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College of the Liberal Arts

THEORETICAL INVESTIGATIONS
OF PARTNERSHIP TURNOVER AND OF MONEY

A Dissertation in
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Abstract

This dissertation consists of four chapters. The first chapter examines the turnover of multiperiod partnerships. The turnover of multiperiod partnerships such as marriages, labor contracts, and joint ventures varies over time and across countries. A model is set out in which these different observations arise as multiple equilibria. In a random pairwise matching model, players are heterogeneous in time preference and each pair plays a prisoners' dilemma game with random payoffs from mutual cooperation. Two steady states are constructed: in one, non-myopic players cooperate even when a match has low payoffs from mutual cooperation; in the other, they cooperate only when a match has high payoffs from it. Transition dynamics across the two steady states are studied. For a numerical example, it is shown that a transition in either direction is an equilibrium.

The last three chapters are contributions to monetary economics. In the second chapter, written jointly with Yu Awaya, a counter-example to the notion that *money is memory* is provided—one that relies on incomplete information. For it, there exists an implementable allocation with money which is not implementable with memory. The result arises because money conveys only a limited amount of information about past actions which can be beneficial in settings with incomplete information.

In the third chapter, I examine a necessary condition for fiat money to be essential. Fiat money, an intrinsically useless object, is said to be *essential* if some good allocations are achieved with it but not without it. It is shown that imperfect monitoring is necessary for money to be essential in a large class of economic environments. This provides a guide for the construction of models in which monetary trade achieves good outcomes.

In the fourth chapter, written jointly with Yu Awaya, it is shown that a seemingly strong condition is not sufficient for money to be essential. Money is thought to be essential when it is difficult to monitor others' behavior. We provide a counterexample to the view. For it, it is shown that even if there is no monitoring, money is inessential—the first best allocation can be attained without money (or

any other form of monitoring).

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Chapter 1

High and Low Turnover Equilibria in a Model of Partnerships

1.1 Introduction

Multiperiod partnerships are present in many situations: marriages, labor contracts, and joint ventures. However, the turnover of these relationships varies. Examples of such diversity include time series and cross sectional variations in marriage turnover (see, for example, Qu and Weston [34] and Paetsch *et al.* [32]) and joint-venture turnover (see, for example, Park and Ungson [33] and Hennart and Zeng [18]). In this paper, a model is set out in which different observations regarding turnover arise as multiple equilibria.

Players randomly meet in pairs, do not see each other's histories in other meetings, and play a two-stage game. In the first stage, they play a nontransferable-utility prisoners' dilemma in which the realized pre-play payoff payoff from mutual cooperation is random—either high or low.[1] Some players are myopic and always choose non-cooperation. At the second stage players choose whether or not to separate, where separation (of either) means that the pair starts the next date in the unmatched pool. Two steady states are constructed: in a *low turnover* steady state, non-myopic players cooperate and continue the present relationship

[1]Non-transferable utility games have been extensively used in various kinds of contexts; see, for example, Smith [35] for marriages, Lazear [26] for performance-pay wage contracts, Ackerberg and Botticini [1] for agricultural contracts, and Legros and Newman [27] for general assortative matching.

even when a match has low payoffs from mutual cooperation; in a *high turnover* steady state, they cooperate only when a match has high payoffs. In the low (high) turnover steady state, relatively few (more) non-myopic players go back to the unmatched pool and the cost of going back to the pool is higher (lower) because there is a higher (lower) fraction of myopic players in the pool. This externality supports the existence of both kinds of steady states, both of which are locally stable. Transition dynamics across the two steady states are studied for numerical examples. The transition from the low turnover steady state to the high turnover steady state exhibits an overshoot in turnover rates due to the outflow from the low payoff matches, whereas the transition in the other direction does not. For each direction, there exist parameters for which both non-myopic and myopic players prefer a transition to the initial steady state.

The model is inspired by Ghosh and Ray [16], which does not have multiple equilibria in terms of turnover rates.[2] It departs from Ghosh and Ray [16] in three respects: it has random payoffs from mutual cooperation; has an endogenous and time-varying state: the probability of meeting a non-myopic partner; and is simplified by having only *cooperate* or *defect* as strategies rather than a continuous choice of cooperation level. My version suffices to give rise to multiple equilibria with different turnover rates and the multiplicity does not rely on the simplification of the strategy space.

Burdett and Coles [11] is similar to the present work in that it also has multiple equilibria in terms of turnover rates. The model and its welfare implications, however, differ from those of the present work in several ways. Their model does not have a cooperation-choice stage; agents choose only whether to separate. And in Burdett and Coles, using the standard utilitarian perspective, an equilibrium with a high turnover rate is always better than another equilibrium with a low turnover rate. In the present model, whether high or low turnover is beneficial depends on parameters.

For my non-stationary analysis, I develop a new computational algorithm. In order to show that a candidate strategy profile constitutes a non-stationary

[2] They explicitly discuss this limitation. See Ghosh and Ray, page 510, and Mailath and Samuelson [29], page 157. Although they focus on occasions in which the state variable is stationary, their strategies are non-stationary in the standard sense as players vary cooperation levels.

equilibrium, it needs to be shown that incentive constraints are satisfied at each date—incentive constraints that depend upon expected discounted values of being unmatched. The main idea of the algorithm is that for a fixed turnover strategy profile, the path of state variables is known. Using this path, I construct a sequence that consists of true states up to a finite date K at which it is assumed that the steady state is attained. Using that sequence, I solve Bellman equations backwards for approximated expected discounted values.[3] I show that this approximation has the following property: as K tends to infinity, the approximation error becomes arbitrarily small. This result allows me to verify existence of the equilibrium path for particular numerical specifications.

The paper is organized as follows. Section 1.2 describes the model. In Section 1.3, I show coexistence and local stability of two steady states. In Section 1.4, I study properties of dynamic transition paths between the two steady states. Section 2.5 concludes. Appendices A.1 and A.2 provide the evolutions of the state variables, Bellman equations, incentive constraints, and equilibrium payoffs. Appendix A.3 provides the proofs.

1.2 Model

The Environment. Time is discrete, lasts forever, and is indexed by $t = 0, 1, 2, \cdots$. There is a non-atomic unit measure of players. There are two types of players, *good* and *bad*. Types are private information to each player. Good players have a common discount factor $\delta \in (0, 1)$. Bad players have a discount factor of zero.

The sequence of actions is as follows. Under circumstances in which bad players have no partner at start of each date, the state of the economy entering date t is a three-tuple: a measure $z_{GG}^{(t)}(H)$ of matches of two good players with high suitability, a measure $z_{GG}^{(t)}(L)$ of matches of two good players with low suitability, and a measure $g^{(t)}$ of unmatched good players. Then, all unmatched players are randomly matched into pairs. Because there are both good and bad players in the unmatched pool, each good player is afraid of meeting a bad player. Whenever

[3] This approach is somewhat reminiscent of the so-called *time path iteration algorithm*. The approximation target and the known variable are however opposite. That is, the time path iteration solves policy functions for approximated state variables (see, for example, Evans and Phillips [14]).

two players are newly matched, match *suitability* v is drawn randomly i.i.d. across matches and it enters the payoff matrix of the game played in the match. The random variable v takes v_H with probability π_H and v_L with probability π_L, is common for two players in a match, and is fixed for the match. I assume that $0 < v_L < v_H \leq 1$.

In each match, two players play a two-stage game. In the first stage of the game, players simultaneously choose from $\{C, D\}$ (C for *Cooperation* and D for *Defection*) and receive payoffs according to Table 1.1. In the table, all parameters are positive. Because bad players have a discount factor of zero, they always play D. In the second stage of the game, two players simultaneously choose whether to continue or terminate the current partnership. A match dissolves if at least one of the players chooses to terminate the partnership. In the game, each player observes her partners' actions and her own actions, but not others'. Then, an exogenous death shock hits with probability $1 - \rho \in (0, 1)$. If a death shock hits, the match will dissolve. At the end of date t, new players enter the economy so that the population is kept stationary, fraction ϕ of whom are bad and $1 - \phi$ good. To prevent the ratio of good and bad players in the entire economy from changing, it is assumed that the initial measure of bad players is equal to ϕ. Because of the continuation probability ρ, the discount factor of good players is effectively $\delta\rho$.

	C	D
C	v	$-l$
D	$1+d$	0

Table 1.1. Payoff

A history of a player is given by own past actions and her partners' past actions (this potentially include many different past partners). A pure strategy of a player is given by two functions: a function from the set of histories into $\{C, D\}$ and a function from the set of histories into $\{Continue, Terminate\}$.

Equilibrium. Each good player maximizes the expected discounted value. The equilibrium concept is sequential equilibrium.

1.3 Steady States

I focus on the following two simple strategies, *low-turnover strategy* and *high-turnover strategy*. Because bad players have a discount factor of zero, they always choose to defect regardless of the match suitability. Hence, it is enough to specify good players' strategies. For the low- (resp. high-) turnover strategy, each good player cooperates in both high and low suitability matches (resp. only in high suitability matches), and each player continues if and only if both players cooperated in the first stage of that date. Starting from an initial state, if the low- (resp. high-) turnover strategy constitutes an equilibrium, I call it the *low- (resp. high-) turnover equilibrium.* A *steady state* is an equilibrium in which the state variables do not depend on time. For steady states, I write variables without time superscripts (t).

For given parameters, these steady states may not exist. In particular, it is easy to see that neither steady state exists when ϕ is close to unity. If ϕ is close to one, almost every player is of the bad type, and good players expect that a partner of an unknown type defects with probability almost one. Hence, a good player has an incentive to deviate.

1.3.1 Coexistence of Two Steady States

The following proposition provides conditions for which the two steady states coexist.

Proposition 1.1. *Suppose that*

$$(1-\delta\rho\pi_L)(1+d) < v_H \tag{1.1}$$

and

$$(1-\delta\rho)(1+d) < v_L. \tag{1.2}$$

Then, there exist $\bar{l} \in (0,\infty)$, $\underline{\phi} \in (0,1)$, and $\bar{\phi} \in (\underline{\phi}, 1)$ such that for any $l < \bar{l}$ and any $\phi \in (\underline{\phi}, \bar{\phi})$, the low- and high-turnover steady states coexist.

All proofs are provided in Appendix A.3. The idea of the proof is that, first, given a fraction of bad players in the unmatched pool at start of a date, denoted by λ, the incentive constraint of good players for the low-turnover steady state can be expressed as a quadratic inequality of it. If this inequality has solutions between 0 and 1, that means that there exists a region of λ such that the low-turnover steady state exists if λ lies in that region. Then, it is shown that when ϕ moves from 0 to 1, λ moves from 0 to 1. Thus, there exists a region of ϕ for which the low-turnover steady state exists. Similarly, for the high-turnover steady state, there exists a region of ϕ for which the high-turnover steady state exists. The last step is to show that these two regions of ϕ have a non-empty intersection under the conditions of the proposition.

The conditions (1.1) and (1.2) are satisfied if $\delta\rho$ is high, d is low, payoffs from mutual cooperation are high, and π_L is high. This is intuitive because otherwise, the gain from a deviation is high relative to the cost of going back to the unmatched pool. These conditions do not guarantee that *given* ϕ, the two steady states exist in the limit of $\delta\rho$. In other words, the set of equilibrium strategies is in general not monotonic in $\delta\rho$, other parameters being fixed.

1.3.2 Local Stability

The two steady states are both locally stable.

Proposition 1.2. *Both high- and low-turnover steady states are locally stable under the conditions of Proposition 1.1.*

The proof consists of two parts. First, given a strategy, the evolution of state variables is described by a system of first-order difference equations

$$(z_{GG}^{(t+1)}(H), z_{GG}^{(t+1)}(L), g^{(t+1)}) = F(z_{GG}^{(t)}(H), z_{GG}^{(t)}(L), g^{(t)}). \tag{1.3}$$

The explicit form of F is provided in Appendices A.1 and A.2. It can be shown that this is a convergent sequence and hence starting from any initial state, the state variables converge to the steady state values.

Then, we need to check that the incentive constraints of good players are satisfied for all $t \geq 0$ if an initial state is close enough to the steady state values. There are two incentive constraints at a date for each equilibrium—one at a low-suitability match and the other at a high-suitability match. However, it can be shown that one of the incentive constraints is either implied by the other or redundant, and hence each equilibrium is guaranteed by a single incentive constraint at each date, denoted as $IC^{(t)}$ (see Appendices A.1 and A.2 for the details). It depends upon the expected discounted value $V^{(t+1)}$ from going back to the unmatched pool and the measure $g^{(t)}$ of unmatched good players. The evolution of $V^{(t)}$ is given by the Bellman equation

$$V^{(t)} = G(g^{(t)}, V^{(t+1)}). \tag{1.4}$$

The explicit form of G is provided in Appendices A.1 and A.2. It can be shown that $V^{(t)}$ is close enough to V if $g^{(s)}$ is close enough to g for all $s \geq t$. Also, $(g^{(t)})_{t=0}^{\infty}$ monotonically converges to the steady state value g. Hence, if $g^{(0)}$ is close enough to g, then $V^{(t)}$ is close to V for all $t \geq 1$. Hence, $IC^{(t)} \geq 0$ are satisfied for all $t \geq 0$ if they are satisfied with strict inequality at steady state, that is, $IC > 0$. The conditions of Proposition 1.1 are given by strict inequalities, and hence the incentive constraint holds strictly at either steady state under the conditions.

1.4 Dynamic Equilibrium Paths

In this section, I show by a numerical example that transition paths between the two steady states exist in both directions. Below, I describe a computational algorithm that I use to calculate these dynamic equilibria and key results that guarantee the work of it.

First, I show that starting from any initial state, $(g^{(t)})_{t=0}^{\infty}$ converges to the steady state value for either steady state.

Proposition 1.3. *For both high- and low-turnover strategies, for any initial state, for any $\epsilon > 0$ there exists T such that for any $t \geq T$, $|g^{(t)} - g| < \epsilon$.*

A key idea here is separation of the state variables from the expected discounted values. That is, given a candidate strategy profile, $(g^{(t)})_{t=0}^{\infty}$ is solely determined by equation (1.3). Then, $(V^{(t)})_{t=1}^{\infty}$ is determined by equation (1.4). Because $V^{(t)}$ depends on $(g^{(s)})_{s=t}^{\infty}$ and incentive constraints depend on $g^{(t)}$ and $V^{(t+1)}$, similar convergence results preserves to $IC^{(t)}$. That is, for any initial state, $(IC^{(t)})_{t=0}^{\infty}$ converges to the steady state value IC for either steady state. Of course, this does not guarantee global stability of steady states (equilibria). It is because even if $IC > 0$ is satisfied at steady state and hence $IC^{(t)} > 0$ are satisfied for large t, we do not know whether $IC^{(t)} \geq 0$ are satisfied for small t. Also, it is not possible to compute the true value of $V^{(t)}$ because it depends upon an infinite sequence $(g^{(s)})_{s=t}^{\infty}$.

However, relying on Proposition 1.3, I can construct an algorithm that approximates true values of $V^{(t)}$ with a finite number K of steps. Below, I describe the algorithm and show convergence of it with respect to K—convergence of the computed value to the true value for any $t \leq K$.

Algorithm 1.1. *1. Fix a strategy profile and some K.*

2. Starting from $g^{(0)}$, use equation (1.3) to obtain $(g^{(t)})_{t=0}^{K}$.

3. For each $t \in \{1, \cdots, K+1\}$, define $\underline{V}^{(t)}$ by $\underline{V}^{(K+1)} = V$ and

$$\underline{V}^{(t)} = G(g^{(t)}, \underline{V}^{(t+1)})$$

for $t = 1, \cdots, K$.

4. Compute incentives constraints using $\underline{V}^{(t)}$ instead of $V^{(t)}$.

Denote incentive constraints constructed by Algorithm 1.1 as $\underline{IC}^{(t)} \geq 0$. Of course, $\underline{IC}^{(t)}$ are different from true $IC^{(t)}$. Hence, $\underline{IC}^{(t)} \geq 0$ may be satisfied even if $IC^{(t)} \geq 0$ are not satisfied. However, the following proposition guarantees that as $K \to \infty$, $\underline{IC}^{(t)}$ converges to $IC^{(t)}$ uniformly for all $t = 0, \cdots, K$, and hence if $\underline{IC}^{(t)} > 0$ for all $t \leq K$, then $IC^{(t)} > 0$ for all $t \leq K$ if K is sufficiently large.

Proposition 1.4. *For both high- and low-turnover equilibria, for any $\epsilon > 0$ there exists K for Algorithm 1.1 such that $|\underline{IC}^{(t)} - IC^{(t)}| < \epsilon$ for all $t = 0, \cdots, K$.*

The key idea of the proof is as follows. By the remark of Proposition 1.3, $\underline{IC}^{(K)}$ is in an ϵ-neiborhood of the true $IC^{(K)}$ for sufficiently large K. Then, it is shown that this approximation error does not increase as t decreases through the backward induction. Hence, $\underline{IC}^{(t)}$ is in the ϵ-neiborhood for all $t = 0, \cdots, K$.

Now, I describe the example. Below, incentive constraints mean what are computed by Algorithm 1.1, $\underline{IC}^{(t)}$. I take $v_L = 0.8$, $v_H = 1$, $\pi_L = 0.5$, $\pi_H = 0.5$, $d = 0.5$, $l = 0.5$, and $\phi = 0.5$, and $\delta = \rho = 0.98$. Parameters are taken for both steady states to exist, and the incentive constraints is satisfied with strict inequality at either steady state.

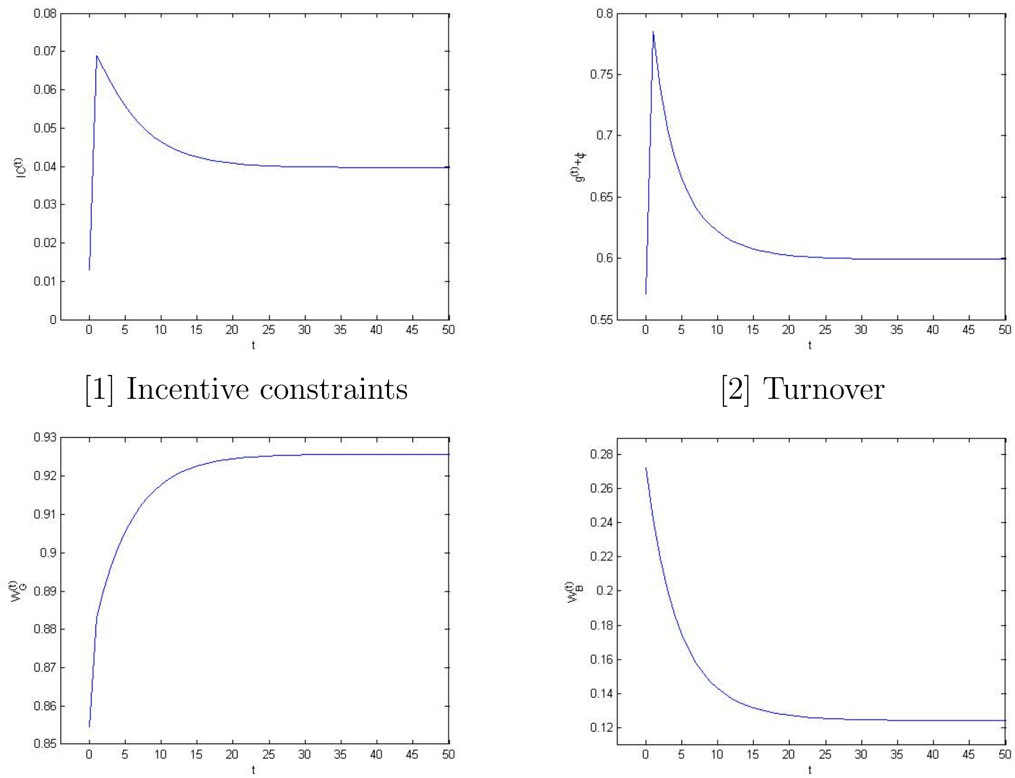
First, I take the low-turnover steady state as an initial state. Suppose that every player plays the high-turnover strategy from date 0 onward (think of this as a strategy change at date 0). I take $K = 500$ and report the first 50 dates[4].

Figure 1.1 shows the transition from the initial state to the high-turnover steady state. As is in (the remark of) Proposition 1.3, the sequence of the incentive constraint globally converges to the steady state value, and it is satisfied with strict inequality at the (high-turnover) steady state. Hence, we know that the incentive constraint is satisfied for sufficiently large t. Also, as is plotted in the figure, it is satisfied for small t too (see in Figure 1.1 [1] that $\underline{IC}^{(t)} > 0$ for all t). The incentive constraint along this transition is the one at a high-suitability match. It is relatively tight at date 0 (compared to large t) because at date 0, good players expect that there will be large inflow of good players into the unmatched pool in the next date and hence the cost of deviating at a high-suitability match today is relatively low.

In Figure 1.1 [2], the turnover rate $g^{(t)} + \phi$, defined as the total population in the unmatched pool at the beginning of each date, starts from the low-turnover steady state value 0.5707 at date 0, jumps up to 0.7854 at date 1, and converges to the high-turnover steady state value 0.5990 (notice that this is higher than the starting value, 0.5707). The value jumps up at date 1 because at the low-turnover steady state there are some low-suitability matches between good players and these matches all endogenously dissolve immediately after players start to play the high-turnover strategy. Hence, there are many players coming back to the unmatched pool at date 1, but over time these good players will be absorbed into high-suitability matches.

Denote the average expected discounted equilibrium payoff from date t onward, normalized by $(1 - \delta\rho)$, to a good player and a bad player as $W_G^{(t)}$ and $W_B^{(t)}$

[4] I did the same computation for $K = 10000$. For the two cases $K = 500$ and $K = 10000$, the differences of values of computed incentive constraints for the first 500 dates are all less than 10^{-4} for the example below.



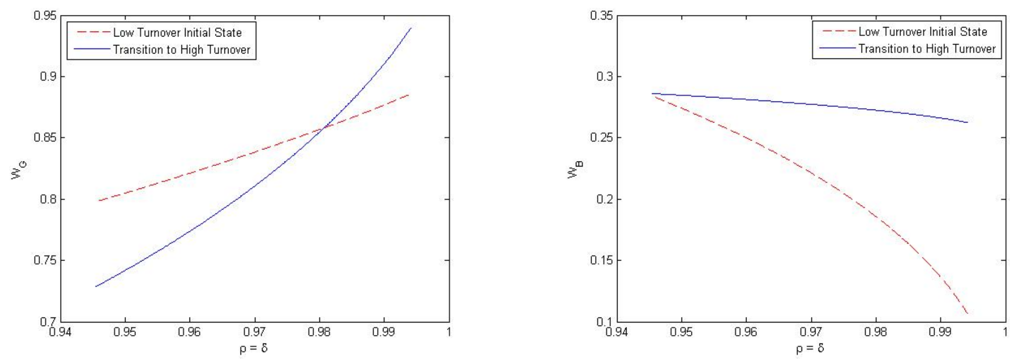
[1] Incentive constraints [2] Turnover

[3] Payoff to good player [4] Payoff to bad player

Figure 1.1. Low Turnover to High Turnover

respectively—*average* is taken over players in the different types of matches (see Appendices A.1 and A.2 for the explicit expressions of $W_G^{(t)}$ and $W_B^{(t)}$). Figure 1.1 [3] and [4] plot $W_G^{(t)}$ and $W_B^{(t)}$, respectively. In Figure 1.1 [3], $W_G^{(t)}$ is increasing over time because at date 1, the turnover rate increases and hence the fraction of good players in the unmatched pool increases, and furthermore because over time, the average payoff in matches between two good players increases by dissolved good players at date 1 being absorbed in high-suitability matches from date 2 onward. In Figure 1.1 [4], $W_B^{(t)}$ is decreasing over time because after the change of the strategy profile, good players start to cooperate only in high-suitability matches and hence bad players cannot exploit good players' cooperative behavior when the match suitability is low, and furthermore because the fraction of good players decreases over time.

To study the effect of the strategy change on the equilibrium payoffs, I compare the expected discounted equilibrium payoff from the low-turnover steady state



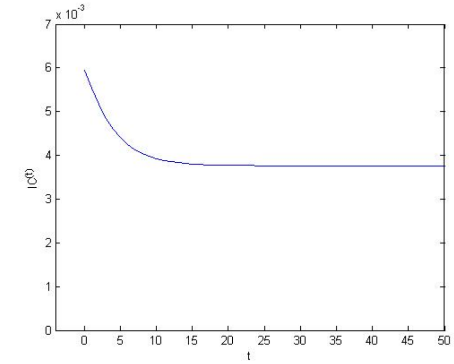
[1] [2]

Figure 1.2. Intial Low Turnover State versus Transition to High Turnover

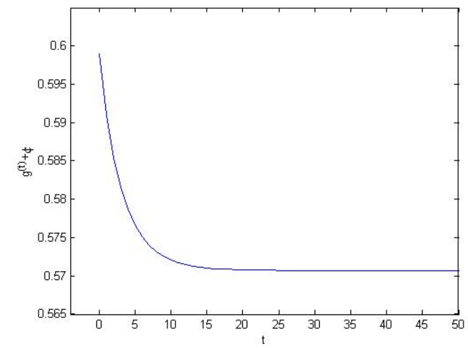
with that from the transition towards the high-turnover steady state at date 0, by changing model parameters. In particular, I do this by changing δ and ρ with restriction $\delta = \rho$ for the region that the low-turnover steady state and the transition towards the high-turnover steady state both exist. Figure 1.2 [1] and [2] plot the average expected discounted payoff from the low-turnover initial state and the one from a transition from the initial state to the high-turnover steady state, for a good player and a bad player respectively. As is shown in Figure 1.2 [1], the ranking between the two average expected discounted payoffs depends on the discount factor. It is intuitive that the ranking depends on the discount factor. If good players in low-suitability matches against a good player abandone the current relationships, then they will have to give up a constant flow of the low suitability and also it will take them some time to find a high-suitability match against a good player. Once they find a high-suitability match against a good player, however, they will be able to enjoy a constant flow of the high suitability over the future.

Next, I take—for the same parameters as in Figure 1.1—the high-turnover steady state as an initial state. Suppose that every player plays the low-turnover strategy from date 0 onward. Again, I take $K = 500$ and report first 50 dates (except for $W_G^{(t)}$ because its convergence to the steady state value is relatively slow).

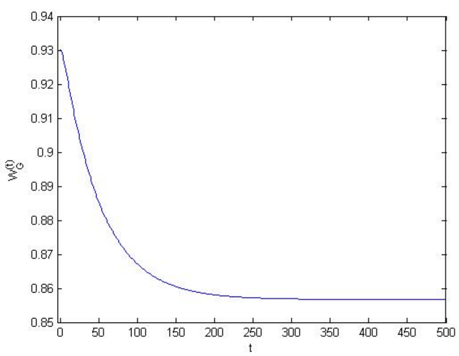
Figure 1.3 shows the transition from the initial state to the low-turnover steady state. The turnover rate $g^{(t)} + \phi$ now starts from the high-turnover steady state value 0.5990 and gradually decreases to the low-turnover steady state value 0.5707.



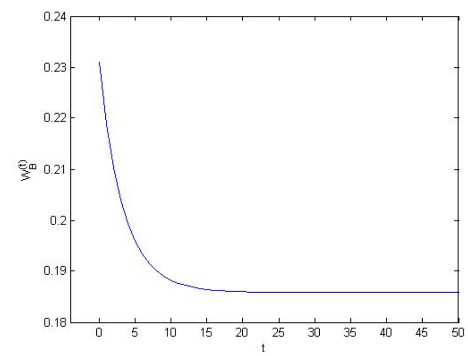
[1] Incentive constraints



[2] Turnover



[3] Payoff to good player

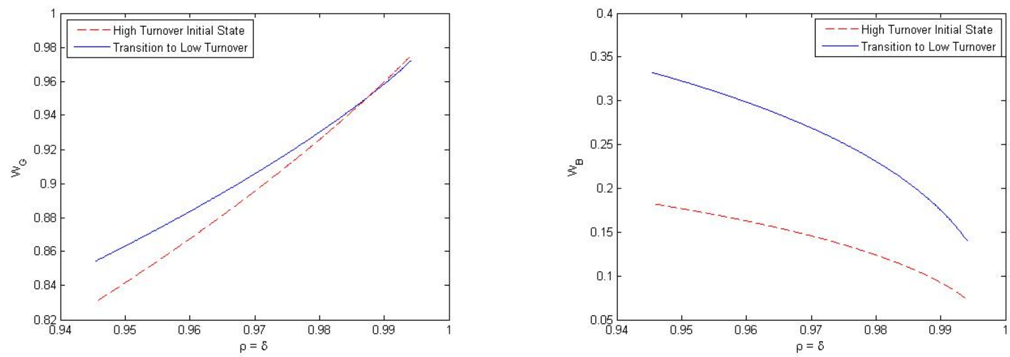


[4] Payoff to bad player

Figure 1.3. High Turnover to Low Turnover

At the high-turnover steady state, there is no low-suitability match. At date 0, some matches between a good player and a bad player dissolve endogenously. As time goes, some of those separated good players will be absorbed in low-suitability matches, and hence the population in the unmatched pool will decrease. For this example, payoffs to both a good player and a bad player monotonically decease as the turnover rate decreases.

Again, I compare the expected discounted equilibrium payoff from the high-turnover steady state with that from the transition towards the low-turnover steady state at date 0, by $\delta = \rho$ for the region that the high-turnover steady state and the transition towards the low-turnover steady state both exist. Figure 1.4 plots the average expected discounted payoff from the high-turnover initial state and the one from a transition from the initial state to the low-turnover steady state.



[1] [2]

Figure 1.4. Initial High Turnover State versus Transition to Low Turnover

1.5 Concluding Remarks

I have provided an explanation for a sharp increase and at the same time a gradual decrease of the divorce rate in the U.S. after the divorce law reform during 1960s. The numerical example I provided is not artificially chosen. The transition from the low-turnover steady state to the high-turnover exhibits an overshoot in the turnover rate. It is because low-suitability matches of the low-turnover steady state endogenously dissolve at the first date and some of the separated good players are absorbed in new long-term relationships from the next date on. This mechanism easily carries over to other parameters and initial conditions for which the economy starts out with some measure of low-suitability matches.

Chapter 2

A Note on ‘Money-Is-Memory’: A Counterexample

2.1 Introduction

Kocherlakota [22] studies the role of money in a class of economies in which people are matched into groups and then trade. In such economies, *memory* means that there is knowledge of past actions of direct and indirect partners, while *money and no memory* means that players observe none of the past actions of other players and there is a fixed supply of fiat money, an intrinsically useless object. For that class, he shows that any implementable allocation with money and no memory is implementable with memory. The class of environments he studies does not contain information asymmetries. Moreover, he conjectures that the result would not extend to an environment with *persistent asymmetries of information.* We confirm his conjecture. We provide an example with persistent information asymmetries about a player's type and show that there is an allocation that is implementable with money and no memory, but not with memory.

In this paper, as in Kocherlakota [22], money conveys coarser information about past actions than memory does. As is well-known in the reputation literature (see for example, Fudenberg and Levine [15] and the survey in Mailath and Samuelson [29]), in the presence of persistent incomplete information an equilibrium outcome under no memory may not be an equilibrium under memory. We show that money, by providing an intermediate level of information between memory and no memory, gives rise to an equilibrium outcome that cannot arise under either memory or no-memory.[1]

Our counter-example is related to analysis in Fudenberg and Levine [15] with a long-lived player who has private information about her permanent type and a sequence of short-lived players. The long-lived player is either a strategic type or a behavioral type. They show that the set of equilibria shrinks relative to the case where there is no behavioral type, because the strategic type has an option to mimic the behavioral type. We adopt those features of Fudenberg and Levine [15], and add an absence-of-double-coincidence feature so that money plays a role when our economy has no memory.

Each date is divided into two parts, day and night. In the day part, the short-lived player can produce, but has no consumption opportunity, while at night he has a potential consumption opportunity, but cannot produce. The long-lived player is essentially in the opposite situation. In the economy with memory, as in Fudenberg and Levine [15], the strategic long-lived player wants to mimic the behavioral type. In an economy with money and no memory, when a new-born short-lived player observes that the long-lived player has money, he cannot distinguish between two histories: (i) trade has occurred in the past using money; (ii) trade has not occurred. This feature of money prevents the strategic long-lived player from mimicking the behavioral type.

Two remarks are in order about our result. First, Kocherlakota [22] provides an example which shows that his main result does not hold if memory is replaced by *perfect monitoring*—knowledge of the past actions of *all* players (not only direct and indirect trading partners). Our counter-example is not of that type because memory and perfect monitoring coincide in it. Second, there is a different conjecture which states that *imperfect monitoring is necessary for money to play a role* (see, for example, Wallace [40]). This conjecture says that in an economy with perfect monitoring, the presence of money does not enlarge the set of implementable allocations. Our economy is *not* a counter-example to that claim.

[1] Kahn et al. [19] also study a positive role of money related to the coarsening of information. However, in their setting there is no persistent incomplete information and some players can choose what to reveal.

Table 2.1. Payoffs

	joint in day and solo at night		solo in day and solo at night	
	long-lived	short-lived	long-lived	short-lived
day	$\nu - \gamma$	$-c_j$	ν	$-c_s$ $(< -c_j)$
night	$-\kappa$	u	$-\kappa$	u
net	$\nu - \gamma - \kappa > 0$	$u - c_j > 0$	$\nu - \kappa$	$u - c_s > 0$

2.2 Model

Time is discrete, lasts forever, and is indexed by $t = 0, 1, 2, \cdots$. There is one long-lived ("she") player and a sequence of short-lived ("he") players, each of whom lives for one date, a day followed by a night. The long-lived player, who has discount factor $\beta \in (0, 1)$, is one of two types, *strategic* or *behavioral*. The long-lived player's type is permanent and privately known to her. Her type is drawn at the beginning of date 0. She is the strategic type with probability $1 - \pi$ and is the behavioral type with probability π.

In each of day and night, there is one indivisible, perishable, produced good. In each day (resp. night), only the long- (resp. short-) lived player can consume the good. In day, there are three technologies: joint production by both players, solo production by the short-lived player, and non-production.[2] At night, there are two technologies: solo production by the long-lived player and non-production. With the cost and utility of non-production normalized to be zero at day and at night, the payoffs are given in Table 1, where all the parameters in the table are positive. Production is costly and there are gains from trade for both players at each date under either day-time technology. However, conditional on production at night, the long-lived player prefers solo production during the day, while the short-lived player prefers joint production.

We also assume that

$$\pi \leq \frac{c_s - c_j}{u - c_j}$$

[2] Here, in the joint production the long-lived player helps the short-lived player in production by incurring a cost. Solo production in day by the long-lived player is not possible.

While the strategic type maximizes her lifetime payoff multiplied by $(1-\beta)$ (which is just normalization, following a convention of the literature on repeated games and reputation), we assume that the behavioral type (i) never agrees to joint production, and (ii) produces the good in the night of a date whenever she consumed the good in the day of the date.

2.3 Mechanisms

We take into account all possible ways of trading. Thus, both mechanisms with one-stage commitment and those without commitment are allowed. As is standard, a mechanism is defined to be a sequence of games that can depend on the history.

Mechanisms with No Commitment

The day-time stage of a mechanism with no commitment has three sub-stages.

1. A lottery over use of the technologies is proposed.

2. (i) If joint production is realized, then simultaneously both players choose from $\{yes, no\}$. If both players choose yes, then joint production is executed. If at least one player chooses no, then joint production is rejected. (ii) If solo or non-production is realized, then only the short-lived player chooses from $\{yes, no\}$.[3]

3. If any rejection occurs, then the short-lived player chooses from $\{solo, non\}$. If *solo* is chosen, then it is executed.

The night-time stage proceeds in the same way except that there is no joint production and *short-lived* is replaced by *long-lived* in the above description.

Mechanisms with One-date Commitment

The day-time stage of a mechanism with one-date commitment proceeds as follows.

1. A lottery over use of the technologies is proposed.

[3]In general, when solo or non-production is realized, the long-lived player could also reply yes or no. In that case, however, there is trivially a dominant strategy for the long-lived player. Hence, our formulation is without loss of generality.

2. (i) If joint production is realized, then simultaneously the long-lived player chooses from $\{yes, no\}$ and the short-lived player chooses from $\{yes, no\} \times \{solo, non\}$. Again, if both players choose yes then joint production is executed, while if at least one player chooses no then joint production is rejected. (ii) If solo or non-production is realized, then only the short-lived player chooses from $\{yes, no\} \times \{solo, non\}$.

3. If any rejection occurs, then the short-lived player does what he has chosen from the set $\{solo, non\}$.

Again, the night-time stage proceeds in the same way except that there is no joint production and *short-lived* is replaced by *long-lived* in the above description.

In mechanisms with one-stage commitment, the second element of the Cartesian product $\{yes, no\} \times \{solo, non\}$ represents the decision conditional on any rejection. That is, before knowing whether rejection occurs, the short-lived player commits to whether he executes solo or non-production if rejection occurs. For either case, each short-lived player is free to execute solo production during the day-time.

Note that the class of one-date commitment mechanisms includes take-it-or-leave-it offers by both the long and short-lived players. We will use this fact in the proof with money. Note also that, because each player can always reply no, any mechanism we consider is sequentially individually rational for each player.

Memory

Following Kocherlakota [22], we mean by memory complete observation of the history. A history consists of (i) previous planner proposals, (ii) previous lottery realizations, (iii) previous actions including whether people have executed solo or non-production when rejections have occured. A planner can choose a proposal that depends on the history up to that time, though no commitment between day and night stages is possible.

A short-lived player's strategy consists of two things: (i) a reply to each day-time proposal and (ii) a day-time choice after any rejection. Each short-lived player has a belief about the long-lived player's type. The long-lived player's strategy consists of three things: (i) a reply to each day-time proposal, (ii) a reply to each night-time proposal, and (iii) a night-time choice after any rejection.

Money

Fiat money is an indivisible, durable, intrinsically useless object. We give the long-lived player one unit of money at the beginning of the initial date. Money is disposable. In particular, if a short-lived player holds money at the end of a night, he simply disposes of the money when he dies. We assume that people publicly observe who has money.

Because there is no memory, the short-lived player at a given date only knows money holdings. The long-lived player remembers what people have done in the past. A public history with money at some moment of a date consists only of (i) who is the current money holder and (ii) what has happened in the earlier stages of the date. Also, players now have strategies about money disposal, too. At any point in time, each player chooses from $\{dispose, not\}$.

Equilibrium and Implementability

A profile of strategies and beliefs is a *(perfect Bayesian) equilibrium for a mechanism* if at any time and any stage and for any history, each player maximizes his or her objective given the other's strategy and belief, and the belief is updated using Bayes' rule whenever possible. We say that an outcome is *implementable* if for some mechanism, there exists an equilibrium which achieves the outcome on the equilibrium path.

2.4 Result

Our result involves a particular outcome called a *cooperative outcome*.[4] A cooperative outcome is defined to be joint production in every day and solo production (by the long-lived player) in every night whenever the long-lived player is strategic. Notice that if the long-lived player is the behavioral type, she, by assumption, never accepts joint production. We have

Theorem 2.1. *Suppose the strategic long-lived player is sufficiently patient. Then, a cooperative outcome is implementable with money, but not with memory.*

Proof. There are two parts of the theorem. We first show that a cooperative

[4]Because the behavioral-type long-lived player does not have a utility function, direct welfare comparison between a cooperative outcome and a non-cooperative outcome, would not be plausible. For our purpose, however, it suffices to show that there is *an* implementable outcome with money that is not implementable with memory.

outcome is implementable with money. The proof of that part uses a commitment proposal, one in which, in effect, the short-lived player makes a take-it-or-leave-it offer involving joint production. The proof of the second part shows that if there is memory, then the strategic long-lived player wants to mimic the behavioral type if joint production is realized. This is profitable for her (if she is patient enough) because all future short-lived players will, then, believe that the long-lived player is the behavioral type and will want to execute solo production. Here are the details for each part.

Implementability with Money

This part of the proof uses a commitment proposal, one in which, in effect, the short-lived player makes a take-it-or-leave-it offer involving joint production. We first describe the mechanism and then a candidate for equilibrium strategies. Then, we show that the strategies are an equilibrium.

(i) The mechansim. In day, the short-lived player proposes joint production and a monetary transfer if the long-lived player has money. The short-lived player proposes non-production and no monetary transfer if the long-lived player does not have money. At night, the long-lived player proposes solo production and monetary transfer if the short-lived player has money. The long-lived player proposes non-production and no monetary transfer if the short-lived player does not have money.

(ii) Candidate strategies. In day, (d.i) the strategic long-lived player replies yes (the short-lived player also replies yes) if a short-lived player proposes joint production and monetary transfer. By assumption, the behavioral type replies no to this proposal, and neither production nor monetary transfer takes place, (d.ii) the strategic long-lived player replies yes if a short-lived player proposes solo production and monetary transfer, and (d.iii) the long-lived player replies no if the short-lived player proposes non-production. At night, (n.i) a short-lived player replies yes if the long-lived player proposes solo production and monetary transfer, and (n.ii) a short-lived player disposes of money if the long-lived player makes any proposal involving non-production. Otherwise he gives money back to the long-lived player.

Now we show the optimality of the candidate strategy profile.

Short-lived player: A short-lived player cannot distinguish between the following two histories in both of which the long-lived player has money at the beginning of a date: (i) the long-lived player accepted joint production in day and produced good at night, at some point in time, and (ii) the long-lived player rejected joint production, at some point in time (and of course, any combination of these). Therefore, if the long-lived player has money, a short-lived player has the prior belief $(1-\pi, \pi)$ about her type.

Suppose that a short-lived player proposes joint production in day (which is supposed to happen on the equilibrium path). If the long-lived player is the strategic type, then the proposal is accepted in day and solo production is executed at night. In that case, the short-lived player gets $u - c_j$. If the long-lived player is the behavioral type, then the proposal is rejected in day and non-production is executed at night. In that case, the short-lived player gets 0. Therefore, given his belief, the short-lived player's expected payoff is $(1-\pi)(u-c_j)$.

Suppose that the short-lived player proposes solo production in day. Then, either type of long-lived player accepts this proposal and the short-lived player's expected payoff is given by $u-c_s$. If the short-lived player proposes non-production in day, then his expected payoff is 0.

Because $\pi \leq (c_s-c_j)/(u-c_j)$ by assumption, $(1-\pi)(u-c_j) \geq u-c_s$. Therefore, neither proposing non-production (which gives 0) nor proposing solo production (which gives $u-c_s > 0$) is a profitable deviation for the short-lived player.

Notice also that, at night, short-lived players are indifferent between disposing of money and giving it to the long-lived player, so any action on monetary transfer (on and off path) is optimal.

Long-lived strategic player: Suppose that the long-lived player accepts joint production (which is supposed to happen on the equilibrium path). In that case, the stage-game payoff to the long-lived player is given by $\nu-\gamma-\kappa$. And, she starts the next date with money. Suppose that the long-lived player rejects joint production. In that case, non-production is executed in day and the short-lived player cannot buy the good at night. In that case, the stage-game payoff to the long-lived player is given by 0, and she starts the next date with money (and thus, the future payoff is the same as the above case). So, this is not a profitable deviation.

Next consider her night-time incentives. Suppose that the long-lived player proposes solo production at night (which is supposed to happen on the equilibrium path). In that case, she incurs cost κ for production today, and she gets $\nu - \gamma - \kappa$

from the next date on. So, the continuation payoff to her is given by $-(1-\beta)\kappa + \beta(\nu-\gamma-\kappa)$. Suppose that the long-lived player proposes non-production at night. In that case, she evades cost κ, but instead, the short-lived player disposes of money and the stage-game payoff to the long-lived player is 0 from the next date on. So, the continuation payoff to her is given by 0. For β sufficiently close to one, $-(1-\beta)\kappa + \beta(\nu - \gamma - \kappa) \geq 0$, which implies that non-production at night is not a profitable deviation for the long-lived player.

Non-implementability with Memory

Suppose, by way of contradiction, that a cooperative outcome is implementable. That is, suppose that for some mechanism there is an equilibrium which supports the cooperative outcome. It will be shown that the strategic long-lived player has a profitable deviation irrespective of the mechanism.

If the strategic type deviates by rejecting joint production at $t = 0$, then all future short-lived players believe that they are facing the behavioral type—because only the behavioral type is supposed to reject joint production along the equilibrium path. Given that belief, the best response of all future short-lived players is to execute solo production whether or not one-date commitment is possible. Thus, the strategic type gets $\nu - \kappa$ from date 1 on regardless of the mechanism. The date-0 payoff to the strategic type is bounded from below by $-\kappa$ irrespective of the mechanism. This is because the worst case scenario for the long-lived player is to get no good in day and produce at night which would give her $-\kappa$. Therefore, the continuation payoff to the strategic type from the deviation is bounded below by $-(1-\beta)\kappa + \beta(\nu - \kappa)$, while that from not deviating is given by $\nu - \gamma - \kappa$. Because $\gamma > 0$, the deviation is profitable for all β sufficiently close to one. $\square$

2.5 Concluding remarks

In our example, incomplete information plays a crucial role. If, instead, the long-lived player's type were observable, then Kocherlakota's money-is-memory result would hold. In particular, a cooperative outcome would be implementable with memory, because, with complete information, the strategic long-lived player could not mimic the behavioral type any longer.

The class of mechanisms we study includes take-it-or-leave-it offers by the short-lived players—which of course are the best mechanism for the short-lived players. This assumption plays a crucial role in showing that the cooperative outcome is implementable with money. For example, if one simply imposes that the *long-lived* player makes a take-it-or-leave-it offer at day stage, then, even with money, the strategic long-lived player can mimic the behavioral type by proposing solo production.

Our result also holds when goods are divisible. Suppose the long-lived player chooses an effort level from $[0, \infty)$ and the short-lived players chooses a production level from $[0, \infty)$ and that the higher the costly effort level of the long-lived player, the lower is the short-lived players' cost of production. The behavioral type long-lived player is assumed to always choose zero effort level, an analogue of the assumption that the behavioral type always rejects joint production.

It also holds when there are many long-lived players and many short-lived players at a date and when they are randomly paired off with each other. It is crucial for our result to have a long-lived player and a sequence of short-lived players à la Fudenberg and Levine [15]. Thus, we take the minimal element of modeling, one long-lived player and one short-lived player at a date, to obtain our main result. Note however that, in such an extention with many agents, it is also crucial to assume that the *same* two people interact for two sub-dates and production technologies in day and night are like the ones that we studied above.

As a final remark, the money-is-memory claim is different from the claim which says that imperfect monitoring is necessary for money to be essential (see, for example, Wallace [40]). Our economy is *not* a counterexample to that essentiality claim. A proof of it would take an economy with memory *and* money and show that the presence of money does not enlarge the set of implementable allocations. In our economy, if there is both memory and money, then no cooperative outcome is implementable. The proof is the same as we gave above for the memory part of Theorem 1.

Chapter 3

The Necessity of Imperfect Monitoring for Essential Money

3.1 Introduction

Previous work by Ostroy [31], Townsend [38], Kocherlakota [22], and Wallace [40] suggests, but does not prove, that imperfect monitoring is necessary for (fiat) money to be essential. This paper provides a proof for a class of environments that includes those studied by Kocherlakota [22], and that also allows for incomplete information and divisible money. As is standard, money is said to be *essential* (in Hahn [17]'s terminology) if some good allocations are achieved with it but not without it. Also, as is standard, perfect monitoring means common knowledge of past actions and imperfect monitoring means anything else. The logic of the proof is straightforward. Consider an economy with perfect monitoring and any equilibrium allocation for a mechanism in that economy—one that could have trades using money. Then show that the allocation can be achieved as an equilibrium allocation for a non-monetary mechanism—one that does not have trades using money.

The necessity proposition in this paper should not be confused with Kocherlakota's [22] widely cited money-is-memory proposition. First,
the money-is-memory proposition compares two economies: one with money and no monitoring and the other with something close to perfect monitoring and no money. The necessity proposition compares monetary and non-monetary mecha-

nisms in an economy with perfect monitoring. Second, as Kocherlakota [22] suggests and as Awaya and Fukai [9] verify, the money-is-memory proposition does not hold when there is incomplete information. I show that the necessity proposition holds even if there is incomplete information. Third, and, perhaps, most importantly, the necessity result provides a guide for the construction of models in which money is essential: any such model must include imperfect monitoring. The money-is-memory propostion does not provide such a guide.

Of course, even the necessity proposition provides only a partial guide. The literature does not contain general sufficient conditions for essentiality of money. Indeed, it has been shown that seemingly strong conditions are not sufficient to give rise to a role for money. Araujo and Camargo [5] show that the first best can be achieved without using money in a class of environments in which trade occurs in a decentralized market with pairwise meetings and in which monitoring is limited to only the last period action of the partner and the preferences of the last period partner of that partner. Awaya and Fukai [8] show inessentiality of money in an environment in which no past action is observed, but in which agents use locally observed sunspots as a coordination device.

The rest of the paper is organized as follows. I describe the class of environments in Section 3.2. I define mechanisms in Section 3.3. Section 3.4 is devoted to the formal statement and the proof of the necessity claim. Section 3.5 discusses the robustness of the proof.

3.2 Environments

Time is discrete, lasts forever, and is indexed by $t = 1, 2, \cdots$. An environment consists of a finite set of players at each date, sets of unobservable types of the players, their endowments, utility functions, production sets, and trade meetings.

Formally, Let $I \subseteq [0, 1]$. Let $I_t \subseteq I$ be a finite set, which represents the set of players who are alive at date t. Each player has private information about her preference type. For player $i \in I_t$, her date-t type θ_{it} is drawn from a finite set Θ_{it} according to some distribution.

At each date, there is a finite number l of perishable consumption goods. At date t, player i is endowed with a vector of goods $e_{it} \in \mathbb{R}^l_+$ and has a production set $Y_{it} \subseteq \mathbb{R}^l$. If player i consumes $c_{it} \in \mathbb{R}^l_+$ and if her type is θ_{it}, then her date-t utility is $u_{it}(c_{it}, \theta_{it})$. Each player maximizes the discounted expected utility with discount factor δ. An allocation is accomplished through a mechanism, which will be described in the next section, and people cannot commit to future actions.

At each date t, players are physically separated, according to a stochastic process $\{z_t\}_{t=1}^{\infty}$, into trade groups in which they can trade with players in the same group. Each z_t is a partition of I_t and is drawn from the set of all partitions of I_t according to some distribution.[1] Each element of such a partition represents one trade group (or a group of players) and the players in a trade group cannot trade with players in any other trade group.

This class of environments, of course, includes environments with complete information by taking sets of types as arbitrary singletons. Examples of environments that fit in this class are Araujo [4], a finite-agents version of Trejos and Wright [39], overlapping generations models, and Townsend [38].

Money is intrinsically useless and durable. No player can produce money and everyone can freely dispose of it. The set of money holdings is $B \subseteq \mathbb{R}_+$. There is no durable object other than money. An initial money distribution, denoted $(m_{i0})_{i \in I_1}$, is chosen as a part of a mechanism. Let m_{it} denote player i's money holdings at the beginning of date $t+1$.

3.3 Mechanisms and Equilibrium

Under perfect monitoring, a public history at date t is given by past matchings, past actions of all players, and the current money holdings. A mechanism consists of initial money holdings and mappings from each public history to an action set and a feasible allocation for each player. Formally, let $z^t = (z_s)_{s=1}^t$ be the past matchings up through date t, and $m_t = (m_{it})_{i \in I_t}$ be money holdings of all players at date t. Also, let a_{it} be player i's action at date t and $a^t = (a_{is})_{i \in I_s, 1 \leq s \leq t}$ be the past actions of all players up through date t. Then, a mechanism consists of initial money holdings m_{i0}, action sets $A_{it}(z^t, a^{t-1}, m_{t-1})$, feasible goods allocation $c_{it}(z^t, a^t, m_{t-1})$, and feasible money allocation $m_{it}(z^t, a^t, m_{t-1})$. Feasibility means physically possible reallocation of goods and money within each trade

[1] As is standard, a partition z_t is a set of non-empty disjoint subsets of I_t whose union is I_t.

group, namely, for any t, at any public history, and for any trade group $J \in z_t$,

$$\sum_{i \in J} c_{it}(z^t, a^t, m_{t-1}) - \sum_{i \in J} e_{it} \in \sum_{i \in J} Y_{it},$$
$$\sum_{i \in J} m_{it}(z^t, a^t, m_{t-1}) \leq \sum_{i \in J} m_{i,t-1}, \text{ and}$$
$$m_{it}(z^t, a^t, m_{t-1}) \in B \text{ for all } i \in I_t$$

As a special class of mechanisms, I define non-monetary mechanisms as ones which always choose previous money holdings as money allocations.

Definition 3.1. A *non-monetary mechanism* is a mechanism such that for any t, at any public history, for any $i \in I_t$, and for any action $a_{it} \in A_{it}(z^t, a^{t-1}, m_{t-1})$,

$$m_{it}(z^t, a^t, m_{t-1}) = m_{i,t-1}.$$

Under a non-monetary mechanism, no player can change her money holdings. Note that there are various ways to define non-monetary mechanisms and the definition above is just one way. For example, defining non-monetary mechanisms as ones that always choose zero money holdings is another way and is equivalent to defining as above.

Equilibrium. Given a mechanism, a (pure) strategy σ_i of player i is a mapping from the set of all information sets of her into her action set.[2] An information set of player i at date t is induced by $(z^t, a^{t-1}, m_{t-1}, (\theta_{is})_{s=1}^t)$. Note that, while the social planner only knows a public history, each player can depend her strategy on her private history including her realized types $(\theta_{is})_{s=1}^t$. A belief μ_i of player i is a probability distribution over the set $\prod_{j \in I_t \setminus \{i\}} \Theta_{jt}$ of types of the other players.

Definition 3.2. A *(perfect Bayesian) equilibrium (in pure strategies) for a mechanism* is a strategy profile $\sigma = (\sigma_i)_i$ and a belief profile $\mu = (\mu_i)_i$ such that

1. Given the belief profile, for any t, for any $i \in I_t$, and at every information set of i's, player i's continuation strategy is a best response to all other players' continuation strategies at current information sets of theirs.

[2] Let H_i^t denote the set of private histories of player i at date t, including her private types. Then, as is standard, the domain of a strategy of the player is $\bigcup_{t=1}^{\infty} H_i^t$.

2. Given the strategy profile and all other players' beliefs, for any t and for any $i \in I_t$, player i's belief is updated by Bayes rule whenever possible.

Because money is intrinsically useless, we are only interested in goods allocations. Given a sequence of exogenous shocks (z^t, θ^t), where $\theta^t = (\theta_{is})_{i \in I_s, 1 \leq s \leq t}$, for an equilibrium (σ, μ) for a mechanism, an *equilibrium outcome* is a goods allocation achieved on the equilibrium path induced by the strategy profile σ.

3.4 Result

The main result, the necessity proposition, is stated as follows:

Proposition 3.1. *Suppose monitoring is perfect. Then, any equilibrium outcome for a mechanism is also an equilibrium outcome for some non-monetary mechanism.*

Thus, for some mechanism to achieve an equilibrium outcome that cannot be achieved by a non-monetary mechanism, some imperfect monitoring is necessary.

To illustrate the logic of the proof, let us consider a mechanism that involves money exchanges in a version of Araujo [4] and duplicate any equilibrium outcome for it by a non-monetary mechanism under perfect monitoring.

Here is the environment. Time is discrete and lasts forever. There are N players, each of whom maximizes her expected discounted utility with discount factor $\beta \in (0, 1)$. Assume N is even. There are $K \geq 3$ indivisible perishable goods. Assume $N/K \in \mathbb{N}$. There are random pairwise meetings at each date, in which production and consumption occur. In a meeting, with probability $1/K$, a player can consume but cannot produce (call her a consumer). With probability $1/K$, she can produce but cannot consume (call her a producer). With probability $1 - 2/K$, she can do neither. Because a producer cannot consume, trade can potentially occur only when a consumer and a producer meet (call such meetings *trade meetings*).[3] A producer can produce (at most) one unit of a good at cost $c > 0$. A consumer derives utility $u > 0$ from consuming one unit of a good. There is no incomplete information. There is fixed supply of fiat money, and random M

[3] Araujo [4] allows producers to consume, and therefore barter exchanges can be possible. This makes no essential difference.

players ($M < N$) start with one unit of money at the initial period. The set of money holdings is $\{0, 1\}$.[4]

Consider the following mechanism that allows trades to depend on current money holdings (call this mechanism (a)): In any trade meeting, a consumer and a producer play a simultaneous move game. The action set of a consumer is $\{consume, not\ consume\}$. The action set of a producer is $\{produce, not\ produce\}$. The mechanism transfers a good from a producer to a consumer and money from a consumer to a producer if a consumer plays *consume* and has money and if a producer plays *produce*. Otherwise, no trade occurs.

Now, consider the following non-monetary mechanism (call this mechanism (b)): Given any history, any non-monetary mechanism has to assign $m_{it}(z^t, a^t, m_{t-1}) = m_{i0}$ for all t and all i. Assign a variable $w_{it} \in \{0, 1\}$ in exactly the same way as mechanism (a) does for its money allocation. For example, assign a player to $w_{i0} = 1$ if she receives money initially and $w_{i0} = 0$ if not. Assign $w_{it} = 1$ if a player produced a good at date t for a consumer and if $w_{i,t-1} = 0$. Then, transfer a good from a producer to a consumer in any trade meeting at date t if and only if the produce plays *produce* and the consumer is assigned to $w_{it} = 1$. Otherwise no trade occurs.

The key is that thanks to perfect monitoring, w_{it} can be reproduced from common knowledge among the players. For example, if either a consumer got money initially or she produced a good for someone at some point of time (say time $s' < s$) and if she has not played *consume* after then till now (for all $t \in \{s' + 1, \cdots, s-1\}$), we know that she is assigned to $w_{is} = 1$. In this way, mechanism (b) does not change incentives of the players and duplicates any equilibrium outcome for mechanism (a). Generalizing this idea gives a proof.

Proof. Suppose that there is an equilibrium outcome for a mechanism in an environment. That is, there exist some mechanism $(m^\star_{i0}, A^\star_{it}, c^\star_{it}, m^\star_{it})_{i,t}$ and some equilibrium $(\sigma^\star, \mu^\star)$ for the mechanism that generates the outcome on the equilibrium path.

Define functions w_{it} of the past matchings, the past actions, and initial money

[4]See Araujo [4] for its derivation and for the essentiality of money in this model when there is no public record keeping and the population is large enough relative to the discount factor.

holdings by, for any (z^1, a^1),

$$w_{i1}(z^1, a^1, m_0^\star) = m_{i1}^\star(z^1, a^1, m_0^\star)$$

and for any $t \geq 1$ and any (z^{t+1}, a^{t+1}),

$$w_{i,t+1}(z^{t+1}, a^{t+1}, m_0^\star) = m_{i,t+1}^\star(z^{t+1}, a^{t+1}, w_t(z^t, a^t, m_0^\star)),$$

where $w_t(z^t, a^t, m_0^\star) = (w_{it}(z^t, a^t, m_0^\star))_{i \in I}$.

Then, define the following non-monetary mechanism $(m_{i0}^\star, A_{it}^{\star\star}, c_{it}^{\star\star}, m_{it}^{\star\star})_{i,t}$. First, take the same initial money holdings $m_0^\star$. The mechanism has to assign $m_{it}^{\star\star}(z^t, a^t, m_{t-1}) = m_{i0}^\star$ at any history by definition. Define $A_{it}^{\star\star}$ by for any z^1,

$$A_{i1}^{\star\star}(z^1, m_0^\star) = A_{i1}^\star(z^1, m_0^\star)$$

and for any $t \geq 1$ and any (z^{t+1}, a^t),

$$A_{i,t+1}^{\star\star}(z^{t+1}, a^t, m_0^\star) = A_{i,t+1}^\star(z^{t+1}, a^t, w_t(z^t, a^t, m_0^\star)).$$

Here, $m_0^\star$ as an argument of $A_{i,t+1}^{\star\star}$ comes from the fact that players' money holdings have to be initial holdings $m_0^\star$ at any date and any history. Define $c_{it}^{\star\star}$ by for any (z^1, a^1),

$$c_{i1}^{\star\star}(z^1, a^1, m_0^\star) = c_{i1}^\star(z^1, a^1, m_0^\star)$$

and for any $t \geq 1$ and any (z^{t+1}, a^{t+1}),

$$c_{i,t+1}^{\star\star}(z^{t+1}, a^{t+1}, m_0^\star) = c_{i,t+1}^\star(z^{t+1}, a^{t+1}, w_t(z^t, a^t, m_0^\star)).$$

I show that $(\sigma^\star, \mu^\star)$ still constitutes an equilibrium under the non-monetary mechanism. Under perfect monitoring, the information set of player i induced by $\langle m_{i0}^\star, z^t, a^{t-1}, (\theta_{is})_{s=1}^t \rangle$ is the same as the information set induced by $\langle m_{i0}^\star, z^t, a^{t-1}, w_{t-1}(z^{t-1}, a^{t-1}), (\theta_{is})_{s=1}^t \rangle$. By construction, for any history, w_{it} coincides with money holdings under the original mechanim. Hence, $\langle m_{i0}^\star, z^t, a^{t-1}, (\theta_{is})_{s=1}^t \rangle$ induces the same information set that

$\langle m_{i0}^{\star}, z^t, a^{t-1}, m_{t-1}^{\star}, (\theta_{is})_{s=1}^{t} \rangle$ does under the original mechanism. By construction of $A_{i,t}^{\star\star}$, at any such information set the same set of actions is available to each player. Thus, every action taken at each information set under the original mechanism is still optimal under the constructed non-monetary mechanism. And, every player's belief is updated in the same way as under the original mechanism. Therefore, $(\sigma^{\star}, \mu^{\star})$ is an equilibrium for the constructed non-monetary mechanism. □

In the argument above, perfect monitoring is needed in order for players to be able to keep track of w_{it}. If no action is observable as an extreme example, then the construction of such w_{it} may not be possible. Most importantly, the information set induced by $(z^t, m_{t-1}, (\theta_{is})_{s=1}^{t})$ can be different from that induced by $(z^t, (\theta_{is})_{s=1}^{t})$ only. In this way, money can change the set of equilibria by changing information sets that players face.

The necessity proposition is about fundamental models of money in which a role of money does not rely on a particular way of trading. Some reduced-form models generate a role for money by assuming a particular mechanism. In the so-called money-in-utility models, in which utility depends on real money holdings, the necessity proposition does not hold because w_{it} is not a real object. In cash-in-advance models, on the other hand, the necessity proposition holds because a non-monetary mechanism can be constructed in which each player receives a non-autarkic allocation only when she accumulates a non-zero amount of w_{it}.

3.5 Robustness of the Proof

In this section, I provide several extensions in which the necessity proposition holds.

3.5.1 Policy Interventions

In section 3.2, I assumed that there is no pre- or post-trade monetary interventions (such as injection, withdrawal, and transfer beyond trade groups) by the social planner pre- nor post-trade. The necessity proposition holds when the social planner can intervene money holdings as a part of a mechanism. For example, consider a form of intervention in which money holdings become $\tau_t(m_{it}) = a_t + b_t m_{it}$ with

$a_t \in \mathbb{R}$ and $b_t > 0$, where m_{it} is post-trade money holdings before the intervention and $\tau_t(m_{it})$ is money holdings after the intervention. Then, define w_{it} in the proof now by for any (z^1, a^1),

$$w_{i1}(z^1, a^1, m_0^\star) = a_1 + b_1 m_{i1}^\star(z^1, a^1, m_0^\star)$$

and for any $t \geq 0$ and any (z^{t+1}, a^{t+1}),

$$w_{i,t+1}(z^{t+1}, a^{t+1}, m_0^\star) = a_{t+1} + b_{t+1} m_{i,t+1}^\star(z^{t+1}, a^{t+1}, w_t(z^t, a^t, m_0^\star)).$$

The rest of the proof goes through in the same way.

3.5.2 Equilibrium Concept

Key elements of the proof are that (1) given an identical sequence of actions and an identical sequence of exogenous realizations, each player reaches the effectively equivalent node of the entire game under the two compared mechanisms, (2) any such node belongs to the same information set under the two mechanisms, and (3) at each of such information sets, the same set of actions is available to each player (in this regard, it is only important that the sets of actions under the two mechanisms have the same cardinality because we can reinterpret the labels of the actions).

It is *not* important, however, how optimally actions are chosen and how beliefs are updated at each of such information sets. Hence, the proposition does not rely on a choice of equilibrium notion. Needless to say, we should adopt the same equilibrium notion under the two compared mechanisms for a valid comparison.

Chapter 4

Essentiality-of-Money Is Delicate

4.1 Introduction

In an influential paper, Kocherlakota [22] proposed a novel view of fiat money—money is memory.[1] Since then, the essentiality of money has come to one of the main concerns in monetary economic theory. The essentiality of money here means that some socially desirable outcomes can be achieved with money but cannot without it.[2] For it, a lack of knowledge of past actions among economic agents is necessary to prevent a folk theorem from holding (see Wallace [40]). This observation opens a natural question—how much and what kind of information do we need for money to be inessential? Some recent work is devoted to show how *little* information can be to achieve some good outcomes *without* money—to make money *inessential*. These papers include Araujo [4], Araujo et al. [6], and Araujo and Camargo [5]. All these papers, however, assume that agents in the economy get *some* information about past actions of other agents in certain ways.[3] In this paper, in contrast, we consider an environment in which there is *no* monitoring of histories of actions, and show that even in this case money is not necessarily essential—without money the first best can be achieved.

Our environment is a version of Bewley [10]. Time is discrete and lasts forever. There is a continuum of agents. At each date, a half of the agents are endowed

[1]See also Ostroy [31] and Townsend [38].

[2]Showing the essentiality of money formally is a tough task, because *all* the other ways of achieving allocations have to be considered (see Wallace [40]).

[3]We will elaborate this point further in the end of Introduction.

with perishable goods and the other half are not. The types are determined i.i.d. across time and are observable. Each agent has a strictly concave utility function from consumption, and hence consumption has to be smoothed out among the agents in a first best allocation. There is a continuum of meeting places, where agents meet and resources are reallocated.[4] There is a *guard* at the entrance of each meeting place. The guard can prohibit an agent to enter it if she goes without bringing her endowments. The guard in each meeting place confiscates and reallocates the goods. Agents are free to go to one of the meeting places at no cost. Agents have an option not to go anywhere but cannot go to two or more places at a date. Although each agent can observe her payoff—an outcome of a reallocation at a meeting place—no one can observe histories of actions of the agents—*no monitoring*. A twist is that, we assume that inside each meeting place, the participants to the meeting place observe a payoff irrelevant random variable—which we label a *local sunspot*. The draws are independent across the meeting places. Each agent can observe the variable if and only if she enters the meeting place. The observability of local sunspots is, of course, consistent with no monitoring of past actions of other agents. Because of no monitoring of histories of actions, if an endowed agent simply stays at home and eats her endowment, no one can detect the behavior. The central question in this paper is how we can deter them from such a non-cooperative behavior.

How do we achieve the first best *without* money? Everyone goes to the same meeting place at the first date. There, the participants—but only the participants– observe a common local sunspot. Agents use a draw of a local sunspot as a *co-ordination device*. In particular, at the second date, they go to the meeting place which the observed local sunspot at the first date specifies. From the next date on, they choose where to go in the same manner. If an endowed agent deviates (and eats her endowment at home), then she does not observe the local sunspot at the date. This means that she cannot know where all the other agents will be and hence she will lose all the trading partners in the future. This deviation is not profitable in the long run.

Three remarks are in order. First, each agent can *infer* the others' actions

[4]This structure is similar to Matsui and Shimizu [30]. See also Araujo et al. [6] and Lagos and Wright [25].

on the equilibrium path because strategies are common knowledge. This does not imply that each agent can observe the others' actions and change future behavior based on the information. In other words, the strategies we constructed is independent of the past actions.[5] Second, we assume that no past *actions* can be monitored by the agents in the economy. This is different from saying that *nothing* from the past can be remembered. In our equilibrium, agents have to remember the local sunspot of the last date (which is of course drawn independently from past actions of agents). If nothing from the past can be remembered, money becomes essential in our economy. Third, our result relies on complete information about the endowment types. If the endowment types are private information, money becomes essential in our economy, because an endowed agent eats at home and comes to a meeting place, claiming that she is a non-endowed agent.[6]

Araujo [4], Aliprantis et al. [3, 2], and Araujo et al. [6] consider community enforcement environments à la Kandori [20]—agents can only observe actions in their participating matches[7] but cannot observe actions of any other match. They construct *contagion equilibria* and show that money is inessential in them under certain conditions. The idea to support a good outcome without money is that a bad behavior spreads to the economy in equilibrium—if a producer does not produce, then the victim of the producer stops producing from then on, and through the process, the producer will be suffered in the long run. For this, it is important that each agent can get information about actions of some (although very limited) other agents.[8][9]

Araujo and Camargo [5] assume some very limited monitoring in a sense that

[5]This does not say that with additional information about past actions, strategies still constitute an equilibrium. In general, it is known that more information about past actions can shrink the equilibrium payoff set. See for example recent work by Sugaya and Wolitzky [36].

[6]A similar phenomenon occurs in many models of monetary economics. Araujo [4] considers a random matching model in which the roles of agents—producers and consumers—are determined stochastically. In his construction without money, it is a key that a consumer can recognize that her partner is a producer—otherwise producers could pretend to be consumers to save their production costs. See also Araujo and Camargo [5], and Araujo et al. [6].

[7]In Aliprantis et al. [3, 2] and Araujo et al. [6], these matches include a centralized meeting.

[8]For it, Araujo [4]—like Kandori [20], assumes a finite number of agents and so bad behavior of a single agent spreads to the economy within a finite period. Aliprantis et al. [3, 2] and Araujo et al. [6] assume some form of a centralized meeting where agents can observe actions of all other agents.

[9]These papers also claim that their environments have *no monitoring*, the definitions of monitoring are different.

is close to that of Takahashi [37]. That is, they consider a random matching environment with a continuum of agents and assume agents are heterogeneous in preference, and that each agent can observe only the last-date action of her partner and the preference of the last-date partner of her partner.[10] They show that even with this limited information, money is inessential in it.

In contrast to those papers, in our model agents get *no* information about past actions of *any* other agent, and money is still inessential in it.

4.2 The Model

Physical Environment. Time is discrete, lasts forever, and is indexed by $t = 1, 2, 3, \cdots$. There is a unit measure of agents with a common discount factor $\beta \in (0, 1)$. In each date, a half of the agents are endowed with 2 units of a perishable consumption good, and the other half do not receive any endowment. The endowment types are determined i.i.d. across time and are observable. There is no aggregate uncertainty. Each agent i has a strictly concave instantaneous utility function $u(c_{i,t})$ for date-t consumption $c_{i,t}$.

We take a standard notion of ex-ante welfare with equal welfare weights. Because the utility function is strictly concave, under this criterion, goods must be transferred from almost every endowed agent to almost every non-endowed agent in a first best allocation, and hence almost every endowed agent must go to a meeting place in it.[11]

Meeting Places. There are *meeting places* where agents meet and goods are reallocated. There is a continuum of them, each of which is indexed by $z \in [0, 1]$. At the entrance of each meeting place, there is a *guard* who can prohibit an agent to enter the meeting place. These guards can observe agents' types but cannot observe agents' histories of actions, because it is too costly to keep track of histories of actions. They are not strategic agents. All the resources that agents bring are confiscated and will be allocated among agents inside the meeting place. Inside

[10] Wiseman [42] points out that the type of equilibria without money that Araujo and Camargo constructed may not be robust if a small production cost shock is introduced, and so money can still be essential with this small perturbation.

[11] The word *almost* simply comes from the fact that there is a continuum of agents. We omit the word hereafter.

each meeting place, the guard allocates goods in a way that maximizes participants' welfare subject to the resource and incentive constraints.[12]

Each agent can freely choose which meeting place to go at no cost, but can go to only one meeting place at a date. Agents have an option not to go anywhere. The action space of each agent at every date is hence simply $A_i = [0,1] \cup \{NG\}$. An action $a_i \in [0,1]$ stands for the meeting place that she goes to. For example, $a_i = 0.35$ means that she goes to the meeting place $z = 0.35$. Not going anywhere is denoted as $a_i = NG$.

Notice in a one-shot version of the game, as long as guard transfers from the endowed to the non-endowed, NG is a strictly dominating action for endowed agents. It is because an endowed agent can stay at home and eat all of her endowed goods, instead of going to a meeting place and providing some of them for non-endowed agents. Hence, in that one-shot game, no trade can occur.

Local Sunspots. A number in $[0,1]$ is assigned in each meeting place. Formally, for each meeting place $z \in [0,1]$, there is a payoff-irrelevant random variable $\lambda(z)$. The random variable is specific to the meeting place z, is drawn once for all at date $t = 1$, and is identically independently uniformly distributed on $[0,1]$ across the meeting places.[13] An agent can observe $\lambda(z)$ if and *only if* she enters the meeting place z. In other words, she does not observe $\lambda(z)$ if she does not enter the meeting place z. We label these random variables *local sunspots*. As we will see later, the local sunspots serve as a coordination device.

No Monitoring. By no monitoring, as is standard, we mean that each agent only observes her own past actions, own past payoffs, and local sunspots at meeting places where she went, but nothing else. Guards can observe agents' endowment types, and hence if an endowed agent eats her endowments and comes to a meeting place to only observe a local sunspot there, the guard at the meeting place can prohibit her to enter it. On the other hand, if a single agent simply stays at home and eats all her endowments by herself, no monitoring implies that this action can never be detected.

Money. Money is an intrinsically useless, durable, and uncounterfeitable object.

[12]In the next section, we will explicitly derive the allocation. For the explicit form of the allocation rule, see page 39. Lemma 2 proves that the allocation rule maximizes social welfare.

[13]It is not important that the realized value is fixed over time. One could instead assume that a value is redrawn at start of every date.

We assume that money is divisible and disposable. At the beginning of date 1, one unit of money is initially distributed to each non-endowed agent.[14]

Equilibrium. The equilibrium concept is sequential equilibrium (simply *equilibrium* hereafter). Each agent maximizes the expected discounted payoff at each date after any history, where the expectation is taken over her beliefs about what the other agents do.

4.3 Main Result

The following theorem is our main result, the inessentiality proposition.

Theorem 4.1. *For any discount factor, money is inessential in the economy.*

The heart of the theorem is that with local sunspots (but without money nor any other record keeping device), a non-trivial equilibrium can be supported for any positive discount factor. The construction of a non-trivial equilibrium with local sunspots is as follows. In the initial date, go to a meeting place, say $z = 0$. Agents use the draw of the local sunspot there as a coordination device to determine where they will meet for the second date. That is, at date 2, they go to the meeting place $\lambda(0)$. In general, go to the meeting place $\lambda(z_{t-1})$, where z_{t-1} is the meeting place where the agent went at date $t-1$. That is, go to the meeting place that the local sunspot observed in the previous meeting place specifies.

Importantly, $\lambda(0)$ can be observed only by those who went to the meeting place 0 at the initial date. Thus, if an endowed agent does not go to the meeting place 0 at the initial date, she will never find the other agents from the next date on. Hence, she cannot get the good when she does not have endowment—punishment for not going to the meeting place when she has endowment. In other words, with local sunspots, any deviation is punished by permanent autarky—the minmax payoff.

Next, we compare the local sunspots to *perfect monitoring*—perfect records of the past actions of all agents. In the perfect monitoring case, applying a similar technique developed by Kocherlakota [21], the best equilibrium is characterized by using the fact that reversion to permanent autarky is the most severe punishment.

[14]It is easy to see that our result does not rely on how money is initially distributed.

Notice that the punishment is the same as that with local sunspots, and so the best equilibrium payoff with local sunspots is no worse than that with perfect monitoring.

Finally, Kocherlakota [22]'s money-is-memory theorem can be shown to be applied here, and any equilibrium with money can be achieved when there is perfect monitoring. Although the definitions of memory and perfect monitoring are in general different, the two notions coincide in our economy when agents utilize only one of the meeting places. Then, it is easily shown that under perfect monitoring, for any equilibrium, there is another equilibrium which utilizes only one meeting place and whose payoffs are the same. Hence, it suffices to consider perfect monitoring instead of memory in this comparison.

Formally, for a given $\beta \in (0,1)$, define ϵ^*_β to be the solution to

$$\begin{aligned} \max_{\epsilon \in [0,1]} \quad & u(2-\epsilon) + u(\epsilon) \\ \text{subject to} \quad & (1-\beta)u(2-\epsilon) + \beta\left[\frac{1}{2}u(2-\epsilon) + \frac{1}{2}u(\epsilon)\right] \\ & \geq (1-\beta)u(2) + \beta\left[\frac{1}{2}u(2) + \frac{1}{2}u(0)\right] \end{aligned}$$

Note that ϵ^*_β uniquely exists because u is strictly increasing and strictly concave.

Now, consider the following allocation and denote it as $\mathbf{c}^*$: at each date, transfer ϵ^*_β unit of the good from every endowed agent to every non-endowed agent, and hence every endowed agent receives $2 - \epsilon^*_\beta$ unit and every non-endowed agent receives ϵ^*_β unit. Then, in the local sunspots case, we have

Lemma 4.1. $\boldsymbol{c}^*$ *is an equilibrium allocation with local sunspots.*

In particular, the first best allocation is an equilibrium allocation with local sunspots if

$$\beta \geq 2\frac{u(2) - u(1)}{u(2) - u(0)}$$

By strict concavity of u, the right hand side is strictly less than 1.

Proof of Lemma 4.1. Candidate Strategy. In the initial date, go to meeting place

0. At date $t \geq 2$, go to the meeting place $\lambda(z_{t-1})$, where z_{t-1} is the meeting place where the agent went at date $t-1$.

Optimality. Consider an endowed agent at date $t = 1$ who is supposed to go to meeting place 0. If an agent follows the candidate strategy above given that the other agents do, she gets $2 - \epsilon_\beta^*$ unit of the consumption good whenever she is endowed and ϵ_β^* whenever she is not. Thus, her discounted payoff, normalized by $(1-\beta)$, is

$$(1-\beta)u(2-\epsilon_\beta^*) + \beta\left[\frac{1}{2}u(2-\epsilon_\beta^*) + \frac{1}{2}u(\epsilon_\beta^*)\right]$$

Suppose she deviates at date 1 and does not go to meeting place 0. Then, her short-term payoff at the date is $u(2)$. However, she does not observe $\lambda(0)$ and therefore she does not know where the other agents will be from date 2 onward. Because the probability that she will find the other agents somewhere by chance is zero, her expected payoff from the deviation is given by

$$(1-\beta)u(2) + \beta\left[\frac{1}{2}u(2) + \frac{1}{2}u(0)\right]$$

By definition of ϵ_β^*, the deviation is not profitable. □

In the perfect monitoring case, we have

Lemma 4.2. *$\mathbf{c}^*$ is the best equilibrium allocation with perfect monitoring.*

Proof of Lemma 4.2. First, it is easy to show that with perfect monitoring, $\mathbf{c}^*$ is an equilibrium allocation. This is sustained by a *grim trigger* strategy—in each date every agent goes to a meeting place, say 0. Any deviation, which can be detected for sure by perfect monitoring, is punished by the reversion to permanent autarky. Then, the same calculation as Lemma 4.1 shows that this constitutes an equilibrium.

To see that this is the best equilibrium, we employ a technique similar to that in a risk-sharing model of Kocherlakota [21] (see also Ljungqvist and Sargent [28]). It is well-known that the best equilibrium is sustained by a grim-trigger strategy where any deviation is punished by reversion to permanent autarky. Hence the best equilibrium is characterized by maximizing

$$\sum_{t=1}^{\infty} \beta^{t-1} u(c_{i,t})$$

subject to the resource constraint $\int_{i\in[0,1]} c_{i,t} \mathrm{d}i \leq 2$ and the incentive constraints of both types of agents where any deviation is punished by reversion to permanent autarky. It is well-known that welfare is maximized when the incentive constraint of the endowed agents is satisfied with equality, that is, for agents with endowment at date τ,

$$(1-\beta)\sum_{t=\tau}^{\infty} \beta^{t-\tau} u(c_{i,t}) = (1-\beta)u(2) + \beta\left[\frac{1}{2}u(2) + \frac{1}{2}u(0)\right]$$

The solution to this problem is $\mathbf{c}^*$. □

This completes the proof of Theorem 4.1.

One observation is that in our environment with i.i.d. endowment shocks, money does not achieve the first best.[15] The reason is that some agents who are always endowed accumulate too much money over time to incentivize them to come to a meeting place at some point of time, and on the other hand, some agents who are always not endowed run out of money at some point of time and nothing is affordable for them (see for example Wallace [41]). This suggests that in this environment, local sunspots perform *strictly* better than money.

4.4 Concluding Remarks

In this paper, we show that no monitoring, when combined with local sunspots, does not guarantee the essentiality of money, and whether money is essential or not depends on the very details of an environment.

In an environment we study, unlike Araujo [4], Araujo et al. [6], and Araujo and Camargo [5], money is inessential even with *no* monitoring about past actions of other agents. There is a debate on when money is essential in the Lagos-Wright

[15] Kocherlakota [23], however, argues that if some of the assumptions on money—for example, disposability—are violated or if there are two monies, then the first best can be achieved.

[24] environment (see Aliprantis, Camera, and Puzzello [3, 2] and Lagos and Wright [25]). We point out that inessentiality still arises in a different environment even if there is no monitoring of histories of actions, and that the essentiality of money relies on the very details of modeling. More generally, Kocherlakota [22] shows that money can be replaced by memory. We point out that in an environment, money can be replaced by some information that is independent of the agents' actions.

Money can have several roles other than a record keeping device. One of such roles is a coordination device. Camera and Casari [12], Duffy and Puzzello [13], and Araujo and Guimaraes [7] shed light on this role of money in both theory and experiments. It is unclear whether local sunspots play a better role than money does in their senses.

For the construction, we assume there is a continuum of identical meeting places. With this assumption, it is shown that once an agent deviates, she can never meet the other agents again. If instead there are only finite meeting places, an agent might not go to the meeting place when she is endowed and go to a random meeting place when she is not endowed expecting that she will meet other agents with some luck. However, when there are sufficiently many (but finite) meeting places, then the first best allocation can still be achieved with local sunspots for a sufficiently high discount factor.

We assume that everyone who goes to the same meeting place observes exactly the same sunspot. The construction is robust to a small noise on local sunspots. Consider a case where each agent gets a wrong sunspot with a small probability. For a sufficiently high discount factor, the same strategy under no noise still constitutes an equilibrium even when there is a noise, and welfare is arbitrarily close to that from the first best when the error is arbitrarily small.

In our model, we assume that types of the agents are drawn independently across dates. The result is also robust when there is persistence. If types are not too persistent, the first best can be achieved with local sunspots for a sufficiently high discount factor. If types are sufficiently persistent relative to the discount factor, the first best cannot be achieved with local sunspots any longer, but welfare from the best outcome with local sunspots decreases in the same degree as that with perfect monitoring, and hence money is still inessential.

Appendix

Chapter 1

A.1 The Low-Turnover Equilibrium

In this section, I provide the explicit expressions for F and G in equations (1.3) and (1.4), the incentive constraints, and the on-equilibirum expected payoffs. Consider first the low-turnover equilibrium. Let $g^{(t)}$ and $b^{(t)}$ be the measure of good players and that of bad players in the unmatched pool at the beginning of date t. Let $(z_{GG}^{(t+1)}(H), z_{GG}^{(t+1)}(L), z_{GB}^{(t+1)}, z_{BB}^{(t+1)})$ be the measures of four kinds of matches formed at date t. Those matches are, respectively, ones between two good players with the high suitability, ones between two good players with the low suitability, ones between a good player and a bad player, and ones between two bad players. The evolution of these six variables are given by the following system.

$$g^{(t)} = \underbrace{(1-\phi)(1-\rho)}_{\text{new entrants to the economy}} + \underbrace{\rho z_{GB}^{(t)}}_{\text{dissolved matches}}, \tag{A.1}$$

$$b^{(t)} = \phi(1-\rho) + \rho(z_{GB}^{(t)} + 2z_{BB}^{(t)}), \tag{A.2}$$

$$z_{GG}^{(t+1)}(H) = \underbrace{\rho z_{GG}^{(t)}(H)}_{\text{remaining matches}} + \underbrace{\pi_H (1/2)(g^{(t)} + b^{(t)})(1-\lambda^{(t)})^2}_{\text{newly formed matches}}, \tag{A.3}$$

$$z_{GG}^{(t+1)}(L) = \rho z_{GG}^{(t)}(L) + \pi_L (1/2)(g^{(t)} + b^{(t)})(1-\lambda^{(t)})^2, \tag{A.4}$$

$$z_{GB}^{(t+1)} = (1/2)(g^{(t)} + b^{(t)})2\lambda^{(t)}(1-\lambda^{(t)}), \text{ and} \tag{A.5}$$

$$z_{BB}^{(t+1)} = (1/2)(g^{(t)} + b^{(t)})(\lambda^{(t)})^2, \tag{A.6}$$

where $\lambda^{(t)}$ is the fraction of bad players in the unmatched pool, that is,

$$\lambda^{(t)} = \frac{b^{(t)}}{g^{(t)} + b^{(t)}}.$$

For example, $g^{(t)}$ depends upon $z_{GB}^{(t)}$. There will be $(1-\phi)(1-\rho)$ of new players entering the economy. Among the good players who did not receive a death shock, $z_{GB}^{(t)}$ players come back to the unmatched pool because bad players do not cooperate and thus those good players choose to terminate the partnerships.

Also, $z_{GG}^{(t+1)}$ depends upon $z_{GG}^{(t)}$, $g^{(t)}$, and $b^{(t)}$. There will be ρ fraction of remaining matches between two good players with the high suitability from the previous date t. There will be newly formed matches from the unmatched pool. The total population of the pool is $g^{(t)} + b^{(t)}$ and the probability of two good players being chosen is $(1-\lambda^{(t)})^2$. With probability π_H, those matches will be of the high suitability.

Under the assumption that $b^{(0)} = \phi$, it is easy to show that $b^{(t)} = \phi$ for all $t \geq 0$ and the evolution of the state variables is given by the following system F,

$$\begin{aligned}
g^{(t+1)} &= (1-\phi)(1-\rho) + \rho\frac{g^{(t)}\phi}{g^{(t)}+\phi}, \\
z_{GG}^{(t+1)}(H) &= \rho z_{GG}^{(t)}(H) + \pi_H(1/2)\frac{\{g^{(t)}\}^2}{g^{(t)}+\phi}, \text{ and} \\
z_{GG}^{(t+1)}(L) &= \rho z_{GG}^{(t)}(L) + \pi_L(1/2)\frac{\{g^{(t)}\}^2}{g^{(t)}+\phi}.
\end{aligned}$$

Consider incentives of a player who engages in a newly formed match with match suitability v and who chooses between action C and D. Because she knows nothing about her partner and because the fraction of bad players in the unmatched pool is common knowledge, her belief that the partner is of the bad type is

$$\lambda^{(t)} = \frac{\phi}{g^{(t)}+\phi}.$$

Let $V_C^{(t)}(v)$ and $V_D^{(t)}(v)$ be the expected value from playing C in a newly formed match with match suitability v and that from playing D. Then, Bellman equations are given by

$$V_C^{(t)}(v) = (1-\lambda^{(t)})v + \lambda^{(t)}[(1-\delta\rho)(-l) + \delta\rho V^{(t+1)}] \text{ and} \tag{A.7}$$

$$V_D^{(t)}(v) = (1-\lambda^{(t)})[(1-\delta\rho)(1+d) + \delta\rho V^{(t+1)}] + \lambda^{(t)}\delta\rho V^{(t+1)}, \tag{A.8}$$

where

$$V^{(t)} = \pi_H V_C^{(t)}(v_H) + \pi_L V_C^{(t)}(v_L). \tag{A.9}$$

Note that in this equilibrium, $V_D^{(t)}(v)$ does not depend on v because a partner of the good type always cooperates. These equations are reduced to the following system G,

$$V^{(t)} = \frac{g^{(t)}}{g^{(t)}+\phi}\{\pi_H v_H + \pi_L v_L\} + \frac{\phi}{g^{(t)}+\phi}[(1-\delta\rho)(-l) + \delta\rho V^{(t+1)}]. \tag{A.10}$$

Incentive constraints for those newly formed matches are given by

$$V_C^{(t)}(v_L) \geq V_D^{(t)}(v_L) \tag{A.11}$$

and

$$V_C^{(t)}(v_H) \geq V_D^{(t)}(v_H). \tag{A.12}$$

Below, I will show that equation (A.11) implies equation (A.12) and the incentive constraints for matches remaining from previous dates.

First, because $V_D^{(t)}(v_H) = V_D^{(t)}(v_L)$, we have

$$\begin{aligned}
& V_C^{(t)}(v_H) - V_D^{(t)}(v_H) \\
= \quad & (1-\lambda^{(t)})v_H + \lambda^{(t)}[(1-\delta\rho)(-l) + \delta\rho V^{(t+1)}] - V_D^{(t)}(v_H) \\
> \quad & (1-\lambda^{(t)})v_L + \lambda^{(t)}[(1-\delta\rho)(-l) + \delta\rho V^{(t+1)}] - V_D^{(t)}(v_L) \\
= \quad & V_C^{(t)}(v_L) - V_D^{(t)}(v_L).
\end{aligned}$$

Hence, equation (A.11) implies (A.12).

Next, consider matches remaining from previous dates. Under this strategy, beliefs are updated in a trivial way. Because only good players play C, once a partner plays C, the belief of a player (that the partner is of the bad type) drops down to 0. If a partner plays D, the match will dissove by the nature of the strategy. In a remaining match with the low suitability in which a player has learnt that her partner is of the good type, the incentive constraint is given by

$$v_L \geq (1-\delta\rho)(1+d) + \delta\rho V^{(t+1)}. \tag{A.13}$$

However, equation (A.11) implies that

$$(1-\lambda^{(t)})\{v_L - [(1-\delta\rho)(1+d) + \delta\rho V^{(t+1)}]\} \geq \lambda^{(t)}(1-\delta\rho)l.$$

Because the right hand side and $1-\lambda^{(t)}$ are positive, this implies that equation (A.13) (with strict inequality).

In a remaining match with the high suitability against a good-type partner, the incentive constraint is given by

$$v_H \geq (1-\delta\rho)(1+d) + \delta\rho V^{(t+1)}.$$

This is implied by equation (A.13).

In summary, we need only to check equation (A.11) for the low-turnover equilibrium, and I define $IC^{(t)}$ as

$$IC^{(t)} = V_C^{(t)}(v_L) - V_D^{(t)}(v_L)$$

for each date t.

I turn to the equilibrium payoffs to a good player and a bad player. The average expected discounted equilibrium payoff to a good player from date t onward (before new matches are formed), normalized by $(1-\delta\rho)$, is equal to the sum of the expected values from three kinds of matches that good players can potentially engage in, each multiplied by the probability of the kind of matches and divided by the total measure of good players in the economy. That is,

$$W_G^{(t)} = \frac{2z_{GG}^{(t+1)}(H)v_H + 2z_{GG}^{(t+1)}(L)v_L + z_{GB}^{(t+1)}[(1-\delta\rho)(-l) + \delta\rho V^{(t+1)}]}{1-\phi}.$$

For example, there will be measure $2z_{GG}^{(t+1)}(H)$ of good players that engage in matches between two good players with the high suitability, in which they receive a payoff of v_H. There will be measure $2z_{GG}^{(t+1)}(L)$ of good players engaging in those with the low suitability. There will be $z_{GB}^{(t+1)}$ good players who engage in matches between a good player and a bad player, in which they receive $(-l)$ today, because bad players play D, and they go back to the unmatched pool.

The average payoff to a bad player *at* date t is given by

$$W_B^{(t)} = \frac{z_{GB}^{(t+1)}(1+d)}{\phi}.$$

There will be $z_{GB}^{(t+1)}$ matches between a good player and a bad player, in which bad players receive $(1+d)$ because good players play C. There will be $z_{BB}^{(t+1)}$ matches between two bad players too, but in those matches bad players receive a payoff of 0 because both players in the matches play D.

A.2 The High-Turnover Equilibrium

For the high-turnover strategy, equation (A.1) is replaced by

$$g^{(t)} = (1-\phi)(1-\rho) + \rho\{2z_{GG}^{(t)}(L) + z_{GB}^{(t)}\} \tag{A.14}$$

and equation (A.4) is replaced by

$$z_{GG}^{(t+1)}(L) = \pi_L(1/2)(g^{(t)} + b^{(t)})(1-\lambda^{(t)})^2. \tag{A.15}$$

The evolution of the state variables is similarly given by the following system F,

$$\begin{aligned} g^{(t+1)} &= (1-\phi)(1-\rho) + \rho\frac{g^{(t)}}{g^{(t)}+\phi}\{\pi_L g^{(t)} + \phi\} \text{ and} \\ z_{GG}^{(t+1)}(H) &= \rho z_{GG}^{(t)}(H) + \pi_H(1/2)\frac{\{g^{(t)}\}^2}{g^{(t)}+\phi}. \end{aligned}$$

Bellman equations are given by

$$
\begin{aligned}
V_C^{(t)}(v_L) &= (1-\delta\rho)(-l) + \delta\rho V^{(t+1)}, \\
V_C^{(t)}(v_H) &= (1-\lambda^{(t)})v_H + \lambda^{(t)}[(1-\delta\rho)(-l) + \delta\rho V^{(t+1)}], && \text{(A.16)} \\
V_D^{(t)}(v_L) &= \delta\rho V^{(t+1)}, \text{ and} && \text{(A.17)} \\
V_D^{(t)}(v_H) &= (1-\lambda^{(t)})[(1-\delta\rho)(1+d) + \delta\rho V^{(t+1)}] + \lambda^{(t)}\delta\rho V^{(t+1)}, && \text{(A.18)}
\end{aligned}
$$

where

$$V^{(t)} = \pi_H V_C^{(t)}(v_H) + \pi_L V_D^{(t)}(v_L). \tag{A.19}$$

Equations (A.16), (A.17), and (A.19) are reduced to the following system G,

$$
\begin{aligned}
V^{(t)} = \pi_H &\left\{ \frac{g^{(t)}}{g^{(t)}+\phi} v_H + \frac{\phi}{g^{(t)}+\phi}(1-\delta\rho)(-l) \right\} \\
&+ \left\{ \pi_H \frac{\phi}{g^{(t)}+\phi} + \pi_L \right\} \delta\rho V^{(t+1)}.
\end{aligned} \tag{A.20}
$$

The incentive constraints for newly formed matches at date t are given by

$$V_C^{(t)}(v_H) \geq V_D^{(t)}(v_H) \tag{A.21}$$

and

$$V_D^{(t)}(v_L) \geq V_C^{(t)}(v_L).$$

The second inequality is automatically satisfied because $(1-\delta\rho)(-l) < 0$.

Now, consider matches remaining from previous dates. In a remaining match with the high suitability in which a player has learnt that her partner is of the good type, the incentive constraint is given by

$$v_H \geq (1-\delta\rho)(1+d) + \delta\rho V^{(t+1)}.$$

However, equation (A.21) implies that

$$(1-\lambda^{(t)})\{v_H - [(1-\delta\rho)(1+d) + \delta\rho V^{(t+1)}]\} \geq \lambda^{(t)}(1-\delta\rho)l,$$

which implies that

$$v_H - [(1-\delta\rho)(1+d) + \delta\rho V^{(t+1)}] > 0.$$

Hence, we need only to check equation (A.21), and I define $IC^{(t)}$ as

$$IC^{(t)} = V_C^{(t)}(v_H) - V_D^{(t)}(v_H)$$

at each date t.

The average expected discounted equilibrium payoff to a good player from date t onward, normalized by $(1-\delta\rho)$, is

$$W_G^{(t)} = \frac{2z_{GG}^{(t+1)}(H)v_H + 2z_{GG}^{(t+1)}(L)\delta\rho V^{(t+1)} + z_{GB}^{(t+1)}[(1-\delta\rho)\pi_H(-l) + \delta\rho V^{(t+1)}]}{1-\phi}$$

and the average equilibrium payoff to a bad player is

$$W_B^{(t)} = \frac{z_{GB}^{(t+1)}\pi_H(1+d)}{\phi}.$$

A.3 Proofs

This section provides proofs.

Proof of Proposition 1.1. Proposition 1.1 is shown by proving the following lemmata.

Lemma A.1. *Suppose that $(1-\delta\rho)(1+d) < v_L$. Then, there exist $\bar{l}_1 \in (0,\infty)$, $\underline{\phi} \in (0,1)$, and $\bar{\phi}_1 \in (\underline{\phi}, 1)$ such that for any $l < \bar{l}_1$ and any $\phi \in (\underline{\phi}, \bar{\phi}_1)$, the low-turnover steady state exists.*

Lemma A.2. *Suppose that $(1-\delta\rho\pi_L)(1+d) < v_H$. Then, for $\underline{\phi}$ in Lemma A.1, there exists $\bar{l}_2 \in (0,\infty)$ and $\bar{\phi}_2 \in (\underline{\phi}, 1)$ such that for any $l < \bar{l}_2$ and any $\phi \in (0, \bar{\phi}_2)$, the high-turnover steady state exists.*

Suppose $(1-\delta\rho)(1+d) < v_L$ and $(1-\delta\rho\pi_L)(1+d) < v_H$. Then, for $\underline{\phi}$, $\bar{\phi} = \min\{\bar{\phi}_1, \bar{\phi}_2\}$, and $\bar{l} = \min\{\bar{l}_1, \bar{l}_2\}$, Proposition 1.1 is proved. □

Proof of Lemma A.1. First, from equations (A.2), (A.5), and (A.6), we have $b = \phi$. From equations (A.1) and (A.5), we have

$$g = \frac{(1-2\phi)(1-\rho) + \sqrt{(1-2\phi)^2(1-\rho)^2 + 4\phi(1-\phi)(1-\rho)}}{2}.$$

Thus, as $\phi \to 0$, $b \to 0$ and $g \to (1-\rho)$. Thus, $\lambda = b/(g+b) \to 0$. As $\phi \to 1$, $b \to 1$ and $g \to 0$. Thus, $\lambda \to 1$. Thus, λ moves from 0 to 1 as ϕ moves from 0 to 1, is a continuous function of ϕ, and does not depend on l.

Next, at the low-turnover steady state, given λ, equation (A.11) is a quadratic function of λ of the form

$$f(\lambda) = A\lambda^2 + B\lambda + C \leq 0. \tag{A.22}$$

If the two roots of equation (A.22), denoted as $\underline{\lambda}$ and $\bar{\lambda}$, are between 0 and 1, the low-turnover steady state exists for $\lambda \in [\underline{\lambda}, \bar{\lambda}]$. I will characterize the roots when $l \to 0$.

It is easily checked that

$$A = \delta\rho[\pi_H v_H + \pi_L v_L - v_L + (1-\delta\rho)(1+d)] > 0,$$
$$f(0) = \delta\rho(\pi_H v_H + \pi_L v_L) - v_L + (1-\delta\rho)(1+d) > 0,$$

and

$$f(1) = -(1-\delta\rho)^2(-l) > 0.$$

Hence, as $l \to 0$, $f(1) \to 0$ and hence one of the roots, if two exist, converges to 1.

Also, from equations (A.7)-(A.9), we have

$$V = \frac{(1-\lambda)(\pi_H v_H + \pi_L v_L) + \lambda(1-\delta\rho)(-l)}{1-\lambda\delta\rho}.$$

Hence, as $l \to 0$, we have

$$V \to \frac{(1-\lambda)(\pi_H v_H + \pi_L v_L)}{1-\lambda\delta\rho}.$$

Then, equation (A.11) becomes

$$\{1-\lambda\delta\rho\}\{v_L-(1-\delta\rho)(1+d)\} \geq \delta\rho(1-\lambda)(\pi_H v_H+\pi_L v_L),$$

which implies that

$$\lambda \geq \frac{(1-\delta\rho)(1+d)+\delta\rho(\pi_H v_H+\pi_L v_L)-v_L}{\delta\rho\{(1-\delta\rho)(1+d)+\pi_H v_H+\pi_L v_L-v_L\}}. \tag{A.23}$$

The right hand side of equation (A.23), the other root, is independent of ϕ, is always larger than 0, but is not automatically smaller than 1. It is between 0 and 1 if and only if

$$(1-\delta\rho)(1+d) < v_L. \tag{A.24}$$

Hence, if inequality (A.24) holds, there exist $\bar{l}_1 \in (0,\infty)$, $\underline{\lambda}_1 \in (0,1)$, and $\bar{\lambda}_1 \in (\underline{\lambda}_1, 1)$ such that for any $l < \bar{l}_1$ and any $\lambda \in (\underline{\lambda}_1, \bar{\lambda}_1)$, the low-turnover steady state exists. Hence, from the first observation, if (A.24) holds, there exist $\bar{l}_1 \in (0,\infty)$, $\underline{\phi} \in (0,1)$, and $\bar{\phi}_1 \in (\underline{\phi}, 1)$ such that for any $l < \bar{l}_1$ and any $\phi \in (\underline{\phi}, \bar{\phi}_1)$, the low-turnover steady state exists. □

Proof of Lemma A.2. First, we have $b = \phi$ similarly to the low-turnover case. From equations (A.14), (A.15), and (A.5), we have

$$g = \frac{(1-2\phi)(1-\rho)+\sqrt{(1-2\phi)^2(1-\rho)^2+4\phi(1-\phi)(1-\rho)(1-\rho\pi_L)}}{2(1-\rho\pi_L)}.$$

Thus, as $\phi \to 0$, $b \to 0$ and $g \to (1-\rho)/(1-\rho\pi_L)$. Thus, $\lambda \to 0$. As $\phi \to 1$, $b \to 1$ and $g \to 0$. Thus, $\lambda \to 1$.

At the high-turnover steady state, given λ, equation (A.21) is a quadratic function of λ of the form

$$f(\lambda) = A\lambda^2 + B\lambda + C \leq 0. \tag{A.25}$$

As in Lemma A.1, I will characterize the roots when $l \to 0$.

It is easily checked that

$$A = \pi_H \delta\rho(1-\delta\rho)(1+d) > 0$$

and

$$f(1) = -(1-\delta\rho)^2(-l) > 0.$$

As $l \to 0$, $f(1) \to 0$ and hence one of the roots, if two exist, converges to 1.

Also, from equations (A.16), (A.17), and (A.19), we have

$$V = \frac{\pi_H\{(1-\lambda)v_H + \lambda(1-\delta\rho)(-l)\}}{1-\delta\rho(\pi_H\lambda+\pi_L)}.$$

As $l \to 0$, we have

$$V \to \frac{\pi_H(1-\lambda)v_H}{1-\delta\rho(\pi_H\lambda+\pi_L)}$$

and hence equation (A.21) becomes

$$\{1-\delta\rho(\pi_H\lambda+\pi_L)\}\{v_H-(1-\delta\rho)(1+d)\} \geq \delta\rho(1-\lambda)\pi_H v_H,$$

which implies that

$$\lambda \geq \frac{(1-\delta\rho\pi_L)(1+d)-v_H}{\delta\rho\pi_H(1+d)}. \tag{A.26}$$

If $(1-\delta\rho\pi_L)(1+d) < v_H$, then the right hand side of equation (A.26) is less than 0. Then, there exist $\bar{l}_2$ and $\bar{\lambda}_2$ such that for any $l < \bar{l}_2$ and any $\lambda \in (0, \bar{\lambda}_2)$, the high-turnover steady state exists. Hence, from the first observation, there exist $\bar{l}_2$ and $\bar{\phi}_2$ such that for any $l < \bar{l}_2$ and any $\phi \in (0, \bar{\phi}_2)$, the high-turnover steady state exists. Furthermore, notice that $\bar{\phi}_2$ can be taken as is larger than $\underline{\phi}$ in Lemma A.1, because as l tends to infinity, λ, and correspondingly ϕ, is arbitrarily close to 1. This completes the proof. □

Proof of Proposition 1.2. Consider the high-turnover equilibrium. By the assumption that $b^{(0)} = \phi$, we have $b^{(t)} = \phi$ for all $t \geq 0$ from equations (A.2), (A.5), and (A.6). From equations (A.14), (A.15), and (A.5), we have

$$g^{(t+1)} = (1-\phi)(1-\rho) + \rho \frac{\pi_L g^{(t)} + \phi}{g^{(t)} + \phi} g^{(t)}.$$

At steady state, we have

$$g = (1-\phi)(1-\rho) + \rho \frac{\pi_L g + \phi}{g + \phi} g.$$

Hence, we have

$$\begin{aligned}
& g^{(t+1)} - g \\
&= \rho \frac{\pi_L g^{(t)} + \phi}{g^{(t)} + \phi} g^{(t)} - \rho \frac{\pi_L g + \phi}{g + \phi} g \\
&= \rho \left\{ \frac{\pi_L g^{(t)} + \phi}{g^{(t)} + \phi} g^{(t)} - \frac{\pi_L g + \phi}{g + \phi} g^{(t)} + \frac{\pi_L g + \phi}{g + \phi} g^{(t)} - \frac{\pi_L g + \phi}{g + \phi} g \right\} \\
&= \rho \left\{ \frac{\pi_L g + \phi}{g + \phi} - (1-\pi_L) \frac{g^{(t)}}{g^{(t)} + \phi} \frac{\phi}{g + \phi} \right\} (g^{(t)} - g).
\end{aligned}$$

Hence, $|g^{(t+1)} - g| < \rho |g^{(t)} - g|$. Hence, we have $\lim_{t \to \infty} g^{(t)} = g$.

Similarly, from equation (A.3), we have

$$z_{GG}^{(t+1)}(H) - z_{GG}(H) = \rho(z_{GG}^{(t)}(H) - z_{GG}(H)) + \pi_H (1/2) \left(\frac{(g^{(t)})^2}{g^{(t)} + \phi} - \frac{g^2}{g + \phi} \right).$$

Notice that

$$\lim_{t \to \infty} \frac{(g^{(t)})^2}{g^{(t)} + \phi} = \frac{g^2}{g + \phi}$$

because

$$\begin{aligned}
& \frac{(g^{(t)})^2}{g^{(t)} + \phi} - \frac{g^2}{g + \phi} \\
&= \frac{(g^{(t)})^2}{g^{(t)} + \phi} - \frac{g^2}{g^{(t)} + \phi} + \frac{g^2}{g^{(t)} + \phi} - \frac{g^2}{g + \phi} \\
&= \frac{g^{(t)} + g}{g^{(t)} + \phi} (g^{(t)} - g) - \frac{g^2}{(g^{(t)} + \phi)(g + \phi)} (g^{(t)} - g).
\end{aligned}$$

Hence, we have $\lim_{t\to\infty} z_{GG}^{(t)}(H) = z_{GG}^{(t)}(H)$. From equation (A.15), we have $\lim_{t\to\infty} z_{GG}^{(t)}(L) = z_{GG}(L)$.

Now, I turn to incentive constraints. Suppose incentive constraints at steady state hold with strict inequality. I show that for any $\epsilon > 0$ there exists $g^{(0)}$ such that

$$|IC^{(t)} - IC| < \epsilon$$

for all $t \geq 0$ and hence $IC^{(t)} > 0$ for all $t \geq 0$.

First, we have

$$\begin{aligned} |\lambda^{(t)} - \lambda| &= \left| \frac{\phi}{g^{(t)} + \phi} - \frac{\phi}{g + \phi} \right| \\ &\leq \frac{\phi}{(g^{(t)} + \phi)(g + \phi)} |g^{(t)} - g| \\ &< \frac{1}{g + \phi} |g^{(t)} - g|. \end{aligned} \tag{A.27}$$

As is shown above, $|g^{(t)} - g|$ is a decreasing sequence, and hence if $|g^{(0)} - g| < (g + \phi)\epsilon$, then $|\lambda^{(t)} - \lambda| < \epsilon$ for all $t \geq 0$.

Second, by applying G, equation (A.20), infinitely many times forward, we have

$$\begin{aligned} V^{(t)} = \; & (1 - \lambda^{(t)})\pi_H v_H + \lambda^{(t)}(1 - \delta\rho)\pi_H(-l) \\ & + \sum_{j=0}^{\infty} \left(\prod_{k=0}^{j} \{\pi_H \lambda^{(t+k)} + \pi_L\} \right) (\delta\rho)^{j+1}(1 - \lambda^{(t+j+1)})\pi_H v_H \\ & + \sum_{j=0}^{\infty} \left(\prod_{k=0}^{j} \{\pi_H \lambda^{(t+k)} + \pi_L\} \right) (\delta\rho)^{j+1}(1 - \lambda^{(t+j+1)})(1 - \delta\rho)\pi_H(-l) \end{aligned} \tag{A.28}$$

and also at steady state we have

$$\begin{aligned} V = \quad & (1 - \lambda)\pi_H v_H + \lambda(1 - \delta\rho)\pi_H(-l) \\ & + \sum_{j=0}^{\infty} (\pi_H \lambda + \pi_L)^{j+1} (\delta\rho)^{j+1}(1 - \lambda)\pi_H v_H \\ & + \sum_{j=0}^{\infty} (\pi_H \lambda + \pi_L)^{j+1} (\delta\rho)^{j+1}(1 - \lambda)(1 - \delta\rho)\pi_H(-l). \end{aligned}$$

Hence, if $|g^{(0)} - g| < (g + \phi)\epsilon$, then because $|\lambda^{(t)} - \lambda| < \epsilon$ for all $t \geq 0$, we have

$$
\begin{aligned}
|V^{(t)} - V| &< \sum_{j=0}^{\infty}(j+1)(\delta\rho)^{j+1}\pi_H v_H \epsilon \\
&\quad + \sum_{j=0}^{\infty}(j+1)(\delta\rho)^{j+1}(1-\delta\rho)\pi_H l \epsilon \\
&= \frac{\pi_H v_H + (1-\delta\rho)\pi_H l}{(1-\delta\rho)^2}\epsilon
\end{aligned}
$$

for all $t \geq 1$.

Finally, from equations (A.16) and (A.18), we have

$$
\begin{aligned}
IC^{(t)} = {} & (1-\lambda^{(t)})v_H + \lambda^{(t)}[(1-\delta\rho)(-l) + \delta\rho V^{(t+1)}] \\
& - \left\{(1-\lambda^{(t)})[(1-\delta\rho)(1+d) + \delta\rho V^{(t+1)}] + \lambda^{(t)}\delta\rho V^{(t+1)}\right\}
\end{aligned}
\tag{A.29}
$$

and at steady state we have

$$
\begin{aligned}
IC^{(t)} = \quad & (1-\lambda)v_H + \lambda[(1-\delta\rho)(-l) + \delta\rho V] \\
& -\{(1-\lambda)[(1-\delta\rho)(1+d) + \delta\rho V] + \lambda\delta\rho V\}.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
& IC^{(t)} - IC \\
&= [-v_H + (1-\delta\rho)(-l) + (1-\delta\rho)(1+d) + \delta\rho V^{(t+1)}](\lambda^{(t)} - \lambda) \\
&\quad - (1-\lambda)\delta\rho(V^{(t+1)} - V).
\end{aligned}
\tag{A.30}
$$

Hence, if $|\lambda^{(t)} - \lambda| < \epsilon$ for all $t \geq 0$, then because $V^{(t+1)} \leq 1 + d$, we have

$$
|IC^{(t)} - IC| < \{v_H + (1-\delta\rho)l + 1 + d\}\,\epsilon + \frac{\delta\rho}{(1-\delta\rho)^2}\pi_H\,\{v_H + (1-\delta\rho)l\}\,\epsilon
$$

for all $t \geq 0$. This completes the proof for the high-turnover equilibrium.

Consider the low-turnover equilibrium. Again, $b^{(t)} = \phi$ for all $t \geq 0$. From equations (A.1) and (A.5), we have

$$g^{(t+1)} - g = \rho \frac{\phi}{g^{(t)} + \phi} \frac{g}{g + \phi} (g^{(t)} - g).$$

Hence, $|g^{(t+1)} - g| < \rho |g^{(t)} - g|$. Hence, we have $\lim_{t \to \infty} g^{(t)} = g$. The evolution of $z_{GG}^{(t)}(H)$ is the same as that of the high-turnover equilibrium and the evolution of $z_{GG}^{(t)}(L)$ is paralell to it. Hence, $\lim_{t \to \infty} z_{GG}^{(t)}(H) = z_{GG}(H)$ and $\lim_{t \to \infty} z_{GG}^{(t)}(L) = z_{GG}(L)$.

I show incentive constraints. First, in the same argument as in the high-turnover case, if $|g^{(0)} - g| < (g + \phi)\epsilon$ then $|\lambda^{(t)} - \lambda| < \epsilon$ for all $t \geq 0$.

Second, from equation (A.10), we have

$$\begin{aligned} V^{(t)} = \quad & (1 - \lambda^{(t)})\{\pi_H v_H + \pi_L v_L\} + \lambda^{(t)}(1 - \delta\rho)(-l) \\ & + \sum_{j=0}^{\infty} \left(\prod_{k=0}^{j} \lambda^{(t+k)} \right) (\delta\rho)^{j+1} (1 - \lambda^{(t+j+1)}) \{\pi_H v_H + \pi_L v_L\} \\ & + \sum_{j=0}^{\infty} \left(\prod_{k=0}^{j} \lambda^{(t+k)} \right) (\delta\rho)^{j+1} (1 - \lambda^{(t+j+1)})(1 - \delta\rho)(-l) \end{aligned}$$

and

$$\begin{aligned} V = \quad & (1 - \lambda)\{\pi_H v_H + \pi_L v_L\} + \lambda(1 - \delta\rho)(-l) \\ & + \sum_{j=0}^{\infty} \lambda^{j+1} (\delta\rho)^{j+1} (1 - \lambda) \{\pi_H v_H + \pi_L v_L\} \\ & + \sum_{j=0}^{\infty} \lambda^{j+1} (\delta\rho)^{j+1} (1 - \lambda)(1 - \delta\rho)(-l), \end{aligned}$$

where, λ^{j+1}, lambda to the power $j+1$, should not be confused with $\lambda^{(j+1)}$. Hence, if $|g^{(0)} - g| < (g + \phi)\epsilon$ then

$$\begin{aligned} |V^{(t)} - V| \quad < \quad & \sum_{j=0}^{\infty} (j + 1)(\delta\rho)^{j+1} \{\pi_H v_H + \pi_L v_L\} \epsilon \\ & + \sum_{j=0}^{\infty} (j + 1)(\delta\rho)^{j+1} (1 - \delta\rho) l \epsilon \end{aligned}$$

$$= \frac{\pi_H v_H + \pi_L v_L + (1-\delta\rho)l}{(1-\delta\rho)^2}\epsilon$$

for all $t \geq 1$.

Finally, from equations (A.7) and (A.8), we have

$$\begin{aligned} & IC^{(t)} - IC \\ = \; & [-v_L + (1-\delta\rho)(-l) + (1-\delta\rho)(1+d) + \delta\rho V^{(t+1)}](\lambda^{(t)} - \lambda) \\ & -(1-\lambda)\delta\rho(V^{(t+1)} - V). \end{aligned}$$

Hence, if $|\lambda^{(t)} - \lambda| < \epsilon$ for all $t \geq 0$ then

$$|IC^{(t)} - IC| < \{v_L + (1-\delta\rho)l + 1 + d\}\,\epsilon + \frac{\delta\rho}{(1-\delta\rho)^2}\pi_H\,\{v_H + (1-\delta\rho)l\}\,\epsilon$$

for all $t \geq 0$. This completes the proof for the low-turnover equilibrium. □

Proof of Proposition 1.3. I prove a stronger claim, that is, for both high- and low-turnover strategies, for any initial state, $(IC^{(t)})_{t=0}^{\infty}$ converges to IC. Consider the high-turnover strategy. First, from the proof of Proposition 1.2, $(g^{(t)})_{t=0}^{\infty}$ converges to g from any initial value. Hence, by equation (A.27), $(\lambda^{(t)})_{t=0}^{\infty}$ converges to λ from any initial state. Second, from equation (A.28), $(V^{(t)})_{t=0}^{\infty}$ converges to V because $(\lambda^{(t)})_{t=0}^{\infty}$ converges to λ. Finally, from equation (A.30), $(IC^{(t)})_{t=0}^{\infty}$ converges to IC. The low-turnover strategy case is similar. □

Proof of Proposition 1.4. Consider the high-turnover equilibrium. First, by equation (A.27), we have

$$|\lambda^{(K)} - \lambda| < \frac{1}{g+\phi}|g^{(K)} - g|.$$

Because $g^{(K)} \to g$, $\lambda^{(K)} \to \lambda$ as $K \to \infty$.

Second, by equation (A.28) and $\lambda^{(K)} \to \lambda$, we have $V^{(K+1)} \to V$ as $K \to \infty$. This implies that for any $\epsilon > 0$ there exists K such that $|\underline{V}^{(K+1)} - V^{(K+1)}| < \epsilon$, because $\underline{V}^{(K+1)} = V$.

Third, I will show that if $|\underline{V}^{(K+1)} - V^{(K+1)}| < \epsilon$ then $|\underline{V}^{(t)} - V^{(t)}| < \epsilon$ for all

$t = 1, \cdots, K+1$. To see this, notice from equation (A.20) that

$$\begin{aligned} |\underline{V}^{(t)} - V^{(t)}| &= |G(g^{(t)}, \underline{V}^{(t+1)}) - G(g^{(t)}, V^{(t+1)})| \\ &= |\{\pi_H \lambda^{(t)} + \pi_L\}\delta\rho(\underline{V}^{(t+1)} - V^{t+1})| \\ &< \delta\rho|\underline{V}^{(t+1)} - V^{(t+1)}|. \end{aligned}$$

Hence, we have $|\underline{V}^{(t)} - V^{(t)}| < |\underline{V}^{(t+1)} - V^{(t+1)}|$ for all $t \leq K$.

Finally, I will show that incentive constraints are close. By definition, we have, as an analogue to equation (A.29),

$$\begin{aligned} \underline{IC}^{(t)} &= (1-\lambda^{(t)})v_H + \lambda^{(t)}[(1-\delta\rho)(-l) + \delta\rho\underline{V}^{(t+1)}] \\ &\quad - \left\{(1-\lambda^{(t)})[(1-\delta\rho)(1+d) + \delta\rho\underline{V}^{(t+1)}] + \lambda^{(t)}\delta\rho\underline{V}^{(t+1)}\right\}. \end{aligned}$$

Hence, we have

$$\begin{aligned} & |\underline{IC}^{(t)} - IC^{(t)}| \\ = \ & |-(1-\lambda^{(t)})\delta\rho(\underline{V}^{(t+1)} - V^{(t+1)})| \\ \leq \ & (1-\lambda^{(t)})\delta\rho|\underline{V}^{(t+1)} - V^{(t+1)}| \\ < \ & \delta\rho|\underline{V}^{(t+1)} - V^{(t+1)}| \end{aligned}$$

for all $t = 0, \cdots, K$. Hence, because $|\underline{V}^{t+1} - V^{(t+1)}| < \epsilon$ for all $t = 0, \cdots, K$, we have $|\underline{IC}^{(t)} - IC^{(t)}| < \epsilon$ for all $t = 0, \cdots, K$.

Consider the low-turnover equilibrium. Similarly to the high-turnover case, $\lambda^{(K)} \to \lambda$ and $\underline{V}^{(K+1)} \to V$ as $K \to \infty$. Hence, for any $\epsilon > 0$ there exists K such that $|\underline{V}^{(K+1)} - V^{(K+1)}| < \epsilon$.

From equation (A.10), we have

$$\begin{aligned} |\underline{V}^{(t)} - V^{(t)}| &= |G(g^{(t)}, \underline{V}^{(t+1)}) - G(g^{(t)}, V^{(t+1)})| \\ &= |\lambda^{(t)}\delta\rho(\underline{V}^{(t+1)} - V^{t+1})| \\ &< \delta\rho|\underline{V}^{(t+1)} - V^{(t+1)}|. \end{aligned}$$

Hence, $|\underline{V}^{(t)} - V^{(t)}| < |\underline{V}^{(t+1)} - V^{(t+1)}|$ for all $t \leq K$.

Finally, we have

$$
\begin{aligned}
& |\underline{IC}^{(t)} - IC^{(t)}| \\
= \; & |-(1-\lambda^{(t)})\delta\rho(\underline{V}^{(t+1)} - V^{(t+1)})| \\
\leq \; & (1-\lambda^{(t)})\delta\rho|\underline{V}^{(t+1)} - V^{(t+1)}| \\
< \; & \delta\rho|\underline{V}^{(t+1)} - V^{(t+1)}|
\end{aligned}
$$

for all $t = 0, \cdots, K$. Hence, $|\underline{IC}^{(t)} - IC^{(t)}| < \epsilon$ for all $t = 0, \cdots, K$. □

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Vita

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