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PREFERENCE SIGNALING IN MATCHING MARKETS

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Abstract

This dissertation examines a natural signaling mechanism in a two-sided matching market between firms and workers. We consider a basic game of incomplete information. Each agent knows her own preferences over matches, but uncertain about other agent preferences. Each worker can send a limited number of private costless signals to firms indicating her interest in positions there; workers send signals simultaneously. Then, each firm makes an offer to a fixed number of workers; firms make offers simultaneously. Finally, workers can choose a fixed number of offers from those available to them. We study the impact of the signaling mechanism for different environments that vary in market size, agent preference distribution, the number of signals, firm and worker positions, and periods of interaction.

Chapter 1 discusses the motivation for our analysis. It describes the necessity for a signaling mechanism in practice and instances of real markets that already use some form of signaling mechanisms.

Chapter 2 analyzes the basic model where each firm can hire at most one worker, each worker can be matched with at most one firm, and each worker can send at most one signal. We analyze the environment where agent preferences are quite dispersed. Specifically, workers have idiosyncratic (and uniformly distributed) preferences over firms. Firms have also idiosyncratic (and uniformly

distributed) preferences over workers. We prove the existence of symmetric equilibrium in pure strategies as well as that multiple symmetric equilibria, with varying responsiveness to signals, may exist. These equilibria can be welfare ranked: workers prefer equilibria where firms respond more to signals, while firms prefer the equilibria where they respond less. Finally, we show that, on average, introducing a signaling mechanism increases both the expected number of matches as well as the expected welfare of workers for this environment. The welfare of firms, on the other hand, changes ambiguously.

Chapter 3 presents extensions to our main model. First, we show that the welfare results of the basic model carry over to a setting in which workers have correlated preferences. Second, the effects of a signaling mechanism persists if we consider another extension where each firm can hire several workers, each worker can be matched to several firms, and workers can send several signals. Finally, we analyze a simpler environment where agents only care about getting a match, but not the quality of the match, defining the value of a signaling mechanism as the expected increase in the number of matches from the introduction of a signaling mechanism. For such an environment, the value of a signaling mechanism is maximal for markets markets where the number of firms and workers are of roughly the same magnitude. Furthermore, additional periods of interaction between firms and workers decrease the impact of signaling. Finally, the optimal number of signals—the number of signals that maximizes the expected increase in the number of matches—increases when workers can be matched to more firms.

Chapter 4 analyzes a version of our basic model where agent preferences are tightly distributed. Workers have almost aligned preferences over firms: each worker has "typical" commonly known preferences with probability close to one and "atypical" idiosyncratic preferences with the complementary probability close to zero. Firms have some commonly known preferences over workers that may vary across firms. Though signals transmit previously unavailable information, they also facilitate information asymmetry in this environment. Prior to the signaling, all firms have identical beliefs about worker preferences. However, after the signals are received they may have diverse beliefs. This disparity in beliefs leads to coordination failure. As a result, the introduction of a signaling mechanism may decrease the total number of matches and the welfare of agents.

Finally, Chapter 5 summarizes our findings of previous chapters and analyzes the roles of preference signaling in matching markets. Signals play two important roles in match formation: they transmit information and they facilitate information asymmetry. When there is only a small amount of information about agent preferences available, as in Chapter 2 and 3, information transmission plays a more important role in match formation. However, when there is almost complete information about agent preferences, as in Chapter 4, the introduction of signaling may lead to coordination failure.

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Preface

The original idea regarding the basic model of preference signaling in matching markets initially belongs to Coles et al. (2007). I have been able to relax some of their assumptions and extend the analysis to more general models. These developments led to Kushnir (2009). The current version of Chapter 2 is the main part of the joint paper Coles et al. (2010b) that is the synthesis of two papers mentioned above. Sections3.1 and Subsections 3.4.1 and 3.4.2 are also included in Coles et al. (2010b).

Although most of the text written by a single author, except the portions mentioned above, the dissertation uses the plural "we" rather than the singular "I." This is not to indicate any royal lineage, but only to make the exposition more conversational.

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Dedication

To Katya and my parents.

Chapter 1

Introduction

Labor markets often have the feature that job seekers apply for many positions, as the low cost of application relative to the high value of being employed induces job seekers to perform broad searches. For example, each year in the job market for new economists, roughly 1000 doctoral students graduating from U.S. schools compete together with other applicants for the around 3000 jobs posted on Job Opening for Economists, and send an average 80 applications (Coles et al., 2010a). Many employers, consequently, face the task of evaluating hundreds of applications, which makes it almost impossible to review them carefully. This means that employers may not only need to assess the quality of an applicant, but also whether the applicant is worth pursuing; that is, whether the applicant would in fact be likely to accept a job offer. In this paper we study a mechanism that allows applicants to signal information about their preferences for positions.

In practice, many markets have the feature that applicants communicate special interest for a select number of places. For example, in college admissions in the United States, many universities have early admission programs, where high school seniors may apply to exactly one college before the general application period (Avery et al., 2003).[1] In another example, in the market for entry-level clinical psychology jobs (Roth & Xing, 1997), a norm developed in which candidates would express to a single program the willingness to accept an offer, should one be made. On "match day," program directors called applicants to make offers and candidates, as per the match rules, were allowed to hold at most one offer during the day. It was common for program directors to make offers out of order to applicants who indicated they would accept an offer immediately.[2]

Some markets have formal mechanisms that allow participants to signal preferences. The American Economic Association (AEA) has operated a signaling service to facilitate the job search for economics graduate students since 2006. Students can send signals to up to two employers to indicate their interest in receiving an interview at the Allied Social Science Associations meeting. Coles et al. (2010a) provide suggestive evidence that sending a signal of interest increases the chances of receiving an interview. Similarly, some online dating websites allow participants to send signals to potential partners. For example in the matchmaking service of the website "Hot or Not," participants can send each other virtual flowers that purportedly increase the chances of receiving a positive response (in this case the number of flowers one may send is unlimited,

[1]Under single early action programs, applicants can apply to exactly one college early, which is often enforced by high school guidance counselors. It means that students may receive an offer early, without any commitment to enroll. Furthermore, there seems to be ample circumstantial evidence that it is somewhat easier to get into a college through early action programs (see Avery et al., 2003). Furthermore, many colleges take great care to note whether applicants visit the campus and show interest, which presumable is costly for parents, at least in terms of time, if not money, and once more seems to taken into account when deciding whom to admit. In a similar, though somewhat more restrictive way, Japanese college graduates indicate somewhat early whether they are interested in certain jobs (see Roth & Xing, 1994).

[2]As an example, the directors of one internship program decided to make their first offers (for their five positions) to numbers 1, 2, 3, 5, and 12 on their rank-order list of candidates, with the rationale that 3, 5, and 12 had indicated that they would accept immediately and that 1 and 2 were so attractive as to be worth taking chances on. Program directors also indicated that their general strategy was "don't tie up offers with people who will hold them all day." They worried having offers "hanging out there" and getting rejected just before the end of match day, and hence not being able to make offers to further candidates (see Roth & Xing, 1997).

but each flower is costly).[3] In a field experiment on a major Korean online dating website, Lee et al. (2009) study the effect of a user attaching one of a limited number of "virtual roses" to a date request. They find that users of both genders are more likely to accept a request when a virtual rose is attached, and that roses are most helpful for sends of "below average popularity."[4]

These examples all share several important features. First, in each case substantial frictions lead to market congestion: employers (or colleges and dating partners) are unable to give full attention to all possible candidates when making decisions. For example, since interviews in the economics job market take place over a single weekend, departments typically interview on average only about 20 candidates, which implies that most departments must strategically choose from among their candidates that are above the bar (Coles et al., 2010a).[5] Second, applicants are ready to provide information about their preferences over employers. Third, employers value this preference information and are prepared to act on it.

But in many markets, preference signaling bears almost no cost, and job seekers have an incentive to indicate particular interest to many employers, regardless of how strong their preferences towards these employers actually are. Employers, therefore, must discern which preference information is sincere and which is simply cheap talk. So while candidates may wish to signal their preferences, and employers may value learning candidate preferences, inability to credibly convey information may prevent any gains from preference signaling from being realized.

[3]See http://www.hotornot.com/m/?flowerBrochure=1.

[4]The dating website focuses largely on people who look for marriage partners, rather than people who want to have lots of dates. Dates may hence seem costly, so, deciding with whom to spend a date is something to be thought of carefully. Especially medium attractive candidates, who may worry that offers are only "safety" offers, seem to respond to signals of special interest.

[5]Similarly, colleges can't make offers to students one by one to see who accepts, and dating site users can only go on so many dates in a given period of time.

A standard interpretation of signaling and its effectiveness is that applicants have private information that is pertinent to how valuable an employee they would be. For example, in Spence's signaling model (Spence, 1973), applicants use wasteful costly signals (such as education) to signal their type (such as their ability).[6] More recently, Avery & Levin (2009) model early application in US college admissions as a way for students to signal college-specific quality, such as enthusiasm for a particular college. In their model, colleges explicitly derive more utility from having such enthusiastic students in their freshman class than they do from other, equally able students. In our model we abstract away from such motives, and instead show how congestion, stemming from the costs of making offers, can generate room for useful signaling.

A more closely related strand of literature is that of strategic information transmission ('cheap talk') between a sender and receiver, introduced in Crawford & Sobel (1982). In our model, however, we consider a multi-stage game with many senders (workers) and many receivers (firms), where the structure of allowable signals plays a distinctive role. Senders must choose the receivers to whom they will send one of their limited, identical signals, and the scarcity of signals induces credibility. Each receiver knows only whether a sender has sent a signal to it or not, and receives no additional information. Nevertheless, some features of Crawford & Sobel persist in our model. Signals are "cheap" in the sense that they do not have a direct influence on agent payoffs. Each agent has only a limited number of signals; so there is an opportunity cost associated with sending a signal. Finally, in our model there always exist babbling equilibria where agents ignore signals; hence, the introduction of a signaling mechanism always enlarges the set of equilibria.

[6] Hoppe et al. (2009) extend this idea to an environment where agents on both sides of the market may send signals. Among other findings, they identify general conditions under which the potential increase in expected output due to the introduction of signaling is offset by the costs of signaling.

While we are the first to introduce preference signaling in decentralized markets, two papers deal with preference signaling in the presence of centralized clearinghouses. Abdulkadiroglu et al. (2008) show that the introduction of a signaling technology improves efficiency of the deferred acceptance algorithm (Gale & Shapley, 1962) in case of weak preferences. Lee & Schwarz (2007) analyze preference signaling in a match formation process between firms and workers that consists of three steps: preference signaling, investments in information acquisition, and formation of matches via centralized clearinghouse. They construct a centralized mechanism where workers communicate their complete preferences to an intermediary, and the intermediary recommends to each firm a subset of workers to interview.

Chapter 2

Preference Signaling in Matching Markets

In this chapter, we investigate how a signaling mechanism that limits the number of signals a job seeker may send can overcome the credibility problem and improve the welfare of market participants. We develop a model that can account for the three stylized facts mentioned Chapter 1. In our model, firms make offers to workers, but the number of offers they may make is limited, so that firms must carefully select the workers to whom they make offers. To focus on the strategic question of offer choice (and abstract away the question of acquiring information that determines preferences), we assume that each agent knows her own preferences over agents on the other side of the market, but is uncertain of the preferences of other agents.

In the basic version of our model, we assume that both worker and firm preferences are idiosyncratic and uniformly distributed. Workers have the opportunity to send a signal to one firm before firms make offers. Each signal has a binary nature: either a firm receives a signal or not. Each signal does not transmit any other information. Firms observe their signals (but do not observe the signals to other firms) and make offers to workers simultaneously. Finally, workers choose offers from those available to them.

We show that, in expectation, introducing a signaling mechanism increases both the number of matches as well as the welfare of workers. Intuitively, when firms make offers to workers who send them signals, these offers are unlikely to overlap, leading to a higher expected number of matches. Furthermore, workers are not only more likely to be matched, but are also more likely to be matched to a firm they prefer the most. On the other hand, when a firm makes an offer to a worker who has signaled it, this creates strong competition for firms who would like to make an offer to that same worker (because they might rank that worker highest). Hence, by responding to signals (making more offers to workers who have signaled them), firms may generate a negative spillover on other firms. Consequently, the effect on firm welfare from introducing a signaling mechanism is ambiguous; welfare for a firm depends on the balance between individual benefit from responding to signals and the negative spillover generated by other firms responding to signals. How much a firm responds to signals is a case of strategic complements. When one firm responds more to signals, it becomes riskier for other firms to make offers to workers who have not sent them signals. Consequently, multiple equilibria, with varying responsiveness to signals, may exist. Furthermore, these equilibria can be welfare ranked: workers prefer equilibria where firms respond more to signals, while firms prefer the equilibria where they respond less.

Finally, we compare the performance of signaling mechanisms across market settings. To do this, we focus on a simpler environment where agents care about getting a match, but not the quality of the match, defining the value of a signaling mechanism as the expected increase in the number of matches from the introduction of a signaling mechanism. For such an environment, the value of a signaling mechanism is maximal for balanced markets; that is markets where the number of firms and workers are of roughly the same magnitude.

The chapter proceeds as follows. Section 2.1 begins with a simple example, and Sections 2.2 and 2.3 discuss the offer game with and without a signaling mechanism, respectively. Section 2.3.2 considers the impact of a signaling mechanism on the welfare of agents. Section 3.4 analyzes the robustness of the welfare results across markets with various sizes. Section 2.4 concludes.

2.1 A Simple Example

In this section we lay out a simple example that shows the effects of introducing a signaling mechanism and highlights some of our main findings. Consider a market with two firms $\{f_1, f_2\}$ and two workers $\{w_1, w_2\}$. For each agent, a match with one's most preferred partner from the other side of the market yields payoff 1, while a match with one's second choice partner yields $x \in (0, 1)$. Remaining unmatched yields payoff 0.

Ex-ante, agent preferences are random, uniform and independent. That is, for each firm f, the probability that f prefers worker w_1 to worker w_2 is one half, as is the probability that f prefers w_2 to w_1. Worker preferences over firms are similarly symmetric. Agents learn their own preferences, but not the preferences of other agents.

We first examine behavior in a game where firms make offers to workers without receiving any additional information. Once agent preferences are realized, each firm may make a single offer to a worker. Workers then accept at most one of their available offers.

In the unique symmetric equilibrium of this game where firm strategies do not depend on the name of the worker, each firm simply makes an offer to its most preferred worker.[1] This is because firms cannot discern which worker is more likely to accept an offer. Note that in this congested market there is a fifty percent chance that both firms make an offer to the same worker, that is that there will only be one match. Hence, on average there are 1.5 matches, and the average expected payoff of a firm is 0.75. For workers, if they receive one offer, they have an equal chance to receive their first or second choice firm. There is also a fifty percent chance that one worker receives two offers, hence receives a payoff of one, while the other worker receives zero. Hence, the expected payoff of workers is: $(2 + x)/4$.

We now introduce a signaling mechanism: before firms make offers, each worker may send a *signal* to a single firm. We focus on non-babbling equilibria where firms interpret a signal as a sign of being the most preferred firm of that worker and workers send a signal to their most preferred firm. Note that there is no equilibrium where firms expect signals from workers, but interpret them as a lack of interest (and hence reduce the probability to make an offer to this worker). This is because workers then simply would not send any signal, an action that can't be observed by any individual firm. There are, however, babbling equilibria where no information is transmitted, though we will not focus on those in this paper, as they are equivalent to not having a signaling device.[2]

To analyze firm behavior, note that a firm that receives a signal from its top worker, makes this worker an offer, since it will be accepted. If on the other hand a firm receives no signals, it again optimally makes an offer to its top worker, as symmetry implies that workers are equally likely to accept an offer. The only interesting strategic decision a firm must make is when it receives a signal only from its second ranked worker. In this case the other firm also received exactly

[1]We consider sequential equilibrium as a solution concept.

[2]Babbling equilibria can take two forms. Either workers randomize over whom to send a signal, and firms ignore signals, or workers do not send a signal, and firms interpret any out of equilibrium signal as a sign of lack of interest. The latter equilibrium does not survive standard equilibrium refinements.

one signal. We say a firm “responds” to signals if it makes the signaling worker an offer, and “ignores” the signal if it instead makes an offer to its top worker, which did not send it a signal.

Suppose f_1 prefers w_1 to w_2 and only w_2 sent a signal to f_1, which implies w_1 sent a signal to f_2. Clearly, whenever f_1 makes an offer to w_2, f_1 receives x. Suppose f_1 instead makes an offer to w_1 who sent a signal to f_2. If f_2 responds to signals, then f_2 makes w_1 an offer, which w_1 will accept, hence f_1 receives a payoff of 0. If f_2 ignores signals, then there is still a fifty percent chance that w_1 is actually f_2's first choice, in which case an offer is tendered and accepted, and f_1, receives 0, otherwise, f_1 receives 1. Table 1 summarizes f_1's payoffs conditional on receiving a signal from its second ranked worker, and the strategies of f_2.

$f_1 \setminus f_2$	**Respond**	**Ignore**
Respond	x	x
Ignore	0	1/2

Table 2.1: A simple example. Firm f_1's payoffs conditional on receiving a signal from its second ranked worker.

Table 2.1 shows that strategies of firms are strategic complements. If a firm responds more to signals, then the other firm is weakly better off from responding more to signals as well. In this example, if f_2 switches from the action ignore (not making an offer to) a second choice worker who has signaled to the safe action of responding (making an offer) to a second choice worker who has signaled, then f_1 optimally also takes the safe action of responding.

Turning to equilibrium analysis, note that if $x > 0.5$ there is a unique equilibrium in which both firms respond to signals. When $x < 0.5$, that is when the value of the first choice worker is much greater than of the second ranked worker there exist two equilibria in pure strategies. In the first, both firms respond to signals (Respond-Respond) and in the second both firm ignore signals (Ignore-Ignore).[3] Table 2.2 summarizes describes properties of these equilibria. The expected firm and worker payoff, as well as the number of matches when signals are ignored are the same as when there is no signaling mechanism.[4]

	Firm Profits	Worker Profits	Number of Matches
Respond-Respond	$(5+2x)/8$	$3/4$	$7/4$
Ignore-Ignore	$3/4$	$(2+x)/4$	$3/2$

Table 2.2: A simple example. Firm payoffs, worker payoffs, and number of matches in symmetric equilibrium

Whenever there are multiple equilibria ($x < 0.5$), we can rank them in terms of firm welfare, worker welfare, and the expected number of matches. Workers and firms are opposed in their preferences over equilibria: workers prefer the equilibrium in which both firms respond to signals while firms prefer the equilibrium in which they both ignore signals. Intuitively, while one firm may privately gain from responding to a signal, such an action may negatively affect the other firms. The expected number of matches in the equilibrium when both firms respond to signals is always greater than in the equilibrium when both firms ignore the signals.

These welfare results enable us to study the effects of introducing a signaling

[3]There is also a mixed strategy equilibrium whenever there are two pure strategy equilibria. However, properties of this equilibrium coincide with those in the equilibrium where both firms respond to signals.

[4]When both firms respond to signals, note that each firm has a fifty percent chance to receive a signal from its first choice worker, and hence receiving one. Otherwise a firm has a $1/4$ chance to receive a signal from its second choice worker only, yielding a payoff of x. With a $1/4$ chance the firm receives no signal, in which case it makes an offer to its first choice worker, who will accept with fifty percent chance (whenever she is not the first choice worker of the other firm). Furthermore, when one firm receives all signals (which happens half the time) there is a fifty percent chance of firms making offers to the same worker, hence, of only one match occurring. Payoffs for workers can be calculated given those results.

mechanism, as outcomes in the offer game without signals are identical to those when both firms ignore signals (even if the Ignore-Ignore equilibrium does not exist). The expected number of matches and the welfare of workers in the offer game with signals in any non-babbling equilibrium is greater than in the offer game with no signals. The welfare of firms changes ambiguously with the introduction of a signaling mechanism. Note that when $x > 0.5$ the equilibrium when both firms ignore the signals does not exist and the welfare of firms in the offer game with signals is greater than in the offer game with no signals.

2.2 The Offer Game with No Signals

2.2.1 General Notation

In this paper we aim for a simple hiring model in which we can highlight the role of agents being able to credibly signal preferences in the presence of congestion. We have a market with a set of firms, a set of workers, and a distribution over firm and worker preferences. Each firm has the capacity to hire at most one worker, and each worker can fill at most one position.[5] We examine an extreme form of congestion: Each firm can only make exactly one offer to hire a worker. In the offer game with no signals, firms make an offer based on limited knowledge of worker preferences. In the second setting, the offer game with signals, before offers are made, each worker has the opportunity to send a costless signal to a firm, who may use this signal to infer worker preferences.

Let $\mathcal{F} = \{f_1, \ldots, f_F\}$ be the set of firms, and $\mathcal{W} = \{w_1, \ldots, w_W\}$ be the set of workers, with $|\mathcal{F}| = F$ and $|\mathcal{W}| = W$. We consider markets with at least two

[5]In Chapter 3 we consider a setup in which each worker may occupy more than one position and can send more than one signal.

firms and two workers. Firms and workers have preferences over each other. For each firm f, let Θ_f be the set of all possible preference lists over workers, where $\theta_f \in \Theta_f$ is a vector of length W. We use the convention that the worker of rank one is the most preferred worker, while the worker of rank W is the least preferred worker. The set of all firm preference profiles is $\Theta_F = (\Theta_f)^F$. Similarly, we define θ_w, Θ_w and Θ_W for workers. Let $\Theta \equiv \Theta_F \times \Theta_W$, and $t(\cdot)$ be the distribution over preference list profiles.

Firm f with preference list θ_f values a match with worker w as $u(\theta_f, w)$, where $u(\theta_f, \cdot)$ is a von-Neumann Morgenstern utility function. In our model, firms will be symmetric in the following sense: we assume that a firm's utility for a match depends only on a worker's rank. That is, for any permutation ρ of worker indices, we have $u(\rho(\theta_f), \rho(w)) = u(\theta_f, w)$.[6] Furthermore, all firms have the same utility function $u(.,.)$. Worker w with preference list θ_w values a match with firm f as $v(\theta_w, f)$, where match utility again depends only on rank, and all workers share the same utility function. Though not essential for our results, we will assume that workers and firms derive zero utility from being unmatched, and that any match is preferable to remaining unmatched for all participants. A market is given by the 5-tuple $\langle \mathcal{F}, \mathcal{W}, t, u, v \rangle$.

We will focus on a simple preference structure: each firm f has preferences over the workers chosen uniformly and randomly from the set of all strict preference orderings over all workers. Worker preferences are analogously chosen; that is, there is no correlation in preferences. This will make the problem symmetric and easy to analyze. In Section 3.1 of Chapter 3 we will relax this assumption, and consider the case in which preferences of workers over firms may exhibit correlation.

[6]Let $\rho : \{1, \ldots, W\} \to \{1, \ldots, W\}$ be a permutation. Abusing notation, we apply ρ to preference lists, workers, and sets of workers, such that the permutation applies to the worker indices. For example, suppose $W = 3$, $\rho(1) = 2$, $\rho(2) = 3$, and $\rho(3) = 1$. Then we have $\theta_f = (w_1, w_2, w_3) \Rightarrow \rho(\theta_f) = (w_2, w_3, w_1)$ and $\rho(w_1) = w_2$.

2.2.2 The Offer Game with No Signals

We first examine behavior in the absence of a signaling mechanism. Play proceeds as follows. After preferences of firms and workers are realized, each firm simultaneously makes an offer to at most one worker. Workers then choose at most one offer from those available to them. Sequential rationality ensures that workers will always select the best available offer. Hence, we take the workers' behavior in the last stage as given and focus on the reduced game with only firms as strategic players.

Once its preference list θ_f (*f*'s type) is realized, firm f decides whether and to whom to make an offer. Firm f may use a mixed strategy denoted by σ_f which maps the set of preference lists to the set of distributions over the union of workers with the no-offer option, denoted by $\mathcal{N}$, that is $\sigma_f : \Theta_f \to \Delta(\mathcal{W} \cup \mathcal{N})$.[7] We denote a profile of all firms' strategies as $\sigma_F = (\sigma_{f_1}, ...\sigma_{f_F})$, and the set of firm f's strategies as Σ_f .

Let the function $\pi_f : (\Sigma_f)^F \times \Theta \to \mathbf{R}$ denote the payoff of firm f as a function of firm strategies and realized agent types. We are now ready to define the Bayesian Nash equilibrium of the offer game with no signals.

Definition 1. Strategy profile $\hat{\sigma}_F$ is a Bayesian Nash equilibrium in the offer game with no signals, if for all $f \in \mathcal{F}$ and $\bar{\theta}_f \in \Theta_f$ the strategy $\hat{\sigma}_f$ maximizes the profit of firm f of type $\bar{\theta}_f$, that is

$$\hat{\sigma}_f(\bar{\theta}_f) \in \arg\max_{\sigma_f \in \Sigma_f} \mathbb{E}_{\theta_{-f}}(\pi_f(\sigma_f, \hat{\sigma}_{-f}, \theta) \mid \bar{\theta}_f).$$

We focus on equilibria in which firm strategies depend only on workers' ranks within a firm's preference list. That is, we rule out strategies that rely on worker

[7] In other words, f selects elements of a W-dimensional simplex; $\sigma_f(\theta_f) \in \Delta^W$, where $\Delta^W = \{x \in R^{W+1} : \sum_{i=1}^{W+1} x_i = 1$, and $x_i \geq 0$ for each $i\}$.

indices, eliminating any coordination linked to the identity of workers.

Definition 2. Firm f's strategy σ_f is *anonymous* if for any permutation ρ, and for any preference profile $\theta_f \in \Theta_f$, we have $\sigma_f(\rho(\theta_f)) = \rho(\sigma_f(\theta_f))$.

As an example, "always make an offer to my second-ranked worker" is an anonymous strategy, while "always make an offer to the worker called w_2" is not.

When deciding whom to make an offer, firms have to consider both the utility from hiring a specific worker and the chance that this worker will accept an offer. Because preferences of both firms and workers are independently and uniformly chosen from all possible preference orderings, and since firms use anonymous strategies, the chances of hiring any worker, conditional on making her an offer, are the same. Hence, each firm optimally makes an offer to the highest-ranked worker on its preference list. Indeed, this is the unique equilibrium when firms use anonymous strategies.

Proposition 1. *The unique equilibrium of the offer game with no signals when firms use anonymous strategies and workers accept the best available offer is* $\sigma_f(\theta_f) = \theta_f^1$ *for all* $f \in \mathcal{F}$ *and* $\theta_f \in \Theta_f$.

Note that the above statement requires that firm strategies be anonymous only in equilibrium. Firm deviations that do not satisfy the anonymity assumption are still allowed. As seen in the example in Section 4.1, there might be quite some lack of coordination, leaving many firms and workers unmatched.

2.3 The Offer Game with Signals

We now modify the game so that each worker may send a "signal" to exactly one firm. A signal is a fixed message, that is the only decision of workers is whether and to whom to send a signal, no decision can be made about the content of the signal. Note that the signal does not directly affect the utility a firm derives from a worker, as the firm's utility from hiring a worker is determined by how high the firm ranks that worker. However, the signal of a worker may affect a firm's beliefs whether that worker is likely to accept an offer. Since we have a congested market where firms can only make one offer, these beliefs may affect the firm's decision to whom to make an offer. The offer game with signals proceeds in three stages:

1. Agents' preferences are realized. Each worker decides whether to send a signal, and to which firm. Signals are sent simultaneously, and are observed only by firms who have received them.

2. Each firm makes an offer to at most one worker; offers are made simultaneously.

3. Each worker accepts at most one offer from the set of offers she receives.

Once again, sequential rationality ensures that workers will always select the best available offer. Hence, we take this behavior for workers as given and focus on the reduced game consisting of the first two stages.

In the first stage, each worker sends a signal to a firm, or else chooses not to send a signal. A mixed strategy for worker w is a map from the set of all possible preference lists to the set of distributions over the union of firms and the no-signal option, denoted by $\mathcal{N}$; that is, $\sigma_w : \Theta_w \to \Delta(\mathcal{F} \cup \mathcal{N})$. In the second stage, each firm observes the set of workers that sent it a signal, $\mathcal{W}^S \subset \mathcal{W}$, and based on these signals forms beliefs $\mu_f(\cdot|\mathcal{W}^S)$ about the preferences of workers. Each firm, based on these beliefs as well as its preferences, decides whether and to whom to make an offer. A mixed strategy of firm f is a map from the set of all possible preference lists, Θ_f, and the set of all possible combinations of received signals,

$2^{\mathcal{W}}$, which is the set of all subsets of workers, to the set of distributions over the union of workers and the no-offer option. That is, $\sigma_f : \Theta_f \times 2^{\mathcal{W}} \to \Delta(\mathcal{W} \cup \mathcal{N})$. We denote a profile of all worker and firm strategies as $\sigma_W = (\sigma_{w_1}, ...\sigma_{w_W})$ and $\sigma_F = (\sigma_{f_1}, ...\sigma_{f_F})$ respectively.

The payoff to firm f is a function of firm and worker strategies and realized agent types, which we again denote as $\pi_f : (\Sigma_w)^W \times (\Sigma_f)^F \times \Theta \to \mathbf{R}$. Similarly, define the payoff of workers as $\pi_w : (\Sigma_w)^W \times (\Sigma_f)^F \times \Theta \to \mathbf{R}$. As the offer game with signals is a multi-stage game of incomplete information, we consider sequential equilibrium as the solution concept.

Definition 3. The strategy profile $\hat{\sigma} = (\hat{\sigma}_W, \hat{\sigma}_F)$ and posterior beliefs $\hat{\mu}_f(\cdot|\mathcal{W}^S)$ for each firm f and each subset of workers $\mathcal{W}^S \subset \mathcal{W}$ are a sequential equilibrium if

- for any $w \in \mathcal{W}$, $\bar{\theta}_w \in \Theta_W : \hat{\sigma}_w(\bar{\theta}_w) \in \arg\max_{\sigma_w \in \Sigma_w} \mathbb{E}_{\theta_{-w}}(\pi_w(\sigma_w, \hat{\sigma}_{-w}, \theta) \mid \bar{\theta}_w)$,
- for any $f \in \mathcal{F}$, $\bar{\theta}_f \in \Theta_f$, $\mathcal{W}^S \subset \mathcal{W}$:
$\hat{\sigma}_f(\bar{\theta}_f, \mathcal{W}^S) \in \arg\max_{\sigma_f \in \Sigma_f} \mathbb{E}_{\theta_{-f}}(\pi_f(\sigma_f, \hat{\sigma}_{-f}, \theta) \mid \bar{\theta}_f, \mathcal{W}^S, \hat{\mu}_f)$,

where $\hat{\sigma}_{-a}$ denotes the strategies of all agents except a, for $a = m, f$ and beliefs are defined using Bayes' rule.[8]

We again focus on equilibria where agents use anonymous strategies.

Definition 4. Firm f's strategy σ_f is *anonymous* if for any permutation ρ, preference profile $\theta_f \in \Theta_f$, and subset of workers $\mathcal{W}^S \subset \mathcal{W}$ who send f a signal, we have $\sigma_f(\rho(\theta_f), \rho(\mathcal{W}^S)) = \rho(\sigma_f(\theta_f, \mathcal{W}^S))$. Worker w's strategy σ_w is *anonymous* if for any permutation ρ and preference profile $\theta_w \in \Theta_w$, we have $\sigma_w(\rho(\theta_w)) = \rho(\sigma_w(\theta_w))$.

[8]As usual in a sequential equilibrium, permissible off-equilibrium beliefs are defined by considering the limits of completely mixed strategies.

2.3.1 Equilibrium Analysis

To analyze the behavior of firms and workers, we first turn to the workers choice whether and to whom to send a signal. In any symmetric equilibrium in which workers send signals and signals are interpreted as a sign of interest by firms and hence increase the chance of receiving an offer, each worker sends her signal to her most preferred firm. Since sending a signal to any firm will lead to identical probabilities of receiving an offer, it is optimal for each worker to simply send its signal to its highest ranked firm (see Proposition 4 in Section 3.1 of Chapter 3, which provides the analog of this statement for a more general setup).

Note that babbling equilibria in which no information is transmitted via signals may also exist. In one form of such equilibria, firms ignore signals and workers randomize any signals they send across firms. In another version, workers do not send signals, and firms interpret unexpected signals negatively. Note however that equilibria where firms interpret off-equilibrium signals negatively fail to survive standard equilibrium refinements (see Section 3.1 of Chapter 3 for details).

Finally, "perverse" equilibria, where firms interpret signals negatively,e.g. as a sign of a particular lack of interest in such a position, and workers nevertheless send such signals do not exist. This is because workers may always opt against sending a signal. We focus on non-babbling equilibria, in which each worker sends a signal only to her most preferred firm.

Hence, we have pinned down worker equilibrium behavior: workers send a signal to their highest ranked firm, and workers accept the best available offer. We now examine offers of firms in the second stage of the game, taking the strategies of workers and beliefs of firms about interpreting signals as given.[9]

[9]Note that in any non-babbling symmetric equilibrium, all information sets for firms are

Call f's most preferred worker T_f (f's top-ranked worker). Consider a firm f that has received signals from a subset of workers $\mathcal{W}^S \subset \mathcal{W}$. Call f's most preferred worker in this subset S_f (f's most preferred signaling worker).

Whenever workers signal to their most preferred firm, and other firms use anonymous strategies, f's offer choice is reduced to a binary decision between making an offer to the top ranked worker, T_f, and the most preferred (potentially) lower-ranked worker who has signaled it, S_f. When the two coincide, that is when $T_f = S_f$, there is no tradeoff, and firm f will make an offer to this worker. The expected payoff to f from making an offer to T_f or S_f is strictly greater than the payoff from making an offer to any other worker. This follows from the symmetry of worker preferences and strategies and the anonymity of firm strategies: for any two workers who sent a signal, f's expectation that these workers will accept an offer is identical. Hence, if f makes an offer to a worker who sent a signal, it should make that offer to the worker it prefers the most among them. The same logic holds for any two workers who have not sent a signal. (Propositions B.12 and B.13 in Appendix B.1 provide a rigorous argument for the above statements).

This suggests a special kind of strategy for firms, which we will call a cutoff strategy.

Definition 5. Strategy σ_f is a *cutoff strategy* for firm f if $\exists j_1, \ldots, j_W \in \{1, \ldots, W\}$, such that for any $\theta_f \in \Theta_f$ and any set $\mathcal{W}^S$ of workers who sent a signal,

$$\sigma_f(\theta_f, \mathcal{W}^S) = \begin{cases} S_f & \text{if } \mathrm{rank}_{\theta_f}(S_f) \leq j_{|\mathcal{W}^S|} \\ T_f & \text{otherwise.} \end{cases}$$

We call $(j_1, \ldots, j_W)$ f's *cutoff vector*, which has as its components *cutoffs* for each positive number $|\mathcal{W}^S|$ of received signals.

realized with positive probability. Hence, firm beliefs are determined by Bayes' Law: if a firm receives a signal from a worker, it believes that worker ranks the firm first in her preference list.

A firm f which employs a cutoff strategy need only look at the rank of the most preferred worker who sent it a signal, conditional on the number of signals *f* has received. If the rank of this worker is below a certain cutoff (lower ranks are better since one is the most preferred rank), then the firm makes an offer to this most preferred signaling worker S_f. Otherwise the firm makes an offer to its overall top ranked worker T_f. Cutoffs may in general depend on the number of signals the firm receives. This is because the number of signals received provides information about the signals the other firms received. This in turn affects the behavior of other firms and hence the optimal decision for firm f. Note however that any cutoff strategy is, by definition, an anonymous strategy.

While we defined cutoffs as integers, we can extend the definition to include all real numbers in the range $(1, W)$ by letting a cutoff $j + \lambda$, where $\lambda \in (0, 1)$, correspond to mixing between cutoff j and cutoff $j + 1$ with probabilities $1 - \lambda$ and λ respectively. This is equivalent to f making offers to S_f when S_f is ranked better than j, randomizing between T_f and S_f when S_f has rank exactly j, and making offers to T_f otherwise.

Cutoff strategies are not only intuitive but also optimal strategies for firms. Whenever other firms use anonymous strategies and workers signal to their most preferred firms, for any strategy of firm f there exists a cutoff strategy that provides firm f with a weakly higher expected payoff (see Proposition B.13). This is due to the fact that the preferences of firms and strategies of workers are symmetric. Consequently, the probability that firm f's offer to T_f or S_f will be accepted depends only on the number of signals firm f receives, and not on the identity of the signaling workers. Hence, if f finds it optimal to make an offer to S_f, it will certainly make an offer to a more preferred S_f, provided the number of signals it receives is the same.

Since cutoff strategies can be represented by cutoff vectors, we can impose a natural partial order on them. Firm f's cutoff strategy σ'_f is greater than cutoff strategy σ_f if all cutoffs of σ'_f are weakly greater than all cutoffs of σ_f and at least one of them is strictly greater. We say that firm f responds more to signals than firm f' when σ_f is greater than $\sigma_{f'}$. From now on we focus attention on cutoff strategies of firms.

We now examine how a firm should adjust its behavior in response to changes in the behavior of opponents. We find that responding to signals is a case of strategic complements.

Proposition 2. *Suppose workers send signals to their most preferred firms and accept their best available offer, suppose all firms use cutoff strategies and firm f uses a cutoff strategy that is a best response. If one of the other firms responds more to signals, then the best response for firm f is to also weakly respond more to signals.*

When all other firms make offers to workers who have signaled to them, it is risky for firm f to make an offer to a worker who has not signaled to it. Such a worker has signaled to another firm, which is more inclined to make her an offer. The greater this inclination on the part of the firm's opponents, the riskier it is for firm f to make an offer to its most preferred overall worker T_f. Hence as a response, firm f is also more inclined to make an offer to its most preferred worker among those who sent a signal, namely S_f.

The strategic complements result allows us to apply Theorem 5 from Milgrom & Roberts (1990) to demonstrate the existence of symmetric equilibria in pure cutoff strategies with smallest and largest cutoffs.

Theorem 1. *In the offer game with signals, there exists a symmetric equilibrium in pure cutoff strategies where 1) workers signal to their most preferred firms and accept their best available offer and 2) firms use symmetric cutoff strategies. Furthermore, there exist pure symmetric equilibria with smallest and largest cutoffs.*

2.3.2 The Effects of a Signaling Mechanism

We have analyzed the unique equilibrium in the offer game with no signals, and we studied symmetric equilibria when we introduce a signaling mechanism. We focused on non-babbling equilibria where firms interpret signals of workers as a sign of interest, and hence each worker sends a signal to her most preferred firm. In this section we address the effect of introducing a signaling mechanism on the market outcome. We consider three outcome measures: the welfare of firms, the welfare of workers and the number of matches in the market.

Our analysis begins with an incremental approach: we first study the effect of a single firm increasing its cutoff, that is, responding more to signals. We then rank various signaling equilibria in terms of their outcomes. Finally, we address how introduction of a signaling mechanism impacts our three measures of welfare.

The expected welfare for a firm f and a worker w is captured by π_f and π_w respectively, where $\pi_f, \pi_w : \Sigma_w^W \times \Sigma_f^F \times \Theta \to \mathbf{R}$. Let the function $m : (\Sigma_w)^W \times (\Sigma_f)^F \times \Theta \to \mathrm{R}$ denote the expected total number of matches in the market as a function of agent strategies and types. In this section we restrict the analysis to cutoff strategies.

Consider the offer game with signals, in which workers send their signal to their first choice firms, and firms interpret these signals as signs of interest. Fix the strategies of all other firms, and assume that one firm changes its strategy to respond more to signals. How does this affect the number of matches, and the workers and firms welfare?

Proposition 3. *Consider any strategy profile in which firms use cutoff strategies, workers send signals to their most preferred firms, and workers accept their best available offer. Fix the strategies of all firms but f as σ_{-f}. Let firm f's strategy σ'_f*

differs from σ_f only in that σ'_f responds more to signals, that is has higher cutoffs then σ_f. Then

- *The expected number of matches increases, that is $\mathbb{E}_\theta[\, m(\sigma'_f, \sigma_{-f}, \theta)\,] \geq \mathbb{E}_\theta[\, m(\sigma_f, \sigma_{-f}, \theta)\,]$.*
- *The expected payoffs of each worker increases, that is for each $w \in \mathcal{W}$, $\mathbb{E}_\theta[\, \pi_w(\sigma'_f, \sigma_{-f}, \theta)\,] \geq \mathbb{E}_\theta[\, \pi_w(\sigma_f, \sigma_{-f}, \theta)\,]$.*
- *The expected payoffs of all firms but f decreases, that is for each $f' \in -f$, $\mathbb{E}_\theta[\, \pi_{f'}(\sigma'_f, \sigma_{-f}, \theta)\,] \leq \mathbb{E}_\theta[\, \pi_{f'}(\sigma_f, \sigma_{-f}, \theta)\,]$ (negative spillover on opponent firms).*

To understand the first result, observe that when firm f switches its offer from the first choice worker T_f to the signal worker S_f, it is the other offers received by these two workers that determine the impact on the total number of matches. If both workers have other offers, or if neither has another offer, the number of matches is unaffected. Only when exactly one of these two workers has another offer does *f*'s switch from T_f to S_f affect the number of matches. However, conditional on exactly one of these having another offer, it is weakly more likely to be T_f, as this worker has signaled to another firm, while S_f has not. Furthermore, T_f is strictly more likely than S_f to have another offer when at least one other firm responds to signals. Hence, making an offer to S_f leads to a greater expected total number of matches.

In addition to creating more matches in expectation, a firm responding more to signals unambiguously increases expected worker welfare. Note that when firm f changes its offer from T_f to S_f, then worker S_f receives an offer from her first choice firm, while worker T_f loses an offer from a firm she ranks second or worse. Hence, when the number of matches is unchanged, average worker welfare increases. Furthermore, it is more likely that the number of matches increases rather than decreases, and once more each match 'gained' is one where a worker receives her first choice firm, while each match 'lost' is one where a worker receives a firm of her second choice or worse. It follows that in expectation, each worker gains when a firm starts responding more to signals.

In contrast, a firm f responding more to signals has a negative effect on the welfare of other firms. When firm f makes an offer to T_f, this offer may be rejected, as T_f may prefer other firms to firm f. But an offer from f to S_f creates "stiff" competition for competing firms, since this worker will accept *f*'s offer with certainty, and offers from other firms will be rejected. Additionally, *f*'s switch from T_f to S_f may only be pivotal for opponent firms making "risky" offers to their top ranked workers. Such offers are more likely to be made to S_f than to T_f since by signaling, S_f has indicated she prefers *f*. Hence, in addition to creating stiffer competition for $-f$, *f*'s switch from T_f to S_f creates more competition for $-f$. The combination of these two effects gives the negative spillover result.

We now use the incremental welfare results to compare welfare across equilibria. The following corollary states that for all three of our welfare measures, there is a clear ranking of any two symmetric equilibria that can be ordered by their cutoffs.

Corollary 1. *Consider any two symmetric cutoff strategy equilibria where in one equilibrium firms have greater cutoffs (respond more to signals). Compared to the equilibrium with lower cutoffs, in the equilibrium with greater cutoffs we have the following:*

- *the expected number of matches is weakly greater,*
- *workers have weakly higher expected payoffs, and*
- *firms have weakly lower expected payoffs.*

Corollary 1 states that firms and workers are opposed in their preferences over equilibria. When multiple equilibria exist, workers prefer the equilibrium that involves firms responding the most to signals, that is the greatest cutoffs, while firms prefer the equilibrium with the lowest cutoffs.

We can now address the effect of adding a signaling mechanism to an offer game with no signals. We will assume that the equilibrium once the signaling mechanism is introduced is one of the symmetric non-babbling equilibria. Using the results above, we can show that introducing a signaling mechanism weakly increases the welfare of workers and the expected number of matches. Furthermore, the inequality is strict if firms respond to signals; that is, if for at least some number of signals, firms use strategies that call for an offer to a worker who signaled, S_f, even when she is not the first choice worker T_f. In contrast, firm welfare cannot be compared. As the example in Section 2.1 illustrates, firm welfare may be higher with or without a signaling mechanism. The following theorem encapsulates these results.

Theorem 2. *Consider any non-babbling symmetric equilibrium of the offer game with signals in which for at least some number of signals, firm strategies call for an offer to the signaling worker, S_f, even when she is not the first choice worker T_f. Then the following three statements hold.*

i. The expected number of matches is strictly greater than in the unique equilibrium of the offer game with no signals.

ii. The expected welfare of workers is strictly greater than in the unique equilibrium of the offer game with no signals.

iii. The welfare of firms may be greater or smaller than in the unique equilibrium of the offer game with no signals.

Note that when there is a symmetric non-babbling equilibrium where firms ignore signals, then this equilibrium is outcome equivalent to a market without a signaling mechanism. When introducing a signaling mechanism hurts firm welfare, it has to be that the negative externality outweighs the individual firm benefit from responding to signals.

2.4 Conclusion

We introduced the basic models of the offer game with and without signals in this chapter. In the game with signals each worker can send one costless private signal to firms indicating her interest in positions there; workers send signals simultaneously. Then, each firm makes an offer to at most one worker; firms make offers simultaneously. Finally, workers choose at most one offer from those available to them.

We considered the environment where workers and firm preferences are uniformly distributed. We showed that, on average, introducing a signaling mechanism increases both the expected number of matches as well as the expected welfare of workers for this environment. The welfare of firms, on the other hand, changes ambiguously. In addition, the signaling mechanism adds the most value for markets wherein the number of firms and the number of workers are of roughly the same magnitude.

We emphasize that signaling mechanisms have the potential to improve outcomes in congested markets in a reasonably non-invasive manner. As opposed to a central clearinghouse (as in the National Resident Matching Program (Roth & Peranson, 1999)), a centralized signaling mechanism is significantly less drastic. As a consequence, market designers may find it easier to get consensus from participants to introduce such a mechanism, which nevertheless may provide substantial benefit.

Chapter 3

Preference Signaling in Matching Markets: Extensions

This chapter considers extension of the basic model of Chapter 2 as well as derives some additional results regarding the effects of a signaling mechanims for various market structures. Section 3.1 studies the environments when with worker preference being correlated. Sections 3.2 and 3.3 consider an extension where each firm has the capacity to hire several workers, and each worker may send up to multiple identical costless private signals and can be matched to several firms. Finally, Section 3.4 analyzes the impact of a signaling mechanism on the number of matches in the market for various market structures. We analyze a pure coordination model where agent care only about being matched, but not about the quality of a match and study how the expected increase in the number of matches from the introduction of a signaling mechanism depends on market size, the number of periods of interaction, signals, and worker positions.

3.1 Block Correlation

In this section we relax the assumption that worker preferences are uncorrelated. More precisely, we consider a market where firms can be partitioned in blocks, so that all workers agree which block contains the most desirable firms, which block the second most desirable set of firms and so on. However, within a block, workers may have idiosyncratic preferences over firms. Hence, for this section we consider markets where agent preferences are *block-correlated*.

Definition 6. A *block-correlated market* is a market $\langle \mathcal{F}, \mathcal{W}, t, u, v \rangle$ such that for a partition $\mathcal{F}_1, \ldots, \mathcal{F}_B$ of the firms into blocks, ordinal preferences (as encompassed in $t(\cdot)$) are such that

1. For any $b < b'$, where $b, b' \in \{1, \ldots, B\}$, each worker prefers every firm in block $\mathcal{F}_b$ to any firm in block $\mathcal{F}_{b'}$;

   (a) Each worker's preferences within each block $\mathcal{F}_b$ are uniform and uncorrelated; and

   (b) Each firm's preferences over workers are uniform and uncorrelated.

We call distributions $t(\cdot)$ that satisfy the criteria in Definition 6 *block uniform*. The environment analyzed in previous sections is a special case of block-correlated markets, where there is only one block of firms. Block-correlated markets are meant to capture the notion that many two-sided markets are segmented. That is, workers may largely agree on the ranking of blocks on the other side of the market, but vary in their preferences within each block. For example, workers might agree on the set of firms that constitute the "top tier" of the market; however within that tier, preferences are influenced by factors specific to each worker.

We again focus on equilibria where agents use anonymous strategies. For firms we maintain the notion of anonymous strategies introduced in Definitions 2 and 4. For workers we only consider permutations P^b that permute firm orderings within blocks; that is, permutation $\rho \in P^b$ if for any firm f and any block b, if $f \in \mathcal{F}_b$ then $\rho(f) \in \mathcal{F}_b$

Definition 7. Worker w's strategy σ_w is *anonymous* if for any permutation $\rho \in P^b$ and preference profile $\theta_w \in \Theta_w$, we have $\sigma_w(\rho(\theta_w)) = \rho(\sigma_w(\theta_w))$.

As previously, let us first consider the offer game with no signals. Since worker preferences are still uniformly distributed there is again a unique equilibrium where firms use anonymous strategies:each firm optimally makes an offer to the highest-ranked worker on its preference list.

We now turn to the offer game with signals, where we will be interested in equilibria where firms within each block play symmetric, anonymous strategies. That is, if firm f and firm f' belong to the same block $\mathcal{F}_b$, for some $b \in \{1, ..., B\}$, they play the same anonymous strategies and have the same beliefs. We call such firm strategies and firm beliefs *block-symmetric*. We denote equilibria where firm strategies and firm beliefs are block-symmetric and worker strategies are anonymous and symmetric as *block-symmetric equilibria*.Before we can characterize the set of block-symmetric equilibria, we discuss the strategies of workers, who must choose whether to send a signal, and if so, to which firm. In block-symmetric equilibria, firms within each block $\mathcal{F}_b$ use the same anonymous strategies. Hence, we can denote the ex-ante probability of a worker w receiving an offer from a firm in block $\mathcal{F}_b$, conditional on w sending and not sending a signal to it as p_b^s and p_b^{ns} correspondingly. We also denote the equilibrium probability that a worker sends her signal to a firm in block $\mathcal{F}_b$ as α_b, where $\alpha_b \in [0, 1]$ and $\sum_{b=1}^{B} \alpha_b \leq 1$.

The following proposition characterizes worker strategies in all block-symmetric sequential equilibria that satisfy criterion *D1* of Cho & Kreps (1987).[1]

Proposition 4. *Consider a block-symmetric sequential equilibrium that satisfies criterion D1. Then either*

1. *for every* $b \in \{1, ..., B\}$, $p_b^s = p_b^{ns}$ *or*
2. *there exists* $b_0 \in \{1, ..., B\}$ *such that* $p_{b_0}^s > p_{b_0}^{ns}$ *and*
   (a) *for any* $b \in \{1, ..., B\}$ *such that* $\alpha_b > 0$, *we have* $p_{b_0}^s > p_{b_0}^{ns}$ *and if a worker sends her signal to block* $\mathcal{F}_b$, *she sends her signal to her most preferred firm within* $\mathcal{F}_b$, *and*
   (b) *for any* $b' \in \{1, ..., B\}$ *such that* $\alpha_{b'} = 0$, *workers' strategies are optimal for any off-equilibrium beliefs of firms from block* $\mathcal{F}_{b'}$.

Proposition 4 states that there are two types of block-symmetric equilibria that satisfy criterion *D1*. Equilibria of the first type are babbling, where firms ignore signals. The outcomes of these equilibria coincide with the outcome in the offer games with no signals. Consequently, the signaling mechanism adds no value in this case. In equilibria of the second type, workers send signals only to their most preferred firm in each block, possibly mixing across these top firms. We call such worker strategies *best-in-block strategies*. Moreover, if in equilibrium worker w is not prescribed to signal to some block $\mathcal{F}_{b'}$, then w's choice of $\alpha_{b'} = 0$ would be optimal for any beliefs of firms in block $\mathcal{F}_{b'}$. In particular, this strategy would be optimal even if firms in block $\mathcal{F}_{b'}$ interpreted signals in the most favorable way for worker w; i.e., upon receiving a signal from worker w each firm f in $\mathcal{F}_{b'}$ believes that it is w's most preferred firm within block $\mathcal{F}_{b'}$. We call such beliefs *best-in-block beliefs*.

[1]See proof of Proposition 4 for the definition of criterion D1 of Cho & Kreps (1987). Criterion $D1$ could be also replaced by "never a weak best response" of Cho & Kreps (1987) and "universal divinity" of Banks & Sobel (1987) without making a change to the statement of Proposition 4.

We will assume that workers use symmetric best-in-block strategies and that firms have best-in-block beliefs, and examine firm offers in the second stage of the game.[2]

An important difference between the single block and multi-block settings is that when there are multiple blocks, offers to workers who have signaled are no longer guaranteed to be accepted. The reason for this is that a firm that receives a signal knows that while it is the worker's most preferred firm in the block, the worker may receive an offer from a firm in a superior block. Nevertheless, several firm strategy results carry over when we introduce block correlation. In a block-correlated market, firm f's offer choice is again reduced to a binary decision between T_f and S_f, provided workers use symmetric best-in-block strategies and firms $-f$ use anonymous strategies. Under these same conditions, cutoff strategies are again optimal for f. The strategic complements result of Proposition 2 also carries over; if firms $-f$ use cutoff strategies and workers use symmetric best-in-block strategies, then when $f' \in -f$ responds more to cutoffs, f optimally responds more to cutoffs as well (see Propositions B.12, B.13, and B.14).

The next result establishes the existence of equilibria in block correlated settings in the offer game with signals. To prove the theorem, we first demonstrate equilibrium existence while requiring firms to use only cutoff strategies.We then invoke the optimality of cutoffs result to show that this step is not restrictive.

Theorem 3. *There exists a block-symmetric equilibrium where 1) workers play symmetric best-in-block strategies, and 2) firms play block-symmetric cutoff strategies.*

[2]Note that firms have best-in-block beliefs on the equilibrium path in any block-symmetric equilibrium. In addition, a block-symmetric equilibrium satisfies criterion D1 if and only if worker strategies remain optimal if firm off-equilibrium beliefs were best-in-block beliefs. Hence, we will focus on equilibria where firms have best-in-block beliefs even off the equilibrium path. See the proof of Proposition 4 in Appendix B.1 for details.

In contrast to Theorem 1 which established equilibrium existence when there is a single block, equilibria here may involve mixed strategies for workers; that is, each worker may signal with positive probability to multiple blocks.

The final result of the section extends the welfare results of Theorem 2. Note that for the comparisons in the theorem to be strict, we require a block with at least two firms where in equilibrium, workers send signals with positive probability. Without this condition, weak comparisons continue to hold.

Theorem 4. *Consider any non-babbling block-symmetric equilibrium of the offer game with signals, in which there is a block $\mathcal{F}_b$ with at least two firms such that $\alpha_b > 0$. Then,*

i. The expected number of matches is strictly greater than in the unique equilibrium of the offer game with no signals.

ii. The expected welfare of workers is strictly greater than in the unique equilibrium of the offer game with no signals.

iii. The welfare of firms may be greater or smaller than in the unique equilibrium of the offer game with no signals.

Note that while the welfare comparisons with and without a signaling mechanism generalize to block correlated markets, the welfare comparisons across equilibria (see Corollary 24) do not generalize. In particular, when there are multiple blocks, when a single firm responds more to signals, firms in lower ranked blocks may benefit. Hence, we no longer see a purely negative spillover on other firms, which was a key step in establishing the welfare ranking.[3]

[3] Since offers to workers who have signaled are no longer guaranteed to be accepted, firms making offers to signaling workers may be affected by f's switch from T_f to S_f. In particular, firms in the *same or higher* blocks responding to signals will not be affected, but firms in lower blocks responding to signals *prefer* that f switch from T_f to S_f. There is a positive spillover on these firms, and negative spillover on all other firms.

3.2 Many-to-One Matching Markets with Multiple Signals

In this section we consider matching markets similar to in Sections 2.2 and 2.3. The difference of our analysis in this section is that each firm now has the capacity to hire at most L workers, and each worker may send up to K identical costless private signals and can fill at most one position.[4].

We assume that firm utilities are additive, i.e. firm f with preference list over individual workers θ_f values a match with a subset of workers $\mathcal{W}_0 \subset \mathcal{W}$ as $u(\theta_f, \mathcal{W}_0) = \sum_{w \in \mathcal{W}_0} u(\theta_f, w)$, where $u(\theta_f, \cdot)$ is a von-Neumann Morgenstern utility function. We again assume that a firm's utility for a match depends only on a worker's rank: for any permutation ρ of worker indices, we have $u(\rho(\theta_f), \rho(w)) = u(\theta_f, w)$. Furthermore, all firms have the same utility function $u(.,.)$. Worker w with preference list θ_w values a match with firm f as $v(\theta_w, f)$, where match utility again depends only on firm's rank, and all workers share the same utility function. We again assume that workers and firms derive zero utility from being unmatched, and that any match is preferable to remaining unmatched for all participants.

As previously we define Θ_f be the set of all possible firm f preference lists and $\Theta_F = (\Theta_f)^F$. Similarly, we define Θ_w and Θ_W for workers. In addition we denote $\Theta \equiv \Theta_F \times \Theta_W$ and $t(\cdot)$ be the distribution over preference list profiles.

We assume that the number of signals, each worker may send, is less than the number of firms, $K < F$, and that each worker can send maximum one signal to a particular firm. Each worker may send any number of signals between 0 and K. Signals are still voluntary, and have the binary nature.

[4]See discussion of the case when workers can hold several positions that we call interviews in Section 3.3.

We further describe the timing and agent strategy space of the offer game with signals. The timing and agent strategy space of the offer game without signals can be adopted from Section 2.2 with necessary modifications in a similar way.

1. Agents' preferences are realized. Each worker sends up to K private costless signals to firms. Signals are sent simultaneously, and are observed only by firms who have received them.

2. Each firm makes an offer to at most L workers; offers are made simultaneously.

3. Each worker accepts at most one offer from the set of offers she receives.

Once again, sequential rationality ensures that workers will always select the best available offer. Hence, we take this behavior for workers as given and focus on the reduced game consisting of the first two stages.

As in Sections 2.2 and 2.3 we analyze agent mixed strategies. In the offer game with signals, each worker now sends up to K signal to firms in the first stage. A mixed strategy for worker w is a map from the set of all possible preference lists to the set of distributions over firm subsets of the power smaller than K that we denote as $\Delta(2^{\mathcal{F}_{\mathrm{K}}})$, i.e. $\sigma_w : \Theta_w \to \Delta(2^{\mathcal{F}_{\mathrm{K}}})$. Similarly, a mixed strategy of firm f is a map from the set of all possible preference lists, Θ_f, and the set of all possible combinations of received signals, $2^{\mathcal{W}}$, to the set of the set of distributions over worker subsets of the power smaller than L that we denote as $\Delta(2^{\mathcal{W}_{\mathrm{L}}})$. That is, $\sigma_f : \Theta_f \times 2^{\mathcal{W}} \to 2^{\mathcal{W}_{\mathrm{L}}}$.

Similarly to Sections 2.2 and 2.3 we define σ_W, σ_F, Σ_w,Σ_f, π_w, andπ_f. Preferences of both firms and workers are independently and uniformly chosen from all possible preference orderings. The definition of sequential equilibrium and anonymous strategies can also be adopted in a similar way.

We first consider an offer game without signals. If firms use anonymous strategies, the chances of hiring any worker, conditional on making her an offer, are the same. Therefore, each firm optimally makes its offers to the L highest-ranked worker on its preference list in the unique symmetric equilibrium of the offer game without signals when firms use anonymous strategies.

Proposition 5. *The unique equilibrium of the offer game with no signals when firms use anonymous strategies and workers accept the best available offer is* $\sigma_f(\theta_f) = (\theta_f^1, ..., \theta_f^L)$ *for all* $f \in \mathcal{F}$ *and* $\theta_f \in \Theta_f$.

We now turn to the analysis of the offer game with signals. In any symmetric equilibrium in which workers send signals and signals are interpreted as a sign of interest by firms and hence increase the chance of receiving an offer, each worker sends her K signals to her K most preferred firms.

Proposition 6. *In any symmetric non-babbling equilibrium of the offer game with signals each worker sends signals to her* K *top firms.*

As in the case of one signal and one firm position, we have pinned down worker equilibrium behavior: workers send their signals to their highest ranked firms, and workers accept the best available offer. We now examine offers of firms in the second stage of the game, taking the strategies of workers and beliefs of firms about interpreting signals as given.[5]

In contrast to Chapter 2 it is not sufficient to consider the choice of firm f between f's most preferred worker and f's most preferred worker in the subset of signaled workers. Firm f can make several offers and potentially invite other

[5]Note that in any non-babbling symmetric equilibrium, all information sets for firms are realized with positive probability. Hence, firm beliefs are determined by Bayes' Law: if a firm receives a signal from a worker, it believes that it is kth top firm, $k \in \{1, ..., K\}$, in the worker preference list; the probability of having rank k is identical across ranks $\{1, ..., K\}$.

workers. However, the choice of firm f still very intuitive. If $-f$ firms use anonymous strategies and workers send their signals to top K firms, firm f makes offers to her L^{NS} top workers who has signaled to it and to her $L^S = L - L^{NS}$ top workers who has not signaled to it in any non-babbling symmetric sequential equilibrium (see Proposition B.25). This again allows us to define the notion of cutoff strategies.

Definition 8. Strategy σ_f is a *cutoff strategy* for firm f if there are L vectors $J^l = (j^l_1, ..., j^l_W)$, $l = 1, ..., L$ such that for any $\theta_f \in \Theta_f$ and any set $\mathcal{W}^S$ of workers who sent a signal to firm f if firm f makes an offer to $w \notin \mathcal{W}^S$ with $r = rank_{\theta_f}(w)$, firm f makes an offer to any $w' \in \mathcal{W}^S$ with $rank_{\theta_f}(w') \leq j^r_{|\mathcal{W}^S|}$.

We call $(J^1, \dots, J^L)$ cutoff matrix that has cutoff vectors for top L firm positions as its components. Note that the probability of firm's offer being accepted by any worker who has signaled to it is the same in a symmetric equilibrium. As well the probability of firm's offer being accepted by any worker who has not signaled to the firm is also the same across such workers (Lemma B.22). The intuition behind firm cutoff strategy is the following. Let us consider a firm that has not decided regarding any of its offers. For the first offer the firm compares the expected payoff from making an offer to its top signaled worker to its top ranked worker. However, if firm has decided to make an offer to its top ranked worker, in order to fill other positions it should compare the payoff from making an offer to its second top ranked worker and its top signaled worker. Since the payoff from second ranked worker is smaller than from top ranked worker the corresponding cutoff will be greater. The same logic extends to L top positions.

Using the logic similar to the case of one position we show that cutoff strategies are also optimal strategies for firms.

Proposition 7. *Suppose workers send their signals to top K firms. Then for any strategy σ_f of firm f, there exists a cutoff strategy that provides f with a weakly* *higher expected payoff than σ_f for any anonymous strategies σ_{-f} of opponent firms $-f$.*

We can also impose a partial order on the cutoff strategies as in Chapter 2. However, we cannot guarantee that firm strategies are strategic complements in the case when workers can send multiple signals. If other firms respond more to signals, this decreases the payoff from making an offer to both workers who has and workers who has not signaled to the firm. Compare to Chapter 2, we can only assure the existence of mixed strategy equilibrium.

Theorem 5. *There exists a block-symmetric equilibrium of the offer game with signals where 1) workers send their signals to top K firms, and 2) firms play symmetric cutoff strategies.*

We now address the welfare implications from the introduction of a signaling mechanism. The Proposition 8 and Theorem 6 formally restate our welfare results from previous chapters for the case when firms have many positions and workers can send multiple signals. The logic of their proof again begins with an incremental approach: we first study the effect of a single firm increasing its cutoff, that is, responding more to signals. We then rank various signaling equilibria in terms of their outcomes. Finally, we show how the introduction of a signaling mechanism impacts our three measures of welfare.

Proposition 8. *Consider any two symmetric cutoff strategy equilibria where in one equilibrium firms have greater cutoffs. Compared to the equilibrium with lower cutoffs, in the equilibrium with greater cutoffs we have the following:*

- *the expected number of matches is weakly greater,*
- *workers have weakly higher expected payoffs, and*

- *firms have weakly lower expected payoffs.*

Theorem 6. *Consider any non-babbling symmetric equilibrium of the offer game with signals. Then the following three statements hold.*

i. The expected number of matches is strictly greater than in the unique equilibrium of the offer game with no signals.

ii. The expected welfare of workers is strictly greater than in the unique equilibrium of the offer game with no signals.

iii. The welfare of firms may be greater or smaller than in the unique equilibrium of the offer game with no signals.

3.3 Many-to-Many Matching Markets with Many Signals

As in Section 3.2 we consider the environment where each firm has the capacity to hire at most L workers, and each worker may send up to K identical costless private signals. The timing of the game with and without signals as in the basic model of Chapter 2.

We assume here that each worker can be matched with at most P firms, such that the number of positions each worker can have is smaller than the number of firms, i.e. $P < F$. In order the assumption regarding uniform distribution of preferences be tractable, we again assume that worker utility is additive, i.e. worker w with preference list over individual workers θ_w values a match with a subset of firms $\mathcal{F}_0 \subset \mathcal{F}$ as $u(\theta_w, \mathcal{F}_0) = \sum_{f \in \mathcal{F}_0} u(\theta_w, f)$, where $u(\theta_w, \cdot)$ is a von-Neumann Morgenstern utility function. Worker utility for a match depends only on a firm's rank as well as we preserve the other assumptions of Section 3.2 regarding agent utility funcitions. These assumptions ensures that each worker derives zero utility from being unmatched, and that each worker preferes to be hired by the maximum number of firms as well as any additional firm delivers a positive payoff to each worker.

Although we do not provide the riguourous proofs the results of Section 3.2 still carry over. Under the above assumptions each worker still prefers to send signals to top K firms, firms still optimally use cutoff strategies, and sequential rationality again ensures that workers always select the best available offers (up to P) at the last stage of the game. Finally, the effects of a signaling mechanism from Section 3.2 still persist as soon as the number of worker positions is limited and some worker cannot accept offers from all firms.

3.4 Market Structure and The Value of a Signaling Mechanism

In this section, we analyze how the expected increase in the number of matches from the introduction of a signaling mechanism, a measure we term the *value of the signaling mechanism*, differs across market structures.

To isolate the impact of a signaling mechanism on the number of matches in the market, we consider the *pure coordination model*, where firms and workers want to match, but are almost indifferent over the identity of the match. Specifically, we consider the cardinal utility from being matched to a partner as being almost the same across partners. If agent a has a preference profile θ_a, it prefers to be matched with partner θ_a^k, rather than with partner $\theta_a^{k'}$, $k' > k$, though the difference between utility intensities is very small. In addition, there is only one block of firms, so that agent preferences are uniformly distributed. For expositional clarity, we postpone the precise formulation of assumptions and propositions to Appendix B.3.

Under these assumptions, there is a unique non-babbling symmetric equilibrium in the offer game with signals. Each worker sends a signal to her most preferred firm. Each firm makes an offer to its most preferred worker that has signaled provided the firm receives at least one signal; otherwise, it makes an offer to its top-ranked worker (see Proposition B.36). Proposition 1 also applies in this setting; that is, there is a unique equilibrium of the offer game with no signals.

We define the expected number of matches in the unique equilibrium in the pure coordination model with signals and with F firms and W workers as $m^S(F,W)$, and without a signaling mechanism as $m^{NS}(F,W)$. Denote the increase in expected number of matches from the introduction of the signaling mechanism, the value of the signaling mechanism, as $V(F,W) \equiv m^S(F,W) - m^{NS}(F,W)$.

3.4.1 Balanced Markets

In this subsection, we analyze how the value of the signaling mechanism changes for markets of various sizes. Under the equilibrium behavior in the offer games with and with no signals, described above, we calculate $V(F,W)$ for these markets. Figure 3.4.1 graphs $100\% * V(F,W)/W$ as a function of F for fixed $W = 10$ and $W = 100$, and $100\% * V(F,W)/F$ as a function of W for fixed $F = 10$ and $F = 100$.

The figures suggest that the value of a signaling mechanism is single peaked when varying one side of the market and holding the other constant. Furthermore, it seems that a signaling mechanism is the most useful when the the

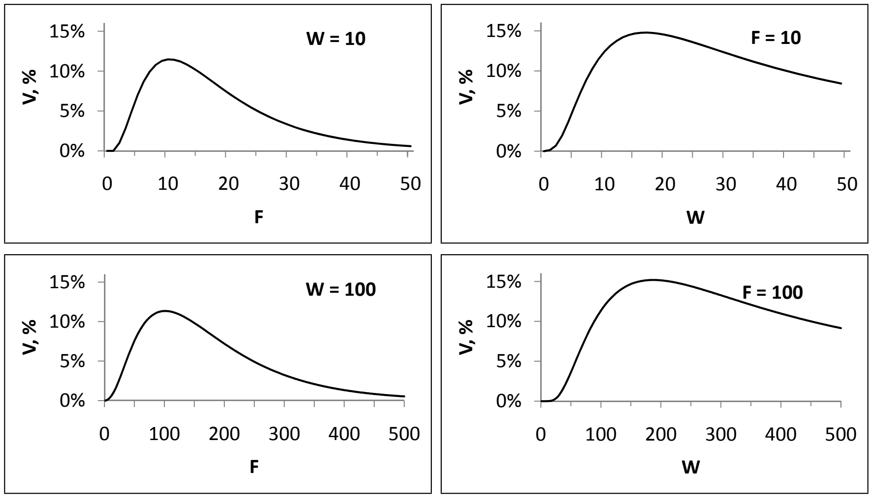

Figure 3.4.1: The proportional increase in the number of matches due to a signaling mechanism as we vary the number of firms for a fixed number of workers (left graphs) and vice versa (right graphs)

number of firms and the number of workers are roughly of the same magnitude. To understand why signaling is most useful in balanced markets, it is helpful to think about the endpoints. With many workers and very few firms, firms will almost certainly match with or without the signaling mechanism, as there is no large coordination problem. With many firms and few workers, the reverse holds: most workers will get offers with or without the signaling mechanism. Hence, the signaling mechanism offers little benefit at the extremes.Furthermore, Figure 3.4.1 suggests that the increase in the expected number of matches remains steady as market size increases, holding constant the ratio of workers to firms. Denote with $O_W(1)$ and $O_F(1)$ functions that are smaller than a constant for large W and for large F respectively. Proposition 9 describes these observations precisely.

Proposition 9. *Let us consider markets with F firms and W workers. Then,*

- *for fixed* W*,* $V(F,W)$ *attains its maximum value at* $F = x_0 W + O_W(1)$*, where* $x_0 \approx 1.01211$;

- *for fixed F, $V(F,W)$ attains its maximum value at $W = y_0 F + O_F(1)$, where* $y_0 \approx 1.8442$.

The proof of Proposition 9 involves the calculation of an explicit formula for $V(F,W)$. The expected increase in the number of matches can be represented as

$$V(F,W) = \alpha(\frac{W}{F})F + O_F(1)$$

or as

$$V(F,W) = \beta(\frac{F}{W})W + O_W(1),$$

where $\alpha(\cdot)$ and $\beta(\cdot)$ are particular functions. Hence, $V(F,W)$ is "almost" homogeneous of degree zero for large markets. Therefore, we can evaluate the introduction of the signaling mechanism for a sample market, and its properties will be preserved for markets of other sizes, but with the same ratio of firms to workers.

For example, we can use Figure 3.4.1 to investigate maximal quantitative gains from the introduction of the signaling mechanism in large markets. For a fixed number of workers, the maximum increase in expected number of matches is approximately 15%. Furthermore, the returns to the signaling mechanism are substantial over a wide range of market conditions. For example, only when the number of firms outweighs the number of workers by more than fourfold do the gains from introducing the signaling mechanism drop to below 1%.

3.4.2 Markets with Many Periods of Interactions

We now consider markets wherein agents can interact in multiple periods. Each worker can send only one signal and has only one interview position to fill. There are $L+1$ periods in the offer game with signals. Workers send signals to firms in period 0. In the other L periods, agent can interact and be matched.

Period 0: Workers send signals.

1. Agent preferences are realized. Each worker sends a signal to at most one firm; signals are sent simultaneously. Signals are observed only by firms who have received them.

Periods $1 - L$**: Agents interact.** Each period consists of two stages:

1. Each firm makes an offer to at most one worker; offers are made simultaneously.

2. Each worker may accept at most one offer from the set of offers she receives.

Offers are binding, and workers can hold offers from period to period. When a worker accepts an offer, the firm-worker pair leaves the market, and this is observed by all agents. The other agents participate in the remaining periods. As a point of comparison, we will also be interested in the L-period offer game with no signals, which is identical to the game described above except that period 0 is excluded. There is a discount factor $\delta \in (0, 1)$ for being matched in later periods; that is, a match in period period T reduces utility by a factor of δ^{T-1}.

Under the assumptions of the pure coordination model and with a sufficiently small discount factor, there is no incentive to delay offers or offer acceptances, since agents care (almost) only about being matched. There is a unique symmetric sequential equilibrium in the offer game with no signals, wherein each firm makes an offer to its most preferred worker and each worker accepts its best offer in each period. Similarly, there is a unique symmetric sequential non-babbling equilibrium in the offer game with signals. Each worker sends her signal to her most preferred firm in period 0. Each worker accepts the best available offer at each period. In any period, each firm makes an offer to its most preferred worker (among those remaining in the market) who has sent it a signal, provided such a worker exists. Otherwise the firm makes an offer to its most preferred worker among those still in the market (Proposition B.37 in Appendix B.3 formally proves these results).

We may now compare the offer game with and without the signaling mechanism when there are multiple periods of interaction. Additional periods of interaction reduce market congestion and provide an opportunity for a greater number firms and workers to be matched. Therefore, the value of the signaling mechanism decreases as the number of periods of interaction increases. The next proposition shows this relationship formally. Since its proof is intuitive, we provide it in-text.

Proposition 10. *The expected percentage increase in the number of matches from the introduction of the signaling mechanism,* $V_{1,1}^{L}(F,W)$*, is a decreasing function of the number of periods of interactions,* L*.*

For clarity of the argument, we compare markets with one and two periods of interaction. Let us consider a market with two periods. Since workers can send only one signal and firms respond to all signals, all firms that receive at least one signal leave the market in period 1 (signals indicate that offers will be accepted for sure). Therefore, no firms remaining in period 2 have received signals, so the second period of the offer game with signals is almost identical to a single period offer game with no signals. The games differ only in the expected number of participants at period 2.

Since the introduction of signaling increases the expected number of matches, the expected number of remaining market participants in period 2 is greater in the offer game with no signals than it is in the offer game with signals. As Proposition 9 shows, the number of matches in a market with one period is proportional to the size of the market. Therefore, the expected number of matches in the second period in the offer game with no signals is greater than in the offer game with signals. In other words, second period plays more significant matching role in the offer game without signals rather in the offer game with signals. Hence, the difference between the expected number of matches in the offer game with signals and the offer game with no signals decreases upon adding the second period of interaction. This logic extends to L periods of interaction.

Using simulations, we investigate how the value of the signaling mechanism declines with the introduction of additional periods of interaction for balanced markets. Figure 3.4.2 graphs the results for markets with $F = 100$ firms, $W = 100$ workers and $L = 1, \ldots, 5$ periods of interaction.

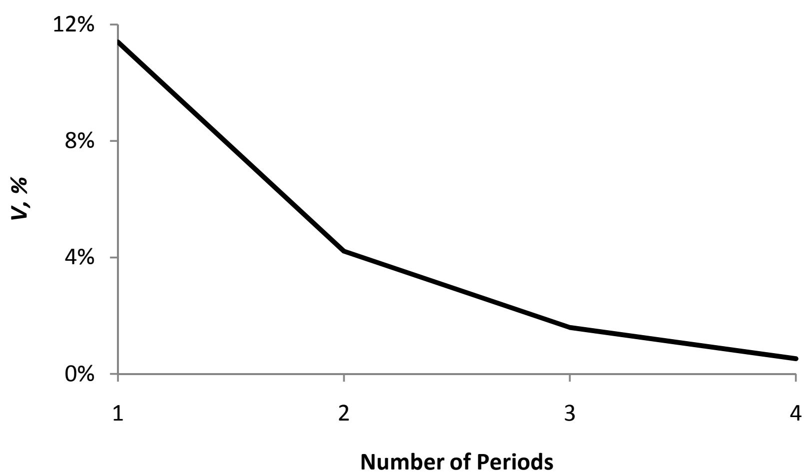

Figure 3.4.2: Value of a signaling mechanism when there are multiple periods of interaction.

Note that for markets with the same number of firms and workers the percentage increase in the expected number of matches due to the introduction of the signaling mechanism is less than 5% when a second period is added. Moreover, the value of the signaling mechanism is less than 0.5% in markets with four periods of interaction.

3.4.3 The Optimal Number of Signals and the Number of Interviews

In this subsection we analyze markets in which workers can send several identical signals and can "occupy" several positions, which we call *interviews*. Each worker may send up to K signals and has I interview slots it may fill. Firms may make up to one offer. We assume $F > K \geq 1$ and $F > I \geq 1$, and each worker utility function is additive with respect to the number of interviews. We maintain the preference assumptions of the pure coordination model and the timing is the same as in our base model in Section 2.2.

Similar to previous analysis, there is a unique equilibrium in the offer game with no signals: each firm f makes an offer to T_f, and each worker accepts the best I offers among those she receives. There is also a unique symmetric non-babbling equilibrium in the offer game with signals: each worker signals to her K most preferred firms; each firm f makes an offer to S_f, if it receives at least one signal and to T_f if it receives no signals. Again, each worker accepts the best I offers among those available (see Proposition B.38 in Appendix B.3).

Denote as $V_{I,K}(F, W)$ the percentage increase in the expected number of matches from the introduction of the signaling mechanism when each worker has I interview positions and can send up to K signals.

Fixing the equilibrium behavior of agents, described above, we use simulations to analyze $V_{I,K}(F, W)$. We analyze markets that are "balanced," in that the number of firms equals the number of available positions. Figure 3.4.2 shows the results of simulations for markets with $W = 50$ workers, $I = 1$, $I = 2$, and $I = 3$ interviews, and $F = 50$, $F = 100$, and $F = 150$ firms correspondingly (solid, dashed, and dot-dashed lines).

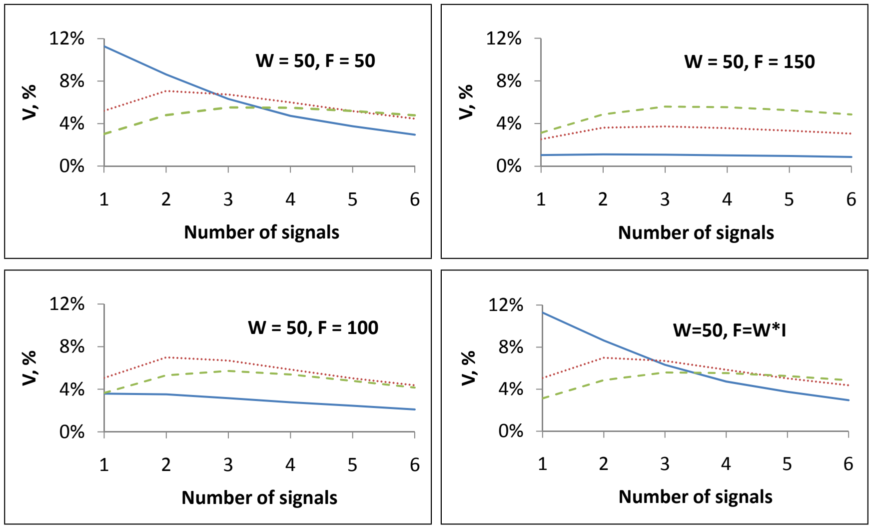

Figure 3.4.3: Multiple signals and interview positions.

The simulations suggest that the value of the signaling mechanism is single-peaked in the number of signals and that the optimal number of signals is increasing in I.

3.5 Conclusion

We have shown that the effects of a signaling mechanism studied in Chapter 2 can be extended to much more general matching markets where agents can be matched to several agents of the opposite side of the market, workers can send multiple signals, as well as worker preferences may exhibit some form of correlation. We want to stress here that most of the result of Chapter 2 carry over to these extensions except to establish the existence of pure strategy equilibrium. Though firm strategies are still strategic complements (for fixed strategies of workers) in block-correlated model in Section 3.1, workers may potentially change their signaling strategies when firms respond to signals in a different way. If workers send multiple signals to firms, firms $-f$ responding more to signals decreases the payoff of firm f from making an offer to both its top signaled and its top ranked worker. This leads to a potential failure of the strategic complementarity result in Section 3.2.

The analysis of the pure coordination model shows that a signaling mechanism adds the most value to balanced markets, where the number of firms and workers approximately of the same magnitude. Additionally, we show that a signaling mechanism is most valuable in congested markets, where agents can interact only small number of periods. Lastly, we analyze pure coordination markets in which workers can send several signals and can "occupy" several positions. We show via simulations that the optimal number of signals—the number of signals that maximizes the value of a signaling mechanism—increases with additional worker positions.

Chapter 4

Harmful Signaling in Matching Markets

In this chapter, we analyze the offer games with and without signals similar to ones introduced in Chapter 2. However, agent preferences are now almost complete. Each worker has either "typical" commonly known preferences with a probability close to one or "atypical" preferences taken from some distribution with the complementary probability close to zero. The preferences of workers are ex-ante independently distributed.[1] Firms have some fixed and commonly known preferences over workers. Firm preferences may vary across firms.

We show that if firms respond to signals in this environment, i.e. treat signals informatively, the introduction of signals decreases the expected number of matches. Prior to the signaling, all firms have almost identical beliefs about worker preferences. However, after the signals are received firms may have disparate beliefs. This disparity in beliefs leads to coordination failure. As a result,

[1]We assume that typical workers rank firms according to some public ranking. For example, typical candidates in the job market for new Ph.D. economists rank departments of economics in their field according to the *U.S. News and World Report* ranking.

the introduction of a signaling mechanism may decrease the total number of matches and the welfare of agents.

We then examine the implications from the introduction of a signaling mechanism with public signals, i.e. signals observed by all firms. Though the expected number of matches increases compared to the game with private signals, public signals still impede match formation when agent preferences are almost complete. Public signals do not transmit enough information about worker preferences. This induces some firms to compete for the same workers, which creates mismatches and decreases the expected total number of matches.

Our negative results on welfare of agents, measured by the total expected number of matches, in new cheap talk equilibria are unexpected and new to the literature. To the best of our knowledge onlyFarrell & Gibbons (1989) in the previous literature have results similar to ours, though they differ in its intuition.[2] Costless communication in two-agent bargaining model of Farrell & Gibbons (1989) gives the buyer an opportunity to pretend to have a lower value and the seller an opportunity to pretend to have a higher value (compared to the truthful information transmission in our model). This enhances their bargaining positions at the cost of the risk of no trade. New cheap talk equilibria are characterized by both less trade and a reduction in the expected gains from trade.

This chapter proceeds as follows. Section 4.1 presents a simple example that illustrates why signals can facilitate coordination failure. Section 4.2 outlines our general model and introduces some notations. Equilibrium analysis is presented in Section 4.3. Section 4.4 analyzes the welfare of agents in the model with and without signals. A public signaling mechanism is considered in Section 4.5. Finally, Section 2.4 discusses some assumptions of our model and concludes.

[2]We are thankful for Lones Smith who drew our attention to this comparison.

4.1 A Simple Example

Let us consider a market with three firms and three workers. Each firm ranks the workers in the same way (w_1, w_2, w_3), i.e. each firm strictly prefers worker w_1 to worker w_2 to worker w_3. Each worker's preference is either typical (f_1, f_2, f_3) with probability $1 - \varepsilon$ or atypical with the complementary probability ε, where ε is small. The atypical preferences are independently uniformly distributed among all possible preference order lists. All workers are acceptable to all firms and vice versa.

We first examine behavior in the game in the absence of a signaling mechanism. The only possible match in a sequential equilibrium of the game without signals is the assortative match, in which each firm is matched to the corresponding worker. Let now us analyze the game with the signaling mechanism, described above. We consider the following equilibrium strategies of agents.[3] Each worker with typical preferences sends her signal to the corresponding firm, i.e. worker w_i sends her signal to firm f_i. Each worker with atypical preferences sends her signal to the best firm worse or equal to the corresponding one (according to typical preferences). Each firm makes its offer to a worker better or equal to the corresponding one, only if it receives a signal from her. Each firm ignores all signals from workers worse than the corresponding one. If a firm receives no signals, it makes an offer to the best worker worse than the corresponding one.

Let us consider the realization of preference profiles when only worker w_1 is atypical and firm f_3 is her favorite firm. Worker w_2 and worker w_3 are typical. Figure I illustrates the equilibrium behavior. Worker w_1 sends her signal to firm f_3. Worker w_2 and worker w_3 send their signals to firm f_2 and firm f_3 correspondingly. Firm f_3 makes an offer to worker w_1, and firm f_1 anticipates that worker w_1 is atypical and makes an offer to worker w_2. Firm f_2 also makes its

[3]See Theorem 7 for the proof that these strategies constitute a sequential equilibrium.

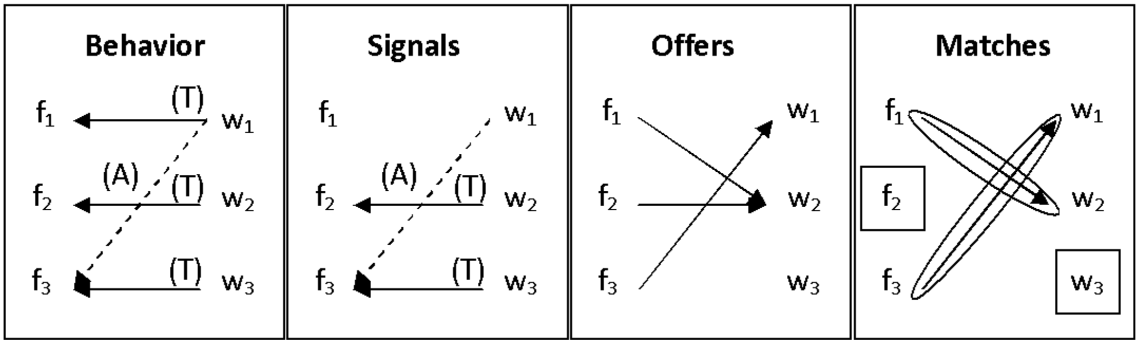

Figure 4.1.1: A simple example of harmful signaling.

offer to worker w_2 and eventually ends up unmatched because worker w_2 accepts firm f_1's offer. The coordination failure arises because firm f_2 has no information about worker w_1's type and cannot anticipate firm f_1's behavior. Thus, the number of matches for some realization of preferences is smaller than the number of matches when the signals are not allowed. Therefore, the expected number of matches is also smaller.

4.2 Model

We consider a two-sided matching model with W workers and F firms, $W \geq F$. The set of workers and the set of firms are denoted as $\mathcal{W}$ and $\mathcal{F}$ correspondingly. Both $\mathcal{W}$ and $\mathcal{F}$ include the empty set. Each worker w orders firms according to some strict preference list θ_w. Similarly, each firm f orders workers according to some preference list θ_f. $\Theta_{\mathcal{W}}$ and $\Theta_{\mathcal{F}}$ together comprise the set of all possible workers' and firms' preference lists.

Each agent a has cardinal utility compatible with her/its preference list θ_a.[4] If worker w with preferences θ_w is matched with firm f, she receives cardinal utility $u_w(f, \theta_w)$. Similarly, if firm f with preferences θ_f is matched with worker w,

[4]We employ cardinal utilities compatible with ordinal ranking similar to Bogomolnaia & Moulin (2001).

it receives cardinal utility $u_f(w, \theta_f)$. We assume that agent utility depends only on the rank of an agent with which it is matched. Specifically, the utility of an agent from being matched with an agent on the kth position in her/its preference list equals $u_a(k)$. We assume that agents have the same utility function; i.e. for any agent a, $u_a(k) = u(k)$. Our results do not depend on the last assumption; however, this assumption simplifies the exposition.

Additionally, agent's cardinal utility from being unmatched is normalized to zero. We also assume that there is no worker whom firms do not want to hire, and there is no worker who prefers being unemployed to being matched with some firm; i.e. for any k, $u(k) > 0$.

Each agent knows only her/its preferences and has some ex-ante common beliefs about the other agents' preferences. We consider an environment where each firm f has some fixed publicly known preference list θ_f. Each worker is one of two types: "typical" or "atypical". A "typical" worker w is denoted as $w(T)$. All workers of typical type have the same commonly known preference list θ_0. An "atypical" worker w is denoted as $w(A)$. The preferences of atypical workers are identically and independently distributed according to some distribution $A(\Theta_{\mathcal{W}})$. Each worker is ex-ante typical with probability $1-\varepsilon$ and atypical with probability ε, for some $\varepsilon \in (0, 1)$. Our main analysis considers the case when ε is small.[5] We also assume that the distribution of atypical preferences, $A(\Theta_{\mathcal{W}})$, has a full support, i.e. each firm can be the top firm of an atypical worker with positive probability.[6]

To model the influence of signals on congested markets, we consider a model with three periods:

[5]The exact bound on ε depends on the parameters of distribution $A(\Theta_{\mathcal{W}})$. However, for each distribution $A(\Theta_{\mathcal{W}})$, one could find an upper bound of ε. We provide a more detailed discussion in Section 4.6.

[6]Formally, for any $f \in \mathcal{F}$ and any $w \in \mathcal{W}$ $\Pr(f = \max_{\theta_w}(f' : f' \in \mathcal{F})) > 0$.

1. *Agents' preferences are realized. Each worker sends a signal to at most one firm; signals are sent simultaneously. Signals are observed only by firms who have received them.*

2. *Each firm makes an offer to at most one worker; offers are made simultaneously.*[7]

3. *Each worker may accept at most one offer from the set of offers she receives.*

We restrict our analysis to pure strategies.[8] A strategy of worker w is a duple $\sigma_w = (\sigma_w^1, \sigma_w^2)$ that represents her decisions at the first and third stages. A strategy of a worker at the first stage is to choose a firm she sends her signal to, $\sigma_w^1 : \Theta_{\mathcal{W}} \to \mathcal{F}$. A strategy of a worker on the last stage is to choose an offer among those available to her, $\sigma_w^2 : \Theta_{\mathcal{W}} \times 2^F \to \mathcal{F}$. A strategy of firm f is its decision at the second stage. Firm f chooses the worker to whom it makes an offer based on a set of signals it receives, $\sigma_f : 2^W \to \mathcal{W}$. The dependence of firm strategy on preferences is omitted, because we assume that each firm has some fixed preferences.

For a given strategy profile of agents $\sigma = (\sigma_w, \sigma_f)$ and realized agents' types $\theta \in (\Theta_{\mathcal{W}})^W \times (\Theta_{\mathcal{F}})^F$ one can determine the final matching and agents' utilities. We denote the utility of agent a given a strategy profile σ and a profile of types θ as $\pi_a(\sigma, \theta)$. The interim expected payoff of worker w with preferences θ_w from strategy σ_w when the other agents follow a strategy profile σ_{-w} equals

$$u_w(\sigma_w|\sigma_{-w}, \theta_w) = \sum\nolimits_{\theta_{-w}} t(\theta_{-w}) \pi_w((\sigma_w, \sigma_{-w}), (\theta_w, \theta_{-w})),$$

[7] *In practice, some firms should rationally make several offers, anticipating that some workers probably reject their offers. We do not model these strategic decisions.*

[8] The analysis of the offer game in which agents can use mixed strategies does not give additional intuition to our main result that signals could impede match formation for some environments. However, this analysis is available upon request.

where $t(\theta_{-w})$ denotes the joint distribution of all agents except worker w preferences. The interim expected payoff of firm f given a subset of received signals $\mathcal{W}^S \subset \mathcal{W}$, beliefs $\mu_f(\cdot|\mathcal{W}^S)$, and other agents' strategy profile σ_{-f} is

$$u_f(\sigma_f|\sigma_{-f}, \mathcal{W}^S) = \sum\nolimits_\theta \mu_f(\theta|\mathcal{W}^S)\pi_f(\sigma_f, \sigma_{-f}, \theta).$$

We employ the concept of sequential equilibrium for multi-stage games with observed actions and incomplete information in order to solve the game (see Fudenberg & Tirole, 1991).

Definition 9. *A strategy profile* (σ_w, σ_f) *and posterior beliefs* $\mu_f(\cdot|\mathcal{W}^S)$ *for each firm* f *and each subset of workers* $\mathcal{W}^S \subset \mathcal{W}$ *is a sequential equilibrium if*

- for any $w \in \mathcal{W}$, $\theta_w \in \Theta_{\mathcal{W}}$: $\sigma_w^1(\theta_w) \in \arg\max_{\alpha \in \mathcal{F}} u_w(\alpha|\sigma_{-w}, \theta_w)$,
- for any $f \in \mathcal{F}$, $\mathcal{W}^S \subset \mathcal{W}$: $\sigma_f(\mathcal{W}^S) \in \arg\max_{\beta \in \mathcal{W}} u_f(\beta|\sigma_{-f}, \mathcal{W}^S)$, and
- for any $w \in \mathcal{W}$, $\theta_w \in \Theta_{\mathcal{W}}$, $\mathcal{F}_0 \subset \mathcal{F}$: $\sigma_w^2(\mathcal{F}_0, \theta_w) \in \arg\max_{\gamma \in \mathcal{F}_0} u_w(\gamma, \theta_w)$,

where beliefs are defined using Bayes' rule.[9]

Now we introduce some notations that will be useful in our further discussion. Though worker strategy is a duple $\sigma_w = (\sigma_w^1, \sigma_w^2)$, we will talk mainly about worker strategies at the first stage. The reason is that each worker has a strictly dominant strategy at the last stage$-$accept the best offer available$-$since she knows her preferences and the preferences are strict. To simplify notation, we omit the upper index and write $\sigma_w(\theta_w)$ instead of $\sigma_w^1(\theta_w)$.

For convenience, we name firms according to the typical preference list $\theta_0 = (f_1, ..., f_F)$; i.e. f_1 is the best firm, f_2 is the second best, etc. Similarly, we name

[9]Off-equilibrium beliefs are defined by considering the limits of completely mixed strategies.

workers in the following way: worker w_1 is the best worker among all workers $\mathcal{W}$ according to firm f_1's preferences, $w_1 = \max_{\theta_{f_1}}(w|w \in \mathcal{W})$; worker w_2 is the best worker among $\mathcal{W}\backslash\{w_1\}$ according to firm f_2's preferences, $w_2 = \max_{\theta_{f_2}}(w|w \in \mathcal{W}\backslash\{w_1\})$; and so on. Generally, worker $w_i = \max_{\theta_{f_i}}(w|w \in \mathcal{W}\backslash\{w_1, ..., w_{i-1}\})$ if $i \leq F$. The other workers $\mathcal{W}\backslash\{w_1, ..., w_F\}$ are named according to some prespecified order.[10]

We say a subset of workers $\mathcal{W}^S \subset \mathcal{W}$ is *reached for firm f when workers follow strategy profile* $\sigma_\mathcal{W}$ if ex-ante probability that only workers from set $\mathcal{W}^S$ send their signals to firm f strictly more than zero.

Definition 10. A subset of workers $\mathcal{W}^S \subset \mathcal{W}$ is reached for firm f when workers follow strategy profile $\sigma_\mathcal{W}$ if

$$\Pr(\mathcal{W}_f^S = \mathcal{W}^S) = \sum\nolimits_\theta t(\theta) \prod\nolimits_{w \in \mathcal{W}^S} I_{\sigma_w(\theta_w)=f} \prod\nolimits_{w' \in \mathcal{W}\backslash\mathcal{W}^S} (1 - I_{\sigma_{w'}(\theta_{w'})=f}) > 0,$$

where $I_{\sigma_w(\theta_w)=f} = \begin{cases} 1 \text{ if } \sigma_w(\theta_w) = f \\ 0 \text{ otherwise} \end{cases}$ and $t(\theta)$ denotes the joint distribution of all agents' preferences.

We also say that firm f responds to worker w's signal, when workers follow strategy profile $\sigma_\mathcal{W}$, if her signal changes the strategy of firm f with positive probability.

Definition 11. Firm f responds to worker w's signal, when workers follow strategy profile $\sigma_\mathcal{W}$, if there exists a subset of workers $\mathcal{W}^S$, $w \notin \mathcal{W}^S$, such that both $\mathcal{W}^S$ and $\mathcal{W}^S \cup w$ are reached for firm f, and $\sigma_f(\mathcal{W}^S) \neq \sigma_f(\mathcal{W}^S \cup w)$.

We proceed with equilibrium analysis in the next section.

[10]For instance, if all firms have the same preferences θ^*, workers are named according to this preference list $\theta^* = \{w_1, ..., w_W\}$.

4.3 Equilibrium Analysis

As a benchmark, we first consider an environment in which workers cannot send signals. Then, the model outlined above is a static game of incomplete information. Therefore, the notion of sequential equilibrium coincides with the notion of Bayesian equilibrium and agents' beliefs are irrelevant. There is a unique equilibrium match in this case.

If signals are not allowed and ε is small, the only optimal strategy of firm f_1 is to make an offer to its best worker $w_1 = \max_{\theta_{f_1}}(w|w \in \mathcal{W})$. The second top firm anticipates that worker w_1 is likely to accept firm f_1's offer. Hence, the only optimal strategy of firm f_2 is to make an offer to its best worker among $\mathcal{W}\setminus\{w_1\}$, $w_2 = \max_{\theta_{f_2}}(w|w \in \mathcal{W}\setminus\{w_1\})$ and so on. Workers accept the best available offer. Overall, there is the maximum number of matches, F (since $F \leq W$), in the equilibrium when signals are not allowed.

Proposition 11. *For sufficiently small ε, there is a unique equilibrium when signals are not allowed: firm f_j, $j = 1, ..., F$, makes an offer to worker w_j; worker w_i, $i = 1, ..., F$, accepts the best available offer.*

We further call the match in our benchmark model as "no signaling" match.

Now, we analyze the set of equilibria in the matching market with signals. Though signals are voluntary in our model, they could still play a negative role and draw away firm offers. In order to eliminate such equilibria, we assume that if firm f makes an offer to worker w when it does not receive her signal, firm f makes an offer to worker w when it receives her signal. See Example C.01 in Appendix C for an example of an equilibrium in which Assumption PRS is violated.

Assumption PRS (Positive Role of Signals). For any firm $f \in \mathcal{F}$ and any worker $w \in \mathcal{W}$ and any $\mathcal{W}^S \subset \mathcal{W}, w \notin \mathcal{W}^S$, if $\sigma_f(\mathcal{W}^S) = w$ then $\sigma_f(\mathcal{W}^S \cup w) = w$.

We further distinguish three types of equilibria in the matching model with signals.

Definition 12. We define the following equilibrium notions.

- An equilibrium is "babbling" if no firm responds to any signal.
- An equilibrium is "informative", if at least one firm responds to some worker's signal.
- An equilibrium is "very informative", if each firm responds to all signals from workers better or equal to its no signaling match.

The set of the first and second type equilibria, i.e. babbling and informative, exhaust the set of all possible equilibria in our model. The set of equilibria of the last type is a subset of the set of informative equilibria.

A babbling equilibrium always exists in our model because signals are costless. If firms do not respond to signals, signals play no role in equilibria. Hence, the only possible match in a babbling equilibrium is no signaling match.

Proposition 12. *For sufficiently small* ε, the only possible match in a babbling equilibrium is no signaling match.

If some firms respond to signals, then signals transmit information about workers' preferences in an equilibrium, which changes the overall matching outcome. However, there is a great multiplicity of informative equilibria. One may suggest to use refinements proposed by (Cho & Kreps, 1987)and(Banks & Sobel, 1987).[11] However, these criteria are very powerful in the case of one sender and one receiver. The situation with many senders and receivers is more difficult. Though these criteria significantly reduce the number of equilibria, they do not guarantee uniqueness.

However, it is sufficient to restrict ourselves to the case in which each firm responds to all signals from workers better or equal to its no signaling match, i.e. very informative equilibria, in order to guarantee uniqueness. This equilibrium consists of the following strategies. Worker w_i sends her signal to the best firm among the firms that prefer worker w_i to their no signaling match $\Delta(w_i) = (f_j \in \mathcal{F} : w_i \succeq_{f_j} w_j)$. If firm f_j receives at least one signal from the set of workers $\Delta(f_j) = (w \in \mathcal{W} : w \succeq_{f_j} w_j)$, i.e. workers better or equal to worker w_j, it makes its offer to the best such worker; otherwise, it makes an offer to its best worker among $\mathcal{W} \backslash \{w_1, ..., w_j\}$.

Theorem 7. *For a sufficiently small ε, under Assumption PRS the set of strategies,*

- $\sigma_{w_i}(\theta_{w_i}) = \max_{\theta_{w_i}}(f \in \Delta(w_i)),$
- $\sigma_{f_j}(\mathcal{W}^S) = \begin{cases} \max_{\theta_{f_j}}(w : w \in \mathcal{W}^S) & \textit{if } \mathcal{W}^S \cap \Delta(f_j) \neq \varnothing \\ \max_{\theta_{f_j}}(w : w \in \mathcal{W}\backslash\{w_1, ..., w_j\}) & \textit{if } \mathcal{W}^S \cap \Delta(f_j) = \varnothing \end{cases},$

and the set of firms' beliefs consistent with agents' strategies constitute a unique very informative equilibrium.[12]

The above theorem is remarkable because it shows that the equilibrium of the model is unique, if we restrict our attention to the case in which firms use signals most extensively. However, we should point out that we do not intend to

[11] Cho & Kreps (1987) analyze *never a weak responce, intuitive criterion, D1,* and *D2* refinements. Banks & Sobel (1987) analyze *divinity* and *universal divinity* refinements.

[12] We should point out that there is a multiplicity of beliefs that could support this equilibrium on off-equilibrium path.

eliminate all other equilibria. First, the theorem illustrates typical agents' behavior in an informative equilibrium. Workers do not just send signals to the best firms. They send their signals to the best firms that respond to these signals, which is in line with AEA advice to participants in the job market for new Ph.D. economists(see AEA, 2005). Similarly, firms do not respond to all signals. Instead they respond to the signals from workers better than those they could secure in the no signaling equilibrium. Second, our results of welfare comparison do hold for most other sequential equilibria.

4.4 Welfare Properties of Equilibria

We evaluate the effect of signals on the matching market from an ex-ante perspective. We mainly use the following quantitative characteristics: the expected number of matches, the expected total welfare of firms, and the expected total welfare of workers.

Let us denote the ex-post number of matches for the profile of preferences $\theta \in \Theta_{\mathcal{W}} \times \Theta_{\mathcal{F}}$, when agents follow the profile of strategies σ as $m(\sigma, \theta)$. Then, the expected number of matches can be expressed as

$$E[M(\sigma)] = \textstyle\sum_{\theta} t(\theta) m(\sigma(\theta), \theta),$$

where $t(\theta)$ denotes the joint distribution of all agents' preferences. Similarly, the expected total welfare of workers and firms can be expressed as

$$E[W_{\text{firm}}(\sigma)] = \textstyle\sum_f \sum_{\theta} t(\theta) \pi_f(\sigma(\theta), \theta), \text{ and}$$
$$E[W_{\text{worker}}(\sigma)] = \textstyle\sum_w \sum_{\theta} t(\theta) \pi_w(\sigma(\theta), \theta)$$

correspondingly.

Proposition 11 shows that the expected number of matches in any no signaling equilibrium is the maximum one. Hence, it is impossible that the expected number of matches in any informative equilibrium exceeds the expected number of matches in any "no signaling" equilibrium. Example 1 and Example 2 demonstrate the case of strict inequality and equality for this welfare criterion.

Example 1 is presented in the introduction and considers the very informative equilibrium with three firms and three workers. To avoid a repetition, we do not discuss it here.

Example 1. There are three firms and three workers. Firms have the same ranking over workers (w_1, w_2, w_3). The typical worker preference list is $\theta_0 = (f_1, f_2, f_3)$. Atypical worker preferences are uniformly distributed. Firm f_j, $j = 1, 2, 3,$ and worker w_i, $i = 1, 2, 3,$ equilibrium strategies are

- $\sigma_{w_i}(\theta_{w_i}) = \max_{\theta_{w_i}}(f \in \Delta(w_i)),$
- $\sigma_{f_j}(\mathcal{W}^S) = \begin{cases} \max_{\theta_{f_j}}(w : w \in \mathcal{W}^S) & \text{if } \mathcal{W}^S \cap \Delta(f_j) \neq \varnothing \\ \max_{\theta_{f_j}}(w : w \in \mathcal{W} \backslash \{w_1, ..., w_j\}) & \text{if } \mathcal{W}^S \cap \Delta(f_j) = \varnothing \end{cases},$

and the set of firms' beliefs consistent with agents' strategies.

Example 2 shows that some informative equilibria could have the maximum expected number of matches. Intuitively, it is possible that if some firm f_j secures a better match with some atypical worker w_i, firm f_i always makes its offer to firm f_j's no signaling match, worker w_j, in an equilibrium. Therefore, firms exchange their matches and it does not decrease the number of matches.

Example 2. Let us consider three firms and three workers. All firms have the same preferences $\theta_{f_j} = \{w_1, w_2, w_3\}$. Let us consider the following equilibrium strategies:

- $\sigma_{w_1}(\theta_{w_1}) = \max_{\theta_{w_1}}(f : f \in \{f_1, f_2\})$ and $\sigma_{w_i}(\theta_{w_i}) = f_i$, $i = 2, 3$;
- $\sigma_{f_j}(\mathcal{W}^S) = \begin{cases} \max_{\theta_{f_j}}(w : w \in \mathcal{W}^S) & \text{if } \mathcal{W}^S \cap \Delta(f_j) \neq \varnothing \\ \max_{\theta_{f_j}}(w : w \in \mathcal{W} \backslash \{w_1, ..., w_j\}) & \text{otherwise} \end{cases}$, for $j = 1, 2$;
- $\sigma_{f_3}(\mathcal{W}^S) = \begin{cases} \max_{\theta_{f_3}}(w : w \in \mathcal{W}^S) & \text{if } \mathcal{W}^S \cap \Delta(f_3) \neq \varnothing \\ w_3 & \text{otherwise} \end{cases}$.

The set of equilibrium beliefs is such that if firm f_1 or f_2 receives a signal from worker w_1, it believes that it is worker w_1's top firm. If firm f_3 receives a signal from worker w_1, its belief coincides with her prior, i.e. worker w_1 is typical with probability $1-\varepsilon$ and atypical with probability ε. Similarly, if any firm f_j receives a signal from worker w_2 or w_3, its belief coincides with its prior. To put it briefly, only firm f_1 and firm f_2 respond to worker w_1's signal. All other signals are ignored. One may check that the described strategies constitute an informative equilibrium.

The results about the expected total welfare of firms and the expected total welfare of workers are not so straightforward and depend on the relative magnitudes of $u(k)$. The intuition is that signals in an informative equilibrium play two roles. On the one hand, signals help to secure "better" matches between some atypical workers and firms. On the other hand, the introduction of signals leaves some workers and firms unmatched. Example 3 illustrates that the introduction of signals is beneficial for a matching market according to egalitarian welfare criterion if and only if the decrease in the number of matches is offset by better matches of atypical workers. A similar example can show that the total welfare of firms changes ambiguously.

Example 3. Let us again consider the game of Example 1. Workers' cardinal utilities from being matched to first, second, and third choice are $\delta+\lambda$, δ, and $\delta-\lambda$

correspondingly. The expected total welfare of workers in no signaling match

$$E[W_{\text{worker}}^{\text{nosignals}}] = \sum_{i=1}^{3} \left[(1-\varepsilon)\, u(i) + \varepsilon \tfrac{1}{3} \sum_{l=1}^{3} u(l) \right] = 3\delta.$$

One may check that the expected total welfare of workers in very informative equilibrium is[13]

$$E[W_{\text{worker}}^{\text{signals}}] = 3\delta + \left(-\tfrac{1}{3}\delta + \tfrac{19}{6}\lambda\right)\varepsilon$$

Hence, the expected total welfare of workers increases, if and only if the difference in utilities between adjacent firms is large enough, $\lambda > \frac{2}{19}\delta$.

The theorem below summarizes the results derived above.

Theorem 8. *For a sufficiently small* ε :

- *the expected number of matches in any informative equilibrium is weakly fewer than in any no signaling equilibrium;*
- *the effect of signals on the expected total welfare of firms and the expected total welfare of workers is ambiguous.*

We have compared above the properties of any informative and no signaling equilibrium. However, more strict result holds for very informative equilibrium. Under the assumption that there are at least three workers and there exists a worker w, such that there are at least three firms that weakly prefer worker w to their no signaling matches, $|\{f_j \in \mathcal{F} : w \succeq_{f_j} w_j\}| \geq 3$, the expected total number of matches is strictly fewer in very informative equilibrium than in the corresponding no signaling match. The intuition for this result is similar to the one for Example 1. If firms respond to signals, some of the realized

[13]Terms of the order of ε^2 and ε^3 are ignored.

matches differ from no signaling match. Moreover, if at least three firms respond to some worker's signal the exchange of matches$-$the situation presented in Example 2$-$is impossible for each realization of preferences. Therefore, the expected number of matches is smaller than the maximum one in this case.

Theorem 9. *For sufficiently small ε, if there are at least three workers and for some worker w, $|\Delta(w)| \geq 3$,* the expected number of matches is strictly smaller in the very informative equilibrium than in the corresponding no signaling equilibrium.

Theorem 8 proves that the expected total welfare of workers changes ambiguously with the introduction of signals. However, the following proposition shows that signals are harmful to workers only because they deprive them of matches. Workers receive weakly better offers conditional on the fact that they receive any offer.

Proposition 13. *If a worker receives an offer in any informative equilibrium, this offer is from a firm weakly better than her no signaling match.*

It is easy to see that the above statement is not true for firms, because some firms may have to make offers to workers worse than their no signaling match if she is atypical.

4.5 Public Signals

One could conjecture that should signals be public, they would always benefit match formation. Public signals introduce no asymmetry of information among firms. Firms have the same beliefs about the distribution of workers' preferences and the same beliefs about the strategies other firms use. Therefore, firms should be able to make offers that are more likely to be accepted. Unfortunately, this intuition is incorrect. This section illustrates that the expected number of matches in an equilibrium of the offer game with public signals could be smaller than the expected number of matches in the offer game without signals.

We consider a market with three firms and three workers. The distribution of agents' preferences is the same as in Section 4.2. Each worker can send at most one signal and accept at most one offer. Each firm has only one vacant position and can make at most one offer. The timing of the game is as follows:

1. Agents' preferences are realized. Each worker sends a signal to at most one firm; signals are sent simultaneously. All agents observe what signals each firm receives.

2. Each firm makes an offer to at most one worker; offers are made simultaneously.

3. Each worker chooses an offer to accept from the set of offers she receives.

The only difference from the game we considered previously is that all agents observe the signals each firm receives. The strategies of workers are the same as in Section 4.2. However, a strategy of firm f now depends on the set of signals each firm receives, $\sigma_f : \mathcal{F}^W \rightarrow \mathcal{W}$.[14]

As previously, the only equilibrium outcome of the offer game with signals is a full match. However, the expected number of matches could be smaller than three if we allow workers to send public signals. Intuitively, public signals do not transmit enough information about workers' preferences. This could introduce a considerable amount of uncertainty about workers' preferences. Therefore,

[14]Note that we again omit the dependence of strategies on firms' preferences, as we assume that each firm has fixed commonly known preferences.

some firms can optimally engage in a competitive behavior for some workers; i.e. firms make their offers to the same worker in an equilibrium. This produces mismatches.

Example 4. There are three firms and three workers. Firms have the same ranking over workers, which we denote as (w_1, w_2, w_3). The typical worker preference list is (f_1, f_2, f_3). The atypical worker preferences are uniformly distributed. We assume that all firms have the same cardinal utility and their utility from being matched to second top worker, i.e. $u(2)$, is at least twice as great as the cardinal utility from being matched to the third top worker, i.e. $u(3)$.

We consider the following strategies of agents in the offer game with public signals. Worker w_i sends her signal to the best firm among the firms that weakly prefer worker w_i to their no signaling match $\Delta(w_i) = (f_j \in \mathcal{F} : w_i \succeq_{f_j} w_j)$:

$$\sigma_{w_i}(\theta_{w_i}) = \max_{\theta_{w_i}}(f \in \Delta(w_i)).$$

Firms use the following strategies.

1. Firm f_1 makes an offer to worker w_1, if it receives a signal from her; otherwise, it makes an offer to worker w_2.
2. Firm f_2 makes an offer to worker w_1, if it receives a signal from her. Firm f_2 makes an offer to worker w_3, if either worker w_1 sends a signal to firm f_1 and worker w_2 sends a signal to firm f_3 or worker w_1 sends a signal to firm f_3 and worker w_2 sends a signal to firm f_2. In all other cases, firm f_2 makes an offer to worker w_2.
3. Firm f_3 makes an offer to the best worker from whom it receives a signal. If it receives no signals, it makes an offer to worker w_3.

Each firm's beliefs on the equilibrium path are consistent with agents' strategies and each firm off-equilibrium beliefs coincide with priors.

Let us consider the strategies outlined in Example 4. Mismatches happen when both worker w_1 and worker w_2 are atypical. If at least two atypical workers send their signals to the same firm, only one worker receives an offer from it. Since, signals are public, all other firms infer that the other worker is atypical. This creates a considerable amount of uncertainty about the worker preferences. Since, this worker could be a good one, firms have incentives to compete for her.

Another reason for excessive competition among firms is that signals may not transmit information about workers' top firms. Some workers send their signals to firms that differ from their top ones in an equilibrium, because they want to attract feasible offers. Therefore, several firms could have incentives to make an offer to a given worker. This creates competition among firms, which again lead to mismatches.

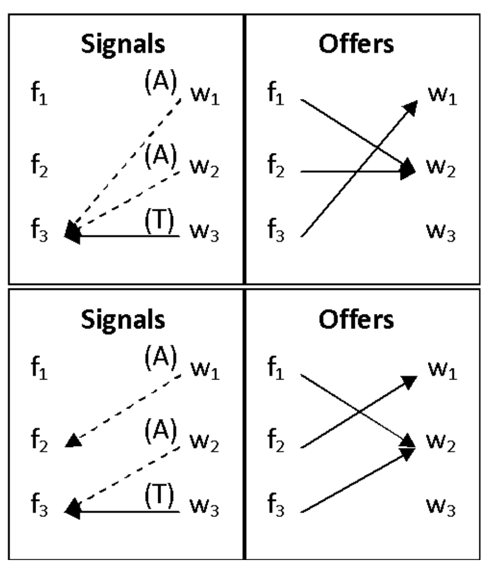

Figure 4.5.1: Public Signals

Proposition 14 formally proves that the set of strategies in Example 4 constitutes a sequential equilibrium.

Proposition 14. *The set of strategies in Example 4 constitutes a sequential equilibrium.*

The implications of the above example can be summarized by way of two observations. First, public signals do not transmit enough information about workers' preferences. This could introduce uncertainty about workers preferences and induce excessive firm competition for the same workers. This results in mismatches.

In addition, mismatches in the offer game with public signals occur only if there at least two atypical workers, which happens only with probability of the order ε^2. In contrast, mismatches in the offer game with private signals occur with the probability of the order ε. Therefore, mismatches happen less often when signals are public.

4.6 Conclusion

There is a general belief that preference signaling should facilitate match formation (see Crawford & Sobel, 1982; Roth, 2008; AEA, 2005). This belief is also supported by the analysis of Chapter 2 and 3. We show in this chapter that this belief can be erroneous for some matching markets. We exemplify an environment in which the introduction of signals harms matching markets: it decreases the expected number of matches and ambiguously affects the welfare of firms and the welfare of workers. Finally, we show that making signals observable to all agents does not change the welfare applications.

In conclusion, we want to discuss some assumptions of our model. We analyze the introduction of signals to congested decentralized matching markets, as we believe the job market for new Ph.D. economists to be. The fact that we do not analyze any centralized clearinghouse mechanism and stable matches captures the idea of decentralized markets. The fact that we analyze a one-period model captures the idea of congestion. Moreover, several (but finite) periods of interactions between firms and workers would give an opportunity for firms to secure better matches; but signals would still introduce information asymmetry. If each worker sent several signals, these would transmit information to a greater number of firms, but each signal would be less informative. Several vacant positions would make only the preferences of firms more complicated and would not influence the results. Overall, the roles of signals in match formation are robust to these modifications.

Chapter 5

Role of Signals in Matching Markets

In previous chapters, we have examined a natural signaling mechanism: allowing workers to send costless private signals to a finite set of firms. While participation is costless and voluntary, this mechanism nevertheless provides workers with a means of expressing preferences that is both credible and equitable. The introduction of such mechanism leads to the increase in the expected number of matches and welfare of workers for a wide range of environments. However, there are instances when the effect of such mechanism may be harmfull. In this chapter we will try to analyze why the effects may be controversal and what roles signaling plays in matching markets.

In a setting where workers agree on the ranking of blocks of firms but vary in their preferences within each block, workers will signal to their most preferred firm within each block. Firms use this information as guidance, optimally using cutoff strategies to make offers. We find that on average, introducing a signaling technology increases both the expected number of matches as well as the expected welfare of workers. The welfare of firms, on the other hand, changes ambiguously, because firms responding more to signals imposes a negative externality on other firms. If the preferences of workers are almost complete and the preferences of firms are fixed and commonly known, the introduction of signals decreases the expected number of matches. The effect of signals on the expected total welfare of agents is ambiguous. Overall, Table 5.1 presents the effects from the introduction of the signals for the two different environments: almost complete and block-uniform distribution of preferences.

Preferences	No signals	$Matches$	W_{worker}	W_{firm}
Almost complete	0	$-$	$\pm$	$\pm$
Block-uniform	0	$+$	$+$	$\pm$

Table 5.1: Welfare effects in models with almost complete and block-uniform distribution of agent preferences

A natural question is why signals influence matching markets in different ways. Our explanation is that the signals play two different roles: transmit information and facilitate information asymmetry. On the one hand, the introduction of signals helps atypical workers to transmit information about their preferences and locate a better match. On the other hand, signals transmit information only to some firms, thus facilitating information asymmetry. This information asymmetry leads to coordination failures that decrease the number of matches.

When there is ex-ante small amount of information about agents' preferences, information transmission plays a more important role in match formation. This happens when agents' preferences are disperse, as in Chapter 2 and 3. However, when there is almost complete information about agents' preferences, as in Chapter 4, the introduction of signals leads to coordination failures. Table 5.2 presents the comparison.

Preferences	Transmit information	Facilitate information asymmetry
Almost complete	Small	**Large**
Block-uniform	**Large**	Small

Table 5.2: The roles of signals in matching markets

The aim of this dissertation is to examine the value that a signaling mechanism might offer congested markets, and how that value varies across market structures and signal designs. We hope that our approach will serve as a benchmark for examining settings with alternative market assumptions and a point of comparison for alternative signaling mechanisms.

Appendix A

Appendix: Preference Signaling in Matching Markets

Proof of Proposition 1.

This proposition is a special case of Proposition B.11. □

Proof of Proposition 2.

This proposition is a special case of Proposition B.14. □

Proof of Theorem 1.

In any symmetric non-babbling equilibrium each worker sends its signal to its most preferred firm. Since all information sets for firms are realized with positive probability firm beliefs are determined by Bayes' Law: if a firm receives a signal from a worker, it believes that worker ranks the firm first in its preference list. Taking worker strategies and firm beliefs as fixed, we show that the second stage of the game when firms choose their strategies is a game with strategic complementarities of Milgrom & Roberts (1990).

We denote the set of firm cutoff strategies as Σ_{cut} with $\sigma = (\sigma_1, ..., \sigma_F)$ being its typical element. The strategy of firm f is a vector $\sigma_f = (j_f^1, ..., j_f^W)$ where j_f^k corresponds to a cutoff when firm f receives k signals. We consider each cutoff being a natural number, i.e. $j_f^k \in \{1, ..., W\}$. As defined on p.21 a natural partial order on Σ_{cut} is defined as: $\sigma \geq_{\Sigma_{cut}} \sigma' \Leftrightarrow \sigma_f \geq \sigma'_f \Leftrightarrow j_f^k \geq \left(j_f^k\right)'$ for any $f \in \mathcal{F}$ and $k \in \{1, ..., W\}$. This partial order is reflexive, antisymmetric, and transitive.

In order to show that the second stage of the game is a game with strategic complementarities we need to check whether $E_\theta(\pi_f(\sigma_f, \sigma_{-f}, \theta))$ is supermodular in σ_f and $E_\theta(\pi_f(\sigma_f, \sigma_{-f}, \theta))$ has increasing differences in σ_f and σ_{-f}. The former one follows from the fact that the change of payoff from a shift of one cutoff vector component does not influence the change of payoff from a shift of another cutoff vector component. Namely, if we consider $\sigma_f^1 = (..., j_l, ..., j_k, ...)$, $\sigma_f^2 = (..., j'_l, ..., j_k, ...)$, $\sigma_f^3 = (..., j_l, ..., j'_k, ...)$, and $\sigma_f^4 = (..., j'_l, ..., j'_k, ...)$ for some $l, k \in \{1, ..., W\}$, then

$$E_\theta(\pi_f(\sigma_f^1, \sigma_{-f}, \theta)) - E_\theta(\pi_f(\sigma_f^2, \sigma_{-f}, \theta)) = E_\theta(\pi_f(\sigma_f^3, \sigma_{-f}, \theta)) - E_\theta(\pi_f(\sigma_f^4, \sigma_{-f}, \theta))$$

The fact that $E_\theta(\pi_f(\sigma_f, \sigma_{-f}, \theta))$ has increasing differences in σ_f and σ_{-f} follows from Proposition 2. Namely, for any σ_f, σ_{-f}, σ'_f, and σ'_{-f} such that $\sigma'_f \geq \sigma_f$ and $\sigma'_{-f} \geq \sigma_{-f}$ we have that

$$E_\theta(\pi_f(\sigma'_f, \sigma'_{-f}, \theta)) - E_\theta(\pi_f(\sigma_f, \sigma'_{-f}, \theta)) \geq E_\theta(\pi_f(\sigma'_f, \sigma_{-f}, \theta)) - E_\theta(\pi_f(\sigma_f, \sigma_{-f}, \theta))$$

As a consequence, the second stage of thegame when firms choose their strategies is a game with strategic complementarities. Since our game is symmetric (unchanged by permutations of firms indices), theorem 5 of Milgrom & Roberts (1990) establishes the existence of the largest and the smallest symmetric pure strategy equilibrium in our environment.□

Proof of Proposition 3.

The first two results, increase in the expected number of matches and positive spillover on workers, directly follows from Theorem 4 in Section 3.1 that considers a more general assumption on agent preferences. The result that if at least one firm in $-f$ responds to signals, then all inequalities are strict also follows from Theorem 4. We do not present the proofs of these results here in order to avoid repetitions. However, the negative spillover on opponent firms result is unique to the case when agent preferences are uniformly distributed. We present the proof of the latter statement below.

Firm f strategy σ_f differs from σ'_f only in that σ'_f has weakly greater cutoffs. Let us consider some firm $f' \in -f$. For each profile of preferences $\theta_{f'}$ and a set of signals $\mathcal{W}^S$, firm f' either makes an offer to $S_{f'}(\theta_{f'}, \mathcal{W}^S)$ or $T_{f'}(\theta_{f'}, \mathcal{W}^S)$. Since each worker sends her signal to her best firm, the change in firm f strategy does not influence the decision of worker $S_{f'}$ about acceptance of firm f' offer. However, as shown in the proof of Proposition 2, the probability that $T_{f'}$ accepts firm f' offer weakly decreases. Overall, the expected payoff of firm $f' \in -f$ weakly decreases when firm f responds more to signals.

$$E_\theta(\pi_{f'}(\sigma_f, \sigma_{-f}, \theta)) \geq E_\theta(\pi_{f'}(\sigma'_f, \sigma_{-f}, \theta)).$$

□

Proof of Corollary 1.

The result that the expected number of matches and the expected welfare of workers is higher in the equilibrium with higher cutoffs is an immediate consequence of the first and the second statement of Proposition 3. In order to show that firms have lower expected payoffs in the equilibrium with greater cutoffs we use the third result of Proposition 3. Let us consider two symmetric equilibria where firms play cutoff strategies σ and σ' correspondingly, such that $\sigma' \geq \sigma$. From the definition of an equilibrium strategy we have:

$$E_\theta[\pi_f(\sigma_f, \sigma_{-f}, \theta)] \geq E_\theta[\pi_f(\sigma'_f, \sigma_{-f}, \theta)]$$

Using the third result of Proposition 3 we proceed with

$$E_\theta[\pi_f(\sigma'_f, \sigma_{-f}, \theta)] \geq E_\theta[\pi_f(\sigma'_f, \sigma'_{-f}, \theta)]$$

Therefore

$$E_\theta[\pi_f(\sigma_f, \sigma_{-f}, \theta)] \geq E_\theta[\pi_f(\sigma'_f, \sigma'_{-f}, \theta)]$$

□

Proof of Theorem 2.

The result immediately follows from Theorem 4 in Section 3.1 that considers a more general setup.

Appendix B

Appendix: Preference Signaling in Matching Markets: Extensions

B.1 Block-Correlated Preferences

Proposition B.11. *The unique equilibrium of the offer game with no signals when firms use anonymous strategies and workers accept the best available offer is $\sigma_f(\theta_f) = \theta_f^1$ for all $f \in \mathcal{F}$ and $\theta_f \in \Theta_f$.*

Proof.

Let us consider some realization of agent preference profile $\theta \in \Theta$. We define function $m_f : (\Sigma_w)^W \times (\Sigma_f)^F \times \Theta \to \mathbf{R}$ being the probability of firm f being matched as a function of agent strategies and types. We compare two strategies of firm f: making an offer to its top worker $\sigma_f = \theta_f^1 = w$ and making an offer to its nth top worker $\sigma'_f = \theta_f^n = w^n$, $n > 1$. We denote a permutation that changes the ranks of w and w^n in a firm preference profile as

$$\rho : (..., w, ..., w^n, ...) \longrightarrow (..., w^n, ..., w, ...)$$

Let us consider a profile of preferences $\theta' \in \Theta$ such that firm f preferences is the same as for profile θ, $\theta'_f = \theta_f$; the ranks of workers w and w^n are exchanged in the preference lists of firms $-f$

$$\text{for any firm } f' \in -f,\ \theta'_f = \rho(\theta_f),$$

worker w and worker w^n preference profiles are exchanged

$$\theta'_w = \theta_{w^n},\ \theta'_{w^n} = \theta_w,$$

and $\theta_{w'} = \theta^{'}_{w'}$ for any other $w' \in \mathcal{W} \setminus \{w, w^n\}$.

Since, firm $-f$ strategies are anonymous we have that

$$\sigma_{-f}(\theta'_{-f}) = \sigma_{-f}(\rho(\theta_{-f})) = \rho\left(\sigma_{-f}(\theta_{-f})\right)$$

This means that the probability of firm f', $f' \in -f$, making an offer to worker w for profile θ equals the probability of making an offer to worker w^n for profile θ'. Moreover, since we exchange worker w and w^n preference lists for profile θ', whenever it is optimal for worker w to accept firm f offer for profile θ, it is optimal for worker w^n to accept firm f's offer for profile θ'. Therefore,

$$m_f(\sigma_f, \sigma_{-f}, \theta) = m_f(\sigma'_f, \sigma_{-f}, \theta')$$

Therefore, for each θ_{-f} there exists θ'_{-f} such that the probability of getting an offer from the top worker equals the probability of getting an offer from nth top worker. Moreover θ'_{-f} is different for different θ_{-f} by the construction. Since, θ_{-f} and θ'_{-f} are equally possible:

$$E_{\theta_{-f}} m_f(\sigma_f, \sigma_{-f}, \theta|\theta_f) = E_{\theta_{-f}} m_f(\sigma'_f, \sigma_{-f}, \theta|\theta_f)$$

and

$$E_\theta m_f(\sigma_f, \sigma_{-f}, \theta) = E_\theta m_f(\sigma'_f, \sigma_{-f}, \theta)$$

Therefore, the expected probability of getting a match from firm f's top choice equals the expected probability of getting a match from firm f's nth top choice. Since, the utility from obtaining a former match is greater, the strategy of firm f of making an offer to its top worker is optimal. □ **Proof of Proposition 4.**

We first define criterion $D1$ of Cho and Kreps in our setting[1]. Let us consider some block-symmetric sequential equilibrium. We fix strategies of agents except worker w and firm f, which we denote as $\sigma_{-f,w}$. We also fixed the beliefs of firms other than firm f, which we denote as μ_{-f}.

We now analyze strategies of worker w and strategies and beliefs for firm f. Worker w sends some off-equilibrium message to firm f whenever she sends a signal when the equilibrium strategy prescribes zero probability of sending a signal to firm f or she does not send a signal when the equilibrium strategy prescribes sending a signal to firm f with probability equal 1. According to the definition of anonymous strategies, the latter one can happen in block-symmetric equilibrium only if firm f is the only one firm in its block. However, the symmetry of workers strategies ensures that all workers send their signals to firm f with probability 1 in this case. Since, signals do not transmit information about workers type, this equilibrium is outcome equivalent to babbling equilibrium. We further concentrate on the first type of off-equilibrium messages.

Consider worker w who sends some off-equilibrium signal to firm f. We denote the expected equilibrium payoff of firm f as u^*_f and the expected equilibrium payoff of worker w as u^*_w. For each firm f's type $\bar{\theta} \in \Theta_f$ and each set of signals that firm f could receive, $\mathcal{W}^S \subset \mathcal{W}$, we denote the mixed best response of firm f

[1]See Cho & Kreps (1987) for the original definition.

that has beliefs $\bar{\mu}$ as

$$MBR_f(\bar{\theta}, \mathcal{W}^S \cup w, \bar{\mu}) = \arg\max_{\sigma_f \in \Sigma_f} E_{\theta_{-f}}(\pi_f(\sigma_f, \sigma_{-f}, \theta) \,|\theta_f = \bar{\theta},\ \mathcal{W}_f^S = \mathcal{W}^S \cup w,\ \mu_f = \bar{\mu}).$$

We then denote the mixed best response of firm f for any possible types and profiles of signals it may receive conditional on receiving worker w's signal as

$$MBR_f(w, \bar{\mu}) = \{MBR_f(\bar{\theta}, \mathcal{W}^S \cup w, \bar{\mu}) \text{ for all } \bar{\theta} \in \Theta_f,\ \mathcal{W}^S \subset \mathcal{W}\}$$

We denote the set of best responses of firm f to probability assessments concentrated on set $\Omega \subset \Theta_w$ as

$$MBR_f(w, \Omega) = \bigcup_{\{\mu_f : \mu_f(\Omega)=1\}} MBR_f(w, \mu_f)$$

We also denote for any worker's type $t \in \Theta_w$

$$\begin{aligned} D_t &= \{\phi \in MBR_f(w, \Theta_w) :\ u_w^*(t) < E_{\theta_{-w}}(\pi_w(\sigma_w, \phi, \sigma_{-w,f}, \theta) \,|\theta_w = t)\} \\ D_t^0 &= \{\phi \in MBR_f(w, \Theta_w) :\ u_w^*(t) = E_{\theta_{-w}}(\pi_w(\sigma_w, \phi, \sigma_{-w,f}, \theta) \,|\theta_w = t)\} \end{aligned}$$

Intuitively, set D_t (D_t^0) is the set of firm f strategies such that worker w of type t, receives an expected payoff that is greater than (equal to) the equilibrium one. We say that the type t may be pruned from firm f's beliefs if firm f's off-equilibrium beliefs should put zero probability on worker w's type t upon receiving a signal from her. Using the above notations criterion $D1$ in our setting can be stated as follows.

Criterion D1. Fix strategies of workers $-w$ and strategies and beliefs of firms $-f$. If for some worker w's type $t \in \Theta_w$ there exists a second worker w's type $t' \in \Theta_w$ with $D_t \cup D_t^0 \subseteq D_{t'}$, then t may be pruned from the domain of firm f's beliefs.

The intuition behind this criterion is that whenever type t of worker w either wishes to defect and send an off-equilibrium signal to firm f or indifferent, some other type t' of worker w strictly wishes to defect. When we prune t of worker w from firm f's beliefs, we believe that firm f puts infinitely more likely probability that off-equilibrium signal has come from type t' than from type t.

We first show that there cannot be a *block-symmetric* sequential equilibrium that satisfies criterion $D1$ such that the ex-ante probability of receiving an offer by some worker from a firm within block $\mathcal{F}^b$, $b \in \{1, ..., B\}$, is smaller when the worker sends her signal to this firm compare to the case when she does not send her signal to this firm, i.e. $p_b^s < p_b^{ns}$.

Let us assume that such block-symmetric sequential equilibrium exists. If there are at least two workers, agents use anonymous block-symmetric strategies, and agents' types are uncorrelated, each worker is unmatched with positive probability. Then, it is optimal for each worker to avoid sending her signal to any firm within block $\mathcal{F}^b$ on the equilibrium path (she want to increase the probability of being matched to each firm). Therefore, some firm $f \in \mathcal{F}^b$ can receive a signal from some worker w only on off-equilibrium path. If it is beneficial for some type $\theta_w \in \Theta_w$ to deviate from the equilibrium path and send her signal to firm f (in this case firm f makes an offer to worker w), then it is beneficial for any type of worker w, $\theta'_w \in \Theta_w$, such that firm f is the top firm within block $\mathcal{F}^b$, to deviate. Therefore, the only types (preference profiles) of worker w that are not pruned in firms believes according to criterion $D1$ are such that firm f is the top firm within block $\mathcal{F}^b$ for worker w. Therefore, if it is optimal for firm f to make an offer to worker w when it does not receive her signal, it is optimal for firm f to make an offer to worker w when it receives her signal. This means that $p_{b_0}^s < p_{b_0}^{ns}$ cannot be part of a block-symmetric sequential equilibrium that satisfies criterion $D1$.

As a consequence of the above argument, we have that for any $b = 1, ..., B$ $p_b^s \geq p_b^{ns}$. It is easy to observe, that there is exist a block-symmetric sequential equilibrium that satisfies criterion $D1$, when for any $b = 1, ..., B$, $p_b^s = p_b^{ns}$. For example, each worker uses the strategy that prescribes sending her signal to firms with equal probability independently on their preferences and firms play the equilibrium strategies of the offer game with no signals. The equilibrium beliefs are block-uniform, i.e. if firm receives a signal from worker w its beliefs coincide with the priors. As one could see there are no off-equilibrium paths need to be specified. Therefore, this sequential equilibrium satisfies criterion $D1$.

Let us now consider the case when there exists $b_0 \in \{1, ..., B\}$, such that $p_{b_0}^s > p_{b_0}^{ns}$ in some block-symmetric sequential equilibrium. Recall that the probability that a worker sends her signal to firms within block $\mathcal{F}^b$ is denoted as α_b, and we have that $\alpha_b \in [0, 1]$ and $\sum_{b=1}^{B} \alpha_b \leq 1$. Let us consider some block $\mathcal{F}^b$, such that $\alpha_b > 0$. As we mentioned above, if there are at least two workers, agents use anonymous block-symmetric strategies, and agents' types are uncorrelated, each worker is unmatched with positive probability in equilibrium. Therefore, $\alpha_b > 0$ and $p_b^s = p_b^{ns}$ are incompatible in an equilibrium (worker w can benefit by signaling to block $\mathcal{F}^{b_0}$ rather than block $\mathcal{F}^b$). Then, if $p_b^s > p_b^{ns}$ worker w should send her signal to her top firm within block $\mathcal{F}^b$, as it delivers the greatest expected payoff to her.

Let us now consider some block $\mathcal{F}^{b'}$, $b' \in \{1, ..., B\}$, such that $\alpha_{b'} = 0$, and some firm $f \in \mathcal{F}_{b'}$ that receives a signal on off-equilibrium path from some worker w. Therefore, either there exists type $t \in \Theta_w$ of worker w such that $D_t \neq \varnothing$ or for any type $t \in \Theta_w$ we have that $D_t = \varnothing$. For the former case, if worker w sends a signal to firm f, firm f offer delivers expected payoff to worker w of type t greater than the equilibrium one. However, whenever firm f offer delivers greater expected payoff than equilibrium one to worker w of type t, it delivers greater expected payoff than equilibrium one to worker w of type t', which prefers firm f to any other firm in block $\mathcal{F}_{b'}$. Therefore, the only firm f off-equilibrium beliefs that survive criterion $D1$ are such that

$$\mu_f(\{\theta_w \in \Theta_w : f = \max_{\theta_w}(f' \in \mathcal{F}_{b'})\}|w \subset \mathcal{W}_f^S) = 1 \tag{B.1.1}$$

Since $D_{t'}$ and $D^0_{t'}$ consist of firm f's best responses, it is optimal for firm f to make an offer to worker w upon receiving her signal if one restricts her beliefs to (B.1.1). This means that the equilibrium strategy of worker w of type t' (not sending a signal to firm f) is not optimal if firm f has beliefs (B.1.1). Therefore, there cannot exist type $t \in \Theta_w$ of worker w such that $D_t \neq \varnothing$ in a block-symmetric sequential equilibrium that satisfies criterion $D1$.

Let us now consider the case when for any type $t \in \Theta_w$ we have that $D_t = \varnothing$. Therefore, it is not beneficial for any type of worker to send an off-equilibrium signal. Therefore, $\alpha_{b'} = 0$ is an equilibrium strategy of worker w independently on off-equilibrium beliefs of firm f. Worker w strategy is optimal for any off-equilibrium beliefs of firms from block $\mathcal{F}^{b'}$, even if each firm f has most favorable for worker w beliefs, such as in (B.1.1).

Note that if there are at least two workers, the interaction between worker w and some firm f (fixing the strategies and beliefs of other agents) is a monotonic signaling game of Cho & Sobel (1990). The assumption of monotonicity is satisfied in our environment because each type of worker w prefers the same action of firm f, i.e. firm f making an offer to worker w. As a consequence, criterion $D1$ is equivalent to “never a weak best response” of Cho & Kreps (1987) and “universal divinity” of Banks & Sobel (1987) in our setting. More detailed discussion of monotonic signaling games can be found in Cho & Sobel (1990).□

Proposition B.12. *Suppose firms $-f$ use anonymous strategies and workers use symmetric best-in-block strategies. Consider a firm f that receives signals from* *workers $\mathcal{W}^S \subset \mathcal{W}$. Then the expected payoff to f from making an offer to S_f is strictly greater than the payoff from making an offer to any other worker in $\mathcal{W}^S$. The expected payoff to firm f from making an offer to T_f is strictly greater than the payoff from making an offer to any other worker from set $\mathcal{W}/\mathcal{W}^S$.*

Proof.

Let us consider firm f from some block $\mathcal{F}_b$, $b \in \{1, ..., B\}$ that has the realized preference profile $\theta^* \in \Theta_f$ and that receives signals from the set of workers $\mathcal{W}^S \subset \mathcal{W}$. We denote worker S_f as w and some other worker from $\mathcal{W}^S$ as w'. We first prove that the expected payoff to f from making an offer to worker w is strictly greater than the expected payoff from making an offer to worker w'. We denote the corresponding strategies of firm f as $\sigma_f(\theta^*, \mathcal{W}^S) = w$ and $\sigma'_f(\theta^*, \mathcal{W}^S) = w'$.

According to Proposition 4, firm f believes that it is the top firm within block $\mathcal{F}_b$ in workers w and w' preference lists. Let us denote the set of possible agents profiles consistent with firm f beliefs as

$$\bar{\Theta} \equiv \{\theta \in \Theta | \theta_f = \theta^* \text{ and } f = \max_{\theta_w}(f' \in \mathcal{F}_{b'}) \text{ for each } w \in \mathcal{W}^S\}$$

Similar to the proof of Proposition B.11, we denote a permutation that changes the ranks of w and w' in a firm preference profile as

$$\rho : (..., w, ...w', ...) \rightarrow (..., w', ...w, ...)$$

Let us consider a profile of agents preferences $\theta' \in \Theta$ such that $\theta'_f = \theta^*$, the ranks of workers w and w' are exchanged in the preference lists of firms $-f$

$$\text{for any firm } f' \in -f,\ \theta'_f = \rho(\theta_f),$$

worker w and worker w' preference profiles are exchanged

$$\theta'_w = \theta_{w'},\ \theta'_{w'} = \theta_w,$$

and for any other $w^0 \in \mathcal{W} \backslash \{w, w'\}$, $\theta_{w^0} = \theta'_{w^0}$. Since firm f preference list is unchanged, $\theta'_f = \theta^*$, and w,$w' \in \mathcal{W}^S$ profile θ' belongs to $\bar{\Theta}$. Since firm $-f$ strategies are anonymous for any $f' \in -f$ and for any $\mathcal{W}_0^S \subset \mathcal{W}$ we have that

$$\sigma_{f'}(\rho(\theta_{f'}), \rho(\mathcal{W}_0^S)) = \rho\left(\sigma_{f'}(\theta_{f'}, \mathcal{W}_0^S)\right)$$

Worker w and w' send their signals to firm f for both profiles θ and θ'. Therefore, they do not send their signals to firms $-f$, i.e. $\rho(\mathcal{W}_0^S) = \mathcal{W}_0^S$. Since $\theta'_f = \rho(\theta_f)$ we have that

$$\sigma_{f'}(\theta'_{f'}, \mathcal{W}_0^S) = \rho\left(\sigma_{f'}(\theta_{f'}, \mathcal{W}_0^S)\right)$$

This means that the probability of firm f' making an offer to worker w for profile θ equals the probability of making an offer to worker w' for profile θ'. Moreover, since we exchange worker w and w' preference lists for profile θ', whenever it is optimal for worker w to accept firm f offer for profile θ, it is optimal for worker w' to accept firm f's offer for profile θ'. Since firms types are independent, the probability of firm f being matched when it uses strategy σ_f for profile θ equals the probability of firm f being matched when it uses strategy σ'_f for profile θ'

$$m_f(\sigma_f, \sigma_{-f}, \theta) = m_f(\sigma'_f, \sigma_{-f}, \theta').$$

Therefore, for each $\theta \in \bar{\Theta}$ there exists $\theta' \in \bar{\Theta}$ such that the probability that firm f gets an offer from worker w equals the probability that firm f gets an offer from worker w'. Moreover, profile θ' is different for different θ by the construction.

Since, θ and θ' are equally possible

$$E_\theta m_f(\sigma_f, \sigma_{-f}, \theta | \theta \in \bar{\Theta}) = E_\theta m_f(\sigma'_f, \sigma_{-f}, \theta | \theta \in \bar{\Theta})$$

Therefore, the expected probability that firm f gets a match if it makes an offer to some worker in $\mathcal{W}^S$ is the same across all workers in $\mathcal{W}^S$. Since, the utility from obtaining a match from S_f is the largest, the expected payoff to f from making an offer to S_f is strictly greater than the payoff from making an offer to any other worker in $\mathcal{W}^S$.

A similar construction is valid for the workers in set $\mathcal{W} \backslash \mathcal{W}^S$. Therefore, the probability that firm f's offer is accepted is the same across any worker in $\mathcal{W} \backslash \mathcal{W}^S$. Hence, firm f prefers making an offer to its most valuable worker - T_f - than to any other worker in $\mathcal{W} \backslash \mathcal{W}^S$.[2] □

Proposition B.13. *Suppose workers use symmetric best-in-block strategies. Then for any strategy σ_f of firm f, there exists a cutoff strategy that provides f with a weakly higher expected payoff than σ_f for any anonymous strategies σ_{-f} of opponent firms $-f$.*

Proof.

Let us first note that Proposition 4 and Proposition B.12 established that the optimal choice of firm f for each set of received signals is either S_f, T_f, or some lottery between them. We break the proof into two parts. First we show that the identities of workers that have sent a signal to firm f influence neither the expected payoff of making an offer to S_f nor making an offer to T_f provided that the total number of signals firm f receives is constant. Second we prove that

[2] Note that it can happen that $T_f = S_f$. In this case the statement of the proposition is still valid since firm f believes that it is the T_f's top firm within block $\mathcal{F}_b$. Therefore, firm f prefers making an offer to T_f rather to any other worker in set $\mathcal{W}$.

if it is optimal for firm f to choose S_f when it receives signals from some set of workers, then it still optimal for firm f to choose S_f if the number of received signals does not change and S_f has a smaller rank (S_f is more valuable).

Let us consider some firm f from block $\mathcal{F}_b$, $b \in \{1, ..., B\}$ and some realization of its preferences, θ^*. Assume that it is optimal for firm f to make an offer to S_f if it receives set of signals $\mathcal{W}^S \subset \mathcal{W}$. We want to show that if firm f receives the set of signals $\mathcal{W}^{S'}$ such that $S_f(\theta^*, \mathcal{W}^S) = S_f(\theta^*, \mathcal{W}^{S'})$ and $\left|\mathcal{W}^{S'}\right| = \left|\mathcal{W}^S\right|$, it is still optimal for firm f to make an offer to S_f. For simplicity, we only consider the case when $\mathcal{W}^S$ and $\mathcal{W}^{S'}$ differ only in one signal. General case directly follows. There exist worker w and worker w' such that the former one belongs to set $\mathcal{W}^S$, but it does not belong to set $\mathcal{W}^{S'}$; while the latter one belongs to $\mathcal{W}^{S'}$, but it does not belong to $\mathcal{W}^S$. We consider two firm f strategies

$$\sigma_f(\theta^*, \cdot) = S_f(\theta^*, \cdot)$$
$$\sigma_f'(\theta^*, \cdot) = T_f(\theta^*, \cdot)$$

According to Proposition 4, firm f believes that it is the top firm within block $\mathcal{F}_b$ for workers who have sent signals to it. We denote the set of possible agents' profiles that are consistent with firm f received signals from $\mathcal{W}^S$ and $\mathcal{W}^{S'}$ as

$$\bar{\Theta}^S \equiv \{\theta \in \Theta | \theta_f = \theta^* \text{ and } f = \max_{\theta_w}(f' \in \mathcal{F}_{b'}) \text{ for each } w \in \mathcal{W}^S\}$$
$$\bar{\Theta}^{S'} \equiv \{\theta \in \Theta | \theta_f = \theta^* \text{ and } f = \max_{\theta_w}(f' \in \mathcal{F}_{b'}) \text{ for each } w \in \mathcal{W}^{S'}\}$$

correspondingly. We now construct a bijection between $\bar{\Theta}^S$ and $\bar{\Theta}^{S'}$. We denote a permutation that changes the ranks of w and w' in a firm preference profile as

$$\rho : (..., w, ...w', ...) \longrightarrow (..., w', ...w, ...)$$

For any profile $\theta \in \bar{\Theta}^S$ we construct profile $\theta' \in \Theta$ such that $\theta_f' = \theta^*$, the ranks of workers w and w' are exchanged in the preference lists of firms $-f$

$$\text{for any firm } f' \in -f,\ \theta'_f = \rho(\theta_f),$$

worker w and worker w' preference profiles are exchanged

$$\theta'_w = \theta_{w'},\ \theta'_{w'} = \theta_w,$$

and for any other $w^0 \in \mathcal{W} \backslash \{w, w'\}$, $\theta_{w^0} = \theta^{'}_{w^0}$.

Since firm f preference list is unchanged, $\theta'_f = \theta^*$, and firm f has symmetric beliefs about workers w for profile θ and about worker w' for profile θ', profile θ' belongs to $\bar{\Theta}^{S'}$. By construction, profile θ' is different for different θ. Since, the powers of sets $\bar{\Theta}^S$ and $\bar{\Theta}^{S'}$ are the same, the above correspondence is a bijection.

Since, firm $-f$ strategies are anonymous, for any $f' \in -f$ and $\mathcal{W}_0^S \subset \mathcal{W}$

$$\sigma_{f'}(\rho(\theta_{f'}), \rho(\mathcal{W}_0^S)) = \rho\left(\sigma_{f'}(\theta_{f'}, \mathcal{W}_0^S)\right)$$

This means that the probability of firm f'making an offer to worker w for profile θ equals the probability of firm f' making an offer to worker w' for profile θ'. Moreover, since we exchange worker w and w' preference lists for profile θ', whenever it is optimal for worker w to accept firm f offer for profile θ, it is optimal for worker w' to accept firm f's offer for profile θ'. Since firms types are independent the probability of firm f being matched when it uses strategy $\sigma_f(\theta^*, \cdot)$ for profile θ equals the probability of firm f being matched when it uses strategy $\sigma_f(\theta^*, \cdot)$ for profile θ'

$$m_f(\sigma_f, \sigma_{-f}, \theta) = m_f(\sigma_f, \sigma_{-f}, \theta')$$

Similar for strategy $\sigma'_f(\theta^*, \cdot)$

$$m_f(\sigma'_f, \sigma_{-f}, \theta) = m_f(\sigma'_f, \sigma_{-f}, \theta')$$

Since, we constructed a bijection between $\bar{\Theta}^S$ and $\bar{\Theta}^{S'}$ and θ and θ' are equally likely

$$\begin{aligned} E_\theta m_f(\sigma_f, \sigma_{-f}, \theta | \theta &\in \bar{\Theta}^S) = E_\theta m_f(\sigma_f, \sigma_{-f}, \theta' | \theta' \in \bar{\Theta}^{S'}) \\ E_\theta m_f(\sigma'_f, \sigma_{-f}, \theta | \theta &\in \bar{\Theta}^S) = E_\theta m_f(\sigma'_f, \sigma_{-f}, \theta' | \theta' \in \bar{\Theta}^{S'}) \end{aligned}$$

Therefore, if firm f optimally makes an offer to S_f (T_f) for the set of signal $\mathcal{W}^S$, it also should optimally make an offer to S_f (T_f), which is the same worker, for the set of signals $\mathcal{W}^{S'}$.

Now we prove that if firm f optimally chooses $S_f(\theta^*, \mathcal{W}^S)$ when it receives signals from $\mathcal{W}^S$, then it should still optimally choose $S_f(\theta^*, \mathcal{W}^{S'})$ for set of signals $\mathcal{W}^{S'}$, if the number of received signals is the same $\left|\mathcal{W}^{S'}\right| = \left|\mathcal{W}^S\right|$ and $S_f(\theta^*, \mathcal{W}^{S'})$ has a smaller rank ($S_f(\theta^*, \mathcal{W}^{S'})$ is more valuable). We consider set $\mathcal{W}^{S'}$ that differs from $\mathcal{W}^S$ only in the best (for firm f) worker and the difference between the ranks of top signaled workers equals one. General case directly follows.

$$\text{for any } w \in \mathcal{W}^S / S_f(\theta^*, \mathcal{W}^S) \Leftrightarrow w \in \mathcal{W}^{S'} / S_f(\theta^*, \mathcal{W}^{S'})$$
$$rank_f(S_f(\theta^*, \mathcal{W}^{S'})) = rank_f(S_f(\theta^*, \mathcal{W}^S)) - 1$$

The construction of the first part of the proof also works in this case. We consider sets of profiles and correspondence similar to the one above. Similar, we can show that the probabilities of firm f being matched with S_f (T_f) are the same for $\mathcal{W}^S$ and $\mathcal{W}^{S'}$. Now if firm f offer to T_f is accepted, firm f gets the same payoff for sets $\mathcal{W}^S$ and $\mathcal{W}^{S'}$. If firm f offer to S_f is accepted, firm f gets strictly greater payoff for set $\mathcal{W}^{S'}$ compare to set $\mathcal{W}^S$, because $S_f(\theta^*, \mathcal{W}^{S'})$ has smaller rank than

$S_f(\theta^*, \mathcal{W}^S)$. Hence, if it is optimal for firm f to make an offer to $S_f(\theta^*, \mathcal{W}^S)$ when it receives set of signals $\mathcal{W}^S$, it is optimal for firm f to make an offer to $S_f(\theta^*, \mathcal{W}^{S'})$ when firm f receives set of signals $\mathcal{W}^{S'}$.

The two statements we have just proved allows us to conclude that if firm $-f$ use anonymous strategies, firm f's optimal strategy could be represented as some cutoff strategy.[3] $\square$

Proposition B.14. *Suppose workers play symmetric best-in-block strategies, and firms $-f$ use cutoff strategies. If firm $f' \in -f$ increases its cutoffs (responds more to signals), firm f will also optimally weakly increase its cutoffs.*

Proof.

Let us consider some firm f from some block $\mathcal{F}_b$, $b \in \{1, ..., B\}$. We consider two sets of other firms strategies, σ_{-f} and σ'_{-f}, that vary only in firm f' strategies. For simplicity, we assume that σ_f differs from $\sigma_{f'}$ only for some profile $\bar{\theta}_{f'}$ and some set of received signals $\overline{\mathcal{W}^S_{f'}}$

$$
\begin{aligned}
\sigma_{f'}(\bar{\theta}_{f'}, \overline{\mathcal{W}^S_{f'}}) &= \alpha S_{f'} + (1-\alpha) T_{f'} \\
\sigma'_{f'}(\bar{\theta}_{f'}, \overline{\mathcal{W}^S_{f'}}) &= \alpha' S_{f'} + (1-\alpha') T_{f'}
\end{aligned}
$$

such that $\alpha' > \alpha$. Formally, this means either σ'_f or σ_f is not a cutoff strategy, because cutoff strategy should be the same for any profile of preferences (anonymity) and when firms receives the same number of signals. The extension to the change in firm f' cutoff strategy, i.e. for any profile of preferences and any set of signals of the same size, is immediate.

[3] Note that there can be other optimal strategies. If firm f is indifferent between making an offer to S_f and making an offer to T_f for some $\mathcal{W}^S$, firm f could change its offer (make offer to S_f or T_f) when it receives the sets of signals with the same S_f and the same power.

Let us consider some realization of firm f preference profile $\theta^* \in \Theta$ and some set of signals $\mathcal{W}^S \subset \mathcal{W}$. We want to show that firm f's payoff from making an offer to T_f decreases whereas firm f's payoff from making an offer to S_f increases when firm f' responds more to signals, i.e. plays strategy $\sigma'_{f'}$ instead of $\sigma_{f'}$.

$$
\begin{aligned}
I)\ E_\theta(\pi_f(T_f, \sigma_{-f}, \theta)|\theta_f &= \theta^*, \mathcal{W}_f^S = \mathcal{W}^S) \geq E_\theta(\pi_f(T_f, \sigma'_{-f}, \theta)|\theta_f = \theta^*, \mathcal{W}_f^S = \mathcal{W}^S) \\
II)\ E_\theta(\pi_f(S_f, \sigma_{-f}, \theta)|\theta_f &= \theta^*, \mathcal{W}_f^S = \mathcal{W}^S) \leq E_\theta(\pi_f(S_f, \sigma'_{-f}, \theta)|\theta_f = \theta^*, \mathcal{W}_f^S = \mathcal{W}^S)
\end{aligned}
$$

Since firm f's offer can only be either accepted or declined, the above statements are equivalent to

$$
\begin{aligned}
I)\ E_\theta(m_f(T_f, \sigma_{-f}, \theta)|\theta_f &= \theta^*, \mathcal{W}_f^S = \mathcal{W}^S) \geq E_\theta(m_f(T_f, \sigma'_{-f}, \theta)|\theta_f = \theta^*, \mathcal{W}_f^S = \mathcal{W}^S) \\
II)\ E_\theta(m_f(S_f, \sigma_{-f}, \theta)|\theta_f &= \theta^*, \mathcal{W}_f^S = \mathcal{W}^S) \leq E_\theta(m_f(S_f, \sigma'_{-f}, \theta)|\theta_f = \theta^*, \mathcal{W}_f^S = \mathcal{W}^S)
\end{aligned}
$$

Namely, the probability of being matched to T_f decreases, and the probability of being matched to S_f increases.

I. We first prove the first statement. We define the sets of agents profiles that lead to the increase and decrease in the probability of getting a match given the change in firm f' strategy as

$$
\begin{aligned}
\bar{\Theta}_+ &\equiv \{\theta \in \Theta | \theta_f = \theta^*, \mathcal{W}_f^S = \mathcal{W}^S, \text{ and } m_f(T_f, \sigma_{-f}, \theta) < m_f(T_f, \sigma'_{-f}, \theta)\} \\
\bar{\Theta}_- &\equiv \{\theta \in \Theta | \theta_f = \theta^*, \mathcal{W}_f^S = \mathcal{W}^S, \text{ and } m_f(T_f, \sigma_{-f}, \theta) > m_f(T_f, \sigma'_{-f}, \theta)\}
\end{aligned}
$$

correspondingly. If set $\bar{\Theta}_+$ is empty, the statement has been proved. Otherwise, we take some $\theta \in \bar{\Theta}_+$ and denote $T_f = w$. Then, we should have

$$
\begin{cases}
T_{f'}(\bar{\theta}_{f'}, \overline{\mathcal{W}_{f'}^S}) = w \\
S_{f'}(\bar{\theta}_{f'}, \overline{\mathcal{W}_{f'}^S}) = w' \neq w
\end{cases}.
$$

and

$$\begin{aligned}\sigma_{f'}(\bar{\theta}_{f'},\overline{\mathcal{W}^S_{f'}}) &= \alpha w' + (1-\alpha)w \\ \sigma'_{f'}(\bar{\theta}_{f'},\overline{\mathcal{W}^S_{f'}}) &= \alpha' w' + (1-\alpha')w\end{aligned}$$

Note that the situation when firm f is from a better group than firm f' cannot happen, i.e. $f' \in \mathcal{F}_{b'}$where $b' > b$. In this case the offer of firm f' is always worse than the offer of firm f and cannot influence the probability that firm f obtains a match. Therefore, firm f is from a group that is weakly worse than $\mathcal{F}_{b'}$, i.e. $b' \leq b$.

Note that worker w has sent a signal neither to firm f nor to firm f'. This allows us to construct $\theta' \in \bar{\Theta}_-$. Namely, we consider a permutation that changes the ranks of w and w' in a firm preference profile

$$\rho : (..., w, ...w', ...) \longrightarrow (..., w', ...w, ...)$$

For any profile $\theta \in \bar{\Theta}_+$ we construct profile $\theta' \in \Theta$ such that $\theta'_f = \theta^*$, the ranks of workers w and w' are exchanged in the preference lists of firms $-f$

$$\text{for any firm } f' \in -f,\ \theta'_f = \rho(\theta_f),$$

worker w and worker w' preference profiles are exchanged

$$\theta'_w = \theta_{w'},\ \theta'_{w'} = \theta_w,$$

and for any other $w^0 \in \mathcal{W}\backslash\{w, w'\}$, $\theta_{w^0} = \theta'_{w^0}$. Note that firm f has the same preferences θ^* and receives the same set of signals for preference profile.

Since firm strategies are anonymous we have that

$$\begin{aligned}\sigma_{f'}(\theta'_{f'}, \mathcal{W}^{S'}_{f'}) &\equiv \sigma_{f'}(\rho(\theta_{f'}), \rho(\mathcal{W}^{S}_{f'})) \\ &= \alpha\rho(w') + (1-\alpha)\rho(w) \\ &= \alpha w + (1-\alpha)w'\end{aligned}$$

and

$$\sigma'_{f'}(\theta'_{f'}, \mathcal{W}^{S'}_{f'}) = \alpha' w + (1-\alpha')w'$$

We assumed that firm f' prevents firm f from being matched with worker w for profile θ. Firm f' makes an offer to worker w more frequently for strategy $\sigma'_{f'}$ rather than σ_f for profile θ'. Therefore, the probability that firm f offer to worker w is accepted is smaller when firm f' uses strategy $\sigma'_{f'}$ rather than σ_f. Worker w prefers firm f' to firm f for profile θ'. Specifically, if firm f is from the group $\mathcal{F}_b$, $b > b'$, worker w always prefers firm f' to firm f. If firm f and firm f' are from the same group, $b = b'$, worker w prefers f to f' (since worker w sends a signal to firm f' for profile of preferences θ').

We should also investigate the behavior of a firm that receives worker w's signal for profile θ, say firm f_y. If firm f_y makes an offer to worker w for profile θ, since the change of firm f' strategy changes firm f payoff, firm f_y should be worse than both firms f and f' in worker w preferences. Hence, firm f_y's offer cannot change the incentives of worker w. If worker w' sends her signal to firm f_y then firm f_y either makes an offer to worker w' or to worker T_{f_y}, which are different from worker w. Hence, firm f_y does not change the incentives of the agents. Overall, we conclude that $\theta' \in \bar{\Theta}_-$.

Note that the above construction gives us for different profiles of $\bar{\Theta}_+$ different profiles of $\bar{\Theta}_-$. Therefore, we have constructed an injective function $\bar{\Theta}_+$ to $\bar{\Theta}_-$

and $|\bar{\Theta}_-| \geqslant |\bar{\Theta}_+|$[4]. Since profiles θ and θ' are equally likely

$$E_\theta(m_f(T_f, \sigma_{-f}, \theta)|\theta_f = \theta^*, \mathcal{W}_f^S = \mathcal{W}^S) \geq E_\theta(m_f(T_f, \sigma'_{-f}, \theta)|\theta_f = \theta^*, \mathcal{W}_f^S = \mathcal{W}^S)$$

II. Let us now show that if firm f' responds more to signals, the probability of firm f being matched to S_f increases. If firm f, $f \in \mathcal{F}_b$, receives a signal from worker w it believes being the best firm in group $\mathcal{F}_b$ according to worker w preferences. Therefore, worker w prefers the offer of firm f to an offer from any other firm f' from group $\mathcal{F}_{b'}$ such that $b' \geq b$. Therefore, the change of the behavior of any firm f' from group $\mathcal{F}_{b'}$, $b' \geq b$, does not influence firm f's payoff.

If we consider some firm f' from group $\mathcal{F}_{b'}$, $b' < b$, it can draw away worker w's offer from firm f only if it makes an offer to worker w. However, firm f' makes an offer to worker w, conditionally on firm f receiving a signal from worker w, only when worker w is $T_{f'}$. However, if firm f' responds more to signals, it makes an offer to its T_f more rarely. This means that firm f' draw worker w away from firm f less often. Therefore, the probability that firm f offer is accepted by S_f increases.

$$E_\theta(m_f(S_f, \sigma_{-f}, \theta)|\theta_f = \theta^*, \mathcal{W}_f^S = \mathcal{W}^S) \leq E_\theta(m_f(S_f, \sigma'_{-f}, \theta)|\theta_f = \theta^*, \mathcal{W}_f^S = \mathcal{W}^S)$$

As a corollary of I) and II), if firm f' increases its cutoff point for some set of signals, firm f will also optimally (weakly) increase its cutoff points. Since, the above logic is valid for the change of cutoff points for any set of signals of the same size and any profile of preferences, the statement of the proposition immediately follows. □

Proof of Theorem 3.

[4] One may show that generally it is impossible to have a correspondence in the other direction.

We consider the set of cutoff strategies of firms and the set of best-in-block strategies of workers. We denote its typical element as $\sigma = (\sigma_F, \sigma_W)$ that consists of firms cutoff strategies $\sigma_F = (\sigma_{f_1}, ..., \sigma_{f_F})$ and workers best-in-block strategies strategies $\sigma_W = (\sigma_{w_1}, ..., \sigma_{w_W})$.

A strategy of firm f is a vector of real numbers of size W that specifies cutoff points for each positive number of signals firm f could receive, $\sigma_f = (j_f^1, ..., j_f^W)$, where j_f^l is a real number from interval $[1, W]$ for each $l = 1, ..., W$. We denote the set of possible firm cutoff strategies as $\Sigma_f^{cut} = [1, W]^W$.

A best-in-block strategy of worker w is a vector of size B that specifies the probability that she sends her signal to her top firm of specific block $\sigma_w = (\alpha_w^1, ..., \alpha_w^B)$, where $\alpha_w^b \geq 0$ for each $b = 1, ..., B$ and $\sum_{b=1}^B \alpha_w^b \leq 1$. We denote the set of possible worker best-in-block strategies as $\Sigma_w^{block} = \{(\alpha^1, ..., \alpha^B) : \alpha^b \geq 0 \text{ and } \sum_{b=1}^B \alpha^b \leq 1\}$.

Let us also denote the expected payoff of worker w when she uses best-in-block strategy σ_w and the other agents use strategy σ_{-w} as[5]

$$U_w(\sigma_w, \sigma_{-w}) = E_\theta(\pi_w(\sigma_w, \sigma_{-w}, \theta))$$

and the expected payoff of firm f when it uses strategy σ_f and the other agents use strategy σ_{-f} as

$$U_f(\sigma_f, \sigma_{-f}) = E_\theta(\pi_f(\sigma_f, \sigma_{-f}, \theta)).$$

We introduce best reply correspondence $g : (\Sigma_f^{cut})^F \times (\Sigma_w^{block})^W \to 2^{(\Sigma_f^{cut})^F \times (\Sigma_w^{block})^W}$ such that

$$g_f(\sigma) = \arg\max_{\beta \in \Sigma_f^{cut}} U_w(\beta, \sigma_{-w})$$

for each $f \in \mathcal{F}$ and

$$g_w(\sigma) = \arg\max_{\beta \in \Sigma_w^{block}} U_a(\beta, \sigma_{-w})$$

[5]Note that the strategy of agents are anonymous. Therefore, they do not depend on particular realization of preferences.

for each $w \in \mathcal{W}$.

The immediate consequence of the above definitions is that Σ_f^{cut} and Σ_w^{block} are *non-empty*, *convex,* and *compact*. Also, $U_w(\sigma_w, \sigma_{-w})$ is a linear function of its first argument. Namely, let us denote the expected payoff of worker w from sending a signal to some block $\mathcal{F}_b$ given the strategies of agents σ_{-w} as $\Pi_b(-\sigma_w)$. If worker w employs strategy $\sigma_w = (\alpha_w^1, ..., \alpha_w^B)$, her payoff equals

$$U_w(\sigma_w, \sigma_{-w}) = \sum\nolimits_{b=1}^{B} \alpha_b \Pi_b(-\sigma_w)$$

Therefore, $g_w(\sigma)$ is a continuous correspondence with closed graph.

Let us now consider function $U_f(\sigma_f, \sigma_{-f})$. Similarly, let us consider some realization of preference profile θ when firm f receives $\left|\mathcal{W}^S\right|$ signals. Given the strategies of other agents σ_{-f}, we denote the expected payoff of firm f from making an offer to T_f as Π_T, and the expected payoff of firm f from making an offer to S_f as Π_S. Then, the payoff of firm f from using cutoff strategy $j_{|\mathcal{W}^S|}$, $\sigma_f = (..., j_{|\mathcal{W}^S|}, ...)$, equals

$$\pi_f(\sigma_f, \sigma_{-f}, \theta) = \begin{cases} \Pi_T \text{ if } j_{|\mathcal{W}^S|} \leq rank(S_f) - 1 \\ (\lceil j_{|\mathcal{W}^S|} \rceil - j_{|\mathcal{W}^S|})\Pi_S + (j_{|\mathcal{W}^S|} - \lfloor j_{|\mathcal{W}^S|} \rfloor)\Pi_T \text{ if } rank(S_f) > j_{|\mathcal{W}^S|} > rank(S_f) - 1 \\ \Pi_S \text{ if } j_{|\mathcal{W}^S|} \geq rank(S_f) \end{cases}$$

where $\lceil j_{|\mathcal{W}^S|} \rceil$ and $\lfloor j_{|\mathcal{W}^S|} \rfloor$ denote the closest integer larger and smaller than $j_{|\mathcal{W}^S|}$ correspondingly.

Function $\pi_f(\sigma_f, \sigma_{-f}, \theta)$ is quasi-concave function of cutoff $j_{|\mathcal{W}^S|}$. Therefore, the expected payoff from using cutoff $j_{|\mathcal{W}^S|}$, $E_\theta[\pi_f(\sigma_f, \sigma_{-f}, \theta)||\mathcal{W}_f^S| = |\mathcal{W}^S||]$, is also a quasi-concave function of cutoff $j_{|\mathcal{W}^S|}$ as it is a linear combination of quasi-concave functions. Therefore, $U_f(\sigma_f, \sigma_{-f})$ is a quasi-concave function of its first argument. Therefore, $g_f(\sigma)$ is a continuous correspondence with closed graph.

Overall, we get that $g(\sigma)$ is a continuous correspondence with closed graph. Hence, $g(\sigma)$ has a fixed point by Kakutani's theorem (see Kakutani, 1941).

We have considered above only the set of cutoff strategies. However, Proposition 4 and Proposition B.13 allow us to conclude that the above equilibrium is an equilibrium when we allow any deviations, not only deviation in cutoff strategies. Overall, we have established the existence of equilibrium when workers use symmetric best-in-block strategies and firms use symmetric cutoff strategies and have best-in-block beliefs. □

Proof of Theorem 4.

In order to prove the statements of the theorem we first prove the following lemma.

Lemma B.11. *Assume firms use cutoff strategies and workers use best-in-block strategies. Fix the strategies of firms $-f$ as σ_{-f}. Let firm f's strategy σ_f differ from σ'_f only in that σ'_f has greater cutoffs (responds more to signals). Then, we have*

$$E_\theta(m(\sigma'_f, \sigma_{-f}, \theta)) \geq E_\theta(m(\sigma_f, \sigma_{-f}, \theta))$$
$$E_\theta(\pi_w(\sigma'_f, \sigma_{-f}, \theta)) \geq E_\theta(\pi_w(\sigma_f, \sigma_{-f}, \theta))$$

Proof.

We prove the first statement first. Let us consider firm f cutoff strategies σ_f and σ'_f such that σ'_f has weakly greater cutoffs. We consider two sets of preference profiles

$$\bar{\Theta}_+ \equiv \{\theta \in \Theta | \; m(\sigma_f, \sigma_{-f}, \theta) < m(\sigma'_f, \sigma_{-f}, \theta)\}$$
$$\bar{\Theta}_- \equiv \{\theta \in \Theta | \; m(\sigma_f, \sigma_{-f}, \theta) > m(\sigma'_f, \sigma_{-f}, \theta)\}$$

For each profile θ from set Θ^+, it must be the case that without firm f offer T_f has an offer from another firm, and worker S_f does not

$$m(\sigma'_f, \sigma_{-f}, \theta) - m(\sigma_f, \sigma_{-f}, \theta) = 1. \tag{B.1.2}$$

Similarly, if profile θ is from set Θ^-, it must be the case that without firm f offer, S_f has an offer from another firm, and T_f does not

$$m(\sigma'_f, \sigma_{-f}, \theta) - m(\sigma_f, \sigma_{-f}, \theta) = -1. \tag{B.1.3}$$

We will now show that $|\bar{\Theta}_+| \geq |\bar{\Theta}_-|$. Equations (B.1.2) and (B.1.3) along with the fact that each $\theta \in \Theta^+ \cup \Theta^-$ happens equally likely will then be enough to prove the result.

Let us denote $T_f = w'$ and $S_f = w$. We construct function $\psi : \Theta \to \Theta$ as follows. Let $\psi(\theta)$ be the profile in which workers have preferences as in θ, but firms $-f$ all swap the positions of workers w' and w in their preference lists. If profile θ belongs to $\bar{\Theta}_-$, without firm f's offer, worker w has an offer from another firm, and worker w' does not. Therefore, when preferences are $\psi(\theta)$, without firm f's offer two following statements should be true i) worker w' ***must*** have another offer and ii) worker w ***cannot*** have another offer.

To see i), note that under θ, worker w sends a signal to firm f, so his outside offer must come from some firm f' who has ranked him first. Under profile $\psi(\theta)$, firm f' ranks worker w' first. If worker w' has not sent a signal to firm f', then by anonymity, w' gets the offer of firm f'. If worker w' has signaled to firm f', worker w' again gets firm f' offer.

To see ii), suppose to the contrary that under $\psi(\theta)$, worker w does in fact receive an offer from some firm $f' \neq f$. Since worker w sends a signal to firm f, worker w

must be $T_{f'}$ under $\psi(\theta)$, so that worker w' is $T_{f'}$ under θ. But then by anonymity w' receives the offer of firm f' under θ, a contradiction.

From i) and ii), we have

$$\theta \in \bar{\Theta}_- \Rightarrow \psi(\theta) \in \bar{\Theta}_+.$$

Since function ψ is injective, we have $|\bar{\Theta}_+| \geq |\bar{\Theta}_-|$.

In order to prove the second statement note that the expected number of matches of each worker increases when firm f responds more to signals. Using the construction presented above, one could show whenever worker w looses a match with firm f for profile θ (worker w is T_f) it is possible to construct profile θ' when worker w obtains the match (worker w is S_f). The function that matches these profiles is again injective. Moreover, worker w values more the match with firm f when she is S_f rather when she is T_f. Therefore, ex-ante utility of worker w increases when firm f responds more to signals. $\square$

Let us denote firm strategies in the unique equilibrium of the offer game with no signals as σ_F^0. Now, consider a block-symmetric equilibrium of the offer game with signals when agent use strategies (σ_F, σ_W). If agents employ strategies (σ_F^0, σ_W), the expected number of matches and the welfare of workers equal the corresponding parameters in the offer game with no signals. Therefore, the result that the expected number of matches and the expected welfare of workers in a block-symmetric equilibrium in the offer game with signals are *weakly* greater than the corresponding parameters in the unique equilibrium of the offer game with no signals is a consequence of sequential application of Proposition B.11.

Let us now consider some non-babbling block-symmetric equilibrium (σ_F, σ_W) of the offer game with signals such that there exists block $\mathcal{F}_b$ with at least two firms such that $\alpha_b > 0$. Proposition 4 shows that firms from block $\mathcal{F}_b$ should respond to signals in the equilibrium, $p_b^s > p_b^{ns}$, i.e. make offers to top signaled workers with positive probability.

Let us take some firm f from block $\mathcal{F}_b$. Using a construction similar to the one in the proof of Lemma B.11 we consider two sets

$$\bar{\Theta}_+ \equiv \{\theta \in \Theta |\ m(\sigma_f^0, \sigma_{-f}, \theta) < m(\sigma_f, \sigma_{-f}, \theta)\}$$
$$\bar{\Theta}_- \equiv \{\theta \in \Theta |\ m(\sigma_f^0, \sigma_{-f}, \theta) > m(\sigma_f, \sigma_{-f}, \theta)\}$$

Let us consider some realized profile of preferences, $\theta \in \Theta$, and denote $T_f = w'$ and $S_f = w$. We also consider $\psi : \Theta \to \Theta$, such that $\psi(\theta)$ is the profile in which workers have preferences as in θ, but firms $-f$ all swap the positions of workers w' and w in their preference lists. Note that $\psi(\psi(\theta)) = \theta$ and ψ is a bijection on Θ. Direct consequence of Lemma B.11 is that $|\bar{\Theta}_+| \geq |\bar{\Theta}_-|$. Let us now show there exist $\theta \in \bar{\Theta}_+$ such that $\psi(\theta) \notin \bar{\Theta}_-$.

There are at least two firms, f and f', in block $\mathcal{F}_b$ that responds to signals. Let us consider some profile θ from $\bar{\Theta}_+$. We again denote $T_f = w'$ and $S_f = w$. Therefore, worker w does not have an offer from any other firm for profile θ from $\bar{\Theta}_+$, but worker w' has at least two offers. Since worker w' sends her signal to firm f' with positive probability and firm f' responds to signals, i.e. makes offers to its top signaled workers, there exist $\theta^* \in \bar{\Theta}_+$ such that worker w' is top signaled worker of firm f', and firm f' makes an offer to worker w'.

However, worker w for profile $\psi(\theta^*)$ does not have any other offer, because she is neither T_f nor S_f for profile $\psi(\theta^*)$. Therefore, $\psi(\theta^*)$ cannot belong to $\bar{\Theta}_-$. Therefore, we have found a profile from $\bar{\Theta}_+$ such that it does not belong to $\bar{\Theta}_-$. As a result, $|\bar{\Theta}_+| > |\bar{\Theta}_-|$ and we have that

$$E_\theta[m(\sigma_f^0, \sigma_{-f}, \theta)] < E_\theta[m(\sigma_f, \sigma_{-f}, \theta)]$$

In addition, we know that

$$E_\theta[m(\sigma_f^0, \sigma_{-f}^0, \theta)] \leq E_\theta[m(\sigma_f^0, \sigma_{-f}, \theta)],$$

which gives us

$$E_\theta[m(\sigma_f^0, \sigma_{-f}^0, \theta)] < E_\theta[m(\sigma_f, \sigma_{-f}, \theta)].$$

Overall, the expected number of matches in the offer game with signals when agents use strategies (σ_F, σ_W) is strictly greater than the expected number of matches in the offer game with no signals.

Using the above construction and the logic of the proof of Lemma B.11 one could get the result for worker welfare. The example presented in Section 2.1 illustrates that signals can ambiguously influence the welfare of firms. Specifically, Table 2.1 shows that firms welfare increases after the introduction of a signaling mechanism only if the value of a second ranked worker is sufficiently high, $x > 0.5$.□

B.2 Many-to-One Matching Market with Multiple Signals

Proof of Proposition 5.

The proof repeats the argument of Proposition 1. □

Proof of Proposition 6.

Let us consider some worker. Firms use symmetric anonymous strategies, signals are identical, and the worker can send at most one signal to a given firm. Hence, from the worker perspective the probability of getting an offer from a firm depends only on whether the worker has sent a signal to this firm or not. Similar to the argument of the proof of Proposition 4 the probability of getting an offer from a firm that receives worker's signal is greater than the probability of getting an offer from a firm that does not receive worker's signal. Since this probability does not depend on the identity of the firm in a symmetric equilibrium we conclude that the worker optimally sends her signals to her K top firms. □

Proposition B.25. *Suppose firms $-f$ use anonymous strategies and workers send their signals to top K firms.* Then firm f makes offers to *its* $L^{NS} \in \{0, ..., L\}$ top workers who has signaled to it and to its $L^S = L - L^{NS}$ top workers who has not signaled to it in any non-babbling symmetric sequential equilibrium.

Proof.

Note that firm use anonymous strategies, workers send their signal to top K firms, and workers accept the best available offer. We first proof lemma that states that from point of view of firm f the probability that a worker who has (has not) signaled to it accepts its offer depends only on the number of signals firm f receives.

Lemma B.22. *Suppose firms $-f$ use anonymous strategies and workers send their signals to top K firms. We consider two events A and B. Event A is that firm f receives the set of signals $\mathcal{W}^S$. Event B is that firm f receives the set of signal $\check{\mathcal{W}}^S$ and —$\mathcal{W}^S| = |\check{\mathcal{W}}^S|$. Then*

- *the probability that worker $w \in \mathcal{W}^S$ accepts firm f offer conditional on event A equals the probability that worker $w' \in \check{\mathcal{W}}^S$ accepts firm f offer conditional on event B;*

- *the probability that worker $w \in \mathcal{W} \backslash \mathcal{W}^S$ accepts firm f offer conditional on event A equals the probability that worker $w' \in \mathcal{W} \backslash \check{\mathcal{W}}^S$ accepts firm f offer conditional on event B.*

Proof.

Let us consider firm f with the realized preference profile $\theta^* \in \Theta_f$ and that receives signals from the set of workers $\mathcal{W}^S$. We first prove that the probability that a worker from $\mathcal{W}^S$ accepts firm f offer conditional on event A equals the probability that a worker from $\check{\mathcal{W}}^S$ accepts firm f offer conditional on event B.

Note that firm f believes that it is one of the top K firms in worker preference list if it receives an offer from her. Let us denote the set of possible agents profiles consistent with firm f beliefs in both events as

$$\Theta^A \equiv \{\theta \in \Theta | \theta_f = \theta^* \text{ and } rank_{\theta_{w^s}}(f) \in \{1, ..., K\} \text{ for each } w^s \in \mathcal{W}^S\}$$

$$\Theta^B \equiv \{\theta \in \Theta | \theta_f = \theta^* \text{ and } rank_{\theta_{w^s}}(f) \in \{1, ..., K\} \text{ for each } w^s \in \check{\mathcal{W}}^S\}$$

Since firm receives the same number of signals for both events, i.e. —$\mathcal{W}^S| = |\check{\mathcal{W}}^S|$, for each worker $w_a \in \mathcal{W}^S$ we take some worker $w'_a \in \check{\mathcal{W}}^S$, $a = 1, ..., |\mathcal{W}^S|$. Let us denote $rank_{\theta_{w_a}}(f) = k_a$ and $rank_{\theta_{w'_a}}(f) = k'_a$. Therefore, $k_a, k'_a \in \{1, ..., K\}$ for each a. We denote a permutation that changes k_a an k'_a positions in worker preference list as

$$\rho^{w_a} : (..., k_a, ..., k'_a, ...) \to (..., k'_a, ..., k_a, ...).$$

We also denote a permutation that changes the ranks of w_a and w'_a for every a in a firm preference lists as

$$\rho^f : (..., w_a, ..., w'_a, ...) \to (..., w'_a, ..., w_a, ...).$$

Let us consider some profile of preferences $\theta \in \Theta^A$ and construct a profile of preferences θ'. We do not change firm f preference list, i.e. $\theta'_f = \theta^*$, the ranks of workers w_a and w'_a are exchanged in the preference lists of firms $-f$ for each a

$$\text{for any firm } f' \in -f,\ \theta'_f = \rho^f(\theta_f),$$

k_a and k'_a position in worker w_a and worker w'_a preference profiles are exchanged for each a

$$\theta'_{w_a} = \rho^{w_a}(\theta_{w_a}),\ \theta'_{w'_a} = \rho^{w_a}(\theta_{w'_a}),$$

and for any other $w^0 \in \mathcal{W} \backslash (\mathcal{W}^S \bigcup \check{\mathcal{W}}^S)$, $\theta_{w^0} = \theta'_{w^0}$. Since firm f preference list is unchanged, $\theta'_f = \theta^*$, and firm f receives signals from the set $\check{\mathcal{W}}^S$ for profile θ' this profile belongs to Θ^B. Since firm $-f$ strategies are anonymous for any $f' \in -f$ and for any $\mathcal{W}^S_{f'} \subset \mathcal{W}$ we have that

$$\sigma_{f'}(\rho^f(\theta_{f'}), \rho^f(\mathcal{W}^S_{f'})) = \rho^f\left(\sigma_{f'}(\theta_{f'}, \mathcal{W}^S_{f'})\right)$$

Workers in $\mathcal{W}^S$ and $\check{\mathcal{W}}^S$ send their signals to the same firms among $-f$ for both profiles θ and θ'. Therefore, i.e. $\rho^f(\mathcal{W}^S_{f'}) = \mathcal{W}^S_{f'}$. Since $\theta'_f = \rho^f(\theta_f)$ we have that

$$\sigma_{f'}(\theta'_{f'}, \mathcal{W}^S_{f'}) = \rho\left(\sigma_{f'}(\theta_{f'}, \mathcal{W}^S_{f'})\right)$$

This means that the probability of firm f' making an offer to worker $w_a \in \mathcal{W}^S$ for profile θ equals the probability of making an offer to a worker in $w'_a \in \check{\mathcal{W}}^S$ for profile θ'. Moreover, since we exchange worker w_a and w'_a preference lists for profile θ', whenever it is optimal for worker w_a to accept firm f offer for profile θ, it is optimal for worker w'_a to accept firm f's offer for profile θ'.

Since firms types are independent the probability of firm f being matched when it makes an offer to w_a for profile θ equals the probability of firm f being matched when it makes an offer to worker w'_a for profile θ'. Therefore, for each $\theta \in \Theta^A$

there exists $\theta' \in \Theta^B$ such that the probability that firm f gets an offer from worker w_a equals the probability that firm f gets an offer from worker w'_a. Moreover, profile θ' is different for different θ by the construction. Therefore, we have constructed a bijection among sets Θ^A and Θ^B. Since θ and θ' are equally possible the probability that firm f offer is accepted by worker w_a in the event A equals the probability that firm f offer is accepted by worker w'_a in the event B.

Note that a similar construction works for the proof of the second statement that involves workers in sets $\mathcal{W}\backslash\mathcal{W}^S$ and $\mathcal{W}\backslash\check{\mathcal{W}}^S$. Therefore, the probability that worker $w \in \mathcal{W}\backslash\mathcal{W}^S$ accepts firm f offer conditional on event A equals the probability that worker $w' \in \mathcal{W}\backslash\check{\mathcal{W}}^S$ accepts firm f offer conditional on event B. $\square$

The statement of the proposition directly follows from the lemma. Since the probability that worker who has sent a signal to firm f accepts its offer independent on the identity of the worker firm f prefers to make offers to its top workers among those who signaled to it. Similar firm f prefers to make offers to its top workers among those who has not signaled to it. Finally, firm f prefers to make all L offers.$\square$

Proof of Proposition 7.

Let us consider two sets of workers that firm f might receive $\mathcal{W}^S$ and $\check{\mathcal{W}}^S$ such that $\mathcal{W}^S = \check{\mathcal{W}}^S$. Firm f makes an offer to workers $\mathcal{W}_{offer} = \mathcal{W}^S_{offer} \bigcup \mathcal{W}^{NS}_{offer}$ such that$\mathcal{W}^{NS}_{offer} \subset \mathcal{W}^S$ and $\mathcal{W}^{NS}_{offer} \subset \mathcal{W}\backslash\mathcal{W}^S$ in equilibrium. Lemma B.22 proves that identities of workers who have sent a signal to firm f do not influence the probability that workers accept the firm's offer provided that the total number of signals firm f receives is constant. Therefore, if workers $\mathcal{W}^S_{offer}$ are among$\check{\mathcal{W}}^S$, i.e. $\mathcal{W}^S_{offer} \subset \check{\mathcal{W}}^S$, it is still optimal for firm f to make its offers to workers $\mathcal{W}_{offer}$.

Let us again consider two sets of signals with the same power, i.e. $\mathcal{W}^S$ and $\check{\mathcal{W}}^S$ such that $\mathcal{W}^S = \check{\mathcal{W}}^S$. However, these sets differ now in one worker: there exist $w \in \mathcal{W}^S$ and $w' \in \check{\mathcal{W}}^S$ such that $\mathcal{W}^S \backslash w = \check{\mathcal{W}}^S \backslash w'$. Moreover, firm f prefers worker w' to worker w, i.e. $rank_{\theta_f}(w') > rank_{\theta_f}(w)$. As a consequence of Lemma B.22, if firm f makes an offer to worker w when it receives the set of signals $\mathcal{W}^S$ in equilibrium, it should make an offer to w' when it receives the set of signals $\check{\mathcal{W}}^S$. Let us consider the case when sets $\mathcal{W}^S$ and $\check{\mathcal{W}}^S$ differ in more than one worker. There are some workers in $\check{\mathcal{W}}_0 \subset \check{\mathcal{W}}^S$ who are better than workers in $\mathcal{W}_0 \subset \mathcal{W}^S$ who receive an offer from firm f when it receives signals from $\mathcal{W}^S$. Similar argument shows that firm f should then optimally make an offer to $\check{\mathcal{W}}_0$ when it receives signals from $\check{\mathcal{W}}^S$.

The two arguments presented above allows us to conclude that if firm $-f$ use anonymous strategies, firm f's optimal strategy could be represented as some cutoff strategy. □

Proof of Theorem 5.

The proof repeats the steps of the proof of Theorem 3. □

Lemma B.23. *Assume firms use cutoff strategies and workers send their signals to their top K firms. Fix the strategies of firms $-f$ as σ_{-f}. Let firm f's strategy σ_f differ from σ'_f only in that σ'_f has greater cutoffs (responds more to signals). Then we have*

$$E_\theta(m(\sigma'_f, \sigma_{-f}, \theta)) \geq E_\theta(m(\sigma_f, \sigma_{-f}, \theta))$$
$$E_\theta(\pi_w(\sigma'_f, \sigma_{-f}, \theta)) \geq E_\theta(\pi_w(\sigma_f, \sigma_{-f}, \theta))$$

where $m(\cdot)$ denotes the total number of matches.

Proof.

Let us consider firm f cutoff strategies σ_f and σ'_f such that σ'_f has weakly greater cutoffs for profile θ_f:

$$\begin{aligned}\sigma_f(\theta_f, \mathcal{W}_f^S) &= \mathcal{W}_{offer}^S \bigcup \mathcal{W}_{offer}^{NS} \\ \sigma'_{f'}(\theta_f, \mathcal{W}_f^S) &= \check{\mathcal{W}}_{offer}^S \bigcup \check{\mathcal{W}}_{offer}^{NS}\end{aligned}$$

In order to preserve anonymity firm f also should have the corresponding increase in cutoff strategies for any profile of preferences and any set of received signals of the same power. Firm f responds more to signals for profile θ_f means that $\mathcal{W}_{offer}^S \subset \check{\mathcal{W}}_{offer}^S \subset \mathcal{W}_f^S$ and $\check{\mathcal{W}}_{offer}^{NS} \subset \mathcal{W}_{offer}^{NS} \subset \mathcal{W} \backslash \mathcal{W}_f^S$. Proposition B.25 shows that $|\mathcal{W}_{offer}^S \bigcup \mathcal{W}_{offer}^{NS}| = |\check{\mathcal{W}}_{offer}^S \bigcup \check{\mathcal{W}}_{offer}^{NS}| = L$. We consider only the case when $\mathcal{W}_{offer}^S \backslash \check{\mathcal{W}}_{offer}^S = w^S$ and $\check{\mathcal{W}}_{offer}^{NS} \backslash \mathcal{W}_{offer}^{NS} = w^{NS}$. More general case directly follows.

We denote two sets of preference profiles

$$\begin{aligned}\Theta_+ &\equiv \{\theta \in \Theta | \; m(\sigma_f, \sigma_{-f}, \theta) < m(\sigma'_f, \sigma_{-f}, \theta)\} \\ \Theta_- &\equiv \{\theta \in \Theta | \; m(\sigma_f, \sigma_{-f}, \theta) > m(\sigma'_f, \sigma_{-f}, \theta)\}\end{aligned}$$

For each profile θ from set Θ^+ it must be the case that without firm f offer w^{NS} has an offer from another firm, and worker w^S does not

$$m(\sigma'_f, \sigma_{-f}, \theta) - m(\sigma_f, \sigma_{-f}, \theta) = 1. \tag{B.2.1}$$

Similarly, if profile θ is from set Θ^-, it must be the case that without firm f offer w^S has an offer from another firm and w^{NS} does not

$$m(\sigma'_f, \sigma_{-f}, \theta) - m(\sigma_f, \sigma_{-f}, \theta) = -1. \tag{B.2.2}$$

We will now show that $|\Theta_+| \geq |\Theta_-|$. Equations (B.2.1) and (B.2.2) along with the fact that each $\theta \in \Theta_+ \cup \Theta_-$ happens equally likely will then be enough to prove the result.

If profile θ belongs to Θ_{-}, without firm f's offer, worker w^S has an offer from another firm, name this firm f', and worker w^{NS} does not. We construct function $\psi : \Theta \to \Theta$ as follows. Let us considerLet $\psi(\theta)$ be the profile such that

- firms swap the positions of workers w^{NS} and w^S in their preference lists.
- if both w^S and w^{NS} send signals to firm f' for profile θ their preferences remain unchanged
- if worker w^S (w^{NS}) sends her signal to firm f' but worker w^{NS} (w^S) does not for profile θ, find a firm f_y such that worker w^S(w^{NS}) does not send her signal to firm f_y, and worker w^{NS} (w^S)does. Exchange the positions of firm f' and firm f_y in worker w^{NS} and worker w^S preference lists.

Note that firm f_y exists because each worker sends exactly K signals in any non-babbling symmetric equilibrium. We need the latter modification because each worker can send several signals, and the fact that worker w^Ssends her signal to firm f does not guarantee that she does not send another signal to firm f'.

If profile θ belongs to Θ_{-}, without firm f's offer, worker w^S has an offer from firm f', and worker w^{NS} does not. Therefore, when preferences are $\psi(\theta)$, without firm f's offer the following two statements should be true i) worker w^{NS} ***must*** have another offer and ii) worker w^S ***cannot*** have another offer.

To see i), note that under θ, worker w^S his outside offer comes from firm f'. Under $\psi(\theta)$ worker w^{NS} take position of worker w^S in firm f' preference list, and worker w^{NS} sends a signal to firm f' for profile $\psi(\theta)$ whenever worker w^S sends a signal to firm f' for profile θ. Anonymity of firm strategies guarantee that firm f' makes an offer to worker w^{NS}.

To see ii), suppose to the contrary that under $\psi(\theta)$, worker w does in fact receive an outside offer from some firm f''. This cannot be firm f'. Otherwise worker w^{NS} should get an offer from firm f' for profile θ by anonymity. This cannot be firm f_y because worker w^{NS} would get an offer from firm f_y for profile θ.

The main idea of the construction preserves the logic of Theorem 4. Specifically, if a worker receives firm's offer when she does not send a signal to the firm, she will definitely receives an offer if she sends a signal to the firm.

From i) and ii), we have

$$\theta \in \Theta_- \Rightarrow \psi(\theta) \in \Theta_+.$$

Since function ψ is injective, we have $|\Theta_+| \geq |\Theta_-|$.

In order to prove the second statement note that the expected number of matches of each worker increases when firm f responds more to signals. Using the construction presented above, one could show whenever worker w looses a match with firm f for profile θ (worker w ranks firm f low) it is possible to construct profile θ' when worker w obtains the match (worker w ranks firm f high). The function that matches these profiles is again injective. Moreover, worker w values more the match with high ranked firms. Therefore, ex-ante utility of worker w increases when firm f responds more to signals. $\square$

Proof of Proposition 8.

The result that the expected number of matches and the expected welfare of workers is higher in the equilibrium with higher cutoffs is an immediate consequence of Lemma B.23.

In order to show that firms have lower expected payoffs in the equilibrium with greater cutoffs we first consider the following situation. We take some firm f

such that its strategy σ_f differs from σ'_f only in that σ'_f has weakly greater cutoffs. Let us consider some firm $f' \in -f$. For each profile of preferences $\theta_{f'}$ and a set of signals $\mathcal{W}^S$, firm f' either makes an offer to $S_{f'}(\theta_{f'}, \mathcal{W}^S)$ or $T_{f'}(\theta_{f'}, \mathcal{W}^S)$. If firm f responds more to signals this decreases the probability that both $T_{f'}$ and $S_{f'}$ accept firm f' offer. Therefore, the expected payoff of firm $f' \in -f$ weakly decreases when firm f responds more to signals.

$$E_\theta(\pi_{f'}(\sigma_f, \sigma_{-f}, \theta)) \geq E_\theta(\pi_{f'}(\sigma'_f, \sigma_{-f}, \theta)).$$

Let us now consider two symmetric equilibria where firms play cutoff strategies $\tilde{\sigma}$ and $\bar{\sigma}$ correspondingly such that $\tilde{\sigma} \geq \bar{\sigma}$. From the definition of an equilibrium strategy we have:

$$E_\theta[\pi_f(\bar{\sigma}_f, \bar{\sigma}_{-f}, \theta)] \geq E_\theta[\pi_f(\tilde{\sigma}_f, \bar{\sigma}_{-f}, \theta)]$$

Using the result proved above we proceed with

$$E_\theta[\pi_f(\tilde{\sigma}_f, \bar{\sigma}_{-f}, \theta)] \geq E_\theta[\pi_f(\tilde{\sigma}_f, \tilde{\sigma}_{-f}, \theta)]$$

Therefore

$$E_\theta[\pi_f(\bar{\sigma}_f, \bar{\sigma}_{-f}, \theta)] \geq E_\theta[\pi_f(\tilde{\sigma}_f, \tilde{\sigma}_{-f}, \theta)]$$

□

Proof of Theorem 6.

Let us denote firm strategies in the unique equilibrium of the offer game with no signals as σ_F^0. Now, consider a symmetric equilibrium of the offer game with signals when agent use strategies (σ_F, σ_W). If agents employ strategies (σ_F^0, σ_W), the expected number of matches and the welfare of workers equal the corresponding parameters in the offer game with no signals. Therefore, the result that the expected number of matches and the expected welfare of workers in a symmetric equilibrium in the offer game with signals are *weakly* greater than the corresponding parameters in the unique equilibrium of the offer game with no signals is a consequence of sequential application of Lemma B.23. The result for worker and firm welfare, and the argument that the comparison is actually strict is similar to the one in Theorem 9. □

B.3 Market Structure and the Value of a Signaling Mechanism.

We will use the following notation below. We denote $u(j)$ utility of a firm from matching with its jth ranked worker.

Proposition B.36. *Under the assumption that*

$$u(W) > \tfrac{W}{F}\left(1 - \left(1 - \tfrac{1}{W}\right)^F\right) u(1) \tag{B.3.1}$$

there is a unique non-babbling equilibrium in the offer game with signals. Each worker sends her signal to her top firm. Each firm f makes an offer to S_f if it receives at least one signal; otherwise, firm f makes an offer to T_f.

Proof.

Let us show that under condition (B.3.1) even if S_f is the worst worker in firm f preferences, firm f still optimally makes her an offer.

Proposition 2 shows that if firms $-f$ respond more to signals, i.e. increase their cutoffs, it is also optimal for firm f to respond more to signals. Therefore, if firm f responds to signals when no other firm does, it will optimally responds to signals when other firm respond. Hence, it is enough to consider the incentives of firm f when firms $-f$ do not respond to signals and always make an offer their top ranked workers.

Let us consider some realized profile of preferences of firm f and denote T_f as w. If firms $-f$ does not respond to signals, then some firm among $-f$ makes an offer to worker w with probability $q = \frac{1}{W}$. Therefore, the probability that the offer of firm f to worker w is accepted equals

$$(1-q)^{F-1} + ... + C^j_{F-1} q^j (1-q)^{F-1-j} \tfrac{1}{j+1} + ... + q^{F-1} \tfrac{1}{F} \tag{B.3.2}$$

where we denote $C^y_x = \frac{x!}{y!(x-y)!}$. Intuitively, j firms among other $F-1$ firms simultaneously make an offer to worker w with probability $C^j_{F-1} q^j (1-q)^{F-1-j}$. Therefore, firm f is matched with worker w only with probability $\frac{1}{j+1}$ because worker w preferences are uniformly distributed. The sum over all possible j from 0 to $F-1$ gives us the overall probability of firm f offer being accepted. We can simplify this expression as

$$\begin{aligned}
& \sum_{j=0}^{F-1} C^j_{F-1} q^j (1-q)^{F-1-j} \tfrac{1}{j+1} && \text{(B.3.3)}\\
= \; & \sum_{j=0}^{F-1} \tfrac{(F-1)!}{j!(F-1-j)!} q^j (1-q)^{F-1-j} \tfrac{1}{j+1} && \text{(B.3.4)}\\
= \; & \sum_{j=0}^{F-1} \tfrac{1}{Fq} \tfrac{F!}{(j+1)!(F-(1+j))!} q^{j+1} (1-q)^{F-(1+j)} && \text{(B.3.5)}\\
= \; & \tfrac{1}{Fq} \sum_{j=1}^{F} \tfrac{F!}{j!(F-j)!} q^j (1-q)^{F-j} && \text{(B.3.6)}\\
= \; & \tfrac{1}{Fq} \left(\sum_{j=0}^{F} \tfrac{F!}{j!(F-j)!} q^j (1-q)^{F-j} - (1-q)^F \right) && \text{(B.3.7)}\\
= \; & \tfrac{1}{Fq} \left(1 - (1-q)^F \right) = \tfrac{W}{F} \left(1 - \left(1 - \tfrac{1}{W}\right)^F \right) && \text{(B.3.8)}
\end{aligned}$$

Alternatively, if firm f makes an offer to its top signaled worker, its offer is accepted with probability equal one. Therefore, it is optimal for the firm to make an offer to the signaled worker only if $u(W) > \frac{W}{F}\left(1-\left(1-\frac{1}{W}\right)^{F}\right)u(1)$. Using the discussion above, we conclude that under assumption B.3.1 there is no other non-babbling symmetric equilibrium in the offer game with signals. □

B.3.1 Balanced Markets

Proof of Proposition 9.

We first calculate an explicit formula for the increase in the expected number of matches from the introduction of the signaling mechanism.

Lemma B.34. *Let us consider a market with W workers and $F > 2$ firms. The expected number of matches in the offer game with no signals equals*

$$m^{NS}(F,W) = W\left(1-\left(1-\tfrac{1}{W}\right)^{F}\right) \tag{B.3.9}$$

The expected number of matches in the offer game with signals equals

$$m^{S}(F,W) = F\begin{pmatrix} 1-\left(\frac{F-1}{F}\right)^{W} + \frac{W(F-1)^{2W-2}}{F^{W}(F-2)^{W-1}}\left(1-\frac{F-1}{W}\left(1-\left(\frac{F-2}{F-1}\right)^{W}\right)\right)* \\ *\left(1-\left(1-\frac{1}{W}\left(\frac{F-2}{F-1}\right)^{W-1}\right)^{F-1}\right)\end{pmatrix} \tag{B.3.10}$$

Proof of Lemma B.34.

Let us first calculate the expected number of matches in the pure coordination game with no signals. Proposition B.11 establishes that the unique symmetric non-babbling equilibrium when agents use anonymous strategies is the following. Each firm makes an offer to its top worker and each worker accepts the best offer among available ones. We have already calculated the probability of firm f being matched to its top worker in Proposition B.36. This probability equals

$$\frac{W}{F}\left(1-\left(1-\frac{1}{W}\right)^F\right)$$

Therefore, the expected total number of matches in the game with no signals equals

$$m^{NS}(F,W) = W\left(1-\left(1-\frac{1}{W}\right)^F\right) \tag{B.3.11}$$

Let us now calculate the expected number of matches in the offer game with signals. Proposition B.36 derives agent strategies in the unique symmetric non-babbling equilibrium in the pure coordination game with signals. Each worker sends her signal to her top firm and each firm makes its offer to top signaled worker if it receives at least one signal, otherwise it makes an offer to its top ranked worker.

We first calculate ex-ante probability of being matched by some firm f. We denote the set of workers that send her signal to firm f as $\mathcal{W}_f^S \subset \mathcal{W}$. If firm f receives at least one signal, $|\mathcal{W}_f^S| > 0$, it guarantees itself a match because each worker sends her signal to her top firm. If firm f receives no signals, it makes an offer to its top ranked worker T_f. This worker accepts firm f offer only if this offer is the best one among those she receives. Let us denote the probability that T_f accepts firm f's offer under the condition that firm f receives no signals as

$$P_{T_f,|\mathcal{W}_f^S|=0} \equiv P(T_f \text{ accepts firm } f'\text{s offer}|\left|\mathcal{W}_f^S\right| = 0)$$

Then ex-ante probability that firm f is matched equals

$$Prob\_match_f(F,W) = P(|\mathcal{W}_f^S| > 0) * 1 + P(|\mathcal{W}_f^S| = 0) * P_{T_f,|\mathcal{W}_f^S|=0} \tag{B.3.12}$$

If firm f receives no signals, $|\mathcal{W}_f^S| = 0$, it makes an offer to T_f that we denote as worker w. Worker w receives an offer from its top ranked firm, say firm f_0, conditional on firm f receiving no signals, $|\mathcal{W}_f^S| = 0$, with probability equal

$$
\begin{aligned}
G &= P(|\mathcal{W}_{f_0}^S| = 1 || \mathcal{W}_f^S| = 0) * 1 + ... + P(|\mathcal{W}_{f_0}^S| = W || \mathcal{W}_f^S| = 0) * \tfrac{1}{W} && \text{(B.3.13)} \\
&= \sum_{j=0}^{W-1} C_{W-1}^{j} \left(\tfrac{1}{F-1}\right)^j (1 - \tfrac{1}{F-1})^{W-j-1} \tfrac{1}{j+1} && \text{(B.3.14)}
\end{aligned}
$$

Intuitively, firm f_0 receives a signal from a particular worker with probability $\frac{1}{F-1}$ (note that firm f receives no signals). Then, if firm f_0 receives signals from j other workers, worker w receives an offer from firm f_0 with probability $\frac{1}{j+1}$. Similarly to equation (B.3.3) the expression for G simplifies to

$$
G = \tfrac{F-1}{W} \left(1 - (1 - \tfrac{1}{F-1})^W\right) \qquad \text{(B.3.15)}
$$

Firm f can be matched with worker w only if worker w does not receive an offer from its top firm, which happens with probability $1 - G$. If worker w does not receive an offer from her top firm – firm f_0 – firm f competes with other firms that have received no signals from workers. The probability that some firm f' among firms $\mathcal{F} \backslash \{f, f_0\}$ receives no signals conditional on the fact that worker w sends her signal to firm f_0 and firm f receives no signals $|\mathcal{W}_f^S| = 0$ equals $r = (1 - \frac{1}{F-1})^{W-1}$. Note that the probability that firm f' does not receive a signal from a worker equals $1 - \frac{1}{F-1}$, because firm f receives no signals. There are also only $W - 1$ workers that can send a signal to firm f', because worker w sends her signal to firm f_0.

Therefore, the probability that some firm f' among firms $\mathcal{F} \backslash \{f, f_0\}$ receives no signals and makes an offer to worker w, conditional on the fact that worker w sends her signal to firm f_0, equals $\frac{r}{W}$. Therefore, the probability that worker w prefers offer of firm f to other offers (conditional on the fact that firm f receives no signals and worker w sends her signal to firm f_0) equals[6]

$$\sum_{j=0}^{F-2} C_{F-2}^{j} \left(\tfrac{r}{W}\right)^{j} \left(1-\tfrac{r}{W}\right)^{F-2-j} \tfrac{1}{j+1} = \tfrac{W}{(F-1)r} \left(1-\left(1-\tfrac{r}{W}\right)^{F-1}\right) \tag{B.3.16}$$

Then the probability that worker w accepts firm f's offer equals

$$P_{T_f,|\mathcal{W}_f^S|=0} = (1-G)\left(\tfrac{W}{(F-1)r}\left(1-\left(1-\tfrac{r}{W}\right)^{F-1}\right)\right)$$

Taking into account that firm f receives no signals with probability equal

$$P(|\mathcal{W}_f^S|=0) = (1-\tfrac{1}{F})^W$$

the probability of firm f being matched in the offer game with signals equals

$$\begin{aligned} Prob\_match_f(F,W) &= 1-(1-\tfrac{1}{F})^W + (1-\tfrac{1}{F})^W * P_{T_f,|\mathcal{W}_f^S|=0} && \text{(B.3.17)}\\ &= 1-(1-\tfrac{1}{F})^W + (1-\tfrac{1}{F})^W \tfrac{W}{(F-1)r}\left(1-\tfrac{F-1}{W}\left(1-(1-\tfrac{1}{F-1})^W\right)\right) * && \text{(B.3.18)}\\ &\quad * \left(1-\left(1-\tfrac{r}{W}\right)^{F-1}\right) && \text{(B.3.19)} \end{aligned}$$

where $r = (1-\frac{1}{F-1})^{W-1}$. The expected total number of matches in the offer game with signals equals $m^S(F,W) = F * Prob\_match_f(F,W)$. $\square$

Lemma B.34 establishes the expected total number of matches in the offer game with and with no signals. Let us first fix W and calculate where the increase in the expected number of matches from the introduction of the signaling mechanism, $V(F,W) = m^S(F,W) - m^{NS}(F,W)$, attains its maximum. In order to obtain the result of the proposition we consider large markets, where the number of firms and the number of workers are large, and we use Taylor's expansion

[6] Note that the maximum number of offers worker w could get equals to $M-1$ as it does not receive an offer from its top firm f_0.

formula

$$(1-a)^b = \exp(-ab + O(a^2 b)) \tag{B.3.20}$$

where $O(a^2b)$ is a function that is smaller than a constant for large values of a^2b. If we denote $x = \frac{F}{W}$ the expected number of matches in the offer game with no signals can be approximates as

$$m^{NS}(F,W) = W\left(1 - \left(1 - \frac{1}{W}\right)^F\right) = W(1 - e^{-x+O(x/W)})$$

Let us now consider the expected number of matches in the offer game with signals. Using the result of Lemma B.34 we get

$$m^S(F,W) = Wx\left(1 - e^{-1/x+O\left(1/(x^2W)\right)} + A * B\right)$$

where

$$\begin{aligned} A &= \left(1 - \frac{F-1}{W}\left(1 - \left(\frac{F-2}{F-1}\right)^W\right)\right) \\ B &= \frac{W(F-1)^{2W-2}}{F^W(F-2)^{W-1}}\left(1 - \left(1 - \frac{1}{w}\left(\frac{F-2}{F-1}\right)^{W-1}\right)^{F-1}\right) \end{aligned}$$

We first calculate an approximation of A for large markets. Using (B.3.20) we get that

$$1 - (1 - \frac{1}{F-1})^W = 1 - e^{-x+O(1/(x^2W))}$$

and

$$A = 1 - x\left(1 - e^{-1/x+O(1/(x^2W))}\right) + O(1/(xW))$$

We now calculate an approximation of B for large markets.

$$\begin{aligned} \frac{W(F-1)^{2W-2}}{F^W(F-2)^{W-1}} &= \frac{W}{F}\left(\frac{F-1}{F}\right)^{W-1}\left(\frac{F-1}{F-2}\right)^{W-1} \\ &= \frac{1}{x}e^{-(W-1)/F+O(1/(x^2W))}e^{(W-1)/(F-1)+O(1/(x^2W))} \\ &= \frac{1}{x}e^{O(1/(x^2W))} \end{aligned}$$

Also, we have that

$$\begin{aligned}\left(1-\left(1-\tfrac{Z}{W}\right)^{F-1}\right) &= 1-e^{-Z(F-1)/W+O(x/W)}\\ &= 1-e^{-Zx+O(x/W)}\end{aligned}$$

where $Z = (\frac{F-2}{F-1})^{W-1} = e^{-1/x+O(1/(x^2W))}$. Then, we have that

$$\begin{aligned}B &= \tfrac{W(F-1)^{2W-2}}{F^W(F-2)^{W-1}} * \left(1-\left(1-\tfrac{1}{W}(\tfrac{F-2}{F-1})^{W-1}\right)^{F-1}\right)\\ &= \tfrac{1}{x}e^{O(1/(x^2W))}(1-e^{-xe^{-1/x}+O(x/W)})\end{aligned}$$

Overall, we have

$$\begin{aligned}V(F,W) &= Wx\left(\begin{array}{c}1-e^{-1/x+O(1/W)}+\left(1-x\left(1-e^{-1/x+O(1/W)}\right)+O(1/W)\right)*\\ *\tfrac{1}{x}e^{O(1/W)}(1-e^{-xe^{-1/x}+O(1/W)})\end{array}\right)-\\ &\quad -W(1-e^{-x+O(1/W)})\\ &= W\left(x-xe^{-1/x}+\left(1-x\left(1-e^{-1/x}\right)\right)(1-e^{-xe^{-1/x}})-1+e^{-x}\right)+O(1)\\ &= W\alpha(x)+O(1)\end{aligned}$$

where $\alpha(x)$ is a positive quasi-concave function that attains maximum at $x_0 \simeq 1.012113$. Therefore, for fixed W, $V(F,W)$ attains its maximum value at $F = x_0W + O(1)$.

Similar to the previous derivation, we fix F and calculate the value of W where $V(F,W)$ attains its maximum.

$$\begin{aligned}V(F,W) &= F\left(\begin{array}{c}1-e^{-1/x+O(1/W)}+\left(1-x\left(1-e^{-1/x+O(1/W)}\right)+O(1/W)\right)\\ *\tfrac{1}{x}e^{O(1/W)}(1-e^{-xe^{-1/x}+O(1/W)})\end{array}\right)\\ &\quad -\tfrac{F}{x}(1-e^{-x+O(1/F)})\\ &= F\left(1-e^{-1/x}+\left(1-x\left(1-e^{-1/x}\right)\right)\tfrac{1}{x}(1-e^{-xe^{-1/x}})-\tfrac{1}{x}\left(1-e^{-x}\right)\right)+O(1)\\ &= F\beta(x)+O(1)\end{aligned}$$

where $\beta(x)$ is a positive quasi-concave function that attains maximum at $x_{00} \simeq 0.53074$. Therefore, for fixed F, $V(F,W)$ attains its maximum value at $W = y_0 F + O(1)$, where $y_0 = 1/x_{00} = 1.8842$. □

B.3.2 Markets with Many Periods of Interaction

Proposition B.37. *Consider the following assumptions on the utility functions and the discount factor.*

$$\begin{aligned} u(W) &> \tfrac{W}{F}\left(1-\left(1-\tfrac{1}{W}\right)^F\right)u(1) \\ u(W) &> \delta u(1),\ v(W) > \delta v(1) \end{aligned}$$

1. *Then, there is the unique symmetric sequential equilibrium in the offer game with no signals and L periods of interaction: each firm makes an offer to its most preferred worker and each worker accepts its best offer in each period.*

2. *Then, there is the unique symmetric sequential non-babbling (in each period) equilibrium in the offer game with signals and L periods of interaction: each worker sends her signal to her top firm at period 0. Each worker accepts the best available offer at each period. Each firm makes an offer at period $l = 1, ..., L$ to its top signaled worker among workers who are in the market; otherwise the firm makes an offer to its top ranked worker among workers who are in the market.*

Proof.

Let us first consider the offer game with no signals and L periods of interaction. Using backward induction we consider the last stage of the game. Since the last stage of the game is one period offer game with no signals the only symmetric equilibrium in this subgame: each firm makes an offer to its top ranked worker and each worker accepts best available offer.

Assumptions $u(W) > \delta u(1)$ and $v(W) > \delta v(1)$ guarantee that there is no incentives to hold offers or make dynamically strategic offers. Since firms $-f$ use symmetric anonymous strategies at stage $L-1$ and stage L the only optimal strategy of firm f at stage $L-1$ is to make an offer to its T_f. Each worker, who receives at least one offer at stage $L-1$, optimally accepts the best available offer immediately. Similar logic applies to the other stages.

Let us now consider the offer game with signals and L periods of interaction. The symmetry of workers $-w$ strategies and anonymous property of firms strategies guarantee that the probability that each firm makes an offer to worker w (considering all L periods), conditionally on receiving her signal/not receiving her signal is the same across firms. Therefore, workers should still optimally send their signals to top firms in the offer game with signals and L periods of interaction.

In addition, there is only first stage of the game when signals play role. Since $u(W) > \delta u(1)$ and $u(W) > \frac{W}{F}\left(1-\left(1-\frac{1}{W}\right)^F\right)u(1)$ each firm f makes an offer to S_f if it receives at least one signal at stage 1. Since $v(W) > \delta v(1)$ workers accept the best available offers immediately. At stage 2, there are only firms that received no signals and whose offers were rejected at stage 1. Therefore, the logic of backward induction in the offer game with no signals and many periods applies to period 2 through period L in the offer game with signals. □

B.3.3 The Optimal Number of Signals and the Number of Interviews

Proposition B.38. *Consider a market when each worker has I interview positions. There is the unique equilibrium of the pure coordination model with no signals. Each firm f makes an offer to T_f and each worker accepts the best I offers from those it receives.*

In addition, assume that each worker can send up to K signals and

$$H(K, W, I)u(W) > H(F, W, I)u(1),$$

where

$$H(F, W, I) = \sum_{j=0}^{I-1} C_{F-1}^{j}(\tfrac{1}{W})^{j}\left(1 - \tfrac{1}{W}\right)^{F-1-j} + \sum_{j=I}^{F-1} C_{F-1}^{j}(\tfrac{1}{W})^{j}\left(1 - \tfrac{1}{W}\right)^{F-1-j}\tfrac{I}{j+1}$$

then, there is also the unique symmetric non-babbling equilibrium in the offer game with signals. Each worker send K signals to her most preferred K firms and accepts the best I offers among those available. Each firm f makes an offer to S_f if it receives at least one signal; otherwise it makes an offer to T_f.

Proof.

Since workers make decision only on the last stage the statement about the offer game with no signals follows directly from Proposition B.11.

Let us now consider the pure coordination game with signals. If firms employ symmetric anonymous strategies, then the probability of a worker w receiving an offer from a firm conditional on w sending a signal to it is the same across all firms. Similarly, the probability of w receiving an offer from a firm conditional on w not sending a signal to it is the same across all firms. We denote the former one as p^s and the latter one as p^{ns}. The argument similar to the one in Proposition 4 show that $p^s \geq p^{ns}$. Therefore, it is optimal for each worker to send K signals to her top firms in any non-babbling symmetric equilibrium, i.e. $p^s > p^{ns}$.

Let us consider some firm f. Since workers send identical signals firm f prefers to make an offer to S_f rather than to any other worker who sends a signal to it. Similar, a firm f prefers to make an offer to T_f rather than to any other worker who do not send a signal to it (similar to Proposition B.12). As one could see the statement of Proposition B.13 also holds.

We now show that condition

$$H(K, W, I)u(W) > H(F, W, I)u(1)$$

is sufficient for a firm to make an offer to S_f even if S_f is the worst worker in firm's preference list.

We employ the logic similar to the one of Proposition B.36. Let us consider some firm f profile of preferences and denote T_f as w. If firms $-f$ do not respond to signals, each firm among $-f$ makes its offer to worker w with probability $\frac{1}{W}$. Therefore, the probability that firm f offer to worker w is accepted equals:

$$H(F, W, I) = \sum_{j=0}^{I-1} C_{F-1}^{j}(\tfrac{1}{W})^j \left(1 - \tfrac{1}{W}\right)^{F-1-j} + \sum_{j=I}^{F-1} C_{F-1}^{j}(\tfrac{1}{W})^j \left(1 - \tfrac{1}{W}\right)^{F-1-j} \tfrac{I}{j+1}$$

If there are fewer than $I - 1$ firms among $-f$ make an offer to worker w, firm f is matched with unit probability. Otherwise firm f is matched with worker w only with probability $\frac{I}{j+1}$. Unfortunately, this expression cannot be simplified as in Proposition B.36. However, when firm f makes an offer to S_f it only knows that it is among K top firms. Therefore, the probability that worker S_f accepts firm f offer equals $H(K, W, I)$. Therefore, if $H(K, W, I)u(W) > H(F, W, I)u(1)$, it is always optimal for firm f to make an offer to S_f. $\square$

Appendix C

Appendix: Harmful Signaling in Matching Markets

Proof of Proposition 11.

Let us show this statement for each firm sequentially. Each worker w_i has preferences $\theta_0 = (f_1, ..., f_F)$ with probability $1-\varepsilon$ and with complementary probability some preferences distributed according to $A(\Theta_{\mathcal{W}})$. Let us consider firm f_1 which has some preferences θ_{f_1}. If it makes an offer to worker $w_1 = max_{\theta_{f_1}}(w|w \in \mathcal{W})$, its offer will be the best worker w_1's offer with probability at least $1 - \varepsilon$. Hence, its expected utility from making an offer to worker w_1 equals at least $(1 - \varepsilon)u(1)$ which is greater than $u(2)$ for sufficiently small ε. Hence, independently on other firms' strategies, firm f_1's optimal strategy is to make an offer to its best worker.

Let us assume that each firm f_k, $k < j$, makes its offer to worker w_k. Now we consider the decision of firm f_j. The expected payoff from making an offer to some worker among $\{w_1, ...w_{j-1}\}$ is less than $\varepsilon u(1)$. In the same time the expected payoff from making offer to some worker among $\mathcal{W} \backslash \{w_1, ...w_{j-1}\}$ is at least $(1 - \varepsilon)u(j)$. Hence, given the strategies of other firms and sufficiently small

ε, the optimal strategy of firm f_j is to make an offer to its best worker among $\mathcal{W}\backslash\{w_1, ...w_{j-1}\}$.□

Proof of Proposition 12.

The only undominated strategy of a worker at the last stage is to choose the best offer among available ones. Then, under the condition that firm f does not respond to any signal, for any $\mathcal{W}^S \subset \mathcal{W}$ reached in an equilibrium $\sigma_f(\mathcal{W}^S) = const$. Let us assume that there exists a realization of agents' preferences such that firm f_1 is matched to some worker w_i, $i > 1$, in the equilibrium. Hence, for any $\mathcal{W}^S \subset \mathcal{W}$, reached in the equilibrium, $\sigma_{f_1}(\mathcal{W}^S) = w_i$. Hence, the expected firm 1's payoff equals at most $u(2)$. However, the strategy $\sigma_{f_1}(\mathcal{W}^S) = w_1$ for any $\mathcal{W}^S \subset \mathcal{W}$ is compatible with assumption that firm f_1 does not respond to any signals and gives payoff $(1-\varepsilon)\,u(1)$ independently of strategies of other firms. Hence, $\sigma_{f_1}(\mathcal{W}^S) = w_i$ cannot be an equilibrium strategy. Similar argument could be applied to any other firm f_j, $j = 2, ..., F$.□

Proof of Theorem 7.

We prove the theorem by way of several lemmata. In the proof of the lemmata we presume that ε is sufficiently small. First, we establish that a firm believes about a particular worker is typical with probability more than $1-\varepsilon$ either when it receives her signal or when it does not receive her signal. Second, we show that firms do not make their offers to a worker better than no signaling match if they do not receive her signal. The third lemma proves that if a firm does receive a signal from a worker better than its no signaling match, it makes its offer to the best such worker. Finally, using the statements of lemmata we show that the set of strategies stated in the theorem constitutes a unique very informative equilibrium.

First two lemmata do not require the assumption that each firm f_j, $j = 1, ..., F$, responds to all signals from workers better or equal to worker w_j according to its preferences.

Lemma C.01. *For any worker $w \in \mathcal{W}$, any firm $f \in \mathcal{F}$, and any $\mathcal{W}^S \subset \mathcal{W}$ either $\mu_f(\theta_w = \theta_0|\mathcal{W}^S \cup w) \geq 1 - \varepsilon$ or $\mu_f(\theta_w = \theta_0|\mathcal{W}^S \backslash w) \geq 1 - \varepsilon$. Similarly, either $\mu_f(\theta_w \neq \theta_0|\mathcal{W}^S \cup w) \leq \varepsilon$ or $\mu_f(\theta_w \neq \theta_0|\mathcal{W}^S \backslash w) \leq \varepsilon$.*

Proof.

Let us denote as α_T and α_A the probabilities that typical and atypical type of worker w correspondingly send a signal to firm f. Then, if worker w sends her signal to firm f, $(1-\varepsilon)\alpha_T + \varepsilon\alpha_A > 0$, we derive its beliefs using Bayes' rule

$$\begin{cases} \mu_f(\theta_w = \theta_0|\mathcal{W}^S \cup w) = \frac{(1-\varepsilon)\alpha_T}{(1-\varepsilon)\alpha_T + \varepsilon\alpha_A} \\ \mu_f(\theta_w = \theta_0|\mathcal{W}^S \backslash w) = \frac{(1-\varepsilon)(1-\alpha_T)}{(1-\varepsilon)(1-\alpha_T) + \varepsilon(1-\alpha_A)} \end{cases}$$

One can verify that

$$\begin{cases} \mu_f(\theta_w = \theta_0|\mathcal{W}^S \cup w) \geq 1 - \varepsilon \quad \Leftrightarrow \quad \alpha_T \geq \alpha_A \\ \mu_f(\theta_w = \theta_0|\mathcal{W}^S \backslash w) \geq 1 - \varepsilon \quad \Leftrightarrow \quad \alpha_T \leq \alpha_A \end{cases}$$

Hence, either $\mu_f(\theta_w = \theta_0|\mathcal{W}^S \cup w) \geq 1 - \varepsilon$ or $\mu_f(\theta_w = \theta_0|\mathcal{W}^S \backslash w) \geq 1 - \varepsilon$. If worker w never sends her signal to firm f, $(1-\varepsilon)\alpha_T + \varepsilon\alpha_A = 0$, firm f's beliefs are $\mu_f(\theta_w = \theta_0|\mathcal{W}^S \backslash w) = 1 - \varepsilon$ and $\mu_f(\theta_w = \theta_0|\mathcal{W}^S \cup w)$ is arbitrary. The second statement directly follows from the first one. □

Lemma C.02. *If* firm f_j does not receive a signal from worker w *strictly better than worker* w_j*,* $w \succ_{f_j} w_j$ it does not make an offer to her in an equilibrium.

Proof.

We prove this statement for firms sequentially. Let us first show its validity for $j = 2$. The only worker that could be better than worker w_2 for firm f_2 is worker w_1 by construction. If $w_2 \succ_{f_2} w_1$ we are done. Assume that $w_1 \succ_{f_2} w_2$.

There are two possibilities: either worker $w_1(T)$ sends her signal to firm f_1, i.e. $\sigma_{w_1}(\theta_0) = f_1$, or she does not send her signal to firm f_1, i.e. $\sigma_{w_1}(\theta_0) \neq f_1$, in an equilibrium.

Assume worker w_1 employs strategy $\sigma_{w_1}(\theta_0) = f_1$. If firm f_2 does not receive worker w_1 signal, firm f_2 believes she is atypical with probability less than ε, $\mu_{f_2}(\theta_{w_1} \neq \theta_0 | \mathcal{W}^S \backslash w_1) \leq \varepsilon$ (Lemma C.01). According to assumption $F \leq W$, firm f_2 can secure a match with some worker w_i, $i \geq 2$, with probability at least $1 - \varepsilon$. Hence, firm f_2 does not make an offer to worker w_1 in an equilibrium.

Worker w_1 employs strategy $\sigma_{w_1}(\theta_0) \neq f_1$ in an equilibrium only if firm f_1 makes its offer to worker w_1 with probability equals to one, and firm f_2 has a chance to be matched with worker w_1 only if she is atypical. Assume firm f_2 makes an offer to worker w_1 when it does not receive her signal. If $w_1(T)$ sends her signal to firm f_2 in an equilibrium, according to Assumption PRS firm f_2 should also make an offer if it receives a signal from w_1. However, if it receives a signal from w_1, the probability that worker w_1 is atypical less than ε (Lemma C.01), which contradicts equilibrium behavior.

Now, we assume that worker $w_1(T)$ does not send her signal to firm f_2 in an equilibrium. If firm f_2 does not receive worker w_1's signal, firm f_2 believes that she is atypical with probability less or equal ε, $\mu_{f_2}(\theta_{w_1} \neq \theta_0 | \mathcal{W}^S \backslash w_1) \leq \varepsilon$ (Lemma C.01). Therefore, it is again suboptimal for firm f_2 to make an offer to worker w_1 if it does not receive a signal from her.

We have shown above that it is suboptimal for firm f_2 to make an offer to worker w_1 if it does not receive a signal from her. Let us assume that it is suboptimal for any firm f_j, $j < k$ to make its offer to a worker w_t, $t < j$, if firm f_j does not receive a signal from it and show that the claim for firm f_k.

We consider some worker w_i, $i < k$. Firm f_i makes its offer to workers $\{w_1, ..., w_{i-1}\}$ with probability less than $\varepsilon(i-1)$. In addition, worker w_i is atypical with probability ε. Hence, firm f_k can secure a match with worker w_i with probability equals at most $i\varepsilon$ if it does not receive a signal from her. For small enough ε firm f_k's offer to worker w_i is suboptimal. □

Now, we assume that each firm f_j, $j = 1, ..., F$, responds to all signals from workers better or equal to worker w_j according to its preferences. The following lemma shows that firm f_j makes its offer to some worker w better or equal to worker w_j if worker w's signal is the best signal firm f_j receives.

Lemma C.03. *Assume that $F > W$. Then, for any $\mathcal{W}^S \subset \mathcal{W}$ $\sigma_{f_j}(\mathcal{W}^S) = max_{\theta_{f_j}}(w : w \in \mathcal{W}^S)$ if $\mathcal{W}^S \cap \Delta(f_j) \neq \varnothing$ in very informative equilibrium*[1].

Proof.

We prove this statement for firms sequentially. Let us consider firm f_1 and worker w_1. Assume that worker w_1 employs strategy $\sigma_{w_1}(\theta_0) \neq f_1$. Then, firm f_1 believes that for any $\mathcal{W}^S \subset \mathcal{W}$ $\mu_{f_1}(\theta_{w_1} = \theta_0|\mathcal{W}^S \backslash w_1) \geq 1 - \varepsilon$. Therefore, for sufficiently small ε, firm f_1 always makes its offer to worker w_1, which contradicts to our assumption that it responds to worker w_1's signal. Therefore, under the assumption that firm f_1 responds to a signal from worker w_1, the only possible worker w_1's equilibrium strategy is $\sigma_{w_1}(\theta_0) = f_1$. In this case, for any $\mathcal{W}^S \subset \mathcal{W}$ firm f_1's belief is $\mu_{f_1}(\theta_{w_1} = \theta_0|\mathcal{W}^S \cup w_1) \geq 1 - \varepsilon$. Hence, firm f_1's highest expected payoff when it receives worker w_1's signal is from making an offer to worker w_1. Hence, for any $\mathcal{W}^S \subset \mathcal{W}$, firm f_1's strategy $\sigma_{f_1}(\mathcal{W}^S \cup w_1) = w_1$ is optimal.

[1]If $F = W$ the claim is still valid with the same assumption for all firms except firm f_F. Firm f_F should respond to a signal from any worker strictly better than the corresponding one.

Assume now that for any $t \leq j < k$, and for any $\mathcal{W}^S \subset \mathcal{W}$, firm f_j employs strategy for $\sigma_{f_j}(\mathcal{W}^S) = max_{\theta_{f_j}}(w : w \in \mathcal{W}^S)$ if $\mathcal{W}^S \cap \Delta(f_j) \neq \varnothing$. We prove below that firm f_k's optimal strategy for any $\mathcal{W}^S \subset \mathcal{W}$ and $\sigma_{f_k}(\mathcal{W}^S) = max(w : w \in \mathcal{W}^S)$ if $\mathcal{W}^S \cap \Delta(f_k) \neq \varnothing$.

There are two possibilities: either $\sigma_{w_k}(\theta_0) \neq f_k$ or $\sigma_{w_k}(\theta_0) = f_k$. For the former case, for any $\mathcal{W}^S \subset \mathcal{W}$ $\mu_{f_k}(\theta_{w_k} = \theta_0 | \mathcal{W}^S \backslash w_k) \geq 1 - \varepsilon$. Hence, it is optimal for firm f_k to make an offer to worker w_k when it receives no signals from workers better or equal to worker w_k, i.e. for any $\mathcal{W}^{S\prime} \subset \mathcal{W}$ such that $\mathcal{W}^{S\prime} \cap \Delta(f_k) = \varnothing$, $\sigma_{f_k}(\mathcal{W}^{S\prime}) = w_k$. Hence, it is also optimal for firm f_k to make an offer to worker w_k when worker w_k's signal is the best signal it receives, i.e. for any $\mathcal{W}^{S\prime\prime} \subset \mathcal{W}$ such that such that $\mathcal{W}^{S\prime\prime} \cap \Delta(f_k) = w_k$, $\sigma_{f_k}(\mathcal{W}^{S\prime\prime}) = w_k$. Therefore, firm f_k does not respond to worker w_k's signal. Contradiction.

For the latter case, $\sigma_{w_k}(\theta_0) = f_k$, if firm f_k does not receive a signal from worker w_k, it anticipates that she is atypical. Therefore, firm f_k does not make its offer to her. If firm f_j receives signals from any worker $w_i \succeq w_k$ no other firm f_p, $p \neq j$ and $p > i$, makes its offer to worker w_i according to Lemma C.02. The only offers that compete with firm f_j's offer could be the ones from the set $\{f_p, p < i\}$. However, any firm f_p, $p < i$, could make an offer to worker w_i only if worker w_p is atypical, which happens with probability ε. Hence, the interim expected payoff for firm f_j from making its offer to worker w_i equals at least $(1 - (i-1)\varepsilon)u'$, where $u' = u_{f_j}(w_i, \theta_{f_j})$. Firm f_j expected payoff from making an offer to any other worker from set $\Delta(f_j)$ is smaller than $(1 - (i-1)\varepsilon)u'$ as this worker either has not sent a signal to firm f_j or has a smaller rank in firm f_j' preferences. The expected payoff from making an offer to some worker $\mathcal{W} \backslash \Delta(f_j)$ is smaller either. Therefore, firm f_j optimal strategy is, $\sigma_{f_j}(\mathcal{W}^S) = max_{\theta_{f_j}}(w : w \in \mathcal{W}^S)$ if $\mathcal{W}^S \cap \Delta(f_j) \neq \varnothing$. □

Now we are ready to prove the theorem. Let us show that the set of strategies, stated in the theorem, constitutes an equilibrium. We first prove that if all agents, except firm f_l, follow the strategies, stated in the theorem, firm f_l's strategy is optimal given its belief is consistent with the other agents' strategies. If firm f_l receives a signal from worker w_t, $t < l$, firm f_l believes that itself is the best firm among $\Delta(w_t) = \{f_j \in \mathcal{F} : w_t \succeq_{f_j} w_j\}$. Let us assume that worker w_t is the best worker who sends a signal to firm f_l. Worker w_t does not accept firm f_l's offer only if she receives an offer from some firm $f_k \in \mathcal{W} \backslash \Delta(w_t)$. However, it happens only if worker w_k is atypical, i.e. with probability less than ε. Hence, firm f_l interim expected payoff from making an offer to worker w_t equals at least $(1 - (n-1)\,\varepsilon)\, u'$, where $u' = u_{f_l}(w_t, \theta_{f_l})$. Firm f_l's offer to a worker better than worker w_t is not optimal according to Lemma C.01. Firm f_l's expected payoff from making an offer to some worker w, $w_t \succ_{f_l} w$, is also smaller than making an offer to worker w_t for sufficiently small ε. Overall, firm f_l's strategy is optimal.

Let us show that, if all agents, except worker w_t, follow the strategies, stated in the theorem, worker w_t's strategy is optimal. Firm f_t does not make an offer to worker w_t when it receives a signal from a better worker. Therefore, if worker w_t is typical, her payoff from sending a signal to firm f_t equals at least $[1 - (l-1)\varepsilon]\, u(t)$. If worker w_t does not send her signal to firm f_t it loses her offer and she could get payoff at most $u(t-1)$, which is less than $[1 - (l-1)\varepsilon]\, u(t)$ for sufficiently small ε. There is also no reason for worker w_t to send her signal to a firm better than firm f_t, because this firm does not respond to her signal according to its equilibrium strategies. Hence, worker $w_t(T)$'s strategy is optimal. Using similar logic one can show that worker $w_t(A)$'s strategy is also optimal.

Now we show that the above strategies constitute the unique very informative equilibrium. Lemmata C.02 and C.03 imply that each firm f_l, $l = 1, ..., F$, has to follow the following strategies in an equilibrium:

$$\text{for any } \mathcal{W}^S \subset \mathcal{W}, \begin{cases} \sigma_{f_l}(\mathcal{W}^S) \neq w_l \text{ if } \mathcal{W}^S \cap \Delta(f_l) = \varnothing \\ \sigma_{f_l}(\mathcal{W}^S) = w_l \text{ if } \mathcal{W}^S \cap \Delta(f_l) = w_l \end{cases}$$

Straightforwardly, the only worker $w_l(T)$'s optimal strategy is to send her signals to firm f_l, $\sigma_{w_l}(\theta_0) = f_l$, otherwise, firm f_l's does not make an offer to student w_l.

Let us consider firm $f^* = \max_{\theta_{w_l}}(f' \in \Delta(w_l))$. Firm f^* responds to signals from workers better or equal than no signaling match and its equilibrium beliefs are $\mu_{f^*}(\theta_{w_l} = \theta_0|\mathcal{W}^S \backslash w_l) \geq 1 - \varepsilon$ and $\mu_{f^*}(\theta_{w_l} \neq \theta_0|\mathcal{W}^S \cup w_l) = 1$. Therefore, if firm f^* does not receive a signal better than worker w_l's one, it's optimal strategy is to make an offer to worker w_l. Taking into account that firm f^* can receive a signal from a better worker with probability less than $(l-1)\varepsilon$, worker $w_l(A)$'s optimal strategy is to send her signal to firm f^* (for sufficiently small ε). Hence, the strategies, stated in the theorem, constitute the unique equilibrium. □

Proof of Theorem 9.

Assumption that $A(\Theta_{\mathcal{W}})$ has a full support and that the strategies of the very informative equilibrium guarantee that some worker w_i sends her signals to each firm in the set $\Delta(w_i) = |\{f_j \in \mathcal{F} : w_i \succeq_{f_j} w_j\}|$ with positive probability. Then, using logic of Example 1 and Example 2 it is straightforward to show that when there are at least three firms in the set $\Delta(w_i)$ and there are at least three workers the mismatch happens with positive probability. Therefore, the expected number of matches strictly smaller than in the corresponding no signaling equilibrium. □

Proof of Proposition 13. The statement directly follows from the strategies of the very informative equilibrium. □

Example C.01. [An equilibrium when assumption PRS does not hold]Let us consider two firms and two workers. We assume that all firms have the same preferences over workers $\theta_{f_1} = \theta_{f_2} = \{w_1, w_2\}$. Also we assume that each typical worker has preferences $\theta_0 = (f_1, f_2)$ and each atypical worker has preferences $\theta_A = (f_2, f_1)$ with probability equal to one. Firms prefer worker w_1 to worker w_2. Agents employ the following strategies:

- $\sigma_{w_1}(\theta_0) = f_2$, $\sigma_{w_1}(\theta_A) = f_1$
- $\sigma_{w_2}(\theta_0) = f_1$, $\sigma_{w_2}(\theta_A) = f_2$
- for any $\mathcal{W}^S \subset \mathcal{W}$ $\sigma_{f_1}(\mathcal{W}^S) = \begin{cases} w_1 \text{ if } w_1 \notin \mathcal{W}^S \\ w_2 \text{ if } w_1 \in \mathcal{W}^S \end{cases}$, $\sigma_{f_2}(\mathcal{W}^S) = \begin{cases} w_1 \text{ if } w_1 \notin \mathcal{W}^S \\ w_2 \text{ if } w_1 \in \mathcal{W}^S \end{cases}$

Agents' believes are:

- for any $\mathcal{W}^S \subset \mathcal{W}$ $\mu_{f_j}(\theta_{w_i} : f_j = \max_{\theta_{w_i}}(f \in \mathcal{F})|\mathcal{W}^S \backslash w_i) = 1$ and $\mu_{f_j}(\theta_{w_i} : f_j = \min_{\theta_{w_i}}(f \in \mathcal{F})|\mathcal{W}^S \cup w_i) = 1$

It is easy to show that the above strategies and the set of beliefs constitute a sequential equilibrium. One may extend this example for the environment with more firms and workers.

Proof of Proposition 14.

Let us first prove that firms' strategies are optimal. Note that if firm f receives a signal from worker w_1 it believes that it is her top firm. Therefore, it is optimal for her to make her an offer. Now, if firm f_1 that does not receive a signal from worker w_1, firm f_1 believes that worker w_1 is atypical and will not accept its offer. Then, firm f_1 strategy of making an offer to worker w_2 is optimal for any signaling pattern, because it believes that her offer will be accepted at least with probability $\frac{1}{2}$. The worst case is when worker w_2 is atypical and sends her signal to firm f_3.

Now we consider firm f_2 optimal strategy. Let us consider the case when worker w_1 sends her signal to firm f_1 and worker w_2 sends signal to firm f_3. Firm f_2 believes worker w_2 prefers firm f_3 to itself. Since, firm f_3 makes an offer to worker w_2, and firm f_1 makes an offer to worker $w_1,$ the only worker that could accepts firm f_2 offer is worker w_3. If worker w_1 sends he signal to firm f_3 and worker w_2 sends her signal to firm f_2. In this case firm f_1 makes an offer to worker w_2, who is typical with probability $(1 - \frac{1}{3}\varepsilon)$. Since, worker w_1 most preferred firm is firm $f_3,$ the optimal strategy of firm f_2 to make an offer to worker w_3.

Let us consider the case firm f_3 receives signals from all workers. In this case firm f_3 makes an offer to worker w_1. It is optimal for firm f_1 and firm f_2 to make an offer to worker w_2 because her preferences over these firms could be equally likely. Hence, the payoff from making an offer to worker w_2 equal $\frac{1}{2}u(2) > u(3)$. Similar, one could show that in other cases it is optimal for firm f_2 to make an offer to worker w_2.In a similar way one could show that it is always optimal for firm f_3 to make an offer to the best worker it receives a signal from.

Let us now show each worker uses optimal strategy. Worker w_1 strategy is optimal, because any firm makes her an offer upon receiving her signal. There is no incentive for worker w_2 to make an offer to firm f_1 since, all firms upon observing such behavior has believes about workers preferences that coincides with the priors. Therefore, worker w_2 optimal strategy is to send her signal to the best firms among f_1 and f_2. Since firms do not put attention to worker w_3 signals, there is no reason for her to deviate from the equilibrium strategy. $\square$

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