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ORIENTATIONS TOWARD MATHEMATICAL PROCESSES OF PROSPECTIVE SECONDARY MATHEMATICS TEACHERS AS RELATED TO WORK WITH TASKS

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Tenille Cannon

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The dissertation of Tenille Cannon was reviewed and approved* by the following:

Rose Mary Zbiek  
Professor of Education (Mathematics Education)  
Department Head, Curriculum and Instruction  
Dissertation Advisor  
Chair of Committee

Fran Arbaugh  
Associate Professor of Education (Mathematics Education)

M. Kathleen Heid  
Distinguished Professor of Education (Mathematics Education)

Gwendolyn M. Lloyd  
Professor of Education (Mathematics Education)

Andrew Belmonte  
Professor of Mathematics

William S. Carlsen  
Professor of Education (Science Education)  
Director of Graduate and Undergraduate Studies in Curriculum and Instruction

*Signatures are on file in the Graduate School
ABSTRACT

Mathematics can be conceptualized in different ways. Policy documents such as the National Council of Teachers of Mathematics (NCTM) (2000) and the Common Core State Standards Initiative (CCSSI) (2010), classify mathematics in terms of mathematical content (e.g., quadratic functions, Pythagorean theorem) and mathematical activity in the form of mathematical processes or mathematical practices (e.g., justifying, representing). A situated cognition perspective (Lave & Wenger, 1991) positing that how one learns mathematics and what one learns about mathematics are dependent on the context in which one engages in mathematical activity (Boaler, 2002; J. S. Brown, Collins, & Duguid, 1989) is used to think about learning mathematics and learning to teach mathematics. Mathematical activity in which learning is situated can be framed using mathematical processes. Four targeted processes of defining, generalizing, justifying, and representing from a processes and actions framework (Heid et al., 2015; Zbiek, Heid, & Blume, 2012; Zbiek et al., 2014) are used to study opportunities for mathematical activity in classrooms.

Students’ experiences with mathematics in the classroom, particularly with the mathematical tasks in which they engage, shape their views about mathematics. Teachers can help students form positive process conceptualizations of mathematics by selecting, modifying, and sequencing mathematical tasks (collectively referred to as task work) that provide opportunities for students to engage in mathematical processes. Yet, not all teachers provide such opportunities. Of particular interest are prospective mathematics teachers (PMTs), who might have personal experiences engaging in mathematical processes yet struggle to include such activity in their classrooms.

The construct of orientations toward mathematical processes (OMP) is used to
understand the nature of PMTs work with tasks. A mathematics teacher’s orientation toward mathematical processes (OMP) in school mathematics is a set of conceptions about mathematics, its teaching and learning related to mathematical processes, and engagement in mathematical processes that influences how a teacher provides opportunities for students to engage in mathematical processes. A situated view of teacher education (Borko et al., 2000) serves as a lens for investigating PMTs’ OMPs and how they might be related to PMTs’ task work. Specific research questions, briefly stated, are: What are PMTs’ OMPs, and how are PMTs’ OMPs related to how PMTs’ select, modify and sequence mathematical tasks?

Five secondary mathematics prospective teachers who had completed at least some of their student teaching experiences participated in the study. Data were collected using three qualitative interviews focused on components of OMP and the teaching activities of selecting, modifying and sequencing mathematical tasks to gather data for each of the participants. The definition of OMP with the processes and actions framework (Heid et al., 2015; Zbiek & Heid, 2012; Zbiek et al., 2014) guided the data analysis, which involved the creation of OMP profiles. An OMP profile is a description of an individual participants’ OMP organized around a theme and paralleling research questions about the nature of OMP and its relationship to task work.

An organization of profiles into types based on pedagogical and mathematical differences structures findings to the first research question. The construct of task template as a means of simplifying task work forms the basis for findings to the second research question. The notion of discovery figured prominently in participants’ OMPs and explains observations about PMTs’ OMPs. Insights into PMTs’ OMPs and their task work has potential implications for undergraduate teacher education and secondary mathematics curricula.
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Chapter 1
Rationale

Mathematics is more than memorizing facts and routine procedures; mathematics involves critical thinking, problem solving, as well as reasoning and sense making. Rich mathematical activities, such as problem solving and reasoning, are valuable in a mathematics classroom (National Council of Teachers of Mathematics [NCTM], 2000, 2007). Mathematical tasks are one method of incorporating mathematical activity into school mathematics (Borko et al., 2000; NCTM, 2007), but mathematical tasks evolve as they are implemented in the classroom (Stein, Remillard, & Smith, 2007; Stein, Grover, & Henningsen, 1996; Stein, Smith, Henningsen, & Silver, 2009) and the opportunities for mathematical activity in a task may vary. What and how teachers think about mathematics is one variable that influences the evolution of a task. In this study, I seek to understand relationships between prospective secondary mathematics teachers’ perceptions of mathematical activity and those teachers’ work with planning mathematical tasks.

In developing my research plan, I began with a consideration of learning and knowledge, as illustrated in the first box in Figure 1-1. I used that perspective of learning to consider how mathematics is learned and the means by which such learning occurs in the classroom. Learning is not at the foreground of my study, but it is useful to consider how teachers learn to teach and how students learn mathematics in a study involving teachers’ approaches to teaching and planning. Situated cognition captures learning through activity, positing that knowledge is situated in activity, and transcends content. Consequently, situated cognition could be used to describe both how PMTs learn to teach mathematics and how students learn mathematics.
The first box in Figure 1-1 represents the theory of learning that grounds the study and leads to a focus on mathematical activity and mathematical tasks. If knowledge and learning are situated in activity as suggested by situated cognition, I needed to consider the activity I wanted to study. PMTs’ knowledge about teaching mathematics might be situated in various activities. This study focuses on the pedagogical activity of planning in which PMTs engage with an additional focus on the possible mathematical activity of students in the classroom. Mathematical tasks can be used to study PMTs’ planning and possible student mathematical activity. They can provide opportunities for students to engage in mathematical activity. My study considers how PMTs’ work with mathematical tasks as they plan lessons and how PMTs’ work with those tasks impacts the kinds of mathematical activity students will encounter in their classrooms.

How PMTs’ work with mathematical tasks likely depends on sets of beliefs, perspectives, knowledge, and conceptions related to mathematics, teaching, and learning. The final box in Figure 1-1 is the focus of this study: orientations toward mathematical processes (OMP). I use this construct to investigate PMTs’ work with tasks and opportunities for students to learn mathematics through various kinds of mathematical activity.

Mathematical Activity in School Mathematics

I use the term mathematical activity to capture the doing of mathematics that extends beyond a particular content area, and includes activities such as reasoning, proving, problem solving, and conjecturing. I use a broad term in this rationale to establish importance of studying...
mathematical activity in school mathematics, and I use a processes and actions framework (Heid et al., 2015; Zbiek & Heid, 2012; Zbiek, Peters, Boone, Johnson, & Foletta, 2009; Zbiek et al., 2014) to formalize mathematical activity into four targeted processes in a later chapter. Figure 1-2 shows how the mathematical processes targeted in this study along with actions on products are one of several frameworks used to capture mathematical activity. Other examples of mathematical activity are included as sections of the circle graph. The four specific processes and common actions on products that are part of the processes and actions framework are listed in the rectangle. Elaborations of other frameworks of mathematical activity are not included in the figure. While the elements of the different frameworks for mathematical activity might overlap elements, the details of these overlaps are omitted from the figure and a simple recognition of overlap is represented using broken lines between sections of the circle graph. Some examples of how these different frameworks overlap and relate to the processes and actions framework are included in Chapter 2.
Figure 1-2. Frameworks for mathematical activity, with details of the processes and actions used in this study.

Mathematical activity is often included in mathematics education policy documents as an important part of school mathematics classrooms. NCTM (1989, 2000) included a list of process standards—focused on mathematical entities such as multiple representations, reasoning, and proof—as part of recommended mathematical standards for school mathematics. Similarly, the recent Common Core State Standards for Mathematics (CCSSM) (Common Core State Standards Initiative [CCSSI], 2010) identified eight mathematical practices for school mathematics to be incorporated in elementary, middle, and secondary mathematics classrooms. Writers of CCSSM developed the mathematical practices by consulting others’ work involving
mathematical activity, such as mathematical habits of mind (Cuoco, Goldenberg, & Mark, 1996) and mathematical proficiencies (National Research Council, 2001). Following the release of CCSSM, revised recommendations for the preparation of mathematics teachers, The Mathematical Education of Teachers II (MET II) (Conference Board of the Mathematical Sciences, 2012) through its focus on mathematical practices in prospective mathematics teachers’ (PMT) preparation, also illustrates the value of mathematical activity in mathematics education. Such standards and recommendations highlight the importance of and support the incorporation of mathematical activity in secondary mathematics classrooms.

Yet, mathematics teaching practice may not reflect policy documents and ideals. Despite past recommendations to incorporate mathematical activity in mathematics classrooms, we can find evidence that not all mathematics teachers adopted these recommendations, which means that mathematical activity may not be part of students’ mathematical experiences. For instance, The Mathematics Curriculum Framework for California Public Schools (California State Department of Education, 1985) advocated for the use of mathematical activity in mathematics classrooms, yet pertinent research on these California reforms (e.g., Cohen & Hill, 2001) indicates that policy may not be enacted as intended by policy writers. Even teachers who view themselves as successful enactors of policy may not incorporate mathematical activity into their practice as described in the framework (Cohen, 1990).

Similarly, research indicates that NCTM’s (2000) process standards may not be enacted in some secondary classrooms. Using data from the TIMMS study, Jacobs and colleagues (2006) found little evidence of NCTM’s process standards in eighth-grade mathematics classrooms. One could argue that teachers in the TIMMS study may have only had a peripheral awareness of the process standards at the time of the study. Years after the release of NCTM’s standards,
Frykholm (1999) found a gap between the ways in which PMTs talked about the standards and PMTs’ teaching practice. Such research suggests that policy is not always enacted as it is written or intended, so it would be reasonable to assume that mathematical activity was not part of many PMTs’ secondary mathematics experiences as secondary students.

Mathematical activity is also important when we consider how one learns mathematics. How one learns mathematics and what one learns about mathematics are dependent on the context in which one engages in mathematical activity (Boaler, 2002; J. S. Brown, Collins, & Duguid, 1989; Lave & Wenger, 1991). The kinds of mathematical activity in which students engage contribute to students’ learning of mathematics. I chose to use a situated cognition perspective (Lave & Wenger, 1991) because it acknowledges the nature of lived experiences, which in the case of learning mathematics involves engagement in mathematical activity.

A situated cognition perspective (Lave & Wenger, 1991) allows researchers and practitioners to acknowledge and value the experiences of PMTs with mathematics and with mathematics teaching and learning. Peripheral participation in the activity of a group is part of the enculturation process in the group (Borko et al., 2000; Lave & Wenger, 1991). PMTs’ secondary and collegiate mathematical experiences are part of their enculturation as mathematics teachers (Ball, 1990a). If we consider the enculturation of PMTs into the mathematics education community, PMTs from different eras may have would have different enculturation experiences, depending on emphases encountered in their teacher education programs. Thus, it becomes important to understand the influence of the mathematical experiences of PMTs today. PMTs need to be prepared to incorporate mathematical activity in the form of mathematical practices into their classrooms. Yet, incorporating mathematical activity is not an easy charge.
Mathematical Tasks

Consideration of mathematical tasks provides one important explanation for why mathematical activity may not be part of secondary school classrooms. In many ways, mathematical tasks form part of the foundation for how students engage in mathematics and can provide opportunities for students to engage in mathematical activity (Hiebert & Wearne, 1993; Krainer, 1993; National Council of Teachers of Mathematics, 2007). Although other aspects of mathematics classrooms influence opportunities for mathematical activity, mathematical tasks serve as a gateway for mathematical activity; it is difficult to generate mathematical activity from routine tasks. “Tasks provide the stimulus for students’ work in classrooms … As a result, they are an important influence on the mathematics learning that occurs in school” (Borko et al., 2000, p.197). Tasks influence the mathematics that is taught and provide messages to students about what mathematics entails (NCTM, 2007).

The nature of a mathematical task matters. I adopt Doyle’s (1988) definition of task, which emphasizes three components of a task: “(a) a goal or product; (b) a set of resources or ‘givens’ available in the situation; and (c) a set of operations that can be applied to the resources to reach the goal or generate the product” (p. 130, emphasis from author). Different tasks will have different goals, resources, and operations, each of which might affect the kinds of mathematical activity in which students engage in the classroom and what students learn about mathematics.

Furthermore, the goals, resources, and operations of a task might evolve as a task is implemented in the classroom (Doyle, 1988; Stein et al, 1996; Stein et al., 2009). A written task is the task as it is found in curriculum materials or elsewhere, which evolves to an intended task as the teacher incorporates the task into his/her instructional plan, which further evolves to an
enacted task as students and teacher engage in the task while in the classroom (Doyle, 1988; Stein, Grover, & Henningsen, 1996; Stein et al., 2009; Tarr et al., 2008). In this study, implementation of a task refers to the evolution of a task from written form to intended form to enacted form.

One could consider opportunities for students to engage in mathematical activity by examining written tasks (e.g., Haggarty & Pepin, 2002). For example, D. R. Thompson, Senk, and Johnson (2012) identified few opportunities for students to engage in justifying activities—one aspect of mathematical activity—in homework sets of twenty high school textbooks. One could also consider the shift from the intended task to the enacted task. Using a cognitive demand framework, Stein and colleagues (Stein et al., 1996; Stein & Lane, 1996; Stein et al., 2009) found relatively few “doing mathematics” tasks—tasks that would be most likely to provide maximal opportunities for engaging in mathematical activity as compared to “procedures with connections” tasks or “procedures without connections”—enacted in middle school classrooms. Furthermore, they found that mathematical tasks often declined in cognitive demand, meaning that intended tasks which teachers planned to use to provide opportunities for mathematical activity morphed into enacted tasks that did not provide opportunities for mathematical activity.

Opportunities for mathematical activity might also change as tasks shift from written tasks to intended tasks. It would be appropriate for a study involving PMTs to focus on this shift because PMTs would likely have experiences with the shift from written task to intended task as they plan lessons during their methods courses and practicum experiences.
Perceptions of Mathematics

How teachers perceive mathematics and its teaching and learning—including teachers’ beliefs, knowledge, and conceptions—offers possible explanations for changes in mathematical activity from the written task to the intended task (Stein et al., 2007). These perceptions of mathematics and its teaching and learning are related to both the implementation of mathematical tasks (Blömeke et al., 2008; Boston & Smith, 2009; Chapman, 2013; Henningsen & Stein, 1997; Lloyd, 2003; Osana, Lacroix, Tucker, & Desrosiers, 2006; Swan, 2007) and classroom practices (Bray, 2011; Cross, 2009; Dossey, 1992; Franke, Kazemi, & Battey, 2007; McGee, Polly, & Wang, 2003; Morris, Hiebert, & Spitzer, 2009; Philipp, 2007; Stipek, Givvin, Salmon, & MacGyvers, 2001; Wilkins, 2008).

If engaging in mathematical activity is to be an important part of secondary mathematics classrooms, then a study of how PMTs perceive mathematics and its teaching and learning as well as the kinds of mathematical opportunities PMTs provide students through mathematical tasks could provide potential explanations as to the presence of, or in some cases lack of, mathematical activity in secondary classrooms. Such findings could offer implications for teacher education programs, such as understanding of the possible difficulties PMTs have with promoting mathematical activity through mathematical tasks and recommendations for helping PMTs use mathematical tasks to promote mathematical activity.
Chapter 2

Development of the Construct Orientations toward Mathematical Processes

In this chapter I develop the main construct of my dissertation: Orientations toward Mathematical Processes (OMP). As part of developing this construct, I discuss relationships among beliefs, knowledge, and conceptions as well as how those constructs relate to orientations. I then formalize mathematical activity by identifying and defining four targeted mathematical processes—representing, generalizing, justifying, and defining. I state and interpret a definition of orientations toward mathematical processes that connects orientations and mathematical processes. Finally, I identify three activities with mathematical tasks—selecting, modifying and sequencing—that might be related to OMP. A situated cognition perspective serves as a theoretical backdrop for this study, and so I begin with an elaboration of situated cognition and use this perspective to develop the construct of study.

A Situated Cognition Perspective of Learning

Learning is a process of enculturation, by which the “newcomers become part of a community of practice” (Lave & Wenger, 1991, p. 29). A community is formed when a group of individuals interact and engage in common activities, called practices. When newcomers join a community, they begin by participating in some, but not all, of the practices of the community, and gradually participate in more practices until they become full participants in the community. Learning is the process of participating more fully in the community of practice, and as such learning is situated in the practices and context in which one engages (Lave & Wenger, 1991).

Knowledge and skills acquired through participation are intimately connected to the context and activity in which the knowledge and skills are developed and used (J. S. Brown et al., 1989; Peressini, Borko, Romagnano, Knuth, & Willis, 2004; Putnam & Borko, 2000). How
one learns a particular set of knowledge and skills and the situation in which that set is developed contributes to what precisely is learned (Putnam & Borko, 2000). To better understand how learning is situated, consider how knowledge and skills relate to tools (J. S. Brown et al., 1989), such as a hammer or a global positioning system. It is possible to acquire a tool and be unable to use it. As one uses a tool, one gains a deeper understanding of how to apply the tool effectively. Furthermore, how an individual learns to use a tool is dependent on how the tool is used by the community (i.e., the context of activity in which the community uses a tool). Like tools, knowledge and skills, or cognitive tools (J. S. Brown et al., 1989) such as an algorithm, develop richer meaning through use and are connected to the context in which they are used in a community of practice.

To further illustrate how learning occurs through and is situated in participating in community of practice, I consider two pertinent learning scenarios: learning mathematics and learning to teach mathematics. I chose to discuss these scenarios because the former relates to mathematical activity and the latter relates to using mathematical tasks to provide opportunities for engaging in mathematical activity, which are two key components of my research study.

**Learning mathematics from a situated cognition perspective.** From a situated cognition perspective (Boaler, 1999; J. S. Brown et al., 1989), one learns mathematics as one engages in the activity of a mathematical community. For example, consider the mathematical activity of proving mathematical claims. Engagement in such mathematical activity can begin peripherally as students first examine the structure of a completed proof, then move to offering ideas for how to complete a proof, and finally participate in full engagement in the completion of a mathematical proof. As students begin to examine a proof and then move to proving, their
understandings of what it means to prove in mathematics and how to prove in mathematics deepen.

Such understandings of proof are connected to the context in which students engage in the practice of proving. For example, a student who participates in proving in a community in which empirical evidence is considered proof would have a different understanding and skill set related to proof than a student who participates in proving in a community in which empirical evidence is considered insufficient proof and deductive reasoning is emphasized. In this study, I take the position that students learn mathematics best as they engage in mathematical activity and that such mathematical activity is situated in the context of the tasks and classrooms in which students engage (Borko et al., 2000).

**Learning to teach mathematics from a situated cognition perspective.** We can also view learning to teach mathematics from a situated cognition perspective. When a prospective mathematics teacher (PMT) decides to become a mathematics teacher, he or she joins a community of mathematics teachers and begins to participate in that community in peripheral ways. A PMT’s participation in the community increases as a PMT engages in teaching practices (Peressini et al., 2004; Putnam & Borko, 2000). For example, PMTs may begin by planning mathematical tasks, focusing on the shift from written task to intended task, before enacting tasks in a classroom.

What a PMT learns about planning mathematical tasks depends on the context in which planning mathematical tasks is learned. Richer learning will occur if a PMT has opportunities to engage in the practice of planning than if a PMT is only told principles of planning mathematical tasks. Furthermore, what a PMT learns about planning mathematical tasks is related to not only what is learned during the process of planning but also what is emphasized or valued by the
community. For example, a methods course instructor may emphasize or value careful questioning in the planning process whereas a mentor teacher may emphasize or value classroom grouping in the planning process. In this study, I examine PMTs’ work with mathematical tasks in the planning phase of instruction as they select, modify and sequence mathematical tasks.

**Beliefs, Knowledge, or Conceptions**

Learning is situated in a community of practice, and that community and individuals in that community hold sets of beliefs, knowledge, and conceptions that contribute to the practices of the community. In this section, I explore various uses of the terms beliefs, knowledge, and conceptions in mathematics education as part of developing the construct of orientation. I acknowledge that while distinctions between knowledge and beliefs may be valuable depending on the nature of one’s study, a distinction between knowledge and beliefs may not be needed if the nature of the study is to understand how those knowledge and beliefs—which might be collectively referred to as conceptions (Philipp, 2007)—influence teaching.

The terms belief and knowledge are not used consistently throughout mathematics education research (Pajares, 1992; Philipp, 2007). In an effort to establish some consistency, Philipp (2007) proposed the following definition and description of beliefs: “Psychologically held understandings, premises, or propositions about the world that are thought to be true … Beliefs, unlike knowledge, may be held with varying degrees of conviction and are not consensual” (p. 259). Here, Philipp describes beliefs as thoughts or ideas that are considered to be true. Beliefs may be held individually or among a group of individuals, such as a community of practice, and may be held with varying conviction and may not be consensual among all members of a community.
Some studies emphasize the difference between knowledge and beliefs whereas other studies do not distinguish between knowledge and beliefs. I claim that the importance of distinguishing between knowledge and beliefs likely depends upon the nature of the research study. Studies focused on teacher’s knowledge, particularly studies focused on assessing teacher’s knowledge (e.g., Hill, Ball, & Schilling, 2008), seem to be concerned with a match between a teacher’s knowledge and the knowledge held by a community. Distinguishing between knowledge and beliefs is helpful in such a research study because researchers are interested in evaluating a teacher’s knowledge to determine whether a teacher’s ideas are held with conviction and are in consensus with the rest of the community.

In contrast, such a distinction would not be necessary in a research study concerned with what a teacher thinks rather than evaluating a teacher’s demonstrated knowledge (Creswell, 2007). In some studies, “the important question for researchers studying mathematics teachers’ beliefs and knowledge is not whether some conception is true in an ontological way but how a teacher views the conception” (Philipp, 2007, p. 267). Philipp emphasizes that for some studies, it is not requisite to consider whether what an individual believes would be considered true by others, but instead should consider how such beliefs are held; such studies deemphasize the distinction between knowledge and beliefs. When this is the case, it might be useful to use a term, such as conception, that encompasses both knowledge and beliefs (A. G. Thompson, 1992). In this study, I use the term conceptions to collectively refer to both knowledge and beliefs because the focus of my study is on how a PMT conceptualizes mathematics and its teaching and learning.

We can think about the role of conceptions in learning from a situated cognition perspective. Within a community of practice, one learns as one aligns his or her conceptions with
the conceptions of the community. This alignment of conceptions occurs as one participates in community practices and interacts with members of the community. Conceptions are a key part of the learning process as an individual moves toward full participation in the community. Yet, to better capture the role of context in the learning process, I consider another term, orientation.

**Characteristics of Orientation**

Colloquially, orientation is often used to describe one’s tendency toward something or relative positioning of something. By examining definitions and descriptions from mathematics education research and science education research, I identify characteristics of orientations to aid in the development of a definition of orientation. The four descriptions shown in Table 2-1 are a list of formal descriptions of orientation that I could find via internet searches for orientation in mathematics and science education, examining primary mathematics education journals (Educational Studies in Mathematics, Journal for Research in Mathematics Education, Journal of Mathematical Behavior, Journal of Mathematics Teacher Education), and two science education journals (The Journal of Science Teacher Education, Journal of Research in Science Teaching). Using these four definitions, I discuss two characteristics that can be used to distinguish conceptions from orientations. I incorporate these characteristics into a definition of orientation toward mathematical processes, which will be discussed later in this chapter.
Table 2-1  

*Definitions/Descriptions of Orientations*

<table>
<thead>
<tr>
<th>Author(s), Year</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cess-Newsome, 1999</td>
<td><em>Content-specific teaching orientations</em> will be defined as teachers’ beliefs about teaching and learning that are grounded in a framework used for organizing knowledge in their field” (p. 57, emphasis added). “A teacher’s orientation toward his/her content is a complex combination of content knowledge, beliefs, and values that have the potential to impact what and how students learn about their content” (p. 58, emphasis added).</td>
</tr>
<tr>
<td>Magnussen et al., 1999</td>
<td>Orientation toward teaching science is a “component of pedagogical content knowledge [that] refers to teachers’ knowledge and beliefs about the purposes and goals for teaching science at a particular grade level” (p. 97, emphasis added).</td>
</tr>
<tr>
<td>Remillard &amp; Bryans, 2004</td>
<td>“We define this orientation [toward a curriculum] as a set of perspectives and dispositions about mathematics, teaching, learning, and curriculum that together influence how a teacher engages and interacts with a particular set of curriculum materials and consequently the curriculum enacted in the classroom and the subsequent opportunities for student and teacher learning.” (p. 364, emphasis added)</td>
</tr>
<tr>
<td>Schoenfeld, 2010</td>
<td>“I use the term orientations as an inclusive term encompassing a group of related terms such as dispositions, beliefs, values, tastes, and preferences. How people see things (their “worldviews” and their attitudes and beliefs about people and objects they interact with) shapes the very way they interpret and react to them” (p. 29, italicized emphasis from author, bold emphasis added)</td>
</tr>
</tbody>
</table>
From an examination of these descriptions of orientations two important characteristics of orientations emerge: (a) orientations consist of combinations of constructs internal to the person, and (b) orientations involve an element of influence. Each description characterizes orientations as consisting of terms related to beliefs, such as dispositions, perspectives, and preferences, and two of the descriptions (i.e., Cess-Newsome, 1999; Magnusson, Krajcik, & Borko, 1999) include knowledge in addition to beliefs. Cess-Newsome (1999) and Remillard and Bryans (2004) characterize orientation in terms of a set or complex combination of beliefs and knowledge. I use the term conception to capture the combination of beliefs and knowledge of a particular construct. Additionally, each of the descriptions includes an element of influence; that is, the combination of beliefs and knowledge influences the way an individual interprets and reacts to a situation. I incorporate these two characteristics of orientations in my definition of orientations stated toward the end of the chapter.

**Influence within context.** To better illustrate the importance of context in orientation, consider the following situation. A teacher approaches a group of students working on a mathematical task involving uniform motion and notices particular elements of the contextualized situation, such as interactions among students or aspects of written work. Conceptions held by the teacher, such as knowledge of individual students or of common student errors, influence what the teacher attends to and how the teacher interprets the situation. The teacher, aware of the students’ difficulty with understanding of rate in uniform motion, might then ask a question of the group relating distance, speed, and time: what can you tell me about the distances driven if two cars drive at different speeds for the same amount of time? Alternately, the teacher after listening to a group discuss how to represent the given information that one car left two hours later, might recognize different interpretations of time and ask the
group to describe what time might mean in context of the problem. The kind of question asked is influenced by how the teacher interpreted the context, which is influenced by the teacher’s conceptions.

**Subsets of conceptions.** Conceptions influence how one interprets and reacts to a particular context, but not all of an individual’s conceptions influence the interpretation and reaction to a situation. Instead, only a subset of an individual’s conceptions contributes to the influence. Returning to the example in the previous paragraph of a teacher interpreting and reacting to the situation of a group of students working on a mathematical task involving uniform motion, the teacher might interpret the group’s mathematical work based on conceptions of students’ difficulty with rate in general but not necessarily for the particular students in the group. Alternately, the teacher’s decision to listen to the group’s discussion of time prior to asking a question might be based on conceptions about learning mathematics within groups but not necessarily conceptions about learning mathematics in individual settings. Not all of the teacher’s conceptions about teaching and learning influenced the teachers’ reaction to the situation. Instead, a subset of those conceptions that the teacher viewed as pertinent to the situation contributed to the teachers’ positioning with respect to the working students. So, an orientation consists of a **subset** of conceptions that influences how one acts within a particular situation or context.

Within mathematics teaching, a teacher or PMT might have orientations toward different aspects of teaching. For example, a teacher might have an orientation toward a set of curriculum materials (Remillard & Bryans, 2004) that influences how they read, adapt, and enact curriculum. Or a teacher might have an orientation toward discourse (A. G. Thompson, Philipp, Thompson, & Boyd, 1994) that influences the kinds of student thinking elicited by the teacher or
the kinds of student thinking valued by the students. A teacher might also have an orientation
toward student errors (Bray, 2011) that influences the way a teacher might handle observed
student errors. I am interested in studying PMTs’ orientations toward mathematical processes,
and I identify and define four mathematical processes in the next section.

**Four Targeted Mathematical Processes**

As described in the first section of this chapter, learning mathematics from a situated
cognition perspective involves participating in the mathematical activity of a community (Lave
& Wenger, 1991), and that learning is situated in the activity and context in which the activity
takes place (J. S. Brown et al., 1989; Putnam & Borko, 2000). Several different schematics for
organizing mathematical activity (e.g., CCSSI, 2010; Cuoco et al., 1996; NCTM, 2000; Tall,
1999) exist. I selected four mathematical processes—representing, generalizing, justifying, and
defining—based on the processes and actions framework developed by Zbiek and colleagues
(2009, 2014) and Heid and colleagues (2015) to investigate as part of this study. In this section, I
state definitions of each of the processes, interpret and exemplify the definitions, and compare
the four processes to other schematics of mathematical activity.

Within the processes and actions framework (Heid et al., 2015, Zbiek & Heid, 2012;
Zbiek et al., 2009, Zbiek et al. 2014) identified the four mathematical processes by reviewing
literature as part of a large study on beginning teachers’ mathematical knowledge. Their
definitions of each of the four processes are included in Table 2-2. The four processes were
identified as mathematical activities in which secondary students could reasonably engage and
mathematical activities that would span secondary content areas. Experiences with the
Pythagorean theorem could involve all four different processes, and, as will be explained in the
methodology chapter, the Pythagorean theorem is one mathematical context I used to investigate
orientations toward mathematical processes. Here, I use the context of the Pythagorean theorem to exemplify each of the four mathematical processes.

Table 2-2

*Selected Mathematical Processes and Their Descriptions*

<table>
<thead>
<tr>
<th>Mathematical Process</th>
<th>Description of Process</th>
</tr>
</thead>
<tbody>
<tr>
<td>Generalizing</td>
<td>Generalizing is the constructive act of extending the domain to which a set of properties applies, either from multiple instances to a class that includes those instances or from a subclass to a larger class of mathematical entities, perhaps adjusting properties to accommodate the larger class.</td>
</tr>
<tr>
<td>Justifying</td>
<td>Justifying (including proving) is the constructive act of explaining how one knows a mathematical claim is true or producing a rationale for belief of a mathematical claim.</td>
</tr>
<tr>
<td>Representing</td>
<td>Representing is the constructive act of creating external visuals, physical objects, verbiage or movements intended to capture properties of a mathematical concept, procedure, or principle.</td>
</tr>
<tr>
<td>Defining</td>
<td>Defining is the constructive act of identifying and articulating, for a given mathematical entity, a set of mathematical characteristics and mathematical relationship (or relationships) involving these characteristics.</td>
</tr>
</tbody>
</table>

(Zbiek et al., 2009, p. 9)

The definition of *generalizing* captures two ways in which generalizing might occur. One way involves looking across a set for common relationships and extending those relationships to
an entire class. For instance, a student might examine the side lengths of several right triangles, notice a relationship among the squares of those side lengths, and extend that relationship to all right triangles. The activity of producing a generalization about the relationship among the side lengths of a right triangle is generalizing. It is also possible to generalize by adjusting relationships or properties of a subclass to include a larger class. For example, a student could examine the squares of side lengths of several non-right triangles and adjust the relationship among squares of side lengths in the Pythagorean theorem to accommodate all triangles. The activity of loosening the conditions of the Pythagorean theorem is generalizing and results in a generalization about different kinds of triangles. In particular, if a triangle is obtuse, then the sum of the squares of lengths of the two shorter sides would be less than the square of the length of the longest side.

Justifying is a mathematical process that involves providing reasoning for why a mathematical claim is valid, and includes proving mathematical claims. A student might use two diagrams involving squares and four congruent right triangles to provide a justification for why the Pythagorean relationship among side lengths is true for all right triangles.

Representing is the activity of creating an external representation of a mathematical object. The definition of representing in Table 2-2 identifies several different forms of representations (e.g., movements, verbiage, models, written visuals) that might be produced as part of representing as well as several different mathematical entities that might be represented (e.g., properties, procedures, concepts). For example, students might create a right triangle with squares formed along each side of the right triangle as a model of the Pythagorean theorem. Or, a student might arrange paper cutouts of right triangles and squares in a way that illustrates a Pythagorean relationship among the squares and the triangles.
Finally, defining is the mathematical process in which mathematical characteristics and properties are identified and expressed for a particular mathematical object. For instance, the mathematical object of a Pythagorean triple consists of three natural numbers, $a$, $b$, and $c$, that satisfy the Pythagorean relationship, $a^2 + b^2 = c^2$. The process of identifying a set of essential characteristics and properties needed for a set of three numbers to be considered a Pythagorean triple (i.e., the numbers are natural numbers and satisfy the Pythagorean relationship) is defining and results in the creation of a definition of Pythagorean triples.

In addition to being processes in which secondary students can engage, each of these four processes is valued in mathematics education culture. For instance, conceptual understanding, which “refers to an integrated and functional grasp of mathematical ideas,” (National Research Council, 2001, p. 118) is related to defining, in which students create definitions of mathematical ideas using concept images (Tall & Vinner, 1981). The process of representing and the action of linking representations is promoted in NCTM’s (2000) Representation Standard, which encourages students to create and translate among representations of mathematical entities as part of learning mathematics. The more recent push for reasoning and sense-making in mathematics classrooms (NCTM, 2009) as well as several mathematical practices (e.g., construct viable arguments, look for regularity in repeated reasoning) (CCSSI, 2010) are connected to the processes of justifying and generalizing. Because these four processes are related to other concepts in mathematics education, it is reasonable to question what these four processes afford that other characterizations of mathematical activity do not.

**Products and actions.** Each of the four processes results in the creation of a mathematical product on which teachers or students act. Figure 2-1 includes an illustration of relationships among the four processes and products as well as possible actions one might take.
on the different mathematical products. For example, engaging in the process of generalizing produces a generalization that can be applied to other contexts, such as applying the Pythagorean theorem (a generalization) to determine the distance a ladder of a given height should be placed from a wall to reach a specified height along the way.

![Diagram of processes, products, and actions](image)

*Figure 2-1. Diagram of processes, products, and actions.*

The four targeted mathematical processes are generative in that they result in the creation of mathematical products. Other schematics for mathematical activity include processes or practices that do not result in mathematical products, involve multiple processes and products, or focus on actions rather than processes. The Communication Standard (NCTM, 2000) emphasizes the process of communicating mathematical ideas, but engaging in a process of communicating does not lead to the creation of mathematical products. Problem solving (CCSSI, 2010; NCTM, 2000) would likely involve several processes and actions on different products. In comparison to problem solving, which encompasses both processes and actions on products, many of the mathematical practices emphasize actions on products rather than processes. For example, the description of the mathematical practice of modeling with mathematics emphasizes interpreting,
analyzing, and applying different mathematical products (CCSSI, 2010), though the practice could involve the process of representing.

If learning mathematics is intimately connected to activity and context, as is posited by the situated cognition perspective of learning, then whether students engage in mathematical processes or actions on mathematical products determines what students learn about mathematics as well as the kind of mathematics learned. For instance, if students primarily encounter opportunities to act on products then they might learn that mathematics is something created or discovered by others and then used by students. Additionally, students might only learn to act on products rather than to engage in processes. On the other hand, if students also encounter many opportunities to engage in mathematical processes, they might learn that mathematics is something they can explore and create, and they learn how to define, justify, generalize, and represent.

A teacher might provide opportunities for students to engage in mathematical processes or a teacher might provide only opportunities for students to engage in actions on mathematical products. I have developed the construct of orientation toward mathematical processes to characterize a teacher’s or PMT’s tendency to provide such mathematical opportunities.

**A Definition of Orientation toward Mathematical Processes**

In the section on orientation, I used definitions or descriptions of orientation in the literature to identify characteristics of an orientation. Specifically, that an orientation consists of subsets of conceptions that have impact within a context. In my definition of orientation toward mathematical processes, I apply characteristics of orientation to the context of mathematical processes, and define it as follows: A mathematics teacher’s orientation toward mathematical processes (OMP) in school mathematics is a set of conceptions about mathematics, its teaching
and learning related to mathematical processes, and engagement in mathematical processes that influences how a teacher provides opportunities for students to engage in mathematical processes.

**Four components of orientations toward mathematical processes.** The definition of OMP identifies four *components* as they relate to mathematical processes: conceptions of mathematics, engagement in mathematical processes, conceptions of learning mathematics, and conceptions of teaching mathematics. A PMT’s conceptions of mathematics related to the four processes include what a PMT thinks the four processes are and their respective roles in mathematics. How a PMT engages in mathematical processes and his/her tendency to engage in processes also influences the kinds of opportunities for engaging in mathematical processes that he/she provides for students. Hence, engagement in processes is also considered as a component of OMP. Conceptions of learning related to mathematical processes include how a PMT conceptualizes the learning of mathematical processes along with views about students’ abilities to engage in mathematical processes. Finally, a PMT’s conceptions of teaching related to mathematical processes include what a PMT thinks teaching each of the processes involves and how to incorporate mathematical processes in the classroom. Conceptions of teaching could also include conceptions of tasks and curricula related to mathematical processes, such as what makes a good justifying or representing task.

My definition of OMP is consistent with the definitions of orientation identified in Table 2-1, especially with Magnussen and colleagues’ (1999) definition. Each of the four constructs related to mathematical processes identified in the definition of OMP capture elements of pedagogical content knowledge (Magnusson et al., 1999; Shulman, 1986) and mathematical knowledge for teaching (Hill et al., 2008). Such a similarity is not coincidental because
Magnussen and colleagues (1999) based their construct of orientations toward science teaching on the work of Grossman and Stodolsky (1995) and Shulman (1986), and included orientations as part of pedagogical content knowledge, which also includes knowledge of instructional strategies, knowledge of students’ understandings, and knowledge of content. In my definition of OMP, I include conceptions of teaching (including knowledge of instructional strategies), conceptions of learning (including knowledge students’ understandings), conceptions of mathematics (including knowledge of content), and engagement in mathematical practices (which also includes some aspects of knowledge of content).

**Opportunities for students to engage in mathematical processes.** The set of conceptions that make up a PMT’s OMP are not just the set of conceptions related to mathematical processes, but are the set of conceptions that influence opportunities for students to engage in mathematical processes. Opportunities for students to engage in mathematical processes may be influenced by many factors. For example, a teacher’s questioning and emphasis placed on calculations, indicative of a calculational orientation toward mathematics teaching (A. G. Thompson et al., 1994), draws students’ attention toward manipulation of representations and could decrease opportunities for engaging in processes. Or, teachers’ conceptions can influence how teachers choose student work to be presented and the order in which it is presented (Stein, Engle, Smith, & Hughes, 2008), which has the potential for increasing or decreasing opportunities for engaging in mathematical processes.

**Three Targeted Activities in Planning Mathematical Tasks**

Use of mathematical tasks is one way that teachers might influence opportunities for mathematical processes in their classroom. Tasks “convey messages about what mathematics is and what doing mathematics entails” (NCTM, 1991. p. 24). Tasks that provide opportunities for
engaging in mathematical processes convey a message of what it means to do mathematics that is different from the message conveyed through tasks that do not obviously provide opportunities for engaging in mathematical processes.

The ways teachers interact with tasks might change the message conveyed about what it means to do mathematics and possible opportunities for engaging in mathematical processes. The ways in which teachers interact with tasks might seem related to the ways they interact with curriculum materials. Several researchers have suggested the following activities pertinent to teacher work with curriculum materials: reading and interpreting (Behm & Lloyd, 2009; M. Brown, 2009; Remillard, 1999; Sherin & Drake, 2009), evaluating (Roth McDuffie & Mather, 2009; Sherin & Drake, 2009), selecting (M. Brown, 2009), analyzing (Roth McDuffie & Mather, 2009), and adapting (Behm & Lloyd, 2009; M. Brown, 2009, Nicol & Crespo, 2006; Sherin & Drake, 2009). While these actions of reading and interpreting, evaluating, selecting, analyzing and adapting are useful for organizing teacher work with curriculum materials, they do not apply equally as well to individual tasks. For example, reading curriculum materials often refers to reading tasks in addition to instructional suggestions (Remillard, 1999) and adapting curriculum materials is often described as omitting and adding tasks within a curriculum (Drake & Sherin, 2009) rather than just modifying individual tasks. While these curricular activities involve tasks, they apply more broadly to curricular materials rather than individual tasks. Additionally, many of the above curricular activities are not observable or distinct. For example, the activities of interpreting and analyzing may not be observable and a teacher might evaluate curricular materials as they engage in reading curricular materials.

Based on the various curricular activities found in the research, I selected three distinct and observable teaching activities that are related to individual tasks and in which teachers might
engage as they plan mathematics lessons. The planning of mathematical tasks is important because it results in an intended task that influences the task as it is enacted in the classroom (Stein et al., 2000) and matters in a study with PMTs because planning is one practice of teaching with which they likely have experiences. First, a teacher might select individual tasks to include in a lesson or group of lessons (M. Brown, 2009; Tarr, Chavez, Reys, & Reys, 2006). Second, a teacher might modify a selected task to meet the needs of students (M. Brown, 2009) or to appeal to student affect (Nicol & Crespo, 2006). Finally, teachers determine the sequence in which the tasks will be used in the classroom. The activity of sequencing was not noted in the research on curricular activities likely because curriculum materials already suggest a task sequence but it is reasonable that teachers might reject the suggested sequence of tasks within a curriculum in favor of a different sequence (Drake & Sherin, 2009). These task activities may not occur sequentially or with mutual exclusivity. For example, a teacher might sequence an initial set of tasks and then select and modify another task to fit within the sequence.

The ways in which teachers work with tasks—selecting, modifying, and sequencing—could have implications for the kinds of mathematical opportunities afforded to students. As teachers select and modify tasks, a written task evolves into an intended task, which might afford opportunities for engaging in mathematical processes different from the written task. Tasks that provide opportunities for students to engage in mathematical processes would, according to a situated cognition perspective, provide more opportunities for learning mathematics than tasks that only provide opportunities for engaging in actions on products. Consider the three Pythagorean theorem tasks shown in Figure 2-2, Figure 2-3, and Figure 2-4 below, which I use to illustrate how selecting, modifying, and sequencing tasks can impact opportunities for engaging in processes.
Pythagorean Rugs Proof Task

Al and Betty have a game. They began with a right triangle, which has legs of lengths \( a \) and \( b \) and a hypotenuse of length \( c \). Then they made the two square rugs shown below. Each rug has sides of lengths \( a + b \), and the triangles within each square are the same as the single right triangle shown at the right.

When it’s Al’s turn, a dart drops on the square rug on the left. If it hits the shaded area, he wins a point. When it’s Betty’s turn, the dart falls on the square rug on the right. If it hits the shaded area, she wins a point. Assume that the darts always hit the rugs, but that they always fall randomly within the rug. In other words, all points on a rug have the same chance of being hit.

1. Is this a fair game? That is, is the chance of that dart landing on the shaded area the same for the two rugs? Explain your answer.

2. How do the two rugs demonstrate that the Pythagorean theorem holds true in general?

(Fendel, Resek, Alpher, & Fraser, 2003, p. 232)

*Figure 2-2. The Pythagorean Rugs Proof Task.*
Square-in-Square Proof Task
Over the centuries, many people have proven the Pythagorean theorem. In this exploration, you examine one of these proofs.

a. Create two squares as shown in the figure below. The four triangles are all congruent right triangles. The longest sides of the four right triangles form the smaller square.

b. Determine the area of each right triangle.

c. Determine the area of the larger square.

d. Determine the area of the smaller square.

e. Describe how to find the area of the smaller square using the areas of the right triangles and the area of the larger square.

f. Rewrite your formula from part e using the lengths $a$, $b$, and $c$ shown in the figure.

Figure 2-3. The Square-in-Square Proof Task.
Generalizing via Dot Paper Task

1. For each row, draw a right triangle with the given side lengths on dot paper. Then, draw a square on each side of the triangle.

<table>
<thead>
<tr>
<th>Length of leg 1</th>
<th>Length of leg 2</th>
<th>Area of square on leg 1</th>
<th>Area of square on leg 2</th>
<th>Area of square on hypotenuse</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
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<td>3</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. For each triangle, find the areas of the squares on the legs and on the hypotenuse. Record your results.

3. Look for a pattern in the relationship among the areas of the three squares. Use the pattern you discover to make the conjecture about the relationship among the areas.

(Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998, p. 28)

Figure 2-4. The Generalizing via Dot Paper Task.

A PMT or teacher might have access to two or more written tasks with very similar content or purpose and need to select one of the tasks to incorporate into their lesson plan. The Pythagorean Rugs Proof task (Figure 2-2) and Square-in-Square Proof task (Figure 2-3) both involve diagrams involving squares and right triangles that can be used to justify the Pythagorean theorem. Yet, the Pythagorean Rugs Proof task arguably provides more opportunities for students to engage in mathematical processes because students are asked to justify that the Pythagorean theorem holds true with little additional guidance. Although students interpret a justification of the Pythagorean theorem in the Square-in-Square Proof task, the task is broken
down into steps that guide the student through each step of a justification without actually having the students engage in a mathematical process. Instead, students act on mathematical products, such as *applying* a generalization to find the area of a right triangle. Selecting Pythagorean Rugs Proof task over Square-in-Square Proof task would offer different opportunities for engaging in mathematical processes.

Just as selecting tasks impacts opportunities for mathematical processes in the shift from written to intended task, modifying mathematical tasks can also affect opportunities for engaging in mathematical processes. For example, a PMT could modify the Pythagorean Rugs Proof task, which provides opportunity for *justifying*, by adding additional directions, such as “Al’s rug and Betty’s rug have the same area. If the four congruent triangles are removed from each rug, what can be said about the remaining areas of each respective rug?” By adding these more directive instructions, the PMT has modified the task so that students only need to *interpret* a representation rather than *justify* a claim, effectively lessening opportunities to engage in mathematical processes.

A task may be modified in a way that increases opportunity to engage in actions and mathematical processes. Learning how to effectively modify mathematical tasks to promote mathematical opportunity may be valuable for a PMT or teacher working with a curriculum that offers limited opportunities for engaging in mathematical processes. For example, a PMT could modify the Square-in-Square Proof task by omitting the directions for parts a-f along with the labels a, b, and c on the diagram and then adding the following directions: “Explain how the diagram below relates to the Pythagorean theorem and then use that relationship to explain why the Pythagorean theorem is true.” In the written task, there were few opportunities to engage in mathematical processes and the focus of the task was on *applying* generalizations and
interpreting and manipulating representations—actions on mathematical products. Modifying the directions of the task provided opportunities for students to engage in a mathematical process of justifying.

Sequencing tasks is another planning activity that could impact opportunities for students to engage in mathematical processes during the shift from written task to intended task. For example, let’s say that a PMT selects to use the Pythagorean Rugs Proof task and the Generalizing via Dot Paper task (Figure 2-4) in a unit on the Pythagorean theorem. The Pythagorean Rugs Proof task provides an opportunity for students to engage in justifying the Pythagorean theorem, and the Generalizing via Dot Paper task provides students with an opportunity to generalize by forming a conjecture similar to the Pythagorean theorem. If the PMT chooses to sequence the tasks by having students work on Generalizing via Dot Paper task first and Pythagorean Rugs Proof task second, then the respective opportunities for engaging in mathematical activity remain intact. On the other hand, if the PMT chooses to sequence the tasks in the reverse order, then the Generalizing via Dot Paper task would no longer offer an opportunity for generalizing because students will have already been introduced the Pythagorean theorem in the Pythagorean Rugs Proof task.

Selecting, modifying, and sequencing are three distinct planning activities in which a teacher might engage during the shift from written to intended task. These activities have the potential to influence the opportunities for engaging in mathematical processes afforded students in classrooms, as illustrated using the tasks in Figure 2-2, Figure 2-3 and Figure 2-4. How teachers select, modify, and sequence mathematical tasks is likely related to their OMP.

In this chapter, I defined OMP using four components: conceptions of mathematics related to mathematical processes, conceptions of teaching related to mathematical processes,
conceptions of learning related to mathematical processes, and engaging in mathematical processes. A teachers’ OMP influences how they interpret mathematical tasks during implementation and can impact the evolution of a task from its written to intended phase, which ultimately determines in part the mathematical processes opportunities afforded students in the classroom. Understanding PMTs’ OMP and how OMP influences the planning of mathematical tasks can provide explanations for the mathematical opportunities observed in secondary classrooms.
Chapter 3

Teachers’ Orientations toward Mathematical Processes and Planning Mathematical Tasks

Much research has been done on teachers’, including prospective secondary mathematics teachers’ (PMTs’), conceptions of mathematics and its teaching and learning. I use a synthesis of this research to describe both what is already known about orientations toward mathematical processes (OMP) as I have defined the construct in Chapter 2 and ways teachers and PMTs interact with mathematical tasks.

I used Internet searches along with reading abstracts of recent publications from mathematics education journals and consulting referenced works in articles and research synthesis chapters to identify relevant literature. I used search terms such as “teacher mathematics beliefs/conceptions,” “teacher conceptions/beliefs of [mathematical processes1],” “orientations toward teaching,” and “teachers knowledge of [mathematical product2]” to identify literature related to teacher’s OMPs. I used search terms such as “mathematical tasks,” “teacher’s use of mathematics curriculum,” “problems-based curriculum,” “selecting mathematical tasks,” “modifying/adapting mathematical tasks,” “sequencing mathematical tasks,” and “problem-posing” to identify literature related to teachers’ use of tasks. To identify literature related to either OMP or task work, I examined abstracts from top tier mathematics education journals3 published since 2009. I also read related abstracts from conference proceedings published since 2013. Additionally, I used chapters related to teachers’ beliefs, knowledge, and curriculum use

1 Mathematical processes searched for included defining, justifying/proof, representing, generalizing, modeling, abstraction, problem solving, and reasoning and sense making.
2 Mathematical products searched for included definitions, justifications/proof, generalizations, and representations.
from mathematics education research handbooks to identify additional useful literature as well as develop a better understanding of the landscape of related literature. I used referenced works in the identified articles and handbook chapters to find other useful sources such as edited books.

I organize this literature synthesis into two parts: a section on the four individual components of OMP and a section on teachers’ mathematical planning activities. The four individual components of OMP are (a) conceptions of processes and their products, (b) conceptions of teaching related to mathematical processes, (c) conceptions of learning related to mathematical processes, and (d) teachers’ engagement in processes. I structured the section on teachers’ mathematical planning activities around the work of selecting, modifying and sequencing mathematical tasks. I conclude the chapter with a statement of my research questions.

Conceptions of Mathematics Related to Mathematical Processes

Conceptions of mathematics related to mathematical processes is one component of OMP and includes what teachers think a mathematical process is and what they think is the role of that process in mathematics. Studies of teachers’ conceptions of mathematics fall along a static–dynamic continuum. Researchers use various terms to describe conceptions along this continuum, such as absolutist or fallibilist (Lerman, 1990); or instrumentalist, Platonist, and problem solving view (Ernest, 1999). One who views mathematics as consisting of mathematical processes, which by definition involve the creation of mathematical products, would likely have a dynamic conception of mathematics. In comparison, one who views mathematics as consisting of actions on mathematical products would likely have a static conception of mathematics.

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4 Unless specifically noted, the term teachers will include both practicing and prospective mathematics teachers. I am careful to identify the specific population when citing results of a study.
because acting on products does not necessarily create new mathematics. I use a static–dynamic continuum as a framework to describe research on conceptions of each process to highlight conceptions of mathematics as processes in comparison to conceptions of mathematics as actions on products.

**Conceptions of justifying and justifications.** The mathematical community has identified various roles of justifying in general, and proving in particular, in mathematics. These roles include verifying the truth of a mathematical claim, explaining why a claim is true, communicating mathematical knowledge, creating or discovering new mathematical knowledge, and systematizing mathematical statements into an axiomatic system (e.g., de Villiers, 2004; Hanna, 1990; Hersh, 1993; Knuth, 2002a, 2002b; Staples, Bartlo, & Thanheiser, 2012). The first three roles in this list could be viewed as acting on already created mathematics and might be associated more with a static conception of mathematics. Whereas the last two roles in this list, which explicitly deal with the creation and organization of new mathematics, would be associated with a dynamic conception of mathematics. One could argue that a rich conception of justifying and proof in mathematics would incorporate each of these roles in mathematics.

Teachers tend to identify some, but not all, roles of proof in mathematics. A view of proof as verifying mathematical claims seems to dominate teachers’ views of proof in mathematics (Conner, Edenfield, Gleason, & Ersoz, 2011; Knuth, 2002a; Steele & Rogers, 2012), whereas other roles of proof seem to be identified with less frequency among teachers (Conner et al., 2011; Jones, 1997; Knuth, 2002a). Teachers rarely mention the role of proof in the discovery of new mathematics (Conner et al., 2011). The dominant view of justifying as verification of claims, presumably using a justification used by past mathematicians could indicate a prevailing static conception of justifying among teachers.
Researchers have found variability in teachers’ identification of the explanatory role of proof in mathematics. Conner and colleagues (2011) as well as Steele and Rogers (2012) found that teachers identified an explanatory role of proof in mathematics during interviews, and an explanatory role was implied in the concept maps generated by prospective teachers in Jones’ (1997) study. In contrast, Knuth (2002a) found that an explanatory role of proof was notably absent from teachers’ discussions of proof in an interview setting. This discrepancy could be related to the meaning of explain held by the researcher. Knuth (2002a) used explain to mean “the promotion of understanding or insight” (p. 389) and found that although three teachers mentioned proof as useful in explaining why something is true, the focus of their descriptions of proof was on why something works rather than on generating understanding. Yet, Jones (1997) does not elaborate on the meaning of explain as it appears in prospective teachers’ concept maps; it very well could be that the prospective teachers used explain to mean explain why something is true rather than generating student understanding. The observation that teachers’ view the role of proof as explaining why something works further supports the claim that many teachers’ have a static conception of justifying in mathematics.

Teachers’ identification of various roles of proof in mathematics may be related to their views of mathematics in general. Conner and colleagues (2011) found that most prospective teachers in their study held instrumentalist and Platonist views of mathematics (see Ernest, 1991), which might explain the absence of discovery as a role of proof among those prospective teachers. Viewing mathematics as dynamic may be connected to a conception of proof as generating new mathematics.

We can learn about teachers’ conceptions of justifications, as opposed to justifying, in mathematics when we examine opinions about what counts as proof in mathematics. It seems
teachers’ conceptions of justifications and proof focus on form or structure rather than logic. For example, most teachers readily identify empirical arguments as not proof and visual and diagrammatic arguments as non-rigorous proof (Conner et al., 2011; Knuth, 2002a; Weber, 2010). Some teachers indicated that justifications incorporate the application of readily identifiable proof methods, such as proof by contradiction (Jones, 1997; Martin & Harel, 1989). The focus on form as a key characteristic of a justification alludes to a conception of justifications as procedures that involve the manipulation of representations of generalizations.

Just as teachers’ identified roles of proof were indicative of static conceptions of justifying, teachers’ focus on form in their conceptions of justifications seem to indicate that many teachers have static conceptions of justifications and proof. These static conceptions of justifying and justification may be related to static conceptions of mathematics. A dynamic-static continuum is also evident in teachers’ conceptions of defining and definitions.

**Conceptions of defining and definitions.** Much of the literature on conceptions of defining and definitions focus on the product of definition rather than the process of defining. One possible explanation for this observation is that teachers do not view definitions as something that can be created or stipulated (Edwards & Ward, 2004), or motivated or extended from mathematical relationships (Levenson, 2012b), which suggests a static conception of definition in mathematics. Teacher’s conceptions of definitions do not seem to include conceptions of defining as a mathematical process, and consequently research has focused on conceptions of definitions.

Definitions in mathematics are not the same as definitions used in colloquial settings. Definitions in mathematics are hierarchical in nature, contain necessary and sufficient characteristics, bring mathematical entities into existence, can be equivalent, and are preferably
minimal (van Dormolen & Zaslavsky, 2003; Winicki-Landman & Leikin, 2000). Teachers’ conceptions of definitions include some but not all of these mathematical characteristics. Conceptions that definitions need to be hierarchical (Levenson, 2012b; Zandieh & Rasmussen, 2010; Zaslavsky, 2005; Zazkis & Leikin, 2008) and contain necessary conditions (Leikin & Winicki-Landman, 2000; Zandieh & Rasmussen, 2010; Zaslavsky, 2005; Zazkis & Leikin, 2008) were common among teachers, but there was considerable disagreement about whether definitions needed to be minimal (Leikin & Winicki-Landman, 2000; Zaslavsky & Shir, 2005; Zazkis & Leikin, 2008). The desire that a definition be accessible to all students and complete in listing all properties could explain why minimalism was not a necessary characteristic of definitions. Furthermore, teachers’ conceptions of definition did not always include the characteristic of equivalency (Leikin & Winicki-Landman, 2001), suggesting that one definition is sufficient for a mathematical entity. The inclusion of hierarchical relationships and necessary conditions in a conception of definitions along with the exclusion of minimalism and equivalency paints a conception of definitions as listings of properties. If a definition were conceptualized as a listing of all properties of a mathematical entity, then multiple equivalent definitions of the same entity would be unnecessary. Furthermore, definitions would not need to be created beyond a listing of properties, which would be indicative of a static conception of definitions.

Teachers’ conceptions of definitions equated definitions and theorems; exploration of this observation provides additional evidence that teachers have static conceptions of definitions that focus on the product of definition rather than the process of defining. Some teachers viewed definitions as reported mathematical facts (Edwards & Ward, 2004) and as something to prove rather than to create (Leikin & Winicki-Landman, 2001; Levenson, 2012b). Their equating of
definitions and theorems could be related to the similar use of definitions and theorems in mathematics. Both definitions and theorems are applied to prove other theorems or equivalence of definitions. We act on the two types of products in similar ways.

To many teachers, definitions seem to be listings of properties, and their conceptions of defining tend to focus on uses of definitions—acting on a product—rather than the creation of definitions. Applying is the action that seems to be predominantly associated with definitions, whereas other actions are emphasized in teachers’ conceptions of representing and representations.

Conceptions of representing and representations. The literature on conceptions of representation contains semantic difficulties because as some researchers have noted representation is used both in the context of creating a representation of a mathematical entity and acting on a representation (Lesser & Tchoshanov, 2005; Stylianou, 2010). The processes and actions framework overcomes this semantic difficulty by using the terms representing and representation as creating and creation respectively. The use of a single term to express both a process and a product may explain why teachers have difficulty describing representation. When asked to describe representation in mathematics, middle school teachers either gave examples of representations or stated that they did not have a definition (Stylianou, 2010). One middle school teacher responded, “Representation? […] I’m not sure what you mean. […] There, there’s no specific definition […] An example that I’ve used in the classroom would be graphing” (p. 334). In response to middle school teachers’ difficulty with articulating a definition of representation, Stylianou (2010) notes, “In teachers’ definitions it became apparent that teachers focus primarily on the ‘product’ aspect of representation rather than the ‘process’ aspect” (p. 334). Perhaps the focus on acting on representations, predominantly in the translating among multiple
representations, in school mathematics could in part account for teachers’ apparent focus on product in their conception of mathematics.

Yet, teachers’ conceptions of representation do tend to include notions of representing. For example, Lesser and Tchoshanov (2005) found that middle school teachers were twice as likely to interpret representation as creating a representation than as a created representation. It seems that teacher’s conceptions of representing include both product and process, possibly because representing is likely present in school mathematics more than any other process. We represent for a variety of purposes: to understand information in a given problem, to record and communicate ideas, to facilitate exploration, and to monitor and evaluate work while problem solving (Stylianou, 2011). Although middle school teachers tend to emphasize communicative roles of representing—that of understanding and recording information, many middle school teachers also acknowledge more process-oriented roles of exploration and evaluation (Stylianou, 2010, 2011).

Different views of mathematics could explain different conceptions of representations among teachers. Although not directly about representing, a study on teachers’ beliefs about modeling could provide some insight into connections between views of mathematics and views of representing. Kaiser (2006) found that teachers with different views about mathematics held differing beliefs about modeling. Teachers who held views of mathematics as problem solving or mathematics as relevant to life and society tended to view modeling and application in mathematics as useful whereas teachers who held formalistic views of mathematics tended to devalue the role of modeling in mathematics. Extrapolating these results from modeling to representing, I claim that different views about mathematics could be correlated with different views of representing in mathematics. Static views of mathematics would likely be associated
with conceptions of representing that emphasize communicative roles and devalue explorative roles of representing whereas dynamic views of mathematics would likely be associated with conceptions of representing that emphasize explorative roles of representing in problem solving.

**Conceptions of generalizing and generalizations.** Few studies of conceptions of generalizing or generalizations were located in my literature search, which included examining major and minor mathematics education journals and research handbooks as well as internet searches using a variety of terms, yielded no results. I found considerable research on teaching and learning generalizations—particularly in the field of early algebra, but no research on teachers’ conceptions of generalizations or generalizing, even when broadening the search to include conceptions of conjecturing and patterning.

One possible explanation for apparent lack of research on conceptions of generalizing could be the foundational role of generalizing in mathematics in general (Burton, 1984; Harel & Tall, 1991) and algebra in particular (Kieran, 2007). It might be that teachers’ conceptions of generalizing and generalization are the same as teachers’ conceptions of mathematics because to them, mathematics is generalizing. For example, many teachers view mathematics in terms of problem solving, which includes conjecturing and generalizing (e.g., Andrews & Hatch, 1999; Beswick, 2007; Cooney, 1985; Felbrich, Müller, & Blömeke, 2008). A teacher in one study described mathematics as “exploring, conjecturing” (Beswick, 2007, p. 110), which could very well include generalizing. Teachers with a problem solving view of mathematics could very likely have a dynamic conception of generalizing that focuses on the creation of generalizations. On the other hand, it is not uncommon for teachers to describe mathematics as a set of rules (e.g., Andrews & Hatch, 1999; Ball, 1990b; Foss & Kleinsasser, 1996; Speer, 2000). It could be that
teachers’ with conceptions of mathematics as rule-bound would likely have conceptions of generalizing that focus on the application and interpretation of generalizations as rules.

**Summary.** Studies on conceptions of processes tended to examine a process in isolation from other processes. Based on my synthesis of the research, teachers’ conceptions of individual processes are likely related to their conceptions of mathematics in general. Teachers with static conceptions of mathematics are likely to have static conceptions of processes whereas teachers with dynamic conceptions of mathematics are likely to have dynamic conceptions of processes. Yet, it could be possible that conceptions of individual processes contribute to teachers’ conceptions of mathematics in different ways and to different degrees. A study examining conceptions of all four processes in mathematics could provide a more in-depth analysis of teachers’ conceptions of mathematics.

In the cases of justifying, defining, and representing, it was found that mathematicians’ and teachers’ conceptions often differed. Using apparent differences between mathematicians’ conceptions of processes and teachers’ conceptions of processes, one might argue that teachers hold diminished/deficient conceptions. Yet, I argue that observed differences in conceptions could be explained by considering the different uses of and presence of processes and products in the respective communities. The mathematical community seems to focus on the creation of new mathematical knowledge more than we might observe in a secondary mathematics community. In school mathematics, we might observe comparatively less engagement in processes and more engagement in acting on products than in research mathematics. The different dominant forms of activity within in the groups could explain the observed differences between mathematicians’ and teachers’ conceptions of mathematics and individual processes. Hence, we observe primarily static conceptions of individual processes among teachers. These static conceptions of individual
processes contribute to teachers’ OMP, and could ultimately explain how the teacher works with mathematical tasks in the classroom

**Conceptions of Teaching Mathematical Processes**

In the previous section, I focused on the role of each process and corresponding product in *school mathematics*, which I had identified as one component of OMP. I now turn to a discussion of conceptions of *teaching* each process and product in school mathematics, another component of OMP. The processes mean different things to different communities because each community values different activities. Teachers value the activity of mathematics teaching in addition to mathematical activity, so a teachers’ OMP consists of, among other things, conceptions of individual mathematical processes and conceptions of teaching each of those processes.

**Conceptions of teaching justifying.** A verifying role of justifying seems to dominate teachers’ conceptions of justifying in mathematics. However, teachers tend to emphasize the explanatory role of justifying in school mathematics (Conner et al., 2011; Staples et al., 2012; Steele & Rogers, 2012). The different foci of justifying in the respective contexts could be explained by the different activities in which members of the mathematics community and members of the teaching community engage (Staples et al., 2012). Although the mathematics community engages in teaching mathematics, they also engage in the generation of valid mathematical knowledge, and consequently need to justify new claims via mathematical proof. In contrast, teachers do not engage as often in the creation of new mathematical knowledge, but instead engage predominantly in teaching mathematics and would consequently want to use justifying and justifications to explain mathematics. Hence, teacher’s conception of teaching justifying would consider an explanatory role of proof as useful in teaching mathematics.
Teachers may view their teaching role as transmitters of knowledge or facilitators of learning (e.g., Conner et al., 2011); hence, explaining mathematics or developing understanding through justifying would assist in fulfilling such roles. Consequently, varying forms of arguments are acceptable for accomplishing the role of explaining. Teachers indicated that they were less likely to use formal proof in their teaching and viewed informal arguments as more desirable for teaching (Knuth, 2002b; Schwarz et al., 2008). It seems that many teachers view informal justifying as sufficient for fulfilling an explanatory role of justifying within the teaching community, which is why we might see more informal than formal justifying in secondary classrooms. One benefit of conceptions of justifying that include informal arguments is that teachers with such a conception may provide more opportunities for students to engage in justifying rather than merely acting on formal justifications created by others.

Conceptions of teaching justifying include not only the kinds of justifying they will use while teaching, but also how justifying fits within the curriculum they are teaching. Some secondary teachers and prospective teachers felt that it was only appropriate to teach proof in a geometry setting, or viewed proof as an object of study rather than a process to be integrated throughout the curriculum (Knuth, 2002b; Mingus & Grassl, 1999). Conceptions of teaching justifying as a segregated topic within mathematics might be reflective of a static conception of justifying because within this view there is no need for justifying in mathematics outside of geometry or the chapter on proof, and hence new arguments do not need to be created. Conceptions of teaching mathematics that permit informal products and segregation of processes from content can be found in the literature for defining and representing as well.

**Conceptions of teaching defining.** Limited research on teachers’ conceptions of teaching defining could be found in the literature. Instead, the literature focuses on teachers’
conceptions of definitions in the context of teaching. Teachers considered the appropriateness of definitions rather than the creation of definitions, which implies that definitions are chosen for the students rather than created by students (Zazkis & Leikin, 2008). Hence, definitions are given and orchestrating student actions on definitions constitutes mathematics for many teachers.

When discussing the suitability of a definition in the classroom, teachers’ pedagogical considerations outweighed their mathematical considerations (Leikin & Winicki-Landman, 2000; Levenson, 2012b; Zazkis & Leikin, 2008). Although practicing teachers in a professional development program focusing on defining indicated that they preferred definitions that were mathematically correct, they also wanted precise and clear definitions to help avoid student confusion (Leikin & Winicki-Landman, 2000). Prospective secondary mathematics teachers expressed similar preferences for definitions that were clear and accessible to students, even at the expense of mathematical rigor and minimalism (Zazkis & Leikin, 2008). One student teacher constructed “little definitions” that in the mathematical community would be considered incorrect, but were logical within the context of her classroom to avoid student confusion and promote student understanding (Zbiek et al., 2014). In each of these studies, teachers expressed a value for definitions that they thought improved the teaching of mathematics, and so their conceptions of teaching defining would include giving students informal definitions that are clear and on which students can easily act. While the mathematical community values rigor and minimalism of definitions, the teaching community values other characteristics of definitions, such as accessibility, that aide in the dominant activity of the community.

**Conceptions of teaching representing.** In the cases of justifying and defining, there seemed to be strong differences between how mathematicians conceived of these processes and how teachers thought about these processes in the context of school mathematics. Yet, the two
communities’ views of representing appear to be fairly similar. Both groups saw usefulness in representations as problem solving tools (Stylianou, 2010, 2011). One difference was in the dominances of representations as a tool for communicating in teachers’ discussions of representing in school mathematics (Stylianou, 2010). It is not surprising that teachers whose job is to communicate mathematics and evaluate student understanding of mathematics would emphasize a communicative use of representation.

Teachers focusing on a communicative role of representation did not view representation as central to middle school mathematics (Stylianou, 2010). Furthermore, teachers with such conceptions viewed representation as a topic of study (e.g., graphing) rather than as something to be integrated into mathematics content (Stylianou, 2010) and some middle school teachers indicated that particular representations were more appropriate for different mathematics content (e.g., using visual representations in geometry) (Lesser & Tcho, 2005). Teachers with conceptions of teaching representing that focus on using representations to record or communicate mathematics, possibly to the exclusion of problem solving uses of representation, tended to segregate teaching representations within and across various topics.

The segregation of teaching a product from other areas of mathematics was observed in research studies on teachers’ conceptions of teaching justifying. It would be reasonable to wonder if these two observations are related; do teachers who segregate the teaching of representations also segregate the teaching of justifications? Furthermore, is this segregation related to conceptions of individual processes or mathematics in general, or is it merely a reflection of common textbooks that relegate justifying to one chapter in a geometry textbook and graphing to one chapter in an algebra textbook? A study of teachers’ OMP would be able to
explore relationships among conceptions of different processes and conceptions of teaching different processes.

**Conceptions of teaching generalizing.** Considerable research on effective practices of teaching related to generalizing can be found in the literature, but few research studies on teachers’ *conceptions* of teaching generalizing were identified. Teachers’ conceptions of generalizing in teaching mathematics could be related to teachers’ views of inquiry, which seem to be varied with inquiry being favored more at the elementary level than the secondary or high school level (Marshall, Horton, Igo, & Switzer, 2009). In the context of algebra, high school teachers are less likely to agree that students should create their own solution methods whereas middle school teachers are more likely to encourage invented solution methods (Nathan & Koedinger, 2000a).

Yet, there are teachers who value inquiry in the mathematics classrooms and view their role as facilitator rather than transmitter, which could imply that at least some teachers value generalizing in the teaching of mathematics because mathematical inquiry often involves generalizing. One teacher stated, “if we’re investigating some aspect of that, and the kids come up with a ‘what if’ idea or, ‘I wonder what happens if we do this,’ then I’d absolutely grab it all the time” (Beswick, 2007, p. 105). This teacher whose conceptions of mathematics tended to be more process–focused seems to indicate that he or she would encourage the investigation of ideas in his or her mathematics classroom, and generalizing might be part of that investigating.

**Summary.** Teachers’ views of teaching mathematical processes seem to be related to their conceptions of mathematics and closely tied to the activities in which a community participates. For example, teachers who predominantly participate in the activity of teaching mathematics hold conceptions of teaching individual processes that facilitate teaching. Teachers’
conceptions of teaching processes might include the use of informal products and processes as a tool for teaching mathematics because it would make the mathematical activity more accessible to students. In addition to conceptions of mathematics, more general pedagogical views, such as teachers’ views of their roles in the classroom—as facilitators and guides or bearers of knowledge, could impact the incorporation of processes in teaching mathematics. In addition to considering how to teach a particular process or product, teachers also need to consider how students learn a particular process or product.

**Conceptions of Learning Mathematics Related to Mathematical Processes**

Conceptions of learning mathematics related to mathematical processes is another component of OMP that includes teachers’ views about how students learn mathematics through different processes and beliefs about students’ abilities to engage in mathematical processes. Teacher’s views of teaching often reflect their views of learning (Philipp, 2007; A. G. Thompson, 1992). When learning is viewed as receiving knowledge, teachers might view teaching as transmitting knowledge. Teachers who describe their role as one of guide might view learning as constructing knowledge. Furthermore, teachers’ beliefs about learning and how students engage in mathematical activity are critical to how teachers design and implement mathematics curricula and opportunities afforded students in the classroom (Lloyd, 2003). I now examine how teachers conceptualize learning mathematics with respect to each process.

**Conceptions of learning mathematics related to justifying.** Teachers’ conceptions of learning mathematics related to justifying center on two main ideas: the appropriateness of justifying for students and the promotion of mathematical understanding. Many teachers (Knuth, 2002b) and prospective mathematics teachers (Pandiscio, 2002; Schwarz et al., 2008) wonder whether proof is appropriate for all students and view formal proof as appropriate only for
advanced students. Pre-formal or informal proof is viewed as sufficient for most students in school mathematics (Knuth, 2002b; Schwarz et al., 2008). Yet, in contrast to secondary teachers in Knuth’s (2002b) study, middle school teachers in a professional development program, which may have influenced those teachers’ conceptions of learning mathematics related to justifying, indicated that justification was appropriate for all students and closely connected to supporting student learning (Staples et al., 2012). Teachers who consider justifying an appropriate activity for all students would likely have an OMP different from teachers who consider justifying appropriate for only a few students and would provide students with different justifying opportunities in the classroom.

Justifying may be viewed as beneficial for student learning because it develops thinking and argumentation skills and promotes conceptual understanding. In a general sense, some teachers view justifying as useful for developing logical thinking and argumentation skills (Knuth, 2002a; Schwarz et al., 2008; Staples et al., 2012), a life-skill tool applicable to a broad range of students and careers. In the context of mathematics in particular, teachers indicate that justifying helps students develop a deeper understanding of mathematics (Knuth, 2002b; Mingus & Grassl, 1999; Schwarz et al., 2008; Staples et al., 2012; Steele, 2005). One middle school teacher noted, “justification allowed students to build on prior knowledge, connect two ideas in a new way, and review other course material” (Staples et al., 2012, p. 454). Findings such as these indicate that teachers view justifying as a way to learn mathematics.

Conceptions of learning mathematics related to defining. Many teachers’ conceptions of defining in the school context focus on clarity and accessibility of definitions for students (Leikin & Winicki-Landman, 2000). Most teachers viewed definitions as statements given to students (Zazkis & Leikin, 2008), but none of the studies found teachers mentioning students
creating definitions. These studies indicate that many teachers conceptualize the learning of mathematics through teacher-given accessible definitions; students may act on definitions—such as interpreting definitions, but usually do not create their own definitions. Unlike the literature on justifying, which described justifying as useful for developing conceptual understanding, I found no mention of how defining can be used to develop conceptual understanding. It seems that most teachers have not considered defining as a process for learning mathematics. Instead, definitions seem to be controlled by the teacher and acted on by the students.

**Conceptions of learning mathematics related to representing.** Teachers likely view representations as tools for communicating and to some extent doing mathematics (Lesser & Tchoshanov, 2005; Stylianou, 2010). Middle school teachers indicated that multiple representations were useful for communicating with more students (Lesser & Tchoshanov, 2005), but some middle school teachers viewed representing as appropriate only for advanced students (Stylianou, 2010). Conceptions of representing that focus on representation as a tool for communicating that is only appropriate only for a few students are likely not to include a view of representing as central to learning mathematics.

*Principles and Standards for School Mathematics* (NCTM, 2000) states that representations should be used to communicate mathematics, but the document also emphasizes other ways representations can be used in school mathematics. “Representations should be treated as essential elements in supporting students' understanding of mathematical concepts and relationships; in communicating mathematical approaches, arguments, and understandings to one's self and to others; in recognizing connections among related mathematical concepts; and in applying mathematics to realistic problem situations through modeling” (p. 67). Yet, evidence that teachers view representing as a means for learning is sparse.
However, some teachers indicate that the sequencing of representations can impact learning and will use their conceptions of learning representations to structure teaching. Teachers predominantly agree that a sequencing of representations in which concrete representations precede visual representations followed by abstract representations is best for learning mathematics because concrete representations are the easiest to understand and abstract representations are the hardest (Lesser & Tchoshanov, 2005). In the context of algebra, Nathan and Koedinger (2000a, 2000b) found that most high school teachers tended to believe that understanding and using symbolic representations is a prerequisite to understanding verbal representations whereas middle school teachers tended to consider verbal representations and problems as easier for students than symbolic representations and problems. Nathan and Petrosino (2003) found that prospective mathematics teachers believe that students find symbolically represented problems easier than word problems, which is in contrast to the observation that students tend to find symbolically represented problems more difficult. While teachers do not necessarily think about representations as tools for learning mathematics, they do have opinions about how students can best learn mathematics through representations.

**Conceptions of learning mathematics related to generalizing.** Little is known about how teachers view the role of generalizing in learning mathematics or their views about whether students can learn mathematics through generalizing. If generalizing is indeed critical to learning mathematics and if teachers’ beliefs about learning impact opportunities afforded students (Philipp, 2007), then understanding teachers conceptions of learning mathematics with respect to generalizing would benefit the field of mathematics education.

**Summary.** In the literature on teachers’ conceptions of learning mathematics related to each of the four processes, we find evidence that some teachers may view justifying as a means
of learning mathematics. Yet, evidence that other processes are viewed as learning tools is sparse. Furthermore, no mention of students using representing and defining to learn mathematics was found in the teacher studies, but that could simply be because teachers were not asked if those processes could be used as a means for learning mathematics. Additionally, studies investigating conceptions of learning related to each of the processes only examined one process in isolation for others. What more could be learned about teachers’ conceptions of learning mathematical processes and the relationships among those conceptions if four mathematical processes were included in the study instead of just one? Not only could we learn more about how different processes are viewed as learning tools but also relationships among those different views.

**Engagement in Mathematical Processes**

Whether or not teachers provide opportunities for students to engage in mathematical processes depends not only on their conceptions of mathematics and its teaching and learning, but also on their ability and tendency to engage in mathematical processes. Hence, I identified engagement in mathematical processes as a fourth component of OMP. Yet, PMTs may not have experienced engagement in mathematical processes (Cooney & Wiegel, 2003; Frykholm, 1999). Although research studies tend to focus on one process in particular, findings from these studies allude to connections among processes.

This body of literature differs considerably from the conceptions literature discussed in prior sections. One of the main differences is the population of the participants in engagement studies compared to conception studies. Studies of teachers’ conceptions included participants who were predominantly practicing teachers and sometimes PMTs. Participants in studies of
teachers’ engagement in mathematic could be teachers or PMTS, but many studies involved college students in undergraduate mathematics courses, many but not all of whom are PMTs.

**Engaging in justifying.** Although not necessarily justifying, determining validity of justifications is one purpose for interacting with justifications and can provide insights into the ways PMTs justify mathematical claims. Research indicates that practicing and prospective teachers can identify valid proofs, but often evaluate invalid proofs as valid (Knuth, 2002a; Martin & Harel, 1989; A. Selden & J. Selden, 2003) for reasons related to the representation of the proof. Many teachers, at least initially, tend to focus on surface features of the representation of the proof rather than the logical structure behind the proof. Such findings could suggest that teachers may approach justifying in formulaic or ritualistic ways. Schwarz and colleagues (2008) observed that PMTs tend to struggle to prove, even in a school mathematics context. The struggle to prove might be explained by PMTs’ formulaic approach to proof.

Not only are PMTs likely to take a formulaic approach to proof and focus on the form of a proof, but PMTs tendency to justify might be influenced by the presence of multiple representations. Hollebrands, Conner, and Smith (2010) observed that when using technology to solve geometric problems, college geometry students, of which PMTs were a subset, did not offer explicit warrants in the presence of abundant examples. Instead, they would offer qualifiers indicating that the claim was true if the technology was reliable. On the other hand, if the students were not using technology (other than to create a single, generic representation from which to argue) then they offered explicit warrants arguing for why a claim was true (Hollebrands et al., 2010). Representing multiple instances of a claim may decrease the tendency to formally justify. Secondary teachers’ acceptance of empirical arguments as informal proof
(Knuth, 2002a) provides additional evidence that representing multiple instances might influence justifying.

**Engaging in defining.** The process of defining may be closely linked to the process of proving (Larsen & Zandieh, 2005; Zandieh & Rasmussen, 2010). In a college geometry teaching experiment, mathematics majors, including PMTs, engaged in the process of defining as they tried to formulate a definition of a triangle on a sphere that would allow for a parallel to Euclidean geometry’s SAS congruence theorem in spherical geometry. The desire to be able to prove a Euclidean geometry theorem in a spherical geometry setting motivated the need for a definition that would allow for such a proof. To help define the triangle that would meet this criterion, students examined the Euclidean proof of SAS congruence theorem and identified characteristics of a triangle that supported the proof and modified these characteristics to the spherical geometry setting. In this case, proof served as a guide to defining.

The ways in which PMTs act on representations may aid in the defining process. Representing various cases of spherical triangles helped students create a concept image that eventually led to the formalization of a definition (Zandieh & Rasmussen, 2010). Using symmetries on an equilateral triangle, two students (one of which was a PMT) developed a representation of symmetries that was closely connected to the physical movements and fit well with the development of a composition table, which ultimately aided the students in defining a group (Larsen, 2009). Although defining was a struggle for these students, representing contributed to successful creations of a definition.

Despite struggles to define, we find evidence that PMTs can define in both novel settings (e.g., group and isomorphism) and familiar yet modified settings (e.g., spherical geometry). Both of the above existence cases of defining occurred in collegiate mathematics courses taught with
an emphasis on defining. It would be reasonable to wonder whether PMTs would engage in defining in a different setting. Furthermore, PMTs tend to think about defining in the school mathematics context as remembering rather than formulating definitions (Leikin & Zazkis, 2010), which could mean that they would not provide opportunities for students to define.

**Engaging in representing.** In the previous two sections I described connections among representing, justifying, and defining. From these studies we glean that representing can aid in defining and representing can distract from the need to justify. I now consider how PMTs create and use representations in mathematical problem solving.

First, PMTs might use representations to explore a mathematical problem (Zbiek & Conner, 2006). For example, PMTs might use flow-charts and tables to explore relationships among symmetries of a triangle (Larsen, 2009). PMTs tend to prefer to explore using representations and strategies with which they are familiar (Leikin, 2003), with numerical and concrete representations being preferred to abstract representations (Türker, Sağlam, & Umay, 2010). The use of familiar representations to explore new problems may not necessarily be negative. For instance, three seventh-graders used representations of functions of single variables in graph and table format to explore functions of two variables, which eventually led to the production of effective graphical and tabular representations of these two-variable functions (Yerushalmy, 1997). Similarly, PMTs could also use familiar representations to explore mathematical problems that could lead to developing novel mathematical representations.

Second, PMTs could use representations to link among multiple representations of the same entity or connect the context of the problem to mathematics (Zbiek & Conner, 2006). For example, PMTs might link a verbal statement of the Pythagorean theorem to \( a^2 + b^2 = c^2 \). Linking multiple representations of multiple entities of the same type might be useful as PMTs engage in
other processes, such as defining and generalizing. For example, one PMT developed a definition for isomorphism by examining group operation tables of two isomorphic groups presented to the PMT in different contexts (Larsen, 2009). Additionally, three PMTs deepened their concept image of the limit of a sequence and linked various representations of the limit definition to the formal definition of limit by examining multiple visual representations of limit within a dynamic geometry environment (Cory & Garofalo, 2011). With the emphasis of multiple representations in school mathematics (NCTM, 2000), it seems reasonable to expect PMTs to translate among various representations of mathematical objects. Yet, the linking and creating of multiple representations can potentially affect how PMTs engage in other processes aside from representing.

**Engaging in generalizing.** Generalizing can occur through different kinds of extensions. First, generalizing occurs by moving from a set of specific instances to a general class of objects. Second, generalizing occurs by moving from a subset of objects to a more general class of objects. For example, the former occurs in the case of examining instances of sets of right-triangle side lengths to form a general statement of the Pythagorean relationship. The later occurs when the restriction of right triangle is removed to form a more general statement about the relationship among side lengths of acute and obtuse triangles.

Generalizations of the first type in which multiple instances are examined to form a generalization about a class of objects is a common type of generalization in the pattern-searching literature on early algebra (e.g., Lannin, 2005) or conjecturing using dynamic geometry software (e.g., Koedinger, 1998). Such generalizations were common among secondary teachers participating in a professional development targeted at an increased appreciation of structure in mathematics (Vale, McAndrew, & Krishnan, 2010). These teachers
examined patterns to form generalizations about polynomials and function transformations. Generalizations of this type seem to be a natural step in the examination of multiple examples. However, generalizations of the second type—generalizing from a subclass to a class—seem to be less common in the literature. Ellis (2007) observed middle school students generalizing about previously constructed generalizations as they removed particulars about their generalizations. It seems reasonable to conclude that if middle school students engage in this second type of generalizing, then PMTs might as well. Yet, evidence of PMTs engaging in such generalizing is sparse, and may be more cognitively difficult than generalizing from a set of specific instances (Sfard, 1991).

**Summary.** The foci of literature on PMTs’ engagement in mathematical processes are engagement in justifying and, to a lesser extent, engagement in defining. Yet, engagement in isolated processes seems unlikely; it seems more reasonable that engagement in one process is related to engagement in other processes. A study examining how PMTs engage in all four processes along with their conceptions of teaching, learning, and mathematics as they relate to the four processes might be useful in understanding the learning opportunities teachers provide students through mathematical tasks.

Looking across the literature of each of the four components of PMTs’ OMP, one might get the impression that PMTs possess what might be considered a diminished OMP. They tend to view mathematics and individual processes as static. They often conceptualize the teaching of processes as topics to be addressed separately and do not seem to consider mathematics learning through engagement in processes. Additionally, although PMTs can engage in the four processes, it is not always clear if PMTs would engage in these processes in the absence of courses focused on particular processes. However, PMTs are still novices in their mathematics
education communities, and their secondary mathematics experiences may not have been process-rich. Yet, PMTs can have process-rich OMPs, and an understanding of experiences leading to such OMPs would benefit mathematics teacher education as teacher educators try to help PMTs develop process-rich OMPs.

A teacher’s OMPs might influence students’ opportunities for engaging in processes in a variety of ways. For example, a teacher’s OMP might influence when and how the teacher uses student thinking in the classroom. Or, a teacher’s OMP might influence the kinds of formative assessment comments the teacher gives students on assignments. Also, a teacher’s OMP might influence how the teacher uses mathematical tasks in the classroom. The next section of this chapter explores how teachers select, modify, and sequence mathematical tasks.

**Planning Mathematical Tasks**

As teachers plan what mathematical tasks to use and how to implement such tasks in the classroom, they might consider various factors such as pacing, classroom norms, school culture, department expectations, and students’ prior knowledge and experiences. “Teaching is, in many ways, a design activity. Teachers must perceive and interpret existing resources evaluate the constraints of the classroom setting, balance tradeoffs, and devise strategies—all in the pursuit of their instructional goals” (M. Brown, 2009, p. 18). Ideally, teachers foreground their instructional goal as they make decisions about mathematical tasks (Stein et al., 2007). These decisions about tasks and goals can be influenced by teachers’ beliefs, values, perceptions, knowledge, and skills related to mathematics and its teaching and learning (M. Brown, 2009; Cross, 2009; Remillard & Bryans, 2004; Roth McDuffie & Mather, 2009; Stein et al., 2007)—teachers’ OMPs likely influence how they plan mathematical tasks.
In this section of the literature review, I examine relevant literature related to mathematics teachers’ (both practicing and prospective) selecting, modifying, and sequencing of mathematical tasks. As a matter of semantics, I use the term *curriculum* to refer written curriculum materials as a whole and the term *task* to refer to an individual activity used in the classroom. Although others’ use of these terms in similar ways (e.g., Stein et al., 2009), I emphasize the distinction here so that the focus of study among various research reports is clear in my synthesis of the literature. For example, some relevant research focused on teachers’ interactions with curriculum materials as a whole whereas others—considerably fewer in number—focused on teachers’ interactions with individual tasks.

**Selecting mathematical tasks.** Teachers might select tasks from a variety of sources. In years past, textbooks served as the primary source for mathematical tasks (Bush, 1986; Crespo & Sinclair, 2008; Haggarty & Pepin, 2002). Other possible sources for tasks include colleagues, professional development seminars and books, practitioners’ journals, and the Internet. Teachers today might utilize Internet sources more frequently than any other source, including textbooks. The variety of sources used by teachers could mean that they are less likely to closely adhere to the selection and sequencing of tasks in a curriculum and instead spend more time as curriculum planners (M. Brown, 2009)—selecting, modifying, and sequencing tasks from a variety of sources.

Different sources of tasks might contain different opportunities for engaging in mathematical processes. Textbooks in many countries consist predominantly of routine exercises with few opportunities for engaging in mathematical processes (Bieda, 2010; Haggarty & Pepin, 2002; D. R. Thompson et al., 2012) whereas textbooks funded through the National Science Foundation (Coxford et al., 2003) might include tasks with comparatively more opportunities. A
quick Internet search for mathematical tasks related to the Pythagorean theorem produced a list of more than half a million website hits many of which contained routine exercises involving straightforward applications of the Pythagorean theorem. For example, many of the exercises ask students to apply the Pythagorean theorem to find the length of the hypotenuse of a right triangle with leg lengths of 6 and 8. Mathematical processes were noticeably absent from the tasks found using an Internet search. Teachers’ use of a variety of sources may not necessarily provide students with more opportunities to engage in mathematical processes, especially when many of the tasks from which to choose provide few opportunities for engaging in processes. Furthermore, even when selecting from tasks that contain a variety of opportunities to engage in mathematical processes, teachers may choose tasks that provide ample opportunities for acting on products but limited opportunities for engaging in processes. An understanding of why teachers’ select one task over another is one objective of this research study.

Not only do teachers select tasks from a variety of sources, research on teachers’ interactions with curricula indicates that teachers use curriculum materials differently. Some might either pilot all aspects of the curriculum or not use the curriculum much and improvise instead while others might adapt the curriculum, picking and choosing which parts of the curriculum to implement in their classroom (M. Brown, 2009; Remillard & Bryans, 2004; Drake & Sherin, 2009). A teacher might use different curricula differently, adapting with respect to one curriculum and improvising with respect to another (Behm & Lloyd, 2009; Lloyd, 2008). Studies of teachers’ curriculum use can provide insights teachers’ broad use of curriculum materials, but do little to inform teachers’ selection of individual tasks within a curriculum.

**Selecting based on task features.** Teachers might focus on features of a task when deciding whether to implement the task in their classroom such as the use of technology in a task
or multiple parts to a task (Stein et al., 2009). One task feature to which teachers often attend is real-world context. Teachers perceive the presence of real-world context as a favorable task feature (Lee, 2012) and will select tasks based on the inclusion of a real-world context (Nicol & Crespo, 2006). While the presence of a real-world context might be more meaningful to students, a real-world context does not mean that the task provides opportunities for engaging in mathematical processes (Nicol & Crespo, 2006).

 Teachers might also select tasks based on a feature that matches a general notion about teaching and learning mathematics (Duncan, Pilitsis, & Piegaro, 2009). For example, a teacher might select a task because it involves multiple representations or manipulatives (Arbaugh & C. Brown, 2005; Levenson, 2012a), or they might select a task that is longer in length, associating length with mathematical difficulty (Osana et al., 2006). In such cases, it seems that teachers’ broad conceptions of teaching and learning—components of OMP—influence their selection of tasks.

 Teachers’ conceptions about mathematics and their engagement in mathematics might be one reason for why teachers rely on task features to select tasks. Crespo (2003) found that teachers tended not to solve tasks completely before giving the tasks to students. Teachers’ tendency to engage or not to engage in the mathematics themselves might be reflective of a conception of mathematics as unproblematic rather than a conception of mathematics as processes (Lloyd, 2009).

 **Selecting based on affective sensitivity to students.** In addition to selecting tasks based on task features, teachers also selected tasks based on a desired affective response from students to the task. For example, they might select tasks that they think students will find more interesting (Lambdin & Preston, 1995; Nicol & Crespo, 2006) or that will motivate the students
to engage in the task (Crespo, 2003; Levenson, 2012). Teachers might also select tasks to avoid student frustration (Lambdin & Preston, 1995). These teachers’ affective responses to a task, which may have been in response to features of a task, resulted in teachers selecting tasks that they hoped would produce or avoid an affective response from their students. In such cases, it seems that teachers’ affective sensitivity to students (Potari & Jaworski, 2002; Zbiek et al., 2014) influenced their selection of tasks. Tasks selected based on affective sensitivity to students may or may not provide opportunities for students to engage in mathematical processes.

**Selecting based on cognitive sensitivity to students.** When selecting tasks, some teachers considered the understandings of their students as well as the mathematics they would like their students to learn. In some cases, such considerations led to teachers omitting tasks that they thought might lead to student errors (Crespo, 2003). In other instances, teachers’ cognitive sensitivity to students led to selecting of tasks that incorporate more mathematical challenge, as was the case with a teacher who used students’ responses to previous tasks to select tasks that were mathematically challenging for her students (Norton & Kastberg, 2011).

Not only did teachers consider the current understandings of their students when selecting tasks, they also considered what they wanted their students to learn. “Unless teachers are clear about what they intend students to learn … it is difficult to plan instructional activities that would be helpful” (Morris et al., 2009, p. 516). Part of being cognitively sensitive to students in the selection of tasks is considering students’ cognition before, during, and after the students engage in a task. Teachers who select tasks based on cognitive sensitivity to students likely consider the kind of mathematical thinking in which they would like their students to engage. For example, one prospective teacher gave the following rationale for her selection of a mathematical task: “in this problem, the justification for the answer is more important than the answer. In fact, there is
no ‘right’ answer but the opportunity to form a well thought out probable answer” (Nicol & Crespo, 2006, p. 340). It seems that this prospective teacher’s valuing of justifying—one component of her OMP— influenced her selection of a mathematical task that provided opportunities for her students to justify.

Teachers may not spontaneously unpack learning goals (Morris et al., 2009) or naturally select tasks with high levels of cognitive demand (Arbaugh & C. Brown, 2005). Yet, teachers with knowledge of different frameworks for analyzing tasks tend to select tasks with higher cognitive demand (Arbaugh & C. Brown, 2005; Boston & Smith, 2009; Stein et al., 2009) and opportunities for engaging in mathematical processes. Teachers’ conceptions of teaching and learning—organized as frameworks— influence the selection of mathematical tasks.

Experienced teachers tend to rely more on students’ cognition, learning goals, and frameworks whereas novice teachers, who are either new to the craft of teaching or new to a particular curriculum, tend to rely on personal understanding to select tasks. For example, Matt (Nicol & Crespo, 2006) selected procedurally based tasks because his own learning of mathematics had been focused on procedures. Furthermore, teachers don’t select tasks when they don’t understand the task itself or solutions associated with the task (Sherin & Drake, 2009). Teachers’ engagement, or lack of engagement, with tasks influences their selection of tasks.

In summary, the components of teachers’ OMPs influence their selection of tasks. Teachers’ conceptions of mathematics influence whether they select a task based exclusively on features or also unpack the task to identify mathematical opportunities. Teachers’ conceptions of teaching and learning mathematics influence task selection as teachers consider students’ cognition, learning goals, and various frameworks in their selection process. Furthermore, tasks selected based on students’ cognition may be selected for more mathematical reasons when
compared to tasks selected for reasons of affect or task feature. Not only do teachers’ OMPs influence how teachers select tasks, but also how teachers modify tasks for their particular classrooms.

**Modifying mathematical tasks.** While there are many studies on how teachers adapt mathematics curricula, little is known about how teachers modify individual tasks. Sherin & Drake (2009) observed that teachers adapted curricula in one of three ways: adding new tasks to the curriculum; replacing one task with a different task in the curriculum; or omitting tasks. Each of these interactions with a curriculum involves adding, replacing, or omitting entire tasks rather than modifying individual tasks. Yet, parallel interactions with individual tasks are reasonable. For example, adding information in ways that simplify or extend a task, replacing parts of the task to alter the context of the task or the focus of the task, omitting parts of a task for time management, or scaffolding a task for accessibility to students (Stein et al., 2009).

As with selecting tasks, teachers modify tasks for affective and cognitive reasons. For example, teachers might change the context of the task that it is more interesting to students (Anghileri, 2006; Levenson, 2012a; Nicol & Crespo, 2006) or teachers might include hints in the task to keep students motivated and help them persist in problem solving (Anghilri, 2006; Norton & Kastberg, 2011). Nicol, Bragg, and Nihad (2013) found that while prospective teachers often altered tasks via context, content, or questioning, the prospective teachers were less attentive to the mathematical impact of their task modifications. Such modifications focused on altering context or questioning seemed to be a result of teachers’ affective sensitivity toward students and usually did not alter the mathematical opportunities of the task.

Yet, when teachers scaffolded tasks by making solutions easily attainable to ensure student success, students’ mathematical performance decreased (Houssart, 2002). One possible
explanation for this decrease in student performance could be that task simplification to effect attitudinal change in students could have the inadvertent effect of decreasing cognitive demand during task implementation (Stein et al., 2000, 2009) and students’ opportunities to engage in mathematical processes.

In comparison, modifications to tasks as a result of teachers’ cognitive sensitivity to students are more likely than modifications as a result of teachers’ affective sensitivity to student to alter the opportunities for engaging in mathematical processes. Teachers might modify tasks to alleviate student difficulty (Lamdin & Preston, 1995) by encouraging the use of a representational tool to develop conceptual understanding (Anghilri, 2006), or by parsing a task into smaller tasks. Alternately, teachers might analyze past student thinking around a task and modify the task for increased complexity (Choppin, 2011), which may have the effect of providing more opportunities for engaging in mathematical processes.

Teachers modify tasks for affective and cognitive reasons. Usually, modifying tasks to change the context as part of motivating students to engage in the task did not affect the mathematical opportunities afforded in the task. Modifying tasks to ensure student success and accessibility might affect the mathematical opportunities of a task. When focusing on students’ cognition, teachers are more likely to change the mathematical opportunities of a task than if teachers focus on students’ affect.

**Sequencing mathematical tasks.** When compared to the amount of research on selecting and modifying mathematical tasks, the amount of research on teachers’ sequencing of mathematical tasks is minimal. Teachers’ dependence on a written curriculum might be one explanation for the lack of research. Although teachers may add, replace, and omit tasks within a curriculum (Sherin & Drake, 2009), teachers tend to follow the overall mathematical sequencing
of a written curriculum (Bush, 1986; Haggarty & Pepin, 2002). So by default teachers sequence mathematical tasks according to the sequencing of the written curriculum they use.

The relative pedagogical difficulty of sequencing compared to that of selecting or modifying as design activities (M. Brown, 2009) might be another explanation for the apparent minimal research on sequencing mathematical tasks. Additionally, selecting and modifying tasks seem to be more obvious design activities, whereas sequencing tasks may not be a design activity in which many teachers even consider as part of teaching. Yet, when teachers do carefully consider the sequence of mathematical tasks, opportunities for students’ engagement in mathematical processes could increase.

Teachers who used student thinking to select and modify tasks (Choppin, 2011; Norton & Kastberg, 2011) engaged in a form of responsive teaching that led these teachers to sequence subsequent tasks in ways that provided more opportunities for engaging in mathematical processes. Teachers participating in professional development focused on teaching mathematics effectively through the use of tasks designed to engage students in challenging mathematics created sequences of tasks that were appropriately challenging and mathematically engaging for students (Sullivan, D. Clarke, & B. Clarke, 2013). Such examples of teachers creating mathematically engaging sequences of tasks suggest how teachers’ OMPs, which were likely directed by professional development activities, could influence task sequencing because teachers’ sequencing of tasks were motivated by a conception of mathematics teaching and learning through effective task engagement.

As with selecting and modifying tasks, teachers who use student thinking to sequence tasks could increase students’ opportunities for engaging in mathematical processes. Teachers’
OMPs could influence whether or not they attend to student thinking as they select, modify, and sequence mathematical tasks.

Summary. After an examination of literature related to teachers’ use of mathematical tasks, we can reasonably conclude the following: (a) that many teachers select and modify tasks based on task features but rarely on underlying mathematical ideas of tasks; and (b) teachers who use student thinking and task frameworks presented as part of their pre-service or in-service experiences tend to select, modify, and sequence mathematical tasks in ways that provide increased opportunities for students to engage in mathematical processes. Many of these pre-service and in-service experiences focused on changing teachers’ conceptions of mathematics and its teaching and learning through mathematical tasks.

It seems reasonable to assume that teachers’ OMPs—sets of conceptions about mathematics and its teaching and learning related to mathematical processes as well as engagement in mathematical processes— influence teachers’ selection, modification, and sequence of mathematical tasks. Yet, the question of how teachers’ OMPs influence such interactions with tasks remains. In the next chapter, I present methods used to answer research questions: What are prospective secondary mathematics teachers’ (PMT) orientations toward mathematical processes (OMP), and how do PMTs’ OMP influence their selection, modification, and sequencing of mathematical tasks?
Chapter 4

Research Methods

The goal of this study is to understand prospective secondary mathematics teachers’ (PMTs’) orientations toward mathematical processes (OMPs) and how those OMPs influence PMTs’ work with mathematical tasks. Drawing upon a situated cognition perspective, I posit that OMPs are situated in an enculturation experience of teacher education. This chapter begins with a discussion of the situated context of the study and its participants. Following an overview of the chronology of data collection and analysis, I describe methods used to gather and analyze data related to each of my research questions. An OMP interview serves as the primary data source for answering the question related to PMTs’ OMPs and two task interviews are used to gather data related to PMTs’ task work.

A Situated Study

The foci of the research questions are PMTs’ OMPs and OMP influence on PMTs task work. Drawing upon a situated perspective of learning, I assume that OMPs of PMTs are situated in a context of a teacher education program (Borko et al., 2000) as well as within various mathematical contexts. Additionally, PMTs’ task work might be situated in the context of their preservice teaching experiences. In this section, I describe the participants and some of their situated experiences along with chosen mathematical contexts used to explore OMPs and task work.

Participants living an enculturation experience. Grounded in a situated cognition perspective, I posit that orientations are situated in experience. I wanted participants with shared experiences in case there existed commonalities across OMPs. I selected participants from a group of PMTs who share an enculturation experience into the mathematics teaching culture by
participating in mathematics education courses as well as field experiences, but are not yet professional mathematics teachers. The preliminary field experiences are critical because those experiences provide opportunities for PMTs to work with mathematical tasks in more formal teaching settings, which might influence their OMPs as they work with students and mathematical tasks.

During PMTs’ preliminary field experiences, they work with mentors, who presumably have varying degrees of influence over PMTs’ orientations and work with tasks. Although it might be argued that the participating PMTs may not have had complete control over their work with tasks in their preliminary field experiences and that mentors could influence PMTs’ OMPs, I argue that working with a mentor is part of the enculturation process and it is important for PMTs’ to have those experiences as they develop their OMPs. Consequently, I identified completion of some preliminary field experiences as a necessary criterion of participant selection.

I recruited volunteers from student teachers at a large research institutional via email and from a mathematics modeling class via personal invitation. One of the participants was at the beginning of her student teaching experience and four of the participants had recently completed student teaching (see Table 4-1 for pseudonyms and information about the context of their student teaching experiences). Although participants came from different cohorts, we can assume similar mathematics education course experiences because they were from the same university and had many of the same instructors.
Table 4-1

*Participant Pseudonyms and Preliminary Field Experiences*

<table>
<thead>
<tr>
<th>Participant Pseudonym</th>
<th>Student Teaching Experiences</th>
</tr>
</thead>
<tbody>
<tr>
<td>Erin</td>
<td>Middle school, 7th grade math, equations</td>
</tr>
<tr>
<td>Jenny</td>
<td>High school, pre-algebra, linear functions and quadratics</td>
</tr>
<tr>
<td>Heather</td>
<td>High school, pre-calculus, exponential functions and the Pythagorean theorem</td>
</tr>
<tr>
<td>Sarah</td>
<td>Middle school, algebra, factoring</td>
</tr>
<tr>
<td>Brad</td>
<td>High school, algebra 2, quadratic functions</td>
</tr>
</tbody>
</table>

**The Pythagorean theorem and quadratic functions as mathematical contexts.**

Research on teachers’ beliefs and conceptions indicates that beliefs are often situated in contexts and are best studied within a context (Philipp, 2007), and so orientations, which include sets of conceptions, are likely situated and would be best studied within a context. Considering this, along with a learning perspective of situated cognition, I decided to use two mathematical contexts for my study of OMPs and task work—the Pythagorean theorem and quadratics—because a PMT might have different OMPs in different mathematical contexts. In particular, I selected two mathematical content areas that are perceived as coming from different content strands; the Pythagorean theorem often perceived as primarily geometric and quadratics often perceived as being primarily algebraic. By using mathematical contexts from different content strands, I increased the possibility of saturation in data collection.
I selected the mathematical topics of the Pythagorean theorem and quadratics because of their prominence in secondary mathematics. As such, PMTs would have had experiences with these topics as secondary students and most likely as college students, in either a geometry for teachers course, a functions content course, or a mathematics teaching methods course. Four of the participants in my study had experiences with teaching either the Pythagorean theorem or quadratic functions in some way during their field experiences. Assuming orientations are situated in experiences, descriptions of these field experiences would likely be more robust and more revealing with respect to their OMPs. Additionally, these are topics with which PMTs are presumably comfortable, but could be presented in unfamiliar settings, which would be useful for studying how they engage in mathematical processes as they work with mathematical tasks. Additionally, opportunities to engage in all four processes of defining, generalizing, justifying and representing could be provided in the contexts of the Pythagorean theorem and quadratics, a necessary criterion for studying PMTs’ orientations toward mathematical processes.

**Chronology of data collection and analysis.** In this section, I provide an overview of my study by outlining my preliminary work as well as the chronology of data collection and analysis used in my study. In preparation for my dissertation research, I conducted two preliminary studies with the first focusing on developing data collection instruments and the second focusing on developing data analysis methods. Based on results of these preliminary studies, I collected data for my dissertation study using a questionnaire and three interviews.

**Two preliminary studies.** To test the feasibility of interview tasks to produce answers to my research questions, I conducted two preliminary studies. The purpose of the first preliminary study in December 2012 was to trial data collection methods. I administered two possible data collection instruments, a questionnaire (see Appendix A) and one 60-minute interview in which
four PMTs solved two Pythagorean theorem problems and then sorted a set of Pythagorean theorem tasks to design a unit. Based on these reflections from video recorded interviews, I designed two interviews to be used in my second preliminary study. The first interview was designed to focus on collecting data related to OMPs and expanded to 90 minutes to include an additional interview task involving teaching scenarios and more time to solve Pythagorean theorem tasks. The second interview focused on collecting data of PMTs’ task work and was also expanded to 90 minutes to provide adequate time for PMTs to plan a Pythagorean theorem unit given a set of tasks.

The purpose of the second preliminary study occurring in May 2013 was to test the revised interview instruments and to explore possible data analysis methods. I interviewed four PMTs at varying stages of their teacher education programs. Analysis of data revealed that PMTs who had at least some student teaching experience provided richer data related to OMPs because they were able to draw on their teaching experiences to answer interview questions. I used transcriptions of interviews to develop data analysis methods for both OMPs and task work that were used in my dissertation study.

Preliminary studies focused on the mathematical context of the Pythagorean theorem. Because orientations are likely situated in context, I added an additional mathematical context of quadratics to better saturate data collection in my dissertation study.

**Overview of data collection instruments used in dissertation study.** To help organize the structure of my dissertation study, I provide the reader with a brief overview of data collection instruments and timeframes for collection and analysis. I used a questionnaire and one 90-minute interview, called the OMP interview that is referenced as Int1 when quoting participants, to collect data related to PMTs’ OMPs. Two 90-minute interviews, called task interviews that are
referenced as Int2 and Int3 when quoting participants, were used to collect data related to PMTs’ task work. Table 4-2 shows the collection and analyses for various data sources in the study, with source names used later to identify interviews included parenthetically, along with a timeframe for each collection or analysis. The top half of the table focuses on the chronology of data collection via a questionnaire, OMP interview and two task interviews. The bottom half of the table focuses on the chronology of data analysis beginning with analyzing OMP interviews across all five participants followed by analysis of sets of interviews for each participant.

Table 4-2

*Chronology of Data Collection and Analysis*

<table>
<thead>
<tr>
<th>Time Frame</th>
<th>Data Source</th>
<th>Description of Collection or Analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>January 2015</td>
<td>Questionnaire</td>
<td>Participants complete online questionnaire</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Researcher analyzes participant responses</td>
</tr>
<tr>
<td>February 2015 – March 2015</td>
<td>OMP Interview (Int1)</td>
<td>Researcher conducts OMP interviews and begins analysis of OMP interviews to inform upcoming task interviews (Int2 &amp; Int3)</td>
</tr>
<tr>
<td>March 2015</td>
<td>Task Interviews (Int2 &amp; Int3)</td>
<td>Researcher conducts task interviews and begins analysis of Int2 to inform Int3</td>
</tr>
<tr>
<td>September 2015 – December 2015</td>
<td>OMP Interview (Int1)</td>
<td>Research analyzes OMP interviews for all participants and records preliminary findings</td>
</tr>
<tr>
<td>January 2016 – August 2016</td>
<td>Interviews</td>
<td>Researcher analyzes Int1, Int2, &amp; Int3 collectively for each participant to develop OMP profiles</td>
</tr>
</tbody>
</table>
As shown in the table, the data analysis process overlaps with the data collection process, which is ideal in qualitative research to ensure saturation of data (Creswell, 2007). To minimize changes in OMP, the questionnaire and interview data for each participant were collected within a few weeks with preliminary analysis conducted between each collection phase. As noted in Table 4-2, different sets of interviews were analyzed during different timeframes to accomplish different purposes. I first analyzed the set of OMP interviews to find a preliminary answer to my first research question: what are PMTs’ OMPs. I then analyzed five sets of interviews organized by participant to answer my second research question: how do OMPs influence task work.

In this section of the chapter I provided a brief overview of the chronology of my research beginning with two preliminary studies focused on developing data collection instruments and data analysis methods. A description of my dissertation study followed, which consisted of data collected from a questionnaire, OMP interview, and two task interviews. In the next two sections of this chapter, I describe the collection and analysis methods used to develop answers to my research questions. My discussion of the collection and analysis is organized around my research questions and focuses on how the different data sources contribute to answering each research question. First, I describe the development and analysis the OMP interview, which is the primary data source for answering my first research question—what are PMTs OMPs? I include a description of the questionnaire and its use in data analysis as part of my discussion of methods used to answer my first research question. I then describe the development and analysis of the task interviews, which I used to answer my second research question—how do PMTs’ OMPs influence their work with mathematical tasks?
Data Collection and Analysis of Orientations toward Mathematical Processes

To answer my first research question—what are PMTs’ OMPs—I used an OMP interview and a questionnaire to gather data on the four components of OMP (conceptions of mathematics, conceptions of teaching, conceptions of learning, and engaging in mathematics) with respect to each of the four processes (defining, generalizing, justifying, and representing). The questionnaire (see Appendix A) contains general items related to the four components of OMP for each of the four processes and the interview situates OMP in teaching a unit on the Pythagorean theorem. The data from the OMP interview served as the primary source for answering my research question of what are PMTs’ OMPs because it was more situated in a context of teaching than the questionnaire and thus more valuable to a study grounded in situated cognition. I used the questionnaire containing more general questions about secondary mathematics teaching and learning to triangulate data and find apparent inconsistencies that needed further investigation. The focus of this section of the chapter will be on the design and analysis of the OMP interview.

OMP Interview. The purpose of the OMP interview was to gather data surrounding the different components of OMP by eliciting information about participants’ conceptions of mathematics and its teaching and learning with respect to mathematical processes and how they engage in mathematical processes. I provided participants with opportunities to engage in mathematical processes by inviting them to solve mathematical tasks. I also asked participants to describe how they would ideally teach the Pythagorean theorem and to share descriptions of some of their field experiences.

The 90-minute interview follows a semi-structured format (Forsey, 2012) to allow for flexibility in the interview process for individual participants. A sample interview schedule
appears in Appendix B. The basic structure of the interview was the same for all participants, but individual follow-up questions varied depending on participants’ approaches to the given mathematical tasks and responses to given teaching scenarios. The variation in follow-up questions is consistent with a situated cognition perspective because PMTs’ OMPs might be different depending on variations in experiences. A semi-structured interview provided needed flexibility to explore different components of OMPs or different mathematical processes in more depth.

**Design of OMP Interview questions.** The OMP interview consisted of three parts, each situated in the context of the Pythagorean theorem. First, I asked participants to solve two tasks related to the Pythagorean theorem. Second, I asked participants to describe what a 3-day unit on the Pythagorean theorem would look like in their ideal mathematical classroom. Finally, I asked participants to sort a set of teaching scenarios related to the Pythagorean theorem and describe how well the teaching scenarios matched their ideal teaching scenario.

**Tasks with opportunities to engage in each of the four processes.** The interview began with two mathematical tasks that I asked participants to solve. The purpose of these two mathematical tasks in the interview was to gather information about one component of OMP: how PMTs’ engage in each of the four processes.

The first task (see Figure 4-1), labeled Task A in the interview but referred to as the Tri-square Rugs task in this report, introduces the Pythagorean theorem and related generalizations in a context that was likely unfamiliar to the participants (see Fendel et al., 2003, p. 232 for task). Also, the task provided opportunities for participants to engage in each of the four processes. For example, it provides opportunities to represent different examples of tri-square rugs for different kinds of triangles or to represent side lengths, areas, and relationships algebraically or
numerically. The task provides opportunities to define a set of triangles needed to create a fair game, to generalize that the sum of the areas of the two smaller squares is less than the area of the largest square when the triangle forming the rug is obtuse, and to justify this generalization. Although a participant might engage in all, some or none of these processes, the task provided an opportunity to collect data related to each of the four processes.

**Task A**

A rug designer decided to make a rug consisting of three separate square pieces sewn together at the corners, with an empty triangle space between them. The rug was an immediate hit, and the designer decided to make more of them. He called these creations “tri-square rugs.” A sample tri-square rug is shown here.

Al and Betty thought these tri-square rugs could be used to make a great game. They made up these rules.

Let a dart fall randomly on the tri-square rug.

- If it hits the largest of the three squares, Al wins.
- If it hits either of the other two squares, Betty wins.
- If the dart misses the rug, simply let another dart fall.

Your goal in this activity is to decide which tri-square rugs you would prefer if you were Al and which you would prefer if you were Betty, and if there are any rugs that lead to a fair game.

*Figure 4-1. Tri-square Rugs task given to participants during the OMP interview (Fendel et al., 2003, p.232)*

In addition to the Tri-square Rugs task, I wanted to provide participants with other opportunities to engage in processes through a different task. The Pythagorean Puzzle task (Lappan et al., 1998, p. 29), called Task B in the interview (see Figure 4-2), was the second task presented to participants and provided opportunities to engage in representing, generalizing, and
justifying. Considering that the participants had just encountered the Pythagorean theorem in the Tri-square Rug task, it is possible that the opportunity to generalize the Pythagorean theorem in this task may not be authentic for the participants. Yet, with further questioning I used the task to gather additional data on how participants engage in representing and justifying. Additionally, the task includes features to which PMTs might attend such as hands-on learning, visual representations, and exploration. I used the task as an opportunity to gather data on participants’ conceptions of tasks particularly with respect to task features.
Figure 4-2. The Pythagorean Puzzle task that participants were asked to solve during the OMP interview (Lappan et al., 1998, p. 29).

The first part of the interview focused on collecting data on how PMTs engage in mathematical processes by inviting them to engage in mathematical processes while solving two
Pythagorean theorem tasks. I also used the tasks to collect information about the participants’ mathematical experiences related to the Pythagorean theorem and some initial conceptions of teaching and learning mathematics.

**Planning to teach a 3-day unit on the Pythagorean theorem.** The second part of the OMP interview focused on collecting data related to other components of OMP, particularly conceptions of mathematics and its teaching and learning. I asked participants to describe their preservice teaching experiences, including any lessons they thought went particularly well and why. Their descriptions of preservice teaching experiences provided not only insights into participants’ conceptions of mathematics and its teaching and learning but also their role as a student teacher working with a mentor teacher and how their OMP might be situated in their preservice teaching experiences.

After participants described their preservice teaching experiences, I asked them to describe what a 3-day unit on the Pythagorean theorem in a high school geometry class (see Part 2 of OMP Interview in Appendix B) might look like in their ideal classroom setting. Although the Pythagorean theorem is often introduced in middle school, preliminary data collection indicated that participants might avoid certain tasks and processes—particularly justifying—in a middle school setting. Consequently, I opted for a high school setting in which participants might view the Pythagorean theorem as something to be explored in more depth so that I could collect more data about the participants’ conceptions of teaching and learning with respect to all four of the processes rather than just some of the processes. Additionally, I purposely phrased the planning of the 3-day lesson within the context of participants’ ideal classroom settings and restricted the length of the unit to reveal participants’ priorities when teaching the Pythagorean theorem.
Teaching scenarios sorting task. During the third part of the OMP interview, I used a sorting task (Friedrichsen & Dana, 2003; Philipp, 2007) as another method for collecting data about the OMP components of conceptions of mathematics and its teaching and learning. In particular, I designed the sorting task to gather data about how these conceptions related to each of the four mathematical processes of defining, generalizing, justifying and representing. I gave participants a set of eight teaching scenarios—two for each process—and asked participants to describe how similar or different the scenarios were to their teaching. The teaching scenarios and how they are organized with respect to each process are given in Table 4-3. For each of the four processes, I wrote one scenario in which the students and teacher engage in the process and I wrote another scenario in which the teacher and students act on a product of the process. For example, in the justifying scenario the students and teacher create a proof of the Pythagorean theorem and in the justification scenario, the teacher and students interpret a proof of the Pythagorean theorem.

Table 4-3

Teaching Scenarios Used in Sorting Task Organized by Process and Action

<table>
<thead>
<tr>
<th>Product/Process</th>
<th>Action on Product Scenario</th>
<th>Process Scenario</th>
</tr>
</thead>
<tbody>
<tr>
<td>Representation/</td>
<td>Representation Teaching Scenario</td>
<td>Representing Teaching Scenario</td>
</tr>
<tr>
<td>Representing</td>
<td>You display a diagram that has sides of a right triangle labeled a, b, and c and students explain how the diagram is connected to the Pythagorean theorem.</td>
<td>To illustrate the Pythagorean theorem and the relationship among a, b, and c, you have your students create and label diagrams on graph paper or using dynamic geometry software.</td>
</tr>
<tr>
<td>Generalization/</td>
<td>Generalization Teaching Scenario</td>
<td>Generalizing Teaching Scenario</td>
</tr>
<tr>
<td>Generalizing</td>
<td>You introduce the Pythagorean theorem to students by stating that for any right triangle the sum of the squares of the legs is equal to the square of the hypotenuse. You and your students then do several examples of how to use the theorem to solve problems.</td>
<td>Before talking about the Pythagorean theorem, you ask students to do the following, “Here are a bunch of different right triangles. Write a statement to describe a relationship among the sides of those triangles.”</td>
</tr>
<tr>
<td>Justification/Justifying</td>
<td>Justification Teaching Scenario</td>
<td>Justifying Teaching Scenario</td>
</tr>
<tr>
<td>--------------------------</td>
<td>--------------------------------</td>
<td>-----------------------------</td>
</tr>
<tr>
<td>To show students that the Pythagorean theorem is true for all right triangles, you show students an applet that goes through each step of a proof with dynamic illustrations and ask them to describe what they see.</td>
<td>You ask the students, “How can we know that the Pythagorean Theorem is true for all right triangles?” Using their ideas and further questioning, you and your students develop a proof of the Pythagorean theorem.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Definition/Defining</th>
<th>Definition Teaching Scenario</th>
<th>Defining Teaching Scenario</th>
</tr>
</thead>
<tbody>
<tr>
<td>You state a definition of a Pythagorean triple as “a set of three natural numbers ( a, b, ) and ( c ) that satisfy the relationship ( a^2 + b^2 = c^2 ).” You then ask the students, “Based on this definition, what are the properties of a Pythagorean triple?”</td>
<td>You have students examine various examples and non-examples of Pythagorean triples and then ask them to complete the statement, “A Pythagorean triple is ______.”</td>
<td></td>
</tr>
</tbody>
</table>

The purpose of this part of the interview was to gather data related to three of the components of OMP (conceptions of mathematics, conceptions of teaching, and conceptions of learning) for each of the four processes. I intentionally wrote the scenarios to be vague so that the participants would need to elaborate on how they interpret the scenarios and so that I could ask follow-up questions related to these OMP components. For example, following the participant’s discussion of the scenario involving representing (see the Representing Teaching Scenario in Table 4-3), I would ask follow-up questions related to conceptions of representing such as, “What do you think it means to have students create a diagram of the Pythagorean theorem?” A follow-up question related to teaching conceptions might be, “What things might you as the teacher do to guide the students in creating a diagram representing the Pythagorean theorem?” Follow-up questions for each scenario can be found in Appendix B.

The focus on the OMP interview was to collect data to answer my first research question—what are PMTs’ OMPs? The interview consisted of three parts, which were collectively designed to capture data on each of the four components of OMP for each of the four processes.
**Analysis of OMP Interview.** Analysis of the OMP interview began during data collection. Within a few days of the OMP interview, I watched a video recording of the interview, identifying initial observations and further questions so that data analysis ran concurrent with data collection and could inform subsequent data collection, as suggested by Maxwell (2005). For example, following Jenny’s OMP interview I noted that she did not seem willing to engage in mathematical processes or solve mathematical tasks. To test whether her OMP included an avoidance of processes, I decided not to ask many follow-up questions during the first half of the Pythagorean task interview to avoid inadvertently prompting her to engage in a process. I also noticed that I had not collected adequate data on participants’ conceptions of processes and decided to add questions to the quadratic task interview that might elicit more information about conceptions of individual processes. I will discuss these additional questions in my description of the task interviews.

Following my preliminary analysis, I identified important sections of the annotated transcripts (Seidman, 2006) and coded them with the purpose of organizing the data into categories that facilitate connections and comparisons within and across categorizations in the development of thematic conceptions (Maxwell, 2005). My initial codings were substantive (Maxwell, 2005) and descriptive in nature, with great care taken to use the voice of the participant to preserve validity (Butler–Kisber, 2010; Seidman, 2006). I used my theoretical framework to give labels to the important sections of the transcript. When possible, I gave two codes to each section of the transcript that related to OMP. The first code identified the component of OMP (conception, teaching, learning, engaging) evidenced in the section of the transcript; the second code identified the process (representing, definition, generalizing, justifying).
To illustrate this initial coding process, I use an example involving conceptions of teaching and learning related to justifying from Sarah’s OMP interview. In response to the justifying teaching scenario (see Table 4-2), she stated,

If I am doing a proof, I want the students to come up with a proof on their own and use their own knowledge. I think if they come up with it on their own, they then are more likely to take ownership of it and remember it. It also helps me to see where exactly they might be going wrong and what misconceptions they might be having about different ideas. (Int1)

I coded this section of the interview in two different ways. First, I coded it as a conception of learning related to justifying because Sarah described how justifying (“come up with a proof on their own”) aids in the learning process (“more likely to take ownership of it and remember it”). I also coded the section as a conception of teaching related to justifying because she implied that her role as teacher included listening to and assessing student thinking (“see where exactly they might go wrong and what misconceptions they might be having about difficult ideas”) and that proof provided an opportunity to elicit student thinking.

At times it was difficult to determine a specific process related to expressed conceptions about mathematics, teaching, or learning. During those instances, I assigned only one code to the section of the transcript. For example, Jenny shared the following conception of learning in her OMP interview: “I feel like if you discover stuff on your own, they just kind of learn more than having things worked out in the book” (Int1). Her statement expresses a conception that one learns more if they have an opportunity to discover “stuff” on their own. The “stuff” to be discovered is not clear in Jenny’s statement and could refer to a definition, generalization, justification, or representation. Hence, “discover” could refer to any or none of the processes targeted in the study. The ambiguity with respect to process yet the significance of the statement with respect to learning resulted in a single code of learning. Although I often followed-up on
ambiguous statements during the interview process, the density of the expressed ideas as well as the rapid frequency with which ambiguous statements were used by particular participants made it difficult to follow-up on every ambiguity. In such cases, one code related to the OMP component was given.

After coding the transcript, I developed memos (Maxwell, 2005) to facilitate analytic thinking by stimulating connections that aide in the development of emerging themes (Creswell, 2007; Seidman, 2006) across processes and components of OMP. While writing the memos, I transitioned from descriptive analysis of the text to more thematic analysis of the interviews (Maxwell, 2005), adding researcher interpretations and hypotheses to the descriptive coding done in the first two close readings. I then returned to the OMP interview transcript and the task interviews to test emerging themes and hypotheses by looking for first refuting and then confirming evidence using constant comparative analysis. The memos served as analysis tools to increase the depth and rigor of my analysis in pursuit of saturation in the data analysis process and to document my analysis process for subsequent analyses across OMP interviews and task interviews.

After analyzing the OMP interviews for all participants, I reread all OMP interviews and reviewed emerging hypotheses. Synthesizing results across OMP interviews, I created more theoretical categories (Maxwell, 2005) that placed the coded data into a more general and abstract framework by looking for connections and differences among participants. For example, as I placed coded data from the OMP interviews into more general categories of discovery, facilitating, presenting, process and action by looking for connections and differences among participants. I used these general categories to develop and test these more abstract emerging
hypotheses as I analyzed the task interview data. Based on this theoretical categorization of data, I constructed preliminary answers to my first research question of what are PMTs’ OMPs.

**Questionnaire.** In addition to using the OMP interview to collect data regarding OMPs, I also used a 20-item questionnaire (see Appendix B). Results from the questionnaire were used minimally to confirm claims from my analysis of the OMP interview.

**Design of questionnaire items.** The OMP questionnaire was modeled after the Mathematical Proof Survey (Conner et al., 2011; Yoo, 2008), and each item contains a preliminary setup and two contrasting views developed in light of the literature on conceptions of mathematics and processes (see Chapter 3) along with a processes and actions framework (Zbiek et al., 2014) used in a project on which I had worked for several years. A contrasting alternatives design (Halloun & Hestenes, 1996) allows participants to select from two options: a process option and an actions option. For example, item number 6 in the questionnaire (see Figure 4-3) provides participants with two options from which they may choose. Option (a) is associated with acting on a justification whereas option (b) is associated with engaging in justifying. The item contains an space for participants to leave clarifying comments. This question is related to how a PMT perceives himself/herself engaging in a process. Similar questions about engaging in each of the other three processes were included in the questionnaire.
OMP consists of four components: conceptions of mathematics that are related to mathematical processes, conceptions of teaching mathematics as related to mathematical processes, conceptions of learning mathematical processes and learning mathematics through mathematical processes, and engaging in mathematical processes. I designed the questionnaire to contain items involving each of these four components of OMP in each of the four processes of defining, generalizing, justifying, and representing. Table 4-4 shows the distribution of the questionnaire items in Appendix B. The components of OMP are listed in the left column and the processes are listed in the top row. The number in a cell is the number of the questionnaire item. For example, item number 6 in the questionnaire is about how PMTs perceive themselves engaging in justifying. The items related to justifying were taken from Yoo (2008) and Conner et al. (2011), with some of the words in the items changed. For example, Yoo (2008) used the following wording for one questionnaire item: “In order to prove a mathematical statement you need to (a) have seen the proof of a similar statement before or (b) know how to apply various types of reasoning and methods of proof.” I altered the wording by changing “prove” to “justify” and “proof” to “justification” in option (a). I also changed the phrasing of the statement from

**Figure 4-3.** A sample item from the OMP Questionnaire.

6. In order to justify a mathematical statement I need to

(a) have seen the justification of a similar statement before or
(b) know how to apply types of reasoning or methods of proof.

<table>
<thead>
<tr>
<th>Mostly a ← Toward a</th>
<th>Toward b</th>
<th>Mostly b</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

**COMMENTS:**
“you” to “I.” I wrote the remaining items in light of the processes framework distinguishing process from action on product (Zbiek et al., 2014) and using research findings on teachers’ beliefs and conceptions of each of the four processes.

### Table 4-4

*Distribution of OMP Questionnaire Items*

<table>
<thead>
<tr>
<th></th>
<th>Mathematics</th>
<th>Defining</th>
<th>Generalizing</th>
<th>Justifying</th>
<th>Representing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conception</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>Engaging</td>
<td>8</td>
<td>7</td>
<td>9</td>
<td>6</td>
<td>10</td>
</tr>
<tr>
<td>Teaching</td>
<td>11</td>
<td>14</td>
<td>13</td>
<td>12</td>
<td>15</td>
</tr>
<tr>
<td>Learning</td>
<td>16</td>
<td>20</td>
<td>18</td>
<td>17</td>
<td>19</td>
</tr>
</tbody>
</table>

In addition to included items about each of the four processes, I also included items involving mathematics in general for two reasons. First, much of the research on teachers’ conceptions is about mathematics in general rather than about individual processes (aside from justifying) and comparisons between answers to general mathematics questions and particular processes might be useful for arguing the need to study conceptions and OMPs of particular processes. Second, I suspected that PMTs’ enculturation experiences in teacher education would predominantly be in the context of general mathematics rather than particular processes, with justifying being the notable exception. Including items written about mathematics in general might be useful for establishing a need to focus on processes in teacher education.

As a first step in the study, I administered the questionnaire to the participants via a Google form. I then scheduled interviews with participants once they completed the online
questionnaire and conducted a preliminary analysis of the questionnaire results for each participant.

**Analysis of questionnaire.** After my first initial analysis of the OMP interviews, I reviewed participants’ answers to the questionnaire items to triangulate the data as I sought answers to my first research question. If I observed discrepancies between my interpretation of questionnaire responses and my observations from the interview, I tried to reconcile these discrepancies by examining the data more closely. But if discrepancies could not be reconciled, the more situated data from the interview was given more value in my analysis because the more situated data was considered more valuable based on my theoretical perspective of situated cognition. Together the data sources of OMP interviews and questionnaire were used to construct answers to my first research question of what are PMTs’ OMPs, with my OMP framework used to guide both the design of the instruments and analysis of the data.

**Data Collection and Analysis of Orientations and Tasks**

My second research question focused on the influence of OMP on participants’ task work of selecting, modifying and sequencing tasks. To generate data that could be used to answer this question, I created to two task interviews, one situated in the context of the Pythagorean theorem and one situated in the context of quadratic, in which I gave participants opportunities to select, modify, and sequence mathematical tasks. In addition to using the task interviews to gather data that could be used to answer my second research question, I also used the task interviews to gather more data on OMPs based on preliminary data analysis of the OMP interviews.

**Task interviews.** To determine the influence of OMP on selecting, modifying, and sequencing mathematical tasks, I first needed to understand how PMTs select, modify, and sequence tasks (such actions on tasks will be collectively referred to as “task work” in this
This task work served as the focus for both task interviews. An interview schedule for the Pythagorean task interview (Int2) is included as Appendix C and an interview schedule for the quadratic task interview (Int3) is included as Appendix D.

**Design of task interviews.** If I wanted to learn more about how PMTs work with mathematical tasks, I needed to provide participants with opportunities to select, modify and sequence mathematical tasks. In both interviews, participants were given a set of tasks and asked to create a unit using some or all of the tasks along with the option to create their own tasks to include in their unit. I created the sets of tasks for participants to use during the task interviews by examining units, lessons and tasks related to the Pythagorean theorem and quadratics in both traditional textbooks as well as NSF-funded texts which were created in response to the NCTM (1989) Standards. Using a processes and actions framework (Heid et al., 2015; Zbiek et al., 2014), I selected tasks with a variety of opportunities for processes and actions on products because I wanted to gather data about how PMTs included opportunities for students to engage in processes in their task work.

Table 4-5 includes a list of tasks used in the Pythagorean task interview and Table 4-6 includes a list of tasks used in the quadratic task interview. Each table includes the name of the task, the letter assigned to the task for interview purposes, features of the task and opportunities for processes evident in the different tasks (see Appendix C and Appendix D for the text of the tasks). Tasks names are used in the report of findings to convey the topic and possible processes of the tasks. Task names were not given to participants and instead letters were assigned to the tasks during the interviews to avoid the influence of a task’s name on the selection, modification and sequencing of the task. The tables also include a list of possible task features to which a PMT might attend. The opportunities for processes and actions I identified in the tasks were
corroborated by a set of graduate students in mathematics education who had used the processes and actions framework in research projects for at least one year. Graduate students identified opportunities for processes and actions in each task; process opportunities agreed upon by at least 4 of the 5 graduate students are included in the table.
Table 4-5

**Pythagorean Theorem Tasks Examined by PMTs in Int2**

<table>
<thead>
<tr>
<th>Category</th>
<th>Task Name</th>
<th>Letter</th>
<th>Features</th>
<th>Opportunities for Processes/Actions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introductory Tasks</td>
<td>Tri-square Rugs</td>
<td>A</td>
<td>• Real-world context</td>
<td>• Representing triangles whose sides form squares</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>• Open-ended</td>
<td>• Generalizing relationships among areas of squares formed from sides of <em>any</em> triangle</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>• Justifying relationships among areas of squares formed from sides of a triangle</td>
</tr>
<tr>
<td></td>
<td>Pythagorean Puzzle</td>
<td>B</td>
<td>• Hands-on</td>
<td>• Representing relationships among right triangles and squares</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>• Generalizing Pythagorean theorem</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>• Justifying Pythagorean theorem, if modified</td>
</tr>
<tr>
<td></td>
<td>Dot Paper Generalizing</td>
<td>D</td>
<td>• Table</td>
<td>• Representing right triangles and squares</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>• Dot-paper</td>
<td>• Generalizing Pythagorean theorem</td>
</tr>
<tr>
<td>Proof Tasks</td>
<td>Pythagorean Rugs Proof</td>
<td>C</td>
<td>• Real-world context</td>
<td>• Justifying Pythagorean theorem</td>
</tr>
<tr>
<td></td>
<td>Square-in-Square Proof</td>
<td>F</td>
<td>• Visual diagram</td>
<td>• Interpreting and applying actions on representations and generalizations</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>• Guided questions</td>
<td></td>
</tr>
<tr>
<td>Triples Tasks</td>
<td>Examples &amp; Non-examples</td>
<td>H</td>
<td>• Discovery</td>
<td>• Defining Pythagorean triple</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>• Examples</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Odd Triples</td>
<td>J</td>
<td>• Table</td>
<td>• Defining odd Pythagorean triple</td>
</tr>
<tr>
<td></td>
<td>Identifying Triples</td>
<td>L</td>
<td>• Extension</td>
<td>• Justifying divisibility of odd triple</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>• Definition included</td>
<td>• Applying definition of Pythagorean triple</td>
</tr>
<tr>
<td>Converse Tasks</td>
<td>Justifying with Converse</td>
<td>G</td>
<td>• Visual</td>
<td>• Justify that given triangle is a right triangle by applying Pythagorean theorem and its converse</td>
</tr>
<tr>
<td></td>
<td>Verifying Converse</td>
<td>I</td>
<td>• Dynamic</td>
<td>• Interpreting and applying actions on representations, generalizations, and definitions</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>• Technology</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>• Guided</td>
<td></td>
</tr>
<tr>
<td>Application Tasks</td>
<td>Triangle Garden</td>
<td>E</td>
<td>• Real-world context</td>
<td>• Justifying length of altitude in scalene triangle</td>
</tr>
<tr>
<td></td>
<td>Baseball</td>
<td>K</td>
<td>• Real-world context</td>
<td>• Applying Pythagorean theorem</td>
</tr>
</tbody>
</table>
Table 4-6

*Quadratic Tasks Examined by PMTs in Int3*

<table>
<thead>
<tr>
<th>Task Name</th>
<th>Letter</th>
<th>Features</th>
<th>Opportunities for Processes/Actions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exploring $a$ with Defining</td>
<td>P</td>
<td>Dynamic Technology - Exploration - Definition given</td>
<td>- Define “wider” in the context of parabolas - Generalize the affects of parameter $a$</td>
</tr>
<tr>
<td>Defining Quartic Parabola</td>
<td>Q</td>
<td>Answer analysis - Extension</td>
<td>- Evaluate a justification</td>
</tr>
<tr>
<td>Exploring $c$ with Intercepts</td>
<td>R</td>
<td>Technology - Exploration</td>
<td>- Interpret graphs of quadratic functions - Generalize about $x$- and $y$-intercepts of quadratic functions</td>
</tr>
<tr>
<td>Intercepts as Solutions</td>
<td>S</td>
<td>Multiple representations</td>
<td>- Link graphical and symbolic representations of quadratic equations and functions</td>
</tr>
<tr>
<td>Unfactorable Algebra Tiles</td>
<td>T</td>
<td>Hands-on</td>
<td>- Represent relationships among terms in a quadratic expression using algebra tiles - Justify equivalence of quadratic expressions - Manipulate symbolic and diagrammatic representations of quadratic equation to find solutions - Link symbolic and diagrammatic representations of equivalent quadratic expressions</td>
</tr>
<tr>
<td>Factoring with Justifying</td>
<td>U</td>
<td>Real-world context</td>
<td>- Justify equivalence of quadratic expressions and steps in a procedure for solving quadratic equations - Manipulate symbolic representations of quadratic equations</td>
</tr>
<tr>
<td>Novel Set of Quadratics</td>
<td>V</td>
<td>Open question</td>
<td>- Define a subclass of quadratic equations based on roots - Represent an element of a subclass of quadratic equations symbolically - Apply generalized procedures for finding roots of equations</td>
</tr>
<tr>
<td>Generalizing Solutions</td>
<td>W</td>
<td>Length of task</td>
<td>- Generalize values of parameters for forms of quadratic equations that give a certain number of solutions - Justify claims about number of solutions - Represent general quadratic functions with varying parameters graphically</td>
</tr>
<tr>
<td>Factors and Intercepts</td>
<td>X</td>
<td>Technology - Multiple representations</td>
<td>- Link a graphical representation of a quadratic function to a symbolic representation of a quadratic function - Represent quadratic functions graphically - Generalize a method for finding intercepts from a symbolic representation of a quadratic function - Justify your generalization</td>
</tr>
<tr>
<td>Task Name</td>
<td>Letter</td>
<td>Features</td>
<td>Opportunities for Processes/Actions</td>
</tr>
<tr>
<td>------------------------</td>
<td>--------</td>
<td>-------------------------------</td>
<td>------------------------------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>Vertex Application</td>
<td>Y</td>
<td>- Non-math context</td>
<td>- <em>Interpret</em> a graphical representation of a quadratic function</td>
</tr>
<tr>
<td></td>
<td></td>
<td>- Multiple representations</td>
<td>- <em>Manipulate</em> an symbolic representation of a quadratic equation</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>- <em>Link</em> representations of graphical and symbolic solutions of quadratic equation</td>
</tr>
<tr>
<td>Completing the Algebra Tile Square</td>
<td>Z</td>
<td>- Hands-on</td>
<td>- <em>Interpret</em> algebra tile representations of quadratic expressions</td>
</tr>
<tr>
<td></td>
<td></td>
<td>- Guided</td>
<td>- <em>Manipulate</em> symbolic representations of quadratic equations and algebra tile representations of quadratic expressions</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>- <em>Link</em> algebra tile representations of quadratic expressions with symbolic representations of quadratic equations</td>
</tr>
</tbody>
</table>

In addition to selecting tasks with a variety of processes, I also selected tasks with different task features and structures. Research indicates that teachers attend to salient features of a task during the selection process (Levenson, 2012a; Osana et al., 2006). Consequently, I tried to include tasks that varied in length (Osana et al., 2006), context (Arbaugh & C. Brown, 2005; Nicol & Crespo, 2006), and presence of technology. Follow-up interview questions removed these salient features of the tasks to determine whether or not a feature of the task influenced PMTs’ work with the task.

The set of tasks that examine proofs of the Pythagorean theorem is an example of how I varied processes and task features across tasks. Both tasks use a common representation of a proof of the Pythagorean theorem, but the Pythagorean Rugs task asks for an explicit justification of the Pythagorean theorem whereas the Square-in-Square Proof task explicitly indicates that the students will “examine” a common proof of the Pythagorean theorem. Consequently, the former provides opportunities for engaging in the process of justifying whereas the latter provides opportunities for interpreting a justification. The features of the two tasks also vary. The Pythagorean Rugs task is in the context of a dart-throwing rug game.
whereas the context of the Square-in-Square Proof task does not have an outside mathematical context. The Pythagorean Rugs task is relatively short in comparison to the Square-in-Square Proof task, which includes a step-by-step examination of a proof of the Pythagorean theorem. The purpose of including tasks with a variety of process opportunities was to provide participants with opportunities to choose between tasks with opportunities for processes and tasks with only opportunities to perform actions.

The purposes of the task interview were to investigate PMTs’ task work and the influence of OMP on that work. At the beginning of the interview, I gave the participants all of the tasks at once and asked them to create a unit on the Pythagorean theorem or quadratics using any combination of the given tasks or variations of the tasks. Follow-up questions, such as “what was it about that task that made you decide to include the task in your unit,” focused on why PMTs selected, modified, and sequenced tasks in particular ways to determine the influence of OMP. Although research indicated that PMTs may not solve the tasks prior to selecting them (Nicol & Crespo, 2006), I chose not to have participants solve the tasks during their planning process because their tendency not to solve the tasks was indicative of their work with tasks as well as their OMP. Instead, I prepared follow-up questions to investigate why a participant tended not to solve the tasks. For example, after a participant commented on each task, I would put tasks on which the participant worked one set and tasks on which the participant did not work in a different set. I then asked the participant what he thought made the two sets of tasks similar or different, and why he decided to solve some tasks but not others. Toward the end of the interviews, I would ask participants to solve particular tasks to gather more data on their engagement in mathematical processes.
My preliminary analysis of the OMP interviews, which occurred just after the OMP interview but before the task interviews, indicated that I did not have enough data about PMTs’ conceptions of processes. I added additional questions to the quadratic task interview by using tasks from the set of tasks used in the quadratic task interview to elicit more information about their conceptions of processes. I gave the participants four of the quadratic tasks—two involving processes (Unfactorable Algebra Tiles and Factoring with Justifying) and two involving primarily actions (Completing the Algebra Tile Square and Intercepts as Solutions)—and asked the participants to group the tasks and explain their reasoning for the grouping. I then gave them three more tasks with different process and action opportunities one at a time (Generalizing Solutions, Exploring a with Defining and Vertex Application) and asked how these tasks might fit into their grouping scheme, if at all. After participants had sorted the seven tasks, I then grouped the tasks into two categories based on presence of opportunities to engage in processes in the tasks and asked participants if they could describe my sorting scheme. The activity of having participants sort tasks and interpret my sorting of tasks revealed participants’ conceptions of tasks with respect to processes and whether processes were part of the participants’ conceptions of mathematics.

To gather data on participants’ conceptions of specific processes, I asked them to describe a good justifying task, a good generalizing task, a good defining task and a good representing task. I then asked the participants to identify tasks that might make good process tasks among the quadratic and Pythagorean theorem tasks. Asking participants to describe tasks related to each of the processes revealed what they thought about the different processes and if they distinguished among the processes.
In summary, I designed the task interviews to reveal how participants selected, modified, and sequenced mathematical tasks with a variety of opportunities for processes and actions and a variety of task features. I invited them to create two units using sets Pythagorean theorem and quadratic tasks with the option to create their own tasks if desired. After they had created their units, I asked them to compare, contrast, and sort tasks to reveal more about their conceptions of processes and tasks. Throughout the interviews, I focused on why participants selected, modified, and sequenced tasks in the ways that they did so that I could ascertain how OMP was related to their task work.

**Analysis of task interviews.** The purposes of the task interviews were to gather data on participants’ task work, and at the same time gather additional data on participants’ OMPs. The analysis of the task interviews began with a critical reading of the interviews looking for evidence that might contradict hypotheses recorded in the memos created from the OMP interviews. I documented any possible contradictions by adding refuting evidence to the summary document. Using the evidence from all three interviews, I then modified the hypotheses as needed to incorporate the additional evidence.

Following a critical reading of the interviews for additional OMP data, I began to analyze the data to answer questions about participants’ task work. I coded PMTs’ work with mathematical tasks as selecting, modifying, or sequencing. The purpose of this descriptive coding was to identify important segments of the interview (Seidman, 2006) that were evidence of PMTs’ task work. I then added data and evidence about each participant’s task work to their respective memos. Explanations about how participants selected, modified, and sequenced tasks were used to answer part of my second research question—how do PMTs work with mathematical tasks.
To completely answer my second research question, not only did I need to be able to describe PMTs’ task work, I also needed to describe the relationship of OMP to task work. To do this, I used my findings from the first research question along with my findings from the task interviews to understand why participants worked with tasks in the ways that they did and how that related to their OMPs. The memos I had created during my analysis of the OMP interviews and added to during my analysis of the task interviews were used to create profiles to aide in my analysis of participant task work and OMP.

**OMP Profiles as a Connecting Method**

The different memos I had created through analysis of both the OMP interview and the task interviews using descriptive coding and thematic categorizing needed organization that would facilitate connections across participants. Profiles (Seidman, 2006) are one method connecting analysis that aides in the identification of relationships among different elements of text (Maxwell, 2005). As suggested by Seidman (2006), I created each profile by juxtaposing previously coded elements of text that were related thematically and adding researcher interpretations as transitions between elements of the transcript. An OMP profile is a description of a participant’s OMP organized around a guiding theme that captures pedagogical and mathematical aspects related to how each participant would provide opportunities for students to engage in mathematical processes. Each OMP paralleled my research questions by describing a participant’s OMP and how that OMP influenced his or her task work. The OMP profiles were tools that facilitated the making of connections across participants in answering my research questions.

Not only are profiles useful connecting strategies in analysis (Maxwell, 2005), but also for constructing narratives for sharing data and communicating results (Seidman, 2006). The
profiles will be presented in Chapter 5 of this dissertation in prelude to sharing findings and discussions related to the research questions.

**Conclusion**

I used my research questions related to PMTs’ OMPs and task work to drive the design and analysis of my study. I delved into the essence of OMP using the OMP interview augmented by the questionnaire, which I designed to gather data regarding all four components of OMP (conceptions of mathematical processes, conceptions of teaching processes, conceptions of learning processes, and engaging in processes) with respect to each of the four mathematical processes (generalizing, defining, representing, and justifying) based on my theoretical framework. To understand how PMTs’ OMPs influenced how they worked with mathematical tasks, I had participants select, modify, and sequence two sets of mathematical tasks, one related to the Pythagorean theorem and one related to quadratics, during two task interviews. I also asked participants to compare, contrast and sort tasks to reveal more about their conceptions of processes and tasks.

Analysis of my data consisted of three main phases. The first phase focused on using the OMP interview to answer my first research question of what are PMTs’ OMPs by using constant comparative analysis in a cycle of descriptive coding with my theoretical framework as a lens and then thematic categorizing on the OMP interview. Supportive analysis and triangulation from the task interviews added validity to the emerging hypotheses. The second phase focused on PMTs’ task work by once again using descriptive coding of task work (selecting, modifying, and sequencing) and then thematic categorizing of the task interviews. The final phase involved the creation of OMP profiles, which are descriptions of a participants OMP organized around a guiding theme that captures pedagogical and mathematical aspects related to how each
participant would provide opportunities for students to engage in mathematical processes through their task work. I share these profiles in the next chapter as a prelude to the findings and discussion found in a later chapter.
Chapter 5

OMP Profiles

In this chapter, I present orientations toward mathematical processes (OMPs) profiles for all participants. I define a teacher’s OMP to be a set of conceptions about mathematics, its teaching and learning related to mathematical processes, and engagement in mathematical processes that influences how a teacher provides opportunities for students to engage in mathematical processes. An OMP profile is a description of a participant’s OMP organized around a guiding theme that captures pedagogical aspects (conceptions about teaching and learning mathematics) and mathematical aspects (conceptions of mathematics and engaging in mathematical processes) related to how each participant plans to provide opportunities for students to engage in mathematical processes.

My research questions ask, what are prospective secondary mathematics teachers’ OMPs and how are their OMPs related to how they select, modify, and sequence mathematical tasks? The profiles themselves are not the answers to my research questions. They are methodological tools used here as narratives for sharing and communicating evidence related to the research questions for individual participants. The evidence I present in each profile parallels my two research questions in that I first describe conceptions of teaching, learning, and mathematics along with engagement in process that characterize a participant’s OMP followed by how a participant’s OMP relates to his or her task work. In Chapter 6, I organize OMP profiles into categories to answer my research questions regarding PMTs’ collective OMPs and their collective task work.

While attending to conceptualizations of teaching and processes and actions, I noticed a common theme across participants: the notion of discovery (Bruner, 1960). Not only did
participants use discovery as a term to capture *pedagogical* methods of teaching but they also used discovery as a term to refer to more than one process in a way that encapsulated the four *processes* into an activity of discovery. In this chapter, I include insights into how participants conceptualized discovery that I will use in Chapter 6 to explain the nature of PMTs’ OMPs.

I organized this presentation of individual profiles by focusing on process and action differences. The chapter begins with profiles dominated by actions. These profiles are Erin’s Distance OMP, Jenny’s Comfort OMP, and Heather’s Guided OMP. Profiles that support opportunities for students to engage in processes follow. These profiles include Brad’s Destination OMP and Sarah’s Discussion OMP.

**Distance OMP Profile: Erin**

A theme of distance from processes captured Erin’s personal mathematics and her approach to teaching mathematics. She distanced herself mathematically from processes by avoiding opportunities to engage in processes. In her descriptions of classroom experiences, Erin described experiences in which she distanced herself from possible student thinking related to processes by asking questions of students related almost exclusively to actions. The theme of distance from processes continued into her task work as she effectively distanced students from processes by primarily providing opportunities to act on products.

**Distance from mathematical processes.** Erin distanced herself from processes in her personal mathematics by preferring to engage in actions—particularly *applying*—and avoiding processes. In Erin’s reaction to the teaching scenario involving *justifying*, Erin indicated that she did not like to engage in proof. “So I know I don't like proofs. I've always been that person—give it to me and I will apply it in a thousand ways if you want me to. But proving, I don't”

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5 I italicize processes and actions that are the focus/foci of a paragraph to add clarity to use of terms used to describe participants’ mathematical activity.
In this statement, Erin indicated that she preferred the action of applying to the process of justifying. Additionally, Erin underscored her preference for the action of applying by indicating that she would apply a formula to equations for long periods of time. She stated, “Give me a formula and I will sit there and do hours and hours of putting it into equations” (Int3). In this statement, Erin indicated that the action of applying is one mathematical activity in which she willingly engaged. Her apparent preference for the action of applying over the process of justifying distanced her from possible process opportunities and positioned her closer to actions.

In comparison to other processes, Erin most frequently expressed a dislike for the process of justifying and avoided engaging the in process during interviews. She avoided justifying when asked if she could use the Pythagorean Puzzle task to justify the Pythagorean theorem, responding with “Nope, nothing” (Int1). When pressed further to justify, she interpreted the representation of squares and triangles she had created while solving the task (see Figure 5-1). She interpreted parts of the representation, “This one is $a^2$ and we have $b^2$ next to it. That’s all I’ve got.” Erin linked parts of the Pythagorean theorem to the representation she had created, but did not attempt to justify further.
Erin’s representation of squares and triangles created while solving the Pythagorean Puzzle task.

Erin also resisted engaging in other processes during interviews. While examining different tasks related to the Pythagorean theorem, she chose not to solve the Examples and Non-Examples task, which included opportunities to define. She stated, “I have never seen a definition of a useful triangle … Never been told a useful triangle versus not a useful triangle” (Int2). Erin did not engage in defining because she was unfamiliar with the term “useful triangle” and its definition. Additionally, in her statement Erin characterized definitions as something to be “seen” or “told” rather than something created. When describing why she like the Identifying Triples task, she stated, referring to the definition of given at the beginning of the task: “It tells them straight up about the Pythagorean Triples” (Int2). Her preference for a task stating a definition reflected a preference for stating definitions, which distanced her from defining.
Erin resisted engaging in representing when the representations of a task were unfamiliar. When asked why she did not want to solve the Unfactorable Algebra Tiles task, she mentioned the presence of an unfamiliar representation medium—algebra tiles—as a reason for not trying to solve the problem. “I hate these things [algebra tiles]. I can't use them. I never used them in high school or middle school so I can’t use them” (Int3). By not attempting to solve a task that involved an unfamiliar representation medium, Erin avoided engaging in the process of representing, effectively distancing herself from the process of representing.

Rather than engaging in representing, Erin interpreted representations. For example, she interpreted the diagram in the Tri-square Rugs task and stated her interpretation as a solution to the task.

I think I would rather be Betty because even though these [the two smaller squares] are smaller and the combined areas might not be close to this [the larger square], the way that it is arranged, I am just thinking you would have a better chance of hitting them [the two smaller squares] versus only hitting one square. (Int1)

In this passage, she interpreted the diagram of a triangle with squares on each side of the triangle to conclude that Betty would have a better chance of hitting two squares instead of one square. Erin did not engage in any processes to solve the task, despite the task containing opportunities to engage in the processes of representing, generalizing, and justifying. Erin favored the action of interpreting over engaging in possible processes.

Not only did Erin avoid engaging in processes in the interviews but she also expressed a dislike of opportunities to explore in her mathematics education courses. Erin frequently used the word explore when describing teaching and learning mathematics. She described exploring as “hav[ing] to take almost no prior knowledge or very little and apply it and see if they can figure out either the formula, what kind of equation it is, … [or] trying to figure out something from it” (Int3). Erin did not like exploring mathematics in her mathematics education courses. She stated
that her mathematics education course instructors and professors “want you to figure out pretty much everything. I don’t like that” (Int3). The course description for the beginning methods course at the university stated that one purpose of the course was “to improve understanding of the nature of mathematics: what is important, how it is practiced, how mathematical validity is determined” (http://bulletins.psu.edu/undergrad/courses/M/MTHED/411/200607SP).

Opportunities to engage in mathematical processes, as part of the practices of mathematics and developing an understanding of the nature of mathematics, were likely present in Erin’s mathematics education courses and could be part of what Erin considered exploring mathematics. Erin’s dislike of exploring could then mean that she disliked engaging in mathematical processes.

Additionally, Erin’s ubiquitous use of explore could be related to her seemingly blurred conceptions of processes. Erin did not articulate clear distinctions among the four processes and often her descriptions of one process involved elements of other processes. Erin’s descriptions of good justifying tasks and good representing tasks were very similar. For example, she described good justifying tasks as “open-ended problems” such as “Susie went to the market and she bought five apples at $3 and six bananas at $2, you know. She only has $10” (Int3). Her description of a good justifying task seemed unrelated to justifying. When asked why she thought these types of problems were good justifying tasks, she responded that they were “something concrete where they can apply it” (Int3). For Erin, concrete and applicable seemed to mean situated in a non-mathematical context. In her description of a justifying task, Erin described a task in which students represent a concrete situation mathematically. She also described a good representing task as “word problems … because you know we're [teachers] always asking them [students] to represent equations or to … represent this amount, this number,
with, you know, a variable or a letter” (Int3). For Erin, good representing problems involved writing algebraic equations for word problems. Her descriptions of good justifying and representing tasks were similar in that both seemed to involve representing a situation mathematically. The similarity in her statements about justifying and representing suggests that her conceptions of justifying and representing were blurred and not distinct.

When pressed for more information about good justifying tasks, Erin identified the Exploring \( c \) with Intercepts task as a good justifying task due to “the idea of going through … five different examples of a quadratic equation and asking them about the effects of \( c \)” (Int3). Her description of a justifying task involved generalizing rather than justifying, suggesting that her conceptions of justifying and generalizing might be blurred to her. Erin’s descriptions of justifying tasks included representing and generalizing in addition to justifying, indicating that her conception of the process of justifying blurs with other processes such as representing and generalizing.

Erin’s blurred conceptions of processes distanced her from processes because she did not articulate what the individual processes involved or their role in mathematics. By avoiding opportunities to engage in processes, Erin distanced herself from mathematically from processes.

**Distance from student thinking.** Not only did Erin distance herself mathematically from processes, she distanced herself from possible student thinking related to processes by describing how she elicited student answers that seem to be dominated by actions. Erin’s descriptions of student thinking focused on steps in a procedure. For example, she described one of her favorite student teaching experiences as a “popcorn” activity in which students called on each other to provide steps in procedural examples, which likely involved the action of manipulating representations of equations. “I did popcorn but it was after every step in an example, … after
every step I would change it [move to another student]” (Int1). As a student teacher, Erin seemingly gave students opportunities to participate primarily in the action of manipulating. Erin’s focus on procedural steps in student participation had the effect of focusing on student thinking dominated by actions, allowing her to remain distant from student thinking that focus on processes.

**Distancing students from mathematical processes through task work.** Erin’s focus on student thinking involving actions created distance from mathematical processes that is reflected in her task work. She selected tasks dominated by opportunities to act on products with minimal opportunities to engage in processes. The tasks Erin selected for her Pythagorean theorem unit along with her planned sequence are shown in Table 5-1. Similarly, Table 5-2 includes the tasks Erin selected for her quadratic unit. The tables include the day Erin planned to use the task, a task type that she assigned to tasks, and the name of the task.

Table 5-1

*Erin’s Plans for a Pythagorean Theorem Unit*

<table>
<thead>
<tr>
<th>Day</th>
<th>Task Type</th>
<th>Task Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td><em>Pythagorean Theorem</em></td>
<td>Pythagorean Puzzle</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Baseball Application</td>
</tr>
<tr>
<td>2</td>
<td><em>Converse</em></td>
<td>Verifying Converse</td>
</tr>
<tr>
<td>3</td>
<td><em>Triples</em></td>
<td>Identifying Triples</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Odd Triples</td>
</tr>
</tbody>
</table>
Table 5-2

*Erin’s Plans for a Quadratic Unit*

<table>
<thead>
<tr>
<th>Day</th>
<th>Task Type</th>
<th>Task Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td><em>Quadratic Form</em></td>
<td>Exploring (a) with Defining</td>
</tr>
<tr>
<td>2</td>
<td><strong>Intercepts</strong></td>
<td>Intercepts as Solutions</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Factors and Intercepts</td>
</tr>
<tr>
<td>Last</td>
<td><strong>Factoring Examples</strong></td>
<td>Novel Set of Quadratics</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Factoring with Justifying</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Vertex Application</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Generalizing Solutions</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Defining Quartic Parabola</td>
</tr>
</tbody>
</table>

Three of the five selected Pythagorean theorem tasks and four of the eight quadratic selected tasks could be resolved through primarily actions. Even when selecting tasks that included opportunities to engage in processes, the presence of processes was not her motivation for task selection. Instead, she selected a task because it fit with the topic of the day. For example, she selected both *Intercepts as Solutions* (Task S) and *Factors and Intercepts* (Task X) from the quadratic task interview because both addressed the topic of intercepts. She stated, “I would want to do the \(x\)-intercepts together.” The Factors and Intercepts task included opportunities to engage in *representing, generalizing, and justifying*, but the presence of these process opportunities did not factor into her task selection process. Instead, the presence of technology in the Factors and Intercepts task and the potential for showing multiple methods influenced her selection of that task. “In this one [Task S] they have to do it by hand and in this one [Task X] they have to do it on the calculator. I like showing them both ways.” (Int3). Even though a task she selected included opportunities for processes, she selected the tasks for reasons
seemingly unrelated to processes. She used task features such as the presence of technology or demonstration of multiple solution methods to select the tasks.

Erin selected tasks based on similarity of topic, as observed in the above example involving intercepts. She also used task features to select tasks. For example, she selected the **Baseball** task based on the task feature of context. “I like this. My students, they were, from being in Pittsburgh, they were all about the Pirates” (Int2). She selected the **Pythagorean Puzzle** task based on the feature of physical activity of manipulating paper triangles and squares. “I like how it's something physical versus me just telling them what the Pythagorean theorem is” (Int2). The presence of technology and the guided questions of the task that “walk them through it” influenced her selection of the **Verifying Converse** task (Task I). “I think I would use Task I as well because I like that it does some kind of software … [and] that it kind of walks them through it” (Int2). Erin’s descriptions suggest that task features of walking students through certain steps in the use of technology mattered. She seemed oblivious to the missed opportunities for students to engage in processes.

The presence or absence of exploration was another task feature Erin used to select tasks. Erin viewed some tasks as involving too much exploration; these tasks tended to be tasks with multiple opportunities to engage in processes, such as the **Tri-square Rugs** task. She did not select tasks that involved what she perceived as exploration tasks because “I don't want them … struggling with it because it sometimes gets to that point of they have no idea” (Int2). By avoiding selecting tasks with exploration, she avoided opportunities for students to engage in processes and effectively distanced her students from processes through her task selection.

Erin did not show evidence of solving the tasks presented in the task interviews. If she spontaneously solved the tasks, she did not articulate or share the solutions. Erin probably relied
on task features in her task selection because she did not solve the tasks before selecting them. Task features enabled her to select tasks without needing to engage in processes or even actions to solve the tasks, reflective of distance from processes.

**Summary of Erin’s distance OMP.** Distance captured Erin’s OMP mathematically and pedagogically. Erin distanced herself mathematically from processes by preferring to engage in actions on products—such as *applying, manipulating,* and *stating.* She further distanced herself from processes by focusing on student thinking involving actions. Her distance from processes might explain why she did not solve a task before selecting it and instead used task features to select tasks, the majority of which involved actions on products. By selecting tasks dominated by actions with minimal opportunities to engage in processes, Erin’s plans lead to the possible creation of a classroom culture in which students’ mathematical learning is situated in actions on products and not on mathematical processes, effectively distancing her students from mathematical processes.

**Comfort OMP Profile: Jenny**

A theme of comfort characterized Jenny’s OMP. I use the term comfort to capture an affective desire on Jenny’s part to avoid negative feelings on the part of the student, such as frustration, intimidation, or difficulty. She made the mathematics in the classroom comfortable for students by selecting tasks that she thought would not invoke negative responses from students and for which students had adequate prerequisite knowledge to solve.

**Jenny’s desire for comfortable mathematics for her and her students.** Jenny seemed to struggle with solving tasks that involved processes. During the OMP interview, she avoided solving the *Tri-square Rugs* task. While working on the task, she *interpreted* the representation of the given tri-square rug to solve the task—“just visually having the two squares would be
better than just having one” (Int1). She did not engage in opportunities to *represent, justify, define,* and *generalize* that were present in the task, even when prompted. As she avoided engaging in processes, her body language and repeated use of interpreting the given representation of a tri-square rug to try to solve the task suggested that she was uncomfortable with engaging in processes.

Perhaps Jenny’s difficulties trying to solve the *Pythagorean Rugs* task and later the *Pythagorean Puzzle* task stemmed from her conceptions of processes. The mathematical activity of looking for commonalities across a set of examples was common to Jenny’s conceptions of different mathematical processes. She mentioned the “repeated reasoning” mathematical practice (CCSSI, 2010) in both the OMP interview and the Pythagorean task interview, describing the practice as “where they see it happening enough times and come up with a general rule for it” (Int1). At various times in the interviews, she described justifying, generalizing, and defining as involving the activity of looking across a set of examples for a general rule. For example, while reacting to the teaching scenario for *justifying*, she mentioned “repeated reasoning” and said that examples could be used to come up with a proof: “when we're working through the examples and the non-examples, we can use that to come up with a proof” (Int1). Jenny indicated that justifying in the form of proof could be done via repeated reasoning and an aspect of this mathematical practice could be used in the justification process.

The idea of looking across multiple examples was also present in Jenny’s description of a good *generalizing* task. She described a good generalizing task as “any task that would have them look at multiple cases and then they can generalize the main idea of it” (Int3). Similarly, Jenny’s description of a good *defining* task involved looking at different cases. She described defining tasks as “where they have to come up with a definition in their own … having them
look at all the different cases or all the different properties or all the features” (Int3). It is interesting to note that looking for commonalities across sets of examples does not readily solve the two tasks Jenny struggled to solve in the OMP interview: the Pythagorean Puzzle task and the Tri-Square Rugs task. If her conceptions of mathematical processes were dominated by the activity of looking for commonalities across sets of examples, then mathematical activity not requiring other than looking across multiple examples—or repeated reasoning—might be difficult and therefore uncomfortable for Jenny.

Not only did Jenny seem to want mathematics to be comfortable for her but she also seemed to want mathematics to be comfortable for her students. She viewed practice as a method of helping students feel comfortable. During discussions of her unit plans for the second day of the Pythagorean theorem, Jenny planned to show students examples of finding the missing side length in a right triangle and then have them practice that skill. When asked why she wanted to have students practice, Jenny explained, “Having them practice with that [finding the missing side length in a right triangle] so they have a solid understanding of what they were doing before they move to something else, so that they don't get frustrated” (Int1). Jenny wanted to avoid student frustration, seemingly so they would be comfortable with the skill of finding the missing side length in a right triangle. Jenny planned to provide students with adequate practice of an activity that involved application of the Pythagorean theorem and manipulation of symbolic representations. In essence, Jenny made the mathematics comfortable in her classroom by having students engage in actions.

Not only did Jenny plan to provide her students with practice involving actions but she also avoided giving students opportunities to engage in processes. During the teaching scenario sorting task of the OMP interview, Jenny described how giving students opportunities to justify
might make them feel intimidated or stuck. She stated, “but I feel like if you ask students to prove something, they get a little intimidated at first. They don't know where to start, and then they are stuck” (Int1). She indicated that students might not know how to start a proof and they might feel intimidated or get stuck while working on a proof, and classified the scenario as somewhat different from how she wanted to teach mathematics.

Jenny also indicated that she probably would not ask students to generalize the Pythagorean theorem because she did not think they would be able to create the desired generalization. In Jenny’s reaction to a teaching scenario in which students generalize the Pythagorean theorem, she stated, “I'm not sure if they would think to square the sides if they had never seen that before” (OMP Int1). In these examples, Jenny avoided student discomfort by giving students opportunities to engage in the processes of justifying and generalizing because she did not want students to feel intimidated nor did she think students were necessarily capable of engaging in generalizing.

**Methods instruction to make mathematics uncomfortable for students through discovery.** Jenny described how her methods courses shifted her view of teaching to include student discovery. She perceived that her methods courses changed her perspectives on teaching.

[They were] just shifting more away from the traditional style of teaching and more towards like student discovery. Like we had to try problems ourselves in class, we weren't just given the answers. So I think that helped me shift more towards having my own students do something like that. (OMP Int1)

Jenny’s experiences with discovery in her methods courses influenced her desire to include similar experiences for her students. Jenny valued the idea of discovery because “if you find something out by yourself, it is more rewarding in a way and it stays with you longer” (PT Int2). Jenny saw discovery as valuable in the learning process because it “stays with you longer” and planned to incorporate student discovery in her classroom.
Jenny admitted that discovery activities, which for her seemed to be primarily about not telling students, might make her students feel frustrated, and thus uncomfortable at times. She tried to alleviate this frustration by providing students with enough direction in her discovery activities. While describing why she liked the Pythagorean Rugs task, she indicated that the task would “give them enough where I feel like they wouldn’t get frustrated over something that doesn’t pertain to the theorem but also allows them to investigate on their own” (Int2). In addition to giving students enough direction so they won’t get frustrated during discovery activities, Jenny planned to support discovery activities through prepared examples prior to the activity. When asked how she might implement the Tri-square Rugs task in her classroom, Jenny suggested making the task more accessible by “giv[ing] them ones [sample Pythagorean Rugs made from paper] that you have premade, like the examples that you want them to try” (Int1). Jenny made the task more accessible and thus comfortable for students by overly scaffolding tasks, which effectively removed students’ opportunities to represent different tri-square rugs and led students to discover Jenny’s products and conclusions. Providing students with at least some direction in discovery activities might be Jenny’s way of making the students more comfortable with discovery.

**Task work as creating comfort for students.** Jenny’s initial approach to task work was to sort tasks into categories based primarily on topics such as Pythagorean triples, Pythagorean converse, and $x$-intercepts. In addition to task categories related to topics, Jenny included the category of exploration when categorizing tasks. Exploration tasks were those that involved students coming up with a product without the teacher telling the students what to do regardless of the mathematical topic. When asked why she called a set of tasks exploration tasks, Jenny explained, “I saw those as coming up with the Pythagorean theorem without telling them that is
what they are doing” (Int2). Jenny then used these categories to help her select and sequence tasks in a way that would avoid negative student responses. Categories used by Jenny and specific tasks Jenny selected in her planned Pythagorean theorem unit and in her planned quadratic unit are shown in Table 5-3 and Table 5-4 respectively.

Table 5-3

**Jenny’s Plans for a Pythagorean Theorem Unit**

<table>
<thead>
<tr>
<th>Day</th>
<th>Task Category</th>
<th>Task Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Exploration</td>
<td>Pythagorean Rugs</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Generalizing via Dot Paper</td>
</tr>
<tr>
<td>2</td>
<td>Triples</td>
<td>Examples &amp; Non-examples</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Triangle Garden</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Identifying Triples</td>
</tr>
<tr>
<td>3</td>
<td>Converse</td>
<td>Verifying Converse</td>
</tr>
</tbody>
</table>

Table 5-4

**Jenny’s Plans for a Quadratic Unit**

<table>
<thead>
<tr>
<th>Day</th>
<th>Task Category</th>
<th>Task Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Application</td>
<td>Factoring with Justifying</td>
</tr>
<tr>
<td></td>
<td>Investigation</td>
<td>Exploring $a$ with Defining</td>
</tr>
<tr>
<td></td>
<td>Intercepts</td>
<td>Intercepts as Solutions</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Factors and Intercepts</td>
</tr>
<tr>
<td>2</td>
<td>Investigation</td>
<td>Exploring $c$ with Intercepts</td>
</tr>
<tr>
<td></td>
<td>Intercepts</td>
<td>Novel Set of Quadratics</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Generalizing Solutions</td>
</tr>
<tr>
<td>3</td>
<td>Algebra Tiles</td>
<td>Unfactorable Algebra Tiles</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Completing the Algebra Tile Square</td>
</tr>
</tbody>
</table>
Jenny tended to select tasks that she thought students would be able to complete without frustration or difficulty, and would thus be comfortable for the students. For example, she selected the Examples and Non-Examples task and the Identifying Triples task from her Pythagorean triples category because she viewed them as tasks that would not frustrate students. When asked why she selected the tasks, she described, “there isn’t much room for misconception to come up because it gives them the different sides, so students wouldn’t get frustrated” (Int2). Jenny selected the tasks because she thought that students would be able to complete the tasks without misconception or frustration.

Jenny liked tasks that included prompts telling students what to do, thus creating the effect of comfort for students. For example, while describing what she liked about the Verifying Converse task, Jenny focused on the specific and directed questions at the end of the task. She explained, “if you want students to do something and you are expecting an answer, you kind of have to prompt them to what would lead them to that answer” (Int2). Jenny occasionally modified tasks to prompt students toward a particular solution form. For example, she suggested modifying the instructions in the Generalizing via Dot Paper task to include the “the word formula or equation so they know” (Int2) what is expected from the task. Such modifications might make the tasks less frustrating for students and thus more comfortable for Jenny and her students because then the students would know the form of their expected response.

Because Jenny wanted to create for her students a comfortable path through the mathematical content, she carefully sequenced tasks based on required prior knowledge. For example, she thought that students needed to know more about intercepts before completing a parameter exploration task called Exploring $c$ with Intercepts. “I think that they should know what intercepts are first because I don’t think you can make sense of the change [in intercepts] if
you don’t know what they [intercepts] are first” (Int3). To make sure students have the prior knowledge about intercepts she thinks they need, she sequenced two tasks related to intercepts—the Intercepts as Solutions task and the Factors and Intercepts task—immediately before the parameter exploration task. Using this sequencing, Jenny created what she perceived to be a mathematically comfortable task sequence in which all needed prerequisites were met so that students could navigate through the content. A desire to make mathematics comfortable for herself and for her students influenced how Jenny selected, modified, and sequenced mathematical tasks.

**Summary of Jenny’s comfort OMP.** A theme of comfort—referring to a desire to make negative student responses of frustration, intimidation, misconception, or difficulty—characterized Jenny’s OMP. Jenny’s desire to make mathematics comfortable for her students seemed to be related her discomfort with processes while attempting to solve tasks during interviews. To make math more comfortable for students, Jenny selected tasks that she did not perceive to be too difficult for students, she modified tasks to decrease the difficulty or ambiguity of the task, and sequenced tasks by considering required prerequisites for a task and deciding whether another task can be used to meet determined prerequisites.

**Guided OMP Profile: Heather**

I use the theme of guided to capture Heather’s OMP. Heather wanted to guide her students to mathematical content, but not through mathematical processes. She indicated that she would guide her students by helping students make what she called connections, which for Heather seemed to be links at the surface level of two mathematical ideas.

**Connections as guides in Heather’s personal mathematics.** Connections are at the core of Heather’s conception of mathematics. While describing why she tried to connect
mathematical topics for students during her student teaching experience, she made a statement about mathematics in general. “They [mathematical ideas] are not isolated events; they are all connected. That’s what math is, that’s what’s fun about math. They [mathematical ideas] are built on top of each other” (Int1). In this statement, Heather indicated that she believes mathematical ideas are not isolated, but instead connect in that ideas build on other ideas. At a later point during the interview while describing her purpose for using warm-ups in the classroom to connect to previously taught mathematical ideas, Heather stated, “I sound like a broken record, it’s [mathematics] building on top of itself; it’s [mathematics] not an isolated event” (Int1). These statements about mathematics in general were given in the context of describing teaching, which suggests that a view of mathematics as connections is related to her teaching. Her view of mathematics as connections is also related to how she engages in mathematics.

Connections for Heather are an acknowledgement that two ideas are related, though the details of how two ideas are related may not be part of a particular connection for Heather. For example, as part of describing how she introduced the Pythagorean theorem while student teaching, she stated that the Pythagorean theorem was connected to the distance formula and could be derived from the distance formula. When asked for details about the connection and derivation, she drew a graph (shown in Figure 5-2) with points and began to describe what she had done during student teaching.

I had a graph, and I had two different points. … I called that one $a$ and this one $b$. [pause] Let me think. [pause] What did I do? [pause] Now I can’t remember what I did. But it had something to do with I had points up on the board in a graph and I wanted them to find the distance. (Int1)

Heather stated that the Pythagorean theorem and the distance formula were connected, but could not remember the details of how she connected the distance formula to the Pythagorean theorem.
during her student teaching experience. Heather’s connection between the distance formula and the Pythagorean theorem was one of existence at the surface level.

\[ \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]

*Figure 5-2.* Diagram Heather drew to illustrate a connection between the Pythagorean theorem and the distance formula.

Not only did Heather describe surface level connections in her classroom mathematics via descriptions of student teaching but she also described difficulty seeing connections in her personal mathematics. Heather described herself as struggling with proof because she did not have what she perceived to be a conceptual understanding of the topic. “Sometimes I can’t necessarily see it [the logic of a proof], and I won’t lie I tend to memorize. I don’t really see how this goes to this because maybe I don’t have a full conceptual understanding about the topic or something” (Int1). For Heather, a conceptual understanding seemed to be about seeing connections—“see how this goes to this”—and not being able to see connections led to memorizing so that she could state justifications.

When Heather could not solve a problem during interviews, she would search her memory for a possible fact that she could connect to part of the problem. She would then try to apply the fact to the problem. Heather’s attempts to solve the **Tri-square Rugs** task illustrate how hunts for connections characterized how she engaged in mathematics and how a connection
between two mathematical ideas could be at the surface level. Heather’s initial impression when given the Tri-square Rugs task was to connect the visual diagram provided in the task (see Figure 5-3) to the Pythagorean theorem. “So when I first saw this [points to diagram in task] my whole thinking was just, this whole thing, Pythagorean theorem type thing, but this isn’t a right triangle” (Int1). Heather identified the Pythagorean theorem as a fact that might be key to solving the problem because the diagram in the task reminded her of a representation of the Pythagorean theorem. She rejected the usefulness of the Pythagorean theorem when she tried to connect the details of the Pythagorean theorem to the diagram. In this case, the representation of a tri-square rug in the task did not involve a right triangle and the Pythagorean theorem applies to right triangles.

![Figure 5-3. A representation from the Tri-square Rugs Task that Heather connects to the Pythagorean theorem.](image)

After deciding that the Pythagorean theorem would not be useful for solving the task, she connected the triangle inequality theorem to the task because the triangle inequality theorem applies to all triangles and the task does not specify a kind of triangle in the rug. “The triangle inequality states, based on my understanding of it, that no matter what kind of triangle … two side lengths added together are always going to be greater than the third. I’m still trying to think about the area it creates then” (Int1). On the surface, the triangle inequality was a fact that could
apply to the task at the surface level, meaning applicable to more than just right triangles. As she tried to connect the details of the triangle inequality to a task involving area, she rejected the triangle inequality as useful for solving the task.

Heather’s approach to solving the Tri-square Rugs task was to look for mathematical facts that might connect to parts of the task. Her engagement in the action of connecting complements her conception that mathematics has connections. Connections also play a role in how Heather intended to teach mathematics.

**Guiding students to mathematical content.** Heather planned to use a combination of discovery and lecture to guide her students to the desired mathematical content. She frequently mentioned wanting to use what she called discovery activities in her classroom. While planning how she would use the set of Pythagorean theorem tasks given in the second interview, she described discovery activities as “a little bit more introductory” with a purpose “to find out like properties or the general formula of the Pythagorean theorem” (Int2).

Discovery activities could apply to a variety of tasks involving opportunities to engage in different processes. She identified the [Examples and Non-examples](#) task—a task with opportunities to *define*—as a discovery activity. She identified the [Pythagorean Rugs](#) task—a task with opportunities to *justify*—as a discovery activity. She identified the [Generalizing via Dot Paper](#) task—a task with opportunities to *represent* and *generalize*—as a discovery activity.

Heather’s application of the term *discovery activity* to different mathematical processes implies that Heather’s conceptions of the processes may be blurred. At the end of the Quadratic Task Interview, Heather described a good generalizing task as one “that's walking you through it but asking you to identify your own properties and come up with explanations or reasons or maybe lead to like a full definition by the end of it” (Int2).
Heather applied the notion of discovery to multiple tasks involving different processes, suggesting a blurring of processes in Heather’s conceptions of mathematics. Connections, which were at the core of Heather’s conceptions of mathematics, were also a key characteristic of discovery activities for Heather. In the first interview, she described a plan to use a discovery activity to introduce the Pythagorean theorem. The discovery activity involved “having students make a connection” (Int1). Heather planned to help students make connections during discovery activities by exposing them to multiple representations and emphasizing important concepts or representations, which would essentially funnel students to her mathematical connections.

One way she planned to help students see connections was through exposure to multiple representations. During the second interview, she identified the Tri-square Rugs task, the Pythagorean Puzzle task, the Pythagorean Rugs Proof task, and the Square-in-square Proof task as discovery activities that were also “proof tasks.” She thought that these tasks were good tasks because “seeing it [representations of the Pythagorean theorem] in that different form will help retention rates and will help students really get what they're talking about” (Int2). She elaborated on why different representations of the Pythagorean theorem help students retain information and “see it.” “Even if you don't necessarily remember it right off the bat, I see something like this [points to diagram in Square-in-square Proof task] and I know that that's a Pythagorean theorem proof kind of thing” (Int2). Heather saw a connection to the Pythagorean theorem in a diagram that was included in multiple Pythagorean theorem tasks. She thought that helping students see similar connections to the Pythagorean theorem by using different representations would help students “really get it.”

To ensure that discovery activities work in the classroom, Heather believed that the teacher needed to emphasize concepts or representations so that students could complete the
activities. During the Pythagorean task interview, Heather defined a set of relatively prime Pythagorean triples for the Odd Triples task (Task J) using the concept of greatest common factor. She indicated it would not be a good task to use in the classroom if students had not previously learned about greatest common factor.

If you haven't talked about that [greatest common factor] in any time leading up to this [Task J] I don't think this would be a good task. If you have, if you stressed that in some way shape or form prior to this lesson I think that might be a little more helpful because I think students that might be able to do something. At least one of them might be like oh okay what about this fact? Does that work for all of them? (Int2)

In the above quote, Heather described how students might define by searching for a connection to a previously learned fact—the greatest common factor—that had been stressed prior to the activity and deciding if that fact applied to all examples in the task. She implied that emphasizing facts that will be connected to tasks was one teaching method needed for effective implementation of discovery activities.

Discovery activities to Heather seemed to involve Heather leading students through actions on products to discover what she wanted them to know. In some ways, the mathematics in the classroom belongs to Heather with the students performing her intended actions. In her description of introducing the Pythagorean theorem using the distance formula, a lesson she taught while student teaching, she focused on what she did as the teacher, frequently using “I” in her description.

So I taught it actually with this representation [points to diagram in Tri-square Rugs task]. They all essentially knew what that was for the most part. … First I showed how it is derived from the distance formula. … And then I showed how squares come off of the sides and that’s why it’s squared in the formula. (Int1)

As captured in this passage, Heathers’ first-person descriptions of teaching were primarily teacher-focused with Heather determining the mathematical connections to which she wanted to guide her students.
**Guidance as a lens for task selection.** Heather pedagogically categorized and then sequenced tasks based on the amount of guidance and discovery she thought students would need to be given to complete the task. Specific tasks Heather selected along with her categorization of the tasks are listed in Table 5-5 for the Pythagorean task interview and Table 5-6 for the quadratic task interview. The tables show that Heather’s task sequence followed the format of introductory tasks, mid-unit tasks, and extending tasks. This section includes a discussion of each of these task categories with connections to guidance.

Table 5-5

*Heather’s Plans for a Pythagorean Theorem Unit*

<table>
<thead>
<tr>
<th>Day</th>
<th>Task Category</th>
<th>Task Name</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><em>Introductory</em></td>
<td>Generalizing via Dot Paper</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>Pythagorean Puzzle</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>Pythagorean Rugs</td>
</tr>
<tr>
<td>3</td>
<td><em>Extending</em></td>
<td>Verifying Converse</td>
</tr>
</tbody>
</table>

Table 5-6

*Heather’s Plans for a Quadratic Unit*

<table>
<thead>
<tr>
<th>Day</th>
<th>Task Category</th>
<th>Task Name</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><em>Introductory</em></td>
<td>Exploring a with Defining</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>Exploring c with Intercepts</td>
</tr>
<tr>
<td>2</td>
<td><em>Practice</em></td>
<td>Factoring with Justifying</td>
</tr>
<tr>
<td>3</td>
<td><em>Practice</em></td>
<td>Factors and Intercept</td>
</tr>
<tr>
<td>4</td>
<td><em>Extending</em></td>
<td>Completing the Algebra Tile Square</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Defining Quartic Parabola</td>
</tr>
</tbody>
</table>
In the task interviews, Heather planned her units to begin with introductory tasks, which to her were tasks in which students used “the pattern to make some conjecture about the relationship” (Int2). These introductory tasks included “discovery activities to find out like properties or the general formula” (Int2). The tasks did not specifically mention the object to be discovered, such as the Pythagorean theorem. Introductory tasks preceded teacher telling, and so Heather preferred that introductory tasks include substantial guidance so that students might follow them easily and without much teacher input. “I think that they [introductory tasks] help students discover things on their own. I think they’re guided without the teacher having to do all, or providing all, that guidance” (Int2). For Heather, the better tasks included guidance in their structure so that the teacher did not have to provide all the guidance.

Heather seemed to contend, task features that helped students identify patterns were ideal for introductory tasks because they provided the needed scaffolding to guide students to the intended mathematics. Heather explained why the Generalizing via Dot Paper task was a good introductory task. “It [Generalizing via Dot Paper task] gives you instructions. It gives you questions to be looking for and then in that table I can visually see. Okay, well what's happening here? Is there any kind of pattern I can make?” (Int2). In the quote, she identified three task features that she viewed as useful in introductory tasks: clear instructions, visual organizers such as tables, and questioning that directs students to the pattern of interest. Heather used this collective set of features to identify good introductory tasks as part of her task work.

Introductory tasks also needed to include adequate amounts of guidance. Heather described how as a student she needed adequate guidance to solve a task.

I think that a lot of times if you put a discovery activity in front of the student, and I was this way too, I kind of just like if I wasn’t really provided much guidance I’d just kind of sit there and like I don't know. And I will just like write it off in that case. I’ll just be like I don't know. I don't get this. And I won't try very hard. (Int2)
If a task did not have enough guidance, Heather herself did not know how to do the task and would not try hard to solve the task. She thought students would behave similarly to her; she did not try to solve tasks with inadequate amounts of guidance. Additionally, Heather thought adequate guidance was needed to “increase retention rate … because if there is not enough guidance I think students just, I think it will go in one ear and out the other” (Int2). In these two quotes, Heather identified adequate guidance as an important characteristic of introductory tasks.

Heather follows introductory tasks with mid-unit tasks. These mid-unit tasks were those tasks that required prior work with the unit topic. She described these tasks in Day 2 of her Pythagorean theorem unit. “Day 2 is more of a work off of Day 1. Like, now that we know what the Pythagorean theorem is, now that we know the general properties and the general formula of it” (Int2). In her Pythagorean theorem unit she identified the Square-in-Square Proof and Pythagorean Rugs tasks as mid-unit tasks because they required that the students had already encountered the Pythagorean theorem. Similarly, she viewed the Generalizing Solutions task in her quadratic unit as a mid-unit task because students “need to know what a parabola looks like and that \( x^2 \) creates a parabola” (Int3). Mid-unit tasks required students to have worked with the unit topic prior to attempting the task.

Heather ended her unit with tasks that extended the topic of the unit. She identified the Verifying Converse task as an extending task in her Pythagorean theorem unit. “I think it takes everything they know and still is extending their knowledge. For example, … writing the converse, taking the converse, constructing things using digital software, and then moving things around and seeing what happens” (Int2). In the above quote, Heather suggested possible mathematical activities in which students might engage while working on the Verifying Converse task. The description of students “moving things around and seeing what happens”
implied facilitating mathematics and possible process opportunities such as generalizing. However, the guided questioning at the end of task (see Figure 5-4) effectively directs students to engage in the actions of interpreting a dynamic representation of a triangle to answer the questions.

| 1. When the value of $(AB)^2 - [(AC)^2 + (BC)^2]$ is positive, is angle C acute, right, or obtuse? |
| 2. When the value of $(AB)^2 - [(AC)^2 + (BC)^2]$ is zero, is angle C acute, right, or obtuse? |
| 3. When the value of $(AB)^2 - [(AC)^2 + (BC)^2]$ is negative, is angle C acute, right, or obtuse? |

*Figure 5-4. Guided questioning at the end of the Verifying Converse task.*

Heather used task features to identify pedagogical categorizations for tasks. Considering Heather’s tendency toward connections, it was not surprising that she used task features to guide her task work. Task features enabled her to make connections to pedagogical notions about teaching mathematics. Once she has connected tasks to a particular pedagogical use, she could then sequence task categories to follow a general pattern of introductory tasks, mid-unit tasks, and extending tasks.

**Summary of Heather’s guided OMP.** Heather’s OMP can be characterized using a theme of guidance. Connections, which for Heather seemed to be links at the surface of ideas, were central to her conception of mathematics and the way that she engaged in mathematics by hunting for a mathematical fact that might connect to the task at hand. Heather planned to guide students to connections by using discovery activities with adequate amounts of guidance and support for students.

**Discussion OMP Profile: Sarah**

Sarah’s OMP reflected her preference for what she called discussion–based mathematics teaching that promoted the development of conceptual understanding. Conceptual understanding,
to Sarah, included anything that was not procedural. She sought to develop this kind of conceptual understanding through discussions, and in so doing inadvertently planned opportunities for students to engage in processes.

**Discussion as pedagogical preference.** Sarah expressed a preference for teaching mathematics through discussions, which might be done in small or whole groups and were usually directed by Sarah. In describing her student teaching experience, she stated, “I enjoyed having the students work in pairs and walking around and having discussions with 3 kids at a time and then us all talking about it” (Int1). Sarah enjoyed the lessons during student teaching in which she and students had opportunities to discuss mathematics. In contrast, Sarah stated, “The lessons I didn’t like are the ones where I had to lecture a lot.” Sarah did not like lessons in student teaching that involved lecture. She preferred leading discussions to lecturing in her student teaching.

Sarah’s preference for discussions seemed to come from her perception that discussions developed what she called conceptual understanding. When asked what she liked about discussion, Sarah stated, “I think that the discussion helps with the conceptual understanding because it takes it away from the numbers and more about the idea” (Int1). By conceptual understanding, Sarah seemed to refer to any mathematical thinking that was not procedural. When asked what she meant by conceptual understanding, she stated, “I know that with the new standards, they are really pushing for conceptual understanding as opposed to procedural understanding. So thinking about the topic as opposed to ‘you do this, and then you do this, and then you get an answer’” (Int1). She approached defining conceptual understanding by comparing it to procedural understanding as if the two were antithetical. She preferred discussions because they helped develop thinking that was not procedural.
Sarah also perceived discussions as pedagogically beneficial because they provided a lens into student thinking. Sarah mentioned that discussions made it “easier for me … to see what misconceptions they have and what isn’t clear to them” (Int1). Sarah viewed discussions as a pedagogical tool to develop conceptual understanding and to assess student thinking.

**Discussions foster process opportunities.** Sarah inadvertently planned to provide students with opportunities to engage in processes in the service of generating good discussions. She indicated that tasks that elicit explaining and multiple representations provided the best fodder for mathematical discussions. She mentioned that in her student teaching would she modify tasks to require students to explain because “a lot of times when I do give students questions that ask them to explain it usually ends in a class discussion where we can hear everyone’s ideas. One student might have something a little different than another” (Int2). Sarah tried to generate discussions by planning to ask students to explain. These explanations might involve opportunities to engage in processes such as *justifying*.

Sarah also suggested that multiple representations could be used to generate discussions. She described an experience during student teaching in which she used multiple algebra tile representations of polynomials generated by students to generate a discussion. “Each group was coming up with different ways to show it, which was awesome to have two different representations of the same thing. … So that sparked a discussion. If there were two groups that had different ideas I had them both come up to the board” (Int1). Sarah generated a discussion by *linking* and *interpreting* multiple representations that students created using algebra tiles. The students engaged in *representing* prior to the discussion. The process of representing was in service of generating a discussion.
In comparison to tasks that provided opportunities to justify or represent, Sarah was wary of tasks that provide opportunities to define or generalize because she wanted to make sure that students knew the correct definition or generalization.

For me if I was just given a task like that [one involving examples and non-examples] and was never told that [my definition] was correct, I would be nervous that maybe I did something wrong or maybe I’m not seeing what the teacher wants me to see. (Int1)

She indicated that while such tasks might be useful for generating discussion, a teacher needed to be careful to ensure that students are given authoritative clarification at the end of the discussion.

Her apparent preference for process opportunities of justifying and representing might also be related to her personal tendency to engage in these processes and observed difficulties engaging in generalizing and defining. During the interviews, Sarah represented and justified without prompting. For example, she immediately began representing different tri-square rugs while trying to solve the Tri-square Rugs task. In addition to representing spontaneously, Sarah justified claims spontaneously. After she had arranged the shapes to solve the Pythagorean Puzzle task as shown in Figure 5-5, she immediately used the representation she had created to justify the Pythagorean theorem.

I know that the sum of the areas of the two smaller squares will equal the area of the larger square… Because I know that the two frames are equal and that the eight triangles are congruent, and therefore have the same lengths and areas. And if I have four of them in each, then the portion of the frame that is left over has to be equal between these two squares and this one over here. (Int1)

Sarah spontaneously and successfully engaged in the processes of representing and justifying without prompting.
Figure 5-5. Sarah’s arrangement of triangles and squares created while working on the Pythagorean Puzzle task.

In contrast to representing and justifying, Sarah did not spontaneously define or generalize. Rather than define, Sarah tried to remember and state definitions. For example, while trying to solve the Tri-square Rugs task, Sarah had identified that rugs that formed a right triangle led to a fair game when she indicated that she did not know how to continue to solve the task. She stated, “I was thinking acute and obtuse triangles or angles. But I can’t remember enough about them at the moment to try and figure out how to make it work” (Int1). When asked, she identified “properties, definitions” as the information she was trying to remember. Rather than trying to figure out the definitions and properties of acute and obtuse triangles, Sarah tried to remember and state definitions and properties.
Sarah also did not generalize spontaneously. Later in her work to solve the Tri-square Rug task, after having represented various tri-square rugs involving different triangles (see Figure 5-5), she indicated that she was stuck and could not find a solution. “I’m trying to think if this angle being greater than or less than 90 affects the squares. But I can’t figure out how. … I’m stuck” (Int1). Rather than continuing to analyze the representations to form a generalization, Sarah stopped working on the task and effectively avoided an opportunity to generalize.

Sarah demonstrated a tendency to engage in representing and justifying and did so with success, but she did not tend to define or generalize without prompting or sometimes even with prompting. Perhaps differences in how she engaged in processes were related to the ways she generated discussions in her student teaching. She generated discussions using multiple representations and eliciting explanations, which may be related to representing and justifying.

Interestingly, Sarah articulated conceptions of all four processes that did not seem to be blurred with each other. When asked, Sarah described a good defining task as a task in which you “give them something to start with, but maybe not the completed definition. And talk about why this has to be in there and why this doesn’t have to be in there” (Int3). Sarah’s description to give students an incomplete definition to be analyzed and revised implied that defining involves the creation of a definition.

While her descriptions of other processes do not contain this same generative property that was implied in her description of defining, her descriptions do not seem to contain elements that blur with other processes. With respect to generalizing, she stated, “when I think of generalizing, I think of finding patterns” (Int3). While this view of generalizing might be narrow or incomplete, it is consistent with descriptions of generalizing among the mathematics education community. A good representing task, she stated, involving “giving it [a
representation] in multiple forms and giving students the opportunity to see how the different forms relate” (Int3). While focused on links among multiple representations rather than the generative act of representing, Sarah’s description of representing did not include elements of other processes.

Sarah’s description of justifying was less clear than her descriptions of other processes and involved identifying characteristics of a justification such as “shouldn’t give a specific example” and should be “in terms of variables instead of numbers … and with more words” (Int3). These characteristics could also be present in other processes, so Sarah’s conception of justifying could involve aspects of other processes.

**Desire for discussion dominated task work.** Sarah foregrounded the development of conceptual understanding in her sequencing of tasks that she thought would optimize class discussions. Sarah began her task work by sorting tasks pedagogically with the intended sequence beginning with an investigation task, following with practice tasks, and ending the unit with tasks that summarize and extend the topic. Table 5-7 and Table 5-8 list the tasks Sarah selected in the Pythagorean task interview and the quadratic task interview respectively. The ensuing paragraphs contain discussions of the different types of tasks Sarah used and how they connect to discussion and conceptual understanding.
Table 5-7

Sarah’s Plans for a Pythagorean Theorem Unit

<table>
<thead>
<tr>
<th>Day</th>
<th>Task Type</th>
<th>Task Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Investigation</td>
<td>Examples &amp; Non-examples</td>
</tr>
<tr>
<td>1</td>
<td>Practice</td>
<td>Identifying Triples</td>
</tr>
<tr>
<td>2</td>
<td>Practice</td>
<td>Justifying with Converse</td>
</tr>
<tr>
<td>2</td>
<td>Practice</td>
<td>Baseball</td>
</tr>
<tr>
<td>2</td>
<td>Digging Deeper</td>
<td>Pythagorean Rugs Proof</td>
</tr>
<tr>
<td>3</td>
<td>Digging Deeper</td>
<td>Verifying Converse</td>
</tr>
</tbody>
</table>

Table 5-8

Sarah’s Plans for a Quadratic Unit

<table>
<thead>
<tr>
<th>Day</th>
<th>Task Type</th>
<th>Task Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Investigation</td>
<td>Intercepts as Solutions</td>
</tr>
<tr>
<td>1</td>
<td>Investigation</td>
<td>Exploring $a$ with Defining</td>
</tr>
<tr>
<td>1</td>
<td>Investigation</td>
<td>Exploring $c$ with Intercepts</td>
</tr>
<tr>
<td>2</td>
<td>Investigation</td>
<td>Factors and Intercepts</td>
</tr>
<tr>
<td>2</td>
<td>Practice</td>
<td>Factoring with Justifying</td>
</tr>
<tr>
<td>3</td>
<td>Practice</td>
<td>Completing the Algebra Tile Square</td>
</tr>
<tr>
<td>4</td>
<td>Wrap-up</td>
<td>Defining Quartic Parabola</td>
</tr>
<tr>
<td>4</td>
<td>Wrap-up</td>
<td>Generalizing Solutions</td>
</tr>
</tbody>
</table>

For Sarah, investigation tasks were tasks that did not explicitly tell students what something was or how to solve the task. For example, she identified the Examples and Non-Examples task as an investigation task because “it’s not coming out and saying exactly what you
are looking for but instead you have to look for a relationship that goes with this these sets of numbers but not these sets of numbers” (Int2). She sequenced investigation tasks first because she wanted students to understand ideas first and she saw investigation tasks as accomplishing this goal. She stated, “I would rather have the students gain the conceptual understanding first… I want them to first understand what they are doing and why they are doing it” (Int1). She wanted students to investigate ideas first and come up with relationship on their own, which were characteristics of investigation tasks, because she viewed that as more beneficial in the learning process.

In my mind I think it works better and is more beneficial when students come up with mathematical relationships on their own. Because when they come up with it on their own it means they’ve really thought about it and they are putting all of the pieces together and they really understand it. (Int3)

Sarah sequenced the task type of investigation first because those tasks provided opportunities for students to develop mathematical relationships and conceptual understanding, which she viewed as beneficial to the learning process.

Sarah sequenced practice tasks after investigation tasks. While describing her views on conceptual understanding, Sarah stated, “I would rather have the students gain the conceptual understanding first, and then do the problems over and over again” (Int1). After students had developed what Sarah called a conceptual understanding through investigation tasks, Sarah planned to give students practice tasks. “Once they come up with the basic idea, I like having them practice and then assessing where they are, seeing if I need to tweak something as a class or with individuals” (Int2). In this quote, Sarah indicated that one purpose of practice tasks was for her to assess individual students and groups of students.

Following practice tasks, Sarah sequenced tasks that she viewed as summarizing ideas, such as the Generalizing Solutions task, at the end of a unit because it involves “putting things
together … [and] …seems like a wrap up of the unit” (Int3). She also sequenced tasks that she considered to be “digging deeper” tasks at the end of a unit. When describing why she liked the idea of sequencing the Verifying Converse task last in her Pythagorean theorem unit, Sarah stated, “I know that before I can dig deeper, I need to make sure they understand the basic ideas. … If they don’t understand the basics there is no way they can dig deeper” (Int2). Sarah pedagogically sequenced tasks so that investigation tasks that might generate discussions came first, followed by practice tasks, and then ending with summarizing or extending tasks that involved “digging deeper.”

**Summary of Sarah’s discussion OMP.** Sarah privileged discussions in her mathematics teaching because she viewed discussions as opportunities to talk about ideas and develop what Sarah called conceptual understanding. She planned to generate discussions through the use of representing and possibly justifying via requests for explanations. These methods of generating discussions seemed to match the processes that Sarah was the most willing to engage—the processes of representing and justifying. Sarah’s task work was pedagogical in nature in that she parsed all tasks into pedagogical categories that she then sequenced.

**Destination OMP Profile: Brad**

Learning goals were at the core of Brad’s OMP. Brad focused on mathematical content destinations to which he wanted to lead his students along what he considered a challenging mathematical path that occasionally included process opportunities. Pedagogically, Brad remained focused on learning goals as he planned instruction. While these learning goals were primarily about content, Brad’s plan for helping students meet his objectives provided auxiliary opportunities for mathematical processes. Brad’s focus on his students’ mathematical
destinations influenced his use of learning goals to guide his task work in creating sequences of tasks with a logical flow from one learning goal to the next.

**Learning goals as content-focused mathematical destinations.** Throughout the interviews, Brad consistently returned to learning goals or objectives in his discussion of teaching.

It all comes down to what your curriculum is and what your objectives are. I feel like you want to have meaningful objectives and you obviously want your students to meet or exceed them. And so you just want to choose tasks that can help you achieve that. (Int3)

Learning goals were central to Brad’s conceptions of teaching, and as he indicated in the quote, they drove his choice of tasks so that students could meet or exceed his learning goals.

He often expressed learning goals in terms of the topical content he wanted his students to learn for the day. For example, he identified one day of his Pythagorean theorem unit as a “[Pythagorean] triples day” (Int2), and he indicated that he wanted students to learn “completing the square” (Int3) during one day of his planned quadratic unit. These content destinations may or may not include process opportunities. For instance, process opportunities might be involved in one lesson involving the Pythagorean theorem in which Brad wanted students to “discover the formula [Pythagorean theorem] on their own” (Int1). In his description of the lesson he did not articulate how he wanted students to discover the Pythagorean theorem, but it is possible that he could have planned to provide opportunities for student to generalize to develop the Pythagorean theorem. Yet other lessons may not include process opportunities. He described his plan for his completing the square lesson as “more of an example” (Int3) and gave no indication that he might provide opportunities for processes during the lesson. Brad did not view the inclusion of processes in a lesson or learning goal as critical; he was focused on the content destination.
While Brad’s learning goals may or may not include processes, he thought that all learning goals and tasks needed to include identifiable content. He saw little value in tasks that did not have an obvious content related learning goal. For example, Brad strongly disliked the Novel Set of Quadratics task because he could not easily identify a content learning goal for the task. “It’s not that it’s unfamiliar; it’s undefined. I just don’t know what they are supposed to get out of it” (Int3). Because Brad could not identify a content related purpose to the task, he did not find it valuable despite its opportunities for students to engage in defining and to “look for and make use of structure” (CCSSI, 2010, p.8). For Brad, a learning goal needed to include relevant secondary mathematics content.

Brad’s conception of what is mathematical is one possible explanation for the content focus of Brad’s learning goals. His view of what is mathematical did not seem to include processes. For example, Brad did not view the Pythagorean Puzzle task (see Figure 4-2) as very mathematical despite opportunities to represent, generalize, and justify present in the task. Of the task, he stated, “I don't personally think there is a lot of mathematical thinking going on” (Int1). Brad considered it not to be very mathematical because “they don't have to write out formulas and stuff like that” (Int1). The fact that Brad did not consider a task with opportunities to engage in processes as mathematical suggests that Brad’s conception of mathematics did not necessarily include processes. It is not surprising then that his learning goals would focus on content rather than processes.

His learning goals might also be content focused because of his willingness to let the curriculum determine his learning goals. Brad seemed to place curricular authority on the curriculum selected by the district or school and indicated that his learning goals “really depend on what the curriculum is looking for” (Int1). For Brad, what the curriculum was looking for
seemed to be about content and products, such as a particular factoring method (a *generalization* rather than *generalizing* consideration) or a particular proof of the Pythagorean theorem (a *justification* rather than *justifying* consideration). Brad intended to let the curriculum, which for him seemed to be about included classroom content, determine his mathematical destinations. Consequently, his mathematical destinations—or learning goals—were more about content than process for Brad.

**Learning goals fostering process opportunities.** Brad’s preferred method of teaching involved what he called activity based learning. After solving the *Tri-square Rugs* task, Brad described why he thought it would be a good task to use in the classroom. He stated, “I think that activity based learning is really good because it allows them to discover the concept on their own. And I think that it lasts longer, they have a deeper understanding of it” (Int1). Brad valued an activity based pedagogical approach that provided opportunities for students to “discover” because he thought it provided depth and longevity to the learning process.

By approaching teaching through activity based learning and discovery, with a focus on mathematical destinations, Brad provided process opportunities for students. For example, in his description of the third day of his Pythagorean theorem unit, Brad stated a learning goal that involved opportunities to engage in processes. “The focus is more on the Pythagorean triples and they get to define [points to Examples and Non-examples Task] … and now they can create their own triples and draw some conclusions based on those [points to Odd Pythagorean Triples Task]” (Int2). His implied learning goals for this lesson involved the content of Pythagorean triples and the processes of at least *defining* and possibly *representing* (creating), and *generalizing* or *justifying* (draw some conclusions). Brad’s challenging learning goals surrounding Pythagorean triples fostered mathematical processes in his classroom.
Brad’s ability and tendency to engage in the processes of generalizing and justifying suggest that he could support students in these processes. For example, Brad quickly generalized the conditions of the Pythagorean theorem to solve the Tri-square Rugs task and he offered a justification of the Pythagorean theorem using the Pythagorean Puzzle task without prompting. Although Brad might focus on getting his students to a content destination, his ability to engage in generalizing and justifying indicated that he could support a journey that involved at least some of the processes.

Yet, sometimes Brad’s mathematical work resulted in the creation of imprecise products. For example, Brad imprecisely represented squares on the hypotenuses of right triangles (see Figure 5-6) when asked to solve the Generalizing via Dot Paper task. When asked how he drew the squares on the sides of the specified triangles using dot paper, he stated, “I’m just kind of eye-ball ing it” (Int2). His work in the creation of imprecise representations resulted in representations that were not useful for generalizing to develop the Pythagorean theorem later in the task. Brad indicated that his imprecise representations might make the rest of the task more difficult. “It’s also not as easy to see what this area is [points to squares on hypotenuses, see Figure 5-6] so that makes it much more difficult” (Int2). When asked how he would find the area of the square on the hypotenuse, Brad indicated that he would use the Pythagorean theorem. “I would just use the Pythagorean theorem and just find the hypotenuse and then square it” (Int2). Brad’s imprecise representations of squares on the hypotenuses precluded the opportunity of generalizing to develop the Pythagorean theorem so that Brad had to apply the Pythagorean theorem to find the desired areas.
Learning goals as guides in task work. Brad’s approach to using tasks focused on creating a sequence of tasks with a flow from one learning goal to the next. Brad’s focus on learning goals was not only in the sense of attention to standards but also attention to what students might learn by engaging in a task. His initial approach when reading a new task was to determine the purpose of the tasks for the students and then consider how it might fit with the tasks that he had already read. In the end, he created sequences of tasks that flowed from one learning goal to the next within a single mathematical topic. The tasks Brad selected during the task interviews are listed in Table 5-9. Brad did not categorize tasks in ways that other participants did and so this information is not included in the table.
Table 5-9

*Brad’s Plans Created during the Task Interviews*

<table>
<thead>
<tr>
<th>Pythagorean Theorem Unit</th>
<th>Quadratic Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Day</strong></td>
<td><strong>Task Name</strong></td>
</tr>
<tr>
<td>1</td>
<td>Pythagorean Puzzle</td>
</tr>
<tr>
<td></td>
<td>Pythagorean Rugs</td>
</tr>
<tr>
<td>2</td>
<td>Justifying Using Converse</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Examples and Non-examples</td>
</tr>
<tr>
<td></td>
<td>Odd Triples</td>
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<td></td>
<td></td>
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</tbody>
</table>

The task sequence Brad created in the quadratic task interview (Int3) illustrates the flow of his learning goals. He planned to begin his unit with the *Vertex Application* task because it “kind of gets in their head that when you have a quadratic term … it kind of makes a u-shape.” He used the Vertex Application task to accomplish his learning goal to have students understand the overall shape of a quadratic graph. Brad then suggested that the *Factors and Intercepts* task would follow nicely because “they will see that these also make a u-shape” and can then draw conclusions about zeros and factors: “they’ll discover that the zeros are the same thing as the q and the p [in \((x – q)(x – p)\)].” Brad stated that drawing conclusions about factors and zeros set the students up for “look[ing] at different ways to represent a quadratic and hopefully find[ing] the solutions” in the *Factoring with Justifying* task. After students had looked at several quadratics and their graphs, Brad planned to use the *Exploring a with Defining* task because it “flat out starts with [the definition of a] quadratic function and then it gives you the standard form.”
Brad’s plans flow from an exploration of parameter $a$ to a parameter exploration of $c$ in the Exploring $c$ with Intercepts task, in which students also consider “if the minimum occurs above the $x$-axis … and then that kind of introduces them to what happens with one solution, two solutions, zero solutions.” He concluded his unit with the Generalizing Solutions task (Task W) because “I feel like Task W is a great follow-up to that … I feel like I just said everything that sums up Task W.” Brad tried to make sure that what students learned in one task logically connected to what they would learn in the next task, effectively creating a “flowing” sequence of tasks.

In fact, task flow was so important to Brad in sequencing that he chose not to select a task that he liked but interrupted the flow of the other tasks.

D [Generalizing via Dot Paper Task] might fit in there with the Day 1 and Day 2 stuff … but I feel like the flow of what I have right now goes together pretty well … I do like it, I just feel like they are going to draw the same conclusions from these ones [points to the Pythagorean Puzzle task and the Pythagorean Rugs task] as this … and these ones just happen to fit better together. (Int2)

Brad viewed the learning outcomes from the Generalizing via Dot Paper task as similar to the learning outcomes in the other tasks in his sequence. Consequently, he opted not to include the Generalizing via Dot Paper task because it might disrupt the flow he had already created. The exclusion of tasks that interrupted the flow of a set of tasks was further evidence that Brad privileged the use of logically connected learning goals in his task work.

Summary of Brad’s destination OMP. Brad kept his eye on his students’ mathematical destinations and oriented his teaching toward creating opportunities for students to reach those destinations. He did this by creating sequences of tasks with learning goals that flowed from one task to another. His learning goals were focused on mathematical content and usually did not explicitly include processes, but his desire to teach using activity based learning resulted in
auxiliary opportunities for students to engage in processes. For Brad, it was the destination of mathematical content that was the focus, not necessarily the serendipitous journey through mathematical processes.

**Conclusion**

This chapter included descriptions of OMP profiles. Each profile is presented around a theme that captured participants’ mathematical activity as well as pedagogical approaches and how those aspects influenced task work. The evidence presented in the profiles parallel answers to my research questions—what are PMTs’ OMPs and how do OMPs influence task work—for individual participants. The profiles are not intended as answers to my research questions, but are used here as a narrative tool for sharing and communicating evidence related to my research question for individual participants. I will organize the profiles into a schematic of OMP types as partial answer to my research questions.

The five OMP profiles were Erin’s Distance OMP, Jenny’s Comfort OMP, Heather’s Guided OMP, Sarah’s Discussion OMP, and Brad’s Destination OMP. Erin distanced herself from mathematical processes by avoiding opportunities to engage in processes, distanced herself from student thinking by inviting thinking dominated by actions, and distanced her students from mathematical processes by selecting tasks with minimal opportunities to engage in processes. Jenny’s OMP profile was characterized by comfort in that she tried to avoid negative responses from students, which sometimes involved missed opportunities to engage in processes. Heather’s mathematical activity was dominated by connections and she sought to guide her students to mathematical content by helping them see connections, which resulted in students engaging primarily in the action of connecting. Sarah valued mathematical discussions in the classroom, which she created by providing students opportunities to represent and justify, thus process
opportunities presented themselves in service of mathematical discussions. Brad, with an eye toward his students’ mathematical destinations, created sequences of tasks based on learning goals with occasional auxiliary opportunities to engage in processes.

The participants’ OMPs seemed to differ mathematically in terms of whether participants engaged in processes as well as actions and of whether and how they provide opportunities for students to engage in processes. In the next chapter, I use these differences to organize participants’ OMPs in answer to my first research question, what are PMTs’ OMPs, and then to capture how OMPs influence the task work of selecting, modifying, and sequencing mathematical tasks in answer to my second research question.
Chapter 6

Findings and Discussion

The purpose of this chapter is to assimilate the set of orientations toward mathematical processes (OMPs) profiles presented in Chapter 5 and present answers to my two research questions: what are prospective mathematics teachers’ (PMTs’) OMPs and how do PMTs’ OMPs influence how PMTs select, modify, and sequence mathematical tasks? An organization of profiles into types based on pedagogical and mathematical differences structures my findings to the first research question and the construct of task template as a means of simplifying the complex nature of task work forms the basis for my findings to the second research question. The notion of discovery figured prominently in participants’ OMPs; its explanatory role in PMTs’ OMPs will be discussed. Connections to relevant literature are presented within the discussion of the research findings.

In this chapter, I draw connections among OMPs and make claims about common task work. Consequently, I use examples from individual OMPs, but I have omitted detailed evidence regarding individual OMPs from this chapter. More detailed evidence regarding individual OMPs can be found in Chapter 5: OMP Profiles.

Findings and Discussions Related to PMTs’ OMPs

Prior to answering the question, what are PMTs’ OMPs, a restatement of the definition of OMP may be helpful. A mathematics teacher’s orientation toward mathematical processes (OMP) in school mathematics is a set of conceptions about mathematics, its teaching and learning related to mathematical processes, and engagement in mathematical processes that influences how a teacher provides opportunities for students to engage in mathematical
processes. The four mathematical processes targeted in this study are defining, generalizing, justifying, and representing, descriptions of which can be found in Table 4-2.

My definition of OMP has four components that I used to design instruments and analyze data: conceptions about mathematics teaching related to mathematical processes, conceptions about mathematics learning related to mathematical processes, conceptions of mathematics related to mathematical processes, and engagement in mathematical processes. During the analysis process and construction of OMP profiles, I found it useful to treat the four OMP components as two: a pedagogical group that consisted of conceptions of teaching and learning mathematics and a mathematical group that consisted of conceptions of mathematics and engagement in mathematics. Pedagogical differences along a presenter–facilitator dichotomy and mathematical differences along an action–process dichotomy proved to be especially useful in categorizing PMTs’ OMPs.

Using these observed pedagogical and mathematical differences among PMTs’ OMPs, I developed an organizational scheme to answer my first research question. The PMTs’ OMPs can be characterized as predominantly facilitator-driven and action-based. Major influences are PMTs’ desires for students to discover mathematics on their own and PMTs’ difficulties engaging in and distinguishing among mathematical processes.

**Mathematical differences among profiles.** Participants’ OMPs varied mathematically along an action–process dichotomy depending on the degree to which processes in addition to actions on products were part of a PMT’s personal mathematics and classroom mathematics. Personal mathematics (Zbiek et al., 2014) captures a PMT’s mathematical ability and tendency to engage in processes and actions. Classroom mathematics (Zbiek et al., 2014) captures the mathematical opportunities afforded to students in the classroom. When deciding whether a
PMT’s mathematics was action-based or process-based, I considered both the frequency and the intentionality of processes in his/her personal and classroom mathematics. Three of the participants tended to have action–based OMPs while the other two participants had more process–based OMPs.

**Action dominated profiles.** Actions dominated Erin’s, Jenny’s, and Heather’s personal and classroom mathematics. Erin ([Distance OMP])\(^6\) considered herself to be “very much a numbers and formulas person” (Int1) and avoided engaging in processes when asked to solve problems such as the Tri-square Rugs task. Her mathematical identity as a numbers and formulas person is indicative of her preference for applying products in her personal mathematics, “give me a formula and I will sit there and do hours and hours of putting it into equations” (Int3) and is similar to her plans to have students participate in tasks by sharing steps in applications of procedures. Actions, particularly the action of applying, dominated her personal mathematics as well as the opportunities she planned for her students in the classroom.

Jenny ([Comfort OMP]) seemed to be most comfortable with actions on products in her personal mathematics and found her collegiate mathematics classes, which may have had process opportunities, to be “very difficult” (Int1). Additionally, she seemed to struggle when asked to engage in processes during interviews, as observed in her difficulty with generalizing in the Tri-square Rugs task and representing in the Pythagorean Puzzle task. Her mathematical difficulties seemed to be reflected in her desire to create a mathematically comfortable classroom in which her students will not get frustrated and will “know where to start” (Int2).

The action of connecting seemed to dominate Heather’s ([Guided OMP]) personal and classroom mathematics. Her approach to difficult tasks, such as the Tri-square Rugs task was to

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\(^6\) I use hyperlinks in this chapter to link to OMP profiles in Chapter 5 and text for specific tasks used in task interviews. The tasks are located in Appendix C and Appendix D.
look for theorems, definitions, and representations that connect to aspects of the task. Heather also wanted her students to see connections and planned to guide her students to her relevant connections, essentially providing students with opportunities to make connections but not necessarily to engage in processes.

Erin, Jenny, and Heather had mathematically similar OMPs in that they tended to favor actions on products to the exclusion of mathematical processes. In contrast, Brad and Sarah demonstrated abilities to engage in at least some of the processes and provide anticipated opportunities, albeit unintentionally, for students to engage in mathematical processes.

**Possible processes profiles.** While both Brad and Sarah demonstrated an ability to engage in processes and possible opportunities for student engagement in processes, they demonstrated these qualities with respect to only some but not all four processes. Brad (Destination OMP) favored justifying and generalizing, but not necessarily representing and defining. In comparison, Sarah (Discussion OMP) favored justifying and representing, but not necessarily generalizing and defining. Both Brad and Sarah expressed a desire for students to justify or explain in their classrooms and both provided reasonable justifications without prompting when solving tasks during the interviews. For Sarah, these justifications were important in the development of good class discussions and evidence of student understanding. For Brad, these justifications were part of developing a community of learners in which they explain the mathematics to each other so that all students can eventually reach his desired mathematical destination. While Brad and Sarah indicated that they would provide opportunities for students to engage in processes, these opportunities usually came as unintentional byproducts of a desire to help students attain desired learning goals and to create mathematical discussions that promoted conceptual understanding, respectively.
The five participating PMTs had OMPs that differed mathematically depending on whether processes in addition to actions on products were present in their personal and intended classroom mathematics. The participants’ OMPs also differed pedagogically. Despite the differences, types of OMP profiles are identifiable.

**Pedagogical differences among profiles.** Participants’ OMPs differed depending on their view of the primary role of teacher as presenter or facilitator. With the exception of Erin, the participants tended to view their role more as a facilitator than as a presenter and they expressed a desire for students to develop, or what some participants deemed discover, mathematics on their own because they said it increased student ownership of the content and students were more likely to remember the mathematics they created. This notion of “discovery” seemed pervasive throughout the group of participants and was likely situated in their shared teacher education experiences.

**Presenter profile.** Erin ([Distance Profile](#)) primarily viewed her role almost exclusively as presenter. Even for her technology student teaching lesson, a lesson in which there was an expectation in her teacher education program to use technology to facilitate mathematical learning, Erin reported using technology to create individual presentations of mathematics that her students could complete at their own pace. Her desire to present mathematics to her students might be related to her strong dislike of being “made to explore” (Int1) in her mathematics education courses: “You know being here and having classes where they’re very conceptual and like they just want you to, they want you to figure out pretty much everything. I don’t like that … I want to be given [formulas]” (Int3). Erin preferred the teacher role of *presenter* as both a teacher and a student.
**Facilitator profiles.** The other participants viewed their teacher role as one of facilitator, albeit to varying degrees. Each of Jenny, Heather, and Brad indicated that lecture was at times valuable and occasionally saw their role as one of presenter. Jenny and Heather differed with respect to the point during a unit at which they assumed the role of facilitator. Although Jenny (Comfort OMP) expressed a desire to be a facilitator, she only occasionally assumed this facilitator role, usually at the beginning of a unit when giving the students an opportunity to do a discovery activity or at the end of a unit when doing an activity that will let students use ideas learned in the unit. In comparison, Heather (Guided OMP) felt strongly that a “mixture of lecture and discovery” (Int2) was ideal but tended to include more discovery lessons throughout her planned unit rather than only at the beginning and ends of units.

Like Heather, Brad (Destination OMP) valued lecture and discovery, but his descriptions of his role during these lessons were more student–focused than those of Heather. In her description of helping students connect the Pythagorean theorem to the distance formula, Heather’s use of the pronoun “I” to describe her role in the lesson indicated how she was the one in control of the mathematics in the classroom: “I kind of did it in a couple of different representations. First I showed how it is derived from the distance formula. … Then I showed how squares come off the sides and that’s why it’s squared in the formula” (Int1). In contrast, Brad described what students do while trying to “bring together” (Int2) the different things they would have learned:

> They could all be on, you can have different groups on the right track, but no one has brought it all together, and just hearing other ideas, someone might be able to come to the final conclusion. Because they thought one thing, and then they heard something else, and then that drove it home for them. (Int1)
Although both indicated that they viewed themselves as facilitators of mathematics, Brad’s descriptions were more student-focused than Heather’s teacher-focused talk and teacher-centered classroom mathematics.

In contrast to the other participants, Sarah (Discussion OMP) almost exclusively favored her role as facilitator and expressed a dislike for lecture. “The lessons I didn’t like are the ones where I had to lecture a lot. I enjoyed having the students work in pairs … and having discussions with three kids at a time and then us all talking about it” (Int1). She excitedly described the variety of student thinking she saw during a student teaching lesson in which students developed strategies for factoring using algebra tiles. “Each group was coming up with different ways to show it, which was awesome to have two different representations of the same thing. … So that sparked a discussion” (Int1). The only instances in which Sarah indicated that she might need to present mathematics were with respect to students discovering definitions and generalizations. In those cases, she worried students would wonder whether or not the definitions or generalizations created by the students would be the “actual” definition or generalization and would need to be told the “actual” definition and generalization.

Sarah’s preference for facilitating—and possibly her need to make sure students have correct definitions and generalizations—could be situated to her unique high school experience with an integrated curriculum that involved an element of investigation. Unlike other participants who indicated that teachers presenting mathematics dominated their secondary mathematical experiences, Sarah may have had secondary mathematical experiences with a teacher facilitating mathematics.

Not only did participants’ OMPs differ mathematically in terms of actions and processes, but they also differed pedagogically with respect to presenter and facilitator. Yet, the groupings
based on mathematical commonalities differed from the groupings based on pedagogical commonalities. To make sense of both mathematical and pedagogical differences among PMTs’ OMPs, I developed an organizational schema for types of OMPs.

**Types of OMP profiles.** In answer to my first research question, I present a possibly fluid categorization of OMP profiles into four types—action–presenter, action–facilitator, process–presenter, and process–facilitator. It is possible for an OMP profile to waiver between two profile types. The categorization is fluid in that it is possible for a PMT to change profile types, depending on the values of the local community of practice in which a PMT is situated. For example, a PMT might have an action–facilitator OMP if a mathematics teacher education program in which discovery is valued but change to having an action–presenter OMP after working at a school in which presentations of actions is the norm.

Figure 6-1 captures how participants’ OMPs align with the different profile types to capture the observed mathematical and pedagogical differences among participant’s OMPs. Erin’s and Heather’s OMPs seemed to fit with the action–presenter and action–facilitator OMP types, respectively. In comparison, the other participants’ OMPs seemed to fit better between two OMP types, as captured in Figure 6-1 by the boxed OMPs placed on the borders between two OMP types. Jenny’s OMP seemed to be in transition from action–presenter to action–facilitator; Brad’s and Sarah’s OMPs seemed to be in transition from action–facilitator to process–facilitator. I have placed only participants’ OMPs in Figure 6-1; it is possible for an OMP type to have other profiles. In the next sections, I draw upon data and literature to illustrate the different OMP types.
**Figure 6-1.** Organization of participants’ OMPs into different OMP types, with transitional OMPs between two OMP types.

**Action–presenter OMPs.** A PMT with an action–presenter OMP would be oriented towards presenting actions on products to students. Erin’s [Distance OMP](#) exemplifies this OMP type because she distanced herself from processes and preferred actions on products in her personal and classroom mathematics. Additionally, telling dominated her descriptions of her classroom experiences with minimal consideration of student thinking and student participation relegated to students presenting steps in mathematical procedures.
In some ways, Jenny’s Comfort OMP fits the action–presenter type. At times she planned to present actions on products to students, which is likely related to her being most comfortable as a presenter of mathematics and uncomfortable with processes in her personal mathematics. Yet, she described her experiences in her mathematics education courses as shifting her perspective of teaching mathematics from presenter towards facilitator: “[Mathematics education courses were] just shifting more away from the traditional style of teaching and more towards like student discovery … So I think that helped me shift more towards having my own students do something like that” (Int1). Consequently, Jenny’s OMP seemed to waiver between action–presenter and action–facilitator.

**Action–facilitator OMPs.** A PMT with an action–facilitator OMP type creates a classroom in which the teacher facilitates students’ actions on products. Heather’s Guided OMP exemplifies the action–facilitator type. As mentioned earlier, the action of connecting dominated Heather’s personal mathematics and her plans to guide students to desired mathematical content. For Heather, facilitating with an adequate amount of guidance was important in both her personal mathematics and her classroom mathematics:

I think that a lot of times if you put a discovery activity in front of the student, and I was this way too, I kind of just like if I’m not really provided much guidance I just kind of sit there and like I don't know. … But for example that one [Generalizing via Dot Paper task] gives you a table. It gives you instructions. It gives you questions to be looking for and then in that table I can visually see okay well what's happening here. (Int2)

Heather wanted the instructions and questions to be sufficiently clear with enough guidance that students could make desired connections—“bring to light something” (Int2)—and act on products. While Heather predominantly guided students through actions on products, Brad and Sara facilitated students in processes.
**Process–facilitator OMPs.** A PMT with a process–facilitator OMP regularly provides opportunities and supports students in the engagement of mathematical processes. Although none of the participants in the research project had OMPs that fit nicely into the process–facilitator OMP type, two of the participants had OMPs that fit between process–facilitator and action–facilitator types. Brad and Sarah both valued the role of teacher facilitator and described classrooms in which they as teachers drew student thinking into orchestrated mathematical discussions. In their pursuits of good discussions (Sarah) or learning goals (Brad), the two PMTs inadvertently provided opportunities for students to engage in mathematical process. The inadvertent and auxiliary nature of process opportunities in Brad’s classroom and Sarah’s classroom is why Brad’s [Destination OMP](#) and Sarah’s [Discussion OMP](#) are transitional OMPs between action–facilitator and process–facilitator rather than examples of the process–facilitator type.

**Process–presenter OMPs.** Although no process–presenter OMPs were observed in my study, such OMPs are possible when a teacher presents her explorations to the class for students to execute algorithmically (Zbiek, 1995). One possible explanation for the absence of process–presenter OMPs in my data is the inquiry emphasis present in the PMTs’ teacher education program. Another possible explanation is that I gathered data only on participants’ planned classroom experiences rather than on their enacted classroom experiences. It is possible that a teacher might plan to facilitate actions and processes but then presents actions and processes during the enactment of the curriculum, representing a decline in mathematical intensity as observed with research on decline in cognitive demand (Stein et al., 1996) or in the case of a teacher who expressed views of teaching as facilitating but presented explorations in class (Zbiek, 1995).
Discovery dominance discussion. Looking across participants’ OMP profiles, one theme is common to all profiles: the theme of discovery. Participants used different words or phrases—such as discovery, explore, investigation, or activity-based learning—but captured a similar pedagogical idea. The word(s) used by each participant along with a representative quote capturing the intended meaning of the word(s) is included in Table 6-1. I use the term *discovery* because of its relatively pervasive use among participants to capture the pedagogical idea of letting students find a mathematical idea by themselves without explicit instruction. Even Erin, who did not embrace the notion of discovery, mentioned the emphasis on exploring in her teacher education program.

Table 6-1

<table>
<thead>
<tr>
<th>Participant</th>
<th>Word(s)</th>
<th>Representative Quote of Word Use</th>
</tr>
</thead>
<tbody>
<tr>
<td>Erin</td>
<td>Explore</td>
<td>“Not give it to them straight up” (Int2).</td>
</tr>
<tr>
<td>Jenny</td>
<td>Discover, Explore, Investigate</td>
<td>“So they could figure something out without you telling them” (Int1).</td>
</tr>
<tr>
<td>Heather</td>
<td>Discover</td>
<td>“Find out like properties or the general formula” (Int2).</td>
</tr>
<tr>
<td>Sarah</td>
<td>Discover, Investigate</td>
<td>“The whole idea of having students discover it on their own rather than just say, this is the formula this is how you use it now do a bunch of them” (Int2).</td>
</tr>
<tr>
<td>Brad</td>
<td>Discover, Activity-based</td>
<td>“To discover the Pythagorean theorem without being explicitly taught it” (Int2).</td>
</tr>
</tbody>
</table>

It is not surprising that participants situated in the same teacher education program would share commonalities in their OMPs. The notion of discovery was common across all OMPs likely because the teacher education program at the university emphasized inquiry mathematics teaching and learning. OMPs of PMTs situated in other teacher education programs likely have
other unifying conceptions in their OMPs, such as scaffolding, student thinking, or proof. In the next section, I explore how the theme of discovery led to the observed facilitator–driven OMPs and masked distinctions among processes leading to action–based OMPs.

**Discovery as an explanation for facilitator OMPs.** For the participants, discovery meant letting students figure out a mathematical idea on their own without prior teacher instruction. Hence, a teacher would take on a facilitator role in a discovery setting. For example, Jenny assumed a facilitator role in an activity that led students through a human graphing experience (Int2). Heather created slides to guide students through connections between the distance formula and the Pythagorean theorem to help students “see” the intended mathematics (Int1). This pedagogical conception of discovery is consistent with one way that Bruner (1960) used the term as a method of teaching for which “each teacher has his own tricks and approach to stimulating discovery by the student” (p. 611). The PMTs wanted students to participate in discovery activities. Yet, what the PMTs wanted students to discover was not always clear.

**Discovery as an enveloping term for processes.** Although the term *discovery* was used pedagogically by PMTs to describe a method of teaching, PMTs’ focused on providing opportunities for students to *discover* and not necessarily on engaging students in one of the targeted mathematical *processes*. Yet, participants seemed to use discover as if it were a mathematical process. Bruner (1960) used discover in a similar way, stating, “let me propose instead that discovery is better defined not as a product discovered but as a process of working” (p.612). Participants’ use of discover as a process is not unique and has deep roots in early mathematics education. But, by treating discovery as a mathematical process, participants’ effectively masked the differences among the four processes of defining, generalizing, representing, and justifying.
The participants described discovery as an activity of finding a relationship or commonality. However, they did not attend to what exactly the discovery work produced. In particular, it did not seem to matter to the participants if that activity generated a definition, a generalization, or a justification. For example, when asked what makes a good justifying task, Brad described a good justifying tasks as those that “require a little bit more critical thinking and they say why, why does this happen or why does this make sense” (Int3). He then identified Exploring a with Defining—a task that provides opportunities for students to engage in generalizing and defining—as a good justifying task. For Brad, reasoning and sense–making formed the basis of a good justifying task, not necessarily opportunities for students to generate justifications. Similarly, Jenny described coming up with a proof as “when we're working through the examples and the non-examples” (Int1) and identified a task with opportunities to define—the Exploring a with Defining task—as a good justifying task. For the PMTs, discovery was described as “not coming out and saying exactly what you are looking for but instead you have to look for a relationship” (Jenny, Int2) and could include any or none of the processes of defining, justifying, and generalizing.

Thinking of discovery as an activity that involves finding commonalities across a set of objects might explain the observed difficulty four of the participants had with solving the Tri-square Rugs task. Heather, Jenny, and Sarah attempted to find a mathematical relationship involving the tri-square rugs by generating examples and looking for a relationship across the examples. Jenny characterized this thinking as “repeated reasoning” and identified it as one of the mathematical practices (CCSSI, 2010). The attempt to discover a relationship by looking across examples proved unproductive for Heather, Jenny, and Sarah. But Brad, whose generalization approach differed in that he generalized the conditions of the Pythagorean
theorem rather than looking for connections across examples, was able to successfully solve the problem. Brad used a different type of generalization to solve the problem, a type that did not involve looking for commonalities across a set of objects. Perhaps the use of the term *discovery* to capture any mathematical activity not only masks the differences among mathematical processes such as justifying, generalizing and defining but also masks different ways of engaging in the processes.

While participants often mentioned having students engage in discovery, they rarely—with the exception of Brad—identified the mathematics students were to discover through the activity. By not identifying what is to be discovered, PMTs blended processes with each other as well as with actions. What then results when students engage in discovery might be a blend of processes and actions, or even non-mathematical activity.

Rather than making distinctions among different processes, participants were able to use the notion of discovery as an enveloping term that collapsed the processes into the activity of finding mathematical content by making connections or finding commonalities. The tendency to use discovery in place of processes not only affects classroom mathematics in the sense that any activity could be considered a process but also affects personal mathematics in that discovery does not help PMTs approach mathematical problems via processes.

**Summary of findings related to PMTs’ OMPs.** An analysis of participants’ OMPs indicates that they were predominately facilitator–driven in that most of the participants in the study viewed their role of teacher as that of facilitator. An emphasis on teaching through discovery in participants’ teacher education program offers a possible explanation for why most participants’ OMPs were facilitator–driven: the PMTs felt that they needed to be facilitators to allow students opportunities to discover.
Mathematically, the participants’ OMPs differed with respect to the presence of opportunities for students to engage in processes in addition to actions on products. The majority of PMTs in the study had action–based OMPs because they predominantly provided opportunities for students to engage in actions on products. Brad and Sarah, who did offer students opportunities to engage in processes, did so inadvertently in the advancement of an alternate teaching goal such as generating discussion. Rather than providing opportunities for students to engage in processes that generate mathematical products, participants focused on providing opportunities for students to discover.

**Findings and Discussion Related to PMTs’ Task Work**

My first research question focused on developing an understanding of PMTs’ OMPs. In this section I use results from my first research question to answer my second research question: how do PMTs’ OMPs influence how they select, modify, and sequence mathematical tasks? I use the term *task work* to refer to the collective work of selecting, modifying, and sequencing mathematical tasks.

Most of the PMTs in my study used what I label as *task templates* to guide their task work. Brad was unique in that he did not use task templates but instead used learning goals to guide his task work. PMTs’ OMPs influenced the kinds of task templates PMTs used; PMTs with presenter OMPs tended to use topical task templates and PMTs with facilitator OMPs tended to use pedagogical task templates. The task templates provided a way for PMTs as peripheral participators in a community of practice to simplify the complex practice of task work. I next define what I mean by *task templates* and discuss how they are used by PMTs in task work.
**Task templates as guides in task work.** During my analysis of the task interviews, I noticed that like other PMTs (Duncan, Pilitsis, & Piegaro, 2009), participants attended to task features while working with tasks. They tended to group particular task features together in ways that created for them a pattern that could be used to guide not only the selection of tasks as observed in the literature (Crespo, 2003; Duncan, Pilitsis, & Piegaro, 2009; Nicol & Crespo, 2006; Osana et al., 2006) but also the sequencing and modifying of tasks. I call these general patterns used to guide task work *task templates*.

A task template is an abstracted perception of a task based on some features of tasks. For example, a task template for an exploratory task could include such features as not explicitly telling students how to complete the task, kinesthetic activity for exploring mathematics, or sets of examples and non-examples. A PMT might evaluate a task to determine if the task shares one or more of these characteristics and categorize it as an exploratory task if applicable.

Illustrations of how PMTs might use task templates to guide task work are shown in Figure 6-2 and Figure 6-3. The figures separate task templates into two possible groups: topical and pedagogical. It is feasible that other kinds of task templates exist for PMTs depending on their teacher education program and other experiences, but these templates tended to dominate the work of the participants in my study. Participants with presenter OMPs used topical templates and participants with facilitator OMPs used pedagogical templates. Task template names along with possible tasks from the *Pythagorean Task Interview* (see Appendix C) and the *Quadratic Task Interview* (see Appendix D) that fit a template are grouped using rectangles, see Tables 4-5 and 4-6 for task feature and process/action information about each task. Samples of selected tasks within each template are highlighted yellow. Sometimes all possible tasks that fit into a template were selected or only some tasks were selected by a PMT; sometimes none of the
tasks were selected by a PMT using topical templates. Arrows between tasks templates illustrate how templates are ordered when used as guides to sequence tasks—or to sequence templates. Double arrows show when participants interchanged two templates. PMTs with presenter OMPs would interchange the sequence of templates whereas PMTs with facilitator OMPs consistently used an investigation–practice–extend sequence. Elaborations of how task templates are used to guide task work are explained in the following sections.

**Topical Templates as Guides for Task Work in Presenter OMPs**

(a) Example of Topical Templates used in Designing a Pythagorean Theorem Unit

- **Pythagorean Theorem Template**
  - Tri-Square Rugs
  - Pythagorean Puzzle
  - Generalizing via Dot Paper
  - Baseball Application

- **Triples Template**
  - Odd Triples
  - Examples & Non-examples
  - Identifying Triples

- **Converse Template**
  - Justifying with Converse
  - Verifying Converse

(b) Example of Topical Templates used in Designing a Quadratic Unit

- **Parameter Template**
  - Exploring a with Defining
  - Exploring c with Intercepts

- **Intercept Template**
  - Intercepts as Solutions
  - Factors & Intercepts

- **Factoring Template**
  - Novel Set of Quadratics
  - Factoring with Justifying
  - Vertex Application

- **Algebra Tiles Template**
  - Completing the Algebra Tile Square
  - Unfactorable Algebra Tiles

*Figure 6-2.* Example of topical task templates used by the PMTs in selecting and sequencing tasks for (a) the Pythagorean theorem and (b) quadratics.
Pedagogical Templates as Guides for Task Work in Facilitator OMPs

(a) Example of Pedagogical Templates used in Designing a Pythagorean Theorem Unit

<table>
<thead>
<tr>
<th>Investigation Template</th>
<th>Practice Template</th>
<th>Extending Template</th>
</tr>
</thead>
<tbody>
<tr>
<td>- Tri-Square Rugs</td>
<td>- Pythagorean Rugs Proof</td>
<td>- Verifying Converse</td>
</tr>
<tr>
<td>- Pythagorean Puzzle</td>
<td>- Square-in-Square Proof</td>
<td></td>
</tr>
<tr>
<td>- Generalizing via Dot Paper</td>
<td>- Justifying with Converse</td>
<td></td>
</tr>
<tr>
<td>- Examples &amp; Non-Examples</td>
<td>- Baseball</td>
<td></td>
</tr>
</tbody>
</table>

(b) Example of Pedagogical Templates used in Designing a Quadratic Unit

<table>
<thead>
<tr>
<th>Investigation Template</th>
<th>Practice Template</th>
<th>Extending Template</th>
</tr>
</thead>
<tbody>
<tr>
<td>- Intercepts as Solutions</td>
<td>- Factors &amp; Intercepts</td>
<td>- Quartic Parabola</td>
</tr>
<tr>
<td>- Exploring a with Defining</td>
<td>- Unfactorable Algebra Tiles</td>
<td></td>
</tr>
<tr>
<td>- Exploring c with Intercepts</td>
<td>- Completing the Algebra Tile Square</td>
<td></td>
</tr>
<tr>
<td></td>
<td>- Factoring with Justifying</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Wrap-up Template</td>
</tr>
<tr>
<td></td>
<td></td>
<td>- Generalizing Solutions</td>
</tr>
</tbody>
</table>

Figure 6-3. Example of pedagogical task templates used by PMTs in selecting and sequencing tasks for (a) the Pythagorean theorem and (b) quadratics.

**Kinds of task templates used influenced by OMP.** OMPs were related to the kinds of task templates PMTs use as guides in task work, with differences aligning along the presenter–facilitator dichotomy. Erin, who had an action–presenter OMP type, categorized tasks based on mathematical topic. For example, she categorized Intercepts as Solutions and Factors and Intercepts as “intercept tasks” (Int3). In comparison, Heather and Sarah, who had facilitator OMPs, categorized tasks based on pedagogical use regardless of topic. For example, the Pythagorean Puzzle task and the Generalizing via Dot Paper task were categorized as “investigation tasks.” Not surprisingly, Jenny who had an OMP that waivered between presenter and facilitator had task templates that were both topical (e.g., “triples tasks”) and pedagogical
(e.g., “exploration tasks”). PMTs’ OMPs align with the kinds of task templates PMTs used and these task templates are used to guide task work.

PMTs’ task work begins with categorizing tasks using templates. Jenny explicitly described this task work at the beginning of her quadratic interview. “I’m just going to put them in piles … like a category type thing” (Int3). The process of categorizing tasks using templates is a first step in simplifying the complexity of task work for PMTs as new participants in a community of practice. PMTs in my study used two kinds of task templates—topical and pedagogical—to categorize tasks and the kinds of task templates used by the participants seemed to be related to the values of the teacher education program in which the PMTs participated. PMTs participating in a different teacher education program might learn different values and constructs of teaching that might results in different kinds of task templates.

**Task templates as guides for selecting tasks.** Once PMTs have categorized tasks using task templates, they selected tasks within each category. Task templates simplified the work of selecting mathematical tasks by narrowing the scope of possible tasks. For example, rather than trying to select among a set of 12 tasks, a PMT can categorize tasks using templates to reduce the number of tasks under consideration for selection. For example, Sarah identified four tasks that could be used to introduce the Pythagorean theorem. She called these tasks “investigation tasks” because they were “not coming out and saying exactly what you are looking for but instead you have to look for a relationship” (Int2). By categorizing the tasks, she simplified her work of selecting so that she now only needed to select among the four investigation tasks she had identified. Furthermore, sorting tasks was not difficult, resulting in the sort advancing rather than hindering selection.
When selecting among tasks with the same task template, PMTs often chose the task they liked best based on task features. For example, Erin selected the Pythagorean Puzzle task and the Baseball task from her Pythagorean theorem template because she liked tasks in which students could use manipulatives to explore and tasks that involved interesting applications. Heather selected the Generalizing via Dot Paper task and the Pythagorean Puzzle task from her investigation template because she liked the guided questioning and table in the former task and the hands-on exploration in the latter. Once a PMT had categorized tasks using templates, PMTs tended to use task features to select tasks within a template.

A PMT might select all or some of the tasks within a template, depending on the anticipated time needed for the tasks or if they disliked one or more of the tasks. For example, Jenny identified two tasks with a Pythagorean theorem converse topical template but only selected one of the tasks, Verifying Converse, because she thought that a task involving technology would take an entire period and there would not be time for another task. Erin identified three tasks with a Pythagorean triples template, but only selected two of the tasks, rejecting the Examples and Non-Examples task that she found confusing.

I observed some differences between PMTs using topical task templates and PMTs using pedagogical task templates with respect to the omission of an entire task template from a sequence. A PMT using topical task templates might choose to select none of the tasks fitting a topical task template. For example, Erin chose not to select a task from her completing the square task template in the quadratic task interview because she found algebra tiles, which were used in the tasks, to be personally confusing and consequently did not want to use them in the classroom. The omission of an entire topical task template was not problematic because the topical task templates are not necessarily connected nor do they require a specific sequence. In comparison, it
seemed that a PMT using pedagogical task templates must select at least one task from each template to complete the needed sequence of task templates—investigation tasks, practice tasks and then extending or wrap-up tasks. Perhaps omitting a pedagogical task template would be akin to removing a critical step in an algorithm.

**Task templates as organizers for sequencing tasks.** Rather than sequencing tasks, PMTs sequenced task templates. PMTs using pedagogical templates seemed to follow an algorithm for task sequencing by placing investigation tasks at the beginning of a unit, practice tasks in the middle of a unit, and wrap-up or extending tasks at the end of a unit. Using pedagogical templates to guide sequencing might mean that mathematical topics may not be discretely addressed in the unit. For example, Heather sequenced the *Pythagorean Rugs* task and the *Examples and Non-examples* task as mid-unit tasks to be used on the same day. The former task addresses a proof of the Pythagorean theorem and the latter the topic of Pythagorean triples. Consequently, Heather implicitly planned to introduce triples and prove the Pythagorean theorem on the same day of the unit, which is something that would not happen when tasks are sequenced using topical templates.

While PMTs with pedagogical task templates seemed to follow a more formulaic order of task templates, the order of topics did not seem as critical to PMTs using topical task templates to guide their task work. For example, Erin switched the order of two of her topic task templates—Pythagorean triples task template and Pythagorean converse task template—three times during the interview. She switched the order of the task templates because she could not decide whether to teach Pythagorean triples before the converse of the Pythagorean theorem. She seemed to view both topics as auxiliary to the content of the Pythagorean theorem. She viewed the converse task template as auxiliary to the Pythagorean theorem because the converse “talks
about the different angles” (Int2), meaning that the converse might be applied to any triangle whereas the Pythagorean theorem applied only to right triangles. She viewed the Pythagorean triples task template as auxiliary to the Pythagorean theorem because she did not remember learning about Pythagorean triples in high school and thought they would be something “fun for them to explore” (Int2). Erin’s view of the topics of Pythagorean converse and Pythagorean triples as auxiliary to the Pythagorean theorem had the affect of making the sequencing of those task templates interchangeable.

PMTs using pedagogical task templates had an algorithmic sequence of templates they could use to guide their task work. PMTs using topical task templates did not have an algorithmic sequence of templates and could interchange task templates when sequencing tasks.

**Task templates as supplanters for modifying tasks.** In general, participants did not modify tasks during the interviews. Task templates offer a possible explanation for why participants did not tend to modify tasks in the interviews, even when prompted. It is possible that PMTs do not need to modify tasks because they only select tasks that fit one of their task templates. In this way, task templates supplant the need to modify tasks in mathematical ways.

Rather than modifying the mathematics of a task, a task might be modified to alter task features such as context. For example, Heather modified a student teaching task involving amortization schedules by altering the context from buying a home to paying off student debt, a context she thought students might find more interesting and readily applicable in the near future. Heather modified the task feature of context but did not modify the task in a way that alters the processes or actions of the task.

If tasks do not align with a task template, then a PMT might modify the task to align better with the desired task template. Although Sarah did not modify tasks during the interview,
she indicated that she usually modifies tasks to include directions for students to explain. “Normally when I change things it would be to explain or justify” (Int2). The set of tasks Sarah had selected for the Pythagorean theorem unit already requested that students explain or justify and so Sarah did not modify the set of tasks she selected during the interview. She might modify tasks with a request to explain if the tasks do not align with this feature of her pedagogical task templates.

Task templates simplify the work of selecting, modifying and sequencing tasks. Task templates simplify the work of selecting by limiting choices of tasks to consider for selection. Task templates also simplify the work of modifying tasks by eliminating the need to modify unless a task does not align with a particular template in which case the modification needed is simply that of altering the task so that it aligns better with the task template. Using task templates to guide sequencing also simplifies task work. When topical task templates are used, the sequencing order is not critical and order of templates can be easily changed. When learning goals rather than task templates were used to guide task work, sequencing of tasks increased in complexity.

**Learning goals as replacements for task templates.** Unlike the other participants, Brad did not seem to use task templates to guide his work. Instead, he used learning goals to help him select, modify, and sequence mathematical tasks. As described in Brad’s Destination OMP profile, Brad wanted to create a sequence of tasks with learning goals that logically “flowed” from one task to another. He would select or not select a task based on whether or not the task fit with the other tasks.

He also modified tasks to better align with his learning goals. Prior to modifying one of the Pythagorean theorem tasks, Brad stated that when modifying tasks “it is all about what the
objective for that lesson [is]” (Int1), illustrating the importance of learning goals in Brad’s teaching. Sometimes these modifications had the effect of removing possible process opportunities from the task. Brad suggested modifying the Tri-square Rugs task by giving students three different examples of tri-square rug designs (involving right, acute, and obtuse triangles)—effectively removing the opportunity for students to engage in the mathematical process of representing in favor of leading students to the goal of identifying three generalizations about the squares of sides of a triangle. Brad’s modification was consistent with his learning goal to have students learn about the inequalities of squares of side lengths of triangles. His learning goal did not necessarily include the process to generalize conditions of the Pythagorean theorem.

Brad’s work with tasks was qualitatively more complex than the work of PMTs using task templates to select and sequence tasks. While PMTs using task template tended to use task features to select tasks and then an algorithm to sequence the templates, Brad considered details of the tasks to determine possible learning outcomes and how the learning outcomes of one task might fit with the learning outcomes of another task. Using learning goals to guide task work resulted in unit plans with richly connected tasks sequenced by possible student learning.

**Summary of findings related to PMTs’ task work.** PMTs’ use of task templates may simplify the complexity of engaging as novices in the practice of task work. A PMT’s OMP influenced the kind of task templates used in task work, with facilitator OMPs using pedagogical task templates and presenter OMPs using topical task templates.

PMTs use task templates to simplify the work of selecting through seemingly unproblematic sorting of tasks into categories from which tasks can be selected. Task templates eliminate the need to modify tasks because only tasks matching templates are selected. For
PMTs with facilitator OMPs, task templates simplify the sequencing of tasks by creating an algorithmic sequence of investigation tasks, practice tasks, and extending tasks. In comparison, tasks templates simplify the work of sequencing tasks for PMTs using topical task templates because the order of topic templates is not always critical and can often be interchanged without consequence.

The use of task templates simplified task work, yet one PMT in my study did not use task templates to guide task work. Brad, whose OMP focused on learning goals, used learning goals rather than task templates to guide his task work. Task work guided by learning goals was qualitatively more complex than task work guided by task templates.

**Conclusion**

In this chapter, I reported the findings and discussion related to my research questions of what PMTs’ OMPs are, and how PMTs’ OMPs relate to how PMTs select, modify, and sequence mathematical tasks. To answer the first research question, I assimilated participant OMP profiles into an organized schema based on an action–process dichotomy and a presenter–facilitator dichotomy observed in the participants’ OMP profiles. I found that PMTs’ OMPs tended to be primarily facilitator–driven in that three of the five participants primarily described their teaching role as one of facilitator with a fourth participant occasionally viewing her teaching role as a facilitator. Additionally, PMTs’ OMPs also tended to be focused on actions rather than on processes. Three of the five participants primarily planned to provide opportunities for students to engage in actions and minimal opportunities for students to engage in processes. The other two participants incorporated processes more frequently in their classrooms, but the processes were auxiliary to some other classroom purpose such as inviting discussion or accomplishing a learning goal.
In answer to my second research question of how are PMTs’ OMPs related to how they select, modify and sequence mathematical tasks, I found that PMTs used task templates to guide their task work. The kinds of task templates they used were related to their OMP. Participants with presenter OMPs tended to use topical task templates to guide their work in selecting tasks. Participants with facilitator OMPs tended to use pedagogical task templates to guide their work in selecting and sequencing tasks using a task template sequence formula of investigation tasks, practice tasks, and extending or wrap-up tasks. In contrast to using task templates, Brad used learning goals to guide task work, which is not surprising given that learning goals figured prominently in his OMP. The use of learning goals to guide task work resulted in unit plans that were qualitatively richer than those created using task templates.

In the next chapter, I describe some implications related to PMTs’ OMPs and the use of discovery to mumble processes. I also discuss implications related to the use of task templates to simplify mathematical task work and consider plans for further research into OMP and task work.
Chapter 7

Implications, Limitations, and Further Questions

This chapter includes concluding thoughts related to the study. Following a summary of framing and findings, I consider implications of discovery, the processes and actions lens, and task templates to each of research, teacher education, and secondary curriculum. I conclude the chapter with further questions indicative of an anticipated research trajectory.

Summary of Framing and Findings

In this study, I investigated prospective mathematics teachers’ (PMTs’) orientations toward mathematical processes (OMPs) and their task work with the research questions: what are PMTs’ OMPs and how do OMPs influence how PMTs select, modify, and sequence mathematical tasks. A PMT’s OMP is a set of conceptions about mathematics and its teaching and learning related to processes as well as how s/he engages in processes that influence how s/he provides opportunities for students to engage in mathematical processes. I used a processes and actions framework (Heid et al., 2015; Zbiek et al., 2014) to target four mathematical processes of defining, generalizing, justifying, and representing. Drawing upon mathematics and science education literature describing orientations (e.g., Cess-Newsome, 1999; Magnussen et al., 1999; Remillard & Bryans, 2004; Schoenfeld, 2010), I identified four components of OMP to be (a) nature of engaging in mathematical processes, (b) conceptions of mathematics and mathematical processes, (c) conceptions of teaching mathematics and (d) conceptions of learning mathematics. I used my definition of OMP to create a framework to investigate my research questions.

Applying methods of qualitative research, I used primarily interview data to identify conceptions related to processes as well as how participants engaged in processes that influenced
possible opportunities for processes in the classroom. Analysis of the data resulted in the creation of OMP profiles that capture how participants’ pedagogical and mathematical conceptions and practices relate to their work with mathematical tasks. I then used these profiles to answer my research questions.

Results from the study indicate that PMTs have primarily facilitator–driven and action–based OMPs, with possible movements between facilitator and presenter and possible movements between action and process. Participants’ use of an enveloping concept—that of discovery—offers possible explanations for the frequency of observed facilitator OMPs among participants and for participants’ blurring of mathematical processes with each other and with actions. The notion of discovery, which refers to students finding some mathematical idea or skill without prior teacher modeling, implies a teacher role of facilitating as students complete an activity that will hopefully lead to the discovery of some mathematics. Most participants favored including discovery in their classrooms and hence described their role as predominantly that of facilitator rather than presenter.

Importantly, the participants’ foci on the pedagogical method of discovery rather than on what was to be discovered resulted in most participants not distinguishing among the processes of defining, generalizing, and defining. Not only did discovery envelop mathematical processes but also other mathematical activity, including actions on products. The use of discovery to describe all mathematical activity alleviated the need to focus on particular processes and instead encouraged focus on actions with occasional auxiliary opportunities to engage in processes.

Most participants used task templates to aid in their task work. PMTs’ OMPs were related to their work with tasks. PMTs with presenter OMPs tended to use topical task templates to select and sequence mathematical tasks. In comparison, PMTs with facilitator OMPs tended to
use pedagogical task templates to select and sequence mathematical tasks. The use of templates simplified the complex nature of task work by creating a scaffold for categorizing tasks into templates from which PMTs can select, often based on task features, a task from a given category. The use of task templates further simplified the nature of selecting by limiting the number of tasks under consideration for selection. Additionally, task templates simplified task work by eliminating the need to modify tasks because only tasks that matched a template would be considered for selection. Finally, task templates simplified the work of sequencing tasks by providing a general guide for sequencing. PMTs using topical task templates loosely sequenced topics rather than tasks. Comparatively, PMTs with pedagogical task templates sequenced tasks with investigation tasks first, practice tasks next, and extending tasks last.

**Implications of Findings with Considered Limitations**

Results of the study offer several implications in mathematics education research, teacher education, and secondary mathematics curricula. In particular, I consider implications of the predominance of action–facilitator OMPs, participants’ use of discovery, and participants’ use of task templates. I weave implications of the processes and actions framework into discussions of OMPs, discovery, and task templates. I address limitations in methods related to each of these areas within the discussions of implications.

**Action–facilitator OMPs.** Prior research has pointed to an existence of a presenter–facilitator distinction among teachers’ beliefs and approaches to teaching mathematics (e.g., Conner et al., 2011; Franke et al., 2007; Munter, 2014; A. G. Thompson, 1992). While a presenter–facilitator distinction is useful for considering pedagogical differences among teachers, my research findings indicate that an additional distinction between action and process contributes to an understanding of teachers’ orientations toward teaching and the kinds of
opportunities afforded students in the classroom. In particular, my findings suggest that a pedagogical view of teaching as facilitating is not sufficient to provide opportunities for students to engage in mathematical processes. Both Jenny and Heather exhibited views of teaching as facilitating but intended to facilitate *actions* instead of *processes*. The added dimension of action and process (Heid et al, 2015; Zbiek & Heid, 2012; Zbiek et al., 2014) provides a lens to describe the kinds of mathematical activity facilitated in the classroom.

My categorization of OMPs into types of action–presenter, action–facilitator, process–presenter, and process–facilitator can be used as a framework for organizing PMTs’ OMPs in a teacher education program. The types of OMPs can be used reflectively among mathematics teacher educators as they consider the enculturation of PMTs within a program. Consistent with *Principles to Action* (NCTM, 2014), mathematics teacher educators seemingly would like prospective and practicing teachers to have process–facilitator OMPs (Sowder, 2007). PMTs experience a lengthy *apprenticeship of observation* (Ball & Cohen, 1999) during elementary and secondary schooling and continuing into collegiate mathematics courses, and many of their experiences might have been in a traditional mathematics setting (Sowder, 2007). Given these experiences observing traditional mathematics teaching, PMTs likely begin their mathematics education enculturation experience with action–presenter OMPs and a need to “break” with their experiences (Ball, 1990a). Considering that three of the five participants in my study had facilitator OMPs rather than presenter OMPs, the results imply that PMTs’ OMPs change from presenter to facilitator during participation in methods courses and preservice teaching experiences. These implications are limited in part by the qualitative nature of my study, which alone does not generalize to large populations. But, my work joined with other literature that suggests changes in beliefs (e.g., Conner et al., 2011; Lilejedahl, P., Rolka, K., & Rosken, B.,
2007) adds support to possible implications that PMTs’ OMPs can change as they engage more in teaching practices. Further research investigating change in PMTs’ OMPs as they participate in parts of a teacher education program might contribute to suggested progressions of teacher visions (Munter, 2014).

Results from my study indicated that PMTs’ OMPs are predominantly focused on actions. Three of the participants focused on actions without process opportunities while the possible process opportunities planned by the remaining two participants were auxiliary to other teaching goals. Admittedly, the study is limited with respect to the trustworthiness of my analysis. Although throughout my data analysis and report writing I discussed data and preliminary findings with a senior mathematics educator with extensive experience with the processes and actions framework, all data coding has not yet been vetted by a group of researchers. My analysis is limited in that data coding around processes and actions may inadvertently reflect my bias as researcher, affecting findings related to PMTs’ primary focus on actions. I made efforts to minimize this limitation by consulting with a senior researcher regarding preliminary research findings, rereading data with a critical eye toward bias, and questioning the data. Despite these limitations, findings are still informative to teacher educators as they seek to understand PMTs and the kinds of experiences and activities that shape their OMPs.

Teacher educators should consider whether their PMTs are focused on actions or processes. Additionally, they should consider how to support PMTs with opportunities to change, particularly in the move from actions to processes. Such a move from actions to processes might be supported in mathematics and statistics content courses in which PMTs have explicit opportunities to engage in processes and reflect on that engagement. In particular, it would be useful for PMTs to unpack the complex and interconnected nature of the mathematical
processes in which they engage. Professional development programs supporting teachers in
disciplinary explorations around mathematical activity have shown how a richer understanding
of mathematics influenced classroom practice (Schifter, 1998). Similarly, metacognitive analysis
of engagement in processes can help PMTs make distinctions among processes and avoid
reliance on enveloping terms such as discovery that mask differences between processes.

**Discovery as an enveloping term.** Participants in the study were part of a larger context
that included a teacher education program with courses and field experiences. As participants in
this contextualized community focused on inquiry in mathematics, it is not surprising that
participants mentioned discovery as a method of teaching. While participants in my study tended
to use an enveloping term of *discovery* to encompass processes and task templates to guide task
work, this does not prove that the majority of PMTs use discovery to mask distinctions between
processes or that the majority of PMTs use task templates to select, modify and sequence
mathematical tasks. From a situated cognition perspective, OMPs and task work depend on the
activities and experiences in which OMPs are situated. Those experiences include but are not
limited to experiences learning secondary and collegiate mathematics, learning to teach
mathematics in a teacher education program, and participating in the practice of teaching through
preservice teaching opportunities with mentor teachers.

Participants’ use of *discovery* can explain the prevalence of intentions to facilitate
mathematics in the absence of process opportunities. The notion of discovery is not new in
mathematics education (e.g., Anthony, 1973; Bruner, 1960; Goldin, 1990; A. G. Thompson,
1984), so participants’ use of discovery is not surprising. Regarding discovery, Bruner (1960)
said, “Probably we do violence to the subtlety of such technique by labeling it simply the
‘method of discovery,’ for it is certainly more than one method, and each teacher has his own
tricks and approach to stimulating discovery by the student” (p.611). Bruner claimed that discovery as a pedagogical approach can be varied and subtle, suggesting that discovery as merely *not telling* might be an oversimplification of what discovery involves in the classroom. Participants’ descriptions of discovery did not suggest varied and subtle conceptions akin to Bruner’s claim regarding discovery as a method. Instead, participant’s use of the term discovery to imply a pedagogical approach of not telling is akin to Kirschner, Sweller, and Clark’s (2006) description of it as “the minimally guided approach” (p.75), in which students discover a mathematical idea without being told what the idea is or how to apply it.

Bruner’s (1960) statement about discovery captures a varied use of the term discovery as a pedagogical approach to teaching, one in which multiple methods could be couched as discovery. Discovery, with its multifaceted applicability, is one way in which PMTs satisfy suggested policy implementations of CCSSM in ways that may not reflect intended policy (e.g., Cohen, 1990; Cohen and Hill, 2001). By focusing on discovery, PMTs facilitate mathematical activity in the form of action but not necessarily process.

The use of discovery as a pedagogical notion among the participants in the study might serve to sensitize teacher educators to PMTs’ possible use of umbrella terms that envelop multiple processes or mathematical/pedagogical ideas. While participants in this study tended to use the notion of discovery to pedagogically capture the activity of students finding some mathematical idea on their own prior to teacher telling, different umbrella terms and ideas for this or other constructs might be used by other cohorts of PMTs. These ideas might be reflective of values or emphases in their teacher education program, such as differentiated instruction, scaffolding, or conceptual understanding.
In other learning communities, other foci used as enveloping terms could also affect the presence or absence of process opportunities in the classroom. For example, PMTs focusing on differentiated instruction might differentiate by ability in a way that the more difficult problems merely contain more challenging opportunities to engage in actions on products (such as more difficult manipulating of representations) and not necessarily attend to mathematical processes in differentiation. Alternatively, PMTs might scaffold mathematical tasks in a way that unknowingly alters the task (Houssart, 2002) so that opportunities to engage in processes are changed into opportunities to act on products. Additionally, PMTs might use conceptual understanding to envelop any kind of mathematical thinking that is not procedural, such as adaptive reasoning and strategic competence (NRC, 2001).

Additionally, discovery might be a mesh of other notions about teaching mathematics such as problematizing, open-ended problems, construction, and inquiry. PMTs might use these notions to capture a general notion of what they “should” do that is different from their experiences as students and what they anticipated when beginning their teacher education program.

**Processes and actions framework as an elaboration of doing mathematics.** Not only did PMTs use discovery to refer to a pedagogical method of teaching but also to describe various mathematical processes. PMTs used discovery to refer to mathematical processes and actions alike, and tasks identified as discovery by participants did not always involve mathematical processes. The processes and actions framework (Heid et al., 2015; Zbiek et al., 2014) contributes to research on mathematical tasks by offering an elaboration of what “doing mathematics” (Stein et al., 2009) tasks might involve. Furthermore, a delineation of “doing
mathematics” tasks provides a language for describing specific mathematical activity opportunities within a task.

In addition to having PMTs and teachers sort tasks using cognitive demand as a way to provide teachers with opportunities to analyze tasks (Arbaugh & C. Brown, 2005; Boston & Smith, 2009; Stein et al., 2000), teacher educators could have PMTs sort tasks using processes and actions. Not only could PMTs separate tasks involving processes from tasks involving actions but also separate tasks by the opportunities to define, generalize, justify, or represent. A task analysis using processes and actions might be particularly enlightening to prospective teachers when analyzing a task involving multiple processes. Oftentimes, tasks with opportunities to engage in one process contain opportunities to engage in other processes as well. For example, the Tri-square Rugs task and the Pythagorean Puzzle task offer opportunities to represent, generalize, and justify. The Exploring a with Defining task offers opportunities to define and generalize.

Mathematical processes tend to be investigated in isolation of other processes. For example, studies on teachers’ mathematical activity and beliefs often focus on justifying or proof (e.g., Conner et al., 2011; Knuth, 2002a, 2002b) but not on the relationship between justifying and other processes or how other processes facilitate the process of justifying and proving. A few studies, particularly within research on defining and definitions, suggest a possible lack of separation between defining and justifying (Leikin & Winicki-Landman, 2001; Levenson, 2012). My study contributes to this research not only by confirming a lack of separation between justifying and defining but also by introducing an encapsulation of generalizing with justifying and defining, as described in Chapter 6.
**Task templates as guides to emerging task work.** Participants used task templates to guide their emerging task work. Given the complexity of task work (see M. Brown, 2009), it is not surprising that PMTs have a means for simplifying that work as they begin participating in the practices of teaching. Furthermore, pedagogical task templates might be PMTs’ attempts to implement a launch-explore-summarize model (Fendel et al., 2003) for implementing problems-based tasks that had been used in their teacher education program. Task templates provide a lens for capturing PMTs’ initial task work and could be used as a tool among teacher educators as a starting point for PMT task work.

Using learning goals to guide task work might a natural next step following task templates in participating in the practices of task work. As PMTs participate in task work with more frequency and regularity—and as their familiarity with mathematics curricula increases—they might begin using learning goals to guide their task work. Prior work on learning goals (Hiebert, Morris, Berk, & Jansen, 2007; Stein & Smith, 2011) suggests that a focus on learning goals might be beneficial for learning to teach. The use of learning goals rather than task temples might change how PMTs’ select, modify and sequence mathematical tasks. Additionally, the processes and actions framework could be applied to writing learning goals that contain process information and move beyond discovery. Applying processes and actions to learning goals provides a framework for incorporating not only *what* is to be discovered (mathematical products) but also *how* the product is discovered (process).

Interestingly, PMTs in my study did not modify tasks in mathematical ways. This could imply that PMTs have limited experiences with modifying tasks and hence do not engage in modifying tasks. Having opportunities to modify mathematical tasks to better support and align
with learning goals that include processes, PMTs might begin to understand how learning goals support task work and move beyond the use of task templates to guide their task work.

Perhaps the study was limited with respect to data collection methods for gather data on PMTs modifying tasks. In an effort to provide participants with a set of tasks in the task interviews that varied with respect to process opportunities, I may have negated participants’ need to modify. Within any particular topic in the task interviews, the participants typically had the option to choose an action-dominated task or a task with process opportunities. They chose the task that better fit their orientation and thus did not need to modify the task. Altering the set of tasks so that some topics only included opportunities to act on products while other topics included tasks dominated by process opportunities may have incurred more modifying from the participants. Another way to alter the data collection would be to have participants select among a set of already modified tasks.

**Future Research Plans**

My current research lies at the intersection of teacher knowledge and beliefs and classroom practice, with a predominant focus on the planning aspect of practice. Mathematical activity in the form of mathematical processes serves as an additional focus of my research. The notion of mathematical activity spans policy documents over multiple decades in the form of such things as Process Standards (NCTM, 2000), Standards for Mathematical Practices (CCSSI, 2010), and mathematical habits of mind (Cuoco et al., 1996). My research goals combine knowledge and beliefs, classroom practice, and mathematical activity. They are guided by the question *how do prospective secondary mathematics teachers learn to provide opportunities for students to engage in mathematical activity* with a practical goal of supporting prospective
mathematics teachers as they learn to use mathematical processes and practices to teach mathematical content.

Developing an answer to this larger question about how PMTs learn to provide opportunities for mathematical engagement involves several research phases: an understanding of key relationships or PMT tendencies, an understanding of development of such relationships, and an experiment in influencing their development. First, I seek to understand how PMTs’ beliefs, knowledge and conceptions are related to the mathematical opportunities afforded to students in the classroom via mathematical tasks. The current study contributes to this portion of my research trajectory. Second, I shift my focus from an exploratory study of relationships to an understanding of the development of beliefs, knowledge and conceptions that leads to strategic use of mathematical tasks to promote learning through mathematical activity. Finally, I plan to experiment in developing PMTs’ abilities to use tasks to promote learning through mathematical activity.

The current study addresses the first step in this research trajectory by developing a construct for examining PMTs’ beliefs, knowledge and conceptions as they relate to mathematical process opportunities in the classroom—orientations toward mathematical processes. Additionally, this study addresses part of the relationship between beliefs, knowledge, and conceptions to mathematical task work.

Now that I have established a relationship between PMTs’ OMPs and work with mathematical tasks, I am particularly interested in the development of OMPs among PMTs and how PMTs’ experiences in teacher education programs shape their OMPs. While ideally I would like to study PMTs over an extended period of time, I might begin by studying the OMPs of PMTs’ at different places in their mathematics education experiences. I have already studied the
OMPs of PMTs at the end of their teacher education experiences in my study. Other snapshots in time to consider might be the OMPs of PMTs in their first methods course or OMPs of novice teachers in their first few years of teaching secondary mathematics. The context of working with a mentor in a school system might be particularly enlightening as PMTs’ peripheral participation changes to include more activities of teaching as they enter new field experiences. Synthesizing results from studies of PMTs at different moments in time might be useful for understanding the stability of OMPs over time and the identification of pivotal moments in PMTs’ experiences that lead to changes in OMPs.

The above litany of options illustrates the vast opportunities for research on OMPs. To focus these options into a trajectory, I plan to conduct the following studies. First, I plan to conduct a case study of a PMT or practicing teacher with a process–facilitator OMP. The case of a teacher with a strong process–facilitator OMP would be useful for understanding how OMPs relate to task work and the mathematical experiences of students in the classroom. Such a case study would provide insights into the development of not just any type of OMP but the development of the kind of OMP we as teacher educators might like our students to develop.

Second, I will use the results from the current study and the case study to develop a course that blends secondary and advanced mathematical content with mathematics teaching methods—particularly those methods of task work. The purpose of the course would be to provide PMTs with experiences that might change the nature of their OMPs to that of process–facilitator. PMTs would have opportunities to engage in mathematical processes and consider how to teaching mathematics through processes. Such a course would focus in part on engaging PMTs in mathematical processes and helping them unpack their engagement to highlight distinctions and connections among processes. The course would also include a pedagogical
focus on task work to provide students with process opportunities. This experiment in developing PMTs’ OMPs might further inform practice as I seek to answer questions such as, how did PMTs’ OMPs change during the course and what experiences related to the course facilitate that change? How did PMTs’ task work change during the course and what experiences related to the course facilitate the use of learning goals to guide task work?

The above studies might contribute answers to the broader questions of how do PMTs’ OMPs evolve throughout their time in a teacher education program and during their first two years of teaching and what experiences influenced changes in OMPs? I could also consider changes in PMTs’ task work: How do PMTs’ selecting, modifying, and sequencing mathematical tasks change for PMTs during their teacher education and first years of teaching? What experiences might explain these changes? An understanding of how OMPs develop throughout the course of a teacher education program can offer possibilities for designing a course to perturb action–based and presenter–focused OMPs and prompt a change to a process–facilitator OMP.

**Concluding Thoughts**

Despite limitations in generalizability and data collection, my study offers implications for both teacher education and secondary curriculum materials. Findings for my research questions offer a possible organization and lens for assessing PMTs’ OMPs as well as sensitizing mathematics teacher educators to possibly simplified conceptions of mathematics pedagogy in the form of discovery and seeming simplification of task work through the use of task templates.

My work thus far lays the foundation to establish a relationship between orientations toward mathematical processes and mathematical processes opportunities afforded students via mathematical tasks. It also begins a research trajectory to understand how prospective secondary mathematics teachers learn to provide opportunities for students to engage in mathematical
activity. Long-term research plans include further studies into the development of OMP and experiments in developing OMP through mathematical and pedagogical activity.
References


Appendix A: OMP Questionnaire

Teaching and Learning Mathematics Questionnaire

The following questions address your views about mathematics and its teaching and learning along a continuum. The following example clarifies how to respond to each of the questions.

Example

I prefer to eat at restaurants with an atmosphere that is
(a) Loud and exciting, or
(b) Quiet and calm.

Circle option 1 to select alternative (a) exclusively.
Circle option 6 to select alternative (b) exclusively.
Circle options 2, 3, 4, or 5 to select a weighted combination of the two alternatives.

For example, circling option 2 would indicate that I like to eat at restaurants that are mostly loud and exciting, but not too loud and exciting. I would circle option 3 if I liked to eat at restaurants that are somewhat (instead of mostly) loud and exciting, but still more loud than quiet.

COMMENTS: A comment can be included in the comment box if desired.

My preferred atmosphere of a restaurant depends on with whom I am in the restaurant. I prefer a loud and exciting atmosphere when I am with a group of friends, but a quiet atmosphere when I want to have a conversation with a few people.
The first ten items are about how you think about mathematics and doing mathematics.

1. I believe that mathematics is
   (a) a dynamic field in which humans create and construct knowledge or
   (b) a fixed body of absolute facts independent from human invention.

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COMMENTS:

2. I consider proof to be
   (a) a step-by-step method to follow or replicate or
   (b) something I construct based on my understanding and knowledge of the topic.

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COMMENTS:

3. I believe definitions are
   (a) descriptions that I develop as I explore mathematical concepts or
   (b) descriptions of mathematical concepts I am given.

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COMMENTS:
4. In mathematics, representations are
   (a) useful tools for facilitating mathematical exploration or
   (b) primarily useful for sharing mathematical results.

5. Theorems and formulas are mathematical statements
   (a) developed by students and mathematicians through examining relationships or
   (b) developed by mathematicians and accepted and used by students.

6. In order to justify a mathematical statement I need to
   (a) have seen the justification of a similar statement before or
   (b) know how to apply types of reasoning or methods of proof.
7. When learning a new mathematical concept, I
   (a) locate statements related to the concept in a textbook or
   (b) examine examples and properties to create my own definition of the concept.

   ![Diagram]

   **COMMENTS:**

8. For me, solving mathematical problems involves
   (a) primarily using theorems and examples presented in class or
   (b) exploring ideas related to what was presented in class.

   ![Diagram]

   **COMMENTS:**

9. In a mathematics class, I prefer to have the instructor
   (a) provide several examples and then look for a pattern on my own or
   (b) state a formula and then provide me with supporting examples.

   ![Diagram]

   **COMMENTS:**
10. When solving a novel mathematical problem, I usually
(a) use only the given symbols, graphs, or table to find a solution or
(b) represent the problem in my way to find a solution.

11. In a high school mathematics class, the goal of the instruction is to
(a) transmit established mathematical facts, procedures, and concepts to students or
(b) help students to construct their own mathematical knowledge.
12. The primary purpose of teaching proof in a secondary mathematics classroom is to
(a) confirm the truth of mathematical results already known to be true or
(b) promote students’ understanding of why the mathematical results are true.

13. When introducing a new theorem or formula, I plan to
(a) allow students to generate several examples and form a conjecture or
(b) state the theorem or formula and then give examples.

14. When introducing a new definition, I plan to
(a) encourage students to determine which of the properties are redundant or
(b) make sure that all properties are listed so that the concept is accessible to all students.
15. When solving a mathematical problem with the class, I would
(a) create a diagram to help students understand the problem or
(b) have students create their own diagram.

[Diagram]

COMMENTS:

16. In a mathematics classroom, the role of a student is to
(a) absorb mathematical concepts and practice problems for accurate performance or
(b) participate in the learning activity to figure out and discuss solutions with others.

[Diagram]

COMMENTS:

17. In high school mathematics, I expect students to
(a) make conjectures, investigate them, and construct their own proofs or
(b) practice proving statements that have already been proven by an expert.

[Diagram]

COMMENTS:
18. Students can best learn a new rule, theorem, or formula if
   (a) students discover the new rule, theorem, or formula or
   (b) the rule, theorem, or formula is explained and applied to real-world contexts.

![Diagram]

COMMENTS:

19. In a high school mathematics class, I would have students
   (a) practice using representations like those they have been shown or
   (b) produce original representations different from those they have been shown.

![Diagram]

COMMENTS:

20. Students learn new mathematical definitions best when
   (a) students use examples and non-examples to develop an agreed upon classroom
tenition or
   (b) a definition is explained with all related properties and examples and non-examples
   are provided.

![Diagram]

COMMENTS:
Appendix B: OMP Interview Schedule

Italicized text indicates specific questions or instructions spoken by the interviewer.

In this interview, I am going to ask you to solve a couple of mathematics problems and then I am going to ask you to look at some other mathematics problems. I am asking you not to share or talk about these problems (or anything else related to the interview) with anyone else outside this interview. I am even asking you not to say anything to anyone else about how easy or hard the interview was for you.

Your name (or any other names, including the name of the school at which you are doing you field experience) will not be used when I report my results, and your instructors and mentors will not know your information. I will use a pseudonym so that no one (aside from myself and my advisor) will be able to identify you. I will be video recording this interview, but the recording will only be used for this dissertation and only what you are writing will be recorded (your face will not be video recorded).

Do you have any questions or concerns before we begin?

Throughout the interview, you will have access to the following tools that you may use if you desire. You might find some of these tools useful in solving the problems, but you do not need to use them.

- TI-84 Plus Graphing Calculator
- Spreadsheet Software (Excel)
- Dynamic Geometry Software (Geometer’s Sketchpad)
- Graph Paper
- Dot Paper
- Blank Paper
- Markers
- Rulers

Part 1: Solving Two Mathematical Tasks

Interviewer places Task A in front of participant.

This is the first problem. I have labeled the problem Task A. Please read the problem and then try to solve the problem. As you work on the problem, please think aloud so that I hear what you are doing. As you work, I will probably ask you what you are thinking and other questions about what you are doing.
Task A (Tri-square Rug Task)
A rug designer decided to make a rug consisting of three separate square pieces sewn together at the corners, with an empty triangle space between them. The rug was an immediate hit, and the designer decided to make more of them. He called these creations “tri-square rugs.” A sample tri-square rug is shown here. Al and Betty thought these tri-square rugs could be used to make a great game. They made up these rules. Let a dart fall randomly on the tri-square rug.
- If it hits the largest of the three squares, Al wins.
- If it hits either of the other two squares, Betty wins.
- If the dart misses the rug, simply let another dart fall.

Your goal in this activity is to decide which tri-square rugs you would prefer if you were Al and which you would prefer if you were Betty, and if there are any rugs that lead to a fair game.

(Fendel et al., 2003, p.232)

Possible Follow-Up Questions

The first few possible follow-up questions are responses to common questions and situations that occurred during preliminary trials of the OMP interview.

- If a participant struggles to begin, I ask questions that get the participant to interpret the task.
  - What is the task asking you to find?
  - How would you describe a tri-square rug to someone?
  - What needs to happen for Al to win? What needs to happen for Betty to win?
  - Are there different kinds of tri-square rugs?
- If a participant wants to explore the problem using a dynamic geometry environment, but struggles to create a dynamic geometry sketch, I will prompt the participant in the creation of the sketch. If it seems that a significant amount of time will be spent creating the sketch, I will have them use one that I had already created. I found that participants in my preliminary data collection struggled to create a dynamic sketch and spent a significant amount of time on the sketch, and the interview data gathered during this time was not useful to answering my research question.
• If a participant asks about whether equilateral and isosceles triangles apply, I will ask them whether they think those kinds of special triangles make sense in this scenario and why or why not.

The primary purpose of asking participants to work on this problem is to understand how they engage in mathematical processes. The following questions address that component of OMP.

• How would you describe the different game-winning scenarios? (Generalizing)
• How do you know? (Justifying)
• What are Pythagorean triples (or whatever name they use)? (Defining)
• Why did you set up this table (or write the following equation or some other representation)? Did it help you solve the problem? Why do you think so? (Representing)

The following questions address other components of OMP.

• If you were to use this task in your classroom, what kinds of answers would you be looking for from your students? (Conceptions of learning related to processes)
• Do you think students could solve this problem? Are there some students that could solve the problem? Which ones? (Conceptions of learning related to processes)

This task reappears in the Task Interview in which participants are asked to plan a unit on the Pythagorean theorem. I found it useful to begin talking about task work related to this task with the participant while the task is fresh on their minds. The following questions related to task work.

• What kinds of things do you like about this task? Are there things that you don’t like? (Selecting)
• Is this a task you would consider using in the classroom? (Selecting)
• What things about the task would you want to change before using it in the classroom? What things about the task would need to change in order for you to be willing to use it in the classroom? (Modifying)
• When do you think would be a good time to use this task? (Sequencing)
• What do you think of the whole game and rug context of the task? (Selecting)
Here is the second problem. I have labeled the problem Task B. Please read the problem and then try to solve the problem. As you work on the problem, please think aloud so that I hear what you are doing. I will probably ask you questions as you work on the problem. First, read the problem aloud. I will then give you further instructions regarding this problem.

The task involves cutting out 11 different shapes, but I have a set of these shapes that are already cut out and ready for you to use. The cut out shapes are similar to the shapes shown in Task B.

Task B (Pythagorean Puzzle)

A. Cut out the puzzle pieces shown below. Examine a triangular piece and the three square pieces. How do the side lengths of the squares compare to side lengths of the triangle?

![Pythagorean Puzzle Diagram]

B. Arrange the 11 puzzle pieces to fit exactly into the two puzzle frames. Use four triangles in each frame.

![Puzzle Frames]

C. Carefully study the arrangements in the two frames. What conclusion can you draw about the relationship among the areas of the three square puzzle pieces?

D. What does the conclusion you reached in part C mean in terms of the side lengths of the triangles?

(Lappan et al., 1998, p. 29)
Possible Follow-Up Questions

After participants complete the four questions given in the task, I will ask them questions related to their engagement in the processes.

- I know that the task does not ask you do this, but do you think it is possible to use these puzzles to explain why the Pythagorean theorem is true? How? (Justifying)
- Why did you arrange the puzzle pieces in the way that you did? (Representing)
- When did you notice the connection to the Pythagorean theorem? (Linking Representations of the Pythagorean theorem)
- You stated the Pythagorean theorem as _____. Why did you state it that way? What do you see as the important parts of that statement? (Stating Generalizations and Conceptions of Generalizations)

The following questions address other components of OMP.

- Is this a problem that you think students could do? Why or why not? (Conceptions of learning)
- Which parts do you think students could do and which parts do you think they might struggle to complete? (Conceptions of learning)
- How would you support their struggles with those parts? (Conceptions of teaching)
- What do you think students would learn from this task? (Conceptions of learning)
- What kinds of students do you think could do this problem? What kinds of students do you think couldn’t do this problem? (Conceptions of learning)

This task reappears in the Task Interview in which participants are asked to plan a unit on the Pythagorean theorem. I found it useful to begin talking about task work related to this task with the participant while the task is fresh on their minds. The following questions related to task work.

- Are there things about this task that you like? (Selecting)
- Are there things about this task that you do not like? (Selecting)
- Is this a task you could see yourself using in your own classroom? Why or why not? (Selecting)
- Are there things about this task that you would change before using it in the classroom? Why? (Modifying)
- Let’s say you had to use both of Task A and Task B in your classroom, exactly how they written now. Which task would you give students first? Why? (Sequencing)

Participants may have encountered a similar representation of the Pythagorean theorem, which may impact their engagement in the processes. I will ask them the following life history questions related to their mathematics preparation as related to the Pythagorean theorem.
• I noticed that you placed the puzzle pieces so that it looks like we have squares within squares. Have you seen either of these two arrangements before? If so, where? (Representing verses Representation Stating)
• Have you ever used these kinds of puzzles to prove the Pythagorean theorem? (Justifying verses Justification Stating)
• Have you taken or are you taking a geometry course as part of your teacher preparation? If so, which ones? What did you do in the course?
• What other math classes have you taken or are you taking?
• What about mathematics education classes?
• What have been some of your favorite classes? Least favorite classes? Why?
• Have you studied the Pythagorean theorem in any of these classes?
• If so, what do you remember doing in those classes?
• What do you remember about your experiences with the Pythagorean theorem in middle school and high school?

Part 2: Planning a 3-day Unit on the Pythagorean Theorem

You may have noticed that both Task A and Task B are related to the Pythagorean theorem. As a teacher, you will likely have an opportunity to teach a unit on the Pythagorean theorem.

During your pre-service teaching experiences, you have worked with a mentor or possibly a couple of mentors, and you probably had to follow their directions and preferences while teaching in their classrooms.

Let’s suppose that you now have your own high school classroom, and you are going to teach a 3-day unit on the Pythagorean theorem. The students are average tenth-grade students and you have access to any kind of teaching tool you want to use. What kinds of things would you like to have happen during those 3-days? What would you do and what would you have the students do?

Is there anything you would not include in the lessons?

The primary purpose of this part of the OMP interview is to gather data on participants’ conceptions of teaching and learning. Possible follow-up questions will focus on those components of OMP.

• Why would you want to teach the Pythagorean theorem by doing _____? (Conceptions of teaching)
• Why would you want your students to do ______? (Conceptions of learning)

Have you had a chance to teach the Pythagorean theorem in your student teacher or pre-student teaching experiences?
If they answer **YES**, ask,

- Would you mind describing some of the things that you and your students did during those lessons?
- What do you think went well during those lessons?
- Were there any things you would do differently next time or in your own classroom?
- Were you teaching middle school or high school?
- What other topics have you taught?
- Are there any things that you have done in the classroom that you thought work really well? Why? *(Conceptions of teaching)*
- Were there any things that you have done in the classroom that you didn’t think went as well? *(Conceptions of teaching)*

If they answer **NO**, ask,

- What kinds of lessons have you taught?
- Did you teach at a middle school or a high school?
- Are there any things that you have done in the classroom that you thought work really well? Why? *(Conceptions of teaching)*
- Were there any things that you have done in the classroom that you didn’t think went as well? *(Conceptions of teaching)*

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**Part 3: Teaching Scenarios Sorting Task**

*I am going to give you several cards with different lesson scenarios related to introducing the Pythagorean theorem described on the cards. Please read each card and then tell me how similar or different the scenario is from your idea of a unit on the Pythagorean theorem. You can place the card in one of four piles on the table in front of you labeled very similar, somewhat similar, somewhat different, or very different depending on how the scenario reflects the unit that you have in mind. By the way, the number on the card is only for convenience in talking about the scenario; the scenarios are in no particular order.*

The purpose of this part of the OMP interview is to collect data on the **components of OMP**. Each follow-up question was designed to address one of the components of OMP, as indicated in parentheses. For each scenario, I tried to ask one question related to each of the components. As participants reacted to each scenario, I noted which of components of OMP were addressed in their response. I asked additional follow-up questions so that all components of OMP (except engagement) were addressed for each scenario.

**Scenario 1**
To illustrate the Pythagorean theorem and the relationship among $a$, $b$, and $c$, you have your students create and label diagrams on graph paper or using dynamic geometry software.

**Follow-up Questions to Scenario 1**
What do you think it means to create a diagram illustrating the Pythagorean theorem? (Conceptions of representing)

- How is that different from or similar to having students create a diagram of a different mathematical concept? (Conceptions of representing)

How might your answer change if you, the teacher, were to give the students a diagram representing the Pythagorean theorem? (Conceptions of teaching/learning)

- Do you think that it is important to have the students create a diagram in this scenario? Why or why not? (Conceptions of learning)

How might you guide your students in creating a representation? (Conceptions of teaching)

Scenario 2
To show students that the Pythagorean theorem is true for all right triangles, you show students an applet that goes through each step of a proof with dynamic illustrations and ask them to describe what they see.

Follow-up Questions to Scenario 2

- Do you think that something like this qualifies as a proof? Why or why not? (Conceptions of justifying)

- What do you think would be important for students to do during this activity? (Conceptions of learning)

- In this scenario, the teacher/applet provides the basic steps of the proof; do you think students could help prove the theorem? (Conceptions of learning)

- What things might you do as the teacher to help the students prove the theorem? Why would you do these things? (Conceptions of teaching)

- What if instead of an applet, you gave them a note packet with the basic proof given in the packet and then had them use the packet to take additional notes as you talked about each step of the proof? (This question addresses whether or not the presence of technology is driving their reaction to the scenario.)

Scenario 3
You have students examine various examples and non-examples of Pythagorean triples and then ask them to complete the statement, “A Pythagorean triple is ______.”

Follow-up Questions to Scenario 3

- What kinds of examples and non-examples might you give to students?
  - What things would you want to highlight in the examples and non-examples?
  - Do you think that non-examples are important? (Conceptions of teaching)

- In this scenario, you are basically asking the students to come up with a definition after examining some examples and non-examples. What if the sequence of
events was changed so that you gave the students the definition first and then had them come up with examples? (Conceptions of learning)
  o What do you think a definition is in mathematics?
    ▪ What do you think the role of a definition is in secondary school mathematics? (Conceptions of defining)

Scenario 4
You ask the students, “How can we know that the Pythagorean theorem is true for all right triangles?” Using their ideas and further questioning, you and your students develop a proof of the Pythagorean theorem.

Follow-up Questions to Scenario 4
  o The description of the scenario is a little vague. How do you see the development of the proof playing out in your classroom? (Conceptions of teaching)
  o Do you think it is important for students to have a chance to prove the Pythagorean theorem? (Conceptions of learning)
    ▪ What kinds of guidance do you think students might need? (Conceptions of teaching and learning)
  o What if a student suggests that looking at several examples is a proof of the theorem? (Conceptions of justifying and conceptions of learning)

Scenario 5
You introduce the Pythagorean theorem to students by stating that for any right triangle the sum of the squares of the legs is equal to the square of the hypotenuse. You and your students then do several examples of how to use the theorem to solve problems.

Follow-up Questions to Scenario 5
  o You mentioned that the statement of the theorem was kind of wordy. How might you present the theorem to the students? (Conceptions of teaching)
  o Would you consider switching the order around? So, giving the examples first and then stating the Pythagorean theorem? (Conceptions of teaching and learning)

Scenario 6
You display a diagram that has sides of a right triangle labeled $a$, $b$, and $c$ and students explain how the diagram is connected to the Pythagorean theorem.

Follow-up Questions to Scenario 6
  o What kinds of diagrams do you think it is important for students to encounter as they learn about the Pythagorean theorem? Why would those diagrams be important?
• Do you think students could create any of those diagrams? (Conceptions of learning)
  o Would you consider asking students to create a diagram on their own? (Conceptions of learning)
  o If the participant mentions “multiple representations,” ask, why do you think it is important to have multiple representations? (Conceptions of teaching)

Scenario 7
Before talking about the Pythagorean theorem, you ask students to do the following, “Here are a bunch of different right triangles. Write a statement to describe a relationship among the sides of those triangles.”

Follow-up Questions to Scenario 7
  o Let’s say that as a teacher you have access to a computer lab with Geometer’s Sketchpad. And you give students a file, similar to the one that I have open on this computer, of a dynamic triangle with all side lengths and angle measures labeled. Could you fill in the details of the scenario using this GSP file?
  o Do you think the students could find a pattern leading to the Pythagorean theorem? (Conceptions of learning)
  o What kinds of things might you do as the teacher to help the students? (Conceptions of teaching)

Scenario 8
You state a definition of a Pythagorean triple as “a set of three natural numbers $a$, $b$, and $c$ that satisfy the relationship $a^2 + b^2 = c^2$.” You then ask the students, “Based on this definition, what are the properties of a Pythagorean triple?”

Follow-up Questions to Scenario 8
  o What do you think it means by “properties” in this scenario? How are properties different from and similar to definitions and theorems?
    • Is a definition different from a theorem? (Conceptions of definitions and generalizations)
  o Do you think students could identify those properties on their own if you gave them several examples of Pythagorean triples first before stating the definition? (Conceptions of learning)
  o What things would you do as the teacher to help the students understand the definition of Pythagorean triple? (Conceptions of teaching)

In my preliminary trials of this interview, I gained useful data by asking participants to compare the two scenarios related to the same process/product. The following questions ask participants to choose between two contrasting scenarios: one in which students engage in a process and one in which students act on a product.
• Scenario 2 and Scenario 4 both mention proofs of the Pythagorean theorem. Which of these do you think is more similar to the way you would want your classroom to be? (Justifying)

• Scenario 6 and Scenario 1 both mention diagrams. How do you see these scenarios as differing? Are those differences important to you? Which do you think is more similar to the way you want your classroom to be? (Representing)

• Take a look at Scenario 7 and Scenario 5. What do you see as similar and different between the two? Which of these do you think is more similar to the way you want your classroom to be? (Generalizing)

• Scenario 8 and Scenario 3 both mention Pythagorean triples. Which of these scenarios do you think is more similar to the way you want your classroom to be? (Defining)
Appendix C: Pythagorean Task Interview Schedule

As a teacher, you will likely have an opportunity to teach a unit on the Pythagorean theorem. You might find problems to include in your unit by looking in your textbook or a different textbook, from a colleague, or on the Internet. You will need to decide which tasks you might use and in which order. I am going to give you 12 tasks that are related to the Pythagorean theorem. These tasks come from a variety of sources and have all been used in different secondary mathematics classrooms. You might recognize two of the tasks from our last interview.

I would like you to pretend that you are teaching a 3-day unit on the Pythagorean theorem in a secondary mathematics classroom and these are some of the tasks that you could include in your unit. Take a look at each of these tasks and tell me which tasks you would use and why you would use the tasks.

Follow-up Questions
The purpose of this interview is to gather data regarding participants’ task work: selecting, modifying, and sequencing. The following questions are organized by task work.

Selecting
• What is it that you like/dislike about that particular task? Why is that?
• What if task asked students to use paper-and-pencil rather than a computer? Would you be more inclined to use the task then? Why or why not? (Alter a task feature to see if that affects selection.)
• This set of tasks all deal with ____. Why did you select this particular task over the others? (Compare tasks with the same topic to determine how content affects selection.)
• I noticed that you worked to solve these tasks, but not these ones. Why is that? (Compare tasks based on participants’ attempts to solve to determine how engagement in tasks affects selection.)

Modifying
• How would you alter the task for use in your classroom?
• Why would you make those changes?
• Could you give me a specific example of how your proposed change might look?

Sequencing
• Now that you have selected some of the tasks, what order would you use them in the classroom?
• Which ones would you want to use on Day 1? Day 2? Day 3?
• What is it about that order that you like?
Task A: Tri-square Rug

A rug designer decided to make a rug consisting of three separate square pieces sewn together at the corners, with an empty triangle space between them. The rug was an immediate hit, and the designer decided to make more of them. He called these creations “tri-square rugs.” A sample tri-square rug is shown here. Al and Betty thought these tri-square rugs could be used to make a great game. They made up these rules. Let a dart fall randomly on the tri-square rug.

- If it hits the largest of the three squares, Al wins.
- If it hits either of the other two squares, Betty wins.
- If the dart misses the rug, simply let another dart fall.

Your goal in this activity is to decide which tri-square rugs you would prefer if you were Al and which you would prefer if you were Betty, and if there are any rugs that lead to a fair game.

(Fendel et al., 2003, p.232)
Task B: Pythagorean Puzzle

A. Cut out the puzzle pieces shown below. Examine a triangular piece and the three square pieces. How do the side lengths of the squares compare to side lengths of the triangle?

B. Arrange the 11 puzzle pieces to fit exactly into the two puzzle frames. Use four triangles in each frame.

C. Carefully study the arrangements in the two frames. What conclusion can you draw about the relationship among the areas of the three square puzzle pieces?

D. What does the conclusion you reached in part C mean in terms of the side lengths of the triangles?

(Lappan et al., 1998, p. 29)
Task C: Pythagorean Rugs

Al and Betty have a game. They began with a right triangle, which has legs of lengths $a$ and $b$ and a hypotenuse of length $c$. Then they made the two square rugs shown below. Each rug has sides of lengths $a + b$, and the triangles within each square are the same as the single right triangle shown at the right.

When it’s Al’s turn, a dart drops on the square rug on the left. If it hits the shaded area, he wins a point. When it’s Betty’s turn, the dart falls on the square rug on the right. If it hits the shaded area, she wins a point. Assume that the darts always hit the rugs, but that they always fall randomly within the rug. In other words, all points on a rug have the same chance of being hit.

3. Is this a fair game? That is, is the chance of that dart landing on the shaded area the same for the two rugs? Explain your answer.

4. How do the two rugs demonstrate that the Pythagorean theorem holds true in general?

(Fendel et al., 2003)
Task D: Generalizing via Dot Paper

1. For each row, draw a right triangle with the given side lengths on dot paper. Then, draw a square on each side of the triangle.

<table>
<thead>
<tr>
<th>Length of leg 1</th>
<th>Length of leg 2</th>
<th>Area of square on leg 1</th>
<th>Area of square on leg 2</th>
<th>Area of square on hypotenuse</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
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<td>2</td>
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<td></td>
<td></td>
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</tr>
<tr>
<td>1</td>
<td>3</td>
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<tr>
<td>2</td>
<td>3</td>
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<tr>
<td>3</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. For each triangle, find the areas of the squares on the legs and on the hypotenuse. Record your results.

3. Look for a pattern in the relationship among the areas of the three squares. Use the pattern you discover to make the conjecture about the relationship among the areas.

(Lappan et al., 1998, p. 28)
Task E: Triangle Garden

Leslie, a landscape architect, has made a design for her flowerbed for a very important client. The flowerbed will be in the shape of a triangle, with sides of lengths 13, 14, and 15 feet, as shown in the diagram. Leslie needs to know the area of the flowerbed so she can order the correct amount of fertilizer. Suppose you are Leslie’s assistant. In order to find the area, you need to find the length of the altitude (labeled \( h \)).

1. Find the length of the altitude. (Hint: Guess at ways that the side of length 14 might be split up into two parts, and try to figure out \( h \) from that.)

2. Explain how you know that the answer you got in question 1 is correct. In other words, how are you sure that you made the right guess?

3. While you’re thinking about the situation, you might as well calculate the area for Leslie.

(Fendel et al., 2003)
Task F: Square-in-square Proof

Over the centuries, many different people have proven the Pythagorean theorem in many different ways. In this exploration, you examine one of these proofs.

1. Create two squares as shown in the figure below. The four triangles are all congruent right triangles. The longest sides of the four right triangles create the smaller square.

2. Determine the area of each right triangle.
3. Determine the area of the larger square.
4. Determine the area of the smaller square.
5. Describe how to find the area of the smaller square using the areas of the right triangles and the area of the larger square.
6. Rewrite your formula from part e using the lengths \(a\), \(b\), and \(c\) shown in the figure.
Task G: Justifying using Converse

A student thought that the triangle at the right looked like a right triangle but wasn’t sure. Find the length of each side of this triangle, and use your answers to determine with certainty whether or not it is a right triangle. Explain your reasoning.

Task H: Examples and Non-examples

To build a pyramid, the ancient Egyptians needed a way to determine whether or not certain angles in the pyramid were right angles. They used triangles with particular side lengths to help them decide whether an angle was indeed a right angle. The table below lists side lengths for some useful triangles and some not useful triangles.

<table>
<thead>
<tr>
<th>Useful Triangles</th>
<th>Not Useful Triangles</th>
</tr>
</thead>
<tbody>
<tr>
<td>6, 8, 10</td>
<td>9, 16, 20</td>
</tr>
<tr>
<td>7, 24, 25</td>
<td>10, 20, 25</td>
</tr>
<tr>
<td>8, 15, 17</td>
<td>1, 1, √2</td>
</tr>
<tr>
<td>10, 24, 26</td>
<td>12, 22½, 25½</td>
</tr>
</tbody>
</table>

1. Write a definition of a useful triangle.
2. According to your definition, would a triangle with side lengths of 12, 16, and 20 be a useful triangle? What about a triangle with side lengths of 4.5, 6, and 7.5?
Task I: Verifying Converse

a. Write the Pythagorean theorem as a conditional statement. Write the converse of the Pythagorean theorem as a conditional statement.

b. Use dynamic geometry software to test the truth of the converse of the Pythagorean theorem by completing steps 1-5.

1. Construct $\triangle ABC$ and two lines perpendicular to the base $AB$, as shown in the figure below.

2. Create a table with headings like those shown below.

<table>
<thead>
<tr>
<th>$m\angle ACB$</th>
<th>$m\overline{AB}$</th>
<th>$m\overline{CA}$</th>
<th>$m\overline{CB}$</th>
<th>$(m\overline{AB})^2 - (m\overline{CA})^2 + (m\overline{CB})^2$</th>
</tr>
</thead>
</table>

3. Record the appropriate measurements for $\triangle ABC$ in the table.

4. Move point $C$ to another location between the two lines. Record the appropriate the measurements in the table.

5. Repeat step 4 for several other locations of $C$. Include some positions that make angle $C$ acute (less than $90^\circ$), some that make it a right angle (exactly $90^\circ$), and some that make it obtuse (greater than $90^\circ$ but less than $180^\circ$).

c. 1. When the value of $(AB)^2 - [(AC)^2 + (BC)^2]$ is positive, is angle $C$ acute, right, or obtuse?

2. When the value of $(AB)^2 - [(AC)^2 + (BC)^2]$ is zero, is angle $C$ acute, right, or obtuse?

3. When the value of $(AB)^2 - [(AC)^2 + (BC)^2]$ is negative, is angle $C$ acute, right, or obtuse?
Task J: Odd Pythagorean Triples

Theo’s teacher told him that Pythagorean triples could be found by picking a whole number greater than 1, called $m$, and then finding $m^2 - 1$, $2m$, and $m^2 + 1$. Theo calculated several Pythagorean Triples and created a table like the one shown below.

<table>
<thead>
<tr>
<th>Generating number</th>
<th>Pythagorean Triple</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td>$m^2 - 1$</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
</tr>
<tr>
<td>5</td>
<td>24</td>
</tr>
<tr>
<td>6</td>
<td>35</td>
</tr>
<tr>
<td>7</td>
<td>48</td>
</tr>
<tr>
<td>8</td>
<td>63</td>
</tr>
</tbody>
</table>

Theo notices that when $m$ is even he gets Pythagorean triples that are different from the Pythagorean triples he gets when $m$ is odd.

1. What name do you think Theo should give to the Pythagorean triples generated when $m$ is even?

2. Theo wants a definition for these Pythagorean triples that distinguishes them from the triples he gets when $m$ is odd. Write a definition that Theo could use.

3. Theo claims that all Pythagorean triples that are generated by an odd $m$ will be divisible by 2. Do you agree? Explain your reasoning.
**Task K: Baseball**

A baseball diamond is a square with 90 feet between first and second base. What is the distance from home plate to second base?

![Baseball Diamond Diagram]

**Task L: Identifying Pythagorean Triples**

Any three natural numbers that make the equation $a^2 + b^2 = c^2$ true are *Pythagorean triples*. Determine whether each set is a Pythagorean triple.

a. 3, 6, 9  
b. 3, 4, 5  
c. 5, 12, 13  
d. 7, 24, 25  
e. 10, 24, 26  
f. 8, 14, 16
Appendix D: Quadratics Task Interview Schedule

Part 1: Planning a Unit on Quadratic Relations

Have you had a chance to teach anything related to quadratic relations in your student teacher or pre-student teaching experiences?

If they answer YES, ask,

• Would you mind describing some of the things that you and your students did during those lessons?
• What do you think went well during those lessons?
• Were there any things you would do differently next time or in your own classroom?
• Were you teaching middle school or high school?
• Are there any things that you have done in the classroom that you thought work really well? Why?
• Were there any things that you have done in the classroom that you didn’t think went as well?

If they answer NO, ask,

• What kinds of lessons have you taught?
• Did you teach at a middle school or a high school?
• Are there any things that you have done in the classroom that you thought work really well? Why?
• Were there any things that you have done in the classroom that you didn’t think went as well?

Now, ask ALL participants the following:

During your pre-service teaching experiences, you have worked with a mentor or possibly a couple of mentors, and you probably had to follow their directions and preferences while teaching in their classrooms.

Let’s suppose that you now have your own high school classroom, and you are going to teach a unit on quadratic relations. The students are average eleventh-grade students in an Algebra 2 class and you have access to any kind of teaching tool you want to use.

What kinds of things would you like to have happen during the unit? What would you do and what would you have the students do?

• Is there anything you would not include in the lessons?

Part 2: Quadratic Tasks

As a teacher, you will likely have an opportunity to teach a unit on quadratic relations. You might find problems to include in your unit by looking in your textbook or a different textbook, from a colleague, or on the Internet. You will need to decide which tasks you might use and in
which order. I am going to give you tasks that are related to the quadratic relations. These tasks come from a variety of sources and have all been used in different secondary mathematics classrooms.

I would like you to pretend that you are teaching a unit on quadratic relations in an Algebra 2 class and these are some of the tasks that you could include in your unit. Take a look at each of these tasks and tell me which tasks you would use and why you would use the tasks.

Follow-up Questions
The purpose of this interview is to gather data regarding participants’ task work: selecting, modifying, and sequencing. The following questions are organized by task work.

Selecting
- What is it that you like/dislike about that particular task? Why is that?
- What if task asked students to use paper-and-pencil rather than a computer? Would you be more inclined to use the task then? Why or why not? (Alter a task feature to see if that affects selection.)
- This set of tasks all deal with ____. Why did you select this particular task over the others? (Compare tasks with the same topic to determine how content affects selection.)
- I noticed that you worked to solve these tasks, but not these ones. Why is that? (Compare tasks based on participants’ attempts to solve to determine how engagement in tasks affects selection.)

Modifying
- How would you alter the task for use in your classroom?
- Why would you make those changes?
- Could you give me a specific example of how your proposed change might look?

Sequencing
- Now that you have selected some of the tasks, what order would you use them in the classroom?
- What kinds of tasks would you want the students to do before this task? What kinds of tasks would you want to come after this task?
- What is it about that order that you like?
- Do you think you would use this task at the beginning of a lesson, at the end of a lesson, or somewhere in the middle?
- Suppose you wanted to use both of these tasks. Which order would you use them in and why?
- Are there any tasks that are not part of this set that you would want to include in your unit? What might those tasks look like? Where would you add them to your unit?
Part 3: Conceptions of Processes and Tasks

Sorting Tasks

Consider tasks P, R, T, and Z. How would you group these tasks and why would you group them that way?

How would Task S fit with your organization of the other four tasks? What about Task W? Task X?

Describing Researcher Sort

So I’ve had other teachers sort the tasks P, R, T, and Z in different ways. One of the groupings I’ve seen is putting Task P and Task T together and Task R and Task Z together. What do you think the reasoning behind this grouping might be?

What if I were to add more tasks to this grouping so that tasks P, T, and W are grouped together and tasks R, Z, and S together? Why do you think a teacher might group the tasks in this way?

Good Processes Tasks

What makes a task a good justifying task? Why?
What makes a task a good generalizing task? Why?
What makes a task a good defining task? Why?
What makes a task a good representing task? Why?

Using the tasks from our Pythagorean theorem interview and our quadratic interview, do you think any of these tasks make a good justifying task? Generalizing task? Defining task? Representing task?
Task P: Exploring a with Defining

Proposed Definition:
A quadratic function in $x$ is a function in the form $f(x) = ax^2 + bx + c$, where $a$, $b$, and $c$ are real numbers.

1. Using an online graphing calculator, create a graph of a quadratic function $ax^2 + bx + c$ with sliders $a$, $b$, and $c$. Keeping $b$ and $c$ constant, vary the values of $a$.
   a. How would you describe the effects of $a$ on the graphs of quadratic functions? Make sure you address different cases of $a$ in your explanation.

   b. What happens to the graph of a quadratic function when $a = 0$? Why does this make sense?

   c. Based on your findings in Parts a and b, do you think that the proposed definition contains all necessary conditions of a quadratic function? Explain.

2. Tommy described the effects of $a$ on the graphs of quadratic functions as follows: *as you increase the value of $a$, the parabola gets wider*. But, what does “wider” mean? Working together, create a group definition for the width of a parabola.
Task Q: Quartic Parabola

1. Hannah claims that the graph of the function \( g(x) = x^4 \) is a parabola. Do you agree or disagree with this claim?

2. Hannah provides the following justification for why the graph of \( g \) is a parabola. Do you think that her justification is accurate and convincing?

   We can rewrite \( g(x) = x^4 \) as \( g(x) = (x^2)^2 \), which is a quadratic function in \( x^2 \). We know that the graph of a quadratic function is a parabola, so the graph of \( g \) must also be a parabola. Also, the graph of \( g \) has a “U” shape, just like a parabola. Therefore, the graph of \( g(x) = x^4 \) is a parabola.
**Task R: Exploring c in Intercepts**

A general quadratic function has the form \( f(x) = ax^2 + bx + c \). In this investigation, you will explore how different values of \( c \) affect the graph of a quadratic function.

1. Graph the following quadratic functions using a graphing calculator. Record the \( x \)- and \( y \)-intercepts of the graph of each function.
   a. \( f(x) = x^2 \)
      \( x \)-intercepts: _______________ \( y \)-intercepts: _______________
   b. \( f(x) = x^2 - 9 \)
      \( x \)-intercepts: _______________ \( y \)-intercepts: _______________
   c. \( f(x) = x^2 + 9 \)
      \( x \)-intercepts: _______________ \( y \)-intercepts: _______________
   d. \( f(x) = x^2 + 5 \)
      \( x \)-intercepts: _______________ \( y \)-intercepts: _______________
   e. \( f(x) = x^2 - 5 \)
      \( x \)-intercepts: _______________ \( y \)-intercepts: _______________

2. Use your observations for Part 1 to answer the following questions about a quadratic function of the form \( f(x) = x^2 + c \).
   a. What do you know about the \( x \)-intercepts of \( f \) when \( c \) is a positive number?
   b. What do you know about the \( x \)-intercepts of \( f \) when \( c \) is a negative number?
   c. What do you know about the \( y \)-intercepts of \( f \) when \( c \) is a positive number?
   d. What do you know about the \( y \)-intercepts of \( f \) when \( c \) is a negative number?
Task S: Intercepts as Solutions

a. Find the $x$-intercepts of each graph.

\[ y = 2x - 3 \]

\[ y = x^2 + 3x - 4 \]

b. Solve $2x - 3 = 0$. Is the solution of $2x - 3 = 0$ the same as the $x$-intercept of $y = 2x - 3$?

c. Do the $x$-intercepts that you found in Question 1b satisfy the equation $x^2 + 3x + 4 = 0$?

d. Graph $y = x^2 + x - 6$ and find the $x$-intercepts of the graph. Do the values of the $x$-intercepts of $y = x^2 + x - 6$ satisfy the equation $x^2 + x - 6 = 0$?

(Bellman, A. E. et al., 2007, p.565)
**Task T: Unfactorable Algebra Tiles**

Consider the collection of squares and non-squares shown here.

![Algebra Tiles Image]

The four non-squares are congruent two each other as are the five small squares.

- **a.** What expression represents the total area of the ten figures?

- **b.** Explain why the given shapes cannot be arranged to form a larger square without overlaps or gaps.

- **c.** What is the minimal addition or subtraction of unit squares that will make it possible to arrange the new set of pieces to form a larger square?

- **d.** Write an expression involving a binomial square to represent the area of the larger square formed by all the given pieces and the unit squares added or subtracted in Part c.

(Fey et al., 2009, p. 357)

**Specific Follow-up Questions:**
- Task Z and Task T both include algebra tiles. Let’s say you wanted to use one of them. Which do you like better and why?
Task U: Factoring with Justifying

If a soccer player kicks the ball from a spot on the ground with initial upward velocity of 24 feet per second, the height of the ball \( h \) (in feet) at any time \( t \) seconds after the kick will be approximated by the quadratic function \( h = 24t - 16t^2 \). Finding the time when the ball hits the ground again requires solving the equation \( 24t - 16t^2 = 0 \).

a. Check the reasoning in this proposed solution of the equation.
   i. The expression \( 24t - 16t^2 \) is equivalent to \( 8t(3 - 2t) \). Why?
   ii. The expression \( 8t(3 - 2t) \) will equal 0 when \( t = 0 \) and when \( (3 - 2t) = 0 \). Why?
   iii. So, the solutions of the equation \( 24t - 16t^2 = 0 \) will be 0 and 1.5. Why?

b. Adapt the reasoning in Part a to solve these quadratic equations.
   i. \( 0 = x^2 + 4x \)
   ii. \( 0 = 3x^2 + 10x \)
   iii. \( 0 = x^2 - 4x \)
   iv. \( -x^2 - 5x = 0 \)
   v. \( -2x^2 - 6x = 0 \)
   vi. \( x^2 + 5x = 6 \)

(Hirsh et al., 2009, p. 512)

Specific Follow-up Questions:
- The equation \( x^2 + 5x = 6 \) seems to be different from the other equations in the group. Do you agree? Why or why not?
- Would you want to keep that equation as part of the task or remove it from the task?
- What kinds of things would you want the students to learn or do before you give them this task?
Task V: Novel Set of Quadratics

Compare the roots of these three equations. What do you notice?

\[ 4x^2 - 17x + 4 = 0 \quad 6x^2 - 37x + 6 = 0 \quad 8x^2 - 65x + 8 = 0 \]

What could you call these equations? Write a definition to include all equations of this type.

Add another equation that acts the same way.

(adapted from Small & Lin, 2010, p.32)

Task W: Generalizing Solutions

Use what you know about solving quadratic equations and the graphs of quadratic functions to answer these questions.

a. What choice of values for \( a \) and \( d \) will give equations in the form \( ax^2 = d \) that have two solutions? Only one solution? No solutions? Explain the reasoning behind your answers and illustrate that reasoning with sketches of graphs for the related function \( y = ax^2 \).

b. What choices of values for \( a \), \( c \), and \( d \) will give equations in the form \( ax^2 + c = d \) that have two solutions? Only one solution? No solutions? Explain the reasoning behind your answers and illustrate that reasoning with sketches of graphs for the related function \( y = ax^2 + c \).

c. Why must every equation in the form \( ax^2 + bx = 0 \) (neither \( a \) nor \( b \) zero) have exactly two solutions? Explain the reasoning behind your answers and illustrate that reasoning with sketches of graphs for the related function \( y = ax^2 + bx \).

(Hirsh et al., 2009, p. 513)
Task X: Factors & Intercepts

1. Graph the function \( f(x) = (x + 2)(x - 3) \) on a graphing calculator and estimate its \( x \)-intercepts. Substitute the \( x \)-values you found into the function to see if they really give a result of zero.

2. Repeat the steps from Question 1 for the function \( g(x) = (x - 5)(x + 7) \).

3. Repeat the steps from Question 1 for the function \( k(x) = (x +9)(2x - 7) \).

4. Examine your results from Questions 1 through 3.
   a. Find a method for identifying the intercepts without graphing, directly from the expressions that define the functions.
   b. Use the zero property to explain your method.

(Fendel & Resek, 1999, p.19-20)
Task Y: Vertex Application

An archer shoots an arrow straight upward at 64 feet per second. The height of the arrow \( h(t) \) (in feet) at time \( t \) seconds is given by the function
\[ h(t) = -16t^2 + 64t. \]

a) Use the accompanying graph to estimate the amount of time that the arrow is in the air.

b) Symbolically find the amount of time that the arrow is in the air.

c) Use the accompanying graph to estimate the maximum height reached by the arrow.

d) At what time does the arrow reach its maximum height?

(Dogopolski, 2006, p.368)

Specific Follow-up Questions:
- When would you want to use this task in the unit? What kinds of things would you want students to learn before giving them this task?
Task Z: Completing the Algebra Tile Square

Algebra tiles can be used to solve quadratic equations. Use algebra tiles to find the solution to the equation below.

\[ x^2 + 10x = 39 \]

a. Begin with the terms on the left side of the equation. We can arrange one \( x^2 \) tile and 10 \( x \) tiles as shown to the right.

b. Rearrange the \( x \) tiles by putting 5 \( x \) tiles to the right of the \( x^2 \) tile and 5 \( x \) tiles below the \( x^2 \) tile. The new arrangement should look like the diagram to the right.

c. The next step is to complete the square. How might you complete the square on the diagram to the right?

d. What kinds of tiles and how many tiles would you need to complete the square?

e. Algebraically, we have changed our quadratic equation. What is the new equation? Fill in the blanks.

\[ x^2 + 10x + \underline{\hspace{2cm}} = 39 + \underline{\hspace{2cm}} \]

f. Notice that the left side of the equation is a perfect square quadratic. Factor this quadratic.

g. Now solve for \( x \) by taking the square root of both sides of the equation. What are the solutions to the equation? *Don’t forget positive and negative square roots.*

h. Repeat this process to solve the equation \( x^2 + 6x = 40 \)

Specific Follow-up Questions:
- Would you consider removing some of the steps in the task? Which steps do you think are important to keep in the task?
VITA

Tenille Cannon

Education
M.A. Mathematics Education, Brigham Young University, August, 2008
B.A. Mathematics Education, Brigham Young University, August, 2003

Selected Academic and Professional Experience
Mathematics Teacher, State College Area High School, 2013-present
Mathematics Teacher, Payson Junior High School, 2003-2006

Journal Article

Selected Presentations

Fellowships and Awards
Mid-Atlantic Center for Mathematics Teaching and Learning Fellowship
Donald B. and Mary Louise Elder Tait Scholarship in Mathematics Education