DYNAMICAL GAUGE EFFECTS AND HOLOGRAPHIC SCALING
OF NON-EQUILIBRIUM MOTION IN A DISORDERED AND
DISSIPATIVE ATOMIC GAS

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Abstract

In this thesis, we describe both an experimental implementation and a theoretical investigation of gauge field effects in cold atom systems. The experimental part focuses on the motion of atoms in a disordered potential with dissipation. The atomic system under consideration utilizes a generalization of a dark state cooling mechanism, known as Ramman sideband cooling. By coupling near-resonant laser modes, we create an open system which exhibits a non-equilibrium phase transition between two steady-state behaviors, and shows scale-invariant behavior near the transition. This behavior is loosely analogous to dynamical gauge effects in quantum chromodynamics, and can be mapped onto generalized open problems in theoretical understanding of quantized non-Abelian gauge theories. Additionally, the scaling behavior can be understood from the geometric structure of the gauge potential and linked to a measure of information in the local disordered potential, reflecting an underlying holographic principle.

As an extension toward developing fractional quantum Hall (FQH) systems, we consider an experimental strategy of creating many FQH samples along a chain of lattice sites, and coupling them together via tunneling. We calculate a mean-field phase diagram and derive an effective field theory to describe this system and find that such a system support novel insulator and super fluid states. Close to the centrifugal limit with small tunneling strength, 1/2-Laughlin states with N atoms are localized on each lattice site, analogous to Mott-Insulator state. As the tunneling strength increases, the system transitions from the Laughlin-insulator states to superfluid states, whose order parameter reflects the insulator states through momentum and atom number conservation. The transport properties near the phase transition reveal the fractional 'charge' nature of the insulator states, in such that the effective mass of the tunneling quasi-particle is fractionalized. We demonstrate an experimentally feasible pathway to a state describable as a Mott-insulator of 1/2-Laughlin states, and also interrogate the coherence properties of the superfluid phase.
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2.1 Schematic of the apparatus. The vacuum chamber consists of two experimental regions with a center to center distance of 0.8 m. A large pressure difference between the two glass cells is maintained by two flow contrictions and differential pumping. Atoms are first collected in the upper source cell with higher vapor pressure and then transferred down to the lower UHV cell through a guiding beam. Another MOT at the UHV cell catches atoms for further cools them into a three dimensional disordered potential. 

2.2 Atom Sources. There are eight AMDs installed inside the source cell. Five of them are for Rb (B, E, C, F and H) and three are for Cs (A, D, and G).

2.3 A picture of the whole bake-out setup. 20 heater tapes are wrapped around the chamber, with each controlled by a variac. Temperature sensors are attached to different parts of the chamber surfaces to keep track of local temperatures. A turbo pump is connected to the main chamber to pump out gas during bake-out. Several layers of aluminum foil are used to spread the heat evenly and reduce heat loss.

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2.10 Schematic for beat-note lock. An optical fiber that contains light from reference, MOT, and repumper lasers is connected to a high frequency photodiode, which converts the beats of light intensity to microwave signals. These signals are split into two paths with one for locking the MOT laser and the other for locking the repumper laser. After proper filtering and frequency mixing, they are fed back to the ECDL through two parallel loops based on DPFD and APD respectively.

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2.14 Schematic of the timing system. The hardware is centered around a PXI system (NI PXIe - 8133). Analog and digital output cards are installed on the PXI system and provide control signals for experimental equipment. The software is built around a web server that transforms timing sequences written in XML to data arrays executable by Labview. Data generated by the experiment instruments can be analyzed in a GUI.

2.15 Optical configuration in the source cell. Our MOT consists of three pairs of counterpropagating beams and one pair of quadrupole coils. The $\infty$ shape beam path permits using a moving molasses technique if needed. The repumper light frequency is on the $F = 1 \rightarrow F' = 2$ transition, and the MOT beam is red detuned from the $F = 2 \rightarrow F' = 3$ transition with $\delta = 24$ MHz. Four lattice beams (blue) create a 3D lattice potential for RSC. One optical pumping beam (yellow) for RSC is sent in from a side with circular polarization. The temperature of the atoms in the source cell is measured by a time-of-flight (TOF) setup using two light sheets, with the upper one 4 cm below the MOT which measures the absorption of atoms passing through it and the lower one as an intensity reference.

2.16 Fluorescence measurement of atom number in a MOT. At $t = 10$ ms, the MOT and repumper lights are turned off together with the quadrupole coils, while the MOT light is brought into resonance. The atom number is derived from the peak fluorescence level $\Delta V$. This measurement shows $1.7 \times 10^8$ atoms in the MOT after 400 ms of loading.

2.17 Illustration of polarization gradient cooling in one dimension. Two counterpropagating beams of perpendicular linear polarization interfere with each other creating a light field with its polarization varying from $\sigma^-$ to $\sigma^+$ over a half-wavelength distance. Light shifts for an atom in $m = F$ (blue) and $m = -F$ (red) are shown for a transition from $F \rightarrow F' = F + 1$. The atoms that move in this light field convert kinetic energy to potential energy, which then is continuously removed through scattering photons.

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2.21 Simplified diagram for the UHV cell optics. A imaging system consisting of four microscope objectives is built around the UHV cell. Images of the atoms are routed to a scientific camera through two imaging paths - one with magnification of 24 and the other with magnification of 2.5. Both fluorescence and absorption images can be taken of the atoms. A six beam MOT is used to capture atoms falling from the source cell. One pair of MOT beams pass through two objectives, restricting the beam size to a maximum of 3 mm. Optical dipole traps and potential patterns can be projected in through the imaging system.

2.22 Optical diagram for the high power laser system. This system is based on a high power fiber amplifier (Nufern). Typically, light for each path is controlled by AOMs and shutters. A portion of the output is coupled into two large-core multimode fibers that generate random light patterns, and are later used to create a 3D disordered potential.
2.23 (a) A picture of the finger imaging system. (b) Close look at the finger. Beam 1, 2, 3: 1064 nm beams for the rotating lattice, with two EOMs inserted. Beam 4, 5: 1064 nm beams form a cross dipole trap for evaporative cooling; Beam 6, 7: 1064 nm beams form a lattice in the axial direction, also this is a beam path for 1080 nm light; Beam 8, 9, 10, 11: for RSC beams that trap and cool atoms, also to be used for fluorescence imaging; Beam path 8 also contains the guiding beam. (c) Test imaging with 250 nm nano-particles from the imaging system. (d) Fourier transform of the image of separated point-like objects, which reflect the modulation transfer function of the microscope. The dashed circle corresponds to a spatial resolution of 500 nm, which is consistent with a NA=0.8.

3.1 Generation of the disordered potential. An equilibrium distribution of optical fiber modes projects four spatially disordered wavefronts, to form a far-off-resonant potential in a small volume, in which atoms are dissipatively coupled to a single-mode optical pumping beam. Information carried by wavefronts can be controlled by selective launch of optical fibers, yielding altered structure in the aperture plane.
3.2 Characterization of the optical modes that form the disordered potential. a, Solutions of the characteristic equation for the optical fiber for all possible modes denoted by $\nu, K$ pairs. The color scale denotes the deviation $\beta - n_1k$, with $n_1$ the core index and $k = 2\pi/\lambda$, for wavelength $\lambda$, of the propagation constant $\beta$ from that of an axial ray. The green (blue) horizontal line indicates the lower (upper) boundary of the occupied modes for the only high-$K$ (low-$K$) data in Fig. 3.4 and Fig. 3.10 e,f,g (c,d,g). b, The information content projected into a volume of dimension $\ell$ in the experiment chamber is apparent from the contraction of normalized singular values of the expansion matrix $\Lambda$ as the scaling parameter $\ell$ is increased. All singular values of $\Lambda$ constructed to 10th order in the power series expansion are shown for $\ell = 0.32\mu$m (red), $\ell = 1.0\mu$m (orange), and $\ell = 3.0\mu$m(green). c, Most significant singular values for all length-scales $\ell$. The diffraction limit is visible as a closing of the gap between the largest three eigenvalues as the length-scale is increased to $O(1\mu$m). d, Scaling of information content within a volume as determined by the eigenmode structure of the fiber, and mode projections into experiment chamber, demonstrating $F_s \sim \ell^K$, with $K = 0.54 \pm 0.06$. The power series shown contain all terms up to order 6 (squares), 8 (pluses), 10 (crosses), and 12 (circles). The red line shows the best fit between 0.04$\mu$m and 3.6$\mu$m for 12th order, which is used to extract $K$. Its uncertainty corresponds to the standard deviation of pairwise slopes for all points presented here. 48

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3.5 Local gauge transform that rotates the local magnetic field direction to measurement axis $x$.

3.6 Processes contributing to the vacuum energy and stability of Bogoliubov normal mode analysis. (a) The Hartree term, in which a sound mode of the vacuum excitation propagates, and interacts with the vacuum through virtual pair creation of other similar excitations - the interaction is local, and mediated by the propagator for a spin-$1/2$ interaction (squiggly line). (b) The Fock term, in which the excitation interacts with itself by creating a propagating disturbance in the medium. The squiggly line represents any one of three propagators representing spin $j_{ij} = 1/2, 3/2$ or $5/2$. (c) By combining multiple of these processes, one can build up a representation of vacuum state properties as a series of disconnected diagrams. Placing a Hartree term at each vertex $i$, labeled by its spin projection $m_i$, and connecting vertices by edges representing Fock terms of spin $j_{ij}$, one arrives at a simplified diagram in (d). This bears a strong resemblance to structures considered in lattice gauge theories, in which the vertices form an ordered graph, or loop quantum gravity spin networks, in which the graph is disordered.
3.7 Instanton dynamics can be applied to the tunneling dynamics of a single particle in a periodic potential (left) or to the topological dynamics of a gauge field (right). Similar to the way in which a particle tunnels from one stable classical minimum of a potential to the next, the topology of a gauge field (represented by its Chern-Pontyragin number) can change in time. By forming a bounding surface (either taken at large distances from a system or on causal boundaries) one can choose an appropriate gauge in which the instanton number vanishes on time-like surfaces (the outside of the cylinder) and therefore is different on space-like surfaces (endcaps). A singly-charged instanton implies a change in topological index of one. Just as in a periodic lattice, a multiply charged instanton can be decomposed into single charged instantons, provided the constituent collective coordinates are well-separated. In this way, time-evolution can be broken into a series of individual topological tunneling events.

3.8 Discrete Model of Hilbert Space. Geometrically, these are points of convexity in a space in which the corresponding master-equation sets the distance between states; such geometry must be defined covariantly through the effective gauge fields $A^{a\mu}_H$ and $A^{a\mu}_d$, which completely specify the dynamics. Similar ideas used in a continuous sense lead to an analog model for emergent quantum gravity.

3.9 Higgs-like transition to scaling behavior. The number of atoms retained in the disordered potential (corrected for vacuum lifetime) after varied holding times and for varied orientation of magnetic and optical pumping fields are shown as scatter points. The transition to scaling-law behavior (blue line, $\theta = 23 \pm 2^\circ$) is sharp, representing a discrete transition between decay due to massive (exponential, time-constant shown by green line) and massless (scale-invariant) gauge boson modes. Here, $\theta$ represents the angle between the external magnetic field and optical pumping beam propagation axis, which determines degree of decoupling of the dark state. Zero angle corresponds to pure $\sigma_+$ polarization, which maximally decouples the dark state. The mesh surface shows a fit result from a model allowing for a crossover from exponential to power law decay of varying width.
3.10 Scaling behavior vs information content. Plot shows normalized loss curves for different optical fiber mode content (corresponding to fourier-plane images shown in false color), including equilibrated mode content (red, purple, orange $\kappa = 0.54 \pm 0.06$), low transverse-wavenumber excitation (blue $\kappa = 0.41 \pm 0.07$), and high transverse-wavenumber (black $\kappa = 0.52 \pm 0.05$, green $\kappa = 0.53 \pm 0.05$). Inset shows loss scaling parameter $\tilde{\gamma}$ against $\kappa$.

3.11 Images of the intensity distribution in the aperture (a,c,e) and imaging (b,d,f) planes. Red, green, and blue correspond to all, only low-$K$, and only high-$K$ optical modes from the light potential, respectively. Scaling laws of the information content of the corresponding figures in a-f. The exponential of the Shannon information, $F_s$, is calculated for each point as a function of distance from the origin, $\ell/2$, constructed with all terms up to 10th order in the power series. The lines are least-squares linear fits to $\log(F_s)$ for $\ell$ between 40 nm and 3.6$\mu$m, which yield exponents, $\kappa$, of $0.53 \pm 0.05$ (red), $0.52 \pm 0.05$ (green), $0.41 \pm 0.07$ (blue). All possible $\kappa$ from each combination of two points of $\log(F_s)$ are shown in the inset; the error on the fit was taken to be the standard deviation of this spread in slopes. Orange is a simulation with only the lowest 5% modes in $K$ retained with $\kappa = 0.22 \pm 0.03$.

3.12 Variation of dissipative capture with pumping intensity. The number of retained atoms after 5.1s is plotted against optical pumping intensity, calibrated to scattering rate from the (bright) $m_F = 0$ state. The red line is a power law fit of the data, showing an exponent of $0.5 \pm 0.2$. Error bars show the standard deviation in the mean along both axes, and the shaded region uncertainty in the power-law scaling.
3.13 **a** Microwave spectroscopy showing sideband asymmetry of the atoms after cooling. The plot shows 947 measurements of the retained atoms versus $\delta f$ where $f = f_0 + \delta f$ is the applied microwave frequency, and $f_0$ is the hyperfine splitting. The asymmetry of each peak (see inset) can be used to extract kinetic temperature - the red line shows a fit to parameterize temperature. **b** Parametric excitation of the atoms is performed by modulating the light intensity; this couples vibrational levels that differ by two due to even symmetry and is used (inset) to infer the distribution of local trapping frequencies. The plot shows the survived atom number as a function of the center frequency of the modulation. The inset shows the fraction of atoms contained within a 1kHz bin of local vibration frequency.

3.14 Illustration of the microwave transitions. The Zeeman shift is caused by the external magnetic field. We keep the field strength the same as during the cooling period. A horn delivers microwave with frequency centered on the splitting of the two hyperfine ground states, and drives magnetic transitions among $F = 1$ and $F = 2$ magnetic sublevels.

3.15 Amplitude modulation of lattice potential.
4.1 (a) We consider a chain of rotating traps, through which atoms can tunnel at a rate $t$ along the rotation axis. At sufficiently high rotation rates $\Omega$, and with (b) sufficiently strong scattering due to repulsive interactions parameterized by $\eta$, atoms may occupy higher angular momentum eigenstates $m_i$ in the lowest Landau level, forming (c) strongly correlated fractional Hall states within in a two-dimensional well (filling factor $\nu = 1/2$ state is illustrated for $n = 4$ atoms). Such states are describable as composite particles bound to fluid vortices, and we show for moderate $t$ that tunneling between such states reduces to highly collaborative tunneling of composites at a renormalized rate $t'$. (d) Under appropriate conditions, a tunnel-coupled chain will form insulating states of well-defined atom number and FQH filling factor at each site, and superfluid states formed by local superpositions thereof. Such states possess novel transport properties, in which particle flow along the chain is described by motion of composite entities, which for a given axial flow induce a torsional strain in the background insulator. The linkage between flow and strain can be described in a chain with periodic boundary conditions (e) as a topology-preserving insertion (steps i-iii) of motional flux in the presence of intrinsic flux representing the insulating state.

4.2 The energy spectrum $\epsilon_{nf}$ for each of these $n$-body states (A) is shifted downward proportionate to the chemical potential $\mu$ to form the free energy in a grand-canonical ensemble. Experimental control over the many-body state can be exerted through (B) deformation of the rotating trap (strength $\epsilon$), which couples states of differing angular momentum (proportional to slope $d\epsilon_{nf}/d\Omega$) and introduces avoided level crossings (highlighted in yellow), or (C) by controlling the tunneling rate $t$, producing avoided crossings between states of differing particle number (expectation value of $n$ represented by color). Spectra shown here correspond to $\eta = 0.0054$, $\mu = 0.025\hbar \omega$, $\epsilon = 0.002$ (in B), $t = 0.004\hbar \omega$ (in C). Spectra in (A) and (B) were produced with DD, and (C) by minimization of the Gutzwiller form as described in the text.
4.3 (A-C) Mean-field phase diagrams, showing components $\psi_m$ of the order parameter in color, and zero- and finite-temperature phase boundaries (joined symbols) at three rotation rates. The dashed red line shows the analytic result for the zero temperature phase boundary. (A, $\Omega = 0.929\omega$) At low rotation rates, the phase-diagram reduces to that of the Bose-Hubbard model, exhibiting insulator-to-superfluid $nI_\infty - SF$ transitions in the $m = 0$ channel. (B, $\Omega = 0.988\omega$) As the rotation rate is increased, high-occupancy insulating regions $nI_\nu$ transition to higher angular momenta and stronger correlation due to the change in lowest-energy few-body form. (C, $\Omega = 0.997\omega$) Near the centrifugal limit, insulators at all occupancies have transitioned into the $\nu = \frac{1}{2}$ (Laughlin) form, in which the on-site interaction energy is zero; at higher $t$, superfluidity develops as coherent superpositions of these states. When $t$ is sufficiently large, the Laughlin-superfluid is destroyed in favor of a more completely mean-field type state. (D-E) Mean-field ground state probabilities $|a_{n_f}|^2$ are shown as a function of tunneling strength $t$ for two different cuts from (B) and (C). They show two sudden changes, first at the insulator boundary, and a second jump as the FQH-forms are lost at higher $t$ - the latter can be roughly described by calculation of $u$ in an effective field theory, corresponding to saddle points in the free-energy along a second component of $\psi$ (points: numeric Gutzwiller ansatz, solid lines: EFT to fourth order in $\psi$.)
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For My Parents, Sister and Brother,

For Liqing and Yutong.
Chapter 1
Introduction

1.1 Artificial Gauge Fields in Atomic Systems

In cold atom systems, researchers have put intense effort on creating artificial gauge fields for neutral atoms [1]. Compared to other systems, ultracold atom systems are more controllable by manipulating the precisely crafted optical and magnetic potentials, and even the strength and sign of inter-particle interaction may be altered through Feshbach resonance [2]. This makes it a versatile platform to study gauge dynamics and simulate other quantum systems [3, 4]. Some recent developments include realizing Haldane model in cold fermion systems [5] and simulating Hofstadter Hamiltonian in optical lattices [6, 7]. One of the interest in our research is to develop fractional quantum Hall systems in cold atoms using rotating traps [8].

Fractional quantum Hall effects (FQHE), where interacting electrons transition from a nearly free two dimensional electron gas into a new strongly correlated state describable by composite particles of electrons associated with a number of magnetic flux quanta [9–11], have dramatically altered the way we look at ordering in quantum many body systems. The FQH states reveal a new kind of order, known as topological order, and represent a new state of matter, which cannot be characterized by a local order parameter and its long range correlations. Phase transitions into this state cannot be described by Landau symmetry breaking theory, since no symmetry (or symmetry breaking) is associated with this order due to its non local entanglement [12, 13]. Many unique properties are predicted along with this topological order, such as excitations with fractional charge and fractional quantum statistics in particular states. Considerable experimental effort has been
devoted to the observation of these novel properties, and to the study of topological
order [14–18]. Apart from fundamental physics, it also provides a way to generate
many body entangled states, which could apply to quantum computation [19].

A situation that is analogous to the two dimensional electron gases (2DEG)
has been predicted in a gas of interacting bosons that are confined to a rapidly
rotating trap [20,21]. For atoms in the rotating frame, with the rotation frequency
$\Omega$ approaching the harmonic oscillator frequency $\omega$ of the confining trap, the
Hamiltonian is similar in form to that which describes the 2DEG. The single
particle energy levels become highly degenerate, forming a 'lowest Landau level'. In
this analogy, the vorticity of the rotating Bose gas plays the role of the magnetic
field, and quantized vortices take the place of magnetic flux quanta. The many-
body effects are generated by repulsive two-body interactions characterized by
a scattering length, which replaces the coulombic repulsion for electrons. Apart
from the intrinsic statistics of bosons versus fermions, the FQHE occurs when the
weakly interacting Bose gas transition to highly entangled topological states at
quantized, non-zero angular momentum. Additionally, the excitations of this system
are predicted to possess fractional statistical character [22] in certain states, though
experimental verification of this would present a challenging goal. Ultracold atom
system is a promising system to study the topological properties of the fractional
Hall physics.

1.2 A Brief Introduction to Gravitation

A mathematical description of gravitation can be traced back to Issac Newton, who
states that one object attracts other objects in the universe due to a force that is
proportional to the product of their masses and inverse square of their distances.
For centuries, Newton’s law of gravitation has successfully explained the motion
of planets, and even had led to the discovery of new planets such as Neptune. In
Newtonian mechanics, there exists a family of inertial reference frames where the
laws of nature take the same form, known as Galilean invariance. In Newton’s
opinion, these frames exist in an absolute space which always remains similar and
unmovable. However, the theory of electrodynamics presented by James Clark
Maxwell in 1864 does not satisfy the principle of Galilean relativity, since Maxwell’s
equations predict that the speed of light, $c$, is a universal constant in vacuum. For
example, if the light speed in one inertial frame is \( c \), then it will be \( c + v \) in another inertial frame moving with velocity, \( v \), relative to the first one according to Galilean transformation. In fact, it is shown by Michelson and Morley’s experiment that the velocity of light is the same both for light traveling along the earth’s orbit motion and light traveling perpendicular to it. In 1905, Einstein proposed the principle of special relativity where he introduced a Lorentz transformation which leaves Maxwell’s equation and the speed of light invariant. He also modified the laws of motion to make them invariant under Lorentz transformation. The theory of special relativity puts space and time on the same footing and resolves a number of paradoxes that arise when objects are traveling at large speed. The core idea is that both mechanics and electrodynamics should be invariant with respect to Lorentz transformations, instead of Galilean transformations. However, the Lorentz transformations are still only for a class of inertial frames. The question of what determines these inertial frames still remains unclear.

It is clear that Newton’s law of gravitation does not satisfy the principle of special relativity. On the efforts to construct a relativistic theory of gravitation, Einstein introduced the principle of equivalence which states that the gravitational force as experienced locally while standing on a massive body is actually the same as the force experienced by an observer in an accelerated frame. Guided by this principle, he generalized special relativity and Newton’s law of universal gravitation and proposed a unified description of gravity as a geometric property of space and time. The gravitational field can be represented by the 10 components of the metric tensor of Riemannian spacetime geometry. The principle of equivalence requires that physical equations should be invariant under general coordinate transformations, not just Lorentz transformations in the absence of gravity. This general principle of relativity forms the current description of gravity and many of its predictions have been observed. The most recent breakthrough is the measurement of a gravitation wave by LIGO [23,24], where they measured ripples in the curvature of spacetime that propagate as waves at the speed of light due to interactions such as merging of black holes.

It is predicted by general relativity that a sufficient compact mass can deform spacetime into a black hole [25,26], where the gravitational effect is so strong that nothing can escape from inside it. The surface that characterizes the boundary of a black hole is called an event horizon. Due to the property of the black hole, anything
that passes the event horizon loses its state information. This one-way boundary raises an information paradox - a seemingly closed system loses information. This is contradictory to the principle of quantum mechanics, in which the time evolution of a closed system is unitary and information is preserved. However, in 1975, Hawking showed that black holes can experience blackbody radiation due to quantum effects near the event horizon [27], known as Hawking radiation. It is noted that the temperature of the black hole is proportional to the surface area of the event horizon, rather than the volume. The Hawking radiation is completely independent of the material entering the black hole, according to the no-hair theorem - all information about the matter falling into a black hole is permanently inaccessible to external observers. Suppose an object described by a pure quantum state enters a black hole, the information of that state is partly transformed into mixed states of Hawking radiation. Thus, the total information is not preserved at the quantum level.

This information paradox remains an open question, one lies at the heart in combining quantum mechanics and gravity. One of the hypotheses is holographic principle [28], which states that the information of the enclosed volume can be stored on its boundary whose dimensions is lower by one. In the case of a black hole, it suggests that all of the information of matter that enters the hole can be contained on the surface of the event horizon. As mentioned before, the temperature and entropy of a black hole scale with the horizon area instead of the volume. This can be seen as an indication of the holographic principle. Information describing matter can be encoded on the boundary in terms of Shannon entropy, where the surface contains many fundamental “bit” areas characterized by a Plank scale.

A modern gravity theory developed by Erik Verlinde [29, 30] based on the holographic principle and quantum information theory describes gravity as an entropic force. That is to say gravity is not a fundamental force, but rather it arises as an emergent effect driven by the tendency to maximize the total entropy of the universe. One important claim of the emergent gravity theory is that it explains the deviation of galaxy rotation from its prediction by general relativity based on the visible mass of the universe, which usually involves the introduction of dark matter [31]. Recently, this theory has gained great interest from the public due to its impressive agreement with data from observation [32], with no free parameters and without introducing dark matter. The idea of entropic force provides a new perspective to understand gravitation in our universe. It can be treated as a giant
disordered system undergoing non-equilibrium evolution.

One main direction of development for the gravitation theory is to combine general relativity with quantum mechanics. String theory is one way in this direction, and it predicts the quantized gravitation field - a graviton, which can be treated as one vibration mode of a string representing a fundamental particle. However, experimental verification of it is extremely hard due to the low energy scale. An alternative approach is quantizing the gravitation field directly, similar to other quantum field theory. An example is the loop quantum gravity (LQG) theory. Spacetime can be treated as a dynamical field and is discretized, according to LQG. This can be understood from the essence of general relativity, which describes gravitational field as a geometric effect, and is invariant under general coordinate transformations. This resembles the gauge symmetry of a quantum gauge field. There are still open problems in quantizing this field. It is worth noting that challenges in quantization of the gauge field also exits in high energy physics, such as in QCD where it is still unclear whether the gauge boson has mass in low energy limit [33–35].

1.3 Thesis Overview

In this thesis, we describe both an experimental implementation and a theoretical investigation of gauge field effects in cold atom systems. Chapter 2 gives a brief description of the experimental apparatus. We have developed an experimental platform around two vacuum cells that are separated by 80 cm. A high numerical aperture imaging system is built around the atoms which allows multiple directional imaging. Diode laser systems are developed to provided cooling and trapping light. The experiment part of this work focuses on the motion of atoms in a disordered potential under dissipation. The atomic system under consideration here utilizes a generalized dark state cooling mechanism - Raman sideband cooling. By coupling near-resonant laser modes, we create an open system which exhibits a non-equilibrium phase transition between two steady-state behaviors. This behavior is loosely analogous to dynamical gauge effects in quantum chromodynamics, and can be mapped onto generalized open problems in quantized non-Abelian gauge theories. Additionally, the scaling behavior can be understood from the geometric structure of the gauge potential and can be linked to a measure of information.
in the local disordered potential, reflecting an underlying holographic principle. Details of this experiment are described in Chapter 3.

As an extension toward developing fractional quantum Hall (FQH) systems, we consider a experiment strategy of creating many FQH samples along a chain of lattice sites, and coupling them together via tunneling. We calculate a mean-field phase diagram and derive an effective field theory to describe this system and find that such a system supports novel insulator and super fluid states. Close to the centrifugal limit with small tunneling strength, 1/2-Laughlin states with N atoms are localized on each lattice site, analogue to Mott-Insulator state. As the tunneling strength increases, the system transitions from the Laughlin-insulator states to superfluid states, whose order parameter reflects the insulator states through momentum and atom number conservation. The transport properties near the phase transition reveal the fractional 'charge' nature of the insulator states, in such that the effective mass of the tunneling quasi-particle is fractionalized. We demonstrate an experimentally feasible pathway to a state describable as a Mott-insulator of 1/2-Laughlin states, and also interrogate the coherence properties of the superfluid phase. Details are discussed in Chapter 4.
Chapter 2  
Experimental Apparatus

In this chapter, we describe the experimental apparatus and subsystems constructed for the present experiments. The whole experimental system includes a vacuum chamber, a diode laser system, a high power laser system, an imaging system and a timing control system. Details of each system are discussed in the following sections.

2.1 Apparatus Overview

The main structure of the vacuum chamber was transferred from a previous experiment in Stanford [8,36]. Optical systems have been built around it, and a schematic of the main parts is shown in Fig. 2.1. It consists of three experiment regions with two UHV-compatible pyrex glass cells made from fused optical flats which are connected by a stainless-steel vacuum system. The source cell has a pressure, around \( O(10^{-9}) \) torr, to facilitate optical trap loading. Multiple Rubidium (Rb) and Cesium (Cs) getter dispensers are installed to produce alkali vapors. Atoms are first collected by a magneto-optical trap (MOT) and then cooled down to 4.5 \( \mu K \) using polarization gradient cooling (PGC). We further reduce the temperature of atoms by applying Raman sideband cooling in an optical lattice created by interfering four beams in tetrahedral geometry. This step lowers the temperature to below 1 \( \mu K \).

Atoms are transferred down to the UHV cell under free fall. A guiding dipole potential created by far-detuned 1064 nm light connects the two cells and provides transverse confinement for the atoms. This increases the atom source brightness toward the UHV cell by 3 orders of magnitudes compared to the previous experiment.
Figure 2.1. Schematic of the apparatus. The vacuum chamber consists of two experimental regions with a center to center distance of 0.8 m. A large pressure difference between the two glass cells is maintained by two flow contrictions and differential pumping. Atoms are first collected in the upper source cell with higher vapor pressure and then transferred down to the lower UHV cell through a guiding beam. Another MOT at the UHV cell catches atoms for further cools them into a three dimensional disordered potential.
[8]. This guiding beam also helps us directly load a UHV MOT with only a 3 mm width over a distance of 80 cm. This achievement allows us to construct a unique imaging system that employs four high numerical aperture microscope objectives surrounding the UHV cell. It is capable of imaging 3D features down to a length scale of 1 µm. Also, this imaging system allows us to project optical potentials with sub-micron spatial structures onto atoms. In this work, we create a 3D disordered potential by projecting random light patterns generated by multi-mode fibers from four directions. Atoms in the UHV MOT are cooled by PGC and loaded to this potential. It is worth mentioning that at the bottom of the UHV cell lies another “finger” imaging system, which combines a high numerical aperture microscope objective and a plano-convex lens. This finger imaging system is designed to perform site resolved measurements of atoms in lattice traps with a resolution of 500 nm.

2.2 Atom Sources

During the shipment of the vacuum chamber, a leak was developed around the electrodes of the titanium sublimation pump (TSP), which needed to be fixed. We took this opportunity to upgrade the atom sources inside the source cell as well. We installed alkali metal dispensers (AMDs) as atom sources. AMDs are small containers of alkali metal that are normally used during the preparation of photosensitive surfaces of photocathodes.

![Atom Sources](image)

**Figure 2.2.** Atom Sources. There are eight AMDs installed inside the source cell. Five of them are for Rb (B, E, C, F and H) and three are for Cs (A, D, and G).
Generally the alkali metal generating material is a mixture of an alkali metal chromate with a reducing agent. Besides its reducing action, the reducing agent is able to irreversibly sorb almost all the chemically active gases produced during the reduction reaction, which prevents them from contaminating the alkali metal vapor. There are eight getter dispensers installed, including five for Rb (SAES Rb/NF/3.4/12/FT 10+10) and three for Cs (Alvatec AS-2-Cs-40-S) as shown in Fig 2.2. By applying a moderate current (2-5 A) through the stainless steel housing of the compounds, the container releases Rb or Cs vapors which gradually fill the source cell. The normal lifetime for typical usage of a single getter can be one to two years.

![Figure 2.3. A picture of the whole bake-out setup. 20 heater tapes are wrapped around the chamber, with each controlled by a variac. Temperature sensors are attached to different parts of the chamber surfaces to keep track of local temperatures. A turbo pump is connected to the main chamber to pump out gas during bake-out. Several layers of aluminum foil are used to spread the heat evenly and reduce heat loss.](image-url)
After installing AMDs, the chamber is reassembled carefully to attain ultra-high vacuum pressure. The whole chamber is baked out to accelerate diffusion and the de-absorption of molecules within the chamber in order to lower the ultimate outgassing rate at room temperature. The outgassing rate depends exponentially on the temperature through the Boltzmann factor $e^{k_B T}$. During the bake-out, the temperature of the chamber is kept as even as possible for two reasons. One is that the bonding between glass and metal can not survive large temperature gradients due to their different thermal expansion coefficients. The other is that the final pressure of the chamber mainly depends on the coldest point of the chamber body during the bake-out. We attached about 30 thermocouple sensors to the surface of the chamber and glass cells to monitor the local temperatures. To heat the whole chamber evenly, we first covered it with a layer of aluminum foil to improve the thermal conductivity. On top of that, we wrapped about 20 heater tapes, each controlled by a variac, to balance the heat in different regions. As a final step,
several layers of aluminum foil are added on top of the heater tapes to help spread the heat and preserve the temperature. A turbo pump is connected to the chamber to pump away gases released from the bake-out. The whole bake-out setup is shown in Fig 2.3. A cold cathode ionization gauge (IMG-300) is connected with the chamber to measure the pressure.

We first ramp the temperature slowly from room temperature to $\sim 200 \, ^{\circ}C$ in two days and then hold it there for 6 days to allow the pressure to asymptotically bottom out at $10^{-7}$ torr, as shown in Fig 2.4. While at the holding temperature, we degas AMDs by running a moderate current through them one at a time, during which a series pressure bursts are observed as shown in Fig 2.5. After the holding period, we slowly lower the temperature of the chamber to $40 \, ^{\circ}C$, at which point the ionization gauge bottomed out at $7 \times 10^{-10}$ torr, which is close to the pumping limit of the turbo pump. At this point, we turned on two ions pumps and fired the TSP. The ultimate pressure after separating the turbo pump was beyond the bottom measurable pressure of $5 \times 10^{-12}$ torr for a hot-filament ion gauge.

Figure 2.5. Getter bake-out. This is to release gases deposited on AMDs’ surfaces. During bake-out, getter B, C, F and H are degassed with current 0.5 A, 1.0 A, 1.5 A and 2.0 A respectively for 2 hours.
2.3 Diode Laser System

The diode laser system consists of several subsystems, including a laser lock system, a tapered amplifier system and an optical switch-yard. The near detuned laser light used in the experiment is produced by custom built external cavity diode lasers (ECDL). This type of laser is very convenient for atomic physics experiments due to their continuous single-mode tunability over a wide frequency range. The frequencies of these lasers are stabilized by the laser lock system, which utilizes both an absorption lock and a beat-note lock. In the lock system, two ECDLs are locked near the two hyperfine ground state transitions of the Rb $D_2$ line, $F = 1 \rightarrow F'$ and $F = 2 \rightarrow F'$. Due to practical efficiencies of optical components, typical usable power of ECDLs are $\sim 10$ mW. To produce sufficient power for trapping and cooling, we amplify the laser power by a tapered amplifier setup. This experiment utilized a double pass configuration to enhance the power amplification to 400. The typical output of the tapered amplifier setup is about 500 mW, with about 200 mW sent to an optical switch-yard through a polarization maintaining (PM) fiber. The switch-yard split the light into multiple beam paths for different phases of trapping and cooling. It is also responsible for precisely controlling the timing of these beams using acoustic-optical modulators (AOMs). Light beams from the switch-yard are delivered to the main experiment chamber through 10 m PM fibers. This scheme separates the maintenance of diode lasers from the modification of optics on the chamber side. Details of each subsystem are described in the following subsections.

2.3.1 Laser Lock System

The key components of the lock system are four ECDLs which have almost the same configurations. A schematic of an ECDL is shown in Fig 2.6 (a). It consists of a laser diode, a collimating lens, and a diffraction grating which reflects the -1 order peak back into the laser diode (Littrow configuration). The intra-cavity is formed by the reflective front and rear facets of the laser diode, and its optical length can be tuned by changing the injection current, which modifies the index of refraction of the gain material. An external cavity is formed by the laser diode’s rear facet and the diffraction grating, which imprints a frequency profile that depends on the grating’s orientation. A piezoelectric actuator (piezo) is attached to the back of
Figure 2.6. (a) Schematic for an ECDL laser in Littrow configuration. \( \theta = \sin^{-1}(\lambda/2d) \) is the Littrow angle for grating period \( d \). (b) Gain factors that affect the frequency of ECDL in a typical configuration.

the grating for fine tuning the external cavity length. In the present experiment, the laser diode has a gain profile width of about 10 nm centered at 785 nm. The output mode of the laser is determined by the overall gain profile which combines the diode gain, intra-cavity, external cavity, grating and temperature, as shown in Fig 2.6 (b). Temperature affects both the center of the diode gain profile and the cavity length, hence it is stabilized to within 10 mK by a feedback loop using thermal electric cooler (TEC). The lock system stabilizes the frequency by feeding phase error signals back to the injection current of the laser diode and control voltage of the piezo. Typically it allows a single mode tuning range of 10 GHz.
Figure 2.7. Schematic of the optics for laser lock system. Four ECDLs are arranged in the same optical configuration for easy maintenances. The Reference laser (blue) is locked to the 3/4 crossover of $^{85}$Rb through a Doppler free absorption lock using a Rb vapor cell. MOT laser (Orange) and Repumper laser (pink) are stabilized relative to the Reference laser through a beat-note lock.
We adopted a lock scheme, developed by Vuletic [37], that combines the stability of locking to a saturated absorption spectrum with complete tunability over all of the hyperfine components in the excited states of $^{87}$Rb $D_2$ line. Our setup employs four ECDLs with almost the same optical arrangement, which allows each of them to be used interchangeably. This permits fast diagnostics and little down time in case one needs to be replaced. One ECDL, denoted as “reference”, is stabilized to an absorption line. The rest are then locked relative to the reference laser by beating them with it on a fast photodiode. The beatnote frequencies are then further locked to reference oscillators (VCOs) through a phase-locked loop. The wide tuning range of the VCO allows for setting the frequency to be resonant with any transition on $^{87}$Rb $D_2$ line.

The optical schematic of the lock system is shown in Fig 2.7. The ECDLs have almost the same optical setup, such that all of them can be locked to a resonant transition using saturated absorption spectroscopy [38]. In the present experiment, the reference laser is stabilized to the $F = 3 \rightarrow F' = 3/4$ crossover resonance of $^{85}$Rb. A small portion of its output is split into two paths. One of them is used as the pump beam, and goes through a double pass AOM setup that shifts its frequency by 160 MHz ($2 \times f_{AOM}$) and also allows for frequency modulation created by a reference oscillator. The other goes through a cat’s eye setup which maintains the same frequency, and is used as the probe beam as well as the source for beat-note. These two beams are then coupled to two single mode PM fibers separately to clean the mode profiles and improve the stability of the beam paths. They are then sent counterpropagating through a Rb vapor cell which contains a natural isotopic mixture of $^{85}$Rb and $^{87}$Rb. The sub-Doppler absorption resonances convert the frequency modulation of the pump to amplitude modulation of the probe beam. The transmitted probe beam power is measured by a custom built photodiode with a bandwidth of $\sim 1$ MHz, and then mixed with a corresponding reference oscillator to generate a dispersive signal, as shown in Fig 2.8. Finally, this signal is integrated and fed back to both the injection current of the laser diode and the piezo control voltage of the reference laser to stabilize it at the 3/4 crossover. The unity gain bandwidth is measured to be $\sim 10$ kHz, and the RMS frequency excursion is $\sim 600$ kHz while in lock.

A small portion of the reference probe beam combined with the three other probe beams is coupled to a single mode PM fiber and sent to a fast photodiode.
Figure 2.8. Saturation absorption spectra for Rb85. The demodulated signal is shown for transitions for the upper hyperfine manifold $F = 3 \rightarrow F'$. The reference laser is locked to the 3/4 crossover resonance with an offset of 80 MHz to the blue side. (New Focus 1554-LF 12 GHz) for a beat-note lock. The schematic of the lock circuit is shown in Fig 2.10. Out of the rest of the three ECDLs, two of them are denoted as “MOT” and “repumper”, and are locked near the resonance of $F = 2 \rightarrow F' = 3$ and $F = 1 \rightarrow F' = 2$ for the $^{87}$Rb $D_2$ line. The last one (green) is used for backup, and hence is neglected under current discussion. The two beat-notes corresponding to MOT-reference and repumper-reference is 1346.17 MHz and 5031.86 MHz, taking into account AOM shifts from the switch-yard. The output signal from the photodiode is amplified and split into four paths using a four-way splitter (Mini-Circuits ZN4PD1-63W-S+). Two of them are designated to lock MOT and repumper ECDLs, and one is connected to a spectrum analyzer for diagnostic. In order to analyze phases and frequencies, these beat-notes are further shifted down to below 300 MHz. With proper amplification, the MOT beat-note is mixed with a VCO (Mini-Circuits ZX95-2150VW) of frequency 1426 MHz to produce an output around 80 MHz. This signal is then filtered and compared with one 80 MHz reference oscillator (VCO ZOS-150). The repumper beatnote is treated similarly except with a higher frequency VCO (Mini-Circuits ZX95-5400) of 5111 MHz.

The frequency and phase error are detected and fed back to lasers in two parallel paths. One path uses a digital phase frequency discriminator (DPFD) based on AD9901. The DPFD takes in the 80 MHz beat-note signal and compares it with the 80 MHz VCO reference. The output of DPFD is either a 80 MHz pulse train
with its duty cycle proportional to the phase difference in the linear range (0 to 2 \( \pi \)), or a high or low voltage beyond the linear response range. This error signal is then filtered and fed to both the injection current and piezo control voltage of the corresponding ECDL. This whole path has a unity gain bandwidth of \( \sim 50 \) kHz, with a capture range of 100 MHz.

The other feedback path uses an analog phase detector (APD, Mini-Circuits RPD-1) that generates a small voltage depending on the phase errors, which is directly AC coupled to the laser diode. This path has a high bandwidth of \( \sim 2 \) MHz, but has little gain at low frequencies. The DPFD path provides the gain at low frequencies and is capable to compensate thermal drifts and vibrations. The DPFD path keeps the laser frequency near resonance, but is unable to achieve phase lock due to its low gain at high frequencies, where the phase noise is dominated by fluctuations in electronic charge density. The APD path provides the gain over the range of 50 kHz to 2 MHz, that allows the system to acquire phase lock. Two typical power spectra of the MOT and repumper ECDL beat-note signals in lock are shown in Fig 2.9.

![Figure 2.9. Typical beat-note signals for the MOT and repumper ECDLs in lock. The DPFD path and the APD path together provides a narrow peak (100 kHz) in the center. The two shoulders indicate the crossover of the gain for these two paths.](image-url)
### Beat-Note Lock Schematic

**Figure 2.10.** Schematic for beat-note lock. An optical fiber that contains light from reference, MOT, and repumper lasers is connected to a high frequency photodiode, which converts the beats of light intensity to microwave signals. These signals are split into two paths with one for locking the MOT laser and the other for locking the repumper laser. After proper filtering and frequency mixing, they are fed back to the ECDL through two parallel loops based on DPFD and APD respectively.
2.3.2 Double-Pass Tapered Amplifier

The output power from the MOT or repumper ECDL is \( \sim 10 \) mW, which needs to be amplified for laser trapping and cooling. We use a tapered amplifier (EYP-TPA-0780-01000-3006-CMT03-0000) to amplify the light power out of the laser lock setup. A tapered amplifier is a semiconductor device that combines a single mode guiding region and a tapered gain section with a linearly increasing width as the beam propagates through it. In this way, the output power can be much higher without exceeding the maximum intensity limit of the gain material. The two facets of the TA chip are supposedly coated with an anti-reflection (AR) layer to eliminate self-lasing (which we found not to be the case in our setup). Typically, this amplifier is used in a single pass configuration, and requires about 50 mW of injection power to saturate the power gain and output 1 W power. It is known that below saturation, the output light spectrum has a diffuse background from amplified spontaneous emission (ASE). To overcome the shortage of injection power and to suppress the ASE, we utilize a double-pass configuration [39,40], as shown in Fig. 2.11.

![Double-pass tapered amplifier setup](image)

**Figure 2.11.** Double-pass tapered amplifier setup. The injection beam is sent back to the chip through an extinction port of the optical isolator. It passes the gain medium twice so that the output power is enhanced by a factor of 20. (FPI: Fabry-Pérot interferometer)

The MOT and repumper light are combined on the switch-yard (see section 2.3.3) with the same polarization and a power ratio of 10:1. They are then coupled to a single mode PM fiber with an angle polished facet and delivered to the TA injection port. This arrangement decouples the alignment of the beam combining setup on the switch-yard from the TA setup, and also helps pre-align the injection
beam later. The MOT laser is directly coupled to the PM fiber and its injection power is kept constant. The intensity is controlled by AOMs in the switch-yard. Since the repumper laser shares the same beam path with the MOT laser, an AOM and a shutter are inserted in the repumper beam path before combining with the MOT laser to control the intensity of the repumper independently. One portion of the repumper laser is left without going through the TA for other cooling purposes discussed later.

To build the double pass TA, the alignment is carried out in reverse order. We first take advantage of the back going ASE, and couple that to a single mode PM fiber. A 4.51 mm aspherical lens with NA = 0.55 (Thorlabs C230TME-B) is used to collimate the back going beam. Then an auxiliary beam is sent through that fiber to inject the TA. Due to the rectangular shape of the chip’s output facet, the beam profile is first shaped with a 4.03 mm aspherical lens with NA = 0.64 (Thorlabs 354340-B) and then a 22.2 mm cylindrical lens before passing through a two stage optical isolator. The isolator is used for rejecting any light reflected to the output port. To allow light going backward to TA, we inject the light through the middle extinction port instead. To safely do that, we manage to couple a tiny amount of light exiting that extinction port to the injection fiber. It is very critical to control the injection power such that it does not damage the single mode guiding region. Fig 2.12 (a) shows a measurement of the back end power with back injection of 100 µW. We place a mirror to retro-reflect the back going beam such that it goes back through the chip another time. With the same µW injection power, the measured output power is shown in Fig 2.12 (b). Output power can achieve 700 mW, with only 100 µW of injection, after the optical isolator (with transmission 80%) in a double-pass configuration.

In practice, the TA is running at a lower current around 1.3 A with an output power of 400 mW. This is due to a self-lasing problem at high current, where additional frequency modes appears in the spectrum. This mainly results from the imperfect AR coating of the chip facet. To compensate that, the injection power is increased to 1.1 mW, with 10% being repumper light. The output is coupled to a single mode PM fiber with efficiency of 55% and sent to the switch-yard.
Figure 2.12. Power measurement of the tapered amplifier. (a) Power of the backgoing light when injected backward with 100 µW. A measurement of the ASE is also shown for comparison. (b) Power output of the double-pass configuration as a function of current.
2.3.3 Optical Switch-yard

The optical switch-yard takes in the output from the TA and splits it into multiple beam paths, as shown in Fig 2.13. Each path is responsible for one or multiple steps in cooling and trapping of atoms as mentioned in section 2.1. The power of each beam can be controlled independently through a set of AOMs (ISOMET). Inside a AOM, a piezoelectric transducer is attached to a transparent crystal which experiences a modification of its refractive index when applying mechanical pressure (i.e acousto-optic effect). A radio frequency (RF) electric signal typically around 80 MHz drives the transducer and creates a traveling sound wave inside the crystal. Light propagating through the crystal experiences Bragg diffraction with the angle determined by \( \sin \theta = \frac{m \lambda}{\Lambda} \), where \( \lambda \) and \( \Lambda \) represent wavelengths of light and sound wave respectively. \( m \) is the diffracting order and is an integer number \((0, \pm 1, \pm 2, \ldots)\). The frequency of the light is shifted by \( mf_{AOM} \). Usually the \( \pm 1 \) order is chosen as the experiment beam due to its higher diffraction efficiency. By controlling the intensity of the RF signal, we can achieve light switching time of \( \sim 50 \mu s \) with intensity extinction ratio of 1000:1. It also allows for frequency modulation over a range of \( \pm 20 \) MHz which adds more flexibility to the control sequence. To completely turn off the light, we insert into each path a mechanical shutter (nmLaser LST200SLP) which can block the beam in 2 ms. Each of these shutters is driven by running 0.5 A to 1 A current through an electromagnet beam, thus it usually produces lots of heat. Hence, each one is attached to a heat sink and sits on top of a rubber foot to suppress the vibration from closing action and prevent heat from flowing to the optical table.

During a normal experiment sequence, all the beam frequencies are dynamically tuned either through the laser lock point or through frequency modulation of the AOM driver. A shift in frequency changes the diffraction angle of the AOM, thus alters the optical alignment. To improve the stability, most of these AOMs are in a double pass configuration [41], where a lens is placed one focal length away from the AOM with a mirror reflecting the light back through the AOM another time. A quarter waveplate is used to rotate the back-going beam polarization by 90° to separate it from the input beam on a polarization beam splitter (PBS). Finally, the output for each beam path is coupled to a single mode PM fiber and delivered to the main chamber side, which completely separates the optical alignment in the
The purpose of each beam path is listed below:

- **Source MOT**: this path contains both MOT and repumper laser frequencies, and is responsible for loading MOT and polarization gradient cooling (PGC) in the source cell. This path is in a double pass configuration using the +1 order with the MOT AOM frequency set at 100 MHz. It takes 100 mW from the TA output.

- **Source Cell Raman Sideband Cooling (RSC)**: RSC in the source cell uses four beams in a tetrahedral geometry. One AOM controls the vertical lattice beam (Vertical Lattice AOM) and another AOM (Source Cell Lattice AOM) controls the other three beams. These two AOMs share the same RF source to ensure a common frequency. They are in single pass configurations at 80 MHz using the +1 order. All beams are derived from the zeroth order of the MOT AOM path.

- **Detection**: this path is used for diagnostics in the source cell and UHV cell, including time of flight (TOF) and absorption imaging. The AOM (Detection AOM) frequency is the same as the MOT AOM. The power in this path is around 10 mW.

- **UHV MOT**: this path is used for MOT loading and PGC in the UHV cell. The UHV MOT AOM frequency is 100 MHz.

- **RSC Optical Pumping**: this path is for the optical pumping light \((F = 1 \rightarrow F' = 0)\) during RSC. It is taken from the repumper laser before combining with MOT laser. The power is around 1 to 2 mW. The RSC OP AOM operates at 80 MHz in double pass using the +1 order.

- **MOT Repumper**: this path is responsible for controlling the repumper laser independently before combining with the MOT laser. A portion of the repumper laser (1 mW) goes through a double-pass AOM (Repumper AOM) setup at 95 MHz (+1 order). Then it combines with the MOT laser through a PBS, following with another PBS to project them onto the same polarization before coupling to a single mode PM fiber for TA injection.
Figure 2.13. A Diagram of the optical switch-yard. Output from the tapered amplifier is split into multiple beam paths, with each path responsible for one or several steps in cooling and trapping atoms. The power of each beam path is dynamically controlled by an AOM and a mechanical shutter. Specific purpose of each path is discussed in the main text.
2.4 Timing System

As mentioned in section 2.1, a typical experiment consists of many steps and operates as a long series of timed events - a time sequence. For example, in a typical experiment, we first load a MOT, second cool atoms with molasses, and then drop atoms to UHV cell, etc. Each of these steps is a time sequence itself, such as detuning a laser at $T_1$, turning on a coil at $T_2$, etc. The timing and control of all these events is managed by a timing system, which is built around a PXI system (NI PXIe-8133) and its associated components. The PXI system has a set of “output cards” that can produce a large number of digital and analog control signals. These signals can be clocked in a well-defined sequence as commanded by a user through a software interface hosted by the PXI system and the controlling computer. Each of these signals physically controls something in the experiment. For example, one analog signal might control the intensity of a laser beam, and a digital signal might trigger the camera. A time series of each channel can be treated as an array of numbers, with each row corresponding to each signal channel and each column a moment in time. A clock signal advances the column by one and updates the output signals.

A typical experiment can last for tens of seconds to tens of minutes, and the output cards normally can be clocked at $\mu$s periods. Thus it is not possible to clock the card with such a high rate, otherwise there will be too much data $O(10^8)$ to feed through the card. A better way is to clock the card is when there needs to be an update - usually on the order of $10^4$ events- while maintaining the card’s time resolution. As shown in Fig 2.14, the clock signal is arranged in a cascaded manner. A field-programmable gate array (FPGA) is used as a base clock signal for all the cards. It has an on-board clock at 80 MHz which allows a timing resolution of 25 ns. The FPGA generates clock signals for a four channel analog IO board (AIO PXIe-6124) and a 32 channel digital IO board (DIO PXIe-6536). It does not directly output a clock pulse at 80 MHz. Instead, it counts how many clock cycles it needs to wait and only outputs a strobe to these two boards when needed. This dramatically reduces the amount of data sent to the board while still keeping the time resolution. The DIO can output digital control signals to experimental equipment. Also, it provides three channels to serve as update strobes for the three analog output boards (AO PXI-6723,6733). In this case, when the AO board needs
an update, it requires the corresponding DIO channel to generate a trigger, which in turn requires a strobe from FPGA to update the DIO channel. The digital board has a time resolution of 200 ns, and the AO board has a time resolution of 2 $\mu$s. Hence, overall any channel can shift an update with time resolution of 25 ns, which is much better than clocks at any of the board’s own time resolution.

**Figure 2.14.** Schematic of the timing system. The hardware is centered around a PXI system (NI PXIe - 8133). Analog and digital output cards are installed on the PXI system and proivde control signals for experimental equipment. The software is built around a web server that transforms timing sequences written in XML to data arrays executable by Labview. Data generated by the experiment instruments can be analyzed in a GUI.

The software that is responsible for creating a time sequence and transforming it into information to command the hardware is centered around a Python web server. It is a routine that listens on a HTTP socket for requests, such as a command to store or to parse time sequences written in the XML GUI, to store any data, and to execute pythonic commands in user-specific data contexts. The web-based XML GUI is the primary user-interface for the server, where one can create time sequences by specifying edges (an edge is a value-time pair) in XML tags. Taking advantage of the flexibility of Python language, one can create complex time series easily, such as a linear/nonlinear ramping of a voltage. In each experiment run,
a Labview program controls the PXI system and tells it when to execute time sequences. It asks for sequence data from the server, and then parses the desired sequences from the XML GUI and sends it to Labview. The server is also able to store and display data after each experiment run, by interacting with drivers of corresponding instruments, such as the cameras or the oscilloscope. A data GUI is used to perform real-time analysis of the experiment results and provides feedback to any recent parameter changes. All the raw data from the experiment is automatically saved in HDF5 formated files and backed up.

2.5 Trapping and Cooling in the source cell

Experiments in the source cell trap and cool atoms before transferring them to the UHV cell. Later we transfer atoms down to the UHV cell through free fall. The source cell and the UHV cell are separated by 0.8 m, which corresponds to a free fall time of 0.4 s. To increase the brightness of atoms falling into the UHV cell, lower temperatures are needed because the cloud radius $r \propto vt \propto \sqrt{T}$. Starting with a Rb vapor at 300 K created by heating the getter dispenser, there are three main steps in trapping and cooling atoms - MOT, molasses (PGC), and Raman sideband cooling (RSC). Each of these steps is discussed in the following subsections in detail. After RSC, the temperature reaches $\sim 1\mu K$, and the atoms are ready to be released.

2.5.1 MOT Loading

A MOT is a standard and widely used technique in trapping atoms to high density due to its ease to set up, large capture velocity, and robustness. Its experimental implementation and properties have been studied extensively in the literature [42–46]. A working MOT depends on two effects: one is the Doppler cooling process which creates a damping force for the atoms and cools them. The other one is a linearly inhomogeneous magnetic field usually created by a pair of quadrupole coils. Doppler cooling in three dimensions typically requires three pairs of counterpropagating beams with their frequency red detuned to a closed electric dipole transition. The two beams in each pair are circularly polarized and are of the opposite polarization. It is arranged such that any nonzero velocity
of an atom shifts a beam’s frequency closer to its resonance, when the beam’s propagation direction opposes that atom’s motion. This effect causes the atom to scatter preferentially from that beam, which slows it down. The quadrupole magnetic field gradient causes a position dependent Zeeman shift of the ground state magnetic sublevels, so that atoms further away from the zero of the magnetic field are closer to resonance with a particular beam. This effectively provides a restoring force towards the field zero. Usually atoms are cooled and trapped within a region of several hundred of micrometers. Their motion resembles that of a strongly damped harmonic oscillator.

![Optical configuration in the source cell. Our MOT consists of three pairs of counterpropagating beams and one pair of quadrupole coils. The $\infty$ shape beam path permits using a moving molasses technique if needed. The repumper light frequency is on the $F = 1 \rightarrow F' = 2$ transition, and the MOT beam is red detuned from the $F = 2 \rightarrow F' = 3$ transition with $\delta = 24$ MHz. Four lattice beams (blue) create a 3D lattice potential for RSC. One optical pumping beam (yellow) for RSC is sent in from a side with circular polarization. The temperature of the atoms in the source cell is measured by a time-of-flight (TOF) setup using two light sheets, with the upper one 4 cm below the MOT which measures the absorption of atoms passing through it and the lower one as an intensity reference.](image)

**Figure 2.15.** Optical configuration in the source cell. Our MOT consists of three pairs of counterpropagating beams and one pair of quadrupole coils. The $\infty$ shape beam path permits using a moving molasses technique if needed. The repumper light frequency is on the $F = 1 \rightarrow F' = 2$ transition, and the MOT beam is red detuned from the $F = 2 \rightarrow F' = 3$ transition with $\delta = 24$ MHz. Four lattice beams (blue) create a 3D lattice potential for RSC. One optical pumping beam (yellow) for RSC is sent in from a side with circular polarization. The temperature of the atoms in the source cell is measured by a time-of-flight (TOF) setup using two light sheets, with the upper one 4 cm below the MOT which measures the absorption of atoms passing through it and the lower one as an intensity reference.

Fig 2.15 shows an optical diagram of our source MOT setup. The light beam is delivered to the main chamber from the switch-yard through a single mode PM fiber. It contains both the MOT and the repumper frequencies. The MOT laser is tuned close to the $F = 2 \rightarrow F' = 3$ transition, with a detuning to the red by 24 MHz. Due to an off-resonance transition from $F = 2 \rightarrow F' = 2$, atoms may end up in the $F = 1$ ground state and will not scatter MOT light any further. The
repumper light is fixed on the $F = 1 \rightarrow F' = 2$ transition which brings the atoms back to the $F = 2$ manifold, creating a closed loop for atoms to continue scattering photons. We create two pairs of counterpropagating laser beams by splitting a laser beam on a PBS into two paths and routing them in a $\infty$ shape. The light is circularly polarized using a series of quarter waveplates. Another beam pair along the quadrupole axis is created by sending a beam through the center of the coils and retro-reflecting it back again. Each beam has an intensity of 7.5 mW/cm$^2$. This scheme allows us to create three beam pairs with better power efficiency compared to splitting one beam into six. Furthermore, this also enables us to implement moving molasses in the future, which requires the upward (downward) propagating two beams to be red-shifted (blue-shifted) relative to the quadrupole axis pair. Each of the quadrupole coils has 30 turns with a diameter of 2.5 inches. The two combined produce a magnetic field gradient of 16.8 G/cm at 10 A along the quadrupole axis.

In the present experiment, the $1/e$ loading time of the MOT is set to 600 ms, which can be tuned by changing the vapor pressure. The atom number is counted through a fluorescence measurement. A photodiode is placed near the cell measures the peak resonance fluorescence power in the absence of the quadrupole field. On resonance, atoms scatter photons at the saturated rate of $\Gamma/2$. The optical power collected by the photodiode with solid angle $d\Omega$ is

$$P_{pd} = N \frac{hc \Gamma}{\lambda} \frac{d\Omega}{4\pi}, \quad (2.1)$$

where $N$ is the number of atoms and $\lambda$ is the wavelength of the scattered light. The photodiode converts the light power to a photocurrent and outputs a voltage based on the trans-impedance gain. With our setup, one atom corresponds to 2.6 nV. Fig 2.16 shows a typical measurement of atom number around $2 \times 10^8$.

Due to the Doppler-cooling mechanism, the temperature of atoms inside a MOT is limited by the Doppler-temperature $T_D$, which is given by

$$k_B T_D = \frac{h \Gamma}{2}, \quad (2.2)$$

where $k_B$ is the Boltzmann constant. For $^{87}$Rb with $\Gamma = 2\pi \times 6.06$ MHz, this temperature is 145 $\mu$K. We apply additional cooling steps to bring the temperature below this limit.
Figure 2.16. Fluorescence measurement of atom number in a MOT. At $t = 10$ ms, the MOT and repumper lights are turned off together with the quadrupole coils, while the MOT light is brought into resonance. The atom number is derived from the peak fluorescence level $\Delta V$. This measurement shows $1.7 \times 10^8$ atoms in the MOT after 400 ms of loading.

### 2.5.2 Optical Molasses

To further lower the atom temperature, we apply polarization gradient cooling (PGC) through optical molasses [47–51]. It uses the same optical configuration for the MOT but in the absence of any magnetic field. A simple illustration of PGC in one dimension is shown in Fig 2.17. In this case, two counterpropagating light beams of perpendicular linear polarization interfere with each other creating a light field with a polarization that varies from $\sigma^-$ to $\sigma^+$ over half-wavelength distance. Consider the transition of $F \rightarrow F' = F + 1$ : the light shifts, $\Delta E_g$, of the ground magnetic sublevels are given by [47],

$$\Delta E_g = \frac{\hbar \delta C_{ge}^2}{1 + (2\delta/\Gamma)^2 I_s},$$

where $C_{ge}$ is the Clebsch-Gordan coefficient that describes the coupling between an atom and a light field, $I_s$ is the saturation intensity, and $\delta$ is the detuning.

Since the value of $C_{ge}$ depends on both magnetic quantum numbers, and also the polarization of light field, the light shift for $m = F$ can be many times larger than that for $m = -F$ when the light field is completely $\sigma^+$, and vice versa. Therefore, in places where the light is $\sigma^+$, the light pumps the atoms to the $m = F$ states which
are strongly negatively shifted, and similarly for positions that are $\sigma -$ polarized in which atoms are driven into $m = -F$ states. As the atoms move through the light field, their potential energy increase due to the change in polarization, while their kinetic energy decrease. At a local maximum, they are pumped to the opposite magnetic states and their potential energy are removed by scattering photons with a higher frequency. This process continues and in general atoms will climb potential hills and on average lose energy. The PGC model is much more complicated in three dimensions, because there are always polarization gradients exist. The final temperature $T_{PGC}$ scales with the light shift $\Delta E_g$ as

$$k_B T_{PGC} = b \Delta E_g,$$  \hspace{1cm} (2.4)

where $b$ is a coefficient that depends on the polarization scheme. As we can see, to lower the final temperature, we can increase the detuning, and/or lower the intensity.

In the experiment, we apply molasses following the MOT by turning off the quadrupole coil and increasing detuning of the light. The PGC time period is over in 3 ms. During the first 2 ms, the detuning of the MOT beam is ramped from 96 MHz to 183 MHz, while the intensity of the MOT beam is ramped down to 1/10 of

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{Fig2_17.png}
\caption{Illustration of polarization gradient cooling in one dimension. Two counterpropagating beams of perpendicular linear polarization interfere with each other creating a light field with its polarization varying from $\sigma -$ to $\sigma +$ over a half-wavelength distance. Light shifts for an atom in $m = F$ (blue) and $m = -F$ (red) are shown for a transition from $F \rightarrow F' = F + 1$. The atoms that move in this light field convert kinetic energy to potential energy, which then is continuously removed through scattering photons.}
\end{figure}
Figure 2.18. Time of flight measurement of the molasses temperature. The time trace shows a FWHM of 4.6 ms, indicating a temperature of 4.5 $\mu$K. The red dashed line shows a single peak Gaussian fit.

its initial value. The atoms are held at these parameters for additional 1 ms and reach a temperature of $\sim 4 \mu K$ as measured by time of flight analysis as shown in Fig 2.18. In order for the PGC to work well, the ambient magnetic field should be zeroed within a few tens of mG, such that the Zeeman shifts between magnetic sublevels do not dominate the light shifts. There are eleven sets of compensation coils around the chamber to optimize the temperature for this reason.

### 2.5.3 Raman Sideband Cooling

To lower the temperature further and facilitate the transfer of atoms into the UHV cell, we further cool the atoms using three dimensional Raman sideband cooling, which, unlike optical molasses, is a dark state cooling scheme and can in principle produce temperatures below a single photon recoil [36,52,53]. Following the molasses period, atoms are pumped into the $F = 1$ hyperfine ground state by turning off the repumper beam and keeping the MOT beam at a low intensity for 1 ms. The atoms are loaded into a 3D optical lattice potential consisting of four beams in a tetrahedral geometry, as shown in Fig 2.15. A small uniform
external magnetic field is applied to the atoms such that the Zeeman splitting \( \Delta E_z = g_F \mu_B B \) equals the vibrational quanta, \( \hbar \omega \), with \( \omega \) the vibrational frequency, as shown in Fig 2.19. A Raman transition, provided by lattice beams, couples atoms with degenerate magnetic sublevels that differ by one vibrational quanta. An optical pumping beam with mostly \( \sigma^+ \) and a little \( \pi \) polarization drives the \( F = 1 \rightarrow F' = 0 \) transition. The \( \sigma^+ \) light pumps atoms from the \( |F = 1, m = -1, \nu \rangle \) states to the \( |F = 0, m = 0, \nu \rangle \) states, followed by a spontaneous emission back to the \( |F = 1, m, \nu \rangle \) hyperfine manifold. In the Lamb-Dicke regime, which is the case for this experiment, the vibrational quantum number \( \nu \) remains unchanged.

As time goes on, most of the atoms accumulate in the \( |F = 1, m = 1, \nu = 0 \rangle \) state, with a small portion that remain in the \( |F = 1, m = 0, \nu = 0 \rangle \) and the \( |F = 1, m = 1, \nu = 1 \rangle \) states. The \( \pi \) polarization component of the pumping light clears atoms in the \( |F = 1, m = 0, \nu = 0 \rangle \) state and drives them to the \( |F = 1, m = 1, \nu = 0 \rangle \) state. Thus, the whole pumping process creates a dark state \( |F = 1, m = 1, \nu = 0 \rangle \), where atoms no longer participate in the pumping process. Atoms can be released from the lattice into free space adiabatically resulting in extremely low temperatures.

The lattice light is delivered from the switch-yard to the source cell using two PM fibers - one for the vertical lattice beam, and the other is split into three beams. All four beams are linearly polarized and are arranged in a tetrahedral geometry. Due to the transverse nature of light wave, the lattice potential not only provides a scalar potential, but also generates a vector light shifts due to the crossed polarizations of the beams. The vector light shift can be treated as an effective magnetic field that drives the degenerate Raman transitions. Furthermore, the lattice light also serves as a depumping beam that clears the atoms off the \( F = 2 \) hyperfine manifold that are there due to off resonant transitions. The frequency of the lattice beam is set at 20 MHz blue of the \( F = 2 \rightarrow F' = 2 \) transition during the cooling period. The optical pumping beam is split from the repumper laser as discussed earlier. The polarization is critical for getting to low temperatures, because a small amount of \( \sigma^- \) light would drive the atoms out of the dark state. The direction of the optical pumping beam is shown in Fig 2.15. The external magnetic field is applied nearly aligned with the optical pumping beam with a small angle in order to add a component of \( \pi \) polarization. The lattice depth is \( \sim 50 \) kHz and the optical pumping intensity is \( \sim 1/100 I_s \).
Figure 2.19. Degenerate Raman sideband cooling. By applying a moderate external magnetic field, the Zeman shift $g_F \mu_B B$ of the magnetic sublevel is matched to the vibrational energy $\hbar \omega$ of the lattice. This allows degenerate Raman coupling (blue lines) to change the atomic state between different magnetic sublevels. Atoms are driven by optical pumping light (red lines) with mostly $\sigma^+$ light and a little $\pi$ polarization into the lowest vibrational state in $m = 1$, where they are free from scattering photons. Frequencies of the lattice and optical pumping beam are shown on the right, with $\delta_l = 20$ MHz and $\delta_{op} = 12$ MHz.

RSC is applied immediately after the molasses period. The lattice beams are pulsed on for 13 ms, during which the optical pumping beam is applied with its frequency blue detuned to the $F = 1 \rightarrow F = 0$ transition by 12 MHz. The temperature is minimized by tuning the compensation coil which generates an external magnetic field of 80 mG. To measure the temperature after RSC, we adiabatically ramp the lattice off in 2 ms to remove the zero point energy and measure a temperature $\sim 1 \mu K$, as shown in Fig 2.20. The loading efficiency from molasses to lattice is about 80%.

2.5.4 Time of Flight

The temperature of the atom cloud in the source cell is measured through a time of flight (TOF) technique. A simple setup is shown in Fig 2.15. During the TOF, the atom cloud is allowed to fall under gravity and freely expand, with a rate that depends on its temperature. There are two light sheets below the MOT/lattice position - the upper one is 4 cm below the MOT position and the lower one is 30
Figure 2.20. Time of flight measurement of RSC temperature. The red dash line shows a two components Gaussian fit that takes into account the atoms falling from the molasses cooling. The fit shows a FWHM of 3 ms for the RSC atoms, indicating a temperature of $1\,\mu K$.

cm below which serves as an intensity reference. The light sheets contain both MOT and repumper laser frequencies, with the MOT tuned to the $F = 2 \rightarrow F' = 3$ resonance. As the cloud passes through the upper light sheet, the atoms absorb a fraction of the light, and the overall transmitted power is measured by a photodiode. According to Beer’s law, the transmitted intensity is given by

$$I(t) = I_0 e^{-n(t)\sigma}, \quad (2.5)$$

where $\sigma$ is the scattering cross section. A logarithmic amplifier takes in signals from both photodiodes and outputs a voltage $V_o$,

$$V_o = \ln \frac{V_{\text{upper}}}{V_{\text{lower}}} \propto \ln \frac{I(t)}{I_{\text{lower}}} \propto n(t), \quad (2.6)$$

where $n(t)$ is the density of atoms passing through the upper light sheet. The velocity of the atoms in a thermal cloud in equilibrium follow a Maxwell-Boltzmann distribution. Assuming the density of the cloud forms a Gaussian spatial distribution with an initial FWHM of $w_0$, then at time $t$ due to thermal expansion, the width becomes
where \( w_v(t) \) is determined by the temperature,

\[
  w_v(t) = \sqrt{\frac{2 \ln 2 k_B T}{m}}.
\]

The time trace of the TOF signal is given by \[54\]

\[
  V(t) \propto \frac{1}{\sqrt{2\pi \sigma_t}} \exp \left[ -\left( \frac{g(t_0^2 - t^2)}{2\sqrt{2\sigma_t}} \right)^2 \right]
\]

where \( \sigma_t = \frac{w_t}{2 \ln 2} \). With a little approximation, the temperature is related to the FWHM of the time trace given by

\[
  k_B T = \frac{M g^2 t_w^2}{8 \ln 2} - \frac{M g w_0^2}{16 d \ln 2},
\]

where \( d \) is the distance, \( M \) is atom mass, and \( t_w \) is the FWHM of the time trace.

### 2.6 UHV Cell Setup

The UHV cell has a pressure of at most \( \sim 10^{-12} \) torr, hence it permits a long trap lifetime. After the RSC in the source cell, atoms are transferred to the UHV cell through a straight path defined by a high power guiding beam along the direction of gravity. The guiding beam provides a confining potential transverse to the direction of the atoms’ motion with a peak depth of \( \sim 2 \mu K \), and improves the transfer efficiency by a factor of 2 per drop. Experiments in the UHV cell center around a high resolution imaging system consisting of four high numerical aperture microscope objectives. Cooling and trapping in the UHV cell includes MOT, molasses and RSC in a disordered potential (discussed in the next chapter). A simplified diagram of the optical setup for the UHV cell is shown in Fig 2.21.

#### 2.6.1 UHV MOT Loading

When atoms reach the UHV cell, they are recaptured by a small volume MOT, which uses three pairs of counterpropagating beams. Due to spatial constraints from the imaging system, the UHV MOT volume cannot be very large. The quadrupole
beam goes through two high NA (0.4) microscope objectives which are part of the imaging system, as shown in Fig 2.21. This limits the beam size to a maximum of 3 mm inside the cell. The guiding beam is aligned to connect the source MOT and the UHV MOT, such that atoms are sent to the UHV MOT directly. The transfer rate is $10^6$ atoms per drop. It is noted that due to the small size of the UHV MOT, the magnetic field zero needs to be carefully positioned inside the volume. We use the surrounding compensation coils to optimize the number of atoms per drop. The benefit by doing so is that we can place the objectives one working distance (20 mm) away from the atom cloud. To load more atoms into the MOT, in the present experiment, we typically accumulate several drops from the source MOT. The UHV MOT is kept on for a period of several seconds, during which multiple repetitions of cooling and trapping in the source cell are performed to create loadings. Typically $10^7$ atoms are loaded in the UHV MOT before preceding to the next step.

### 2.6.2 Imaging System

The imaging system consists of four identical objectives (USMC WM-020NIR) arranged around the four sides of the UHV cell. Each of them are NIR coated with total transmission of around 50% for 780 nm and 1064 nm. This setup in principle can image atoms in four directions, with two images relayed back through a folded telescope. Images from these two perpendicular axes are routed back to a CCD camera (Roper Scientific) with $512 \times 512$ pixels. There are two imaging paths with different image magnifications. On each axis, denoted as quadrupole or non-quadrupole path, a 12 cm 2 inch diameter achromatic lens in combination with the microscope objective produces an image plane before a beam splitter. This intermediate image is magnified 12 times compared to the actual cloud size. On one side of the beam splitter lies a 10 cm and a 20 cm lens that relay the image onto the camera with an additional magnification of 2, resulting in a total magnification of 24. On the other side of the beam splitter, two lenses (15 cm and 16 cm ) are combined together to demagnify the intermediate image by a factor of 5, which results in a total magnification of 2.5. These two paths allow us to switch imaging between a larger field of view (FOV) with fewer details (low magnification path) and a smaller FOV but higher resolution (high magnification path). Furthermore, the optical setup allows us to image atoms using both fluorescence and absorption.
Figure 2.21. Simplified diagram for the UHV cell optics. A imaging system consisting of four microscope objectives is built around the UHV cell. Images of the atoms are routed to a scientific camera through two imaging paths - one with magnification of 24 and the other with magnification of 2.5. Both fluorescence and absorption images can be taken of the atoms. A six beam MOT is used to capture atoms falling from the source cell. One pair of MOT beams pass through two objectives, restricting the beam size to a maximum of 3 mm. Optical dipole traps and potential patterns can be projected in through the imaging system.
An absorption beam is sent back to the cell through a beam splitter in front of the camera. The absorption light can be sent through both the high and the low magnification paths as well. On the other hand, this imaging system also allows us to project potentials and light patterns onto the atoms with fine structures. In the next chapter, we create a three dimensional disordered potential by projecting random light patterns through this system.

2.6.3 High Power Laser System

In our experiment, most of the dipole potentials are derived from a high power fiber amplifier (Nufern). A diagram of the system and some related optics are shown in fig 2.22. A single mode laser (Lightwave Electronics NPRO 126) provides the seed beam (1064 nm) with 80 mW optical power. The fiber amplifier outputs a 40 W single mode beam with linear polarization. This beam enters a switch-yard and is split into multiple paths. One of the paths delivers light into two large-core multimode fibers (Thorlabs - MHP910L02, 0.22 NA). These fibers are used in the experiment to create a disordered potential in three dimensions, by projecting their output onto the atoms through the imaging system. There are three double-pass AOM setups on the table. One of them controls the power of a beam for a dipole potential along the quadrupole path to hold atoms against gravity. The other two are for a rotating lattice setup [8]. To create this lattice, we first interfere three beams with equal intensity at even projection angles to form a triangular lattice of tubes, and then modulate the phase of two of these beams at a much higher frequency compared to the lattice vibrational frequency using electroptic phase modulators (EOMs). One AOM is responsible for controlling the intensity of these three beams. The other AOM is used to control a beam that goes through the center of these three beams. This beam creates a standing wave along the non-quadrupole axis which separates the atoms into an array of two dimensional traps in order to study fractional Hall physics.
Figure 2.22. Optical diagram for the high power laser system. This system is based on a high power fiber amplifier (Nufern). Typically, light for each path is controlled by AOMs and shutters. A portion of the output is coupled into two large-core multimode fibers that generate random light patterns, and are later used to create a 3D disordered potential.

2.6.4 Finger Imaging System

In the UHV cell, the bottom part is a circular shaped glass nipple (“finger”) connected to the square part through a 3 mm diameter glass neck, as shown in Fig 2.23 (a). We construct a high resolution imaging system using a microscope objective (M PLAN 50) with NA = 0.5 combined with a plano-convex lens attached to the bottom surface of the finger. The total NA of this system is 0.8. This imaging system is designed such that the vacuum wall and the lens form a first objective for atoms placed 100 µm away from the inside surface. The microscope approximately
conforms to an aplanatic sphere, which simultaneously minimizes spherical aberration and coma. Fig 2.23 (c) shows a test image of 250 nm nanoparticles taken from this microscope. This system can be used to perform site resolved measurement of FQH states. Rotating lattice beams can be projected in from underneath.

Figure 2.23. (a) A picture of the finger imaging system. (b) Close look at the finger. Beam 1,2,3: 1064nm beams for the rotating lattice, with two EOMs inserted. Beam 4,5: 1064 nm beams form a cross dipole trap for evaporative cooling; Beam 6,7: 1064 nm beams form a lattice in the axial direction, also this is a beam path for 1080nm light; Beam 8,9,10,11: for RSC beams that trap and cool atoms, also to be used for fluorescence imaging; Beam path 8 also contains the guiding beam. (c) Test imaging with 250 nm nano-particles from the imaging system. (d) Fourier transform of the image of separated point-like objects, which reflect the modulation transfer function of the microscope. The dashed circle corresponds to a spatial resolution of 500 nm, which is consistent with a NA=0.8.
Chapter 3
Non-equilibrium Behavior of Atoms in a Disordered Potential

In this chapter, we present a table-top realization of a non-equilibrium quantum system described by a dynamical gauge field propagating on an effectively curved space and time manifold. The system is formed by neutral atoms interacting with both a conservative disordered optical field and a dissipative pumping field. In the presence of a sufficiently dark state, we demonstrate non-equilibrium behavior reminiscent of the information paradox in black hole physics. At a well-defined transition point, the analog of gauge-boson mass is seen to vanish, inducing scale-invariant behavior as a Higgs-like mechanism is removed. The subsequent scaling behavior can be understood using the holographic principle with a tunable analog of the Planck length derived from the scaling of disorder. These effects suggest a range of new phenomena in weakly dissipative quantum systems, including the presence of emergent forms of analog gravitation.

3.1 Introduction

At the heart of both gauge symmetry and the second-law of thermodynamics is a redundant labeling of the quantum states of a system. In the former, this is a consequence of local symmetries, while in the latter it arises from discarding information regarding the state of an external body. A derivative type of equivalence is exploited in recent work on black-hole physics [55], quantum gravity [56] and attacks on long-standing open questions, such as the existence at large length scales of a gauge boson mass gap in Quantum-Chromodynamics (QCD). This suggests that the study of table-top systems in which the nonequilibrium behavior of a
quantized gauge field is sufficiently slow to be observed can shed insight on several problems of broad significance.

To introduce the idea, we first note that for any open single-particle system whose basic interaction can be written as

\[ H = \frac{\vec{p}^2}{2m} + V(q) + V_{\text{int}}(q, Q), \]  

(3.1)
a trivial gauge transformation can take the energy into

\[ H' = \vec{F} \cdot (\vec{p} - \vec{A}_H(q, Q))^2/2m, \]  

(3.2)

where we denote by \( p, q \) the particle momenta and coordinates, \( V \) its potential energy, and \( V_{\text{int}} \) its interaction with its environment with degrees-of-freedom \( Q \). Here, \( \vec{A}_H(q, Q) \) is a gauge field that contains all the interactions between the particle and its environment, and also experiences back action from the particle.

Without further manipulation, the four-vector potential \( A_H^\mu = (\phi, \vec{A}_H) \), with \( \phi \) the scalar potential, is from a geometric picture trivial in that its field curvature \( F^{\mu\nu} = \partial^\mu A_H^\nu - \partial^\nu A_H^\mu \) must vanish everywhere. However, since the \( V_{\text{int}}(q, Q) \) must include degrees-of-freedom \( Q \) from the environment, any probabilistic description using a density-matrix, \( \rho(q, q', t) \), must generally account for ignorance of the state of \( Q \) by incorporating nonconservative dynamics into the evolution. As a result, the curvature \( F^{\mu\nu}(q) \) used to predict probabilities for the reduced system no longer vanishes, and \( A_H^\mu \) lies on the orbit of a fundamentally irreducible and dynamical gauge field degree of freedom \( A_d^\mu(q) \) arising from this (potentially dissipative) interaction, without approximation, and despite the a priori lack of electromagnetic fields, free charges, or spins in either body. Furthermore, in a weak coupling limit, the quantization of \( A_d \) and \( A_H \) become relevant to these dynamics.

While it is not immediately clear such a description is useful, it does suggest that simple geometric aspects underly the statistics of particle motion with dissipation. For example, even at the classical limit, through a recent generalization [57] of Berry’s geometric phase, one then might expect that bringing the particle along a closed loop \( q(t) \) may modify \( \rho(q, q') \) by a “Berry’s entropy” \( s_B \) related to the curvature of the gauge field [57] threading the loop, and given for an initially pure state by the final \( \rho \ln \rho \), or entropy of entanglement with \( Q \). If \( Q \) represent weakly-coupled light-scattering modes for atoms in the presence of a strong conservative
potential, the final state heat energy $k_B T s_B$ may be seen as a topological redefinition of the photon recoil.

For a thermal wave-packet under continuous weak optical cooling, this can result in Unruh-like [58,59] effects, making the photon and dynamical gauge vacuums experienced by slowly-scattering atoms indistinguishable from a thermal reservoir at a temperature $T_u = \hbar a / 2 \pi k_B v$ with $a, v$ the local wave-packet acceleration and velocity. For atoms with sufficient slow-light-like effects induced by weak coupling fields (see Section below), $T_u, a,$ and $v$ can be set on a common scale by the photon recoil. The resulting motion of atoms is a reflection of the underlying geometric relations of $A_H^\mu$ and $A_d^\mu$ and the quantization properties of the dynamical field $A_d^\mu$.

We introduce this type of gauge field based on a well-known dark state cooling mechanism - Raman sideband cooling - in a disordered potential. The disordered environment provides two benefits. One is that it contains a broad range of potential depths for atoms, and forms the basis for non-equilibrium dynamics. The other is that it contains a large amount of information compared to a regular lattice, which is reflected in the behavior of the atoms motion through interaction. The random potential also interacts with atoms' spin states through an effective magnetic field generated by crossed polarizations. The cooling effect happens when we apply a weak optical pumping field together with an external magnetic field. The atom-optical interaction can be treated as a dynamical gauge field living in a curved space. Details of the experiment are described in the following sections.

### 3.2 Generation of a Disordered Potential and its Characterization

The disordered potential in this experiment is generated by projecting random mode profiles through the high numerical aperture imaging system discussed in the last chapter. As shown in Fig 2.21, two multimode fibers (core size 0.91 um, Thorlabs - MHP910L02, 0.22 NA) are used for the two axes, labeled as quadrupole and non-quadrupole axes. We couple a single mode beam into each multimode fiber, which outputs an equilibrium mixture of all supporting modes. Four random profiles are then projected onto atoms to form a far-off resonance potential in a small volume. Information can be controlled by a selective launch of optical fibers.
An illustration of the setup is shown in Fig. 3.1.

**Figure 3.1.** Generation of the disordered potential. An equilibrium distribution of optical fiber modes projects four spatially disordered wavefronts, to form a far-off-resonant potential in a small volume, in which atoms are dissipatively coupled to a single-mode optical pumping beam. Information carried by wavefronts can be controlled by selective launch of optical fibers, yielding altered structure in the aperture plane.

We model the optical properties of the fiber using all solutions of Maxwell’s equations in a cylindrically-symmetric dielectric waveguide [60] described by electric field strengths in cylindrical coordinates,

\[
\bar{E}_{\nu_m, K_m} = \left[ \frac{i}{K_m^2} \left( \beta_m K_m A J'_{\nu_m}(K_m r) + i \nu_m \omega \mu B J_{\nu_m}(K_m r) \right) \hat{r} + \frac{i}{K_m^2} \left( \beta_m i \nu_m \frac{A}{r} J_{\nu_m}(K_m r) - B \omega \mu K_m J'_{\nu_m}(K_m r) \right) \hat{\phi} + A J_{\nu_m}(K_m r) \hat{z} \right] e^{i(\nu_m \phi + \beta_m z)} \tag{3.3}
\]

where \( m \) indexes all solutions to the characteristic equation, \( \nu_m \) is the (integer) azimuthal mode index, \( K_m \) the radial wave-vector, and \( \beta_m \) the longitudinal wave-vector. \( J_{\nu} \) are \( \nu^{th} \)-order Bessel functions of the first kind, and the constants \( A, B \) are provided in reference [60]. Through the course of propagation, an initially well-defined superposition of modes becomes, through random intermodal coupling, a statistical distribution of occupied modes with amplitudes we designate by \( \{a_m\} \). We argue that the most statistically relevant superposition is defined by a
maximum entropy argument to distribute power evenly between modes in the fiber. We represent the collapse of information contained in the many mode amplitudes $a_m$ into a typical region of space where beams interfere by constructing a local power-series representation of the electric field from all four fibers. The power series is computed with some effort by first expanding the electric field for a given beam to all orders, $\sigma$, in cylindrical radius $r_b$ about its own optical axis,

$$\vec{E}_f = \sum_{m,b} a_{m,b} \hat{E}_{\nu_m,K_m,b}$$

(3.4)

where $b$ indexes the beam, $\hat{k}_b \in (\hat{r}_b, \hat{\phi}_b, \hat{z}_b)$ are cylindrical unit vectors defined along each beam axis, and $r_b, \phi_b, z_b$ are the corresponding coordinates. Combining all beams into a common cartesian coordinate system $(x, y, z)$ centered at the beam intersections and scaled by a variable length $\ell$, and expanding the result in a local power series near the origin as $\vec{E}_f = E_{k,l_x,l_y,l_z}(2x/\ell)^{l_x}(2y/\ell)^{l_y}(2z/\ell)^{l_z}\hat{k}$, the coefficients $E_{k,l_x,l_y,l_z}$ may be determined from the mode amplitudes $\{a_{m,b}\}$ through a matrix $\Lambda$ as $E_i = \Lambda_{ij} a_j$. This expansion was checked by confirming that $\vec{E}_{\nu_m,K_m} - E_{k,l_x,l_y,l_z} x^{l_x} y^{l_y} z^{l_z} \hat{k} = 0$ to machine precision within a radius of 1 micron from the origin.

We characterize the effective number of degrees of freedom, $F_s$, in the electric field by taking the exponential of the Shannon entropy of $\Lambda$,

$$F_s = e^{-\sum_i s_i \ln s_i}$$

(3.5)

where $s_i$ are the singular values $\sigma_i$ of $\Lambda$ normalized to their sum, $s_i = \sigma_i / \sum_i \sigma_i$. A plot of the singular values is shown in Fig. 3.2 b and c, illustrating for distances short compared to the optical wavelength three effective degrees of freedom for the three directions of electric field, for longer distances a sub-extensive power-law scaling due to long-range correlations in the field strength, and at larger distances a breakdown of the power-series expansion, which occurs later for a larger number of terms in the expansion. The value of $\kappa$ is extracted from a best fit over the scaling region for a maximum term in the expansion of $l_x + l_y + l_z = 12$, which consists of approximately two decades in length scale.
Figure 3.2. Characterization of the optical modes that form the disordered potential. **a**, Solutions of the characteristic equation for the optical fiber for all possible modes denoted by $\nu$, $K$ pairs. The color scale denotes the deviation $\beta - n_1 k$, with $n_1$ the core index and $k = 2\pi/\lambda$, for wavelength $\lambda$, of the propagation constant $\beta$ from that of an axial ray. The green (blue) horizontal line indicates the lower (upper) boundary of the occupied modes for the only high-$K$ (low-$K$) data in Fig. 3.4 and Fig. 3.10 **e**,f, g (c), d, g. **b**, The information content projected into a volume of dimension $\ell$ in the experiment chamber is apparent from the contraction of normalized singular values of the expansion matrix $\Lambda$ as the scaling parameter $\ell$ is increased. All singular values of $\Lambda$ constructed to 10th order in the power series expansion are shown for $\ell = 0.32\mu m$ (red), $\ell = 1.0\mu m$ (orange), and $\ell = 3.0\mu m$ (green). **c**, Most significant singular values for all length-scales $\ell$. The diffraction limit is visible as a closing of the gap between the largest three eigenvalues as the length-scale is increased to $O(1\mu m)$. **d**, Scaling of information content within a volume as determined by the eigenmode structure of the fiber, and mode projections into experiment chamber, demonstrating $F_s \sim \ell^\kappa$, with $\kappa = 0.54 \pm 0.06$. The power series shown contain all terms up to order 6 (squares), 8 (pluses), 10 (crosses), and 12 (circles). The red line shows the best fit between 0.04$\mu m$ and 3.6$\mu m$ for 12th order, which is used to extract $\kappa$. Its uncertainty corresponds to the standard deviation of pairwise slopes for all points presented here.

To vary the information scaling law described by $\kappa$ and thus interrogate the holographic scaling relation later, non-equilibrium mode distributions were created in the fiber by varying the launch conditions. While equilibrated mode content was created using a strong lens to focus light near the entrance facet of the fiber at a numerical aperture just below that of the fiber, under-filling of modes was accomplished by utilizing the comparatively slow redistribution of mode content between modes with large differences in $K_m$. Non-equilibrated conditions were
generated by changing the angle and inclination of the injection of the beam to the fiber axis. Two types of non-equilibrated conditions were generated by either (I) injecting a collimated beam along the fiber axis which predominantly populates the low-radial modes or (II) injecting a focused beam with an inclination to the fiber axis which populates the high-radial modes.

\( \kappa \) was calculated for each of these conditions using the method described above but with \( \Lambda \) multiplied by an additional mask that sets all high-radial (low-radial) modes greater (less) than \( \max(K_m) \times K_{\text{rad}}/K_M \) to zero. \( K_{\text{rad}} \) is equal to the radius of the beam in the Fourier plane determined by measuring where the average radial intensity falls to 50\% of the intensity at the center, and \( K_M \) is the value of \( K_{\text{rad}} \) with equilibrated mode content. To verify the robustness of the calculation of \( \kappa \), we also weighted each of the modes of \( \Lambda \) by either a gaussian with an envelope whose standard deviation is equal to \( \max(K_m) \times K_{\text{rad}}/K_M \), or to a seventh degree polynomial spline fitted to the radially averaged distribution of intensities of the beam in the Fourier plane. The gaussian weighting yielded \( \kappa \) to within \( \pm 0.02 \), and the spline yielded \( \kappa \) to \( \pm 0.03 \).

### 3.3 Generalized Raman Sideband Cooling

As mentioned above, to introduce a spin dependent interaction for the atoms, we employed a dark state cooling mechanism inspired from degenerate RSC, discussed in section 2.5.3. In degenerated RSC, there are two key factors for optimal cooling. One is the presence of a polarization gradient which serves as an effective magnetic field that couples states with different magnetic sublevels. The other one is working in a regular lattice potential, so that the Zeeman shift from an external magnetic field matches the vibrational quanta. So, to apply RSC in a disordered potential is not trivial. In the way we generate random light profiles, single mode laser light goes through a multi-mode fiber. This propagation changes the pure input polarization to a random ensemble of locally varying polarizations. Thus, the disordered potential naturally provides crossed polarizations to support Raman transitions.

In addition to the Raman coupling, the disordered potential provides a large range of vibrational frequencies due to the wide distribution of local potential depths. This might not be ideal for the best cooling over the whole lattice volume,
but it creates a quantum network where there always exist places for optimal dark state cooling. As shown in Fig 3.3 (a), the RSC works similarly to degenerate RSC with slightly mismatched vibrational levels. A uniform magnetic field separates out many small regions where the vibrational energy splitting matches the Zeeman shift. Atoms outside these regions with higher kinetic energies move into them and are cooled down to local vibrational ground states. Their kinetic energy is removed by the optical pumping process, as shown in Fig 3.3 (b). The atoms in dark states undergo coherent evolutions, except rare photon scattering process from the far detuned speckle potential. Due to the presence of gravity, atoms are preferentially tunning downward slowly. During this process, the kinetic energy they pick can be dissipated by the RSC process as well. Transport of atoms through this disordered quantum network is highly non-equilibrium and complicated.

In this experiment, we first load $O(10^6)$ atoms into the UHV MOT and cool them down in the presence of the disordered potential using far-detuned optical molasses to a temperature $\sim 4K$ and then adiabatically release them into the potential. The normal loading efficiency is $\sim 80\%$, and loss mainly comes from the high energy tail of the thermal distribution. Captured atoms are allowed

![Figure 3.3. Raman sideband cooling in a disordered potential. (a) Optical pumping process during RSC. The Raman transition $\Omega_R$ is provided by the disordered far-detuned optical field. The cooling is on the $F = 1 \rightarrow F' = 0$ transition of $^{87}$Rb $D_2$ line. $\Delta$ is the Zeeman shift from an external magnetic field. Best cooling occurs when $\Delta$ matches the local vibrational energy splitting. (b) Atoms motion in disordered potential. Atoms in various spin states (orange/red) transition between states and associated potential contours (green/grey) via Larmor precession in an externally defined magnetic and optical field, and are optically pumped (blue) into local vibrational ground states.](image)
to dynamically evolve in the disordered potential for a variable time up to 600 seconds, before recollected in the MOT. We then count the number of atoms using fluorescence imaging. During the evolution time, an extremely weak and circularly polarized optical pumping field with an intensity 3 to 300 $nW/cm^2$ is applied to atoms with frequency blue detuned by 12 MHz to the $|F = 1 \rightarrow F' = 0\rangle$ transition. In addition, another weak beam resonant with the $|F = 2 \rightarrow F' = 2\rangle$ transition on the $D_2$ line is applied to depump atoms from $|F = 2\rangle$ hyperfine manifold. An external magnetic field of 57 mG is applied nearly along the propagation direction of the optical pumping beam. In this way, the optical pumping beam is almost $\sigma+$ polarization with slight amount of $\pi$ polarization. The angle between the beam and the magnetic field is a critical parameter for tuning the spin dependent interaction, which can change the cooling to heating when reversing the field direction. Under the optimal cooling conditions, atoms are driven to vibrational ground states of local potential wells that are optically dark to the pumping field. However, no single metastable minimum is completely “dark” due to continued motion due to tunneling and the uniform force of gravity. During this time, atoms are lost due to rare events in which sufficient kinetic energy is gained that particles undergo unbounded motion, and also due to background gas collisions.

To show the effect of the RSC cooling, we measured the atom number decay curve at several conditions shown in Fig 3.4. The optimal condition is tuned by adjusting the magnetic field angle and strength together with optical pumping intensity and detuning to maximize the atom number retained after 5 seconds of holding time. For comparison, we also create a heating condition by flipping the direction of the magnetic field. A situation with twice the optimal field strength is also included to illustrate the elimination of dark state cooling. As we can see from Fig 3.4 aside from optimal cooling, all the conditions result in an exponential decay of the atom number. Under optimal cooling, a power-law decay behavior is observed. This manifests dissipation from the generalized RSC process, which removes kinetic energy of the atoms and pumps them into local vibrational ground states. Furthermore, this scale invariant decay behavior shows an anomalous transport phenomenon under dissipation in the disordered potential. It is noted that under optimized conditions, atoms are detectable under continuous cooling for periods as long as 20 minutes.
Figure 3.4. Atoms remaining in the disordered potential as a function of holding time under conservative (green), dissipative (red, purple), non-optimal dissipative (black), and heating conditions (blue). Dashed lines show power law fits to the dissipative data with optimized cooling for two nominally identical experiment runs $\tilde{\gamma} = 0.31 \pm 0.04$ and $\tilde{\gamma} = 0.41 \pm 0.04$. This data is corrected for vacuum lifetime loss; error bars indicate standard error in the mean due primarily to initial atom number fluctuation. Inset, Raw data without vacuum loss correction.

3.4 Dynamical Phase Transition and Scaling Behavior

Let us discuss more about the scaling behavior shown in Fig 3.4. The RSC separates the motion of atoms into two classes at long timescales - power-law decay or exponential decay. Anomalous transport [61,62] or the presence of a non-equilibrium critical point could explain this scale invariance, the latter suggesting a connection to directed percolation (DP) [63] and absorbing-state phase transitions (ASPT) [64]. In the classical limit, this power-law decay behavior can be seen as a result of Lévy flights [65], where certain momentum states once occupied take a long time to escape. Atoms moving in the disordered potential are optically pumped into vibrational ground states of local potential minimums, which ideally take a long time to get out. This power-law decay behavior has been observed in other systems as well [66–68]. These methods suffer from two shortcomings - (I) ASPT noise-correlations miss topologically-induced (Langevin) noise (as captured by YM), leading to an inapplicable definition of upper-critical dimension. (II)
Necessary modifications of critical point phenomena due to strong disorder through the Harris [69] and Luck [70] criteria provide no clear remedy for obtaining scaling exponents. Similarly, it is unclear how to make simple extensions of Anderson and weak localization phenomena to the present dissipative case in a way to predict scaling laws.

To understand this anomalous transport in quantum limit under weak dissipation, we use a gauge field picture to describe the dynamics of the atoms moving in the disordered potential. Moreover, this picture helps us to address the question whether it is a smooth transition or a sudden change from power-law to exponential decay of the atom number.

### 3.4.1 Light-Atom Dynamics Described by Gauge Transformations

To introduce the gauge field, we start from the atomic Hamiltonian and perform a series of gauge transformations that turn the atom-light interactions into a gauge field. To begin, we take the atom-optical interaction and gravitational potential to have a Hamiltonian of the form

\[
H = \frac{\vec{p}^2}{2m} + U_0 \vec{E}_f^\dagger \cdot \vec{E}_f + (g\vec{B} + d\vec{D}) \cdot \vec{F} + mgz, \tag{3.6}
\]

with \(\vec{p}\) the momentum, \(m\) the mass, \(z\) the vertical coordinate, and \(\vec{F}\) the atomic hyperfine spin operator, \(U_0 = h\Gamma^2 I/12\delta I_s\) the scalar potential with \(\Gamma\) the excited state line-width, \(\delta = (1/2\delta_{1/2} + 1/\delta_{3/2})^{-1}, I_s\) the saturation intensity, \(g_F\) the Landé g-factor, and \(\vec{E}_f(x)\) the far-detuned optical field. The total effective magnetic field

\[
\vec{B} = \vec{B}_{ext} + B_0 \vec{E}_f^\dagger \times \vec{E}_f \tag{3.7}
\]

includes the externally applied field \(\vec{B}_{ext}\) and the vector light shift characterized by \(B_0 = iD U_0/2\hbar\), with the relative detuning \(D = (\delta_{3/2} - \delta_{1/2})/(\delta_{3/2}/2 + \delta_{1/2})\) of the far-detuned light from the \(5^2S_{1/2}\) to \(5^2P_{3/2}\) and \(5^2P_{1/2}\) transitions. We assume the electronic excited state population due to the optical pumping process coupling the \(F = 1\) and \(F' = 0\) is small, and with the excited state adiabatically eliminated, dissipation enters through
\[ \tilde{D} = (\vec{F} \cdot \vec{E}_{\text{op}}) \sum_v (\vec{E}_{\text{op}}^\dagger + \vec{E}^\dagger_v) + (\vec{F} \cdot \sum_v \vec{E}_v + \vec{E}_{\text{op}}) \vec{E}_{\text{op}}^\dagger. \] (3.8)

Here, \( \vec{E}_{\text{op}}(\vec{x}) \), \( \vec{E}_v(\vec{x}, t) \) are the field operators for the coherent optical pumping and vacuum fields in a rotating-wave picture.

**Figure 3.5.** Local gauge transform that rotates the local magnetic field direction to measurement axis \( x \).

Here we notice the third term in Eq (3.6) reflecting an effective magnetic field interacting with the atomic spin. As suggested in [71], measurement of a spin along an axis defined by a particle’s coordinate imparts an effective monopole-like gauge-field. Here, we can perform a similar gauge transformation such that the direction of effective magnetic field aligns with the new \( x \) axis. Let

\[ g\vec{B} + d\vec{D} = |g\vec{B} + d\vec{D}| \hat{n} = Q \hat{n}. \] (3.9)

Under a local gauge transform, we can rotate the \( x \) axis of the reference frame to \( \hat{n} \). The atomic wave-function is transformed by

\[ U_1 = \exp \left( -iF_y(\theta_F - \pi/2)/\hbar \right) \exp \left( -iF_z\phi_F/\hbar \right) \] (3.10)

with \( \theta_F \) and \( \phi_F \) orientation angles for \( \hat{n}_F \), as shown in Fig 3.5. As a result, the transformed Hamiltonian \( H' = U_1^\dagger H U_1 \) acquires a non-abelian gauge field \( A_1^{\alpha \mu} \), which depends on the spin of atom. The interaction between the optical field and atoms becomes \( V_{\text{int}} = QF_x. \) The new Hamiltonian reads

\[ H' = \frac{(\vec{p} - \vec{A}_1)}{2m}^2 + V + QF_x + mgz, \] (3.11)

where \( V = U_0 \vec{E}_f^\dagger \cdot \vec{E}_f \). The gauge field \( A_1^{\alpha \mu} \) varies in time and space due to the direct spatial and temporal-dependence of the vacuum or far-detuned modes.
\( \vec{E}_v(\vec{x}, t), \vec{E}_f(\vec{x}) \), but without further manipulation forms a pure (or trivial) gauge field in that the first two terms of the field curvature

\[
F^{\alpha\mu\nu} = \partial^\mu A^{\alpha\nu} - \partial^\nu A^{\alpha\mu} + g A^{\mu} \epsilon^{abc} A_b A_c
\]

vanish everywhere identically.

Below we will provide a more careful analysis of the effect of open coupling to the external optical fields, but we first give a more physical feel for its origin by considering the effect of measurement back-action due to the effective measurement formed by coupling to the classical degree-of-freedom \( Q \). Under strong coupling of the spin to the optical fields, the spin may be expected to adiabatically follow the locally defined spin basis \( F^x \). As a result of the commutation \([F^a, F^b] = i\hbar \epsilon^{abc} F^c\), the components \( F^y, F^z \) exhibit maximal uncertainty, and any motion of the particle according to the influence \( A^{\alpha\mu} \) occurs as a coherent summation over all possible pathways of \( F^y, F^z \); the rapid dephasing over neighboring pathways effectively fix the \( F^y, F^z \) contributions to zero, and the corresponding field curvature no longer vanishes. To better clarify the latter effect, through the course of interaction, the value of the optical degree of freedom \( \pi_Q \) conjugate to \( Q \) forms an indicator of the time-averaged spin component \( F^x \) of the atom in the local basis, and the non-commuting \( F^y, F^z \) experience measurement back-action.

It is helpful for further calculation to introduce an alternative gauge formulation through an added transform \( U_2 = \exp \left( if_2 \right) \) with \( f_2 = -t(QF^x + V)/\hbar \), such that the atom-optical hamiltonian

\[
H_2 = H_1 \left( U_2^\dagger U_2 \right) HU_2 U_1 = \left( p^i F^i + g A F^i A^{\mu\nu} \right) / 2m,
\]

with \( A^{a\mu}_H = A^{a\mu}_1 + A^{a\mu}_2 \), and \( A^{a\mu}_2 = \partial^\mu f_2 \). It is important to stress that the presence of optical the mode operators in the definition of \( A^{a\mu}_H \) imply that it is a quantum, or dynamical field, which evolves as light is scattered.

The gauge field is specified through the transformation \( U = U_2 U_1 \) by

\[
\tilde{A}_\mu = i[\partial_\mu U]U^{-1} = i\partial_\mu \log U.
\]

Here, \( \tilde{A}_\mu \) indicates that this is one specific transformation within the gauge symmetry.
Locally, this is a member of the Lie group for rotations in SU(2) generated by the generator $F^a$, and extended by U(1) spin-independent transformations, and will later be expressed as

$$
\bar{A}^\mu = \sum_{b=0...3} \bar{A}^{b\mu} F^b / \hbar
$$

(3.15)

where $\bar{A}^{b\mu}$ is a function of the coordinates and the optical fields, and the SU(2) generators have been extended by one index $b = 0$ such that $F^{b=0}$ represents $\hbar$ multiplying the hyperfine spin identity operator. The lie-group algebra must then be extended such that $[F^a, F^b] = i\hbar \epsilon^{abc} F^c$ with the structure constant $\epsilon^{abc}$ now running over four indices, and $\epsilon^{abc} = 0$ for any $a, b, c = 0$. We likewise extend $\rho^{ab}$ such that $\rho^{00} = \sum_a |\psi^a|^2$, and $\rho^{0a} = \rho^{a0} = 0$. This provides a convenient form for expressing both the gauge-fields and particle- and spin-currents, and $j^a_\mu$ can follow a similar extension, which incorporates the total mass flow in the $a = 0$ component.

3.4.2 The Light-Atom Coupling Described by a Dynamical Gauge Field

The Hamiltonian we have can be written in a second quantization form, where we introduce wave function $\psi$ and promote it to be a field operator. This leads to a Schrödinger-Pauli Hamiltonian

$$
H_p = \frac{1}{2m} \psi^* \left[ F^i (p_i + A_i) \right]^2 \psi
$$

(3.16)

where the implied sum over spin indices is taken only over the spatial coordinates $i = 1, 2, 3$, and the atomic wave-function $\psi$ is taken as a three-component spinor. The gauge field we derived contains all the atom-light interactions. Here we treat it as a quantum degree of freedom and allow it to evolve and fluctuate around its equilibrium value $\bar{A}^\mu$ defined by the specific transformation above. To do that we follow a coherent state path integration and work with the Lagrangian, which can be written in the following form

$$
\mathcal{L} = \psi^* (i\hbar \partial_t + A_{\mu=0}) \psi - H_p
$$
\[
\psi^*(-i\hbar \partial_t + \frac{1}{2m} F^i p_i F^j p_j + \frac{1}{2m} F^i A_i F^j A_j) \psi - j^{a\mu} A_{\mu}^a, \quad (3.17)
\]

where the current \( j^{a\mu} \) is defined as

\[
 j^{a\mu} = \psi^*(F^\nu p_\nu F^a) \psi / 2m \quad (3.18)
\]

We fix the gauge field to the value \( \bar{A}_{a\mu} \) through the introduction of a Lagrangian multiplier \( \tilde{j}_b^\mu \) in the action

\[
S_A = \int d^3x dt \left[ \mathcal{L} + i\tilde{j}_b^\mu (A_b^{\mu} - \bar{A}_{b\mu}) \right], \quad (3.19)
\]

Shifting the integration over \( \tilde{j}_b^\mu \) by \( j_b^\mu \),

\[
S_A = \int d^3x dt \left[ \mathcal{L} + i(j_b^\mu + j_b^\mu)(A_b^{\mu} - \bar{A}_{b\mu}) \right]. \quad (3.20)
\]

The field \( \bar{A}_{b\mu} \), as derived above, depends on the amplitudes of optical scattering modes. We incorporate the action for the light fields to represent the full quantum system by adding

\[
S_a = i \int dt \left( \dot{a}_i^* a_i^j - \omega_i^* a_i^j a_i^j \right), \quad (3.21)
\]

where \( a_i^j \) are the coherent state path amplitudes for mode types \( i \) (far-detuned, optical pumping, and vacuum modes class) and individual modes \( j \) within each class. To focus on the behavior of atoms, we trace out the vacuum optical modes (discussed in detail in section 3.7.1), which results in a dissipative gauge field described by an effective action

\[
S_D^{\text{eff}} = \int d^4x \psi^*[x] \left( i\hbar \partial_t - (\not{p} + \bar{A})^2 / 2m \right) \psi[x] - j^{a\mu}_{\mu}[x] A_{a\mu}[x]
+ \int d^4x d^4x_s \left[ (\partial_\mu \partial_\nu^{[s]} V_{\rho}^{bc}[x, x_s]) j_{\rho}^b[x] j_{\nu}^c[x_s] \right], \quad (3.22)
\]

where for temporary convenience we have denoted products of the form \( F^i X_i = X^i \), and written the sum of the ‘Langevin’ current \( \tilde{j} \) and \( j \) simply as \( j \). The quantity \( V_{\rho}^{bc}[x, x_s] \) describes an effective self-interaction term for \( j_{\rho}^b \). This term is special since it introduces terms both nonlinear and nonlocal terms in the equation of motion for the atomic field. This term arises from mechanical effects from the light
atom coupling, and it’s precise form is derived in section 3.7.1 below. In addition to this mechanical action on the atomic state, tracing out scattered light modes leads to entropic effects related to both the number of optical modes traced out and their precise effect on the atomic state. These effects can be understood as a soft type of gauge-fixing of the field $A_\mu^a$. These dynamics are derived explicitly in later sections by utilizing the BRST (or Fadeev-Popov ghost) formalism, and generally contribute more terms to the action.

### 3.4.3 Bogoliubov Picture of Collective Mode Dynamics

The nonlinearity described by $V_{\rho}^{ab}$ or $(f_{\mu}^a$ below), though richer in structure, plays a role similar to that in scalar Gross Pitaevskii (GPE) systems by altering the response to small perturbations away from stable configurations. In each stable configuration, the properties of small perturbations (sound waves in scalar GPE) may generally be expected to change - altering, in particular, the dispersion relation for excitations. This defines a speed of sound for wave-like excitations (and thus an approximately relativistic system for sufficiently low momenta), and masses for gapped excitations. Under appropriate conditions, a given stable configuration can even become dynamically unstable [72,73] to small excitations, signifying development of a new ground state. To understand this in the current context, we can (similar to methods utilizing Bogoliubov transforms or linearization of the scalar GPE) introduce small perturbations on top of a ground state $\psi_0$,

$$\psi = \psi_0 + \psi_{s=0} + \psi_{s=1}$$

(3.23)

where $\psi_{s=0}$ and $\psi_{s=1}$ represent positive and negative frequency perturbations to the ground state $\psi_0$ conserving momentum and energy. In different language, the state $\psi_0$ might be viewed as a “vacuum-state” in which small excitations $\psi_s$ are excited.

First let us recall that the structure of each of the three components $\psi_0, \psi_s$ is itself a three-component spinor for the three spin-states $a = 1, 2, 3$ of the atom. We assume that in some coordinates $\vec{x}[x]$, the vacuum state $\psi_0 = |\psi_0| \exp(i \eta_{\theta \mu} \vec{x}^\mu)$ may be written as a plane-wave structure with energy and momentum $\eta_0$. For perturbations to this state which preserve energy and momentum, we expect plane-wave structures $\psi_s = |\psi_s| \exp(i(-1)^s(\eta_{\theta \mu} + \eta_\mu)\vec{x}^\mu)$. Since the two components $\psi_{s=0,1}$ share a relation between their momenta, it is convenient to package them
into a single six-component object $\delta \psi$ with a single index $a \otimes s$. This notation mimics that of Dirac, in which the negative-frequency components are taken physically to represent alternatively anti-particles or holes in a Dirac sea. In our case, they simply represent a local deficit in energy caused by a fluctuation of the atomic state offsetting a local surplus created in the positive frequency perturbation, and are no more mysterious than the decay of a Bose-Einstein condensate by the emission of energy and momentum conserving sound waves at a point of dynamic instability.

We can rewrite the state definition above then as

$$
\psi = \psi_\theta + (P_{s=0} + P_{s=1}) \delta \psi
$$

where the projection matrices $P_s$ are of dimension three by six, and extract the positive and negative frequency perturbations $\psi_s$. Since any spin/differential operator $F^c \partial_\mu$ acting on $\psi$ extracts momentum $\pm \eta_\mu$ from these components, it is helpful to introduce generalized six-by-six gamma-matrices

$$
\gamma^0 = \begin{pmatrix} 0 & F^0 \\ F^0 & 0 \end{pmatrix}, \quad \gamma^{c=1,2,3} = \begin{pmatrix} 0 & F^c \\ -F^c & 0 \end{pmatrix} \quad \text{and} \quad \gamma^5 = \begin{pmatrix} -F^0 & 0 \\ 0 & F^0 \end{pmatrix}
$$

which are spin-1 analogs of the traditional Dirac gamma-matrices in the Weyl-basis (note that the index 4 is conventionally skipped to avoid confusion with a timelike index 4 instead of 0). One can compare the non-relativistic form of the current defined in the previous section to that which would be obtained for a spin-1/2 particle in a relativistic limit under minimal coupling, where the Dirac Lagrangian would read

$$
\mathcal{L}_D = \psi^* [\gamma^\mu (i \hbar \partial_\mu - \frac{1}{c} A^a_\mu \gamma_a) - mc] \psi.
$$

In this case, the fields $\psi$ are treated as four-component spinors, the gamma-matrices $\gamma^\mu$ are four-by-four matrices built from the spin-1/2 operators $F^\mu$ (Pauli matrices), and the current is coupled to the gauge field through the term

$$
\mathcal{L}_{Dj} = \psi^* \gamma^\mu \gamma_a \psi \times A^a_\mu \equiv j^\mu_a A^a_\mu
$$
Defining the generalized gamma matrices allows use of the relations $F^c \partial_\mu P_s \delta \psi = P_s \gamma^c \partial_\mu \delta \psi$. With considerable abuse of Feynman’s slash notation we can define $\vec{\partial}_\mu = \gamma^c \partial_\mu$, or in the simpler case that we need to sum over all $c = \mu$ to form a scalar, $\vec{\partial} = \gamma^\mu \partial_\mu$. The projectors can be conveniently written as $P_s = \text{Diag}[(1+(-1)^s \gamma^5)/2]$, where Diag[] takes the block two-by-two diagonal. Note that the anti-commutation relations $\{\gamma^a, \gamma^b\} = 2 \eta^{ab}$ hold for the standard gamma matrices with $h^{ab}$ a flat Minkowski metric, are not valid for the spin-1 generalization. This follows from the fact that $\{F^a, F^b\} \neq 2 \delta^b_a$, whereas the equality holds for the spin-1/2 Pauli matrices.

Special care must be taken for creating ‘plane-wave’ wave-functions in the spinor representation. In order to create a plane-wave for a spin-1 particle, we first start from a zero-momentum state, and imagine promoting it by application the operator $\exp (i \delta \eta^\mu \bar{x}_a \sigma^a \partial^\nu / \partial x^\nu).$ To operate on the spinor $\psi_\theta$ or $\psi_b$, the $\bar{x}_a$ must be multiplied by the appropriate spin matrix $F^a$. Computing a finite-rotation operator for large $\eta$ then involves raising such spin operators to large powers. Let’s imagine we have gone through this process (since the spin matrices are not commutative, this would be laborious), and now wish to compute a spatial derivative of the wave-function. Certainly one component of this derivative should include the partial derivatives of the new coordinates $\epsilon_a^\mu = \partial \bar{x}_a^\mu / \partial x^\nu$. However, we note that this returns a three-component object $\eta_{\nu a} \epsilon^\mu_\nu$ to account for the difference between spin components $a$, and is not in the form of a spin-operator (a matrix). Instead, we can borrow techniques from quantum electrodynamics and chromodynamics - under a small transformation of the coordinate system corresponding to a ‘local lorentz transformation’, the wave function should transform as $\exp (i \epsilon_{ab} \sigma_{ab} \partial^\mu / 4) \exp (i \eta_{a} \bar{x}_a^\mu)$, where the tensor $\sigma_{ab} = [\gamma^a, \gamma^b]$ in the spin-1 generalization of the gamma matrices. We take then the process of constructing the coordinate system as first producing a common coordinate system $\bar{x}[x]$, followed by local spin-rotations, through an operation $\exp (-i \theta_{ab} \sigma_{ab} \partial^\mu [x]/4) \exp (i \eta_{a} \bar{x}_a^\mu)$. Differentiation then should extract two components - $\partial_\mu \to i \epsilon_{\nu} \epsilon^\mu_\nu - i \theta_{ab} \omega^a_\mu$, where the quantity $\omega_{ab}^\mu = \partial_\mu \sigma_{ab}$ corresponds to the so-called ‘spin-connection’. Choosing the natural extension of the matrix $\sigma_{ab}^\mu [x] = \sigma^a_\nu (e^a_{\nu} e^b_{\nu} - e^b_{\nu} e^a_{\nu})$ allows localized twists introduced into the scalar coordinate transform to be smoothly incorporated into the spin-rotation, and we find the $\omega^a_\mu = \sigma^a_\nu \epsilon^b_{\nu} \partial_\mu e^b_{\nu}$. For the current discussion, it is best to ignore this complication as best possible without introducing approximation. To do so, we will
generally ignore the spin-connection, realizing that we can take the appropriate replacement of \( \eta \) at the end. Before continuing, we briefly note that this definition of the spin-connection differs from its usual place in general relativity, in which it plays the role of a gauge degree-of-freedom in the Minkowski metric. Here it arises simply from a tensor construction of the atomic wavefunction.

In a standard Bogoliubov treatment of collective modes (for example, in a scalar condensate), the action’s term fourth-order in the field-operator \( \psi \) is of central interest, and typically the separation into a classical part \( \psi_\theta \) and a fluctuation \( \delta \psi \) is retained to second-order in the fluctuation \( \delta \psi \). Above, we discussed this nonlinearity in terms of a quadratic dependence on currents \( j^\mu \). For a product of currents \( j^b_\mu [x_1] j^c_\nu [x_2] = \psi^*[x_1] \hat{j}^b_\mu \psi [x_1] \psi^*[x_2] \hat{j}^c_\nu \psi [x_2] \), one then obtains two important terms. The first (suppressing for a moment the spatial arguments),

\[
\psi^*_s \hat{j}^b_\mu \psi^*_s \hat{j}^b_\mu \psi \hat{j}^c_\nu \psi \hat{j}^c_\nu,
\]

is the analog of the term of Gross-Pitaevskii descriptions of condensates which determines the chemical potential. Here, this term determines the space- and time-dependent behavior of the vacuum state of the wavefunction \( \psi \). The second term involves all second-order products in \( \delta \psi \), and can be written as a sum of elements in the two-by-two matrix

\[
\delta \psi^*_s \ T_{i j}^{b c} \ \delta \psi_s = \delta \psi^*_s \left( \begin{array}{cc}
\hat{j}^c_\nu \psi^*_\theta \psi \hat{j}^b_\mu & (\psi^*_\theta \hat{j}^b_\mu \psi \theta) \hat{j}^c_\nu \\
(\psi^*_\theta \hat{j}^b_\mu \psi \theta) \hat{j}^c_\nu & \hat{j}^b_\mu \psi^*_\theta \psi \hat{j}^c_\nu
\end{array} \right) \delta \psi_s
\]

in which each entry is itself two-by-two for the \( s, s' \) indices. Each element can be viewed as a sum over matrix elements \( \langle j_1 m_1 | T | j_3 m_3 \rangle \). The off-diagonal components \( i \neq j \ T_{i j}^{b c} \) represent the interaction of a ground-state spin-current with a current due to the small fluctuations. The diagonal components of the tensor, \( i \ T_{i i}^{b c} \), are constructed from the outer-product of the ground state spinor \( \psi_\theta \). This outer-product can be decomposed into three contributions consisting of \( S = 0, 1, 2 \) total spin. Alternatively, we can work with the Dirac-like representation of \( \delta \psi \),

\[
\delta \psi^*_s \ T_{i j}^{b c} \ \delta \psi = \delta \psi^*_s \ P_s \left( \begin{array}{cc}
\hat{j}^c_\nu \psi^*_\theta \psi \hat{j}^b_\mu & (\psi^*_\theta \hat{j}^b_\mu \psi \theta) \hat{j}^c_\nu \\
(\psi^*_\theta \hat{j}^b_\mu \psi \theta) \hat{j}^c_\nu & \hat{j}^b_\mu \psi^*_\theta \psi \hat{j}^c_\nu
\end{array} \right) P_s \delta \psi
\]

It is interesting to ask how correctly to approach the indices \( s, s' \) on the interaction tensor. Considered as a single index formed by the outer product of \( s \otimes s' \), one might consider decomposition into total-spin components 0 and 1 acting on \( \delta \psi \). Alternatively, one could view the indices as representing two-by-two matrices which...
could be expanded in a basis covering $U(2)$. The appropriate choice is determined by how such objects transform under gauge or frame transformations. Below, in discussion of the geometry of the Unruh effect, we will argue the latter is more appropriate, as under local gauge transformation associated with an accelerating frame, the Unruh mechanism can be understood as an analytic continuation of local Lorentz transformations. Expanding $s^{\prime}_i T_{\mu \nu}^{bc}$ in spin-1/2 Pauli-matrices $\sigma^i$ and the identity $\sigma^0$, we can write $s^{\prime}_i T_{\mu \nu}^{bc} = \sum a^{ij}_T \mu^{bc} \sigma^a_{s^{\prime}}$.

In this way, we can see that the second-order terms in the products of two fluctuating currents can be expanded in the fluctuation of the wave-function as outer-products of spin-0 and spin-1/2 states (off-diagonal), or as spin-0,1,2 and spin-1/2 (diagonal). Representing these again by superposition, the off-diagonal terms are spin-1/2 objects, while the diagonal terms have components decomposable into spin-1/2,3/2 and 5/2. Restoring the position arguments,

$$\delta \psi^*_i s^{\prime}_j T_{\mu \nu}^{bc}[x_1, x_2] \delta \psi = \int d^4x' \delta \psi^*_i[x_1] P_s^*$$

$$\times \left( \hat{\gamma}_\nu^\theta \psi_\theta[x_2] \hat{\gamma}_\nu^\theta \psi_\theta[x_2] \hat{\gamma}_\mu^\theta \psi_\theta[x_2] \hat{\gamma}_\mu^\theta \psi_\theta[x_2] \delta^4(x_1 - x_2) \right) P_s \delta \psi[x_2]$$

we see the off-diagonal terms are local, while the diagonal terms are non-local. These represent two separate processes in Bogoliubov physics - the local (spin-1/2) terms are Hartree terms, and the non-local (spin-1/2,3/2,5/2) are Fock terms in which the vacuum-state $\psi_\theta$ mediates an interaction separated in space in time through a propagating interaction (see Fig 3.6).

Another convenient way to visualize this interaction is to represent the influence of the vacuum on co-located current fluctuations at points $x_1$ or $x_2$ as a spin-1/2 object residing at each point (the spin-state labels a particular Hartree interaction). The diagonal terms can be represented (See Fig. 3.6(d)) by an edge with an associated element of spin $j = 1/2..5/2$ joining any two points (whose spin labels Fock-term it represents). This description of the effect of the 'vacuum state' $\psi_\theta$ bears a cosmetic similarity to so-called ‘spin-network’ states of lattice gauge theory (which are discrete and ordered networks like the above) and similar constructs in loop quantum gravity (disordered, discretized versions), although we will not
pursue the analogy further here.

In Bogoliubov approaches, one traditionally strives to create superpositions of positive- and negative-frequency states which diagonalize the total energy, including both the interaction above, and the kinetic energy. Most often this is done in cases where the Fock terms are ignored, and diagonalization is performed on the Hartree plus kinetic energy. The kinetic energy (as shown in the previous section) can be written in terms of the dynamical gauge field $\mathcal{A}$ using the operator

$$K = (\hbar \gamma_{00} + A_{\mu=0}) \mathbb{I} + \hbar \gamma^0 \eta_{00} + (\mathcal{A} + \hbar \psi + \hbar \phi)^2 / 2m,$$

where the slash is extended only over spatial indices. To accommodate for the coordinate system $\bar{x}[x]$, a factor of $e^a_{\mu}[x] = \partial \bar{x}_a/\partial x^\nu$ is included in the definition of the gamma-matrices $\gamma^\mu[x] = e^a_{\mu} \gamma^a$.

The kinetic contribution to the action $K$ is a manifestly local interaction between $A^a_\mu$ and the atomic field, and therefore adds only to the Hartree terms - returning to the suppressed spatial indices,

$$\delta \psi^* \left( \begin{array}{c} P^a_s \hat{c}_s \psi_\theta \psi_\theta \hat{c}_s \hat{c}_s^b P_s \\ K + P^a_s V^{bc\mu}(\psi_\theta \hat{c}_c \psi_\theta) \hat{c}_c^b P_s \\ P^a_s \hat{b}_s \psi_\theta \psi_\theta \hat{b}_s P_s \\ P^a_s \hat{b}_s \psi_\theta \psi_\theta \hat{b}_s P_s \end{array} \right) \delta \psi \quad (3.31)$$

Figure 3.6. Processes contributing to the vacuum energy and stability of Bogoliubov normal mode analysis. (a) The Hartree term, in which a sound mode of the vacuum excitation propagates, and interacts with the vacuum through virtual pair creation of other similar excitations - the interaction is local, and mediated by the propagator for a spin-1/2 interaction (squiggly line). (b) The Fock term, in which the excitation interacts with itself by creating a propagating disturbance in the medium. The squiggly line represents any one of three propagators representing spin $j_{ij} = 1/2, 3/2$ or 5/2. (c) By combining multiple of these processes, one can build up a representation of vacuum state properties as a series of disconnected diagrams. Placing a Hartree term at each vertex labeled by its spin projection $m_i$, and connecting vertices by edges representing Fock terms of spin $j_{ij}$, one arrives at a simplified diagram in (d). This bears a strong resemblance to structures considered in lattice gauge theories, in which the vertices form an ordered graph, or loop quantum gravity spin networks, in which the graph is disordered.
In the language introduced above, this amounts to performing a (localized) rotation in the effective spin-1/2 space associated with the \( s, s' \) indices to diagonalize the Hartree terms. This could be accomplished by dressing the fields \( \delta \psi \) with a spin-rotation \( U = 3 \otimes \exp (i \sigma^z \phi_b) \), a six-by-six matrix which rotates components in the two-by-two subspace associated with the positive and negative frequency components by a Bogoliubov angle \( \phi_b \). Note that the angle \( \phi_b \) must then be chosen purely complex in order to maintain the commutation relations \( [\hat{\psi}^a[x], \hat{\psi}^{b\dagger}[x']] = \delta_{ab} \delta^4(x - x') \). Instead of doing precisely this, we will just introduce a general six-by-six transformation matrix \( U \), and ask what it must be to extract eigenmodes. Focusing on only the upper-right Hartree term, this demands that

\[
K + \sum_{ss'} P^*_s V^{bc\mu\nu} \theta_{j\mu}^{b} j_{\nu}^{c} P_s = U \Lambda U^{-1}
\]

for some diagonal \( \Lambda \). Note that we have denoted the vacuum state current \( \theta_{j\mu}^{b} = (\psi^{\dagger}_{\hat{0}} j_{\mu}^{b} \psi_{\hat{0}}) \) for brevity. Defining a vacuum field as a six-by-six tensor \( \theta \hat{A} = 2 \otimes F^c F^\nu V^{bc\mu\nu} \theta_{j\mu}^{b} \), the eigenvalue equation can be written

\[
K + \bar{P}^*_s \theta \hat{A}^{\nu\gamma} j_{\nu}^{\gamma} P_s = K - h_\theta \hat{A} / 2m = U \Lambda U^{-1}.
\]

For this to be possible, we must have

\[
\det(K - h_\theta \hat{A} / 2m - \Lambda) = 0.
\]

The eigenvalues \( \Lambda_s \) represent the propagator for the eigenmodes of fluctuation \( \delta \psi = U \delta \psi \). Recognizing that the kinetic terms in \( K \) only enter along the diagonal of the \( ss' \) indices aside from the term \( h \gamma^0 \eta \theta^0 \), and that the Hartree energy \( h_\theta \hat{A} \) contributes solely on the diagonal, the determinant can be handled in blocks in which the elements in one row or column commute. To maintain a reasonably simple notation, we will write the quantities \( \eta_{\mu} e^\mu_{\nu} \) simply as \( \eta_{\nu} \). The secular equation becomes

\[
\det(\tilde{\Gamma}^+ \tilde{\Gamma}^- - (hF^0 \eta \theta^0)^2) = \det \begin{pmatrix} \tilde{\Gamma}^+ & hF^0 \eta \theta^0 \\ hF^0 \eta \theta^0 & \tilde{\Gamma}^- \end{pmatrix} = 0
\]

(3.35)
\[ \Gamma^\pm = (\hbar \eta_0 - \Lambda_{\pm}) I + A_{\mu=0} + (F^i(A_i + \hbar \eta_i + \hbar \eta_{\theta i}))^2 / 2m + \hbar F^\nu \theta A_\mu F^{\nu \eta_\nu} / 2m \] 

(3.36)

At this point, the scalar GPE result would be finished. In a scalar GPE problem, the two solutions \( \pm \) to the secular equation (equivalent to the two branches \( \Gamma^\pm \) of the secular equation here) correspond to phase and amplitude distortions of a condensate. The spin-1 version instead has left behind a reduced secular equation in the spin-indices, which represents the additional degrees of freedom involved in not just sound propagation, but spin waves as well. Note that the vacuum state has two effects on the dispersion - it offsets the spin-independent energy of the perturbation in quadratures through the second term of the determinant, and it splits the Pauli-energy \( \Gamma \) into a doublet \( \pm \) through the field \( \theta A_\mu \). Since this term adds to that obtained through expanding the square of the Pauli energy, we see that the two states couple with opposite charge to the vacuum, but with the same charge to the dynamical field \( A \):

\[ \Gamma^\pm = (\hbar \eta_0 - \Lambda_{\pm}) I + (A \pm \theta A \frac{\hbar F^{\nu \eta_\nu}}{2m})_{\mu=0} + (F^i(A_i \pm \frac{\theta A_i}{2} + \hbar \eta_i + \hbar \eta_{\theta i}))^2 / 2m + \Gamma^\pm_{\theta} \] 

(3.37)

\[ \Gamma^\pm_{\theta} = \frac{1}{2m} (F^i \theta A_i F^j \theta A_j / 4 \mp F^i \theta A_i F^j (\hbar \eta_{\theta j} + A_j)) \] 

(3.38)

The contribution of \( \theta A_\mu \) is reminiscent of the intrinsic moment (and field) of a quantized spin and charge. This is a direct result of the fact that the vacuum field \( \theta A_\mu \) appears to have broken the symmetry implied by the slash notation. For all other terms quadratic in slashed quantities, the result is diagonal in the \( s, s' \) indices of like sign, and thus contribute equally to the Pauli doublet. The vacuum energy \( \Gamma^\pm_{\theta} \) is independent of the energy and momentum of the excitation.

In this light it is helpful to note that we defined the vacuum field above as \( \theta A = I_2 \otimes F^c F^\nu V^{bc\mu \nu} \theta j^b_j \), but could equally-well formulate it in a more well-posed Feynman slash notation as \( \theta A = \gamma^c \gamma^\nu V^{bc\mu \nu} \theta j^b_j \). The appearance of a pair of generalized gamma matrices suggests something somewhat simple in terms of the diagrams for Hartree and Fock processes - while it is easy to see the Fock diagrams have two connections to an odd half-integral spin line, the Hartree terms should be interpreted similarly, in that the single line is actually the limit of two closely
split. This allows an interpretation of the vacuum field $\theta A$ as comprised of paired fermionic disturbances, and the oppositely charged response of the two components of the Bogoliubov spinor is a direct consequence of this.

It is then also interesting to ask if there are situations in which the spin- and ‘charge’- (mass-) degrees-of-freedom can separate. Rare instances of this are known to occur in one-dimensional condensed matter systems and some cold atom systems, and form an interesting case of ‘fractionalization’ of excitations. To this end, we note that the vacuum energy $\eta_{00}$ is responsible for mixing the Pauli-doublet, and as a consequence creates states with ill-defined sign of coupling to the vacuum field $\theta A$. One can ask what the equivalent of the Bogoliubov ‘mixing-angle’ $\phi_b$ is for the spinful system here, and how it varies spatially. We will not pursue this line further here, other than to say that since such effects are generally accompanied by a change in exchange phase, they should be evident by examining (anti-)commutator brackets such as $[P_a U^{-1} \delta \psi, (P_b U^{-1} \delta \psi)]^\dagger$, where the $P_a$ project onto one component of the six-component spinor space. Since these effects depend critically on the ground-state structure, we will first try to understand it.

Before moving on to study the ground state structure, we make some last comments on the Bogoliubov model above.

1. **The Bogoliubov Treatment has a Residual Gauge Symmetry.** In analyzing the effect of the interaction term determined by $V_{\mu}^{\nu}$ above, we consider quadratic fluctuations of the current $j^{\mu}_{\nu}$. In doing so, we originally ignored the influence of the Lagrange multiplier $\tilde{j}^{\mu}_{\nu}$. We now consider its inclusion by adding it to the vacuum current $\theta j^{\mu}_{\nu}$. Since this in turn generates the field $\theta A$, one can see that the intrinsic field attached to the Bogoliubov collective mode is altered by the Langevin current. One might suspect that this effect is artificial, since the addition of currents, and indeed the nonlinearity in the Bogoliubov picture itself are generated by the arbitrary decision to shift the integration over $\tilde{j}^{\mu}_{\nu}$ in equation 3.20. It seems we have two equivalent descriptions (in fact an infinite range of descriptions if we shift by some factor $\alpha j^{\mu}_{\nu}$) based on this decision. This is true, and represents an additional type of gauge symmetry, where we have chosen to work with the choice $\alpha = 1$. Since the result of this choice is a nonlinearity driven by the vacuum field $\theta A$, it is tempting to view $\theta A$ itself as the result of this transformation $U^{-1} \partial_{\mu} U$, and interpret $\alpha^2$ as a type of coupling-constant.

In sec 3.7.1, we show that the Langevin current $\tilde{j}^{\mu}_{\nu}$ is determined by the gradient
of a Nakanish-Lautrap (NL) field (linked to the BRST symmetry), which in turn determines the strength of fluctuations of $A$ from a fixed gauge. This implies that the intrinsic field or moment of the excitation $\delta \psi$ must be allowed to fluctuate. In the next sections, we approach this from a different perspective by considering the topology of the vacuum state. Under the BRST transformation, the NL field is left unchanged, so the vacuum-field itself is unmodified. However, the gauge-field $A$ is replaced by a pair of fermionic ghost modes. In light of the interpretation above of the vacuum as formed by paired fermionic modes, this is no longer a surprising type of (super)-symmetry; both gauge fields may be considered as bound states of odd-exchange-symmetry fields. Since $A$ participates in the Bogoliubov-diagonalization procedure described above, under the BRST transform odd-symmetry fields enter into the eigenstate amplitudes in pairs.

It is tempting then to interpret the vacuum currents $\theta j^b_\mu$ in the same way as the Langevin current, and associate them with the divergence of a separate NL field which drives a second gauge fixing problem associated with the choice of $\alpha$. By this association, we must address the issue of what anti-commuting fields correspond to the “ghost” fields associated with the gauge-fixing symmetry. The natural choice is $\delta \bar{\psi}$ found below.

(2) The Bogoliubov Solution Determines the Stability of a Local Ground State. We have not explicitly calculated the dispersion relation for excitations. To understand this quickly, one can compare to the analogous secular equation determining the Dirac equation in electrodynamics:

$$\det \begin{pmatrix} (mc^2 - E + e\phi) & c\sigma \cdot (p - eA) \\ -c\sigma \cdot (p - eA) & (mc^2 + E - e\phi) \end{pmatrix} = 0 \quad (3.39)$$

which suggests that the speed of sound can be read off from the secular equation by separating spin-dependent and -independent terms in the secular matrix, and performing a unitary transformation to place the spin-identity component on the diagonal. This is tantamount to decomposing $\Gamma^\pm = \sum \Gamma^\pm_a F^a$ into spin-components, applying a localized spin-rotation to the coordinate system through the spin-connection $\omega^{ab}_\mu$, and solving for the overall transform.
for some mass tensor $m_{ij}$, spatial metric $c_{ij}$ (both three by three matrices decomposable into spin-matrices) and energy $E$. Since finding $V$ is somewhat tedious and dependent on the vacuum state through $\eta_0$ and $\theta A$, we will not explicitly carry this out. Suffice to say, this process involves separating the $U(1)$ and $SU(2)$ components of the $U(2)$ representation of $\Gamma$ through a local rotation of the coordinates. If this is possible, a diagonal matrix of rest energies $c^*mc$ and speed of ‘light’ (or sound) through the matrix $c$ may be defined. In this case, $\delta \bar{\psi} = VU\delta \psi$ represents a preferred representation for collective excitations in which the equation of motion is a relativistic Dirac equation in a curved space-time,

$$ (i\gamma^\mu[x]D_\mu - m)\delta \bar{\psi} = 0. \quad (3.41) $$

After restoring the shorthand notations above, the local generalized gamma matrix $\gamma^\mu[x] = \gamma^\nu e^\mu_\nu[x]$, and the covariant derivative $D_\mu = \partial_\mu + A_\mu - (i/4) \omega^{ab}_\mu \cdot [\gamma_a[x], \gamma_b[x]]$. The metric is given by $g^{\mu\nu} = e^\mu_\sigma h^{\sigma\sigma'} e^\nu_{\sigma'}$, with $h^{\sigma\sigma'}$, a flat Minkowski metric.

This procedure can fail if no such rotation is possible. In general, this is not a crisis, but signifies that the chosen vacuum state $\psi_0$ is simply unstable to decay (referred to as dynamic instability in scalar GPE). Since this is a local condition, one can understand the failure by considering a region of space in which the solution does exist and continuing to a point of instability. In this case, the metric $g^{\mu\nu}$ begins well-defined, and signals instability once the condition on local spin-rotations cannot be met.

At these points, presumably, the local speed of sound decreases to zero, and a so-called ‘Killing-horizon’ is obtained, which is represented by a surface in space-time coordinates. All such surfaces taken together form a patchwork of volumes in which the local vacuum state must either contribute a vanishing field curvature or discontinuously vary. These surfaces are critical for understanding the topological state of the vacuum below.
### 3.4.4 The ground state $\psi_\theta$

In the previous section, we saw that the excitations of the atomic state out its preferred time- and spatially-dependent ground state are governed by two principle effects, local interactions introduced through Hartree terms in the Bogoliubov expansion, and Fock terms representing propagating disturbances of the ground state. The latter effects suggest that the ground state response to a fluctuation can be understood by introducing an effective field to parameterize the vacuum response. Like other polarizable or permeable media, one would expect that this response could be understood by introducing a non-Abelian generalization of the dielectric tensor familiar from Abelian electromagnetism in the presence of bound charges. In a non-Abelian theory, this response can be richer. Below, we will use the language of topology to delineate the different types of vacuum polarizability possible in the current system by analyzing the Fock terms from the previous section. We will find that there are a discrete number of such topologies possible, labeled by an integer $q$ representative of a topological invariant (a number which cannot be modified by smooth redefinitions of the coordinates). There is in general no reason to expect that the true ground state sits in just a single of these topological states, and it is for this reason that we included the parameter $\theta$ in the definition of the vacuum state $\psi_\theta$ in the previous section. The parameter $\theta$ represents a generalized superposition state of different topologies, which we will define explicitly below.

The effects of including fluctuations of vacuum state topology have been under intense scrutiny in quantum chromodynamics (QCD) in recent years [74] as a means for understanding the development of confining phases and attainment of gauge-boson mass at long length scales. This is closely related to fundamental challenges in gauge-fixing a quantum systems - even after utilizing the BRST method resulting in the introduction of ghost fields, Gribov [75,76] pointed out that multiple discrete copies of a physical gauge configuration can remain. This “Gribov ambiguity” arises when the ghost field propagator $\kappa[\eta]$ exhibits poles. Early attempts in QCD to account for this postulated a prescription (the Gribov-Zwanziger propagator) to limit the values the gauge field is taken over in path integration lie in the “first Gribov region” prior to this pole. This was seen to affect the gauge boson propagator by introducing a gauge boson mass at large length-scales. In the physical system described here, no such ansatzes or prescriptions should be necessary - the allowance
for a superposed state like $\psi_\theta$, however, does allow us to interpret physical effects using the language developed in QCD.

To understand the topological structures determining the self energy of an atomic state, we return the notation developed before the Bogoliubov picture. Through variation of the action in eq. 3.22 with respect to $\psi^*$, we get the equation of motion in the following form

$$
(\partial_t + A_0 - p^2)\psi[x] = \int d^4x_s (\partial^{(s)}_\mu \varphi^{bc}_\rho [x_s, x]) j^c_{\mu} [x_s] j^c_{\nu} \psi[x],
$$

(3.42)

where $\varphi^{bc}_\rho [x_s, x] = V^{bc}_\rho [x_s, x] + V^{cb}_\rho [x, x_s]$. It is helpful to first understand the origin of this nonlinear term. From equation 3.82, it is clear that this term can also viewed as a source current interacting with its own field, that is, a self-interaction. Equivalently this could be expressed as a total energy from the field alone by defining a field curvature

$$
f^{bc}_{\mu \nu} [x_1, x_2, \eta] = \partial_\mu \varphi^{bc}_i [x_1, \eta] \kappa^{-1/2} [\eta] j^c_{\nu} [x_2].
$$

(3.43)

This field curvature tensor, in a simpler Abelian (electromagnetic) system interacting with external charges and currents, would describe the electric and magnetic fields produced at an observation point $x_1$ due to a point charge or current $j$ at $x_2$ oscillating at frequency $\eta^\mu$ (both temporal and spatial). The equation above (aside from being a non-Abelian generalization) differs in two essential ways. First, it was obtained by considering the nonlinearity introduced by integrating out detailed information in scattered light interacting with a spinful atom, and describes the self-interaction of a current as described by the atomic state $\psi$. For this reason, it is natural to consider this term as describing the energy of the atomic state $\psi$, including its interaction with the light. Second, the response of the fields to the currents $j$ is determined in part by the inverse square-root of the ghost propagator $\kappa^{-1/2} [\eta]$. This is also the case, for example, in calculation of the dielectric properties of the QCD vacuum [77]. Taken together, it is then natural to describe this action as arising from the ‘vacuum energy’ of the ground state due to the fields $f^{bc}_{\mu \nu}$. In comparison to the Abelian case of electromagnetism, the field structure is richer, involving both spin-currents and fields which are representative of the richer gauge symmetry.

It is helpful to reduce the action above to a simpler form. Using the field
curvature, we can rewrite the second term in equation 3.22 as

\[ S_{A_j} = \int d\eta dx_1 dx_2 \cdot f_{bc}^{[x_1, x_2, \eta]} s f_{\mu}^{\nu}[x_2, x_1, \eta] \\
= \int dx f_{\mu}^a[x] f_{\mu}^{a\nu}[x], \quad \text{(3.44)} \]

where the field strength \( f_{\mu}^a[x] \) is obtained from the \( f_{\mu}^{bc}[x_1, x_2, \eta] \) by

\[ f_{\mu}^a[x] = \int d^4(\Delta x) d^4 \eta s' f_{\mu}^{ac'}[x + \Delta x, x - \Delta x, \eta] \left( \gamma^\mu \right)_{s's}. \quad \text{(3.45)} \]

This manipulation calculates the local field structure by summing over contributions at different frequencies \( \eta \) and source-field separations \( \Delta x \). The index manipulations are natural if one first considers equation 3.43 with the relativistic form of the current \( j^\mu_\psi \), and projects into the non-relativistic form by dropping the negative frequency terms in the Dirac spinors. This is accomplished above by choosing the \( \gamma^\mu \) to be expressed in the Weyl-basis, where the projection operators take the form \( (1 \pm \gamma^5)/2 \).

The expression for \( S_{A_j} \) above, in Abelian electromagnetism, corresponds to a vacuum field energy density \( (\varepsilon_0 E_0^2 + \mu_0^{-1} B_0^2)/2 \) for electric and magnetic fields \( E_0 \) and \( B_0 \), which combines the vacuum permittivity and permeability with field strengths to form an energy. This can be understood by viewing the vacuum as a polarizable medium, in which the vacuum fluctuation of the field \( E_0 \) interacts with itself as well as a self-induced field described by \( \varepsilon_0 \).

For the self-interaction in the non-Abelian case to be finite, the field curvature must vanish at large distances \( x \) from any sources \( \psi \), and therefore approach a pure gauge \( A_{j\mu} = g^{-1} \partial_{\mu} g \), with \( g \) an element of the gauge group described by the index \( a \). It is easy therefore to establish a lower bound on this action by expanding

\[ S_{A_j} = \int d^4x \left[ \frac{1}{2}(f_{\mu}^a[x] \pm f_{\mu}^{a\nu}[x])^2 \right]_{\pm} f_{\mu}^a[x] f_{\mu}^{a\nu}[x] \\
\geq \int d^4x f_{\mu}^a[x] f_{\mu}^{a\nu}[x] \\
= \int d^4x 32\pi^2 Q(x) = 32\pi^2 q \quad \text{(3.46)} \]
where the \( \star \) operator computes the Hodge dual of the field curvature \( \star f^a_{\rho \sigma}[x] = \epsilon_{\rho \sigma \mu \nu} f^{a \mu \nu}[x] / 2 \) [78]. This defines a topological constant called the Chern-Pontryagin number \( q \), an integer which can also be expressed (as above) as the volume integral of a topological charge density \( Q(x) \)

\[
Q(x) = \mp f^a_{\mu \nu}[x, \eta] \star f^{a \mu \nu}[x], \tag{3.47}
\]

or equivalently as an integral over the bounding surface

\[
q = \int_{S^3} d^3 x \epsilon^{\mu \rho \nu \sigma} (\partial_\mu g)^{-1} (\partial_\nu g)^{-1} (\partial_\sigma g)^{-1} \tag{3.48}
\]

The invariant \( q \) written in this form represents the number of "twists" the gauge group undergoes on a bounding surface \( S^3 \) of \( \mathbb{R}^4 \), and separates classes of field configurations which cannot be deformed into one another under any smooth change of coordinates. In this way, the topology of the bulk is determined by the gauge field on a bounding surface. While one such surface might naturally be taken at large time and space coordinates, it is more natural to take these at the horizons described in previous sections, where the Bogoliubov conditions indicate the instability of a particular vacuum state to decay. For the remainder of this discussion, we will consider the boundary to be chosen at some arbitrary location, and consider the vacuum state defined within its interior. Before continuing, it is useful to get a more physical feel for what the topology physically represents.

### 3.4.5 Topology of Non-Abelian Gauge Fields as Dynamic Evolution of a Non-equilibrium State

The topological charge density \( Q(x) \) formed from product of the field curvature and its dual, has, as its Abelian electromagnetic counterpart, an interpretation as the product of electric and magnetic fields \( \vec{E} \cdot \vec{B} \). This permits a simple physical interpretation [74] of the effect its presence has on the dynamical evolution of the system - since the latter can be expressed as the time-derivative of \( h(x) = \vec{A} \cdot \vec{B} = \vec{A} \cdot \vec{\nabla} \times \vec{A} \). This magnetic helicity density \( h(x) \), once integrated over volume, represents a type of knot invariant calculated by linkages between field lines of \( \vec{A} \) [79].
Thus the topological charge can be loosely viewed as driving the reconnection of vorticity lines in the dynamical gauge field $A_\mu^a$, in that it represents a time-derivative of the non-Abelian analog of magnetic helicity. Since the non-Abelian version contains additional indices $a$ for spin, it is helpful to understand more physically the generalization of helicity to the optical case we consider here. The introduction of $A_\mu^a$ was originally made as a short-hand description for optical scattering processes, in which the effect of many optical scattering modes is represented minimally by the gauge potential. Since the scattered light contains both spin (polarization) and orbital (phase-front) angular momentum, the overall conservation of angular momentum during a localized scattering process must involve both forms of angular momentum transfer into the field whose detailed structure was removed by introduction of $A_{j\mu}^a$. The larger gauge structure of $A_{j\mu}^a$ represents the separation of these two forms of angular momentum, and the different topological classes $q$ represent different rates of transfer of total angular momentum into the vacuum modes. This is one advantage of a gauge-formalism treatment of the light-scattering problem - local conservation laws (and their violation) can be understood by topological invariants.

For example, in the special case of the state defined with $q = 0$, we expect this helicity is conserved, and topological class defined by the atomic state $\psi$ remains constant. Within any one topological class $q$, the action obtains the minimum value for a field which is either self-dual or anti-self-dual $\pm q f_{\mu\nu}[x, \eta] = \pm q f_{\mu\nu}[x, \eta]$ - each minimum in the action represents a separate classical solution to the equation of motion, with corresponding fields $\psi_q$ and gauge structure $A_{j\mu}^{a(q)}$. Classically, one would then expect the time-evolution of the atomic state to follow one of these solutions, and correspond a given “rule” concerning the conservation, growth, or decay of the topological state with time. This classical rule is codified in the “vacuum-energy” determined by $S_{A_j}$.

### 3.4.6 Topology in Fundamental Gauge-Field Theories

In fundamental gauge theories prior to the mid-nineteen-seventies, this (the assumption of a well-defined topological state $q$) was the default starting point for perturbative treatments of gauge theories, in which quantum fluctuations are assumed small, and do not couple between states of different topology. From the
point of view of (non-gauge) field theory, this is a reasonable assumption, since “tunneling” between two classical minima with an overall change in topology requires a macroscopic redistribution of field, and a corresponding WKB barrier extensive in the volume [80]. For gauge-theories however, it is possible to circumvent this argument, since in principle two minima at distinct $A_j$ may be connected by a pathway of non-physical variations of the gauge-field. We might expect these to be eliminated once the BRST prescription is introduced - however, as pointed out by Gribov [75], discrete degeneracies can still remain. The tunneling amplitudes between two such minima are then not necessarily vanishingly small.

Ignoring the Gribov ambiguity led to gauge theories with manifestly massless gauge bosons - furthermore, the phenomenon of confinement illustrated an inability to understand the evolution of gauge fields at long (infrared) length-scales. The subsequent understanding of the Gribov problem began to resolve this. Treating it as a counting-problem in path integration over values of the gauge-field, Gribov, and later Zwanziger, limited variation of the gauge-field over a single Gribov region, in which no degenerate classical minima appear. The Gribov region (or horizon) is determined by demanding that the ghost propagator ($\kappa[\eta]$ above) exhibit only a single pole in the variation of $A$. This prescription breaks the original BRST symmetry, and in particular, leads to a massive gauge-boson at large length-scales. To some extent, this begins to explain the phenomenon of confinement, and provides concrete predictions for the variation of mass with length-scale which are currently in rough agreement with large-scale lattice gauge theory calculations. An alternative, and less prescriptive approach, was later provided by t’Hooft, in which superpositions of degenerate minima of the action were constructed, and the vacuum assumed to be in a state of lowest energy according to the tunneling probabilities between topological states. This led (in QCD) to the definition of ‘topological susceptibility’, and an understanding for the origin of large mass for certain mesons. It was originally hoped that this would also lead to a better understanding of confinement, however as discussed in detail below, a well-regularized version of this theory is infrared divergent, and fails precisely at long length-scales. In a physical system like the one here, the gauge-field fluctuates between topological classes according to the dynamics of the external environment, and thus prescriptions like those given by Zwanziger or the scaling arguments for topological theories like t’Hooft’s are replaced by well-defined methods of counting optical modes. Thus we
can use these as a case study of how a fundamental gauge theory might arise as an effective description of a microscopic system, and perhaps utilize it to understand confinement better.

### 3.4.7 Superpositions of Topological States and Topological Susceptibility

In a quantum system, fluctuation between classical minima of the action are allowed, and superposition states such as $\psi_\theta = \sum_q e^{iq\theta} \psi_q$ would generally be expected to lower the energy. A loose analogy can be drawn with the motion of a particle tunneling between the wells of a one-dimensional periodic potential. There, a similar superposition of the classical solution near each minimum position $x_i$ (analagous to $q$) defines a Bloch state with a periodic parameter defining the quasi-momentum. In that analogy, $\theta$ plays a role similar to quasi-momentum, and is clearly periodic. The preferred value of $\theta$, as determined by the band energy $\epsilon(\theta) = -2K \cos(\theta)$, depends on the magnitude and phase of the tunneling amplitude $K$ between degenerate minima, favoring $\theta = 0$ when the tunneling amplitude is positive $K > 0$, $\theta = \pi$ when negative $K < 0$ and degenerate (a flat band) when $K = 0$. The second-order variation of the action with $\theta$ should then play a role similar to the effective mass in a periodic system, and is reflective of the amplitude $K$. A convenient picture to understand both the periodic potential and the dynamics of the gauge field is to utilize the language of “instantons” [78], in which the amplitude $K$ can be calculated by extending the action to a Euclidean form, and analyzing the dynamics of a particle in the inverted potential $-V(x)$. Classical solutions to this problem correspond to the motion of a particle from one maximum to another, since this occurs over a short time interval, the dynamics are referred to as an ‘instanton’. Longer range motion can be viewed as repeated events of this type, either rightward (instanton) or leftward (anti-instanton). The amplitude for a single process is given by $K = \sqrt{S_0/(2\pi\hbar)} |\det G/\det V|$, where $S_0$ is a fixed action, $G$ is a propagator for small fluctuations about the stable minimum in the original potential, and $V$ describes small excitations along the classical pathway joining them. The factor of $\hbar^{1/2}$ appears as a result of time-translational invariance for the process.

In the current model, $q$ and $K$ represent distinct topological minima of the action, and the rate and phase with which the gauge field changes between them,
respectively. For this reason, the curvature of the action in $\theta$ is often referred to as a topological susceptibility [74] - one expects the preferred value of $\theta$ changes according to the sign and magnitude of this susceptibility, and that this condition is determined by the amplitude $K$. The instanton dynamics for topological changes in gauge fields is a well-studied [74,80] problem, and is a crucial element in understanding certain features in QCD (such as the anomalously large mass difference in $\eta$ and $\eta'$ mesons via the chiral anomaly, or Witten-Veneziano mechanism [81,82]).

To begin, it is simple to see that the gauge field $A_{a(q=1)}^{j\mu}$ describes a localized dynamical change in topology. To see this, it is helpful to consider a large cylindrical boundary in space and time at which the field approaches pure-gauge, and work in a gauge such that $A_{a(q=1)}^{j\mu(=\theta)} = 0$. In this case, the Chern-Pontryagin number may be written as an integration of the gauge configuration over three surfaces - the two space-like end-caps of the closed cylinder, and the time-like cylinder wall. For the chosen gauge, the wall contributes nothing to the integral, leaving only the space-like endcaps at distant past and future. The value $q = 1$ then implies that the Chern-Pontryagin integration over the endcap surfaces results in a change of homotopy class between distant past and future. Higher instanton classes $|q| \geq 2$ reflect a larger change in topological number over the interval from distant past to future, and, similar to the case of a particle tunneling in a periodic potential, can be decomposed into well-separated singly-charged $|q| = 1$ instantons. In this picture, the dynamical evolution of the topology of the state is given by some number of discrete instanton structures; in exponentiating the process of moving from a state of well-defined topological index $m$ to $n$ over a time $T$, one expects [80]

$$\langle \theta | e^{-iHT/\hbar} | \theta \rangle \propto \sum_{n,n'} (KVT e^{-S_0})^{n+n'} e^{i(n-n')\theta}/n!n'! = e^{2KVT e^{-S_0} \cos \theta}$$

and the ‘$\theta$-vacuum’ energy density is given by

$$E(\theta)/V = -2K \cos \theta e^{-S_0},$$

similar to the case of a periodic potential. Thus an understanding of $K$, and its dependence on external parameters, allows one to understand the topological susceptibility. Similar to the case for a periodic potential, this involves analysis of the small oscillations about both the equilibrium and instanton pathways.
3.4.8 Scaling forms for the Topological Susceptibility in Fundamental Gauge Theories

To understand the rate at which topological transitions occur, the topological solutions \( A^q_j \) must first be given an explicit form. The self-dual and anti-dual instanton solutions for different \( q \) are given for SU(2) gauge fields by the forms [78]

\[
\begin{align*}
 g^{q=0} &= 1 \\
 g^{q=1} &= (x_0 + ix_i F^i)/r \\
 g^{q>1} &= (g^{q=1})^q
\end{align*}
\] (3.51)

For \( q = 1 \), these lead to a field structure in an appropriate gauge

\[
A^q_\mu = \frac{\rho^2 (x - X)_\nu}{(x - X)^2 ((x - X)^2 + \rho^2) \tilde{\eta}^i_{\mu \nu} (g F^i g^{-1})}
\] (3.52)

where \( \tilde{\eta}^i_{\mu \nu} \) are the anti-self-dual t’Hooft matrices, and the parameters \( X_\mu, \rho \) and \( g \) are collective coordinates or zero modes, whose variation does not change the topological class \( q \), and therefore leaves the classical (lower minimum) action unchanged. The \( X_\mu \) can be viewed as center-of-mass coordinates, \( \rho \) a scaling parameter, and \( g \) a global rotation of the spin-axes. In the interpretation above as an instanton structure, these determine the instant-at-which, the duration-of, and the global gauge rotation of a topological transition. In that sense, they represent a breaking of the original symmetries of the action, and may be considered Goldstone modes of excitation.

In fundamental gauge theories, the topological tunneling amplitude is fixed by conditions of regularization of ultraviolet divergences. The tunneling amplitude then takes the form [80]

\[
K = e^{-8\pi^2/\hbar} \int_0^\infty \frac{d\rho}{\rho^3} f(\rho M)
\] (3.53)

The prefactor of \( (\sqrt{\hbar})^{-8} \), similar to the periodic potential (in which a factor of \( 1/\sqrt{\hbar} \) appears due to integration over the instant at which a tunneling event occurs), reflects symmetries in the location and structure of the instanton. As described
above, there are four such symmetries (zero modes) for the location of a $q = 1$ instanton $X_\mu$, another three representing global gauge transformations $g$, and one for the scale-factor $\rho$, for a total of eight symmetries. Since these correspond to no change in the action, integration over these modes contribute a single factor of $1/\sqrt{\hbar}$ to the partition function each. Only the last of these zero-modes, representing scale factors $\rho$, is nontrivial, since it take into account dynamics at arbitrarily short and long time- and length-scales. For this reason, we leave it in a form consistent with dimensional analysis, an integration of a dimensionless function $f$ over $\rho$, with $M$ an undetermined mass scale.

The introduction of a scale $M$ is necessary in fundamental gauge theories to prescriptively regularize the appearance of an infinite number of perturbations to the classical instanton pathway (see e.g. [80]) at short length-scales $\rho$. This is typically introduced by explicitly summing over variations of the gauge field, and canceling the divergence by renormalizing the non-Abelian coupling constant

$$g_A^2 = \frac{g_A^2}{1 + \frac{\pi^2}{12} g_A^2 \log(M\sqrt{\tau_0})}$$

(3.54)

where $\tau_0$ is a proper-time scale cut-off. Expanding perturbatively in the gauge-parameter $g_A$ [83], leads to a form

$$f(\rho M) \sim A(\rho M)^{8\pi^2 \beta_1} (1 + O(g_A^2)),$$

(3.55)

with $\beta_1$ and $M$ linked by the scaling relation

$$\beta_1 \log M = g_A^{-2} + O(g_A^2).$$

(3.56)

The parameter $\beta_1 = 11/12\pi^2$ can be determined from perturbations in small $g_A$, and leads to an infrared divergence for large instanton structures $\rho \to \infty$ in the expression for $K$ above. This is due to the growth of the renormalized coupling constant $g_A$ for large instanton structures, in which the perturbative picture for scaling is likely to break down. This is a common occurrence for pictures of non-Abelian gauge theories at large length-scales, where confinement is likely to play a role - it is typically assumed that some new effects at strong coupling introduce an infrared cutoff, beyond the reach of perturbation theory. This is also directly related to the assumption that individual instanton structures may be considered
well-separated, and that only $A^{\lvert q \rvert =1}$ is necessary to understand the evolution. For this reason, it is not sensible to apply the form 3.53 to arbitrarily large $\rho$, but the contributions of $A^{\lvert q \rvert \geq 2}$ should be included. Below, we will first consider how the regularization and renormalization of 3.53 work in a physical system, after which we will address the problem of higher $\lvert q \rvert$ instanton classes. In a fundamental theory, one also typically demands that the regularization process leaves no remnant of the arbitrarily chosen cut-off $M$. Since the simple instanton picture leaves an infrared divergence, this renormalization process for this picture is a moot point. Below we will argue that in the physical system, these two problems, tunneling in higher instanton classes and renormalization, are closely related.

3.4.9 Topological Susceptibility and Non-Perscriptive Regularization in the Atom-Optical System

Since the problem at hand represents a physical system in which the quantum mechanical behavior is assumed a priori well-understood, such prescriptive techniques for renormalization and regularization should not be necessary. The problem of small oscillations about the instanton pathway at a given scale factor $\rho$ cannot correspond physically to an infinite number of normal modes. This is due to the fact that the atom-optical evolution of the system contains both a finite and decreasing number of degrees of freedom at short length-scales and a defined scaling of degrees-of-freedom at large length-scales. Since the evolution of the combined system at short-lengthscales requires the redistribution of quanta in optical fields, changes below the scale of an optical wavelength correspond to a vanishing number of normal variations. At large scales, the scaling of number of effective degrees-of-freedom in the optical potential limit the rate at which the normal variations can grow. For this simple reason, the scaling relation $\beta_1 \log M = g_A^{-2} + O(g_A^2)$ above cannot be valid for the problem at hand, nor can the implicit infrared divergence.

At the same time, it is not necessary to entirely abandon the regularization process described above. We note that a rescaling of the coupling constant $g_A$ for long timescales is not an unreasonable method to approach a coarse-grained picture of the experimental system, provided that we do not compromise the microscopic picture in the process. Since at the microscopic level there is no need to depart from $g_A = 1$, one can utilize the regularized form in equation 3.54 above with the
choice \( M = \tau_0^{-1/2} \). The proper-time cut-off \( \tau_0 \) and the associated instanton mass scale \( M \) should be defined at the scale below which only a single normal mode of the gauge field remains.

This can be understood more clearly by considering the method of calculation used to arrive at equation 3.53 above [83], which also clarifies how to understand the physical origin of the prefactor \( \mathcal{A} \). This problem was originally studied in the context of QCD by \( \text{'t} \text{Hooft} \) [84], and refined by Polyakov [83]. Small fluctuations of the gauge field \( A_j \) from a given topological state \( q \) can be understood by expanding it as \( A_j = A_j^{(q=1)} + \zeta_n \alpha_n \), where the \( \alpha_n \) are a set of normal modes, and \( \zeta_n \) the corresponding amplitude.

These modes are meaningless, however, without fixing the gauge freedom. This is typically handled in fundamental theories by utilizing the Fadeev-Popov method again, introducing another set of ghost degrees-of-freedom \( \phi \) to fix the gauge such that \( \nabla_i \alpha_i = 0 \), where the covariant derivative is defined such that \( \nabla_i \alpha_k = \partial_i \alpha_k + [A_j^{(q=1)} \alpha_k] \). We will first discuss how this is treated in fundamental theories before discussing its replacement in a physical system like the atom-optical system described in this thesis.

Writing the gauge-field contribution to the lagrangian \( \alpha^* \mathcal{G} \alpha \) with operator \( \mathcal{G} \) and ghost as \( \phi^* \mathcal{V} \phi \), the contribution to the action for each of these can be understood by path-integrating over the amplitudes \( \zeta_n \). This results in a topological tunneling amplitude \( K = (\det \mathcal{G} / \det \mathcal{V})^{1/2} / K_0 \), where \( K_0 = (\det \mathcal{G}_0 / \det \mathcal{V}_0)^{1/2} \) is determined from the extension to the abelian case \( g_A = 0 \). The determinants can be calculated from the spectrum of solutions to the eigenmode equations obtained by substituting the expansion for \( A_j^{(q=1)} \) into the Yang-Mills form for the action, yielding

\[
\nabla_i \nabla_i \phi_n = \mathcal{V} \phi_n = \mu_n \phi_n
\]

\[
\nabla_i (\nabla_i \alpha_{nk} - \nabla_k \alpha_{ni}) + [A_j^{(q=1)} f_{km}, \alpha_{nm}] = \mathcal{G} \alpha_{nk} = -\lambda_n \alpha_{nk}
\]

with eigenvalues \( \mu_n \) and \( \lambda_n \). Ascribing a degeneracy \( \rho_n^{(\mathcal{G},\mathcal{V})} \) to each eigenvalue \( \lambda_n \) or \( \mu_n \),

\[
(\det \mathcal{G} / \det \mathcal{V})^{1/2} = e^{W_\mathcal{G} - W_\mathcal{V}}
\]

with
\[ W_G = -(1/2) \sum_n \rho_n^{(G)} \log \lambda_n \]
\[ W_V = -(1/2) \sum_n \rho_n^{(V)} \log \mu_n \]

The detailed calculation of the spectrum of \( \lambda_n \) is tedious and will not be reproduced here (it can be found in [83]). By dimensional analysis, for each eigenvalue \( \lambda_n \), we can ascribe an inverse squared length-scale. As \( \lambda_n \) increases, the length-scale on which variation of the gauge fluctuation \( \alpha_n \) is relevant decreases. In fundamental theories, without introducing a cut-off scale through \( M^{-1} \) as above, the \( W_G \) represent divergent contributions to the tunneling scale, since both \( \lambda_n \) and \( \rho_n \) scale with positive powers of \( n \).

Thankfully, in a physical system the cut-off \( M \) is better defined, and the prescriptive regularization process is unnecessary. This is true physically because below a spatial length-scale of roughly one optical wavelength, the number of scattering modes accessible drops to unity. Equivalently, shorter than a single photon scattering time, only a single quantized time-evolution process is permitted. The ratio of these two scales, temporal and spatial, defines a characteristic velocity \( c \) which tacitly appears in the four-vector notation above. This velocity scale will be discussed below - for now, we will simply assume such a scale exists, and leave \( M \) as an overall scaling. This implies that if the “mass” or length scale \( M \) is taken to represent a combined spatial and temporal scale of a single optical mode and a single scattering time, the number of normal modes of the gauge fluctuation below this scale drops to one. A characteristic velocity also allows the definition of proper time, and meaning can be given to the parameter \( \tau_0 \) above.

Less formally, one can recognize that the gauge fluctuations in the eigenvalue problem above are not entirely free variables, since \( \alpha \) is determined through the variation of the atomic \( \psi \). This means that the eigenvalue problem as treated in fundamental gauge theories is missing a component due to the change of variables (or change in measure) in moving from \( \psi \) to \( \alpha \), which appears in the form of a prefactor \( \det(\partial \alpha/\partial \psi) \) in path integration. Since \( \alpha \) is determined by \( \psi \) quadratically through the current, the determinant is not independent of \( \alpha \), and the prefactor to the path-integration can be absorbed as a coupling between revised Grassmann
(ghost) fields $\phi$ and $\alpha$. This would have the effect of replacing the first line in 3.57 above with an equation of motion for $\phi$ coupled to $\alpha$, and a corresponding coupling term introduced in the second line of 3.57 above. While this should provide a more accurate version of argument above, we will not explicitly construct this argument, but rely on the physical intuition of mode counting to link $M$ to $\tau_0$.

3.4.10 Topological Susceptibility and Higher Order Topological Hopping

It is clear that simply having a physical cut-off $M$ for the regularization process above still leaves an infrared divergence in the large-instanton limit. This is because the origin of this divergence is more closely tied to a breakdown of the separated-instanton assumption. Higher order instantons $|q| \geq 2$ are necessary to understand this limit, and can be handled with methods similar to those above. It is helpful to consider again the analogy between the topological instanton problem and tunneling of a particle in a periodic lattice again. The restriction to $|q| = 1$ instanton classes is in this picture very similar to a tight-binding approximation used to understand motion in a crystalline lattice. The tight-binding model uses a single hopping amplitude $K$, and will always result in a band-structure given by $\epsilon(\theta) = -2K \cos(\theta)$. Going beyond a tight-binding model can be viewed as adding longer-range hopping terms $K_{|q|}$ to the model, and generally results in a band-structure whose fourier coefficients in $\theta$ are given by the $K_{|q|}$, such that $\epsilon(\theta) = \sum_{|q|} -2K_{|q|} \cos(q\theta)$. This has substantial ramifications for the effective mass (or by analogy the topological susceptibility), since $\epsilon''(\theta) = \sum_{|q|} q^2 K_{|q|}$. Depending on the amplitudes and phases of the $K_{|q|}$, this can favor $\theta$-vacuum superpositions with nonzero $\theta$.

To understand how $K_{|q|}$ vary with $q$, one can first consider the collective degrees of freedom in a multiply-charged instanton. As above, for every collective coordinate, we expect a prefactor of $\sqrt{\hbar}$ to appear in the partition function determining $K_{|q|}$. For higher $q$ instantons, there are $8|q|$ such collective coordinates, which together form the ‘Modulus-space’ of the instanton. From this, one can conclude that the scaling $K_{|q|} = e^{8\pi^2/\hbar} H_{|q|}$, where again we separate out a factor, $H_{|q|}$ to handle the overall scaling(s) of the instanton structure. Before analyzing the role of cut-offs in $H_{|q|}$, one should recognize that the scaling with $\hbar$ can effectively be
Figure 3.7. Instanton dynamics can be applied to the tunneling dynamics of a single particle in a periodic potential (left) or to the topological dynamics of a gauge field (right). Similar to the way in which a particle tunnels from one stable classical minimum of a potential to the next, the topology of a gauge field (represented by its Chern-Pontryagin number) can change in time. By forming a bounding surface (either taken at large distances from a system or on causal boundaries) one can choose an appropriate gauge in which the instanton number vanishes on time-like surfaces (the outside of the cylinder) and therefore is different on space-like surfaces (endcaps). A singly-charged instanton implies a change in topological index of one. Just as in a periodic lattice, a multiply charged instanton can be decomposed into single charged instantons, provided the constituent collective coordinates are well-separated. In this way, time-evolution can be broken into a series of individual topological tunneling events.

replaced by scaling with the square of the coupling constant (see e.g. [80]), such that $K_{|q|} = e^{8\pi^2/g_A^2 g_A^{-8|q|}} \times H_{|q|}$, and thus the scaling is nominally strongly decreasing with $|q|$ at large coupling constant without consideration of $H_{|q|}$. In certain limits, the instanton structures for $|q| \geq 2$ can viewed as multiple instantons, in which case the collective coordinates resemble a set of $|q|$ coordinates like those from the $q = 1$ case. However, once the separation is comparable to the scale-factors (which becomes manifestly true in the infrared limit), it becomes less clear how to handle calculation of $H_{|q|}$.

To understand what is entailed in the calculation of $H_{|q|}$, we first try to understand the calculation of $H_{|q|=1}$ above in the language of its modulus-space of collective coordinates (this is actually what was done in ref [83], though the language used was not explicitly geometric). The study of the moduli-space, or space of collective coordinates for instanton structures is now well-developed [78]. In general, the $8|q|$ collective coordinates for a charge $q$ instanton form a manifold of a Kähler type. For $q = 1$, this space is isomorphic to $M = \mathbb{R}^4 \times \mathbb{R}^4 / \mathbb{Z}_2$ [78], where the first $\mathbb{R}^4$
denotes the coordinates $X_\mu$, and the second $\mathbb{R}^4/\mathbb{Z}_2$ corresponds to the scale freedom $\rho$ (a positive real number) combined with the gauge rotation’s three coordinates (the modulus with $\mathbb{Z}_2$ arises due to invariance with the gauge group’s center). The scale factor $\rho$ plays an important role in this context. The natural metric $h$ for the space $\mathbb{M}$ can be found by considering the small variation of two of the instanton field’s collective coordinates $X = (X^\mu, \rho, g)$, and computing the resulting overlaps of the gauge field variation. All metric components are independent of the collective coordinates $X$ except for those involved in global gauge rotation of the instanton structure. For these, the metric components scale as $\rho^2$, and thus the Moduli-space metric $h_z$ on the subspace $\mathbb{R}^4/\mathbb{Z}_2$ scales with a conformal parameter $\rho$. Interestingly, the complete action for the instanton structure can be written in a simple Yang-Mills form

$$S_{zm} = \int d^8X^{(8)} f^{a\mu
u} f_\mu f_\nu = \int d^4X^{(4)} f^{a\mu
u} f_\mu f_\nu \int d^4z \sqrt{h_z} R_{h_z}$$

where $R_h$ is the Ricci-scalar for the metric $h$.

Seen in this way, two things are clarified. First, it is clear the cut-off $M$ is a breaking of the conformal invariance [83], and can be associated with an ‘internal’ set of coordinates $z$ for the instanton structure associated with its global gauge invariance and scale-factor. Second, the action contributing to the amplitude of a topological tunneling term can be understood by a Yang-Mills type action promoted to a higher-dimensional space corresponding to the collective coordinates of an instanton. Thus, to understand a given topological sector $q$, one might suspect that a higher-dimensional Yang-mills action in the corresponding collective coordinates will suffice.

The calculation of the spectrum of eigenvalues above involved the interrogation of operators $G[A^{(q=1)}]$ and $V[A^{(q=1)}]$ for gauge and ghost degrees of freedom, and entered in the tunneling element $K_{q=1}$ through their determinants. The determinant can be viewed geometrically as a type of volume element for the fluctuations of the gauge- and ghost-fields in a space consisting of degrees of freedom $\zeta_n$. However, these modes are also functional solutions of the differential operators $G[A^{(q=1)}]$ and $V[A^{(q=1)}]$, and therefore the ‘volume element’ $\det G$ is equivalent to a type of zero-point energy for the operator $G$ (and similar for $V$). The operator $G$ is derived
from the action $S_{sm}$ above as the equation of motion from variation of $\alpha$ - to understand $V$, we must add to the action either (in the fundamental gauge theory picture) a Fadeev-Popov gauge-fixing determinant, or (in the physical system) a change-of-variables determinant as discussed above. One might wonder why the zero-point energy of the ghost operator $V$ reduces that from $G$. In fundamental theories, this occurs because only a fraction of the gauge field variations found in the spectrum of $G$ are physical (inequivalent to others under gauge freedom), and the ghost degrees-of-freedom work to reduce this count to only physical degrees-of-freedom. In the physical atom-optical system, only a fraction of the gauge field configurations are accessible by choice of atomic ground state $\psi$. For either purpose, we can add to the lagrangian $\bar{\phi}(x)V\phi(x)$ with the grassmann field $\phi$.

In fact, it is possible to demand that the physical requirement for $V$ as determined by change-of-variables and its form as a BRST or Fadeev-Popov determinant ($V = \partial\nabla$) coincide, as shown in section 3.7.1. It is important to consider whether these modes’ contributions can also be brought under the geometric picture of a modulus-space as above. The calculation of $\det V$ is therefore tantamount to counting the number of zero-modes a Grassmann field inherits in the presence of (here a singly-charged) instanton. The form above, as determined by BRST, maintains an important symmetry under exchange of gauge and ghost (and Nakanishi-Lautrap) degrees-of-freedom.

For the sake of counting degrees-of-freedom in the modulus space, it is also important to ask if additional modes are introduced by the variation of the atomic state described in the Bogoliubov model above as $\delta\bar{\psi}$. It is well-known [78] that this indeed the case, and follows a simple form in the case that the modes $\delta\bar{\psi}$ participate in a super-symmetry like that described for different choices of gauge $\alpha$. All such fermionic modes introduce an additional $8|q|$ fermionic (anti-commuting) coordinates.

### 3.4.11 Topological Susceptibility, Renormalization, and the Higgs Transition in the Atom-Optical System

We have not in the previous section addressed the potential for divergences in $H_{|q|}$ as we saw for $H_{|q|=1}$ before at large scale factors $\rho$, nor have we addressed the aspect of whether a picture based on any of these is at the end of the day
renormalizable. Since we previously associated the cut-off scale $M$ as a breaking of conformal invariance, the issue of renormalization is reduced to one of maintaining conformal invariance.

Let's try to understand when two conditions are met: (1) the resulting picture in a given instanton sector $q_0$ is conformally-invariant (or renormalizable), and (2) the contribution to the tunneling amplitude $K_{|q_0|}$ from the factor $\det G/\det V = 1$ for some $q_0$. In fact, these two conditions are equivalent. This is easiest to see by borrowing a result from a perhaps at first surprising place, string theory. A simple reason for this connection is that it was string theorists who first investigated the structure of super-manifolds like $\mathcal{M}_{p_e,p_o}$ in detail, specifically with an eye toward manifolds which exhibit conformal invariance. Typically, this is versed in the language of conformal charge $c$, which is essentially the count of normal modes of excitation described above. For theories with anti-commuting degrees-of-freedom like the ghost modes $\phi$ above (or those used to gauge-fix the moduli-space metric), the contribution to conformal charge is negative, and conformal invariance is achieved only for a total conformal charge $c = 0$.

In light of this, we quickly summarize the geometric picture found in the last section. In the geometric language, we can associate each topological sector $q$ with a $8|q|$-dimensional Kähler manifold $\mathcal{M}_{|q|}$ whose internal metric is cut-off with a conformal scale $M$, and whose contribution to the tunneling scale $K_{|q|}$ is the associated zero-point energy of a gauge-field defined on it. Accounting for the BRST degrees-of-freedom from gauge-fixing, the manifold is promoted to a super-manifold $\mathcal{M}_{(p_e,p_o)}$ with even/odd dimensions $(p_e,p_o) = (8,16) \ast (|q| + N_f)$, where $N_f$ is the number of fermionic Bogoliubov modes. The tunneling scale $K_{|q|}$ is reduced by the zero-point energy of Grassmann-fields (odd) defined on it. The construction of manifolds for general instanton spaces is referred to as the Atiyah, Drinfeld, Hitchin, and Manin (ADHM) process, and is well-studied. From this we can reason that the only renormalizable theory, and one which retains conformal invariance despite a naturally defined scale cut-off $M$, is one in which the ADHM construct provides a modulus-space without an infrared divergence.

Many-instanton theories in the ADHM picture can be described by a Yang-Mills type action for the collective coordinate manifold. In these cases, the action is known to follow a form \[85\]
\[ S_{MI} = \int d^4 x \, Tr \left[ \frac{1}{2} F_{\mu \nu} F^{\mu \nu} + \frac{i g^2}{16 \pi^2} F_{\mu \nu}^* F^{\mu \nu} + 2i \bar{\chi}_A \gamma^A - D_{\mu} X_a D_{\mu} X_a \right] \]  

(3.61)

which is a direct generalization of the single-instanton case given previously, but with bosonic coordinates written as \( X \) and Grassmann coordinates as \( \chi \). There are two cases in which the many-instanton picture is known to give renormalizable results, remaining finite in the infrared limit. The first is in specific super-symmetric theories such as Yang-Mills field theories with four super-symmetries in specific dimensionalities. The theory at hand has only two identified symmetries of this sort and does not qualify. The second case is when there are gauge-fields present which acquire non-zero expectation values in the vacuum state (a vacuum expectation value or VEV). This is the typical case considered for particle physics in which a Higgs mechanism is specified, which typically postulates the existence of a non-zero field directly coupled to the dynamical field which breaks the gauge symmetry and gives mass to the gauge boson. In our case, the instanton physics is modified by the jacobian taking the atomic field \( \psi \) into the vacuum gauge field \( g_A \) defined by the relation 3.43; this introduces a coupling between the associated odd (ghost) and even (gauge) degrees of freedom by modifying the \( G \) and \( V \) operators to introduce a coupling term. In general, this produces collective coordinates for instanton structures which are non-commutative. In the language of instanton physics, the action above is modified by an additional interaction between instanton collective coordinates given by

\[ S_{MI_{\text{int}}} = \int d^4 x \, \bar{\chi} \, Tr \left[ \frac{g^2}{2} [X_a, X_b]^2 + g_A \sum_{aAB} (\bar{\chi}_A [X_a, \bar{\chi}_B] + \chi_A [X_a, \chi_B]) \right]. \]  

(3.62)

Normally, we would expect that in the vacuum state the expectation values of the bosonic collective coordinates \( X^\mu \) are all zero, and the commutation relations between fields vanish, contributing no potential energy to the action. However, in the presence of the coupling introduced by the physical system, they obtain a nonzero expectation value, and a non-commutative potential in the action above. In the instanton analogy to a periodic potential, this implies that long-range hopping processes \( K_{|q|} \) are such that \( \log(K_{|q|} \approx q_0 e^{-8\pi^2/g^2 A g^2_A |q|}) \leq 0 \). The resulting band-theory
energy $\epsilon(\theta)$, including long-range hopping elements with this sign crossover, now has a nontrivial dependence on $\theta$. In the atom-optical problem, this leads to a change in the topological susceptibility, and grants mass to the gauge boson.

In later sections we will try to approach this problem more comprehensively and simply by introducing a generalized set of non-commuting coordinates (similar to instanton collective coordinates) to describe the atom-optical system from the top-down, leading to a picture of gauge-fields and disorder that bears a strong resemblance to gauge fields propagating on a gravitationally curved space-time metric. But we can see already the essential physics of a Higgs-mechanism from the instanton picture, in which the dynamic reconnection of optical vorticity and spin is taken between two limits at the longest length- and time- scales. It is surprising that the instanton picture, which in many ways has been of limited utility in theoretical approaches to fundamental gauge physics due to infrared divergences, provides a natural, and physical, intuition for the existence and character of a Higgs transition in an open quantum system.

### 3.4.12 Geometrical Viewpoint

We first provide a simple argument that power-law decay and the role of disorder can be understood more simply geometrically, and from an approximate discretized model more familiar in laser-cooling methods. On long timescales, the diagonal components of a density matrix $\hat{\rho}_a(t + \Delta t) = \sum_{Q',Q} \langle Q' | \hat{u}^\dagger(\Delta t) | Q \rangle \hat{\rho}_a(t) \langle Q | \hat{u}(\Delta t) | Q' \rangle$ should suffice to describe transport between discretized quantum states. The evolution is Liouvillian due to the projective action $|Q\rangle\langle Q|$ tracing over optical states $|Q\rangle$; in general the master equation matrix element $w_{ij} = \partial \rho_{a,ii}(t + \Delta t) / \partial \rho_{a,jj}(t) \neq w_{ji}$. For long measurement times, the loss time constant reflects the smallest relevant eigenvalue of the operator $\omega_{ij} = \lim_{\Delta t \to 0} (w_{ij} - \delta_{ij}) / \Delta t$; power-law behavior results when this eigenvalue vanishes in the long-time limit as $\tilde{\gamma}/t$, and indicates an absence of dynamic scale. The change in phase-volume in a single time-step is given by $\det(w)$ - its second largest eigenvalue $\lambda_2$ (representing the slowest process away from equilibrium) is bounded by Cheeger’s- [86] and Bruger’s- inequalities through the isoperimetric constant $h$ as $1 - h^2/2 \geq \lambda_2 \geq 1 - 2h$ - the vanishing constant $\gamma$ implies sub-exponential decay $h \sim t^{-\gamma_h}$, with $1 \leq \gamma_h \leq 2$. The isoperimetric constant represents the smallest surface-area to volume ratio of a discrete space.
(Fig. 3.8) of atomic states describing directed transport, and suggests a similar underlying continuous geometric model without discretization.

![Discrete Model of Hilbert Space](image)

**Figure 3.8.** Discrete Model of Hilbert Space. Geometrically, these are points of convexity in a space in which the corresponding master-equation sets the distance between states; such geometry must be defined covariantly through the effective gauge fields $A_{H}^{\alpha\mu}$ and $A_{d}^{\alpha\mu}$, which completely specify the dynamics. Similar ideas used in a continuous sense lead to an analog model for emergent quantum gravity.

This geometric bottle-necking can be understood on physical grounds as a non-equilibrium quantum effect. The thermal length scale for the gauge-field dynamics is set by the thermal deBroglie wavelength $\lambda_{db}$ for a cooling atomic wavepacket. Within $\lambda_{db}$, excitations of the gauge-field should be related by unitary dynamics, whereas points separated by more than $\lambda_{db}$ are related predominantly through classical statistical correlations. The boundary between these regions, including time as a coordinate, defines a type of horizon similar to that considered in black-hole physics, where the holographic principle was first discussed. This boundary becomes sharp if, as there, the speed of propagation (group velocity) for unitary dynamics vanishes. In both systems, this boundary must accelerate at some rate $a$; the resonant vacuum modes in the atomic system then appear thermally occupied at an Unruh temperature $T_{u} = (\hbar a / 2\pi c k_{B})$, and a balance is achieved in the free energy changes due to thermal entropy and the entropy of disorder. The relation to gravity is not just cosmetic - through detailed modeling of this behavior (see section 3.7), one may arrive at a Palatini formulation for general relativity in a set of non-commutative coordinates describing optical modes and excitations of the gauge field. The Higgs mechanism can be understood by mapping the dynamics of the gauge-field onto a related two-dimensional conformal field theory, in which
the dark-state cooling conditions appear effectively as a gauge potential acting on
the dynamics of a two-dimensional sheet. This gives mass to the gauge boson with
sufficiently compromised dark-state cooling, and exhibits a sudden transition to
masslessness with sufficiently dark terminal states, corresponding to vanishing flux
on the sheet. This transition is observable by varying the angle subtending the
optical pumping propagation axis and the external magnetic field (Fig. 3.9).

Figure 3.9. Higgs-like transition to scaling behavior. The number of atoms
retained in the disordered potential (corrected for vacuum lifetime) after varied holding
times and for varied orientation of magnetic and optical pumping fields are shown as
scatter points. The transition to scaling-law behavior (blue line, $\theta = 23 \pm 2^\circ$) is sharp,
representing a discrete transition between decay due to massive (exponential, time-
constant shown by green line) and massless (scale-invariant) gauge boson modes. Here,
$\theta$ represents the angle between the external magnetic field and optical pumping beam
propagation axis, which determines degree of decoupling of the dark state. Zero angle
corresponds to pure $\sigma_+$ polarization, which maximally decouples the dark state. The
mesh surface shows a fit result from a model allowing for a crossover from exponential to
power law decay of varying width.
3.5 Information Scaling

In the massless case, the Cardy formula [87] can be used to relate the variation of thermal entropy and that of disorder, showing a type of dynamic equilibration which relates the associated disorder and thermal temperatures to a local gravitational field strength (a dilaton field) - under such conditions, a holographic scaling law is a natural outgrowth, relating scaling powers for degrees of freedom in the optical potential and the dynamic evolution time. In fig. 3.10, we demonstrate such an effect, observing a relation between the atom loss constant $\tilde{\gamma}$ and the information scaling $\kappa$ roughly consistent with $\kappa - 2 = -z(\tilde{\gamma} + 1)$, with a dynamic critical exponent $z = 1$, suggesting the relevant scaling quantity is the informational degrees of freedom per unit area. In the experiment we vary the mode content of the multimode fiber by changing the angle and divergence of the input beam. Some images of the intensity distribution is shown in Fig 3.11.

Figure 3.10. Scaling behavior vs information content. Plot shows normalized loss curves for different optical fiber mode content (corresponding to fourier-plane images shown in false color), including equilibrated mode content (red, purple, orange $\kappa = 0.54 \pm 0.06$), low transverse-wavenumber excitation (blue $\kappa = 0.41 \pm 0.07$), and high transverse-wavenumber (black $\kappa = 0.52 \pm 0.05$, green $\kappa = 0.53 \pm 0.05$). Inset shows loss scaling parameter $\tilde{\gamma}$ against $\kappa$. 

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Figure 3.11. Images of the intensity distribution in the aperture (a,c,e) and imaging (b,d,f) planes. Red, green, and blue correspond to all, only low-$K$, and only high-$K$ optical modes from the light potential, respectively. g, Scaling laws of the information content of the corresponding figures in a-f. The exponential of the Shannon information, $F_s$, is calculated for each point as a function of distance from the origin, $\ell/2$, constructed with all terms up to 10th order in the power series. The lines are least-squares linear fits to $\log(F_s)$ for $\ell$ between 40 nm and 3.6$\mu$m, which yield exponents, $\kappa$, of $0.53 \pm 0.05$ (red), $0.52 \pm 0.05$ (green), $0.41 \pm 0.07$ (blue). All possible $\kappa$ from each combination of two points of $\log(F_s)$ are shown in the inset; the error on the fit was taken to be the standard deviation of this spread in slopes. Orange is a simulation with only the lowest 5% modes in $K$ retained with $\kappa = 0.22 \pm 0.03$. 
3.6 Additional Scaling

The scale invariance of the classical YM dynamics should be lost in the quantum limit, as the coupling constant $g_A$ now introduces a preferred scale for the dynamic evolution. To explore this regime, the experiment was repeated at a series of optical pumping intensities for fixed experiment duration, see figure 3.12. The scaling of retained number for sufficiently low intensities follows an apparent power-law form $N_v \sim I_{op}^{\lambda}$, with $\lambda = 0.5 \pm 0.2$ over two decades in intensity $I_{op}$. While not fully understood, we note that the number of atoms retained follows roughly the same dependence as the dissipative flux density $\vec{\nabla} \times \vec{A}_d$, which suggests the relevance of a non-equilibrium version of “filling-factor.” This is consistent with the picture above, in which the dissipative flux associated with the dynamical gauge-field sets a relevant scale for disorder.

![Figure 3.12. Variation of dissipative capture with pumping intensity. The number of retained atoms after 5.1s is plotted against optical pumping intensity, calibrated to scattering rate from the (bright) $m_F = 0$ state. The red line is a power law fit of the data, showing an exponent of $0.5 \pm 0.2$. Error bars show the standard deviation in the mean along both axes, and the shaded region uncertainty in the power-law scaling.](image-url)
3.7 Gauge Field Model

3.7.1 Gauge Fixing in Path Integral

The dynamical gauge field can be fixed to this specific transformation using path-integration

\[ W[A_{\mu}] = \int D[a] \prod_b \delta[A_{\mu} - \bar{A}_{\mu}] \]

\[ \prod_i \delta[\bar{E}_i - \sum_j a_j^i E_j^i(\vec{x},t)]W[a] \],

(3.63)

where we express the electric fields in terms of individual mode amplitudes \( a \), which are taken to represent coherent state amplitudes, and the action in \( W[a] \) is given by the \( S_a = \int dt (\dot{a}_i^j a_i^j - \omega_i^j a_i^j a_i^j) \) where a sum is implied over all three mode types \( i \) (far-detuned, optical pumping, and vacuum modes class) and individual modes \( j \) within each class. For the far-detuned modes \( E_{fi} \) and optical pumping mode \( E_{op} \) we set \( \omega_i^{j,op} = 0 \), but for the vacuum modes the frequency of \( E_{\nu} \) is measured relative to the coherent optical pumping mode frequency. Expressing the delta function above through a fourier-transform, and absorbing a constant,

\[ W[A_{\mu}] = \int D[a, \tilde{j}_\mu^b] \exp[i \int dt (\dot{a}_i^j a_i^j - \omega_i^j a_i^j a_i^j)] \]

\[ \times \exp(i \int d^3x dt \tilde{j}_\mu^b (A_{\mu} - \bar{A}_{\mu})) \]

(3.64)

we have, an Euler-Lagrange equation of motion from variation of \( A_{\mu} \) as

\[ \partial^\mu F_{\mu\nu}^a + \hbar g A^e_{abc} A_{\mu}^b F_{\mu\nu}^c = \tilde{j}_\nu^a - j_\nu^a - 2g_{\rho\mu}^{ab} A_{\nu}^b \]

(3.65)

These dynamics are not complete in the sense that they depend on as yet unspecified dynamics of the mode amplitudes \( \{a_i\} \) through \( \tilde{j}_\mu^b \). Rather than develop a second equation of motion, which would result in full deterministic evolution of the coupled atom-optical system (with a vanishing field curvature as a consequence of the
"pure gauge" represented by $\bar{A}_\mu^b$, we integrate-out the modes $\{a_\nu\}$ corresponding to the vacuum modes of the pumping process. We separate the pumping-dependent terms from the gauge term $\bar{A}_\mu^b$, such that the pumping-dependent portion of the Lagrangian density becomes

$$L_D \approx \bar{j}_\mu^b \bar{A}_\mu^b[\{a_\nu\}] + \sum_j |E_j^\nu|^2 a_j^\nu (i \partial_t - \omega_j^\nu) a_j^{\nu*} / c,$$  \hspace{1cm} (3.66)$$

where $\bar{A}_\mu^b$ is the first order correction to $\bar{A}_\mu$ at low coherent optical pumping mode strength $a_{\text{op}}$.

It is helpful to first define

$$\bar{A}_{\mu} = a_{\text{op}} \lim_{a_{\text{op}} \to 0} \frac{\partial \bar{A}_\mu}{\partial a_{\text{op}}} = i a_{\text{op}} [\partial_\mu \partial_{a_{\text{op}}} \log U]_{a_{op}=0} \quad \text{(3.67)}$$

and split the rotation $U_1 = U_1^\delta U_1^B$ into two parts, first rotating the hyperfine spin axis locally into the direction of $\vec{B}$, and subsequently into the direction $\hat{n}_F$ through $U_1^\delta$. A similar split for $U_2$ leads to a simpler expression for $\bar{A}_{D\mu}$ as

$$\bar{A}_{D\mu} = a_{\text{op}} \partial_\mu [\partial_{a_{\text{op}}} U_2^B U_2^B (U_1^B)^{-1}]_{a_{op}=0} \quad \text{(3.68)}$$

which reduces to

$$\bar{A}_{D\mu} = \partial_\mu \left[ - \frac{2t}{\hbar} (\bar{D} \cdot \bar{B})(\bar{F} \cdot \bar{B}) - \tilde{\Omega} \right] \quad \text{(3.69)}$$

where $\tilde{\Omega}$ represents the effect of measurement back-action on the non-commuting components $F^{y,z}$ in an "interaction-representation," due to the local differential rotation $\delta \theta_F, \delta \phi_F$, of the hyperfine spin axis away from the direction specified by $\bar{B}$ due to $\bar{D}$. Explicitly,

$$\tilde{\Omega} = e^{-ig\bar{B} \cdot \bar{F}_t / \hbar} e^{2i(F^\nu \delta \theta_F + F^z \delta \phi_F) / \hbar} e^{ig\bar{B} \cdot \bar{F}_t / \hbar} - 1 \quad \text{(3.70)}$$
\[
\equiv e^{-i\vec{B}\cdot\vec{F}t/\hbar}e^{2i\vec{F}\cdot\delta\vec{r}/\hbar}e^{i\vec{g}\vec{B}\cdot\vec{F}t/\hbar} - 1
\]

\[
e^{2i(\delta\vec{r} + g_t\vec{B}\times\delta\vec{r})\cdot\vec{F}t/\hbar} - 1
\]

and

\[
\tilde{A}_{D\mu} = \frac{2d}{\hbar}\vec{F}\cdot\partial_\mu[\vec{D} \times \vec{\hat{B}} - \vec{\hat{B}} \cdot \vec{D}\vec{B}t - \frac{\vec{D} - \vec{\hat{B}} \cdot \vec{D}\vec{B}}{g\vec{B}}]
\]

(3.71)

Due to the SU(2) and U(1) symmetries, the local gauge transformation may be written using the same “extended” index \( b = 0 \ldots 3 \), with \( F^0/\hbar \) representing the hyperfine spin identity operator, and the remaining three indices the SU(2) generators \( F^x,y,z \) in a form

\[
\tilde{A}_{D\mu} = \sum_{b=0 \ldots 3} \tilde{A}^b_{D\mu} F^b/\hbar
\]

(3.72)

This can be written concisely as

\[
\tilde{A}^c_{D\mu} = \partial_\mu[G^c_{rk}\Sigma^{rk}_{op}]
\]

(3.73)

where \( \Sigma^{rk}_{op} = \sum_j \Sigma^{rk}_{op,j} = \sum_j (a^i_{v}E^r_{op}E^{i|k}_{v,j} + a^j_{v}E^k_{v,j}E^{i|r}_{op}) \) is the spatial components of the stress-energy tensor for the optical pumping field, and

\[
G^c_{rk} = \frac{2d}{g\vec{B}}(\epsilon^{kr} + \delta_{kr}\delta_{c0})
\]

\[
- 2d\hbar(\epsilon^{rc} + \delta_{rc}\delta_{00})(t\epsilon^{ijk}\vec{B}^j + \frac{1}{g\vec{B}} - t)\vec{B}^i\vec{B}^k)
\]

(3.74)

The field \( \tilde{A}^b_{D\mu} \) depends linearly on the vacuum mode amplitudes \( a^i_{v}(t) \), and the path integration can be written as a gaussian in \( a^i_{v} \). To understand the contributions from \( j^b_{\mu} \), it is convenient to first integrate the action by parts based on the Eq 3.66, obtaining
\[
S_D = \int d^4x G_{rk}^b \Sigma_{op}^r (\delta S_\mu - \partial_\mu) \tilde{j}^{bu} + \sum_j |E_j|^2 a_j^i (i \partial_t - \omega_j) a_j^i / c
\]  
(3.75)

where \( \delta S_\mu \) is the \( \mu \)th component of the unit normal at the bounding surface of the integration.

The quantity \( \delta \tilde{\rho}^b = (\delta S_\mu - \partial_\mu) \tilde{j}^{bu} \) can be interpreted within the bulk as the deviation of the Langevin current \( \tilde{j}^b_\mu \) from local conservation (and on the bounding surface \( S \) as a flux). We intend to create a description of the dynamics of this system which does not include detailed information regarding the scattered optical pumping light - integrating out the dynamics of the mode amplitudes \( a_\nu^j \) achieves this, and gives an alternate description using a condensed set of physical parameters, including a dynamical gauge field and its evolution according to a modified view of the underlying coordinate system. We implement this by reformulating the field in terms of a local ‘Bogoliubov-Fourier’ transform

\[
a_j^i(t) = \int d^4 \eta d^4 x' E_{ij}^{k*} (x') \delta(t - t') u_k^i (\eta_\mu) \beta_i (\eta_\mu)
\]  
(3.76)

where the \( u_k^i [\eta_\mu, x_\mu] \) with \( i = (0, 1) \) represent spatial ‘wave-functions’ of generalized momenta \( \eta_\mu \). We take \( \beta_1^* = \beta_0^* \) to represent anomalous coupling of the vacuum annihilation operator \( a^i_\nu \) to creation operators \( \beta_i [\eta_\mu] \), and will later perform path-integration over the new amplitudes \( \beta_i \).

In order to account for Unruh-like effects using the mode structures \( u_j^k \) (defined explicitly below), in which unoccupied modes appear occupied in an accelerated coordinate system, we choose the dependence on the coordinates to enter through a new, smooth and conformally-mapped, complex coordinate system \( \bar{x}_\mu [x_\nu] \) - we choose this in the spirit of understanding Unruh effects through analytic continuation of the action of the hamiltonian into Lorentz boosts, and will take the complex part of the new coordinates to generate anomalous couplings using a specific form of \( u_j^k [\eta_\mu, x_\mu] \) below.

We further choose the corresponding metric tensor \( h_{\mu\nu}^i \) (with \( i = 0, 1 \) representing...
the metric and its conjugate) to satisfy

\[ (\det h^i) \frac{1}{2} = \bar{G}_b^b \delta \rho^b \]  

(3.77)

where \( \bar{G}_0^b = G_{rk}^b (E_{op}^r u_1^k + E_{op}^r u_2^k) \) and \( \bar{G}_1^b = \bar{G}_0^{b*} \).

The quantity \( \bar{G}_1^b \) then represents a sum of ‘cooling’ and anomalous absorption processes, in which either a pumping photon is absorbed and re-emitted into an unoccupied vacuum mode, or a photon is absorbed from the apparently thermally occupied vacuum mode by an accelerating atom and emitted into the pumping beam. The quantity \( \bar{G}_0^b \) represents the unitary opposite of these processes. With this definition, the dissipative part of the action can then be written in a simple, more geometric form

\[ S_D = \int d^4 \eta \left[ v_i \beta_i + \beta_i \kappa_{ij}[\eta] \beta_j \right] \]

with \( v_i = \int d^4 \bar{x}_i = \int d^4 x \sqrt{-h^i} \exp (\phi^i) \) the volume in the new coordinates \( \bar{x}_i \), and under a condition

\[
\int d^4 x \sigma_{ij}^\tau [u_i^k(\eta_1 x) \partial_t u_i^k(\eta_2 x) - u_i^{k*}(\eta_2 x) \partial_t u_i^k(\eta_1 x)] \\
= -i \kappa_{ij}[\eta_1] \delta(\eta_1 - \eta_2)
\]

(3.78)

for some function \( \kappa \), which ensures the “diagonality” in \( \eta \), and permits the \( \beta \) to be integrated out of the path integral (conditions under which this is possible are discussed below). This leads to an effective action

\[ S_D^{\text{eff}} = \int d^4 \eta (v_i \kappa_{ij}^{-1} v_j - \frac{1}{2} \bar{c}_i \kappa_{ij} \bar{c}_j) \]

(3.79)

where we have included the change-in-measure associated with the determinant of the operator \( \kappa \) through anti-commuting Grassmann numbers \( c_i[\eta] \) and \( \bar{c}_i[\eta] \), representing so-called ‘ghost’ modes - while these are decoupled from the dynamics at the classical level, we include them in order to retain the well-behaved quantized path integration we began with - the function \( \kappa_{ij}[\eta] \) can now be recognized as the
propagator for ghost modes with momenta and energy $\eta_\mu$ in a Yang-Mills picture. The inverse of this operator also relates the dynamical gauge field to the Langevin current. Restoring the original coordinates in the integration,

$$ S^{\text{eff}}_D = \int d^4x_1 d^4x_2 \delta \bar{\rho}^b[x_1] V^{bc}_\rho[x_1, x_2] \delta \bar{\rho}^c[x_2] $$

with an effective interaction

$$ V^{bc}_\rho[x_1, x_2] = \int d^4\eta \bar{G}^b_i[x_1, \eta] \kappa_i^{-1} G^c_s[x_2, \eta] \quad (3.80) $$

returning to $\tilde{j}_\mu$, and integrating again by parts,

$$ S^{\text{eff}}_D = \int d^4x_1 d^4x_2 \left( \partial_\mu^{(1)} \partial_\nu^{(2)} V^{bc}_\rho[x_1, x_2] \tilde{j}^b_\mu[x_1] \tilde{j}^c_\nu[x_2] \right) $$

$$ \equiv \int d^4x \tilde{j}^b_\mu A^{bu}_j \quad (3.81) $$

where one can view the dynamical gauge field $A_j$ generated by the Langevin current $\tilde{j}$ as

$$ A^{bu}_j[x] = \int d^4x_\delta (\partial_\mu^{(x)} \partial_\nu^{(x)} V^{bc}_\rho[x, x_\delta] \tilde{j}^\rho_\nu[x_\delta] $$

$$ \quad (3.82) $$

### 3.7.2 Soft Fixing and BRST Symmetry

Generally, one expects the preferred field $\bar{A}^{bu}_c$ breaks gauge symmetry and introduces massive bosonic excitations for the fields $A^{bu}_c$. While this is generally true, under certain cases closely connected to the presence of a dark state, a limited form of gauge symmetry can be restored. To see this, it is helpful to bring the action found above into a more standard form by scrutinizing the change in measure induced by removal of the detailed dynamics of the optical pumping fields. The fluctuation of the Langevin current $\tilde{j}$ is reminiscent of the additional current [88] associated with
softly fixed gauge conditions - this can be seen by associating the Langevin current
\[ \tilde{j}_\mu^a[x] \equiv \partial_\mu b^a[x] \] (3.83)

with the divergence of a set of scalar fields \( b^a[x] \) (Nakanishi–Lautrup fields). Integrating equation 3.81 by parts and combining with the original action,
\[
S_\xi = \int d^4 x \left[ b^a \partial_\mu (A^{a\mu} - \bar{A}_{c}^{a\mu}) - \bar{c}^a \partial_\mu (\delta_{ac} \partial^\mu + g_A \epsilon^{abc} (A^{b\mu} - \bar{A}_{c}^{b\mu})) c^c + b^a[x] \int d^4 x' b^b[x'] \xi^{ab}[x, x'] \right]
\] (3.84)

where we have introduced transformed ghost fields
\[
\bar{c}^a[x] = \int d^4 \eta \, \varepsilon u^a_i[\eta, x] \bar{c}_i[\eta]
\]
\[
c^a[x] = \int d^4 \eta \, \varepsilon u^a_i[\eta, x] c_i[\eta]
\] (3.85)

and the functions \( \varepsilon u^a_i \) and \( \varepsilon u^a_i \) are chosen such that
\[
\kappa_{ij}[\eta] \delta(\eta - \eta') = \int d^4 x \, \varepsilon u^a_i[x, \eta] \partial_\mu (\delta_{ac} \partial^\mu + g_A \epsilon^{abc} (A^{b\mu}[x] - \bar{A}_{c}^{b\mu}[x])) \varepsilon u^c_j[x, \eta']
\] (3.86)

When such a choice is possible, BRST symmetry [88–91] is satisfied, such that the action is invariant under the variation
\[
\delta \Psi = \delta (A^{a\mu} - \bar{A}_{c}^{a\mu}, \bar{c}^a, c^a, b^a) = \lambda Q \Psi
\]
\[
= \lambda (\partial_\mu c^a + g_A \epsilon^{abc} (A^{b\mu} - \bar{A}_{c}^{b\mu}) c^c, b^a, g_A \epsilon^{abc} b^c, 0)
\] (3.87)

which mixes bosonic fields \( (b^a, A^a) \) with fermionic fields \( (c^a, \bar{c}^a) \) with a scale set by an infinitesimal anti-commuting parameter \( \lambda \).

Provided BRST symmetry can be maintained for the complete action, renormalization to large length and time scales may be achieved, even in the presence of the symmetry-breaking term \( \bar{A}_{c}^{a\mu} \). Note that the action is invariant (closed) under
BRST, such that $QL = 0$, which can be seen from the fact that these terms are BRST-exact

$$L = Q\varepsilon^a(\partial_\mu(A^a_\mu - \bar{A}^a_\mu)) + \int d^4x'b^a[x][x']\xi^{ab}[x, x']$$

combined with the fact that the BRST operator is nilpotent, $Q^2 = 0$.

The final term in the action 3.84 may be interpreted as a soft condition setting the gauge $\partial_\mu(A^a_\mu - \bar{A}^a_\mu)$ to zero through the auxiliary fields $b^a[x]$, which fluctuate against a ‘stiffness’ given by

$$\xi^{ab}[x_1, x_2] = \Box_1 \Box_2 V^{ab}[x_1, x_2] = \int d^4\eta \Box_i \tilde{G}^a_{\eta} \kappa^{-1}_{is} \Box_2 \tilde{G}^b_{\eta}$$

with the d’Alembertian (combined with a surface term) for the $x_i$ coordinate $\Box_i = (\delta_{\nu\sigma} - \partial^{(i)}_\mu)\partial^{(i)}_\mu$. It is important to note that this stiffness is itself a function of the $b^a[x]$ through $\tilde{j}^a_\mu$ in the functions $\tilde{G}^a_{\eta}$ and $\kappa_{is}[\eta]$, and the gauge-fixing contains higher powers than quadratic in $b^a[x]$ and its derivatives. If the $b^a[x]$ were integrated-out of the action, this would lead to terms higher than quadratic in $A^a_\mu[x]$ and its derivatives, and would produce a contribution to the kinetic energy associated with the non-abelian field $A^a_\mu[x] = A^a_\mu[x] - A^a_{\mu}[x]$.

As a simple example, in the case $\xi^{ab}[x, x'] \rightarrow \xi_0 \delta(x - x')$, with small $\xi_0$, Landau gauge conditions would be strongly enforced with a locally preferred gauge condition $\partial_\mu A^{b\mu} = \partial_\mu \bar{A}^{b\mu}$ in a more-or-less standard picture. The general case can be seen as a type of $R\xi$-gauge fixing suitable for gauge fields produced by a natural system, in which correlation is to be expected over some length-scale, and gauge-fixing conditions are naturally softened by the dynamics of the external body. Unlike the case for the standard model electroweak interaction, off-diagonal elements in $\xi^{ab}$ couple the $U(1)$ symmetry to $SU(2)$, and despite the fact that that the $U(1)$ sector is abelian and decoupled from the $SU(2)$ through the structure constant $\epsilon^{abc}$, all excitations can acquire mass through this term unless $\xi^{0b} = \xi^{b0} = 0$ for all $b \neq 0$. Under this latter condition, the $U(1)$ fields are completely decoupled, and partial gauge symmetry may be restored, in which case a single component of the gauge
field would lose its mass.

It is helpful to understand this on a more geometric footing - the gauge-fixing action represented by $\xi^{ab}$ has its origin in the action determined by products of ‘volume’ elements $v_i[\eta]$ corresponding the coordinates $\bar{x}_i[\eta]$ - the tensor field $\xi^{ab}[x, x']$ reflects the (non-)local contributions to the total squared-volume $v_i v_j$ due to spin components $a, b$ as well as their correlations - one can view the tensor aspect as necessary to describe the local volume as due to an ‘internal space’ representing spin dynamics and whose value reflects a contribution to the Fadeev-Popov determinant. This is made clear by the constraint assumed above for the volume form $v_i$.

$$\sqrt{h_i} = \mathcal{G}^b_i \Box b^b$$

(3.90)

The volume can be calculated from the coordinates $\bar{x}_i[x]$ by first using a Nambu-Goto form and introducing an auxiliary dynamical metric $\bar{g}^{\mu\nu}[\eta, x]$ to convert it into the form of a Polyakov action

$$v_j = \int d^4 x \sqrt{\bar{g}_j} \bar{h}^{\mu\nu}_j \bar{h}_j^i \partial_\mu \bar{x}^\sigma_j \partial_\nu \bar{x}^\sigma'_j = \int d^4 x \sqrt{\bar{g}_j} h^\mu_j$$

(3.91)

where $\bar{h}^{\sigma\sigma'}_j$ is a flat metric in the higher-dimensional Riemannian space with $\sigma = 1...N$, the metric determinant $g_j = \det(j g_{\mu\nu})$, and $j h^\mu_j$ represents the contraction of $j g^{\mu\nu}$ with the transform metric $j h_{\mu\nu}$ induced from the $\sigma$-coordinates as

$$j h_{\mu\nu} = \frac{\partial \bar{x}^\sigma_j}{\partial x^\mu} \bar{h}_j^{\sigma\sigma'} \frac{\partial \bar{x}^\sigma'_j}{\partial x^\nu}$$

(3.92)

The constraint can be imposed by introducing a dilaton field $\phi_j[\eta, x]$ through a Lagrange multiplier $\chi_j[\eta, x] = \exp(\phi_j)$ and adding to the action

$$S_\phi = i \int d^4 x d^4 \eta \left[ \sqrt{\bar{g}_j} h^\mu_j - \mathcal{G}^b_j \Box b^b \right] \chi_j[\eta, x]$$

(3.93)

showing that the NL-fields $b^b[x]$ couple linearly to the exponential of the dilaton field $\phi_i[\eta, x]$. This has an important consequence, in that the natural $R_\xi$ gauge-fixing
couples the fluctuations of the dissipative gauge field $A^b_{d\mu}[x]$ within its gauge orbit to the dilaton field. This suggests that, within this model, the modification of measure in a path-integration associated with the Fadeev-Popov volume of the gauge orbit is tied closely to the strength of the dilaton field. This will in turn be linked to local effects which resemble the dilaton-like components of gravitation. It is important to stress that these effects are at least partially entropic in origin, associated with the disregard of detailed dynamics of light-scattering modes.

It is natural then to ask if the dynamical gauge field $A^b_{d\mu}$ and dynamical metric $g^j_{\mu\nu}$ obey the known equations of motion of gauge fields and gravity - for example, whether the Einstein-Hilbert form of the action is obtained in any limit. To answer this, we return in the next section to the remaining constraints introduced by the coordinate transform above - under an appropriate redefinition of the action, these constraints may also be automatically satisfied according to the associated Euler-Lagrange equations.

Before continuing, we first interpret physically the appearance of two auxiliary metrics $g^j_{\mu\nu}$ for $j = 0, 1$ associated to the coordinates and their conjugation. This is related to the appearance of the volume-volume interaction appearing above. The interpretation of this term changes somewhat if we consider the combined set of coordinates as a higher-dimensional Hermitian manifold. In this case, the dimensionality is in a sense doubled, and $v_i$ represent roots of the total volume of the manifold. Likewise, the auxiliary metrics $g^j_{\mu\nu}$ resemble complex roots of the Hermitian metric, similar to vielbeins, and the inverse of the ghost-propagator appears effectively as a curvature tensor. As such, the $g^j_{\mu\nu}$ are insufficient to fully determine the structure of the higher-dimensional manifold - with $N$ total coordinates, it remains to specify another $N$ quantities related to the spin-connections. We will see these enter below through conditions fixing the coordinate transforms $\tilde{x}$ in the form of Lagrange multipliers $\Lambda^N_\mu$. Finally, for a Hermitian manifold, there is a natural sense of orientation to the volume form - this is provided by a final set of four multipliers $\lambda_{is}$ introduced to fix the form of the ghost propagator $\kappa_{is}$. In this way, a single natural Kähler manifold structure is assembled through the introduction of constraints above.
3.7.3 The Unruh Effect and Gauge-Field Dynamics

The Unruh effect describes the apparent thermal occupation of a vacuum by an accelerating observer - for an observer in a frame accelerated at a rate $a$, the thermal energy scale is set by $T_u = \hbar a/2\pi k_B c'$, with $c'$ the maximal speed of propagation of excitations of the vacuum. Typically, Unruh effects are considered unobservably small due to the extreme accelerations necessary to achieve Kelvin-scale temperatures with excitations at the speed of light. For the experiment here, extremely low optical scattering rates are used, such that dispersive effects on the slower $\lambda$-like transition for $m_f = 0, 1$ lead to an effective speed of propagation [92] naively on the order of $\Gamma c/\Omega^2_c \sim 10^{-12} c$ at the highest bright-state Rabi-frequencies $\Omega_c$ used. For Unruh temperatures comparable to the ultra-cold thermal scales on the order of $100\text{nK}$, this yields Unruh accelerations ranging over $0.1 - 10m/s^2$. These are comparable to the forces generated by the optical potential, and thus should be included in the description. To be clear, the application of the Unruh effect should not be considered surprising, since the combination of the slow-light-based calculation above and the Unruh calculation simply state that a single photon recoil of momentum is absorbed over a duration of order $1 - 100\text{ms}$, which is roughly the bright-state scattering rate.

Unruh effects can be considered geometric in nature, and can be captured naturally by simple geometrically-motivated scattering modes as we describe below. Thus, the dynamical gauge field is in some sense just an alternative method for describing EIT-like interference effects in a multilevel system that takes into account local variation of the optical fields. This treatment is helpful in cases, like here, that considerable conservative motion occurs during the spontaneous photon scattering process.

In the analysis above, both the interaction $V^{bc}_{\rho}$ and the gauge stiffness $\xi^{bc}$ above are determined by the structure of $\det \kappa_{is}[\eta]$, which in turn is limited by the Bogoliubov-Fourier wave-functions $u^k_i(\eta, x)$ through the constraints above. It is therefore helpful that explicit $u^k_i$ be chosen. We provide here first a somewhat incomplete description, starting from simple plane-wave structures, and then introducing arguments based on the symmetries present in the light-scattering problem constrained by conformality. We introduce the conformal constraint in order to preserve scale-invariant dynamics already present in the action above.
The most intuitive start is to use “plane-waves’ in the new coordinates $\tilde{x}[x]$ introduced above, which imply definition of the $\eta_{\mu}$ as conserved quantities (energy and momenta) associated with translation in the new coordinates. The complex form of the new coordinates permits smooth connection to the anomalous Bogoliubov amplitudes $u_2$ as the coordinates are varied, in line with the Unruh effect and associated analytic continuations. For simplicity, we first propose a short form to illustrate the idea,

$$u^k_l = \frac{\alpha_k}{2} \exp \left( i\eta_{\mu}(\sigma_{ij})^s \tilde{x}_j^{\mu} + i\pi s l \right) \times \sqrt{d\mu}, \quad (3.94)$$

where $s$ is summed over the values 0, 1, $\alpha_k$ is an amplitude, and $d\mu$ is a normalizing measure. To understand this more easily, it is helpful to see these have the form

$$u^k_0 \propto \cosh(\theta) \exp(i\phi) \quad (3.95)$$
$$u^k_1 \propto \sinh(\theta) \exp(i\phi) \quad (3.96)$$

where the Bogoliubov angle $\theta = i\eta_{\mu}(\tilde{x}_0^{\mu} - \tilde{x}_1^{\mu})$ is given by the imaginary part of the coordinates and $\phi = \eta_{\mu}(\tilde{x}_0^{\mu} + \tilde{x}_1^{\mu})$ by the real.

It is helpful to bring mode expansions for the ghost and scattered light modes into a similar form; for this reason we will consider below the quantity $a u^b_i = \tilde{G}^b_i$, which is ‘dressed’ by the gauge form to transform the polarization index into one for spin, similar to the ghost mode structures $c, \bar{c} u^b_i$. Furthermore, for simplicity, we will consider the real and complex parts of the coordinates $A, \bar{A}$ as a set of eight real-valued coordinates defining a higher-dimensional system, and condense the set of indices $(i, \phi = c/\bar{c}/a, b)$ with a single letter $A$.

Restricting the coordinate transforms $\tilde{x}$ to be smooth and conformal, however, places surprisingly strong constraints on the form above. We will explore the complex nature in a moment - first, for purely real coordinates, all such transformations would be members of the conformal group $SO(4,2)$. While this group includes infinitesimal generators $P_{\mu}$ for translation, and therefore suggests conserved momenta $\eta_{\mu}$ as used above, it also includes generators $M_{\mu\nu}$ for rotations and boosts to moving frames, dilations $D$, and special conformal translations $K_{\mu}$ related to frame acceleration. The three added generators $M, D, K$ form a subgroup isomorphic to the Weyl group.
$W$, such that the full conformal group is a product of an internal Weyl subspace and a Minkowski space $M_4 = SO(4,2)/W$ - it is convenient to write the spatial coordinates using a matrix form

$$X_{\alpha\alpha'} = \begin{pmatrix} x_0 + x_1 & x_2 + ix_3 \\ x_2 - ix_3 & x_0 - x_1 \end{pmatrix} \tag{3.97}$$

in which case all conformal transformations may be written in the form

$$\tilde{X} = \frac{AX + B}{CX + D} \tag{3.98}$$

for $2 \times 2$ complex matrices $A, B, C, D$.

This representation is not unusual - for instance, it is used as a convenient starting point for instanton and twistor formalisms, and is simply related to the fact that all conformal mappings can be understood by manipulating a ‘product’ of the local coordinates and the Weyl group representing an internal space. This also suggests inclusion in the above of a larger set of conserved charges $\bar{\eta}_\sigma$ associated with incorporating the irreducible representation of the internal space, and extension of the ‘coordinates’ to a larger set $\bar{x}_i\sigma$ (where we still retain an index $i$ for conjugation of the complex coordinates, equivalent to a parity flip of a subset of coordinates in the higher-dimensional representation). The plane wave expansion above may then be generalized to a pair of Fourier expansions in a basis dual (in the Peter-Weyl sense) to $SO(4,2)$ and represented by the $\bar{\eta}_\sigma$.

To incorporate the Unruh-like effects, the coupling between complex and real coordinates used above must be accounted for, in a way which connects the special conformal generator $K_\mu$ (representing acceleration) to the anomalous angle $\theta$. A similar approach is discussed in refs. [93,94], the first of which interestingly interprets the Unruh effect as a ‘spontaneous breakdown of conformal symmetry’. There, the $SO(4,2)$ group is extended into a complex representation $U(2,2)$, such that a complex (Cartan) space $D_4$ is obtained as a continuation of $M_4$. The modified form of the conformal transformation 3.98 above, irreducible representations (depending on a conformal dimension $\lambda$ and two spin indices $s_1, s_2$), and the transformation of an associated wave-function with the associated normalization are reported.
there. The complex conformal group exhibits a symmetry under the combined replacements $P_\mu \leftrightarrow K_\mu$ and $D \leftrightarrow -D$; together these imply a (CPT) symmetry under simultaneous exchange of proper time, parity, and conjugation of wavefunctions [94], which the authors have argued is also tantamount to Born-Reciprocity, a point to which we will return in different language below.

The mode expansions for scattered pumping light $a u_i^b$ and (anti)-ghost modes $(c)\bar{c} c_i^b$ can each be expanded in a separate set of preferred coordinates as $\phi u_i^b \propto \exp(i\bar{\eta}_\sigma \bar{x}_{\mu}\sigma^\mu)$, for each field $\phi = (a, a^*, c, \bar{c})$. To bring the optical pumping modes into a form similar to the ghost modes, we define

$$a u_s^b = G_{rk}^{bc} E_{op,t}^{\sigma_f} a u_g^k[\bar{x}_f] \sigma^x_{gs}$$  \hfill (3.99)$$

and define all fields through coordinate transformations $A \bar{x}$,

$$\phi u_i^b[\bar{\eta}, \bar{x}] = e^{i\bar{\eta}\sigma \bar{x}}$$ \hfill (3.100)$$

We include constraints that the physical momenta follow the ‘induced’ form $\eta_\mu = \bar{\eta}_\sigma A \partial A \bar{x} / \partial x^\mu$, and that the coordinates $A \bar{x}^\sigma$ be determined by a fixed mapping $A \bar{x}^\sigma[x]$ defining a four-dimensional ‘surface’ $S$ in the higher-dimensional space. It is formally convenient to absorb these constraints as two additional ‘coordinates’

$$\bar{x}^N_A = -i \log[\delta^N(A \bar{x}^\sigma - A \bar{x}^\sigma[x])]$$ \hfill (3.101)

$$\bar{x}^{N+1}_A = -i \log[\delta^4(\eta_\mu - \bar{\eta}_\sigma \partial A \bar{x}^\sigma / \partial x^\mu)]$$ \hfill (3.102)$$

with corresponding momenta $\bar{\eta}^N = \bar{\eta}^{N+1} = 1$, such that the implied sum over $\sigma$ in 3.100 extends over $\sigma = 0...N + 1$, and define

$$\phi u_i^b[\bar{\eta}, \bar{x}] = \int d^N \bar{\eta} d^N \bar{x} \phi u_i^b[\bar{\eta}, \bar{x}]$$ \hfill (3.103)$$

(note that the final coordinates $\bar{x}^{N(+1)}$ are not integrated).

The constraints above, along with the desired volume forms, reduce to a Palatini
form for the action in terms of the partial derivatives $A e^\sigma_\mu = \partial_A \bar{x}^\sigma / \partial x^\mu$. It is helpful to first break the transformations $A \bar{x}^\sigma [x_\mu]$ into two steps, $A \bar{x}^\sigma [A M^j [x_\mu]]$ consisting of fixed ‘lifts’ $A M^j$ of $x_\mu$ to the internal ($j = 0$), external ($j = 1$), and constraint ($j = 2$) spaces, followed by continuous mappings $A \bar{x}^\sigma$ of those spaces onto themselves. The existence of a covering group like the above assure the transformation may be broken into these three sectors. The partial derivatives can then be written $A e^\sigma_\mu = A M^j_\mu A \tilde{e}^\sigma_j$, which allows for an interpretation of the $A \tilde{e}^\sigma_j = \partial_A \bar{x}^\sigma / \partial A \bar{x}^j$ as frame-fields or ‘vielbeins’, through the partial derivatives of the lifts $A M^j_\mu = \partial_A \bar{x}^j / \partial x^\mu$. This allows the action found above to be written in a modified Palatini form

$$S_P = \int d^N \bar{x} \, d^4 \bar{\eta} [A e^\sigma_\mu B \tilde{e}^\sigma_\rho A P^{\rho \rho'} A B \Omega^{\rho \rho'} + A e^\sigma_\mu A F^{\rho \rho'}]$$

where $A, B$ now represent sets $(i, j, \phi)$ condensing the labels for coordinate conjugation $i$, sector $j$ and field $\phi$. The quantity $\Omega$ may be viewed as representing a curvature tensor, and takes its form from expansion of the ghost action above as

$$A A \Omega^{\rho \rho'} = \bar{\eta}_{j \sigma} \eta_{j' \sigma} \tilde{c}_a^i \tilde{c}_c^i \delta^{\rho \rho'} \delta_{\phi, c} \delta_{\phi', c}$$

and the contribution to the curvature from the volume constraint above is added through the action of the operator $P$ as

$$P^{\rho \rho'} = \delta^A_{A'} \delta^\rho_{\rho'} + \delta^A_{\rho'} \delta^\rho_{\rho'} + i \chi_j \sqrt{j} g^{\rho \rho'} \bar{h}_{\rho \rho'} \hat{p}_{c c \rightarrow a a} \partial_{\bar{\eta}_{j \sigma}} \partial_{\eta_{j' \sigma}} \partial_{\tilde{c}_a^i} \partial_{\tilde{c}_c^i}$$

which exchanges the stress-energy associated with the ghost modes with a complex curvature determined by the auxiliary metric acting in the optical-pumping mode space. Here, $\hat{p}_{c c \rightarrow a a}$ shifts the $AB$ indices from ghost modes to optical pumping.

Classically, independent variation of the vielbeins and the spin-connections (contributing to the form of $\Omega$) for $j = 0$ are well-known to lead to an action equivalent to the Einstein-Hilbert form, and thus the classical equations of motion for general relativity. In the $j = 1$ sector, the Yang-Mills kinetic energy terms arise
through spin-connects in a way similar to the Kaluza-Klein compactification. The operator $P$, if its second term is of an appropriate form, leads to a self-dual or anti-dual formulation of gravity and gauge fields. If in this case, $P$ is to be interpreted as a type of dualization, it must be viewed as acting in the higher-dimensional manifold specified the totality of all coordinate systems $A$.

The final term $F$ in the action above goes beyond the Palatini form, arising from two sources. The first is contributed by the ghost propagator’s covariant derivative,

$$ (\phi=\bar{c},i,j) F^\rho_\sigma = M^i_{\rho} \bar{\eta}_\sigma \epsilon^{abc} A^b_{\mu} c^a c^c $$

and represents three-leg gauge-ghost vertices, with the attachment of an antighost vielbein.

A second contribution arises from fixing the form of the ghost propagator $\kappa_{is}$ to the form $\bar{\kappa}_{is}$ determined above by the optical pumping fields. To implement this, we introduce a lagrange multiplier $\lambda_{is}[\eta,x]$ and add to the action $i\lambda_{is}(\kappa_{is} - \bar{\kappa}_{is})$. The first term is equivalent to replacing the ghost amplitudes $c_i^a c_s^s$ with $c_i^a c_s^s + i\lambda_{is}$, and the second introduces a term linear in the vierbeins to the action through

$$ A=(\phi=a,i,j) F^\rho_\sigma = -i\lambda_{is}A M^0_{\rho} \bar{\eta}_\sigma $$

In total, the action is then constructed from

$$ S = S_P + S_\phi + S_\xi' + S'_{\psi} $$

where $S_\xi'$ discards the middle term of $S_\xi$ as defined above, and $S'_{\psi}$ is second-order in the covariant derivatives of $\psi$. While it would be possible to study this model in detail (including necessities we have not included here, such as accounting for gauge-equivalence in the vielbein definitions), we will find it easier to map this problem onto boundaries of a two-dimensional space formed by consideration of the $j = 2$ coordinates.
3.7.4 Constraint Space as a Two-Dimensional Conformal Field Theory

In Palatini formulations of gravity, the vielbeins $\tilde{e}$ are treated as degrees-of-freedom. This is natural for the internal and external coordinates $j = 0, 1$, but merits special consideration in the constraint space $j = 2$, where we have given the coordinates values defining, for $\sigma = N$, a specific coordinate transform, and, for $\sigma = N + 1$, conservation of momentum.

At the outset, defining coordinates this way appears ill-posed, as only the “point” $A\tilde{x}^N \to -i\infty$ carries direct meaning for the coordinate transform $A\tilde{x}$, and labels the entirety of the surface $A\tilde{x}[x]$. Likewise, only divergent values of the coordinate $A\tilde{x}^{N+1}$ appear a priori relevant, representing strict momentum conservation. Though only their common divergence then would seem to have meaning, Heisenberg uncertainty suggests otherwise; fixing $A\tilde{x}^N$ would imply all normal coordinates are known, and thus lead to arbitrarily large fluctuations of the normal momentum. Thus non-divergent values of $A\tilde{x}^N$ and $A\tilde{x}^{N+1}$ must also be considered in their motion.

Treating the constraint-space vielbeins on equal footing as the other two sectors requires introduction of a constraint in the form of an additional Lagrange multiplier $A\Lambda^\mu_{N(+1)}$ in the action

$$S_\Lambda = i A \Lambda^\mu_N (A e^N_\mu - \partial_\mu A \tilde{x}^N [x_\mu])$$
$$S_\Lambda = i A \Lambda^\mu_{N+1} (A e^{N+1}_\mu - \partial_\mu A \tilde{x}^{N+1} [x_\mu])$$

which fixes the vielbeins to the values

$$\partial_\mu A \tilde{x}^N = -i A e^\sigma_\mu \frac{\delta^N_\sigma (A \tilde{x}^\sigma - A \tilde{x}^\sigma [x])}{\delta (A \tilde{x}^\sigma - A \tilde{x}^\sigma [x])}$$
$$\partial_\mu A \tilde{x}^{N+1} = -i \bar{\eta}_\sigma \partial_\nu A e^\sigma_\mu \frac{\delta^4_\nu (\eta_\mu - \bar{\eta}_\sigma A e^\sigma_\mu)}{\delta (\eta_\mu - \bar{\eta}_\sigma A e^\sigma_\mu)}$$

It is revealing that the first of these relates the normal vector of the surface $A\tilde{x}^\sigma [x]$ to the vielbeins in all sectors, while the second relates spin-connections defined through the derivatives $\partial_\nu A e^\sigma_\mu$ to a four-dimensional vector tangent to the surface. These are reminiscent of Dirichlet and Neumann boundary conditions.
familiar from string-theoretical pictures, in which the gauge field is determined by
the motion of the endpoint of an open string tied to the surface of a membrane. With some effort, it can be shown this is not just cosmetic. The delta-function
derivatives above may be viewed more generally as expectation values for differential
operators using a common distribution function $\Delta$,

$$\partial_{\mu A} \bar{x}^N = -i A^\sigma \epsilon_\mu^\sigma \langle \frac{\partial}{\partial A \bar{x}^\sigma} \rangle \Delta$$  \hspace{1cm} (3.113)$$

$$\partial_{\mu A} \bar{x}^{N+1} = -i \bar{\eta}_\sigma \partial_{\nu A} \epsilon_\mu^\sigma \langle \frac{\partial}{\partial A \bar{p}_\nu} \rangle \Delta$$  \hspace{1cm} (3.114)$$

$$\Delta = \prod_A \delta^N(A \bar{x}^\sigma - A \bar{x}^\sigma[x]) \times \delta^4(\eta_\mu - \bar{\eta}_\sigma A^\sigma)$$

$$= \int D[\prod_A A \bar{p}_\sigma, A \bar{x}_\sigma, A \bar{p}_\nu] \delta(A \bar{p}_\nu - \eta_\nu + \bar{\eta}_\sigma A^\sigma) \exp \sum_A 2\pi i [A \bar{p}_\sigma A \bar{x}^\sigma + A \bar{x}^\nu A \bar{p}_\nu]$$

where we have defined the quantity $A \bar{p}_\nu$ and constrained it to the value $\bar{\eta}_\nu - \bar{\eta}_\sigma A^\sigma$, and offset the $A \bar{x}^\sigma$ by the value of $A \bar{x}^\sigma[x]$. We have chosen the names for the
Lagrange multipliers $p$ and $x$, and the form of action above, to be suggestive of
Born-reciprocity.

A more convenient coordinate system can be constructed by bringing the
divergences of $A \bar{x}^N$ to finite values through $\sigma_A = \tan^{-1}(i_A \bar{x}^N)$. While the endpoint $\sigma_A = \pi/2$ is maximally relevant in determining the surface location, the endpoint at $\sigma_A = -\pi/2$ is maximally irrelevant; we consider connection of this irrelevant
point to another for a second surface $B$ by forming paired coordinates

$$\sigma_{AB} = \theta(\sigma_{AB})(\pi + \sigma_A + \sigma_B) - (\pi/2 + \sigma_A)$$  \hspace{1cm} (3.115)$$

where $\theta$ is a heaviside function. The new coordinate $\sigma_{AB}$ takes the value $-\pi$ on the
surface $A$ and $\pi$ on surface $B$, and otherwise represents either of $A/B \bar{x}^N$ according
to its sign. In this way, the coordinates $\sigma$ form a (Chan-Patton-like) matrix of
affine parameters for curves connecting surfaces in pairs. A similar construct $\tau_{AB}$
can be made using the $\bar{x}^{N+1}$ coordinates, taking the values $\pm \pi$ when momentum is
conserved on the $A/B$ surfaces.
It is convenient to group the variables in the action also pairwise by introducing functions of the form

\[
AB\bar{x}^\sigma[\sigma_{AB}, x] = A\bar{x}^\sigma[\sigma_{AB} + \pi] + B\bar{x}^\sigma[\sigma_{AB} - \pi]
\]

\[
AB\bar{p}^\nu[x, \tau_{AB}] = A\bar{p}^\nu[\tau_{AB} + \pi] + B\bar{p}^\nu[\tau_{AB} - \pi]
\]

\[
ABx^\sigma[x, \tau_{AB}] = Ax^\sigma[\tau_{AB} - \pi] + Bx^\sigma[\tau_{AB} + \pi]
\]

\[
ABp^\nu[\sigma_{AB}, x] = Ap^\nu[\sigma_{AB} - \pi] + Bp^\nu[\sigma_{AB} + \pi]
\]

defined now only at the points \(\sigma = \pm \pi\) and \(\tau = \pm \pi\). The distribution can now be written more simply as

\[
\Delta = \int D[p_\sigma, x_\mu, \bar{p}_\nu] \Delta_p^{-1} \Delta_x^{-1} \delta_{\tau=\pm \pi}(\bar{p}_\nu - \eta_\nu \mathbb{I} + \bar{\eta}_\sigma e_\sigma^\nu) \exp\left(\frac{i}{4N_a^2} \int d\sigma d\tau \text{Tr}\left[\frac{d}{d\sigma}(p_\sigma \bar{x}^\sigma) + \frac{d}{d\tau}(x^\nu \bar{p}_\nu)\right]\right)
\]

where we have suppressed the \(A/B\) matrix indices, \(\mathbb{I}\) is the matrix identity in the \(A/B\) space, \(N_a\) its dimension, and the trace is taken over the matrix products of \(p_\sigma \bar{x}^\sigma\) and \(x^\nu \bar{p}_\nu\). Note also that we have taken advantage of the full derivative to introduce integration over \(\tau\) and \(\sigma\). The factors of \(\Delta_p = \int D[p_\sigma]\) account for the increase in measure of the functional integral by introducing interior points \(p_\sigma[\sigma]\) for \(-\pi < \sigma < \pi\), which do not affect the integrand. All such functions must by definition be antisymmetric in \(\sigma\) under transposition of the Chan-Patton indices. Finally, note that the partial derivative \(e_\sigma^\nu\) (only presently defined at the values \(\tau = \pm \pi\)) has also been extended to a matrix form - this implies generally noncommutative relations \([e_\mu^\sigma, e_\mu'^\sigma] \neq 0\), and becomes relevant if the same form is also used to represent the Palatini action.

Since the surface normals form an \(N - 4\) dimensional subspace, it is simpler to group the multipliers \(\pi_\sigma = (x_\mu, p_\sigma)\) into an \(N\)–dimensional vector, and form a corresponding vector \(q_\sigma = (\eta_\nu \mathbb{I} - \bar{\eta}_\sigma e_\sigma^\nu, \bar{x}_\sigma)\). In these variables, the distribution can be written concisely as
\[ \Delta = \int D[\pi_\sigma, \tilde{g}] \Delta_g \exp \int \text{Tr} \left[ \frac{i}{4N^2_\sigma} d(\pi_\sigma q^\sigma) - \frac{1}{2} \chi \ast (d\pi^\sigma) \wedge d\pi_\sigma \right] \]

with \( d \) the exterior derivative on the \((\sigma, \tau)\) space, \( \ast \) the hodge dual, and \( \wedge \) the exterior product. Note that integration is now also taken over the two-dimensional metric \( \tilde{g} \), which in combination with the final term of the exponent, accounts for the measure of integration over interior points; we will assume \( \det (\tilde{g}) = 1 \), and introduce Weyl-like scalings through the dilaton field \( \chi \), which must coincide with that introduced above to match the boundary values. The Fadeev-Popov factor \( \Delta_g \) appears due to gauge symmetry in \( \tilde{g} \). Aside from the matrix-nature of the variables, the form for \( \Delta \) above is equivalent to that described in [95] for a bosonic string. There, it was noted that completing integration over \( \pi^\sigma \) results in the standard Polyakov action with coordinates \( q^\sigma \) and an inverted coupling constant \( \chi \), while introducing an integration over \( q^\sigma \) results in a Polyakov action in \( \pi^\sigma \) - a realization of T-duality. Therefore, perhaps not surprisingly, there is motivation for use of these ideas without \textit{a priori} postulation of string-like object, but only beginning from a description of gauge field dynamics and considering the simultaneous definition of preferred coordinates and momenta. The advantage is that non-perturbative techniques in string theory, such as renormalization-group methods for background fields, T- and S-duality, are now directly applicable to these problems; it is also convenient that two-dimensional conformal field theory may be applied to the fields \( \tilde{x}^\sigma \) defined on the \((\sigma, \tau)\) coordinate space.

The two ends of this problem, the gravitational-/gauge dynamical- picture based on path-integration \( \mathcal{Z} = \int D[\tilde{e}] \exp (S) \) of the action \( S[\Delta] \) above, and the two-dimensional dynamical distribution \( \Delta[\tilde{e}] \) are coupled through the evolution of the vielbeins \( \tilde{e} \), such that the two distributions \( \mathcal{Z}[\Delta] \) and \( \Delta[\mathcal{Z}] \) are co-dependent. Since the coupling of \( \mathcal{Z} \) to \( \Delta \) is through expectation values of momenta and position associated with the \( A \tilde{x} \), and the coupling of \( \Delta \) to \( \mathcal{Z} \) is through the associated vielbeins \( A \tilde{e} \), a natural method of approximately decoupling these problems is to introduce expansions of these quantities in their moments defined by each distribution.
3.7.5 Holography and Effects on Excitation Masses

The picture of T-duality is particularly useful in understanding how dark-state conditions can lead to a discrete (topological) change in non-equilibrium behavior from the microscopic level. Generally, the energy-momentum relation of excitations of a string are given discretely by its winding numbers $m$ along compact dimensions, and its integral $n$ number of phase windings in the fourier-decomposition of $A_B \tilde{x}^\sigma$ along its length $\sigma$. T-duality asserts (for the bosonic string) that the same energy-momentum relation will be found on combined exchange $m \leftrightarrow n$, exchange of the $q^\sigma$ fields for a dual set, and a specific inversion of the coupling-constant $\chi$ and length of the compact dimensions. Finally, for open strings as above, Neumann and Dirichlet boundary conditions are exchanged. The field $\pi^\sigma$ above can be seen as the field dual to $q^\sigma$, and thus if it experiences a change which would alter these discrete numbers, the excitation spectrum is changed.

The coordinate transform introduced for normal-mode scattering of optical pumping light above introduces an extra factor in the constraint sector for the coordinate $A_N \tilde{x}^N$ related to the amplitude $O_a$ of the prefactors $G_{rk}$ and $E_{r_op}$. The corresponding coordinate $A_N \tilde{x}^N$ with $A = (\phi = a, i)$ therefore enters everywhere offset by the value $i \log O_a$, making it convenient to absorb the offset into the constraint-space description. Since the effect enters only through the expectation values against the distribution $\Delta$, it is clear this is equivalent to a local shift of the value of the multiplier

$$\pi_\sigma[\sigma, \tau] \rightarrow \pi_\sigma + \frac{i}{O_a} \frac{\partial O_a}{\partial x^\mu} e^\mu_\sigma$$ (3.117)

which is tantamount to a gauge field derived from the extension of $O_a[x^\mu]$

$$\alpha_{\sigma \beta} = \partial_{\beta} \frac{i}{O_a} \frac{\partial O_a}{\partial x^\mu} e^\mu_\sigma[\sigma, \tau]$$ (3.118)

introduced into the two-dimensional conformal field theory for the fields $\pi^\sigma[\sigma, \tau]$ ($\beta$ indexes the 2D coordinates $(\sigma, \tau)$). Since the only physically-relevant points for these expectation values lie on the boundary at $\sigma, \tau = \pm \pi$, it is natural to expect the impact of this term can be understood through a line-integral of this quantity
tracing the boundary of the constraint space. This is apparent above, as the first term in the ‘action’ defining $\Delta$ is a boundary term. Due to the shift of $\pi^\sigma$, this contributes $S_h = -\sum_\sigma i\hbar^\sigma / 4N_a^2$, where

$$h^\sigma = \oint_{\partial \Sigma} \alpha^\sigma_\beta \cdot d\ell^\beta$$

(3.119)

Through the definition of $q^\sigma$, we can see that the integral over the boundary $\partial \Sigma$ of the constraint space $\Sigma$ is equivalent to some set of loops $A_{\partial \Sigma}(\sigma, \tau)$ in the full coordinate systems $A_{\bar{x}^\sigma}$. This demarcates the boundary of the two-dimensional system, and lies within one (or two for off-diagonal elements of $q$) of the surfaces $A$. Since these are themselves parameterized by $x^\mu$, the loops can be projected into the original coordinates as $\partial \Sigma^\mu(\sigma, \tau)$ by inverting the original transforms $A_{\bar{x}^\sigma}[x^\mu]$.

Equivalently then, the contribution from the constraint space to the full action is given by integrals of the gauge-like quantity $\alpha$ related to the dynamical gauge-field $\bar{A}$. These may be viewed equivalently as (Wilson-like) loops in real space, or as loops along the boundary of the two-dimensional field theory in the constraint space. The geometry of the loops arise dynamically according to the variation of the vielbeins $\bar{e}$. The field structure $\alpha^\sigma_\beta$, however, is defined by the arrangement of optical pumping fields and external magnetic fields through $O_a$.

Alternatively, the effect of these loops can be understood from the interior of the constraint space $\sigma, \tau$. To do this, one must extend not only $e^\alpha_\mu$ to the interior points, but also $O_a[x^\mu]$ by some suitably smooth mapping, and thereby define some area $\Sigma^\mu[\sigma, \tau]$ filling the interior of the loop $\partial \Sigma^\mu$, and parameterizing it by $\sigma$ and $\tau$. Naturally, we will require that the area chosen, and its parameterization, are immaterial to our results. In this way, we might expect discrete values of $h^\sigma$ to represent flux quanta of the gauge-field $\alpha^\sigma_\beta$, and we might expect their effect to enter through a ‘filling factor’ of flux through $\partial \Sigma$. The natural choice for $\Sigma^\mu[\sigma, \tau]$ is one which follows the form of $e^\sigma_\mu[\sigma, \tau]$, such that $\partial_\mu \Sigma^\sigma[\sigma, \tau] = e^\sigma_\mu[\sigma, \tau]$.

At a base level, the Virasoro algebra representing the two-dimensional field theory is altered by the introduction of additional terms in the stress-energy tensor $T(z)$ due to topological defects in $\alpha^\sigma_\beta$

$$T_{\alpha\beta}(z) = T^0_{\alpha\beta}(z) + \alpha^\sigma_\alpha \alpha_{\beta\sigma} + \alpha_{\beta\sigma} \partial_\alpha \pi^\sigma + \alpha^\sigma_\alpha \partial_{\beta} \pi_\sigma$$

(3.120)
Where $T^0$ is the standard result from the unmodified Polyakov action. As a result, the Noether-currents associated with any operator $O$ are modified along with the associated Ward identities. As a result, the generators $L_n$

$$L_n = \frac{1}{2\pi i} \oint dz \, z^{n+1} T_{zz}(z) \tag{3.121}$$

are shifted and the ground state $|0\rangle$ modified such that $L_0|0\rangle \neq 0$ (here $T_{zz}$ is decomposed in the basis of $z$).

Not all flux of $\alpha^\sigma_\beta$ are necessarily relevant to the gauge-boson dynamics. The excitation modes of a string corresponding to gauge-field dynamics are given by vertex operators of the fields $q^\sigma$ [95] as

$$V \sim \int_{\partial \Sigma} ds \, \zeta_a : \partial_z q^a \exp (i \eta^*_a q^\sigma) : \tag{3.122}$$

where $\zeta_a$ is a polarization vector, the sum over index $a$ is restricted to directions parallel to the surface, the derivative $\partial_z$ is taken with respect to $z = \exp (i \sigma - \tau)$, and the $: :$ imply normal ordering in $\tau$. While the constraint of Weyl-invariance normally restricts gauge-bosons to zero-mass $\eta^*_a \eta^{a\sigma} = 0$ [96], the presence of $\alpha^\sigma_\beta$ with non-vanishing field curvature implies a shift to the corresponding $n$ and $m$, generally lifting this condition and giving the gauge boson mass. If, through manipulation of $O_a$, this defect is removed, the massless condition is restored. A similar statement may be made concerning conditions on transversivity of polarization $\zeta^a \eta^*_a = 0$, which is, in the absence of the offset, a result of requiring the operator above to be primary. An excellent viewpoint for this was suggested by ref. [95], in which a T-dual vertex was introduced, equivalent here to

$$V^* \sim \int_{\partial \Sigma} ds \, \zeta_b : \partial_z \pi^b \exp (i \eta^*_a \oint_{z_i} \star d\pi^a) : \tag{3.123}$$

The effect of a puncture introduced by the shift of $\pi_q$ at $z_i$ is then to introduce an associated winding number about $z_i$.

All such winding numbers are dynamically determined through the product of vielbeins through $e^\sigma_\mu$ and the form of $O_a$. Since the $e^\sigma_\mu$ are dynamical, one
might speculate that the presence of defects in $\alpha$ is simply canceled by considering integration over a different class of $e_\mu^\sigma$ with opposing defects. To affect the dominant form of mass, there must be some form of preference for the vielbeins $e_\mu^\sigma$ according to their defect structure. This preference can be seen by considering the effective change in variables introduced in the representation of pumping scattering modes through eq. 3.99. We assume for the moment, that this relation is achieved through a change-of-variable in writing the $u^k_i$ and $u^b_i$ in simple exponentiated forms as described above, and consider the change in measure associated in moving from dynamical variables $\bar{A}e_\mu^\sigma$ to $A e_\mu^\sigma$, where the $\bar{A}$ system represents a set of coordinates $\bar{A}\bar{x}$ indexed by vacuum-mode polarization $k$ instead of the spin-index $b$. The relation can be found by differentiating eq. 3.99 once each with respect to $x^\mu$ and $\bar{\eta}^\sigma'$,

$$A e_\mu^\sigma = (\bar{A} e_\mu^\sigma - i(1 + \bar{\eta}_\sigma^r \partial_\bar{\eta}_\sigma^r)^{-1} \partial_{\bar{\eta}_\sigma^r} \partial_\mu) J^b_k$$

(3.124)

where $J^b_k$ is a local jacobian written in operator form as

$$J^b_k = G^b_{r k} F^r_{op, f} \sigma_f^e \sigma_{g a} e^{\bar{\eta}_g} (\bar{x}^g - A x^g)$$

(3.125)

In this way, the measure of the path-integration $Z$ (proportional to $1/\det J$) can be directly tied to the effective gauge field $\alpha^\sigma_\beta$ - the second term in 3.124 for $\sigma = N$ contributes a shift of the constraint vielbein through

$$\alpha^\sigma_\beta = e_\mu^\sigma i \partial_\beta (1 + \bar{\eta}_\sigma^r \partial_\bar{\eta}_\sigma^r)^{-1} \partial_{\bar{\eta}_\sigma^r} \partial_\mu J^b_k$$

(3.126)

It is important to note that the coordinates $q_\sigma$ are not dynamical variables of the action defining $\Delta$, but rather $\Delta[q_\sigma]$ is functionally dependent on them. Generically, this represents the interaction between the two-dimensional field theory defined by $\Delta$ and the full action $S$ above. In general, this is complicated to solve. To understand the coarse behavior of the system from $\Delta$ alone, we assume that the effect of the ‘mean values’ of the fluctuating $q_\sigma$ can be understood as a nonlinear-sigma model arising from a ‘background’ fields corresponding to a metric $G_{\mu\nu}$, dilaton $\Phi$ and torsion fields $H_{\mu\nu\rho}$ (modified by a Born-Infeld term for the gauge fields described
below) in a more-or-less standard picture of a bosonic string [96]. This replacement would be consistent with the dynamics determined by $S$ only if its low-energy equivalent action coincided with that predicted by the variation of $S$ over the variables in $q_\sigma$. Since both pictures represent formulations of gauge-fields coupled to gravity, it is likely this can be made true. Unlike more fundamentally-inspired pictures, this consistency can be directly checked by demanding the means predicted by the distributions $\Delta$ and $Z$ agree, and constraints placed on the model through them - in this way, the corresponding background fields defined through $g$, $\Lambda$ and $\lambda$ above should be directly related to background fields $G, \Phi, H$. We reserve this connection for future studies.

To produce the equivalent field $\alpha^\sigma_{\beta}$ affecting the two-dimensional theory, the background torsion must be chosen appropriately, accounting for the offset $S_h$ to the effective action. Due to the presence of a natural short length scale cutoff $\mu$ at the scattered light wavelength, there is reason to believe that arbitrarily small spatial loops $C_{\bar{z}}$ will eventually fail to provide the same results as large loops, and thus short-time behavior of the evolution will depend on this cutoff. In the opposite limit of the longest time-scale behavior, however, one can ask if a well-defined limit point can be achieved in which the dynamic behavior is not dependent on this cutoff. For this, one can demand that the associated beta functions [96] vanish at a fixed point of flow to larger $\mu$. At the fixed point, the background metric $G_{\mu\nu}$ and associated Ricci scalar $\mathcal{R}$, dilaton $\Phi$ and torsion tensor $H_{\mu\nu\rho}$ must satisfy the equations of motion of the effective Einstein-Hilbert action

$$S^{(1)}_\Delta \propto \int d^N \pi \sqrt{-G} e^{-2\Phi} (R - H_{\mu\nu\rho}H^{\mu\nu\rho} + 4\partial_\mu \Phi \partial^\mu \Phi)$$

combined with a low-energy action generalizing the Born-Infeld action for a non-Abelian gauge field with curvature $F^{\mu\nu}_d$, which for small fields may be written [96]

$$S^{(2)}_\Delta \propto \int d^4 \pi^\sigma \quad (3.127)$$

$$\operatorname{Tr}[F^{\mu\nu}_d F_{d\mu\nu}/4 + D_{\mu} \pi^{\sigma'} D^{\mu} \pi^{\sigma'} - V_\pi(\pi^{\sigma'})]$$

where $D_\mu$ is the covariant derivative associated with $A^{b\mu}_d$.
It is worth mentioning that a similar mechanism is known to occur in supersymmetric Yang-Mills and instanton [78] pictures, in which the Higgs mechanism is associated with the separation of otherwise coincident D-branes, and does not rely on the presence of gravitational terms as were unavoidable above. In those cases, the potential energy of the surfaces corresponding to $A_B \pi^\sigma$ is restricted by T-duality, and is, as above, equal to $V_\pi = \sum_{\sigma'\sigma''} \text{Tr}[\pi^{\sigma'}, \pi^{\sigma''}]^2/2$ for summation over directions $\sigma', \sigma''$ normal to the surface. Within the zero-energy subspace obtained by choosing commuting coordinates, the gauge boson mass is determined by the middle term of $S^{(2)}_{\Delta}$ as proportional to the surface separations $|A_\pi \sigma - B_\pi \sigma|$, where $\sigma$ is chosen along the direction of separation, and $A, B$ represent the relevant coordinates for the given field component (in the case of instanton physics, the relevant separation is of the instanton and D-brane). Under the CPT symmetry discussed above, and consistent with the gauge-symmetry of $A^{b_{\mu}}_d$, all but two coordinate systems $A, B$ must be identical. The gauge boson mass must then be determined by the interplay of $H_{\mu\nu\rho}$ and the vacuum expectation values for the $A_B \pi^\sigma$ fields. However, these must be constrained by the same relation of measure ($J^b_\epsilon$) and constraint-space gauge field $\alpha^\sigma_\beta$ described in the microscopic picture above.

Finally, for the effective action $S^{(1)}_{\Delta} + S^{(2)}_{\Delta}$ to represent a consistent picture of gravitation at the fixed point, the effective four-dimensional gravitational constant $G_0$ must be inversely proportional to the internal volume $V_{\text{int}}$ associated with the $N - 4$ additional dimensions in $S^{(1)}$ [96]:

$$G_0 \propto \frac{1}{V_{\text{int}}} \quad (3.128)$$

In the next section, we will see that this volume can be modified in the presence of thermal and disorder-driven effects.

### 3.7.6 Disorder and Thermal Averaging

It is helpful to put holographic effects on a more physical footing before considering the quantitative details entailed by the model above. We consider first the boundary separating any given region of an atomic wave-packet (near which its dynamics are related by unitary motion) from regions in which only thermal correlations are present, which occurs roughly at the thermal deBroglie wavelength. The
slowest dynamics of this system, those measured in the experiment, occur at low temperatures through the continuous motion of this boundary. In a disordered system, increasing the radius $r_{dB}$ of this boundary changes free energy associated with the unitary region in two distinct ways. Trivially, as the packet cools, the thermal entropy decreases, and the wavepacket expands. In a disordered system, however, the informational entropy of disorder also increases as a larger region of disorder is sampled, and thus there is a balance required between these rates of entropy change. A non-trivial outgrowth of the above is that these two effects interact with a third effect associated with the change in a local analog gravitational potential through the dilaton-field.

Thermal averaging can be understood in the usual way by adding a complex ‘time’ coordinate $t_t$ to the action, and assuming all dynamical fields are periodic in the new direction over a scale $\beta_t$ with $1/\beta_t = k_B T$, and $T$ the temperature. Likewise, to understand the role of a disordered optical field formed by the far-detuned light, we introduce an additional parameter, or coordinate, $t_d$ to those describing the mode amplitudes $a_{fi}|x_\mu, t_t, t_d|$ for far-detuned light, as well as all other dynamical variables. We will assume this coordinate also to imply periodicity, though at an independent inverse ‘temperature of disorder’ $\beta_d$. Corresponding to each are additions to the original action proportional to $|a_{fi}|^2$ and the associated kinetic energy. Physically, the disorder temperature is related to the variance $\delta a^2_{fi}$ of the far-detuned modes by $k_B T_d = \delta a^2_{fi}$. Though one could follow similar steps to those for integrating-out the optical pumping modes above, we will forgo that complication, and concentrate instead on changes to the constraint space.

Since we assume no particular relation to the prior set of preferred coordinates $A\bar{x}^{j\sigma}$ for $j = 0, 1, 2$, it is sufficient to consider the additional coordinates $(t_t, t_d)$ as an isolated sector $j \equiv -1$ covered by the other coordinates. In this way, the entirety of the previous analysis may be retained, with the exception of expanding the indices to include the larger range of $j$, and with coordinate integrations in the action extending over all $j \leq 1$. We therefore anticipate the addition of a new set of vielbeins $e^{j}_{\rho=-1,\sigma}$ to our dynamical variables. We consider here the change to the constraint space dynamics due to the presence of the additional associated partial derivatives $e^{j}_{\mu=-1,\sigma}$ and constraint-space fields $q^{j=-1,\sigma}$.

An immediate consequence can be seen from T-dualities, under which the coordinate lengthscales $\beta_t$ and $\beta_d$ independently map onto dual pictures in which
the corresponding Polyakov actions appear with coupling constant and dimension related by \( \beta_i \rightarrow \chi/\beta_i \) with \( i = t \) or \( d \). Applying the T-dual (integrating out \( q^{j=1, \sigma=t} \)) to one \((i=t)\) of the two shows that the action for any particular string state is altered by \( Z_t \chi/\beta_t \), where the integer \( Z_t \) is determined by the defect(s) introduced by \( O_a \) above. In addition, the Dirichlet and Neumann boundary conditions are exchanged, leading to a total of \( N + 2 \) fields corresponding to each.

Without modifying the action, another type of exchange can be made to the same dimension \( t \) exploiting modular invariance to exchange coordinates \( \sigma \) and \( \tau \) in the argument of \( \pi^\sigma[\sigma, \tau] \). Similar arguments, using modular invariance alone, lead to the Cardy-Verlinde relation [29,87] for the entropy of a single field component \( q^\sigma \) or \( \pi^\sigma \) as \( S^\sigma(L^\sigma_0) = 2\pi \sqrt{(c^\sigma/6)(L^\sigma_0 - c^\sigma/24)} \) with \( c^\sigma \) the central charge for the field with index \( \sigma \), and \( L^\sigma_0 \) the virasoro generator representing the ground state.

The combined operations allow the highly-disordered and low-temperature actions to be mapped onto one another, in that the distribution \( \Delta[\beta_d, \beta_t] = \Delta[\beta_d, \chi/\beta_t] \). As a result, the action for \( \Delta \) is minimized (that is, the entropy is maximized under exchange of energy within the \( j = -1 \) sector with all other variables held fixed) when the dilaton field assumes the value

\[
\chi = \beta_t \beta_d
\]  

(3.129)

This suggests the ‘local strength of gravity’ is affected by both the strength of disorder and the thermal energy scale. It is not difficult to ascribe a simple meaning behind this result. It is interesting to note that if one associates a type of inverse ‘temperature’ \( \beta_\chi \) with \( \sqrt{\chi} \), one can interpret the above as a type of equilibration for \( \beta_\chi = \beta_t = \beta_d \). This makes sense only if one can associate some form of entropy \( S_\chi \) with the dynamics behind \( \chi \); since \( \chi \) itself was introduced as a result of disregarding the detailed behavior of the scattered optical pumping light, it is natural to associate this with a form of entanglement entropy. Under conservation of the exchanged energy and momenta \( \eta_\mu \), this would then represent a maximization of the combined thermal, disorder and entanglement entropy.

The introduction of \( \beta_t \) and \( \beta_d \) have a simpler further effect, serving to increase the volume of the internal space. In this way, both the entropies of disorder and thermodynamics effect the local effective gravitational constant. Since the Planck-
area is defined using the gravitational constant and effective speed-of-light, this suggests that holographic scales will be set by the same physics. A simple scaling argument can be created for this - since at low $\beta_d$, the Cardy-Verlinde formula $S^\sigma(L_\sigma^0)$ simply counts the effective degrees-of-freedom $F_s$ in the disorder, the fixed point described above should scale with a power related to $F_s$. Assuming $F_s \sim \ell^\kappa$ scales with the length scale $\ell$ to a power $\kappa$, the fixed point indicates a dynamical scaling $\kappa - 2 = -z(\bar{\gamma} + 1)$.

3.8 Characterization of RSC in Disordered Potential

To further investigate the cooling effect and the properties of the disordered potential, we present microwave-spectroscopic measurements of sideband asymmetry exhibited by atoms after cooling, as well as vibrational spectroscopy measurements, as shown in Fig 3.13. The two together indicates a kinetic temperature after cooling for short durations around 50nK, well above the Anderson localization temperature [97], though likely dominated by a thermal tail. The details of spectroscopy experiments are described below.

3.8.1 Microwave Spectroscopy

We performed a microwave spectroscopy measurement to verify the cooling effect in the disordered potential with a wide range of vibrational frequency distribution. Atoms are first cooled for one second under optimized conditions, with the optical pumping beam 12 MHz blue detuned to the $|F = 1\rangle \rightarrow |F = 0\rangle$ transition on the $D_2$ line and the depumping beam resonant with the $|F = 2\rangle \rightarrow |F' = 2\rangle$ transition. At the end of the cooling period, the optical pumping and depumping beams are turned off, while the magnetic field remains unchanged. Shortly (25 ms) after the optical pumping beam is off, a microwave field is applied with varied frequency for 50ms, together with weak (“blow-off”) light resonant with the $F = 2 \rightarrow F' = 3$ transition to remove atoms from the potential. Negligible loss of atoms is observed without application of microwave radiation, indicating atoms are well de-pumped during cooling to the $F = 1$ manifold. The microwave frequency is tuned to $f = f_0 + \delta f$, where $f_0$ is the hyperfine splitting, and $\delta f$ is a frequency offset chosen in the range $\pm 150$ kHz, causing transfer of atoms to $F' = 2$ and subsequent loss.
Figure 3.13. **a** Microwave spectroscopy showing sideband asymmetry of the atoms after cooling. The plot shows 947 measurements of the retained atoms versus $\delta f$ where $f = f_0 + \delta f$ is the applied microwave frequency, and $f_0$ is the hyperfine splitting. The asymmetry of each peak (see inset) can be used to extract kinetic temperature - the red line shows a fit to parameterize temperature. **b** Parametric excitation of the atoms is performed by modulating the light intensity; this couples vibrational levels that differ by two due to even symmetry and is used (inset) to infer the distribution of local trapping frequencies. The plot shows the survived atom number as a function of the center frequency of the modulation. The inset shows the fraction of atoms contained within a 1kHz bin of local vibration frequency.

due to resonant light scattering. The remaining atoms are recaptured in a MOT to measure the atom number through fluorescence imaging.

As shown in Fig. 3.13 **a**, loss occurs at evenly spaced peaks corresponding to microwave coupling on allowed transitions between different states in the $F = 1, 2$ hyperfine manifolds. The corresponding transitions are illustrated in Fig 3.14. The separation is measured to be 40.5 kHz, corresponding to a Zeeman shift from external magnetic field of 57.8 mG. The slight asymmetry in peak shape shows a cooling signature [98]. Though hard to define an equilibrium temperature for atoms, we extract a kinetic temperature by fitting the peak shape with a Boltzmann weighted Lorentzian as
Figure 3.14. Illustration of the microwave transitions. The Zeeman shift is caused by the external magnetic field. We keep the field strength the same as during the cooling period. A horn delivers microwave with frequency centered on the splitting of the two hyperfine ground states, and drives magnetic transitions among $F = 1$ and $F = 2$ magnetic sublevels.

\[
N = N_0 - A \sum_{n_1, n_2} C_{n_1, n_2} \int d\nu P(\nu)e^{-\hbar n_1/k_B T}(1 - e^{-\hbar \nu/k_B T}) \frac{1}{1 + 4\delta_f^2/\gamma^2} \tag{3.130}
\]

where $P(\nu)$ is the measured vibration frequency distribution (described below), $\gamma$ is dominated by broadening from the “blow-off” light, and the detuning $\delta_f = f - (n_2 - n_1)\nu - \delta_z$ with $\delta_z = g \mu_b \Delta m_F$ the Zeeman shift, with $g \mu_b = 700$ kHz/G. Here, $C_{n_1, n_2}$ is the coupling coefficient between two different vibrational states

\[
C_{n', n} = \sum_{l=0}^{\min(n', n)} \frac{\sqrt{n! \cdot n'!} e^{-\frac{1}{2} \alpha^2} (-\alpha)^{(n'-l)} \alpha^{(n-l)}}{(n-l)! \cdot (n'-l)! \cdot l!} \tag{3.131}
\]

where $\alpha = \sqrt{m \omega \delta x^2/2 \hbar} = \delta x/\sqrt{2} x_0$, with $x_0$ the harmonic oscillator length scale. $\delta x$ denotes a position shift of potential minimum for two different spin states. The temperature is estimated to be 50 nk.
3.8.2 Vibrational Spectroscopy

To measure the vibrational frequency distribution of atoms in the disordered potential, we parametrically excite atoms by modulating the laser intensity with small amplitude, shown in Fig 3.15. For far off resonance dipole trap, the heating rate $\Gamma$ is given by [99],

$$\Gamma = \frac{\pi^2 \nu^2 S(2\nu)}{2}.$$  \hspace{1cm} (3.132)

Here $\nu$ is the local trap frequency and $S(\omega)$ is the one-sided power spectrum of the fractional intensity noise. Due to the even symmetry of the parametric modulation, the heating rate depends on twice the trap frequency. The average energy of atoms in the modulation bandwidth increase exponentially,

$$\langle E(t) \rangle = \langle E_0 \rangle \exp(\Gamma t).$$

![Figure 3.15. Amplitude modulation of lattice potential.](image)

The measurement start by cooling atoms for 2.1s at optimum condition. After that, we modulate the speckle beam intensity for 100 ms with a chirped sinusoidal form

$$\Delta I/I = A \sin((2\pi(f_A - 0.5 - t/T)t))$$  \hspace{1cm} (3.133)

where $A$ is the modulation depth, $f_A$ is the modulation center frequency and $T$ is the modulation time. The modulation depth $A$ is set to be 2% to minimize the power broaden effect. The chirp frequency bandwidth is 1 kHz. Due to the heating of this modulation, atoms possessing vibrational frequencies within this modulation bandwidth are heated and lost from the potential when their energy are sufficiently higher than the potential depth. After that, atoms are holding in the disordered potential for extra 100 ms and then collected in a MOT to measure the atom number. The result is shown in Fig. 3.13b in the main text, as the modulation frequency $f_A$ is scanned. The plot is average over 591 measurements, with each frequency chosen randomly in the range of 1 to 125 kHz to minimize effect.
from atom number drift over time. We further extract a normalized vibrational frequency distribution by taking account the modulation power spectrum,

\[ N = N_0 - N_0 \int P(\nu)H(e^{\pi^2 \nu^2 S_M(2\nu)t}, \langle E_{th} \rangle / \langle E_0 \rangle) \, d\nu, \tag{3.134} \]

where \( H \) is the Heaviside step function, \( \langle E_{th} \rangle \) is a threshold energy, and \( S_M(\omega) \) is the modulation power spectrum. The vibrational frequency distribution \( P(\nu) \) is shown in the inset of Fig. 3.13 b.
We describe the phase diagram and thermodynamic properties of a chain of axially-tunnel-coupled fractional quantum Hall systems realized by rotating a series of optical dipole traps about their center. We demonstrate not only a experimentally feasible pathway to a state describable as a Mott-insulator of composite bosons, but also describe the nature of the coherent states at higher tunnel coupling strength, and identify a series of new superfluid phases with rich behavior. The phase diagram directly reveals not only characteristic features of the few-body systems, including the effective mass of composite particle- and hole-like excitations and their interactions, but emergent properties of the chain also reveal a fundamental mapping between the adiabatic dynamics of two dimensional systems governed by particle braiding and the hydrostatic response of the gas in the conducting phases.

4.1 Introduction

Shortly after the discovery of fractional quantum Hall effects in electronic systems, it was realized certain thermodynamic ground states possess excitations with fractionalized mass and statistics. Despite intense efforts, direct observation of these features has proved elusive in experiments on electronic systems. Recently, it was predicted [20,21,100–102] that rapidly rotating gases of ultracold Bosonic atoms exhibit states similar to fractional quantum Hall ground states with emergent topological and quantum order, and experiments have begun to probe gases in this regime [8]. The question then arises as to how one can create and probe excitations in these systems which may shed light on the structure of the fractional Hall ground
states, or perform other experiments indicative of this novel type of order. In a recent letter [103], it is shown that hole-like excitations in a fractional Hall state could be created and probed by introducing impurity atoms of a second atomic species, and that the pair-correlations of two such excitations reveal fractionalized angular momentum as a result of their fractionalized statistics. Here we consider a second method of interrogating fractional Hall effects in cold gases, in which many such samples are created along a chain of lattice sites, and coupled together via tunneling. We find that such systems support novel insulating and superfluid states, and that an interplay of the conserved quantities, particle number and angular momentum, arises, affecting the character and dynamics of quantum and thermal fluctuations, and leading to novel transport properties. We calculate a mean-field phase diagram and derive effective field theory to describe this system.

4.2 A Chain of Rotating Atomic Gases

As shown in Fig 4.1 (a), we consider the thermodynamic ground state of the Hamiltonian

\[ \hat{H} = \hat{H}_\Omega + \hat{H}_|| - \mu \hat{N}, \]  

(4.1)

where

\[ \hat{H}_\Omega = (1 - \Omega/\omega) \hat{L} + \eta \hat{V} + \hat{H}_\epsilon \]  

(4.2)

is the on-site hamiltonian for gas harmonically trapped and rotating at the frequency \( \Omega \), projected onto the lowest-landau-level (LLL), and

\[ \hat{H}_|| = -t \sum_{m,i} (\hat{a}_{i+1,m}^\dagger \hat{a}_{i,m} + \text{c.c.}) \]  

(4.3)

describes tunneling along the chain. Here, the \( \hat{a}_{i,m} \) destroy a particle of angular momentum \( m \) at site \( i \), and obey bosonic commutation relations \( [\hat{a}_{i,m}, \hat{a}_{i',m'}^\dagger] = \delta_{i,i'} \delta_{m,m'} \), and all energy scales are measured relative to the harmonic trap energy \( \hbar \omega \) in the plane of rotation. The chemical potential \( \mu \) is introduced to allow variation of the total particle number \( \hat{N} = \sum_{i,m} \hat{a}_{i,m}^\dagger \hat{a}_{i,m} \). The total angular momentum \( \hat{L} = \sum_{i,m} m \hat{a}_{i,m}^\dagger \hat{a}_{i,m} \), and the interaction energy \( \hat{V} \) is assumed to be given by contact interactions, whose form is given by
Figure 4.1. (a) We consider a chain of rotating traps, through which atoms can tunnel at a rate $t$ along the rotation axis. At sufficiently high rotation rates $\Omega$, and with (b) sufficiently strong scattering due to repulsive interactions parameterized by $\eta$, atoms may occupy higher angular momentum eigenstates $m_i$ in the lowest Landau level, forming (c) strongly correlated fractional Hall states within in a two-dimensional well (filling factor $\nu = 1/2$ state is illustrated for $n = 4$ atoms). Such states are describable as composite particles bound to fluid vortices, and we show for moderate $t$ that tunneling between such states reduces to highly collaborative tunneling of composites at a renormalized rate $t'$. (d) Under appropriate conditions, a tunnel-coupled chain will form insulating states of well-defined atom number and FQH filling factor at each site, and superfluid states formed by local superpositions thereof. Such states possess novel transport properties, in which particle flow along the chain is described by motion of composite entities, which for a given axial flow induce a torsional strain in the background insulator. The linkage between flow and strain can be described in a chain with periodic boundary conditions (e) as a topology-preserving insertion (steps i-iii) of motional flux in the presence of intrinsic flux representing the insulating state.
\[ \hat{V} = \sum_{\{m\}} V_{\{m\}} \hat{a}_{i,m_1}^\dagger \hat{a}_{i,m_2}^\dagger \hat{a}_{i,m_3} \hat{a}_{i,m_4} \]  
\[ (4.4) \]

with

\[ V_{\{m\}} = 2\pi \delta_{m_3+m_4}(m_1 + m_2)! 2^{m_1+m_2} (\prod_i m_i !)^{-1/2}, \]  
\[ (4.5) \]

which describes the scattering of particles between angular momentum orbitals \( m_{1...4} \) in the LLL. Finally, we include the effect of a small rotating quadrupolar moment to the local trapping potential through

\[ \hat{H}_\epsilon = \sum_{m,i} \epsilon (\hat{a}_{i,m+2}^\dagger \hat{a}_{i,m} + \text{c.c.}), \]  
\[ (4.6) \]

such that the major and minor trap frequencies \( \omega_{\pm} \) give

\[ \epsilon = 2(\omega_+ - \omega_-)/(\omega_+ + \omega_-). \]  
\[ (4.7) \]

The few-body eigenstates for the Hamiltonian \( \hat{H}_\Omega \) are calculated by direct diagonalization (DD), similar to the methods used in references \([8,104]\). In figure 4.2, we show the lowest eigen-energy of \( H_\Omega \) as a function of \( \Omega \) for a number of atoms \( n \leq 5 \). As the rotation rate is increased, the kinetic energy penalty for single atoms to occupy higher angular momentum eigenstates is reduced, and interaction begins to mix the non-interacting eigenstates. As a result, at high \( \Omega \), each occupancy enters into a series of progressively more strongly entangled ground states of higher angular momentum. Previously, we have identified the first and last states in this sequence as an \( |L = n\rangle \) single-vortex, and \( |L = n(n-1)\rangle \) \( \frac{1}{2} \)-Laughlin state, respectively, for all particle numbers (for \( n = 2 \) these states are equivalent); other states have been tabulated extensively in previous literature, and experiments have now entered a regime in which individual few-body samples may be brought with reasonable fidelity into many of them. The \( \frac{1}{2} \)-Laughlin state is a generalization of the \( \frac{1}{q} \)-Laughlin state known from 2DEGs, whose wavefunction can be written in complex coordinates as \( \phi_L(z_i) = \prod_{ij} (z_i - z_j)^2 \exp(-\sum_i |z_i|^2/2) \). If the particle number in the \( \frac{1}{2} \)-Laughlin state is made indefinite through a coherent superposition of such states, it can be considered as a condensate of composite objects, in which \( q \) quanta of vorticity are attached to each particle coordinate prior to condensation.
We interrogate whether such a state can be directly created in a chain of cold-atom FQH systems, and whether its dynamics are best described by the tunneling of composite bosons through the chain, or by the motion of bare particles. We find that not only does the composite language form a good description, but that relations between emergent hydrostatic properties (such as compressibility and rotational moment and superfluid density and torsional stiffness) reflect a basic mapping of topology from two spatial plus one time (2+1) dimensions into a static three-dimensional version (3+0).

**Figure 4.2.** The energy spectrum $\epsilon_{nf}$ for each of these $n$-body states (A) is shifted downward proportionate to the chemical potential $\mu$ to form the free energy in a grand-canonical ensemble. Experimental control over the many-body state can be exerted through (B) deformation of the rotating trap (strength $\epsilon$), which couples states of differing angular momentum (proportional to slope $d\epsilon_{nf}/d\Omega$) and introduces avoided level crossings (highlighted in yellow), or (C) by controlling the tunneling rate $t$, producing avoided crossings between states of differing particle number (expectation value of $n$ represented by color). Spectra shown here correspond to $\eta = 0.0054$, $\mu = 0.025\hbar\omega$, $\epsilon = 0.002$ (in B), $t = 0.004\hbar\omega$ (in C). Spectra in (A) and (B) were produced with DD, and (C) by minimization of the Gutzwiller form as described in the text.
4.3 Phase Diagram

We form a mean-field model first by assuming an approximate Gutzwiller form for the many-body state of the system $|\Psi\rangle = \prod_j |\Psi\rangle_j$ with

$$|\Psi\rangle_j = \sum_{n,f} a_{n,f} |n,f\rangle_j ,$$

(4.8)

where $a_{n,f}$ represents the amplitude of the $f^{th}$ lowest energy-eigenstate $|n,f\rangle_j$ of the $n$-atom solution to $\hat{H}_\Omega$, localized on site $j$. Forming the energy $\langle \Psi | \hat{H} | \Psi \rangle$, and minimizing its variation with $a_{n,f}$ subject to the constraint $\sum_{n,f} |a_{n,f}|^2 = 1$, one finds the condition

$$\sum_{n',f'} M_{n,n'} a_{n',f'} = \mathcal{E} a_{n,f} ,$$

(4.9)

with

$$M_{n,n'} = \delta_{n,n'} (\epsilon_{n,f'} - \mu n') - 2t \delta_{n',n-1} \sum_m \psi_m \tau^{(m)}_{n',f',f} + c.c. \quad (4.10)$$

Here, the quantity $\mathcal{E}$ is a Lagrange multiplier, and $\epsilon_{n,f}$ represents the energy of the state $|n,f\rangle_j$ due to $\hat{H}_\Omega$. The nonlinearity is represented through the fields $\psi_m = j \langle \Psi | \hat{a}_{j,m} | \Psi \rangle_j$, and the quantity $\tau^{(m)}_{n,f_1,f_2} \equiv \langle n,f_1 | \hat{a}_m | n+1,f_2 \rangle$ reflects the overlap of two few-body states when a single particle of angular momentum $m$ is removed, describing the effectively lower rate of tunneling between strongly correlated states on neighboring sites. A numeric solution of these equations can be performed by choosing an arbitrary set of coefficients $a_{n,f}$, calculating the fields $\psi_m$, and solving for the lowest eigenvector of $M$, iterating until the fields converge [105] - a representative sample of field values obtained this way are shown in figure 4.3.

In the decoupled limit $t = 0$, the many-body states are simultaneous eigenstates of the number operators $\hat{n}_j$ at each site $j$, and for $\epsilon = 0$, are also eigenstates of angular momentum $\hat{L}_j$; and thus the ground-state wavefunction is a product state $\prod_j |n,L\rangle_j$ with $n$ and $L$ chosen to minimize the total energy $E = N_s (\epsilon_{n,L} - \mu n)$, reflecting a generalized Mott-insulator state, which we label as $n \nu I_{\nu}$ with filling factor $\nu = n(n-1)/2L$. For each particle number $n$, the ground-state crossing sequence
Figure 4.3. (A-C) Mean-field phase diagrams, showing components $\psi_m$ of the order parameter in color, and zero- and finite-temperature phase boundaries (joined symbols) at three rotation rates. The dashed red line shows the analytic result for the zero temperature phase boundary. (A, $\Omega = 0.929\omega$) At low rotation rates, the phase-diagram reduces to that of the Bose-Hubbard model, exhibiting insulator-to-superfluid $n \sigma_\infty - SF$ transitions in the $m = 0$ channel. (B, $\Omega = 0.988\omega$) As the rotation rate is increased, high-occupancy insulating regions $n \sigma_\nu$ transition to higher angular momenta and stronger correlation due to the change in lowest-energy few-body form. (C, $\Omega = 0.997\omega$) Near the centrifugal limit, insulators at all occupancies have transitioned into the $\nu = 1/2$ (Laughlin) form, in which the on-site interaction energy is zero; at higher $t$, superfluidity develops as coherent superpositions of these states. When $t$ is sufficiently large, the Laughlin-superfluid is destroyed in favor of a more completely mean-field type state. (D-E) Mean-field ground state probabilities $|a_{nf}|^2$ are shown as a function of tunneling strength $t$ for two different cuts from (B) and (C). They show two sudden changes, first at the insulator boundary, and a second jump as the FQH-forms are lost at higher $t$ - the latter can be roughly described by calculation of $u$ in an effective field theory, corresponding to saddle points in the free-energy along a second component of $\psi$ (points: numeric Gutzwiller ansatz, solid lines: EFT to fourth order in $\psi$.)
consists of $n - 1$ transitions, and thus below the chemical potential corresponding to $n_m$ filling, there are $n_m(n_m + 1)/2$ such insulating states which can be reached through control of $\Omega$ and $\mu$. For sufficiently high rotation rates, the few-body ground state for any particle number $n$ is a $\nu = \frac{1}{2}$ ($\frac{1}{2}$-Laughlin) form, with zero interaction energy, and the total energy can be written as $\epsilon_{n,L=n(n-1)} = (1 - \Omega/\omega)n(n - 1)$.

The boundaries between insulators $n_{I_1/2}$ therefore occur at the critical chemical potentials of $\mu_c = 2n\hbar(\omega - \Omega)$. Adiabatic evolution or cooling of a sample into these or other insulator phases by dynamic control of $\Omega$, $\mu$ and $t$ provides an experimental method for producing large numbers of pure few-body FQH states of definite occupancy and filling factor.

One might naively guess that the transition at the insulator boundary signifies the onset of superfluidity of composite particles along the length of the chain, forming a locally coherent state. In fact, if one retains only the lowest few-body eigenstates just below the centrifugal limit, a mean-field calculation would support this picture somewhat trivially, as described in reference [105]. However, retaining the full few-body spectrum for each particle number, the mean-field calculation presented in figure 4.3C does not support this picture beyond sufficiently high $t$, nor does it strictly support the picture of the superfluid phase as a locally coherent state (with a statistical spread of occupancy $\sqrt{n}$ about $n$) even at intermediate values of $t$.

At intermediate $t$, a superposition of locally correlated states of different $n$ does develop, but due to the linkage between the angular momentum and particle number for the lowest energy FQH states, only two FQH states are superposed in any given superfluid state. The superfluid resembles a hard-core Bose gas of composite particles or holes in the neighboring FQH insulator state, with only a single nonzero component $\psi_m$ of the order parameter, with $m = \Delta L$, the difference in angular momentum of the superposed few-body states. At such a transition with $\epsilon = 0$, the bosonic gauge and rotational symmetries are simultaneously spontaneously broken, as the phase of the order parameter determines both a measurable superfluid phase and a rotational orientation to the corresponding few-body superposition of angular momentum eigenstates.

The phase boundary for any $n_o I_\nu$ insulator can be located semi-analytically by expanding the expectation value of the full many-body Hamiltonian to second order in the fields $\psi_m$, determining when energy is lowered by nonzero fields. Using a
mean-field Hamiltonian $\hat{H}_{MF} = \hat{H}_\Omega + \hat{H}_c - \mu \hat{N}$, where

$$\hat{H}_c = -2t \sum_{n,f,f',m} (\psi_m \tau^{(m)*}_{n,f',f} |n+1,f\rangle \langle n,f'| + \text{c.c.})$$

The energy $\langle \hat{H} \rangle \approx E_0 + \sum_{mm'} \psi_m \mathcal{R}_{mm'} \psi_{m'}^*$, where $E_0$ is independent of the $\psi_m$ and with the matrix $\mathcal{R} = 8N_s t^2 (\gamma - 2t \gamma^2)$, where

$$\gamma_{mm'} = \sum_f \frac{\tau^{(m')*}_{n_0-1,f,0} \tau^{(m)}_{n_0-1,f,0}}{\epsilon_{n_0,0} - \epsilon_{n_0-1,f} - \mu} + \frac{\tau^{(m')*}_{n_0,f,0} \tau^{(m)}_{n_0,f,0}}{\epsilon_{n_0,0} - \epsilon_{n_0+1,f} + \mu}$$

The mean-field energy is lowered when the lowest eigenvalue of $\mathcal{R}$ becomes negative, signifying the phase boundary illustrated by the dotted lines in figure 4.3a-c. At high $\Omega$ and small but nonzero $t$, the terms contributing most strongly to $\gamma$ correspond to $\nu = 1/2$ (Laughlin) states at particle numbers $n_0 \pm 1$; $\mathcal{R}$ has an eigenvalue which first inverts sign for the lower value of $t$ such that

$$2t \alpha_{n_0-1/2+1/2}^2 = 2(n_0 - 1/2 \mp 1/2)(1 - \Omega/\omega) \mp \mu \quad (4.11)$$

corresponding to the only nonzero tunneling matrix elements between $\nu = 1/2$ states $\tau^{2n}_{n_0\nu \nu}$ (we calculate using DD $\tau^{2n}_{n_0\nu \nu} = 0.707, 0.603, 0.546, 0.508$, and note that this matrix element asymptotes to $1/2$ as $n$ increases). Thus the boundary from the Laughlin-insulator into the symmetry-broken state has a fixed slope on the $\mu - t$ plane, determined by the structure of the on-site few-body wavefunction, independent of both rotation rate and interaction strength $\eta$, provided $(1 - \Omega/\omega) \ll \eta$, and $n$ is sufficiently large. Analysis of the single-particle Green’s function (see section 4.5) shows that the longitudinal masses of particle- and hole-excitations in these insulators are determined by the same parameters as $M_{\pm} = \hbar^2/t |\alpha_n|^2$, and the energetic gap for their creation at the minimum in their dispersion disappears at these boundaries. These conclusions should persist to larger values of $n$, provided rotation is brought sufficiently close to $\Omega \approx \omega$.

We note that within normalization factors, $\tau^{(m)}_{n,f_1,f_2}$ is equivalent to the matrix element in first quantized form $\int d^2n \zeta d^2 \zeta \Phi^*_{f_1}(\zeta_1...\zeta_n) \Phi_{f_2}(\zeta_1...\zeta_n) \phi_{m}(\zeta)$, where $\Phi_{f_2}(\zeta_1...\zeta_n|\zeta)$ represents the $(n+1)$-body state with one particle coordinate symmetrically chosen and set equal to the classical coordinate $\zeta$. Carrying out the sum
over \( m \) in the chain-coupling Hamiltonian,

\[
\hat{H}_c = -2t \int d^2\zeta \psi^*(\zeta) \hat{\tau}(\zeta) + c.c , \tag{4.12}
\]

where \( \psi(\zeta) \equiv \sum_m \psi_m \phi_m(\zeta) \), and \( \hat{\tau}(\zeta) = \sum_j \delta(\zeta_j - \zeta) \). Thus the total energy is reduced by creating states of strong overlap between the mean-field and individual particles, tying dynamics of \( \psi \) to the transverse motion of “coordinate-fixing” excitations in the local state. For example, the dynamics and adiabatic manipulation of excitations along the chain in the \( nI_\nu \) insulators and \( nS_\nu \) superfluids determine the nonzero elements \( \tau_{n\nu\nu}^{(m)} \) such that

\[
\langle \hat{H}_c \rangle \propto -2n \text{Re}[\tilde{t} \int d^2\zeta \prod_j (\zeta_j - \zeta)^{1/\nu} \phi^*_m(\zeta) \psi_m(\zeta)] \tag{4.13}
\]

the expectation value of a creation operator for \( \nu^{-1} \) counter-circulating quasi-holes in the \( m = n/\nu \) orbital in the \( |n\nu\rangle \) state. More generally, the operator \( \hat{\tau} \) therefore appears as a type of fusion operator, creating a (fractionalized) hole by statistically distinguishing an individual particle, and in \( \hat{H}_c \), reversibly transferring weight to the coherent field.

Since the energy to promote the mean-field component to a nonzero longitudinal momentum \( k \) can be calculated by adiabatically taking \( t \rightarrow \tilde{t} \equiv te^{ika} \), one can see that equivalently the quasi-hole wavefunction is taken to \( \phi_m(\zeta) \propto \zeta^m \rightarrow (\zeta e^{ika/m})^m \), rotating each quasi-hole location through an angle \( ka/m = k\alpha \nu/n \). The corresponding quasi-classical orbit of this excitation is therefore screw-like, and includes a component equal to the geometric phase corresponding to a single orbit of a quasi-hole center \( \zeta \) around a closed-loop, which defines its fractionalized charge (or mass) \([106]\). The longitudinal current \( j = \langle \partial \hat{H}_c(t \rightarrow te^{i\phi})/\partial \phi|_{\phi=0} \rangle \).

In the superfluid phase at high \( t \), the lowest energy few-body FQH eigenstates are no longer weighted strongly due to their more even distribution among orbitals \( m \), and the superfluid phase reflects a more completely mean-field type state. This is consistent with a picture in which particles must first pay a chain-localization energy \( t \) before participating in a strongly correlated on-site state; for modest tunneling strengths the localization energy penalty is sufficiently low to allow atoms to first localize to a lattice site and participate in a strongly correlated state, but at high \( t \), this kinetic energy cost is too high, and a delocalized superfluid state is favored.

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4.4 Transport Properties Near Criticality

In this section, we consider novel transport phenomena and properties near criticality through an effective field theory. For finite temperatures, the free energy density

\[ F/N_s = -k_B T \ln Z \quad (4.14) \]

can be constructed using a coherent-state path integral representation of the partition function

\[ Z = \int \mathcal{D}\{b_{im}, \psi_{im}\} e^{-\int_0^\beta \mathcal{L} d\tau}, \quad (4.15) \]

with \( \beta = 1/k_B T \), and Lagrangian

\[ \mathcal{L} = \sum_i \mathcal{L}_i^0 - \sum_{i,m} (\psi_{im}^* b_{im}^* + \psi_{im}^* b_{im}) + \sum_{ijm} \frac{\psi_{im}^* \sigma_{ij}^{-1} \psi_{jm}}{t} \quad (4.16) \]

where the complex numbers \( b_{im}(\tau) \) label coherent state amplitudes for the \( m^{th} \) orbital at site \( i \), and \( \mathcal{L}_i^0 \) is the site-decoupled few-body Lagrangian given by

\[ \mathcal{L}_i^0 = \sum_m \frac{db_{im}^*}{d\tau} b_{im} + ((1 - \Omega/\omega)m - \mu)|b_{im}|^2 \]

\[ + \sum_{\{m\}} \mathcal{V}_{\{m\}} b_{im1}^* b_{im2}^* b_{im3} b_{im4} \quad (4.17) \]

We use a generalized Hubbard-Stratonovich transformation [107] to couple the mean-field components \( \psi_{im} \), using the tunneling matrix \( \sigma_{ij} = \delta_{i,j \pm 1} \), and integrate out the \( b_{im}(\tau) \) to obtain the partition function

\[ Z = z_0 \int \mathcal{D}\{\psi_{im}\} e^{-\int_0^\beta \mathcal{L}_\psi d\tau}. \quad (4.18) \]

The effective lagrangian can be expanded in powers of slowly varying fields \( \psi = (\psi_m(z, \tau)) \) and gradients as

\[ \mathcal{L}_\psi \approx \int dz [\psi^* r \psi + \psi^* \psi^* u \psi \psi + \psi^* k_1 \partial_{\tau} \psi + \partial_{\tau} \psi^* k_2 \partial_{\tau} \psi + \partial_z \psi^* K \partial_z \psi] \quad (4.19) \]
where the (matrix) coefficients can be determined from the microscopic parameters by expanding the field-dependent portion of the path-integral, yielding expressions (detailed in section 4.5) dependent on the single- and two-particle Green’s functions confined to a single lattice site. The decoupled few-body partition function \( z_0 = \sum_{n_f} \exp \left( \frac{(\mu_n - \epsilon_{nf})}{k_B T} \right) \), and one finds generally the matrix \( r \) (which forms a complex Hessian for the free energy of the insulator),

\[
\begin{align*}
    r_{mm'} &= \delta_{mm'}/2t - z_0^{-1} \sum_{n_f} \frac{\tau_{nff'}^{(m)} \tau_{nff'}^{(m')*}}{\epsilon_{nf} - \epsilon_{n+1,f'} + \mu} \\
    &\quad \times \left( e^{(\epsilon_{n+1,f'} - \mu(n+1))/k_B T} - e^{(\epsilon_{nf} - \mu)/k_B T} \right)
\end{align*}
\]

Reaching insulator states in the chain requires temperatures and adiabatic manipulation timescales more demanding than does producing FQH states in isolated wells - the finite temperature insulator boundaries are found by setting the lowest eigenvalue of \( r \) to zero as illustrated in figure 4.3c, displaying a memory of the insulator boundary structure up to temperatures of \( 0.5\eta h/k_B \) for \( \Omega = 0.997\omega \), corresponding to \( \approx 800\text{pK} \) for physical parameters consistent with the experiment in ref. [8], though we note that this scale increases with stronger interactions and stronger planar confinement. The parameter \( u \) can be extracted from the few-body spectrum as described in the section 4.5, and directly probes two-particle physics - its explicit calculation permits calculation of critical points in the superfluid phases at higher \( t \), as illustrated by the solid lines in figure 4.3f, which correspond to continuous transitions at saddle points along a new direction of \( \psi \) in the free energy introduced by nonzero fields \( \psi_{m_0} \).

The remaining coefficients are most transparently recovered from symmetries under two global time-dependent rotation of phases \( (b \text{ and } \psi)_{im} \rightarrow (b \text{ and } \psi)_{im} e^{im\phi_s(\tau)} \), and corresponding shifts \( (\mu, \Omega)_s \rightarrow (\mu, \Omega)_s + i(-1)^s \partial \phi_s / \partial \tau \) for either \( s = 0 \) (bosonic gauge rotation) or \( s = 1 \) (physical rotation about \( z \)) with \( \epsilon = 0 \). Similar to the single-orbital Bose-Hubbard case \( k_\sigma = -\partial^\sigma r / \partial \mu^\sigma \), and for \( \epsilon = 0 \), \( k_{1m'm} = (\partial r_{m'm}/\partial \Omega)/m \).

The parameter \( k_1 \) is given by the inverse slope of the phase boundary in the \( \mu - t \) plane. Near the centrifugal limit, \( k_1 \) never vanishes by this calculation, and thus all transitions to the coherent state lie in the universality class of a dilute gas, in contrast to the single-orbital case at low \( \Omega \). More insight into the transition can
be found from the single-particle Green’s function for the chain using a cumulant expansion [108] in powers of \( t \), similar to the single-orbital case [109,110], but with diagrams carrying orbital indices (this can be performed non-pertubatively in \( t \) by summing an infinite set of diagrams [108,110]).

Finally, we note a unique feature of superfluidity in the fractional Hall chain related to interplay of bosonic gauge and rotational symmetries. Using the expansion 4.19 above, one can alternately consider the stiffness \( \partial^2 F/\partial \phi_s^2 \) of the superfluid to “twists” of either the phase of the order parameter \( (\phi_0) \), or a “torsional” twist \( (\phi_1) \) of the gas over its physical length or in time. By standard arguments, the increase in free energy from the former can be used to infer the superfluid density [111], while the latter defines a torsional stiffness of the chain. The responses to these two deformations are linked in a manner characteristic of each phase. In the \( \nu = 1/2 \) superfluid at occupation \( n_0 \), for example, the two twists are physically indistinguishable due to the presence of a single nonzero component \( \psi_{m=2n_0} \), with a superfluid phase twist of \( \phi_0 \) equivalent to a torsional twist of \( \phi_1 = \phi_0/2n_0 \) - with periodic boundary conditions, such as would occur in a chain with its ends joined, this connects a discrete fractional twist of the chain about its axis to its quantized superfluid velocity. This belongs to a family of relations between transport parameters which can be understood by finding the change of free energy due to spatial and temporal gradients of the order parameter phases \( \phi_s \), relating the shifts from time-gradients to chemical potential and rotation rate, and spatial gradients to superfluid density and torsional stiffness.

The free energy can be expanded using the fields \( \psi_m \) which minimize the action given by the lagrangian in the main text as

\[
\beta \Delta F = \sum_{m'ms} \frac{\partial \phi_s}{\partial \tau} (im^s)e^{i(m^s-m'^s)\phi_s}\psi_{m'}|k_1m'm|\psi_m \\
+ \sum_{m'ms} \left( \frac{\partial \phi_s}{\partial z} \right)^2 (m'm)^s \psi_{m'}|K_m'm|\psi_m \\
+ \sum_{m'ms} \left( \frac{\partial \phi_s}{\partial \tau} \right)^2 (m'm)^s \psi_{m'}|k_2m'm|\psi_m
\]

At the same time, the free energy can be related to changes in chemical potential and rotation rate as
\[ \Delta F \approx \sum_s (\partial F/\partial \chi_s) \Delta \chi_s + (\partial^2 F/\partial \chi_s^2) \Delta \chi_s^2 / 2 \]  \hspace{1cm} (4.21) 

with \( \chi = (\mu, \Omega) \). The changes \( \Delta \chi_s \) can be related to temporal gradients of the twist angles \( \partial \phi_s / \partial \tau \) using the symmetries under gauge and rotational transforms, and thus each prefactor in the expansion 4.21 can be tied to a derivative of the free energy. Since the particle and angular momentum density are \( \rho = -\partial F / \partial \mu \) and \( \ell_z = \partial F / \partial \Omega \) respectively, and the compressibility and moment of inertia \( \kappa = \partial^2 F / \partial \mu^2 \) and \( I = \partial^2 F / \partial \Omega^2 \), we have the following relations for physical parameters

\begin{align*}
\rho &= -\frac{\partial F_0}{\partial \mu} - \sum_{m' m} \frac{1}{\hbar} |\psi_{m'}| k_{m'm} |\psi_m| \quad (4.22) \\
\ell &= \frac{\partial F_0}{\partial \Omega} + \sum_{m' m} \frac{m}{\hbar} e^{i(m-m')\phi_1} |\psi_{m'}| k_{m'm} |\psi_m| \quad (4.23) \\
\kappa &= \sum_{m' m} |\psi_{m'}| k_{2m'm} |\psi_m| \quad (4.24) \\
I &= \sum_{m' m} |\psi_{m'}| m' k_{2m'm} m |\psi_m| \quad (4.25)
\end{align*}

For a spatially varying phase with constant gradient, \( \phi_s = 2\pi z / L \), the change in free energy for \( s = 0 \) defines a superfluid density \( \rho_s \) through \( \Delta F = \hbar^2 \Delta \phi_0^2 \rho_s / 2M \), and for \( s = 1 \) defines a torsional stiffness through \( \Delta F = S \Delta \phi_1^2 / 2 \). One has then also

\begin{align*}
\rho_s &= \sum_{m' m} \frac{1}{\hbar} |\psi_{m'}| K_{m'm} |\psi_m| \quad (4.26) \\
S &= \sum_{m' m} |\psi_{m'}| m' K_{m'm} m |\psi_m| \quad (4.27)
\end{align*}

### 4.5 Landau Expansion of Free Energy

Terms in the Landau expansion of free energy can be calculated from partial derivatives of the partition function; the derivatives of order \( q \) can be related to
the 2q-point thermal Green’s functions

\[
\Pi_{\{m\}} \left( \frac{\partial}{\partial \psi_{m_1}}, \ldots, \frac{\partial}{\partial \psi_{m_q}}, \frac{\partial}{\partial \psi^*_{m_1}}, \ldots, \frac{\partial}{\partial \psi^*_{m_q}} \right) Z_{\{\psi \to 0\}} \tag{4.28}
\]

The first two terms in the expansion are found to be

\[
Z_0 = \left[ \sum_{nf} e^{-(\epsilon_n - \mu)/k_B T} \right]_{N_f} = z_{0}^{N_f} \tag{4.29}
\]

\[
r_{mm'} = \left( \frac{\delta_{mm'}}{2t} - \mathcal{G}_{mm'}^{(1)} \right) \tag{4.30}
\]

where \( \mathcal{G}_{mm'}^{(1)} = \langle f b_{m'}(\tau) b_{m'}(\tau') d\tau d\tau' \rangle \) integrates the single-particle thermal Green’s function on a lattice site, which we find from the spectral function

\[
\rho_{mm'}(\omega) = z_{0}^{-1} \sum_{nff'} \langle nf|\hat{a}_m|n'f'\rangle \langle n'f'|\hat{a}^\dagger_{m'}|nf\rangle \left( e^{(\epsilon_n + 1 - \mu(n+1))/k_B T} - e^{(\epsilon_f - \mu)/k_B T} \right)
\]

\[
\times \delta(\epsilon_n - \epsilon_{n'} - \mu(n - n') - \omega)
\]

\[
= z_{0}^{-1} \sum_{nff'} \tau_{nf'}^{(m)} \tau_{nf'}^{(m')*} \delta(\epsilon_n - \epsilon_{n+1,f'} + \mu - \omega)
\]

\[
\times \left( e^{(\epsilon_{n+1,f'} - \mu(n+1))/k_B T} - e^{(\epsilon_f - \mu)/k_B T} \right) \tag{4.31}
\]

to be

\[
\mathcal{G}_{mm'}^{(1)} = z_{0}^{-1} \sum_{nff'} \tau_{nf'}^{(m)} \tau_{nf'}^{(m')*} \left( e^{(\epsilon_{n+1,f'} - \mu(n+1))/k_B T} - e^{(\epsilon_f - \mu)/k_B T} \right)
\]

The second-order term in the expansion of the partition function can be expressed as an integral over the two-particle Green’s function isolated on a site:

\[
u_{\{m\}} = \int_0^\beta d^4 \tau \mathcal{G}_{\{m\}}^{(2)} (\tau_1 \tau_2 | \tau_3 \tau_4)
\]

\[
= z_{0}^{-1} \sum_{\{n,f\}} \int_0^\beta d\tau_4 \int_0^\beta d\tau_3 \int_0^\beta d\tau_2 \int_0^\beta d\tau_1 e^{-\beta h_1}
\]

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which are permuted over pairs of creation and annihilation operators. To reduce
where for brevity $h_i = (\epsilon_{n_i f_i} - \mu n_i)$, and the $\hat{\sigma} = \hat{a}$ or $\hat{a}^\dagger$ according to the $\sigma$, which are permuted over pairs of creation and annihilation operators. To reduce
the computational burden, the $n_i, f_i$ states are chosen to match “rings” of 4-tuples of nonzero elements $\tau_{n_f f_2}^{(m)}$ and $\tau_{n_f f_2}^{(m)*}$ found in the few-body spectrum for each set of indices $\{m\} = (m_1..m_4)$.

In order to determine the universality classes of transitions, the expansion
must also include spatial and temporal gradients in the $\psi_m$, for which we expand
$\psi_m(z, \tau) \approx \psi_m + z \partial \psi_m / \partial z + \tau \partial \psi_m / \partial \tau$. The remaining terms in the Landau
expansion of the lagrangian can be found using symmetry under gauge rotation.

$$k_{1mm'} = - \frac{\partial r_{mm'}}{\partial \mu}$$

$$= z_0^{-1} \sum_{n_f f'} \frac{\tau_{nff}^{(m)} \tau_{nff'}^{(m)*}}{(\epsilon_{nf} - \epsilon_{n+1, f'} + \mu)^2} \times (e^{(\epsilon_{n+1, f'} - \mu(n+1))/kbT} - e^{(\epsilon_{nf} - \mu)/kbT})$$

$$- \frac{\tau_{nff}^{(m)} \tau_{nff'}^{(m)*}}{\epsilon_{nf} - \epsilon_{n+1, f'} + \mu} \times \left( -\frac{n-1}{kbT} e^{(\epsilon_{n+1, f'} - \mu(n+1))/kbT} + \frac{n}{kbT} e^{(\epsilon_{nf} - \mu)/kbT} \right)$$

$$k_{2mm'} = - \frac{\partial^2 r_{mm'}}{\partial \mu^2}$$

$$= z_0^{-1} \sum_{n_f f'} \frac{-2 \tau_{nff}^{(m)} \tau_{nff'}^{(m)*}}{(\epsilon_{nf} - \epsilon_{n+1, f'} + \mu)^2} \times (e^{(\epsilon_{n+1, f'} - \mu(n+1))/kbT} - e^{(\epsilon_{nf} - \mu)/kbT})$$

$$+ \frac{2 \tau_{nff}^{(m)} \tau_{nff'}^{(m)*}}{(\epsilon_{nf} - \epsilon_{n+1, f'} + \mu)^2} \times \left( -\frac{n+1}{kbT} e^{(\epsilon_{n+1, f'} - \mu(n+1))/kbT} + \frac{n}{kbT} e^{(\epsilon_{nf} - \mu)/kbT} \right)$$

$$+ \frac{\tau_{nff}^{(m)} \tau_{nff'}^{(m)*}}{\epsilon_{nf} - \epsilon_{n+1, f'} + \mu} \times \left( (n+1)^2 e^{(\epsilon_{n+1, f'} - \mu(n+1))/kbT} + (n/kbT)^2 e^{(\epsilon_{nf} - \mu)/kbT} \right)$$

A cumulant expansion can also be used to determine the single particle Green’s function

$$\times \prod_{j=1}^4 \langle n_j f_j | \hat{\sigma}_{m_j}^j | n_{(j+1)mod4} f_{(j+1)mod4} \rangle e^{(h_j - h_{(j+1)mod4}) \tau_j}$$ (4.32)
\[ G^0_{1m'm}(\omega_s) = \sum_{nff'} (\tau^m_{n-1,ff'} \tau^{m'}_{n-1,ff'}) \epsilon_{nf} - \epsilon_{n-1,f'} - \mu + i\omega_s - \tau^m_{n,f'} \tau^{m'}_{n,ff'} \epsilon_{nf} - \epsilon_{n+1,f'} - \mu + i\omega_s) e^{(\mu n - \epsilon_n f)^2} \]

where the indices \( m, m' \) represent orbitals, and the matsubara frequency is labeled as \( \omega_s \). One can build a series expansion for the coupled system in powers of \( t \) as a sum over connected diagrams of the form shown in Fig 4.4.

Figure 4.4. Feynman diagram representation of Green’s function.

The summation over internal sites \( j_k \) is most easily performed in some type of reciprocal space; for reasons which will become clear below, it is illustrative to first perform a \( j \)-dependent gauge transformation corresponding to a weak twist of frame or bosonic gauge, changing both the underlying basis states \( \phi_{jm} \rightarrow \phi_{jm} e^{ijm^* \phi_s} \) and \( t\sigma_{jk} \rightarrow t^j_k = t e^{id_{jk} m^* \phi_s} \sigma_{jk} \) with \( d_{jk} = \pm \delta_{j,k\pm 1} \). One can transform to a fourier-basis \( \phi_{km} = \sum_j e^{i k m d} \phi_{jm} \), which for the nearest-neighbor coupling yields a dispersion relation \( t_{km} = t \cos(k_m d - m^* \phi_s) \) with \( d \) the lattice period, and \( k_m \) the momentum in the \( m \)th orbital in the limit \( \phi_s = 0 \). Associating a factor of \( t_{km} \) with each hopping line and performing the sum,

\[ G_{1m'm}(k,\omega_s) = G^0_{1m'm} + G^0_{1m'm} t_{km} G^0_{1m'm} + \ldots = G^0_{1} \sum_{n=0}^{\infty} g^n = G^0_{1} (Q^{-1} \tilde{\Gamma} Q) \] (4.36)

where \( g_{m'm} = t_{km} G^0_{1m'm} = (Q^{-1} \Gamma Q)_{m'm} \) is an eigenvalue decomposition with eigenvalues \( \Gamma_m \), and the matrix \( \tilde{\Gamma}_{m'm} = \delta_{m'm}/(1 - \Gamma_m) \). The representation of \( G_1 \) can be converted through the expansion \( G_1(\chi', \chi) \equiv \text{Tr}(\tilde{\phi} G_1) \) using the matrix \( \tilde{\phi}_{m'm} = \tilde{\phi}_{m'}^{\ast} (\chi') \tilde{\phi}_m (\chi) \) formed from basis states in various representations \( \chi \), for example with \( \chi = (p, p_z) \) and \( \phi_s = 0 \), one has \( \tilde{\phi}_m (p, p_z) = p^m \delta(p_z - \hbar k_m) / \pi 2^{m/2} \). Permuting the trace,
\[ G_1(\chi', \chi) = \text{Tr}[(Q \tilde{\phi} G_1^0 Q^{-1}) \tilde{\Gamma}] \] (4.37)

\[ = \frac{1}{t} \text{Tr}[(q_2^{-1} \phi q_2 q_1^{-1} \Gamma_0 q_1) \tilde{\Gamma}] \]

where the subscript 0 indicates setting the \( k = (k_m) = 0 \), the unitary matrices \( q_1 = Q_0 Q^{-1} \) and \( q_2 = S Q^{-1} \), with \( \tilde{\phi} = S^{-1} \phi S \) the eigenvalue decomposition of \( \tilde{\phi} \).

In the cylindrically symmetric limit \( \epsilon = 0 \), the matrices \( Q, Q_0 \) and \( q_1 \) reduce to the identity, and \( q_2 = S \), reducing the calculation of the Green’s function to

\[ G_1(\chi', \chi) = \frac{1}{t} \text{Tr}[\phi (ST_0 S^{-1})] \] (4.38)

Poles in the Green’s function, once analytically continued to real time, describe various excitations. In the decoupled limit, all such excitations have a finite energetic penalty. Once tunnel coupling is introduced, each excitation can delocalize, and the energy as determined by the corresponding pole of \( G_1 \) becomes dependent on the longitudinal momenta \( k = (k_m) \); determining the minimum in energy as a function of \( k \) yields the gap, and the lowest order expansion coefficient of \( \delta k^2 \) away from this point in the pole frequency allows for determination of the mass of the excitation. Presumably at some value of \( t \), the mode becomes gapless; the first such mode to become gapless determines the location of the phase boundary, reproducing the condition on the matrix \( r \) given in the main text, and defining the lowest energy excitations near the phase boundary.

At low temperature, for \( \epsilon = 0 \), and \( \mu \) and \( \Omega \) chosen to place the equilibrium thermodynamic ground state in the \( n_0 I_{\nu_0} \) insulating phase, the dynamical single-particle Green’s function for an isolated well reduces to

\[ G^0_{1,m',m} \approx -i \left( \frac{\tau_{n_0-1, \nu_0 \nu_-} \tau_{n_0-1, \nu_0 \nu_-}'}{\epsilon_{n_0 \nu_0} - \epsilon_{n_0-1, \nu_-} - \mu + \omega + i \eta} \right. \]

\[ - \left. \frac{\tau_{n_0, \nu_0 \nu_+} \tau_{n_0, \nu_0 \nu_+}'}{\epsilon_{n_0+1, \nu_+} - \epsilon_{n_0 \nu_0} - \mu + \omega + i \eta} \right) \] (4.39)

where \( \nu_\pm \) refer to filling factors of the neighboring particle number states of lowest

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thermodynamic potential, representing particle- and hole-like excitations in the
insulator. The matrix elements
\[
\tau_{m_0,\nu_0} = \langle n_0 - 1, \nu_0 | \hat{a}_{m_0} | n_0, \nu_0 \rangle \neq 0 \text{ only for } m_0 = n_0(n_0-1)/2\nu_0 - (n_0-1)(n_0-2)/2\nu_0 \text{ and } \tau_{m_0,\nu_0} = \langle n_0, \nu_0 | \hat{a}_{m_0+1} | n_0+1, \nu_+ \rangle \neq 0 \text{ only for } m_+ = n_0(n_0+1)/2\nu_+ - n_0(n_0-1)/2\nu_0 \text{ by conservation of angular momentum.}
\]

For the general case \( m_+ \neq m_- \), and thus the two terms in the sum contribute two relevant eigenvalues \( \Gamma_{m_0} \), but for the special case \( m_+ = m_- = m_0 \), such as occurs at low rotation rates in all \( n_0I_\infty \) insulators, only a single eigenvalue \( \Gamma_{m_0} \) is relevant.

The Green’s function \( G_{1m'm} \) in the tunnel-coupled case then exhibits two poles, either from the two terms in \( \Gamma_{m_0} \) or each term \( \Gamma_{m_0} \), defining two low-energy excitations dependent on a single \( k_{m_0} \) or two \( k_{m_0} \) respectively. Choosing the most compact representation of \( G_{1m'm} \) through \( \chi = (\ell_z, p_z) \), the eigenvalues \( \phi_{m'm} = \delta(p_z - h\ell_{m_0})\delta(\ell_z - mh) \), and \( S \) is the identity. Performing the sums over values of \( m \) and \( k_{m_0} \), we find the dispersion from the condition \( 1 - \Gamma_m = 0 \) for cases in which poles reside in different orbitals \( m_\pm \) as

\[
\omega_\pm = \mu \mp (\epsilon_{n_0\pm1,\nu_\pm} - \epsilon_{n_0,\nu_0}) \\
\mp t|^\epsilon_{n_0(1\mp1)/2,\nu_\pm}|^2 \cos \left( \frac{p_z d}{h} - \left( \frac{\ell_z}{h} \right)^s \phi_s \right) \\
\approx \Delta_\pm + \frac{\hbar^2}{2M_\pm}(p_z - p_{z,0})^2 \tag{4.40}
\]

where \( \Delta_\pm \) represent the particle/hole gaps at the minima in the dispersion located at \( p_{z,0} \), and the \( M_\pm \) represent the effective masses. The gaps vanish along a line in the \( \mu - t \) plane governed by

\[
\pm \mu - (\epsilon_{n_0\pm1,\nu_\pm} - \epsilon_{n_0,\nu_0}) \\
= -t|^\epsilon_{n_0(1\mp1)/2,\nu_\pm}|^2 \tag{4.41}
\]

while the masses

\[
M_\pm = \hbar^2/t|^\epsilon_{n_0(1\mp1)/2,\nu_\pm}|^2 \tag{4.42}
\]

show the effect of correlations in slowing the tunneling processes, but never vanish
for finite $t$.

The vanishing of a gap $\Delta_\pm$ (in any reference frame determined by the $\phi_s$) is sufficient to predict the location of all phase boundaries separating neighboring $nI_{1/2}$ insulators from coherent phases, but it is clear from its general form that it cannot predict the location of curved phase boundaries, such as occur for the $nI_\infty$ phases (whose poles inhabit the same orbital $m_0 = 0$ and thus escape the form above), nor other transitions, such as illustrated in figure 4.3 of the main text for the region between $4I_{3/4}$ and $3I_{1/2}$, nor the tip of the $2I_{1/2}$ phase.

Moreover, in no instance do the gap and the mass vanish at the same point, invariant of Galilean and/or statically twisted reference frames. In the single-orbital Bose-Hubbard model, such a case does arise at the tri-critical point, and this fact can easily be recovered from the multi-orbital case described here by considering cases where two poles reside in the same orbital $m_0$, wherein the particle and hole excitations hybridize according to equation 4.39 to create the modified dispersions

$$\omega_\pm^{(c)} = \omega_\pm \pm (\omega_+ - \omega_-)[\sqrt{1 - h^2} - 1]$$

where $\omega_\pm$ are the particle/hole energies found above for the uncoupled case, and

$$h^2 = \frac{4t^2|\tau_{n_0 \nu_0 \nu_+}^{\ell_z/h}|^2|\tau_{n_0 - 1 \nu_0 \nu_-}^{\ell_z/h}|^2}{(\omega_+ - \omega_-)^2} \cos^2(p_z d/h - (\ell_z/h)^s \phi_s)$$

### 4.6 Topological Interpretation

The time-evolution described by the path-integral in the main text obtained by setting all $\psi_{im} = \psi_m$ can be described by self-consistent evolution of a single two-dimensional system in the presence of a statically defined $\psi$. This leads to a picture of time-evolution in the insulating phases which is (neglecting fluctuations) equivalent to that of an interacting two-dimensional system, in which the domain of the path-integral can be broken into classes of homotopically-equivalent paths. The same separation of paths can be made in the coherent case. However, since the homotopy groups are a function of the configuration space of the few-body system, we first analyze how this space changes for the cases $\psi = 0$ and $\psi \neq 0$. We find
that while in the insulator phases paths can be described by the standard braid
group, the weakly coherent case leads to a entirely different first homotopy group.

The mapping $p_\phi : (\zeta, z) \rightarrow (\zeta e^{i\phi z/L})$ of the chain coordinates $(\zeta, z)$ onto the
base space $(\zeta)$ is continuous and surjective, such that the chain represents a covering
space of the on-site coordinates, and each site a sheet of its base space. In this case,
$p_0$ represents a specific covering map, whose fiber connecting neighborhoods on-site
as seen embedded in $\mathbb{R}^3$ is straight. If periodic boundary conditions are imposed
on the chain, the twist of any given fiber over the chain becomes discretized by an
integer number of revolutions over its length $L$.

Any path in coordinates $\zeta$ can be homotopically lifted to discrete paths in the
chain, and the first homotopy (fundamental) group $\pi_1(B)$ describing equivalent
paths within the base space $B$ can be related to that of the covering space, the
latter a subgroup $p_*(\pi_1(B))$ of the former. For a single particle, whether in two or
higher dimensions $d$, this fact is trivial, as the fundamental group of the simply
connected $\mathbb{C}$ or $\mathbb{R}^d$ is simply the identity.

### 4.6.1 Path Integral Formulation for Isolated Two-Dimensional
Systems and the Insulating Phases

For a multiparticle system, the decomposition of the mean-field $\psi$ in the coordinates
$(\zeta, z)$, however, based on a dynamical mapping from the original symmetrized
multiparticle coordinate configuration space $B'_n \equiv (\zeta_{i=1 \ldots n}) = (\mathbb{C}^n - \mathbb{D})/S_n$, formed
by $n$ factors of the complex plane with all points $\mathbb{D}$ removed where any two particle
coordinates coincide, properly symmetrized (glued) by forming the quotient with the
symmetric group $S_n$ [112,113]. Closed paths in this space correspond to trajectories
in any few-body path integral similar to that described by $Z$ in the main text, which
can be divided into (homotopy) classes of paths which can be smoothly deformed
into one another [112]. The fundamental group $\pi_1(B'_n)$ describing equivalent loops
in this space is isomorphic to the $n-$particle braid group $B_n$ generated by the
operations $\sigma_i$, clockwise exchanging the particles $i$ and $i + 1$. The only relations
restricting these operations (the presentation of the braid group) are

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad (4.45)$$
\[ \sigma_i \sigma_{j \neq i \pm 1} = \sigma_{j \neq i \pm 1} \sigma_i \]

and the group has the one-dimensional unitary representations \( \chi(\sigma_i) = e^{-i\theta} \), with \( 0 \leq \theta < 2\pi \), extrapolating continuously from bose (\( \theta = 0 \)) to fermi (\( \theta = \pi \)) statistics. As described in Ref. [113], any propagator \( K \) for a fixed number \( n \) particles between initial and final points \( q \) and \( q' \) in \( \mathbb{B}'_n \), can be expressed as a sum over paths in each equivalence class:

\[
K(q', \tau' | q, \tau) = \sum_{\Upsilon \in \pi_1(\mathbb{B}'_n)} \chi(\Upsilon) \int_{q(\tau) \subset \Upsilon} \mathcal{D}[q(\tau)] e^{i \int_{\tau}^{\tau'} L[q(\tau)] \, d\tau} \]

Here \( \chi(\Upsilon) \) is a phase-factor characterizing each equivalence class \( \Upsilon \), which to satisfy composition rules of path integration must form a one dimensional unitary representation of \( \pi_1(B'_n) \). Equivalently [113], the phase factors can be absorbed into the path integral by recognizing that

\[
\chi(\Upsilon) = e^{-i \frac{\theta}{\pi} \int d\tau \partial_\tau (\sum_{i < j} \angle \zeta_{ij}(\tau))} \]

with \( \angle \zeta_{ij} \) the orientation of the relative coordinate between particles \( i \) and \( j \) in the plane, and the member of the braid group \( \Upsilon \) specified by any representative path \( q(\tau) = \{ \zeta(\tau) \} \) in the class. Absorbing these into \( K \), they take the form of an additive total time-derivative of the statistical gauge generator \( \Lambda \equiv (\theta/\pi) \sum_{i < j} \angle \zeta_{ij}(\tau) \), resulting in the alternate expression for propagators

\[
K(q', \tau' | q, \tau) = \int \mathcal{D}[q(\tau)] e^{i \int_{\tau}^{\tau'} (L - \partial_\tau \Lambda) \, d\tau} \]

Following ref. [113], it is also possible to consider this path integral in the universal covering space \( \tilde{\mathbb{B}}'_n \) of \( \mathbb{B}'_n \), with final points \( q' \) lifted onto separate sheets. This exploits the isomorphism between the fundamental group \( \pi_1(\mathbb{B}'_n) \) and the deck transformation group of the mapping from \( \tilde{\mathbb{B}}'_n \) to \( \mathbb{B}'_n \) to introduce a monodromy action. In this picture, the propagation of an initial wavefunction \( \Phi(q, \tau) \) described
by $K$ leads to a multivalued final wavefunction $\Phi(q', \tau')$, with values corresponding to each sheet projected back onto the original configuration space $\mathbb{B}'_n$.

This formulation effectively describes a statistical gauge transformation of the few-body wavefunction $\Phi \rightarrow e^{i\Lambda}\Phi$ leading to the definition of a dynamical gauge field

$$\alpha(\zeta_i) \equiv (\partial_{\text{Re}\zeta_i} + i\partial_{\text{Im}\zeta_i})\Lambda \quad (4.49)$$

experienced by each particle arising from its statistical interaction with all others. In second-quantized form, the gauge field can equivalently be expressed as a solution of the equation

$$\partial_{\zeta} \alpha_i - \partial_{\zeta} \alpha_r = \frac{\theta}{\pi} \rho(\zeta) \quad (4.50)$$

where the subscripts $r, i$ denote the real and imaginary components, and $\rho$ is the density, expressing the composition of flux and particles. Using the continuity equations for particle currents [114], the time derivative $\partial_t \alpha = -\left(\frac{\theta}{\pi}\right)(j_i + ij_r)$, with $j$ representing particle current. These equations of motion correspond to an action of the Chern-Simons form

$$\mathcal{L}_\alpha = \left(\pi/2\theta\right)(\vec{\alpha} \cdot \nabla \times \vec{\alpha}) - \vec{\alpha} \cdot \vec{j} \quad (4.51)$$

where $\vec{\alpha} = (\alpha_0, \text{Re}\alpha, \text{Im}\alpha)$ is a generalization of $\alpha$ above including a time-component $\alpha_0$ as a Lagrange multiplier to enforce condition 4.50, and $\vec{j} = (\rho, j_r, j_i)$. The full system is then described by $\mathcal{L}_0 = \mathcal{L}_\alpha + \mathcal{L}_m$, where $\mathcal{L}_m$ can be found from $\mathcal{L}_0$ by the substitution $\vec{A} \rightarrow \vec{A} + \vec{\alpha}$, with $\vec{A}$ the original vector potential,

$$\mathcal{L}_m = \hat{b}^i(\zeta)(i\hbar \partial_\tau - A_0 - \alpha_0)\hat{b}(\zeta) - \hat{b}^i(\zeta)\frac{1}{2m}(i\hbar \partial_\zeta - A - \alpha)^2\hat{b}(\zeta) - \hat{b}^i(\zeta)[\frac{m}{2}(\omega^2 - \Omega^2)|\zeta|^2 - \mu]\hat{b}(\zeta) \quad (4.52)$$
Under the choice \( \vec{\alpha} = -\vec{A} \), representing a statistical gauge transform under which the new particles experience no net gauge potential, we see that the number of external flux attached per particle is \( \nu = b/n = (\pi/\theta) \) - with the condition \( \theta = 2\pi, \nu = 1/2 \) and \( \mathcal{L}_m \) represents a field-free interacting Bose gas.

Before concluding, we note a intuitive picture of the action \( S_\alpha = \int d^2 \zeta d\tau \mathcal{L}_\alpha \), obtained by gluing the endpoints of the path integral trajectories \( q \) and \( q' \) to visualize the space as isomorphic to a solid torus. Integrated over this volume, the action can be expressed as an abelian Chern-Simons three-form \( S_\alpha \propto \int \alpha \wedge d\alpha \) - this form is similar to that discussed in the context of hydrodynamic helicity in three-dimensional fluids [108] and plasmas, with \( \vec{\alpha} \) playing the role of a velocity field in three spatial dimensions. In these cases, the integration over the three-form forms an integer topological invariant \( h = \kappa_i \kappa_j \sum_{ij} \Theta_{ij} + \kappa_j^2 \sum_j \Xi_j \) related to the number of links \( \Theta_{ij} \) and self-links \( \Xi_j \) of individual vortex lines \( i,j \) with winding numbers \( \kappa_{i,j} \).

4.6.2 Path Integral Formulation for Weakly-Coupled Two Dimensional Systems

Incorporation of the mean-field based on the coupling Hamiltonian \( \hat{\mathcal{H}}_c \), and determination of its hydrodynamic properties as described in the main text breaks this process into two topological steps. In the first, described by the elements \( \tau(\zeta) \), the \( n+1 \)-particle states are coupled to the \( n \)-particle states through a (covering) mapping of the underlying manifold from \( B'_{n+1} \) to \( B'_n \), resulting in alternately the matrix elements \( \tau^{(m)} \) between states, or their representation as a complex function \( \tau(\zeta) \) defined on a simpler manifold \( \mathcal{T} \). Once \( \tau \) is determined, self-consistent solution with \( \psi \) results in a local mean-field model. Finally, lifting this local (two-dimensional) system into the larger space including the chain coordinate \( z \) allows for determination of the response of the system to slow spatial and temporal variation of \( \psi(\zeta, z) \). Identifying the form of the manifold \( \mathcal{T} \) and the function \( \tau \) self-consistently with \( \psi \) then forms a full solution to the hydrodynamical problem.

The mean field can be incorporated by introducing the field \( \psi(\zeta) \) through a modified lagrangian density according to a spatially continuous representation of the same Hubbard-Stravantovich transform used above to obtain the finite temperature phase diagram numerically,
\[ \mathcal{L}[[\psi_i, b_i]] = \sum_i \mathcal{L}_0[b_i] + \sum_i \mathcal{L}_\psi[\psi_i, b_i] + \sum_{ij} \mathcal{L}_{\psi}^{(c)}[\psi_i, \psi_j] \]

with

\[ \mathcal{L}_\psi[\psi_i, b_i] = -b_i^\dagger(\zeta)\psi_i(\zeta) - b_i(\zeta)\psi_i^*(\zeta) \quad (4.53) \]

\[ \mathcal{L}_{\psi}^{(c)}[\psi_i, \psi_j] = \psi_i(\zeta)(t\sigma_{ij})^{-1}\psi_j^*(\zeta) \]

As above, integrating over all values of \( \psi_i(\zeta, \tau) \),

\[ Z = \int \mathcal{D}[[\psi_i(\zeta, \tau), b_i(\zeta, \tau)]] e^{-S} \quad (4.54) \]

with the action

\[ \mathcal{S}[[\psi_i(\zeta, \tau), b_i(\zeta, \tau)]] = \sum_i \int_0^\beta d\tau d^2\zeta \mathcal{L}[[\psi_i, b_i]] \quad (4.55) \]

recovers the full microscopic model and therefore is an exact representation of the original problem in the form of a coherent state path integral with continuous fields.

In order to connect to the picture based on the few-body first-quantized propagator above, and understand how a statistical gauge field or fields may be introduced in the coherent phase, it is helpful to convert the coherent state path integral into a sum over paths in first-quantized form. To do so, we break the action integral into infinitesimal time steps, replace the coherent fields with \( \hat{\psi}_i(\zeta, \tau) = \langle \psi_j(\zeta, \tau), \phi_j(\zeta, \tau) \rangle |\psi_i(\zeta, \tau)\rangle \), with \( |\psi_j(\zeta, \tau), \phi_j(\zeta, \tau)\rangle \) the coherent state, in which the dynamical auxiliary field \( \psi \) has been added to the state label. Since the path integral 4.54 is formally equivalent to one in which the auxiliary field is represented by an operator \( \hat{\psi}_i \) with eigenvalues and states \( \hat{\psi}_i|\psi_j(\zeta, \tau), \phi_j(\zeta, \tau)\rangle = \psi_i(\zeta, \tau)|\psi_j(\zeta, \tau), \phi_j(\zeta, \tau)\rangle \), we treat both fields identically.

We then expand the path in intermediate states at all times \( \tau \) using the identity
\[ \mathbb{I} = \sum_{n_b} \sum_{n_\\psi} \int d^{2n} [\{\zeta, \bar{\zeta}\}] |\zeta_1 \ldots \zeta_{n_b}, \bar{\zeta}_1 \ldots \bar{\zeta}_{n_\\psi} \rangle \langle \zeta_1 \ldots \zeta_{n_b}, \bar{\zeta}_1 \ldots \bar{\zeta}_{n_\\psi}| \] (4.56)

where the second sum is restricted such that \( n_b + n_\\psi = n \). The states \( |\zeta_1 \ldots \zeta_{n_b}, \bar{\zeta}_1 \ldots \bar{\zeta}_{n_\\psi}\rangle \) represent Bose-symmetrized, spatially localized states of definite numbers \( n_b \) with coordinates \( \zeta_1 \ldots n_b(\tau) \) and \( n_\\psi \) with coordinates \( \bar{\zeta}_1 \ldots n_\\psi(\tau) \).

The \( n_\\psi \) quanta are introduced here only as a means for calculation of the path integral above - while they can loosely be interpreted as particles with a longitudinal dependence to their wave function with a distinguishable form to that represented by the \( \hat{b}_i \), there is no need to make this association as the problem is fully defined by the choice of action in 4.55. Without further additions to the action, namely topological terms which modify exchange processes, it is clear one must choose the commutators \( [\hat{b}_i(\zeta), \hat{b}_j^\dagger(\zeta')] = [\hat{\psi}_i(\zeta), \hat{\psi}_j^\dagger(\zeta')] = \delta_{ij} \delta(\zeta - \zeta') \) and \( [\hat{b}_i(\zeta), \hat{\psi}_j^\dagger(\zeta')] = 0 \).

While the \( n_\\psi \) quanta are defined here formally through the Hubbard-Stravantovich using the auxiliary coherent states \( \psi \), they are reminiscent of the introduction of \( n_\\psi \) “worms” in quantum Monte Carlo algorithms [115]. In the latter, the introduction of one or more such entities through “cuts” of existing or insertions of new world-lines and subsequent movement and joining of their endpoints can connect topologically distinct multi-particle trajectories contributing to the partition function. Before re-joining endpoints, the trajectories obtained through a single cut/insert and endpoint manipulation represent contributions to the single-particle Green’s function, and the inclusion of \( n_\\psi > 1 \) such operations represent contributions to higher-order Green’s functions. Given the role of the single-particle Green’s function \( G_1^{(0)} \) in determining the position of the first phase boundary for fixed chemical potential and increasing tunneling, there is inspiration to consider the classification of its contributing terms according to their topological properties.

To that end, we consider the configuration space of the states \( |\zeta_1 \ldots \zeta_{n_b}, \bar{\zeta}_1 \ldots \bar{\zeta}_{n_\\psi}\rangle \). Addition of a fixed, transversely localized field \( \psi(\zeta) \neq 0 \) punctures the complex base space for each quanta of \( n_\\psi \) at one additional point \( \mathbb{P} \), and alters the homotopy classes from \( \pi_1(\mathbb{B}'_{n}) \) to \( \pi_1(\mathbb{B}'_{n}[\mathbb{P}_{n_\\psi}]) \), where

\[ \mathbb{B}'_{n}[\mathbb{P}_{s}] \equiv \frac{(\mathbb{C} - \mathbb{P}_{s})^n - \mathbb{D}}{\mathbb{S}_n} \] (4.57)

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are new configuration spaces in which \( s \) punctures are placed in the complex-plane for each particle.

For \( s = 1 \), \( \pi_1(\mathbb{B}_n'\mathbb{P}_1) \) is isomorphic to the circular braid group \( \mathcal{CB}_n \) of \( n \)-braidings on an annulus, whose generators \( \sigma_i^c \) wind the \( i \)th and \((i + 1)\)th particles as before, and \( \gamma_c \) cyclically permutes the particles around the puncture [116]. In addition to the relations of Eq. 4.45 applied to the \( \sigma_i^c \), for this group

\[
\gamma_c^{-1}\sigma_i^c\gamma_c = \sigma_{i+1}^c
\]

it is possible to understand this group in a slightly different way - \( \mathcal{CB}_n \) has also previously been noted [116,117] to be isomorphic to \( D_{n+1} \), a finite index \((n + 1\)-coset) subgroup of the \((n + 1)\)-particle braid group \( \mathcal{B}_{n+1} \), in which a single strand of the original configuration is required to end on its original position [117]. One can thereby alternatively view this as making one particle of the \((n + 1)\)-particle path integral statistically distinguishable such that its beginning and end points are fixed - indeed, under the application of the operator \( \hat{\tau}(\zeta) \), any state of \((n + 1)\) particles is projected in this way to an \( n \)-particle state. This reflects the fact that the longitudinal delocalization of a single quanta along the chain effectively identifies a single particle from the few-body system. From this connection it is clear the homotopic classes of the (weak) mean-field case can correspond to neither braid group \( \mathcal{B}_n \) nor \( \mathcal{B}_{n+1} \). A unitary one dimensional representation of \( \mathcal{CB}_n \) can be found by expanding the mapping \( \chi \) to admit \( \chi(\gamma_c) = e^{-i\theta_0} \), with \( 0 \leq \theta_0 < 2\pi \). This permits tracking windings of particle trajectories about each other and the auxiliary field separately according to any representative path \( q_1(\tau) = \{\zeta_1(\tau)\ldots\zeta_{n_0}(\tau), \bar{\zeta}_1(\tau)\} \in \Upsilon_1 \) (similar to the case above) as

\[
\chi(\Upsilon_1) = e^{\frac{i}{\pi} \int d\tau \partial_\tau (\theta \sum_{i<j} \angle \zeta_{ij} + \frac{i}{2}\theta_0 \sum_i \angle (\zeta_i - \zeta))}
\]

The right-hand side can be seen as a new homotopic invariant corresponding either to self-linkages and cross-linkages between the \( \zeta \) and \( \bar{\zeta} \) trajectories, or as self-linkages and windings within a thickened toroidal surface. Setting
\[ \Lambda = \frac{\theta}{\pi} \sum_{i<j} \angle \zeta_{ij} + \frac{\theta_0}{2\pi} \sum_i \angle (\zeta_i - \bar{\zeta}_i) \]  
\( (4.60) \)

and absorbing the phase factors \( \chi(\Upsilon) \) into the path integral, a new auxiliary statistical gauge-field emerges as

\[ \beta = (\partial_{\bar{\zeta}_1,r} + i\partial_{\bar{\zeta}_1,i})\Lambda \]  
\( (4.61) \)

Equivalently, \( \alpha \) and \( \beta \) can be calculated in second-quantization as

\[ \partial_{\zeta,r} \alpha_i - \partial_{\zeta,i} \alpha_r = \frac{\theta}{\pi} \rho(\zeta) + \frac{\theta_0}{2\pi} \rho_s(\zeta) \]  
\( (4.62) \)

\[ \partial_{\zeta,r} \beta_i - \partial_{\zeta,i} \beta_r = \frac{\theta_0}{2\pi} \rho(\zeta) \]

It will be helpful for reasons which are later apparent to separate \( \alpha = \alpha_1 + \alpha_2 \) into two components

\[ \partial_{\zeta} \alpha_{1i} - \partial_{\zeta} \alpha_{1r} = \frac{\theta}{\pi} \rho(\zeta) \]  
\( (4.63) \)

\[ \partial_{\zeta} \alpha_{2i} - \partial_{\zeta} \alpha_{2r} = \frac{\theta_0}{2\pi} \rho_s(\zeta) \]

representing the attachment of \( \alpha_1 \)-flux to particles with density \( \rho \) and \( \alpha_2 \)-flux to the coherent field \( \rho_s = |\psi|^2 \), in quanta determined by \( \theta \) and \( \theta_0 \). Only the \( \alpha \) potentials contribute to the effective vector potential seen by particles, while the \( \beta \) potential affects the coherent field, and should be included in any ascribed canonical momenta - since it couples to the particle density \( \rho \), it represents an interaction between the coherent field and the few-body structure. Differentiating the above, applying continuity \( \partial_r \rho = -(\partial_{\zeta,j_r} + \partial_{\zeta,j_i}) \),

\[ \partial_r \alpha_1 = -\frac{\theta}{\pi} (j_i + i j_r) \]  
\( (4.64) \)
\[
\partial_r \beta = -\frac{\theta_0}{2\pi} (j_i + ij_r)
\]

within additive constants. Equations 4.62-4.64 can be obtained by minimizing action due to the Lagrangian

\[
\mathcal{L} = \mathcal{L}_{\alpha_1} + \mathcal{L}_\beta + \mathcal{L}_m
\]

with

\[
\mathcal{L}_\beta = \left(\frac{\pi}{\theta_0}\right) \vec{\beta} \cdot (\nabla \times \vec{\beta}) - \vec{\beta} \cdot \vec{j} \\
\mathcal{L}_{\alpha_1} = \left(\frac{\pi}{2\theta}\right) \vec{\alpha_1} \cdot (\nabla \times \vec{\alpha_1}) - \vec{\alpha_1} \cdot \vec{j}
\]

in notation similar to the case for \(\alpha\) above. In producing 4.64 from 4.63, it is necessary to make use of Stoke’s theorem, treating 4.63 as the \(z\)-component of the curl of a three dimensional vector lying in the plane of \(\zeta\). The equivalent expression for \(\alpha_2\) is more subtle, if one were to insist here that \(\psi\) allow for axial variation \(\psi(\zeta, z)\), the time rate of change of coherent density \(\partial_r \rho_s\) can involve axial currents, and the continuity equation

\[
\partial_r \rho_s = -\nabla \cdot \vec{j}_s
\]

must be formed from the divergence of a true three-dimensional current \(\vec{j}_s\). Instead, we will make a slowly-varying approximation for the longitudinal coordinate, first treating continuity for \(\rho_s\) without the axial component to understand the lagrangian for a homogeneous mean-field case, then proceeding by generalizing that result for axial variation. In this case, the contribution to the Lagrangian due to \(\alpha_2\) is

\[
\mathcal{L}_{\alpha_2} = \left(\frac{\pi}{\theta_0}\right) \vec{\alpha_2} \cdot (\nabla \times \vec{\alpha_2}) - \vec{\alpha_2} \cdot \vec{j}_s
\]

in similar notation.

The values of \(\theta\) and \(\theta_0\) have as yet not been chosen, and are in some sense
arbitrary choices of statistical gauge. The ground-state wave-function of the few-body system in the centrifugal limit is simplest under the choice $\theta = 2\pi$ when $\rho_s = 0$, as this leads to “condensation” of composites according to the $\frac{1}{2}$-Laughlin form. For $\rho_s \neq 0$, however both $\theta$ and the angle $\theta_0$ must be chosen. For a single distinguishable trajectory as described here, this is tantamount to finding the ground state and adiabatic dynamics of a single immersed impurity atom with identical characteristics (mass and interaction strengths) to the host atom. As discussed previously [103], near the centrifugal limit this state closely resembles a $1/2$–quasihole, leading to the natural choice of $\theta_0 = 2\pi$ for those parameters. It is illuminating to consider the effect of adiabatically moving the coordinate $\zeta$ in $\psi(\zeta)$ through a closed loop in the plane. Under such a process, it is well-known that the full many-body wave-function may accumulate a geometric phase independent of the rate of change of $\zeta$ - in the case of a single quasi-hole in a filling factor $\nu$ state, this corresponds to the Aharanov-Bohm phase accumulated by a single fractionally-charged object in the external magnetic field. For more general values of $\Omega, \eta, \mu$, the geometric phase may be different or may not exist. Nevertheless, it is clear that many-body wave-function need not be a single-valued function of $\zeta$, with the domain of $\zeta$ taken to be the complex plane.
Bibliography


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PUBLICATIONS


CONFERENCES

1. Jianshi Zhao, Craig Price, Qi Liu and Nathan Gemelke, “Dynamical Gauge Effects and Holographic Scaling of Non-Equilibrium Motion in a Disordered and Dissipative Atomic Gas”, APS March Meeting 2016, B50.00005, Talk


3. Louis Jacome, Jianshi Zhao, and Nathan Gemelke, “Probing Fractional Quantum Hall Physics with Rotating Bose Gases”, APS March Meeting 2015, Volume 60, Number 1, Poster

4. Louis Jacome, Jianshi Zhao, and Nathan Gemelke, “Fractional Quantum Hall Physics with Rotating Bose Gases”, 45th Annual Meeting of the APS DAMOP, Volume 59, Number 8, Poster

5. Jianshi Zhao, Louis Jacome, and Nathan Gemelke, “Fractional Quantum Hall Effects in a Two-Dimensional Atomic Gas”, APS March Meeting 2014, Volume 59, Number 1, Poster