The Pennsylvania State University
The Graduate School

THE BAUM-CONNES CONJECTURE AND GROUP ACTIONS ON AFFINE BUILDINGS

A Thesis in
Mathematics
by
Dmitry Matsnev

Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Doctor of Philosophy

August 2005

The thesis of Dmitry Matsnev was reviewed and approved* by the following:

Nigel Higson
Distinguished Professor of Mathematics
Thesis Advisor, Chair of Committee
Head of Department of Mathematics

John Roe
Professor of Mathematics

Victor Nistor
Professor of Mathematics

Murat Gunaydin
Professor of Physics

*Signatures are on file in the Graduate School.

Abstract

Guentner, Higson, and Weinberger proved using Hilbert space techniques that for any countable linear group the Baum-Connes assembly map is split-injective; for the case of a countable linear group of matrices of size 2 they showed that the Baum-Connes assembly map is an isomorphism.

In this thesis we study the the possibility of applying a finite-dimensionality argument in order to prove part of the Baum-Connes conjecture for finitely generated linear groups.

For any finitely generated linear group over a field of characteristic zero we construct a proper action on a finite-asymptotic-dimensional $CAT(0)$-space, provided that for such a group its unipotent subgroups are "boundedly composed". The $CAT(0)$-space in our construction is a finite product of symmetric spaces and affine Bruhat-Tits buildings.

For the case of finitely generated subgroup of $SL(2, \mathbb{C})$ the result is sharpened to show that the Baum-Connes assembly map is an isomorphism.

Table of Contents

Acknowledgments

I would like to express my deep gratitude to my adviser Nigel Higson for his invaluable guidance and encouragement. Without his help and support this thesis would not be finished at all.

Also I would like to thank all the members of the geometric functional analysis group at Penn State for making my study there pleasant and productive.

Last, but certainly not least, I want to thank Becky Halpenny for her efforts in organizing graduate students life.

Chapter 1

Introduction

The Baum-Connes conjecture was introduced by Paul Baum and Alain Connes in the early 80's (for a formal account see [BC88a, BC88b]). It connects the K-theory of the reduced crossed product of a C^*-algebra by a group acting on such algebra and the K-homology of the corresponding classifying space of proper actions of that group.

In this work we study the possibility of applying a finite-dimensionality argument in order to prove part of the Baum-Connes conjecture for finitely generated linear groups, and in this chapter we review the entire work from the large scale.

In Section 1.1 we define the assembly map, formulate the Baum-Connes conjecture and discuss the present state of it. In Section 1.2 we formulate the results which are proven in this thesis, with necessary explanations. Finally, in Section 1.3 we give a brief outline of the argument.

1.1 Motivations and known results

Let Γ be a discrete group acting on a C^*-algebra A by automorphisms. One can define a morphism (also known as the "assembly" map, see [BCH94])

$$\mu : KK^{\Gamma}(\underline{E}\Gamma, A) \to K(A \rtimes_r \Gamma)$$

from the K-homology of the classifying space $\underline{E}\Gamma$ of proper actions of Γ to the K-theory of the reduced crossed product of A by Γ.

Paul Baum and Alain Connes introduced the following conjecture about this morphism:

Conjecture 1.1. *The assembly map μ is an isomorphism.*

This statement is known as the Baum-Connes conjecture.

Remark 1.2. Some authors actually consider two ingredients of the conjecture separately: the injectivity and surjectivity. We shall be mostly concerned with injectivity.

While Conjecture 1.1 is formulated in terms of pairs (Γ, A), it is possible to state it purely in terms of group Γ. Namely we can ask whether the original conjecture hold for the group Γ and any C^*-algebra A, on which Γ acts. This version of the conjecture is called *the Baum-Connes conjecture with coefficients*[1], while the version with $A = \mathbb{C}$ is called *the Baum-Connes conjecture without coefficients*. In this work our main concern will be the conjecture with coefficients. The conjecture with coefficients is stable under passing to subgroups (see [CE01]).

In fact, even the injectivity part of the conjecture is a rather difficult problem. It is known, for example, that the injectivity of the Baum-Connes assembly map implies the Novikov's higher signature conjecture [FRR95].

There are different approaches to the understanding of the Baum-Connes conjecture. Here we will discuss only some of them, directly referring to the notions of proper action and uniform embeddability.

Theorem 1.3 (Guentner, Higson, and Weinberger, [GHWar]). *For any field K, the Baum-Connes conjecture with coefficients holds for any countable subgroup of $GL(2, K)$.*

Theorem 1.4 (Guentner, Higson, and Weinberger, [GHWar]). *For any field K and any natural number n, the injectivity portion of the Baum-Connes conjecture with coefficients holds for any countable subgroup of $GL(n, K)$.*

The original proof of Theorem 1.3 in [GHWar] relies on the following result:

[1]Some authors refer to it as to the Baum-Connes *property* with coefficients, in the view of the counterexamples (modulo a statement due to Gromov) by Higson, Lafforgue, and Skandalis in [HLS02].

Theorem 1.5 (Higson and Kasparov, [HK01]). *If a group Γ admits a metrically proper isometric action on a Hilbert space, then the Baum-Connes conjecture with coefficients holds for Γ.*

Then, via the techniques of valuations and positive-definite functions, the existence of such action is established, which proves the statement of Theorem 1.3.

It was proven by Yu (see [Yu00]) that if a discrete group Γ uniformly embeds into a Hilbert space, then the Baum-Connes assembly map is injective. This proves Theorem 1.4.

Unfortunately, in the proofs of Theorems 1.3 and 1.4 the group actions were not constructed explicitly, and Hilbert spaces on which the group acts can easily be infinite-dimensional.

There is another approach, which reflects the coarse geometric view of discrete groups. It is based on the following result:

Theorem 1.6 (Yu, [Yu98]). *If a finitely generated group Γ has finite asymptotic dimension as a discrete metric space with the word-length metric corresponding to a finite set of generators and its classifying space is of finite homotopy type, then the coarse Baum-Connes conjecture holds for Γ, or, alternatively, the Baum-Connes assembly map for such Γ is injective.*

This theorem was proven in [Yu98] by means of extensive ϵ-δ calculations and is based on the elementary facts from the asymptotic dimension theory only. Higson in [Hig00] proved that the finiteness assumption on the classifying space can be relaxed. There exists an alternative proof by Wright in [Wri02], which is based on the introduction of a more delicate coarse structure on a metric space.

1.2 Statement of results

We are interested in proving Theorem 1.4 or at least many cases of it, by reducing it to Theorem 1.6. Also we are interested in giving a more elementary proof of Theorem 1.3, without appealing to the Hilbert space actions.

The following theorems are proven in this thesis.

Theorem. *Let K be any field of characteristic 0, and Γ be a finitely generated subgroup of $SL(n, K)$. If there exists a natural number m, such that all unipotent* *subgroups of* Γ *are finitely composed with parameter* m *(informally, this condition means that they are built out of "layers" of asymptotic dimension not more than* m*), then* Γ *acts properly on a finite-asymptotic-dimensional* $CAT(0)$ *space.*

For the case $n = 2$ we sharpen this result and prove

Theorem. *Let* K *be any field of characteristic 0, and* Γ *be a finitely generated subgroup of* $SL(2, K)$*. Then* Γ *satisfies the Baum-Connes conjecture (with coefficients).*

1.3 Outline of the argument

In this section we outline the main ideas involved in our construction and explain the methods being used.

Starting from a finitely generated group Γ, a subgroup of $SL(n, \mathbb{C})$, we can regard it as a subgroup of $SL(n, K)$, where the field K is a finitely generated extension of $\mathbb{Q}$ by the entries of the matrices from the finite generating set.

We want to construct a proper action of our group on a space of finite asymptotic dimension in order to approach Theorem 1.6 as close as possible.

At the first step we apply the following construction of Alperin and Shalen (see [AS82]). An element x of K is *integral* if and only if it is integral with respect to every discrete valuation on K. Since the field K is finitely generated, we can test the integrality using only finitely many valuations. To each discrete valuation on a field K one can associate an affine building, a contractible simplicial complex of nonpositive curvature, on which the group $SL(n, K)$ acts by isometries. The important property of that action is that its isotropy consists of the matrices whose spectrum is integral with respect to the discrete valuation which defines the building.

Take the product X of all the buildings corresponding to a finite set of discrete valuations on K used to test integrality of the field elements. It has a finite asymptotic dimension, since each building does. Consider the diagonal action of Γ on X. The isotropy of this action will consist of matrices whose spectrum is integral for each discrete valuation, that is those matrices will have integral in K eigenvalues. We will say that such matrices have *integral characteristic* and define a class of subgroups of Γ (which we will call *groups of integral characteristic*) consisting of matrices of integral characteristic. Reformulating the last statement in this language, the isotropy groups of the action we constructed at this step are all of integral characteristic.

Now at the second step our hope would be to construct an action of Γ on some space of finite asymptotic dimension, with the property that the restriction of this action to any subgroup of integral characteristic is proper. Then, taking the diagonal action of Γ, comprised of the action constructed before and this one, we would obtain a proper action of Γ in some space of finite asymptotic dimension. Unfortunately this is not possible in general.

Example 1.7. Let t be a transcendental complex number. Consider a group G generated by the elements

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

This group contains a subgroup of integral characteristic

$$\Gamma = \left\{ \begin{pmatrix} 1 & p(t,t^{-1}) \\ 0 & 1 \end{pmatrix} \;\middle|\; p(t,t^{-1}) \in \mathbb{Z}[t,t^{-1}] \right\},$$

which is a free abelian group of infinite rank and this prevents G from acting properly on a finite-asymptotic-dimensional space.

We will show, however, that the subgroups of this kind are the only obstructions to the construction of a proper action. More precisely, we prove, using a trace technique of Zimmer, that any subgroup G of $SL(n,K)$ of integral characteristic admits a faithful representation within $SL(m,\mathbb{Q})$, for some m, if the natural action of G on $\mathbb{C}^n$ is irreducible. The fact that such a group resembles a discrete subgroup of the arithmetic group $SL(m,\mathbb{Q})$ allows us to construct an action of G on a product of $(m-1)$-dimensional affine buildings, corresponding to the p-adic valuations on $\mathbb{Q}$, and a symmetric space $SL(m,\mathbb{R})/SO(m)$. Due to the opposite nature of the notion of boundness on these spaces, the action will be proper. More careful study shows that in fact we can extend such an action to Γ and its restriction onto any integral characteristic subgroup of Γ, possessing an irreducible action on $\mathbb{C}^n$, remains proper.

For the case of a subgroup of integral characteristic with reducible action, we consider the Jordan-Hölder series and apply the procedure described before to each irreducible component. We will be left with the integral characteristic subgroups which are *unipotent*, that is, the characteristic polynomial of every element of which is $\chi(\lambda) = (1 - \lambda)^n$. Note that the group Γ from Example 1.7 is exactly of this kind.

At the last step of our construction we restrict our attention to the unipotent subgroups of Γ which are boundedly composed. We construct an action of G, such that for every boundedly composed unipotent subgroup of Γ of parameter not more than some fixed bound m the restriction of the action is proper. The main idea in this construction is based on the fact that, given a finitely generated field $K \subseteq \mathbb{C}$, there exist enough embeddings $K \to \mathbb{C}$ that metrically separate matrix entries of different elements of G in the usual metric on $\mathbb{C}$. Each embedding $K \to \mathbb{C}$ gives rise to a homomorphism $SL(n, K) \to SL(n, \mathbb{C})$, so that Γ, as a subgroup of $SL(n, K)$, has different realizations in $SL(n, \mathbb{C})$. Each of these copies of Γ acts on a symmetric space $SL(n, \mathbb{C})/SU(n)$, and the combined twisted diagonal action on the product of the symmetric spaces is proper.

Finally, the proper action for the first theorem in Section 1.2 can be taken to be the diagonal action of the actions constructed at all three steps.

Note that since $GL(n, K)$ embeds into $SL(n + 1, K)$, any finitely generated subgroup of $GL(n, K)$ can be realized as a finitely generated subgroup of $SL(n + 1, K)$, for which we know how to construct a proper action. Thus our result extends to all finitely generated linear groups over K.

To sharpen the result for $n = 2$ and prove the full Baum-Connes conjecture directly, we follow essentially the same steps and show that the isomorphism of the assembly map can be carried over each step in the construction, as follows.

For the Alperin-Shalen technique at the first step, note that the affine buildings for $SL(2, K)$ are just trees, and if we allow Γ act on each tree from the finite set one at a time, repeated invocations of the theorem of Oyono-Oyono in [OO98] (see also [Tu01] for an alternative argument), which says that the countable group acting on a tree satisfies the Baum-Connes conjecture if and only if the isotropy subgroups do, reduces the verification of the conjecture from Γ to its subgroups of integral characteristic.

In the second step of our construction we will be dealing with subgroups of integral characteristic acting irreducibly on $\mathbb{C}^2$. Alternatively we can say that these subgroups are Zariski-dense in $SL(2,\mathbb{C})$ and appeal to work of Zimmer and Margulis (see [Zim84b]) on the rigidity of Kazhdan groups. It will be shown that the subgroup under consideration faithfully embeds into the group of 4×4 matrices over the field of algebraic numbers, and then, similar to the construction for a general n, we use the fact that the ground field is finitely generated to reduce the discussion to the case of a subgroup of $SL(m,\mathbb{Q})$ for some m, which can be treated explicitly by [CEN01].

We come to the third step with subgroups of integral characteristic which are not Zariski-dense in $SL(2,\mathbb{C})$. The dimension of the Zariski closure of such group is not more than 2, and we can afford to classify all such groups explicitly. As it turns out, each discrete subgroup under consideration is a semi-direct product of abelian groups, so that it is amenable, and we appeal to the direct argument involving an explicit treatment of the classifying space for such group in [MV03], which proves the Baum-Connes conjecture with coefficients.

Again, we can extend our result to any finitely generated subgroup Γ of $GL(2,K)$. consider the following exact sequence:

$$1 \to \Gamma \cap SL(2,K) \to \Gamma \to K^{\times},$$

where the only nontrivial maps are the natural inclusion and the determinant respectively. Our argument proves the conjecture for $\Gamma \cap SL(2,K)$, the group $K^{\times}$ is abelian and so is any subgroup of it, and we can apply the extension result from [CEOOar] to conclude that Γ itself satisfies the Baum-Connes conjecture with coefficients.

Chapter 2

Preliminaries on fields, valuations, and buildings

In this chapter we collect some technical tools to be used later in the discussion. In Section 2.1 we state some facts about field extensions and field homomorphisms. Section 2.2 introduces the notion of a discrete valuation on a ring or a field, Section 2.3 gives a brief account of Bruhat-Tits buildings, and finally Section 2.4 describes the canonical affine building associated to a discrete valuation.

2.1 Field extensions and embeddings

Definition 2.1. Given a field extension $F \subseteq E$, we call a set $\vec{t} = \{t_i\}_{i \in I}$ of elements of E a *transcendence base* of the extension if $\vec{t}$ is algebraically independent over F and E is an algebraic extension of $F(\vec{t})$.

In general, any field extension $F \subseteq E$ can be obtained via a two-step process: first we extend F to $F(\vec{t})$ by adjoining a transcendence base $\vec{t}$ of the original extension E over F (we will refer to $F(\vec{t})$ as to purely transcendental extension) and then making an algebraic extension E of $F(\vec{t})$.

Lemma 2.2. *Let E be a finitely generated extension of the field F. Then the algebraic closure $\bar{F}^E$ of F in E is a finite extension of F.*

Proof. Let $\vec{t}$ be a transcendence base for E over $\bar{F}^E$. It can be constructed in the following fashion: take a finite generating set of E over F. This set generates E

over $\bar{F}^E$ as well. Then $\vec{t}$ can be taken to be a maximal algebraically independent over $\bar{F}^E$ subset of this set.

Note that $\vec{t}$ is also a transcendence base for E over F. Then $\bar{F}^E(\vec{t})$ is a finitely generated algebraic extension of $F(\vec{t})$ and thus is a finite extension. This means that any element of $\bar{F}^E(\vec{t})$ is a finite $F(\vec{t})$-linear combination of some set Θ of elements of $\bar{F}^E(\vec{t})$. Those elements are composed out of rational expressions with $\vec{t}$, with coefficients in $\bar{F}^E$. We have only finitely many such elements, and they generate $\bar{F}^E$ over F, whence the lemma. □

Recall that for any field homomorphism $\sigma : K \to L$ one has $\ker \sigma = \{0\}$, therefore it is customary to refer to field homomorphisms as to field *embeddings*.

Definition 2.3. Let σ be a field homomorphism $K \to L$. Let F be a common subfield of K and L. We call σ an *embedding over* F if $\sigma(x) = x$ for all $x \in F$.

Lemma 2.4. *Let $\sigma_1, \dots, \sigma_n : K \to L$ be distinct field embeddings, and let*

$$F = \{x \in K | \sigma_1(x) = \cdots = \sigma_n(x)\} .$$

Then $n \leqslant [K : F]$.

Proof. This is classical (see, for instance, [McC76, Theorem 2.2]). Seeking a contradiction, suppose that $[K : F] = m < n$, so that one can furnish a basis $\{e_1, \dots, e_m\}$ of K over F. Consider the following system of linear equations over L:

$$\sum_{j=1}^{n} \sigma_j(e_i)x_j = 0, \qquad i = 1, \dots, m. \tag{2.1}$$

It has nontrivial solution, say $(c_1, \dots, c_n) \in L^n$.

Any $y \in K$ is $y = \lambda_1 e_1 + \cdots + \lambda_m e_m$, where $\lambda_i \in F$ for every $i = 1, \cdots$ and for each $i = 1, \dots, m$ we have:

$$\sigma_1(\lambda_i) = \cdots = \sigma_n(\lambda_i).$$

Multiply (2.1) term-wise by these equalities and obtain:

$$\sum_{j=1}^{n} \sigma_j(\lambda_i e_i)x_j = 0, \qquad i = 1, \dots, m$$

Plug $x_j = c_j$ and add all the equations up:

$$\sum_{j=1}^{n} c_j \sigma_j(y) = 0 \tag{2.2}$$

Since this holds for all y in K, we conclude that the embeddings $\{\sigma_j\}$ are linearly dependent. We show that this can not happen by induction.

If $n = 1$ then $c_1\sigma_1(x) = 0$ for every x means $c_1 = 0$, which contradicts nontriviality of c_1.

For the inductive step, suppose that this could not happen for $(n-1)$ embeddings. If in (2.2) some $c_{j_0} = 0$, then we can rewrite (2.2) with embeddings $\sigma_1, \ldots, \sigma_{j_0-1}, \sigma_{j_0+1}, \ldots, \sigma_n$ and obtain a contradiction. So we need to check the case when all $c_j \neq 0$.

Choose some $c \in K$ with $\sigma_1(c) \neq \sigma_2(c)$. We have:

$$c_1^{-1}\sigma_1(c^{-1}) \sum_{j=1}^{n} c_j \sigma_j(cx) = 0 \qquad \forall x \in K.$$

On the other hand,

$$c_1^{-1} \sum_{j=1}^{n} c_j \sigma_j(x) = 0 \qquad \forall x \in K.$$

Subtracting one from another, we get

$$\sum_{j=2}^{n} c_1^{-1} c_j (\sigma_1(c^{-1})\sigma_j(c) - 1)\sigma_j(x) = 0 \qquad \forall x \in K.$$

By the inductive hypothesis, this means that all coefficients are zeros, in particular, $\sigma_1(c^{-1})\sigma_j(c) - 1 = 0$, but this contradicts our assumption $\sigma_1(c) \neq \sigma_2(c)$. □

Lemma 2.5. *If K is a finitely generated subfield of $\mathbb{C}$, then there are finitely many embeddings*

$$\sigma_i : K \to \mathbb{C} \qquad i = 1, \ldots, n,$$

such that

$$\{x \in K | \sigma_i(x) = x, \quad i = 1, \ldots, n\} = \mathbb{Q}.$$

Proof. We write $K = (\mathbb{Q}(t_1, \dots, t_r))(a_1, \dots, a_s)$, where $\{t_1, \dots, t_r\}$ (which we shortly abbreviate as $\vec{t}$) is a transcendence base for K, while $a_1, \dots, a_s$ all are algebraic over $\mathbb{Q}(\vec{t})$.

Since a finitely generated algebraic extension is finite, there is some number θ, algebraic over $\mathbb{Q}(\vec{t})$, with minimal polynomial

$$p_\theta \in \mathbb{Q}(\vec{t})[x], \qquad \deg p_\theta = n,$$

such that $K = \mathbb{Q}(\vec{t})(\theta)$.

Choose $\{T_1, \dots, T_r\} = \vec{T}$, an algebraically independent set of transcendental numbers, disjoint from $\vec{t}$.

Define a field isomorphism $\sigma : \mathbb{Q}(\vec{t}) \to \mathbb{Q}(\vec{T})$ by sending t_i to T_i. These are embeddings over $\mathbb{Q}$.

Define $P_\theta(x)$ to be the image of $p_\theta(x)$ under σ, that is, the result of application of σ to the coefficients of p_θ. As $P_\theta(x)$ is a polynomial of degree n, it has complex roots $\Theta_1, \dots, \Theta_n$.

Let $L = \mathbb{Q}(\vec{T})(\Theta_1, \dots, \Theta_n)$ and define the field homomorphisms

$$\sigma_i : K \to L \qquad i = 1, \dots, n$$

by

$$\sigma_i(x) = \sigma(x), \qquad x \in \mathbb{Q}(\vec{t})$$
$$\sigma_i(\theta) = \Theta_i$$

We claim that $\sigma_1, \dots, \sigma_n$ are the desired embeddings. We will show that if $\sigma_i(x) = x$ for every $i = 1, \dots, n$, then x is in $\mathbb{Q}$. Since all σ_i's are all distinct isomorphisms,

$$F = \{x \in K | \sigma_1(x) = \cdots = \sigma_n(x)\}$$

is a subfield of K with $[K : F] \geqslant n$. We know that $\mathbb{Q}(\vec{t}) \subseteq F$ by the previous lemma, so that $F = \mathbb{Q}(\vec{t})$. Now if $\sigma_1(x) = \cdots = \sigma_n(x) = x$, then x is in $F \cap L$, which is $\mathbb{Q}$. $\square$

2.2 Discrete valuations

Definition 2.6. Let R be an integral domain (a commutative ring without zero divisors in which $0 \neq 1$). A map $\nu : R \to \mathbb{Z} \cup \{+\infty\}$ is called a *discrete valuation* if it satisfies the following properties for any $a, b \in R$:

- $\nu(a) = +\infty$ if and only if $a = 0$
- $\nu(ab) = \nu(a) + \nu(b)$
- $\nu(a + b) \geq \min(\nu(a), \nu(b))$

Given a discrete valuation ν of an integral domain R, it can be extended to the field of fractions $\mathrm{frac}(R)$ by

$$\nu\left(\frac{a}{b}\right) = \nu(a) - \nu(b), \qquad a, b \in R.$$

This indeed gives a well-defined extension, for if $\frac{a}{b} = \frac{c}{d}$ for some elements $c, d \in R$, then $ad = bc$ in R, so that $\nu(a) - \nu(b) = \nu(c) - \nu(d)$.

Starting from a field K, one can regard K as a ring and define the corresponding notion of a discrete valuation, following Definition 2.6 (see [Cas86]). Clearly this approach is consistent with the extension to fractions defined above.

To any discrete valuation ν one can associate its *ring of integers*, usually denoted by $\mathcal{O}_\nu$. It consists of all elements of the field K with non-negative valuation, while the operations are induced from the field.

Any element of $\mathcal{O}_\nu$ with valuation 1 is called a *uniformizer* of the valuation ν. Usually it does not matter which particular element is chosen for the uniformizer, and commonly it is denoted by π.

Note that the ideal $\pi\mathcal{O}_\nu$ is maximal in $\mathcal{O}_\nu$ [Cas86, Chapter 4]. The *residue field* of the valuation ν is the quotient of $\mathcal{O}_\nu$ by its maximal ideal $\pi\mathcal{O}_\nu$.

Example 2.7. One of the typical examples of discrete valuation is a p-adic valuation on $\mathbb{Q}$. Namely, we fix prime number p and represent any nonzero rational number q as

$$q = p^n \frac{a}{b}, \text{ with } n \in \mathbb{Z} \text{ and } (a, p) = (b, p) = 1.$$

Then define the p-adic valuation of this q as $\nu_p(q) = n$. The ring of integers of ν_p consists of all rational numbers without any occurrence of p in the denominator, p itself serves as a uniformizer, and the residue field is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

Example 2.8. Another valuation which we will employ is a discrete valuation on $K[x]$ given by $\nu(f) = -\deg f$. The extension of this valuation to the field of fractions $K(x)$ is given by

$$\nu\left(\frac{q(x)}{r(x)}\right) = \deg r - \deg q, \text{ where } q, r \in K[x].$$

The ring of integers here consists of all ratios of polynomials $\frac{q}{r}$ with $\deg q \leqslant \deg r$, the residue field in this case is infinite.

2.3 Affine buildings

The notion of a *building* as a special kind of a simplicial complex was developed by Bruhat and Tits in the early 1970's (see [BT72, BT84, Tit75]). We will employ a certain type of buildings, namely affine buildings, the definition and basic properties of which are discussed here. Most of this section follows [Gar97].

Definition 2.9. A simplicial complex X is called a *chamber complex* if every simplex in X is contained in a maximal simplex, with respect to inclusion relation, and for any pair x, y of maximal simplices in X there is a sequence of maximal simplices $x_0 = x, x_1, x_2, \ldots, x_n = y$ such that for all $i = 1, 2, \ldots, n$ the simplices x_{i-1} and x_i are adjacent, that is share a subsimplex of codimension 1.

The maximal simplices of a chamber complex are called *chambers*, and the codimension 1 subsimplices are called *facets*. Chamber complexes in which each facet is a facet of exactly two chambers are called *thin*, while chamber complexes in which each facet is a facet of at least 3 chambers are called *thick*. Note that according to Definition 2.9 any chamber complex is connected and all maximal simplices have the same dimension.

Definition 2.10. An *affine building* is a thick chamber complex X which admits a set of chamber subcomplexes (called *apartments*) satisfying the following axioms:

1. Each apartment is a thin chamber complex, isomorphic to a Euclidean space.

2. For any two simplices x, y in X there is an apartment A containing both of them.

3. If A_1 and A_2 are two apartments containing simplices x and y, then there is an isomorphism $A_1 \to A_2$ fixing x and y pointwise.

Several comments are in order. If we omit the requirement that each apartment is isomorphic to a Euclidean space in the first axiom, we would have a general definition of a *building*. The last axiom implies that all apartments are isomorphic (see [Gar97, Corollary (4.3)]). We will discuss two geometric properties of affine buildings: canonical apartment retracts and contractibility.

Lemma 2.11. *The entire building X is retractable to any apartment A.*

Proof. Fix an apartment A and some chamber C within A. For any x in X axiom 2 provides apartment B containing both C and x, while axiom 3 provides an isomorphism $f : B \to A$. Define the retract of x to be $f(x)$. We need to show that $f(x)$ does not depend on the choice of B and f.

Suppose that we have two isomorphisms $f, g : B \to A$, constructed using axiom 3. They fix C pointwise, and if we choose some chamber D in B, adjacent to C along some facet, this facet will be fixed by f and g pointwise. Consider three chambers in A: $C = f(C) = g(C)$, $f(D)$, and $g(D)$. They all share a common facet, therefore, due to the thinness of chamber complex A, at least two of them should coincide. Since f and g are isomorphisms, they can not map different chambers C and D to the same one, therefore $f(D)$ and $g(D)$ coincide. We claim that they coincide pointwise, for they coincide along the facet common to C and D pointwise by construction, and the only vertex in D which is not in this facet has to be mapped to the same vertex by f and g. Using the fact that any point in B can be connected to C by a sequence of chambers in the way that the previous one and the next one are adjacent along some facet, this argument shows that f and g are the same on the entire apartment B.

Now suppose that we have two apartments B_1 and B_2, both containing C and x. We will show that the corresponding isomorphisms $f_1 : B_1 \to A$ and $f_2 : B_2 \to A$ are the same on $B_1 \cap B_2$ (we already proved in the previous paragraph that the isomorphisms f_1 and f_2 are unique). Consider an isomorphism $g : B_1 \to B_2$ fixing C pointwise. If we take a simplex y in $B_1 \cap B_2$, the argument in the previous paragraph says that g and identity map coincide on y, therefore g fixes $B_1 \cap B_2$ pointwise. Now compare the two isomorphisms from B_1 to A, $f_2 \circ g$ and f_1. Due to the uniqueness discussed above, they coincide on B_1, while on $B_1 \cap B_2$, $f_2 \circ g = f_2$. Thus f_1 and f_2 agree on the overlap $B_1 \cap B_2$, and the retraction constructed above does not depend on the choice of apartment. □

As usual, we write $|X|$ for a geometric realization of a simplicial complex X. The affine building X admits a *canonical* metric on its geometric realization $|X|$, defined in the following way. For any two points $x, y \in |X|$ find an apartment A whose geometric realization contains them and fix an isomorphism of A onto the Euclidean space $|A|$ (it is convenient to assume that all maximal simplices in A give rise to subsets of $|A|$ of diameter 1). Then define the canonical distance $\operatorname{dist}(x, y)$ to be the usual Euclidean distance between x and y in $|A|$. This construction is independent of the choice of apartment A.

Similarly, the line segment $[x, y]$ is intrinsically defined (independently of the choice of apartment containing x and y), and for any t between 0 and 1 we can define a point $z_t = tx + (1-t)y$ to be the unique point in $|X|$ so that $\operatorname{dist}(z_t, x) = t \operatorname{dist}(x, y)$ and $\operatorname{dist}(z_t, y) = (1-t) \operatorname{dist}(x, y)$.

In the language of geometric realizations the contraction f defined in Lemma 2.11 can be understood as a contraction $|f| : |X| \to |A|$. It is essential that $|f|$ is a Lipschitz contraction:

$$\operatorname{dist}(|f|x, |f|y) \leqslant \operatorname{dist}(x, y).$$

If either of x or y lies in the geometric realization of the base chamber C of contraction this inequality becomes equality.

Theorem 2.12. *The geometric realization $|X|$ of the affine building X is contractible.*

A detailed proof of this theorem can be found in [Gar97, (14.3) and (14.4)], here we will outline the key ideas.

Sketch of a proof. We start with the *Negative Curvature Inequality* for affine buildings. Pick two points x, y in $|X|$ and a number t between 0 and 1. Then the point

$z_t = tx + (1-t)y$ also lies in $|X|$, in the same apartment as x and y. For any point z from this apartment the following equality holds:

$$\mathrm{dist}^2(z, z_t) = t\,\mathrm{dist}^2(z, x) + (1-t)\,\mathrm{dist}^2(z, y) - t(1-t)\,\mathrm{dist}^2(x, y) \tag{2.3}$$

This formula for Euclidean space is an easy consequence of the inner product and associated norm calculation if we take z to be the affine origin of the geometric realization of the apartment.

For a point z not necessarily sharing an apartment with x and y the following inequality holds instead of (2.3):

$$\mathrm{dist}^2(z, z_t) \leqslant t\,\mathrm{dist}^2(z, x) + (1-t)\,\mathrm{dist}^2(z, y) - t(1-t)\,\mathrm{dist}^2(x, y) \tag{2.4}$$

To prove this formula, consider the retraction $|f|$ of $|X|$ to some apartment $|A|$ containing x and y. We can arrange the base chamber $|C|$ of the retraction to contain z_t. Then (2.3) holds for $|f|z$ instead of z, and we obtain (2.4) by combining it with inequalities $\mathrm{dist}(|f|(z), x) \leqslant \mathrm{dist}(z, x)$ and $\mathrm{dist}(|f|(y), x) \leqslant \mathrm{dist}(y, x)$.

Consider a map $[0,1] \times |X| \times |X| \to |X|$ defined by $(t, x, y) \mapsto z_t = tx + (1-t)y$. This map is continuous. Indeed, if we take (t_1, x_1, y_1) close to (t, x, y) and let $z_1 = t_1 x_1 + (1-t_1)y_1$, then $\mathrm{dist}(z_1, x)$ is close to $\mathrm{dist}(z_1, x_1) = (1-t_1)\,\mathrm{dist}(x_1, y_1)$ and $\mathrm{dist}(z_1, y)$ is close to $\mathrm{dist}(z_1, y_1) = t_1\,\mathrm{dist}(x_1, y_1)$.

Therefore in the negative curvature inequality (2.4) for x_1, y_1, z_1

$$\mathrm{dist}^2(z_1, z_t) \leqslant t_1\,\mathrm{dist}^2(z_1, x_1) + (1-t_1)\,\mathrm{dist}^2(z_1, y_1) - t_1(1-t_1)\,\mathrm{dist}^2(x_1, y_1) \tag{2.5}$$

the right-hand side is close to

$$t_1\,\mathrm{dist}^2(z_1, x_1) + (1-t_1)\,\mathrm{dist}^2(z_1, y_1) - t_1(1-t_1)\,\mathrm{dist}^2(x_1, y_1),$$

which is close to

$$t(1-t)^2\,\mathrm{dist}^2(x, y) + t^2(1-t)\,\mathrm{dist}^2(x, y) - t(1-t)\,\mathrm{dist}^2(x, y) = 0.$$

Thus the left-hand side in (2.5) tends to 0 as well, which means that $\mathrm{dist}(t_1 x_1 + (1-t_1)y_1, tx + (1-t)y)$ is small, as was desired.

Putting everything together, fix some point y in $|X|$ and consider a map $(t, x) \mapsto tx + (1 - t)y$. This maps gives the contraction of $|X|$ to $\{y\}$. □

2.4 Affine buildings for $SL(n, k)$

Now we review the construction of the affine Bruhat-Tits building for $SL(n, K)$, where K is a discretely valued field.

Suppose K is equipped with a discrete valuation ν and denote by $\mathcal{O}$ the ring of integers of ν and by π some uniformizer.

Definition 2.13. An $\mathcal{O}$-*lattice* in a K-vector space K^n is a free $\mathcal{O}$-submodule containing a K-basis for K^n.

Let $\mathcal{L}$ be the set of $\mathcal{O}$-lattices in K^n, and let $\bar{\mathcal{L}}$ be the set of homothety classes of lattices, i.e. $\bar{\mathcal{L}} = \mathcal{L}/K^*$. One defines an $(n-1)$-dimensional simplicial complex, the Bruhat-Tits building X, in the following way:

- The vertices of X are the elements of $\bar{\mathcal{L}}$.
- A set of $(m+1)$ distinct vertices $\bar{L}_0, \bar{L}_1, \ldots, \bar{L}_m$ forms an m-simplex in X if

$$L_0 \subset L_1 \subset \cdots \subset L_m \subset \pi^{-1} L_0$$

  for some representatives $L_0, L_1, \ldots, L_m$ of $\bar{L}_0, \bar{L}_1, \ldots, \bar{L}_m$ respectively.

To shorten notation we will write L_{m+1} for $\pi^{-1}L_0$ in the chain of inclusions above.

In particular, any maximal simplex in X, a chamber, is given by a chain of lattices

$$L_0 \subset L_1 \subset \cdots \subset L_n \subset \pi^{-1} L_0, \tag{2.6}$$

such that for any i the quotient L_{i+1}/L_i is a one-dimensional vector space over $\varkappa = \mathcal{O}/\pi\mathcal{O}$, the residue field of the valuation ν.

This chamber has $(n+1)$ facets, obtained by omitting one lattice in the chain above, say the i^{th} facet is given by

$$L_0 \subset L_1 \subset \cdots \subset L_{i-1} \subset L_{i+1} \subset \cdots \subset L_n \subset \pi^{-1} L_0.$$

Any other chamber sharing this facet with the former one is given by

$$L_0 \subset L_1 \subset \cdots \subset L_{i-1} \subset L \subset L_{i+1} \subset \cdots \subset L_n \subset \pi^{-1} L_0, \tag{2.7}$$

where L/L_{i-1} is a one-dimensional subspace of the two-dimensional $\varkappa$-vector space L_{i+1}/L_{i-1}.

Theorem 2.14. *The simplicial complex X defined above is an affine building in the sense of Definition 2.10.*

The reader is referred to [Gar97, (19.2)] for the proof of this theorem.

To understand apartments in X, we introduce the notion of a *frame* in K^n, namely, an unordered set $\{f_1, f_2, \ldots, f_n\}$ of one-dimensional subspaces in K^n, which span the entire space altogether: $f_1 + f_2 + \cdots + f_n = K^n$. To each frame $F = \{f_1, f_2, \ldots, f_n\}$ we associate an apartment A_F consisting of all simplices with vertices whose representative lattices are expressible as

$$L = M_1 + M_2 + \cdots + M_n,$$

where M_i is an $\mathcal{O}$-lattice in f_i.

The natural action of $SL(n, K)$ on K^n gives rise to an action of $SL(n, K)$ on $\bar{\mathcal{L}}$ and therefore to a simplicial action on X. This enables us to talk about the action of any subgroup of $SL(n, K)$ on X, but for now we discuss the transitivity of the action of the entire group.

Lemma 2.15. *The action of $SL(n, K)$ on X is strongly transitive on the pairs $(apartment, chamber)$.*

Proof. Consider two apartments A_F and $A_{F'}$, which correspond to the frames $F = \{f_1, f_2, \ldots, f_n\}$ and $F' = \{f'_1, f'_2, \ldots, f'_n\}$. The weak transitivity on apartments follows from the fact that there is an element g in $SL(n, k)$ which sends f_i to f'_i for each i. This g will send the frame F to F', hence the apartment A_F to $A_{F'}$.

To gain the strong transitivity, we show that the stabilizer of an apartment acts transitively on the chambers within the apartment. Note that since the apartments are connected chamber complexes, it is enough to show that for each pair of adjacent chambers there is a group element sending one to another.

Suppose we have two adjacent chambers, given by (2.6) and (2.7) within an apartment corresponding to frame $F = \{f_1, f_2, \dots, f_n\}$. Then there exist two indices i_1 and i_2 such that the difference between the two chambers, the lattices L_i and L, are given in L_{i+1}/L_{i-1} by f_{i_1} and f_{i_2} respectively. The linear transformation of k^n, which interchanges f_{i_1} and f_{i_2}, but keeps any other f_i untouched, transforms one chamber to another. This transformation stabilizes the set F and therefore the apartment A_F as well. □

We finish this discussion with the following simple observation.

Remark 2.16. The stabilizer of a vertex $x \in X$ is a subgroup of $SL(n, K)$ conjugate in $SL(n, K)$ to $SL(n, \mathcal{O})$ (this follows, for example, from the discussion in [Gar97, (19.3)]). This means, in particular, that the coefficients of the characteristic polynomial of any element of such stabilizer are in $\mathcal{O}$.

Chapter 3

Asymptotic dimension

In this chapter we study the notion of asymptotic dimension, the coarse geometry counterpart of Ostrand's covering dimension. We start with the definition and basic properties in Section 3.1. In Section 3.2 we apply the notion of asymptotic dimension to the case of a group. Finally, we close with Section 3.3, where we give examples of spaces with finite asymptotic dimension: ultrametric spaces, affine Bruhat-Tits buildings associated to a discretely valued field, and symmetric spaces.

3.1 Asymptotic dimension: the definition

The notion of *asymptotic dimension* of a metric space was introduced by Gromov in [Gro93] as an asymptotic property of a metric space. We shall use the following definition taken from [Roe03].

Definition 3.1. A metric space X is said to have asymptotic dimension not more than n if for any $d > 0$ there exists a uniformly bounded cover of X with the property that any ball of radius d in X meets not more than $(n+1)$ elements of the cover.

The minimal bound n from the definition above is called the *asymptotic dimension* of X and denoted $\operatorname{asdim} X$.

Now we record a few simple properties of this notion.

Lemma 3.2. *If* $Y \subseteq X$, *then* $\operatorname{asdim} Y \leqslant \operatorname{asdim} X$.

Proof. If $\{U_i\}_{i\in I}$ is a uniformly bounded cover of X, then $\{Y \cap U_i\}_{i\in I}$ is a uniformly bounded cover of Y. If the d-ball meets some element of the latter cover, it meets the corresponding element of the former as well, whence the lemma. $\square$

Lemma 3.3. $\operatorname{asdim}(X \cup Y) = \max\{\operatorname{asdim} X, \operatorname{asdim} Y\}$.

The proof of this equality can be found in [BD01, Finite Union Theorem]. Since only the fact of the finiteness of the asymptotic dimension is of immediate interest to us here, we will prove a weaker result:

$$\operatorname{asdim}(X \cup Y) \leqslant \operatorname{asdim} X + \operatorname{asdim} Y + 1. \tag{3.1}$$

Proof. Let $d > 0$ be given. According to Definition 3.1, construct a uniformly bounded cover $\mathcal{U} = \{U_i\}_{i\in I}$ of X and a uniformly bounded cover $\mathcal{V} = \{V_j\}_{j\in J}$ of Y. Then $\mathcal{U} \cup \mathcal{V}$ is a uniformly bounded cover of $X \cup Y$. Any d-ball in $X \cup Y$ meets not more than $(\operatorname{asdim} X + 1)$ elements of $\mathcal{U}$ and also not more than $(\operatorname{asdim} Y + 1)$ elements of $\mathcal{V}$. Thus this ball meets at most $(\operatorname{asdim} X + \operatorname{asdim} Y + 2)$ elements of $\mathcal{U} \cup \mathcal{V}$, which proves (3.1). $\square$

Lemma 3.4. $\operatorname{asdim}(X \times Y) \leqslant \operatorname{asdim} X + \operatorname{asdim} Y$.

The proof of this formula can be found in [Roe03, Proposition 9.11]. Here we will prove that

$$\operatorname{asdim}(X \times Y) \leqslant (\operatorname{asdim} X + 1)(\operatorname{asdim} Y + 1) - 1.$$

Proof. Let $d > 0$ be given and the uniformly bounded covers $\mathcal{U}$ and $\mathcal{V}$ of X and Y respectively are arranged in accordance with Definition3.1. Consider

$$\mathcal{U} \times \mathcal{V} = \{U \times V \mid U \in \mathcal{U}, V \in \mathcal{V}\}.$$

This is a uniformly bounded cover of $X \times Y$. Any d-ball in $X \times Y$ centered at (x, y) lies in the product of the d-ball in X centered at x and the d-ball in Y centered at y. These balls meet not more than $(\operatorname{asdim} X + 1)$ and $(\operatorname{asdim} Y + 1)$ elements of the covers $\mathcal{U}$ and $\mathcal{V}$ respectively, therefore their product meets not more than $(\operatorname{asdim} X + 1)(\operatorname{asdim} Y + 1)$ elements of $\mathcal{U} \times \mathcal{V}$. This proves the desired inequality. $\square$

3.2 Asymptotic dimension of groups

Given a finitely generated group Γ, one may regard it as a (discrete) metric space, namely, fix a symmetrized finite set of generators $S = S^{-1}$ and define the word-length metric on Γ to be

$$\mathrm{dist}(\gamma_1, \gamma_2) = \text{length of a shortest word in } S \text{ representing } \gamma_1^{-1}\gamma_2. \tag{3.2}$$

Different metrics for the same group, which arise this way, are Lipschitz equivalent. Indeed, if we start with two different finite generating sets $S = S^{-1}$ and $T = T^{-1}$, then we have a dictionary expressing any element of the first generating set as a word in the second one and a similar dictionary for the second generating set. Both those dictionaries are finite, and one can take the Lipschitz constant of the equivalence to be the length of the longest word in those dictionaries. Bearing this in mind, we give

Definition 3.5. A finitely generated group Γ has asymptotic dimension n if the underlying discrete metric space, equipped with the word metric, has asymptotic dimension n.

Note that the metric, defined via (3.2), restricts to a metric on any subset of Γ, in particular, we can equip any subgroup of Γ (not necessarily finitely generated) with such a compatible metric.

Definition 3.6. A (not necessarily finitely generated) group G has asymptotic dimension not more than n if for any finitely generated subgroup H of G, $\operatorname{asdim} H \leqslant n$. The infimum of all such n over all possible finitely generated subgroups of G is called the asymptotic dimension of G.

Remark 3.7. Definition 3.6 is compatible with Definition 3.5 in the following sense. If H is a finitely generated subgroup of a finitely generated group G, then $\operatorname{asdim} H \leqslant \operatorname{asdim} G$. Indeed, the word metrics, induced on H by its own set of generators and by the restriction of the corresponding metric on G, coming from its set of generators, are coarsely equivalent. This means that the "intrinsic" $\operatorname{asdim} H$ and the one, calculated using the metric on G, coincide. The latter therefore does not exceed $\operatorname{asdim} G$.

Now we recall the notion of a proper group action.

Definition 3.8. An action of a metrizable group Γ on a metrizable space X is called *proper* if there exist finite subgroups H_i of G, $i \in I$, and the H_i-invariant subspaces X_i of X such that the natural maps $p_i : G \times_{H_i} X_i \to X$ are G-homeomorphisms onto their images with

$$\bigcup_{i \in I} p_i(G \times_{H_i} X_i) = X.$$

Remark 3.9. It is more natural to define the notion of a proper action for a topological group, acting on a topological space, but within the scope of this work we appeal to discrete groups and metric spaces only.

Remark 3.10. In what follows we will use the fact that the isotropy subgroups of a proper group action are finite.

Among all spaces on which Γ acts properly, there is a special one, defined up to homotopy (see [BCH94]):

Definition 3.11. A metrizable space $\underline{E}\Gamma$ is called universal for Γ, if Γ acts properly on it, and for any other space X on which Γ acts properly, there exists a Γ-equivariant continuous map $X \to \underline{E}\Gamma$, unique up to Γ-equivariant homotopy.

Along with a notion of a proper action, there exists a notion of a metrically proper action.

Definition 3.12. An isometric action of a discrete group G on a metric space X is called *metrically proper* if for all bounded subsets $Y \subseteq X$ the set

$$\{g \in G |\, g.Y \cap Y \neq \emptyset\}$$

is finite.

Remark 3.13. If a discrete group G admits a metrically proper action on a space X of finite asymptotic dimension, then G has finite asymptotic dimension in its own metric, and its dimension does not exceed the one of X. To see this note that in the setup above G is coarsely equivalent to the orbit Gx_0, $x_0 \in X$, because of properness, while the latter is a subspace of X, and so its asymptotic dimension does not exceed the one of X.

3.3 Examples

In chapters which follow we shall use affine buildings and symmetric spaces in our constructions of proper actions. Here we collect some results about their asymptotic dimension.

3.3.1 Ultrametric spaces

Definition 3.14. A metric space X is called *ultrametric* if its distance function $\mathrm{dist}(\cdot,\cdot)$ satisfies the following strong triangle property:

$$\mathrm{dist}(x,z) \leq \max(\mathrm{dist}(x,y), \mathrm{dist}(y,z)) \qquad \text{for any } x,y,z \in X. \tag{3.3}$$

Example 3.15. Starting from a discrete valuation ν on a field F as discussed in Section 2.2, one can define a norm on F

$$|x|_\nu = \left(\frac{1}{2}\right)^{\nu(x)}, \qquad x \in F,$$

where we let $\left(\frac{1}{2}\right)^\infty = 0$ for the case $x = 0$. (One can take any real number between 0 and 1 instead of $\frac{1}{2}$ in this definition, for the p-adic valuation on $\mathbb{Q}$ it is customary to take $\frac{1}{p}$.)

With respect to this norm F becomes a metric space if we define the metric in the following fashion:

$$\mathrm{dist}(x,y) = |x-y|_\nu.$$

The last condition in Definition 2.6 of a discrete valuation corresponds to condition (3.3), which means that a discretely valued field is an ultrametric space with regard to the associated metric. Now we will calculate its asymptotic dimension.

Theorem 3.16. *For an ultrametric space X its asymptotic dimension is* 0.

Proof. Suppose $d > 0$ is chosen. We will construct a cover of X with the property that any d-ball lies entirely in one of the elements of the cover.

According to the ultrametric condition (3.3) the closed d-ball centered at y has the property that any two points x, z inside such a ball are not more than distance d apart. This means that any point not in this ball has distance strictly more than

d from any point inside the ball, and, moreover, any two d-balls with different centers are at least d-disjoint.

The collection of all d-balls in X is a d-disjoint cover of X with required properties. □

3.3.2 Affine buildings

The goal of this section is to prove that affine buildings corresponding to discrete valuations have finite asymptotic dimension in the intrinsic metric. Our argument will be based on the following result of Bell and Dranishnikov:

Theorem 3.17 (cf. [BD04, Theorem 1]). *Let $f : X \to Y$ be a Lipschitz map of a (geodesic) metric space to a metric space. If for every $R > 0$ the inverse images of the R-balls $f^{-1}(B_R(y))$ are isometric for all $y \in Y$ and have asymptotic dimension m, then* $\operatorname{asdim} X \leqslant \operatorname{asdim} Y + m$.

Throughout this section we fix a discretely valued field K with valuation ν and denote by X the building associated to $SL(n, K)$.

We shall show that X is coarsely equivalent to a certain matrix group equipped with a pseudometric coming from a length function and thus reduce the computation of the asymptotic dimension of X to the one of that matrix group. The basic definition follows.

Definition 3.18. A *length function* on a group G is function $l : G \to [0, \infty)$ such that

1. $l(1) = 0$,
2. $l(g) = l(g^{-1})$, and
3. $l(gh) \leqslant l(g) + l(h)$

for any g, h in G.

Following [GHWar], we introduce a length function on $SL(n, K)$ in the following way:

$$l(g) = -\min_{1\leqslant i,j\leqslant n} \left\{\nu(g_{ij}), \nu(g^{ij})\right\}, \qquad g \in SL(n, K). \tag{3.4}$$

Here we use the standard notation: g_{ij} is the (i,j)-th entry of matrix g, and g^{ij} is the (i,j)-th entry of its inverse. This length function defines a pseudometric on $SL(n,K)$ and on its subgroups by

$$dist(g,h) = l(g^{-1}h), \qquad g,h \in SL(n,K). \tag{3.5}$$

We start with some technical computations. Let G be the group $SL(n,K)$ and let B denote its upper-triangular subgroup:

$$B = \{g \in G \mid g_{ij} = 0 \text{ for } 1 \leqslant j < i \leqslant n\}.$$

Further, let N be the uni-upper-triangular subgroup of B, that is its elements satisfy $g_{ii} = 1$ for all $i = 1, \dots, n$. Let A denote the subgroup of diagonal matrices in B. All these subgroups inherit the pseudometric (3.5) from G.

Lemma 3.19. *Let a and b be two real numbers. Then*

$$-\min\left\{0, a, \frac{1}{2}b\right\} \leqslant -\min\{0,a,b\} \leqslant -2\min\left\{0, a, \frac{1}{2}b\right\}.$$

Proof. Note that each term of the inequality is a non-negative number. If $b \geqslant 0$, then the inequalities in question become

$$-\min\{0,a\} \leqslant -\min\{0,a\} \leqslant -2\min\{0,a\},$$

which is obviously true. If $a \geqslant 0$, while $b < 0$, the inequalities become $-\frac{1}{2}b \leqslant -b \leqslant -b$, which is again true. Thus we concentrate on the case when both a and b are negative. We split the rest of the proof into three cases.

If $a < b < \frac{1}{2}b$, then the inequalities in our lemma become $-a \leqslant -a \leqslant -2a$, and this is a true system of inequalities.

If $b \leqslant a \leqslant \frac{1}{2}b$, then we have $-a \leqslant -b \leqslant -2a$. This is true.

Finally, if $b < \frac{1}{2}b < a$, then we want to check $-\frac{1}{2}b \leqslant -b \leqslant -b$. Again, this is a true system of inequalities. □

Lemma 3.20. $\operatorname{asdim} N = 0$.

Proof. Define a length function $\tilde{l}$ on N in the following fashion. For $g \in N$ let

$$\tilde{l}(g) = -\min_{1\leqslant i<j\leqslant n}\left\{0, \frac{1}{2^{j-i-1}}\nu(g_{ij}), \frac{1}{2^{j-i-1}}\nu(g^{ij})\right\}. \tag{3.6}$$

Before going further, we mention that $\tilde{l}$ is just a slight modification of the length function l in (3.4), where the effect of each matrix entry diminishes exponentially as we move away from the diagonal. In fact, for 2×2 matrices $\tilde{l}$ and l coincide. To see this, recall that $\nu(1) = 0$ and $\nu(0) = +\infty$. For any element $g = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ of N we have

$$l(g) = -\min\{\nu(1), \nu(z), \nu(0), \nu(-z)\} = -\min\{0, \nu(z)\} = \tilde{l}(g).$$

Note that we do not claim that $\tilde{l}$ defines a length function on the entire G, but rather on its subgroup N only.

We shall show that $\tilde{l}$ is coarsely equivalent to the restriction of the original length function l onto N and that N becomes an ultrametric space with respect to $\tilde{l}$. This will allow us to appeal to Theorem 3.16.

To simplify notation, we shall concentrate on the case of 3×3 matrices; the general case is similar.

We start by checking that $\tilde{l}$ is indeed a length function on N. The only non-trivial condition from Definition 3.18 which $\tilde{l}$ does not satisfy right away is the last one. Take two elements of N:

$$g = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} 1 & u & w \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix}, \; x, y, z, u, v, w \in K.$$

For convenience, let us rewrite (3.6) for the case of g, h, and their product explicitly:

$$\tilde{l}(g) = -\min\left\{0, \nu(x), \nu(y), \frac{1}{2}\nu(z), \frac{1}{2}\nu(xy - z)\right\},$$
$$\tilde{l}(h) = -\min\left\{0, \nu(u), \nu(v), \frac{1}{2}\nu(w), \frac{1}{2}\nu(uv - w)\right\},$$

$$gh = \begin{pmatrix} 1 & x+u & xv+z+w \\ 0 & 1 & y+v \\ 0 & 0 & 1 \end{pmatrix}, \text{ and}$$

$$\tilde{l}(gh) = -\min\left\{0, \nu(x+u), \nu(y+v), \frac{1}{2}\nu(xv+z+w), \frac{1}{2}\nu(xy+uv+yu-z-w)\right\}.$$

Let us show that *each* of the quantities

$$0, \nu(x+u), \nu(y+v), \frac{1}{2}\nu(xv+z+w), \text{ and } \frac{1}{2}\nu(xy+uv+yu-z-w) \tag{3.7}$$

is greater or equal to

$$m = \min\left\{0, \nu(x), \nu(y), \frac{1}{2}\nu(z), \frac{1}{2}\nu(xy-z), \nu(u), \nu(v), \frac{1}{2}\nu(w), \frac{1}{2}\nu(uv-w)\right\}. \tag{3.8}$$

It is sufficient to show that *each* quantity in (3.7) is greater or equal to *some* quantity from the list in (3.8).

1. $0 \geqslant 0$.

2. $\nu(x+u) \geqslant \min\{\nu(x), \nu(u)\}$ in accordance with the additive property of a discrete valuation. Obviously $\min\{\nu(x), \nu(u)\} \geqslant m$.

3. $\nu(y+v) \geqslant m$ in a similar fashion.

4. $\frac{1}{2}\nu(xv+z+w) \geqslant \frac{1}{2}\min\{\nu(xv), \nu(z), \nu(w)\}$ to start with. Note that both $\frac{1}{2}\nu(z)$ and $\frac{1}{2}\nu(w)$ occur in (3.8), and we only need to check that $\frac{1}{2}\nu(xv) \geqslant m$. If the last inequality were not true, then $\frac{1}{2}\nu(xv)$ would be strictly less than any element from the list in (3.8), in particular, $\nu(x)$ and $\nu(v)$. But this gives us a contradiction, for the average $\frac{1}{2}\nu(xv) = \frac{1}{2}(\nu(x)+\nu(v))$ can not be strictly less than both of its components $\nu(x)$ and $\nu(v)$.

5. $\frac{1}{2}\nu(xy+uv+yu-z-w) \geqslant \frac{1}{2}\min\{\nu(xy), \nu(uv), \nu(yu), \nu(z), \nu(w)\}$ in accordance with the additive property of a discrete valuation. Now $\frac{1}{2}\nu(xy) = \frac{1}{2}(\nu(x)+\nu(y))$, and this quantity is greater or equal at least one of $\nu(x)$ and $\nu(y)$ from the list in (3.8). Similarly $\frac{1}{2}\nu(uv)$ and $\frac{1}{2}\nu(yu)$ are greater or equal at least one quantity among $\nu(u), \nu(v)$, and $\nu(y)$ respectively. Valuations of

z and w occur in (3.8) themselves, so that we even do not need to do any arithmetics for them.

The statement that we have just proven can be written as

$$\min\left\{0, \nu(x+u), \nu(y+v), \frac{1}{2}\nu(xv+z+w), \frac{1}{2}\nu(xy+uv+yu-z-w)\right\} \geqslant$$
$$\min\{\min\left\{0, \nu(x), \nu(y), \frac{1}{2}\nu(z), \frac{1}{2}\nu(xy-z)\right\},$$
$$\min\left\{0, \nu(u), \nu(v), \frac{1}{2}\nu(w), \frac{1}{2}\nu(uv-w)\right\}\}.$$

Multiplying both sides of this inequality by -1 and using the identity $-\min\{a,b\} = \max\{-a,-b\}$, one obtains

$$\tilde{l}(gh) \leqslant \max\left\{\tilde{l}(g), \tilde{l}(h)\right\}.$$

Thus $\tilde{l}$ is indeed a length function on N, and, moreover, an ultrametric one. It gives rise to an ultrametric pseudodistance $\widetilde{dist}(\cdot,\cdot)$:

$$\widetilde{dist}(g,h) = \tilde{l}(g^{-1}h) = \tilde{l}((g^{-1}k)(k^{-1}h)) \leqslant \max\left\{\tilde{l}(g^{-1}k), \tilde{l}(k^{-1}h)\right\} =$$
$$\max\left\{\widetilde{dist}(g,k), \widetilde{dist}(k,h)\right\}, \quad g,h,k \in N.$$

Finally, we want to show that l and $\tilde{l}$ are coarsely equivalent on N. In fact, for any $g \in N$ we have $\tilde{l}(g) \leqslant l(g) \leqslant 2\tilde{l}(g)$. To see this, let

$$g = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x,y,z \in K,$$

so that

$$l(g) = -\min\{0, \nu(x), \nu(y), \nu(z), \nu(xy-z)\} \quad \text{and}$$
$$\tilde{l}(g) = -\min\left\{0, \nu(x), \nu(y), \frac{1}{2}\nu(z), \frac{1}{2}\nu(xy-z)\right\}.$$

An invocation of Lemma 3.19 with $a = \min\{\nu(x), \nu(y)\}$ and $b = \min\{\nu(z), \nu(xy-z)\}$

gives us the desired inequalities. Note that for the general case of $n \times n$ matrices we need to use Lemma 3.19 repeatedly $(n-2)$ times. $\square$

Lemma 3.21. $\operatorname{asdim} A = n - 1$.

Proof. Take an element g in A and define its "coarse version" $\tilde{g}$ by letting $\tilde{g}_{ij} = \pi^{\nu(g_{ij})}$ (here π denotes a uniformizer of the valuation ν). We have:

$$dist(g, \tilde{g}) = l(g^{-1}\tilde{g}) = -\min_{1\leqslant i\leqslant n} \left\{\nu((g_{ii})^{-1}\tilde{g}_{ii}), \nu((\tilde{g}_{ii})^{-1}g_{ii})\right\} = 0.$$

Thus A is coarsely equivalent to the set of all such $\tilde{g}$, that is, to its own subgroup

$$\tilde{A} = \left\{ \left. \begin{pmatrix} \pi^{\nu_1} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \pi^{\nu_{n-1}} & 0 \\ 0 & \cdots & 0 & \pi^{-\nu_1-\cdots-\nu_{n-1}} \end{pmatrix} \right| \nu_1, \ldots, \nu_{n-1} \in \mathbb{Z} \right\}.$$

This subgroup, however, is isomorphic to $\mathbb{Z}^{n-1}$ (the matrix shown above corresponds to a point $(\nu_1, \ldots, \nu_{n-1})$ in $\mathbb{Z}^{n-1}$) with the length function

$$\tilde{l}((\nu_1, \ldots, \nu_{n-1})) = \max\left\{|\nu_1|, \ldots, |\nu_{n-1}|, |\nu_1 + \cdots + \nu_{n-1}|\right\}.$$

This length function is bounded from below by the l^∞ length function and from above by the l^1 length function; for a finite-dimensional space all these functions are Lipschitz equivalent and furnish a metric space of dimension $(n-1)$ (see [Gro93] for details).

It is known that asymptotic dimension is a coarse invariant (see [Gro93]), therefore $\operatorname{asdim} A = n - 1$ as well. $\square$

Now we are in a position to state

Theorem 3.22. *The affine building X has finite asymptotic dimension.*

Proof. First we note that our building X is coarsely equivalent to the group $G = SL(n, K)$. Then if C is the maximal compact subgroup of G, then we can represent G as CB, where B consists of upper-triangular matrices, so that G itself is coarsely equivalent to B.

Since asymptotic dimension is a coarse invariant, it will be sufficient to show that B has finite asymptotic dimension in order to prove that X does.

Define a homomorphism $f : B \to A$ in the following fashion:

$$f : \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ 0 & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & g_{nn} \end{pmatrix} \mapsto \begin{pmatrix} g_{11} & 0 & \cdots & 0 \\ 0 & g_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & g_{nn} \end{pmatrix}.$$

We claim that f is a 1-Lipschitz map. Since f is a homomorphism, we need to compare $l(f(g))$ and $l(g)$ for $g \in B$. According to (3.4),

$$l(g) = -\min_{1 \leqslant i \leqslant j \leqslant n} \left\{ \nu(g_{ij}), \nu(g^{ij}) \right\},$$

whilst

$$l(f(g)) = -\min_{1 \leqslant i = j \leqslant n} \left\{ \nu(g_{ij}), \nu(g^{ij}) \right\},$$

which is obviously not more than the former quantity.

Given an $R > 0$, the R-ball in A centered at a is

$$B_R(a) = \{ g \in A | \, |\nu(g_{ii}) - \nu(a_{ii})| \leqslant R, i = 1, \dots, n \}.$$

Therefore

$$f^{-1}(B_R(a)) = \{ g \in B | \, |\nu(g_{ii}) - \nu(a_{ii})| \leqslant R, i = 1, \dots, n \}.$$

Left multiplication by a^{-1} is an isometry between $f^{-1}(B_R(a))$ and the inverse of the R-ball centered at $1 \in A$. The latter set is

$$\{ g \in B | \, |\nu(g_{ii})| \leqslant R, i = 1, \dots, n \} = \bigcup_{\substack{g_{ii} \in K \\ |\nu(g_{ii})| \leqslant R}} \begin{pmatrix} g_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & g_{nn} \end{pmatrix} N =$$

$$\bigcup_{\substack{\nu_1, \dots, \nu_n \in \mathbb{Z} \\ -R \leqslant \nu_1, \dots, \nu_n \leqslant R}} \begin{pmatrix} \pi^{\nu_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \pi^{\nu_n} \end{pmatrix} \bigcup_{\substack{g_{ii} \in K \\ \nu(g_{ii}) = 0}} \begin{pmatrix} g_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & g_{nn} \end{pmatrix} N.$$

We claim that the subgroup

$$\tilde{N} = \bigcup_{\substack{g_{ii} \in K \\ \nu(g_{ii})=0}} \begin{pmatrix} g_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & g_{nn} \end{pmatrix} N$$

is coarsely equivalent to N. Indeed, for any element of $\tilde{N}$, say

$$g' = \begin{pmatrix} g_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & g_{nn} \end{pmatrix} g, \text{ where } g \in N, \text{ and } \nu(g_{ii}) = 0 \text{ for } i = 1, \dots, n,$$

we have an element

$$g'' = \begin{pmatrix} g_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & g_{nn} \end{pmatrix} g \begin{pmatrix} g_{11}^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & g_{nn}^{-1} \end{pmatrix}$$

in N (here we are using the fact that N is a normal subgroup of B) with

$$dist(g', g'') = l\left(\begin{pmatrix} g_{11}^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & g_{nn}^{-1} \end{pmatrix}\right) = 0.$$

We know that by the virtue of Lemma 3.20 that $\operatorname{asdim} N = 0$, hence $\operatorname{asdim} \tilde{N} = 0$, that is the inverse of the R-ball is a finite union of sets of asymptotic dimension 0. According to Lemma 3.3, its asymptotic dimension is 0 as well, and so is the asymptotic dimension of every $f^{-1}(B_R(a))$.

Now we are ready to apply Theorem 3.17 and conclude that $\operatorname{asdim} B$ does not exceed $\operatorname{asdim} A$, which is $(n-1)$, according to Lemma 3.21, but since B contains A as a subgroup, asymptotic dimension of B is precisely $(n-1)$. $\square$

3.3.3 Symmetric spaces

The following theorem was proven in [BD04] by means of connecting the Hirsch length and the asymptotic dimension of a nilpotent group:

Theorem 3.23 (cf. [BD04, Theorem 12]). *Let G be a connected Lie group and let K be its maximal compact subgroup. Then (in the G-invariant metric)* $\operatorname{asdim} G/K = \dim G/K$.

We will need the following consequence of this theorem:

Corollary 3.24. *The symmetric spaces $SL(n, \mathbb{C})/SU(n)$ and $SL(n, \mathbb{R})/SO(n)$ have finite asymptotic dimension for any natural n.*

Chapter 4

Groups of integral characteristic

In this chapter we study subgroups of $SL(n, K)$ over a finitely generated field K with the property that all the eigenvalues of every element are algebraic integers. In places, we shall elaborate on the case of $SL(2, K)$. We show that any finitely generated subgroup of $SL(n, K)$ admits an action on a finite-asymptotic-dimensional space (a finite product of symmetric spaces $SL(n, \mathbb{C})/SU(n, \mathbb{C})$ and $SL(n, \mathbb{R})/SU(n, \mathbb{R})$ and the affine p-adic buildings presented in Chapter 2), such that the restriction of this action on any subgroup with this integral property on its spectrum is proper, provided that the unipotent parts of such group are composed out of parts with uniformly bounded asymptotic dimension. This finite-dimensional space will depend on the field and the bounding dimension, but not on the particular group itself. In Section 4.1 we give the precise definition of a group of integral characteristic. In Section 4.2 we describe a proper action for a linear group over the rationals. Section 4.3 is devoted to the construction of an action for groups acting irreducibly on $\mathbb{C}^n$. Section 4.4 applies the irreducible technique from the previous section to the reducible action case and shows that it fails precisely for unipotent subgroups. Finally, in Sections 4.5 and 4.5 respectively we construct an action which is proper for boundedly composed unipotent subgroups, first for subgroups of $SL(2, K)$ and then for the general case of $SL(n, K)$.

4.1 Definition and basic properties

Definition 4.1. A subgroup Γ of $SL(n,\mathbb{C})$ is said to have an *integral characteristic* if the coefficients of the characteristic polynomial of every element of Γ are algebraic integers.

Remark 4.2. This is equivalent to the condition that the eigenvalues of every element of Γ are algebraic integers, since the ring of algebraic integers is integrally closed. Recall that an integral domain A is called integrally closed if any element of the fraction field of A, which is a root of a monic polynomial with coefficients in A, belongs to A itself.

Typical examples of groups of integral characteristic are unipotent groups, namely subgroups of the conjugates of the upper-triangular matrix groups with 1's along the diagonal. However, the entire class of groups of integral characteristic is much richer: for example, it contains all subgroups of $SL(n,\mathbb{Z})$.

4.2 Subgroups of $SL(n,\mathbb{Q})$

Even though not every subgroup of $SL(n,\mathbb{Q})$ is of integral characteristic, we start by studying this case, one of the simplest cases of a linear group. Later we will embed most of our groups of integral characteristic into $SL(n,\mathbb{Q})$ for an appropriate dimension n.

Let Γ be a finitely generated subgroup of $SL(n,\mathbb{Q})$. Taking the finite symmetrized set of generators of Γ, we can generate the entire group as a semigroup, and all the entries of any element of Γ are therefore obtained from a finite number of entries of the generating set by additions and multiplications. This observation allows us to treat Γ as defined over the ring $A=\mathbb{Z}[\frac{1}{s}]$, where s is the l.u.b. of all the denominators of the fractions participating in the entries of the generating set of Γ. If necessary, we will enlarge A so that s is the product of distinct primes and therefore we may assume that prime factorization of s is $p_1\cdot p_2\cdots p_m$ with nonrepeating terms.

For any index k between 1 and m the natural inclusions

$$\Gamma\subseteq SL(n,A)\subseteq SL(n,\mathbb{Q}_{p_k})$$

allow us to consider the action of Γ on the p_k-adic building of equivalence classes of lattices in $\mathbb{Q}_{p_k}^n$. Denote this affine building by X_{p_k}.

The action α_{p_k} of $SL(n, \mathbb{Q}_{p_k})$ (and, abusing notation, of Γ as well) on X_{p_k} is defined via the natural action on lattices.

The action α_∞ of Γ on the symmetric space $X_\infty = SL(n,\mathbb{R})/SO(n,\mathbb{R})$ is defined via the natural isometric action of $SL(n,\mathbb{R})$ and the inclusion of $SL(n, A)$ into $SL(n,\mathbb{R})$.

Finally, Γ acts on the product X of buildings and the symmetric space:

$$X = X_{p_1} \times X_{p_2} \times \cdots \times X_{p_m} \times X_\infty$$

via the diagonal action

$$\alpha = \alpha_{p_1} \times \alpha_{p_2} \times \cdots \times \alpha_{p_m} \times \alpha_\infty$$

Theorem 4.3. *The action α is proper.*

In the proof we will be using the following:

Lemma 4.4. *Let Γ be a group acting on a metric space X by isometries. Then the condition*

$$\forall x \in X \qquad \forall C > 0 \qquad \#\{g \in \Gamma | \operatorname{dist}(g.x, x) < C\} < \infty$$

implies the condition

$$\forall x \in X \qquad\qquad \#\{g \in \Gamma | g.B \cap B \neq \emptyset\} < \infty$$

for any ball B in X.

Proof. Given ball B, we take x to be its center, and C to be 3 times its radius. Then B and the translated ball $g.B$ do not meet each other, as long as $\operatorname{dist}(g.x, x) \geqslant C$, so that the second set is empty. But if $\operatorname{dist}(g.x, x) < C$, then the first condition guarantees that there are only finitely many group elements with such property. $\square$

Proof of the theorem. We have to show that for any compact K

$$\#\{g \in \Gamma | g.K \cap K \neq \emptyset\} < \infty.$$

Since it is enough to prove the statement for sufficiently large compact sets K only, we will enlarge the set K in the following way: let K_{p_k} be the projection of K into the k-th building, and K_∞ be the projection of K into the symmetric space. Clearly

$$K \subseteq K_{p_1} \times \cdots \times K_{p_m} \times K_\infty,$$

and we will be working with the latter set instead of K.

Now the set

$$\{g \in \Gamma | g.(K_{p_1} \times \cdots \times K_{p_m} \times K_\infty) \cap (K_{p_1} \times \cdots \times K_{p_m} \times K_\infty) \neq \emptyset\}$$

is actually

$$\bigcap_{k=1}^{m} \{g \in \Gamma | g.K_{p_k} \cap K_{p_k} \neq \emptyset\} \cap \{g \in \Gamma | g.K_\infty \cap K_\infty \neq \emptyset\},$$

therefore it is enough to show that the intersection of all $\{g \in \Gamma | g.B \cap B \neq \emptyset\}$ is finite (here B denotes one of $K_{p_1}, \dots, K_{p_m}$, or K_∞, and we may assume, by enlarging the compact sets, if necessary, that each B is indeed a closed ball).

Then it suffices to show that for any number C and any points $v_k \in X_{p_k}$ $(k = 1, \dots, m)$ and $x \in X_\infty$

$$\#\left(\bigcap_{k=1}^{m} \{g \in \Gamma | \operatorname{dist}_{X_{p_k}}(g.v_k, v_k) < C\} \cap \{g \in \Gamma | \operatorname{dist}_{X_\infty}(g.x, x) < C\}\right) < \infty \quad (4.1)$$

Indeed, note that the action of g on all our spaces is isometric, therefore Lemma 4.4 can be engaged for each building and the symmetric space, respectively.

By the triangle inequality it is sufficient to check condition (4.1) for the "base" vertices v_0 of the buildings and the "center" point x_0 of the symmetric space.

Now let $g = (g_{ij})$ be an element from the intersection of those sets. Here each matrix entry g_{ij} belongs to $\mathbb{Z}[\frac{1}{p_1}, \dots, \frac{1}{p_m}]$, namely

$$g_{ij} = \frac{a_{ij}}{\prod_{k=1}^{m} p_k^{n_{ijk}}}, \qquad a_{ij}, n_{ijk} \in \mathbb{Z}, \qquad (a_{ij}, p_k) = 1, \qquad k = 1, \dots, m.$$

The distance from $g.v_0$ to v_0 in the building X_{p_k} is bounded only if each matrix entry has bounded-from-above powers n_{ijk} in the denominator, which means the denominator itself should be bounded.

Since $\text{dist}_{X_\infty}(g.x_0, x_0) < C$ means $\cosh(\text{dist}_{X_\infty}(g.x_0, x_0)) < \cosh C$, and, according to [BH99],

$$\cosh(\text{dist}_{X_\infty}(g.x_0, x_0)) = \sum_{i,j=1}^{n} g_{ij}^2,$$

each matrix entry should be bounded. Together with the previous observation, this leads to the finite number of choices for a_{ij} and n_{ijk}, that is there are finitely many such elements g in the intersection. □

4.3 Groups with an irreducible action

In the discussion of the next results we fix an algebraically closed field K and a multiplicative monoid G in the matrix algebra $M_n(K)$.

The classical Burnside lemma (see, for example, [Bas80]) claims that:

Lemma 4.5. *If a group G acts irreducibly on K^n, then G contains a linear basis $\{g_1, g_2, \dots, g_{n^2}\}$ of $M_n(K)$.*

Proof. Consider the linear span of G with coefficients in K. Naturally it is a K-algebra with faithful module K^n, therefore Schur's lemma says that the division algebra $End_{KG}(K^n)$ is K and further $KG = End_K(K^n) = M_n(K)$. □

Now we assume that G is a subgroup of $SL(n, K)$ and Γ is a subgroup of G of integral characteristic. We will construct a matrix representation of G with the property that the image of Γ under that representation is arithmetic in most cases. The following result resembles the "trace" technique used by Zimmer in his study of Property T groups (see [Zim84b]), simplified by Higson (unpublished).

Lemma 4.6. *Let G be a subgroup of $SL(n,K)$. There exists a representation $\alpha : G \to GL(N,\mathbb{C})$ with $N = n^2$ and such that for every subgroup Γ of G, which has integral characteristic and acts irreducibly on K^n, its image $\alpha(\Gamma)$ is conjugate in $GL(N,\mathbb{C})$ to a subgroup of $GL(N,\mathbb{A})$. Here $\mathbb{A}$ denotes the field of algebraic numbers.*

Proof. For every $g \in M_n(K)$ define a complex-valued linear functional f_g on $KM_n(K) = M_n(K)$ by

$$f_g(h) = \mathrm{tr}(gh).$$

Take some basis $\{g_1, g_2, \ldots, g_N\}$ of $M_n(K)$ and consider

$$V = \mathrm{Span}\{f_{g_1}, \ldots, f_{g_N}\}.$$

Since the conditions defining f_g are linear with respect to g, λf_g can be formally written as $f_{\lambda g}$, and $f_g + f_h$ as f_{g+h}. According to our assumptions V contains every f_{g_j}, $j = 1, 2, \ldots, N$, therefore it contains their linear span, that is, every f_g with $g \in M_n(K)$. We have $\dim V = N = n^2$ and

$$V = \mathrm{Span}\{f_{g_1}, \ldots, f_{g_N}\} = \mathrm{Span}\{f_g\}_{g \in M_n}.$$

Any element $g \in G$ acts on V by

$$g.f_h = f_{gh}.$$

With respect to the basis $\{g_1, g_2, \ldots, g_N\}$ of $M_n(K)$ and the corresponding basis $\{f_{g_1}, f_{g_2}, \ldots, f_{g_N}\}$ of V this action is given by a matrix (α^g_{ij}), such that

$$g.f_{g_i}(h) = \sum_{j=1}^{N} \alpha^g_{ij} f_{g_j}(h) \tag{4.2}$$

Thus we have a representation $\alpha : G \to M_N(\mathbb{C})$.

To prove the injectivity of α it is enough to show that if

$$g.f_1(h) = f_1(h) \tag{4.3}$$

for some $g \in G$ and all $h = g_1, \ldots, g_N$, then $g = 1$. Taking the linear span over all such h, the condition is equivalent to the one with $h \in M_n(K)$. Plugging standard elementary matrices for h in (4.3), we deduce that g is indeed equal to 1.

In our construction the representation α depends on the choice of the basis $\{g_1, \ldots, g_N\}$. Suppose $\{\tilde{g}_1, \ldots, \tilde{g}_N\}$ is another basis of $M_n(K)$ which gives rise to a representation $\tilde{\alpha}$. Then, since the space V is the same for both bases, and any group element is represented as a linear operator on V, the images of any subgroup of G under α and $\tilde{\alpha}$ are conjugate in $GL(N, \mathbb{C})$. This allows us to choose any convenient basis in the proof of the last statement of the lemma.

Suppose that Γ is a subgroup of G, has integral characteristic and acts irreducibly on K^n. Then the basis $\{g_1, g_2, \ldots, g_N\}$ can be taken to be the one furnished by Lemma 4.5, that is, consisting of elements of Γ.

For $\gamma \in \Gamma$ write (4.2) as

$$\operatorname{tr}(\gamma g_i h) = \sum_{j=1}^{N} \alpha_{ij}^{\gamma} \operatorname{tr}(g_j h). \tag{4.4}$$

Then α_{ij}^{γ} are the solutions of the system of linear equations with algebraic coefficients, as long as $\gamma \in \Gamma$ (obviously each g_j is in Γ and we can test using h from the basis $\{g_1, g_2, \ldots, g_N\}$ only), and therefore have to be algebraic as well. This proves the last statement of the lemma. □

Corollary 4.7. *Suppose that there exists a finitely generated subring A of $\mathbb{A}$, such that trace of every element in Γ lies in A. Then α represents Γ within $GL(N, \tilde{A})$, where $\tilde{A}$ is a finitely generated subring of the field of fractions of A.*

Proof. We know that coefficients α_{ij}^{γ} are uniquely determined as solutions of (4.4), and it is enough to check these conditions for $h = g_1, g_2, \ldots, g_N$. Therefore (4.4) is equivalent to a system of linear equations on α_{ij}^{γ} with nonzero coefficients accompanying the unknowns $\operatorname{tr}(g_j g_k)$. Define

$$\tilde{A} = A[(\operatorname{tr}(g_1 g_1))^{-1}, (\operatorname{tr}(g_1 g_2))^{-1}, \ldots, (\operatorname{tr}(g_N g_N))^{-1}].$$

Then (4.4) can be solved by Gauss-Jordan elimination process, involving only ring operations in $\tilde{A}$, that is all α_{ij}^{γ} should belong to $\tilde{A}$. □

Now we are in a position to prove

Theorem 4.8. *Given G, a finitely generated subgroup of $SL(n, \mathbb{C})$, there exists an action of G on a finite-asymptotic-dimensional space X, such that for every subgroup Γ of G of integral characteristic acting irreducibly on $\mathbb{C}^n$ the induced action of Γ on X is proper.*

Proof. Construct an embedding $\alpha : G \to GL(N, \mathbb{C})$, as in Lemma 4.6. We know that G is finitely generated, thus we can assume that $\alpha : G \to GL(N, F)$, where F is a finitely generated field (generated, say, by all entries of the images of all generators of G). The images of elements of any subgroup Γ under α are algebraic, and invoking Lemma 2.2 we conclude that $\alpha(\Gamma) \subseteq GL(N, K)$, where K is an extension of $\mathbb{Q}$ of degree m.

Now we can identify

$$GL(N, K) \cong End(K^N) \subseteq End(\mathbb{Q}^{mN}) \cong GL(mN, \mathbb{Q})$$

and think of Γ as of a subgroup of $GL(mN, \mathbb{Q})$. According to Theorem 4.3 it acts properly on a finite-dimensional space.

We know by the virtue of Lemma 4.6 that any other subgroup $\tilde{\Gamma}$ of integral characteristic within G is conjugate to a subgroup of $GL(mN, \mathbb{Q})$ in our representation, so that it also acts properly. □

4.4 Diagonal Parts

Keep all the notations from Section 4.3.

Theorem 4.9. *Let Γ be a subgroup of G of integral characteristic and such that the restriction of the action α to Γ is improper. Then Γ is unipotent, that is conjugate to a subgroup of the uni-upper triangular matrices.*

Proof. If we have $\Gamma \subseteq G \subset SL(n, K)$, a subgroup of integral characteristic, which does not act irreducibly on $\mathbb{C}^n$, then we can find a tower of linear subspaces of K^n

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_r = K^n,$$

such that the induced action of Γ on each V_i/V_{i-1} is irreducible, and therefore in a suitable basis every element g of Γ is of the form

$$g = \begin{pmatrix} g_1 & * & \dots & * \\ 0 & g_2 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & g_r \end{pmatrix},$$

where each g_i is the restriction of the natural action of g onto the space V_i/V_{i-1}. Consider the map $g \mapsto g_i$. This map is a homomorphism

$$f_i : \Gamma \to \Gamma_i \subseteq GL(\dim(V_i/V_{i-1}), K),$$

such that Γ_i acts irreducibly on $K^{\dim(V_i/V_{i-1})}$, and the results of Section 4.3 apply to each Γ_i.

The map

$$f : \begin{pmatrix} g_1 & * & \dots & * \\ 0 & g_2 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & g_r \end{pmatrix} \mapsto (g_1, g_2, \dots, g_r)$$

is a homomorphism $\Gamma \to \Gamma_1 \times \cdots \times \Gamma_r$ whose kernel consists of unipotent matrices. Applying the composition of α and f gives us an action which is proper on each Γ_i, by dint of Lemma 4.6.

The isotropy of this action comes from the kernel of the homomorphism f, that is it consists of the unipotent matrices. □

We will proceed to construct a proper action for them in the next section.

4.5 Unipotent subgroups of $SL(2, \mathbb{C})$

In the following discussion all fields are supposed to be subfields of $\mathbb{C}$, that is, in particular of characteristic 0.

The following lemma reflects a standard fact from linear algebra.

Lemma 4.10. *Let K be a subfield of $\mathbb{C}$, θ an algebraic number over K of degree n, and $\theta_1 = \theta, \dots, \theta_n$ be its conjugates. Then*

$$\det \begin{pmatrix} 1 & \theta_1 & \dots & \theta_1^{n-1} \\ 1 & \theta_2 & \dots & \theta_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \theta_n & \dots & \theta_n^{n-1} \end{pmatrix} \neq 0.$$

Proof. Note that since the minimal polynomial of θ does not have repeated roots, all θ_i's are different, and then the lemma follows from the well known formula for the Vandermonde determinant. □

Now we show that the idea of the Vandermonde determinant can be generalized for the case of polynomials with independent coefficients in the following way:

Lemma 4.11. *Let K be a field and $p_1, \dots, p_n \in K[\vec{t}]$ be linearly independent over K. Then for any choice of mutually disjoint families $\vec{t}_1, \dots, \vec{t}_n$ of indeterminates*

$$\det \begin{pmatrix} p_1(\vec{t}_1) & \dots & p_n(\vec{t}_1) \\ \vdots & \ddots & \vdots \\ p_1(\vec{t}_n) & \dots & p_n(\vec{t}_n) \end{pmatrix} \neq 0 \tag{4.5}$$

Proof. We will argue by induction on the number n of polynomials. If $n = 1$, then the linear independence of $\{p_1\}$ means $p_1 \neq 0$ and so the determinant in question is nonzero as well.

Now suppose that the statement of the lemma was proven for any choice of $n-1$ linearly independent polynomials. Impose the lexicographic order on the set $\vec{t}$ of indeterminants. Then, working in the K-span of $p_1, \dots, p_n$, we can choose a new basis $\tilde{p}_1, \dots, \tilde{p}_n$ for the K-span of $p_1, \dots, p_n$, with the property that the leading term of each $\tilde{p}_2, \dots, \tilde{p}_n$ has order strictly less than that of $\tilde{p}_1$. The new system will also be linearly independent over K, and the matrix

$$\begin{pmatrix} \tilde{p}_1(\vec{t}_1) & \dots & \tilde{p}_n(\vec{t}_1) \\ \vdots & \ddots & \vdots \\ \tilde{p}_1(\vec{t}_n) & \dots & \tilde{p}_n(\vec{t}_n) \end{pmatrix} \tag{4.6}$$

will be obtained from the original matrix by elementary operations on the matrix columns, so that the determinants of two matrices coincide up to a possible sign discrepancy. The determinant of the latter matrix can be decomposed along the first row as

$$\tilde{p}_1(\vec{t}_1)\det\begin{pmatrix}\tilde{p}_2(\vec{t}_2) & \dots & \tilde{p}_n(\vec{t}_2)\\ \vdots & \ddots & \vdots\\ \tilde{p}_2(\vec{t}_n) & \dots & \tilde{p}_n(\vec{t}_n)\end{pmatrix}+\sum_{i=2}^{n}(-1)^i\tilde{p}_i(\vec{t}_1)(\text{polynomial in }\vec{t}_2,\dots,\vec{t}_n) \quad (4.7)$$

According to the induction hypothesis the coefficient of $\tilde{p}_1(\vec{t}_1)$ of the first term is nonzero, and, since no other term with $\vec{t}_1$ could possibly cancel $\tilde{p}_1(\vec{t}_1)$, the entire determinant is nonzero. □

Remark 4.12. The conclusion of this lemma remains true even if $p_1,\dots,p_n$ lie not in $K[\vec{t}]$, but in the fraction field $K(\vec{t})$. Indeed, there are finitely many denominators in $p_1,\dots,p_n$. After multiplying each p_i by all of them, we obtain $\tilde{p}_1,\dots,\tilde{p}_n$ in $K[\vec{t}]$, linearly independent over K if the original $p_1,\dots,p_n$ were. Now the lemma can be applied to the modified set $\tilde{p}_1,\dots,\tilde{p}_n$, yielding a nonzero determinant. The determinant for $p_1,\dots,p_n$ will then be that nonzero quantity, divided by the product of denominators of $p_1,\dots,p_n$.

We are ready to start the construction of a proper action. Given a finitely generated group $G\subset SL(2,K)$, where K is a finitely generated subfield of $\mathbb{C}$, we can think of K as of a two-step extension of $\mathbb{Q}$: first we pick some transcendence base $\vec{t}$ of K and obtain a purely transcendental extension $\mathbb{Q}(\vec{t})$, and then K itself is an algebraic extension of $\mathbb{Q}(\vec{t})$, and we know by the virtue of Lemma 2.2 that this is a finite extension, and so it is generated by an element θ, algebraic over $\mathbb{Q}(\vec{t})$ of degree n.

Given such K, fix a natural number m which will serve as a common bound on the dimensions of unipotent subgroups of G and pick $mn\#\{\vec{t}\}$ complex numbers, algebraically independent over K. We will group these numbers into mn tuples of $\#\{\vec{t}\}$ elements and denote these tuples as

$$\vec{t}_1,\vec{t}_2,\dots,\vec{t}_{mn},$$

and so set up a component-wise bijection between each $\vec{t_j}$ and the transcendence base $\vec{t}$. The existence of these numbers follows from the following inductive procedure: take the algebraic closure of K; it is countable since K is. Now pick the first number from the complement of the closure and extend the closure by adding this element. The new field is again countable, and so is its closure, and we can pick the second complex number from its complement, and so on.

Let $\theta = \theta_1, \theta_2, \ldots, \theta_n$ be conjugates of θ. Define n embeddings

$$\sigma_i : K = \mathbb{Q}(\vec{t})(\theta) \to \mathbb{Q}(\vec{t})(\theta_i) \subset \mathbb{C}, \qquad i = 1, \ldots, n$$

by letting σ_i to be an identity on $\mathbb{Q}(\vec{t})$ and sending θ to θ_i.

Also define mn embeddings

$$\sigma_{i+n} : K = \mathbb{Q}(\vec{t}) \to \mathbb{Q}(\vec{t_i}) \subset \mathbb{C}, \qquad i = 1, \ldots, mn$$

by identifying $\vec{t}$ with $\vec{t_i}$ component-wise. Extend these embeddings to K by mapping θ to some number, algebraic over $\mathbb{Q}(\vec{t_i})$ of degree n.

Now let $X \times X \times \cdots \times X$ be a product of $mn+n$ copies of the symmetric space $SL(2, \mathbb{C})/SU(2)$ and define an action of $SL(2, K)$ on this space by

$$g.(x_1, x_2, \ldots, x_{mn+n}) = (\sigma_1(g).x_1, \sigma_2(g).x_2, \ldots, \sigma_{mn+n}(g).x_{mn+n}), \tag{4.8}$$

where $\sigma_i(g).x_i$ on the right-hand side means applying σ_i to every entry of $g \in SL(2, K)$ and making the resulting matrix in $SL(2, \mathbb{C})$ act on the point x_i of the i-th symmetric space in the usual way.

Theorem 4.13. *The restriction of the action* (4.8) *to any unipotent subgroup of* G *of dimension* m *is proper.*

Proof. Any unipotent subgroup of dimension m is conjugate to

$$\begin{pmatrix} 1 & u_1\mathbb{Z} + \cdots + u_m\mathbb{Z} \\ 0 & 1 \end{pmatrix},$$

where $u_1, \dots, u_m$ are linearly independent over $\mathbb{Q}$ elements of K. Let

$$u_j = \sum_{l=0}^{n-1} p_{j,l}(\vec{t})\theta^l, \qquad p_{j,l}(\vec{t}) \in \mathbb{Q}(\vec{t}), j = 1, \dots, m.$$

Seeking a contradiction, assume that the restriction of the action (4.8) is improper. Then there exist infinitely many different m-tuples of integers $(z_1, \dots, z_m)$, such that the quantities

$$\sigma_i(z_1 u_1 + \cdots + z_m u_m), \qquad i = 1, \dots, mn + n \tag{4.9}$$

are all bounded. For $i = 1, \dots, n$ write this condition as

$$\sum_{j=1}^{m} z_j \sum_{l=0}^{n-1} p_{j,l}(\vec{t})\theta_i^l = \text{bounded}, \qquad i = 1, \dots, n,$$

that is

$$\begin{pmatrix} 1 & \theta_1 & \dots & \theta_1^{n-1} \\ 1 & \theta_2 & \dots & \theta_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \theta_n & \dots & \theta_n^{n-1} \end{pmatrix} \begin{pmatrix} \sum_{j=1}^m z_j p_{j,0}(\vec{t}) \\ \sum_{j=1}^m z_j p_{j,1}(\vec{t}) \\ \vdots \\ \sum_{j=1}^m z_j p_{j,n-1}(\vec{t}) \end{pmatrix} = \text{ bounded.}$$

According to Lemma 4.10, the θ-matrix has a (fixed) inverse, so that the sum $\sum_{j=1}^m z_j p_{j,l}(\vec{t})$ must be bounded for every $l = 0, \dots, n-1$.

Write this condition as

$$\begin{pmatrix} p_{1,0}(\vec{t}) & \cdots & p_{m,0}(\vec{t}) \\ \vdots & \ddots & \vdots \\ p_{1,n-1}(\vec{t}) & \cdots & p_{m,n-1}(\vec{t}) \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_m \end{pmatrix} = P(\vec{t}) \begin{pmatrix} z_1 \\ \vdots \\ z_m \end{pmatrix} = \text{ bounded.}$$

Transform the matrix $P(\vec{t})$ into an echelon form $\tilde{P}(\vec{t})$ by means of elementary operations with the rows over $\mathbb{Q}(\vec{t})$. Note that the linear independence of $u_1, \dots, u_m$ over $\mathbb{Q}$ means linear independence of columns of such a matrix, even in echelon form. In other words, the obtained echelon form has at least one nonzero row.

It could happen that some nonzero entries of the resulting matrix are rational. In this case, multiply the rows where this happens by some elements of $\mathbb{Q}(\vec{t})$ to ensure that all nonzero entries are in $\mathbb{Q}(\vec{t})\backslash\mathbb{Q}$ and continue to denote the resulting matrix by $\tilde{P}(\vec{t})$.

Now apply σ_{i+n} to the i-th row of $\tilde{P}(\vec{t})$ (that is, write $\vec{t_i}$ instead of $\vec{t}$ in that row) and add all the rows together to obtain a row which we denote as

$$(\tilde{p}_1(\vec{t_1},\ldots,\vec{t_n}),\ldots,\tilde{p}_m(\vec{t_1},\ldots,\vec{t_n})). \tag{4.10}$$

We claim that the elements $\tilde{p}_1,\ldots,\tilde{p}_m$ are linearly independent over $\mathbb{Q}$. Indeed, suppose that there exist m rational numbers $\lambda_1,\ldots,\lambda_m$, not all of them zeros, such that $\lambda_1\tilde{p}_1(\vec{t_1},\ldots,\vec{t_n})+\cdots+\lambda_m\tilde{p}_m(\vec{t_1},\ldots,\vec{t_n})=0$. Write this condition as

$$\sum_{j=1}^{m}\lambda_j\sum_{i=1}^{n}\tilde{p}_{j,i-1}(\vec{t_i})=0.$$

Note that each $\tilde{p}_{j,i-1}(\vec{t_i})$ is either 0 or algebraically independent from other $\tilde{p}_{j,i-1}$'s with the same i. This means

$$\sum_{j=1}^{m}\lambda_j\tilde{p}_{j,i-1}(\vec{t_i})=0$$

for all indices i for which not all $\tilde{p}_{j,i-1}(\vec{t_i})$ are zeros. This implies that the columns of $\tilde{P}$ and P as well are linearly dependent with the same choice of coefficients $\lambda_1,\ldots,\lambda_m$, contradictory to our assumption.

Now we apply $\sigma_{i+2n},\sigma_{i+2n},\ldots,\sigma_{i+mn-n}$ to the i-th row of $\tilde{P}(\vec{t})$ and add rows together. As a result, we obtain m rows, similar to (4.10), with the only difference that the variables in the j-th row are $\vec{t}_{n(j-1)+1},\ldots,\vec{t}_{nj}$. Thus we have:

$$\begin{pmatrix} p_1(\vec{t_1},\ldots,\vec{t_n}) & \cdots & p_m(\vec{t_1},\ldots,\vec{t_n}) \\ \vdots & \ddots & \vdots \\ p_1(\vec{t}_{mn-n+1},\ldots,\vec{t}_{mn}) & \cdots & p_m(\vec{t}_{mn-n+1},\ldots,\vec{t}_{mn}) \end{pmatrix}\begin{pmatrix} z_1 \\ \vdots \\ z_m \end{pmatrix} = \text{bounded}. \tag{4.11}$$

Lemma 4.11 says that the matrix on the left-hand side of (4.11) has a fixed inverse, which means $z_1,\ldots,z_m$ are all bounded, yielding a contradiction. □

Remark 4.14. Let us summarize that in the proof of the previous theorem we have shown that if for any unipotent subgroup the $(1,2)$ matrix entry of every element of it contains only m linearly independent over $\mathbb{Q}$ numbers, then the boundness of this matrix entry under all embeddings implies that this matrix entry actually takes a finite number of possible values.

This discussion culminates in

Theorem 4.15. *If Γ is a finitely generated subgroup of $SL(2,\mathbb{C})$, such that there exists a uniform bound on* asdim *of all unipotent subgroups of Γ, then there is a finite-asymptotic-dimensional space, on which Γ acts with the restriction of the action on any unipotent subgroup of Γ being proper.*

Proof. Let m be the common bound on asdim of all unipotent subgroups of Γ, and $X \times \cdots \times X$ be the product of symmetric spaces, equipped with the action (4.8). Then for any unipotent subgroup of Γ of dimension not more than m we have enough embeddings to apply Theorem 4.13. □

4.6 Unipotent subgroups of $SL(n,\mathbb{C})$

For the case of unipotent subgroups of linear groups of $n\times n$ matrices we will reduce the construction of a proper action to the case of $SL(2,K)$, already discussed in the previous section.

Suppose we have G, a unipotent subgroup of some finitely generated subgroup of $SL(n,K)$. Assume for now that G is uni-upper-triangular:

$$G \subseteq \{[g_{ij}] \mid g_{ii} = 1, g_{ij} = 0 \text{ for all } i = 1,\dots,n, \quad j < i\}. \tag{4.12}$$

We start by splitting G into separate "layers" $G_0 = G, G_1, \dots, G_{n-2}$ in the inductive fashion which we will describe in a moment, but first we give an informal illustration of this process for the case $n = 4$. Our goal is to get the following sequence:

$$G_0 = G = \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \rhd G_1 = \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rhd G_2 = \begin{pmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(The subscript of G corresponds to the thickness of the band with zeros above the diagonal with units.)

Now we give the formal definitions. Let G_1 be the normal subgroup of G comprised of matrices $[g_{ij}]$ for which $g_{ij} = 0$ for $j = i + 1$.

Define a group homomorphism

$$\phi_1 : G \to \prod_{i=1}^{n-1} \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| \, x \in K \right\} = H_1,$$
$$\phi_1([g_{ij}]) = \left(\begin{pmatrix} 1 & g_{12} \\ 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & g_{n-1,n} \\ 0 & 1 \end{pmatrix} \right).$$

Since $\ker \phi_1 = G_1$, ϕ_1 factors through G/G_1.

Suppose that we have already defined $G_1, \ldots, G_{k-1}$ for some $k \leqslant n - 2$. Let G_k be the (normal) subgroup of G_{k-1} of the following kind:

$$G_k = \{[g_{ij}] \in G_{k-1} \quad | \quad g_{ij} = 0 \text{ for } j = i + k\}.$$

Also define a group homomorphism

$$\phi_k : G_{k-1} \to \prod_{i=1}^{n-k} \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| \, x \in K \right\} = H_k,$$
$$\phi_k([g_{ij}]) = \left(\begin{pmatrix} 1 & g_{1,1+k} \\ 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & g_{n-k,n} \\ 0 & 1 \end{pmatrix} \right).$$

Again, ϕ_k factors through G_{k-1}/G_k.

Definition 4.16. We say that G is *boundedly composed* if there exists a natural number m, such that, in the notation introduced above,

1. $\operatorname{asdim} G_{n-2} \leqslant m$.
2. $\operatorname{asdim} \phi_k(G_{k-1}) \leqslant m$ for $k = 1, \ldots, n - 2$.

The number m is called parameter of the composition.

The idea of this definition is to introduce an effective way to test finite asymptotic dimensionality of G. The following paragraphs describe the structure of a boundedly composed group and also introduce some notation which we will use later.

Start with G_{n-2}. Since it has asymptotic dimension at most m, there are m elements of K, which we will call $u_1^{(1,n)}, \dots, u_m^{(1,n)}$, such that

$$G_{n-2} = \left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 & u_1^{(1,n)}\mathbb{Z} + \cdots + u_m^{(1,n)}\mathbb{Z} \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \right\}.$$

(more precisely, we should say that there are not more than m linearly independent over $\mathbb{Q}$ elements of K with such representation, but we will allow some "fake" generators $u_*^{(*,*)}$ to simplify the notation.)

In the same time, the image ϕ_{n-2} of G_{n-3} in H_2 has asymptotic dimension not more than m by definition, which means that there exist two m-tuples

$$u_1^{(1,n-1)}, \dots, u_m^{(1,n-1)} \quad \text{and} \quad u_1^{(2,n)}, \dots, u_m^{(2,n)}$$

of elements of K (again, we let them have the same cardinality m without requiring linear independence), such that $\phi_{n-2}(G_{n-3})$ lies in

$$\begin{pmatrix} 1 & u_1^{(1,n-1)}\mathbb{Z} + \cdots + u_m^{(1,n-1)}\mathbb{Z} \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & u_1^{(2,n)}\mathbb{Z} + \cdots + u_m^{(2,n)}\mathbb{Z} \\ 0 & 1 \end{pmatrix}.$$

Treating each "layer" G_k of G in this fashion, we obtain $\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$ m-tuples

$$\{u_1^{(i,j)}, \dots, u_m^{(i,j)}\}, \qquad i = 1, \dots, n-1, \qquad j = n-i+1, \dots, n, \tag{4.13}$$

such that for $j = 2, \dots, n$ the image $\phi_{j-1}(G_{j-2})$ is in

$$\begin{pmatrix} 1 & u_1^{(1,j)}\mathbb{Z} + \cdots + u_m^{(1,j)}\mathbb{Z} \\ 0 & 1 \end{pmatrix} \times \cdots \times \begin{pmatrix} 1 & u_1^{(n-j+1,n)}\mathbb{Z} + \cdots + u_m^{(n-j+1,n)}\mathbb{Z} \\ 0 & 1 \end{pmatrix}.$$

Now we are ready to describe the matrix entries of elements of G.

Lemma 4.17. *A uni-upper-triangular group G, boundedly composed with parameter m, is encoded by the m-tuples* (4.13) *in the following sense. For any group element $[g_{ij}] \in G$ its matrix entries satisfy the following conditions:*

$$\begin{aligned} g_{ij} &= 0 && \textit{for } i > j, \\ g_{ij} &= 1 && \textit{for } i = j, \\ g_{ij} &\in u_1^{(i,j)}\mathbb{Z} + \cdots + u_m^{(i,j)}\mathbb{Z} + P_{ij} && \textit{for } i < j, \end{aligned}$$

where

$$P_{ij} = \sum \left\{ u_k^{(\tilde{i},\tilde{j})} u_l^{(\hat{i},\hat{j})} \mathbb{Z} \;\middle|\; k, l = 1, \ldots, m, \quad \tilde{j} - \tilde{i} < j - i > \hat{j} - \hat{i} \right\}.$$

Proof. The first two conditions are trivial.

For the last one note that the condition describing P_{ij} means all possible $\mathbb{Z}$-combinations of the pairwise products of $u_*^{(*,*)}$ taken from $\phi_{\tilde{k}}(G_{\tilde{k}-1})$ for all $\tilde{k}$ less than k mapping g_{ij} into its spot in H_k. Now proceed by induction. For elements g_{ij} with $j = i + 1$ (that is lying along the first diagonal above the 1's) we can only pick a $\mathbb{Z}$-combination of the corresponding $u_*^{(i,j)}$, due to the nature of matrix multiplication. For the next diagonal g_{ij} can still get elements $u_*^{(i,j)}$, plus the portion coming from the *previous* diagonal (that's what P_{ij} stands for). For the diagonal after that one we can get entries from $u_*^{(i,j)}$ and also from two previous diagonals, and so on. □

It is natural to ask how the properties of being boundedly composed and having finite asymptotic dimension are connected. We will appeal to the following result of Bell and Dranishnikov.

Theorem 4.18 ([BD04, Theorem 7]). *Let $\phi : G \to H$ be a surjective homomorphism of a finitely generated group with kernel K. Then* $\operatorname{asdim} G \leqslant \operatorname{asdim} H + \operatorname{asdim} K$.

Using this theorem, we can prove

Theorem 4.19. *Boundedly composed groups have finite asymptotic dimension.*

Proof. Keep all the notation used in Definition 4.16. We will prove that $\operatorname{asdim} G_k < \infty$ inductively for $k = n - 2, \ldots, 0$.

Condition 1 of Definition 4.16 means that $\operatorname{asdim} G_{n-2} < \infty$.

For the inductive step suppose that $\operatorname{asdim} G_k < \infty$ has been proven for some k. Let H be the image of G_{k-1} under ϕ_k. According to condition 2 of Definition 4.16, $\operatorname{asdim} H < \infty$. Applying Theorem 4.18 for homomorphism ϕ_k, its image H and kernel G_k we conclude that $\operatorname{asdim} G_{k-1} < \infty$.

Since $G_0 = G$, we are done. □

Finally we are in a position to present the construction of a proper action. We will employ the same field embeddings and the twisted action, as in Section 4.5. For convenience of the reader a brief reminder of that construction follows, with a few cosmetic changes.

We are given a finitely generated subfield K of $\mathbb{C}$ and the uniform bound m on the asymptotic dimension of all unipotent subgroups of a given finitely generated subgroup of $SL(n, K)$.

Let $\vec{t}$ be a transcendence base of K over $\mathbb{Q}$, so that K is an algebraic extension of degree r of $\mathbb{Q}(\vec{t})$. Pick $mr\#\{\vec{t}\}$ algebraically independent over K complex numbers and group them into mr tuples, each of $\#\{\vec{t}\}$ elements:

$$\vec{t}_1, \ldots, \vec{t}_{mr}. \tag{4.14}$$

Define r embeddings of K into $\mathbb{C}$, fixing $\mathbb{Q}(\vec{t})$ and mapping the generator of K over $\mathbb{Q}(\vec{t})$ to its conjugates in this extension. Call these embeddings $\sigma_1, \ldots, \sigma_r$.

Define mr embeddings $\mathbb{Q}(\vec{t})$ into $\mathbb{C}$ by fixing $\mathbb{Q}$ and sending $\vec{t}$ to $\vec{t}_i$ from the list (4.14). Extend each of them to K by sending the algebraic generator of K over $\mathbb{Q}(\vec{t})$ to some complex number of degree r over the corresponding $\mathbb{Q}(\vec{t}_i)$. Call the extensions $\sigma_{r+1}, \ldots, \sigma_{mr+r}$.

Finally, let $X \times X \times \cdots \times X$ be a product of $mr + r$ copies of the symmetric space $SL(n, \mathbb{C})/SU(n)$, each equipped with a natural action of $SL(n, \mathbb{C})$. Define a twisted action of $SL(n, K)$ on the product of symmetric spaces in the following way:

$$g.(x_1, x_2, \ldots, x_{mr+r}) = (\sigma_1(g).x_1, \ldots, \sigma_{mr+r}(g).x_{mr+r}), \quad g \in SL(n, K). \tag{4.15}$$

Theorem 4.20 (cf. Theorem 4.13). *The restriction of the action* (4.15) *to any* *unipotent subgroup of G is proper.*

Proof. As in the proof of Theorem 4.13 we assume that the action is improper and deduce that this implies that there are infinitely many unipotent group elements with uniformly bounded under all embeddings $\sigma_1, \ldots, \sigma_{mr+r}$ matrix entries.

Every unipotent subgroup of G is conjugate to some uni-upper-triangular subgroup, and since such a conjugation is given by a fixed matrix, we have infinitely many uni-upper-triangular group elements with entries, uniformly bounded under all embeddings.

Keep the notation from Lemma 4.17. The boundness of the (i, j)-th entry for $j - i = 1$ in the light of Remark 4.14 means that this entry is expressible as

$$z_1 u_1^{(i,j)} + \cdots + z_m u_m^{(i,j)}, \quad (z_1, \ldots, z_m) \in \text{bounded subset of } \mathbb{Z}^m.$$

This means that we can expect only finitely many different numbers for this line $j - i = 1$.

Consider the entries along the next line, $j - i = 2$. We know that each of them is bounded. In addition, the P_{ij} portion of each entry comes from the previous diagonal, in our case $j - i = 1$, and, since we have already proven that there are finitely many options there, the set of possible offsets P_{ij} is bounded. Hence the linear part

$$z_1 u_1^{(i,j)} + \cdots + z_m u_m^{(i,j)}, \quad (z_1, \ldots, z_m) \in \mathbb{Z}^m$$

is bounded as well, that is, invoking Remark 4.14 again, the set of all possible coefficients $(z_1, \ldots, z_m)$ is bounded for every entry along this line. Summarizing, only finitely many different entries along the line $j - i = 2$ are possible.

Inductively we show that there are only finitely many options for each matrix entry, but this contradicts the assumption that we start with an infinite subset of group elements. □

Chapter 5

Proof of the main theorem

In this chapter we collect all the results from the previous chapters together and prove that, under suitable conditions on the unipotent subgroups, every finitely generated subgroup of $SL(n, \mathbb{C})$ admits a proper action on a finite-asymptotic-dimensional space. The proof is given in Section 5.2, while the reduction to groups of integral characteristic via the Alperin-Shalen hierarchies is discussed in Section 5.1.

5.1 Alperin-Shalen hierarchies

Let Γ be a finitely generated subgroup of $SL(n, \mathbb{C})$. Since Γ is finitely generated, we can think of it as of a subgroup of $SL(n, A)$, where A is a finitely generated subring of $\mathbb{C}$. Let K be the quotient field of A.

The following construction was introduced by Alperin and Shalen in their study of the cohomological dimension on linear groups (see [AS82]). We start by recalling some terminology and basic facts about integral ring extensions. Within the scope of this discussion all rings under consideration are subrings of $\mathbb{C}$.

Definition 5.1. Given two commutative rings A and B, B is called an *extension* of A if A is a subring of B. Such an extension is called *integral* if any element of B is *integral* over A, that is, it is a root of some monic polynomial with coefficients in A.

The following result is known as the Noether Normalization Lemma.

Lemma 5.2. *Let K be a field and A an integral domain, finitely generated by the elements $y_1, \dots, y_n$ as an algebra over K. Then there are elements $x_1, \dots, x_r \in A$, algebraically independent over K, such that A is integral over $K[x_1, \dots, x_r]$.*

Proof. We will proceed by induction. If $y_1, \dots, y_n$ are algebraically independent over K, we are done. Otherwise there exists a polynomial p with coefficients in K, such that

$$p(y_1, \dots, y_n) = 0. \tag{5.1}$$

Assume that y_1 has a nontrivial occurrence in p. Consider a linear change of variables $\tilde{y}_2 = y_2 - \lambda_2 y_1, \dots, \tilde{y}_n = y_n - \lambda_n y_1$ with $\lambda_2, \dots, \lambda_n \in K$ (we will specify the coefficients later). It is clear that $y_1, \tilde{y}_2, \dots, \tilde{y}_n$ generate A over$K$. We will prove that there exists a choice of coefficients, such that y_1 is integral over the ring generated by $\tilde{y}_2, \dots, \tilde{y}_n$ over K and then appeal to the induction hypothesis.

Write (5.1) as

$$p(y_1, \tilde{y}_2 + \lambda_2 y_1, \dots, \tilde{y}_n + \lambda_n y_1) = 0. \tag{5.2}$$

Regard this equation as a polynomial equation for y_1 with coefficients in the ring generated by $\tilde{y}_2, \dots, \tilde{y}_n$ (over K). The highest power of y_1 in (5.2) comes from the highest degree term in (5.1) in the following fashion: if the highest term in (5.1) is

$$\sum_{m_1+\dots+m_n=m} c_{(m_1,\dots,m_n)} y_1^{m_1} \dots y_n^{m_n}, \qquad c_{(m_1,\dots,m_n)} \in K,$$

then the term with the highest power of y_1 in (5.2) is

$$y_1^m \sum_{m_1+\dots+m_n=m} c_{(m_1,\dots,m_n)} \lambda_2^{m_2} \dots \lambda_n^{m_n}.$$

Note that the coefficient of y_1^m in this equation is an element of K (it does not contain any of $\tilde{y}_1, \dots, \tilde{y}_n$). Since K has infinitely many elements, we can choose $\lambda_2, \dots, \lambda_n$ in a way that this coefficient is nonzero, and therefore (5.2) is an integral equation for y_1 in the ring generated over K by $\tilde{y}_2, \dots, \tilde{y}_n$. □

Note that the number r in Lemma 5.2 corresponds to the transcendence degree of A over K. We will use this lemma in the form of

Corollary 5.3. *Let A be a finitely generated integral domain. Then there exists a nonzero integer s and algebraically independent over $\mathbb{Q}$ elements $x_1, \dots, x_r \in A[s^{-1}]$, such that $A[s^{-1}]$ is an integral extension of $\mathbb{Z}[s^{-1}][x_1, \dots, x_r]$.*

Proof. Let $y_1, \dots, y_n$ be the generating set for A and let B be the ring generated by $y_1, \dots, y_n$ over $\mathbb{Q}$. Apply Lemma 5.2 for the ring B and the field $\mathbb{Q}$ and obtain $x_1, \dots, x_r \in B$ with the property that B is an integral extension of $\mathbb{Q}[x_1, \dots, x_r]$.

Each y_i, as an element of B, is a root of a monic polynomial with coefficients in $\mathbb{Q}[x_1, \dots, x_r]$. Take $s_1 \in \mathbb{Z}$ to be the common denominator of all rationals, participating in the coefficients of the corresponding polynomials for all y_i's. Then each y_i is integral over $\mathbb{Z}[s_1^{-1}][x_1, \dots, x_r]$.

It could happen that not all x_j's lie in A. We know that each of them is a $\mathbb{Q}$-linear combination of the products of y_i's, so let $s_2 \in \mathbb{Z}$ be the common denominator of all rationals participating in such expressions. Then each x_j lies in the ring extension of $\mathbb{Z}[s_2^{-1}]$ by $y_1, \dots, y_n$, which is $A[s_2^{-1}]$.

Finally, let $s = s_1 s_2$. Then each generator y_i of A is integral over $\mathbb{Z}[s^{-1}][x_1, \dots, x_r]$, moreover $\mathbb{Z}[s^{-1}][x_1, \dots, x_r] \subseteq A[s^{-1}]$, and so the latter is an integral extension of the former. □

Lemma 5.4. *Let A be a finitely generated ring with field of fractions K. Then there exist finitely many discrete valuations $\nu_1, \dots, \nu_m$ on A with*

$$A \cap \mathcal{O}_{\nu_1} \cap \dots \cap \mathcal{O}_{\nu_m} \subset \mathcal{O},$$

where $\mathcal{O}$ is the ring of algebraic integers in K.

Proof. We start by finding $s \in \mathbb{Z}$ and $x_1, \dots, x_r$ in $A[s^{-1}]$ such that the polynomial ring $(\mathbb{Z}[s^{-1}])\,[x_1, \dots, x_r]$ has an integral extension $A[s^{-1}]$. For each $i = 1, \dots, r$ the ring $\mathbb{Z}[s^{-1}][x_1, \dots, x_r]$, as an extension of $\mathbb{Z}[s^{-1}][x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_r]$, admits a discrete valuation μ_i, given by

$$\mu_i(f) = -\deg f, \qquad f(x) \in \mathbb{Z}[s^{-1}][x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_r][x].$$

These valuations can be extended to $A[s^{-1}]$ and also to K in finitely many ways (see, for example, [McC76, Chapters 3 and 4]) to produce discrete valuations $\nu_1, \dots, \nu_t$.

If $x \in A[s^{-1}] \cap \mathcal{O}_{\nu_1} \cap \cdots \cap \mathcal{O}_{\nu_t}$, its minimal polynomial over $\mathbb{Q}(x_1, \ldots, x_r)$ has coefficients in $(\mathbb{Z}[s^{-1}])\,[x_1, \ldots, x_r] \cap \mathcal{O}_{\mu_1} \cap \cdots \cap \mathcal{O}_{\mu_r}$. Thus x is integral over $\mathbb{Z}[s^{-1}]$.

Finally add p-adic valuations $\nu_{t+1}, \ldots, \nu_m$, which correspond to all prime divisors of s.

Now if $x \in A \cap \mathcal{O}_{\nu_1} \cap \cdots \cap \mathcal{O}_{\nu_m}$, it must be integral over $\mathbb{Z}$ and therefore $x \in \mathcal{O}$. □

Theorem 5.5. *For any finitely generated group $\Gamma \subset SL(n, A)$ there exists a finite product of $(n-1)$-dimensional affine buildings, each of finite asymptotic dimension, on which Γ acts in such a way that the isotropy groups lie among the subgroups of Γ of integral characteristic.*

Proof. Starting from a finitely generated $\Gamma \subset SL(n, A)$, one can find the discrete valuations $\nu_1, \ldots, \nu_m$ on K, described in Lemma 5.4 above. Let

$$X = X_{\nu_1} \times X_{\nu_2} \times \cdots \times X_{\nu_m},$$

where X_{ν_i} is the $(n-1)$-dimensional affine building, corresponding to a discrete valuation ν_i.

The entire group $SL(n, K)$ acts on each X_{ν_i} in the obvious way, via the natural action on the lattices in K^n. Hence Γ acts on X_{ν_i} as well. Finally, define the diagonal action of Γ on X:

$$g.(x_1, x_2, \ldots, x_m) = (g.x_1, g.x_2, \ldots, g.x_m), \qquad g \in \Gamma, x_i \in X_{\nu_i}. \tag{5.3}$$

The isotropy subgroups of this action are the subgroups of Γ which stabilize a vertex in each X_{ν_i}. We know that the stabilizer of any vertex in X_{ν_i} stabilizes the associated class of $\mathcal{O}_{\nu_i}$-lattices in K^n, thus the coefficients of its characteristic polynomial are in $\mathcal{O}_{\nu_i}$. Therefore the isotropy elements of the action (5.3) have the coefficients of their characteristic polynomials in $\mathcal{O}_{\nu_1} \cap \cdots \cap \mathcal{O}_{\nu_m}$, and according to Lemma 5.4 they are algebraic integers in A, which means that the isotropy groups have integral characteristic. □

Remark 5.6. The term "hierarchies" refers to the details of the original construction in [AS82], where instead of the diagonal action (5.3) a hierarchy of collections of subgroups of Γ was introduced, each level of such hierarchy acting on a building, corresponding to one of the valuations, in the way that the isotropy of each action belongs to the next level of the hierarchy.

5.2 The proof

Now we are able to prove

Theorem 5.7. *Let Γ be a finitely generated subgroup of $SL(n, \mathbb{C})$. If there is a uniform bound on the dimension of unipotent subgroups of Γ, then there exists a finite-dimensional space on which Γ acts properly.*

Proof. Take X to be the product of the buildings furnished in the proof of Theorem 5.5 and the finite-dimensional spaces from Theorem 4.8 and Theorem 4.15. Then the diagonal action of Γ on X is proper. Indeed, the stabilizer of such action should be of integral characteristic (Theorem 5.5), without any irreducible diagonal parts (Theorem 4.8), that is unipotent. But all such subgroups act properly because of Theorem 4.15. □

Chapter 6

A second look at $SL(2,\mathbb{C})$: the Baum-Connes conjecture

In this chapter we give a more detailed treatment of linear groups of 2×2 matrices. While the results of Chapter 5 provide a proper action of any such group on a finite-dimensional space (under the constraints on the unipotent subgroups), they only prove that these groups have finite asymptotic dimension and therefore satisfy the coarse Baum-Connes conjecture. Here we prove the full Baum-Connes conjecture by pushing the techniques developed in the previous chapters further.

In Section 6.1 we present a direct proof that the Baum-Connes conjecture holds for any finitely generated subgroup of $SL(2,\mathbb{Q})$ by explicitly constructing a proper action on a finite product of simplicial trees and a hyperbolic plane. This section is a repetition of Section 4.2 with a few cosmetic changes.

In Section 6.2 we reduce the conjecture for an arbitrary finitely generated subgroup of $SL(2,\mathbb{C})$ to the case of groups of integral characteristic by introducing the hierarchies of families of subgroups and applying the inductive argument.

Finally, for groups of integral characteristic we have two cases: in Section 6.3 we prove the conjecture for Zariski-dense subgroups of integral characteristic using ergodic theory arguments, while in Section 6.4 we employ the algebraic group theory to reduce the conjecture for Zariski-non-dense subgroups to amenable groups.

Remark 6.1. We will be using the fact that the Baum-Connes conjecture (with coefficients) passes to subgroups, without explicitly mentioning it.

6.1 Subgroups of $SL(2, \mathbb{Q})$

Let Γ be a finitely generated subgroup of $SL(2, \mathbb{Q})$. Taking the finite symmetrized set of generators of Γ, we can generate the whole group as a semigroup, and all the entries of any element of Γ are therefore obtained from a finite number of entries of the generating set by additions and multiplications. This observation allows us to treat Γ as defined over the ring $A = \mathbb{Z}[\frac{1}{s}]$, where s is the l.u.b. of all the denominators of the fractions participating in the entries of the generating set of Γ. If necessary, we will enlarge A so that s is the product of distinct primes and therefore we may assume that prime factorization of s is $p_1 \cdot p_2 \cdots p_n$ with nonrepeating terms.

As in Section 4.2, consider the 1-dimensional buildings (trees) T_{p_k} corresponding to each p_k-adic valuation on A and denote by Γ_{p_k} the induced actions of Γ on these trees.

Define an action γ_H of Γ on the 2-dimensional real hyperbolic space $\mathbb{H}_2$ via the natural isometric action of $SL(2, \mathbb{R})$ on $\mathbb{H}_2$.

Finally, Γ acts on the product of trees and a hyperbolic space $\mathbb{H}_2$:

$$T = T_{p_1} \times T_{p_2} \times \cdots \times T_{p_n} \times \mathbb{H}_2$$

via the diagonal action

$$\gamma = \gamma_{p_1} \times \gamma_{p_2} \times \cdots \times \gamma_{p_n} \times \gamma_H.$$

Theorem 6.2. *The action γ is proper.*

Proof. The proof of Theorem 4.3 is applicable here, with a minor change: instead of a symmetric space $SL(n, \mathbb{C})/SU(n)$ we have a hyperbolic space $K_{\mathbb{H}_2}$.

Since for the hyperbolic space and its "center" point x_0 the property

$$\cosh(\mathrm{dist}_{\mathbb{H}_2}(g.x_0, x_0)) = \sum_{i,j=1}^{2} g_{ij}^2, \qquad g \in SL(n, \mathbb{R})$$

still holds, the argument from the proof of Theorem 4.3 works without any alteration. □

6.2 Reduction to groups of integral characteristic

In this section we are constructing a hierarchy of families of subgroups of $SL(2,\mathbb{C})$ in such a way that its base consists of groups of integral characteristic, and any finitely generated subgroup of $SL(2,\mathbb{C})$ belongs to some level of this hierarchy.

6.2.1 Actions on trees and reduction to isotropy subgroups

The main motivation for the study of the isotropy of the group action on a tree is the following

Theorem 6.3 (Oyono-Oyono, [OO98]). *Let Γ be a discrete countable group acting on a tree. Then the Baum-Connes conjecture holds for Γ if and only of it holds for all the isotropy subgroups of the action on vertices of the tree.*

The idea of Theorem 6.3 suggests that, having an action of a group on a tree, we can try to reduce the verification of the Baum-Connes conjecture for the entire group to the isotropy subgroups, which may be easier.

Now we define a sequence

$$\mathcal{H}_0, \mathcal{H}_1, \ldots, \mathcal{H}_i, \ldots$$

of families of subgroups of $SL(2,\mathbb{C})$ in the following way:

- $\mathcal{H}_0$ consists of groups of integral characteristic.
- $\mathcal{H}_i$ consists of all groups acting on trees, with isotropy in $\mathcal{H}_{i-1}$.

The main theorem which will allow us to reduce the Baum-Connes Conjecture for any finitely generated subgroup of $SL(2,\mathbb{C})$ to the groups of integral characteristic will then be:

Theorem 6.4. *Every finitely generated subgroup of $SL(2,\mathbb{C})$ lies in $\cup_{i=0}^{\infty}\mathcal{H}_i$.*

Indeed, the repeated applications of Theorem 6.3 allow one to reduce the conjecture for any group in $\mathcal{H}_i$ to groups in $\mathcal{H}_0$, that is groups of integral characteristic.

The rest of this section is devoted to the proof of Theorem 6.4.

6.2.2 Hierarchies of subgroups

We start with a finitely generated subgroup Γ of $SL(2, \mathbb{C})$ and notice that we can treat it as a subgroup of $SL(2, A)$, where A is a finitely generated subring of $\mathbb{C}$.

The ultimate goal is to show that Γ lies somewhere in the sequence $\{\mathcal{H}_i\}$, defined in the Section 6.2.1.

Now we construct a special hierarchy of subgroups of Γ, resembling the technics of Alperin and Shalen in [AS82] (see Chapter 5 for a detailed discussion) and show that this hierarchy fits inside the general one, described above. Starting from $\Gamma \subseteq SL(2, A)$, construct a sequence $\{\tilde{\mathcal{H}}_0, \ldots, \tilde{\mathcal{H}}_{m+1}\}$ of families of subgroups of Γ in the following fashion:

- $\tilde{\mathcal{H}}_{m+1} = \{\Gamma\}$
- $\tilde{\mathcal{H}}_i = \{$ subgroups of Γ with the coefficients of the characteristic polynomial of every element in $\cap_{j=i}^{m} \mathcal{O}_{\nu_j}\}$ for $i = m, m-1, \ldots, 0$

(As before, we use the following notation: $\nu_0, \ldots, \nu_m$ are the discrete valuations furnished by Lemma 5.4 for the field of fractions of A; $\mathcal{O}_{\nu_i}$ for $i = 0, \ldots, m$ are the corresponding rings of integers, and we consider an action of Γ and its subgroups on the trees of the equivalence classes of $\mathcal{O}_{\nu_i}$-lattices.)

It is easy to see that for each $i > 0$ any group G in $\tilde{\mathcal{H}}_i$ acts on the $\mathcal{O}_{\nu_{i-1}}$-tree with isotropy in subgroups of conjugates of $SL(2, \mathcal{O}_{\nu_{i-1}})$, that is in $\tilde{\mathcal{H}}_{i-1}$. On the other hand, Lemma 5.4 shows that the intersection of all rings of integers O_{ν_i} is a subring in the algebraic integers, that is to say any group in $\tilde{\mathcal{H}}_0$ has integral characteristic. Thus we have shown that $\tilde{\mathcal{H}}_i \subseteq \mathcal{H}_i$ for any $i = 0, 1, \ldots, m$, whence Theorem 6.4 for our particular group Γ.

6.3 Zariski-dense subgroups

In this section we present the argument for groups Γ of integral characteristic whose Zariski closure is the entire $SL(2, \mathbb{C})$.

Let Γ be a subgroup of $SL(2, \mathbb{C})$ of integral characteristic, and $G = SL(2, \mathbb{C})$ be its Zariski closure. The following result is a modification of Lemma 4.6 and it goes back to Zimmer (cf. [Zim84a, Lemma 6.1.7]).

Lemma 6.5. *There exists a faithful representation*

$$\alpha : SL(2,\mathbb{C}) \to GL(4,\mathbb{C}),$$

such that $\alpha(\Gamma) \subset GL(4,\mathbb{A})$.

Proof. We will use the construction, similar to the one in the proof of Lemma 4.6. Let f_g be a complex-valued map $G \to \mathbb{C}$ defined by

$$f_g : h \mapsto \operatorname{tr}(gh), \qquad h \in G.$$

Note that $f_{g_1}(h) + f_{g_2}(h) = \operatorname{tr}((g_1+g_2)h)$ and $\lambda f_g(h) = \operatorname{tr}((\lambda g)h)$ for any $g_1, g_2, h \in G$ and $\lambda \in \mathbb{C}$. This allows us to consider $f_g(h)$ as a short-hand notation for $\operatorname{tr}(gh)$ for any $g \in \mathbb{C}G$, $h \in G$. Let

$$V = \operatorname{Span}_{\mathbb{C}} \{f_g\}_{g\in\mathbb{C}G}.$$

Since the conditions defining f_g are linear with respect to the entries of g, the linear space V has finite dimension. More precisely, this dimension is 4, for G contains 4 linearly independent in $M_2(\mathbb{C})$ matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}. \tag{6.1}$$

Consider the following action of G on V:

$$g.f_h = f_{gh}, \qquad g \in G, h \in \mathbb{C}G. \tag{6.2}$$

Since this action is linear, we have a representation of G.

Let

$$W = \operatorname{Span}_{\mathbb{C}} \{f_g\}_{g\in\mathbb{C}\Gamma}.$$

This subspace is Γ-invariant and, since Γ is Zariski-dense in G, is also G-invariant. Thus $W = V$.

Let $g_1, g_2, g_3, g_4 \in \Gamma$ be such that $\{f_{g_1}, f_{g_2}, f_{g_3}, f_{g_4}\}$ is a basis of V (we can arrange this because V is generated by f_g for $g \in \Gamma$). With respect to this basis the action (6.2) is given by a matrix (α_{ij}^g), such that

$$g.f_{g_i}(h) = \sum_{j=1}^{4} \alpha_{ij}^g f_{g_j}(h), \qquad g, h \in G, i = 1, \ldots, 4. \tag{6.3}$$

Thus we obtain a representation $\alpha : G \to GL(4, \mathbb{C})$.

We need to check that α is faithful. Similar to Lemma 4.6, we want to show that if

$$g.f_1(h) = f_1(h)$$

for some $g \in G$ and all $h \in G$, then $g = 1$. If we test this condition using matrices from (6.1) as h, g will coincide with the identity matrix entry-wise.

If $g \in \Gamma$, (6.3) means that in particular

$$\operatorname{tr}(g g_i g_k) = g.f_{g_i}(g_k) = \sum_{j=1}^{4} \alpha_{ij}^g f_{g_j}(g_k) = \sum_{j=1}^{4} \alpha_{ij}^g \operatorname{tr}(g_j g_k), \qquad i, k = 1, \ldots, 4.$$

Then (α_{ij}^g) are the solutions of this system of linear equations with algebraic coefficients, and therefore the matrix entries (α_{ij}^g) of the representation α are algebraic. $\square$

Since the conclusion of Lemma 6.5 is the same as the one of Lemma 4.6, for any other Zariski–dense subgroup $\tilde{\Gamma}$ of integral characteristic its image $\alpha(\tilde{\Gamma})$ is conjugate in $GL(4, \mathbb{C})$ to a subgroup whose matrix entries belong to a finitely generated subring of $\mathbb{A}$. In the discussion which follows we continue to work with Γ.

By the Primitive Element Theorem $\alpha(\Gamma) \subset GL(4, K)$, with $[K : \mathbb{Q}] < \infty$. Take $H = \alpha(G)$. We see that $\alpha(\Gamma) \subseteq H \cap GL(4, K)$, and both $\alpha(\Gamma)$ and $H \cap GL(4, K)$ are Zariski–dense in H, thus they are both defined over K by [Zim84a, Proposition 3.1.8]. This means $\alpha(\Gamma)$ is locally isomorphic to $SL(2, K)$ (see [Zim84b, Theorem 7]), and we can apply Theorem 4.8 to $\alpha(\Gamma)$ and obtain a proper action.

6.4 Zariski-non-dense subgroups

Finally, we discuss the case of Zariski-non-dense subgroups of integral characteristic.

Let us start with some preliminary remarks on algebraic Lie groups. Suppose G is a Zariski-closed proper subgroup of $SL(2,\mathbb{C})$. We write G_0 for the Zariski-connected component of the unit of G. It is well known that G_0 is a normal subgroup of G of finite index [Bor91, I.1.2].

That is to say, G is almost connected in this case, and the Baum-Connes conjecture for almost connected groups is proven in [CEN01] and [Laf03]. This general result requires quite heavy KK-theory technique and is proven for the Baum-Connes conjecture with trivial coefficients only. Here we will present a more elementary argument for the version of the conjecture with coefficients, by explicitly describing the structure of the groups we are dealing with.

Since for algebraic groups the notions of connected and irreducible components coincide [Bor91, AG.17.2], G_0 is abelian if and only if its Lie algebra is commutative [Hoc81, IV.4.3]. Since G is a proper subgroup of $SL(2,\mathbb{C})$, its dimension is strictly less than 3. In the subsections below we will address each dimension case separately.

6.4.1 Dimension 0

In this case $\dim G_0 = 0$ as well, and, since G_0 is connected, we conclude that it is trivial. The group G itself, being a finite extension of G_0, is finite, whence the Baum-Connes conjecture for G holds trivially.

6.4.2 Dimension 1

The Lie algebra of G (and G_0 as well) has to be 1-dimensional. In particular, it has to be commutative, thus G_0 is abelian. There are only two (up to conjugacy) connected abelian 1-dimensional groups, namely $\left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle| \, a \in \mathbb{C}^\times \right\}$ and $\left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| \, b \in \mathbb{C} \right\}$. We will treat them separately.

Suppose $G_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\}$ (up to conjugacy). Then, since G_0 is normal in G, any conjugate of $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ by any element in G, say $\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$, has to have the same diagonal form:

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} = \begin{pmatrix} * & (a^{-1} - a)g_{11}g_{12} \\ (a - a^{-1})g_{21}g_{22} & * \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}.$$

This means that $g_{11}g_{12} = 0$ and $g_{21}g_{22} = 0$. To satisfy the first condition, we need to take either g_{11} to be zero (in this case g_{21} has to be nonzero, since the matrix has determinant 1, and the second condition leads to $g_{22} = 0$) or g_{12} to be zero (in this case g_{22} has to be nonzero, since the matrix has determinant 1, and the second condition leads to $g_{12} = 0$). Thus G may contain only matrices with zeros on the diagonal, or off the diagonal:

$$G \subseteq \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} 0 & a \\ -a^{-1} & 0 \end{pmatrix} \right\} = H.$$

This group H is amenable, and, modifying Theorem 6.3, it is possible to show (see [MV03, Theorems 5.18 and 5.23]) that any countable subgroup of H satisfies the Baum-Connes conjecture with coefficients[1].

Now suppose $G_0 = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\}$ (again, up to conjugacy). Since G_0 is normal in G, any conjugate of $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ by an arbitrary element $\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ element of G should have the same form:

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} = \begin{pmatrix} * & * \\ -bg_{21}^2 & * \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

[1]Theorem 5.18 in [MV03] shows that H satisfies the Baum-Connes conjecture with trivial coefficients, while Theorem 5.23 proves that any countable subgroup of such group satisfies the conjecture with arbitrary coefficients.

Thus we have $g_{21} = 0$, which means that G contains only matrices of the form $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$, and we will discuss this group in the following subsection.

6.4.3 Dimension 2

Let H denote the subgroup of $SL(2,\mathbb{C})$ consisting of all the matrices of the form $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$, where $a \in \mathbb{C}^{\times}, b \in \mathbb{C}$. Note that its Lie algebra consists of the matrices of the form $\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$, $a, b \in \mathbb{C}$.

Lemma 6.6. *Any subgroup K of $SL(2,\mathbb{C})$, which includes H and some element not in H, coincides with the whole group $SL(2,\mathbb{C})$.*

Proof. Suppose K contains some element $\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ with $g_{21} \neq 0$. Then we can multiply this element by an element $\begin{pmatrix} g_{21}^{-1} & -g_{22} \\ 0 & g_{21} \end{pmatrix}$ of H on the right to get $\begin{pmatrix} g_{11}g_{21}^{-1} & -1 \\ 1 & 0 \end{pmatrix}$.

Now for any complex numbers a and b with $a \neq 0$ we can multiply $\begin{pmatrix} g_{11}g_{21}^{-1} & -1 \\ 1 & 0 \end{pmatrix}$ on the left by $\begin{pmatrix} a & b - ag_{11}g_{21}^{-1} \\ 0 & a^{-1} \end{pmatrix}$ to get $\begin{pmatrix} b & -a \\ a^{-1} & 0 \end{pmatrix}$. Since K is a group, it ought to contain all inverses as well, in particular $\begin{pmatrix} 0 & a \\ -a^{-1} & b \end{pmatrix}$. This means that K contains all matrices of determinant 1 with 0 in the upper left corner.

Finally, let us take an arbitrary element of $SL(2,\mathbb{C})$, say $\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$. Since we already know that all matrices with $s_{21} = 0$ belong to H, and therefore to K, the essential part of the argument is to show that any such matrix with $s_{21} \neq 0$ belongs to K. The identity

$$\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} = \begin{pmatrix} 1 & s_{11}s_{21}^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -s_{21}^{-1} \\ s_{21} & s_{22} \end{pmatrix}$$

completes the proof, since all matrices on the right-hand-side belong to K. $\square$

Now we provide some technical results about Lie subalgebras of $\mathfrak{sl}(2,\mathbb{C})$.

Lemma 6.7. *For any 2-dimensional noncommutative Lie algebra there exists a basis* $\{x_1, x_2\}$ *with multiplication table* $[x_1 x_2] = x_1$.

Proof. Consider some basis $\{e_1, e_2\}$ of such an algebra, and suppose $[e_1 e_2] = a_1 e_1 + a_2 e_2$. Since the algebra is noncommutative, at least one of the coefficients a_1 and a_2 is not zero, let us say $a_1 \neq 0$, for definiteness. It is easy to see that two elements $x_1 = a_1 e_1 + a_2 e_2$ and $x_2 = a_1^{-1} e_2$ are linearly independent, so that they constitute a basis as well. With respect to such basis the multiplication is given by $[x_1 x_2] = x_1$. $\square$

Lemma 6.8. *The Lie algebra* $\mathfrak{sl}(2,\mathbb{C})$ *contains only one (up to conjugation) 2-dimensional Lie subalgebra, namely* $\left\{ \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} \middle| \, a, b \in \mathbb{C} \right\}$.

Proof. Suppose we have a 2-dimensional noncommutative Lie subalgebra $\mathfrak{h}$ of $\mathfrak{sl}(2,\mathbb{C})$. Let $\{x_1, x_2\}$ denote the basis of $\mathfrak{h}$ constructed in the lemma above. A priori there could be two possibilities: both eigenvalues of the matrix x_2 coincide (and therefore are zeros) or they are distinct. In the first case x_2 is conjugate to its Jordan form, namely $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and if x_1 after same conjugation has the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the multiplication condition $[x_1 x_2] = x_1$ is

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} -c & a-b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

from where we conclude that $a = b = c = d = 0$, which means $x_1 = 0$ and therefore can not serve as basis element, so that the case where both eigenvalues of the matrix x_2 coincide can not happen. Now suppose that the eigenvalues of x_2 are distinct, say λ and $-\lambda$. Conjugating x_1 and x_2, we write the multiplication condition as

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \right] = \begin{pmatrix} 0 & -2b\lambda \\ 2c\lambda & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

so that we have $a = d = 0$ and $2c\lambda = c$, $-2b\lambda = b$. We are looking for solutions with at least one of the coefficients b and c being non-zero, therefore we end up with two possibilities:

1. $b = 0 \neq c$, $\lambda = \frac{1}{2}$
2. $c = 0 \neq b$, $\lambda = -\frac{1}{2}$

Thus any non-commutative 2-dimensional Lie subalgebra of $\mathfrak{sl}(2, \mathbb{C})$ is conjugate-equivalent to $\mathfrak{h}_1 = \mathbb{C}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$ or $\mathfrak{h}_2 = \mathbb{C}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$. Finally, $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are conjugate to each other via the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. By scaling the second matrix in $\mathfrak{h}_2$, we obtain the representation $\left\{ \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} \middle| \, a, b \in \mathbb{C} \right\}$.

Now we show that $\mathfrak{sl}(2, \mathbb{C})$ does not contain any commutative 2-dimensional subalgebras. Suppose one such exists and has a basis $\{x, y\}$. Conjugating by some matrix, we can put y into Jordan form, and let x be represented by $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{pmatrix}$ under the same conjugation. We have two possibilities: the eigenvalues of the matrix, representing y, coincide (and therefore are zeros) or they are distinct, and by scaling the matrix we assume that they are 1 and -1. In the first case the commutativity condition can be written as

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{pmatrix},$$

which leads to $x_{11} = x_{21} = 0$, so that x is a scalar multiple of y, and this can not happen. In the second case we have

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{pmatrix},$$

this means $x_{12} = x_{21} = 0$, and again we have a contradiction with linear independence of x and y. □

Getting back to the group G_0, we see that Lemma 6.8 describes the Lie algebra of G_0, up to conjugacy. Therefore G_0 and H are conjugate to each other. Lemma 6.6 suggests that there are no proper subgroups of $SL(2,\mathbb{C})$, larger than H, therefore we conclude that $G = G_0$.

Finally, G is a semidirect product

$$\left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| \, b \in \mathbb{C} \right\} \rtimes \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle| \, a \in \mathbb{C}^{\times} \right\}$$

of two abelian groups, hence it is amenable, and we conclude that any finitely generated subgroup of G satisfies the Baum-Connes conjecture with coefficients by applying [MV03, Theorem 5.23].

Chapter 7

Conclusion

In this thesis we have studied the finite-dimensionality arguments which concern different aspects of the Baum-Connes conjecture with coefficients for linear groups over the fields of characteristic 0. In this chapter we comment on the limitations of the method developed in this thesis and possible directions of further investigations.

7.1 Proper vs. metrically proper

In this work we have shown that under some assumptions on the unipotent subgroups any finitely generated linear group over a field of characteristic zero admits a proper action on a finite-asymptotic-dimensional space. We are tempted to feed this result into Theorem 1.6, but for that we need to make sure that the action in question is actually metrically proper.

In general, our construction produces an action on a finite product of symmetric spaces and affine buildings. Notably, while proving properness of the action on a symmetric space we actually proved metrical properness, however that was not the case for the action on the affine buildings part. It could happen that the stabilizer of any vertex in a building is finite or even trivial, while there are infinitely many group elements moving this vertex distance 1 away.

To illustrate this phenomenon, we recall Example 1.7. The linear group G in that example is generated by matrices $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. This group is a subgroup of $SL(2, A)$, where we take the finitely generated ring A to be $\mathbb{Z}[t, t^{-1}]$

so that its field of fractions $\mathbb{Q}(t)$ is just a simple transcendental extension of the rationals. Lemma 5.4 asserts that A is an integral extension of $\mathbb{Z}[t - t^{-1}]$ and produces two discrete valuations on A. Since the degree of this extension is 2, the discrete valuation on $\mathbb{Z}[t - t^{-1}]$ which assigns to any polynomial in $t - t^{-1}$ the opposite of the polynomial's degree has two extensions to A. These valuations are

$$\nu_1(x) = m \quad \text{and} \quad \nu_2(x) = -n,$$
$$x = z_m t^m + \cdots + z_n t^n, \quad z_i \in \mathbb{Z}, \quad -\infty < m \leqslant n < \infty, \quad z_m \neq 0 \neq z_n.$$

An element x of A which is integral with respect to both of them has to be a rational integer, so that $\mathcal{O} = \mathbb{Z}$ in our notation.

A natural action of G on the product of two trees (with respect to valuations ν_1 and ν_2) has isotropy conjugate to subgroups of $SL(2, \mathbb{Z})$ (Theorem 5.5), while an action of G on a hyperbolic plane (in accordance with Theorem 6.2) is proper on $SL(2, \mathbb{Z})$. While the latter action is metrically proper, the former one is not, due to the fact that there are infinitely many elements of A with same valuations ν_1, ν_2 and therefore infinitely many elements of G moving vertices of the trees finite distance away.

We see that the Alperin-Shalen technique of Chapter 5 is not sharp enough to provide a metrically proper action for G. However, the technique developed in Chapter 4 says that the unipotent subgroup Γ in the notation of Example 1.7 is not boundedly composed and thus we have a reason to reject G from that perspective.

7.2 Fields of positive characteristic

Let K be an infinite field of positive characteristic and let Γ be a finitely generated subgroup of $SL(n, K)$. Can we apply the argument presented in this thesis to Γ and deduce that Γ acts properly on a space of finite asymptotic dimension (provided we imposed the boundedness restriction on the unipotent subgroups of Γ)?

In our construction we reduce the question to the subgroups of Γ of integral characteristic in Chapter 5 via the invocation of Noether Normalization Lemma and construction of finitely many valuations based on the normalized set of generators. This part of the construction applies flawlessly, for the only assumption we took is the infinite number of elements of the ground field K.

Then for subgroups of Γ of integral characteristic we either embed them into $GL(n, \mathbb{Q})$ for a suitably chosen m or embed them in different ways into $SL(n, \mathbb{C})$ and consider the actions on a product of affine buildings and a symmetric space. Either way, subgroups of Γ do not have to embed faithfully into a linear group over a field of characteristic 0 ($\mathbb{Q}$ or $\mathbb{C}$ in our construction), and this part of the argument fails.

It is reasonable to seek for conditions which guarantee that the subgroups of integral characteristic of a given finitely generated matrix group admit a proper action of a finite-asymptotic-dimensional space, and this action extends to the action of Γ (for example, finite order of those subgroups, but this is too restrictive).

The case of linear groups over a finite field is trivial, since they are finite and therefore have asymptotic dimension 0 and satisfy the full Baum-Connes conjecture.

7.3 Unipotent subgroups

In Theorem 4.20 we prove that if all unipotent subgroups of a finitely generated linear group G are uniformly boundedly composed, then there is an action of G on a finite-asymptotic-dimensional space, such that its restriction to any unipotent subgroup of G is proper. Then we deduce that the only obstruction that could prevent a finitely generated linear group from acting properly on a space of finite asymptotic dimension is the possible existence of unipotent subgroups which are not uniformly boundedly composed. Note that it could happen that some group G has unboundedly composed unipotent subgroups (or a sequence of boundedly composed unipotent subgroups with parameter of the composition tending to infinity), but these subgroups do not appear among the isotropy subgroups of the Alperin-Shalen-type action from Chapter 5. In such a case the action of G that we constructed would still be proper.

Another question concerns the notion of a boundedly composed unipotent subgroup. It is easy to show that any group of this kind has finite asymptotic dimension (Theorem 4.20). The converse is certainly true for 2×2 matrices (Theorem 4.15). While the property of being boundedly composed is an efficient way of checking the finiteness of asymptotic dimension, one can ask whether there exists a sharper condition which is actually equivalent to the uniform boundness of the asymptotic dimension of the unipotent subgroups.

7.4 The γ-element

There is a better K-theoretic description of the injectivity of the Baum-Connes assembly map, which involves the notion of a γ-element. The group Γ is said to have a *γ-element* if there exists a C^*-algebra A on which it acts properly and elements $D \in KK^{\Gamma}(A, \mathbb{C})$ and $\eta \in KK^{\Gamma}(\mathbb{C}, A)$, called Dirac and dual-Dirac, such that $\gamma = \eta \otimes_A D$ satisfies $p^*\gamma = 1$ for the projection $p : \underline{E}\Gamma \rtimes \Gamma \to \Gamma$. If such an element exists, the assembly map is split-injective and is easily described (see [Tu04]). After the original work [Kas88] of Kasparov, introducing the γ-element for groups acting properly on a simply connected manifold with non-positive sectional curvature, there have been quite a few attempts to translate his result to analogous geometric situations. Julg and Valette in [JV84] constructed the γ-element for groups acting properly on a tree, and Kasparov and Skandalis constructed it for groups acting on affine buildings in [KS91]. There appears a natural question, whether the γ-element for a group acting on the affine buildings used in Chapter 5 can be "lifted" from the isotropy subgroups to the group itself.

Bibliography

[AS82] Roger Alperin and Peter Shalen, *Linear groups of finite cohomological dimension*, Invent. Math. **66** (1982), 89–98.

[Bas80] Hyman Bass, *Groups of integral representation type*, Pacific J. Math. **86** (1980), 15–52.

[BC88a] Paul Baum and Alain Connes, *Chern character for discrete groups*, A fête of topology, Academic Press, 1988, pp. 163–232.

[BC88b] ———, *K-theory for discrete groups*, Operator algebras and applications, London Math. Soc. Lecture Notes, no. 135, 1988, pp. 1–20.

[BCH94] Paul Baum, Alain Connes, and Nigel Higson, *Classifying space for proper action and K-theory of group C^*-algebras*, Contemp. Math. **167** (1994), 241–291.

[BD01] Gregory Bell and Alexander Dranishnikov, *On asymptotic dimension of groups*, Algebraic and Geometric Topology **1** (2001), 57–71.

[BD04] ———, *A Hurewicz-type theorem for asymptotic dimension and applications to geometric group theory*, preprint, 2004.

[BH99] Martin Bridson and André Haefliger, *Metric spaces of nonpositive curvature*, Grundlehren der mathematischen Wissenschaften, vol. 319, Springer-Verlag, 1999.

[Bor91] Armand Borel, *Linear Algebraic Groups*, Springer-Verlag, 1991.

[BT72] François Bruhat and Jacques Tits, *Groupes réductifs sur un corps local, I. Données radicielles valuées*, I.H.E.S. Publ. Math. **41** (1972), 5–251.

[BT84] ———, *Schemas en groupes et immeubles des groupes classicues sur un corps local*, Bull. Soc. Math. Fr. **112** (1984), 259–301.

[Cas86] John William Scott Cassels, *Local Fields*, London Mathematical Society Student Texts, no. 3, Cambridge University Press, 1986.

[CE01] Jérôme Chabert and Sigfried Echterhoff, *Permanence properties of the Baum-Connes conjecture*, Documenta Math. **6** (2001), 127–183.

[CEN01] Jérôme Chabert, Sigfried Echterhoff, and Ryszard Nest, *The Connes-Kasparov Conjecture for almost connected groups*, preprint, 2001.

[CEOOar] Jérôme Chabert, Sigfried Echterhoff, and Hervé Oyono-Oyono, *Going-down functors, the Künneth formula and the Baum-Connes conjecture*, Geom. Funct. Anal. (to appear).

[FRR95] Steven Ferry, Andrew Ranicki, and Jonathan Rosenberg, *A history and survey of the Novikov conjecture*, Novikov Conjectures, Index Theorems and Rigidity, vol. 1, 1995, pp. 7–66.

[Gar97] Paul Garrett, *Buildings and classical groups*, Chapman & Hall, 1997.

[GHWar] Erik Guentner, Nigel Higson, and Shmuel Weinberger, *The Novikov Conjecture for Linear Groups*, I.H.E.S. Publ. Math. (to appear).

[Gro93] Mikhail Gromov, *Asymptotic invariants of infinite groups*, Geometric Group Theory, vol. 2, Cambridge University Press, 1993.

[Hig00] Nigel Higson, *Bivariant K-theory and the Novikov conjecture*, Geom. Funct. Anal. **10** (2000), no. 3, 563–581.

[HK01] Nigel Higson and Guennadi Kasparov, *E-theory and KK-theory for groups which act properly and isometrically on Hilbert space*, Invent. Math. **144** (2001), no. 1, 23–74.

[HLS02] Nigel Higson, Vincent Lafforgue, and George Skandalis, *Counterexamples to the Baum-Connes Conjecture*, Geom. Funct. Anal. (2002).

[Hoc81] Gerhard Hochschild, *Basic Theory of Algebraic Groups and Lie Algebras*, Springer-Verlag, 1981.

[JV84] Piérré Julg and Alain Valette, *K-theoretic amenability for $SL_2(\mathbb{Q}_p)$, and the Action on the Associated Tree*, J. Func. Anal. (1984), no. 58, 194–215.

[Kas88] Guennadi Kasparov, *Equivariant KK-theory and the Novikov conjecture*, Invent. Math. **91** (1988), 147–201.

[KS91] Guennadi Kasparov and George Skandalis, *Groups Acting on Buildings, Operator K-theory and Novikov's Conjecture*, K-theory (1991), no. 4, 303–337.

[Laf03] Vincent Lafforgue, *Banach KK-theory and the Baum-Connes conjecture*, arXiv:math.OA/0304342 (2003).

[McC76] Paul Joseph McCarthy, *Algebraic Extensions of Fields*, Chelsea Publishing company, 1976.

[MV03] Guido Mislin and Alain Valette, *Proper Group Actions and the Baum-Connes Conjecture*, Advanced Courses in Mathematics, Birkhäuser, 2003.

[OO98] Hervé Oyono-Oyono, *La Conjecture de Baum-Connes pour les groupes agissant sur les arbres*, C.R. Acad. Sci. Paris **326** (1998), no. 1, 799–804.

[Roe03] John Roe, *Lectures on coarse geometry*, University Lecture Series, vol. 31, American Mathematical Society, 2003.

[Tit75] Jacques Tits, *On buildings and their applications*, Proc. Intern. Cong. Math., Vancouver 1974, Montreal, 1975, pp. 209–220.

[Tu01] Jean-Louis Tu, *Remarks on Yu's property A for discrete metric spaces and groups*, Bull. Soc. Math. France **129** (2001), 115–139.

[Tu04] ______, *The Gamma Element for Groups which Admit a Uniform Embedding into Hilbert Space*, Operator Theory **153** (2004), 271–286.

[Wri02] Nicholas Wright, *C_0-Coarse geometry*, Ph.D. thesis, The Pennsylvania State University, 2002.

[Yu98] Guoliang Yu, *The Novikov Conjecture for groups with finite asymptotic dimension*, Ann. of Math. **147** (1998), no. 2, 325–355.

[Yu00] ______, *The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space*, Invent. Math. **139** (2000), no. 1, 201–240.

[Zim84a] Robert Zimmer, *Ergodic theory and semisimple groups*, Birkhäuser, 1984.

[Zim84b] ______, *Kazhdan groups acting on compact manifolds*, Invent. Math. **75** (1984), 425–436.

Vita

Dmitry Matsnev

Personal:

Born on February 28, 1977 to Anatoly Matsnev and Nina Matsneva

Education:

1999-2005	Ph.D. in Mathematics, Pennsylvania State University
1994-1999	Diploma in Mathematics and Applied Mathematics, Moscow State University
1984-1994	School #444, Moscow

Honors and Awards:

2005	Charles H. Hoover Teaching Award, Pennsylvania State University
2005	ZZRQ Alumnae Award, Pennsylvania State University
2003	Vollmer-Kleckner Fellowship, Pennsylvania State University
1999-2000	Eberly College of Sciences Scholarship, Pennsylvania State University
1999	Medal of Distinction, Moscow State University
1994	Mathematical Olympiad Diploma, Moscow State University
1994	Moscow Intellectual Marathon, First place in Eastern Educational District of Moscow
1994	Tournament of Towns Award, Russian Academy of Sciences, Quantum Magazine
1994	Medal of Distinction, Eastern Educational District of Moscow

Publications:

2002	D. A. Matsnev, *On the recognition of the finite definiteness of automaton monomial algebras*, Fundam. Prikl. Mat. **8**, 503–516.
1998	D. Korotaev, D. Matsnev, A. Akulov, *Informatics Tests*, Mozhaisk Publishing Company.
1997	D. A. Matsnev, *Verification of the Direct Sum Conjecture*, Math. Modeling and Control in Complex Systems, MGAPI, 18–19.

Professional Membership:

2000-	American Mathematical Society