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Abstract

This dissertation consists of three chapters. The first two consider stochastic games. Stochastic games are understood as games in which agents interact in the long-run and receive stochastic shocks to payoffs. The main focus is to try to analyze how the persistence of the shocks affects the players' ability to generate high payoffs. The final chapter proposes a way for agents in an exchange economy to fairly distribute gains from trade.

In the first chapter, I study a dynamic Bertrand-duopoly model in which price leadership patterns arise. Price leadership is when a firm initiates price changes while its competitor always matches the change with a lag. The firms produce a homogeneous product and are identical except for the information they possess about demand. The market size follows a Markov process and its realizations are observed by one of the firms but not the other. Price leadership patterns appear in equilibrium for a wide range of parameters. For high and fixed discount rates, price leadership with the informed firm acting as a leader allows firms to jointly approximate monopolistic profits in equilibrium as the the market size becomes more persistent.

The second chapter, written with Yu Awaya, studies a infinite-horizon game in which the same two players play a Prisoners' dilemma with unknown stochastic shocks to payoffs and imperfect monitoring. We argue that more persistent shocks facilitate cooperation.

Finally, the third chapter presents a new way to fairly distribute welfare gains derived from trade in exchange economies. The imposed fairness condition is based on the balanced marginal contributions condition satisfied by the Shapley value in transferable utility games. The solution is defined for cardinal finite exchange economies and satisfies many desirable properties. Then, solutions for ordinal economies are created by mapping ordinal economies onto cardinal economies and then applying the original solution.

Table of Contents

List of Figures

List of Tables

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Chapter 1

Barometric Price Leadership

1.1 Introduction

Price leadership is an industry pricing pattern in which periods of uniform (across firms) and constant prices are disrupted by one firm, the leader, whose new price soon becomes the uniform price. Among the industries in which such patterns have been claimed to be observed are gasoline (Stigler, 1947), rayon (Markham, 1951), and airlines (Rotemberg and Saloner, 1988). Stigler (1947) and Markham (1951) suggested that such patterns could emerge without explicit collusion in situations in which the leader is better informed about industry demand conditions. Stigler labeled such price leadership *barometric price leadership* and went on to suggest that it should be more prevalent the more persistent are industry demand conditions (See Stigler (1947), p. 446).

Previous attempts to produce explicit models of price leadership following the Stigler-Markham ideas have interpreted price leadership as firms opting for sequential rather than simultaneous pricing within each discrete period and before demand is realized (Cooper, 1997, Rotemberg and Saloner, 1990). Having the firms *choose* between sequential and simultaneous pricing can be interpreted as explicit collusion or firms making price announcements that are not immediately effective. Explicit collusion is inconsistent with the above ideas of Stigler and Markham. Also, those models fail to explain price leadership instances in which non-immediate price announcements are not used like the British supermarkets case in the late 2000's (Seaton and Waterson, 2013). Moreover, there are two shortcomings of such models. First, a time series of prices generated by such models would not allow an observer to distinguish between sequential and simultaneous price setting models. Second, such models have nothing to say about the role of persistence of demand conditions.

I develop a model of barometric price leadership that builds on the Stigler-Markham ideas. There is an infinite-horizon in discrete time and there are two firms; one is informed and the other uninformed. The market demand that the firms face can be either high or low and follows a symmetric persistent Markov process. The informed firm sees the market realization before setting its price; the uninformed firm never sees it. At each period, the two firms engage in price (Bertrand) competition; they set their prices simultaneously and are not able to engage in overt communication. If the two prices are different, then all sales go to the firm with the lower price; otherwise, they share sales equally. At the end of each period, each firm sees both prices, but only its own sales. In this model, price leadership is a sequence of prices in which the uninformed firm always matches the informed firm's previous price, while the informed firm sets different prices for different states.[1]

I show that price leadership is an equilibrium outcome for a broad set of parameters. The conditions can be summarized in terms of two parameters: the persistence of market demand and the common discount rate of the firms. The intuition is simple: as the market size becomes more persistent, the uninformed firm is able to infer more about tomorrow's state from today's state. Therefore, the informed firm's price becomes more informative as the market size becomes more persistent. These observations are consistent with Stigler's idea that prices in industries with price leaders are more persistent than those of industries without price leadership. Although the set of equilibria is large when firms are patient, price leadership is especially interesting because for high and fixed discount factors, price leadership in which joint profits approximate monopoly profits is a limiting

[1]After formulating my model, it came to my attention that Escobar and Llanes (2015) were developing a similar idea. They present a model of general repeated games in which one player possess private information about her type. Their model can be applied to generate price leadership patterns if the costs of one firm are constant and known while the other firm's costs are private and evolve according to a persistent Markov process. Compared to Escobar and Llanes, our model not only delivers a different story but also allows for both payoffs to depend on the stochastic shock.

equilibrium as persistence approaches perfect persistence.

The remainder of the paper is organized as follows. The next section contains the model. In Section 1.3, price leadership is defined and the results are presented. In Section 1.4, I discuss possible extensions and a potential application of the main idea in the model to a recent practice in the supermarket industry called *category management* (see Federal Trade Commission (FTC, 2001)).

1.2 Model

Consider a market with two firms, I and U, I stands for informed while U stands for uninformed. These firms interact in an infinite horizon game. At each period $t = 0, 1, 2, \ldots$, the firms compete in a homogeneous product Bertrand model. A state s^t, interpreted as the market size, is drawn at the beginning of each period from the set $\{s_l, s_h\}$ where $0 < s_l < s_h$. The state follows a Markov process with transition matrix

$$\begin{bmatrix} \Pr(s' = s_l | s = s_l) & \Pr(s' = s_h | s = s_l) \\ \Pr(s' = s_l | s = s_h) & \Pr(s' = s_h | s = s_h) \end{bmatrix} = \begin{bmatrix} 1 - \phi & \phi \\ \phi & 1 - \phi \end{bmatrix}$$

for some $\phi < \frac{1}{2}$ and initial distribution $\Pr(\cdot | s_h)$. The results will not depend on the selection of the initial distribution. We say that the state is persistent because the probability of the state changing, ϕ, is always less than one half.

While the Markov process is commonly known, only Firm I observes the realization of the state s^t.[2] After Firm I learns the state, the firms simultaneously set prices p_I^t and p_U^t from the support $[0, \overline{p}]$ where $\overline{p} > s_h$. For a given state s^t and prices p_I^t and p_U^t, the quantity demanded at period t is

$$\max\{s^t - \min\{p_1^t, p_2^t\}, 0\}.$$

If the firms set different prices, the firm with the lowest price gets the whole demand. Otherwise, firms share sales equually. Therefore, for prices p_I^t and p_U^t

[2]The model resembles a two-firm and two-state version of Kandori (1991). The information asymmetries is were the models differ. Firms are able to perfectly observe the state in Kandori's model.

and state $s^t \in \{s_l, s_h\}$, firm i's stage payoff at period t is given by

$$u_i(p_i, p_j; s) = \begin{cases} \pi(p_i; s) & \text{if } p_i < p_j, j \neq i; \\ \frac{\pi(p_i;s)}{2} & \text{if } p_i = p_j, j \neq i; \\ 0 & \text{otherwise.} \end{cases}$$

where $\pi(p; s) = \max\{p(s-p), 0\}$. At the end of each period, firms observe both prices and their own quantity but are unable to observe their competitor's quantity. That is, at the of period t, firm i observes p_I^t, p_U^t and q_i^t.

The firms have a common discount factor $\delta \in (0,1)$. Firm i's payoff from a sequence of prices $\{(p_I^t, p_U^t)\}_{t=0,1,\ldots}$ and a sequence of states $\{s^t\}_{t=0,1,\ldots}$ is

$$(1-\delta)\sum_{t=0}^{\infty} \delta^t u_i(p_I^t, p_U^t; s^t).$$

Irrespective of the state s^t, the only stage game Nash equilibrium prices are given by $p_I^t = p_U^t = 0$ and the unique stage Nash equilibrium payoffs are therefore zero. Similarly, at a period t in which the market size is $s^t = s_x$ for $x \in \{l, h\}$, a monopolist that knows s^t would set a price

$$p_x^M = \frac{s_x}{2}.$$

At each period t, when firms are about to set prices, they possess different information. The uninformed firm knows the sequence of prices and its own quantity up to period t, that is, its period t history is the sequence $h_U^t \equiv \{p_I^\tau, p_U^\tau, q_U^\tau\}_{\tau=0}^{t-1}$. Let $\mathcal{H}_U^t$ denote the set of all possible period t histories for the uninformed firm.

On the other hand, the informed firm knows the sequence of prices up to period t and the whole sequence of states including s^t. That is, by the time the informed firm sets p_I^t, it knows the history $h_I^t \equiv \{p_I^\tau, p_U^\tau, s^\tau\}_{\tau=0}^{t-1} \times s^t$. Denote the set of all period t histories for the informed firm as $\mathcal{H}_I^t$. Let $\mathcal{H}_j$ be the set all possible histories for firm j, or $\mathcal{H}_j = \bigcup_{t=0}^{\infty} \mathcal{H}_j^t$. A pure strategy for firm j is a mapping

$$P_j : \mathcal{H}_j \to [0, \overline{p}].$$

Note that the uninformed firm does not possess any private information. That is, for each history h_I^t that the informed firm observes, there is only one possible history h_U^t for the uninformed firm. The opposite is not true. Knowing both prices and its own quantity is not always enough for the uninformed firm to infer the market size. For example, the uninformed firm learns s^t if $p_U^t < p_I^t$ and $p_U^t < s^t$ because firm U knows that it captured the whole demand and therefore the quantity it sold was equal to $s^t - p_U^t$. On the contrary, firm U would not be able to learn s^t if $p_I^t < p_U^t$.

1.2.1 Monopolistic Profits

The maximum profits that can be attained in this environment are obtained by an informed monopolist, or a monopolist that at every period knows the state before setting its price. For that reason, we use monopolistic as the reference for high profits. In this subsection, we calculate those profits and establish necessary conditions for firms to be able jointly implement those profits.

Let $V^M(s_l)$ and $V^M(s_h)$ denote the expected payoff of a monopolist before it learns the realization of the current state given that the previous state was s_l and s_h respectively. Then, if the previous state was s_l: the current state is s_l with probability $(1-\phi)$ and the informed monopolist obtains a stage payoff of $\pi(p_l^M; s_l)$ and a continuation value $V^M(s_l)$; and, the current state is s_h with probability ϕ in which case the monopolist obtains a stage payoff of $\pi(p_h^M; s_h)$ and a continuation value of $V^M(s_h)$. That is,

$$V^M(s_l) = (1-\delta)\big[(1-\phi)\pi(p_l^M; s_l) + \phi\pi(p_h^M; s_h)\big] + \delta\big[(1-\phi)V^M(s_l) + \phi V^M(s_h)\big]. \tag{1.1}$$

Similarly,

$$V^M(s_h) = (1-\delta)\big[\phi\pi(p_l^M; s_l) + (1-\phi)\pi(p_h^M; s_h)\big] + \delta\big[\phi V^M(s_l) + (1-\phi)V^M(s_h)\big]. \tag{1.2}$$

It is possible for firms to obtain joint profits equal to those of the informed monopolist but the restrictions on parameters are stringent. If firms are exactly implementing joint monopolistic profits, the uninformed firm is never setting a price below p_h^M because at every period the high demand state occurs with positive probability. Then, in such an equilibrium, the informed firm sets prices like a monopoly. Also, to be deterred from price cutting, the uninformed firm sometimes should sell in such an equilibrium. But at any period in which the uninformed firm is supposed to make a sale, the informed firm is tempted to pretend that the demand is low. For this deviation not to be profitable, we require that sharing the high-demand monopolistic profits is better that getting the whole high demand at the low price. That condition is established in the next proposition.

Proposition 1. *Firms can exactly implement joint profits equal to the informed monopolist profits in equilibrium only if and only if*

$$\frac{s_l}{s_h} \leq \frac{4-\sqrt{8}}{4} \approx 0.2929. \tag{1.3}$$

In the next section, a class of strategy profiles that generates price leadership patterns is introduced. Those strategies allow firms to obtain joint high profits even when the inequality (1.3) does not hold.

1.3 Price Leadership

Next, price leadership is formally defined as a class of strategy profiles that generate price leadership patterns.

Definition 1 (Price Leadership)**.** For any pair of prices, $\mathbf{p} = (p_l, p_h)$ with $p_l \neq p_h$, the price leadership pricing rules are defined as follow:

I. At any period $t \geq 0$, the informed firm sets a price p_I^t equal to p_l if the state is s_l and equal to p_h if the state is s_h;

U. the uninformed firm starts by setting the price p_h at $t = 0$ and after that always sets a price p_U^t that matches the informed firm's previous price, that is, $p_U^t = p_I^{t-1}$.

If a firm detects a deviation from the previous pricing rules in the past, then both firms set a price equal to 0 forever.

In this environment in which the market size is persistent, an strategy profile satisfying the previous definition will generate price leadership patterns. If the state changes, the informed firm changes its price accordingly and the uninformed firm follows in the next period.

Previous works have interpreted price leadership as firms opting for sequential (Stackelberg) rather than simultaneous (Nash) pricing within each discrete period before the demand is realized (Cooper, 1997, Deneckere and Kovenock, 1992, Mouraviev and Rey, 2011, Rotemberg and Saloner, 1990, Yano and Komatsubara, 2006, 2012). This interpretation differs from the price leadership introduced in this paper because the prices generated by those models do not distinguish between simultaneous and sequential pricing. At each period in those models, all firms have set the same prices by the time they start selling.

The price leadership strategy profile with prices $\mathbf{p}$ is simple in the sense that the period t's action only depends on the previous period actions for the uninformed firm and on the previous period actions and the current state for the informed firm. The remainder of this section shows that price leadership equilibria perform well in terms of joint profits despite its simplicity.

Because our interest lie in supporting price leadership outcomes in which firms attain high profits, we will restrain our attention to cases where $p_l < p_h$.

1.3.1 Payoffs from Price Leadership

The expected payoffs from the price leadership strategy profile are derived as follows. We start with the expected payoffs of the uninformed firm when both are playing according to the price leadership profile with prices $\mathbf{p} = (p_l, p_h)$ where $p_l < p_h$. In that case, Firm U's information is summarized by the informed firm previous price. Hence, let $V^U_{\mathbf{p}}(p)$ for $p \in \{p_l, p_h\}$ be the uninformed firm expected discounted payoff given that the informed firm previous price was equal to p. Then, provided that both firms are following the price leadership strategy profile,

- if the informed firm's previous price was p_l, the uninformed is setting the price p_l today. Also, it must be the case that the market size was s_l in the previous period so the state today is s_l with probability $(1 - \phi)$ and s_h with probability ϕ. If the current state is s_l again, both firms set a price p_l and split the market today and the uninformed firm derives a continuation value of $V^U_{\mathbf{p}}(p_l)$. On the other hand, if the current state is s_h, the informed firm sets a price p_h and the uninformed gets the whole market today at a price p_l and a continuation value of $V^U_{\mathbf{p}}(p_h)$. That is,

$$V^U_{\mathbf{p}}(p_l) = (1-\delta)\left[(1-\phi)\frac{\pi(p_l;s_l)}{2} + \phi\pi(p_l;s_h)\right] + \delta\left[(1-\phi)\,V^U_{\mathbf{p}}(p_l) + \phi\,V^U_{\mathbf{p}}(p_h)\right] \quad (1.4)$$

- if the informed firm's previous price was p_h, the expected discounted payoff for the uninformed firm is given by

$$V^U_{\mathbf{p}}(p_h) = (1-\delta)\left[(1-\phi)\frac{\pi(p_h;s_h)}{2}\right] + \delta\left[\phi V^U_{\mathbf{p}}(p_l) + (1-\phi)V^U_{\mathbf{p}}(p_h)\right]. \quad (1.5)$$

Similarly, assuming both firm are following the price leadership strategy profile, we will derive the expected payoffs for firm I. Remember that the informed firm knows the state by the time the prices are set. All the information that firm I needs to calculate its expected payoff is the previous and the current state. Let $V^I_{\mathbf{p}}(s,s')$ with $s,s' \in \{s_l,s_h\}$ be the informed firm's expected discounted payoff provided that the current state is s' and the previous state was s. Then,

- when both the previous and the current states are low, the informed firm knows that the uninformed is going to set a price equal to p_l because the previous state (and the informed firm previous price) was low. Hence, because the state today is also s_l, the informed firm also sets a price equal to p_l and both firms equally split the market today. The next state is s_l with probability $(1-\phi)$ in which case the continuation value of firm I is $V^I_{\mathbf{p}}(s_l,s_l)$. With probability ϕ, the next state is s_h in which case the informed gets a continuation payoff of $V^I_{\mathbf{p}}(s_l,s_h)$. Then,

$$V^I_{\mathbf{p}}(s_l,s_l) = (1-\delta)\frac{\pi(p_l;s_l)}{2} + \delta\left[(1-\phi)V^I_{\mathbf{p}}(s_l,s_l) + \phi V^I_{\mathbf{p}}(s_l,s_h)\right] \quad (1.6)$$

- when the previous state was low and the current state is high, the expected discounted payoff for the informed firm is

$$V^I_{\mathbf{p}}(s_l,s_h) = \delta\left[\phi V^I_{\mathbf{p}}(s_h,s_l) + (1-\phi)V^I_{\mathbf{p}}(s_h,s_h)\right] \quad (1.7)$$

- when the previous state was high and the current state is low, the expected discounted payoff for the informed firm is

$$V^I_{\mathbf{p}}(s_h, s_l) = (1-\delta)\pi(p_l; s_l) + \delta\left[(1-\phi)V^I_{\mathbf{p}}(s_l, s_l) + \phi V^I_{\mathbf{p}}(s_l, s_h)\right] \tag{1.8}$$

- when both the previous and the current states are high, the expected discounted payoff for the informed firm is

$$V^I_{\mathbf{p}}(s_h, s_h) = (1-\delta)\frac{\pi(p_h; s_h)}{2} + \delta\left[\phi V^I_{\mathbf{p}}(s_h, s_l) + (1-\phi)V^I_{\mathbf{p}}(s_h, s_h)\right]. \tag{1.9}$$

1.3.2 Price Leadership as an Equilibrium

Conditions for the price leadership strategy profile to be a PBE are derived in this section. First, we need to specify the beliefs on the equilibrium path. At any $t \in \mathbb{N}_0$, let $\mu^t(\cdot|h^t_U) \in \Delta(S)$ be the belief that the uninformed firm has about the state at period t given a history $h^t_U \in \mathcal{H}^t_U$. The initial belief $\mu^0(\cdot)$ is given by $\Pr(\cdot|s_h)$. If firms are following price leadership with prices p_l and p_h with $p_l < p_h$, then μ^t only depends on the action that the informed firm took at period $t-1$. Then, given a history $h^t_U \in \mathcal{H}^t_U$, the uninformed firm beliefs about s^t are given by

$$\mu^t(\cdot|h^t_U) = \begin{cases} \Pr(\cdot|s^{t-1} = s_l) & \text{if } p^{t-1}_I = p_l \\ \Pr(\cdot|s^{t-1} = s_h) & \text{if } p^{t-1}_I = p_h \end{cases}$$

Next, we need to show that there are no profitable deviations from the price leadership profile. Firm can potentially make a profit by deviating to charge a lower price than the price prescribed by the strategy profile. All price cuts except one are immediately detected and therefore trigger a Nash reversal. Hence, price cuts that are immediately detected are not profitable for patient firms. The only price cut that is not detected occurs when both firms are supposed to set a price p_h but firm I deviates by setting a price p_l. In that case, the uninformed firm does not sell and cannot distinguish if there was a deviation or the state was s_l. That deviation is depicted in the next figure.

Figure 1.1 represents a situation in which firms have played according to the price leadership strategy profile up to period t and $s^{t-1} = s^t = s_h$. Because the

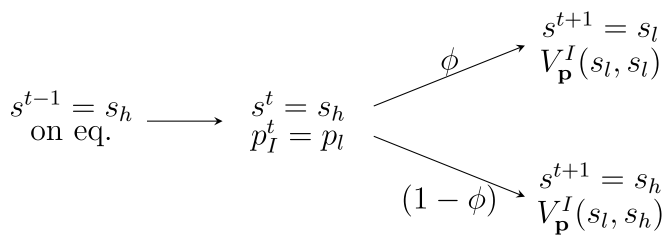

Figure 1.1. Undetected Deviation.

price of the informed firm was p_h in the previous period, the uninformed sets a price p_h at period t. Then, firm I can deviate by setting a price p_l and getting the whole market instead instead of sharing the demand at price p_h. Hence, the informed firm's stage payoff from such deviation at period t is given by $\pi(p_l; s_h)$. The continuation values also change because the deviation leads to the uninformed firm setting a price p_l at period $t+1$. With probability ϕ the state at $t+1$ will change to s_l and in that situation the informed firm faces an identical problem as when on equilibrium the state is going from s_l to s_l therefore obtaining a expected discounted payoff of $V_{\mathbf{p}}^I(s_l, s_l)$. Similarly, with probability $1-\phi$ the state remains s_h at period $t+1$ and the informed firm obtains a discounted payoff of $V_{\mathbf{p}}^I(s_l, s_h)$.

Although high discount factors are enough to discourage all other price cuts in a price leadership profile, the firms being patient is not sufficient to discourage the informed firm from pretending the state is low when it is high. Conditions to ensure that the inform firm does not want to set the price p_l when it is supposed to set the price p_h are presented next.

1.3.3 Jointly approaching monopolistic profits

It is natural to start by deriving conditions that guarantee that price leadership with the monopolistic prices can be sustained as a PBE. Remember that an informed monopolist will set a price $p_l^M = \frac{s_l}{2}$ whenever the market size is low and a price $p_h^M = \frac{s_h}{2}$ if the market size is high. The next proposition establishes a condition on s_l, s_h and ϕ, so that price leadership with monopolistic prices is a PBE for patient enough firms.

Proposition 2. *If the following inequality holds,*

$$\frac{s_l}{s_h} < \frac{2-\phi}{2+\phi} \qquad (\star)$$

there exists $\overline{\delta} \in (0,1)$ *such that price leadership with monopolistic prices is a PBE for any* $\delta > \overline{\delta}$.

The condition $(\star)$ is intuitive given that the inequality holds for low $\frac{s_l}{s_h}$ and low ϕ. The informed does not want to deviate and set a price p_l^M when both firms are supposed to set a price p_h^M and split the high demand because:

i. when $\frac{s_l}{s_h}$ is low, the monopolistic price for the low state, p_l^M, is low relative to p_h^M and that makes the option of deviating to set p_l^M less desirable.

ii. when ϕ is low, the demand is persistent, the market size is likely to stay high in the next period and deviating would lead to a low price by the uninformed firm.

Also, condition $(\star)$ is consistent with Stigler's conjecture that "the prices of industries with price leaders are less flexible than those of industries without price leaders, despite the larger fluctuations of output of the former group."[3] Note that condition $(\star)$ implies that price leadership is more likely to arise if ϕ is low, a condition that generates more persistent prices. In a similar fashion, price leadership is more likely to arise if s_l/s_h is not close to 1 which generates larger output fluctuations.

Moreover, there is a minimum $\overline{\delta}$ satisfying Proposition 2. In the Figure 1.2, we represent that minimum $\overline{\delta}$ satisfying Proposition 2 as a function of the parameters.

We can observe that as the market size becomes more persistent we can sustained monopolistic prices at a price leadership equilibrium with lower discount factors. Also, note that when $\frac{s_l}{s_h} \leq 0.6$, monopolistic prices can be sustained if firms are patient enough for any $\phi \in (0, 1/2)$. Also, as $\frac{s_l}{s_h}$ approaches $(2-\phi)/(2+\phi)$ firms need to be more patient to support monopolistic prices in a price leadership equilibrium. Moreover, monopolistic profits cannot be sustain in a price leadership equilibrium in the white area in the top right corner regardless of the discount factor.

[3]In Stigler (1947), on p. 446.

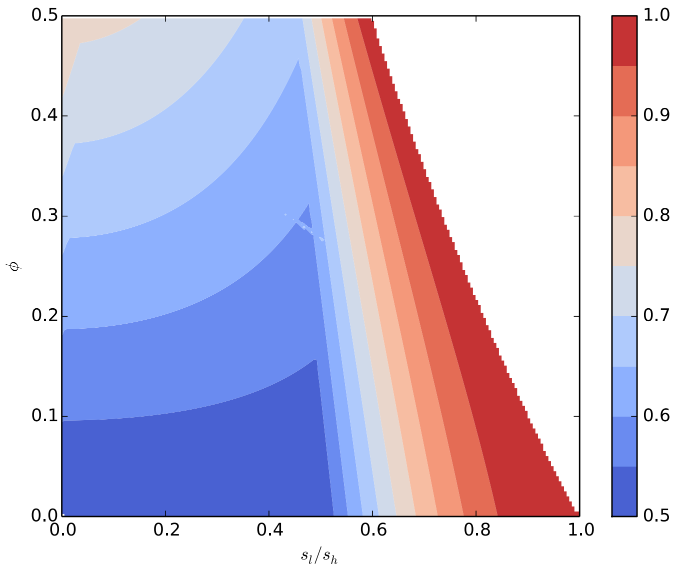

Figure 1.2. $\overline{\delta}$ as a function of ϕ and $\frac{s_l}{s_h}$.

As we previously showed in Proposition 1, whenever inequality (1.3) is not holding, the firms cannot exactly implement monopolistic joint profit. But Proposition 2 implies that for a wide range of parameters in which firms cannot exactly implement joint monopolistic profits, price leadership with monopolistic prices is still an equilibrium as can be seen in the next figure.

Looking at Figure 1.3, patient firms are only able to exactly obtain joint monopolistic profits when parameters are in the darker shaded area. But price leadership with monopolistic prices appears in equilibrium when firms are patient for parameters in both the dark and light shaded area.

If the firms are able to sustain price leadership with monopolistic prices, joint profits are below the monopolistic profits only when there was a change of state and the the uninformed firm is not able to adjust its price. Therefore, as the market size becomes more persistent the joint profits must approximate monopolistic profits. The next lemma contains a closed form solution for the difference between the informed monopolist expected profits and the joint expected profits from following price leadership with monopolistic prices.

We compare the joint profits derived from price leadership with monopolistic prices to those obtained by an informed monopolist in the next lemma. The

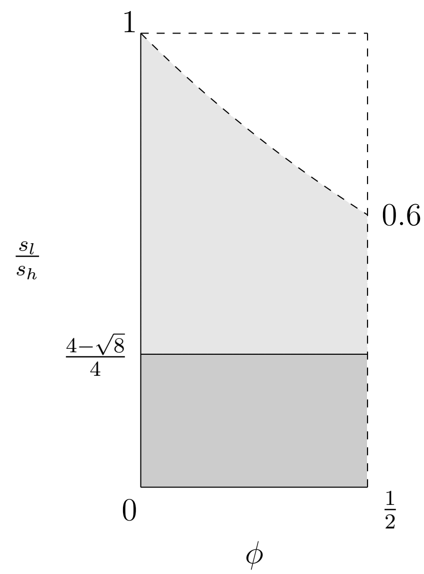

Figure 1.3. Results in the parameter space

comparison will consider ex-ante profits, that is, before the informed firm or the informed monopolist know the realization of the current state.

Lemma 1. *The difference in ex-ante discounted payoffs between the informed monopolist and the joint profits of firms following price leadership with monopolistic prices are*

$$V^M(s_l) - [V^U_{\mathbf{p}^M}(p^M_l) + V^I_{\mathbf{p}^M}(s_l)] = \left(\frac{1-\delta+\delta\phi}{1-\delta+2\delta\phi}\right)\phi\left(\frac{s_h - s_l}{2}\right)^2 \qquad (1.10)$$

and

$$V^M(s_h) - [V^U_{\mathbf{p}^M}(p^M_h) + V^I_{\mathbf{p}^M}(s_h)] = \left(\frac{\delta\phi^2}{1-\delta+2\delta\phi}\right)\left(\frac{s_h - s_l}{2}\right)^2 \qquad (1.11)$$

where $V^I_{\mathbf{p}^M}(s_l) = (1-\phi)V^I_{\mathbf{p}^M}(s_l, s_l) + \phi V^I_{\mathbf{p}^M}(s_l, s_h)$*, the ex-ante expected payoff for the informed firm provided that the previous state was low, and similarly* $V^I_{\mathbf{p}^M}(s_h) = \phi V^I_{\mathbf{p}^M}(s_h, s_l) + (1-\phi)V^I_{\mathbf{p}^M}(s_h, s_h)$ *is its expected value when the previous state was high.*

From Lemma 1, we conclude that the joint profits approach those of an informed monopolist as ϕ goes to 0. That is not necessarily the case when δ goes to 1 and everything else is fixed.

Proposition 3. *Fix s_l and s_h, then there exist a $\overline{\delta}_{s_l,s_h} \in (0,1)$ such that for any fixed $\delta > \overline{\delta}_{s_l,s_h}$,*

- *exists a $\overline{\phi} > 0$, such that for any $\phi < \overline{\phi}$, price leadership with monopolistic profits is a PBE.*
- *the ex-ante expected joint profits go to the monopolistic profits as ϕ goes to 0.*

It is important to point out that when ϕ goes to 0, a price leadership equilibrium approximates monopolistic profits for high and *fixed* discount factors. In contrast, it is unclear whether that would be the case for revision strategies, or strategy profiles that rely on a statistical test to enforce equilibrium play. For example, suppose we try to implement joint profits that are near monopolistic profits by asking the informed firm to always set a price p_h^M while the informed firm sets a price p_l^M when the state is s_l and a price p_h^M when the state is s_h. If the uninformed firm wants to test whether the informed firm is playing according to the specified profile, a reliable test would likely need to be longer and more complicated as ϕ becomes smaller and that would require a larger δ in order to implement the desired outcome.

1.3.4 Extending the Equilibrium Results

The next proposition shows that when price leadership with monopolistic prices is not an equilibrium, price leadership still is an equilibrium for other prices if firms are patient. As previously argued, if monopolistic prices cannot be sustained for patient firms, it must be the case that is too tempting for the informed firm to pretend that the demand is low when it is actually high. In one such case, the payoff derived from that deviation would be lower if firms decide to set a lower price p_l than the monopolistic one. The previous intuition leads to the next result.

Proposition 4. *If condition ($\star$) does not hold, there exists a price $\overline{p}_l$ with $0 < \overline{p}_l < p_l^M$, such that for any $p_l \in [0, \overline{p}_l)$, the price leadership strategy profile with prices p_l and p_h^M is a PBE provided that firms are patient enough.*

Therefore, price leadership can be sustained as an equilibrium for any s_l, s_h and $\phi < 1/2$ as long as firms are patient enough.

1.4 Extensions

Here I discuss extending the model to multiple states, to the case in which firms can only observe prices and a possible application of the model to a practice called *category management*.

Multiple states

The model can be extended to allow for more than two states. Now assume that at each period t, the market size s^t is drawn from the set $\{s_1, \ldots, s_n\}$ for some $n > 2$. Let $s_j < s_{j+1}$ for $j = 1, \ldots, n-1$. Again the market size follows Markov process with transition matrix $\mathbf{P}$ and initial distribution μ. Let $\mathbf{P}(s'|s)$ be the probability that the current state is s' given that the previous state was s. To show that the intuition behind the results holds for more than two states, we capture the idea that the market size is persistent by assuming that $\mathbf{P}(s_j|s_j) = 1 - \phi$ for $j = 1, \ldots, n$, and consider the case in which ϕ becomes small.

Now, let's consider price leadership with monopolistic prices, $p_j^M = s_j/2$, and derive the incentive constraints that capture the idea that the informed firm does not want to price according to the wrong state. The informed firm has an incentive to do so only when the current market size is greater or equal than the previous market size. Suppose that $s^{t-1} = s_j$ and $s^t \geq s_j$ for some $j > 1$, then at period t the uninformed firm would not want to set a price equal to p_{j-1}^M if

$$V^I_{\mathbf{p}^M}(s_j, s^t) \geq (1-\delta)\pi(p^M_{j-1}; s^t) + \delta \mathbb{E}_{s^{t+1}}\left[V^I_{\mathbf{p}^M}(s_{j-1}, s^{t+1})|s^t\right].$$

Compared to the two states case, extra assumptions on the state space are required for these incentive constraints to hold.

Considering the case with $n = 3$ is enough to get an idea. The incentive constraints that guarantee that the informed firm does not price according to the wrong state are,

- when the previous and the current state are s_j for $j = 2, 3$, the informed firm does not want to deviate and set a price p^M_{j-1} if

$$V^I_{\mathbf{p}^M}(s_j, s_j) \geq (1-\delta)\pi(p^M_{j-1}; s_j) + \delta \sum_{k=1}^{3} \mathbf{P}(s_k|s_j) V^I_{\mathbf{p}^M}(s_{j-1}, s_k). \tag{1.12}$$

- when the previous state was s_2 and the current state is s_3, the uninformed firm does not want to deviate and set a price equal to

$$V^I_{\mathbf{p}^M}(s_2, s_3) \geq (1-\delta)\pi(p^M_1; s_3) + \delta \sum_{k=1}^{3} \mathbf{P}(s_j|s_3) V^I_{\mathbf{p}^M}(s_1, s_k). \qquad (1.13)$$

Again, values and incentive constraints are continuous on ϕ and taking limits when ϕ goes to 0, the incentive constraint (1.12) turns into

$$\frac{\pi(p^M_j; s_j)}{2} \geq (1-\delta)\pi(p^M_{j-1}; s_j) + \delta^2 \frac{\pi(p^M_j; s_j)}{2}$$

or

$$(1+\delta)\frac{\pi(p^M_j; s_j)}{2} > \pi(p^M_{j-1}; s_j).$$

Because $\pi(p^M_j; s_j) > \pi(p^M_{j-1}; s_j)$, the previous inequality always hold for high discount factors.

Similarly, taking limits when ϕ goes to 0, the incentive constraint (1.13) turns into

$$\delta\frac{\pi(p^M_3; s_3)}{2} > (1-\delta)\pi(p^M_1; s_3) + \delta^2 \frac{\pi(p^M_3; s_3)}{2}$$

or

$$\delta\frac{\pi(p^M_3; s_3)}{2} > \pi(p^M_1; s_3).$$

The previous inequality holds as long as,

$$\frac{s_1}{s_3} < 1 - \sqrt{1 - \frac{\delta}{2}}.$$

Consequently, to obtain an analogous result to Proposition 3 for the $n = 3$ case, we would require that

$$\frac{s_1}{s_3} < \frac{4-\sqrt{8}}{4}.$$

But it is also important to notice that when the previous inequality does not hold, we can still sustain price leadership with prices $\mathbf{p} = (p_1, p^M_2, p^M_3)$ as a PBE for some $p_1 < p^M_1$ if the firms are patient and the market size is persistent.

Firms only observing prices

The fact that the uninformed firm is sometimes able to infer the state does not drive the results. This can be observed by assuming that the uninformed firm observes prices but does not observe quantities at all. The modified model allows to better understand the intuition behind the results. Consider the environment from Section 2 with firms only observing prices at the end of each period as the only difference. Therefore, when firms are setting prices at period t, the uninformed firm has only observed the sequence $\{p_I^\tau, p_U^\tau\}_{\tau=0}^{t-1}$ while the informed firm knows the sequence $\{p_I^\tau, p_U^\tau, s^\tau\}_{\tau=0}^{t-1} \times s^t$. Consequently, the uninformed firm is never able to infer s^t. In that case, when firms are following the price leadership strategy profile, the uninformed firm is not able to detect any deviation in which the informed firm sets a price p_l or p_h.

Compared to the case in which firms are able to observe their own quantities, there is one more incentive constrained that needs to be verified for the price leadership strategy profile to be a PBE: when the previous state was s_l and the current state is s_h, the informed firm does not want to set a price p_l. Expected payoff do not change and the expected payoff from following price leadership is the same as in equation (1.7),

$$V_{\mathbf{p}}^I(s_l, s_h) = \delta \left[\phi V_{\mathbf{p}}^I(s_h, s_l) + (1-\phi) V_{\mathbf{p}}^I(s_h, s_h)\right].$$

Note that the informed firm does not sell in the current period when it follows price leadership. Instead, the informed firm can deviate to set p_l and share the market without being detected. But if the informed firm sets a price p_l today, the uninformed firm will also set a price p_l tomorrow. Therefore, the incentive constraint is given by,

$$V_{\mathbf{p}}^I(s_l, s_h) \geq (1-\delta)\frac{\pi(p_l; s_h)}{2} + \delta\big[\phi V_{\mathbf{p}}^I(s_l, s_l) + (1-\phi) V_{\mathbf{p}}^I(s_l, s_h)\big]. \tag{1.14}$$

Therefore, in a price leadership equilibrium with monopolistic prices, when the state goes from low to high, the informed firms does not want to deviate to set a price p_l because doing so will imply another period of low prices tomorrow when a high price could be charge by following price leadership. The following lemma simplifies the previous incentive constraint for the case in which firms are using monopolistic prices.

Lemma 2. *When firms are following price leadership with monopolistic prices* $\mathbf{p}^M = (p_l^M, p_h^M)$, *the incentive constraint*(1.14) *holds if and only if*

$$\left[1+\delta\phi\right]\left(\frac{s_l}{s_h}\right)^2 - 2\left(\frac{s_l}{s_h}\right) + \delta(1-\phi) \geq 0. \tag{1.15}$$

The left-hand-side of the previous inequality is continuous on ϕ and δ. When $\delta \in (0,1)$ and $\phi \in (0,1/2)$, the left-hand-side of inequality (1.15) is increasing on δ and decreasing on ϕ.

Hence, Proposition 3 holds without any extra assumptions because when ϕ goes to 0, the inequality (1.15) holds in the limit if $\delta \geq k(2-k)$ where $k = s_l/s_h$ and $k(2-k) < 1$ for $k \in [0,1)$.

Proposition (2) would hold if firms cannot observe quantities if condition ($\star$) is replaced by

$$\frac{s_l}{s_h} < \frac{1-\phi}{1+\phi}. \tag{1.16}$$

That is the case because (1.16) implies ($\star$) and, as δ goes to 1, the left-hand-side of inequality (1.15) goes to $(1-\phi) - 2k + (1+\phi)k^2$ where $k = \frac{s_l}{s_h}$ and the expression is positive as long as inequality (1.16) holds.

Category management

Category management refers to "an organizational approach in which the management of a retail establishment is broken down into categories of like products" (See FTC (2001), p. 47). A common way of managing a category is by assigning a "category captain," usually the category leading manufacturer, as the primary advisor for the category. In some instances, the category captains may advise the retailer on the prices for all category brands and the shelving of the category products (See American Antitrust Institute (2003), p. 4). Some have argued that the motive behind the practice is that all the parties benefit from the information and expertise that the captain possess, including information about demand, as a leading manufacturer (See p. 204 in Desrochers et al. (2003) or p. 46-47 in FTC (2001)).

The use of a category captain raises concern because a manufacturer that is selected as a category captain can use its position to increase profits at the expense of its competitors. However, the practice's prevalence suggests that captains do not abuse their positions. Still, it is unclear as to why manufacturers other than the category captains comply with the practice. The intuition behind price leadership in this model may provide an explanation: If the category captain has better information, then the competitors may benefit from delegating decisions to the informed firm. On the other hand, the captains allow their competitors to attain high profits because doing so is better than unleashing a retaliation or manufactures and retailers independently setting prices.

1.5 Concluding Remarks

In this model, I study an infinite-horizon duopoly model in which firms engage in Bertrand competition at each period. The market size follows a Markov process, and at each period the realization of the current state is only known by one firm. In that environment, I show the existence of equilibria that generates price leadership patterns for a wide set of parameters. Moreover, firms can derive joint profits that approach the monopolistic profits from this type of equilibria as the market size becomes more persistent. In such cases, given that overt communication is not feasible, the informed firm leads the uninformed firm towards joint profit maximization.

Moreover, compared to some previous models of barometric price leadership, we dispose of the assumption that firms have the option of pricing sequentially at each stage. We can understand pricing sequentially as a firm communicating their price intentions to competitors before making the price change, or as firms making non-immediately-effective price announcements as in the vitamins industry (Marshall et al., 2008). Our results suggest that firms can attain high profits through price leadership with no need for overt communication or price announcements.

Although our model differs from the "secret price cuts models"[4] because firms are able to observe prices, the informed firm can cut prices without being detected by pretending that the demand is low when it is actually high. Price leadership discourages this type of deviation with no need for on-equilibrium punishments because if the informed firm lowers its price the uninformed will match the low price in the next period.

[4] See subsection 6.7.1 in Tirole (1988) for an example.

This paper provides theoretical support for Stigler's barometric price leadership observations. The patterns can emerge in the presence of asymmetric information about persistent market conditions because the price of the better informed firm can be used to infer information about the unknown market conditions.

1.6 Appendix

1.6.1 Proofs

Proposition 1. Let us start with the "if" part. Assuming that the condition (1.3) is satisfied, we will propose an equilibrium that exactly achieves joint monopolistic profits. The proposed strategy profile is one in which the uninformed firm always set the price p_h^M while the informed sets the price p_l^M when the state is s_l and p_h^M when the market size is s_h. When a deviation is detected they trigger a stage Nash reversal. Let $\widetilde{V}^I(s)$ be the discounted payoff when the state is s for $s \in \{s_l, s_h\}$. If both firms are following the prescribed strategy profile, then

$$\widetilde{V}^I(s_l) = (1-\delta)\pi(p_l^M; s_l) + \delta\left[(1-\phi)\widetilde{V}^I(s_l) + \phi\widetilde{V}^I(s_h)\right]$$

and

$$\widetilde{V}^I(s_h) = (1-\delta)\frac{\pi(p_h^M; s_h)}{2} + \delta\left[\phi\widetilde{V}^I(s_l) + (1-\phi)\widetilde{V}^I(s_h)\right].$$

Detectable deviations trigger a Nash reversal and as a consequence are not profitable for patient firms. Then, we just need to verify that deviations that are not detected are not profitable. The only deviation that is not detected is one in which the informed firm plays as if the market size is s_l when it is actually s_h. Then, we require that

$$\widetilde{V}^I(s_h) \geq (1-\delta)\pi(p_l^M; s_h) + \delta\left[\phi\widetilde{V}^I(s_l) + (1-\phi)\widetilde{V}^I(s_h)\right]$$

or

$$(1-\delta)\frac{\pi(p_h^M; s_h)}{2} + \delta\left[\phi\widetilde{V}^I(s_l) + (1-\phi)\widetilde{V}^I(s_h)\right] \geq$$
$$(1-\delta)\pi(p_l^M; s_h) + \delta\left[\phi\widetilde{V}^I(s_l) + (1-\phi)\widetilde{V}^I(s_h)\right]$$

which holds as long as

$$\pi(p_h^M; s_h) \geq 2\pi(p_l^M; s_h).$$

Just plugging in the functions, we can see that the informed firm will not have incentives to lie about the state as long as,

$$\frac{s_h^2}{4} \geq s_l\left(s_h - \frac{s_l}{2}\right).$$

Finally, the previous inequality holds if,

$$2\left(\frac{s_l}{s_h}\right)^2 - 4\left(\frac{s_l}{s_h}\right) + 1 \geq 0,$$

or

$$\frac{s_l}{s_h} \leq \frac{4 - \sqrt{8}}{4}.$$

We need to show the "only if" part. Note that condition (1.3) implies that a myopic informed firm will prefer to share the high demand at a price p_h^M to get the whole high demand at a price p_l^M. That is, $\pi(p_h^M; s_h)/2 \geq \pi(p_l^M; s_h)$.

Also, at each period, the probability that the market size is always positive, if the uninformed firm sets a price below p_h^M with positive probability the demand will be high and the lowest price in the industry will be below the monopolistic price which leads to profits below the monopolistic ones.

As a consequence, if firms are exactly achieving monopolistic profits jointly, it must be the case that the uninformed firm never sets a price below p_h^M and that the informed is always setting the monopolistic price according to the state of the demand. Also, the uninformed firm has to make enough sales otherwise it will be tempted to price cut the informed firm and here is were the problem lays. Every time that the demand is high and the uninformed is supposed to sell, the informed firm can set a price p_l^M and the uninformed firm will not be able to distinguish if this was in fact a deviation or the demand was low. That deviation would be profitable if the condition (1.3) is not satisfied.

Whenever this is the case, firms cannot exactly implement joint monopolistic profits. □

Before getting into the proof of the propositions, we start by presenting close-form solutions for expected discounted payoffs when both firms are playing according to the price leadership strategy profile.

The following lemma will prove useful to calculate the values that firm derive from price leadership.

Lemma 3. *The solutions V_1 and V_2 to the system of equations*

$$V_1 = (1-\delta)\Pi_1 + \delta[(1-\phi)V_1 + \phi V_2] \tag{1.17}$$

and

$$V_1 = (1-\delta)\Pi_2 + \delta[\phi V_1 + (1-\phi)V_2] \tag{1.18}$$

with $\Pi_1, \Pi_2 \in \mathbb{R}_+$, $\delta \in (0,1)$ and $\phi \in (0, 1/2)$ are given by

$$V_1 = \left[\frac{1-\delta}{[1-\delta(1-\phi)]^2 - \delta^2\phi^2}\right]([1-\delta(1-\phi)]\Pi_1 + \delta\phi\,\Pi_2)$$
$$V_2 = \left[\frac{1-\delta}{[1-\delta(1-\phi)]^2 - \delta^2\phi^2}\right](\delta\phi\,\Pi_1 + [1-\delta(1-\phi)]\Pi_2)\,.$$

Proof. The system of equations (1.17) and (1.18) can be written as

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = (1-\delta)\begin{pmatrix} \Pi_1 \\ \Pi_2 \end{pmatrix} + \delta\begin{bmatrix} 1-\phi & \phi \\ \phi & 1-\phi \end{bmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}.$$

Then,

$$\begin{bmatrix} 1-\delta(1-\phi) & -\delta\phi \\ -\delta\phi & 1-\delta(1-\phi) \end{bmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = (1-\delta)\begin{pmatrix} \Pi_1 \\ \Pi_2 \end{pmatrix},$$

or

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = (1-\delta)\begin{bmatrix} 1-\delta(1-\phi) & -\delta\phi \\ -\delta\phi & 1-\delta(1-\phi) \end{bmatrix}^{-1}\begin{pmatrix} \Pi_1 \\ \Pi_2 \end{pmatrix}.$$

By inverting the matrix in the right hand side of the equation,

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \left(\frac{1-\delta}{[1-\delta(1-\phi)]^2 - \delta^2\phi^2}\right)\begin{bmatrix} 1-\delta(1-\phi) & \delta\phi \\ \delta\phi & 1-\delta(1-\phi) \end{bmatrix}\begin{pmatrix} \Pi_1 \\ \Pi_2 \end{pmatrix}.$$

That completes the proof. □

Using the lemma, we can see a close form solution for the uninformed firm values.

Corollary 1. *The uninformed firm discounted expected utilities* $V^U(p_l)$ *and* $v^U(p_h)$ *that solve equations* (1.4) *and* (1.5) *are given by*

$$V^U_{\mathbf{p}}(s_l) = \left[\frac{1-\delta}{[1-\delta(1-\phi)]^2 - \delta^2\phi^2}\right]\left([1-\delta(1-\phi)]\Pi^U_l + \delta\phi\,\Pi^U_h\right)$$
$$V^U_{\mathbf{p}}(s_h) = \left[\frac{1-\delta}{[1-\delta(1-\phi)]^2 - \delta^2\phi^2}\right]\left(\delta\phi\,\Pi^U_l + [1-\delta(1-\phi)]\Pi^U_h\right)$$

where $\Pi^U_l = (1-\phi)\frac{\pi(p_l;s_l)}{2} + \phi\pi(p_l;s_h)$ *and* $\Pi^U_h = (1-\phi)\frac{\pi(p_h;s_h)}{2}$.

In a similar fashion we can obtain the values for the informed firm.

Corollary 2. *The discounted expected values for the informed firm,* V^I_{ll}*,* V^I_{lh}*,* V^I_{hl} *and* V^I_{hh} *that are solutions to the system of equations (1.6 - 1.9) are given by*

$$V^I_{\mathbf{p}}(s_l,s_l) = \left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right]\left[(c-\delta^2\phi^2)\Pi^I_{ll} + \delta\phi c\Pi^I_{lh} + \delta^2\phi^2\Pi^I_{hl} + \delta^2\phi(1-\phi)\Pi^I_{hh}\right] \tag{1.19}$$

$$V^I_{\mathbf{p}}(s_l,s_h) = \left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right]\left[\delta^2\phi(1-\phi)\Pi^I_{ll} + c^2\Pi^I_{lh} + \delta\phi c\Pi^I_{hl} + \delta(1-\phi)c\Pi^I_{hh}\right] \tag{1.20}$$

$$V^I_{\mathbf{p}}(s_h,s_l) = \left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right]\left[\delta(1-\phi)c\Pi^I_{ll} + \delta\phi c\Pi^I_{lh} + c^2\Pi^I_{hl} + \delta^2\phi(1-\phi)\Pi^I_{hh}\right] \tag{1.21}$$

$$V^I_{\mathbf{p}}(s_h,s_h) = \left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right]\left[\delta^2\phi(1-\phi)\Pi^I_{ll} + \delta^2\phi^2\Pi^I_{lh} + \delta\phi c\Pi^I_{hl} + (c-\delta^2\phi^2)\Pi^I_{hh}\right] \tag{1.22}$$

where $c = [1-\delta(1-\phi)]$, $\Pi^I_{ll} = \frac{\pi(p_l;s_l)}{2}$, $\Pi^I_{lh} = 0$, $\Pi^I_{hl} = \pi(p_l;s_l)$ *and* $\Pi^I_{hh} = \frac{\pi(p_l;s_l)}{2}$.

Proof. Again, let

$$V^I_{\mathbf{p}}(s_l) \equiv (1-\phi)V^I_{\mathbf{p}}(s_l,s_l) + \phi V^I_{\mathbf{p}}(s_l,s_h)$$

and

$$V^I_{\mathbf{p}}(s_h) \equiv \phi V^I_{\mathbf{p}}(s_h,s_l) + (1-\phi)V^I_{\mathbf{p}}(s_h,s_h).$$

Then, plugging equations (1.6) to (1.9) in the previous two equations we obtain,

$$V_{\mathbf{p}}^{I}(s_l) = (1-\delta)\Big[(1-\phi)\Pi_{ll}^{I} + \phi\Pi_{lh}^{I}\Big] + \delta\Big[(1-\phi)V_{\mathbf{p}}^{I}(s_l) + \phi V_{\mathbf{p}}^{I}(s_h)\Big]$$

and

$$V_{\mathbf{p}}^{I}(s_h) = (1-\delta)\Big[\phi\Pi_{hl}^{I} + (1-\phi)\Pi_{hh}^{I}\Big] + \delta\Big[\phi V_{\mathbf{p}}^{I}(s_l) + (1-\phi)V_{\mathbf{p}}^{I}(s_h)\Big].$$

Therefore, following exactly the same steps as in lemma 3 we obtain

$$V_{\mathbf{p}}^{I}(s_l) = \left[\frac{1-\delta}{[1-\delta(1-\phi)]^2 - \delta^2\phi^2}\right]\big([1-\delta(1-\phi)]\Pi_l^I + \delta\phi\Pi_h^I\big) \tag{1.23}$$

and

$$V_{\mathbf{p}}^{I}(s_h) = \left[\frac{1-\delta}{[1-\delta(1-\phi)]^2 - \delta^2\phi^2}\right]\big(\delta\phi\Pi_l^I + [1-\delta(1-\phi)]\Pi_h^I\big) \tag{1.24}$$

where $\Pi_l^I = (1-\phi)\Pi_{ll}^I + \phi\Pi_{lh}^I$ and $\Pi_h^I = \phi\Pi_{hl}^I + (1-\phi)\Pi_{hh}^I$.

Note that equation (1.6) is equal to,

$$\begin{aligned}
V_{\mathbf{p}}^{I}(s_l, s_l) =& (1-\delta)\Pi_{ll}^I + \delta V_{\mathbf{p}}^{I}(s_l) \\
=& (1-\delta)\Pi_{ll}^I + \delta\left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right]\big(c[(1-\phi)\Pi_{ll}^I + \phi\Pi_{lh}^I] + \delta\phi[\phi\Pi_{hl}^I \\
&+ (1-\phi)\Pi_{hh}^I]\big) \\
=& \left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right]\Big(\big[c^2 - \delta^2\phi^2 + c(1-c)\big]\Pi_{ll}^I + \delta\phi c\,\Pi_{lh}^I + \delta^2\phi^2\Pi_{hl}^I \\
&+ \delta^2\phi(1-\phi)\Pi_{hh}^I]\Big) \\
=& \left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right]\big([c-\delta^2\phi^2]\Pi_{ll}^I + \delta\phi c\,\Pi_{lh}^I + \delta^2\phi^2\Pi_{hl}^I + \delta^2\phi(1-\phi)\Pi_{hh}^I]\big).
\end{aligned}$$

Similarly, equation (1.7)is equal to,

$$\begin{aligned}
V_{\mathbf{p}}^{I}(s_l, s_h) =& (1-\delta)\Pi_{lh}^I + \delta V_{\mathbf{p}}^{I}(s_h) \\
=& (1-\delta)\Pi_{lh}^I + \delta\left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right]\big(\delta\phi[(1-\phi)\Pi_{ll}^I + \phi\Pi_{lh}^I] + c[\phi\Pi_{hl}^I \\
&+ (1-\phi)\Pi_{hh}^I]\big)
\end{aligned}$$

$$= \left[\frac{1-\delta}{c^2 - \delta^2\phi^2}\right]\big(\delta^2\phi(1-\phi)\Pi^I_{ll} + [c^2 - \delta^2\phi^2 + \delta^2\phi^2]\Pi^I_{lh} + \delta\phi c\Pi^I_{hl} + \delta(1-\phi)c\Pi^I_{hh}\big)$$

$$= \left[\frac{1-\delta}{c^2 - \delta^2\phi^2}\right]\big(\delta^2\phi(1-\phi)\Pi^I_{ll} + c^2\Pi^I_{lh} + \delta\phi c\Pi^I_{hl} + \delta(1-\phi)c\Pi^I_{hh}\big)$$

The same argument be done with equations (1.8) and (1.9) to get $V^I_{\mathbf{p}}(s_h, s_l)$ and $V^I_{\mathbf{p}}(s_h, s_h)$. □

Corollary 3. *The expected ex-ante discounted payoff for an informed monopolist, solutions to the system of equations (1.1-1.2), are given by*

$$V^M(s_l) = \left[\frac{1-\delta}{[1-\delta(1-\phi)]^2 - \delta^2\phi^2}\right]\big([1-\delta(1-\phi)]\Pi^M_l + \delta\phi\,\Pi^M_h\big)$$
$$V^M(s_h) = \left[\frac{1-\delta}{[1-\delta(1-\phi)]^2 - \delta^2\phi^2}\right]\big(\delta\phi\,\Pi^M_l + [1-\delta(1-\phi)]\Pi^M_h\big)$$

where $\Pi^M_l = (1-\phi)\pi(p^M_l; s_l) + \phi\pi(p^M_h; s_h)$ *and* $\Pi^M_h = \phi\pi(p^M_l; s_l) + (1-\phi)\pi(p^M_h; s_h)$.

Finally, before getting into the proof of Proposition 2, we provide a list of all potentially profitable deviations from price leadership with monopolistic prices.

Deviations from price leadership with monopolistic prices.

- Lets start by analyzing the potential deviations by the uninformed firm.
  - If the informed firm previous price was p^M_l, the uninformed firm believes that the previous market size was s_l and that the current state is s_l with probability $(1-\phi)$ and s_h with probability ϕ. Then, the uninformed firm can deviate by
    - ∗ charging a slightly lower price than p^M_l. In that case, the uninformed firm gets the whole market and believes that with probability $(1-\phi)$ its stage payoff will be arbitrarily close to $\pi(p^M_l; s_l)$ and with probability ϕ its stage payoff will be arbitrarily close to $\pi(p^M_l; s_h)$. This deviation triggers a Nash reversal. Then, for that deviation not to be profitable we require that

$$V^U_{\mathbf{p}^M}(p^M_l) \geq (1-\delta)[(1-\phi)\pi(p^M_l; s_l) + \phi\pi(p^M_l; s_h)]. \tag{1.25}$$

* charging a slightly lower price than p_h^M (and above p_l^M). In that case, the uninformed firm sells only if the current state is s_h, a scenario the uninformed firm believes to occur with probability ϕ and will lead to a stage payoff of $\pi(p_h^M; s_h)$. Again, this deviation triggers a Nash reversal and is not profitable as long as,

$$V^U_{\mathbf{p}^M}(p_l^M) \geq (1-\delta)\phi\pi(p_h^M; s_h). \tag{1.26}$$

– Similarly, if the informed firm previous price was p_h, the uninformed firm believes that the current state is s_l with probability ϕ and s_h with probability $(1-\phi)$ and can deviate by

* charging a slightly lower price than p_l^M. This type of deviation is not profitable as long as,

$$V^U_{\mathbf{p}^M}(p_h^M) \geq (1-\delta)[\phi\pi(p_l^M; s_l) + (1-\phi)\pi(p_l^M; s_h)]. \tag{1.27}$$

* charging a slightly lower price than ${}^M p_h$. This type of deviation is not profitable if

$$V^U_{\mathbf{p}^M}(p_h^M) \geq (1-\delta)(1-\phi)\pi(p_h^M; s_h). \tag{1.28}$$

• Informed firm.

– The demand goes from s_l to s_l. Because firms are following the price leadership profile, the informed firm previous price was p_l^M implying that the uninformed firm current price is also p_l^M. In this case, the only potentially profitable deviation is for the informed firm is to charge a price slightly below p_l^M, a scenario in which the informed firm gets a stage payoff arbitrarily close to $\pi(p_l^M; s_l)$.

$$V^I_{\mathbf{p}^M}(s_l, s_l) \geq (1-\delta)\pi(p_l^M; s_l) \tag{1.29}$$

– The demand goes from s_l to s_h. The informed firm knows that the the uninformed is setting a price equal to p_l^M so the only potentially profitable deviation is for the informed firm is to charge a price slightly below p_l^M and get a stage payoff close to $\pi(p_l^M; s_h)$.

$$V^I_{\mathbf{p}^M}(s_l, s_h) \geq (1-\delta)\pi(p_l^M; s_h) \tag{1.30}$$

– The demand goes from s_h to s_l. In that case, the informed firm is obtaining the whole monopolistic profits in that period so there is no potential profitable deviation.

– The demand goes from s_h to s_h. In that case, the uninformed sets a current price of p_h^M because the informed previous price was p_h^M. Then, there are two potential profitable deviations for the informed firm,

  * it can charge a price slightly below p_h^M (and above p_l^M) and get a stage payoff close to $\pi(p_h^M; s_h)$. Such a deviation is not profitable as long as

$$V^I_{\mathbf{p}^M}(s_h, s_h) \geq (1-\delta)\pi(p_h^M; s_h) \tag{1.31}$$

  * it can deviate by pretending the state is s_h by charging a price p_l^M as in Figure 1.1. The informed firm gets the whole market deriving a stage payoff of $\pi(p_l^M; s_h)$. The uninformed firm does not make a sale and therefore is not able to distinguish whether the market size was low or there was a deviation. Therefore, the next period the uninformed firm will set a price equal to p_l^M. Also, next period market size is s_l with probability ϕ and the informed firm will face an identical problem as the case in which market size went from s_l to s_l. Similarly, next period market size is s_h with probability $(1-\phi)$ and the informed firm will face an identical problem to the case in which the market size went from s_l to s_h. Consequently, if price leadership with monopolistic prices is an equilibrium, it must be the case that

$$V^I_{\mathbf{p}^M}(s_h, s_h) \geq (1-\delta)\pi(p_l^M; s_h) + \delta[\phi V^I_{\mathbf{p}^M}(s_l, s_l) + (1-\phi)V^I_{\mathbf{p}^M}(s_l, s_h)] \tag{1.32}$$

The following lemma rewrites the incentive constraint (1.32) in a way that not only simplifies the proof of Proposition 2 but also provides some intuition over the necessary conditions for the informed firm not to want to pretend that the demand is low when it is actually high.

Lemma 4. *The incentive constraint* (1.32) *holds if and only if*

$$[1+\delta(1-\phi)]\pi(p_h^M;s_h) - 2\pi(p_l^M;s_h) + \delta\phi\pi(p_l^M;s_l) \geq 0. \tag{1.33}$$

Proof. We start with the incentive constraint (1.32),

$$V^I_{\mathbf{p}^M}(s_h,s_h) \quad \geq \quad (1-\delta)\pi(p_l^M;s_h) + \delta\big[\phi V^I_{\mathbf{p}^M}(s_l,s_l) + (1-\phi)V^I_{\mathbf{p}^M}(s_l,s_h)\big]$$

and if we plug equation (1.9),

$$(1-\delta)\frac{\pi(p_h^M;s_h)}{2} + \delta\big[\phi V^I_{\mathbf{p}^M}(s_h,s_l) + (1-\phi)V^I_{\mathbf{p}^M}(s_h,s_h)\big] \geq$$
$$(1-\delta)\pi(p_l^M;s_h) + \delta\big[\phi V^I_{\mathbf{p}^M}(s_l,s_l) + (1-\phi)V^I_{\mathbf{p}^M}(s_l,s_h)\big].$$

The previous inequality can be rewritten as,

$$(1-\delta)\left[\frac{\pi(p_h^M;s_h)}{2} - \pi(p_l;s_h)\right] \geq$$
$$\delta\left\{\phi\left[V^I_{\mathbf{p}^M}(s_l,s_l) - V^I_{\mathbf{p}^M}(s_h,s_l)\right] + (1-\phi)\left[V^I_{\mathbf{p}^M}(s_l,s_h) - V^I_{\mathbf{p}^M}(s_h,s_h)\right]\right\}$$

If we plug equations (1.6)-(1.9) in the right hand side of the previous inequality, we obtain

$$(1-\delta)\left[\frac{\pi(p_h^M;s_h)}{2} - \pi(p_l^M;s_h)\right] \quad \geq \quad -\delta(1-\delta)\left[\frac{\phi\pi(p_l^M;s_l) + (1-\phi)\pi(p_h^M;s_h)}{2}\right].$$

The last inequality can be written in the following way,

$$[1+\delta(1-\phi)]\pi(p_h^M;s_h) - 2\pi(p_l^M;s_h) + \delta\phi\pi(p_l^M;s_l) \geq 0$$

which completes the proof. □

Proposition 2. Price leadership is a PBE as long as all the incentive constraints, (1.25)-(1.32), hold. Fix the prices to the respective monopolistic prices, that is,

$p_l^M = s_l/2$ and $p_h^M = s_h/2$. First note that the first seven incentive constraints, (1.25)-(1.31), correspond to price cutting deviations that are immediately detected. Therefore, any such deviation triggers a Nash reversal. As a result, any of those deviations is profitable if firms are patient enough because price leadership has a positive continuation value. Then, there exists a $\overline{\delta}_1 < 1$ such that the incentive constraints (1.25)-(1.31) hold for any $\delta \in (\overline{\delta}_1, 1)$.

Now, it remains to show that the incentive constraint (1.32) holds if firms are patient enough given the assumption ($\star$). To do so, we start by plugging the monopolistic prices in equation (1.33).

$$[1 + \delta(1-\phi)]\pi(s_h/2; s_h) - 2\pi(s_l/2; s_h) + \delta\phi\pi(s_l/2; s_l) \geq 0.$$

Just plugging the profits,

$$[1 + \delta(1-\phi)]\frac{s_h^2}{4} - s_l\left(s_h - \frac{s_l}{2}\right) + \delta\phi\frac{s_l^2}{4} \geq 0$$

or

$$[1 + \delta(1-\phi)]\frac{s_h^2}{4} - s_l s_h + (2 + \delta\phi)\frac{s_l^2}{4} \geq 0.$$

Then, multiplying by $\frac{4}{s_h^2}$, we can conclude that (1.33) with monopolistic prices holds if and only if

$$[1 + \delta(1-\phi)] - 4k + (2 + \delta\phi)k^2 \geq 0$$

where $k = \frac{s_l}{s_h}$.

Denote the left-hand side of the previous inequality as the function f, that is, $f(\delta, \phi, k) = [1 + \delta(1-\phi)] - 4k + (2 + \delta\phi)k^2$. The function f is continuous and differentiable in all arguments. Moreover, f is increasing on δ because

$$\frac{\partial f}{\partial \delta}(\delta, \phi, k) = (1-\phi) + \phi k^2 > 0$$

and decreasing on k whenever ($\star$) holds since

$$\frac{\partial f}{\partial k}(\delta, \phi, k) = 2(2+\phi)k - 4 < 2(2+\phi)\left(\frac{2-\phi}{2+\phi}\right) - 4 = -2\phi < 0.$$

Then, because f is decreasing on k when $(\star)$ holds, for any k and ϕ satisfying $(\star)$,

$$\begin{aligned} f(1,\phi,k) &> f\left(1,\phi,\frac{2-\phi}{2+\phi}\right) \\ &= (2-\phi) - 4\left(\frac{2-\phi}{2+\phi}\right) + (2+\phi)\left(\frac{2-\phi}{2+\phi}\right)^2 \\ &= \left(\frac{2-\phi}{2+\phi}\right)[(2+\phi) - 4 + (2-\phi)] \\ &= 0. \end{aligned}$$

Because f is continuous on δ, if $(\star)$ holds, there exists a $\bar{\delta}_2$ such that for any $\delta \in (\bar{\delta}_2, 1)$, $f(\delta; k, \epsilon) > 0$. Moreover, because f is increasing on δ,

$$\bar{\delta}_2 = \max\left\{0, -\left(\frac{1-4k+2k^2}{1-\phi+\phi k^2}\right)\right\}.$$

Therefore, for any $\delta \in (\bar{\delta}_2, 1)$, (1.32) holds if $(\star)$ holds.

As a result, if $(\star)$ holds and we let $\bar{\delta} = \max\{\bar{\delta}_1, \bar{\delta}_2\}$, for any $\delta \in (\bar{\delta}, 1)$, the incentive constraints (1.25)-(1.32) hold and therefore PL is a PBE. □

Lemma 1. From equations (1.23) and (1.24),

$$V_{\mathbf{p}}^I(s_l) = (1-\phi)V_{\mathbf{p}}^I(s_l, s_l) + \phi V_{\mathbf{p}}^I(s_l, s_h) = \left[\frac{1-\delta}{[1-\delta(1-\phi)]^2 - \delta^2\phi^2}\right]\left([1-\delta(1-\phi)]\Pi_l^I + \delta\phi\Pi_h^I\right)$$

and

$$V_{\mathbf{p}}^I(s_h) = \phi V_{\mathbf{p}}^I(s_h, s_l) + (1-\phi)V_{\mathbf{p}}^I(s_h, s_h) = \left[\frac{1-\delta}{[1-\delta(1-\phi)]^2 - \delta^2\phi^2}\right]\left(\delta\phi\Pi_l^I + [1-\delta(1-\phi)]\Pi_h^I\right)$$

where $\Pi_l^I = (1-\phi)\frac{\pi(p_l;s_l)}{2}$ and $\Pi_h^I = \phi\pi(p_l; s_l) + (1-\phi)\frac{\pi(p_h;s_h)}{2}$.

From Corollary 1,

$$V_{\mathbf{p}}^U(s_l) = \left[\frac{1-\delta}{[1-\delta(1-\phi)]^2 - \delta^2\phi^2}\right]\left([1-\delta(1-\phi)]\Pi_l^U + \delta\phi\,\Pi_h^U\right)$$

$$V^U_{\mathbf{p}}(s_h) = \left[\frac{1-\delta}{[1-\delta(1-\phi)]^2 - \delta^2\phi^2}\right]\left(\delta\phi\,\Pi^U_l + [1-\delta(1-\phi)]\Pi^U_h\right)$$

where $\Pi^U_l = (1-\phi)\frac{\pi(p_l;s_l)}{2} + \phi\pi(p_l;s_h)$ and $\Pi^U_h = (1-\phi)\frac{\pi(p_h;s_h)}{2}$.

Then, letting $c = 1-\delta(1-\phi)$

$$\begin{aligned}V^I_{\mathbf{p}}(s_l)+V^U_{\mathbf{p}}(p_l) &= \left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right]\left(c\big[\Pi^I_l+\Pi^U_l\big] + \delta\phi\left[\Pi^I_h+\Pi^U_h\right]\right)\\ &= \left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right]\left(c\big[(1-\phi)\pi(p_l;s_l)+\phi\pi(p_l;s_h)\big]\right.\\ &\qquad\left. +\,\delta\phi\big[\phi\pi(p_l;s_l)+(1-\phi)\pi(p_h;s_h)\big]\right)\end{aligned}$$

and

$$\begin{aligned}V^I_{\mathbf{p}}(s_h)+V^U_{\mathbf{p}}(p_h) &= \left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right]\left(\delta\phi\big[\Pi^I_l+\Pi^U_l\big] + c\big[\Pi^I_h+\Pi^U_h\big]\right)\\ &= \left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right]\Bigg(\delta\phi\big[(1-\phi)\pi(p_l;s_l)+\phi\pi(p_l;s_h)\big]\\ &\qquad +\,c\big[\phi\pi(p_l;s_l)+(1-\phi)\pi(p_h;s_h)\big]\Bigg)\end{aligned}$$

From Corollary 3,

$$\begin{aligned}V^M(s_l) = \left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right]&\left(c\big[(1-\phi)\pi(p_l;s_l)+\phi\pi(p_h;s_h)\big]\right.\\ &\left.+\,\delta\phi\left[\phi\pi(p_l;s_l)+(1-\phi)\pi(p_h;s_h)\right]\right)\end{aligned}$$

$$\begin{aligned}V^M(s_h) = \left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right]&\left(\delta\phi\left[(1-\phi)\pi(p_l;s_l)+\phi\pi(p_h;s_h)\right]\right.\\ &\left.+\,c\big[\phi\pi(p_l;s_l)+(1-\phi)\pi(p_h;s_h)\big]\right)\end{aligned}$$

Therefore,

$$V^M(s_l) - \left(V^I_{\mathbf{p}}(s_l)+V^U_{\mathbf{p}}(p_l)\right) = \left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right]c\phi\big[\pi(p_h;s_h)-\pi(p_l;s_h)\big]$$

and

$$V^M(s_h) - \big(V^I_{\mathbf{p}}(s_h) + V^U_{\mathbf{p}}(p_h)\big) = \left[\frac{1-\delta}{c^2 - \delta^2\phi^2}\right]\delta\phi^2\big[\pi(p_h; s_h) - \pi(p_l; s_h)\big].$$

Also,

$$\left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right] = \left[\frac{1-\delta}{[(1-\delta)+\delta\phi)]^2 - \delta^2\phi^2}\right] =$$

$$= \left[\frac{1-\delta}{(1-\delta)^2 + 2\delta(1-\delta)\phi}\right] = \left[\frac{1}{1-\delta+2\delta\phi}\right]$$

and plugging the monopolistic prices,

$$\pi(p^M_h; s_h) - \pi(p^M_l; s_h) = \frac{s_h^2}{4} - \frac{s_l}{2}\Big(s_h - \frac{s_l}{2}\Big) = \frac{1}{4}\big(s_h^2 - s_l s_h + s_l^2\big) = \frac{1}{4}\big(s_h - s_l\big)^2.$$

Finally,

$$V^M(s_l) - \big(V^I_{\mathbf{p}^M}(s_l) + V^U_{\mathbf{p}^M}(p_l)\big) = \frac{1}{4}\left[\frac{c\phi}{1-\delta+2\delta\phi}\right]\big(s_h - s_l\big)^2$$

and

$$V^M(s_h) - \big(V^I_{\mathbf{p}}(s_h) + V^U_{\mathbf{p}}(p_h)\big) = \frac{1}{4}\left[\frac{\delta\phi^2}{1-\delta+2\delta\phi}\right]\big(s_h - s_l\big)^2.$$

□

Proposition 3. Fix s_l and s_h and let $k = \frac{s_l}{s_h}$. We define

$$\overline{\delta}_{s_l,s_h} \equiv \max\left\{\frac{1}{2}\,,\ \frac{4k-2k^2}{1+4k-2k^2}\,,\ -2k^2+4k-1\right\}.$$

We will show that for any fixed $\delta > \overline{\delta}_{s_l,s_h}$, there exists a $\overline{\phi} \in (0, 1/2)$, such that for any $\phi < \overline{\phi}$ price leadership with monopolistic prices is a PBE.

We fixed any $\delta > \overline{\delta}_{s_l,s_h}$. Now, given that we are fixing the discount factor, we must pay attention to those incentive constraints that correspond to deviations that trigger a Nash reversal. Therefore, we need to verify that all all the incentive constraints (1.25) to (1.32) are satisfied when ϕ goes to 0. First note that,

$$\begin{aligned}
\lim_{\phi\downarrow 0} V^U_{\mathbf{p}^M}(p_l) &= \frac{\pi(p_l^M; s_l)}{2} \\
\lim_{\phi\downarrow 0} V^U_{\mathbf{p}^M}(p_h) &= \frac{\pi(p_h^M; s_h)}{2} \\
\lim_{\phi\downarrow 0} V^I_{\mathbf{p}^M}(s_l, s_l) &= \frac{\pi(p_l^M; s_l)}{2} \\
\lim_{\phi\downarrow 0} V^I_{\mathbf{p}^M}(s_l, s_h) &= \delta\frac{\pi(p_h^M; s_h)}{2} \\
\lim_{\phi\downarrow 0} V^I_{\mathbf{p}^M}(s_h, s_l) &= (1-\delta)\pi(p_l^M; s_l) + \delta\frac{\pi(p_l^M; s_l)}{2} \\
\lim_{\phi\downarrow 0} V^I_{\mathbf{p}^M}(s_h, s_h) &= \frac{\pi(p_h^M; s_h)}{2}
\end{aligned}$$

Next, we show that as ϕ goes to 0, the incentive constraint (1.25) is satisfied as long as $\delta > 1/2$. Note that (1.25) is satisfied if,

$$V^U_{\mathbf{p}^M}(p_l) - (1-\delta)[(1-\phi)\pi(p_l^M; s_l) - \phi\pi(p_l^M; s_h)] \geq 0.$$

The left hand side of the previous inequality is continuous on ϕ for $\phi \in [0, 1/2]$. Also, when ϕ goes to 0, the left hand side goes to

$$\frac{\pi(p_l^M; s_l)}{2} - (1-\delta)\pi(p_l^M; s_h).$$

Therefore, the incentive constraint (1.25) holds when ϕ goes to 0 as long as $\delta > 1/2$. The same argument applies for incentive constraints (1.26) to (1.29), and (1.31).

We are left with verifying that the incentive constraints (1.30) and (1.32) hold. Let's look at (1.30), it holds if

$$V^I_{\mathbf{p}^M}(s_l, s_h) - (1-\delta)\pi(p_l^M; s_h) \geq 0.$$

When ϕ goes to 0, the left hand size of the previous inequality goes to

$$\delta\frac{\pi(p_h^M; s_h)}{2} - (1-\delta)\pi(p_l^M; s_h).$$

The previous expression can be written as

$$\delta\frac{s_h^2}{8} - (1-\delta)\frac{s_l s_h}{2} + (1-\delta)\frac{s_l^2}{4}$$

which is greater than 0 if

$$\delta > \frac{4k-2k^2}{1+4k-2k^2}.$$

Therefore, the incentive constraint (1.30) holds as ϕ goes to 0 as long as $\delta > \frac{4k-2k^2}{1+4k-2k^2}$.

Finally, using lemma 4, we know that the incentive constraint (1.32) holds as long as

$$[1+\delta(1-\phi)]\pi(p_h^M; s_h) - 2\pi(p_l^M; s_h) + \delta\phi\pi(p_l^M; s_l) \geq 0$$

and plugging the profits, that turns into,

$$[1+\delta(1-\phi)]\frac{s_h^2}{4} - s_l\Big(s_h - \frac{s_l}{2}\Big) + \delta\phi\frac{s_l}{4} \geq 0.$$

If we multiply the previous inequality by $\frac{4}{s_h^2}$, we would know that the inequality holds as

$$[1+\delta(1-\phi)] - 4k + 2k^2 + \delta\phi k^2 \geq 0.$$

Note that the left hand side of the previous inequality is continuous on ϕ. Also, as ϕ goes to 0, the left hand side goes to,

$$(1+\delta) - 4k + 2k^2.$$

Consequently, the incentive constrain (1.32) holds when ϕ goes to 0 as long as

$$\delta > 4k - 2k^2 - 1.$$

Therefore, for any $\delta > \overline{\delta}_{s_l,s_h}$, there exists a $\overline{\phi} > 0$, such that for any $\phi < \overline{r}$, price leadership with monopolistic prices is a PBE.

In the next figure, we can observe $\overline{\delta}_{s_l,s_h}$ as a function of s_l/s_h.

As a corollary of lemma 1, we can see that

$$\lim_{\phi \downarrow 0}\left[V^M(s_h) - (V^U_{\mathbf{p}^M}(s_h) + V^I_{\mathbf{p}^M}(p_h))\right] = 0.$$

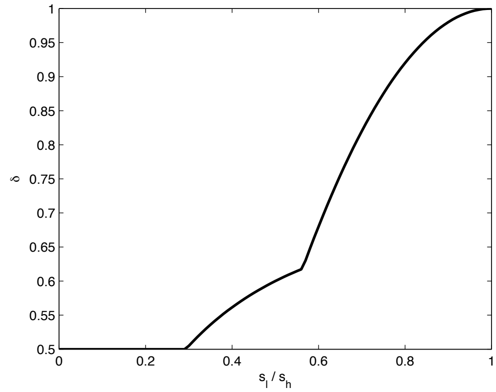

Figure 1.4. $\overline{\delta}_{s_l,s_h}$ as a function of s_l/s_h.

This concludes the proof. □

Proposition 4. The proof follows an argument similar to the one in the proof of Proposition 2. We need to show that no firm wants to price cut its competitor provided that both firms are following the price leadership profile with an arbitrary price $p_l < p_l^M$ for the low demand and the monopolistic price p_h^M for the high demand. As we have argued before the only price cut that is not immediately detected is when the both firms are supposed to set a price p_h^M and the informed firm deviates by setting a price p_l. Any other price cut would not be profitable for patient firms because it would be immediately detected and would trigger a Nash reversal.

Hence, we need to show that even when condition ($\star$) is not satisfied, there exists a $\overline{p_l} < p_l^M$ such that for any $p_l < \overline{p}_l$, a patient enough informed firm would not want to set a price p_l when both firms are supposed to set a price p_h^M. That is, for $p_l < \overline{p}_l$, exists a $\overline{\delta} < 1$ such that for any $\delta > \overline{\delta}$,

$$V_{\mathbf{p}}^I(s_h, s_h) \geq (1-\delta)\pi(p_l; s_h) + \delta\big[\phi V_{\mathbf{p}}^I(s_l, s_l) + (1-\phi)V_{\mathbf{p}}^I(s_l, s_h)\big].$$

Following exactly the same steps as in lemma (4), we can show that the previous inequality holds if

$$[1 + \delta(1-\phi)]\pi(p_h^M; s_h) - 2\pi(p_l; s_h) + \delta\phi\pi(p_l; s_l) \geq 0.$$

When plugging $p_h^M = s_h/2$ in the previous inequality, we obtain

$$[1+\delta(1-\phi)]\frac{s_h^2}{4} - 2p_l(s_h - p_l) + \delta\phi p_l(s_l - p_l) \geq 0.$$

Define the function $f(p_l, \delta)$ as the left-hand side of the previous inequality. That is,

$$\begin{aligned} f(p_l,\delta) =&[1+\delta(1-\phi)]\frac{s_h^2}{4} - 2p_l(s_h - p_l) + \delta\phi p_l(s_l - p_l) \\ =&[1+\delta(1-\phi)]\frac{s_h^2}{4} - (2s_h - \delta\phi s_l)p_l + (2-\delta\phi)p_l^2. \end{aligned}$$

First, note that

$$f(0,\delta) = [1+\delta(1-\phi)]\frac{s_h}{2} > 0.$$

Hence, the informed firm would never want to deviate in such a way if $p_l = 0$. Therefore, irrespective of s_l, s_h and ϕ, price leadership where firms set a price equal to 0 for the low demand and a monopolistic price for the high demand is always an equilibrium for patient firms.

Moreover, f is a continuous and differentiable function and

$$\frac{\partial f}{\partial p_l}(p_l,\delta) = 2(2-\delta\phi)p_l - (2s_h - \delta\phi s_l).$$

Then, for any $0 \leq p_l < p_l^M$,

$$\frac{\partial f}{\partial p_l}(p_l,\delta) < (2-\delta\phi)s_l - (2s_h - \delta\phi s_l) = 2(s_l - s_h) < 0$$

and

$$\frac{\partial f}{\partial \delta}(p_l,\delta) = (1-\phi)\pi(p_h^M; s_h) + \delta\phi\pi(p_l; s_l) > 0.$$

Therefore, because of the continuity of f, as long as $f(p_l, 1) > 0$, it will exist a $\overline{\delta} < 1$ such that for any $\delta \in (\underline{\delta}, 1)$ price leadership with prices p_l and p_h^M is a PBE. Hence, because f is decreasing in p_l, there is a maximum $\overline{p}_l$ such that there exists exist a $\overline{\delta} < 1$ such that for any $\delta \in (\underline{\delta}, 1)$ price leadership with prices $p_l < \overline{p}_l$ and p_h^M is a PBE. That maximum $\overline{p}_l$ is the one that solves $f(\overline{p}_l, 1) = 0$ and it is well defined since $f(0,1) > 0$ and $f(p_l^M, 1) < 0$.

Therefore, we have shown that we can always support a price leadership equilibrium if firms are patient. □

Lemma 2. We start with the incentive constraint (1.14),

$$V^I_{\mathbf{p}^M}(s_l, s_h) \geq (1-\delta)\frac{\pi(p_l^M; s_h)}{2} + \delta\big[\phi V^I_{\mathbf{p}^M}(s_l, s_l) + (1-\phi)V^I_{\mathbf{p}^M}(s_l, s_h)\big].$$

and if we plug equation (1.7) in the left-hand-side of the previous inequality,

$$\delta\big[\phi V^I_{\mathbf{p}^M}(s_h, s_l) + (1-\phi)V^I_{\mathbf{p}^M}(s_h, s_h)\big] \geq (1-\delta)\frac{\pi(p_l^M; s_h)}{2} + \delta\big[\phi V^I_{\mathbf{p}^M}(s_l, s_l) + (1-\phi)V^I_{\mathbf{p}^M}(s_l, s_h)\big].$$

The previous inequality can be rewritten as,

$$\delta\big\{\phi\left[V^I_{\mathbf{p}^M}(s_h, s_l) - V^I_{\mathbf{p}^M}(s_l, s_l)\right] + (1-\phi)\left[V^I_{\mathbf{p}^M}(s_h, s_h) - V^I_{\mathbf{p}^M}(s_l, s_h)\right]\big\} \geq (1-\delta)\frac{\pi(p_l^M; s_h)}{2}.$$

If we plug equations (1.6)-(1.9) in the left hand side of the previous inequality, we obtain

$$\delta(1-\delta)\left[\frac{\phi\pi(p_l^M; s_l) + (1-\phi)\pi(p_h^M; s_h)}{2}\right] \quad \geq \quad (1-\delta)\frac{\pi(p_l^M; s_h)}{2}.$$

or

$$\delta(1-\phi)\pi(p_h^M; s_h) - \pi(p_l^M; s_h) + \delta\phi\pi(p_l^M; s_l) \geq 0.$$

Plugging the profits, the previous inequality turns into,

$$\delta(1-\phi)\frac{s_h^2}{4} - \frac{s_l s_h}{2} + \frac{s_l^2}{4} + \delta\phi\frac{s_l^2}{4} \geq 0$$

and multiplying by $\frac{4}{s_h^2}$,

$$\big[1+\delta\phi\big]\Big(\frac{s_l}{s_h}\Big)^2 - 2\Big(\frac{s_l}{s_h}\Big) + \delta(1-\phi) \geq 0.$$

□

Chapter 2

Cooperation with Persistent Shocks

with Yu Awaya

2.1 Introduction

Agents interacting in a long-run relationship are often exposed to unobserved stochastic shocks to payoffs and are not always able to perfectly monitor others' actions. For example, firms in an oligopoly, are affected by market fluctuations and typically face some degree of uncertainty about the market conditions when making decisions. At the same time, they are not always able to perfectly monitor other firms. Yet, not much is known about the effects of persistent shocks in games with imperfect monitoring.

In this paper, we study a model in which two agents repeatedly interact in the presence of imperfect monitoring and unobserved persistent shocks to payoffs. At each period, agents play a prisoners' dilemma and their payoffs are affected by an unobserved stochastic shock. At the end of each period, players are able to observe only their own payoff and are unable to infer the other player's action or the realization of the period's shock. The main message is that more persistent shock dynamics facilitate cooperation, therefore allowing agents to obtain larger expected discounted payoffs than stage Nash repetition. To make the argument, we compare two environments with shocks having the same mean and variance, but in one, the shocks are persistent, while in the other, they are independent and identically distributed (*i.i.d.*) across time. We introduce a version of the grim trigger profile for the persistent environment and show that the profile is able to sustain cooperation for persistent shocks even when cooperation is not possible with *i.i.d.* shocks.

There is little work examining stochastic games with both unobserved shocks and imperfect monitoring. Many have studied imperfect monitoring with *i.i.d.* shocks. Applications of such type of models to collusion in oligopolies include Green and Porter (1984) and Abreu et al. (1986). There are also other studies examining stochastic shocks with agents being able to observe the shock realizations before making decisions. For example, Rotemberg and Saloner (1986), Kandori (1991), and Bagwell and Staiger (1997) study price-setting oligopolies in which demand follows a stochastic process but firms are able to observe all other firms' previous prices and the demand realization. In one of the closest works to ours, Yamamoto (2015) studies a general class of games in which there are unknown stochastic shocks but agents are able to observe others' actions while having a different focus: a folk theorem. In contrast, our results are for fixed discount factors and focus on whether persistence helps or hinders agents' ability to cooperate.

The structure of the chapter is as follows: In section 2.2, we introduce the model. Before analyzing the persistent environment, we present results from previous studies for the *i.i.d.* environment in Section 2.3. In Section 2.4, we present our results for the persistent case. Finally, Section 2.5 contains a discussion based on a numerical example to illustrate the nature of our results.

2.2 Model

In our model, there are two players interacting in an infinite horizon in discrete time. At each period $t = 0, 1, 2, \ldots$, each player $i \in \{0, 1\}$ chooses an action $a_i^t \in \{C, D\}$. Given an action profile $a^t = (a_1^t, a_2^t)$, player i's stage payoff are given by

$$u_i^t(a^t, \theta^t) = g_i(a^t) + \theta^t, \tag{2.1}$$

where θ^t is a random shock to payoffs and $g(\cdot)$ represents the following prisoners' dilemma:

	C	D
C	$2, 2$	$-1, 3$
D	$3, -1$	$0, 0$

Table 2.1. Prisoners' dilemma.

There is a common discount factor $\delta \in (0, 1)$ and given a sequence of stage payoff realizations $\{u_i^t\}_{t=0}^{\infty}$, player i's discounted payoff is given by

$$(1-\delta)\sum_{t=0}^{\infty}\delta^t u_i^t.$$

At the end of each period, each player is only able to observe her own payoff realization. That is, player i only observes $u_t^i(a^t, \theta^t)$ but is not able to observe the other player's action, a_{-i}^t, or the shock realization, θ^t. Hence, at the beginning of period t, player i's private history is given by:

$$h_i^t = \{a_i^t, u_i^t\}_{\tau=0}^{t-1}$$

Note that this is an imperfect monitoring game, from her own payoff, a player is not able to learn the other player's action.

Regarding the shocks, θ^t, we consider two cases. The first one is the *i.i.d.* case, which has been widely studied and we are using as a benchmark. In that case, θ^t is independent and normally distributed at each period t. In the second case, the shocks are persistent. More precisely, they follow an auto-regressive process.

2.3 *i.i.d.* shocks

This section covers the widely studied *i.i.d.* shocks case. We decide to present this case and some of the results from previous studies because it will allow us to better expose the nature and significance of our results for the persistent case.[1]

[1] For a more thorough discussion of this environment, look at section 7.2 in Mailath and Samuelson (2006).

At each period $t = 0, 1, 2, \ldots$,

$$\theta^t \sim \mathcal{N}(0, \sigma_0^2)$$

where $\sigma_0 > 0$. That is, θ^t is drawn a normal distribution with mean 0 and variance σ_0^2.

Next, we will argue when and how can players sustain high payoffs. For simplicity, we will restrict attention to *strongly symmetric pure-strategy equilibria*, that is, equilibria in which both players choose the same action after each history. In particular, the two points we would like to highlight from previous studies are:

i. Punishments occur in equilibrium and players' expected discounted payoffs are bounded away from 2, the best symmetric payoff that is attained by permanently playing (C, C).

ii. If σ_0 is *large*, always playing (D, D) is the only equilibrium.

The strongly symmetric equilibrium that attains the highest payoffs takes the form of a grim trigger profile.

In a grim trigger profile, players start playing C and continue doing so as long as their stage payoff is above some threshold, $\overline{u}_0$; once a stage payoff falls below the threshold, the player switch to play D forever. Formally, for each player i:

- At $t = 0$, $a_i^0 = C$.
- At $t > 0$,

$$a_i^t = \begin{cases} C & \text{if} \quad a_i^{t-1} = C \text{ and } u_i^{t-1} > \overline{u}_0 \\ D & \text{otherwise} \end{cases}$$

First, note that once a Nash reversal is triggered, each player expected discounted payoff is 0.

When players are following the grim trigger profile and are still cooperating, at the beginning of each period t, the probability that they will continue to cooperate in $t + 1$ is given by:

$$\begin{aligned} p(\overline{u}_0) &\equiv \Pr\left(u_i^t > \overline{u}_0 \middle| a^t = (C, C)\right) \\ &= \Pr\left(2 + \theta^t > \overline{u}_0\right) \end{aligned}$$

$$
\begin{aligned}
&= \Pr\big(\theta^t > \overline{u}_0 - 2\big) \\
&= 1 - \Phi\bigg(\frac{\overline{u}_0 - 2}{\sigma_0}\bigg).
\end{aligned}
$$

Given a threshold $\overline{u}_0$, each player's expected discounted payoff at the beginning of a period provided that they are still playing C, $V(\overline{u}_0)$ satisfies:

$$
V(\overline{u}_0) = (1-\delta)2 + \delta p(\overline{u}_0)V(\overline{u}_0). \tag{2.2}
$$

Then,

$$
V(\overline{u}_0) = \frac{(1-\delta)2}{1-\delta p(\overline{u}_0)}. \tag{2.3}
$$

We proceed to derive conditions that guarantee that the grim trigger profile is an equilibrium. First note that once Nash reversal occurs, there is no profitable deviation. Then, the grim trigger profile is an equilibrium if a player does not want to play D when both are supposed to play C. If a player deviates, the probability of triggering a Nash reversal increases. Formally, if player 1 deviates and play $a_1^t = D$ at period t, then the probability that player 2 continues playing C in period $t+1$ is:

$$
\begin{aligned}
q(\overline{u}_0) &\equiv \Pr\big(u_2^t > \overline{u}_0 \big| a^t = (D,C)\big) \\
&= \Pr\big(-1 + \theta^t > \overline{u}_0\big) \\
&= \Pr\big(\theta^t > \overline{u}_0 + 1\big) \\
&= 1 - \Phi\bigg(\frac{\overline{u}_0 + 1}{\sigma_0}\bigg).
\end{aligned}
$$

Note that for any threshold $\overline{u}_0$, $p(\overline{u}_0) > q(\overline{u}_0)$. Then, a player would not want to deviate as long as

$$
V(\overline{u}_0) \geq (1-\delta)3 + \delta q(\overline{u}_0)V(\overline{u}_0),
$$

plugging (2.2) in the left-hand side of the previous inequality,

$$
(1-\delta)2 + \delta p(\overline{u}_0)V(\overline{u}_0) \geq (1-\delta)3 + \delta q(\overline{u}_0)V_(\overline{u}_0)
$$

$$
\delta\big[p(\overline{u}_0) - q(\overline{u}_0)\big]V(\overline{u}_0) \geq (1-\delta)
$$

and plugging (2.3),

$$\delta\big[p(\overline{u}_0) - q(\overline{u}_0)\big]2 \geq 1 - \delta p(\overline{u}_0).$$

Finally, we obtain the following incentive constraint:

$$\delta\big[3p(\overline{u}_0) - 2q(\overline{u}_0)\big] \geq 1. \tag{2.4}$$

For the incentive constraint (2.4) to hold:

(a) it is necessary that $\delta > \frac{1}{3}$ because $0 < 3p(\overline{u}_0) - 2q(\overline{u}_0) < 3$.

(b) σ_0 cannot be too large because it is necessary that $3p(\overline{u}_0) - 2q(\overline{u}_0) > 1$. And, regardless of $\overline{u}_0$, when σ_0 grows:

$$\lim_{\sigma_0 \uparrow \infty} p(\overline{u}_0) = \lim_{\sigma_0 \uparrow \infty} q(\overline{u}_0) = \frac{1}{2}.$$

Hence, as σ_0 goes to infinity, the incentive constraint goes to

$$\frac{\delta}{2} \geq 1,$$

which cannot hold. As a consequence, for large σ_0, the grim trigger profile cannot be an equilibrium and repetition of (D, D) is the only equilibrium.[2]

In the next section, we introduce the persistent shocks case and analyze how observations (a) and (b) change in the persistent environment.

2.4 Persistent shocks

In this section, the payoff shocks will follow an $AR(1)$ process. That is, for some $\varphi \in (0, 1)$ and $\sigma_\epsilon > 0$:

$$\theta^t = \varphi\theta^{t-1} + \epsilon^t \qquad \forall\, t > 0,$$

[2]This result is not dependant of restricting attention to the strongly symmetric equilibria. Meaning, if σ_0 is too large, the repetition of (D, D) is the only equilibrium.

where ϵ^t is drawn from a normal distribution $\mathcal{N}(0, \sigma_\epsilon^2)$ independently across periods. In the initial period, $t = 0$, assume that

$$\theta^0 \sim \mathcal{N}\left(0, \frac{\sigma_\epsilon^2}{1-\varphi^2}\right).$$

Consequently, note that for every $t = 0, 1, 2, \ldots$:

$$\mathbb{E}(\theta^t) = 0 \quad \text{and} \quad \mathbb{V}(\theta^t) = \frac{\sigma_\epsilon^2}{1-\varphi^2}. \tag{2.5}$$

Throughout this work, we will compare different processes in terms of how persistent they are. We say that the process is more persistent, the larger φ is. For consistency, when comparing different processes, we will keep the variance of the shocks, $\mathbb{V}(\theta^t)$, constant and equal to σ_0.

2.4.1 Adapting the grim trigger profile to the persistent shocks case

The strategies from the *i.i.d.*-environment can be adapted for the persistent environment. In our adaptation players start choosing D in the first period. After that, players start playing C and continue doing so as long as stage payoff are above a threshold. A Nash reversal is triggered after a period in which a player receives a stage payoff below the threshold, that is, the player switches to play D forever. The main difference with respect to the *i.i.d.* case is that the threshold will depend on previous periods payoffs for reasons that will become evident later. In the next definition, we present our interpretation of the grim trigger profile.

Definition 2. For the persistent environment, we present the following adaptation of the grim trigger profile:

- At $t = 0$, $a_i^0 = D$.
- At $t = 1$, $a_i^1 = C$.
- At $t = 2$,

$$a_i^2 = \begin{cases} C & \text{if} \quad a_i^1 = C \text{ and } u_i^1 > \varphi u_i^0 + 0.5 \\ D & \text{otherwise} \end{cases}$$

- For $t > 2$,

$$a_i^t = \begin{cases} C & \text{if} \quad a_i^{t-1} = C \text{ and } u_i^{t-1} > \tau(u_i^{t-2}) \\ D & \text{otherwise} \end{cases}$$

where the threshold is given by the function:

$$\tau(u_i^{t-2}) = \varphi\big[u_i^{t-2} - 2\big] + 0.5. \tag{2.6}$$

Our goal is to establish conditions for the grim trigger profile to be a PBE. Now on, we will assume that as long as players continue playing C, they will believe that their competitor has played C and update beliefs about the shocks accordingly.

We will calculate the expected discounted values that players obtain from following the grim trigger profile in the next subsection. In subsection 2.4.1.2, we analyze the potentially profitable deviations from the grim trigger profile. After that, in subsection 2.4.1.3, we present results that show that regardless of σ_0, the grim trigger profile is an equilibrium in which firms obtain payoffs larger than the stage Nash repetition if the shocks are persistent enough. Moreover, as the shocks become more persistent, the probability of a Nash reversal in equilibrium diminishes.

2.4.1.1 Values

We start by calculating, $V^N(\theta^{t-1})$, the expected discounted value at period $t > 1$ when players are in a Nash reversal given that the realization of the previous shock was θ^{t-1}.

$$\begin{aligned} V^N(\theta^{t-1}) =& (1-\delta)\sum_{\tau=0}^{\infty} \delta^\tau \mathbb{E}[\theta^{t+\tau}|\theta^{t-1}] \\ =& (1-\delta)\sum_{\tau=0}^{\infty} \delta^\tau \varphi^{\tau+1}\theta^{t-1} \\ =& (1-\delta)\varphi\theta^{t-1}\sum_{\tau=0}^{\infty}(\delta\varphi)^\tau \end{aligned}$$

Then,

$$V^N(\theta^{t-1}) = \left(\frac{(1-\delta)\varphi}{1-\delta\varphi}\right)\theta^{t-1}. \tag{2.7}$$

Now, we consider the case in which players are still cooperating. We start by calculating the probability that players continue playing C tomorrow provided that they are playing C today. As will be shown, thresholds are chosen so that this probability does not depend on time or shock realizations. At the beginning of period $t=1$, the ex-ante probability that players continue playing C in period $t=2$ given $a^1=(C,C)$ and θ^0 is equal to:

$$\begin{aligned}
&\Pr\left(u_i^1 > \varphi u_i^0 + 0.5 \,\middle|\, \theta^0, a^0=(D,D), a^1=(C,C)\right)\\
=&\Pr\left(2+\theta^1 > \varphi\theta^0 + 0.5 \,\middle|\, \theta^0\right)\\
=&\Pr\left(2+\varphi\theta^0+\epsilon^1 > \varphi\theta^0 + 0.5 \,\middle|\, \theta^0\right)\\
=&\Pr\left(\epsilon^1 > -1.5\right)\\
=&1-\Phi\left(\frac{-1.5}{\sigma_\epsilon}\right)
\end{aligned}$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function.

For $t>1$, the ex-ante probability that players continue playing C tomorrow given that they are playing C today and the previous shock realization was θ^{t-1} is equal to:

$$\begin{aligned}
&\Pr\left(u_2^t \geq \tau(u_2^{t-1}) \mid \theta^{t-1}, a^{t-1}=a^t=(C,C)\right)\\
=&\Pr\left(u_2^t \geq \varphi[u_2^{t-1}-2] + 0.5 \mid \theta^{t-1}, a^{t-1}=a^t=(C,C)\right)\\
=&\Pr\left(u_2^t \geq \varphi\theta^{t-1} + 0.5 \mid \theta^{t-1}, a^t=(C,C)\right)\\
=&\Pr\left(2+\theta^t \geq \varphi\theta^{t-1} + 0.5 \mid \theta^{t-1}\right)\\
=&\Pr\left(2+\varphi\theta^{t-1}+\epsilon^t \geq \varphi\theta^{t-1} + 0.5\right)\\
=&\Pr\left(\epsilon^t \geq -1.5\right)\\
=&1-\Phi\left(\frac{-1.5}{\sigma_\epsilon}\right).
\end{aligned}$$

Consequently, the transitions do not depend on the shock realizations. Let

$$p \equiv 1 - \Phi\left(\frac{-1.5}{\sigma_\epsilon}\right). \tag{2.8}$$

Because those probabilities do not depend in the shock realizations, the values are recursive. Let $V^C(\theta^{t-1})$ denote the expected discounted payoff at the beginning of period t given that players are playing $a_i^t = C$. Then, $V^C(\theta^{t-1})$ must satisfy:

$$V^C(\theta^{t-1}) = \mathbb{E}_{\theta^t,\theta^{t+1}}\left[(1-\delta)(2+\theta^t) + \delta\{pV^C(\theta^t) + (1-p)V^N(\theta^t)\}\middle|\theta^{t-1}\right] \tag{2.9}$$

Our first result, presented next, shows that for any previous shock realization and fixed parameters: the expected discounted value when still playing C is equal to the value when in a Nash reversal plus a constant.

Claim 1. *For any $\theta \in \mathbb{R}$,*

$$V^C(\theta) = V_0^C + V^N(\theta) \tag{2.10}$$

where

$$V_0^C = \frac{(1-\delta)2}{1-\delta p}. \tag{2.11}$$

2.4.1.2 Deviations

We proceed to analyze potentially profitable deviations. Note that in the first period, $t = 0$, there are no profitable deviations since they are playing a static Nash and deviating will not improve their continuation payoffs. Now we proceed to calculate the probability that a player does not trigger a Nash reversal by deviating.

At period $t = 1$, if player 1 deviates, $a_1^t = D$, then the probability that player 2 continues to cooperate in period 2 is given by:

$$\begin{aligned}
&\Pr\left(u_2^1 \geq \varphi u_2^0 + 0.5\middle|\theta^0, a^0 = (D,D), a^1 = (D,C)\right)\\
=&\Pr\left(2+\theta^1 \geq \varphi\theta^0 + 0.5\middle|\theta^0\right)\\
=&\Pr\left(2+\varphi\theta^0 + \epsilon^1 \geq \varphi\theta^0 + 0.5\middle|\theta^0\right)
\end{aligned}$$

$$= 1 - \Phi\left(\frac{1.5}{\sigma_\epsilon}\right).$$

For $t > 1$, given that players are still playing C and no deviations have occurred up to period t, if player 1 deviates at period t, the ex-ante probability that player 2 will continue playing C in period $t+1$ is given by:

$$\begin{aligned}
&\Pr\left(u_2^t \geq \tau(u_2^{t-1}) |\, \theta^{t-1}, a^{t-1} = (C, C), a^t = (D, C)\right) \\
=& \Pr\left(u_2^t \geq \varphi\theta^{t-1} + 0.5 \,|\, \theta^{t-1}, a^t = (D, C)\right) \\
=& \Pr\left(-1 + \theta^t \geq \varphi\theta^{t-1} + 0.5 \,|\, \theta^{t-1}\right) \\
=& \Pr\left(-1 + \varphi\theta^{t-1} + \epsilon^t \geq \varphi\theta^{t-1} + 0.5\right) \\
=& \Pr\left(\epsilon^t \geq 1.5\right) \\
=& 1 - \Phi\left(\frac{1.5}{\sigma_\epsilon}\right).
\end{aligned}$$

Note that this probabilities do not depend on shock realizations or in the time. Let:

$$q \equiv 1 - \Phi\left(\frac{1.5}{\sigma_\epsilon}\right). \tag{2.12}$$

Then, player 1's expected discounted payoff from such deviation conditional on θ^{t-1} would be equal to

$$\pi^D(\theta^{t-1}) \equiv \mathbb{E}_{\theta^t}\left[(1-\delta)(3+\theta^t) + \delta\{qV^D(\theta^t) + (1-q)V^N(\theta^t)\}\middle|\theta^{t-1}\right] \tag{2.13}$$

where $V^D(\theta^t)$ is a player's expected discounted payoff after deviating at period t and a reversal was not triggered conditional on θ^t. So in order to calculate the expected discounted payoff of such a deviation we need to calculate the value $V^D(\theta^t)$. To do so, first note that after deviating at period t without triggering a reversal, Player 1 can either play (i) C, or (ii) D, at period $t+1$. Let's calculate the maximum expected discounted payoff for both cases:

(i) If player 1 chooses $a_1^{t+1} = C$, then both players obtain a $t+1$ stage payoff equal to $u_i^{t+1} = 2 + \theta^{t+1}$, and the probability that he does not trigger a Nash reversal is given by:

$$\begin{aligned} q_C =& \Pr\left(u_2^{t+1} \geq \tau_C(u_2^t)|\theta^t, a^t = (D,C), a^{t+1} = (C,C)\right) \\ =& \Pr\left(u_2^{t+1} \geq \tau_C(-1+\theta^t)|\theta^t, a^{t+1} = (C,C)\right) \\ =& \Pr\left(2+\theta^{t+1} \geq \varphi\theta^t - 3\varphi + 0.5)|\theta^t\right) \\ =& \Pr\left(\epsilon^t \geq -1.5 - 3\varphi)\right) \\ =& 1 - \Phi\left(\frac{-1.5\,(1+2\varphi)}{\sigma_\epsilon}\right). \end{aligned} \tag{2.14}$$

If the Nash reversal is triggered, player 1's continuation payoff is given by $V^N(\theta^{t+1})$. If the Nash reversal is not triggered, then player 1 faces exactly the same problem as if no deviation had occurred, therefore the highest continuation value for player 1 is equal to $V^C(\theta^{t+1})$ if the proposed strategy is an equilibrium. Then, the highest expected discounted payoff conditional on $a_1^{t+1} = C$ and θ^t is given by:

$$h^C(\theta^t) = \mathbb{E}\Big[(1-\delta)(2+\theta^{t+1}) + \delta\{q_C V^C(\theta^{t+1}) + (1-q_C)V^N(\theta^{t+1})\}\Big|\theta^t\Big] \tag{2.15}$$

(*ii*) Now we will consider $h^D(\theta^t)$, the highest player 1's expected discounted payoff conditional on $a_1^{t+1} = D$ and θ^t. If player 1 chooses $a_1^{t+1} = D$, then period $t+1$ stage payoffs are given by $u_1^{t+1} = 3 + \theta^{t+1}$ and $u_2^{t+1} = -1 + \theta^{t+1}$, and the probability that he does not trigger a Nash reversal is given by

$$\begin{aligned} q_D =& \Pr\left(u_2^{t+1} \geq \tau(u_2^t)|\theta^t, a^t = a^{t+1} = (D,C)\right) \\ =& \Pr\left(u_2^{t+1} \geq \tau(-1+\theta^t)|\theta^t, a^{t+1} = (D,C)\right) \\ =& \Pr\left(-1+\theta^{t+1} \geq \varphi\theta^t - 3\varphi + 0.5|\theta^t\right) \\ =& \Pr\left(-1+\varphi\theta^t + \epsilon^{t+1} \geq \varphi\theta^t - 3\varphi + 0.5|\theta^t\right) \\ =& \Pr\left(\epsilon^{t+1} \geq -1.5(2\varphi - 1)\right) \\ =& 1 - \Phi\left(\frac{1.5(1-2\varphi)}{\sigma_\epsilon}\right). \end{aligned} \tag{2.16}$$

Hence, if choosing $a_1^{t+1} = D$ triggers a Nash reversal, her continuation value is given by $V^N(\theta^{t+1})$. If choosing $a_1^{t+1} = D$ does not trigger a Nash reversal, player 1 faces the same problem at $t+2$ than at $t+1$, hence her highest continuation value is given by $h^D(\theta^{t+1})$ if choosing $a_1^{t+1} = D$ is optimal in the first place. Then, the highest expected discounted payoff conditional on $a_1^{t+1} = C$ and θ^t is given by:

$$h^D(\theta^t) = \mathbb{E}\Big[(1-\delta)(3+\theta^t) + \delta\{q_D h^D(\theta^{t+1}) + (1-q_D)V^N(\theta^{t+1})\}\Big|\theta^t\Big] \quad (2.17)$$

Then,

$$V^D(\theta) = \max\{h^D(\theta),\, h^C(\theta)\}. \quad (2.18)$$

Similar to the value of the grim trigger profile, the expected discounted payoff after deviating are equal to the value of a Nash reversal plus a constant, by constant we mean that it does not depend on the shock realizations.

Claim 2. *For* $\theta \in \mathbb{R}$,

$$h^C(\theta) = h_0^C + V^N(\theta) \quad (2.19)$$

where

$$h_0^C = (1-\delta)2 + \delta q_C V_0^C \quad (2.20)$$

Claim 3. *For* $\theta \in \mathbb{R}$,

$$h^D(\theta) = h_0^D + V^N(\theta) \quad (2.21)$$

where

$$h_0^D = \frac{(1-\delta)3}{1-\delta q_D} \quad (2.22)$$

Therefore, $V^D(\theta) = V_0^D + V^N(\theta)$ with $V_0^D = \max\{h_0^D, h_0^C\}$.

Corollary 4. *For* $\theta \in \mathbb{R}$,

$$\pi^D(\theta) = \pi_0^D + V^N(\theta) \quad (2.23)$$

where

$$\pi_0^D = (1-\delta)3 + \delta q V_0^D. \quad (2.24)$$

Hence, the grim trigger profile will be an equilibrium as long as the incentive constrain $V_0^C \geq \pi_0^D$ holds.

2.4.1.3 Existence

We proceed to establish conditions for the incentive constraint $V_0^C \geq \pi_0^D$ to hold. First, in the next lemma, the incentive constraint is written in terms of the discount factor δ and the probabilities p, q, q_C and q_D.

Lemma 5. • *if $h_0^D \geq h_0^C$, then the incentive constraint $V_0^C \geq \pi_0^D$ holds if and only if:*

$$\big[1 - \delta q_D\big]2 \geq \big[1 - \delta p\big]\big[1 - \delta(q_D - q)\big]3; \tag{2.25}$$

• *if $h_0^C > h_0^D$, then the incentive constraint $V_0^C \geq \pi_0^D$ holds if and only if:*

$$\big[1 - \delta q\{1 + \delta q_C - \delta p\}\big]2 \geq (1 - \delta p)3 \tag{2.26}$$

Remember that the probabilities, p, q, q_C and q_D, depend on the parameters of the stochastic process, φ and σ_ϵ, but do not depend on the shock realizations. In the next proposition, we argue that under some conditions, when the shocks become more persistent while keeping $\mathbb{E}(\theta^t)$ and $\mathbb{V}(\theta^t)$ constant, (i) the grim trigger profile is an equilibrium, and, (ii) the probability of a Nash reversal goes to 0.

Proposition 5. *Fix $\delta \in (\frac{1}{3}, 1)$ and $\sigma_0 > 0$. For $\varepsilon > 0$ there exists $\overline{\varphi}_\epsilon \in (0,1)$ such that for any $\varphi > \overline{\varphi}$ and $\sigma_\epsilon^2 = (1-\varphi^2)\sigma_0^2$:*

i. the grim trigger profile is a PBE.

ii. $1 - p < \varepsilon$.

Note that condition $\sigma_\epsilon^2 = (1-\varphi^2)\sigma_0^2$ guarantees that as φ changes, $\mathbb{V}(\theta^t)$ remains constant and equal to σ_0^2.

2.5 Discussion and comparison of the *i.i.d.* and the persistent case

When covering the *i.i.d* case we establish that for a strongly symmetric equilibrium to exist (equation (2.4) to hold), δ has to be larger than $\frac{1}{3}$ and σ_0 being sufficiently small. In Proposition 5, the requirement on the players discount factor does not change but now the grim trigger profile is a PBE for any $\sigma_0 > 0$ as long as the shocks are persistent enough. Moreover, as the shocks become more persistent, the probability of a Nash reversal goes to 0 in the grim trigger equilibrium leading to profits well above the profits generated by the repetition of the stage Nash.

It is important to point out that all this results hold without requiring that the discount factor goes to 1.

Numerical example. Let $\delta = 0.5$. Then, the incentive constraint from the *i.i.d.* case, equation (2.4), in the next figure.

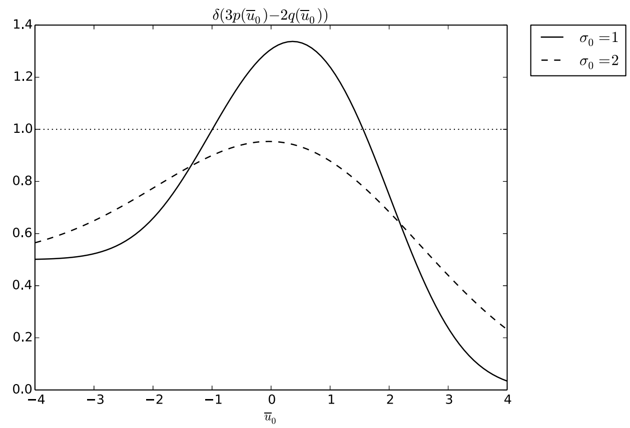

Figure 2.1. Incentive Constraint for the *i.i.d.* case

The lines represent the expression $\delta\big(3p(\overline{u}_0)-2q(\overline{u}_0)\big)$ as a function of the threshold $\overline{u}_0$ for σ_0 equal to 1 and 2. Again, the incentive constraint is satisfied when the expression is larger than 1. It can be observed that when $\sigma_0 = 1$, many values of the threshold satisfy the incentive constraint. On the other hand, when $\sigma) = 2$, a grim trigger profile equilibrium does not exist because there is no threshold that satisfy the incentive constraint.

However, that is not the case for the persistent case. To argue so, we can take a look at the persistent case with $\delta = 0.5$, $\sigma_0 = 2$ and $\varphi = 0.6$. In Figure 2.2, we present a simulation of both the *i.i.d* and the persistent shocks.

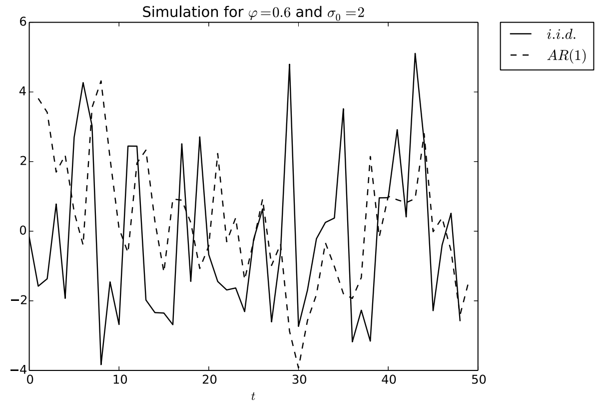

Figure 2.2. A simulation for the $i.i.d.$ and persistent shocks

As we have argued, the grim trigger profile is an equilibrium whenever $V_0^C - \pi_0^D > 0$ which is represented in Figure 2.3 as a function of φ for $\delta = 0.5$ and $\sigma_0 = 2$. As can be observed, the grim trigger profile is an equilibrium whenever $\varphi > 0.49$.

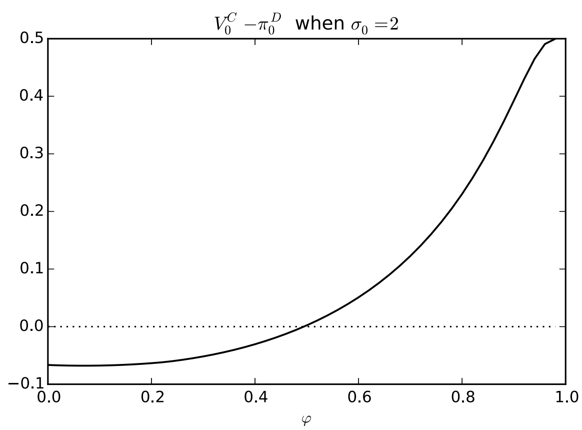

Figure 2.3. Incentive constraint for the persistent case

Also, Figure 2.4 shows that as φ goes to 1, the probability that players will continue playing C when following the grim trigger profile, p, goes to 1. Conversely, q, the probability that players will continue playing C after a first deviation goes to 0. As a consequence, when φ increases so does p and therefore the players' expected discounted payoffs from the grim trigger profile.

Concluding, for fixed δ, as the shocks are more persistent, players can punish with smaller probability and still sustain the grim trigger profile as an equilibrium. In particular, the grim trigger profile is an equilibrium for $\varphi = 0.6$ and for that

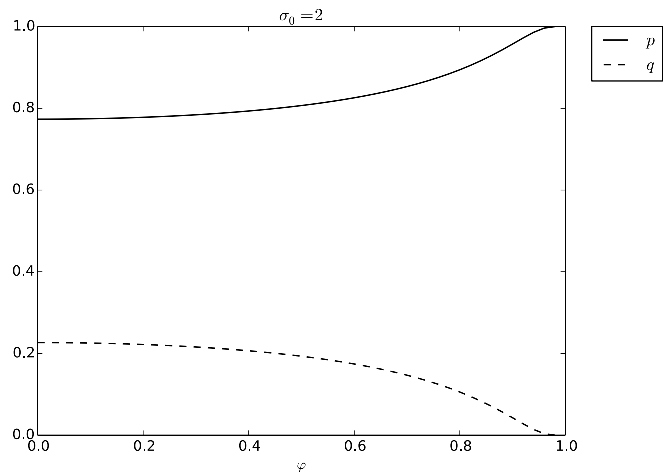

Figure 2.4. Probabilities of not triggering a Nash reversal

case $p = 0.8257$, $V_0^C = 1.70314$ and the expected discounted payoff at $t = 0$ is 0.8516.

2.6 Appendix

2.6.1 Proofs

Claim 1. We want to show that (2.10, 2.11) is a solution to (2.9). To do so, first note from (2.7) that $V^N(\cdot)$ is a linear mapping and therefore:

$$\mathbb{E}\left[V^N(\theta^{t+1})\middle|\theta^t\right] = \varphi V^N(\theta^t).$$

Plugging equation (2.10) on the right hand side of (2.9), we obtain:

$$\begin{aligned} V^C(\theta^t) =& \mathbb{E}_{\theta^{t+1}}\left[(1-\delta)(2+\theta^{t+1}) + \delta\{pV^C(\theta^{t+1}) + (1-p)V^N(\theta^{t+1})\}\middle|\theta^t\right] \\ =& \mathbb{E}_{\theta^{t+1}}\left[(1-\delta)(2+\theta^{t+1}) + \delta\{pV_0^C + V^N(\theta^{t+1})\}\middle|\theta^t\right] \\ =& (1-\delta)(2+\varphi\theta^t) + \delta\{pV_0^C + \varphi V^N(\theta^t)\}. \end{aligned} \tag{2.27}$$

Also, from (2.11), we know that:

$$(1-\delta)2 = (1-\delta p)V_0^C.$$

Similarly, from (2.7), we know that:

$$(1-\delta)\varphi\theta^t = (1-\delta\varphi)V^N(\theta^t).$$

Then, plugging the last two equations in (2.27):

$$\begin{aligned}
V^C(\theta^t) &= (1-\delta)(2+\varphi\theta^t) + \delta\{pV_0^C + \varphi V^N(\theta^t)\} \\
&= (1-\delta p)V_0^C + (1-\delta\varphi)V^N(\theta^t) + \delta\{pV_0^C + \varphi V^N(\theta^t)\} \\
&= V_0^C + V^N(\theta^t).
\end{aligned}$$

□

Claim 2. We want to show that (2.19,2.20) solves equation (2.15). Remember that:

$$\mathbb{E}\Big[V^N(\theta^{t+1})\Big|\theta^t\Big] = \varphi V^N(\theta^t).$$

Then, plugging (2.19) in the right-hand side of equation (2.15) we obtain:

$$\begin{aligned}
h^C(\theta^t) &= \mathbb{E}\Big[(1-\delta)(2+\theta^{t+1}) + \delta\{q_C^C V^C(\theta^{t+1}) + (1-q_C^C)V^N(\theta^{t+1})\}\Big|\theta^t\Big] \\
&= \mathbb{E}\Big[(1-\delta)(2+\theta^{t+1}) + \delta\{q_C^C V_0^C + V^N(\theta^{t+1})\}\Big|\theta^t\Big] \\
&= (1-\delta)(2+\varphi\theta^t) + \delta\{q_C^C V_0^C + \varphi V^N(\theta^t)\}. \qquad (2.28)
\end{aligned}$$

Remember that, from (2.7), we know that:

$$(1-\delta)\varphi\theta^t = (1-\delta\varphi)V^N(\theta^t).$$

Then, plugging the last equation on (2.28):

$$\begin{aligned}
h^C(\theta^t) &= (1-\delta)2 + (1-\delta\varphi)V^N(\theta^t) + \delta\{q_C^C V_0^C + \varphi V^N(\theta^t)\} \\
&= (1-\delta)2 + (1-\delta\varphi)V^N(\theta^t) + \delta\{q_C^C V_0^C + \varphi V^N(\theta^t)\} \\
&= (1-\delta)2 + \delta q_C^C V_0^C + V^N(\theta^t) \\
&= h_0^C + V^N(\theta^t).
\end{aligned}$$

□

Claim 3. The approach is analogous to the previous proof. Plugging (2.21) in the right-hand side of equation (2.17) we obtain:

$$\begin{aligned} h^D(\theta^t) =& \mathbb{E}\Big[(1-\delta)(3+\theta^{t+1}) + \delta\{q_C^D h^D(\theta^{t+1}) + (1-q_C^D)V^N(\theta^{t+1})\}\Big|\theta^t\Big] \\ =& \mathbb{E}\Big[(1-\delta)(3+\theta^{t+1}) + \delta\{q_C^D h_0^D + V^N(\theta^{t+1})\}\Big|\theta^t\Big] \\ =& (1-\delta)(3+\varphi\theta^t) + \delta\{q_C^D h_0^D + \varphi V^N(\theta^t)\}. \end{aligned} \tag{2.29}$$

Again, from (2.7), we know that:

$$(1-\delta)\varphi\theta^t = (1-\delta\varphi)V^N(\theta^t)$$

and from (2.22):

$$(1-\delta)3 = (1-\delta q_C^D)h_0^D.$$

Plugging the last two equations on (2.29):

$$\begin{aligned} h^D(\theta^t) =& (1-\delta q_C^D)h_0^D + (1-\delta\varphi)V^N(\theta^t) + \delta\{q_C^D h_0^D + \varphi V^N(\theta^t)\} \\ =& h_0^D + V^N(\theta^t). \end{aligned}$$

□

Lemma 5.

- If $h_0^D \geq h_0^C$, then the incentive constraint $V_0^C \geq \pi_0^D$ turns into:

$$V_0^C \geq (1-\delta)3 + \delta q h_0^D.$$

Plugging (2.11) and (2.22), the previous inequality becomes:

$$\frac{(1-\delta)2}{1-\delta p} \geq (1-\delta)3 + \delta q \frac{(1-\delta)3}{1-\delta q_D}$$

$$\frac{2}{1-\delta p} \geq \frac{[1+\delta(q-q_D)]3}{1-\delta q_D}$$

$$[1-\delta q_D]2 \geq (1-\delta p)[1+\delta(q-q_D)]3.$$

- If $h_0^C > h_0^D$, then the incentive constraint $V_0^C \geq \pi_0^D$ turns into:

$$V_0^C \geq (1-\delta)3 + \delta q h_0^C.$$

Plugging (2.20), the previous inequality becomes:

$$V_0^C \geq (1-\delta)3 + \delta q\big[(1-\delta)2 + \delta q_C V_0^C\big]$$

$$\big[1-\delta^2 q q_C\big]V_0^C \geq (1-\delta)3 + \delta q(1-\delta)2$$

Plugging (2.11):

$$\big[1-\delta^2 q q_C\big]\frac{2}{1-\delta p} \geq 3 + \delta q 2$$

$$\big[1-\delta^2 q q_C\big]2 \geq (1-\delta p)\big[3 + \delta q 2\big]$$

$$\big[1-\delta^2 q q_C - \delta q + \delta^2 q p\big]2 \geq (1-\delta p)3$$

$$\big[1-\delta q\{1 + \delta q_C - \delta p\}\big]2 \geq (1-\delta p)3$$

□

Corollary 4. Plugging into (2.13):

$$\begin{aligned}
\pi^D(\theta^{t-1}) &= \mathbb{E}_{\theta^t}\left[(1-\delta)(3+\theta^t) + \delta\{qV^D(\theta^t) + (1-q)V^N(\theta^t)\}\middle|\theta^{t-1}\right] \\
&= \mathbb{E}_{\theta^t}\left[(1-\delta)(3+\theta^t) + \delta\{qV_0^D + V^N(\theta^t)\}\middle|\theta^{t-1}\right] \\
&= (1-\delta)3 + \delta q V_0^D + \mathbb{E}_{\theta^t}\left[(1-\delta)\theta^t + \delta V^N(\theta^t)\middle|\theta^{t-1}\right]
\end{aligned}$$

□

Proposition 5. Fix $\delta \in (\frac{1}{3}, 1)$ and $\sigma_0 > 0$. By plugging the assumption that $\sigma_\epsilon = (1-\varphi^2)^{\frac{1}{2}}\sigma_0$ into the probabilities of not triggering a Nash reversal (equations 2.8, 2.12, 2.14, 2.16), we obtain:

$$p = 1 - \Phi\left(\frac{-1.5}{(1-\varphi^2)^{\frac{1}{2}}\sigma_0}\right)$$

$$q = 1 - \Phi\left(\frac{1.5}{(1-\varphi^2)^{\frac{1}{2}}\sigma_0}\right)$$

$$q_C = 1 - \Phi\bigg(\frac{-1.5(2\varphi+1)}{(1-\varphi^2)^{\frac{1}{2}}\sigma_0}\bigg)$$

$$q_D = 1 - \Phi\bigg(\frac{-1.5(2\varphi-1)}{(1-\varphi^2)^{\frac{1}{2}}\sigma_0}\bigg).$$

All those probabilities are a function of φ. Taking limits when φ goes to 1:

$$\begin{aligned}
\lim_{\varphi\uparrow 1} p &= 1 - \lim_{\varphi\uparrow 1}\Phi\bigg(\frac{-1.5}{(1-\varphi^2)^{\frac{1}{2}}\sigma_0}\bigg) = 1 \\
\lim_{\varphi\uparrow 1} q &= 1 - \lim_{\varphi\uparrow 1}\Phi\bigg(\frac{1.5}{(1-\varphi^2)^{\frac{1}{2}}\sigma_0}\bigg) = 0 \\
\lim_{\varphi\uparrow 1} q_C &= 1 - \lim_{\varphi\uparrow 1}\Phi\bigg(\frac{-1.5(2\varphi+1)}{(1-\varphi^2)^{\frac{1}{2}}\sigma_0}\bigg) = 1 \\
\lim_{\varphi\uparrow 1} q_D &= 1 - \lim_{\varphi\uparrow 1}\Phi\bigg(\frac{-1.5(2\varphi-1)}{(1-\varphi^2)^{\frac{1}{2}}\sigma_0}\bigg) = 1.
\end{aligned}$$

Those limits imply that the probability that the other player will continue cooperating goes to 1 as long as you cooperate, and the probability of a Nash reversal after the first deviation goes to 1. That is the intuition behind the proposed automaton being an equilibrium.

Showing that both incentive constraints, (2.26) and (2.25), hold at the same time is sufficient to show that both players using the proposed automaton is an equilibrium. Next, we will prove that both incentive constraints hold for φ close to 1. To do so, note that incentive constraint (2.25) holds as long as:

$$g_D(\varphi) \equiv \big[1-\delta q_D\big]2 - \big[1-\delta p\big]\big[1-\delta(q_D-q)\big]3 \geq 0$$

and the incentive constraint (2.26) holds as long as:

$$g_C(\varphi) \equiv \big[1-\delta q\{1+\delta q_C-\delta p\}\big]2 - (1-\delta p)3 \geq 0.$$

Taking limits when φ goes to 1:

$$\lim_{\varphi\uparrow 1} g_D(\varphi) = (1-\delta)2 - (1-\delta)^2 3$$

and

$$\lim_{\varphi \uparrow 1} g_C(\varphi) = 2 - (1 - \delta)3.$$

The previous two limits are strictly positive because $\delta > \frac{1}{3}$. Therefore, there exists a $\overline{p}$ such that for $\varphi > \overline{p}$, both players using the proposed automaton is an equilibrium. Therefore, we have shown that as the φ goes to 1, the grim trigger profile is an equilibrium while the probability of continuing cooperation goes to 1. □

Chapter 3

Balanced Contributions and Fairness in Exchange Economies

3.1 Introduction

Trade in exchange economies leads to welfare gains, yet one of the lingering questions in economics is how to fairly distribute these gains. Our purpose is to propose a new definition of fairness to evaluate allocations in exchange economies with individual endowments. The proposed solution takes into consideration that initial endowments and trade possibilities among subgroups of the economy lead to agents possessing different rights.

Multiple conceptions of fairness have been suggested throughout the years. The concepts of envy-freeness (Foley, 1967, Varian, 1974) and egalitarian equivalence (Pazner and Schmeidler, 1978) propose ways to fairly distribute a social endowment. However, when considering an economy with *individual* endowments, fairness is not the only property that an allocation should satisfy. At a minimum, we should require allocations to be individually rational. Schmeidler and Vind (1972) extend the envy-free notion to economies with individual endowments by considering envy-free trades. In a work similar to ours, Pérez-Castrillo and Wettstein (2006) propose a way to extend egalitarian equivalence to the case of economies with individual endowments.

Any notion of fairness implies interpersonal comparisons. Pérez-Castrillo and

Wettstein (2006) propose a solution, the *ordinal Shapley value*, which defines interpersonal comparisons in terms of a reference bundle. This is reminiscent of the way the Nash bargaining solution can be considered a social welfare function through implicit interpersonal comparisons based on the status quo point. Pérez-Castrillo and Wettstein (2006) measure agents' trading contributions to the welfare of others in terms of the reference bundle. Then, as a fairness condition, they ask for these contributions to be balanced, that is, an agent contributes to the other agents in the economy in the same amount that the others contribute to her. The ordinal Shapley value receives its name because balance contributions is a property that characterizes the Shapley value in transferable utility games (Myerson, 1980).

We follow an approach similar to Pérez-Castrillo and Wettstein (2006) by also directly defining the balance contributions property in exchange economies but by interpreting contributions in a different manner. The solution is initially defined for finite cardinal exchange economies, and interpersonal comparisons are made in terms of utility gains. An agent's welfare gain when going from consuming one bundle to another is interpreted as the change in utility. Then, an agent's contribution to another agent is the utility gain that the latter obtains by having the former in the economy. Our solution maps each cardinal economy onto the set of efficient allocations that balance contributions. By defining a solution entirely based on properties that characterize the Shapley value, we also extend it to a non-transferable utility environment. Our approach differs from those of the non-transferable utility values in Harsanyi (1963), Shapley (1969) and Maschler and Owen (1992) because those three values associate each non-transferable utility game with transferable utility games to then use their Shapley value to select a distribution for the original environment.

The solution always satisfies uniqueness in utility levels, two kinds of monotonicity in endowments, symmetry in contributions and equal treatment of equals. Also, under minimal assumptions, the solution is non-empty. Although the solution is not invariant to affine transformation of utility functions, solutions for ordinal economies can be constructed by mapping ordinal economies into cardinal economies and then applying our solution. A solution for ordinal economies that is constructed in such a way will unambiguously assign a set of allocations to each ordinal economy as long as it maps each ordinal economy into one and only one cardinal economy. We present two such solutions for ordinal economies that inherit all the properties of the cardinal solution. The first solution measures contributions in terms of a common reference bundle, similar to ordinal Shapley value and reminiscent of egalitarian equivalence. Following the guidelines of previous works, the second solution measures changes by means of compensating variations.[1] Compared to the ordinal Shapley value, these solutions are easier to calculate and contributions are measured in a more intuitive way.

The next section describes the environment and the solution. Section 3.3 presents the solution's properties. Finally, Section 3.4 discusses the cardinality of the solution and makes some comparisons to the ordinal Shapley value.

3.2 The Environment and the Solution

We work in the space of **finite cardinal exchange economies** in Sections 3.2 and 3.3. A cardinal exchange economy consists of l divisible goods and a nonempty set of agents $N = \{1, \ldots, n\}$ where each agent $i \in N$ is represented by her utility function representation $u_i : \mathbb{R}^l_+ \to \mathbb{R}$ and her initial endowment $\omega_i \in \mathbb{R}^l_+$. Altogether, a cardinal exchange economy is given by $(u_i, \omega_i)_{i \in N}$. The environment differs from the one in Pérez-Castrillo and Wettstein (2006); the commodity space is the nonnegative orthant, R^l_+, as opposed to the whole $\mathbb{R}^l$. As a result, there is no need to assume that the feasible utility profiles are bounded.

For $(u_i, \omega_i)_{i \in N}$, an **allocation** is a vector $x = (x_i)_{i \in N} \in \mathbb{R}^{|N| \times l}_+$ which assigns the bundle $x_i \in \mathbb{R}^l_+$ to agent i. An allocation x is **feasible** if $\sum_{i \in N} x_i \leq \sum_{i \in N} \omega_i$. An allocation x is **Pareto efficient** if it is feasible and there is no other feasible allocation y such that $u_i(y_i) \geq u_i(x_i)$ for all $i \in N$ and $u_j(y_j) > u_j(x_j)$ for some $j \in N$. A utility function u is **monotone** if for $x, x' \in \mathbb{R}^l_+$, $x' \gg x$ implies $u(x') > u(x)$.[2] For notational simplicity, it will be assumed that for each $i \in N$, $u_i(0) = 0$.

A **solution** will be a mapping that assigns a set of feasible allocations to each economy $(u_i, \omega_i)_{i \in N}$. A solution is said to be efficient if every economy is mapped

[1]See Chipman and Moore (1980) for more on using compensating variations as measures of welfare change.

[2]For two vectors of the same dimension $a = (a_1, \ldots, a_m)$ and $b = (b_1, \ldots, b_m)$, we will write $a \gg b$ if and only if $a_n > b_n$ for $n = 1, \ldots, m$.

into a set of efficient allocations.

3.2.1 The solution

Next, we define a new solution, μ, that is based on the notion of balanced contributions satisfied by the Shapley value in TU-games.[3] Contributions capture the welfare gains derived from trade in terms of utility gains. As the fairness condition we require contributions to be balanced.

Definition 3. Given $(u_i, \omega_i)_{i\in N}$, the solution μ is inductively defined as follows,

a. For an economy with just one agent (u_1, ω_1), the solution μ is just the endowment, that is,
$$\mu(u_1, \omega_1) = \{\omega_1\}.$$

b. For an economy with $|N| \geq 2$, assume that the solution has been defined for each economy with $N' \subsetneq N$.Then, an allocation x is in the solution if it is Pareto efficient and for each $k \in N$ there exists a $y^{-k} \in \mu((u_i, \omega_i)_{i\in N\setminus\{k\}})$ such that,

$$\sum_{i\neq j} \left[u_i(x_i) - u_i(y_i^{-j})\right] = \sum_{i\neq j} \left[u_j(x_j) - u_j(y_j^{-i})\right] \qquad \forall j \in N. \tag{3.1}$$

For any allocation x in the solution, we will say that the allocations y^{-j} for $j \in N$ **support** x in the solution if they are the allocations in the solution for the economies with $|N| - 1$ agents satisfying (3.1).

Note that if agent j was not in the economy, then the solution would assign a bundle y_i^{-j} to agent i. By adding agent j to the economy, the solution assigns the bundle x_i to agent i. Then, player j's contribution to agent i is defined as i's utility gain of having agent j, that is, $u_i(x_i) - u_i(y_i^{-j})$. Condition (3.1) requires contributions to be balanced as in the Myerson's characterization of the Shapley value.

[3] Characterization due to Myerson (1980). Appendix 3.5.1 contains the formal characterization.

3.2.2 Example

Consider the economy with two commodities and three agents represented by $u_1(x) = u_2(x) = x_1^{1/2} x_2^{1/2}$ and $u_3(x) = x_1 + x_2$ and $\omega_1 = (9, 1)$, $\omega_2 = (1, 9)$, and $\omega_3 = (5, 5)$. The solution set has to be computed inductively. For the economies with just one agent the solution sets are just their endowments.

For the economies with two agents, take the economy with agent 1 and 2 as an example: the allocation $y^{-3} = (y_1^{-3}, y_2^{-3})$ is in its solution if it is efficient and

$$u_1(y_1^{-3}) - u_1(\omega_1) = u_2(y_2^{-3}) - u_2(\omega_2).$$

The following table contains the unique allocations in the solution.

	Agent 1		Agent 2		Agent 3	
Allocation	bundle	utility	bundle	utility	bundle	utility
y^{-1}	-	-	$(13/3, 13/3)$	$13/3$	$(5/3, 29/3)$	$34/3$
y^{-2}	$(13/3, 13/3)$	$13/3$	-	-	$(29/3, 5/3)$	$34/3$
y^{-3}	$(5, 5)$	5	(5,5)	5	-	-

Table 3.1. Example: solutions for the two-agent economies

Finally, we consider the three agents economy. For an allocation x, agent 1's contributions are balanced if

$$[u_1(x_1) - u_1(y_1^{-2})] + [u_1(x_1) - u_1(y_1^{-3})] = [u_2(x_2) - u_2(y_2^{-1})] + [u_3(x_3) - u_3(y_3^{-1})].$$

The balanced contributions allocations condition can be written analogously for the other agents. The following table contains the only allocation in the solution and the corresponding utility levels for each of the agents.

x_1	$u_1(x_1)$	x_2	$u_2(x_2)$	x_3	$u_3(x_3)$
$(\frac{71}{15}, \frac{71}{15})$	$\frac{71}{15}$	$(\frac{71}{15}, \frac{71}{15})$	$\frac{71}{15}$	$(\frac{83}{15}, \frac{83}{15})$	$\frac{166}{15}$

Table 3.2. Example: solution for the three-agents economy

The corresponding contributions are:

$$\begin{bmatrix} - & u_1(x_1) - u_1(y_1^{-2}) & u_1(x_1) - u_1(y_1^{-3}) \\ u_2(x_2) - u_2(y_2^{-1}) & - & u_2(x_2) - u_2(y_2^{-3}) \\ u_3(x_3) - u_3(y_3^{-1}) & u_3(x_3) - u_3(y_3^{-2}) & - \end{bmatrix} = \begin{bmatrix} - & \frac{6}{15} & -\frac{4}{15} \\ \frac{6}{15} & - & -\frac{4}{15} \\ -\frac{4}{15} & -\frac{4}{15} & - \end{bmatrix}$$

All of agent 3's contributions are negative. Consequently, agents 1 and 2 are worse off with the inclusion of agent 3 and the solution is not in the core of the economy. Agents 3 contributions to the others are negative because the solution considers the case in which agent 1 (resp. 2) leaves the economy and in that situation, agent 2 (resp. 1) will benefit from having agent 3 in the economy.

3.3 Properties of the solution

In this section, we show some properties of the solution. Our main result, Theorem 4, states necessary conditions for the solution set to be non-empty.

Theorem 4. *Consider $(u_i, \omega_i)_{i\in N}$ with $1 \leq n < \infty$ and $\omega_i \gg 0$ for each $i \in N$. Moreover, assume that u_i is continuous and monotone for each $i \in N$, then $\mu((u_i, \omega_i)_{i\in N}) \neq \varnothing$.*

The next Proposition states the main properties of the solution set. At any allocation in the solution set, the contributions are symmetric (S), the utility levels are unique (U), each agent is at least as good as her endowment (IR), and equals are treated equally (ET). Moreover, if the utility functions are monotone, the solution satisfies two types of monotonicity on initial endowments, (M1) and (M2). All those properties are formally presented next.

(S) For any economy $(u_i, \omega_i)_{i\in N}$ with $|N| \geq 2$, and for any $x \in \mu((u_i, \omega_i)_{i\in N})$ supported by the allocations $y^{-i} \in \mu((u_j, \omega_j)_{j\neq i})$ for each $i \in N$, the following equation holds for any pair $j, k \in N$ with $j \neq k$,

$$u_j(x_j) - u_j(y_j^{-k}) = u_k(x_k) - u_k(y_k^{-j}).$$

(U) For any cardinal economy $(u_k, \omega_k)_{k\in N}$ and for any two allocations in its solution set, $\{x, x'\} \subset \mu((u_k, \omega_k)_{k\in N})$,

$$u_i(x_i) = u_i(x'_i) \qquad \forall\, i \in N.$$

(IR) For any economy $(u_i, \omega_i)_{i\in N}$ and any allocation in its solution set, $x \in \mu((u_i, \omega_i)_{i\in N})$, $u_j(x_j) \geq u_j(\omega_j)$ for each $j \in N$.

(ET) For any $(u_i, \omega_i)_{i\in N}$ and $x \in \mu((u_i, \omega_i)_{i\in N})$, if there are two agents $j, k \in N$ such that $u_j = u_k$ and $\omega_j = \omega_k$, then $u_j(x_j) = u_j(x_k)$.

(M1) For any economy $(u_i, \omega_i)_{i\in N}$ and any two agents $j, k \in N$ with $u_j = u_k$ and $\omega_j \geq \omega_k$,

$$u_j(x_j) \geq u_k(x_k)$$

for any $x \in \mu((u_i, \omega_i)_{i\in N})$.

(M2) For any two economies, $(u_i, \omega_i)_{i\in N}$ and $(u_i, \omega'_i)_{i\in N}$, with the same set of agents N such that there exist one and only one agent k such that $\omega_k \geq \omega'_k$ and $\omega_j = \omega'_j$ for each $j \neq k$. Then,

$$u_k(x_k) \geq u_k(x'_k)$$

for any $x \in \mu((u_i, \omega_i)_{i\in N})$ and any $x' \in \mu((u_i, \omega'_i)_{i\in N})$.

Proposition 6. *The solution satisfies (U), (S), (IR), and (ET). Also, if all utility functions are monotone, the solution satisfies (M1) and (M2).*

All these properties are particularly appealing when we think of the solution as an extension to exchange economies of the transferable utility Shapley value because even the equal treatment of equals is a property that the non-transferable utility values fail to satisfy.[4]

[4] Yannelis (1985) provides an example of a simple exchange economy in which the Shapley non-transferable utility value (Shapley, 1969) even fails to treat equals equally.

3.4 Discussion of the Solution's Cardinality

Let us now consider **finite ordinal exchange economies**. An ordinal exchange economy consists of l divisible goods and a set of agents $N = \{1, \ldots, n\}$ where each agent $i \in N$ is described by her binary preference relation $\succeq_i$ on $\mathbb{R}^l_+$ and her initial endowment $\omega_i \in \mathbb{R}^l_+$. Throughout this section we will say that a preference relation is **regular** if it is complete and transitive. Altogether, an ordinal exchange economy is a tuple $(\succeq_i, \omega_i)_{i \in N}$.

There are reasons to be concerned about the fact that the solution is not unique up to affine transformations of the utility function representations; an example of this is if we put ourselves in a situation in which agents' preferences must be inferred from choices. But by defining the solution for the space of cardinal exchange economies we obtain some versatility.

We define a solution for ordinal economies by first mapping preference relations into utility function representations to then apply the solution μ. Note that a sufficient requirement for all the properties to hold is that agents with the same preferences end up represented by the same utility functions.

The solution can easily be adapted in at least two ways to obtain solutions for ordinal economies that continue to satisfy all the properties from the previous section. The two new solutions initially map each ordinal economy into one and only one cardinal economy to then select the balanced contributions allocations. By doing so, there is no room for ambiguity about which utility function is used to represent each preference. The first solution, in subsection 3.4.1, uses a reference bundle interpretation similar to the ordinal Shapley value. The second solution, in subsection 3.4.2, measures contributions in terms of compensating variations.

Compared to the ordinal Shapley value, these solutions are easier to calculate and measure contributions in a more intuitive way. To illustrate this, consider the way the ordinal Shapley value is inductively defined. For a given reference bundle $a \in \mathbb{R}^l_{++}$, an allocation x satisfies its conditions if it is efficient and there exist balanced contributions, c_i^{-j} for $i \neq j$, such that for each agent k, all others are indifferent between the bundle they are receiving at x and the one they would receive at an allocation in the ordinal Shapley value of the hypothetical economy $(\succeq_i, \omega_i + c_i^{-k} a)_{i \in N \setminus \{k\}}$. In the construction of Pérez-Castrillo and Wettstein (2006), if an agent j were to leave the economy he would contribute proportions of the reference bundle to the others' endowments to then apply the ordinal Shapley value to the new economy without j. As a result, agent j's contributions do not measure the entire welfare gains the others obtain by adding j into the induction process because there is also a welfare gain from applying the solution after the contributions are made.

To conclude, subsection 3.4.3 contains an example of a two agents and two commodities economy in which the allocations satisfying the ordinal Shapley value conditions change when the reference bundle change. Furthermore, for the same simple economy, it is shown that the solutions in 3.4.1 and 3.4.2 do not depend on the selection of the reference bundle and prices respectively.

3.4.1 A reference bundle approach

The balanced contributions allocations admit a reference bundle interpretation similar to the one in the ordinal Shapley value in the following way. As long as a preference relation $\succeq$ is regular, continuous and monotone, we can fix an arbitrary bundle $a \in \mathbb{R}^l_{++}$ and represent the preferences through the continuous and monotone utility function $u^a_i : \mathbb{R}^l_+ \to \mathbb{R}$ where $u^a_i(x)$ is the unique real such that $x \sim_i u^a_i(x) \cdot a$ for any $x \in \mathbb{R}^l_+$.

So for any ordinal economy $(\succeq_i, \omega_i)_{i\in N}$ if preferences are regular, continuous and monotone, we can fix a common reference bundle $a \in \mathbb{R}^l_{++}$, and let the well-defined solution μ^a for the economy $(\succeq_i, \omega_i)_{i\in N}$ be the set of allocations

$$\mu^a((\succeq_i, \omega_i)_{i\in N}) = \mu((u^a_i, \omega_i)_{i\in N}).$$

The solution μ^a measures the welfare gain an agent gets when going from a bundle x to a bundle y is the proportion of the reference bundle that the agent needs to be added to $u^a_i(x) \cdot a$ to make her indifferent to y. Provided that the same reference bundle is used for each agent, μ^a possesses all the properties from the previous section plus ordinality, but the suggested outcome now may depend on the selection of the reference bundle.

3.4.2 An expenditure approach

We can also try to use the compensating variation as a measure of welfare change to tackle the cardinality issue. To do so, we will use the fact that if a preference relation $\succeq$ is regular, continuous and monotone, then the expenditure function $e_i(p,\cdot) : \mathbb{R}^l_+ \to \mathbb{R}$ is a utility function representation for any price $p \in \mathbb{R}^l_{++}$.[5]

Consider an ordinal exchange economy $(\succeq_i, \omega_i)_{i\in N}$. If preferences are regular, continuous and monotone, we can fix a strictly positive set of prices $p \in \mathbb{R}^l_+$ and using the expenditure function as a utility representation for every agent, we can define the solution, μ^p, for the ordinal economy $(\succeq_i, \omega_i)_{i\in N}$ as the set of allocations

$$\mu^p((\succeq_i \omega_i)_{i\in N}) = \mu((e_i(p,\cdot), \omega_i)_{i\in N}).$$

The solution μ^p satisfies all the properties of the solution plus ordinality. Note that the symmetry of μ^p is equivalent as i's compensating variation for having j in the economy being equal to j's compensating variation of having i in the economy. Also, using expenditures functions to represent agents' preferences guarantees that any two agents with the same preferences are represented by the same utility function irrespective of the price.

3.4.3 Example

Let us look back at the example from section 3.2.2, and just consider an economy composed of agent 1 and agent 2. Again, the initial endowments are $\omega_1 = (9,1)$ and $\omega_2 = (1,9)$, and $u_i(x) = x_1^{1/2}x_2^{1/2}$ for each $i \in \{1,2\}$. First, consider the ordinal Shapley value for this economy. Given a reference bundle $a \in \mathbb{R}^2_{++}$, an allocation x is in the ordinal Shapley value if it is efficient and there exists $c \in \mathbb{R}$ such that $u_i(x_i) = u_i(\omega_i + c \cdot a)$ for $i = 1, 2$. Different reference bundles may lead to different ordinal Shapley value allocations. For example, the only allocation in the ordinal Shapley value is both agents getting the bundle $x = (5,5)$ if the reference bundle is $a = (1,1)$. But if we use $a = (9,1)$ as reference bundle then the only allocation in the ordinal Shapley value is agent 1 getting the bundle $x_1 = (187/31, 187/31)$ while agent 2 gets $x_2 = (123/31, 123/31)$.

[5]The expenditure function $e_i(p,\cdot)$ is defined as $e_i(p,x^0) = \min\{\, p\cdot x \,|x \in \mathbb{R}^l_+ \text{ and } x \succeq_i x^0\}$.

Now, we will compute the reference bundle interpretation of the balanced contributions allocations covered in subsection 3.4.1. Let $a \in \mathbb{R}^2_{++}$ be the reference bundle. Then, for each $i = 1, 2$ and any bundle $x_i \in \mathbb{R}^2_+$, $u^a(x_i)$ is the unique number such that $u^a(x_i) \cdot a \sim x_i$, or equivalently, $u^a(x_i) = (a_1 a_2)^{-1/2}(x_{i1}x_{i2})^{1/2}$. Then, $x^* \in \mu^a((u_i, \omega_i)_{i \in \{1,2\}})$ if x^* is efficient and

$$(a_1 a_2)^{-1/2}[(x^*_{11}x^*_{12})^{1/2} - 3] = (a_1 a_2)^{-1/2}[(x^*_{21}x^*_{22})^{1/2} - 3].$$

Consequently, both agents consuming the bundle $(5, 5)$ is the only allocation in the solution set irrespective of the reference bundle.

Finally, we calculate the solution with contributions measured according to compensations from subsection 3.4.2 with $p \in \mathbb{R}^2_{++}$ being the reference prices. For $i = 1, 2$, the expenditure function is given by $e_i(p, x_i) = 2\left(p_1 p_2 x_{i1} x_{i2}\right)^{1/2}$. Therefore, $x^* \in \mu^p((u_i, \omega_i)_{i \in \{1,2\}})$ if x^* is efficient and

$$2(p_1 p_2)^{1/2}\left[(x_{11}x_{12})^{1/2} - 3\right] = 2(p_1 p_2)^{1/2}\left[(x_{21}x_{22})^{1/2} - 3\right].$$

Again, both agents consume the bundle $(5, 5)$ at the only allocation in the solution set irrespective of the reference prices.

3.5 Appendix

3.5.1 The Shapley value

Consider a TU-game (N, v) where N is the set of agents and $v : 2^N \to \mathbb{R}^{|N|}$ is the characteristic function. A **value** is a function f that maps every game (N, v) to a vector in $\mathbb{R}^{|N|}$ such that $\sum_{i \in N} f_i(N, v) = v(N)$. The next theorem presents a characterization of the Shapley value (Shapley, 1953) due to Myerson (1980).

Theorem 5. *A value ξ is the Shapley value if and only if it satisfies*

$$\sum_{i \in N \setminus \{j\}} (\xi_i(N, v) - \xi_i(N \setminus \{j\}, v)) = \sum_{i \in N \setminus \{j\}} (\xi_j(N, v) - \xi_j(N \setminus \{i\}, v))$$

for all (N, v) with $|N| \geq 2$ and all $j \in N$.

For a given value ξ, the difference $\xi_i(N, u) - \xi_i(N \setminus \{j\}, u)$ is commonly known as the marginal contribution of player j to the value of player i. Therefore, Theorem 5 states that the Shapley value is the only one that balances contributions in the sense that for each agent, the sum of her marginal contributions to others add up to sum of others' marginal contributions to her.

3.5.2 Proofs

Theorem 4. The proof proceeds by induction on the number of agents in the economy. We would also use the fact, proven in the next proposition, that the solution satisfies uniqueness of utility levels (U) and symmetry (S) to simplify the proof.

For an economy with just one agent, the solution set is non-empty by definition.

Now, consider an economy $(u_i, \omega_i)_{i \in N}$ with $|N| \geq 2$. Assume that the solution exists for any economy with less than $|N|$ agents. For each $j \in N$, let v^{-j} be the unique utility levels induced by the solution for the economy $(u_i, \omega_i)_{i \in N \setminus \{j\}}$. Similar to Pérez-Castrillo and Wettstein (2006), we will rely on a fixed-point argument to show the existence of an allocation in the solution set but before we will introduce some notation. For any utility profile $v \in \mathbb{R}_+^{|N|}$ and for each pair of agents i and j with $i \neq j$, denote the contribution of agent j to agent i, $c_i^{-j}(v)$, as

$$c_i^{-j}(v) = v_i - v_i^{-j}.$$

Moreover, define the function $C_i : \mathbb{R}^{|N|} \to \mathbb{R}$ as

$$C_i(v) = \sum_{j \neq i} c_j^{-i}(v) - \sum_{j \neq i} c_i^{-j}(v).$$

$C_i(v)$ can be understood as the net contribution of agent i. If $C_i(v) < 0$, then the sum of the contributions that agent i is receiving is larger than the sum of the contributions that i is giving to the others. Note that $\sum_{i \in N} C_i(v) = 0$.

To apply the fixed point argument, we will proceed to construct the mapping. The domain will be the set

$$U^P = \left\{ u \in \mathbb{R}^{|N|} \middle| \text{exists } x \text{ Pareto efficient such that } u_i = u_i(x_i) \text{ for each } i \in N \right\}.$$

The set U^P is homeomorphic to the $(n-1)$-unit simplex because $\sum_{i\in N} \omega_i \gg 0$ and preferences are regular, continuous and monotone (Proposition 4.6.1 in Mas-Colell (1990)).

Let $C_i^-(v) = \min\{0, C_i(v)\}$. Then, the function $p : U^P \to \mathbb{R}^n$ will punish all the agents with $C_i^- < 0$ by assigning them the utility level $p_i(v) < v_i$ while leaving the others the same. The utility levels $p_i(v)$ are defined as follows

$$p_i(v) = \left(1 + \frac{C_i^-(v)}{\max_{j\in N}\{|C_j(v)|\} + 1}\right) v_i.$$

Given that $p(v)$ may not be efficient and therefore not in U^P, the function f will map $p(v)$ into $f(v) = (f_i(v))_{i\in N} \in U^P$ in the following manner,

$$f_i(v) = p_i(v) + \delta(v) \qquad \forall i \in N,$$

where $\delta(v)$ is the unique real number such that $f(v)$ is efficient.Note that the function f is continuous since each contribution is a continuous function of v.

The mapping f has a fixed point $v^* \in U^P$ because U^P is homeomorphic to the $(n-1)$-unit simplex and f is a continuous map from U^P into itself.

Moreover, at any fixed point $v^* = f(v^*)$ it must be the case that $v^* = f(v^*) = p(v^*)$. Otherwise, if $v^* \neq p(v^*)$ then $v^* = f(v^*) \gg p(v^*)$ which is a contradiction because $\sum_{i\in N} C_i(v^*) = 0$ implies that there exists an agent j such that $C_j^-(v^*) = 0$ which means that $v_j^* = f_j(v^*) > p_j(v^*) = v_j^*$. Therefore, exists $v^* \in U^P$ such that $v^* = p(v^*)$.

Also, $v^* = p(v^*)$ if and only if $v_i^* C_i^-(v^*) = 0$ for each $i \in N$. Therefore, $v_j^* > 0$ implies $C_j(v^*) \geq 0$. The proof would be complete if $v_i^* > 0$ for each $i \in N$.

We will show that it cannot be the case that $v_j^* = 0$ for some $j \in N$, because that would imply $C_j(v^*) > 0$ contradicting $\sum_{i\in N} C_i(v^*) = 0$. Note that if there is an agent $j \in N$ such that $v_j^* = 0 < u_j(\omega_j)$, there must be an agent with $k \in N\backslash\{k\}$ with $v_k^* > v_k^{-j}$ otherwise agent j getting her endowment while the others get an allocation in the solution set for the economy without j Pareto dominates v^*. Then, $c_k^{-j}(v^*) > 0$ and $c_j^{-k}(v^*) = -v_j^{-k} \leq 0$. Also, since $v_k^* > 0$, $C_k(v^*) = 0$. The following claim implies that $C_j(v^*) > 0$ and therefore completes the proof because any efficient allocation x^* satisfying $v^* = u(x^*)$ also balances contributions.

Claim 1 If the solution satisfies (U) and (S) for any economy with less than $|N|$ agents, then the following equation holds for any $v \in \mathbb{R}^{|N|}$ and any pair of agents $i, j \in N$,

$$C_j(v) - C_i(v) = |N| \, (c_i^{-j}(v) - c_j^{-i}(v)). \tag{3.2}$$

The equation can be easily verified for the case of $|N| = 2$. For an economy with $|N| > 2$ and for any three different agents i, j, k let $v_i^{-j,k}$ be the utility level attained by agent i at an allocation in the solution set for the economy with set of agents $N \setminus \{j, k\}$. Note that symmetry of the solution for economies with $|N| - 1$ agents implies

$$v_i^{-k} - v_i^{-j,k} = v_j^{-k} - v_j^{-i,k}.$$

Without loss of generality, we will show that the equation holds for agent 1 and agent 2. Note that for any $k \neq 1, 2$ the following holds for any $v \in \mathbb{R}^{|N|}$,

$$v_1 - v_1^{-2} + v_1^{-2} - v_1^{-2,k} = v_1 - v_1^{-k} + v_1^{-k} - v_1^{-2,k} \tag{3.3}$$

$$v_2 - v_2^{-k} + v_2^{-k} - v_2^{-1,k} = v_2 - v_2^{-1} + v_2^{-1} - v_2^{-1,k} \tag{3.4}$$

$$v_k - v_k^{-1} + v_k^{-1} - v_k^{-1,2} = v_k - v_k^{-2} + v_k^{-2} - v_k^{-1,2} \tag{3.5}$$

Adding up equations (3.3), (3.4) and (3.5) and using the symmetry of the solution in the economies with $|N| - 1$ agents, we obtain

$$\left[v_1 - v_1^{-2}\right] - \left[v_2 - v_2^{-1}\right] = \left[v_1 - v_1^{-k}\right] - \left[v_k - v_k^{-1}\right] + \left[v_k - v_k^{-2}\right] - \left[v_2 - v_2^{-k}\right].$$

Writing the last equation in terms of contributions,

$$(c_1^{-2}(v) - c_2^{-1}(v)) = (c_1^{-k}(v) - c_k^{-1}(v)) - (c_2^{-k}(v) - c_k^{-2}(v)).$$

Then, adding up for all $k \neq 1, 2$,

$$(|N| - 2)(c_1^{-2}(v) - c_2^{-1}(v)) = \sum_{k \geq 3}(c_1^{-k}(v) - c_k^{-1}(v)) - \sum_{k \geq 3}(c_2^{-k}(v) - c_k^{-2}(v))$$

and adding $2(c_1^{-2}(v) - c_2^{-1}(v))$ on both sides of the equation,

$$|N|(c_1^{-2}(v) - c_2^{-1}(v)) = \sum_{k \neq 1}(c_1^{-k}(v) - c_k^{-1}(v)) + \sum_{k \neq 2}(c_k^{-2}(v) - c_2^{-k}(v)).$$

But $-C_1(v) = \sum_{k \neq 1}(c_1^{-k}(v) - c_k^{-1}(v))$ and $C_2(v) = \sum_{k \neq 2}(c_k^{-2}(v) - c_2^{-k}(v))$, hence

$$C_2(v) - C_1(v) = |N| \, (c_1^{-2}(v) - c_2^{-1}(v)).$$

The argument can be repeated for any other pair of agents and this completes the proof of the claim. □

Proposition 6. To proof proceeds by induction on the number of agents.

By definition, the solution satisfies (U) and (IR) for economies with one agent and (S) and (ET) for economies with two agents. The solution also satisfies (M2) for any economy with only one agent with monotone preferences. Moreover, the solution satisfies (M1) for any economy with two agents represented by the same monotone utility function u and $\omega_1 \geq \omega_2$ because

$$u(x_1) = u(x_1) - u(\omega_1) + u(\omega_1) = u(x_2) - u(\omega_2) + u(\omega_1) \geq u(x_2).$$

Also, (U) is satisfied for any economy with two agents because otherwise, if there are two allocations, x' and x'', in the solution set such that $u_1(x_1') > u_1(x_1'')$, then balanced contributions implies

$$u_2(x_2') - u_2(\omega_2) = u_1(x_1') - u_1(\omega_1) > u_1(x_1'') - u_1(\omega_1) = u_2(x_2'') - u_2(\omega_2).$$

But that implies that the allocation x' Pareto dominates the allocation x'' contradicting the fact that x'' is efficient and therefore cannot be in the solution set.

Now, let us consider economies with $|N| \geq 2$. Assume that the solution solutions satisfies (U), (IR), (S) and (ET) for all economies with less than $|N|$ agents. Moreover, provided that all utility functions are monotone, the solution satisfies (M1) and (M2) for all economies with less than n agents.

Again, for any $j \in N$, let $v^{-j} \in \mathbb{R}^{|N|-1}$ be the unique utility levels at any allocation $y^{-j} \in \mu((u_i, \omega_i)_{i \in N \setminus \{j\}})$.

(S) Suppose that $x \in \mu((u_i, \omega_i)_{i \in N})$. Since the solution satisfies properties (U) and (S) for economies with $|N| - 1$ agents, the claim in the existence proof applies and letting $v = (u_i(x_i))_{i \in N}$ and using the fact that $C_i(v) = 0$ for each $i \in N$ because x is in the solution set, we can write write (3.2) as

$$|N| \left(c_j^{-k}(v) - c_k^{-j}(v)\right) = C_j(v) - C_k(v) = 0 \quad \forall \, j, k \in N$$

implying

$$u_j(x_j) - v_j^{-k} = u_k(x_k) - v_k^{-j} \quad \forall \, .j, k \in N.$$

Therefore, contributions are balanced one to one.

(U) Suppose towards a contradiction that exists $x, x' \in \mu((u_i, \omega_i)_{i \in N})$ such that $u_j(x_j) > u_j(x'_j)$ for some agent $j \in N$. Using Claim 1, we can see that for every agent $i \neq j$,

$$u_i(x_i) - v_i^{-j} = u_j(x_j) - v_j^{-i} > u_j(x'_j) - v_j^{-i} = u_i(x'_i) - v_i^{-j}.$$

But x would Pareto dominate x' and therefore contradict x' being in the solution set. Hence, the utility levels must be unique.

(IR) Suppose towards a contradiction that there exists $x \in \mu((u_i, \omega_i)_{i \in N})$ such that $u_j(x_j) < u_j(\omega_j)$ for some $j \in N$. Then, since x is efficient, there must be an agent $k \in N$ with $u_k(x_k) > v_k^{-j}$ otherwise (ω_j, y^{-j}) would Pareto dominate x. But this implies

$$u_k(x_k) - v_k^{-j} > 0 > u_j(x_j) - v_j^{-k}$$

contradicting the claim. Therefore, the solution must satisfy individual rationality.

(ET) Let $(u_i, \omega_i)_{i \in N}$ be such that there are two agents $j, k \in N$ such that $u_j = u_k$ and $\omega_j = \omega_k$. Then, the two $|N| - 1$ agents economies that result from removing j and k are identical, therefore property (U) guarantees that $v_k^{-j} =$

v_j^{-k}. Also, for any $x \in \mu((u_i, \omega_i)_{i \in N})$, we know from property (S) that

$$u_j(x_j) - v_j^{-k} = u_k(x_k) - v_k^{-j},$$

and therefore,

$$u_j(x_j) = u_k(x_k).$$

(M1) Take an economy $(u_i, \omega_i)_{i \in N}$ with set of agents $|N| \geq 2$ such that $u_1 = u_2$, $\omega_1 \geq \omega_2$ and all utility functions are monotone. The economy with set of agents $N \setminus \{2\}$ has at least as many resources than the economy with set of agents $N \setminus \{1\}$. Then, there must be an agent $j \in N$ with $v_j^{-2} \geq v_j^{-1}$ because of monotonicity of the utility functions and efficiency of the solution for the $|N| - 1$ agents economies. Also, for any $x \in (u_i, \omega_i)_{i \in N}$, using the fact that contributions are symmetric, we obtain

$$u(x_1) - v_1^{-j} = u_j(x_j) - v_j^{-1} \geq u_j(x_j) - v_j^{-2} = u(x_2) - v_2^{-j},$$

and since $v_1^{-j} \geq v_2^{-j}$ because the solution satisfies (M1) for the economies with $|N| - 1$ agents,

$$u(x_1) \geq u(x_2).$$

(M2) Consider a set of agents N with $|N| \geq 2$ and two economies $(u_i, \omega_i)_{i \in N}$ and $(u_i, \widetilde{\omega}_i)_{i \in N}$ differ only in $\omega_1 \geq \widetilde{\omega}_1$. Let v^{-j} and $\widehat{v}^{-j}$ be the utility levels attained at the solution sets of the economies $(u_i, \omega_i)_{i \in N \setminus \{j\}}$ and $(u_i, \widetilde{\omega}_i)_{i \in N \setminus \{j\}}$ respectively. Let $x \in \mu((u_i, \omega_i)_{i \in N})$ and $\widetilde{x} \in \mu((u_i, \widetilde{\omega}_i)_{i \in N})$. Suppose towards a contradiction that $u_1(x_1) < u_1(\widetilde{x}_1)$. But, $v_1^{-i} \geq \widetilde{v}_1^{-i}$ for each $i \in N \setminus \{1\}$ because of the induction hypothesis and,

$$u_i(x_i) - v_i^{-1} = u_1(x_1) - v_1^{-i} < u_1(\widetilde{x}_1) - \widetilde{v}_1^{-i} = u_i(\widetilde{x}_i) - \widetilde{v}_i^{-1}.$$

But $v^{-1} = \widetilde{v}^{-1}$ because the economies $(u_i, \omega_i)_{i \in N \setminus \{1\}}$ and $(u_i, \widetilde{\omega}_i)_{i \in N \setminus \{1\}}$ are the same and therefore $u_i(x_1) < u_i(\widetilde{x}_i)$ for each $i \in N$ but this is a contradiction since x is efficient. Therefore, the solution μ satisfies (M2).

□

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Research

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Working Papers

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