COMBINATORICS AND REPRESENTATION THEORY FOR GENERALIZED PERMUTOHEDRA

A Dissertation in Mathematics

by
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ABSTRACT

In this thesis, we study the representation theory of the symmetric group on a new categorification of the theory of generalized permutohedra. The vector spaces in the categorification are tightly constrained, arising as solution spaces to certain continuity relations which appeared first in Quantum Field Theory in the mid 20th century. The generators of the vector space are characteristic functions of certain polyhedral cones called plates, due to A. Ocneanu.

In combinatorics, the Eulerian numbers count the number of permutations with a given number of ascent and descents. The classical Worpitzky identity in combinatorics expands a power $r^p$ as a sum of Eulerian numbers, with binomial coefficients. The main combinatorial result is the categorification of Worpitzky’s identity to an isomorphism between two graded representations of the symmetric group, corresponding geometrically to the decomposition of a scaled simplex into a direct sum of translated unit hypersimplices. We recover the classical Worpitzky identity by evaluating the characters on the identity permutation. The character values in general can be interpreted as relative volumes of generalized hypersimplices.

The proof involves a new algebra of commuting operators which act by translation on plates in a scaled simplex. We show that the spectrum of this algebra is labeled by the subset

$$\{ x \in (\mathbb{Z}/r)^n : \sum x_i \equiv 1 \},$$
on which $S_n$ acts by permuting positions. The values of the character of a permutation $\sigma$ is thus given by the number of fixed points on this set. This number is obtained by counting solutions to a certain modular Diophantine equation involving the cycle lengths of $\sigma$. 
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Preface

In this thesis we study the action of the symmetric group on certain finite-dimensional vector spaces constructed from families of convex polyhedral regions in Euclidean space, called plates. The vector spaces are the linear spans of the characteristic functions of families of plates.

Our main theorem is a formula for the characters of the symmetric group that we obtain in this way. It generalizes a well-known formula in combinatorics, called the Worpitzky identity, that from our point of view is obtained by evaluating characters on the identity permutation. We shall summarize some of our other theorems later in this introduction.

Our work extends a larger program being developed by Adrian Ocneanu, in which the concept of a plate (which is due to him) is used in the development of so-called higher representation theory. In fact the theory of plates appears to touch on a wide variety of structures in mathematics, and perhaps even physics.

Chapter 1 reviews definitions and results due to Ocneanu, as well as giving definitions of rational and quantum, or q-plates, which may find use in connection with physics, an explicit, computation results and a combinatorial change of basis formula which involves a new use of a permutation statistic.

Chapter 2 first lays the groundwork, formulates and then solves the main problem, which is to prove the character formula for a new symmetric group module. We present axioms which characterize an algebra $C_n^r$ of commuting translation operators on q-plates, together with a noncommutative deformation $A_n^r$ which is related to a generalized Clifford algebra. By counting the monomial basis elements in $C_n^r$, we show that the trace of a permutation is either $r^{k-1}$ or 0, depending on the cycle lengths and $r$. We find a partition of unity of $C_n^r$ which diagonalizes the translation operators and makes the action of permutations set-theoretic. We then show that the character value $r^{k-1}$ counts the solutions to a linear Diophantine equation $\sum c_i \lambda_i \equiv 1 \mod r$, for fixed $\lambda_i$.

Chapter 3 is independent of the rest of the thesis, as it contains an independent project which was inspired by discussions with Ocneanu on tensor product invariants. For the main result, we obtain a new formulation of the Segre cubic variety which exhibits an additional $S_5$-symmetry.

Chapter 4 summarizes on-going work and states some conjectures. We define several automorphism groups which act on eigenvectors of permutations, and we compute the linear dimension of the space of plates in the $3 \times 3$ Birkhoff polytope.

Chapter 5, which consists of the very difficult proof of the linear independence of the set of q-plates about a point, follows an algorithm of Ocneanu and is included for completeness from our joint work. The algorithm and several connected topics are fully implemented in Mathematica.

In Chapter 6 we give include Mathematica code which can be used to reproduce certain computations.
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When I asked Adrian Ocneanu for a thesis problem, he gave me a conjecture and suggested that I develop the subject as far as I could. While it started as an almost purely geometric question, by the end I had reduced it to problem in algebra which I eventually showed to be equivalent to a number theoretic computation. I want to express my deepest gratitude to him for giving me a thesis problem with such a rich structure, for fruitful collaboration, and in particular for dedicating his time, energy and great enthusiasm to countless hours of intensive discussions with me over the past four years.

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CHAPTER 1

Introduction

We shall begin by describing the Worpitzky identity and relevant surrounding ideas in combinatorics. Then we shall summarize results from [39] on the essential properties of plates, along with several examples and illustrations. Following that we formulate our main result, the character formula, along with some further results.

1. The Worpitzky Identity

The Eulerian numbers $E_{i,j}$, or $A_{n,j}$ in the usual notation, were defined by Euler in 1755. They count permutations of $\{1, \ldots, n\}$ with $i$ ascents and $j = (n - 1) - i$ descents. Our notation differs slightly from the standard one in order to emphasize a correspondence with plates in hypersimplices, as discussed below. The following table starts at the top with $E_{0,0}$ and continues with $E_{1,0}, E_{0,1}, \ldots$.

\[
\begin{array}{cccc}
1 \\
1 & 1 \\
1 & 4 & 1 \\
1 & 11 & 11 & 1 \\
1 & 26 & 66 & 26 & 1 \\
& & & & \\
\end{array}
\]

For example, the third row counts the permutations

$$321, ~ 132, 213, 231, 312, ~ 123.$$  

In 1883 Worpitzky discovered an identity which expresses a power as a sum of Eulerian numbers with binomial coefficients. For example,

\[
x^2 = \binom{x}{2} \cdot 1 + \binom{x+1}{2} \cdot 1 \\
x^3 = \binom{x}{3} \cdot 1 + \binom{x+1}{3} \cdot 4 + \binom{x+2}{3} \cdot 1 \\
& & \\
\]

See for example [42] for a discussion of the Eulerian numbers and the Worpitzky identity in particular and various proofs.

In this thesis, we generalize this combinatorial identity to an identity of characters of the symmetric group $S_n$. These are characters of plate modules for the simplex and hypersimplices in $n$ coordinates, described below.
2. Plates

From this section on, by a plate we shall mean the characteristic function of the polyhedral cone defined by a flag of inequalities of the form

\[ x_{S_1} \geq s_1 \]
\[ x_{S_1} + x_{S_2} \geq s_1 + s_2 \]
\[ \vdots \]
\[ x_{S_1} + \cdots + x_{S_{k-1}} \geq s_1 + \cdots + s_{k-1} \]
\[ x_{S_1} + \cdots + x_{S_k} = s_1 + \cdots + s_k, \]

where the last line is an equality. Here \( x_S = \sum_{i \in S} x_i \) with \( x_1, \ldots, x_n \) real variables, \( S_1 \sqcup \cdots \sqcup S_n = \{1, \ldots, n\} \) is an ordered set partition and \( s_1 + \cdots + s_k = r \in \mathbb{Z} \) is an ordered partition. In most of this thesis we will assume \( x_i \geq 0 \) and \( r \geq 1 \). The dataset for the above equations is encoded in a set partition with lumps \( S_i \) decorated by positions \( s_i \). We will surround it with brackets as \([[(S_1)_s_1, \cdots (S_k)_{s_k}]\].

In more detail, a plate is the characteristic function of the cone about a face of the permutohedron, the polytope formed by the convex hull of permutations of \((0, 1, 2, \ldots, n-1) \in \mathbb{R}^n\). The permutohedron itself was studied and generalized previously in [43]. See the plates in Figure 1 below for the nondegenerate case when the face is 0-dimensional. In this case, from the inequalities above one can verify that the plate is the convex span of a simple root system of type \( A_{n-1} \). Remark that the nondegenerate plate \([1, 2, \ldots, n]\) has been studied before in [19] as a polyhedral cone, though not as a characteristic function. There it is constructed as the cone generated at the origin 0 by the positive roots \( e_i - e_j \) of type \( A_{n-1} \), for \( 1 \leq i < j \leq n \).

Plates satisfy elaborate relations which are proved in [39] and illustrated in Figure 2 in the case \( n = 3 \).

For positive integers \( a, b \) with \( a + b = n \), let

\[ B_{a,b} = \{ x \in [0,1]^n : \sum_{i=1}^n x_i = a \}, \]

and for each \( r \geq 1 \) let

\[ \Delta^n_r = \{ x \in [0,r]^n : \sum_{i=1}^n x_i = r \}. \]

Here \( B_{a,b} \) is the well-known hypersimplex, equivalently realized as the convex hull of permutations of the vector \((1, \ldots, 1, 0, \ldots, 0)\) with \( a \) 1’s and \( b \) 0’s.

We denote by \( \text{Pl}(\Delta^n_r) \) and \( \text{Pl}(B_{a,b}) \) the complex-linear span of plates which have support in \( \Delta^n_r \) and respectively \( B_{a,b} \).

2.1. The Equivariant Worpitzky Identity. Using generating function techniques, we generalize the Worpitzky identity to an identity of characters, between the module \( \text{Pl}(\Delta^n_r) \) of plates in a simplex, which has dimension \( r^{n-1} \), and modules \( \text{Pl}(B_{a,n-a}) \) for \( a = 1, \ldots, n-1 \) of plates in hypersimplices \( B_{a,b} \).

In Theorem 68 we prove the equivariant Worpitzky isomorphism of \( S_n \)-modules

\[ \text{Pl}(\Delta^n_r) \simeq \bigoplus_{a=1}^{n-1} \text{Sym}^{r-a}(\mathbb{C}^n) \otimes \text{Pl}(B_{a,n-a}), \]

where \( \text{Sym}^{r-a}(\mathbb{C}^n) \) is the degree \( r - a \) component of the polynomial ring in \( n \) variables. See Figure 5.
In this way we have categorified the classical Worpitzky identity. We replace a positive counting formula with an identity of characters of symmetric group representations, realized concretely as complex linear spaces spanned by characteristic functions of regions called plates. Our vector spaces, however, are far from arbitrary. They capture the “linear” properties of a quantum field theory, as mentioned in Section 5 of this chapter.

3. Formulation of the Main Theorem

There was a conjecture due to Ocneanu about the characters of the \( S_n \)-representation of plates as functions on the simplex \( \Delta^n \). The characters are remarkable due to the fact that the values are natural numbers with number-theoretic properties. The conjecture, which we prove, is the following.

**Theorem 1.** If \( \sigma \) is a permutation with cycle lengths \( \lambda_1, \ldots, \lambda_k \), acting on plates in \( \Delta^n \), then the character value of \( \sigma \) is \( r^{k-1} \) if \( \gcd(\lambda_1, \ldots, \lambda_k, r) = 1 \) and 0 otherwise.

We give two independent proofs of this result.  
**Method 1.** Prove the character formula given above using the monomial basis of \( \mathcal{C}_r^n \) defined below. We do this in Theorem 58.

**Method 2.** Using a partition of unity of \( \mathcal{C}_r^n \), the character value of a permutation \( \sigma \) in the representation is equal to the number of fixed points of the \( \sigma \) acting on the Diophantine set
\[
\mathcal{I}_r^n = \{ I \in (\mathbb{Z}/r)^n : \sum i_j \equiv 1 \mod r \},
\]
which labels the eigenvalues of the generators of \( \mathcal{C}_r^n \) acting on the partition of unity.

See Theorem 61 and surrounding discussion for precise statements and details.

In this thesis, we use both Method 1 and Method 2 separately to prove the character formula, by constructing a new algebra structure which acts on cyclic linear combinations of plates called \( q \)-plates, defined below, with a partition of unity into idempotents labeled with the elements in \( \mathcal{I}_r^n \). We provide in the diagram an explicit formula for these idempotents.

3.1. The Translation Algebra. Fix natural numbers \( r \) and \( n \) and let \( q = e^{-2\pi i/r} \). We define a commutative algebra \( \mathcal{C}_r^n \) over \( \mathbb{C} \) which we call the translation algebra. It is given explicitly in terms of generators and relations as
\[
\mathcal{C}_r^n = \langle e_1, \ldots, e_n : e_ie_j = e_je_i, e_1^r = 1, e_1 \cdots e_n = q \rangle.
\]
This has basis the set of all monomials \( \{e_1^{i_1} \cdots e_n^{i_n} : i_j \in \{0, \ldots, r-1\} \} \) normalized such that \( e_1 \) does not appear and is thus of dimension \( r^{n-1} \).

3.2. More on Plates. A basis plate is a plate \([(S_1)_{s_1}(S_2)_{s_2} \cdots (S_{k-1})_{s_{k-1}}(S_k)_{s_k}] \) such that \( 1 \in S_1 \), which means geometrically that it contains the direction towards the vertex labeled 1. A q-plate has the notation \( \{\cdots\} \) and is defined by the cyclic sum
\[
\{(S_1)_{s_1} \cdots (S_k)_{s_k}\} = q^0[[S_1]_{s_1}[S_2]_{s_2} \cdots [S_{k-1}]_{s_{k-1}}[S_k]_{s_k}] + q^{-s_k}[[S_k]_{s_k}[S_1]_{s_1}[S_2]_{s_2} \cdots C_c] + q^{-(s_{k-1}+s_k)}[[S_{k-1}]_{s_{k-1}}[S_k]_{s_k}[S_1]_{s_1}[S_2]_{s_2}] + \cdots
\]
in which the power of \( q \) is the \((-1)\) times the sum of the positions of the lumps moved to the front.

A basis q-plate is a q-plate with the sum normalized to have coefficient \( q^0 = 1 \) when 1 is in the first lump, in which case we have the basis q-plate. Note that q-plates are basis q-plates, up to scaling by a root of unity.

Examples are given below to illustrate what is going on geometrically.
3. FORMULATION OF THE MAIN THEOREM

In Figure 1 the standard nondegenerate plates are given for \( n = 3 \) and \( 4 \) as unbounded polyhedral cones generated by simple roots of type \( A_2 \) respectively \( A_3 \). For example, in the \( n = 4 \) coordinate case on the right, the plate is identified with the region

\[
\{(1,1,1,1) + t_1 \cdot (1,-1,0,0) + t_2(0,1,-1,0) + t_3(0,0,1,-1) : t_i \geq 0\},
\]

opening toward the vertex \((4,0,0,0)\).

Figure 2 contains a sample of the plate relations in dimension 2. These express an arbitrary plate as a linear combination with coefficients \( \pm 1 \) of basis plates.

In Figure 3 we use the notation \( 1 = q^a, 2 = q^b \) and \( 3 = q^c \), where \( q = e^{-2\pi i/r} \) and \( r = a + b + c \), to describe both the coefficients in the expansion of the q-plate \( \{1a2b3c\} \) in terms of basis plates and conversely. Note that \( 1 \) does not appear due to the relation \( q^aq^bq^c = 1 \). The subscripts \( a, b, c \) are implicit in Figure 3.

Finally, Figure 4 shows the graph of the q-plate

\[
\{1_12_13_14_1\} = [1_12_13_14_1] + q^{-1} [4_11_12_13_1] + q^{-(1+1)} [3_14_11_12_1] + q^{-(1+1+1)} [2_13_14_11_1]
\]
as a cyclic sum of four plates in the simplex \( \Delta^4 \), each weighted with a power of \( q = e^{-2\pi i/4} \).
3.3. Method of Proof. The logical structure of the results proved in this thesis is presented in the diagram below, where each $\leftrightarrow$ represents an isomorphism. Our goal is to prove the character formula for the bottom row, for plates. Our first proof involves a computation on the monomial basis in the third row from the bottom. For our second proof, we count fixed points in the top row.

The bottom double arrow $\leftrightarrow$ involves a quite intricate algorithm due to Ocneanu and appears in the Appendix A.
4. Additional Results

4.1. A Signed Decomposition of the Standard Plate. In addition to the two approaches outlined above, we emphasize a combinatorial expression which follows from the Appendix in Appendix A for the change of basis between plates and q-plates. Our formula expresses the coefficients in the expansion of the standard plate $\pi_0 = [[s_1 \cdots s_n]]$ in terms of basis q-plates $\{\pi\}$.

Conjecture 2. For each $j = 2, \ldots, n$, let $\tilde{q}_j = \prod_{i=j}^{n} q_i$, where $q_i = q^{s_i} = e^{-2\pi is_i/r}$ and $s_i$ is the position of $i$ in the standard plate $\pi_0 = [[s_1 \cdots s_n]]$. Then we have

$$
(1 - \tilde{q}_2^{-1}) \cdots (1 - \tilde{q}_n^{-1}) \pi_0 = \sum_{\pi} (-1)^{\text{length}(\pi)} \prod_{i \in \text{Inv}(\pi)} \tilde{q}_i^{-1} \{\pi\},
$$

where $\text{Inv}(\pi)$ is the set of irreducible components of $\pi$.
5. Further Aspects of the Theory of Plates

Table 1. Decompositions of simplex modules into irreducibles

<table>
<thead>
<tr>
<th>$n \setminus r$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1^1$</td>
<td>$2^1$</td>
<td>$3^1$</td>
<td>$4^1$</td>
</tr>
<tr>
<td>2</td>
<td>$2^1$</td>
<td>$2^1 \cdot 1^2$</td>
<td>$2^1 \cdot 1^2$</td>
<td>$2^1 \cdot 1^2$</td>
</tr>
<tr>
<td>3</td>
<td>$3^1$</td>
<td>$2 \cdot 3^1 + 2^1 \cdot 1^1$</td>
<td>$3 \cdot 3^1 + 3 \cdot 2^1 \cdot 1^2$</td>
<td>$5 \cdot 3^1 + 5 \cdot 2^1 \cdot 1^4 + 1^3$</td>
</tr>
<tr>
<td>4</td>
<td>$4^1$</td>
<td>$2 \cdot 4^1 + 2 \cdot 3^1 \cdot 1^1$</td>
<td>$5 \cdot 4^1 + 2 \cdot 2^2 + 5 \cdot 3^1 \cdot 1^2 + 2^1 \cdot 1^2$</td>
<td>$8 \cdot 4^1 + 4 \cdot 2^4 + 12 \cdot 3^1 \cdot 1^4 + 4 \cdot 2^1 \cdot 1^2$</td>
</tr>
</tbody>
</table>

where the sum is over all plates $\pi$ in the plate basis in $n$ variables, and where $\text{Inv}(\pi)$ labels the set of inverted pairs of variables $(i, i+1)$ in $\pi$.

For precise statements see Theorem 27 and surrounding discussion.

4.2. Multiplicities of the Irreducible Representations. It is an obvious question to ask for the multiplicities of the irreducible representations in $\text{Pl}(\Delta^n_r)$ and $\text{Pl}(B_{a,b})$. We have computed the multiplicities for the trivial representation, but a full combinatorial interpretation of the general case, for arbitrary irreducible representations, appears to be a nontrivial and important question which we shall pursue in the near future.

The first few cases of the multiplicities of the irreducible representations in $\text{Pl}(\Delta^n_r)$ are given in the table below. Here for example $4 \cdot 2^1 \cdot 1^2$ means that the irreducible $S_4$ representation labeled by the partition $2 + 1 + 1$ of 4 occurs with multiplicity 4.

By explicit enumeration of invariant basis plates we found the generating functions for the multiplicities of the trivial representations to be

$$\frac{1}{(1-x)^3(1+x+x^2)} = 1 + 2x + 3x^2 + 5x^3 + 7x^4 + 9x^5 + \cdots$$

$$\frac{1}{(1-x)^4(1+x)^2} = 1 + 2x + 5x^2 + 8x^3 + 14x^4 + 20x^5 + \cdots,$$

where the coefficients of the series can be seen in columns labeled 3 and 4 in the table.

5. Further Aspects of the Theory of Plates

We believe that we are still only just scratching the surface, but let us point out some objects in Mathematics which we either know or strongly suspect are equivalent or related to plates, along with suggested references.

1. Matroids and polymatroids.
   (a) Plates in hypersimplices $B_{a,b}$ are closely related to matroids, as in [21, 22]. In [39] Ocneanu develops elaborate combinatorial techniques which show how to localize generalized permutohedra to hypersimplices. It is shown that these localizations are matroids, and any matroid is a linear combination of such localizations.
   (b) Plates in simplices $\Delta^n_r$ are closely related to polymatroids, in the sense of [12].

2. The module of nondegenerate plates about a point is isomorphic to the free Lie module.
   (a) In classical representation theory, according to the standard construction one works in the regular representation to obtain the irreducible representations of the symmetric group, by projecting with the Young idempotents onto the irreducible submodules. It is well-known that the regular representation of $S_{n-1}$ is the restriction of an $S_n$-module called the free Lie module, see [46]. It is spanned by the $n!$ alternating binary trees, or bracketings, where each leaf is labeled with an integer.
1, \ldots, n, and has basis the \((n-1)\) “combed” antisymmetric trees \([1, [i_2, [i_3 \cdots, [i_{n-1}, i_n] \cdots]]\)
for which 1 is outermost.

(b) The fundamental observation in [39] is that the module of plates about a point
to the free Lie module in brackets of \(n\) (distinct) generators. Here \(S_n\) acts on the
space of complex-linear combinations of plates by permuting variables.
Here the \(S_n\)-module of nondegenerate plates has basis the \((n-1)\)!
permutations of 2, 3, \ldots, \(n\) of the standard plate
\([[[1_0 2_0 \cdots, n_0]]]\),
which has support the cone generated by the simple \(A_{n-1}\) roots \(e_i - e_{i+1}\) extending
out from the point \((0, \ldots, 0)\). Note that the \(S_n\)-action crucially fixes the position
of the plate, \((0, \ldots, 0)\), which would not be the case for a nondegenerate plate with
generic position \((s_1, \ldots, s_n)\).

(3) Descent statistics for permutations and toric geometry.
(a) The \(S_n\) hypersimplex plate modules \(Pl(B_{a,b})\) are isomorphic to the modules in
[24] which describe the \(S_n\)-action on the cohomology of a toric variety associated
to a Coxeter arrangement [55]. In particular, the characters values for plates
in hypersimplices can be obtained as coefficients of the quasi-Eulerian symmetric
functions which appear in [24]. See [30, 45, 55] for the related toric geometry.

(4) Word-quasi-symmetric functions and Écalle’s mould calculus.
(a) In [33] rational functions in \(n\) formal indeterminants \(y_1, \ldots, y_n\) are given to pro-
vide a functional model for the so-called word-quasisymmetric functions, WQSym.
These functions, which are given as examples of Écalle’s so-called moulds, when
modified by imposing the condition \(y_1 \cdots y_n = 1\) give conjecturally an isomorphic
copy of the module of plates about a point. We have tested this thoroughly us-
ing a computer, but we postpone the proof to future work. See the discussion
surrounding Conjecture 16.

(5) Statistical physics and Wightman’s Axiomatic quantum field theory.
(a) In [60], a matrix \(A(q)\) of scalar products of \(n\)-particle states is proved to be positive
definite, leading to the realizability for the “\(q\)-mutator relations” on a Hilbert space.
The permutation statistics involved in the construction of \(A(q)\) are quite close to
those used in the formula of Theorem 27 for the expansion of a plate in terms of
\(q\)-plates.
(b) In the review paper [53], p. 827 gives the so-called Steinmann identities, which are
properties that embody the “linear” properties of a quantum field theory. Property
(iii) is the same as the condition found by Ocneanu to determine when a charac-
teristic function is a linear combination of plates. See [39] for the characterization
of plate linearity, and [54, 59] for more physics reading.
CHAPTER 2

Polyhedral Cone Geometry for Permutohedra

1. Plates

1.1. Introduction. The all-subset hyperplane arrangement

$$\left\{ \sum_{i \in I} x_i = 0 : I \subseteq \{1, \ldots, n\} \right\},$$

of $2^n - 1$ hyperplanes appears in various disjoint areas of science, including physics, economics and psychometrics, and has been studied at least implicitly since the mid 20th century in quantum field theory in connection with generalized retarded functions [2, 17, 47].

In contrast with the famous Coxeter arrangement of $\binom{n}{2}$ hyperplanes of the form

$$H_{i,j} = \{x \in \mathbb{R}^n : x_i = x_j\},$$

a direct, comprehensive analysis of the full decomposition of the all-subset arrangement in chambers would be extremely difficult, and even a formula for the number of non-empty chambers has not been found, though bounds for the growth do exist [7]. The number of non-empty chambers has been computed only for $n \leq 8$, according to the Online Encyclopedia of Integer Sequences.

One would like to find a first-order approach, or entry point, and initiate a systematic study of the all-subset arrangement.

Adrian Ocneanu has found this first-order approach.

The fundamental building blocks, which Ocneanu calls plates, are constructed from polyhedral cones which are in correspondence with faces of generalized permutohedra. Generalized permutohedra were introduced and studied by Postnikov in [43].

Further, from Postnikov in [44] we learned of a very interesting likely connection with prior work [19], and subsequently [32], on constructions related to triangulations of root polytopes.

In future work, we shall investigate connections with a family of polyhedral cones closely related to plates has been studied independently in [33] as a functional model for the so-called mould calculus of Écalle. Additionally, in [50], characters for certain $S_n$ modules coming from toric geometry were obtained using symmetric functions. In particular, in [24] abstract generators and relations are given for $S_n$ modules which we note are isomorphic to the modules of plates in hypersimplices.

The idea is to use characteristic functions of polyhedral cones which are defined up to sets of measure zero. Looking somewhat ahead, it is by working in $\{x \in \mathbb{R}^n : \sum x_i = 0\}$ rather than all of $\mathbb{R}^n$ that we obtain the rich combinatorics.

Following Ocneanu, let us call a member of the all-subset arrangement a special hyperplane.

In more detail, the core structure in Ocneanu’s systematic study of the all-subset hyperplane arrangement is obtained as follows. We define a model space to be the vector space generated linearly by characteristic functions of polyhedral cones which are bounded by certain flags of special hyperplanes, in correspondence by translation with the faces of the permutohedron of all dimensions. The characteristic functions of these cones are called plates.
First, we shall restrict ourselves to the ambient space
\[ \{ x \in \mathbb{R}^n : \sum x_i = 0 \}, \]
which is invariant under coordinate permutation.

Second, we shall adopt the convention that plates are not defined on special hyperplanes.

These two constraints lead to the rich combinatorial and representation-theoretic structures
which we investigate in this thesis.

1.2. Plates.

**Definition 3.** A **plate** is a real-valued function on the space \( \{ x_1, \ldots, x_n \in \mathbb{R}^n : \sum x_i = N \} \)
defined on the complement of the **special hyperplanes** \( \{ x_I = \sum_{i \in I} x_i = m \in \mathbb{Z} \} \) for \( I \) a subset
of \( \{1, \ldots, n\} \). It will be clear from the context whether we refer to the set or to its characteristic
function.

A plate, denoted \( [(S_1)_{s_1} \cdots (S_k)_{s_k}] \), is encoded by a set composition of the variable labels
\( \{1, \ldots, n\} = S_1 \cup \cdots \cup S_k \) and a composition \( s_1, \ldots, s_k \) of an integer \( r \), and is the characteristic
function of the set described by the equations
\[
\begin{align*}
x_{S_1} & \geq s_1 \\
x_{S_1} + x_{S_2} & \geq s_1 + s_2 \\
& \vdots \\
x_{S_1} + \cdots + x_{S_{k-1}} & \geq s_1 + \cdots + s_{k-1} \\
x_{S_1} + \cdots + x_{S_k} & = s_1 + \cdots + s_k
\end{align*}
\]
in which the last inequality is an equality which defines the ambient hyperplane. The subsets
\( S_i \subset \{1, \ldots, n\} \) as above are called the **lumps** of the plate, and \( s_1, \ldots, s_k \) are called the **positions**
of their respective lumps.

In much of what follows we will work mainly with the scaled simplex
\( \Delta^n_r = \{ x \in [0, r]^n : \sum x_i = r \} \),
and the hypersimplices
\( B_{a,b} = \{ x \in [0, 1]^n : \sum x_i = a \} \),
where \( a + b = n \), extracting results about hypersimplices from generating function techniques,
as \( r \to \infty \).

We shall need the following result of Ocneanu [39], observed independently by Postnikov [44].

**Proposition 4.** **Translations of the coordinate hyperplanes** \( \{ x \in \mathbb{R}^n : \sum x_i = 0, x_k = j \} \) for
\( j = 0, \ldots, r \) and \( k = 1, \ldots, n \), decompose \( \Delta^n_r \) into a union of hypersimplices \( B_{a,b} \), where \( a, b \geq 1 \)
and \( a + b = n \), which coincide only on their boundaries.

Figure 1 depicts the two hypersimplices \( B_{1,2}, B_{2,1} \) in three coordinates, defined set-theoretically as
\[ \{ x \in [0, 1]^3 : x_1 + x_2 + x_3 \in \{1, 2\} \}. \]
In what follows, following Ocneanu we shall call the collection of all \( n - 1 \) hypersimplices in \( n \)
coordinates the **period solid**.
1.3. Plate Relations. As observed by Ocneanu in [39], to give generators and relations for plates in any simplex or hypersimplex it suffices to take the set of all plates passing through a given point.

In [39], Ocneanu establishes fundamental properties of plates, which allow one to use representation theory to study the action of the symmetric group $S_n$. We shall need a subset of these results, which we reproduce without proof in Theorems 7, 8 and 9.

**Definition 5.** Let $\pi, \pi'$ be plates. We shall say that $\pi'$ is a lumping of $\pi$ if for each lump $A'_a$ of $\pi'$, $A'$ is a union of consecutive lumps of $\pi$ and if $a'$ is the sum of their respective positions.

For example, $\pi' = [[1a24,b+c,3d]]$ is a lumping of $\pi' = [[1a2b4,c,3d]]$.

The lumping of $x_2$ and $x_4$ into $x_{24} = x_2 + x_4$ corresponds to a projection of the ambient simplex along the edge with equation $x_2 + x_4 = b + c$, onto a lower dimensional space parametrized by the variables $x_1, x_{24}, x_3$. Under this projection, the two simplex vertices labeled with 2 and 4, which are the endpoints of the projection edge, are identified.

Recalling that plates in $n$ variables are characteristic functions of flags of at most $n - 1$ inequalities, intersected with a hyperplane $\sum x_i = r$, we see now that lumping is easiest to understand in terms of the defining equations.

**Proposition 6.** The lumpings of a plate $\pi$ are obtained from $\pi$ by removing one or more bounding hyperplanes, i.e. we delete a subset of the $\leq n - 1$ inequalities which define $\pi$.

According to Theorem 7, a plate $\pi$ with $l$ lumps determines a decomposition of its ambient space into a sum of the $l$ cyclic rotations of $\pi$ which coincide only on their bounding special hyperplanes.
1. PLATES

**Theorem 7.** Let $r = a + b + \cdots + c$ and $n \geq 1$ be given. Let $(A, B, \ldots, C, D)$ be an ordered set partition of $\{1, \ldots, n\}$. Then we have the cyclic sum relation
\[
[(S_1S_2\cdots S_{k-1}S_k)] = \left[[((S_1)_{s_1}(S_2)_{s_2}\cdots(S_{k-1})_{s_{k-1}}(S_k)_{s_k})] + \left[[((S_k)_{s_k}(S_1)_{s_1}(S_2)_{s_2}\cdots(S_{k-1})_{s_{k-1}})]] + \cdots + \left[[((S_2)_{s_2}\cdots S_{k-1})_{s_{k-1}}(S_k)_{s_k}(S_1)_{s_1})]ight].
\]

All computations involving plates are made possible by the formula given in Theorem 9 below, which characterizes the linear relations which arise due to the fact that plates are not defined on special hyperplanes. In particular, Theorem 8 provides a basis, and using Theorem 9 one can build matrices for the action of permutations. To prove our character formula, however, we introduce a basis which involves roots of unity in an essential way. Using this basis, it is possible to parametrize explicitly the diagonal of the matrix of any given permutation, from which it is immediate to obtain its trace. Additionally, using this **q-basis** inspires interesting algebraic structures which we introduce in Definitions 32 and 42.

We come now to one of the main results of [39], stated in what follows without its proof. However, in Definition 14 we introduce a generating function representation of plates which we discovered in order to verify the plate relations by hand in three coordinates, and then using a CAS such as Mathematica for higher dimensions.

**Theorem 8.** The set of all plates which have 1 in the first lump is linearly independent. In particular, it is a basis.

In what follows, we call the basis of Theorem 8 the **standard basis.**

In [39], Ocneanu proves the fundamental Theorem 9, giving the module of relations for plates, using homological arguments involving rooted trees.

The proof involves properties of hypergeometric functions of Euler Beta type for special parameter configurations.

**Theorem 9.** Let
\[
\pi = \left[(S_m)_{s_m}(S_{m-1})_{s_{m-1}}\cdots S_{1}S_{m+1}\cdots (S_k)_{s_k}\right],
\]
labeled so that $1 \in S_1$. This expands in the standard basis as
\[
\pi = \sum_{\pi' \in shL((S_1)_{s_1},\ldots,(S_m)_{s_m},(S_{m+1})_{s_{m+1}},\ldots,(S_k)_{s_k}))} (-1)^{m-1}(-1)^{k-nL}[\pi'],
\]
where $n_L$ is the number of lumps of $\pi'$ and where the sum is over all lumped shuffles of $(S_1, \ldots, S_m)$ and $(S_{m+1}, \ldots, S_k)$ so that the lumps $S_{m+1}, \ldots, S_k$, i.e. those after $S_1 \ni 1$, are not lumped together.

**Remark 10.** The symmetric group $S_n$ acts on plates by permuting the variables $x_1, \ldots, x_n$ which are labeled by the elements of the lumps of a plate.

For completeness we record the following easy but crucial observation.

**Corollary 11.** The action of the symmetric group is well-defined with respect to the plate relations. It follows that the vector space of plates is an $S_n$-module.

**Proof.** From a close examination of the formula given in Theorem 9 we see that the plate relations involve only permutations and unions of lumps, as sets. Both operations are $S_n$-equivariant, and the result follows.

**Example 12.** We give the expansions of two plates in the standard basis.
\[
[[35_a124_b6_c]] = [[124_b356_a+c]] + [[12345_{a+b}6_c]] - [[124_b6_c35_a]] - [[124_a35_b, 6_c]]
\]
Example 13. In Figure 2 two more plate relations are given.

\[
[[35b124_a6_c7_d]] = [[124_a6_c35_d+b]] + [[124_a356_a+b+c7_d]] + [[12345_a+b, 6_c7_d]] - [[124_a6_c7_d35_b]] - [[124_a6_c35_b7_d]] - [[12345_a+b, 6_c7_d]]
\]

Figure 2. Plate relations in dimension 2

In the next section, describe how we were able to verify the relations given in Theorem 9, by introducing two generating function representations for the set of integer points in a polyhedral cone.

1.4. Generating Function Verification of Plate Relations. The genesis for the computations which led to this thesis was Ocneanu’s Mathematica notebook, in which the “shuffle-lump” plate relations were implemented on linear combinations of formal functions

\[
\text{plate}([\{s_1, \ldots, s_k\}, \{S_1, \ldots, S_k\}]),
\]

where \( S_1 \sqcup \cdots \sqcup S_k \) is a set composition of \( \{1, \ldots, n\} \) and \( p_1, \ldots, p_k \) is a composition of an integer \( r \). Recall that the standard basis of plates consists of all plates of the above form which satisfy \( 1 \in S_1 \). Thus, the Mathematica notebook contained an algorithm which expands an arbitrary plate as a linear combination of plates in the standard basis.

We were able to verify the plate relations by hand, geometrically, in only three variables using Figure 2, but one needs more data in order to formulate conjectures in full generality, which requires one to be able to compute for \( n \geq 6 \) variables.

We introduce generating functions for the lattice of integer points in a plate. This led to an independent computer-assisted verification of the plate relations for small dimension \( n \).

Definition 14. Let \( S_1 \sqcup \cdots \sqcup S_k \) be a set composition of \( \{1, \ldots, n\} \). Let

\[(y_1, \ldots, y_n) = (e^{-x_1}, \ldots, e^{-x_n})\]

for real parameters \( x_1, \ldots, x_n \) satisfying \( x_1 + \cdots + x_n = 0 \), and put \( y_{S_j} = \prod_{s \in S_j} y_s \). We define the rational plate \( [[S_1, \ldots, S_k]]_R \), where we have omitted the position \( (p_1, \ldots, p_k) = (0, \ldots, 0) \), to
be the rational function
\[
[[S_1, \ldots, S_k]]_R = \frac{1}{(1 - y_{S_1}) (1 - y_{S_1}y_{S_2}) \cdots (1 - y_{S_1} \cdots y_{S_{k-1}})}
\]

The following convenient observation presumably has applications, which we shall return to in later work.

**Proposition 15.** The series expansion of the rational plate \([[S_1, \ldots, S_k]]_R\) converges on the support of the plate \([[S_1, \ldots, S_k]]\) itself, namely
\[
\begin{align*}
x_{S_1} & \geq 0 \\
x_{S_1} + x_{S_2} & \geq 0 \\
& \vdots \\
x_{S_1} + \cdots + x_{S_{k-1}} & \geq 0 \\
x_1 + \cdots + x_n & = 0.
\end{align*}
\]

**Proof.** The rational plate is a product of geometric series, and the domain of convergence of a product of multi-variable geometric series equals the intersection of the respective domains of convergence, each of which determines one of the inequalities. \(\square\)

We have checked rigorously on the computer that rational plates as above satisfy the plate relations. We postpone proofs of that and faithfulness of the representation to later work.

**Conjecture 16.** The complex linear space of rational plates \(\pi_\mathcal{R}\) is isomorphic to the module of plates about a point.

**Example 17.** Let \(\pi = [[3, 2, 1]]\). Restrict to \(\sum_{i=1}^{3} x_i = 0\), and set \(y_i = e^{-x_i}\). Then
\[
\pi_\mathcal{R} = [[3, 2, 1]]_R = \frac{1}{(1 - y_3)(1 - y_2y_3)},
\]
and so
\[
\frac{1}{(1 - y_3)(1 - y_2y_3)} = \frac{1}{(1 - (y_1y_2)^{-1})(1 - y_1^{-1})} = \frac{y_1(y_1y_2)}{(y_1y_2 - 1)(y_1 - 1)}
\]
\[
= \frac{(y_1 - 1)(y_1y_2 - 1) + y_1 + y_1y_2 - 1}{(y_1y_2 - 1)(y_1 - 1)}
\]
\[
= 1 - \frac{1}{1 - y_1y_2} - \frac{1}{1 - y_1} + \frac{1}{(1 - y_1)(1 - y_1y_2)}
\]
\[
= [[123]]_R - [[12, 3]]_R - [[1, 23]]_R + [[1, 2, 3]]_R,
\]
which we recognize termwise as the linear combination
\[
[[123]] - [[12, 3]] - [[1, 23]] + [[1, 2, 3]],
\]
which we get from Theorem 9. See also the second row of Figure 2. Note that in the figure, the plate relations are for the place \([[1, 2, 3]]\), which is centered at \((1, 1, 1)\) and intersected with the triangle, whereas in the example immediately above the plate is centered implicitly at \((0, 0, 0)\) and is unbounded. For more detail we defer to \([39]\), where Ocneanu develops some sophisticated combinatorics which expresses the intersection of any plate with any hypersimplex as a linear combination of unbounded plates in the standard basis.
2. A Quantum Plate Basis

The lumps of a plate can be rotated cyclically. In what follows, we introduce a set of elements in the simplex plate module $Pl(\Delta^n)$ which are the common eigenvectors for such rotations.

We call these elements q-plates. In joint work with Ocneanu, which we include as an appendix, we prove that they are linearly independent and together form a basis of the plate space consisting of complex-linear combinations of plates.

**Definition 18.** Let $\pi = [[A_a B_b \cdots C_c D_d]]$ be a plate in $Pl(\Delta^n)$ with $r = a + b + \cdots + c + d$. Put $q = e^{-2\pi i/r}$. We define the q-plate $\{\pi\} = \{A_a B_b \cdots C_c D_d\}$ to be the q-weighted cyclic sum

$$\{A_a B_b \cdots C_c D_d\} = [[A_a B_b \cdots C_c D_d]] + q^{-d}[[D_d A_a B_b \cdots C_c]] + q^{-(e+d)}[[C_c D_d A_a B_b \cdots]] + \cdots + q^{-(r-a)}[[B_b \cdots C_c D_d \cdots A_a]].$$

The important property of q-plates is the following.

**Proposition 19.** Q-plates are invariant up to a root of unity under cyclic rotation of the lumps, and in the cyclic expansion of a q-plate in terms of plates each lump appears in the front exactly once.

$$\{C_c \cdots D_d A_a \cdots B_b\} = q^{-(a+\cdots+b)}\{A_a \cdots B_b C_c \cdots D_d\} = q^{(c+\cdots+d)}\{A_a \cdots B_b C_c \cdots D_d\}$$

**Proof.** This can be seen directly from the definition. □

Thus, because of the rotation property, the basis q-plates are those which have 1 in the first lump.

More generally, any permutation of the variables in a q-plate yields, up to a root of unity, a basis q-plate.

Corollary 20 suggests that the set of q-plates is well-suited to study the action of the symmetric group. This compatibility is central to our proof of the character formula in Chapter 3.

**Corollary 20.** The set of q-plates in $Pl(\Delta^n)$ is invariant under permutations in $S_n$, up to possibly a root of unity.

**Proof.** Let $\pi$ be a plate and let $\{\pi\}$ be the q-plate which arises as the cyclic symmetrization of $\pi$. Plates by definition have the property that each variable $x_1, \ldots, x_n$ appears in a unique lump $L$ of $\pi$, and each lump of $\pi$ is in the front for precisely one summand of the q-plate $\{\pi\}$. It follows that if $\sigma \in S_n$ is any permutation, then $\sigma \cdot \{\pi\} = q^m \{\pi'\}$ for a unique $m \in \{0, \ldots, r-1\}$ and a unique cyclic rotation $\pi'$ of $\pi$. □

**Example 21.** Fix where $r = a+b+c$ and put $q = e^{-2\pi i/r}$. Let $\sigma = (14)$. Let $\{\pi\} = \{12, 3b4c\}$. Then

$$\sigma \cdot \{12, 3b4c\} = \{24, 3b1c\}$$

$$= [[24, 3b1c]] + q^{-c}[[1, 24, 3b]] + q^{-(b+c)}[[3b1c24a]]$$

$$= q^{-c} \left( [[1, 24, 3b]] + q^{-b}[[3b1c24a]] + q^{-(a+b)}[[24a3b1c]] \right)$$

$$= q^{-c} \{1, 24a3b\}.$$
3. Linear Independence of q-plates via Change of Basis

In what follows, we establish the formula which expresses the standard non-degenerate plate \( \pi_0 = [[[1, 2, \ldots, n]]] \) as a linear combination of q-plates. Replacing the variables \( x_1, \ldots, x_n \) in \( \pi_0 \) with arbitrary lumps gives the expansion of any basis plate in terms of q-plates.

We first provide experimental support for our assertion that the set of q-plates is a basis for the module of plates. We include the proof in the Appendix.

3.1. Experimental Verification of q-plate Linear Independence in \( \Delta_n^r \). In joint work with Ocneanu, included in Appendix A, we prove Theorem 22. The problem is to show that the product of the two conjecturally inverse linear transformations between bases is the identity. The proof involves a delicate pairwise cancellation mechanism for off-diagonal entries of the product.

Additionally, by way of an apparently new, useful variant of the inversion permutation statistic we have an explicit, extremely compact formula for the expansion of a plate in terms of q-plates. This is the content of Theorem 27. The proof will involve Theorem 22.

**Theorem 22.** Any plate in \( n \) variables about a point has a unique expression as a linear combination of q-plates, where the coefficients are rational functions in \( q \) and are defined when \( q = e^{2\pi i / r} \).

In Figure 4, a more general result is obtained, in which we replace the root of unity \( q \) by \( n \) nonzero complex variables \( q_1, \ldots, q_n \) which satisfy \( q_1 \cdots q_n = 1 \). In the figure, we use the notation \( j = q_j \) for \( j = 1, 2, 3 \), though \( j = 1 \) is not present.

In what follows, we present computations in support of Theorem 22. Namely, we check for \( n \leq 4 \) and various values of \( r \) that the coefficient matrix for the expansion of the set of q-plates in \( \mathbb{P}(\Delta_n^r) \) into the standard plate basis is nonsingular when
Figure 4. Basis change: Plates $\Leftrightarrow$ q-plates

$q = e^{2\pi i/r}$. Thinking of the coefficient matrix equivalently as a map $\pi \mapsto \{\pi\}$ suggests the possibility of computing its eigenvalues.

Indeed, we are able to verify explicitly for small $n$ that the eigenvalues of the coefficient matrix are nonzero when $q = e^{2\pi i/r}$.

Remark 23. From Section 3 we already know that the change of basis matrix will have $\det \neq 0$. However, the eigenvalues have a finer structure which may be interesting on their own merits.

In what follows, we present the results of our computations. The Mathematica code involves Ocneanu’s notebook and is therefore quite extensive and shall appear elsewhere.

For $\Delta_3^3$ we computed the block upper-triangular $3^3-1 \times 3^3-1$ matrix

\[
\begin{pmatrix}
1 & \frac{1}{q^2} & \frac{1}{q} & \frac{1}{q} & \frac{1}{q} & \frac{1}{q} & \frac{1}{q} & \frac{1}{q} & \frac{1}{q} \\
0 & \frac{q-1}{q^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{q-1}{q} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{q^2-1}{q^2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{q-1}{q} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{q^2-1}{q^2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{q-1}{q} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{q^2-1}{q^2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{q-1}{q} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

where each column contains the coefficients in the expansion of the standard basis q-plates

\[
\{\{123\}, \{123\}, \{123\}, \{123\}, \{132\}, \{132\}, \{123\}, \{123\}, \{123\}, \{123\}\}
\]
in terms of standard basis plates

\[
\{[[123]], [[123]], [[123]], [[123]], [[123]], [[123]], [[123]], [[123]], [[123]], [[123]]\}.
\]

For example, the first column expresses the fact that $\{123\} = [[123]]$. 

```
It straightforward to check that this matrix can has polynomial eigenvalues (with some multiplicity)
\[1, 1 - \frac{1}{q^2}, 1 - \frac{1}{q}, -\frac{2}{q^2} + \frac{1}{q} + 1\]
and becomes
\[\{1, 1 - q, (q - 1)^2, 1 - q^2\}\]
after imposing the relation \(q^3 = 1\). Note that the zeros of these polynomials are now roots of unity.

The matrix for general \(r, n\) becomes quickly far too big to reproduce here, but some additional sets of eigenvalues, as follows.

For \(\Delta_3^3\) we obtained polynomial eigenvalues with some multiplicity
\[\{1, 1 - q, (q - 1)^2, 1 - q^2\}\]
which are zero precisely on all roots of unity of order \(\leq 5 - 1\), with multiplicity.

For \(\Delta_4^4\) we similarly get for eigenvalues polynomials in \(q\) whose roots range over roots of unity of all orders
\[\{1, 1 - q, (q - 1)^2, -(q - 1)^3, (q - 1)^2(q + 1), 1 - q^2, 1 - q^3\}\]

We optimistically make the obvious generalization, and we will focus on proving the following conjecture in future work.

**Conjecture 24.** The eigenvalues of the change of basis matrix as above between q-plates and plates in \(\Delta_n^r\) are, after imposing the relation \(q^r = 1\), polynomials in \(q\) whose zeros are precisely all roots of unity of order \(\leq r - 1\), with multiplicity.

By Theorem 22, Conjecture 24 and the following Corollary both hold true.

**Corollary 25.** The \(r^{n-1} \times r^{n-1}\) change of basis matrix whose rows are q-plates as linear combinations of plates, as above, is invertible whenever \(q\) is a primitive \(r^th\) root of unity, which we take as usual to be \(q = e^{-2\pi i/r}\). From this it follows that the set of q-plates in a simplex is a basis.

In the Appendix, we include the proof from our joint work with Ocneanu that the set of q-plates about a point is a basis for the module of plates about a point.

## 4. Decomposing the Standard Plate and Nearest Neighbor Inversions

In this section we use the change of basis formula to obtain \(2^{n-1}\) decompositions (with multiplicity) of the standard plate, each of which has a conjectural expression as a signed sum of simplices. This suggests a new refinement of the Eulerian numbers, by decomposing hypersimplex plates into classes which have the same numbers of nearest neighbor inversions. This is work in progress.

### 4.1. Nearest Neighbor q-Commutation.

By construction, a nondegenerate plate about the origin is determined by a permutation in line notation. Thus, it is reasonable to expect that a permutation statistic could prove useful.

**Definition 26.** Let \(\alpha \in S_n\) and set
\[\text{Inv}(\alpha) = \#\{i \in \{1, \ldots, n\} : \alpha^{-1}(i) > \alpha^{-1}(i + 1)\},\]
the number of nearest neighbor inversions, i.e. the number of pairs \((i, i + 1)\) such that \(i\) appears before \(i + 1\) in \(\alpha\) when \(\alpha\) is written in line form. If \(\pi\) is a plate with lumps possibly of
size $\geq 2$, then let $\text{Inv}(\pi)$ be the number of nearest neighbor inversions of $\pi'$, the permutation obtained by putting each lump in decreasing order in $\pi$ and flattening the result.

For example, the nondegenerate plate $[[7, 2, 6, 1, 4, 5, 3]]$ has four nearest neighbor inversions $(1, 2), (3, 4), (5, 6), (6, 7)$, while its lumping $[[7, 26, 145, 3]]$ has five nearest neighbor inversions $(1, 2), (3, 4), (4, 5), (5, 6), (6, 7)$, as can be read from $\pi' = [[7, 6, 2, 5, 4, 1, 3]]$.

In Theorem 27, which we were able to verify explicitly in Mathematica through $n = 6$ variables, we give an explicit formula for the expansion of the standard plate $[[1, 2, \ldots, n]]$ as a linear combination of q-plates, where the coefficients are determined up to sign by the number of nearest neighbor inversions. Let $\bar{q}_i = q_i q_{i+1} \cdots q_n$, $i = 2, \ldots, n$, where $q_1, \ldots, q_n$ are variables satisfying only $q_1 \cdots q_n = 1$.

**Theorem 27.** The standard nondegenerate plate $\pi_0 = [[1, 2, \ldots, n]]$ in $n$ variables expands in terms of q-plates as

$$(1 - \bar{q}_2^{-1}) \cdots (1 - \bar{q}_n^{-1}) \pi_0 = \sum_{\pi} (-1)^{\text{length}(\pi)} \prod_{i \in \text{Inv}(\pi)} \bar{q}_i^{-1} \{\pi\},$$

where the sum is over the plate basis in $n$ variables.

The proof will appear in [15].

The examples which follow aim to substantiate Theorem 27.

**Example 28.** The formula for the change of basis for the case $n = 3$ is given by

$$(1 - \bar{q}_2^{-1}) (1 - \bar{q}_3^{-1}) [[1, 2, 3]] = \{1, 2, 3\} + \bar{q}_3^{-1}\{1, 3, 2\} - \bar{q}_2\{21, 3\} - \bar{q}_3^{-1}\{1, 32\} - \bar{q}_3^{-1}\{31, 2\} + \bar{q}_2^{-1}\bar{q}_3^{-1}\{321\}.$$

Expanding and collecting terms on the right hand side and then applying the plate relations we obtain

$$\text{RHS} = [[[1, 2, 3], \ldots, 1, 2, 3]] + \bar{q}_3^{-1}\{1, 3, 2\} - \{31, 2\} - ([1, 32]) = -\bar{q}_3^{-1}\{1, 2, 3\}$$

$$+ \bar{q}_2^{-1}\{21, 3\} + \{2, 31\} - \{21, 3\} - \{2, 31\} = -\bar{q}_2^{-1}\{1, 2, 3\}$$

$$+ \bar{q}_2^{-1}\bar{q}_3^{-1}\{21, 3\} + \{32, 1\} - \{32, 1\} - \{1, 32\} + \{321\} = \bar{q}_2^{-1}\bar{q}_3^{-1}\{1, 2, 3\}$$

$$= \left(1 - \bar{q}_2^{-1}\right) \left(1 - \bar{q}_3^{-1}\right) [[1, 2, 3]].$$

Expanding each q-plate in terms of plates and collecting terms, we observed that the coefficients of the $2^{n-1}$ monomials in the variables $\bar{q}_i^{-1}$ each sums to the standard plate $[[1, 2, \ldots, n]]$ modulo the plate relations. Each such expression is equal to the standard plate modulo the plate relations. This follows from Theorem 27.

**Example 29.** Collecting coefficients of the $\bar{q}_i$ monomials in the expression in Theorem 27 according to the nearest neighbor class for the orientation

$$\text{pos}_1 > \text{pos}_2 > \text{pos}_3 < \text{pos}_4$$

we obtain the signed sum

$$[[3, 2, 1, 4]] + [[3, 2, 4, 1]] + [[3, 4, 2, 1]]$$

$$- [[3, 24, 1]] - [[23, 1, 4]] - [[23, 4, 1]] - [[3, 2, 14]] - [[3, 2, 14]] - [[3, 12, 4]] - [[3, 4, 12]]$$

$$+ [[3, 124]] + [[3, 124]] + [[23, 14]]$$
which we regroup into inclusion/exclusion classes, lumping only when the variables are already written in descending order so as to preserve the orientation, as

\[
[[3, 2, 1, 4]] - [[32, 1, 4]] - [[3, 21, 4]] + [[321, 4]]
+ [[324, 1]] - [[3, 2, 41]] - [[32, 41]] + [[32, 41]]
+ [[3, 42, 1]] - [[3, 421]] - [[3, 421]],
\]

which one can check is equal to the standard plate \([[[1, 2, 3, 4]]]\), modulo the plate relations. In [15] we shall prove that each line is the characteristic function of a simplex contained in the standard plate.

\[
[[3, 2, 1, 4]] - [[32, 1, 4]] - [[3, 21, 4]] + [[321, 4]] = \{ x \in \mathbb{R}^4 : x_3 \leq 0, x_{32} \leq 0, x_{321} \geq 0, x_{1234} = 0 \}
[[3, 2, 4, 1]] - [[3, 2, 41]] - [[32, 41]] + [[32, 41]] = \{ x \in \mathbb{R}^4 : x_3 \leq 0, x_{32} \geq 0, x_{324} \leq 0, x_{1234} = 0 \}
[[3, 4, 2, 1]] - [[3, 4, 21]] - [[3, 42, 1]] + [[3, 421]] = \{ x \in \mathbb{R}^4 : x_3 \geq 0, x_{34} \leq 0, x_{321} \leq 0, x_{1234} = 0 \}
\]

**Example 30.** The sum

\[
[[1, 2, 3, 4]] = (-[[3, 1, 2, 4]] + [[31, 2, 4]])
+ (-[[1, 3, 4, 2]] + [[1, 3, 42]])
+ (-[[3, 1, 4, 2]] + [[31, 4, 2]] + [[3, 1, 42]] - [[31, 42]])
+ (-[[1, 3, 2, 4]] + [[1, 32, 4]])
+ (-[[3, 4, 1, 2]] + [[3, 41, 2]])
\]

corresponds to the orientation \((\text{pos}_1 < \text{pos}_2 > \text{pos}_3 < \text{pos}_4)\) for the positions of nearest neighbors. We have broken the sum into pieces such that each line is an alternating sum of lumpings of a single nondegenerate plate. The sum of plates in each line vanishes outside the standard nondegenerate plate \([[1, 2, 3, 4]]\).

The lines in Example 30 can be shown to be regions which are defined by the following inequalities.

\[
(-[[3, 1, 2, 4]] + [[31, 2, 4]]) = \{ x \in \mathbb{R}^4 : x_3 \leq 0, x_{31} \geq 0, x_{312} \geq 0, x_{1234} = 0 \}
(-[[1, 3, 4, 2]] + [[1, 3, 42]]) = \{ x \in \mathbb{R}^4 : x_1 \geq 0, x_{13} \geq 0, x_{134} \leq 0, x_{1234} = 0 \}
(-[[3, 1, 4, 2]] + [[31, 4, 2]] + [[3, 1, 42]] - [[31, 42]]) = \{ x \in \mathbb{R}^4 : x_3 \leq 0, x_{31} \leq 0, x_{314} \leq 0, x_{1234} = 0 \}
(-[[1, 3, 2, 4]] + [[1, 32, 4]]) = \{ x \in \mathbb{R}^4 : x_1 \geq 0, x_{13} \leq 0, x_{123} \geq 0, x_{1234} = 0 \}
(-[[3, 4, 1, 2]] + [[3, 41, 2]]) = \{ x \in \mathbb{R}^4 : x_3 \geq 0, x_{34} \leq 0, x_{341} \geq 0, x_{1234} = 0 \}.
\]

From Theorem 27 we obtain the following result by expanding the left-hand side of the formula

\[
(1 - \bar{q}_2^{-1}) \cdots (1 - \bar{q}_n^{-1}) \pi_0 = \sum_{\pi} (-1)^{n-\text{length}(\pi)} \prod_{i \in \text{Inv}(\pi)} \bar{q}_i^{-1} \{\pi\},
\]

and comparing coefficients of the monomials in the variables \(\bar{q}_i\), where as above the sum is over the standard basis in \(n\) variables.

**Corollary 31.** Each of the \(2^{n-1}\) summations is equal to the standard plate \(\pi_0 = [[[1, 2, \ldots, n]]]\), modulo the plate relations.

The general expressions shall appear in [15].
CHAPTER 3

The Combinatorics and Symmetry of Plates

1. Two Helpful Algebras

In what follows, we introduce two algebras, the translation algebra, which we need to prove
the character of the simplex plate module, and as a bonus a q-deformation, the area algebra.

Our geometric intuition is that the set of plates in a simplex has “asymptotically” a multi-

plicative structure, which we axiomatize in Definition 32.

We show in Theorem 45 that the translation algebra admits a q-deformation, which can be
formulated as a particular generalized Clifford algebra. In the first nontrivial case the structure
coefficients depend on the quantum area $q^{ad-bc}$ of a parallelogram with edge vectors $(a, b), (c, d)$,
where $q$ is a root of unity. Thus we call it the area algebra $A^n$.

The translation algebra is used below to implement a constructive proof of the simplex
character formula, giving an explicit parametrization of the diagonal of the matrix of each
permutation.

In what follows we shall assume that the ground field is $\mathbb{C}$.

1.1. Translation Algebra. Let $q = e^{-2\pi i/r}$. We may identify set of q-plates in a simplex
scaled by an integer $r \geq 1$ with an algebra $C^n_r$ of commuting operators which, roughly speaking,
translate special hyperplanes by integer vectors compatibly with the plate module, such that
the operator which increases all weights by 1 acts by $q$ on all of $C^n_r$.

Definition 32. The translation algebra is the commutative algebra over $\mathbb{C}$ given in terms
of generators and relations as

$$C^n_r = \langle e_1, \ldots, e_n : e_i^r = 1, e_1 \cdots e_n = q \rangle.$$

Consider the set of q-plates, forgetting for now the complex linear structure.

If we identify the identity element $1 \in C^n_r$ with the characteristic function of the whole
simplex, i.e. the q-plate $\{12 \cdots n_r\}$, then we shall see that $C^n_r$ acts freely and transitively on
the set of q-plates, i.e. on the q-basis. Remark that in the generic case, i.e. for nondegenerate plates
in $\text{Pl}(\Delta^n_r)$ in which $r \gg n$, then the $e_i$ act on the position of the lumps of a generic q-plate by
translation in the root directions $(0, \ldots, 1, \ldots, 0, \ldots, -1, \ldots, 0) \in \mathbb{R}^n$. Here generic means that
the q-plate is nondegenerate and that all of its $n$ lumps have position $\gg 1$.

Proposition 33. The symmetric group $S_n$ acts on the translation algebra $C^n_r$ by permuting
the variables. Moreover, as a complex vector space, $C^n_r$ has dimension $r^n - 1$, and the set

$$\{ e_1^{j_2} \cdots e_n^{j_n} : 0 \leq j_i \leq r - 1 \}$$

consisting of monomials normalized so that $e_1$ does not appear, is a basis, which we call the
standard monomial basis.

Proof. Since $e_1 \cdots e_n - q$ is fixed by variable permutation, the $S_n$-action is well-defined.

Modulo the defining relation $e_1 \cdots e_n = q$, each monomial has a unique normalization to a
standard monomial in which $e_1$ does not appear. There are no further relations, so the basis is
parametrized by the $n - 1$ integers $j_2, \ldots, j_n \in \mathbb{Z}/r$. This completes the proof. □
We now show that $C_n^r$ acts naturally on q-plates in the simplex $\Delta_n^r$. In the generic case, $e_i \in C_n^r$ acts on a q-plate $\{\pi\}$ by translation in a root direction $(0, \ldots, 1, \ldots, -1, \ldots, 0)$ which is determined by the set composition which defines $\{\pi\}$.

**Theorem 34.** There exists an $S_n$-equivariant set bijection $C_n^r \to Pl(\Delta_n^r)$.

**Proof.** Any given monomial $m = e_i^2 \cdots e_i^n$ in the standard basis factors uniquely as a decreasing flag of monomials

$m = e_{A_2}^a e_{A_3}^a \cdots e_{A_k}^a$,

such that

1. $e_{A_i} = \prod_{i \in A} e_i$ for $A \subseteq \{1, \ldots, n\}$
2. $A_2 \supseteq A_3 \supseteq \cdots \supseteq A_k$
3. $1 \leq a_i \leq r - 1$ and $a_2 + \cdots + a_k \leq r - 1$.

Then put

$\{\pi_m\} = \{(A_1)_{a_1} \cdots (A_k)_{a_k}\}$,

where we define

$A_i := \tilde{A}_i \setminus \tilde{A}_{i+1}$

and

$a_k := r - \sum_{i=1}^{k-1} a_i$.

For the inverse map, it suffices to solve for the $\tilde{A}_i$ in terms of $A_j$, as

$\tilde{A}_i = \cup_{j=i}^k A_j$.

□

**Corollary 35.** The translation algebra $C_n^r$ and the simplex plate module $Pl(\Delta_n^r)$ are isomorphic as $S_n$-modules.

**Proof.** By the result of Appendix A, the set of standard q-plates in a simplex is a basis. Thus, extending linearly the map in Theorem 34 the result follows.

□

**Example 36.** For example, by the bijection in Theorem 34 we map the $(n-1)$-uple $(5, 1, 4, 1) \in (\mathbb{Z}/7)^{5-1}$ into the monomial

$e^{(0, 5, 1, 4, 1)} = e_5^2 e_3 e_4^4 e_5 \in C_7^5$,

which factors successively the smallest powers as

$e_5^2 e_3 e_4^4 e_5 = (e_2 e_3 e_4 e_5)(e_4^2)(e_2 e_4)^3 e_2$.

Arranging the factors in the order of increasing powers, we have

$e_5^2 e_3 e_4^4 e_5 = (e_1)^0 (e_3 e_5)^4 (e_4)^4 (e_2)^5$.

We keep each parenthesis lumped together, with a position, written as a subscript, given by the jump from its power to the next power, to finally obtain the q-plate

$\{(1)_{1-0}(35)_{4-1}(4)_{5-4}(2)_{7-5}\} = \{13534122\} \in Pl(\Delta_7^5)$,

recalling that $r = 7$.

The mechanism of the last step is due to Ocneanu in [39] and is called by him the clock, as the powers and names of variables can be arranged as clock hands in $\mathbb{Z}/r$, viewed as a circle, and the jumps are the distances between consecutive hands.
Example 37. Figure 1 illustrates our geometric intuition for Theorem 34, which reconstructs the q-plate
\[ \{\pi\} = \{1,3,2,1\} = q^0[[1,3,2]] + q^{-1}[2,1,3] + q^{-2}[3,2,1] \]
\[ = [[1,3,2]] + q^2[2,1,3] + q^1[3,2,1] \]
from its monomial $e_2^2e_3 \in C_3^2$.

\begin{center}
\begin{tikzpicture}
\node (1) at (0,0) {$q^0=1$};
\node (2) at (1.5,-0.5) {$q^1$};
\node (3) at (3,0) {$q^2$};
\node (4) at (1.5,0.5) {$q^2$};
\node (5) at (0,1.5) {$q^1$};
\node (6) at (3,1.5) {$q^1$};
\node (7) at (2.25,0.5) {$1$};
\node (8) at (1,0) {$1$};
\node (9) at (1.5,1.5) {$1$};
\node (10) at (0,0) {$\{1,3,2,1\}$};
\node (11) at (3,0) {$\{1,3,2,1\}$};
\node (12) at (1.5,0) {$e^0_1e^2_2e^3_1$};
\end{tikzpicture}
\end{center}

\textbf{Figure 1.} The set of q-plates is uniquely labeled by values in corner unit simplices.

From the isomorphism of Corollary 35 it follows that the action of $C^n_r$ on itself induces an action on $\text{Pl}(\Delta^n_r)$.

**Proposition 38.** For a generic plate $\{\pi\}$, i.e. each lump $A_a$ has length $|A| = 1$ and position $a > 1$, the element $e_i \in C^n_r$ acts by translation in some root direction $f_i - f_j$, if $\{f_1, \ldots, f_n\}$ is the standard basis for $\mathbb{R}^n$.

**Proof.** Let
\[ \{\pi\} = \{(A_1)_{a_1} \cdots (A_n)_{a_n}\}, \]
where each $|A_i| = 1$ and $a_i > 1$.

It suffices to check what happens on the monomial
\[ m = e^{a_1}_{A_2}e^{a_2}_{A_3} \cdots e^{a_k-1}_{A_k}, \]
where $\tilde{A}_i = \cup_{j=i}^n A_j$ and each $a_i > 1$. Assuming $i \in A_i$ then
\[ e_i m = e^{a_1}_{\tilde{A}_2} \cdots e^{a_i+1}_{\tilde{A}_i} e^{a_{i+1}-1}_{\tilde{A}_{i+1}} \cdots e^{a_k-1}_{\tilde{A}_k}. \]
That is, if $i \in A_i$ then
\[ e_i \cdot \{(A_1)_{a_1} \cdots (A_{i-1})_{a_{i-1}}(A_i)_{a_i} \cdots (A_n)_{a_n}\} = \{(A_1)_{a_1} \cdots (A_{i-1})_{a_{i-1}}(A_i)_{a_i-1} \cdots (A_n)_{a_n}\}, \]
which corresponds to the translation of the center of the q-plate $\{\pi\}$ in the root direction $f_{A_i} - f_{A_{i+1}}$, where $f_1, \ldots, f_n$ is the standard basis for $\mathbb{R}^n$. \hfill \Box

**Example 39.** Let $\{\pi\} = \{1,3,2,4,5,3,1\} \in \text{Pl}(\Delta^4_{14})$. Then by the clock bijection we have
\[ e_2 \cdot \{1,3,2,4,5,3,1\} \simeq e_2 e^{(0,3,3+2+5,3+2)} = e^{(0,4,10,5)} \simeq \{1,4,2,1,4,5,3,1\} \]
which changes the differences of increasing powers accordingly.
In what follows, we present a convenient show that it is possible recover the q-plate \( \{ \pi \} \in \text{Pl}(\Delta^n_r) \) by making experimental observations “at infinity,” as described in what follows.

**Proposition 40.** Each q-plate \( \{ \pi \} \) in a scaled \((n - 1)\)-dimensional simplex \( \Delta^n_r \) can be recovered from its values on the \( n \) corner unit simplices.

**Proof.** By the cyclic sum relation of Theorem 7, the set of cyclic rotations of a plate \( \pi \) overlap only on higher codimension faces. Therefore, the value of the q-plate \( \{ \pi \} \) at a corner unit simplex of \( \Delta^n_r = \{ x \in [0,r]^n : \sum x_i = r \} \) determined by the equation \( x_j \geq r - 1 \) is obtained is the coefficient of the unique cyclic rotation of \( \pi \) such that \( j \) is in its first lump. \( \square \)

From Theorem 34 we can also obtain geometrically the corresponding element of the translation algebra from the values of the q-plate at the corners.

**Corollary 41.** The exponents of the monomials \( e_1^{i_1} \cdots e_n^{i_n} \) are the values of \( \{ \pi \} \) at the corners of \( \Delta^n_r \).

**Proof.** This follows from Theorem 34 together with Proposition 40. \( \square \)

1.2. Geometry of the Translation Algebra. By the discussion culminating in Proposition 40, the exponents in each monomial \( e_2^{j_2} \cdots e_n^{j_n} \) encode roots of unity \( e^{2\pi i j_k / r} \) placed in the corner unit simplices of the simplex \( \Delta^n_r \), cut out by the inequality \( r - 1 \leq x_k \leq r \). We adopt the perspective that corner values of a q-plate in \( \Delta^n_r \) extend uniquely into the interior to give a q-plate.

Based on what we learned previously from Ocneanu [39] about how to extend a plate from the simplex to the period solid, we illustrate in \( n = 3 \) coordinates the extension of a q-plate in the simplex \( \Delta^3_4 \) to a periodic tiling with permutohedra, modulo a period \( r \), where each generalized permutohedron is weighted with a root of unity.

See Figure 2 for a periodic tiling of the plane with weighted generalized hexagons.

![Figure 2](image-url)  

**Figure 2.** Extending a q-plate to a permutohedral tiling

1.3. Area Algebra. Here we introduce a q-deformation of the translation algebra \( C^n_r \) which we call the **area algebra** \( A^n_r \). Throughout, we fix the standard cyclic order \((12\cdots n)\) for the generators and the structure constants, hence the symmetric group \( S_n \) does not act.

In future work we shall ask whether the area algebra can be used to inspire and organize deformations of scaling and tiling of permutohedral cones.
For general theory related to possible group-theoretic applications of what follows, see for example [11] on twisted group algebras, and [1, 26, 34] on generalized Clifford algebras.

Fix integers \( n \geq 3 \) and \( r \geq 2 \), and let \( q = e^{-2\pi i/r} \), as usual.

The idea is to use a finite abelian group to keep track of the corner values of a q-plate “at infinity” in the simplex \( \Pi(\Delta^n_r) \), and to define structure coefficients for our associative and q-commutative product of q-plates \( \{\pi_1\}, \{\pi_2\} \in \Pi(\Delta^n_r) \) that depend quadratically on the values at infinity of \( \{\pi_1\}, \{\pi_2\} \) respectively.

**Definition 42.** The area algebra \( A^n_r \) is generated by formal linear combinations of elements \( [(a,b,\ldots,c)] \), where \( a,b,\ldots,c \in (\mathbb{Z}/r) \), which are subject to the relation

\[
[(1,\ldots,1)] \equiv q[(0,\ldots,0)]
\]

and which multiply as

\[
[(i_1,\ldots,i_n)] \cdot [j_1,\ldots,j_n] = q^{-v_1^t M v_2}[(i_1 + j_1,\ldots,i_n + j_n)]
\]

where

\[
v_1 = (i_1,\ldots,i_n)
\]

\[
v_2 = (j_1,\ldots,j_n)
\]

and

\[
M = \begin{bmatrix}
0 & 1 & 0 & \cdots & -1 \\
-1 & 0 & 1 & 0 & \vdots \\
0 & -1 & 0 & \ddots & 0 \\
\vdots & 0 & \ddots & 0 & 1 \\
1 & \cdots & 0 & -1 & 0
\end{bmatrix}
\]

**Example 43.** In \( A^3_r \), since \( [a + 1, b + 1, c + 1] = q[a,b,c] \) up to scaling we have

\[
[(0,a,b)] \cdot [(0,c,d)] = q^{-(ad-bc)}[(0,a+c,b+d)].
\]

Thus, we think of the coefficient \( q^{-(ad-bc)} \) as the **quantum area** of the parallelogram spanned by the vectors \((a,b)\) and \((c,d)\).

**Remark 44.** The structure matrix \( M \) as in Definition 42 is clearly not uniquely determined by the compatibility requirement. Indeed, it is likely to be interesting to substitute for \( M \) other (possibly not antisymmetric) structure matrices for which all rows and columns sum to 0, so that \((1,\ldots,1)\) is annihilated by \( M \) on both sides.

Thus, our main result for the area algebra is that the structure coefficients are well-defined on the translation algebra.

**Theorem 45.** The area algebra \( A^n_r \) is a well-defined deformation of the translation algebra.

**Proof.** By construction, the identification of the monomial \( e^{i_1} \cdots e^{i_n}_n \in C^n_r \) with its exponent vector \( [(i_1,\ldots,i_n)] \in A^n_r \) respects the relations on the two algebras. Moreover, the coefficient \( q^{-v_1^t M v_2} \) is well-defined because the vector \((1,\ldots,1)\) is in the kernel of the matrix \( M \).

What makes this an intriguing result is its geometric origin and physical interpretation. The area algebra is well-defined with respect to over all scaling of the q-plate by \( q = e^{-2\pi i/r} \), and the structure coefficients are quadratic polynomial functions of quantities which could conceivably be measured in a laboratory.

The structure coefficients for \( A^n_r \) are potentially very interesting.
Example 46. In $A^n_r$, the structure matrix is
\[
M = \begin{bmatrix}
0 & 1 & 0 & 0 & -1 \\
-1 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 1 \\
1 & 0 & 0 & -1 & 0 
\end{bmatrix},
\]
so we obtain, for clarity taking arbitrary generators rather than scaling to basis elements,
\[
[(a_1, a_2, a_3, a_4, a_5)] \cdot [(b_1, b_2, b_3, b_4, b_5)] = q^{-C}[(a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4, a_5 + b_5)]
\]
where
\[
C = -a_2b_1 + a_5b_1 + a_1b_2 - a_3b_2 + a_2b_3 - a_4b_3 + a_3b_4 - a_5b_4 - a_1b_5 + a_4b_5.
\]
We recognize that this can be organized as a cyclic product of exponentiated minors
\[
[(a_1, a_2, a_3, a_4)] \cdot [(b_1, b_2, b_3, b_4)] = \frac{1}{q^{\Delta_{12}q^{\Delta_{23}}q^{\Delta_{34}}q^{\Delta_{45}}q^{\Delta_{51}}}}[(a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4, a_5 + b_5)],
\]
where $\Delta_{ij}$ is the determinant of columns $i, j$ of the matrix
\[
\begin{bmatrix}
a_1 & a_2 & a_3 & a_4 & a_5 \\
b_1 & b_2 & b_3 & b_4 & b_5
\end{bmatrix}.
\]

From Theorem 45, by explicit computation we obtain an intriguing observation.

Corollary 47. If $n$ is even then the scaling element splits into two commuting pieces as
\[
q[0, \ldots, 0] = [(1, \ldots, 1)] = [(1, 0, 1, 0 \ldots, 1, 0)] \cdot [(0, 1, 0, 1, \ldots, 0, 1)] = [(0, 1, 0, 1, \ldots, 0, 1)] \cdot [(1, 0, 1, 0 \ldots, 1, 0)].
\]

2. Result I

2.1. Symmetric Group Character of the Translation Algebra. In what follows, we compute the character of the action of $S_n$ on the translation algebra $C^n_r$, by explicitly parametrizing the diagonal of each permutation matrix.

Let $q = e^{-2\pi i/r}$, as usual.

The idea is to approach $C^n_r$ as if it were a polynomial ring in $n$ variables $e_i$, except that each generator has a finite order $r$ and together satisfy one relation $e_1 \cdots e_n = q$. The character value of a permutation $\sigma \in S_n$ on the usual polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ can be computed by counting monomials whose exponent vector is constant over the cycles of $\sigma$, see for example [46]. Each such monomial is then an eigenvector of $\sigma$ with eigenvalue 1.

This method of proof can be applied to the translation algebra, but with the important caveat that monomials in the standard basis now have eigenvalues which are $r^{th}$ roots of unity. Remarkably, the character values on $C^n_r$ are always either 0 or a power $r^{k-1}$, depending on the number of cycles $k$ and gcd($\lambda_1, \ldots, \lambda_k, r$).

We factor a monomial into a product $m = X \cdot R$ where $X$ is a certain part of $m$ which is fixed by $\sigma$ and depends on a choice of normalization, and $R$ is the remainder monomial. In the case that $m$ is an eigenvector of $\sigma$, then $R$ is a power of a certain monomial "$\Omega$" and $m$ factors as $m = X \cdot \Omega^l$ for some $l \in \{0, \ldots, r - 1\}$ such that $\Omega^l$ is rotated by $\sigma$ by an $r^{th}$ root of unity which we determine precisely in what follows.
Definition 48. Let $\sigma = (p_1 \cdots p_\lambda) \in S_n$ be a cycle of length $\lambda \geq 1$. Define the special monomial

$$\Omega_\sigma = e_{p_1}^0 e_{p_2}^1 \cdots e_{p_\lambda}^{\lambda-1}.$$ 

Via the bijection of Theorem 34, in the case $r \geq n$ then $\Omega_\sigma$ takes the form

$$\{(1)_{1}(2)_{1} \cdots (n-1)_{1} (n)_{r-p_\lambda}\}.$$

We shall take powers of $\Omega_\sigma$ in the translation algebra to obtain eigenvectors of $\sigma$. In particular, for an $n$-cycle we have

Proposition 49. Let $\sigma = (p_1 \cdots p_m) \in S_n$ be a cycle. Let $g = \gcd(m, r)$. Recall that $q = e^{-2\pi i/r}$. Then $\sigma \cdot \left(\Omega_{\sigma}^{r/g}\right) = X_{\sigma}^{-r/g} \Omega_{\sigma}^{r/g}$.

Proof. First note that if $g = 1$ then $\Omega_{\sigma}^{r/g} = 1$, so $\sigma \left(\Omega_{\sigma}^{r/g}\right) = \sigma (\Omega_{\sigma}^{0}) = \Omega_{\sigma}^{0}$ since $\Omega_{\sigma}^{0}$ is the constant monomial 1. Thus the result is trivially true. Let us now assume that $g > 1$. Then

$$\sigma \cdot \left(\Omega_{\sigma}^{r/g}\right) = (p_1 p_2 \cdots p_m) \cdot (e_{p_2}^1 e_{p_3}^2 \cdots e_{p_m}^{m-2} e_{p_m}^{(m-1)})^{r/g}$$

$$= (e_{p_2}^1 e_{p_3}^2 \cdots e_{p_m}^{(m-2)} e_{p_m}^{(m-1)})^{r/g}$$

$$= (e_{p_2}^{m-r/g} e_{p_3}^{r/g} e_{p_4}^{2r/g} \cdots e_{p_m}^{(m-2)r/g})$$

$$= (e_{p_1} e_{p_2} \cdots e_{p_m})^{-r/g} \left(e_{p_2}^1 e_{p_3}^2 \cdots e_{p_m}^{m-2} e_{p_m}^{(m-1)}\right)^{r/g}$$

$$= X_{\sigma}^{-r/g} \Omega_{\sigma}^{r/g},$$

having used $e_1^{(m-1)r/g} = e_1^{-r/g}$, as $g$ divides $m$. \hfill \Box

As a special but important case we have the following.

Corollary 50. Let $\sigma = (12 \cdots n)$ and $r = n$, and put $g = \gcd(n, r) = n$. Then $\Omega_{\sigma}^{r/g} \in C_r^n$ is an eigenvector of $\sigma$ with eigenvalue $q^{-r/g} = e^{2\pi i/g} = e^{2\pi i/n}$.

Proof. We check explicitly.

$$\sigma \cdot \left(\Omega_{\sigma}^{r/g}\right) = X_{\sigma}^{-1} \Omega_{\sigma}^{1}$$

$$= q^{-1} \Omega_{\sigma}$$

$$= e^{2\pi i/n} \Omega_{\sigma}.$$

\hfill \Box

Let us first fix notation which will be used in the proof of the main result of this section, Lemma 56.

Definition 51. Let $\sigma = \sigma_1 \cdots \sigma_k \in S_n$ be the decomposition of $\sigma$ into disjoint cycles $\sigma_i$, where for simplicity we assume $1 \in \sigma_1$, i.e. that $\sigma_1$ is the cycle containing 1, and let $X_{\sigma_i} = \prod_{j \in \sigma_i} e_j$ be the indicator monomial of $\sigma_i$. For example, if $\alpha = (346)$ then $X_{\alpha} = e_3 e_4 e_6$.

Let $m = e_2^{i_2} \cdots e_6^{i_6} \in C_r^n$ be an arbitrary basis monomial.

Let $m = Y_{\sigma_1} \cdots Y_{\sigma_k}$,
where we define
\[ Y_{\sigma_i} = e^{ip_1} \cdots e^{ip_{\lambda_i}} = (X_{\sigma_i})^{ip_1} \cdot R_{\sigma_i}, \]
for each cycle \( \sigma_i = (p_1 \cdots p_{\lambda_i}) \), and where
\[ R_{\sigma_i} = \left( e_0^{ip_1} e_1^{ip_2-\cdots} \cdots e_0^{ip_{\lambda_i}-ip_1} \right) \]
is called the \textbf{remainder monomial}.

Put simply, we are normalizing \( m \) to a monomial \( R \) whose exponent vector is 0 over the variable corresponding to the beginning of each cycle \( \sigma_i \).

**Example 52.** Fix \( r = 14 \), say. Let \( \sigma = \sigma_1\sigma_2\sigma_3 = (123)(4)(5678) \) and let \( m = e_2^7 e_3^5 e_4^1 e_5^8 e_6^{10} e_7^5 e_8^{12} \) be a basis monomial with exponent vector \( v = (0, 7, 5, 11, 8, 10) \). Then via the above decomposition we have
\[ m = (X_{\sigma_1}X_{\sigma_2}X_{\sigma_3}) \cdot (R) \]
\[ = \left( \left( e_1 e_2 e_3 \right)^0 (e_4)^{11} (e_5 e_6 e_7 e_8)^8 \right) \cdot \left( \left( e_2 e_3 \right)^7 (e_4)^0 \left( e_5 e_6 e_7 e_8 \right)^4 \right) \]
It is easy to check that this is \textit{not} an eigenvector of \( \sigma \).

On the other hand we have

**Example 53.** Let \( r = 12 \), \( \sigma = (1234)(5678)(9 10 11 12) \) and \( m = e_2^3 e_3^6 e_4^9 e_5^{11} e_6^2 e_7^5 e_8^7 e_9^{10} e_{10}^{11} e_{12}^4 \).
Then \[ m = \left( (e_1 e_2 e_3 e_4)^0 (e_5 e_6 e_7 e_8)^8 (e_9 e_{10} e_{11} e_{12})^7 \right) \cdot \left( \left( e_1 e_2 e_3 e_4 \right)^7 (e_5 e_6 e_7 e_8)^0 \left( e_9 e_{10} e_{11} e_{12} \right)^4 \right) \]
\[ = \left( X_{1234} X_{5678} X_{9101112} \right) \cdot (R) \]
where \[ R = \left( \Omega^3_{(1234)} \Omega^3_{(5678)} \Omega^3_{(9 10 11 12)} \right) = \Omega^3_3. \]
Thus, \( m \) is an eigenvector of \( \sigma \) with eigenvalue \( q^{-9} = e^{-9 \cdot 2\pi i/12} = e^{2\pi i/4} \).

In what follows, we shall enumerate the eigenvectors of any permutation \( \sigma \), allowing us to compute the trace of \( \sigma \), hence the character of \( C^n_r \) as a representation of \( S_n \).

**Definition 54.** A monomial \( m \) is a \textbf{weight monomial} of a permutation \( \sigma \) of \( \sigma \)-\textbf{weight} \( c \) if \( \sigma \cdot m = cm \) for a complex number \( c \).

We would like to classify the set of weight monomials for each permutation \( \sigma \in S_n \).

**Lemma 55.** If \( \sigma \cdot m = c \cdot m \), then \( c = e^{2\pi i/r} \) and is an \( r \)-th root of unity.

**Proof.** Fix a permutation \( \sigma \in S_n \). The set of all monomials in \( C^n_r \) decomposes into orbits under multiplication by \( e_1 \cdots e_n = q \). Thus, if \( m \in C^n_r \) is a basis monomial then either \( \sigma \cdot m = (e_1 \cdots e_n)^l \cdot m \) for some \( l \in \{0, \ldots, r-1\} \), in which case we are done, or not. If not, then \( \sigma \cdot m = q^r m' \) for some \( l' \in \{0, \ldots, r-1\} \) with \( m' \neq m \), and hence \( m \) is not a weight monomial.

It follows that all \( \sigma \)-weights are \( r \)-th roots of unity. \( \square \)

The \( \sigma \)-weight of a weight monomial \( m \) can be obtained from the remainder monomial \( R \) of Definition 51. The intuition for the proof of what follows is that all weight monomials are a product of an invariant monomial which is built from indicator functions, times a power of the special monomial \( \Omega \) defined above.

\[ Y_{\sigma_i} = e^{ip_1} \cdots e^{ip_{\lambda_i}} = (X_{\sigma_i})^{ip_1} \cdot R_{\sigma_i}, \]
for each cycle \( \sigma_i = (p_1 \cdots p_{\lambda_i}) \), and where
\[ R_{\sigma_i} = \left( e_0^{ip_1} e_1^{ip_2-\cdots} \cdots e_0^{ip_{\lambda_i}-ip_1} \right) \]
Lemma 56. Let $\sigma = \sigma_1 \ldots \sigma_k \in S_n$ be of cycle type $\lambda_1, \ldots, \lambda_k$. Let $l \in \{0, \ldots, r - 1\}$.

(1) If $m$ has a $\sigma$-weight $q^{-t}$, then $t = l \cdot (r / \gcd(\lambda_1, \ldots, \lambda_k, r))$ for some

$$l \in \{0, \ldots, \gcd(\lambda_1, \ldots, \lambda_k, r) - 1\}.$$ 

Equivalently, all $\sigma$-weights are of the form $q^{-lr/g} = e^{2\pi i l/g}$, where $g = \gcd(\lambda_1, \ldots, \lambda_k, r)$.

(2) A weight monomial $m \in \mathcal{C}$ satisfies $\sigma \cdot m = q^{-vl/g}$ if and only if in Definition 51 we have for the remainder monomial

$$R = (\Omega_{\sigma_1} \cdots \Omega_{\sigma_k})^{lr/g} \cdot (\lambda_{\sigma_1} \ldots \lambda_{\sigma_k}).$$

Proof. For (1), suppose $\sigma \cdot m = q^{-t}m$ where $m$ is a standard basis element and $1 \leq t \leq r - 1$. Putting $m = e_1^{c_1} e_2^{c_2} \cdots e_n^{c_n}$, where $c_1 = 0$, we have

$$q^{-t} e_2^{c_2} \cdots e_1^{c_n} = \sigma(e_2^{c_2} \cdots e_1^{c_n}) = e_1^{c_1(1)} e_2^{c_2(1)} \cdots e_n^{c_n(1)} = (e_1 \cdots e_n)^{c_1(1)} \left( e_2^{c_2(2) - c_2(1)} \cdots e_n^{c_n(2) - c_n(1)} \right) = q^{c_1(1)} \left( e_2^{c_2(2) - c_2(1)} \cdots e_n^{c_n(2) - c_n(1)} \right)$$

Thus, $c_1(1) \equiv -t \mod(r)$, and

$$c_2 \equiv c_2(2) - c_2(1) \equiv c_2(2) + t$$

$$c_3 \equiv c_3(3) + t$$

$$\vdots$$

$$c_n \equiv c_n(1) + t$$

Thus, on each cycle $\sigma_d = (p_1 \cdots p_{\lambda_d})$ we have an equation

$$\lambda_d t \equiv \sum_{i=1}^{\lambda_d} (c_{p_i} - c_{\sigma_d^{-1}(p_i)}) = \sum_{i=1}^{\lambda_d} (c_{p_i} - c_{\sigma_d^{-1}(p_i)}) \equiv 0$$

from which it follows that $r$ divides $\lambda_d t$, hence, defining $l_d = t \lambda_d/r$, then $t = l_d r / \lambda_d$ for all $d = 1, \ldots, k$. Thus,

$$t = (l_i/m_i) r / g,$$

where we define $m_i = \lambda_i / g$ for $g = \gcd(\lambda_1, \ldots, \lambda_k, r)$. Since $t$ is independent of $i$, $l = l_i/m_i$ is as well. Now

$$l = t(g/r) \leq \frac{r - 1}{r} g < g.$$

Since $l$ is an integer, $l \leq g - 1 = \gcd(\lambda_1, \ldots, \lambda_k, r) - 1$. This completes part (1).

For (2), let $g = \gcd(\lambda_1, \ldots, \lambda_k, r)$. Suppose $R$ is obtained from the monomial $m$ and is given as in the statement of the Theorem. By Proposition 49, for each $d = 1, \ldots, k$ we have

$$\sigma_d \cdot (\Omega_{\sigma_d}^{lr/g}) = X_{\sigma_d}^{lr/g} \Omega_{\sigma_d}^{lr/g},$$

and

$$X_{\sigma_1} \cdots X_{\sigma_k} = e_1 \cdots e_n = q,$$
it follows that
\[ \sigma \cdot (R) = \sigma \cdot (Y_{\sigma_1} \cdots Y_{\sigma_k}) \]
\[ = (\sigma_1 \cdot Y_{\sigma_1}) \cdots (\sigma_k \cdot Y_{\sigma_k}) \]
\[ = (X_{\sigma_1} Y_{\sigma_1}) \cdots (X_{\sigma_k} Y_{\sigma_k}) \]
\[ = (X_{\sigma_1} \cdots X_{\sigma_k})^{-lr/g} (Y_{\sigma_1} \cdots Y_{\sigma_k}) \]
\[ = q^{-lr/g} R. \]

Now suppose conversely that \( \sigma \cdot m = q^{-lr/g} m \). From Definition 51 we get
\[ q^{-lr/g} m = \sigma \cdot m \]
\[ = \sigma \cdot \left( \left( X_{\sigma_2} \cdots X_{\sigma_k} \right) \cdot R \right) \]
\[ = \left( X_{\sigma_2} \cdots X_{\sigma_k} \right) (\sigma \cdot R), \]

where we define
\[ R = e_{c_2}^2 \cdots e_{c_n}^n \]
for \( c_i \in \{0, \ldots, r - 1\} \). With \( \sigma_d = (p_1 \cdots p_{\lambda_d}) \), it follows that
\[ q^{-lr/g} R = \sigma \cdot (R) \]
\[ = \sigma \cdot \left( e_{c_2}^2 \cdots e_{c_n}^n \right) \]
\[ = e_{c_{\sigma^{-1}(1)}}^{c_{\sigma^{-1}(1)}} \cdots e_{c_{\sigma^{-1}(n)}}^{c_{\sigma^{-1}(n)}} \]

whence \( c_{\sigma^{-1}(1)} \equiv -lr/g \mod (r) \) and corresponding to each cycle \( \sigma_d = (p_1 \cdots p_{\lambda_d}) \) we have
\[ e_{c_{p_1}^{c_{p_1} - lr/g}} e_{c_{p_2}^{c_{p_2} - lr/g}} \cdots e_{c_{p_d}^{c_{p_d} - lr/g}} = \sigma_d \cdot \left( e_{p_1}^{c_{p_1} p_1} \cdots e_{p_d}^{c_{p_d} p_d} \right) \]
\[ = e_{p_1}^{c_{p_1} p_1} e_{p_2}^{c_{p_2} p_2} \cdots e_{p_d}^{c_{p_d} p_d - 1} \]

and therefore
\[ c_{p_2} \equiv c_{p_1} + lr/g \]
\[ c_{p_3} \equiv c_{p_1} + 2lr/g \]
\[ \vdots \]
\[ c_{p_{\lambda_d}} \equiv c_{p_1} + (\lambda_d - 1)lr/g. \]

That is,
\[ R = R_{\sigma_1} \cdots R_{\sigma_k} \]

where
\[ R_{\sigma_d} = e_{p_1}^{c_{p_1} \Omega_{\sigma_d}^{lr/g}} \]

for \( d = 2 \ldots, k \), and with \( R_{\sigma_1} = \Omega_{\sigma_1}^{lr/g} \).

We are now able to parametrize explicitly the weight monomials in the translation algebra \( C_r^n \) of a given permutation \( \sigma \). This will lead to Theorem 58, which computes the character of the simplex module \( \text{Pl}(\Delta^n_r) \).
Corollary 57. In the notation as above, the weight monomials of a permutation \( \sigma = \sigma_1 \cdots \sigma_k \in S_n \), consist precisely of the set
\[
\left\{ (X_{\sigma_2}^{j_2} \cdots X_{\sigma_k}^{j_k}) \cdot (\Omega_{\sigma_1} \cdots \Omega_{\sigma_k})^{l/r} / g : j_2, \ldots, j_k = 0, \ldots, r - 1 \text{ and } l = 0, \ldots, g - 1 \right\},
\]
where \( g = \gcd(\lambda_1, \ldots, \lambda_k, r) \), and where \( \Omega_\sigma \) and \( X_\sigma \) are given in Definitions 48 and 51.

We are now prepared for our main result for the translation algebra.

Theorem 58. Let \( r, n \geq 1 \). Let \( \sigma = \sigma_1 \cdots \sigma_k \in S_n \) be a permutation of cycle type \( \lambda_1, \ldots, \lambda_k \). Set \( g = \gcd(\lambda_1, \ldots, \lambda_k, r) \). The character value of \( \sigma \) acting on the translation algebra is then
\[
\text{tr}(\sigma) = \delta_{g,1} r^{k-1}.
\]

Proof. With \( \sigma \) as in the statement of the theorem, if \( g = 1 \) then the parametrization of Corollary 57 gives \( r^{k-1} \) weight monomials all of weight 1 and hence \( \text{tr}(\sigma) = r^{k-1} \). If \( g \geq 2 \) then by Theorem 56 and Corollary 57 there are \( r^{k-1} \) weight monomials for each of the \( g^{th} \) roots of unity \( e^{2\pi i / g} \) for \( l = 0, \ldots, g - 1 \). In this case \( \text{tr}(\sigma) = 0 \). \( \square \)

3. Result II

3.1. Diagonalizing the Translation Algebra. In what follows, we diagonalize the \( n \) commuting translation operators \( x \mapsto e_i x \) for \( i = 1, \ldots, n \). The eigenbasis is then completely labeled by the \( n \)-tuples of eigenvalues
\[
\{ I \in (\mathbb{Z}/r)^n : \sum_j i_j \equiv 1 \mod r \}.
\]
The elements \( e_I \) form a partition of unity of \( C_r^n \). Recall that the geometric action of multiplication by \( e_i \) is, for the generic case of the standard nondegenerate q-plate \( \{ \pi_0 \} = \{ 1_{a_1}2_{a_2} \cdots n_{a_n} \} \), where \( a_1 + \cdots + a_n = r \gg n \), translation of the position \( (a_1, \ldots, a_n) \) in the root direction
\[
(0, \ldots, 1, -1, 0, \ldots, 0),
\]
where +1 is on the position \( i - 1 \).

In what follows we present a basis for \( \text{Pl}(\Delta_r^n) \) which is invariant under translation, and which has the property that it is permuted set-theoretically by \( S_n \), i.e. we obtain a monomorphism \( S_n \hookrightarrow S_{r^n-1} \).

Definition 59. Let \( I = I_r^n \) denote the set of all \( (i_1, \ldots, i_n) \in \{ 0, \ldots, r-1 \}^n \) with \( \sum_{j=0}^{r-1} i_j \equiv 1 \mod r \).

For each \( n \)-tuple \( I = (i_1, \ldots, i_n) \in I_r^n \), let
\[
e_I = \frac{1}{r^n} \prod_{j=1}^{n} \left( \sum_{k=0}^{r-1} (q^{-i_j} e_j)^k \right)
\]
where \( k \) is from the monomial, since we get \( e_j^k \) by expanding.

Remark that the condition \( \sum_k i_k \equiv 1 \) is dual to \( e_1 \cdots e_n = q \), and when both conditions are applied, the remaining variables are changed by a Fourier transform.

The monomials \( e_J^k \) are mapped by the clock transform onto the q-plate basis, which in turn is related by an explicit invertible matrix to the standard plate basis.
Theorem 60. The elements \( \epsilon_I \) form a partition of unity into \( r^{n-1} \) nonzero idempotents, that is,

\[
\epsilon_I \epsilon_{I'} = \delta_{I,I'} \epsilon_I
\]

\[
\sum_{I \in \mathcal{I}} \epsilon_I = 1,
\]

which are simultaneous eigenvectors for multiplication by \( e_i \), that is,

\[
e_j \epsilon_I = q^{ij} \epsilon_I,
\]

and so

\[(e_1 \cdots e_n) \epsilon_I = q \epsilon_I.
\]

Proof. We have

\[
\epsilon_I = \frac{1}{r^n} \sum q^{-(i_1,\ldots,i_n) \cdot (k_1,\ldots,k_n)} \epsilon_{k_1^n}.
\]

Note that \( i_1 \equiv 1 - (i_2 + i_3 + \cdots + i_n) \), and \( \epsilon_{k_1^n} = q^{k_1} (e_2 \cdots e_n)^{-k_1} \), so

\[
\epsilon_I = \frac{1}{r^n} \sum q^{k_1 - (i_2,\ldots,i_n) \cdot (k_2,\ldots,k_n) - i_1 k_1} \epsilon_{k_2^n} \epsilon_{k_3^n} \cdots \epsilon_{k_n^n}
\]

\[
= \frac{1}{r^n} \sum q^{k_1 - (i_2,\ldots,i_n) \cdot (k_2,\ldots,k_n) + i_2 k_1 + \cdots + i_n k_1} \epsilon_{k_2^n} \epsilon_{k_3^n} \cdots \epsilon_{k_n^n}
\]

\[
= \frac{1}{r^n} \sum q^{k_1 - (i_2,\ldots,i_n) \cdot (k_2 - k_1,\ldots,k_n - k_1)} \epsilon_{k_2^n} \epsilon_{k_3^n} \cdots \epsilon_{k_n^n}
\]

\[
= \frac{1}{r^{n-1}} \sum q^{-(i_2,\ldots,i_n) \cdot (l_2,\ldots,l_n)} \epsilon_{l_2} \cdots \epsilon_{l_n},
\]

since we summed \( r \) identical terms. Let \( I' = (i_2,\ldots,i_n) \), which determines \( I \) since \( \sum_k k_k \equiv 1 \mod r \). Thus, the change of basis from monomials \( e^{I'} = e_{l_2}^{j_2} \cdots e_{l_n}^{j_n} \) to the idempotents \( \epsilon_{I'} \) is a discrete Fourier transform on \( (\mathbb{Z}/r)^{n-1} \).

For orthogonality, for each \( l = 1,\ldots,n \) we have the standard computation

\[
\left( \frac{1}{r} \sum_{j_1=0}^{r-1} q^{-m_1 j_1} \epsilon_l^{j_1} \right) \left( \frac{1}{r} \sum_{j_2=0}^{r-1} q^{-m_2 j_2} \epsilon_l^{j_2} \right) = \frac{1}{r^2} \sum_{j_1=0}^{r-1} \left( \sum_{j_2=0}^{r-1} q^{-(m_1-m_2) j_1} q^{-m_2 j_2} \epsilon_l^{j_1} \epsilon_l^{j_2} \right)
\]

\[
= \frac{1}{r^2} \sum_{j_1=0}^{r-1} \left( q^{-(m_1-m_2) j_1} \right) \left( \sum_{j_2=0}^{r-1} q^{-m_2 j_2} \epsilon_l^{j_2} \right)
\]

\[
= \delta_{m_1,m_2} \frac{1}{r} \sum_{k=0}^{r-1} q^{-m_2 j_2} \epsilon_l^{j_2},
\]

and by extension the elements \( \epsilon_I \) satisfy \( \epsilon_I \epsilon_{I'} = \delta_{I,I'} \epsilon_I \), where in the equation we take \( j_k = j_1 + j_2 \mod r \).

Finally, we have

\[
\sum_{I \in \mathcal{I}} \epsilon_I = 1,
\]

since on the \( l \)th factor, \( l = 2,\ldots,n \) we can sum and get

\[
\frac{1}{r} \sum_{m=0}^{r-1} \left( \sum_{j=0}^{r-1} q^{-m j} \epsilon_l^j \right) = \frac{1}{r} \sum_{j=0}^{r-1} \left( \sum_{m=0}^{r-1} q^{m j} \right) \epsilon_l^j = \sum_{j=0}^{r-1} \delta_{j,0} \epsilon_l^j = 1.
\]
Consequently we have \( r^{n-1} \) mutually orthogonal idempotents in an \( r^{n-1} \)-dimensional space, so in particular the \( \epsilon_I \) are linearly independent.

This suggests a second proof of the character formula for the action of \( S_n \) on the translation algebra \( C^n_r \).

**Theorem 61.** The value of the character of a permutation \( \sigma = \sigma_1 \cdots \sigma_k \in S_n \) of cycle type \( \lambda_1, \ldots, \lambda_k \), acting on \( C^n_r \), equals the number \( r^{k-1} \) of solutions to the Diophantine equation

\[
\{(c_1, \ldots, c_k) \in (\mathbb{Z}/r)^k : \sum c_j \lambda_j \equiv 1 \ mod \ r\}
\]

if \( \gcd(\lambda_1, \ldots, \lambda_k, r) = 1 \) and 0 otherwise.

**Proof.** Fix a permutation \( \sigma = \sigma_1 \cdots \sigma_k \in S_n \) with cycle lengths \( \lambda_1, \ldots, \lambda_k \). From Theorem 60 we see that the idempotents \( \epsilon_I \) are in \( S_n \)-equivariant bijection with the set

\[
I^n_r = \{ x \in (\mathbb{Z}/r)^n : \sum x_i \equiv 1 \ mod \ r \},
\]

so to compute the trace of \( \sigma \) on \( I^n_r \) we need only count elements of \( I^n_r \) which are fixed by \( \sigma \).

An element \( I = (i_1, \ldots, i_n) \in I^n_r \) satisfies \( \sigma \cdot I = I \) if and only if \( I \) takes a constant value \( c_j \) over each cycle \( \sigma_j \), so

\[
\sum_{j=1}^n i_j = \sum_{j=1}^k c_j \lambda_j \equiv 1 \ mod \ r.
\]

Hence,

\[
\gcd(\lambda_1, \ldots, \lambda_k) = 1 \ mod \ r
\]

or equivalently,

\[
\gcd(\lambda_1, \ldots, \lambda_k, r) = 1
\]

which is precisely the condition given above. By Theorem 58 we have \( \text{tr}(\sigma) = r^{k-1} \), hence there are \( r^{k-1} \) solutions to the Diophantine equation

\[
\{(c_1, \ldots, c_k) \in (\mathbb{Z}/r)^k : \sum c_j \lambda_j \equiv 1 \ mod \ r\}.
\]

Now, if \( \gcd(\lambda_1, \ldots, \lambda_k, r) > 1 \) then there do not exist \( c_1, \ldots, c_k \) such that

\[
\sum c_j \lambda_j \equiv 1 \ mod \ r,
\]

that is, there is no element \( I \in I^n_r \) which has constant values on and is therefore fixed by \( \sigma \), hence in this case \( \text{tr}(\sigma) = 0 \).

**Remark 62.** The necessity of the condition \( \sum_{j=0}^{r-1} i_j \equiv 1 \ mod \ r \) can be seen directly from the relation \( e_1 \cdots e_n = q \), since we have

\[
q \epsilon_I = (e_1 \cdots e_n) \epsilon_I = q^{i_1+\cdots+i_n} \epsilon_I,
\]

for each \( I = (i_1, \ldots, i_n) \).

The character formula can also be obtained directly, independently of Theorem 58, as follows.

**Theorem 63.** Let \( a_1, \ldots, a_k \in \mathbb{Z}/r \). The number of solutions of the equation

\[
\sum_{i=1}^k a_i x_i \equiv 1 \ mod \ r
\]

is equal to \( r^{k-1} \) if \( \gcd(a_1, \ldots, a_k, n) = 1 \) and is 0 otherwise.
4. An Isomorphism, and Characters

4.1. Plate Localization: Geometric Proof of Equivariant Worpsitzky Identity. In what follows, we use a geometric argument to prove the existence of a graded Worpsitzky isomorphism between the simplex plate module $P(\Delta^n)$ and its localization onto unit hypersimplex modules tensored with the polynomial ring in $n$ variables to describe the position of the hypersimplex in the integer lattice of $\Delta^n$. In [39], Ocneanu invents new combinatorial structures which solve the extremely difficult problem of proving the isomorphism explicitly, in the standard plate basis.

First we present a convenient use of polynomials to encode translations. This will allow explicit character computations for plates localized to hypersimplices.

Definition 65. Let $\pi$ be a plate, viewed as a characteristic function. Let $v \in \mathbb{N}^n$. Denote by $\pi_v$ the characteristic function of the plate $\pi$ translated by $v$,

$$\pi_v(x) = \pi(x - v) \in \{0, 1\}.$$
Proposition 66. Let $r - a \geq 1$. The two $S_n$-modules generated by
\[ \{ \pi_v : \pi \in B_{a,b}, \ v_1 + \cdots + v_n = r - a \} \]
and
\[ \{ g(y_1, \ldots, y_n) \otimes \pi : \pi \in Pl(B_{a,b}), \ \deg(g) = r - a \} \]
are canonically isomorphic, where $g$ ranges over all monomials of total degree $r - a$, and where $S_n$ acts from the right as $y_i \mapsto y_{\sigma(i)}$.

Proof. The isomorphism is given explicitly by identifying translation by an integer vector $(v_1, \ldots, v_n)$ with multiplication by the monomial $y_1^{v_1} \cdots y_n^{v_n}$. Equivariance follows because
\[ \sigma(\pi_v) = \sigma(\pi) \sigma(v), \]
where $\sigma(v_1, \ldots, v_n) = (v_{\sigma(1)}, \ldots, v_{\sigma(n)})$, and
\[ \sigma(y_1^{v_1} \cdots y_n^{v_n} \otimes \pi) = y_{\sigma(1)}^{v_1} \cdots y_{\sigma(n)}^{v_n} \otimes \sigma(\pi) = y_1^{v_{\sigma(1)}} \cdots y_n^{v_{\sigma(n)}} \otimes \sigma(\pi), \]
which has exponent vector $(v_{\sigma(1)}, \ldots, v_{\sigma(n)})$, as above. □

Figure 3. Polynomials encode position

Example 67. Figure 3 illustrates the $n = 3$ case for $Pl(\Delta_3^3)$. Here there are two kinds of intersections of $\Delta_3^3$ with unit cubes, up to translation, namely the “down triangle” $B_{1,2}$ and the “up triangle” $B_{2,1}$.

The hypersimplex plate $[[123_1]] \in Pl(B_{1,2})$ is the convex hull of $(1,0,0), (0,1,0), (0,0,1)$, and $[[123_2]] \in Pl(B_{2,1})$ is the convex hull of $(1,1,0), (0,1,1), (1,0,1)$.

In Figure 3, the hypersimplex plates $[[123_1]]$ and $[[123_2]]$ are translated by the vectors $(1,1,0)$ and respectively $(0,0,1)$. The data is stored using the monomials, or symmetric tensors, $(e_1 e_2)$ and $e_3$, where $\{e_1, e_2, e_3\}$ is the standard basis for $\mathbb{C}^3$. 
Theorem 68. There exists an isomorphism of $S_n$ modules

$$\text{Pl}(\Delta^n_r) \simeq \bigoplus_{a=1}^{n-1} \text{Sym}^{r-a}(\mathbb{C}^n) \otimes \text{Pl}(B_{a,n-a}),$$

where by convention $\text{Sym}^k(\mathbb{C}^n) = 0$ if $k < 0$.

Proof. A plate $\pi \in \Delta^n_r$ decomposes uniquely into the sum of its nonempty intersections with the unit hypercubes in $[0,r]^n$. Each nonempty intersection

$$\pi \cap \left( \times_{j=1}^{n}[i_j,i_j+1] \right)$$

is a plate in some (translated) hypersimplex. With respect to the $S_n$ action, translation of a plate $\pi$ by some $(i_1, \ldots, i_n) \in \mathbb{N}^n$ is encoded combinatorially in the tensor product $e_1^{i_1} \cdots e_n^{i_n} \otimes \pi$, as in Figure 3, so it follows from the formula

$$\dim(\text{Sym}^{r-a}(\mathbb{C}^n)) = \binom{n+r-a-1}{n-1}$$

that the classical Worpitzky identity holds. Together with the classical Worpitzky identity that

$$\text{dim}(\text{Sym}^{r-a}(\mathbb{C}^n)) \cdot \text{dim}(\text{Pl}(B_{a,n-a})) = \sum_{a=1}^{n-1} \binom{n+r-a-1}{n-1} E_{a-1,n-a-1},$$

and by dimensionality the injection above is an isomorphism and the result follows.

\[\square\]

4.2. Hypersimplex and Period Solid Character. In this section $q$ will be a formal variable.

For each $n$, let $\text{Pl}(\Delta^n)$ denote the sum of the plate modules of $(n-1)$-dimensional simplices in $n$ variables with all possible edge lengths $\{\text{Pl}(\Delta^n_j)q^j : j = 1, 2, \ldots\}$. Similarly, let $\text{Sym}^\bullet(\mathbb{C}^n)$ denote the set of finite linear combinations of elements of $\{\text{Sym}^j(\mathbb{C}^n)q^j : j = 0, 1, 2, \ldots\}$. Finally, let $\text{Pl}(B^n)$ be the set of all linear combinations of elements from $\{\text{Pl}(B_{j,n-j})q^j : j = 1, \ldots, n-1\}$.

Proposition 69. We have an isomorphism of graded $S_n$-modules,

$$\text{Pl}(\Delta^n) \simeq \text{Sym}^\bullet(\mathbb{C}^n) \otimes \text{Pl}(B^n).$$

Proof. Sum the formula in Theorem 68 over all $r \geq 1$.

The trace of a permutation $\sigma \in S_n$ on the period solid module $\text{Pl}(B^n)$ can be obtained as follows.

Corollary 70. If $\sigma$ has cycle type $\lambda_1, \ldots, \lambda_k$ with $g = \gcd(\lambda_1, \ldots, \lambda_k)$, then

$$\chi_{\text{Pl}(B^n)}(\sigma) = \sum_{a=1}^{n-1} \chi_{\text{Pl}(B_{a,b})}(\sigma) q^a = \frac{\chi_{\text{Pl}(\Delta^n)}(\sigma)}{\chi_{\text{Sym}}(\sigma)} = \frac{\sum_{j=1}^\infty \chi_{\text{Pl}(\Delta^n_j)}(\sigma) q^j}{\prod_{i=1}^k (1 - q^{\lambda_i})^{-1}}$$

$$= \left( \sum_{m: \gcd(m,g) = 1} m^{k-1} q^j \right) \left( \prod_{i=1}^k (1 - q^{\lambda_i}) \right).$$
Conjecture 71. For $\sigma$ as above, we have the following simplification.

$$\chi_{Pl(n)}(\sigma) = P_{g,k}(q)[\lambda_1/g]q^{\lambda_1} \cdots [\lambda_k/g]q^{\lambda_k},$$

where

$$[\lambda/g]q^\lambda = \frac{1 - q^\lambda}{1 - q^g} = \frac{1 - (q^g)^{\lambda/g}}{1 - q^g} = \sum_{j=0}^{(\lambda/g)/g} (q^g)^j,$$

and where $P_{k,n}(q)$ is a symmetric and unimodal polynomial, i.e. the coefficients are symmetric about and have a single peak at the middle value(s).

The generating function given above can be presented explicitly, as follows. Given a permutation $\sigma = \sigma_1 \cdots \sigma_k$ with cycle lengths $\lambda_1, \ldots, \lambda_k$, as usual, let $g = \gcd(\lambda_1, \ldots, \lambda_k)$. The generating series for the character value of $\sigma$ on $\Delta^n$ decomposes into a sum of series, as

$$\chi_{Pl(n)}(\sigma) = \sum_{m \mid \gcd(m,g)} m^{k-1}q^j$$

$$= \sum_{m=0}^{\infty} \left( \sum_{1 \leq j \leq n-1 \mid \gcd(j,g)=1} (mg+j)^{k-1}q^{mg+j} \right)$$

$$= \sum_{1 \leq j \leq n-1 \mid \gcd(j,g)=1} \left( \sum_{m=0}^{\infty} (mg+j)^{k-1}q^{mg+j} \right)$$

Example 72. Suppose $\sigma = \sigma_1 \sigma_2 \sigma_3 \sigma_4 \in S_{12}$ has cycle type $(3, 3, 3)$. Then, by explicit computation we have

$$\chi_{Pl(\Delta^n)}(\sigma) = \sum_{r=0}^{\infty} \sum_{j \in \{1, 2\}} (3r+j)^{4-1}q^{3r+j}$$

$$= \sum_{j \in \{1, 2\}} \left( \sum_{r=0}^{\infty} (3r+j)^{4-1}q^{3r+j} \right)$$

$$= q(8 + 3q^3 + 60q^6 + q^9) + q^2(1 + 60q^3 + 93q^6 + 8q^9)$$

$$= \frac{q + 8q^2 + 60q^4 + 93q^5 + 93q^7 + 60q^8 + 8q^{10} + q^{11}}{(1 - q^3)^4}$$

In forthcoming work we shall study the coefficients of the polynomials $P_{k,n}$ in terms of volumes of generalized hypersimplices, linear dimension of their corresponding plate modules and permutation statistics, where in the latter the question is to show that the coefficients count elements in some group.

We recall from [39] and [40] an important result of Ocneanu, in which he solved the problem of determining the conditions which characterize $Pl(B_{a,b})$ as a submodule of $Pl(\Delta_{a+b})$. Namely, the basis for the hypersimplex module $Pl(B_{a,n-a})$ is determined by a condition on lump size and position, as follows.

Theorem 73. The hypersimplex module $Pl(B_{a,n-a})$, where $1 \leq a \leq n - 1$, is a submodule of the simplex module $Pl(\Delta_n^+)$, determined by the following condition on the lumps $(L)_p$, where $L = \{l_1, \ldots, l_m\}$ has position $p \geq 1$.

Let $\pi \in Pl(\Delta_n^+)$ be in the standard basis, so that $1$ is in the first lump, where $1 \leq a \leq n - 1$. Then, $\pi$ is a basis plate for $Pl(B_{a,n-a})$ if and only if for any lump $(L)_p$ of $\pi$ we have $p < m = |L|$, i.e. the position of $L$ is strictly less than the number of variables in $L$.  

The theorem expresses the fact that for the projection onto the space with lumped coordinates \( x_L = \sum_{l \in L} x_l \), the preimage of a plate in the lumped coordinates intersects the hypersimplex in a nontrivial way, with the preimage of the plate center intersecting the interior of the hypersimplex.

We conclude with two examples.

**Example 74.** Let \( \sigma = (12)(34)(5678) \). The q-plates which contribute to the trace of \( \sigma \) on \( \text{Pl}(B_{2,6}) \) are

<table>
<thead>
<tr>
<th>q-plate</th>
<th>eigenvalue</th>
</tr>
</thead>
<tbody>
<tr>
<td>{12345678_2}</td>
<td>1</td>
</tr>
<tr>
<td>{1234_15678_1}</td>
<td>1</td>
</tr>
<tr>
<td>{123456_178_1}</td>
<td>1</td>
</tr>
<tr>
<td>{12_1345678_1}</td>
<td>1</td>
</tr>
<tr>
<td>{1357_12468_1}</td>
<td>-1</td>
</tr>
<tr>
<td>{1368_12457_1}</td>
<td>-1</td>
</tr>
<tr>
<td>{1457_12368_1}</td>
<td>-1</td>
</tr>
<tr>
<td>{1468_12357_1}</td>
<td>-1</td>
</tr>
</tbody>
</table>

so the trace is 0.

**Example 75.** Let \( \sigma = (1234)(56)(78) \). The corresponding q-polynomial is

\[
[4]_q[2]_q[2]_q = 1 + 3q + 4q^2 + 4q^3 + 3q^4 + q^5
\]

and \( A_2(q) = 1 + q \). Then for example

\[
\chi(B_{3,5})(\sigma) = \text{coef}_q(qA_2(q)[4]_q[2]_q[2]_q) = 7
\]

The 7 fixed q-plates in \( \text{Pl}(B_{3,5}) \) are explicitly

\[
\{12345678_3\}, \{1234_25678_1\}, \{1234_15678_2\}, \{1234_1567_82_1\}, \\
\{123478_156_1\}, \{1234_156_78_1\}, \{1234_178_156_1\}.
\]
CHAPTER 4

Independent Work in Invariant Theory

1. A Symmetric Form of the Segre Cubic

Here we reproduce the results of a side project which is unrelated to the main work which we intend to publish at a later date.

1.1. Two Classical Varieties from Representation Theory of the Symmetric Group. Let us first fix terminology. We shall write $V(\lambda_1,\ldots,\lambda_k)$ for the irreducible representation of the symmetric group $S_n$ which is labeled by the partition $\lambda = (\lambda_1,\ldots,\lambda_k)$ of $n$. Define $v(t) = (1, t, t^2) \in \mathbb{C}^3$. Consider the compound determinant $X(ab,cd,ef) := \det(v(t_a) \times v(t_b), v(t_c) \times v(t_d), v(t_e) \times v(t_f))$, where $v(t_i) \times v(t_j)$ is the cross product $v(t_i) \times v(t_j) = \det(i \ j \ k \ 1 \ t_i \ t_i^2 \ 1 \ t_j \ t_j^2) = ((t_j - t_i) \cdot t_i t_j, -(t_j - t_i) \cdot (t_i + t_j), (t_j - t_i))$.

Lemma 76. The polynomial $X_{(12,34,56)}$ is invariant up to sign under the order $(2^3)(3!) = 48$ wreath product $\text{Wr}(S_2, S_3)$ as a subgroup of $S_6$. The $S_6$-orbit of $X_{(12,34,56)}$ has 15 elements, up to sign.

Proof. The wreath product $\text{Wr}(S_2, S_3)$ can be presented explicitly in terms of generators as, for example, $\langle (12), (34), (56), (13)(24), (35)(46) \rangle$.

By linearity of the determinant, we have $X_{(12,34,56)} = g_{(12,34,56)} \cdot C_{(12,34,56)}$, where $g_{(12,34,56)} = (t_2 - t_1)(t_4 - t_3)(t_6 - t_5)$ and $C_{(12,34,56)} = \det \begin{bmatrix} 1 & 1 & 1 \\ t_1 + t_2 & t_3 + t_4 & t_5 + t_6 \\ t_1 t_2 & t_3 t_4 & t_5 t_6 \end{bmatrix}$.

It follows by the orbit-stabilizer theorem that the $S_6$-orbit $X$ of $X_{(12,34,56)}$ has $6!/48 = 15$ elements, up to sign.

Remark that as elements in a vector space the $X_i$ are linearly dependent.

Definition 77. Let $X_G = \{ X_{(12,34,56)}, X_{(16,23,45)}, X_{(14,26,35)}, X_{(15,24,36)}, X_{(13,25,46)} \} \subset X$

In what follows, we prove that $X_G$ is a basis which behaves optimally with respect to permutation of the coordinate labels. We therefore call the $X_G$ good basis due to the following simple relations.
Proposition 78. The set
\[ \{X_{(12,34,56)}, X_{(16,23,45)}, X_{(14,26,35)}, X_{(15,24,36)}, X_{(13,25,46)} \} \]
spans the orbit space \( \mathcal{X} \).

Proof. By explicit computation, expanding each \( X_I \) as a polynomial in the minors \( \Delta_J \), we have the following list of linear relations.

\[
\begin{align*}
X_{(12,35,46)} &= -X_{(15,24,36)} + X_{(16,23,45)} \\
X_{(12,36,45)} &= X_{(13,25,46)} - X_{(14,26,35)} \\
X_{(13,24,56)} &= X_{(14,26,35)} + X_{(16,23,45)} \\
X_{(13,26,45)} &= -X_{(12,34,56)} - X_{(15,24,36)} \\
X_{(14,23,56)} &= -X_{(13,25,46)} - X_{(15,24,36)} \\
X_{(14,25,36)} &= -X_{(12,34,56)} - X_{(16,23,45)} \\
X_{(15,23,46)} &= X_{(12,34,56)} - X_{(14,26,35)} \\
X_{(15,26,34)} &= X_{(13,25,46)} + X_{(16,23,45)} \\
X_{(16,24,35)} &= X_{(12,34,56)} - X_{(13,25,46)} \\
X_{(16,25,34)} &= -X_{(14,26,35)} - X_{(15,24,36)}
\end{align*}
\]

Proposition 79. The set \( \mathcal{X}_G \) is linearly independent, and its span is the irreducible \( S_6 \)-module \( V_{(2,2,2)} \).

Proof. Let \( \Delta_{i,j,k} = \det(v(t_i), v(t_j), v(t_k)) \).

It follows from standard theory representation theory of the symmetric group, see for example [18], that the set
\[ \{ \Delta_{123}\Delta_{456}, \Delta_{124}\Delta_{356}, \Delta_{125}\Delta_{346}, \Delta_{134}\Delta_{256}, \Delta_{135}\Delta_{246} \} \]
is a basis for \( V_{(2,2,2)} \). Expanding by minors, for example
\[
X_{(12,34,56)} = \det \begin{bmatrix} \Delta_{134} & \Delta_{156} \\ \Delta_{234} & \Delta_{256} \end{bmatrix},
\]
and applying the so-called straightening relation [18]
\[
\Delta_{abc}\Delta_{ijk} = (\Delta_{ajc}\Delta_{ibk} - \Delta_{aic}\Delta_{jbk} - \Delta_{aij}\Delta_{bck} - \Delta_{ajb}\Delta_{ick} + \Delta_{aib}\Delta_{jck})
\]
expresses the usual basis of \( V_{(2,2,2)} \) in terms of our good basis \( \mathcal{X}_G \). By explicit computation we have

\[
\begin{align*}
\Delta_{123}\Delta_{456} &= -\frac{1}{2} (X_{12,34,56} + X_{13,25,46} - X_{14,26,35} + X_{15,24,36} - X_{16,23,45}) \\
\Delta_{124}\Delta_{356} &= \frac{1}{2} (X_{12,34,56} - X_{13,25,46} + X_{14,26,35} - X_{15,24,36} + X_{16,23,45}) \\
\Delta_{125}\Delta_{346} &= \frac{1}{2} (X_{12,34,56} + X_{13,25,46} - X_{14,26,35} - X_{15,24,36} + X_{16,23,45}) \\
\Delta_{134}\Delta_{256} &= \frac{1}{2} (X_{12,34,56} + X_{13,25,46} + X_{14,26,35} + X_{15,24,36} + X_{16,23,45}) \\
\Delta_{135}\Delta_{246} &= -\frac{1}{2} (X_{12,34,56} - X_{13,25,46} - X_{14,26,35} - X_{15,24,36} - X_{16,23,45}).
\end{align*}
\]
Recall that
\[ X_{12,34,56} = g_{12,34,56} C_{12,34,56} = \left( \prod_{i=1}^{3} (t_{i+1} - t_i) \right) \det \begin{bmatrix} 1 & 1 & 1 \\ t_1 + t_2 & t_3 + t_4 & t_5 + t_6 \\ t_1 t_2 & t_3 t_4 & t_5 t_6 \end{bmatrix}, \]
in the notation from Lemma 76.

For an overview of character theory for the symmetric group used in Proposition 80 below, see [18].

**Proposition 80.** The linear spans of the $S_6$ orbits of $C_{12,34,56}$ and of $g_{12,34,56}$ are both isomorphic to the $S_6$ module $V_{(3,3)}$.

**Proof.** It is well known, see again [18], that the polynomials $g_I$ span the irreducible $S_6$ representation $V_{(3,3)}$.

One computes explicitly the matrix representation with respect to the bases respectively
\[ \{g_{12,34,56}, g_{16,23,45}, g_{14,26,35}, g_{15,24,36}, g_{13,25,46}\} \]
and
\[ \{C_{12,34,56}, C_{16,23,45}, C_{14,26,35}, C_{15,24,36}, C_{13,25,46}\}, \]
to obtain the corresponding values for the character of $V_{(3,3)}$ on conjugacy class representatives. □

For example, in the “C” basis the permutation $(12)(34)(56)$ has the matrix expression
\[ (12)(34)(56) = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \]
and thus has trace $-3$.

The relations for the polynomials $C_I$, which are the same up to some signs as those for the $X_I$, are as follows.

\[
\begin{align*}
C_{(12,35,46)} &= C_{(15,24,36)} - C_{(16,23,45)} \\
C_{(12,36,45)} &= C_{(13,25,46)} - C_{(14,26,35)} \\
C_{(13,26,45)} &= C_{(12,34,56)} + C_{(15,24,36)} \\
C_{(14,23,56)} &= -C_{(13,25,46)} - C_{(15,24,36)} \\
C_{(14,25,36)} &= C_{(12,34,56)} + C_{(16,23,45)} \\
C_{(15,23,46)} &= C_{(14,26,35)} - C_{(12,34,56)} \\
C_{(15,26,34)} &= -C_{(13,25,46)} - C_{(16,23,45)} \\
C_{(16,24,35)} &= C_{(13,25,46)} - C_{(12,34,56)} \\
C_{(16,25,34)} &= -C_{(14,26,35)} - C_{(15,24,36)} 
\end{align*}
\]

Note that under cyclic permutation of $t_1, \ldots, t_6$ the good basis decomposes (up to sign) into orbits of sizes 2 and 3:
\[ \{X_{12,34,56}, X_{16,23,45}\} \cup \{X_{14,26,35}, X_{15,24,36}, X_{13,25,46}\}. \]
1.2. A Quartic Relation for the Usual Determinantal Basis of $V_{(2,2,2)}$. In order to determine how many relations to expect between the polynomials $C_I$, a computer computation shows that the Jacobian of the map $\mathbb{C}^6 \to \mathbb{C}^5$ given by

$$(t_1, \ldots, t_6) \mapsto \{X_{12,34,56}, X_{16,23,45}, X_{14,26,35}, X_{15,24,36}, X_{13,25,46}\}$$

has rank 4. We therefore expect one algebraic relation to hold among the elements of the good basis. Indeed, though the computation was too intensive for our laptop, with Oeding’s help [41] Grobner basis techniques in Macaulay2 implemented on a computing cluster yielded the following homogeneous degree 4 polynomial relation.

**Theorem 81.** The polynomials

\[
\begin{align*}
    d_1 &= \Delta_{123}\Delta_{456} \\
    d_2 &= \Delta_{124}\Delta_{356} \\
    d_3 &= \Delta_{125}\Delta_{346} \\
    d_4 &= \Delta_{134}\Delta_{256} \\
    d_5 &= \Delta_{135}\Delta_{246},
\end{align*}
\]

satisfy

$$(d_1(d_1 - d_2 + d_3 + d_4 - d_5) + d_2d_5 + d_3d_4)^2 - 4d_2d_3d_4d_5 = 0.$$  

**Proof.** An explicit computation using any CAS shows that the expression vanishes.  

1.3. A Relation for the Irreducible Cubics $C_{ij,rs,uv}$. Recall that

$$C_{ab,cd,ef} = \det \begin{bmatrix} 1 & 1 & 1 \\ t_1 + t_2 & t_3 + t_4 & t_5 + t_6 \\ t_1t_2 & t_3t_4 & t_5t_6 \end{bmatrix}.$$  

A computer-assisted computation, this time small enough to be run on a laptop, shows that the polynomial relation suggested by the rank computation of the Jacobian of the polynomial map

$$(t_1, \ldots, t_6) \mapsto \{C_{12,34,56}, C_{16,23,45}, C_{14,26,35}, C_{15,24,36}, C_{13,25,46}\},$$

can be expressed in terms of symmetric functions, after a sign twist. Namely, the five polynomials $C_I$ above satisfy the following surprising identity.

**Theorem 82.** Let

$$(f_1, f_2, f_3, f_4, f_5) = (C_{12,34,56}, -C_{16,23,45}, C_{14,26,35}, -C_{15,24,36}, C_{13,25,46}).$$

Then

$$5\sigma_1^3 - 18\sigma_1\sigma_2 + 27\sigma_3 = 0,$$

where $\sigma_i = \sigma_i(f_1, f_2, f_3, f_4, f_5)$ is the degree $i$ elementary symmetric function.

**Proof.** As above, an explicit computation using any CAS shows that the expression vanishes.  

This shows that projective variety defined by the null locus of this polynomial is additionally invariant under permutations of the 5 functions $C_I$.

In personal correspondence, Dolgachev [14] directed our attention to [13], where the polynomials $C_I$ used to describe the Segre cubic. It appears that our formulation in terms of elementary symmetric functions is new, however.
It would be interesting to determine whether this $S_5$ symmetry could be related to the constructions in [37] and its sequels. However, the variables there do not manifest the $S_5$ invariance which is apparent here. To the best of our knowledge, this symmetry is entirely new.
CHAPTER 5

Plans for Future Work and Experimental Results

1. Character Automorphism Group

1.1. Definition of Character Automorphism Group. In what follows, we introduce two automorphism groups which are built from the wreath product \( \text{Wr}(\mathbb{Z}/r, S_n) \) and the multiplicative group \((\mathbb{Z}/r)^\times\). We would like to mention for possible applications the Hecke algebra for the wreath product \( \text{Wr}(\mathbb{Z}/r, S_n) \) [3], due to Ariki-Koike.

We consider the action of a rectangular permutation \( \sigma = \sigma_1 \cdots \sigma_n \) of shape \((r, \ldots, r)\) on the hypersimplices \( B_{a,rn-a} \), for \( a = 1, \ldots, rn - 1 \), embedded in hyperplanes cutting transversely through the cube \([0, 1]^{rn}\).

For any integer \( s \) relatively prime to \( r \), \( \sigma^s \) is a permutation as the same shape as \( \sigma \) and therefore \( \sigma \) and \( \sigma^s \) are members of the same conjugacy class, hence we obtain

**Proposition 83.** For \( \sigma \in S_{rn} \) as above, whenever \( \gcd(r, s) = 1 \) there exists a permutation \( \mu_s \in S_{rn} \) such that

\[ \mu_s \sigma \mu_s^{-1} = \sigma^s. \]

The following is a basic result in group theory.

**Proposition 84.** Let \( \sigma \) be any cycle of order \( r \).

1. The group generated by powers of \( \sigma \) is isomorphic to the cyclic group \( \mathbb{Z}/r \).
2. The group generated by automorphisms of the form \( \sigma \mapsto \sigma^s \), for each \( s \) relatively prime to the order \( r \) of \( \sigma \), is isomorphic to the multiplicative group \((\mathbb{Z}/r)^\times\) of integers modulo \( r \).

In light of the above Propositions, we build a new group \( G_\sigma \) which acts compatibly with the action of \( \sigma \). First we introduce some notation and gather some basic properties of the elements \( \alpha_{i,i+1} \) and \( \mu_s \) in what follows.

We postpone the proof of what follows to a future publication.

**Conjecture 85.** Let \( \sigma = \sigma_1 \cdots \sigma_n \in S_{rn} \) be a rectangular permutation of shape \((r, \ldots, r)\), as above.

1. For each \( 1 \leq i \leq n - 1 \) there exists an involution, which we call \( \alpha_{i,i+1} \in S_{rn} \), such that \( \alpha_{i,i+1} \sigma \alpha_{i,i+1}^{-1} = \sigma_{i+1} \).
2. For each \( s \) relatively prime to \( r \), there exists \( \mu_s \in S_{rn} \) such that \( \mu_s \sigma \mu_s^{-1} = \sigma^s \).
3. The generators \((1, 2), \ldots, (n-1, n)\) of the symmetric group \( S_n \) act on \( \langle \sigma_1, \ldots, \sigma_n \rangle \simeq (\mathbb{Z}/r)^n \) as

\[ (i \ i + 1) \cdot \sigma_i = \alpha_{i,i+1} \sigma_i \alpha_{i,i+1}^{-1} = \sigma_{i+1} \]

is well-defined.
4. Denote by \((\mathbb{Z}/r)^\times\) the multiplicative group of integers modulo \( r \). Then \((\mathbb{Z}/r)^\times\) acts on \( \langle \sigma_1, \ldots, \sigma_n, \alpha_{1,2}, \ldots, \alpha_{n-1,n} \rangle \) as

\[ s \cdot \sigma_i := \mu_s \sigma_i \mu_s^{-1} = \sigma_i^s \]
and
\[ s \cdot \alpha_{i,i+1} = \mu_s \alpha_{i,i+1} \mu_s^{-1} = \alpha_{i,i+1}. \]

(5) The group \(G_\sigma\), defined in terms of its generators as
\[ G_\sigma = \langle \sigma_1, \ldots, \sigma_n, \alpha_{1,2}, \ldots, \alpha_{n-1,n} \rangle \cong \text{Wr}(\mathbb{Z}/r, S_n), \]
for \(\sigma_i\) and \(\alpha_{i,i+1}\) as above, is a (well-defined) subgroup of \(S_n\) for the action defined above, where \(\text{Wr}(\mathbb{Z}/r, S_n)\) is the wreath product of \(\mathbb{Z}/r\) with \(S_n\).

(6) The multiplicative group of integers modulo \(r\), \((\mathbb{Z}/r)^\times\), acts on \(G_\sigma\) by conjugation by \(s \cdot g = \mu_s g \mu_s^{-1}\). This action is well-defined, independent of choice of \(\mu_s\).

(7) The group \(G_\sigma\) acts on the eigenspaces of \(\sigma\) on any module of plates, where the group which fixes all eigenvectors up to a root of unity is the subgroup \(\langle \sigma \rangle\) generated by \(\sigma\) itself.

(8) The order of \(G_\sigma\) is \(r^n n!\).

There is another interesting order \(r^{n-1} n!\) subgroup
\[ G'_\sigma = \left( \langle \sigma_1 \sigma_2^{-1}, \ldots, \sigma_n \sigma_n^{-1} \rangle \rtimes S_n \right), \]
which is transverse to the group generated by \(\sigma\), \(\langle \sigma_1 \cdots \sigma_n = \sigma \rangle \cong \mathbb{Z}/r\).

**Example 86.** With \(\sigma = \sigma_1 \sigma_2 \sigma_3 = (123)(456)(789)\) we have
\[ G_\sigma = \langle (123), (456), (789), (14)(25)(36), (47)(58)(69) \rangle, \]
and
\[ G'_{\sigma} = \langle (123)(465), (456)(798), (14)(25)(36), (47)(58)(69) \rangle, \]
having chosen for example \(\alpha_{1,2} = (14)(25)(36)\) and \(\alpha_{2,3} = (47)(58)(69)\).

Moreover, action of the generator \(2 \in (\mathbb{Z}/r)^\times\) on \(G_\sigma\) can be given on its generators as
\[
\begin{align*}
(123) &\mapsto ((23)(56)(89))) (123) ((23)(56)(89))^{-1} = (132) = (123)^2 \\
(456) &\mapsto ((23)(56)(89)) (456) ((23)(56)(89))^{-1} = (465) = (456)^2 \\
(789) &\mapsto ((23)(56)(89)) (789) ((23)(56)(89))^{-1} = (798) = (789)^2.
\end{align*}
\]
\[
\begin{align*}
\end{align*}
\]
Additionally, as can be verified explicitly, the semi-direct products \(G_\sigma \rtimes (\mathbb{Z}/r)^\times\) and \(G'_{\sigma} \rtimes (\mathbb{Z}/r)^\times\) have respectively orders \(2 \cdot (3^3 3!) = 324\) and \(2 \cdot (3^2 3!) = 108\).

**Example 87.** Let \(\sigma = (123)(456)(789)(10\ 11\ 12)\) be a permutation of cycle type \(3,3,3,3\). Let \(r\) be a multiple of \(3\), so \(g = \gcd(3,3,3,3,3) = 3\). Then
\[ G_\sigma \cong \text{Wr}(\mathbb{Z}/3, S_4) \]
and
\[ G_\sigma \rtimes (\mathbb{Z}/3)^\times \]
are both subgroups of \(S_{12}\), with orders respectively \(3^3 3! = 162\) and \(2 \cdot 3^3 3! = 324\). Explicitly we have the presentation
\[
G_\sigma \rtimes (\mathbb{Z}/3)^\times = \langle (123), (456), (789), (10\ 11\ 12), (14)(25)(36), (47)(58)(69), (7\ 10)(8\ 11)(9\ 12) \rangle
\]
\[ \rtimes \langle (23)(56)(89)(11\ 12) \rangle, \]
where the last element \((23)(56)(89)(11\ 12)\) acts by conjugation on the first factor.
Now let us compute the generating function for the trace of action of $\sigma$ on the graded module $\Delta^{12}_r = \oplus_{r \geq 1} \Delta^{12}_r$. We have

$$\sum_{j \in \{1,2\}} (3r+j)^{4-1} q^{3r+j}$$

$$= \frac{q(8q^9 + 93q^6 + 60q^3 + 1)}{(q^3 - 1)^4} + \frac{q(1 + 60q^3 + 93q^6 + 8q^9)}{(-1 + q^3)^4} + \frac{q^2(8 + 93q^3 + 60q^6 + q^9)}{(-1 + q^3)^4}$$

$$= \frac{q^{11} + 8q^{10} + 60q^8 + 93q^7 + 93q^5 + 60q^4 + 8q^2 + q}{(q^3 - 1)^4}.$$

Note that the sum of the coefficients of the numerator is 324, which is the same as the order of the group $G_\sigma \cong (\mathbb{Z}/3)^\times$.

### 1.2. Eigenspace Automorphism Groups

In what follows, we fix a cyclic orientation $(1 \to 2 \to \cdots \to n \to 1)$ to define an injection from the translation algebra into the set

$$\{ x \in (\mathbb{Z}/r)^n : \sum x_i \equiv 0 \mod (r) \}.$$ 

**Definition 88.** Let $q = e^{-2\pi i / r}$. Given an arbitrary element $e_I = e_i^1 \cdots e_i^n \in C^n_r$, let

$$p(e_I) = [i_2 - i_1, \ldots, i_n - i_{n-1}, i_1 - i_n] \in (\mathbb{Z}/r)^n.$$

We define the **level** of $e_I$ as follows. Select representatives $i_{a,a+1} \in \{0, \ldots, r - 1\}$ for each $i_{a+1} - i_a$. Then define

$$\text{lev}(e_I) = \sum_{a=1}^n i_{a,a+1}.$$

It is easy to see that $p(e_I)$ is independent of multiplication of $e_I$ by $e_1 \cdots e_n$ and is in \{0, r, 2r, \ldots, (r-1)r\}. Thus, it is a well-defined map on the set of monomials in the translation algebra.

**Example 89.** Let $\sigma = (123456)$. In the notation of the previous section, $G_\sigma = \langle (123456) \rangle \simeq \mathbb{Z}/6$, and $\mathbb{Z}/6\times \simeq \mathbb{Z}/2$. Let $\mu_5 = (26)(35)$, so we have

$$\mu_5 (123456) = 156432 = (26)(35)^{-1}.$$ 

The $q$-plates in $\text{Pl}(\Delta^6_6)$ which are eigenvectors of $\sigma$, respectively the weight monomials in $C^6_6$, are

<table>
<thead>
<tr>
<th>q-plate</th>
<th>eigenvalue</th>
<th>$e_I$</th>
<th>$p(e_I)$</th>
<th>level</th>
</tr>
</thead>
<tbody>
<tr>
<td>${123456}_0$</td>
<td>$e^{2\pi i / 6}$</td>
<td>$e^2 e_3 e_4 e_5 e_6$</td>
<td>$[1, 1, 1, 1, 1, 1]$</td>
<td>1 - 6</td>
</tr>
<tr>
<td>${123456}_1$</td>
<td>$e^{4\pi i / 6}$</td>
<td>$e^2 e_3^2 e_4^2 e_5 e_6$</td>
<td>$[2, 2, 2, 2, 2, 2]$</td>
<td>2 - 6</td>
</tr>
<tr>
<td>${123456}_2$</td>
<td>$e^{6\pi i / 6}$</td>
<td>$e^2 e_3^3 e_4^3 e_5 e_6$</td>
<td>$[3, 3, 3, 3, 3, 3]$</td>
<td>3 - 6</td>
</tr>
<tr>
<td>${123456}_3$</td>
<td>$e^{8\pi i / 6}$</td>
<td>$e^2 e_3^4 e_4^4 e_5 e_6$</td>
<td>$[4, 4, 4, 4, 4, 4]$</td>
<td>4 - 6</td>
</tr>
<tr>
<td>${123456}_4$</td>
<td>$e^{10\pi i / 6}$</td>
<td>$e^2 e_3^5 e_4^5 e_5 e_6$</td>
<td>$[5, 5, 5, 5, 5, 5]$</td>
<td>5 - 6</td>
</tr>
</tbody>
</table>
We expect that \((\mathbb{Z}/6)^x\) will permute the levels as follows.

\[
\begin{align*}
0 & \mapsto 0 \equiv 0 \pmod{6} \\
1 & \mapsto 5 \cdot 1 \equiv 5 \pmod{6} \\
2 & \mapsto 5 \cdot 2 \equiv 4 \pmod{6} \\
3 & \mapsto 5 \cdot 3 \equiv 3 \pmod{6} \\
4 & \mapsto 5 \cdot 4 \equiv 2 \pmod{6} \\
5 & \mapsto 5 \cdot 5 \equiv 1 \pmod{6}
\end{align*}
\]

Let us check explicitly that this is what happens. Using \(\mu_5 = (26)(35)\), as above, we have

\[
\begin{align*}
\mu_5 \cdot \{123456\} & = \{123456\} \\
\mu_5 \cdot \{1_21_34_15_16_1\} & = \{1_21_65_14_13_12_1\} \\
\mu_5 \cdot \{14_25_23_6_2\} & = \{14_23_6_22_5_2\} \\
\mu_5 \cdot \{13_5_32_4_6_3\} & = \{13_5_32_4_6_3\} \\
\mu_5 \cdot \{14_23_6_22_5_2\} & = \{14_25_23_6_2\} \\
\mu_5 \cdot \{1_21_65_14_13_12_1\} & = \{1_21_21_65_14_13_12_1\}
\end{align*}
\]

and the levels are permuted exactly as expected.

In the future, we shall investigate the action of \(G^{\text{transverse}}\) on the eigenspaces of \(\sigma\).

Consider the union of hyperplanes of the following form. For each \(j \in \{1, \ldots, rn - 1\}\) such that \(\gcd(j, r) = 1\) define the hyperplane cross-section

\[
\mathcal{H}^n_j = \Delta^n_j \cap [0, r]^n.
\]

Remark that \(\mathcal{H}^n_j\) is not in general a simplex.

In Chapter B we reproduce some of our Mathematica code which supports the conjecture which follows.

**Conjecture 90.** The dimensions of the total plate module

\[
\dim \left( \bigoplus_{j=1}^{rn-1} Pl\left( \mathcal{H}^n_j \right) \right) = r^n(n-1)!
\]

and of its submodule

\[
\dim \left( \bigoplus_{\gcd(j, r)=1} Pl\left( \mathcal{H}^n_j \right) \right) = \varphi(r^n)(n-1)! = \varphi(r)r^{n-1}(n-1)!
\]

are numerically equal to the scaled volumes of the ambient hyperplanes, i.e.

\[
\sum_{j=1}^{rn-1} Vol\left( \mathcal{H}^n_j \right) = r^n(n-1)!
\]

and

\[
\sum_{\gcd(j, r)=1} Vol\left( \mathcal{H}^n_j \right) = \varphi(r^n)(n-1)!
\]
1.3. A Galois Group for the Translation Algebra. Let \( q = e^{-2\pi/r} \), as usual.

It is a standard result in commutative algebra that the vector space obtained by adjoining a primitive \( r \)-th root of unity to the rational numbers has dimension \( \varphi(r) \) over the rational numbers, where \( \varphi \) is the Euler totient function. It is also a standard result that the Galois group \( \text{Gal}(\mathbb{Q}(q)/\mathbb{Q}) \) of automorphisms of \( \mathbb{Q}(q) \) over \( \mathbb{Q} \) is isomorphic to \((\mathbb{Z}/r)^{\times}\), the multiplicative group of integers modulo \( r \).

Let us briefly consider what happens if we restrict the ambient field for the translation algebra \( \mathbb{C}_n^r \) from \( \mathbb{C} \) to \( \mathbb{Q} \), forgetting momentarily about the \( S_n \)-action.

First we define an analog of the action of \( \text{Gal}(\mathbb{Q}(q)/\mathbb{Q}) \) on \( \mathbb{Q}(q) \), on the translation algebra \( \mathbb{C}_n^r \), as follows.

**Definition 91.** For each integer \( s \in (\mathbb{Z}/r)^{\times} \), \( s \) relatively prime to \( r \), define its action on the generators of \( \mathbb{C}_n^r \) by powers

\[
s \cdot e_i = e_i^s \quad \text{and} \quad s \cdot q = q^s.
\]

First, we note that this definition makes sense, since the defining relation is invariant to powers,

\[
0 = s \cdot 0 = s \cdot (e_1 \cdots e_n - q) = e_1^s \cdots e_n^s - q^s = (e_1 \cdots e_n)^s - q^s
\]

which is still zero in \( \mathbb{C}_n^r \).

**Proposition 92.** Let \( q = e^{2\pi i/r} \). Then, the \( \mathbb{Q} \)-vector spaces

\[
U = \left\{ \sum_{I \in \{0, \ldots, r-1\}^{n+1}} c_I q^0 e_1^i_2^i_3^i_4^i_{n} = \sum_{I \in \{0, \ldots, r-1\}^{n+1}} c_I e_2^i_3^i_4^i_{n} : c_I \in \mathbb{Q} \right\}
\]

and its extension by an \( r \)-th root of unity \( q \)

\[
V = \left\{ \sum_{I \in \{0, \ldots, r-1\}^{n+1}} c_I q^{i_1+i_2+\cdots+i_n} e_1^i_2^i_3^i_4^i_{n} = \sum_{I \in \{0, \ldots, r-1\}^{n+1}} c_I e_2^i_3^i_4^i_{n} : c_I \in \mathbb{Q} \right\}
\]

have respectively dimensions

\[
\dim(U) = r^{n-1}
\]

and

\[
\dim(V) = \varphi(r^n),
\]

where \( \varphi \) is Euler’s totient function.

Further, \( U \) is fixed by the Galois group action \( q \mapsto q^s \) and \( e_i \mapsto e_i^s \) for gcd\( (r, s) = 1 \).

**Proof.** The fact that \( \dim(U) = r^{n-1} \) over \( \mathbb{Q} \) follows by enumerating basis monomials, just as for \( \mathbb{C}_n^r \) over the complex numbers.

It is a standard result in number theory that the field extension of the rational numbers by an \( r \)-th root of unity has dimension \( \varphi(r) \) over \( \mathbb{Q} \). Thus,

\[
\dim(V) = \varphi(r) \dim(V) = \varphi(r) \cdot r^{n-1} = \varphi(r^n).
\]

That \( U \) is fixed by the Galois action follows because the coefficients of the basis monomials are by definition real and are therefore fixed by the usual Galois group action on \( \mathbb{Q}(q) \). \( \square \)
2. Plates in the $3 \times 3$ Scaled Birkhoff Polytopes

The $n \times n$ scaled Birkhoff polytope, denoted here $\mathcal{K}_r^n$ is the set of $n \times n$ matrices with nonnegative entries, such that each row and each column has integer sum $r \geq 1$.

It is well known that the $n \times n$ Birkhoff polytope is the convex of the $n!$ permutation matrices in the vector representation $\mathbb{C}^n$.

**Example 93.** The $3 \times 3$ Birkhoff polytope is the convex hull of the $3!$ matrices

\[
(A_1, \ldots, A_6) = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} , \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array} , \begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array} , \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array} , \begin{array}{ccc} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} , \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{array} \right).
\]

We derive the closed form series $\sum_{i=1}^{\infty} c_i x^i$ where each coefficient $c_i$ is a positive integer which counts the number of integer lattice points in the set

\[
\left\{ \sum_{i=1}^{6} a_i A_i : \sum a_i = r \right\}
\]

is then the coefficient of $x^r$. The result is the series

\[
x + 6x^2 + 21x^3 + 55x^4 + 120x^5 + 231x^6 \cdots.
\]

Six terms provides enough information to obtain the closed form generating function

\[
\frac{1 + x + x^2}{(1 - x)^5},
\]

which matches the result from the Online Encyclopedia of Integer Sequences, A002817 [36].

We here consider the case $n = 3$, formulated equivalently as a subset of a Cartesian product,

\[
\{(u_1, u_2, u_3) \in \Delta_r^3 \times \Delta_r^3 \times \Delta_r^3 : u_1 + u_2 + u_3 = (r, r, r)\} \subset \Delta_{3r}^9.
\]

It is not immediately obvious how to define plates in the Birkhoff polytope. However, under some assumptions we are able to derive a reasonable estimate, first by enumerating a local basis by hand enumeration for $n = 2, 3, 4$ and then by a single line of Mathematica code, reproduced in Appendix B.

Namely, we apply the equivariant Worpitzky to localize the basis on each $\Delta_r^3$. We then count triples of embeddings of hypersimplices $B_{1,2}$, $B_{2,1}$ into the simplex $\Delta_r^3$ such that the sum of the three centroids is the point $(r, r, r)$. This is the content of Figure 1, in which we use the fact that any two centroids determines the third. See Appendix B for Mathematica code which extends the computation to large enough edge length to conjecture a generating function.

**Conjecture 94.** The generating function for the dimensions of plate modules in the scaled $3 \times 3$ Birkhoff polytopes is given by the series

\[
\frac{x(1 + x)(1 + x + x^2)}{(1 - x)^5} = x + 7x^2 + 27x^3 + 76x^4 + 175x^5 + \cdots.
\]
2. PLATES IN THE $3 \times 3$ SCALED BIRKHOFF POLYTOPES

Figure 1. Birkhoff Plates, $3 \times 3$ case

\[ \frac{\left(1+\pi^2k^2+\pi^4k^4+\ldots\right)}{1-\pi^2} \]

The first two configurations determine the third.
APPENDIX A

Ocneanu’s Proof of q-Plate Linear Independence

Let us start with an example, the case of $n = 3$ variables.

With respect to the ordered bases of q-plates

$$\{B\} = \{\{1, 2, 3\}, \{1, 3, 2\}, \{1, 23\}, \{13, 2\}, \{12, 3\}, \{123\}\}$$

and of plates

$$[[B]] = \{[[1, 2, 3]], [[1, 3, 2]], [[1, 23]], [[13, 2]], [[12, 3]], [[123]]\}$$

the change of basis matrix from q-plates to plates is

$$\begin{bmatrix}
1, 3, 2 & -q_2 q_3 & q_2 q_3 & -q_2 q_3 & q_2 q_3 & q_2 q_3 & q_2 q_3 \\
1, 23 & 0 & 0 & -q_2 q_3 & q_2 q_3 & q_2 q_3 & q_2 q_3 \\
13, 2 & 0 & 0 & 0 & -q_2 q_3 & q_2 q_3 & q_2 q_3 \\
12, 3 & 0 & 0 & 0 & 0 & -q_2 q_3 & q_2 q_3 \\
123 & 0 & 0 & 0 & 0 & 0 & -q_2 q_3 & q_2 q_3 \\
\end{bmatrix}$$

and its inverse, the change of basis from plates to q-plates is

$$\begin{bmatrix}
1, 3, 2 & q_2 q_3 & q_2 q_3 & q_2 q_3 & q_2 q_3 & q_2 q_3 & q_2 q_3 \\
1, 23 & q_2 q_3 & q_2 q_3 & q_2 q_3 & q_2 q_3 & q_2 q_3 & q_2 q_3 \\
13, 2 & 0 & 0 & q_2 q_3 & q_2 q_3 & q_2 q_3 & q_2 q_3 \\
12, 3 & 0 & 0 & 0 & q_2 q_3 & q_2 q_3 & q_2 q_3 \\
123 & 0 & 0 & 0 & 0 & q_2 q_3 & q_2 q_3 \\
\end{bmatrix}$$

where we use the notation $[x] = 1 - x$.

We shall describe these matrices for an arbitrary number of variables $n$ and prove that their product is the identity.

For both matrices, we shall compute the first row, that is, the expansion of the standard plate, respectively q-plate labeled $1, 2, 3, \ldots$. The remaining rows can be obtained from this standard first row, by replacing labels with lumps of labels.

The first matrix, which we call the direct matrix, is computed in two steps. We start by expressing the standard q-plate in terms of plates, with an expression due to Early. We have

$$\{1, \ldots, n-1, n\} = [[1, \ldots, n-1, n]] + q_n^{-1}[[n, 1, \ldots, n-1]] + q_{n-1}^{-1} q_n^{-1}[[n-1, n, 1, \ldots, n-2]] + \cdots$$

In the second step, the plates in the above expression are brought to the basis, which consists of plates with 1 in the first lump, using an expression due to Ocneanu. Although the coefficients of this expansion are $\pm 1$, the proof requires quite elaborate hypergeometric functions.
To expand the plate 
\[
[[m + 1 \ldots, n, 1, \ldots, m]],
\]
in terms of basis plates, we reverse the first part \(m + 1, \ldots, n\) to \(n, \ldots, m + 1\) and we shuffle it with the remaining second part \(1, \ldots, m\), and lump it in all possible ways such that 1 is in the first lump, and no two labels from the second part are in the same lump. Recall that each lump is unordered. There is an overall lumping sign, which is the parity of the difference between the number \(n\) of initial lumps and the number of resulting lumps. Additionally, there is a sign \((-1)^{n-m}\), or \((-1)^{\text{parity}}\) for each label in the first part.

From the first step, the expression of q-plates in terms of plates, for each label \(i\) in the first part, we have a coefficient \(q_i^{-1}\), so all together we have a coefficient \((-q_i^{-1})\) for each label which was rotated in front, together with an overall lumping sign.

In order to keep track of the two parts, we shall color the first part blue and the second part red, as in 
\[
[[m + 1 \ldots, n, 1, \ldots, m]].
\]

We shall describe now the inverse map, in which we are given a lumped basis plate \(S\) and we have to find the coefficient of \(S\) in the expansion of the standard q-plate \(\{1, 2, 3 \ldots, n\}\). These coefficients are the entries of the first row of the direct matrix.

To find this coefficient, we shall start from the coloring, red and blue, described above. Each lump contains at most one red label, with the remaining labels colored blue. The red labels, \(1, \ldots, m\), are in ascending order, and are smaller than the blue labels, \(m + 1, \ldots, n\). The blue labels are reversed, and there is a coefficient \((-q_i^{-1})\) for each blue label \(i\). There is an overall sign corresponding to the parity of the lumping.

Thus, from each coloring we obtain the labels \(1, \ldots, n\) by unshuffling. As the red labels are increasing and the blue labels are decreasing, there is one possible choice ambiguity in the coloring. The smallest label \(i\) of the last lump of \(S\) can be colored either red or blue. When colored blue, we get an extra coefficient \((-q_i^{-1})\). The only case without such an ambiguity is the case when \(S\) has a single lump.

Thus for instance, we compute a coefficient in the example matrix with \(n = 3\), the coefficient of \(S = [[13, 2]]\) in the expansion of the standard q-plate \(\{1, 2, 3\}\).

There is a sign \(-\) is due to the lumping of \(1, 2, 3\) to \(13, 2\).

The possible colorings for \(S = [[13, 2]]\) are \([[13, 2]],\) which gives the coefficient \((-q_2)^{-1}(-q_3)^{-1}\) due to the blue labels 2, 3, and the coloring \([[13, 2]],\) which gives the coefficient \((-q_3)^{-1}\) for the blue label 3. This gives the total coefficient of \(S = [[13, 2]]\) in the expansion of the standard q-plate \(\{1, 2, 3\}\)

\[
- \left( (-q_2)^{-1}(-q_3)^{-1} + (-q_3)^{-1} \right) = \frac{(-1 + q_2)}{q_2 q_3} = \frac{[q_2]}{q_2 q_3}.
\]

To obtain the general row, for the expansion of a q-plate \(T\), each label \(i\) of the standard plate is replaced with the \(i^{\text{th}}\) lump \(T_i\) of \(T = \{T_1, T_2, \ldots, T_k\}\). The coefficient \(q_i\) is then replaced by the \(\prod_{j \in T_i} q_j\). The lumps of \(S\) are then subdivided, instead of labels, into sublumps \(T_i\), with each sublump colored red or blue, as before.

The following rules are obtained from the description of the standard case. We convene to place the possible unique red sublump first in each lump.

1. The sublump containing 1 should be in the first lump, and colored red.
2. Each lump should contain at most 1 red sublump, positioned first, if it exists.
3. The remaining sublumps are colored blue.
To obtain $T$ by unshuffling and unlumping, we take the red lumps in order, followed by the blue lumps in reverse order.

By this procedure, we can obtain all the colorings and unlumpings of a basis plate $S$, and thus the column of the direct matrix corresponding to $S$, containing the coefficients of $S$ in the expansion of each basis q-plate $T$.

For instance, for $S = [[13, 2]]$, we have the sublump decompositions

\[ [[13, 2]], [[1 \cdot 3, 2]], \]

with possible colorings

\[ [[13, 2]], [[13, 2]], [[1 \cdot 3, 2]], [[1 \cdot 3, 2]], \]

which give after unshuffling, the q-plates $\{13, 2\}$, in which $S$ appears, as

\[ \{13, 2\} + (-q_2^{-1})\{13, 2\} \text{ and } -(-q_3^{-1})\{12, 3\} = (-q_2^{-1})(-q_3^{-1})\{1, 2, 3\}, \]

and thus

\[ \frac{[q_2]}{q_2} \{13, 2\} \text{ and } \frac{[q_2]}{q_2 q_3} \{1, 2, 3\}. \]

These form, in the direct matrix, the column of the coefficients of $[[13, 2]]$ in the expansions of basis q-plates.

For the inverse matrix, we have the following description, which will be proved by showing the product of the direct and inverse matrices is the identity.

We first describe the case of the expansion of the standard plate $S_0 = [[1, 2, \ldots, n]]$ in terms of basis q-plates, the coefficients of which form the first row of the inverse matrix. The general case is obtained by replacing labels with lumps, exactly as for the direct matrix.

All the denominators of the expansion of $S_0$ are the same, equal to $[q_2 q_3 \cdots q_n][q_3 q_4 \cdots q_n] \cdots [q_n]$, where we recall that $[x] = -1 + x$.

The numerators are computed as follows. There is an overall sign due to lumping, as in the direct matrix.

The numerator corresponding to the coefficient of a q-plate $T = \{T_1, \ldots, T_k\}$ in the expansion of the standard plate $S_0 = [[1, 2, \ldots, n]]$ is computed as follows.

1. Put the labels in each lump $T_i$ in descending order, and concatenate the lumps into a sequence $\tau$.
2. Decompose the labels in $\tau$ as a set composition into blocks $B_1, B_2, \ldots, B_l$, defined as follows. Start with $B_1$ as the first element $b$ of $\tau$. If $b - 1$ exists (that is, if $b > 1$) and is located after $b$ in $\tau$, then append it at the end of $B_1$. If $b - 2$ exists and is located in $\tau$ after $b - 1$, then append it at the end of $B_1$. Continue this procedure as long as possible. This defines the first block $B_1$.

Proceed with the remaining labels $\tau'$ in $\tau$ which are not used in $B_1$. Start the second block $B_2$ with the first element $b'$ in $\tau'$, and append $b' - 1$ if it exists and is located after $b'$ in $\tau'$ (or in $\tau$), continue with $b' - 2$ and so on, as long as possible.

Continue the decomposition into such blocks until all labels are exhausted.

Remark that an equivalent description of the algorithm is the following. View $\tau$ as a line permutation and decompose the inverse permutation $\tau^{-1}$ into descending runs, i.e. maximal intervals $I_1, I_2, \ldots, I_l$, each of which is descending, with $\tau^{-1} = I_1 I_2 \cdots I_l$. Apply $\tau$ to obtain the blocks $B_i = \tau(I_i)$ arranged in reverse order. Thus, $I_i$ consists of the positions in $\tau$ of the elements of $B_i$ arranged in reverse order, which are descending and positioned consecutively in $\tau^{-1}$, since they correspond to consecutive labels of $\tau$. 
**Remark 95.** In the sliding algorithm above, which describes the inverse matrix, labels which are not consecutive, i.e. more than 1 apart, slide past each other. This suggests a connection between the inverse matrix labels and braid group generators, which behave the same way.

**Example 96.** The coefficient of \(\{1, 3, 2\}\) in the expansion of the standard plate \([1, 2, 3]\) has denominator, which depends only on \([1, 2, 3]\), defined by

\[
[q_2q_3][q_3] = (-1 + q_3q_2)(-1 + q_3),
\]

and the numerator is obtained as follows.

The lumps 1, 3, 2 concatenate in the sequence \(\tau = (1, 3, 2)\), in which the first block is 1 and the second block is made of the consecutive descending labels 3, 2. The denominator is obtained by raising each block to a power equal to the number of preceding blocks, and thus is

\[
(q_1)^0(q_3q_2)^1 = q_2q_3
\]

for a total coefficient of

\[
\frac{q_2q_3}{[q_2q_3][q_3]}
\]

in the \(n = 3\) inverse matrix.

The fact that the q-plates form a basis will follow from the following theorem.

**Theorem 97.** The product of the inverse matrix with the direct matrix is the identity.

**Proof.** Due to the behavior with respect to lumping of both the direct and inverse matrices, it will be enough to prove the following lemma, asserting that the first row of the product matrix is 1, 0, . . . , 0.

**Lemma 98.** We have

(1) The coefficient of the standard plate \(S_0\) in the expansion of the standard plate \(S_0\) into basis q-plates, using the inverse matrix, followed by the expansion of these basis q-plates into basis plates, using the direct matrix, is 1.

(2) The coefficient of any other basis plate \(S\) in the expansion of the standard plate \(S_0\), as above, is 0.

The first case above has no lumping. For the second case above, we shall show that the coefficients cancel in pairs. The product of the two lumping signs in the direct and inverse matrices is the total lumping sign from \(S_0\) to \(S\), which can be thus skipped for the cancellation.

We shall discuss the case of the cancellation first. The denominators depend only on the standard plate \(S_0\) and can be thus skipped for the proof of the cancellation.

The idea of the proof is that the cancellation occurs when we decompose the coefficient of \(S\) in \(S_0\) after all the sublumpings and colorings of \(S\), which describe the intermediate q-plate \(T\) in the decomposition of \(S_0\) into basis q-plates \(T\) followed by the decomposition of \(T\) into basis plates, among which one is \(S\).

Further, the cancellation occurs if we remove an element \(s\). Namely, for each sublumping and coloring of the quotient \(S' = S \setminus \{s\}\), consider all the sublumpings and colorings of \(S\) which restrict to the given sublumping and coloring of \(S'\).

**Lemma 99.** The contributions of the sublumpings and colorings of \(S\) for a given sublumping and coloring of the quotient \(S' = S \setminus \{s\}\) add up to 0, if the label \(s\) is a switcher defined by the following properties.

(1) \(s > 1\).

(2) The lump of \(s - 1\) in \(S\) is the same or positioned after the lump of \(s\) in \(S\).
(3) If \( s + 1 \) exists, that is if \( s < n \), then the lump of \( s + 1 \) in \( S \) is positioned strictly after the lump of \( s \) in \( S \).

The proof of the cancellation will be finished when we also prove the following.

**Lemma 100.** If the basis plate \( S \) is not the standard plate \( S_0 \), then a switcher label \( s \) for \( S \) exists.

The proof of the Theorem will be finished when we show the following.

**Lemma 101.** The numerators corresponding to the different colorings of \( S_0 \) in the product of the direct and inverse matrices add up to the expansion of the standard denominator; and thus the coefficient of \( S_0 \) in the expansion of \( S_0 \) in the product matrix is 1, the first entry of the identity matrix.

Let us now prove the three Lemmas above. The existence of a switcher element \( s \) for a basis plate \( S \) follows from the following Lemma.

**Lemma 102.** Let \( S = [[S_1, \ldots, S_k]] \), where each lump is ordered ascendingly. Let \( S_i \) be the first lump of \( S \) which is not standard, that is, such that \( S_i = \{j\} \) for \( j < i \), and \( S_i \neq \{i\} \). Such a nonstandard lump exists since \( S \) is not the standard plate \( S_0 \).

Then the last label of \( S_i \) is a switcher.

Note that \( s - 1 \) cannot be in a block earlier than \( S_i \), since, as \( s \) is last in \( S_i \), that would make \( S_i = \{s\} \) standard. Also, as \( s \) is last in \( S_i \), \( s + 1 \), if it exists, will be in a later lump.

This ends the proof of the existence of a switcher. We shall now assume in what follows that \( s \) is a switcher for a given basis plate \( S \), located in a lump \( S_i \) of \( S \), and we shall choose a lumping and coloring for the quotient \( S' = S \setminus \{s\} \).

The lumpings and colorings of \( S \) with the chosen quotient are obtained by inserting \( s \) into the sublumps of \( S_i' = S_i \setminus \{s\} \), or by making \( s \) a new sublump of \( S_i \). If \( S_i' \) has \( m \) blue sublumps, we shall show that the number of such possible insertions of \( s \) into \( S_i' \) is \( 2m + 2 \). We shall show that the coefficients of the product of the direct and inverse matrices, corresponding to these \( 2m + 2 \) insertions of \( s \) cancel in pairs.

Let us recall the axiomatization of the possible colorings of the plate \( S \) or \( S' = S \setminus \{s\} \).

1. 1 is in the first sublump of the first lump, colored red.
2. Each lump of \( S \) or \( S' \) has at most one red sublump, positioned first, with the remaining sublumps blue.

To obtain the intermediate q-plate \( T \) in the expansion of \( S_0 \) in basis q-plates, we unlump \( S \) as follows. We first arrange all the red sublumps in order, followed by all the blue sublumps in reverse order.

Let us now describe the possible insertions and colorings of the switcher label \( s \) into the sublumped and colored quotient \( S_i' = S_i \setminus \{s\} \) or its lump \( S_i \) in \( S \).

These insertions are of two types. Let \( L \) be the red lump of \( S_i' \), if it exists, else take \( L = \emptyset \).

In case (1), \( s \) is inserted into \( L \) as a red label. In case (2), \( \{s\} \) becomes the first blue sublump. We shall show that the contributions of the cases (1) and (2) cancel each other.

We shall use the fact that \( s \) is a switcher label, namely that \( s - 1 \) is in the same lump \( S_i \) as \( s \) or in a later lump, and \( s + 1 \), if it exists, is in a later lump.

1. In the first insertion, since \( s \) is in the red sublump of \( S_i \), after unshuffling \( s - 1 \), if red, will be in the same sublump as \( s \) or, if blue, later.

If \( s + 1 \) exists, \( s + 1 \) is in a later lump. \( s + 1 \) is either red, thus later since reds are kept in order, or blue, thus later as well.
Thus, when we do the sliding algorithm to obtain blocks which give the powers of the variables $q_i$, since blocks consist of consecutive descending elements which become positioned in order after the unshuffling. Thus, since $s - 1$ is positioned after $s$, they will be in the same block, and since $s + 1$ is positioned after $s$. Similarly, $s + 1$ will be in a different block from $s$, with that block positioned after the block of $s$, since blocks are positioned in ascending order. These two blocks will be $\{s, s - 1, \ldots\}$ followed by $\{\ldots, s + 1\}$ if $s + 1$ exists, that is $s$ is a block head and $s + 1$ is the end of the next block.

(2) In the second insertion, $\{s\}$ is the first blue sublump of $S_i$. If $s - 1$ is red, it will be positioned earlier than $s$, after unshuffling, since reds are positioned before blues.

If $s - 1$ is blue, it will be later than $s$, since $\{s\}$ is the first blue sublump of $S_i$, and $s - 1$ is in $S_i$ or in a later lump $S_j$ with $j > i$. After unshuffling, since blues are reversed, $s - 1$ will be again positioned before $s$.

So in both cases, after unshuffling, $s - 1$ will be positioned before $s$, so $s - 1$ and $s$ will be in different blocks.

If $s + 1$ exists and is red, since $s$ is blue, after unshuffling $s + 1$ will be positioned before $s$, since reds are positioned before blues.

If $s + 1$ exists and is blue, then since $s$ is a switcher, $s + 1$ is positioned in a later lump than $s$. After unshuffling, $s + 1$ will be positioned before $s$, since the order of the blues is reversed by unshuffling.

So in both cases, after unshuffling, $s + 1$ will be positioned before $s$, so $s + 1$ and $s$ will be in the same block.

Thus, in the second insertion, the consecutive blocks are $\{s - 1, s - 2, \ldots\}$, nonempty, and $\{\ldots, s + 1, s\}$, with $s + 1$ potentially missing if it does not exist.

Thus, between the two subcases of insertion of $s$ into the quotient, the block structure changed from

$$\{s, s - 1, \ldots\}, \{\ldots, s + 1\}$$

to

$$\{s - 1, \ldots\}, \{\ldots, s + 1, s\}$$

Thus, the only element changing blocks is $s$. The power of each $q_i$ in the numerator of the coefficient is the number of blocks before the block containing $i$, from the inverse matrix, from which power one subtracts 0 or 1, depending on whether $i$ is red or blue, due to the direct matrix. Thus, $q_s$ gains a power 1 for $s$ being in a later block, and loses a power 1 for $s$ switching from red to blue. Overall, the power of every $q_i$ in the first and second insertion remains the same. As there is a sign change between the two cases which is due to the extra blue sublump $\{s\}$ in the second case, the two contributions cancel.

There is a second type of insertion of $s$ in $S_i' = S_i \setminus \{s\}$, corresponding to every blue sublump $L$ of $S_i'$. In case (1), $s$ is inserted into $L$, colored blue. In case (2), $\{s\}$ forms a blue lump after $L$. We shall show that the contributions of cases (1) and (2) cancel each other for every blue lump $L$ of $S_i'$.

As in the previous discussion, since $s$ is a switcher, $s + 1$ is in a later lump, if it exists, so after unshuffling $s + 1$ will be positioned earlier than $s$ regardless of the color of $s + 1$. Thus, $s + 1$ and $s$ will be in the same block, in both subcases.

If $s - 1$ is in the sublump $L$, then in the first subcase, when $s$ is inserted into $L$, after unshuffling $s$ and $s - 1$ will be in $L$, which will be ordered descendingly for the construction of the blocks. Thus, in this case $s + 1, s, s - 1$ are in this order, and thus will be in the same block. In the second subcase, when $\{s\}$ is a blue sublump after $L \ni s - 1$, since the unshuffling reverses
the order of the blue sublumps, again $s$ will be before $s - 1$. Again, $s + 1, s, s - 1$ are in this order and will be in the same block.

If $s - 1$ is not in the sublump $L$, then the two subcases, with sublumps $L \cup \{s\}$ and respectively $L, \{s\}$, make no difference for the relative position of $s - 1$ and $s$, before or after unshuffling. Thus, in this case as well, the block positions of $s - 1$ and $s$ will not change between the two subcases.

In both of the situations discussed above, the block structure of the two subcases will be the same. As the color of $s$ is blue in both subcases, the powers of each $q_i$ in the coefficient will not change between the two subcases. There is a change of sign, due to the extra blue block $\{s\}$ in the second subcase, and thus the contributions of the two subcases cancel each other.

This ends the proof of the cancellation, and thus shows that the entries of the first row in the product of the direct and inverse matrices, other than the first entry, are zero.

We now show that the diagonal element of the first row of the product matrix is 1. This number is the coefficient of the standard plate into the expansion of the standard plate into q-plates and further into plates in the standard basis.

All the elements of the first row of the product matrix have the same denominator, coming from the inverse matrix, which is equal to

$$(−1 + q_2 q_3 \cdots q_n) (−1 + q_3 \cdots q_n) \cdots (−1 + q_n).$$

The expansion of this product gives $2^{n−1}$ terms, each of which has ascending powers for $q_2, q_3, \ldots, q_n$, with the jump in from 0 to the power of $q_2$ and from the power of $q_i$ to the power of $q_{i+1}$ each 0 or 1. The sign of such a product is the parity of $n − 1$ minus the power of $q_n$, since that difference counts the number of terms $(-1)$ used in the product.

We thus compute the coefficient in the product matrix of the standard plate

$$S = [[[1] \{2\} \cdots \{n\}]]$$

into the expansion of the standard plate, so all the lumps we work with have size 1, and there is no nontrivial sublumping. The terms of the numerator will be obtained from the different colorings of these lumps, recalling that the first lump $\{1\}$ is red.

Blue lumps are reversed by unshuffling and positioned after the red ones. Initially the lumps were ascending. After unshuffling, the blue lumps are descending.

We shall group the blue lumps into consecutive segments, which are ascending, each blue segment $\{k\}, \{k+1\}, \ldots \{l\}$ except possibly for the last one, followed immediately by a red lump $\{l + 1\}$. After unshuffling, red lumps are first, followed by blue lumps in reverse order. Thus, $l + 1, l, l − 1, \ldots, k$ will be in this order, and will form a block after sliding, as blocks are made of consecutive descending elements, which are positioned in ascending order after unshuffling.

Red elements are ascending before and after unshuffling, and thus will be in different blocks. Thus, each block will consist of a consecutive segment of blue lumps, possibly empty, and except possibly for the last lump, the red lump following that segment.

For instance,

$${1}, {2}, {3}, {4}, {5}, {6}$$

becomes after unshuffling

$${1}, {2}, {5}, {6}, {4}, {3}, {3},$$

and thus the sequence

$$1 2 5 6 4 3$$

which divides into blocks

$${1}, {2}, {5, 4, 3}, {6},$$

made of consecutive descending elements, positioned ascendingly.
When we remember the coloring, the blocks are
\[ \{1\}, \{2\}, \{5, 4, 3\}, \{6\}. \]
These are precisely the red elements 1, 2, 5, each followed by the blue interval, possibly empty, which precedes it in the initial coloring, and with the last block missing the red element.

Recall now that the power of each \( q_i \) is the number of blocks preceding \( i \), from which we subtract 1 if \( i \) is blue. Thus, the previous example gives the coefficient
\[ q_1^0 q_2^1 q_3^{2-1} q_4^{2-1} q_5^{3-1} q_6 = (q_2 q_3 q_4)^1 (q_5 q_6)^2. \]
Note that the factors in each parenthesis are corresponding precisely to the decomposition of the elements 2, 3, \ldots, \( n \) into a red label, followed by a possibly empty blue interval, here
\[ 2, 3, 4, 5, 6 \Rightarrow (2, 3), (5, 6) \Rightarrow (q_2 q_3 q_4)^1 (q_5 q_6)^2 = (q_2 q_3 q_4 q_5 q_6)(q_5 q_6). \]
The sign of that coefficient is the number of blue elements, which is the parity of \( n - 1 \) minus the number of parentheses, and thus \( n - 1 \) minus the power of \( q_n \).

Thus, the terms of the numerator coincide with the terms obtained by expanding the denominator, and the coefficient is 1.

This ends the proof for the first row of the product matrix, which corresponds to the expansion of the standard plate.

The general case, in which we expand a plate other than the standard plate uses the same proof as above, in which labels are replaced with lumps, as done before, when we showed that the direct and inverse matrices were upper-triangular, with respect to the filtration by lumping. □
APPENDIX B

Mathematica Code

(1) We compute the generating function for the $3 \times 3$ Birkhoff polytope in the code which follows.

\[
\text{FindGeneratingFunction}[
\text{Table}[
\text{Map}[\text{Total}, \text{Tuples}[\text{Permute}[\text{IdentityMatrix}[3], \text{SymmetricGroup}[3]], \{r\}]]/\text{Union}/\text{Length}, \{r, 0, 8\}], x] = 1 + x + x^2 \frac{1 - x}{(1 - x)^5}.
\]

(2) Section 2: To compute the dimension of the vector space of plates in the scaled Birkhoff polytopes $K^3_r$ we used the following Mathematica code. Elements of the Birkhoff polytope are represented by triples of points. The code below counts all ordered triples of centers of hypersimplices $B_{1,2}$ and $B_{2,1}$, each translated in its copy of $\Delta^3$ on the integer lattice, which sum to $(r, r, r)$.

...Section 3.1...

\[
\text{FindGeneratingFunction}[
\text{Table}[
\text{Select}[\text{Tuples}[\text{Select}[\text{Flatten}[\text{Map}[
\{# + \{1, 1, 1\}/3, # + \{2, 2, 2\}/3\}&, \text{Tuples}[\text{Range}[r+1]-1, 3]], 1], \text{Total}[#] == r&, 3]], \text{Total}[#] == \{r, r, r\} /\text{Length}, \{r, 1, 10\}], x] = x(1 + x)(1 + x + x^2) \frac{1 - x}{(1 - x)^5}.
\]

(3) Plate Enumerations. First construct all set compositions which have 1 in the first lump:

...Section 3.1...

(4) For $n = 3$ the output is

\{
\{\{1, 2, 3\}\}, \{\{1, 2\}, \{3\}\}, \{\{1, 3\}, \{2\}\}, \{\{1\}, \{2, 3\}\}, \{\{1\}, \{2\}, \{3\}\}, \{\{1\}, \{3\}, \{2\}\}\},
\]

which coincides with the basis of plates about a point in $n = 3$ variables.

(5) The Mathematica code for the enumeration algorithm follows. We count the number of basis plates in the $r$ hyperplane sections of cube $[0, r]^n$ which are perpendicular to the vector $(1, 1, \ldots, 1)$. The number of coordinates is set to $n = 4$ and can be changed.

...Mathematica Code...
This has output:
\[
\{6, 96, 486, 1536, 3776, 14406, 24576, 39366, 60000\}
\]
\[
= 3! \cdot \{1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000\}
\]

(6) There are two interesting submodules which are enumerated as follows. First select a
subset of the hyperplane sections consisting of those whose coordinate sum is a multiple
of \(r\). To count these, we change \(make\text{PossiblePositions}\) above to what is given below.
\[
make\text{PossiblePositions}[lumps_] := \text{Select}\left[\text{Tuples}\left[\text{Map}[\text{Range}, \text{Map}[\text{rLength}[\#]-1&, lumps]]\right], \text{Mod}[\text{Total}[\#], r] == 0&\right];
\]
The output becomes then a factor of \(r\) smaller:
\[
\{6, 48, 162, 384, 256, 750, 1296, 2058, 3072, 6000\}
\]
\[
= 3! \{1, 8, 27, 64, 125, 216, 343, 512, 729, 1000\}
\]
Second, consider the subset of hyperplanes whose coordinate sum is relatively prime
with \(r\).
\[
make\text{PossiblePositions}[lumps_] := \text{Select}\left[\text{Tuples}\left[\text{Map}[\text{Range}, \text{Map}[\text{rLength}[\#]-1&, lumps]]\right], \text{GCD}[\text{Total}[\#], r] == 1&\right];
\]
For example, from the above, with \(n = 4\) we get
\[
\{6, 48, 324, 768, 3000, 2592, 12348, 12288, 26244, 24000\}
\]
which agrees with the formula \(\varphi(r^n)(n - 1)!\), obtained from the command
\[
\text{Table}\left[\varphi\left(r^4\right)(4 - 1)!, \{r, 1, 10\}\right].
\]

(7) This is the same as the relative volumes of the corresponding hyperplanes, which we
first enumerate separately and then sum, obtaining
\[
\text{Table}\left[\text{Table}\left[\text{RegionMeasure}\left[\text{ImplicitRegion}\left[0<=x1<=r&&0<=x2<=r
\&\&0<=x3<=r&&0<=x4<=r&&x1+x2+x3+x4==m, \{x1, x2, x3, x4\}\right]\right], \text{Select}[\text{Range}[4r-1], \text{GCD}[\#, r] == 1&]\right]\right], \{r, 1, 4\}\right] \cdot 3
\]
\[
\{1, 1, 1\}, \{1, 23, 23, 1\}, \{1, 8, 60, 93, 93, 60, 8, 1\}, \{1, 27, 121, 235, 235, 121, 27, 1\}\}
\]
where the third sequence, for example, is to be compared to the generating function in
Example 87,
\[
\sum_{j \in \{1, 2\}} (3r + j)^{4-1} q^{3r+j} = \frac{q^{11} + 8q^{10} + 60q^8 + 93q^7 + 93q^5 + 60q^4 + 8q^2 + q}{(q^3 - 1)^4}.
\]
The sequences sum respectively to
\[
\{6, 48, 324, 768, 3000, 2592, 12348, 12288\},
\]
as above.
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