Gamma$ can be identified with the unit tangent bundle of the surface $M = PSO(2) \backslash PSL(2, \mathbb{R})/\Gamma$ [6]. To make the identification, we note that $PSL(2, \mathbb{R})$ acts on the upper half plane $H$ by the conformal (Möbius) transformations, $(a \ b \ c \ d) \cdot z = \frac{az + c}{bz + d}$. This action is transitive, and the stabilizer of any point is $PSO(2)$. Therefore the upper half plane can be identified with the homogeneous space $PSO(2) \backslash PSL(2, \mathbb{R})$. If we further identify points which are translates of each other by the discrete subgroup $\Gamma$, which acts without fixed points, then we get an identification of $M = PSO(2) \backslash PSL(2, \mathbb{R})/\Gamma$, with a Riemann surface. If $\Gamma$ has fixed points, then the resulting homogeneous space will only be an orbifold.

Through this identification, the matrices

\[
U = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad X = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}
\]

define respectively, via left multiplication on $PSL(2, \mathbb{R})/\Gamma$, the generators of the (stable) horocycle flow $\{\phi_t^U\}$ and of the geodesic flow $\{\phi_t^X\}$ on the unit tangent bundle $SM$ of the hyperbolic surface $M = H/\Gamma$.

The spectrum $\sigma(\triangle_M)$ of the Laplacian on $M$ has pure point discrete component of finite multiplicity and an absolutely continuous component on the interval $[1/4, \infty)$.
with finite multiplicity equal to the number of cusps of $M$ [8]. The Laplacian $\Delta_M$ has a smallest nonzero eigenvalue, $\mu_0 > 0$. Selberg’s spectral gap conjecture [21] asserts that there are no eigenvalues in $(0, 1/4)$ for any arithmetic lattice. Let $\sigma_{pp}$ denote the pure point spectrum of $\Delta_M$ and $C$ the (finite) set of cusps of $M$. Then of course, if $M$ is compact, the spectrum of the Laplacian $\Delta_M$ is supported on $\sigma_{pp}$.

The Hilbert space $L^2(PSL(2, \mathbb{R})/\Gamma)$ is a unitary representation of $PSL(2, \mathbb{R})$, and the spectral decomposition of $L^2(PSL(2, \mathbb{R})/\Gamma)$ is given in terms of the irreducible unitary representations of $PSL(2, \mathbb{R})$. It is given by,

$$L^2(PSL(2, \mathbb{R})/\Gamma) = \bigoplus_{\mu \in \sigma_{pp}} m_\mu \cdot \mathcal{H}_\mu \oplus \bigoplus_{n \in \mathbb{Z}^+} m_n \cdot (\mathcal{H}_n \oplus \mathcal{H}_{-n}) \oplus \bigoplus_{c \in \mathcal{C}} \mathcal{H}_c \quad (4.2)$$

Here $m_\mu$ is the finite multiplicity of an eigenvalue $\mu \in \sigma_{pp}$, and $m_n$ is the dimension of the space of holomorphic sections of the $n$-th power of the canonical line bundle, and is computable via the Riemann-Roch Theorem [23]. And,

$$m_\mu \cdot \mathcal{H}_\mu = \bigoplus_{i=1}^{m_\mu} \mathcal{H}_\mu, \quad m_n \cdot (\mathcal{H}_n \oplus \mathcal{H}_{-n}) = \bigoplus_{i=1}^{m_n} (\mathcal{H}_n \oplus \mathcal{H}_{-n}), \quad \mathcal{H}_c = \int \mathcal{H}_c(\lambda) d\mathcal{S}_c(\lambda)$$

Note that the trivial representation $\mathcal{H}_0$ corresponding to $\mu = 0$ appears with multiplicity one, and is realized as the space of constant functions, and the Stieltjes measures $d\mathcal{S}_c$ are supported on $[1/4, \infty)$ and are absolutely continuous. All representations $\mathcal{H}_\mu$ for any $\mu \geq 0$ are irreducible unitary representations for $PSL(2, \mathbb{R})$.

First, the $X$-invariant distributions are classified. Let $\mathcal{E}'(L^2(PSL(2, \mathbb{R})/\Gamma))$ be the dual space of the space $C^\infty(L^2(PSL(2, \mathbb{R})/\Gamma))$ of smooth vectors in $L^2(PSL(2, \mathbb{R})/\Gamma)$. Then the subspace

$$\mathcal{I}(PSL(2, \mathbb{R})/\Gamma) = \{D \in \mathcal{E}'(L^2(PSL(2, \mathbb{R})/\Gamma)) | L_X D = 0\}$$

of $X$-invariant distributions is determined by the spectrum of the Laplacian $\Delta$.

**Theorem 4.1.** The space $\mathcal{I}(PSL(2, \mathbb{R})/\Gamma)$ has infinite countable dimension. Furthermore, there is a decomposition

$$\mathcal{I}(PSL(2, \mathbb{R})/\Gamma) = \bigoplus_{\mu \in \sigma_{pp}} m_\mu \cdot \mathcal{I}_\mu \oplus \bigoplus_{n \in \mathbb{Z}^+} m_n \cdot (\mathcal{I}_n \oplus \mathcal{I}_{-n}) \oplus \bigoplus_{c \in \mathcal{C}} \mathcal{I}_c \quad (4.3)$$

where, $(s \in \mathbb{R})$
1. for $\mu = 0$, the space $\mathcal{I}_0$ is spanned by the $\text{PSL}(2, \mathbb{R})$-invariant volume,

2. for $0 < \mu < 1/4$, the space $\mathcal{I}_\mu \subset W^{-s}(\text{PSL}(2, \mathbb{R})/\Gamma)$, provided $s > 1$, and it has dimension equal to 2.

3. for $\mu \geq 1/4$, the space $\mathcal{I}_\mu \subset W^{-s}(\text{PSL}(2, \mathbb{R})/\Gamma)$, provided $s > 1/2$, and it has dimension equal to 2.

4. for $n \in \mathbb{Z}^+$, the space $\mathcal{I}_n \subset W^{-s}(\text{PSL}(2, \mathbb{R})/\Gamma)$, provided $s > 0$, and it has dimension equal to 2. Similarly for $\mathcal{I}_{-n}$.

5. for $c \in \mathbb{C}$, the space $\mathcal{I}_c \subset W^{-s}(\text{PSL}(2, \mathbb{R})/\Gamma)$, provided $s > 1/2$, and it has infinite countable dimension.

**Proof.** All the operators in the universal enveloping algebra $\mathcal{U}(\text{psl}(2, \mathbb{R}))$ are decomposable with respect to the decomposition (4.1), and therefore so are all the spaces which are defined by those operators, such as $C^\infty(\mathcal{H}), \mathcal{E}'(\mathcal{H}), W^s(\mathcal{H})$, and $\mathcal{I}(\mathcal{H})$, for $\mathcal{H} = L^2(\text{PSL}(2, \mathbb{R})/\Gamma)$. Therefore the theorem is established once we determine the $\mathcal{X}$-invariant distributions in each irreducible unitary representation $\mathcal{H}_\mu$ for $\text{PSL}(2, \mathbb{R})$. The details are found in the next section. Of course, this then establishes Theorem 1.1.

The $\mathcal{X}$-invariant distributions of order $s > 1$ form a complete set of obstructions to the existence of smooth solutions of the cohomological equation $Xf = g$ for functions $g \in W^s(\text{PSL}(2, \mathbb{R})/\Gamma)$. Let

$$\mathcal{I}^s(\text{PSL}(2, \mathbb{R})/\Gamma) = \{D \in W^{-s}(\text{PSL}(2, \mathbb{R})/\Gamma)|L_XD = 0\}$$

Then $\mathcal{I}^s(\text{PSL}(2, \mathbb{R})/\Gamma)$ is determined by Theorem 4.1.

**Theorem 4.2.** Let $s > 1$, then there exists a constant $C_{s,t}$ such that, for all $g \in W^s(\mathcal{H})$,

- if $t < -1$, and $g$ has no component on the trivial sub-representation of $L^2(\text{PSL}(2, \mathbb{R})/\Gamma)$,
- or
- if $t < s - 1$ and $D(g) = 0$ for all $D \in \mathcal{I}^s(\text{PSL}(2, \mathbb{R})/\Gamma)$,

then the equation $Xf = g$ has a solution $f \in W^t(\text{PSL}(2, \mathbb{R})/\Gamma)$, which satisfies the Sobolev estimate $\|f\|_t \leq C_{s,t}\|g\|_s$. Solutions are unique modulo the trivial sub-representation iff $t > 0$.

**Proof.** This will follow from Theorem 4.3 by setting $\mathcal{H} = L^2(\text{PSL}(2, \mathbb{R})/\Gamma)$. 
4.1.2 Norms

We now define the Sobolev norms on each irreducible unitary representation for \( \text{PSL}(2, \mathbb{R}) \). First, the Laplacian and Casimir operator are defined by,

\[
\Delta = -(X^2 + Y^2 + \theta^2) \quad \text{and} \quad \Box = -X^2 - Y^2 + \theta^2
\]

(4.4)

We note that these operators differ by a minus sign from how we defined the Laplacian and Casimir operators in section 3.4. We do this in order to conform to the notation of [4].

We recall from section 3.4, that the Sobolev norm on \( H_\mu \) is given by

\[
\| f \|_{W^s(H_\mu)} = \| (I + \Delta)^{s/2} f \|_{H_\mu}.
\]

On the other hand, in section 3.2.1, we found that we could construct an orthonormal basis for \( H_\mu \), which is, in fact, an eigenbasis for the action of the operator \(-iE\), where \( E = 2\pi(\theta) \) is the generator of the maximal compact subgroup \( K = \text{SO}(2) \). In order to use some of the results of [4], we construct an eigenbasis which is orthogonal, but not orthonormal. Namely, let \( v_0 \) be a unit vector of eigenvalue 0 for \( E \). That is, \( Ev_0 = 0 \). Then define \( v_k = R_+ v_0 \) for \( k > 0 \) and \( v_k = R_- v_0 \) for \( k < 0 \). This provides an orthogonal basis for each of the principal or complementary series representations. For the discrete series, we take the basis defined using only the vectors \( v_{n+k} = R_+ v_n \), where \( n \) is the smallest eigenvalue in \( \text{spec}(-i/2E) \), so that \(-i/2Ev_n = nv_n\).

We now form an adapted basis by defining

\[
u_k = c_k(R_+u_{k-1}), \quad \text{where } c_k = \frac{2}{2k - 1 + \nu} \quad \text{for } k > 0
\]

(4.5)

\[
u_k = c_k(R_-u_{k+1}), \quad \text{where } c_k = \frac{2}{-2k - 1 + \nu} \quad \text{for } k < 0
\]

We then compute the action of the Laplacian \( \Delta = \Box - 2\theta^2 \) on this eigenbasis.

\[
\theta u_k = iku_k, \quad \Delta u_k = (\mu + 2k^2)u_k
\]

(4.6)

The norms of the \( u_k \) are given recursively by,

\[
\| u_k \|^2 = \begin{cases} \| u_{k-1} \|^2, & \text{if } \nu \in i\mathbb{R} \\ \frac{2k-1}{2k+1} \| u_{k-1} \|^2, & \text{if } \nu \in \mathbb{R} \end{cases}
\]

(4.7)

By defining \( \Pi_{\nu,k} = \prod_{i=0}^{k} \frac{2i-1-\nu}{2i+1+\nu} \), and noting that \( \| u_0 \| = 1 \) (or \( \| u_n \| = 1 \) for the discrete series), we get that \( \| u_k \| = |\Pi_{\nu,k}| \). The following useful lemma of [4] gives estimates on the size of \( |\Pi_{\nu,k}| \).
Lemma 4.1 (Flaminio-Forni). If \( \nu \in i \mathbb{R} \), then for all \( k \geq 0 \),

\[
|\Pi_{\nu,k}| = 1
\]  

(4.8)

There exists a \( C > 0 \) such that, if \( \nu \in (-1, 1) \setminus \{0\} \), for all \( k > 0 \), we have

\[
C^{-1} \left( \frac{1-v}{1+v} \right) (1+k)^{-\nu} \leq \Pi_{\nu,k} \leq C \left( \frac{1-v}{1+v} \right) (1+k)^{-\nu}
\]  

(4.9)

if \( \nu = 2n - 1 \), for all \( k \geq l \geq n \), we have

\[
C^{-1} \left( \frac{k-n+1}{l-n+1} \right)^{-\nu} \leq \frac{\Pi_{\nu,k}}{\Pi_{\nu,l}} \leq C \left( \frac{k-n+1}{l-n+1} \right)^{-\nu}
\]  

(4.10)

We can now give an expression for the Sobolev norm of a vector \( f = \sum_{k=0}^{\infty} f_k u_k \),

\[
\|f\|_{W^\nu(\Sigma_d)} = \left( \sum_{-\infty}^{\infty} (1 + \mu + 2k^2)^{s} |\Pi_{\nu,k}| |f_k|^2 \right)^{1/2}
\]  

(4.11)

by the above lemma, \( \|u_k\|^2_s \approx (1 + |k|)^{2s-\Re(\nu)} \), and so it follows that,

\[
\|f\|_s \approx \left( \sum_{-\infty}^{\infty} (1 + |k|)^{2s-\Re(\nu)} |f_k|^2 \right)^{1/2}
\]  

(4.12)

4.1.3 Invariant Distributions

In this section we examine the equation \( Xf = g \) for the diagonal element \( X = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{R}) \). We will use the notation of section 3.2.1 and 4.1.2.

We want to describe the \( X \)-invariant distributions. We start by recalling the action of \( X \) on the general adapted basis element \( u_k \), defined by (4.3). By noting that \( U = \theta - i / 2 (R_+ - R_-) \) and \( X = [\theta, U] \), we can compute that,

\[
X u_k = \frac{2k+1+v}{4} u_{k+1} - \frac{2k-1-v}{4} u_{k-1}
\]  

(4.13)

or if we write \( b^+(k) = \frac{2k+1+v}{4} \) and \( b^-(k) = \frac{2k-1-v}{4} \), then we have \( X u_k = b^+(k) u_{k+1} - b^-(k) u_{k-1} \).

Note that for \( k > 0 \), we have \( b^+(k) = -b^-(k) \) and \( b^-(k) = -b^+(k) \). Then we get \( X \cdot D(u_k) = -D(Xu_k) = -b^+(k) d_{k+1} + b^-(k) d_{k-1} \). And so,

\[
X \cdot D = 0 \iff -b^+(k) d_{k+1} + b^-(k) d_{k-1} = 0 \text{ for all } k
\]

\[
\Rightarrow d_{k+1} = \frac{b^-(k)}{b^+(k)} d_{k-1}
\]
It is easily seen that we then get two linearly independent solutions which will be uniquely determined by whatever initial conditions we impose. As always, we analyze the solutions separately for each series. We will make the convention that empty products are equal to one.

**-Principal and Complementary Series.**

We use the initial conditions \( d_0 = 1 \) and \( d_1 = 1 \). We now have the following two independent solutions,

\[
(D_1)_{2k+1} = 0, \quad (D_1)_{2k} = \prod_{j=0}^{|k|-1} \frac{b^- (2j + 1)}{b^+ (2j + 1)}
\]

\[
(D_2)_{2k} = 0, \quad (D_2)_{2|k|+1} = -(D_2)_{-2|k|-1} = \prod_{j=1}^{|k|} \frac{b^- (2j)}{b^+ (2j)}
\]

**-Discrete Series.**

We use the initial conditions \( d_n = 1 \) and \( d_{n+1} = 1 \). We again get two independent solutions,

\[
(D_1)_{n+2k} = \prod_{j=0}^{k-1} \frac{b^- (n + 2j + 1)}{b^+ (n + 2j + 1)}, \quad (D_1)_{n+2k+1} = 0, \quad \text{for } k \geq 0
\]

\[
(D_2)_{n+2k+1} = \prod_{j=1}^{k} \frac{b^- (n + 2j)}{b^+ (n + 2j)}, \quad (D_2)_{n+2k} = 0, \quad \text{for } k \geq 0
\]

### 4.1.4 Sobolev Order

The invariant distributions for the principal and complementary series have positive Sobolev order, while for the discrete series, they are given by measures, in fact functions of growing regularity. To prove this, we use the following lemma.

**Lemma 4.2.** We have that \( \frac{b^- (k)}{b^+ (k)} \leq 1 \), for all \( k \geq 0 \), and \( v \) in the principal or complementary series, or \( k \geq n \), and \( v \) in the discrete series.

**Proof.** First we examine the principal series, i.e. \( v \in i\mathbb{R} \). Computation shows for \( k \geq 0 \) that,

\[
\left| \frac{b^- (k)}{b^+ (k)} \right|^2 = \left| \frac{2k - 1 - v}{2k + 1 + v} \right|^2 = \frac{(2k - 1)^2 + |v|^2}{(2k + 1)^2 + |v|^2} \leq 1 \quad \text{for } k \geq 0
\]
On the other hand, if \( \nu \in \mathbb{R} \), and \( \nu + 1 > 0 \), then we get that,

\[
\left| \frac{b^- (k)}{b^+ (k)} \right| = \left| \frac{2k - 1 - \nu}{2k + 1 + \nu} \right| \leq 1 \quad \text{(for } k \geq 0)\]

This is true of the complementary series, where \(-1 < \nu < 1\), and the discrete series, where \(\nu = 2n - 1\).

**-Principal and Complementary Series.**

\( D_i \) defines a linear functional by setting \( D_i (f) = \sum_{-\infty}^{\infty} f_k (D_i)_k \), whenever the sums converge. These sums are convergent on the Sobolev spaces of appropriate order as follows,

\[
|D_1 (f)| \leq \sum_{-\infty}^{\infty} |f_k| |(D_1)_k|
\]

\[
= \sum_{-\infty}^{\infty} |f_{2k}| \left| \prod_{j=0}^{k-1} \frac{b^- (2j + 1)}{b^+ (2j + 1)} \right| \leq \sum_{-\infty}^{\infty} |f_{2k}|
\]

and

\[
|D_2 (f)| \leq \sum_{-\infty}^{\infty} |f_k| |(D_2)_k|
\]

\[
= \sum_{-\infty}^{\infty} |f_{2k+1}| \left| \prod_{j=0}^{k-1} \frac{b^- (2j)}{b^+ (2j)} \right| \leq \sum_{-\infty}^{\infty} |f_{2k+1}|
\]

Both of these sums converge provided \( t > \frac{1 - \Re (\nu)}{2} \).

**-Discrete Series.**

We treat this case exactly the same, except the sums are taken over \( k \geq 2n - 1 \), rather than over all of \( \mathbb{Z} \). However, the inequality \( \left| \frac{b^- (k)}{b^+ (k)} \right| \leq 1 \) for all \( k \geq n \), is insuffi-

cient. We will make use of the fact that \((D_1)_{2k+n} = \frac{b^- (n+1)}{b^- (2k+1+n)f_{2k+1+n}}\), and \((D_2)_{2k-1+n} = \frac{b^+ (n)}{b^+ (2k+n)f_{2k+n}}\), where \( f_k^{(i)} \) is the (formal) solution to the homogeneous equation \( X f = 0 \) (see section 4.1.5). The asymptotics of \( f_{k}^{(i)} \) are characterized for the discrete series by lemma 4.6. One can then use (4.12) to see that the sum converges for \( f \in W^s (\mathcal{H}_n) \) for every \( s > 0 \).
Note also that since \( |b^{-}(k)|^2 < 1 \), for \( k > 0 \), the above bounds on the order of the distributions might be improved even further. In general, there is an inverse relationship between the (formal) solutions of the homogeneous equation, and the \( X \)-invariant distributions, given in section 4.1.7. Except for the discrete series, we will not make an attempt to utilize this relationship in order to obtain optimal statements concerning Sobolev order.

### 4.1.5 The Homogeneous Equation

Now we study the solutions to the equation \( Xf = g \). To do this we first solve the homogeneous problem \( Xf = 0 \), and then construct a Green’s function to solve the nonhomogenous problem.

We have that,

\[
Xf = \sum_{k \in \mathbb{Z}} \left( b^+(k-1)f_{k-1} - b^-(k+1)f_{k+1} \right)u_k \tag{4.14}
\]

So \( Xf = g \) becomes \( g_k = (Xf)_k = b^+(k-1)f_{k-1} - b^-(k+1)f_{k+1} \).

- **Principal and Complementary Series.**

The homogeneous problem becomes \( b^+(k-1)f_{k-1} - b^-(k+1)f_{k+1} = 0 \), for all \( k \in \mathbb{Z} \). This has two linearly independent solutions which satisfy the initial conditions \( f^{(1)}_0 = 1, f^{(1)}_1 = 0 \) and \( f^{(2)}_0 = 0, f^{(2)}_1 = 1 \).

\[
f^{(1)}_{2k} = \prod_{j=0}^{[k]-1} \frac{b^+(2j)}{b^-(2j+2)}, \quad f^{(1)}_{2k+1} = 0
\]

\[
f^{(2)}_{2|k|+1} = -f^{(2)}_{-2|k|-1} = \prod_{j=1}^{[k]} \frac{b^+(2j-1)}{b^-(2j+1)}, \quad f^{(2)}_{2k} = 0
\]

We need to understand the asymptotic behavior of these solutions.

**Lemma 4.3.** For the principal series, \( \nu \in i\mathbb{R} \), we have,

\[
C_{\nu}^{-1}(4|k| + 1)^2 + |
u|^2)^{-\frac{1}{2}} \leq |f^{(1)}_{2k}|^2 \leq C_{\nu}(4|k| + 3)^2 + |
u|^2)^{-\frac{1}{2}} \tag{4.15}
\]

And,

\[
C_{\nu}^{-1}(4|k| - 1)^2 + |
u|^2)^{-\frac{1}{2}} \leq |f^{(2)}_{2k+1}|^2 \leq C_{\nu}(4|k| + 5)^2 + |
u|^2)^{-\frac{1}{2}} \tag{4.16}
\]

where \( C_{\nu} \) is bounded in \( \nu \).
Proof. We start with $f^{(1)}$. By taking logs, we have
\[
\log \left| \prod_{j=0}^{k-1} \frac{b^+(2j)}{b^-(2j+2)} \right|^2 = \sum_{j=0}^{k-1} \log \left| \frac{b^+(2j)}{b^-(2j+2)} \right|^2
\]
\[
= \sum_{j=0}^{k-1} \log \left( \frac{16j^2 + 8j + 1 + |\nu|^2}{16j^2 + 24j + 9 + |\nu|^2} \right)
\]
\[
= - \sum_{j=0}^{k-1} \log \left( 1 + \frac{8(2j + 1)}{(4j + 1)^2 + |\nu|^2} \right)
\]

We can estimate the logarithms by the inequality,
\[
\frac{x}{1 + x} \leq \log(1 + x) \leq x, \quad \text{for } x \in \mathbb{R}^+
\]

This leads to the inequalities,
\[
\frac{8(2j + 1)}{(4j + 3)^2 + |\nu|^2} \leq \log \left( 1 + \frac{8(2j + 1)}{(4j + 1)^2 + |\nu|^2} \right) \leq \frac{8(2j + 1)}{(4j + 1)^2 + |\nu|^2}
\]

And,
\[
- \sum_{j=0}^{k-1} \frac{8(2j + 1)}{(4j + 1)^2 + |\nu|^2} \leq - \sum_{j=0}^{k-1} \log \left( 1 + \frac{8(2j + 1)}{(4j + 1)^2 + |\nu|^2} \right) \leq - \sum_{j=0}^{k-1} \frac{8(2j + 1)}{(4j + 3)^2 + |\nu|^2}
\]

First, both $\frac{8(2x + 1)}{(4x + 3)^2 + |\nu|^2}$ and $\frac{8(2x + 1)}{(4x + 1)^2 + |\nu|^2}$, have critical points at $x^+_v = \frac{-2 + \sqrt{1 + |\nu|^2}}{4}$, and so each only has at most one critical point in $[0, \infty)$. This critical point is a maxima, and therefore $\frac{8(2x + 1)}{(4x + 3)^2 + |\nu|^2}$ and $\frac{8(2x + 1)}{(4x + 1)^2 + |\nu|^2}$ are monotonic decreasing on $[\frac{-2 + \sqrt{1 + |\nu|^2}}{4}, \infty)$. And we note that $x^+_v = \frac{-2 + \sqrt{1 + |\nu|^2}}{4} \leq 0$ for $|\nu|^2 \leq 3$.

The sum $\sum_{j=0}^{k-1} \frac{8(2j + 1)}{(4j + 3)^2 + |\nu|^2}$ can then be bounded from below by the integral inequality. So let $a_v$ be the smallest integer s.t. $a_v \geq \frac{-2 + \sqrt{1 + |\nu|^2}}{4}$. Then we have the
inequality,

\[
\sum_{j=0}^{|k|-1} \frac{8(2j+1)}{(4j+3)^2 + |v|^2} \geq \sum_{j=a_v}^{|k|-1} \frac{8(2j+1)}{(4j+3)^2 + |v|^2}
\]

\[
\geq \int_{a_v}^{|k|} \frac{8(2x+1)}{(4x+3)^2 + |v|^2} dx
\]

\[
= \frac{1}{2} \left( \log |(4x+3)^2 + |v|^2| \right)_{a_v}^{|k|} - \frac{2}{|v|} \arctan \left( \frac{2u+1}{|v|} \right)_{2a_v+1}^{2|k|+1}
\]

So,

\[
- \sum_{j=0}^{|k|-1} \frac{8(2j+1)}{(4j+3)^2 + |v|^2} \leq - \frac{1}{2} \left( \log |(4x+3)^2 + |v|^2| \right)_{a_v}^{|k|} - \frac{2}{|v|} \arctan \left( \frac{2u+1}{|v|} \right)_{2a_v+1}^{2|k|+1}
\]

While the sum \( \sum_{j=0}^{|k|-1} \frac{8(2j+1)}{(4j+3)^2 + |v|^2} \) can then be bounded from above by the integral inequality,

\[
\sum_{j=0}^{|k|-1} \frac{8(2j+1)}{(4j+3)^2 + |v|^2} \leq \int_0^{a_v-1} \frac{8(2x+1)}{(4x+1)^2 + |v|^2} dx + \sum_{j=a_v}^{a_v} \frac{8(2j+1)}{(4j+1)^2 + |v|^2} + \int_{a_v}^{|k|-1} \frac{8(2x+1)}{(4x+1)^2 + |v|^2} dx
\]

\[
\leq \int_0^{|k|-1} \frac{8(2x+1)}{(4x+1)^2 + |v|^2} dx + C_{a_v}
\]

\[
= \frac{1}{2} \log |(4x+1)^2 + |v|^2|_0^{|k|-1} - \int_1^{2|k|-1} \frac{2du}{4u(u-1) + (1 + |v|^2)} + C_{a_v}
\]

\[
= \frac{1}{2} \left( \log |(4x+1)^2 + |v|^2|_0^{|k|-1} - \frac{2}{|v|} \arctan \left( \frac{2u-1}{|v|} \right)_{1}^{2|k|-1} \right) + C_{a_v}
\]

So that,

\[
- \frac{1}{2} \left( \log |(4x+1)^2 + |v|^2|_0^{|k|-1} - \frac{2}{|v|} \arctan \left( \frac{2u-1}{|v|} \right)_{1}^{2|k|-1} \right) - C_{a_v}
\]

\[
\leq - \sum_{j=0}^{|k|-1} \log \left( 1 + \frac{8(2j+1)}{(4j+1)^2 + |v|^2} \right)
\]
And then,
\[
\left| \prod_{j=0}^{N-1} \frac{b^+(2j)}{b^-(2j+2)} \right|^2 = \exp \left\{ - \sum_{j=0}^{N-1} \log \left( 1 + \frac{8(2j+1)}{(4j+1)^2 + |v|^2} \right) \right\}
\]
\[
\leq \exp \left\{ - \sum_{j=0}^{N-1} \frac{8(2j+1)}{(4j+3)^2 + |v|^2} \right\}
\]
\[
\leq \exp \left\{ \log |(4x+3)^2 + |v|^2|^{-\frac{1}{2}} \left| a_v \right| + \frac{1}{|v|} \arctan \frac{2u+1}{|v|} \right\}
\]

**Lemma 4.4.** For all $|v| > 0$ and $|k| > a_v + 1$, we have $0 \leq \frac{1}{|v|} \arctan \frac{2u+1}{|v|} \leq \frac{\pi}{2}$.

**Proof.** This will follow from the easy fact that $\lim_{|v| \to \infty} \frac{1}{|v|} \arctan \frac{2u+1}{|v|} \left|_{2u+1}^{\infty} \right. = 0$ and that $\lim_{|v| \to 0^+} \frac{1}{|v|} \arctan \frac{2u+1}{|v|} \left|_{2u+1}^{\infty} \right. = -\frac{1}{2}$, in addition to the fact that $\frac{1}{|v|} \arctan \frac{2u+1}{|v|} \left|_{2u+1}^{\infty} \right.$ has a maxima at $|v| = 0$, which gives a value bounded by $\frac{\pi}{2}$.

Note that $C_{a_v}$ is bounded in $v$. A similar result holds for $\frac{1}{|v|} \arctan \frac{2u-1}{|v|} \left|_{1}^{2u-1} \right.$, which then establishes the lemma for $f^{(1)}$. The proof for $f^{(2)}$ is entirely similar.

For the Complementary series, we attempt to estimate using logs in a similar way.

**Lemma 4.5.** For the complementary series, $-1 < v < 1$, $v \neq 0$, we have

\[
\frac{1+v}{3-v} \cdot \left( \frac{4|k|-3+v}{1+v} \right)^{\frac{1}{2}} \leq |f_{2k}^{(1)}| \leq \left( \frac{4|k|+3-v}{3-v} \right)^{\frac{1}{2}}
\]

And,

\[
\frac{3+v}{5-v} \cdot \left( \frac{4|k|-1+v}{3+v} \right)^{\frac{1}{2}} \leq |f_{2k+1}^{(2)}| \leq \left( \frac{4|k|+5-v}{5-v} \right)^{\frac{1}{2}}
\]

**Proof** We first take logs and get that,

\[
\log \left| \prod_{j=0}^{N-1} \frac{b^+(2j)}{b^-(2j+2)} \right| = \sum_{j=0}^{N-1} \log \left| \frac{b^+(2j)}{b^-(2j+2)} \right|
\]
\[
= \sum_{j=0}^{N-1} \log \left( \frac{4j+1+v}{4j+3-v} \right)
\]
\[
= \sum_{j=0}^{N-1} \log \left( 1 + \frac{2(1-v)}{4j+1+v} \right)
\]
Again we use that,
\[ \frac{x}{1+x} \leq \log(1+x) \leq x, \quad \text{for } x \in \mathbb{R}^+ \]

This gives us,
\[ \frac{2(1-\nu)}{4j+3-\nu} \leq \log \left(1 + \frac{2(1-\nu)}{4j+1+\nu}\right) \leq \frac{2(1-\nu)}{4j+1+\nu} \]

and then,
\[ -\sum_{j=0}^{\lfloor k \rfloor - 1} \frac{2(1-\nu)}{4j+1+\nu} \leq -\sum_{j=0}^{\lfloor k \rfloor - 1} \log \left(1 + \frac{2(1-\nu)}{4j+1+\nu}\right) \leq -\sum_{j=0}^{\lfloor k \rfloor - 1} \frac{2(1-\nu)}{4j+3-\nu} \]

We then estimate the sums by the integral inequalities,
\[ \sum_{j=0}^{\lfloor k \rfloor - 1} \frac{2(1-\nu)}{4j+3-\nu} \geq \int_0^{\lfloor k \rfloor} \frac{2(1-\nu)}{4x+3-\nu} \, dx = \log \left(\frac{4\lfloor k \rfloor + 3-\nu}{3-\nu}\right)^{1-\nu} \]

And
\[ \sum_{j=1}^{\lfloor k \rfloor - 1} \frac{2(1-\nu)}{4j+1+\nu} \leq \int_0^{\lfloor k \rfloor - 1} \frac{2(1-\nu)}{4x+1+\nu} \, dx = \log \left(\frac{4\lfloor k \rfloor + 3+\nu}{1+\nu}\right)^{1+\nu} \]

Taking exponentials gives us the upper inequality for \( f^{(1)} \). We get the lower inequality by noting that \( \prod_{j=0}^{\lfloor k \rfloor - 1} \frac{b^+(2j)}{b^-(2j+2)} = \frac{1+\nu}{3-\nu} \cdot \prod_{j=1}^{\lfloor k \rfloor - 1} \frac{b^+(2j)}{b^-(2j+2)} \). The proof for \( f^{(2)} \) is entirely similar.

- **Discrete Series.**

Here the homogeneous problem becomes \( b^+(k)f_k - b^-(k+2)f_{k+2} = 0 \), for all \( k \geq n \). As above, this has two linearly independent solutions which satisfy the initial conditions \( f^{(1)}_n = 1, f^{(1)}_{n+1} = 0 \) and \( f^{(2)}_n = 0, f^{(2)}_{n+1} = 1 \).

\[ f^{(1)}_{n+2k} = \prod_{j=0}^{k-1} \frac{b^+(n+2j)}{b^-(n+2j+2)}, \quad f^{(1)}_{2k+1} = 0 \]

\[ f^{(2)}_{n+2k+1} = \prod_{j=1}^{k} \frac{b^+(n+2j-1)}{b^-(n+2j+1)}, \quad f^{(2)}_{2k} = 0 \]

The asymptotic behavior of these solutions is given by,
Lemma 4.6. For \( \nu = 2n - 1 \), we have that,

\[
\left( \frac{k + n}{n} \right)^{n-1} \leq |f^{(1)}_{n+2k}| \leq n \cdot k^{n-1}
\] (4.19)

And,

\[
\left( \frac{k + n - \frac{1}{2}}{n - \frac{1}{2}} \right)^{n-1} \leq |f^{(2)}_{n+2k+1}| \leq \frac{4n + 2}{6} (2k - 1)^{n-1}
\] (4.20)

Proof. Taking logs we get,

\[
\log \prod_{j=0}^{\lfloor k \rfloor - 1} \frac{b^+(n+2j)}{b^-(n+2j+2)} = \sum_{j=0}^{\lfloor k \rfloor - 1} \log \left( \frac{b^+(n+2j)}{b^-(n+2j+2)} \right) \\
= \sum_{j=0}^{\lfloor k \rfloor - 1} \log \left( \frac{j+n}{j+1} \right) \\
= \sum_{j=0}^{\lfloor k \rfloor - 1} \log \left( 1 + \frac{n-1}{j+1} \right)
\]

Again we use the inequality,

\[
\frac{x}{1+x} \leq \log(1+x) \leq x \quad \text{for} \quad x \in \mathbb{R}^+
\]

This yields, by the integral inequalities,

\[
(n-1) \log\left( \frac{k+n}{n} \right) = \int_0^k \frac{n-1}{x+n} \, dx \leq \sum_{j=0}^{k-1} \log \left( 1 + \frac{n-1}{j+1} \right)
\]

\[
\sum_{j=1}^{k-1} \log \left( 1 + \frac{n-1}{j+1} \right) \leq \int_0^{k-1} \frac{n-1}{x+1} \, dx = (n-1) \log k
\]

Taking exponentials gives us the lower inequality for \( f^{(1)} \). We get the upper inequality by noting that \( \prod_{j=0}^{\lfloor k \rfloor - 1} \frac{b^+(n+2j)}{b^-(n+2j+2)} = n \cdot \prod_{j=1}^{\lfloor k \rfloor - 1} \frac{b^+(n+2j)}{b^-(n+2j+2)} \). The proof for \( f^{(2)} \) is entirely similar.

We recall that,

\[
\|f\|_t \approx \left( \sum_{-\infty}^{\infty} (1 + |k|)^{2t-\Re(\nu)} |f_k|^2 \right)^{\frac{1}{2}}
\]

The above lemmas then imply the following,
PROPOSITION 1. For any irreducible unitary representation of $\text{PSL}(2, \mathbb{R})$, we have (for $i = 1, 2$) that, $\| f^{(i)} \|_t < \infty$ iff $t < 0$.

4.1.6 The Green’s Function

We now are ready to construct the Green’s function for the difference equation,

$$g_k = (X f)_k = b^+(k - 1)f_{k-1} - b^-(k + 1)f_{k+1}$$

The Green’s function $G(k, l)$ is defined using the solutions $f^{(1)}, f^{(2)}$ to the homogeneous equation as follows [2].

$$G(k, l) = \frac{\text{det} \begin{pmatrix} f^{(1)}_k & f^{(2)}_k \\ f^{(1)}_{k+1} & f^{(2)}_{k+1} \end{pmatrix}}{\text{det} \begin{pmatrix} f^{(1)}_k & f^{(2)}_k \\ f^{(1)}_{k+1} & f^{(2)}_{k+1} \end{pmatrix}} \quad (4.21)$$

The explicit computation of the Green’s function, for the principal or complementary series, is given as,

$$G(2k, 2l) = 0$$

$$G(2k, 2l - 1) = \frac{f^{(1)}_{2k}}{f^{(2)}_{2l}} = \begin{cases} \Pi_{j=|l|+1}^{k-1} b^+(2j) & \text{for } |k| \geq |l| \\ \Pi_{j=|k|+1}^{l-1} b^+(2j) & \text{for } |k| < |l| \end{cases}$$

$$G(2k + 1, 2l) = \frac{f^{(2)}_{2k+1}}{f^{(2)}_{2l+1}} = \text{sgn} \left( \frac{2k + 1}{2l + 1} \right) \begin{cases} \Pi_{j=|l|+1}^{k-1} b^+(2j-1) & \text{for } |k| \geq |l| \\ \Pi_{j=|k|+1}^{l-1} b^+(2j-1) & \text{for } |k| < |l| \end{cases}$$

$$G(2k + 1, 2l + 1) = 0$$

4.1.7 General Solution of the Non-homogeneous Equation

We are now ready to give the most general (formal) solution of the non-homogeneous equation.

That is, the (formal) solutions of $X f = g$ are given by,

$$f = \begin{cases} f_{2k} = c_1 f^{(1)}_{2k} - \sum_{l \leq k} \frac{G(2k, 2l - 1)}{b^-(2l)} g_{2l-1} \\ f_{2k+1} = c_2 f^{(2)}_{2k+1} - \sum_{l \leq k} \frac{G(2k+1, 2l)}{b^-(2l+1)} g_{2l} \end{cases} \quad (4.22)$$
Where of course, \( c_1, c_2 \in \mathbb{C} \).

Since we have already characterized the asymptotic behavior of the homogenous solutions \( f^{(1)} \) and \( f^{(2)} \), we only need determine the behavior of \( \sum_{l \leq k} \frac{G(2k, 2l - 1)}{b^{(2l)}} g_{2l-1} \) and \( \sum_{l \leq k} \frac{G(2k+1, 2l)}{b^{(2l+1)}} g_{2l} \).

**Lemma 4.7.** (Principal Series). For all \( s > 1 \) there exists a constant \( C_{s,t} > 0 \) such that, for all \( \mu \geq 1/4 \), and for any \( g \in W^s(\mathcal{H}_\mu) \), we have:

- if \( \mu t < s - 1 \), and \( D_1(g) = D_2(g) = 0 \), then \( \exists f \in W^l(\mathcal{H}_\mu) \), s.t. \( Xf = g \), and \( \|f\|_l \leq C_{s,t} \|g\|_s \), and \( f \) is unique if \( \mu t > 0 \).

**Proof.** In the first case, the lemma is equivalent to saying that, if \( s \geq 1 \), and \( t \leq -1 \), then the operator

\[
g = (g_l)_{l \in \mathbb{Z}} \frac{G_{s,t}}{f} \begin{cases} f_{2k} = -\sum_{l \leq k} \frac{G(2k, 2l - 1)}{b^{(2l)}} \|u_{2l-1}\|_s^2 g_{2l-1} \\ f_{2k+1} = -\sum_{l \leq k} \frac{G(2k+1, 2l)}{b^{(2l+1)}} \|u_{2l}\|_s^2 g_{2l} \end{cases}
\]

is a bounded operator with uniformly bounded norm from \( l^2(\mathbb{Z}) \) to \( l^2(\mathbb{Z}) \). We will show that \( G_{s,t} \) is a Hilbert-Schmidt operator with Hilbert-Schmidt norm which is bounded uniformly with respect to \( \mu \geq 1/4 \). The norm is then,

\[
\|G_{s,t}\|_{HS}^2 = \sum_{k,l \in \mathbb{Z}} \frac{|G(2k, 2l - 1)|^2}{b^{(2l)}} \|u_{2l-1}\|_s^2 \frac{|G(2k+1, 2l)|^2}{b^{(2l+1)}} \|u_{2l}\|_s^2
\]

\[
= \sum_{k,l \in \mathbb{Z}} \frac{1}{b^{(2l)}} \frac{|f_{2k}^{(1)}|^2}{|f_{2k+1}^{(2)}|} \frac{(1 + \mu + 2(2k)^2)^l}{(1 + \mu + 2(2l - 1)^2)^s}
\]

\[
+ \sum_{k,l \in \mathbb{Z}} \frac{1}{b^{(2l+1)}} \frac{|f_{2k+1}^{(2)}|^2}{|f_{2l+1}^{(2)}|} \frac{(1 + \mu + 2(2k + 1)^2)^l}{(1 + \mu + 2(2l)^2)^s}
\]

by lemma 4.3, we have that \( \frac{|f_{2k}^{(1)}|^2}{|b^{(2l)}|} \frac{|f_{2k+1}^{(2)}|^2}{|b^{(2l+1)}|^2} \leq C_1 \frac{(4|k| + 3)^2 + |v|^2}{((4|l| + 1)^2 + |v|^2)^{1/2}} \).

We then get, for all \( k, l \in \mathbb{Z} \),

\[
\frac{|f_{2k}^{(1)}|^2}{|b^{(2l)}|} \frac{|f_{2k+1}^{(2)}|^2}{|b^{(2l+1)}|^2} \leq C_1 \frac{16((4|l| + 1)^2 + |v|^2)^{1/2}}{((4|k| + 3)^2 + |v|^2)^{1/2}((4|l| - 1)^2 + |v|^2)}
\]

\[
\leq \frac{16C_1}{\sqrt{9 + |v|^2}} \max\{\frac{\sqrt{25 + |v|^2}}{9 + |v|^2}, \frac{1}{\sqrt{1 + |v|^2}}\}
\]
So we have,
\[
\sum_{k \in \mathbb{Z}} \sum_{l \leq k} \frac{1}{((4l - 1)^2 + |v|^2)} \frac{1}{((4l + 1)^2 + |v|^2)} \frac{1}{(1 + \mu + 2(2l)^2)^t} \frac{1}{(1 + \mu + 2(2l - 1)^2)^s} 
\]
\[
\leq C' \sum_{k \in \mathbb{Z}} \sum_{l \leq k} \frac{(1 + \mu + 2(2k)^2)^t}{(1 + \mu + 2(2l - 1)^2)^s} 
\]

And then by the integral inequality,
\[
\leq C' \int \frac{1}{(1 + \mu + 2(2y)^2)^t} dy \int \frac{1}{(1 + \mu + 2(2x - 1)^2)^s} dx 
\]
\[
\leq C' \frac{C}{1 + \mu} 
\]

Similarly, we have that \(\frac{|f_{2k+1}^{(2)}|^2}{|b^{-}(2l+1)|^2|f_{2l+1}^{(2)}|^2} \leq \frac{((4|k|+5)^2+|v|^2)^{\frac{1}{2}}}{((4|l|-1)^2+|v|^2)^{\frac{1}{2}}}.\)

This then yeilds,
\[
\frac{|f_{2k+1}^{(2)}|^2}{|b^{-}(2l+1)|^2|f_{2l+1}^{(2)}|^2} \leq \frac{16((4|l|-1)^2 + |v|^2)^{\frac{1}{2}}}{((4|l|+1)^2 + |v|^2)((4|k|+5)^2 + |v|^2)^{\frac{1}{2}}} 
\]
\[
\leq \frac{16C_2}{\sqrt{25 + |v|^2}} \max\left\{ \frac{\sqrt{9 + |v|^2}}{25 + |v|^2}, \frac{1}{\sqrt{1 + |v|^2}} \right\} 
\]

And again we have,
\[
\sum_{k \in \mathbb{Z}} \sum_{l \leq k} \frac{1}{((4l+1)^2 + |v|^2)} \frac{1}{((4l-1)^2 + |v|^2)} \frac{1}{(1 + \mu + 2(2k+1)^2)^t} \frac{1}{(1 + \mu + 2(2l)^2)^s} 
\]
\[
\leq C' \int \frac{1}{(1 + \mu + 2(2y+1)^2)^t} dy \int \frac{1}{(1 + \mu + 2(2x)^2)^s} dx 
\]
\[
\leq C' \frac{C}{1 + \mu} 
\]

Putting these two inequalities together proves the first case of the lemma.
Now we suppose that \( D_1(g) = D_2(g) = 0 \), and that \( t < s - 1 \). Now on the kernel of \( D_1, D_2 \), the operator \( G_{v,s,t} \) can be rewritten. First we write, for \( k > 0 \),

\[
f_{2k} = - \sum_{l \leq k} \frac{f_{2k}^{(1)}}{b^2(2l)^2} g_{2l-1} = - \sum_{l \leq k} \frac{f_{2k}^{(1)}}{b^2(2l)^2} g_{2l-1} = \sum_{k<l} \frac{f_{2k}^{(1)}}{b^2(2l)^2} g_{2l-1} = \sum_{k<l} \frac{f_{2k}^{(1)}}{b^2(2l)^2} g_{2l-1}
\]

And similarly,

\[
f_{2k+1} = - \sum_{l \leq k} \frac{f_{2k+1}^{(2)}}{b^2(2l+1)^2} g_{2l} = - \sum_{l \leq k} \frac{f_{2k+1}^{(2)}}{b^2(2l+1)^2} g_{2l} = \sum_{k<l} \frac{f_{2k+1}^{(2)}}{b^2(2l+1)^2} g_{2l} = \sum_{k<l} \frac{f_{2k+1}^{(2)}}{b^2(2l+1)^2} g_{2l}
\]

Therefore, we can rewrite the Hilbert-Schmidt norm as,

\[
\|G_{v,s,t}\|_{HS}^2 = \sum_{k>0} \sum_{k<l} \frac{1}{|b^2(2l)|^2} \frac{|f_{2k}^{(1)}|^2}{|f_{2l}^{(1)}|^2} (1 + \mu + 2(2k)^2)^t \frac{|f_{2k+1}^{(2)}|^2}{|f_{2l}^{(2)}|^2} (1 + \mu + 2(2l)^2)^s + \sum_{k>0} \sum_{k<l} \frac{1}{|b^2(2l+1)|^2} \frac{|f_{2k+1}^{(2)}|^2}{|f_{2l}^{(2)}|^2} (1 + \mu + 2(2l+1)^2)^t \frac{|f_{2k}^{(1)}|^2}{|f_{2l}^{(1)}|^2} (1 + \mu + 2(2l)^2)^s + \sum_{k \leq 0} \sum_{k<l} \frac{1}{|b^2(2l)|^2} \frac{|f_{2k-1}^{(1)}|^2}{|f_{2l-1}^{(1)}|^2} (1 + \mu + 2(2k-1)^2)^t \frac{|f_{2k}^{(2)}|^2}{|f_{2l}^{(2)}|^2} (1 + \mu + 2(2l)^2)^s + \sum_{k \leq 0} \sum_{k<l} \frac{1}{|b^2(2l-1)|^2} \frac{|f_{2k}^{(2)}|^2}{|f_{2l-1}^{(2)}|^2} (1 + \mu + 2(2k)^2)^t \frac{|f_{2k-1}^{(1)}|^2}{|f_{2l-1}^{(1)}|^2} (1 + \mu + 2(2l-1)^2)^s
\]

Just as above, we have that

\[
\frac{|f_{2k}^{(1)}|^2}{|b^2(2l)|^2} \leq C_3 \quad \text{and} \quad \frac{|f_{2k-1}^{(2)}|^2}{|b^2(2l-1)|^2} \leq C_4
\]

Where \( C_3, C_4 \) do not depend upon \( |v| \).

We can then use the integral inequality to get that,

\[
\|G_{v,s,t}\|_{HS}^2 \leq C \int \int_{x > y} \frac{(1 + y^2)^t}{(1 + (x - 1)^2)^s} dy dx
\]
Now if $s > 1$, then the integral is convergent for any $t < s - 1$. Thus, we get the desired result that,

$$\|f\|_t^2 \leq \|G_{v,s,t}\|_{HS}^2 \|g\|_s^2 \leq C_{s,t} \|g\|_s^2$$

We remark that the condition that $s \geq 1$ and $t \leq -1$, in the first part, and that $t < s - 1$, in the second part, are not optimal. In fact, we settled for less than optimal estimates on the decay rate of $\frac{|G(2k,2l-1)|^2}{|b^r(2l)|^2}$ and $\frac{|G(2k+1,2l)|^2}{|b^r(2l+1)|^2}$. Achieving optimal estimates would be a technical exercise, and we do not feel that much is lost by not treating it.

**Lemma 4.8. (Complementary Series).** For all $s > 1$ there exists a constant $C_{s,t} > 0$ such that, for all $0 < v < 1$, and for any $g \in W^s(H_{\mu})$, we have:

- if $t \leq -1$ or
- if $t < s - 1$, and $D_1(g) = D_2(g) = 0$,
then $\exists f \in W^t(H_{\mu})$, s.t. $Xf = g$, and $\|f\|_t \leq \frac{C_{s,t}}{\sqrt{1-v}}\|g\|_s$, and $f$ is unique if $t > 0$.

**Proof.** We construct a Green’s operator as above, and then analyze the conditions under which it’s Hilbert-Schmidt norm is finite. And as above, for the first part of the lemma we have,

$$\|G_{v,s,t}\|_{HS}^2 = \sum_{k,l \in \mathbb{Z}} \frac{|G(2k,2l-1)|^2}{|b^r(2l)|^2} \left\| \frac{u_{2k}}{\|u_{2k-1}\|_S^2} \right\|^2 + \frac{|G(2k+1,2l)|^2}{|b^r(2l+1)|^2} \left\| \frac{u_{2k+1}}{\|u_{2k}\|_S^2} \right\|^2$$

$$= \sum_{k \in \mathbb{Z}} \sum_{l \leq k} \frac{1}{|b^r(2l)|^2} \left\| \frac{|f_{2k}^{(1)}|^2}{|f_{2l}^{(1)}|^2} \prod_{v',|2l'|} (1 + \mu + 2(2k))^t \right\| \prod_{v',|2l'|} (1 + \mu + 2(2l - 1))^s$$

$$+ \sum_{k \in \mathbb{Z}} \sum_{l \leq k} \frac{1}{|b^r(2l)|^2} \left\| \frac{|f_{2k+1}^{(2)}|^2}{|f_{2l+1}^{(2)}|^2} \prod_{v',|2l'|} (1 + \mu + 2(2k+1))^t \right\| \prod_{v',|2l'|} (1 + \mu + 2(2l))^s$$

By lemma 4.5, we have $\frac{|f_{2k}^{(1)}|^2}{|f_{2l}^{(1)}|^2} \leq \left( \frac{4k}{|3-v|} \right)^{-1} (\frac{4k}{|3-v|})^{1-v}$. And we then get, for all $k,l \in \mathbb{Z}$,

$$\frac{|f_{2k}^{(1)}|^2}{|b^r(2l)|^2 |f_{2l}^{(1)}|^2} \leq C_1 \left| \frac{3 - v}{1 + v} \right|^{-v} \left| \frac{4|l|}{3 + v} \right|^{1-v} \frac{16}{|4l - 1 - v|^2} \leq C_1 \max \left\{ \frac{|3 - v|^{1-v}}{|1 + v|^2}, \frac{|1 + v|^{1-v}}{|3 - v|^2} \right\}.$$
Also, since \(0 < \mu < 1/4\), for all \(t \in \mathbb{R}\) there exists a constant \(C_t > 0\) such that, for all \(k \in \mathbb{Z}\),
\[
C_t^{-1}(1 + |k|)^{2t} \leq (1 + \mu + 2k^2)^t \leq C_t(1 + |k|)^{2t}
\]
But by [4] (see lemma 4.1), we then get for \(t < -1\) and \(s > 1\),
\[
\sum_{k \in \mathbb{Z}} \sum_{l \leq k} \frac{1}{|b^r(2l)|^2} \frac{|f_{2k}^{(1)}|^2}{|f_{2l+1}^{(1)}|^2} \frac{\prod_{|v| \geq 2l-1}}{\prod_{|v| \geq 2l}} (1 + \mu + 2(2k)^2)^t \leq \frac{C_{s,t}}{1 - \nu} \sum_{k \in \mathbb{Z}} \sum_{l \leq k} \frac{(1 + |2k|^2)^{2l-v}}{(1 + |2l - 1|^2)^{2s-v}} \leq \frac{C'_{s,t}}{1 - \nu}
\]
And similarly, we get that,
\[
\sum_{k \in \mathbb{Z}} \sum_{l \leq k} \frac{1}{|b^r(2l+1)|^2} \frac{|f_{2k+1}^{(1)}|^2}{|f_{2l+1}^{(1)}|^2} \frac{\prod_{|v| \geq 2l+1}}{\prod_{|v| \geq 2l}} (1 + \mu + 2(2k + 1)^2)^t \leq \frac{C'_{s,t}}{1 - \nu} \sum_{k \in \mathbb{Z}} \sum_{l \leq k} \frac{(1 + |2k + 1|^2)^{2l-v}}{(1 + |2l|^2)^{2s-v}} \leq \frac{C'_{s,t}}{1 - \nu}
\]
And this then implies the first part of the lemma.

For the second part, we suppose that \(D_1(g) = D_2(g) = 0\), and that \(t < s - 1\).

And as above, on the kernel of \(D_1, D_2\), the operator \(G_{\nu,s,t}\) can be rewritten.

As above, we have for \(k > 0\),
\[
f_{2k} = \sum_{k < l} \frac{f_{2k}^{(1)}}{b^r(2l)f_{2l}^{(1)}} g_{2l-1} \quad \text{and} \quad f_{2k+1} = \sum_{k < l} \frac{f_{2k+1}^{(2)}}{b^r(2l+1)f_{2l+1}^{(2)}} g_{2l}
\]
Again, we can rewrite the Hilbert-Schmidt norm as,
\[
\|G_{\nu,s,t}\|_{HS}^2 = \sum_{k \geq 0} \sum_{k < l} \frac{1}{|b^r(2l)|^2} \frac{|f_{2k}^{(1)}|^2}{|f_{2l}^{(1)}|^2} \frac{\|u_{2l-1}\|_s^2}{\|u_{2l} \|^2} + \frac{1}{|b^r(2l+1)|^2} \frac{|f_{2k+1}^{(2)}|^2}{|f_{2l+1}^{(2)}|^2} \frac{\|u_{2l+1}\|_s^2}{\|u_{2l} \|^2}
\]
\[
+ \sum_{k \geq 0} \sum_{|k| < l} \frac{1}{|b^r(2l)|^2} \frac{|f_{2k}^{(1)}|^2}{|f_{2l}^{(1)}|^2} \frac{\|u_{2l-1}\|_s^2}{\|u_{2l+1} \|^2} + \frac{1}{|b^r(2l+1)|^2} \frac{|f_{2k+1}^{(2)}|^2}{|f_{2l+1}^{(2)}|^2} \frac{\|u_{2l+1}\|_s^2}{\|u_{2l} \|^2}
\]
And as above, we can then use the integral inequality to get that,
\[
\|G_{\nu,s,t}\|_{HS}^2 \leq \frac{C}{1 - \nu} \int_{x_1 > y} \frac{(1 + |y|)^{2l-v}}{(1 + |x - 1|)^{2s-v}} dy dx
\]
And if $s > 1$, then the integral is convergent for any $t < s - 1$. Thus, we get the desired result that,

$$\|f\|^2_t \leq \|G_{v,s,\ell}\|^2_{HS} \|g\|^2 \leq \frac{C_{s,t}}{1 - t} \|g\|^2_s$$

**Lemma 4.9.** (Discrete Series). For all $s > 1$ there exists a constant $C_{s,t} > 0$ such that, for all $n \in \mathbb{Z}^+$ ($v = 2n - 1$, $\mu = -n^2 + n$), and for any $g \in W^s(\mathcal{H}_\mu)$, we have:

- if $t < -1$ or
- if $t < s - 1$, and $D_1(g) = D_2(g) = 0$,

then $\exists! \ f \in W^t(\mathcal{H}_\mu)$, s.t. $Xf = g$, and $\|f\|_t \leq C_{s,t} \|g\|_s$.

**Proof.** The Green’s operator for the discrete series is,

$$g = (G\ell)_{\ell \in \mathbb{Z}} \frac{G_{v,s,\ell}}{f} = \begin{cases} f_{n+2k} = - \sum_{0 < l \leq k} \frac{G(n+2k,n+2l-1)}{b^-(n+2l)} \|u_{n+2k}\|^2_t \|g_{n+2l-1}\|^2 \|g_{n+2l}\|^2 & \\
 f_{n+2k+1} = - \sum_{0 < l \leq k} \frac{G(n+2k+1,n+2l)}{b^-(2l+1)} \|u_{n+2k+1}\|^2_t \|g_{n+2l}\|^2 \|g_{n+2l+1}\|^2 & \\
 (\text{Where } f_n = f_{n+1} = 0)
\end{cases}$$

It has Hilbert-Schmidt norm,

$$\|G_{v,s,\ell}\|^2_{HS} = \sum_{k,l \in \mathbb{Z}} \frac{|G(n+2k,n+2l-1)|^2}{|b^-(n+2l)|^2} \frac{\|u_{n+2k}\|^2_t}{\|u_{n+2l-1}\|^2} + \frac{|G(n+2k+1,n+2l)|^2}{|b^-(n+2l+1)|^2} \frac{\|u_{n+2k+1}\|^2_t}{\|u_{n+2l}\|^2}$$

$$= \sum_{0 < k} \sum_{1 < l \leq k} \frac{1}{|b^-(n+2l)|^2} \frac{|f_{n+2k}^{(1)}|^2}{|f_{n+2l}^{(1)}|^2} \frac{\Pi_{v,n+2k}}{\Pi_{v,n+2l-1}} \frac{1 + \mu + 2(n+2k)^2}{(1 + \mu + 2(n+2l-1)^2)^s}$$

$$+ \sum_{0 < k} \sum_{1 < l \leq k} \frac{1}{|b^-(n+2l+1)|^2} \frac{|f_{n+2k+1}^{(2)}|^2}{|f_{n+2l+1}^{(2)}|^2} \frac{\Pi_{v,n+2k+1}}{\Pi_{v,n+2l}} \frac{1 + \mu + 2(n+2k+1)^2}{(1 + \mu + 2(n+2l)^2)^s}$$

We note that for $0 < l \leq k$, we have,

$$\frac{|f_{n+2k}^{(1)}|}{|f_{n+2l}^{(1)}|} = \prod_{j=1}^{k-1} \frac{|b^+(n+2j)|}{|b^-(n+2j+2)|} = \prod_{j=1}^{k-1} \frac{j + n}{j + 1} = \frac{(k + n - 1)!}{k! \cdot (l + n - 1)!}$$

We also have,

$$\Pi_{v,n+2k} = \prod_{j=n+1}^{n+2k} \frac{2j - 1 - \nu}{2j - 1 + \nu} = \prod_{j=1}^{k} \frac{j}{j + 2n - 1} = \frac{(2k)! \cdot (2n - 1)!}{(2k + 2n - 1)!}$$
By using some basic facts about the Gamma function, we can put this all together and we get that for all \( k \geq 1 > 0 \),

\[
\frac{1}{|b^-(n+2l)|^2} \frac{|f^{(1)}_{n+2l}|^2}{|f^{(1)}_{n+2l}|^2} = l^{-2} \cdot \frac{l!^2 \cdot (k+n-1)!^2 \cdot (2k)! \cdot (2l+2n-2)!}{k!^2 \cdot (l+n-1)!^2 \cdot (2l-1)! \cdot (2k+2n-1)!}
\]

\[
= \frac{2}{l(2k+2n-1)} \cdot \frac{(2k)!}{k!^2} \cdot \frac{2l^2}{(2l)!} \cdot \frac{(2l+2n-2)!}{(l+n-1)!^2} \cdot \frac{(k+n-1)!^2}{(2k+2n-1)!}
\]

\[
= \frac{2}{l(2k+2n-1)} \cdot \frac{\Gamma(k+1/2) \cdot \Gamma(l) \cdot \Gamma(l+n-1/2) \cdot \Gamma(k+n-1)}{\Gamma(k) \cdot \Gamma(l+n-1) \cdot \Gamma(l+n-1/2)}
\]

\[
\leq \frac{2}{l(2k+2n-1)} \cdot \frac{\Gamma(k+1)}{\Gamma(k)} \cdot \frac{\Gamma(l)}{\Gamma(l)} \cdot \frac{\Gamma(l+n)}{\Gamma(l+n-1)} \cdot \frac{\Gamma(k+n-1)}{\Gamma(k+n-1)}
\]

\[
= \frac{2(l+n)(k+1)}{l(2k+2n-1)}
\]

Now since \( \nu = 2n - 1 \geq 1 \) and \( \mu = -n^2 + n \), we have for all \( n \in \mathbb{Z}^+ \) and all \( k \geq 0 \),

\[
2^{-1}(\nu + 2k)^2 \leq 1 + \mu + 2(n + 2k)^2 \leq 2(\nu + 2k)^2
\]

So for \( s > 1 \) and \( t < -1 \), we have,

\[
\sum_{0 < k} \sum_{0 < l \leq k} \frac{1}{|b^-(n+2l)|^2} \frac{|f^{(1)}_{n+2l}|^2}{|f^{(1)}_{n+2l}|^2} \prod_{\nu, n+2l} \prod_{\nu, n+2l-1} (1 + \mu + 2(n + 2l - 1)^2)^t
\]

\[
\leq 2^{l+s} \sum_{0 < k} \sum_{0 < l \leq k} \frac{2(l+n)(k+1)}{l(2k+2n-1)} \frac{(2l+2n-2)^2}{(2l+2n-2)^2t}
\]

\[
\leq 2^{l-s+1} \sum_{0 < k} \left\{ (k+1)(k+n-1/2)^2 \sum_{0 < l \leq k} \frac{(l+n)}{l(l+n-1)^2} \right\}
\]

\[
\leq 2^{l-s+1} \left\{ \sum_{0 < k} (k+1)(k+n-1/2)^2 \sum_{0 < l} \frac{(l+n)}{l(l+n-1)^2} \right\}
\]

Then by the integral inequality, it is easy to see that the sum is finite and bounded in \( n \in \mathbb{Z}^+ \). A similar argument holds for \( \sum_{0 < k} \sum_{1 \leq l \leq k} \frac{1}{|b^-(n+2l+1)|^2} \frac{|f^{(2)}_{n+2l+1}|^2}{|f^{(2)}_{n+2l+1}|^2} \prod_{\nu, n+2l+1} \prod_{\nu, n+2l+1} (1+\mu+2(n+2l+1)^2)^t \).

And this then implies the first part of the lemma.

For the second part, we suppose that \( D_1(g) = D_2(g) = 0 \), and that \( t < s - 1 \). And as above, on the kernel of \( D_1, D_2 \), the operator \( G_{\nu,s,t} \) can be rewritten.
First we write, for \( k > 0 \),

\[
f_{n+2k} = - \sum_{0 < l \leq k} \frac{f_{n+2k}^{(1)}}{b^{-c(n+2l)}} g_{n+2l-1} = - \sum_{0 < l \leq k} \frac{f_{n+2k}^{(1)}(D_2)_{n+2l-1}}{b^c(n)} g_{n+2l-1}
\]

\[
= \sum_{k \leq l} \frac{f_{n+2k}^{(1)}(D_2)_{n+2l-1}}{b^c(n)} g_{n+2l-1} = \sum_{k \leq l} \frac{f_{n+2k}^{(1)}(D_2)_{n+2l-1}}{b^c(n)} g_{n+2l-1}
\]

And similarly for \( f_{n+2k+1} \). The Hilbert-Schmidt norm can now be written as,

\[
\|G_{v,s,t}\|_{HS}^2 = \sum_{0 < k \leq l} \sum_{k \leq l} \frac{1}{|b^{-c(n+2l)}|^2} \frac{|f_{n+2k}^{(1)}|^2}{|f_{n+2l}^{(1)}|^2} \frac{\prod_{t=0}^{n+2k} (1 + \mu + 2(n + 2k)^2)^t}{\prod_{t=0}^{n+2l} (1 + \mu + 2(n + 2l - 1)^2)^s}
\]

\[
+ \sum_{0 < k \leq l} \sum_{k \leq l} \frac{1}{|b^{-c(n+2l+1)}|^2} \frac{|f_{n+2k+1}^{(2)}|^2}{|f_{n+2l+1}^{(2)}|^2} \frac{\prod_{t=0}^{n+2k+1} (1 + \mu + 2(n + 2k + 1)^2)^t}{\prod_{t=0}^{n+2l+1} (1 + \mu + 2(n + 2l + 1)^2)^s}
\]

As before, we have,

\[
\sum_{0 < k \leq l} \sum_{k \leq l} \frac{1}{|b^{-c(n+2l)}|^2} \frac{|f_{n+2k}^{(1)}|^2}{|f_{n+2l}^{(1)}|^2} \frac{\prod_{t=0}^{n+2k} (1 + \mu + 2(n + 2k)^2)^t}{\prod_{t=0}^{n+2l} (1 + \mu + 2(n + 2l - 1)^2)^s}
\]

\[
\leq 2^{l+s} \sum_{0 < k \leq l} \sum_{k \leq l} \frac{2(l + n)(k + 1)(2k + 2n - 1)^2l}{l(2k + 2n - 1)(2l + 2n - 2)^2s}
\]

And this sum converges uniformly in \( n \in \mathbb{Z}^+ \), as long as \( s > t + 1 \). Similarly for the other sum, and we get again the desired result that,

\[
\|f\|_T^2 \leq \|G_{v,s,t}\|_{HS}^2 \|g\|_s^2 \leq C_s \|g\|_s^2
\]

### 4.1.8 Global Solutions

Let \( \mathcal{H} \) be any unitary representation of \( PSL(2, \mathbb{R}) \). We prove that if the Casimir operator \( \Box \) on \( \mathcal{H} \) has a spectral gap, then the only obstructions to the existence of a smooth solution \( f \in \mathcal{H} \) of the equation \( Xf = g \), for any smooth vector \( g \in \mathcal{H} \), are given by \( X \)-invariant distributions. We note that this condition is satisfied for \( L^2(PSL(2, \mathbb{R})/\Gamma) \), for any lattice \( \Gamma \).
Theorem 4.3. If there exists a \( \mu_0 > 0 \) such that the spectrum of the Casimir \( \sigma(\Box) \cap (0, \mu_0) = \emptyset \), then we have the following. Let \( s > 1 \), then there exists a constant \( C_{s,t} \) such that, for all \( g \in W^s(\mathcal{H}) \),

- if \( t < -1 \), and \( g \) has no component on the trivial sub-representation of \( \mathcal{H} \), or
- if \( t < s - 1 \) and \( D(g) = 0 \) for all \( D \in I^s(\mathcal{H}) \),

then the equation \( Xf = g \) has a solution \( f \in W^t(\mathcal{H}) \), which satisfies the Sobolev estimate \( \| f \|_t \leq C_{s,t} \| g \|_s \). Solutions are unique modulo the trivial sub-representation iff \( t > 0 \).

Proof. \( \mathcal{H} \) is the direct integral of non-trivial irreducible unitary representations \( \mathcal{H}_\lambda \) of \( \text{PSL}(2, \mathbb{R}) \) with respect to a Stieltjes measure \( ds \). Every vector \( g \in W^s(\mathcal{H}) \) then has a decomposition

\[
g = \int m(\lambda)g_\lambda ds(\lambda)
\]

with \( g_\lambda \in W^s(\mathcal{H}_\lambda) \), and \( m(\lambda) \) being the finite multiplicity with which the representation \( \mathcal{H}_\lambda \) occurs in \( \mathcal{H} \). Now we have that,

- \( D(g) = 0 \) for all \( D \in I^s(\mathcal{H}) \) iff, for \( ds \) almost all \( \lambda \in \mathbb{R} \), \( D_\lambda(g) = 0 \) for all \( D_\lambda \in I^s(\mathcal{H}) \).

- if there exists a constant \( C_{s,t} > 0 \) such that, for \( ds \) almost all \( \lambda \in \mathbb{R} \), the equation \( Xf_\lambda = g_\lambda \) has a solution \( f_\lambda \in W^s(\mathcal{H}_\lambda) \) which satisfies a uniform Sobolev estimate, \( \| f_\lambda \|_t \leq C_{s,t} \| g_\lambda \|_s \), then the equation \( Xf = g \) has a global solution \( f \in W^s(\mathcal{H}) \) with the same Sobolev bound, \( \| f \|_t \leq C_{s,t} \| g \|_s \).

The first claim follows immediately from basic functional analysis. The second assertion is proved by noting that,

\[
\| f \|_t^2 = \int m(\lambda)\| f_\lambda \|_t^2ds(\lambda) \leq C_{s,t}^2 \int m(\lambda)\| g_\lambda \|_s^2ds(\lambda) = C_{s,t}^2 \| g \|_s^2
\]

But of course, the second condition is satisfied as per the above lemmata for the Principal, Complementary, and Discrete series. Finally, since homogeneous solutions only exist for \( t < 0 \), uniqueness only holds for \( t < 0 \).
4.2 The First Cohomology Group of Higher Rank Abelian Actions for
\((PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})) / \Gamma\)

Let \(A\) be the one parameter group generated by \(Y\). We consider the rank 2 abelian subgroups \(A = A_x \times A_{x_2}, A_{x_1} \times A_{U_2}\), and \(A_{U_1} \times A_{U_2}\) of \((PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})) / \Gamma\). \(A\) and \(U\) are the standard generators of the positive diagonal, and upper-triangular subgroups of \(SL(2, \mathbb{R})\). That is, \(X = \begin{pmatrix} 1/2 & 0 \\ 0 & -1 \end{pmatrix}\) and \(U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\).

Now we discuss cocycles over the \(A\) action on \((PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})) / \Gamma\). We refer the reader to section 3.5 for background on cocycles.

Let \(\Gamma\) be an irreducible (not necessarily cocompact) lattice in \((SL(2, \mathbb{R}) \times SL(2, \mathbb{R})) / \Gamma\). And let \(\beta : A \times (PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})) / \Gamma \to \mathbb{R}^k\) be a one cocycle, where \(A\) acts by left multiplication. Or, alternatively, by writing \(f(x) \cdot h = f(hx)\), we will consider one-cocycles to be measurable functions \(\beta : A \to \mathcal{F}((PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})) / \Gamma, \mathbb{R}^k)\) which satisfy \(\beta(a_1 + a_2) = \beta(a_2) + \beta(a_1) \cdot a_2\), and where \(\mathcal{F}((PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})) / \Gamma, \mathbb{C}^k)\) is the space of measurable functions \((PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})) / \Gamma \to \mathbb{C}^k\). By taking component functions, and real and imaginary parts, we may always assume that \(\beta : A \times (PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})) / \Gamma \to \mathbb{C}\).

A cocycle is called smooth if \(\beta : A \to C^\infty(L^2((PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})) / \Gamma))\) is a smooth map. We can also define \(\beta\) to be of class \(C^r\). And we say that \(\beta\) is smooth cohomologous to a constant cocycle \(c\) if there exists a smooth map \(P : (PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})) / \Gamma \to \mathbb{C}\) such that \(\beta(a, g) = -P(\mu(a)) + c(a, g) + P(g)\). Recall that \(c : A \times (PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})) / \Gamma \to \mathbb{C}\) is a constant cocycle if \(c(a, g) = c(a) \in \mathbb{C}\) is a constant function of \((PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})) / \Gamma\) for every \(a \in A\).

Recall that the space \(L^2((PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})) / \Gamma)\) has a direct integral decomposition as,

\[ L^2((PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})) / \Gamma) = \int_{\hat{\Gamma}} m(\mu_1, \mu_2) \cdot H_{\mu_1} \otimes H_{\mu_2} d\mu(\mu_1, \mu_2) \]

where \(m(\mu_1, \mu_2)\) is the finite multiplicity of the irreducible unitary representation of \(PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})\) appearing in \(L^2((PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})) / \Gamma)\).

We can now state the main result of this section as,

**Theorem 4.4.** Let \(\beta : A \times (PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})) / \Gamma \to \mathbb{R}^k\) be a smooth cocycle. Then \(\beta\) is smooth cohomologous to a constant cocycle.
In order to proceed, we first make a reduction to the infinitesimal version. Recall that, for a $C^\infty$ or $C'$ cocycle $\beta$, we can define the infinitesimal generator $\omega$ as follows.

Let $\omega(v) = \frac{d}{dt}|_{t=0}(\exp tv)|_t$ be the infinitesimal generator for $\beta$. Then the cocycle equation implies that $\omega$ is a closed form on the orbit foliation. In fact, it is a closed one form.

We can also recover $\beta$ from $\omega$ by $\beta(\exp X) = \int_0^1 \omega(X) \cdot \exp tX \, dt$, as the exponential map $\exp : \mathfrak{Lie}(A) \to A$ is onto. Therefore we can restrict our attention to the infinitesimal situation. In particular, the infinitesimal version of the cohomology equation is $\omega = \eta - dP$. Thus the $C^\infty$ or $C'$ first cohomology group is trivial if for every $C^\infty$ or $C'$ one form $\omega$ is exact via a function smooth (or $C'$) function $P : (\PSL(2, \mathbb{R}) \times \PSL(2, \mathbb{R}))/\Gamma \to \mathbb{C}$.

Since $\omega$ is a (one) form on $\mathfrak{Lie}(A) = \mathfrak{a}$, it is determined by the two functions $f = \omega(X_1)$ and $g = \omega(X_2)$, where $X_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \oplus 0$ and $X_2 = 0 \oplus \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$ form a basis for $\mathfrak{Lie}(A)$. The regularity of these two functions determines the regularity of $\beta$. In particular, if $f, g \in C^\infty((\PSL(2, \mathbb{R}) \times \PSL(2, \mathbb{R}))/\Gamma))$, then so is $\beta$, and a similar statement can be made for $f, g \in C((\PSL(2, \mathbb{R}) \times \PSL(2, \mathbb{R}))/\Gamma))$. The cocycle identity then becomes $X_2 f = X_1 g$, and $\omega = dP$ becomes $X_1 P = dP(X_1) = \omega(X_1) = f$, and $X_2 P = dP(X_2) = \omega(X_2) = g$. The infinitesimal version of theorem 4.4 is,

**Theorem 4.5.** Let $f, g \in W^s((\PSL(2, \mathbb{R}) \times \PSL(2, \mathbb{R}))/\Gamma), (s > 2)$, and satisfy the equation $X_2 f = X_1 g$. If $t < s - 1$, then there exists solutions $P, P' \in W^t((\PSL(2, \mathbb{R}) \times \PSL(2, \mathbb{R}))/\Gamma)$ such that $X_1 P = f$ and $X_2 P' = g$. Furthermore, the norms of $P, P'$ must satisfy, $||P||_t \leq C_{p_0,s,t}||f||_{2s}$ and $||P'||_t \leq C_{p_0,s,t}||g||_{2s}$. If $t > 1$, then $P$ and $P'$ must coincide, so that there is a true simultaneous solution.

Since $\cap_{s>0} W^s((\PSL(2, \mathbb{R}) \times \PSL(2, \mathbb{R}))/\Gamma) = C^\infty((\PSL(2, \mathbb{R}) \times \PSL(2, \mathbb{R}))/\Gamma))$, this theorem will imply theorem 4.4. In particular, this theorem does guarantee that if $f, g$ are smooth vectors then so is $P$.

Here we have taken $A = A_{X_1, X_2}$. Entirely similar statements will also be proven for $A = A_{X_1, U_2}$ and $A = A_{U_1, U_2}$.

### 4.2.1 Norms

The irreducible unitary representations of $\PSL(2, \mathbb{R}) \times \PSL(2, \mathbb{R})$ are tensor products $\mathcal{H}_{\mu_1} \otimes \mathcal{H}_{\mu_2}$ of the irreducible unitary representations of $\PSL(2, \mathbb{R})$. Since $\Gamma$ is irreducible lattice, we know that $\mathcal{H}_0 \otimes \mathcal{H}_\mu$ and $\mathcal{H}_\mu \otimes \mathcal{H}_0$ cannot appear in the decomposition of $L^2((\PSL(2, \mathbb{R}) \times \PSL(2, \mathbb{R}))/\Gamma))$. Furthermore, there is a $\mu_0 > 0$ s.t. $\mathcal{H}_{\mu_1} \otimes 0$...
and $0 \otimes \mathcal{H}_{\mu_2}$ appear in the decomposition of $L^2((PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R}))/\Gamma)$ only if $\mu_1, \mu_2 \geq \mu_0$.

We will use the notation $f|_{k_1} = \sum_k f_{k_1,k_2} u_{k_1} \otimes u_{k_2}$ and $f|_{k_2} = \sum_k f_{k_1,k_2} u_{k_1} \otimes u_{k_2}$, and where the basis $\{ u_k \}$ is the adapted basis defined in (4.3).

The Laplacian of $PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$ on $\mathcal{H}_{\mu_1} \otimes \mathcal{H}_{\mu_2}$ is $\triangle_1 + \triangle_2$, which acts on the $K$-finite vectors as follows,

$$(\triangle_1 + \triangle_2)(u_{k_1} \otimes u_{k_2}) = \triangle_1 u_{k_1} \otimes u_{k_2} + u_{k_1} \otimes \triangle_2 u_{k_2} = [(\mu_1 + 2k_1^2) + (\mu_2 + 2k_2^2)]u_{k_1} \otimes u_{k_2}$$

Therefore, the Sobolev norms are given by,

$$\|f\|_{\mu_1,\mu_2,s} = \left( \sum_{k_1,k_2} (1 + (\mu_1 + 2k_1^2) + (\mu_2 + 2k_2^2))^{s} |\Pi_{\nu_1,k_1}||\Pi_{\nu_2,k_2}||f_{k_1,k_2}|^2 \right)^{1/2} \quad (4.23)$$

This norm is related to the norm on the factors by noting that, for any pair of sequences of positive numbers $\{a_{k_1}\}$ and $\{b_{k_2}\}$, we have,

$$(1 + a_{k_1} + b_{k_2}) \leq (1 + a_{k_1})(1 + b_{k_2}) \leq (1 + a_{k_1} + b_{k_2})^2$$

And also that

$$(1 + a_{k_1}) \leq (1 + a_{k_1} + b_{k_2})$$

We then get,

$$\|f\|_{\mu_1,\mu_2,s} \leq \left( \sum_{k_2} (1 + \mu_2 + 2k_2^2)^{s} |\Pi_{\nu_2,k_2}||f|_{k_2}|^2 \right)^{1/2} \leq \|f\|_{\mu_1,\mu_2,2s} \quad (4.24)$$

$$\|f|_{k_2}\|_{\mu_1,s} \leq \|f\|_{\mu_1,\mu_2,s} \quad (4.25)$$

The ability to relate the Sobolev norms on the factors $\mathcal{H}_{\mu_i}$ to the Sobolev norms on $\mathcal{H}_{\mu_1} \otimes \mathcal{H}_{\mu_2}$ will prove extremely useful.

### 4.2.2 Vanishing of Obstructions

We will start with $A = A_{X_1,X_2}$. $\omega$ is linear on the Lie algebra $a$, and so can be recovered from the functions $f = \omega(X_1)$ and $g = \omega(X_2)$. The cocycle identity becomes the simple condition $X_2 f = X_1 g$. This can be expressed in terms of the $K$-finite vectors as follows,

$$b^+ (k_2 - 1)f_{k_1,k_2} - b^- (k_2 + 1)f_{k_1,k_2} = b^+ (k_1 - 1)g_{k_1,k_2} - b^- (k_1 + 1)g_{k_1,k_2} \quad \text{for all } k_1, k_2 \in \mathbb{Z}$$
For any linear functional $D_i$ on $\mathcal{H}_{\mu_i}$ ($i = 1, 2$), define $D_i$ on $\mathcal{H}_{\mu_1} \otimes \mathcal{H}_{\mu_2}$ by $(D_i)(u_{k_i} \otimes u_{k_2}) = D(u_{k_i})$. The relationship between norms above shows that if $D \in W^{-s}(\mathcal{H}_{\mu_i})$ ($i = 1, 2$), then $D_i \in W^{-2s}(\mathcal{H}_{\mu_1} \otimes \mathcal{H}_{\mu_2})$. Of course, it is also easy to see that if $D$ is an $X$-invariant distribution on $W^s(\mathcal{H}_{\mu_i})$, then $D_i$ is an $X_i$-invariant distribution on $W^{2s}(\mathcal{H}_{\mu_1} \otimes \mathcal{H}_{\mu_2})$.

We will show that $D_1(f|_{k_2}) = 0$, for all $k_2$ and for all $X$-invariant distributions on $\mathcal{H}_{\mu_1}$, and $D_2(g|_{k_1}) = 0$ for all $k_1$ and for all $X$-invariant distributions on $\mathcal{H}_{\mu_2}$. That is, obstructions to the solution of the equations $X_1\beta|_{k_2} = f|_{k_2}$, and $X_2\beta|_{k_1} = g|_{k_1}$ simultaneously vanish.

**Lemma 4.10.** Let $f, g \in W^s(\mathcal{H}_{\mu_1} \otimes \mathcal{H}_{\mu_2})$, ($s > 2$), and satisfy the equation $X_2f = X_1g$. Then for any $X$-invariant $D_1 \in W^{-s}(\mathcal{H}_{\mu_1})$, we have $D_1(f|_{k_2}) = 0$ for all $k_2$ and also for any $X$-invariant $D_2 \in W^{-s}(\mathcal{H}_{\mu_2})$, we have $D_2(g|_{k_1}) = 0$ for all $k_1$.

**Proof.** We first notice that, for $i = 1, 2$, $\|f|_{k_2}\|_{\mu_i,s}$ is finite provided $\|f\|_{\mu_i,\mu_2,s}$ is. Therefore, it makes sense to evaluate $D_i(f|_{k_2})$, and $D_i(g|_{k_1})$, for each $i, j = 1, 2$. In particular, since $D_2$ is $X$ invariant, we must have $X \cdot D_2(f|_{k_1}) = 0$ for all $k_1$. But because $X_2f = X_1g$, we have for all $k_1$ that

$$0 = X \cdot D_2(f|_{k_1}) = \sum_{k_2}(b^+(k_2 - 1)f_{k_1,k_2-1} - b^-(k_2 + 1)f_{k_1,k_2+1})(D_2)_{k_2}$$

$$= \sum_{k_2}(b^+(k_1 - 1)g_{k_1-1,k_2} - b^-(k_1 + 1)g_{k_1+1,k_2})(D_2)_{k_2}$$

We then have that

$$b^+(k_1 - 1)\sum_{k_2}g_{k_1-1,k_2}(D_2)_{k_2} = b^-(k_1 + 1)\sum_{k_2}g_{k_1+1,k_2}(D_2)_{k_2}$$

Which implies

$$\frac{b^+(k_1 - 1)}{b^-(k_1 + 1)} \cdot \sum_{k_2}g_{k_1-1,k_2}(D_2)_{k_2} = \sum_{k_2}g_{k_1+1,k_2}(D_2)_{k_2}$$

And then finally yields

$$\sum_{k_2}g_{2k_1+1,k_2}(D_2)_{k_2} = f^{(2)}_{2k_1+1} \cdot \sum_{k_2}g_{1,k_2}(D_2)_{k_2}$$
Similarly, $\sum_{k_2} g_{2k_1,k_2} = f_{2k_1}^{(1)} \cdot \sum_{k_2} g_{0,k_2} (D_2)_{k_2}$.

On the other hand,

$$\sum_{k_2} (b^+(k_1 - 1)g_{k_1 - 1,k_2} - b^-(k_1 + 1)g_{k_1 + 1,k_2})(D_2)_{k_2} = 0$$

For all $k_1$ implies

$$\sum_{k_1,k_2} (b^+(2k_1 - 1)g_{2k_1 - 1,k_2} - b^-(2k_1 + 1)g_{2k_1 + 1,k_2})(D_2)_{k_2} = 0$$

Noting that the sum is taken over both $k_1,k_2$ allows us to conclude that

$$\sum_{k_1,k_2} (b^+(k_1 + 1) - b^-(k_1 + 1))g_{2k_1 + 1,k_2}(D_2)_{k_2} = 0$$

For all $k_1$.

But $b^+(k) - b^-(k) = \frac{2k + 1 - \nu}{4} - \frac{2k - 1 - \nu}{4} = \frac{1 + \nu}{2}$. And this then gives us that

$$\sum_{k_1,k_2} g_{2k_1 + 1,k_2}(D_2)_{k_2} = 0$$

When we combine this with the above results we get

$$\sum_{k_1} f_{2k_1 + 1}^{(2)} \cdot \sum_{k_2} g_{1,k_2} (D_2)_{k_2} = 0$$

And then by noting that both $f_{2k_1 + 1}^{(2)} \neq 0$ for any $k_1$, and the sum $\sum_{k_1} f_{2k_1 + 1}^{(2)} \neq 0$, this finally implies that

$$\sum_{k_2} g_{2k_1 + 1,k_2}(D_2)_{k_2} = 0 \quad \text{for all } k_1$$

And similarly for $\sum_{k_2} g_{2k_1,k_2}(D_2)_{k_2}$. Therefore, we have that $D_2(g|k_1) = 0$ for all $k_1$. An entirely similar argument yields $D_1(f|k_2) = 0$ for all $k_2$.

Because we have suppressed the domain for the $k_1,k_2$, the proof works equally well for $\mathcal{H}_{\mu_1}$ belonging to any of the Principal, Complementary or Discrete series.

For $A = A_{X_1,U_2}$ and $A = A_{U_1,U_2}$ we establish analogous lemmas, but the proof for the unipotent generator requires more care. We now state,

**Lemma 4.11.** Let $f, g \in W^s(\mathcal{H}_{\mu_1} \otimes \mathcal{H}_{\mu_2})$, $(s > 2)$, and satisfy the equation $U_2 f = X_1 g$.

Then for any $X$-invariant $D_1 \in W^{-s}(\mathcal{H}_{\mu_1})$, we have $D_1(f|k_1) = 0$ for all $k_2$ and also for any $U$-invariant $D_2 \in W^{-s}(\mathcal{H}_{\mu_2})$, we have $D_2(g|k_1) = 0$ for all $k_1$. 

LEMMA 4.12. Let \( f, g \in W^s(\mathcal{H}_{\mu_1} \otimes \mathcal{H}_{\mu_2}), (s > 2) \), and satisfy the equation \( U_2 f = U_1 g \). Then for any \( U \)-invariant \( D_1 \in W^{-s}(\mathcal{H}_{\mu_1}) \), we have \( D_1(f|_{k_2}) = 0 \) for all \( k_2 \) and also for any \( U \)-invariant \( D_2 \in W^{-s}(\mathcal{H}_{\mu_2}) \), we have \( D_2(g|_{k_1}) = 0 \) for all \( k_1 \).

Proof. The proof of both lemmas is entirely similar. First we note that there is no essential change in how we prove the first lemma for the diagonal element \( X_1 \). We simply note that, for all \( k_1 \), since \( D_2 \) is \( U \)-invariant, we must have \( U \cdot D_2(f|_{k_1}) = 0 \), and then because \( U_2 f = X_1 g \), we have

\[
0 = U \cdot D_2(f|_{k_1}) = X \cdot D_2(g|_{k_1}) = \sum_{k_2}(b^+(k_1 - 1)g_{k_1-1,k_2} - b^-(k_1 + 1)g_{k_1+1,k_2})(D_2)_{k_2}
\]

We then proceed exactly as above.

For the unipotent generator, we proceed as follows. For lemma 4.11, we again note that \( D_1 \) is \( X \)-invariant. This gives us that, for all \( k_2 \), we have \( X \cdot D_1(g|_{k_2}) = 0 \). Combining this with the equality \( U_2 f = X_1 g \) gives us \( U \cdot D_1(f|_{k_2}) = 0 \) for all \( k_2 \).

For lemma 4.12, we note that \( D_1 \) is \( U \)-invariant. This gives us that, for all \( k_2 \), we have \( U \cdot D_1(g|_{k_2}) = 0 \). Combining this with the equality \( U_2 f = U_1 g \) gives us that \( U \cdot D_1(f|_{k_2}) = 0 \) for all \( k_2 \).

Now define \( \tilde{f} \) by \( \tilde{f}_{k_2} = D_1(f|_{k_2}) \) for each \( k_2 \). Then we have just shown that \( \tilde{f} \) is a (formal) solution of the homogeneous equation \( U \cdot \tilde{f} = 0 \). Because the one parameter group generated by \( U_1 \) is non-compact, and the lattice \( \Gamma \) is irreducible, Moore’s Ergodicity Theorem says that no solution of the homogeneous equation \( U \cdot \tilde{f} = 0 \) can belong to \( \mathcal{H}_{\mu_1} \), other than the trivial solution. In particular, this implies one of the following

\[
\left( \sum_{k_2} |\tilde{f}_{k_2}|^2 \right)^{1/2} = +\infty \quad \text{or} \quad \tilde{f}_{k_2} \equiv 0
\]

We prove that the first situation cannot occur.

By Minkowski’s inequality, we have

\[
\left( \sum_{k_2} |\tilde{f}_{k_2}|^2 \right)^{1/2} = \left( \sum_{k_2} |\sum_{k_1} f_{k_1,k_2}(D_1)_{k_1}|^2 \right)^{1/2} \leq \sum_{k_1} \left( \sum_{k_2} |f_{k_1,k_2}|^2 (D_1)_{k_1} \right)^{1/2} = \sum_{k_1} |(D_1)_{k_1}| \left( \sum_{k_2} |f_{k_1,k_2}|^2 \right)^{1/2}
\]
Now, $|(D_1)_{k_1}| \leq a + b|\Pi_{v_1,k_1}| \leq C_{v_1,a,b}$ is bounded in $k_1$. But we also have that $f \in W^s(\mathcal{H}_{\mu_1} \otimes \mathcal{H}_{\mu_2})$, so

$$|f_{k_1,k_2}|^2 \leq C_\mu_{1,\mu_2}(1 + (\mu_1 + 2k_1^2) + (\mu_2 + 2k_2^2))^{-s}$$

That is, $|f_{k_1,k_2}|^2 = O((k_1^2 + k_2^2)^{-2s})$, and therefore the sum $\sum_{k_1} |(D_1)_{k_1}| \left( \sum_{k_2} |f_{k_1,k_2}|^2 \right)^{1/2}$ must be finite for $s > 2$. This then implies that $D_1(f|_{k_2}) = \tilde{f}_{k_2} \equiv 0$. We similarly obtain that $D_2(g|_{k_1}) \equiv 0$.

### 4.2.3 Simultaneous Solutions

We now show how the simultaneous vanishing of obstructions leads to a simultaneous solution of the inhomogeneous equations. In addition, this solution satisfies a uniform bound over all pairs $(\mu_1, \mu_2)$ that satisfy $\mu_1, \mu_2 \geq \mu_0 > 0$.

**Lemma 4.13.** Let $f, g \in W^2s(\mathcal{H}_{\mu_1} \otimes \mathcal{H}_{\mu_2})$, $(s > 2)$, and satisfy the equation $X_1f = X_1g$. If $t < s - 1$, then there exists solutions $P, P' \in W^t(\mathcal{H}_{\mu_1} \otimes \mathcal{H}_{\mu_2})$ such that $X_1P = f$ and $X_2P' = g$. Furthermore, the norm of $P, P'$ must satisfy, $\|P\|_{\mu_1,\mu_2,t} \leq C_{\mu_0,s,t}\|f\|_{\mu_1,\mu_2,2s}$, and $\|P'\|_{\mu_1,\mu_2,t} \leq C_{\mu_0,s,t}\|g\|_{\mu_1,\mu_2,2s}$. If $t > 1$, then $P$ and $P'$ must coincide, so that there is a true simultaneous solution.

**Proof.** By lemma 4.10, we know that, for any $X$-invariant distribution $D_1 \in W^{-2s}(\mathcal{H}_{\mu_1})$, we must have $D_1(f|_{k_2}) = 0$ for all $k_2$. As we have that $\|f\|_{\mu_1,2s} \leq \|f\|_{\mu_1,\mu_2,2s} < \infty$, we must then have that there exists a solution $P|_{k_2} \in W^{2(s-1)}(\mathcal{H}_{\mu_1})$, which satisfies the uniform estimate $\|P|_{k_2}\|_{\mu_1,t} \leq C_{\mu_0,s,t}\|f|_{k_2}\|_{\mu_1,s}$, for every $k_2$. We note that the constant does not depend upon $k_2$. We then have,

$$\|P\|_{\mu_1,\mu_2,t} \leq \left( \sum_{k_2} (1 + \mu_2 + 2k_2^2)^s |\Pi_{v_2,k_2}| \|P|_{k_2}\|_{\mu_1,t}^2 \right)^{1/2}$$

$$\leq C'_{\mu_0,s,t} \left( \sum_{k_2} (1 + \mu_2 + 2k_2^2)^s |\Pi_{v_2,k_2}| \|f|_{k_2}\|_{\mu_1,s}^2 \right)^{1/2}$$

$$\leq C'_{\mu_0,s,t} \|f\|_{\mu_1,\mu_2,2s}$$

Hence the lemma is established for the equation $X_1P = f$. By a similar argument, we can find a solution $P' \in W^t(\mathcal{H}_{\mu_1} \otimes \mathcal{H}_{\mu_2})$ to the equation $X_2P' = g$, which satisfies $\|P'\|_{\mu_1,\mu_2,t} \leq C_{\mu_0,s,t}\|g\|_{\mu_1,\mu_2,2s}$. We now show that $P = P'$, provided $t > 1$. 

Since $X_1X_2 = X_2X_1$, we get that $X_1X_2(P - P') = X_2f - X_1g = 0$. By Moore’s Ergodicity, we recall that there are no homogeneous solutions $h$ of strictly positive order to the equation $Xh = 0$ in any irreducible unitary representation $\mathcal{H}_\mu$. But since we know that $(X_2(P - P'))|_{k_2}$ has strictly positive Sobolev order, we must have that $(X_2(P - P'))|_{k_2} = 0$, for every $k_2$. Therefore $X_2(P - P') = 0$. But again, $(P - P')|_{k_1}$ has strictly positive Sobolev order as well, so we must have $(P - P')|_{k_1} = 0$, for all $k_1$, and so $P = P'$. This completes the lemma.

By the same arguments, and using the results of [4] (see chapter 2), we get the similar results for $A_{X_1,U_2}$ and $A_{U_1,U_2}$.

**Lemma 4.14.** Let $f, g \in W^s(\mathcal{H}_{\mu_1} \otimes \mathcal{H}_{\mu_2})$, $(s > 2)$, and satisfy the equation $U_2f = X_1g$. If $t < s - 1$, then there exists solutions $P, P' \in W^t(\mathcal{H}_{\mu_1} \otimes \mathcal{H}_{\mu_2})$ such that $X_1P = f$ and $U_2P' = g$. Furthermore, the norm of $P, P'$ must satisfy, $\|P\|_{\mu_1,\mu_2; t} \leq C_{\mu_0, s, t}\|f\|_{\mu_1,\mu_2; 2s}$, and $\|P'\|_{\mu_1,\mu_2; t} \leq C_{\mu_0, s, t}\|g\|_{\mu_1,\mu_2; 2s}$. If $t > 1$, then $P$ and $P'$ must coincide, so that there is a true simultaneous solution.

**Lemma 4.15.** Let $f, g \in W^s(\mathcal{H}_{\mu_1} \otimes \mathcal{H}_{\mu_2})$, $(s > 2)$, and satisfy the equation $U_2f = U_1g$. If $t < s - 1$, then there exists solutions $P, P' \in W^t(\mathcal{H}_{\mu_1} \otimes \mathcal{H}_{\mu_2})$ such that $U_1P = f$ and $U_2P' = g$. Furthermore, the norm of $P, P'$ must satisfy, $\|P\|_{\mu_1,\mu_2; t} \leq C_{\mu_0, s, t}\|f\|_{\mu_1,\mu_2; 2s}$, and $\|P'\|_{\mu_1,\mu_2; t} \leq C_{\mu_0, s, t}\|g\|_{\mu_1,\mu_2; 2s}$. If $t > 1$, then $P$ and $P'$ must coincide, so that there is a true simultaneous solution.

### 4.2.4 Global Solutions

The uniform estimates for the simultaneous solution, allow us to obtain a simultaneous global solution. Let $\mathcal{H}$ be any unitary representation of $PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$.

We prove that if both the Casimir operators of the two factors $\Box_\mu$ on $\mathcal{H}$ have a spectral gap, then there is a simultaneous solution $P \in W^t(\mathcal{H})$ to the equations $X_1P = f$ and $X_2P = g$, for any smooth vectors $f, g \in W^s(\mathcal{H})$. We note that this condition is satisfied for $\mathcal{H} = L^2(PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R}) / \Gamma)$, for any irreducible lattice $\Gamma$.

**Theorem 4.6.** If there exists a $\mu_0 > 0$ such that the spectrum of the each Casimir satisfies $\sigma(\Box_\mu) \cap (0, \mu_0) = \emptyset$, then we have the following. Let $f, g \in W^s(\mathcal{H})$, $(s > 2)$, and satisfy the equation $X_2f = X_1g$. If $t < s - 1$, then there exists solutions $P, P' \in W^t(\mathcal{H})$ such that $X_1P = f$ and $X_2P' = g$. Furthermore, the norms of $P, P'$ must satisfy, $\|P\|_t \leq C_{\mu_0, s, t}\|f\|_{2s}$, and $\|P'\|_t \leq C_{\mu_0, s, t}\|g\|_{2s}$. If $t > 1$, then $P$ and $P'$ must coincide, so that there is a true simultaneous solution.
Proof. $\mathcal{H}$ is decomposable as the direct integral of non-trivial irreducible unitary representations $\mathcal{H}_{\mu_1, \mu_2} = \mathcal{H}_{\mu_1} \otimes \mathcal{H}_{\mu_2}$ of $PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$ with respect to a Stieltjes measure $ds(\mu_1, \mu_2)$. Every vector $g \in W^s(\mathcal{H})$ then has a decomposition

$$g = \int m(\mu_1, \mu_2)g_{\mu_1, \mu_2}ds(\mu_1, \mu_2)$$

with $g_{\mu_1, \mu_2} \in W^s(\mathcal{H}_{\mu_1, \mu_2})$, and $m(\mu_1, \mu_2)$ the multiplicity with which the representation $\mathcal{H}_{\mu_1, \mu_2}$ occurs in $\mathcal{H}$.

Now we have simultaneous solutions in each irreducible unitary representation which satisfy uniform estimates by the above lemmata. So that we have solutions $P_{\mu_1, \mu_2}, P'_{\mu_1, \mu_2}$ to $X_1 P_{\mu_1, \mu_2} = f_{\mu_1, \mu_2}$ and $X_2 P'_{\mu_1, \mu_2} = g_{\mu_1, \mu_2}$ simultaneously. The estimates in each irreducible unitary representation allow us to choose a uniform constant $C_{\mu_0, s, t}$, which only depends upon the spectral gap $\mu_0 > 0$ and $t < s - 1$, which provides the estimate $\|P\|_{\mu_1, \mu_2, t} \leq C_{\mu_0, s, t}\|f\|_{\mu_1, \mu_2, 2s}$. We then use this uniform estimates to get that,

$$\|P\|_t^2 = \int m(\mu_1, \mu_2)\|P_{\mu_1, \mu_2}\|_t^2 ds(\mu_1, \mu_2) \leq C_{\mu_0, s, t}^2 \int m(\mu_1, \mu_2)\|f_{\mu_1, \mu_2}\|_s^2 ds(\mu_1, \mu_2) = C_{\mu_0, s, t}\|f\|_s^2$$

Similar arguments give the estimate for a solution $P'$, and if $t > 1$, we again see that $P = P'$ provides a unique simultaneous solution.

The same arguments then also establish the result for $A = A_{X_1, U_2}$ and $A_{U_1, U_2}$.

**Theorem 4.7.** If there exists a $\mu_0 > 0$ such that the spectrum of the each Casimir satisfies $\sigma(\square_i) \cap (0, \mu_0) = \emptyset$, then we have the following. Let $f, g \in W^s(\mathcal{H})$, $(s > 1)$, and satisfy the equation $U_2 f = X_1 g$. If $t < s - 1$, then there exists solutions $P, P' \in W^t(\mathcal{H})$ such that $X_1 P = f$ and $U_2 P' = g$. Furthermore, the norms of $P, P'$ must satisfy $\|P\|_t \leq C_{\mu_0, s, t}\|f\|_{2s}$, and $\|P'\|_t \leq C_{\mu_0, s, t}\|g\|_{2s}$. If $t > 1$, then $P$ and $P'$ must coincide, so that there is a true simultaneous solution.

**Theorem 4.8.** If there exists a $\mu_0 > 0$ such that the spectrum of the each Casimir satisfies $\sigma(\square_i) \cap (0, \mu_0) = \emptyset$, then we have the following. Let $f, g \in W^s(\mathcal{H})$, $(s > 1)$, and satisfy the equation $U_2 f = X_1 g$. If $t < s - 1$, then there exists solutions $P, P' \in W^t(\mathcal{H})$ such that $U_1 P = f$ and $U_2 P' = g$. Furthermore, the norms of $P, P'$ must satisfy $\|P\|_t \leq C_{\mu_0, s, t}\|f\|_{2s}$, and $\|P'\|_t \leq C_{\mu_0, s, t}\|g\|_{2s}$. If $t > 1$, then $P$ and $P'$ must coincide, so that there is a true simultaneous solution.
4.3 \( (\text{PSL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{C})) / \Gamma \)

We now consider the higher rank abelian subgroups \( A = A_{X_1} \times A_N \) and \( A_{U_1} \times A_N \) of \( \text{PSL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{C}) \). Here \( X \) and \( U \) are the standard generators of the positive diagonal, and upper-triangular subgroups of \( \text{SL}(2, \mathbb{R}) \), and \( N \) is any non-compact abelian subgroup of \( \text{SL}(2, \mathbb{C}) \). One such example is the two dimensional unipotent subgroup of upper triangular matrices, \( N = \exp g_a \), where \( g_a \) is the positive restricted root space of \( \text{SL}(2, \mathbb{C}) \) (see chapter 3). Other examples include the one parameter subgroup \( N_{X_2} \) and the rank two subgroup \( N_{X_2, \theta_2} \) which is generated by the elements \( X_2 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \) and \( \theta_2 = \left( \begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right) \).

We will state the main results only for the unipotent subgroup of upper triangular matrices, noting that generalizing to an arbitrary non-compact abelian subgroup of \( \text{SL}(2, \mathbb{C}) \) requires no substantive modification to the proofs.

Let \( \Gamma \) be an irreducible (not necessarily cocompact) lattice in \( \text{PSL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{C}) \), and let \( \beta : A \times (\text{PSL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{C})) / \Gamma \rightarrow \mathbb{R}^k \) be a one cocycle. As in section 4.2, Let \( \omega(v) = \frac{d}{dt} \beta(\exp tv)|_{t=0} \) be the infinitesimal generator for \( \beta \).

Recall that the space \( L^2((\text{PSL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{C})) / \Gamma) \) has a direct integral decomposition as,

\[
L^2((\text{PSL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{C})) / \Gamma) = \int \bigoplus m(\mu, \rho, n) \mathcal{H}_{\mu, \rho, n} ds(\mu, \rho, n)
\]

where \( m(\mu_1, \rho, n) \) is the finite multiplicity of the irreducible unitary representation of \( \text{PSL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{C}) \) appearing in \( L^2((\text{PSL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{C})) / \Gamma) \).

The main results of this section can be stated,

**Theorem 4.9.** Let \( \beta : A \times (\text{PSL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{C})) / \Gamma \rightarrow \mathbb{R}^k \) be a smooth cocycle. Then \( \beta \) is smooth cohomologous to a constant cocycle.

And the infinitesimal version becomes,

**Theorem 4.10.** Let \( f, g, h \in W^2((\text{PSL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{C})) / \Gamma), (s > 2) \).

- if \( N_1 f = X_1 g, N_2 f = X_1 h, N_2 g = N_1 h \) and \( 1 < t < s - 1 \), then there exists a solution \( P \in W^t((\text{PSL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{C})) / \Gamma) \) such that \( X_1 P = f, N_1 P = g, \) and \( N_2 P = h \).
- if \( N_1 f = U_1 g, N_2 f = U_1 h, N_2 g = N_1 h \) and \( 1 < t < s - 1 \), then there exists a solution \( P \in W^t((\text{PSL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{C})) / \Gamma) \) such that \( U_1 P = f, N_1 P = g, \) and \( N_2 P = h \).

In either case, the norm of \( P \) must satisfy \( \| P \|_t \leq C_{\mu_0, s, t} \| f \|_{2s} \).
4.3.1 Norms

The irreducible unitary representations of \( PSL(2, \mathbb{R}) \times SL(2, \mathbb{C}) \) are tensor products \( \mathcal{H}_{\mu,\rho,n} = \mathcal{H}_\mu \otimes \mathcal{H}_{\rho,n} \) of the irreducible unitary representations of \( PSL(2, \mathbb{R}) \) and \( SL(2, \mathbb{C}) \). Since \( \Gamma \) is irreducible lattice, we know that \( \mathcal{H}_0 \otimes \mathcal{H}_{\rho,n} \) and \( \mathcal{H}_\mu \otimes \mathcal{H}_0 \) cannot appear in the decomposition of \( L^2((PSL(2, \mathbb{R}) \times SL(2, \mathbb{C}))/\Gamma) \).

We will use the notation

\[
f|_{k_2,k_3} = \sum_{k_1} f_{k_1,k_2,k_3} u_{k_1} \otimes u_{k_2,k_3}, \quad \text{and} \quad f|_{k_1} = \sum_{k_2} \sum_{k_3=-k_2}^{k_2} f_{k_1,k_2,k_3} u_{k_1} \otimes u_{k_2,k_3}
\]

where \( \{u_{k_1}\} \) is the adapted basis (4.3), and \( \{u_{k_2,k_3}\} \) is an o.n. basis of \( \mathcal{H}_{\rho,n} \).

Now the Laplacian \( \triangle_2 \) of \( SL(2, \mathbb{C}) \) acts on the o.n. basis \( \{u_{k_2,k_3}\} \) of \( \mathcal{H}_{\rho,n} \) as a multiplicative operator. That is,

\[
\triangle_2 \cdot u_{k_2,k_3} = -p(k_2,\rho,n) \cdot u_{k_2,k_3}, \quad \text{where} \quad p(k_2,\rho,n) = k_2^2 + k_2 - \frac{1}{2}(n^2 - 1 + \rho^2) \quad (4.26)
\]

is a quadratic polynomial in \( k_2 \). Now since in the irreducible space \( \mathcal{H}_{\rho,n} \), the basis is parametrized by \((k_2 \geq n, -k_2 \leq k_3 \leq k_2)\), where \( n \geq 0 \) and either \( \rho \in i \cdot \mathbb{R} \) or \( 0 < \rho < 1 \), the polynomial \( p(k_2,\rho,n) \) is positive for all allowable values of \( k_2,\rho,n \).

The Laplacian of \( PSL(2, \mathbb{R}) \times SL(2, \mathbb{C}) \) on \( \mathcal{H}_{\mu_1} \otimes \mathcal{H}_{\rho,n} \) is \( \triangle_1 + \triangle_2 \), which acts on the \( K \)-finite vectors as follows,

\[
(\triangle_1 + \triangle_2)(u_{k_1} \otimes u_{k_2,k_3}) = \triangle_1 u_{k_1} \otimes u_{k_2,k_3} + u_{k_1} \otimes \triangle_2 u_{k_2,k_3}
\]

\[
= [(\mu + 2k_1^2) + p(k_2,\rho,n))]u_{k_1} \otimes u_{k_2,k_3}
\]

Therefore, the Sobolev norms are given by,

\[
\|f\|_{\mu,\rho,n,s} = \left( \sum_{k_1,k_2,k_3=-k_2}^{k_2} (1 + (\mu + 2k_1^2) + p(k_2,\rho,n))]\Pi_{v_1,k_1} |f_{v_1,k_1,k_2,k_3}|^2 \right)^{1/2}
\]

This norm is related to the norm on the factors by again noting that, for any pair of sequences of positive numbers \( \{a_{k_1}\} \) and \( \{b_{k_2}\} \), we have,

\[
(1 + a_{k_1} + b_{k_2}) \leq (1 + a_{k_1})(1 + b_{k_2}) \leq (1 + a_{k_1} + b_{k_2})^2
\]

And also that

\[
(1 + a_{k_1}) \leq (1 + a_{k_1} + b_{k_2})
\]
We then get,
\[
\|f\|_{\mu, \rho, n, s} \leq \left( \sum_{k_2} \sum_{k_3=-k_2}^{k_2} (1 + p(k_2, \rho, n))^s \|f|_{k_2, k_3}\|_{\mu, s}^2 \right)^{1/2} \leq \|f\|_{\mu, \rho, n, 2s}
\]
And
\[
\|f|_{k_2, k_3}\|_{\mu, s} \leq \|f\|_{\mu, \rho, n, s}
\]

4.3.2 Vanishing of Obstructions

We will start with \( A = A_{X_1, N} \). Now, \( \omega \) is linear on the Lie algebra \( a \), and so can be recovered from the functions \( f = \omega(X_1), g = \omega(N_1), \) and \( h = \omega(N_2) \). The cocycle identity is then expressed by the equations,
\[
N_1 f = X_1 g, \quad N_2 f = X_1 h, \quad N_2 g = N_1 h
\]
For \( A = A_{U_1, N} \), the cocycle identity is expressed similarly by the equations,
\[
N_1 f = U_1 g, \quad N_2 f = U_1 h, \quad N_2 g = N_1 h
\]
For any linear functional \( D \) on \( \mathcal{H}_\mu \), define \( D_1 \) on \( \mathcal{H}_{\mu, \rho, n} \) by \( (D_1)(u_{k_1} \otimes u_{k_2, k_3}) = D(u_{k_1}) \). The relationship between norms above shows that if \( D \in W^{-s}(\mathcal{H}_\mu) \), then \( D_1 \in W^{-2s}(\mathcal{H}_{\mu, \rho, n}) \). Of course, it is also easy to see that if \( D \) is an \( X \)-invariant distribution on \( W^s(\mathcal{H}_\mu) \), then \( D_1 \) is an \( X_1 \)-invariant distribution on \( W^{2s}(\mathcal{H}_{\mu, \rho, n}) \).

We will show that \( D_1(f|_{k_2, k_3}) = 0 \), for all \( k_2, k_3 \) and for all \( X \)-invariant distributions on \( \mathcal{H}_\mu \). The obstructions that arise from the \( SL(2, \mathbb{C}) \) factor will turn out to be unimportant in view of our ability to solve equations on the \( PSL(2, \mathbb{R}) \) factor. Similarly we show that \( D_1(f|_{k_2, k_3}) = 0 \), for all \( k_2, k_3 \) and for all \( U \)-invariant distributions on \( \mathcal{H}_\mu \).

**Lemma 4.16.** Let \( f, g, h \in W^s(\mathcal{H}_{\mu, \rho, n}), (s > 2) \). If \( f, g, h \) satisfy the equations,
- \( N_1 f = X_1 g, N_2 f = X_1 h. \) Then for any \( X \)-invariant \( D \in W^{-s}(\mathcal{H}_\mu) \), we have \( D_1(f|_{k_2, k_3}) = 0 \) for all \( k_2, k_3 \).
- \( N_1 f = U_1 g, N_2 f = U_1 h. \) Then for any \( U \)-invariant \( D \in W^{-s}(\mathcal{H}_\mu) \), we have \( D_1(f|_{k_2, k_3}) = 0 \) for all \( k_2, k_3 \).

**Proof.** The proof of the lemma is similar to the lemmas for the \( PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R}) \) case above.
Since $D_1$ is either $X$-invariant or $U$-invariant. This gives us that, for all $k_2, k_3$, we have $X \cdot D_1(g|_{k_2,k_3}) = 0$ and $X \cdot D_1(h|_{k_2,k_3}) = 0$ in the first case, or $U \cdot D_1(g|_{k_2}) = 0$ and $U \cdot D_1(h|_{k_2,k_3}) = 0$ in the second case. Combining this with the equalities $N_1 f = X_1 g$ and $N_2 f = X_1 h$ in the first case, or $N_1 f = U_1 g$ and $N_2 f = U_1 h$ in the second case, then gives us $N_1 \cdot D_1(f|_{k_2,k_3}) = 0$ and $N_2 \cdot D_1(f|_{k_2,k_3}) = 0$ for all $k_2, k_3$.

Now define $\tilde{f}$ by $\tilde{f}_{k_2,k_3} = D_1(f|_{k_2,k_3})$ for each $k_2 \geq n, -k_2 \leq k_3 \leq k_2$. Then we have just shown that $\tilde{f}$ is a (formal) solution of the homogeneous system $N_i \cdot \tilde{f} = 0$, that is, $\tilde{f}$ is $N$-invariant. Moore's Ergodicity theorem then tells us that there can be no non-trivial $N$-invariant vectors in an irreducible unitary representation, since $N$ is a closed, non-compact subgroup. So as before, we show that $\tilde{f}$ must belong to $\mathcal{H}_{\rho,n}$, and so must be the zero vector, for each $k_2, k_3$, which then proves the lemma.

By Minkowski's inequality, we have

$$
\left( \sum_{k_2}^{k_2} \sum_{k_3=-k_2}^{k_3} |\tilde{f}_{k_2,k_3}|^2 \right)^{1/2} = \left( \sum_{k_2}^{k_2} \sum_{k_3=-k_2}^{k_3} \left| \sum_{k_1} f_{k_1,k_2,k_3}(D_1(k_1) \right|^2 \right)^{1/2} 
\leq \sum_{k_1} \left( \sum_{k_2}^{k_2} \sum_{k_3=-k_2}^{k_3} |f_{k_1,k_2,k_3}|^2 \right) \left| \sum_{k_1} \left( D_1(k_1) \right|^{1/2}
\right) = \sum_{k_1} |(D_1(k_1)| \left( \sum_{k_2}^{k_2} \sum_{k_3=-k_2}^{k_3} |f_{k_1,k_2,k_3}|^2 \right)^{1/2} (4.27)
$$

Now, $|(D_1(k_1)| \leq a + b|\Pi_{\nu_1,k_1}| \leq C_{\nu_1,a,b}$ is bounded. But we also have that $f \in W^s(\mathcal{H}_{\rho,n})$, so

$$
|f_{k_1,k_2,k_3}|^2 \leq C_{\rho,n} (1 + (\mu + 2k_1^2) + p(k_2, \rho, n))^{-s} (4.28)
$$

That is, $|f_{k_1,k_2,k_3}|^2 = O((k_1^2 + k_2^2)^{-s})$, and therefore the sum

$$
\sum_{k_1} |(D_1(k_1)| \left( \sum_{k_2}^{k_2} \sum_{k_3=-k_2}^{k_3} |f_{k_1,k_2,k_3}|^2 \right)^{1/2}
$$

must be finite for $s > 2$. This then implies that $D_1(f|_{k_2,k_3}) = \tilde{f}_{k_2,k_3} \equiv 0$.

4.3.3 Simultaneous Solutions

We now show how the simultaneous vanishing of obstructions leads to a simultaneous solution of the inhomogeneous equations. In addition, this solution satisfies a uniform bound over all parameters $(\mu, \rho, n)$ that satisfy $\mu \geq \mu_0 > 0$. 
Lemma 4.17. Let \( f, g, h \in W^{2s}(\mathcal{H}_{\mu,\rho,n}) \), where \( s > 2 \).

- if \( N_1 f = X_1 g, N_2 f = X_1 h, N_2 g = N_1 h \) and \( t < s - 1 \), then there exists a \( P \in W^t(\mathcal{H}_{\mu,\rho,n}) \) such that \( X_1 P = f \) and \( N_1 P = g, N_2 P = h \).
- if \( N_1 f = U_1 g, N_2 f = U_1 h, N_2 g = N_1 h \) and \( t < s - 1 \), then there exists a \( P \in W^t(\mathcal{H}_{\mu,\rho,n}) \) such that \( U_1 P = f \) and \( N_1 P = g, N_2 P = h \).

In either case, \( P \) must satisfy \( \|P\|_{\mu,\rho,n,t} \leq C_{\mu_0,s,t} \|f\|_{\mu,\rho,n,2s} \).

Proof. For the first part, by lemma 4.16 we know that for any \( X \)-invariant distribution \( D \in W^{-2s}(\mathcal{H}_{\mu}) \), we must have \( D_1(f|_{k_2,k_3}) = 0 \) for all \( k_2, k_3 \). As we have that \( \|f\|_{\mu,2s} \leq \|f\|_{\mu,\rho,n,2s} < \infty \), we must then have that there exists a solution \( P|_{k_2,k_3} \in W^{2(s-1)}(\mathcal{H}_{\mu}) \), which satisfies the uniform estimate \( \|P|_{k_2,k_3}\|_{\mu,t} \leq C_{\mu_0,s,t} \|f|_{k_2,k_3}\|_{\mu,s} \) for every \( k_2, k_3 \). We note that the constant does not depend upon \( k_2, k_3 \). We then have,

\[
\|P\|_{\mu,\rho,n,t} \leq \left( \sum_{k_2} \sum_{k_3=-k_2}^{k_2} (1 + p(k_2, \rho, n))^t \|P|_{k_2,k_3}\|_{\mu,t}^2 \right)^{1/2}
\leq C_{\mu_0,s,t} \left( \sum_{k_2} \sum_{k_3=-k_2}^{k_2} (1 + p(k_2, \rho, n))^s \|f|_{k_2,k_3}\|_{\mu,s}^2 \right)^{1/2} \tag{4.29}
\]

The smoothness of \( P \) then follows since the above is true for all \( t < s - 1 \). Hence the lemma is established for the equation \( X_1 P = f \).

Now we show that \( N_1 P = g \) and \( N_2 P = h \). Again, since all operators commute, we have that \( X_1(g - N_1 P) = X_1 g - N_1 f = 0 \). But then \( g - N_1 P \) is an \( X_1 \)-invariant vector in \( \mathcal{H}_{\mu,\rho,n} \), and then by Moore’s Ergodicity theorem, it must be the zero vector. Therefore \( N_1 P = g \). Similarly we must have \( N_2 P = h \).

The second case follows in exactly the same way, and the lemma is established.

4.3.4 Global Solutions

The uniform estimates for the simultaneous solution, allow us to obtain a simultaneous global solution. Let \( \mathcal{H} \) be any unitary representation of \( PSL(2, \mathbb{R}) \times SL(2, \mathbb{C}) \).

We prove that if the Casimir operator of the first factor \( \Box_1 \) on \( \mathcal{H} \) has a spectral gap, then there is a simultaneous solution \( P \in W^t(\mathcal{H}) \) to the equations \( X_1 P = f \) and \( X_2 P = g \), for any smooth vectors \( f, g \in W^p(\mathcal{H}) \). We note that this condition is satisfied for \( \mathcal{H} = L^2(PSL(2, \mathbb{R}) \times SL(2, \mathbb{C}) / \Gamma) \), for any irreducible lattice \( \Gamma \).
THEOREM 4.11. If there exists a $\mu_0 > 0$ such that the spectrum of the Casimir of the first factor satisfies $\sigma(\Box_1) \cap (0, \mu_0) = \emptyset$, then we have the following. Let $f, g, h \in W^{2s}(\mathcal{H})$, $(s > 2)$.

- if $N_1 f = X_1 g$, $N_2 f = X_1 h$, $N_2 g = N_1 h$ and $1 < t < s - 1$, then there exists a solution $P \in W^t(\mathcal{H})$ such that $X_1 P = f$, $N_1 P = g$, and $N_2 P = h$.
- if $N_1 f = U_1 g$, $N_2 f = U_1 h$, $N_2 g = N_1 h$ and $1 < t < s - 1$, then there exists a solution $P \in W^t(\mathcal{H})$ such that $U_1 P = f$, $N_1 P = g$, and $N_2 P = h$.

In either case, the norm of $P$ must satisfy $\|P\|_t \leq C_{\mu_0, s, t} \|f\|_{2s}$.

Proof. $\mathcal{H}$ is decomposable as the direct integral of non-trivial irreducible unitary representations $\mathcal{H}_{\mu, \rho, n}$ of $\text{PSL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{C})$ with respect to a Stieltjes measure $d\mu, \rho, n)$. Every vector $g \in W^s(\mathcal{H})$ then has a decomposition

$$g = \int_{\mathbb{R}} m(\mu, \rho, n)g_{\mu, \rho, n}d\mu, \rho, n)$$

with $g_{\mu, \rho, n} \in W^s(\mathcal{H}_{\mu, \rho, n})$, and $m(\mu, \rho, n)$ the multiplicity with which the representation $\mathcal{H}_{\mu, \rho, n}$ occurs in $\mathcal{H}$.

Now we have a simultaneous solution in each irreducible unitary representation that satisfies uniform estimates by the above lemmata. So that we have a solution $P_{\mu, \rho, n}$ to $X_1 P_{\mu, \rho, n} = f_{\mu, \rho, n}$, $N_1 P_{\mu, \rho, n} = g_{\mu, \rho, n}$, and $N_2 P_{\mu, \rho, n} = h_{\mu, \rho, n}$ simultaneously. The estimates in each irreducible unitary representation allow us to choose a uniform constant $C_{\mu_0, s, t}$, which only depends upon the spectral gap $\mu_0 > 0$ and $t < s - 1$, which provides the estimate $\|P\|_{\mu, \rho, n, t} \leq C_{\mu_0, s, t} \|f\|_{\mu, \rho, n, 2s}$. We then use this uniform estimates to get that,

$$\|P\|_t^2 = \int m(\mu, \rho, n)\|P_{\mu, \rho, n}\|_t^2d\mu, \rho, n) \leq C_{\mu_0, s, t}^2 \int m(\mu, \rho, n)\|f_{\mu, \rho, n}\|_s^2d\mu, \rho, n) = C_{\mu_0, s, t}^2 \|f\|_s^2$$

The same argument with $U_1$ replacing $X_1$ also establishes the theorem for the second case.

4.4 $(\text{PSL}(2, \mathbb{R}) \times \text{SO}(N, 1))/\Gamma$

We now consider the higher rank abelian subgroups $A = A_{X_1} \times A_N$ and $A_{U_1} \times A_N$ of $\text{PSL}(2, \mathbb{R}) \times \text{SO}(N, 1)$. $X$, and $U$ are the standard generators of the positive diagonal, and upper-triangular subgroups of $\text{SL}(2, \mathbb{R})$, and $N$ is any non-compact, closed, abelian subgroup of $\text{SO}(N, 1)$. For example, the $N - 1$ dimensional unipotent subgroup, $N = \exp g_\alpha$, where $g_\alpha$ is the positive restricted root space of $\text{SO}(N, 1)$ (see chapter 3).
Let $\Gamma$ be an irreducible (not necessarily cocompact) lattice in $PSL(2, \mathbb{R}) \times SO(N,1)$, and let $\beta : A \times (PSL(2, \mathbb{R}) \times SO(N,1))/\Gamma \rightarrow \mathbb{R}^k$ be a one cocycle. As in section 4.2, Let $\omega(v) = \frac{d}{dt} \beta(\exp tv)|_{t=0}$ be the infinitesimal generator for $\beta$.

Recall that the space $L^2((PSL(2, \mathbb{R}) \times SO(N,1))/\Gamma)$ has a direct integral decomposition as,

$$L^2((PSL(2, \mathbb{R}) \times SO(N,1))/\Gamma) = \int \oplus m(\mu, \rho, \sigma) \mathcal{H}_{\mu, \rho, \sigma} ds(\mu, \rho, \sigma)$$

where $m(\mu_1, \rho, \sigma)$ is the finite multiplicity of the irreducible unitary representation of $PSL(2, \mathbb{R}) \times SO(N,1)$ appearing in $L^2((PSL(2, \mathbb{R}) \times SO(N,1))/\Gamma)$. Let $\Gamma$ be a lattice for which there exists a $\mu_0 > 0$ such that the multiplicity $m(\mu, \rho, \sigma) = 0$ for all $0 < \mu < \mu_0$.

The main results of this section can be stated,

**Theorem 4.12.** Let $\beta : A \times (PSL(2, \mathbb{R}) \times SO(N,1))/\Gamma \rightarrow \mathbb{R}^k$ be a smooth cocycle. Then $\beta$ is smooth cohomologous to a constant cocycle.

And the infinitesimal version becomes,

**Theorem 4.13.** Let $f_1, g_i \in W^2((PSL(2, \mathbb{R}) \times SO(N,1))/\Gamma)$ for $i = 1, \ldots, \dim N, (s > N)$.

- if $N_if = X_ig_j, N_ig_j = N_ig_i$ for all $i, j = 1, \ldots, \dim N$ and $1 < t < s - 1$, then there exists a solution $P \in W^1((PSL(2, \mathbb{R}) \times SO(N,1))/\Gamma)$ such that $X_1P = f, N_1P = g_i$ for all $i = 1, \ldots, \dim N$.
- if $N_if = U_1g_j, N_ig_j = N_ig_i$ for all $i, j = 1, \ldots, \dim N$ and $1 < t < s - 1$, then there exists a solution $P \in W^1((PSL(2, \mathbb{R}) \times SO(N,1))/\Gamma)$ such that $U_1P = f, N_1P = g_i$ for all $i = 1, \ldots, \dim N$.

In either case, the norm of $P$ must satisfy $\|P\|_1 \leq C_{\mu_0, \rho, \sigma} \|f\|_{2s}$.

The proof of the general results for $SO(N,1)$ follow along exactly as in the previous case. In fact, the data for the irreducible unitary representation for $SO(N,1)$ plays a fairly inessential role, and so we will briefly summarize it now.

First, we note that the irreducible unitary representations $\mathcal{H}_{\rho, \sigma}$ of $SO(N,1)$ are parametrized by the the pair $(\rho, \sigma)$, where $\rho \in \mathbb{C}$ is either purely imaginary (principal series) or real (complementary series and discrete series), and $\sigma$ is an irreducible unitary representation of $SO(N-1)$, hence finite dimensional. We are only concerned with how
the Laplacian operator $\triangle$ for $SO(N,1)$ acts in this space. Since $\triangle$ is an essential self-adjoint operator on each $\mathcal{H}_{\rho,\sigma}$, the spectral theorem says that there is a basis on which $\triangle$ acts multiplicatively.

We recall that every every irreducible unitary representation of $SO(N,1)$ can be restricted to a unitary representation of $SO(N)$ (a maximal compact subgroup), and hence can be further decomposed under the action by $SO(N)$, into a direct sum of irreducible unitary representations of $SO(N)$, all of them necessarily finite. For the real rank one semisimple Lie groups, each irreducible unitary representation of a maximal compact subgroup can appear with at most multiplicity one.

Since $SO(N)$ is compact, its irreducible unitary representations are known. Indeed, by the Theorem of the Highest Weight, the irreducible unitary representations of $SO(N)$ stand in one to one correspondence with the dominant algebraically integral linear functionals $\lambda \in \mathfrak{h}'$, where $\mathfrak{h}$ is a Cartan subalgebra in $\mathfrak{so}(N,\mathbb{C})$, and $\mathfrak{h}'$ is its dual. We recall that a linear functional is algebraically integral if $2 < \gamma_i, \alpha_j > / |\alpha_j|^2 \in \mathbb{Z}$, for all roots $\alpha$. If we let $\Pi$ be a choice of simple roots, then the dominant algebraically integral linear functionals are completely determined by $\Pi$.

Define $\gamma_i$ by the condition that $2 < \gamma_i, \alpha_j > / |\alpha_j|^2 = \delta_{ij}$ for all simple roots $\alpha_j$. We call such a weight $\gamma_i$ a fundamental weight [19], and note that any dominant algebraically integral linear functional $\lambda = \sum_i k_i \gamma_i$, where the $k_i$ are non-negative integers.

Now, the Casimir operator $\Box$ for the subgroup $SO(N)$, acts as a scalar on each irreducible unitary representation of $SO(N)$, by Schur’s lemma. In fact, it acts by the scalar $\Box = |\lambda|^2 + 2 < \lambda, \delta > - |\lambda|^2 / |\delta|^2$ in the representation of highest weight $\lambda$, where $\delta = \frac{1}{2} \sum \alpha$ is the half sum of the positive roots. Therefore, $\Box = \sum_{ij} c_{ij} \cdot k_i k_j - C_0$, where $c_{ij} = < \gamma_i, \gamma_j >$, and $C_0 = |\delta|^2$, is a polynomial in $k_1, \ldots, k_{[N/2]}$ of order 2.

But again by Schur’s lemma, the Casimir operator $\Box$ for $SO(N,1)$, must act as a scalar in each irreducible unitary representation of $SO(N,1)$. Then the Laplacian can be defined as before as $\triangle = \Box - 2 \Box$. The Sobolev norms are then defined by $\|f\|_{W^s(\mathcal{H}_{\rho,\sigma})} = \|(I - \triangle)^{s/2} f\|_{\mathcal{H}_{\rho,\sigma}}$. We can express this norm using an o.n. basis for the irreducible unitary representation $\mathcal{H}_{\rho,\sigma}$.

Let $[N/2]$ be the integer part of $N/2$, then the dimension of the Cartan subalgebra $\mathfrak{h}$ is $[N/2]$. The dimension of the irreducible representation of highest weight $\lambda = \sum_{i=1}^{[N/2]} k_i \gamma_i$ is given by the Weyl dimension formula, $\dim V_\lambda = \prod_i <\lambda + \frac{1}{2} \delta, \alpha_i > / \prod_i <\delta, \alpha_i >$, where
the products are taken over the set of positive roots. For convenience we will just write $\dim V_\lambda = M_\lambda$. Then we can choose an o.n. basis $\{u_{k_1,\ldots,k_{[N/2]}}^j \mid k_i \geq n_i, j = 1,\ldots,M_\lambda\}$, where the $n_i \geq 0$ are integers determined by $\sigma$. The Sobolev norms are then given by,

$$\|f\|_{W^s(\mathcal{H}_{\mu,\rho,\sigma})} = \left( \sum_{k_0,k_j=1}^{M_\lambda} \sum_{j=1}^{[N/2]} (1 + p(k_1,\ldots,k_{[N/2]}))^s |f_{k_0,k_1,\ldots,k_{[N/2]}}^j|^2 \right)^{1/2}$$

(4.30)

where $(I - \triangle) \cdot u_{k_1,\ldots,k_{[N/2]}}^j = (1 + p(k_1,\ldots,k_{[N/2]}))u_{k_1,\ldots,k_{[N/2]}}^j$ is the scalar by which $(I - \triangle)$ acts on the basis. In fact,

$$1 + p(k_1,\ldots,k_{[N/2]}) = 2 \sum_{i,j=1}^{[N/2]} c_{ij} \cdot k_i k_j - 2C_0 - C_{\rho,\sigma} + 1$$

is a polynomial in $k_1,\ldots,k_{[N/2]}$ or order 2, which is positive for $k_i$ all positive [11], so that $(I - \triangle)$ always acts as a positive scalar on each $u_{k_1,\ldots,k_{[N/2]}}^j$.

As before, we can then determine the Sobolev norms for the irreducible unitary representations $\mathcal{H}_{\mu,\rho,\sigma}$ of $PSL(2,\mathbb{R}) \times SO(N,1)$. The Sobolev norms are given by,

$$\|f\|_{W^s(\mathcal{H}_{\mu,\rho,\sigma})} = \left( \sum_{k_0,k_j=1}^{M_\lambda} \sum_{j=1}^{[N/2]} (1 + (\mu + 2k_0^2) + p(k_1,\ldots,k_{[N/2]}))^s |\Pi_{k_1,k_j} ||f_{k_0,k_1,\ldots,k_{[N/2]}}^j|^2 \right)^{1/2}$$

This norm is related to the norm on the factors by again noting that, for any pair of sequences of positive numbers $\{a_{k_0}\}$ and $\{b_{k_1,\ldots,k_{[N/2]}}\}$, we have,

$$(1 + a_{k_0} + b_{k_1,\ldots,k_{[N/2]}}) \leq (1 + a_{k_0})(1 + b_{k_1,\ldots,k_{[N/2]}}) \leq (1 + a_{k_0} + b_{k_1,\ldots,k_{[N/2]}})^2$$

And also that

$$(1 + a_{k_0}) \leq (1 + a_{k_1} + b_{k_1,\ldots,k_{[N/2]}})$$

We then get,

$$\|f\|_{W^s(\mathcal{H}_{\mu,\rho,\sigma})} \leq \left( \sum_{k_0,k_j=1}^{M_\lambda} \sum_{j=1}^{[N/2]} (1 + p(k_1,\ldots,k_{[N/2]}))^s \|f_{k_0,k_1,\ldots,k_{[N/2]}}^j\|^2 \|W^s(\mathcal{H}_\mu)\right)^{1/2} \leq \|f\|_{W^2(\mathcal{H}_{\mu,\rho,\sigma})}$$

And

$$\|f\|_{W^s(\mathcal{H}_{\mu,\rho,\sigma})} \leq \|f\|_{W^s(\mathcal{H}_\mu)}$$
With the norms now described, the proof of Theorem 4.13 follows the method of proof for Theorem 4.10 with only minor modification. In that proof, the properties of the equation $XP = f$ or $UP = f$ on the $PSL(2, \mathbb{R})$ factor are what give a solution $P$, whose norm satisfies uniform estimates. The points where the properties of the representations $\mathcal{H}_{\rho,\sigma}$ on the second factor play a role are in the decay rate of functions $f \in W^s(\mathcal{H}_{\mu,\rho,\sigma})$, namely

$$|f_{k_0, k_1, \ldots, k_{\lfloor N/2 \rfloor}}|^2 \leq C_{\mu, \rho, \sigma} (1 + (\mu + 2k_0^2) + p(k_1, \ldots, k_{\lfloor N/2 \rfloor}))^{-s} \quad (4.31)$$

which takes the place of (4.27) in Lemma 4.16.

Also, the ability to relate the Sobolev norms on the $\mathcal{H}_{\mu}$ factor with the Sobolev norms on $\mathcal{H}_{\mu,\rho,\sigma}$, allow us to emulate (4.28). The rest of the proofs follow with no essential modification, which then establishes Theorem 4.13.
4.5 \( SL(2, \mathbb{C})/\Gamma \)

### 4.5.1 Statement of Results and Initial Reductions

The Lie group \( SL(2, \mathbb{C}) \), viewed as a real Lie groups, is a semisimple Lie group. It has several useful decompositions. Let \( X = (\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}) \in \text{sl}(2, \mathbb{C}) \). Let \( a = \mathbb{R} \cdot X \). Then the restricted root spaces are \( \mathfrak{g}_a = \{ (\begin{smallmatrix} 0 & c \\ 0 & 0 \end{smallmatrix}) \mid c \in \mathbb{C} \} \).

Now we discuss the case of cocycles over the \( N = \exp \mathfrak{g}_a \) action on \( SL(2, \mathbb{C})/\Gamma \). We refer the reader to section 3.5 for background information on cocycles.

Here we consider the cocycle \( \beta : N \times SL(2, \mathbb{C})/\Gamma \rightarrow \mathbb{R}^k \), where \( N = \exp \mathfrak{g}_a \) and acts via left multiplication on \( SL(2, \mathbb{C})/\Gamma \). Or, alternatively, by writing \( f(x) \cdot h = f(hx) \), we will consider one-cocycles to be measurable functions \( \beta : N \rightarrow \mathcal{F}(SL(2, \mathbb{C})/\Gamma, \mathbb{R}^k) \) which satisfy \( \beta(n_1 + n_2) = \beta(n_2) + \beta(n_1) \cdot n_2 \), and where \( \mathcal{F}(SL(2, \mathbb{C})/\Gamma, \mathbb{C}^k) \) is the space of measurable functions \( SL(2, \mathbb{C})/\Gamma \rightarrow \mathbb{C}^k \). By taking component functions, and real and imaginary parts, we may always assume that \( \beta : N \times SL(2, \mathbb{C})/\Gamma \rightarrow \mathbb{C} \).

A cocycle will be called smooth if the map \( \beta : N \rightarrow C^\infty(L^2((SL(2, \mathbb{C})/\Gamma)) \) is smooth. We can also define \( \beta \) to be of class \( C^r \). And we say that \( \beta \) is smooth cohomologous to a constant cocycle \( c \) if there exists a smooth map \( P : SL(2, \mathbb{C})/\Gamma \rightarrow \mathbb{C} \) such that \( \beta(n, g) = -(P(n g) + c(n, g) + P(g)) \). Recall that \( c : N \times SL(2, \mathbb{C})/\Gamma \rightarrow \mathbb{C} \) is a constant cocycle if \( c(n, g) = c(n) \in \mathbb{C} \) is a constant function of \( SL(2, \mathbb{C})/\Gamma \) for every \( n \in N \).

Recall that the space \( L^2(SL(2, \mathbb{C})/\Gamma) \) has a direct integral decomposition as,

\[
L^2(SL(2, \mathbb{C})/\Gamma) = \int_{\mathbb{C}} m(\rho, n) \cdot \mathcal{H}_{\rho, n} d\mu(\rho, n)
\]

where \( m(\rho, n) \) is the finite multiplicity of the irreducible unitary representation of \( SL(2, \mathbb{C}) \) appearing in \( L^2(SL(2, \mathbb{C})/\Gamma) \). We note that the trivial representation appears with multiplicity one. The volume element of \( L^2(SL(2, \mathbb{C})/\Gamma) \) is an \( L^2 \) distribution that is supported on the trivial representation. We remark, then, that if \( \beta : N \rightarrow L^2(SL(2, \mathbb{C})/\Gamma) \) is cohomologous to a constant cocycle, then that constant cocycle must be of the form \( c(n) = \int_{SL(2, \mathbb{C})/\Gamma} \beta(n, g) d\mu \Gamma \). That is, the first smooth cohomology at least has dimension one, but this obstruction arises in a trivial way and so we consider only the first smooth almost cohomology group.

The main result of this section can now be stated as,
\textbf{Theorem 4.14.} Let $\beta : N \times SL(2, \mathbb{C})/\Gamma \to \mathbb{R}^k$ be a smooth cocycle, such that $\beta$ contains no component in the complementary series representations appearing in $L^2(SL(2, \mathbb{C})/\Gamma)$ (i.e. $\beta_{p,0} = 0$ for all $0 < p < 2$). Then $\beta$ is smooth cohomologous to a constant cocycle.

In particular, if $\Gamma$ is any lattice in $SL(2, \mathbb{C})$ such that the direct (sum) integral decomposition of $L^2(SL(2, \mathbb{C})/\Gamma)$ has no component of the complementary series of $SL(2, \mathbb{C})$, then the above theorem would hold for all smooth cocycles $\beta$, and hence would imply that the first smooth almost cohomology over the $N$ action on $SL(2, \mathbb{C})/\Gamma$. This then immediately implies Theorem 1.4.

Now if $\Gamma$ is any arithmetic lattice in $SL(2, \mathbb{C})$, the Generalized Ramanujan-Selberg conjecture states that the direct (sum) integral decomposition of $L^2(SL(2, \mathbb{C})/\Gamma)$ has no component of the complementary series of $SL(2, \mathbb{C})$ (or more accurately, that it only contains irreducible tempered components), and so no smooth cocycle $\beta$ will have a component in the complementary series.

In order to proceed, we first make a reduction to the infinitesimal version. Recall that, for a $C^\infty$ or $C'$ cocycle $\beta$, we can define the infinitesimal generator $\omega$ as follows.

Let $\omega(v) = \frac{d}{dt}\beta(\exp tv) \big|_{t=0}$ be the infinitesimal generator for $\beta$. Then the cocycle equation implies that $\omega$ is a closed form on the orbit foliation. In fact, it is a closed one form.

We can also recover $\beta$ from $\omega$ by $\beta(\exp X) = \int_0^1 \omega(X) \cdot \exp tX \, dt$, as the exponential map $\exp : \text{Lie}(N) \to N$ is onto. Therefore we can restrict our attention to the infinitesimal situation. In particular, the infinitesimal version of the cohomology equation is $\omega = \eta - dP$. Thus the $C^\infty$ or $C'$ first cohomology group is trivial if for every $C^\infty$ or $C'$ one form $\omega$ is exact via a function smooth (or $C'$) function $P : SL(2, \mathbb{C})/\Gamma \to \mathbb{C}$.

Since $\omega$ is a (one) form on $\text{Lie}(N) = g_\alpha$, it is determined by the two functions $f = \omega(U)$ and $g = \omega(U')$, where $U = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $U' = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}$ form a basis for $\text{Lie}(N)$. The regularity of these two functions determines the regularity of $\beta$. In particular, if $f, g \in C^\infty(L^2(SL(2, \mathbb{C})/\Gamma))$, then so is $\beta$, and a similar statement can be made for $f, g \in C'(L^2(SL(2, \mathbb{C})/\Gamma))$. The cocycle identity then becomes $U'f = Ug$, and $\omega = dP$ becomes $UP = dP(U) = \omega(U) = f$, and $U'P = dP(U') = \omega(U') = g$.

\textbf{Theorem 4.15.} Let $f, g \in C^\infty(L^2(SL(2, \mathbb{C})/\Gamma))$, be such that $U'f = Ug$. If $f_{p,0} = g_{p,0} = 0$ for every element of the complementary series that appears in the decomposition of $L^2(SL(2, \mathbb{C})/\Gamma)$, then there exists a $P \in C^\infty(SL(2, \mathbb{C})/\Gamma)$, which provides simultaneous solution to the equations $UP = f$ and $U'P = g$, and for every $1 \leq k$, satisfies the estimates,
\[ \|P\|_{W^k(SL_2)} \leq C_k \|f\|_{W^{k+2}(SL_2)} \quad \text{and,} \]
\[ \|P\|_{W^k(SL_2')} \leq C'_k \|g\|_{W^{k+2}(SL_2')} \]

Notice that the theorem only guarantees that \( P \in C^\infty(SL(2, \mathbb{C})/\Gamma) \), not the stronger statement that \( P \in C^\infty(L^2(SL(2, \mathbb{C})/\Gamma)) \). Also, an entirely similar theorem holds if we replace \( C^\infty \) by \( C^r, r > 1 \).

4.5.2 Norms

We now describe some norms which we might be able to place on the spaces \( D_\chi \) (see section 3.2.2).

The first norm we will place on \( D_\chi \) will be the one that comes from the Hermitian inner product, and which is unitary with respect to the action that we defined. We get two basic types, the so called principal series, and the complementary Series.

- Principal Series. We take \( \lambda = \frac{1}{2}(i\rho + n - 2), \mu = \frac{1}{2}(i\rho - n - 2) \). We have given two ways to realize the spaces \( D_\chi \). First we have a description of \( D_\chi \) as a space of homogeneous functions on \( \mathbb{C}^2 \cong \mathbb{R}^4 \), which are smooth off of the origin. We can then define an inner product by restricting \( F, G \) to the 3-sphere, \( |z_1|^2 + |z_2|^2 = 1 \), and taking the usual inner product on \( L^2(S^3, dV) \), where \( dV \) is the normalized volume element induced from \( \mathbb{R}^4 \).

\[ \langle F, G \rangle_{i\rho,n} = \int_{S^3} F \cdot \overline{G} \, dV \quad (4.32) \]

When we realize \( D_\chi \) as functions on \( \mathbb{R}^2 \), then the ordinary \( L^2(\mathbb{R}^2) \) inner product is used.

\[ (f, g)_{i\rho,n} = \int_{\mathbb{R}^2} f \cdot \overline{g} \, dx \, dy \quad (4.33) \]

It can be shown that both of these norms are unitary with respect to the action of \( SL(2, \mathbb{C}) \) defined above. It is also possible to show that these norms are equivalent. That is, the densely defined operation of restriction to the plane \( z_2 = 1 \) provides a unitary isomorphism from the completion of \( D_\chi \) in the first realization with respect to the first norm, to the completion of the second realization with respect to the second norm.

- Complementary Series. Here we take \( \lambda = \rho - 1, \mu = \rho - 1 \), where \(-1 < \rho < 1, \rho \neq 0\).
We will only describe the norms for the realization of $D_\chi$ as a space of functions on $\mathbb{R}^2$. Then up to a scalar, the norm may be given then as,

$$ (f, g)_{\rho,0} = \int_C \int_C |z_1 - z_2|^{-2\rho - 2} f(z_1) \cdot \overline{g}(z_2) dz_1 d\bar{z}_1 dz_2 d\bar{z}_2 $$

(4.34)

From these norms, we can define Sobolev norms from the vector fields $X \in \mathfrak{sl}(2, \mathbb{C})$ in the usual way,

$$ \|f\|_k = \left( \sum_{|\alpha| \leq k} \|X^\alpha f\|^2 \right)^{1/2} $$

(4.35)

The family of Sobolev norms then generates the topology on $D_\chi$. In fact, $D_\chi = \bigcap_{k>0} W^k(\mathcal{H}_{\rho,n})$.

We may also consider the Sobolev norms which measure regularity only with respect to certain subgroups. In particular, we consider two subgroups of $SL(2, \mathbb{C})$ that are both isomorphic to $SL(2, \mathbb{R})$. First we consider $SL(2, \mathbb{R})$ as a subgroup of $SL(2, \mathbb{C})$ in the obvious way. We may also form a copy of $SL(2, \mathbb{R})$ by exponentiating the algebra generated by the elements $\{V', X, U'\}$. We will denote these two subgroups simply as $SL_2$ and $SL'_2$.

Each of these subgroups then gives rise to a norm that measures regularity with respect to the generators of that group. The norms are then defined by,

$$ \|f\|_{W^k(SL_2)} = \left( \sum_{|\alpha| \leq k} \|X^\alpha f\|^2 \right)^{1/2}, \quad \text{where } X \in \mathfrak{sl}_2 $$

(4.36)

$$ \|f\|_{W^k(SL'_2)} = \left( \sum_{|\alpha| \leq k} \|X^\alpha f\|^2 \right)^{1/2}, \quad \text{where } X \in \mathfrak{sl}'_2 $$

(4.37)

These latter norms will play a critical role in establishing triviality of the first $C^\infty$ cohomology for the action of $\mathfrak{g}_\alpha = \{ (0 \ c) \mid c \in \mathbb{C} \}$.

### 4.5.3 The Equation $UF = G$.

Let $U_c \in \mathfrak{g}_\alpha$, that is $U_c = (0 \ c \ 0)$, $c \in \mathbb{C}$. Now, let $k_\theta = (e^{i\theta/2} \ 0 \ e^{-i\theta/2})$. Then any arbitrary $U_c = (0 \ c \ 0)$, $c \in \mathbb{C}$, can be conjugated to $U_1 = U = (0 \ 1 \ 0)$ by some $k_\theta \in \mathfrak{m}$. Hence it suffices to just consider the equations of the form $UF = G$ for $U = (0 \ 1 \ 0)$.
The action of $U$ on $C^\infty(\mathcal{H}_{\rho,n})$ takes the form,

$$\pi(U) \cdot F(z_1, z_2) = \frac{d}{dt} \pi(\exp tU) \cdot F(z_1, z_2)|_{t=0} = \frac{d}{dt} F(z_1 - tz_2, z_2)|_{t=0}$$

$$= -x_2 \frac{\partial F}{\partial x_1} - y_2 \frac{\partial F}{\partial y_1}$$

(2.1)

Note: If we then restrict to the complex line $z_2 = 1$, and write $z = x + iy$, then we get that $U = -\frac{\partial}{\partial x}$. Thus any solution to $U f = g$ must have solution of the form, $f(z) = \pm \int g(t + z) \, dt$.

### 4.5.4 Invariant distributions

We describe the $U$-invariant distributions on $C^\infty(\pi_\chi)$. Recall that $D \in C^\infty(\pi_\chi)'$ is $U$-invariant if $U \cdot D(f) = -D(U f) = 0$ for all $f \in C^\infty(\pi_\chi)$.

It is immediate from (2.1) that the dirac point mass at $(1, 0)$ is a $U$-invariant distribution, as we have $U \cdot \delta(1,0)(F) = -\delta(1,0)(-x_2 \frac{\partial F}{\partial x_1} - y_2 \frac{\partial F}{\partial y_1}) = 0$. It is clearly continuous for the topology on $C^\infty(\pi_\chi)$.

We notice that given any other point $(p, 0) \neq (0, 0)$, the homogeneity condition implies that $\delta_{(p,0)}(F) = F(p, 0) = p^\lambda \bar{p}^\mu F(1, 0) = p^\lambda \bar{p}^\mu \delta_{(1,0)}(F)$. Hence any $\delta_{(p,0)}$ is just a scalar multiple of $\delta_{(1,0)}$. Alternatively, we could define $\tilde{\delta}_0$ by $\tilde{\delta}_0(f) = \tilde{f}(0)$, where we recall that $\tilde{f}$ is the inversion of $f$. Then $\tilde{\delta}_0$ is a $U$-invariant distribution.

Of course, the general element $U_\alpha \in g_\alpha$ acts by,

$$\pi(U_\alpha) \cdot F(z_1, z_2) = \frac{d}{dt} \pi(\exp tU_\alpha) \cdot F(z_1, z_2)|_{t=0}$$

$$= \frac{d}{dt} F(z_1 - tcz_2, z_2)|_{t=0}$$

$$= (-c_1x_2 + c_2y_2) \frac{\partial F}{\partial x_1} + (c_1y_2 - c_2x_2) \frac{\partial F}{\partial y_1}$$

(2.2.1)

Thus we clearly have,

**Lemma 4.18.** Let $U_\alpha \in g_\alpha$. Then $U_\alpha \cdot \delta_{(1,0)}(F) = 0$. 

Proof. \( U_k \cdot \delta_{(1,0)} (F) = - \delta_{(1,0)} (c_1 x_2 + c_2 y_2 \frac{\partial F}{\partial x_1} + (-c_1 y_2 + c_2 x_2) \frac{\partial F}{\partial y_1}) = 0. \)

A second class of \( U \)-invariant distributions can be obtained as follows. Let \( D_y (f) = \int_{-\infty}^{\infty} f(t + iy, 1) dt \). Then for each \( y \in \mathbb{R} \), we get a \( U \)-invariant distribution. We first note that, for any homogeneous function \( f \in C^\infty (\mathbb{R}) \), we have \( \lim_{|z| \to \infty} f(z, 1) = z^{-1} \). For the principal series, we have that \( n_1 = \frac{1}{2} (n + i\rho) \), \( n_2 = \frac{1}{2} (-n + i\rho) \), while for the complementary series we have \( n_1 = n_2 = \rho \), where \(-1 < \rho < 0\). So since \( z^{n_1 - 1} z^{n_2 - 1} = |z|^{n_1 + n_2 - 2} e^{(n_1 - n_2) \arg z} \), we have that

\[
\begin{align*}
  z^{n_1 - 1} z^{n_2 - 1} &= |z|^{1/2} e^{i(n - \rho) \arg z} \quad \text{for the principal series} \\
  z^{n_1 - 1} z^{n_2 - 1} &= |z|^{2\rho - 2} \quad \text{for the complementary series}
\end{align*}
\]

Hence, \( \lim_{|z| \to \infty} f(z, 1) = z^{n_1 - 1} z^{n_2 - 1} f(1, z^{-1}) = 0 \), in both the principal and complementary series.

\[
U \cdot D_y (f) = -D_y (Uf)
\]

\[
= -\int_{-\infty}^{\infty} U \cdot f(t + iy, 1) dt
\]

\[
= \int_{-\infty}^{\infty} \frac{\partial f}{\partial x} (t + iy, 1) dt
\]

\[
= \lim_{a, b \to -\infty} \{ f(a + iy, 1) - f(-b + iy, 1) \} = 0
\]

Or alternatively, we could just define \( D_y (\phi) = \int_{-\infty}^{\infty} \phi(t + iy) dt \).

Now since \( g \) is in \( C^\infty (\pi) \), we have an asymptotic expansion

\[
g(z) \sim z^{n_1 - 1} z^{n_2 - 1} \sum_{j,k=0}^{\infty} a_{j,k} z^{-j} z^{-k} \text{ as } |z| \to \infty.
\]

We get then that \( |g(z)| \sim C \cdot |z|^{-2} \) as \( |z| \to \infty \), for some constant \( C \geq 0 \). Hence by the quotient test, we have that \( \int_{-\infty}^{\infty} g(t + iy) dt \) converges absolutely.

If \( \{ \phi_m(z) \} \) converges to zero in the topology of \( C^\infty (\pi) \), then \( \{ \phi_m(z) \} \) converges to zero uniformly on every finite region, and hence \( \{ D_y (\phi_m) \} \) converges to zero whenever \( \{ \phi_m(z) \} \) does. Thus each \( D_y \) is a \( U \)-invariant distribution.

### 4.5.5 The Cohomological Equation
We can now define the formal solution to the cohomological equation $Uf = g$. Let $g \in C^\infty(\pi)$ be given. We could then define two formal solutions, $f_+$ and $f_-$, by

$$f_+(x + iy) = \int_0^\infty g(t + x + iy)dt \quad \text{and} \quad f_-(x + iy) = -\int_{-\infty}^0 g(t + x + iy)dt$$

Again, by the quotient test, we have that $f_+$ and $f_-$ both converge absolutely. We also get that $\lim_{|t| \to \infty} g(t + iy) = 0$.

We first suppose that $g \in C^\infty_c(\mathbb{R}^2)$ is any compactly supported smooth function. We note that $\hat{g}(z) = z^\lambda \bar{z}^\mu g(z^{-1})$ is then also smooth by chain rule. Hence $C^\infty_c(\mathbb{R}^2) \subset C^\infty(\pi)$. Now, the topology of $C^\infty(\pi)$, when restricted to $C^\infty_c(\mathbb{R}^2)$, is just the usual uniform topology, and since $C^\infty_c(\mathbb{R}^2)$ is dense in $C^\infty_0(\mathbb{R}^2)$ and $C^\infty(\pi) \subset C^\infty_0(\mathbb{R}^2)$, we get that $C^\infty_c(\mathbb{R}^2)$ is dense in $C^\infty(\pi)$. The same would apply to $C^r$ rather than $C^\infty$ so that we may consider results of lower regularity.

We note that for any $g \in C^r_c(\mathbb{R}^2)$, we automatically have that $\tilde{\delta}_0(g) = 0$. Also, we have the following lemma.

**Lemma 4.19.** Let $g \in C^r_c(\mathbb{R}^2)$ be such that $D_y(g) = 0$ for all $y \in \mathbb{R}$. Then there exists an $f \in C^r_c(\mathbb{R}^2)$ such that $Uf = g$.

**Proof.** Let $g \in C^r_c(\mathbb{R}^2)$ be such that $D_y(g) = 0$ for all $y \in \mathbb{R}$. Then define $f(z) = \int_0^\infty g(t + z)dt$. Since $f(z) = \int_0^\infty g(t + z)dt = -\int_{-\infty}^0 g(t + z)dt$ it is clear that $f(z) = 0$ for all $|z| > N$. Also, since $g$ is compactly supported, the integrals are not truly improper, and so we can differentiate in $z, \bar{z}$ under the integral sign. So $f$ must have at least the same regularity as $g$.

Now, by Minkowski’s inequality for integrals we have the following,

$$\|f\|_s = \| \int_0^C g(t + z)dt \|_s$$

$$= \| (I - \Delta g)^{s/2} \int_0^C g(t + z)dt \|_0$$

$$= \| \int_0^C (I - \Delta g)^{s/2}g(t + z)dt \|_0$$

$$\leq \int_0^C \| (I - \Delta g)^{s/2}g(t + z)dt \|_0$$

$$= \int_0^C \| g(t + z)\|_s dt = C\|g\|_s$$
Of course, the constant $C$ depends upon $g$ in an essential way, and there is no hope of finding a constant which will work for all $g \in C_c^0(\mathbb{R}^2)$.

However, if we could prove an inequality of the type $\|f\|_s \leq C\|Uf\|_{s+m}$ for all $f \in C_c^0(\mathbb{R}^2)$ and some $C > 0$ (with $s + m < r$), then $U$ could be inverted on $C_c^0(\mathbb{R}^2)$, and so we could apply Hahn-Banach theorem to obtain a solution $f \in W^s(\pi)$ for a given $g \in W^{s+m}(\pi)$ to $Uf = g$, which satisfies the estimate $\|f\|_s \leq C\|g\|_{s+m}$. We could then integrate to obtain a global solution on $G/\Gamma$.

At present, we cannot answer whether such an inequality is possible for individual generators $U \in \text{Lie}(N)$, when we use the Sobolev norm which comes from the whole group.

### 4.5.6 Rigidity of One Cocycles

Now we discuss the case of cocycles over the full $N = \exp g_{\alpha}$ action on $SL(2,\mathbb{C})/\Gamma$. Here we consider smooth cocycles $\beta : N \times SL(2,\mathbb{C})/\Gamma \to \mathbb{R}^k$, where $N = \exp g_{\alpha}$ and acts via left multiplication on $SL(2,\mathbb{C})/\Gamma$. Or, alternatively, by writing $f(x) \cdot h = f(hx)$, we will consider smooth cocycles to be smooth functions $\beta : N \to C^\infty(L^2((SL(2,\mathbb{C})/\Gamma))$ which satisfy $\beta(n_1 + n_2) = \beta(n_2) + \beta(n_1) \cdot n_2$.

Again, let $\omega(v) = \frac{d}{dt}\beta(\exp tv)|_{t=0}$ be the infinitesimal generator for $\beta$ (provided it exists). Then the cocycle equation implies that $\omega$ is a closed form on the orbit foliation. We can also recover $\beta$ from $\omega$ by $\beta(\exp X) = \int_0^1 \omega(X) \cdot \exp tX \, dt$.

We also have that the infinitesimal version of the cohomology equation is $\omega = \eta - dP$. Thus the $C^\infty$ or $C^r$ first cohomology group is trivial if for every $C^\infty$ or $C^r$ one form $\omega$ is exact via a function $P \in C^\infty$ or $C^r$.

We want to consider cocycles $\beta : N \to L^2(SL(2,\mathbb{C})/\Gamma)$ and in particular, cocycles which map into the function spaces $W^{2,p}(SL(2,\mathbb{C})/\Gamma)$, or $C^r(L^2(SL(2,\mathbb{C})/\Gamma))$. Since $L^2(SL(2,\mathbb{C})/\Gamma) = \int_{\mathcal{H}} m_{\rho,n} \cdot \mathcal{H}_{\rho,n}$, we can restrict our attention to cocycles $\beta : N \to C^\infty(\mathcal{H}_{\rho,n})$, which map into irreducible components. More specifically, we can consider their infinitesimal generators $\omega : \text{Lie}(N) \to C^\infty(\mathcal{H}_{\rho,n})$.

Now each $\mathcal{H}_{\rho,n}$ can be realized as a subspace of a function space of $C$ or $\mathbb{R}^2$. In fact, a dense subset of $\mathcal{H}_{\rho,n}$ is $D_\lambda$. Then any basis $U_1, U_2$ which we take for $g_{\alpha}$ define linear coordinates on $\mathbb{R}^2$. That is, we can take $U_1 = \frac{\partial}{\partial x}$ and $U_2 = \frac{\partial}{\partial y}$. And we get that if we write $\omega = f dx + g dy$, then $\omega(U_2) \cdot U_1 - \omega(U_1) \cdot U_2 = 0$ becomes the usual condition $\frac{df}{dy} = \frac{dg}{dx}$. So $\omega$ is a closed one form on $\mathbb{R}^2$ in the usual sense, in each irreducible representation.

We begin with a lemma.
**Lemma 4.20.** Suppose that \( \omega = f \, dx + g \, dy \), where \( f, g \in C'(H_{\rho, n}) \). If \( d\omega = 0 \), then we must have that \( \tilde{\delta}_0(f) = \tilde{\delta}_0(g) = 0 \).

**Proof.** This is just a simple consequence of the fact that we must have \( \frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \). Now \( f, g \in C'(H_{\rho, n}) \), so they have asymptotic expansions,

\[
f(z) \sim z^\lambda z^\mu \sum_{j,k=0}^r a_{j,k}z^{-j}z^{-k} \text{ as } |z| \to \infty
\]

\[
g(z) \sim z^\lambda z^\mu \sum_{j,k=0}^r b_{j,k}z^{-j}z^{-k} \text{ as } |z| \to \infty
\]

We can differentiate the expansions term by term, and we get,

\[
\frac{\partial f}{\partial y} = i\left( \frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} \right)f
\]

\[
\sim \sum_{j,k=0}^r a_{j,k}i\left( \frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} \right)z^\lambda z^\mu z^{-j}z^{-k}
\]

\[
= \sum_{j,k=0}^r ia_{j,k}[(\lambda - j)z^\lambda - (\mu - k)z^\lambda]z^{-j}z^{-k}
\]

while,

\[
\frac{\partial g}{\partial x} = \left( \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \right)g
\]

\[
\sim \sum_{j,k=0}^r b_{j,k}\left( \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \right)z^\lambda z^\mu z^{-j}z^{-k}
\]

\[
= \sum_{j,k=0}^r b_{j,k}[(\lambda - j)z^\lambda + (\mu - k)z^\lambda]z^{-j}z^{-k}
\]

So \( \frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \) implies that \( b_{0,0} = ia_{0,0} \) and that \( b_{0,0} = -ia_{0,0} \). Thus \( b_{0,0} = a_{0,0} = 0 \), and the lemma is established.

The lemma says that \( \omega \) is actually a closed one form on the one point compactification of \( \mathbb{R}^2 \), which vanishes at \( \infty \). But then, if \( d\omega = 0 \), we also get that \( \int_C f \, dx + g \, dy = 0 \) for any curve \( C \) of the form \( C = (-\infty, \infty) \times \{y_0\} \) or \( C = \{x_0\} \times (-\infty, \infty) \). But by definition, this says that \( D_y(f) = 0 \) for all \( y \in \mathbb{R} \), and \( D_x(g) = 0 \) for all \( x \in \mathbb{R} \). We therefore
have that if \( \omega = f \, dx + g \, dy \), then the \( \frac{\partial}{\partial x} \)-invariant distributions vanish for \( f \), and the \( \frac{\partial}{\partial y} \)-invariant distributions vanish for \( g \).

Every closed one form on \( \mathbb{R}^2 \) is exact, and so we can find a solution \( P \) to \( dP = \omega \). Since \( \omega \) is a one form (with values in \( \mathbb{R} \) or \( \mathbb{C} \)), \( P \) must be a function \( P : \mathbb{R}^2 \to \mathbb{R} \) or \( \mathbb{C} \).

We now need to address the regularity properties of \( P \), that is, we need to determine if \( P \in C^\infty(\mathcal{H}_{\rho,n}) \) or \( W^s(\mathcal{H}_{\rho,n}) \).

We define the function \( P(x, y) = - \int_1^\infty xf(tx, ty) + yg(tx, ty) \, dt \). Then we have that \( dP = f \, dx + g \, dy = \omega \). In fact,

\[
\frac{\partial P}{\partial x}(x, y) = -\frac{\partial}{\partial x} \int_1^\infty xf(tx, ty) + yg(tx, ty) \, dt
= -\int_1^\infty \frac{\partial}{\partial x} (xf(tx, ty) + yg(tx, ty)) \, dt
= -\int_1^\infty x f' \, dt + f(tx, ty) + ty \frac{\partial f}{\partial y}(tx, ty) \, dt
= -\int_1^\infty \frac{d}{dt} (tf(tx, ty)) \, dt
= f(x, y) - \lim_{t \to \infty} tf(tx, ty)
= f(x, y) \quad \text{(since } \lim_{t \to \infty} \frac{f(tx, ty)}{(tx + ity)^\lambda (tx - ity)^\mu} = 0)\]

Of course, to justify differentiation under the integral sign, we must prove that

\[
\int_1^\infty (tx) \frac{\partial f}{\partial x}(tx, ty) + f(tx, ty) + (ty) \frac{\partial g}{\partial x}(tx, ty) \, dt
\]

is a uniformly convergent integral. From above, however, we note that

\[
| \int_r^\infty (tx) \frac{\partial f}{\partial x}(tx, ty) + f(tx, ty) + (ty) \frac{\partial g}{\partial x}(tx, ty) \, dt | = |rf(rx, ry)| \quad (4.41)
\]

We have \( f(z) \sim C \cdot z^\lambda z^\mu \) for \( |z| \to \infty \), so for any fixed \( r > 0 \) we can always make \( |rf(rx, ry)| \) uniformly small for any \( (x, y) \) outside of \( B_r(0) \). Therefore we have, for every \( (x, y) \neq 0 \), that

\[
-\frac{\partial}{\partial x} \int_1^\infty xf(tx, ty) + yg(tx, ty) \, dt = -\int_1^\infty (tx) \frac{\partial f}{\partial x}(tx, ty) + f(tx, ty) + (ty) \frac{\partial g}{\partial x}(tx, ty) \, dt
\]

So that \( \frac{\partial P}{\partial x}(x, y) = f(x, y) \) on \( \mathbb{R}^2 \setminus (0, 0) \). And since \( P \) and \( f \) are continuous, we must have the equality on all of \( \mathbb{R}^2 \). A similar computation also shows that \( \frac{\partial P}{\partial y} = g \). It is then clear that \( P \in C'(\mathbb{R}^2) \) provided \( f, g \in C'(\mathcal{H}_{\rho,n}) \).
We remark at this point that we could just have defined $P(x,y) = \int_0^1 xf(tx,ty) + yg(tx,ty) dt$, which is the usual way one defines the primitive of a closed one form on $\mathbb{R}^2$. Since the integral is not indefinite, differentiation under the integral sign is automatic, and we would immediately get that $P \in C^r(\mathbb{R}^2)$. Being able to write $P = -\int_1^\infty xf(tx,ty) + yg(tx,ty) dt$, however, is what will allow us to explore the $L^2(\mathbb{R}^2)$ and $C^r(\mathcal{H}_{\rho,n})$ properties of $P$.

**Lemma 4.21.** Suppose $f, g \in C^2(\mathcal{H}_{\rho,n})$, where $\mathcal{H}_{\rho,n}$ belongs to the principal series, then $P = -\int_1^\infty xf(tx,ty) + yg(tx,ty) dt$ exists and belongs to $\mathcal{H}_{\rho,n}$.

**Proof.** We know from above that $P(x,y) = -\int_1^\infty xf(tx,ty) + yg(tx,ty) dt$ exists, and is $C^r(\mathbb{R}^2)$, provided that $f, g \in C^r(\mathcal{H}_{\rho,n})$. We will use the notation $\phi_t(x,y) = \phi(tx,ty)$. We have,

$$\|P\|_{L^2(\mathbb{R}^2)} = \|\int_1^\infty xf + yg dt\|_{L^2(\mathbb{R}^2)}$$

$$\leq \int_1^\infty \|(xf + yg)\|_{L^2(\mathbb{R}^2)} dt \quad \text{(by Minkowski’s Inequality for integrals)}$$

$$\leq \int_1^\infty \|xf\|_{L^2(\mathbb{R}^2)} + \|yg\|_{L^2(\mathbb{R}^2)} dt$$

We can estimate $\|xf\|_{L^2(\mathbb{R}^2)}$ as follows. First we perform a simple change of variables to see that,

$$\|xf\|_{L^2(\mathbb{R}^2)} \leq \left[ \int_{\mathbb{R}^2} \left| \frac{1}{t}(txf(tx,ty)) \right|^2 dxdy \right]^{1/2}$$

$$= \left[ \frac{1}{t^4} \int_{\mathbb{R}^2} \left| (txf(tx,ty)) \right|^2 \det \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} dxdy \right]^{1/2}$$

$$= \frac{1}{t^2} \left[ \int_{\mathbb{R}^2} \left| xf(x,y) \right|^2 dxdy \right]^{1/2}$$

$$= \frac{1}{t^2} \|xf\|_{L^2(\mathbb{R}^2)}$$

We now get the estimate for the integral as,

$$\int_1^\infty \|xf\|_{L^2(\mathbb{R}^2)} dt \leq \int_1^\infty \frac{1}{t^2} \|xf\|_{L^2(\mathbb{R}^2)} = \|xf\|_{L^2(\mathbb{R}^2)}$$
But we know that $|f(z)| \sim C|z|^{-3}$, and so we clearly have that $\|xf\|_{L^2(\mathbb{R}^2)} < \infty$. An entirely similar argument then shows that $\|yg\|_{L^2} < \infty$. Hence $P \in L^2(\mathbb{R}^2)$.

This lemma uses some weak estimates, and does not treat the complementary series. In order to obtain smoothness results, we must make finer use of the available characterization of smooth vectors. Indeed, we know that functions in $C^\infty(\mathcal{H}_{p,n})$ are characterized by the fact that they are $C^\infty$ at $\infty$. We get similarly that

$$
-\int_1^\infty \frac{1}{2} (z + \bar{z}) f(tz) dt \sim \frac{1}{2} (z + \bar{z}) z^{\lambda \bar{z}^\mu} \sum_{j,k=0}^\infty a_{j,k} z^{-j} \bar{z}^{-k} t^{\lambda + \mu - j - k} dt
$$

so that

$$
-\int_1^\infty \frac{1}{2i} (z + \bar{z}) g(tz) dt \sim \frac{1}{2i} (z - \bar{z}) z^{\lambda \bar{z}^\mu} \sum_{j,k=0}^\infty b_{j,k} z^{-j} \bar{z}^{-k} \int_1^\infty t^{\lambda + \mu - j - k} dt
$$

We get similarly that

$$
-\int_1^\infty \frac{1}{2i} (z - \bar{z}) g(tz) dt \sim \frac{1}{2i} (z - \bar{z}) z^{\lambda \bar{z}^\mu} \sum_{j,k=0}^\infty b_{j,k} z^{-j} \bar{z}^{-k} \int_1^\infty t^{\lambda + \mu - j - k} dt
$$

which implies that

$$
P(z) \sim z^{\lambda \bar{z}^\mu} \sum_{j,k=0}^\infty \left( \frac{a_{j,k} - ib_{j,k}}{2(\lambda + \mu - j - k) z^{1-j} \bar{z}^{-k}} + \frac{a_{j,k} + ib_{j,k}}{2(\lambda + \mu + 1 - j - k) z^{-j} \bar{z}^{1-k}} \right)
$$

$$
= z^{\lambda \bar{z}^\mu} \sum_{j,k=0}^\infty \left( \frac{(a_{j+1,k} + a_{j,k+1}) - i(b_{j+1,k} - b_{j,k+1})}{2(\lambda + \mu - j - k)} z^{-j} \bar{z}^{-k} + \frac{a_{0,0} - ib_{0,0}}{2(\lambda + \mu + 1)} z + \frac{a_{0,0} + ib_{0,0}}{2(\lambda + \mu + 1)} \bar{z} \right)
$$
but \( a_{0,0} = b_{0,0} = 0 \), so we just get that

\[
P(z) \sim z^\lambda z^\mu \sum_{j,k=0}^\infty \frac{(a_{j+1,k} + a_{j,k+1}) - i(b_{j+1,k} - b_{j,k+1})}{2(\lambda + \mu - j - k)} z^{-j} z^{-k}
\]

But how do we prove that \( P \in C^\infty(\mathcal{H}_{\rho,n}) \), if we only know that \( f, g \in P \in C^\infty(\mathcal{H}_{\rho,n}) \)? We already know that \( P \in C^\infty(\mathbb{R}^2) \) provided \( f, g \in C^\infty(\mathcal{H}_{\rho,n}) \). To prove that \( P \in C^\infty(\mathcal{H}_{\rho,n}) \), it suffices to show that \( P \) and all its derivatives possess asymptotic expansions in a neighborhood of \( \infty \). Since \( \frac{\partial P}{\partial x} = f \in C^\infty(\mathcal{H}_{\rho,n}) \), and \( \frac{\partial P}{\partial y} = g \in C^\infty(\mathcal{H}_{\rho,n}) \), it suffices to just show that the asymptotic expansion for \( P \) above holds.

**Theorem 4.16.** Let \( f, g \in C^\infty(\mathcal{H}_{\rho,n}) \), then \( P(x, y) = -\int_1^\infty xf(tx, ty) + yg(tx, ty) \) exists, belongs to \( C^\infty(\mathcal{H}_{\rho,n}) \) and satisfies \( UP = f \) and \( U' P = g \).

**Proof.** To make the above argument precise, we only need prove that if \( f \) has an asymptotic expansion as \( f \sim z^\lambda z^\mu \sum_{j,k=0}^\infty a_{j,k} z^{-j} z^{-k} \), where \( a_{0,0} = 0 \), then it is possible to integrate the expansion as follows,

\[
- \int_1^\infty f(tz) dt \sim z^\lambda z^\mu \sum_{j,k=0}^\infty \frac{a_{j,k}}{\lambda + \mu + 1 - j - k} z^{-j} z^{-k}
\]

But

\[
z^\lambda z^\mu \sum_{j,k=0}^N \frac{a_{j,k}}{\lambda + \mu + 1 - j - k} z^{-j} z^{-k} = - \int_1^\infty (tz)^\lambda (tz)^\mu \sum_{j,k=0}^N a_{j,k} (tz)^{-j} (tz)^{-k} dt
\]

So that for any \( \epsilon > 0 \),

\[
|z|^n \left| \int_1^\infty f(tz) dt + z^\lambda z^\mu \sum_{j,k=0}^n \frac{a_{j,k}}{\lambda + \mu + 1 - j - k} z^{-j} z^{-k} \right|
\]

\[
\leq |z|^n \int_1^\infty \left| f(tz) - (tz)^\lambda (tz)^\mu \sum_{j,k=0}^n a_{j,k} (tz)^{-j} (tz)^{-k} \right| dt
\]

\[
\leq |z|^n \int_1^\infty (\delta \cdot |tz|^{-n + \lambda + \mu}) dt \quad \text{(for } |z| \text{ large enough)}
\]

\[
= \left| \frac{\delta}{\lambda + \mu + 1 - n} \right| |z|^{\lambda + \mu} < \epsilon \quad \text{(for } |z| \text{ large enough)}
\]

In the above discussion, we could have easily replaced \( C^\infty(\mathcal{H}_{\rho,n}) \) with \( C^r(\mathcal{H}_{\rho,n}) \) and obtained similar results. We summarize this as follows,
COROLLARY 4.1. Let \( f, g \in C'(\mathcal{H}_{\rho,n}) \), then \( P(x,y) = -\int_1^\infty xf(tx,ty) + yg(tx,ty) dt \) exists and belongs to \( C^{r-1}(\mathcal{H}_{\rho,n}) \).

We note that although we will always have \( P \in C^{r+1}(\mathbb{R}^2) \) whenever \( f, g \in C'(\mathcal{H}_{\rho,n}) \), it's inversion will lose a derivative. That is, the asymptotic expansion of the integrand \( xf(tx,ty) + yg(tx,ty) \) involves a finite sum which loses the highest order term.

4.5.7 Global Results

We now discuss how to obtain a global solution from the solutions which exist in each irreducible component of the decomposition of \( L^2(SL(2,\mathbb{C})/\Gamma) \).

Recall that if \( \omega = f \, dx + g \, dy \) is a closed form, then all \( \frac{\partial}{\partial x} \)-invariant distributions vanish on \( f \), and all \( \frac{\partial}{\partial y} \)-invariant distributions vanish on \( g \). We then get that \( P \) is the simultaneous solution to the single operator equation \( \frac{\partial}{\partial x} P = f \), and \( \frac{\partial}{\partial y} P = g \). We discussed that for single generators, the cohomology equation \( U\varphi = \psi \) would have a global solution provided we could prove inequalities of the form \( \|\varphi\|_s \leq C_{t,\rho,n}\|\psi\|_s \), where the constants \( C_{t,\rho,n} \) are bounded in \( \rho \) and \( n \). In fact, it would suffice to have such an estimate for only one element in \( Lie(N) \). Unfortunately, we do not have such an estimate with respect to the full Sobolev norm \( \|\varphi\|_{\rho,n,t} \).

We do, however, have the weaker estimates for the Sobolev norms which measure regularity with respect to the subgroups \( SL_2 \) and \( SL_2' \) separately. In particular, on each irreducible unitary representation for \( SL(2,\mathbb{C}) \), we have the norms,

\[
\|f\|_{W^k_{\rho,n}(SL_2)} \quad \text{and} \quad \|f\|_{W^k_{\rho,n}(SL_2')} \n\]

Now each irreducible unitary representation \( \mathcal{H}_{\rho,n} \) of \( SL(2,\mathbb{C}) \) can be decomposed as a direct integral of irreducible unitary representations for \( SL_2 \) and \( SL_2' \). The norms then decompose as,

\[
\|f\|^2_{W^k_{\rho,n}(SL_2)} = \int m(\mu)\|f\|^2_{W^k_{\rho}(\mathcal{H}_\mu)} d\mu \quad \text{and} \quad \|f\|^2_{W^k_{\rho,n}(SL_2')} = \int m(\mu')\|f\|^2_{W^k_{\rho}(\mathcal{H}_{\mu'})} d\mu' \n\]

Denote by \( \Box_{SL_2} \) and \( \Box_{SL_2'} \) the Casimir operators associated to the subgroups \( SL_2 \) and \( SL_2' \), respectively. Then provided that each operator \( \Box_{SL_2} \) and \( \Box_{SL_2'} \) has a 'spectral gap', then we will be able to estimate the above norms. We recall the following theorem of [4].
THEOREM 4.17 (FLAMINIO-FORNI). If there exists a $\mu_0 > 0$ such that the spectrum of the Casimir $\sigma(\Box) \cap (0, \mu_0) = \emptyset$, then the following holds. Let $s > 1$, then there exists a constant $C_{s,t}$ such that, for all $g \in W^s(\mathcal{H})$,
- if $t < -1$, and $g$ has no component on the trivial sub-representation of $\mathcal{H}$, or
- if $0 < t < s - 1$ and $D(g) = 0$ for all $D \in T^s(\mathcal{H})$,
then the equation $Uf = g$ has a solution $f \in W^t(\mathcal{H})$, which satisfies the Sobolev estimate $\|f\|_t \leq C_{s,t}\|g\|_s$. Solutions are unique modulo the trivial sub-representation provided $t > 0$.

We notice that, since $U \in s\mathfrak{l}_2$, $U$ must act on $f \in \mathcal{H}_{\rho,n}$ through the decomposition as $Uf = \int \rho(m) U f_\mu dm$. By Moore’s Ergodicity theorem, if $U f_\mu = 0$ for some $\mu$, then $f_\mu = 0$. This allows us to state the following theorem.

THEOREM 4.18. Let $\mathcal{H}_{\rho,n}$ be in the principal series, and let $f, g \in C^\infty(\mathcal{H}_{\rho,n})$. If there exists a $P \in C^\infty(\mathcal{H}_{\rho,n})$ that satisfies $UP = f$ and $U P = g$, then for any $1 \geq k$ there exists a constants $C_k, C_k' > 0$ such that $P$ satisfies the uniform estimates,
- $\|P\|_{W^k_{\rho,n}(\mathfrak{sl}_2)} \leq C_k \|f\|_{W^k_{\rho,2}(\mathfrak{sl}_2)}$ and
- $\|P\|_{W^k_{\rho,n}(\mathfrak{sl}_2^*')} \leq C_k' \|g\|_{W^k_{\rho,2}(\mathfrak{sl}_2^*)}$

The constants are independent of the representation.

Proof. We first notice that every member of the principal series of $SL(2, \mathbb{C})$ is a tempered representation, and it is easily seen that the restriction of that representation to either of the subgroups $SL_2$ or $SL_2'$, is itself a tempered representation, and so by definition, a.e. component of the direct integral decomposition must also be tempered. Since the complementary series representations of $SL(2, \mathbb{R})$ are not tempered, they cannot appear in the decomposition of $\mathcal{H}_{\rho,n}$ under the restriction of the action to either $SL_2$ or $SL_2'$.

Decompose $U, f, P, \mathcal{H}_{\rho,n}$ under the the restriction to the subgroup $SL_2$. Since $P, f \in C^\infty(\mathcal{H}_{\rho,n})$, we must have $P_\mu, f_\mu \in C^\infty(\mathcal{H}_\mu)$ for a.e. $\mu$. Let $s > 1$, then for all $U_\mu$-invariant distributions $D _\mu \in W^{-s}(\mathcal{H}_\mu)$, we must have $D _\mu (f_\mu) = D _\mu (UP) = -U_\mu \cdot D _\mu (f_\mu) = 0$ for a.e. $\mu$. By [4], for any $t < s - 1$, for a.e. $\mu$, there exists an $\alpha_\mu \in W^t(\mathcal{H}_\mu)$, s.t. $U_\mu \alpha_\mu = f_\mu$ and a constant $C_{s,t} > 0$, independent of $\mu$, s.t. $\|\alpha_\mu\|_{W^t(\mathcal{H}_\mu)} \leq C_{s,t} \|f_\mu\|_{W^t(\mathcal{H}_\mu)}$. But by Moore’s Ergodicity theorem, $P_\mu = \alpha_\mu$ for a.e. $\mu$. In summary,
because of uniqueness of solutions on components, any solution \( P \in C^\infty(\mathcal{H}_{\rho,n}) \) to \( U P = f \), must satisfy uniform estimates in a.e. component. So that we then get,

\[
\|P\|_{W^k_{\rho,n}(SL_2)}^2 = \int_{\oplus_m} m(\mu) \|P_\mu\|_{W^k(\mathcal{H}_\mu)}^2 d\mu
\]

\[
\leq C_{s,t} \int_{\oplus_m} m(\mu) \|f_\mu\|_{W^{k+2}(\mathcal{H}_\mu)}^2 d\mu = C_{s,t} \|f\|_{W^{k+2}_{\rho,n}(SL_2)}^2
\]

The exact same argument applies to the estimates for \( SL_2' \).

The fact that the constants provide uniform estimates over all principal series representations allows us to obtain global estimates as follows,

**Theorem 4.19.** Let \( f, g \in C^\infty(L^2(SL(2,\mathbb{C})/\Gamma)) \), be such that \( U' f = U g \). If \( f_{\rho,0} = g_{\rho,0} = 0 \) for every element of the complementary series that appears in the decomposition of \( L^2(SL(2,\mathbb{C})/\Gamma) \), then there exists a \( P \in C^\infty(SL(2,\mathbb{C})/\Gamma) \), which provides simultaneous solution to the equations \( U P = f \) and \( U' P = g \), and for every \( 1 \leq k \), satisfies the estimates,

- \( \|P\|_{W^k_{\rho,n}(SL_2)} \leq C_k \|f\|_{W^{k+2}_{\rho,n}(SL_2)} \) and
- \( \|P\|_{W^k_{\rho,n}(SL_2)} \leq C_k' \|g\|_{W^{k+2}_{\rho,n}(SL_2')} \), for every \( k \geq 1 \).

**Proof.** Since \( f, g \in C^\infty(L^2(SL(2,\mathbb{C})/\Gamma)) \), satisfy \( U' f_{\rho,n} = U g_{\rho,n} \) in almost every member of the principal series, by theorem 4.16 there exists a simultaneous solution \( P_{\rho,n} \in C^\infty(\mathcal{H}_{\rho,n}) \) to the equations \( U P_{\rho,n} = f_{\rho,n} \) and \( P_{\rho,n} = g_{\rho,n} \) for almost every member of the principal series contained in \( L^2(SL(2,\mathbb{C})/\Gamma) \). But then by theorem 4.18, such a solution must satisfy the uniform estimates \( \|P_{\rho,n}\|_{W^k_{\rho,n}(SL_2)} \leq C_k \|f_{\rho,n}\|_{W^{k+2}_{\rho,n}(SL_2)} \) and \( \|P\|_{W^k_{\rho,n}(SL_2)} \leq C_k' \|g\|_{W^{k+2}_{\rho,n}(SL_2')} \), for every \( k \geq 1 \). But we then have, for every \( k \geq 1 \),

\[
\|P\|_{W^k_{\rho,n}(SL_2)}^2 = \int_{\oplus_m} m(\rho, n) \|P_{\rho,n}\|_{W^k_{\rho,n}(SL_2)}^2 d\lambda(\rho, n)
\]

\[
\leq C_k^2 \int_{\oplus_m} m(\rho, n) \|f_{\rho,n}\|_{W^{k+2}_{\rho,n}(SL_2)}^2 d\lambda(\rho, n) = C_k^2 \|f\|_{W^{k+2}_{\rho,n}(SL_2)}^2
\]

And similarly we get \( \|P\|_{W^k_{\rho,n}(SL_2')} \leq C_k' \|g\|_{W^{k+2}_{\rho,n}(SL_2')}, \) for every \( k \geq 1 \).

These estimates then tell us that \( P \) is infinitely differentiable with respect to the generators of \( SL_2 \) and \( SL_2' \). That is, the differentiability with respect to \( SL_2 \) tells us that \( U^n P, V^n P, X^n P \) exists for every \( n \geq 1 \), and differentiability with respect to \( SL_2' \) tells us in addition that \( (U')^n P, (V')^n P \) exists for every \( n \geq 1 \). The operators \( U, U', X, V, V' \) generate the lie algebra \( \text{sl}(2,\mathbb{C}) \), since \( 2\theta_1 = [U - V, U' + V'] \). We then apply theorem 2.1 in [15] (see chapter 2), to obtain that \( P \in C^\infty(SL(2,\mathbb{C})/\Gamma) \), and the result is thus established.

We remark that the Generalized Ramanujan-Selberg conjecture asserts that \( L^2(SL(2,\mathbb{C})/\Gamma) \) contains no members of the complementary series so long as the lattice \( \Gamma \) is arithmetic.
Chapter 5

Conclusions

The common method of proof of all of the main results is to decompose the problems into statements concerning the way certain operators in the Lie algebra act on each irreducible unitary representation. We then use a ‘gluing’ argument to obtain the appropriate global results on $L^2(G/\Gamma)$, where $G$ is a semisimple Lie group and $\Gamma$ is a lattice. Since the left regular representation of $G$ on $L^2(G/H)$ is a unitary representation of $G$, for any closed subgroup $H$, there is always a direct integral decomposition of $L^2(G/H)$ into irreducible unitary representations. Therefore, all of the results in this thesis can be extended to the more general situation of $A$ acting on $G/H$. For the higher rank cases studied in sections 4.2-4.5, we need only to restrict our attention to subgroups $H$ for which $G/H$ is an irreducible $G$-space, i.e. the action of every non-central normal subgroup $N \subset G$ on $G/H$ is ergodic. In fact, we can extend the results to cocycles over any smooth $A$ action on any smooth irreducible ergodic $G$-space $S$. The ‘gluing’ arguments only then run into trouble when the direct integral decomposition of $L^2(S)$ contains irreducible unitary representations which are arbitrarily close to the trivial rep. In our case, when certain members of the complementary series appear. Determining if that occurs for a given irreducible ergodic $G$-space $S$ is, unfortunately, quite often a difficult problem.

In the overview of the results of Flaminio and Forni [4], we noted that the method of proof is to use the detailed description of the action of the operator $U$ on the space of $K$-finite vectors in each irreducible unitary representation, and then solve a second order finite difference equation (in one parameter). Also, the present thesis utilizes the same approach to describe the action of the operator $X$, which is the generator of the geodesic flow, on the $K$-finite vectors. We then solve a second order difference equation (in one parameter).

It is interesting to note that the same method should yield a solution to a Helgason type problem for $PSL(2, \mathbb{R})/\Gamma$ [7]. Namely, if $P \in U(\mathfrak{sl}(2, \mathbb{R}))$ is an n-th order element in the universal enveloping algebra of $\mathfrak{sl}(2, \mathbb{R})$, then for any $g \in \mathcal{E}'(PSL(2, \mathbb{R})/\Gamma)$, does there exist an $f \in \mathcal{E}'(PSL(2, \mathbb{R})/\Gamma)$ such that $Pf = g$ in the sense of distributions?
In fact, the $P$-invariant distributions should be able to be classified and a positive solution to the smooth problem obtained. That is, if $g \in C^\infty$ is a smooth vector, such that $D(g) = 0$ for all $P$-invariant distributions, then there exists an $f \in C^\infty$ such that $Pf = g$ in the ordinary sense.

Because the action of every $P \in \mathcal{U}(\mathfrak{sl}(2, \mathbb{R}))$ on $K$-finite vectors in each irreducible representation can be written explicitly, and yields an $n$-th order difference equation (in one parameter), a (formal) solution using Green’s operators is possible [2]. The only remaining problem is to provide good estimates for the size of that Green’s operator.

While this result looks achievable, it emphasizes a point of caution about the use of representation theory in such a detailed way. For one thing, we do not have such a detailed description for the action of $g$ on the $K$-finite vectors for most semisimple Lie groups. In fact, other than for the spherical principal series (the ones that are induced from the trivial representation on $M$ in the standard minimal parabolic subgroup), there is little hope of obtaining any kind of useful detailed description of the action of $g$ on the $K$-finite vectors of an irreducible unitary representation of $G$. But even for the group $SL(2, \mathbb{C})$ (or $SO(N, 1)$), where detailed descriptions of the action of $\mathfrak{sl}(2, \mathbb{R})$ on the $K$-finite vectors are available [9], [10], [11], attempts to solve equations of the type $Xf = g$ for general $X \in \mathfrak{sl}(2, \mathbb{R})$ prove to be too inefficient to be useful. The problem stems from the fact that an equation of the form $Xf = g$ gives an 2nd order difference equation in multiple parameters. While solution to the general nth order difference equation in one parameter has a robust theory [2] (akin to the O.D.E theory), there is no general closed form solution to an nth order difference equation in more than one parameter, a partial difference equation [3] (akin to P.D.E theory). In fact, for $SL(2, \mathbb{C})$ there exists an o.n. basis of $K$-finite vectors $\{u^k_v|k \geq 0, \ -k \leq v \leq k\}$. In this basis, the action of the nilpotent element $U = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ can be written as,

$$Uu^k_v = C(k - 1, v + 1)u^{k-1}_{v+1} + A(k, v + 1)u^k_{v+1} + C(k + 1, v + 1)u^{k+1}_{v+1} + C(k - 1, v - 1)u^{k-1}_{v-1} + A(k, v - 1)u^k_{v-1} + C(k + 1, v - 1)u^{k+1}_{v-1}$$

This equation already has the form of an implicit reaction-diffusion equation [3], except with variable (as opposed to constant) coefficients. Solving the equation $Pf = g$ for a general $P \in \mathcal{U}(\mathfrak{sl}(2, \mathbb{C}))$ already becomes effectively intractable.

As for our results for cocycles over the $N$ action on $SL(2, \mathbb{C})/\Gamma$, it is easy to conjecture that this result extends to the whole $SO(N, 1)$ series, at least for arithmetic lattices.
CONJECTURE 1. For an arithmetic lattice, the first smooth almost cohomology group over the $N$ action on $SO(N,1)/\Gamma$ is trivial.

For $SO(N,1)$, there are again three series of representations, the principal, discrete, and complementary, and the discrete series only appear for $N$ even. Let $N = \exp so(N,1)_{\alpha}$ be the positive root space. It is an abelian subgroup of $SO(N,1)$. The principal series representations can be realized again as $L^2(\mathbb{R}^{N-1})$ (by identifying $N$ with $\mathbb{R}^{N-1}$). As with $SL(2,\mathbb{C})$, there is a basis of $Lie(N)$ which act on $\mathbb{R}^{N-1}$ by the standard partial differentiation operators $\frac{\partial}{\partial x_i}$.

The infinitesimal version of a cocycle $\beta : N \times SO(N,1)/\Gamma \to \mathbb{C}$ is a closed one form $\omega$, and again the cohomological equation becomes $\omega = \eta - dP$. And again $\omega$ is cohomologous to zero if it is exact. We solve formally by defining $P = \int_0^1 \sum_{i=1}^{N-1} x_i f_i(tx) dt$. The difficult step is to show that $P$ is a smooth vector in $C^\infty(\mathcal{H})$, and not just a smooth function in $C^\infty(\mathbb{R}^{N-1})$. This should involve the vanishing of distributions of the form $\int_\mathbb{R} f_1(x_1, \ldots, x_i + t, \ldots, x_{N-1}) dt$, as well as a distribution which comes from a "point mass at $\infty$".

Once it is established that $\omega : Lie(N) \to C^\infty(\mathcal{H})$ in each tempered irreducible unitary representation, then the arguments of section 4.4.7 should apply with little modification, noting that there will be $N-1$ embedded $SL(2,\mathbb{R})$ subgroups which we get uniform estimates for. This in turn will show that we have derivatives of all orders in a set of directions which generates the algebra $so(N,1)$. 


References


Vita

David J. Mieczkowski was born in Corning, New York on July 22, 1976. He is the youngest son of Dr. John and Marion Mieczkowski. In 1999 he received the B.S. degree in Mathematics, with honors, from the University of Michigan at Ann Arbor. In 2001 he married Jill L. Stelmack. From 1999 to 2002 he attended the State University of New York at Stony Brook, where he was enrolled in the Ph. D. program in mathematics. In 2000 he received his M.A. degree in Mathematics from that institution. In 2002 he enrolled in the Ph. D. program in mathematics at the Pennsylvania State University. While at Penn State, he was supported by an NSF VIGRE Fellowship. In the past he has worked as an adjunct faculty at SUNY Old Westbury, and as a teaching assistant at SUNY Stony Brook. On October 1, 2004, his son Ethan was born. David Mieczkowski is a member of the American Mathematical Society.