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THE MACKEY ANALOGY FOR $\mathrm{SL}(n, \mathbb{R})$

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Abstract

Let G be a connected semisimple Lie group with finite center, and let K be a maximal compact subgroup. Mackey suggested that there should be an analogy between almost all the unitary representations of G and almost all of the unitary representations of an associated semidirect product group $G_0 = K \ltimes \mathfrak{g}/\mathfrak{k}$, where $\mathfrak{g}$ and $\mathfrak{k}$ are the Lie algebras of G and K, respectively.

Perhaps because Mackey typically viewed the dual of a Lie group as a Borel space, the analogy he proposed was between almost all unitary representations of G and G_0. In the case of complex semisimple Lie groups, Higson completed Mackey's analogy at the level of parameters, constructing a reasonably natural bijection between all irreducible tempered representations of G and all irreducible unitary representations of G_0. In this dissertation, following Higson, and relying heavily on Vogan's classification of representations by minimal K-types, we conjecture that there is a very natural bijection between $\widehat{G}$ and $\widehat{G}_0$ for any connected semisimple Lie group with finite center, and prove this conjecture for $G = \mathrm{SL}(n, \mathbb{R})$.

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Chapter 1

Introduction

In this dissertation we study an analogy proposed by George Mackey [Mac75] between a semisimple Lie group and an associated semidirect product group (the Cartan motion group) following an approach similar to that presented in [Hig07].

Mackey suggested that there should be an analogy between unitary representations of a semisimple group and unitary representations of its associated semidirect product group [Mac75]. For example, let $G = PGL(2, \mathbb{C})$ and let G_0 be the semidirect product of $SO(3)$ and $\mathbb{R}^3$ that is constructed using the natural action of $SO(3)$ on $\mathbb{R}^3$. Algebraically, these groups have very different structures. However, G_0 is the group of orientation-preserving isometries of three-dimensional Euclidean space, while G is the group of orientation-preserving isometries of three-dimensional hyperbolic space. In some sense, we can view G_0 as a limiting case as the curvature of hyperbolic space approaches zero [Hig07].

The irreducible unitary representations of G_0 have a well-known interpretation as particle states in quantum mechanics. Since hyperbolic space, particularly with curvature close to zero, is a plausible model for physical space, Mackey thought there should be corresponding particle states on this curved space, and hence a one-to-one correspondence between *almost all* of the irreducible unitary representations of G_0 and those of G [Mac75]. In this dissertation we will extend Mackey's analogy from a suggested measure-theoretic correspondence between $\widehat{G}$ and $\widehat{G}_0$ to a an actual bijection, but between the tempered representations of G and the unitary representations of G_0.

To begin this investigation, let G be a connected semisimple Lie group with finite center, and K a maximal compact subgroup of G with corresponding Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$, respectively. Form the semidirect product group $K \ltimes \mathfrak{g}/\mathfrak{k}$, with the group operation given by

$$(k_1, v_1) \cdot (k_2, v_2) = (k_1 k_2, \mathrm{Ad}_{k_2^{-1}}(v_1) + v_2).$$

Following Mackey, we would like to explore the connection between representations of G and G_0.

In [Hig07], Higson constructed a reasonably natural bijection between the tempered dual of a complex semisimple Lie group and the unitary dual of the associated semidirect product group. Though the bijection is not a homeomorphism, his key result is that both duals can be partitioned by minimal K-types into locally closed parts, and the bijection maps each part of the dual of G homeomorphically onto a corresponding part of the dual of G_0.

In order to extend this result beyond a "mere coincidence of parametrizations" in which Mackey was not interested, Higson used this bijection to construct a continuous field of group C^*-algebras associated with the deformation from G to G_0. As it turns out, the K-theory remains constant along this field [Hig07, p.25], and it was noted in [BCH94, p.263] that this is equivalent to the Connes-Kasparov conjecture.

The fundamental theme of this dissertation, Conjecture 4.0.7, is that there should, in fact, be a very natural bijection between $\widehat{G}$ and $\widehat{G}_0$ for all connected semisimple Lie groups with finite center, where here, and throughout, $\widehat{G}$ denotes the tempered dual of G. While we stop short of proving this in general, the main result is Theorem 4.0.11, in which we establish this bijection for the group $\mathrm{SL}(n, \mathbb{R})$.

In Chapter 2 we summarize Higson's approach for complex semisimple Lie groups. The first step in building a bijection between $\widehat{G}$ and $\widehat{G}_0$ is to have a thorough understanding of each set. To that end, we begin with an overview of the Mackey machine (see [Mac49], for example) for computing the unitary dual of various semidirect product groups. In particular, if K is a compact Lie group acting on an abelian Lie group V by automorphisms, we have (see Theorem 2.1.2)

$$\widehat{K \ltimes V} \cong \{(\sigma, \varphi) \mid \varphi \in \widehat{V} \text{ and } \sigma \in \widehat{K}_\varphi\}/K,$$

where K_φ is the isotropy subgroup of φ in K, and the pair (σ, φ) corresponds to the representation

$$\mathrm{Ind}_{K_\varphi \ltimes V}^{G_0} (\sigma \otimes \varphi).$$

Now, let G be a connected complex semisimple Lie group, with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{a} \subseteq \mathfrak{p}$ be a maximal abelian subalgebra, and let $A = \exp(\mathfrak{a})$. Then, we have the Iwasawa decomposition $G = KAN$, where K is a maximal compact subgroup of G. Let $P = MAN$, where $M = N_K(A)$ is a maximal torus for K, be a minimal parabolic subgroup of G. The representations induced unitarily from P to G are called the unitary principal series, and exhaust the tempered dual of G. If we let W denote the Weyl group of G, then we have (see Theorem 2.2.3)

$$\widehat{G} \cong (\widehat{M} \times \widehat{A})/W$$

where the pair $(\sigma, \varphi) \in \widehat{M} \times \widehat{A}$ corresponds the the representation $\mathrm{Ind}_{MAN}^{G} (\sigma \otimes \varphi \otimes 1)$.

Next we attempt to parametrize $\widehat{G}_0$ in a manner compatible with that of $\widehat{G}$. We use the Cartan-Weyl theory to understand the representations of the various isotropy subgroups of K, culminating in the key result here (Proposition 2.5.2), that

$$\widehat{G}_0 \cong (\widehat{M} \times \widehat{\mathfrak{a}})/W,$$

where the pair $(\sigma, \varphi) \in \widehat{M} \times \widehat{\mathfrak{a}}$ corresponds the the representation $\mathrm{Ind}_{K_\varphi \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0} (\sigma \otimes \varphi)$.

Thus, by identifying $\widehat{A}$ with $\widehat{\mathfrak{a}}$ via the exponential map, we see that $\widehat{G} \cong \widehat{G}_0$ in a reasonably natural way, as desired.

Note also as motivation for what follows, we can write

$$\widehat{G} \cong (\widehat{M} \times \widehat{A})/W \cong \bigsqcup_{\delta \in \widehat{M}/W} \widehat{A}/W_\delta,$$

where W_δ is the isotropy subgroup of δ in W. Similarly, we have

$$\widehat{G}_0 \cong (\widehat{M} \times \widehat{\mathfrak{a}})/W \cong \bigsqcup_{\delta \in \widehat{M}/W} \widehat{\mathfrak{a}}/W_\delta.$$

In addition, since the representations of K are parametrized by $\widehat{M}/W$, we see that the bijection preserves minimal K-types.

As we move out of the realm of complex semisimple Lie groups, constructing the bijection becomes significantly more difficult because both $\widehat{G}$ and $\widehat{G}_0$ can, in general, be quite complicated. A general semisimple Lie group G may have many series of representations (beyond the principal series), and more importantly, they may not all be irreducible. The dual of G_0 is similarly complicated by the fact that the various isotropy subgroups, K_φ, may be disconnected. Thus, the powerful and elegant Cartan-Weyl theory does not directly apply.

In Chapter 3 we outline an alternative approach to describing $\widehat{G}$ due to Vogan (see [Vog77], [Vog79], [Vog81], and [Vog85]). To each $\tau \in \widehat{K}$, Vogan associates a certain cuspidal parabolic subgroup $P(\tau) = M(\tau)A(\tau)N(\tau)$, and a family of representations $\{\overline{\pi}(\tau, \varphi) \,|\, \varphi \in \widehat{A}\} \subseteq \widehat{G}$ (see Definition3.2.7) induced from P, all with τ as a minimal K-type. There are issues with equivalence and reducibility, which will be discussed in more detail in Section 3.2, but postponing those for now, Vogan then parametrizes the representations in $\widehat{G}$ with minimal K-type τ by $\widehat{A}(\tau)$.

We begin our own contributions in Chapter 4, where we describe a method (which is still conjectural, in general) that is directly analogous to Vogan's for classifying $\widehat{G}_0$ by minimal K-types. Here we define a specific subgroup of G_0 in analogy with Vogan's parabolic subgroup P from which all representations with a given minimal K-type may be induced. We take $K(\tau) = M \cap K$ and $\mathfrak{a}(\tau) = \mathrm{Lie}(A(\tau))$, and we construct a family of irreducible unitary representations $\{\overline{\pi}_0(\tau, \varphi) \mid \varphi \in \widehat{\mathfrak{a}}(\tau)\} \subseteq \widehat{G}_0$ (see Definition4.0.6) induced from $K(\tau) \ltimes \mathfrak{a}(\tau)$, all with τ as a minimal K-type. Once again, there are issues with equivalences but, interestingly, not reducibility, which are discussed in more detail in Chapter 4. So, in analogy with Vogan's method for G, we can parametrize the representations in $\widehat{G}_0$ with minimal K-type τ by $\widehat{\mathfrak{a}}(\tau)$. Identifying $\widehat{\mathfrak{a}}(\tau)$ with $\widehat{A}(\tau)$ via the exponential map yields a bijection between $\widehat{G}_0$ and $\widehat{G}$. In particular, we arrive at the main organizing idea of this dissertation:

Conjecture 4.0.7. *Let G be a connected semisimple Lie group with finite center and maximal compact subgroup K. Let $G_0 = K \ltimes \mathfrak{g}/\mathfrak{k}$ be the associated semidirect product group. Let $\tau \in \widehat{K}$, and let $\overline{\pi}(\tau, \varphi)$ and $\overline{\pi}_0(\tau, \varphi)$ be the representations of G and G_0 from Definitions 3.2.7 and 4.0.6, respectively. The correspondence which* *associates to the representation* $\overline{\pi}(\tau, \varphi) \in \widehat{G}$ *the representation* $\overline{\pi}_0(\tau, \varphi) \in \widehat{G}_0$ *is a well-defined bijection preserving minimal* K*-types.*

Remark. While this bijection will not be a homeomorphism, for each K-type τ, the bijection ought to be a homeomorphism from the set of representations in $\widehat{G}$ with minimal K-types in $\mathscr{C}(\tau)$ to those representations in $\widehat{G}_0$ also with minimal K-types in $\mathscr{C}(\tau)$.

We take a slight departure from the main theme of this dissertation in Chapter 5 to summarize the various branching rules and related results necessary to classify representations of the semidirect product groups associated to $\mathrm{SL}(n, \mathbb{R})$ by minimal K-types.

In Chapter 6 we begin the program set forth in Chapter 4 by examining first complex semisimple Lie groups, and then the groups $\mathrm{SL}(2, \mathbb{R})$ and $\mathrm{SL}(3, \mathbb{R})$ from this new perspective. In Section 6.1 we describe the tempered dual of a connected complex semisimple Lie group, G, using Vogan's minimal K-type approach as outlined in Section 3.2. In this case, G has only one cuspidal parabolic subgroup $P = MAN$, and M is a maximal torus for K. As such, the representations $\overline{\pi}(\tau, \varphi)$ take on a very simple form. To be precise,

$$\overline{\pi}(\tau, \varphi) = \mathrm{Ind}_{MAN}^{G}\,(\sigma \otimes \varphi \otimes 1),$$

where $\sigma \in \widehat{M}$ is a highest weight of τ. Similarly, in Section 6.1.1, we follow our methods from Chapter 4 for classifying $\widehat{G}_0$, and define

$$\overline{\pi}_0(\tau,\varphi) = \operatorname{Ind}_{M\ltimes\mathfrak{g}/\mathfrak{k}}^{G_0}(\sigma\otimes\varphi),$$

where once again, $\sigma \in \widehat{M}$ is a highest weight of τ. This section concludes with a proof that the correspondence $\overline{\pi}(\tau,\varphi) \leftrightarrow \overline{\pi}_0(\tau,\varphi)$ is a bijection for connected complex semisimple Lie groups (see Theorem 6.1.9, establishing the validity of Conjecture 4.0.7 for this class of groups.

In the following sections of Chapter 6 we examine both $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(3,\mathbb{R})$. Each section begins with a description of the tempered duals of these groups following Vogan. The group $\mathrm{SL}(2,\mathbb{R})$ is particularly interesting, as this will be our first encounter with the aforementioned issues of reducibility, encoded in the R-groups. Each section concludes with a description of $\widehat{G}_0$ using our methods from Chapter 4. Here, some of the isotropy subgroups K_φ which appear in the Mackey machine do turn out to be disconnected, and so we make heavy use of the branching rules presented in Chapter 5 to construct the necessary representations of G_0 following the minimal K-type approach. Just as with connected complex semisimple Lie groups, we are able to establish Conjecture 4.0.7 for the groups $\mathrm{SL}(2,\mathbb{R})$ (see Theorem 6.2.9) and $\mathrm{SL}(3,\mathbb{R})$ (see Theorem 6.3.9).

In Chapter 7, we study the group $G = \mathrm{SL}(n,\mathbb{R})$. The representation theory of G takes on a slightly different character depending on whether n is even or odd, and so we treat the two cases separately. In Section 7.1 we compute the tempered dual of $G = \mathrm{SL}(2k, \mathbb{R})$ following Vogan's method. Just as with the group $\mathrm{SL}(2, \mathbb{R})$ issues surrounding the reducibility of representations arise in the form of nontrivial R-groups. Once we have a description of $\widehat{G}$, we turn our attention to G_0. As in Chapter 6 we use our minimal K-type approach to classify $\widehat{G}_0$, and once again establish the validity of Conjecture 4.0.7 for $G = \mathrm{SL}(2k, \mathbb{R})$. In the following section, we repeat the process for the group $\mathrm{SL}(2k+1, \mathbb{R})$ following very similar (though somewhat simpler) calculations, once again verifying Conjecture 4.0.7 for $\mathrm{SL}(2k+1, \mathbb{R})$, and hence for the group $\mathrm{SL}(n, \mathbb{R})$, in general.

Chapter 2

The Mackey Analogy for Complex Semisimple Lie Groups

We will begin our study of the Mackey analogy with complex semisimple Lie groups. In this setting, the relationship between the tempered unitary irreducible representations of the group G and the unitary irreducible representations of its associated semidirect product group G_0 is fairly straightforward, and will provide us with motivation for the real semisimple case (see [Hig07] for more details).

To begin, let G be a connected semisimple Lie group with finite center (not necessarily complex), and let K denote a maximal compact subgroup. The associated semidirect product group which we will be interested in is $G_0 = K \ltimes \mathfrak{g}/\mathfrak{k}$, where K acts on $\mathfrak{g}/\mathfrak{k}$ via the adjoint action. Then the group operation for G_0 is given by

$$(k_1, v_1) \cdot (k_2, v_2) = (k_1 k_2, Ad_{k_2^{-1}}(v_1) + v_2).$$

While there may, in fact, be a relationship between all (admissible) representations of G and G_0, we will restrict our attention to certain kinds of unitary representations.

Definition 2.0.1. For any Lie group G, let $\widehat{G}$ denote the set of unitary equivalence classes of those irreducible unitary representations of G that are weakly contained in the regular representation. $\widehat{G}$ is known as the *reduced dual* of G.

See [Dix77, Chapter 18] for more on the reduced dual. Note that for the semisimple groups of interest in this dissertation, the reduced dual is the same as the tempered dual [CHH88]. On the other hand, for the semidirect product group, $\widehat{G}_0$ is the entire unitary dual of G_0.

In order to draw comparisons between $\widehat{G}$ and $\widehat{G}_0$, we begin by a complete description of each dual.

2.1 Semidirect Product Groups

Let V be an abelian Lie group, and K a compact Lie group acting on V by automorphisms. The Mackey machine (see [Mac49, Theorem 3]) allows us to compute the unitary dual of the semidirect product group $K \ltimes V$.

To begin, let φ be a unitary character of V (ie. a homomorphism $A \to \mathbb{T}$), and let K_φ be the isotropy group

$$K_\varphi = \{k \in K \mid \varphi(k \cdot v) = \varphi(v) \, \forall \, v \in V\}$$

Now, take $\sigma \in \widehat{K}_\varphi$, and form the representation $\sigma \otimes \varphi$ of $K_\varphi \ltimes V$, defined by $(\sigma \otimes \varphi)(k, v) = \varphi(v)\sigma(k)$. This representation may then be induced up (see [Mac52, Part IV]), to obtain a representation of $K \ltimes V$. The Mackey machine tells us the following about such representations (see [Mac49, Section 7] or [Mac76, Chapter 3]).

Proposition 2.1.1. *Let V be an abelian Lie group, and K a compact Lie group acting on V by automorphisms. In addition, let $\varphi \in \widehat{V}$, and $\sigma \in \widehat{K}_\varphi$. Then,*

i) The representation $Ind_{K_\varphi \ltimes V}^{K \ltimes V} \sigma \otimes \varphi$ is irreducible,

ii) Up to unitary equivalence, all irreducible unitary representations of $K \ltimes V$ arise from this construction, and

iii) The representations $\mathrm{Ind}_{K_\varphi \ltimes V}^{K \ltimes V}(\sigma_1 \otimes \varphi_1)$ and $\mathrm{Ind}_{K_\varphi \ltimes V}^{K \ltimes V}(\sigma_2 \otimes \varphi_2)$ are unitarily equivalent if and only if $\varphi_2(v) = \varphi_1(k \cdot v)$ and σ_2 is unitarily equivalent to $\sigma_1 \circ Ad_k$, for some $k \in K$.

As a result, we get the following description of the unitary dual:

Theorem 2.1.2. *Let V be an abelian Lie group, and K a compact Lie group acting on V by automorphisms. Then*

$$\widehat{K \ltimes V} \cong \{(\sigma, \varphi) \,|\, \varphi \in \widehat{V} \text{ and } \sigma \in \widehat{K}_\varphi\}/K. \quad \blacksquare$$

2.2 Complex Semisimple Groups

Let G be a connected complex semisimple group with maximal compact subgroup K. We will let $\mathfrak{g}$ and $\mathfrak{k}$ denote their respective Lie algebras, and let $\mathfrak{p}$ denote the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$ (see [Kna86, Chapter 1], for example). Take $\mathfrak{a}$ to be a maximal abelian subalgebra of $\mathfrak{p}$. For each linear functional $\alpha : \mathfrak{a} \to \mathbb{C}$, let (see [Kna86, Section 5.2]

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \,|\, [Y, X] = \alpha(Y)X \,\forall Y \in \mathfrak{a}\}.$$

If $\mathfrak{g}_\alpha \neq 0$, we call α a *root* of $\mathfrak{g}$, and $\mathfrak{g}_\alpha$, the α *root space*. If we let Δ denote the set of all nonzero roots, then we can write the root space decomposition of $\mathfrak{g}$ (see [Kna86, Proposition 5.9], for example)

$$\mathfrak{g} = \mathfrak{g}_\circ \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

and $\mathfrak{g}_\circ = \mathfrak{a} \oplus \mathfrak{m}$, where $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$. An ordering can be put on Δ, and let Δ^+ denote the positive roots. (see [Kna86, p.118]. Then define

$$\mathfrak{n} = \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha.$$

This gives rise to the Iwasawa decomposition [Kna02, p. 373]:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$$

$$G = KAN$$

Now, denote by M the centralizer of A in K, which is a maximal torus in K. Then $P = MAN$ is a *minimal parabolic subgroup* of G (also known as a Borel subgroup), and any closed subgroup of G containing P is called a *parabolic subgroup.* Any parabolic subgroup, P' has a Langlands decomposition (see [Kna86, Section 5.5], for example)

$$P' = M'A'N'$$

analogous to the decomposition of P above. Finally, a parabolic subgroup, P', is called *cuspidal* if the M' in its Langlands decomposition has discrete series representations (those relevant for the Plancherel formula for G (see [Kna86, p.135]).

Returning to the minimal parabolic subgroup $P = MAN$ discussed above, let $\sigma \in \widehat{M}$ and $\varphi \in \widehat{A}$, and form the representation $\sigma \otimes \varphi$ and extend it trivially to N to yield a representation of P. We may then induce this up unitarily to obtain a representation of G (see [Kna86, Section 7.1], for example).

Definition 2.2.1. Let G be a connected complex semisimple Lie group, and $P =$

MAN a minimal parabolic subgroup of G. Let $\sigma \in \widehat{M}$ and $\varphi \in \widehat{A}$, and define

$$\pi(\sigma, \varphi) = \mathrm{Ind}_{MAN}^{G}(\sigma \otimes \varphi \otimes 1).$$

These representations are known as the *unitary principal series* of G.

Much like 2.1.1, we have the following results about the principal series:

Proposition 2.2.2. *Let G be a connected complex semisimple Lie group with a maximal compact subgroup K, and $P = MAN$ a minimal parabolic subgroup of G. Then,*

i) All principal series representations of G are irreducible [Wal71, Theorem 4.1],

ii) Up to unitary equivalence, all tempered representations of G arise as a principal series representation (see [HC54]), and

iii) $\pi_{(\sigma_1,\varphi_1)}$ is unitarily equivalent to $\pi_{(\sigma_2,\varphi_2)}$ if and only if (σ_1, φ_1) and (σ_2, φ_2) are conjugate under the Weyl group W of G, where $W = N(A)/Z(A)$) (see [Bru56, Section 4], for example).

These results give us a complete description of the tempered dual of a complex semisimple Lie group.

Theorem 2.2.3. *Let G be a connected complex semisimple Lie group, with $P = MAN$ a minimal parabolic subgroup. The correspondence which associates to a pair $(\sigma, \varphi) \in \widehat{M} \times \widehat{A}$ the unitary principal series representation $\pi_{(\sigma,\varphi)}$ gives rise to* *a bijection*

$$\widehat{G} \cong (\widehat{M} \times \widehat{A})/W. \quad \blacksquare$$

2.3 The Mackey Analogy for Complex Semisimple Lie Groups

Now that we have a description of the reduced duals of certain semidirect product groups and complex semisimple groups, we now focus on finding analogies between representations of G and G_0, as in [Mac75] and closely following [Hig07, Section 2.3].

The key idea is to describe the dual of G_0 in a manner compatible with that of G. To that end, consider $\mathfrak{a}$, the Lie algebra of the subgroup A in the Iwasawa decomposition of G, as a subspace of the vector space $\mathfrak{g}/\mathfrak{k}$. Let $\mathfrak{a}^{\perp}$ be the unique M-invariant complement of $\mathfrak{a}$ in $\mathfrak{g}/\mathfrak{k}$.

Definition 2.3.1. A character of $\mathfrak{g}/\mathfrak{k}$ is called *balanced* if it is trivial on $a^{\perp}$.

Note that a balanced character factors through the projection of $\mathfrak{g}/\mathfrak{k}$ onto $\mathfrak{a}$, and can therefore be thought of as a character of $\mathfrak{a}$. In addition, every character of $\mathfrak{g}/\mathfrak{k}$ is conjugate under W to a unique balanced character.

For a balanced character φ, we have $M \subseteq K_\varphi$, and generically, $K_\varphi = M$. Thus, a generic representation of G_0 is of the form $\mathrm{Ind}_{K_\varphi \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0}(\sigma, \varphi)$, where φ is a generic unitary balanced character of $\mathfrak{g}/\mathfrak{k}$ (which can be thought of as a unitary character of $\mathfrak{a}$) and $\sigma \in \widehat{M}$. In addition, the unitary characters of $\mathfrak{a}$ may be identified with those of A (see Section 2.4). As a result, a generic representation of G_0 is indexed by such a pair (σ, φ) and is determined up to conjugacy by W. On the other hand, *all* tempered representations of G are indexed by a pair (σ, φ) with $\varphi \in \widehat{A}$ and $\sigma \in \widehat{M}$, determined up to conjugacy by W. Thus, generically, the irreducible unitary representations of G_0 are parametrized in the same way as the irreducible tempered representations of G.

To carry the analogy beyond a "mere coincidence of parameters," Mackey noted that the generic representations of G_0 are induced from the subgroup $M \ltimes \mathfrak{g}/\mathfrak{k}$, and the principal series representations of G are induced from the minimal parabolic subgroup P. As it turns out, the semidirect product group associated to P is $P_0 = M \ltimes \mathfrak{g}/\mathfrak{k}$, and thus $M \ltimes \mathfrak{g}/\mathfrak{k} \subseteq G_0$ stands in relation to $P \subseteq G$ in the same way G_0 stands in relation to G. It follows that (generically) the representation spaces may be identified as unitary representations of the common subgroup K of G.

Mackey did not extend this analogy to all representations of G and G_0. There are a number of reasons why this may be the case. To begin with, there is a serious difference between the representation spaces of non-generic representations of G and G_0; some representations of G_0 are finite-dimensional, however *all* representations of G are infinite-dimensional. On the other hand, since Mackey typically viewed the dual as a Borel space, extending the analogy to the full reduced dual may not have seemed important as the non-generic representations have zero Plancherel measure.

2.4 Review of Cartan-Weyl Theory

In order to extend the Mackey analogy to all representations of G and G_0, we first need a complete description of $\widehat{G}_0$ when G is a connected complex semisimple Lie group. The key ingredient here is to understand the representations of the various isotropy subgroups K_φ. In order to describe $\widehat{K}_\varphi$, we quickly review the Cartan-Weyl theory for compact groups, following Vogan's exposition in [Vog85, Section 1], which classifies representations of compact groups by their highest weights.

Let K be a compact connected Lie group, and let T be a *maximal torus* of K (which is a maximal abelian connected subgroup of K). If we use $K = U(n)$ as a running example, we have

$$T = \left\{ \left(\begin{array}{cccc} e^{i\theta_1} & & & \\ & e^{i\theta_2} & & \\ & & \ddots & \\ & & & e^{i\theta_n} \end{array} \right) \middle| \, \theta_j \in \mathbb{R} \right\}$$

Each element of T acts on $\mathbb{C}^n$ by a rotation in each of the n copies of $\mathbb{C}$ separately. Clearly, this is an abelian group, and any element of $U(n)$ which commutes with T lies in T, as well. So T is, in fact, a maximal torus in $U(n)$.

Now in general, since T is abelian, any irreducible representation $\chi \in \widehat{T}$ must be one-dimensional, and thus is a map from T into the unit circle in the complex plane (ie. a unitary character). Let $d\chi$ denote the differential of χ, which is a map from $\mathfrak{t}$ (the Lie algebra of T) in to the complex numbers, and is given by

$$\chi[exp(tY)] = e^{t[d\chi(Y)]}, \quad \forall Y \in \mathfrak{t} \text{ and } t \in \mathbb{R}. \tag{2.4.1}$$

Now, in order for χ to be unitary, we see that $d\chi$ must take on purely imaginary values. That is,

$$d\chi : \mathfrak{t} \to i\mathbb{R}$$

Definition 2.4.2. Let T be a Lie group with Lie algebra $\mathfrak{t}$. Those functionals on $\mathfrak{t}$ which exponentiate to unitary characters of T will be called *unitary characters* of $\mathfrak{t}$. The set of all unitary characters of $\mathfrak{t}$ will be denoted by $\widehat{\mathfrak{t}}$.

Note that by equation 2.4.1, the character χ is determined by its differential, $d\chi$. Thus, we have

Lemma 2.4.3. *Let T be a compact connected abelian Lie group. Then $\widehat{T}$ may be identified with a subset of $\widehat{\mathfrak{t}}$ by identifying a representation with its differential. In the following we will make this identification, and write χ for both a representation of T, and its differential.*

In the following we will make this identification, and write χ for both a representation of T, and its differential.

In the case of $G = U(n)$, the elements of $\widehat{\mathfrak{t}}$ which correspond to elements of $\widehat{T}$ are the functionals of the form $d\chi(m_1, \ldots, m_n)$ defined by

$$d\chi(m_1, \ldots, m_n)(\theta_1, \ldots, \theta_n) = \sum_{j=1}^{n} i m_j \theta_j.$$

Then the corresponding character of T is given by

$$\chi(m_1, \ldots, m_n)(\mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})) = \prod_{j=1}^{n} e^{i m_j \theta_j}.$$

We now return to the general case.

Definition 2.4.4. Let $\pi \in \widehat{K}$ with representation space V, and let $\chi \in \widehat{T}$. The χ-*weight space* of V is

$$V_\chi = \{v \in V \mid \pi(t)v = \chi(t)v \;\forall t \in T\}.$$

Evidently, V can be written as the direct sum of its weight spaces. This decomposition

$$V = \bigoplus_{\chi \in \widehat{T}} V_\chi$$

is called the *weight space decomposition* of π.

Definition 2.4.5. If V_χ is nontrivial, we call χ a *weight* of π, and the multiplicity of χ in π is the dimension of V_χ.

While any representation of K is determined by its weight space decomposition, computing this may be very complicated. However, a representation of K can be determined with much less information than the complete weight space decomposition.

To begin, choose a basis, say $\{e_1, \ldots, e_n\}$, of $\widehat{\mathfrak{t}}$. Impose a lexicographic ordering on $\widehat{\mathfrak{t}}$ by defining

$$\sum a_i e_i < \sum b_i e_i$$

if and only if

$$a_1 < b_1,$$

or

$$a_1 = b_1 \text{ and } a_2 < b_2, \text{ etc.}$$

Definition 2.4.6. The largest weight of π in this ordering is called the *highest weight* of π.

The idea of highest weights is the key to understanding representations of K (see [Vog85, Proposition 1.10], for example):

Proposition 2.4.7. *(Cartan and Weyl). Let K be a compact connected Lie group. If two representations of K have the same highest weight, then they are equivalent. In addition, the highest weight if any irreducible representation occurs with multiplicity one.*

We can use this result to identify $\widehat{K}$, in a fairly concrete way, with a subset of

$\widehat{\mathfrak{t}}$. In order to complete our picture of $\widehat{K}$, we must specify which elements of $\widehat{\mathfrak{t}}$ can occur as highest weights of irreducible representations of K.

Let $W = N(T)/T$ be the Weyl group of K. Since W acts on T, id also induces an action on $\mathfrak{t}$, $\widehat{T}$, and $\widehat{\mathfrak{t}}$.

Definition 2.4.8. Let $\chi \in \widehat{\mathfrak{t}}$. Then χ is called *dominant* if

$$w \cdot \chi \leq \chi, \ \forall w \in W.$$

This brings us to a complete description of $\widehat{K}$ (see [Vog85, Proposition 1.12], for example):

Theorem 2.4.9. *(Cartan and Weyl) Let $\chi \in \widehat{\mathfrak{t}}$. Then χ is the highest weight of an irreducible representation of K if and only if χ is dominant and χ is the differential of some representation of T. Thus,*

$$\widehat{K} \cong \widehat{T}/W. \quad \blacksquare$$

For $G = U(n)$, we have $W = S_n$ (the symmetric group on n elements) and W acts by permuting all the diagonal entries of an element of T. As a result, we see that

$$\widehat{U}(n) \cong \{\chi(m_1, \ldots m_n) \in \widehat{T} \,|\, m_1 \geq m_2 \geq \cdots \geq m_n\}.$$

Before we return to complex semisimple Lie groups, there is another characterization of the highest weight, due to Cartan and Weyl, which will serve as motivation for our study of minimal K-types (see [Vog85, Proposition 1.15].

Proposition 2.4.10. *Let $\pi \in \widehat{K}$ with highest weight χ. Let $\langle , \rangle$ be a W-invariant positive definite inner product on $\widehat{\mathfrak{t}}$. Then, if λ is another weight of π,*

$$\langle \lambda, \lambda \rangle \leq \langle \chi, \chi \rangle,$$

where equality holds if and only if $\lambda = w \cdot \chi$ for some $w \in W$.

2.5 The Mackey Analogy Revisited

In this section we will use the Cartan-Weyl theory discussed above to complete Mackey's analogy, at the level of parameters, for connected complex semisimple Lie groups. The results presented in this section are due to Higson, and can be found in [Hig07, Section 2.3].

The extension of Mackey's analogy from generic representations of G and G_0 to *all* representations follows fairly easily from the following fact (see [Hig07, Lemma 2.2].

Proposition 2.5.1. *Let G be a connected complex semisimple Lie group. For all $\varphi \in \widehat{\mathfrak{g}/\mathfrak{k}}$, the isotropy subgroup K_φ is connected.*

As a result, we can easily classify all the irreducible unitary representations of the various isotropy subgroups using the Cartan-Weyl theory. This allows us to characterize the dual of G_0 in a very convenient way.

Proposition 2.5.2. *Let G be a connected complex semisimple Lie group with Iwasawa decomposition $G = KAN$, Weyl group W, and let M be the centralizer of A in K. Let $G_0 = K \times \mathfrak{g}/\mathfrak{k}$ be the associated semidirect product group. Identify the unitary characters of $\mathfrak{a}$ with the balanced characters of $\mathfrak{g}/\mathfrak{k}$. For each pair*

$$(\sigma, \varphi) \in \widehat{M} \times \widehat{\mathfrak{a}},$$

fix $\tau_\sigma \in \widehat{K}_\varphi$ with highest weight σ. The correspondence

$$(\sigma, \varphi) \mapsto Ind_{K_\varphi \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0} \tau_\sigma \otimes \varphi$$

determines a bijection

$$(\widehat{M} \times \widehat{\mathfrak{a}})/W \cong \widehat{G}_0.$$

Proof. By Proposition 2.5.1, for each balanced character φ of $\mathfrak{g}/\mathfrak{k}$ the isotropy subgroup K_φ is connected. In addition $M \subseteq K_\varphi$ as a maximal torus, and the Weyl group of K_φ is W_φ, the isotropy subgroup of φ in W. The Cartan-Weyl theory from Section 2.4 then implies

$$\widehat{K}_\varphi \cong \widehat{M}/W_\varphi.$$

Combining this with the Mackey machine yields

$$\widehat{G}_0 = \widehat{K \ltimes \mathfrak{g}/\mathfrak{k}} \cong \bigsqcup_{\varphi \in \widehat{\mathfrak{a}}/W} \widehat{M}/W_\varphi \cong (\widehat{M} \times \widehat{\mathfrak{a}})/W. \quad \blacksquare$$

As we have already noted, we may identify $\widehat{\mathfrak{a}}$ with $\widehat{A}$ via the exponential map. Thus we can complete Mackey's analogy as follows.

Theorem 2.5.3. *Let G be a connected complex semisimple Lie, and G_0 the associated semidirect product group. Identifying $\widehat{A}$ with $\widehat{\mathfrak{a}}$ gives rise to a bijection between $\widehat{G}$ and $\widehat{G}_0$.*

Proof. We have

$$\widehat{G} \cong (\widehat{M} \times \widehat{A})/W \cong (\widehat{M} \times \widehat{\mathfrak{a}})/W \cong \widehat{G}_0. \quad \blacksquare$$

Note also as motivation for what follows, we can write

$$\widehat{G} \cong (\widehat{M} \times \widehat{A})/W \cong \bigsqcup_{\delta \in \widehat{M}/W} \widehat{A}/W_\delta,$$

where W_δ is the isotropy subgroup of δ in W. Similarly, we have

$$\widehat{G}_0 \cong (\widehat{M} \times \widehat{\mathfrak{a}})/W \cong \bigsqcup_{\delta \in \widehat{M}/W} \widehat{\mathfrak{a}}/W_\delta.$$

While the bijection between $\widehat{G}$ and $\widehat{G}_0$ is not a homeomorphism, we see that on each strata the correspondence

$$\widehat{A}/W_\delta \cong \widehat{\mathfrak{a}}/W_\delta$$

is, in fact, a homeomorphism. In addition, since the representations of K are parametrized by $\widehat{M}/W$, we see that the bijection preserves minimal K-types.

Chapter 3

Classifying Representations by Minimal K-types

The Mackey analogy is particularly tractable in the case of a complex semisimple Lie group because both $\widehat{G}$ and $\widehat{G}_0$ are relatively simple to describe. In the case of G_0, this is due to the fact that all isotropy subgroups K_φ are connected. The accessibility of $\widehat{G}$ is due to the existence of only one (up to conjugacy) cuspidal parabolic subgroup of G, and that the principal series representations induced from it are all irreducible, and exhaust the reduced dual.

In this dissertation we will be interested in the Mackey analogy for the group $G = \mathrm{SL}(n, \mathbb{R})$. In this case (and for real semisimple groups, in general), both $\widehat{G}$ and $\widehat{G}_0$ are considerably more complicated. The difficulty in describing $\widehat{G}_0$, as we will see in Chapter 6, comes down to the fact that the isotropy subgroups are no longer connected.

The main difficulty in describing $\widehat{G}$ for $G = \mathrm{SL}(n, \mathbb{R})$, or more general semisimple Lie groups for that matter, comes down to the fact the G may have many cuspidal parabolic subgroups. This complicates the computation of $\widehat{G}$, as for every $\pi \in \widehat{G}$, there is a cuspidal parabolic subgroup $P = M_p A_p N_p \subseteq G$ such that π is equivalent to a constituent of some P-series representation (see [KZ82, p.390], for example). That is, there is a $\sigma \in \widehat{M_p}$ and $\varphi \in \widehat{A}_p$ such that $\pi \in \mathrm{Ind}_{M_p A_p N_p}^{G}(\sigma \otimes \varphi \otimes 1)$. Further complicating matters is the fact that these general P-series representations may be reducible. So, in general, the method for determining $\widehat{G}$ is to find all cuspidal parabolic subgroups $P \subseteq G$, calculate the corresponding P-series for each, and analyze all their irreducible constituents.

In this Chapter, we will describe an alternate method due to Vogan (see [Vog77], [Vog79], [Vog81], and [Vog85]) for describing $\widehat{G}$, by classifying all irreducible representations of G with a given minimal K-type. The main idea in this approach is to use the maximal compact subgroup $K \subseteq G$ to understand representations of G, in much the same way we used the maximal torus $T \subseteq K$ in Section 2.4 to understand representations of K.

3.1 Minimal K-types

Let G be a connected semisimple Lie group with maximal compact subgroup K. As discussed above, in this section we will analyze representations of G via their restrictions to K, in a similar manner to analyzing representations of K via their restrictions to a maximal torus T. We begin with the following definition.

Definition 3.1.1. Let $\pi \in \widehat{G}$, and $\tau \in \widehat{K}$. When π is restricted to K, it will decompose into a direct sum of irreducible representations of K. We call τ a *K-type* of π if τ occurs (with nonzero multiplicity) in the decomposition of $\pi|_K$.

The Cartan-Weyl theory described in Section 2.4 tells us that each representation of K has a distinguished weight, namely the highest one. In a similar manner, for each representation of G, we would like to find a distinguished K-type. This suggests imposing an order structure on $\widehat{K}$, and finding an extreme K-type. Since representations of K are indexed by their highest weights, it seems only natural to carry over the lexicographic ordering of $\widehat{\mathfrak{t}}$. Unfortunately, different orderings of $\widehat{\mathfrak{t}}$ give rise to very different orderings of $\widehat{K}$, which further complicates the description of $\widehat{G}$. The next logical step is to use the lengths of highest weights, much like Proposition 2.4.10. However, Vogan uses a slight modification (see [Vog85, p.277], for example). To begin, we need some notation. Let $\Delta(\mathfrak{k}, \mathfrak{t})$ be the set of roots of $\mathfrak{t}$ in $\mathfrak{k}$ – that is, the set on nonzero weights of T in the adjoint representation of K on $\mathfrak{k}$.

Definition 3.1.2. Let $\Delta^+ = \Delta^+(\mathfrak{k}, \mathfrak{t})$ denote the positive roots (in the chosen ordering of $\widehat{\mathfrak{t}}$). Then, set

$$\rho_c = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha,$$

the half-sum of the positive roots.

Now, let $\langle , \rangle$ denote the Killing form on $\mathfrak{g}$, which can be viewed as a W-invariant inner product on $\widehat{\mathfrak{t}}$ (after identifying $\mathfrak{t}$ and $\widehat{\mathfrak{t}}$).

Definition 3.1.3. *(Vogan, [Vog77])* Let $\tau \in \widehat{K}$ with highest weight χ. Then define the norm of τ by

$$\|\tau\| = \langle \chi + \rho_c, \chi + \rho_c \rangle.$$

In analogy with the Cartan-Weyl theory, for a given representation of G, we want to look for an extreme K-type.

Definition 3.1.4. *(Vogan, [Vog77])* Let $\pi \in \widehat{G}$ and τ a K-type of π. Then τ is a called a *minimal K-type* of π if its norm is minimal among all K-types of π.

A representation $\pi \in \widehat{G}$ may not have a unique minimal K-type. Two representations, τ_1 and $\tau_2 \in \widehat{K}$, are called *associate* if there is a parabolic subgroup $P = MAN$ of G and a discrete series representation $\delta \in \widehat{G}$, such that

$$\operatorname{Ind}_{M \cap K}^{K} \delta$$

contains both τ_1 and τ_2 as minimal K-types.

Definition 3.1.5. *(Vogan, [Vog85])* Let $\mathscr{C}(\tau)$ denote the *associate class* of τ*:*

$$\mathscr{C}(\tau) = \{\tau' \in \widehat{K} \,|\, \tau' \textit{ is associate to } \tau\}.$$

While we certainly cannot classify a representation of G solely on the basis of its minimal K-types, we do have the following very fundamental result, analogous to Proposition 2.4.7.

Proposition 3.1.6. *([Vog79, Theorem 1.2]) Let $\pi \in \widehat{G}$ with τ as a minimal K-type. Then τ has multiplicity one in $\pi|_K$.*

We conclude this section with another result due to Vogan, which gives us a picture of a representation of G in terms of its K-types.

Proposition 3.1.7. *([Vog85, Theorem 3.6]) Let $\tau_1, \tau_2 \in \widehat{K}$. Then either $\mathscr{C}(\tau_1) = \mathscr{C}(\tau_2)$ or $\mathscr{C}(\tau_1) \cap \mathscr{C}(\tau_2) = \emptyset$. The set $\mathscr{C} = \mathscr{C}(\tau)$ has 2^l elements, for some l. There is a group $R = R(\mathscr{C}) \cong (\mathbb{Z}/2\mathbb{Z})^l$, which acts simply transitively on $\mathscr{C}$. Let $\pi \in \widehat{G}$ with minimal K-type τ. Then there is a subgroup $R_\pi \subseteq R$, such that the set of minimal K-types of π is simply the orbit of τ under R_π. Thus, the set of minimal K-types of π has 2^k elements, for some $k \leq l$.*

3.2 Semisimple Lie Groups

We now turn our attention to classifying all irreducible tempered representations of G by their minimal K-types. Let G be a connected semisimple Lie group with finite center, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition. Then $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$, and thus any Lie subalgebra of $\mathfrak{p}$ is abelian (see [Kna86, Chapter 1]. Let $\mathfrak{a}$ be a such a subalgebra of $\mathfrak{p}$, and $A = \exp(\mathfrak{a})$ the corresponding closed vector subgroup of G.

Definition 3.2.1. *(Vogan, [Vog85, 4.2]* Set

$$L = Z_G(A),$$

$$\mathfrak{l} = Lie(\mathfrak{l}) = Z_{\mathfrak{g}}(\mathfrak{a}),$$

$$\tilde{\mathfrak{a}} = Z(\mathfrak{l}) \cap \mathfrak{p}$$

We will call $\mathfrak{a}$ a *special subalgebra* of $\mathfrak{p}$ whenever $\tilde{\mathfrak{a}} = \mathfrak{a}$, and A a *special vector subgroup* of G.

Definition 3.2.2. *(Vogan, [Vog85, 4.3a])* Let A be a special vector subgroup of G, let L be as in Definition 3.2.1, and define

$$\mathfrak{m} = \mathfrak{a}^{\perp} \cap \mathfrak{l},$$

$$M = \text{the group generated by } \exp(\mathfrak{m}) \text{ and } L \cap K.$$

We are now prepared to catalog representations of G by their minimal K-types. Let $\mathscr{C} \in \widehat{K}$ be an associate class of representations. The main idea of Vogan (see [Vog85, Section 4]) is to attach a parabolic subgroup $P = P(\mathscr{C})$ to the associate class $\mathscr{C} \subseteq \widehat{K}$ such that every representation of G with a minimal K-type in $\mathscr{C}$ occurs in a representation induced from P. Vogan begins by constructing a special vector subgroup, A, attached to an associate class of representations of K. While the group A is constructed in a concrete way [Vog81, p.270], it is quite involved and will not be discussed here. Instead, we will find an indirect way to characterize

A, which will be sufficient for our needs. The following result is the first step in constructing representations with a given minimal K-type.

Proposition 3.2.3. *(Vogan, [Vog85, Proposition 4.6]) Let $\mathscr{C} \subseteq \widehat{K}$ be an associate class of representations of K, and let $M = M(\mathscr{C})$ denote the group constructed from $A(\mathscr{C})$ in Definition 3.2.2. Then there is a $\delta = \delta(\mathscr{C}) \in E_2 M$ such that $\mathscr{C}$ is the set of a minimal K-types of*

$$\mathrm{Ind}_{M\cap K}^{K}\,(\delta|_{M\cap K}).$$

The representation δ is determined up to conjugation under the normalizer of A in K.

If A is a special vector subgroup of G and M is constructed from A as in Definition 3.2.2, then we can find a nilpotent subgroup $N \subseteq G$ (as described in the previous chapter) such that $P = MAN$ is a parabolic subgroup of G. Then, we can define the P-series representations as follows.

Definition 3.2.4. Let $P = MAN$ be a parabolic subgroup of G, with $\delta \in E_2 M$ and $\varphi \in \widehat{A}$. Then define

$$\pi(\delta \otimes \varphi) = \mathrm{Ind}_{MAN}^{G}\,(\delta \otimes \varphi \otimes 1),$$

the *P-series* representation with parameter (δ, φ).

Note that these representations are not necessarily irreducible. The minimal K-types of these representations are then determined by the following result.

Proposition 3.2.5. *(Vogan, [Vog79, p.34]) Let $P = MAN$ be a parabolic subgroup of G, and $\pi(\delta \otimes \varphi)$ a P-series representation. Then*

$$\pi(\delta \otimes \varphi)|_K \cong \mathrm{Ind}_{M\cap K}^{K}(\delta|_{M\cap K}).$$

This, in combination with Proposition 3.2.3, gives the following result.

Proposition 3.2.6. *(Vogan, [Vog85, Proposition 4.11] Let $\mathscr{C} \subseteq K$ be an associate class of representations of K, $A = A(\mathscr{C})$ a corresponding special vector subgroup of G, and $\delta = \delta(\mathscr{C})$ the representation of $M(\mathscr{C})$ associated to $\mathscr{C}$ by Proposition 3.2.3. Then there is a continuous family*

$$\{\pi(\varphi) = \pi(\delta \otimes \varphi) \mid \varphi \in \widehat{A} \cong \widehat{\mathfrak{a}}\}$$

of (possibly reducible) representations of G, each with $\mathscr{C}$ as its set of minimal K-types, and each of these minimal K-types occurs with multiplicity one. For almost all φ, $\pi(\varphi)$ is irreducible.

Now, fix $\tau \in \widehat{K}$, let $\mathscr{C} = \mathscr{C}(\tau)$, and let $\{\pi(\tau, \varphi)\}$ be the family of representations built from $\mathscr{C}$ as in Proposition 3.2.6. Let $\mathcal{H}$ denote the Hilbert space of $\pi(\varphi)$. Let $\mathcal{H}_1$ be the intersection of all the closed, G-invariant subspaces of H containing the

K-type τ. Note that since τ has multiplicity one in π, it also has multiplicity one in $\mathcal{H}_1$. Let $\mathcal{H}_0$ be the largest closed, G-invariant subspace of $\mathcal{H}_1$ not containing τ.

Definition 3.2.7. Define $\overline{\pi}(\tau, \varphi)$ to be the subquotient of $\pi(\tau, \varphi)$ on

$$\overline{\mathcal{H}} = \overline{\mathcal{H}}_{\tau,\varphi} = \mathcal{H}_1/\mathcal{H}_0.$$

Then $\overline{\pi}(\tau, \varphi)$ is an irreducible representation of G containing the K-type τ as a minimal K-type, with multiplicity one.

Using this construction, Vogan proved a version of the Langlands classification of $\widehat{G}$ – that any irreducible representation of G can be realized in a specific way inside a standard representation.

Proposition 3.2.8. *(Vogan, [Vog85, Theorem 5.2]) Let π be an irreducible representation of G containing τ as a minimal K-type. Then there is a $\varphi \in \widehat{A}$ such that π is equivalent to $\overline{\pi}(\tau, \varphi)$.*

Now, in order to use Theorem 3.2.8 to establish a classification of $\widehat{G}$, we need to describe the equivalences among all of the $\overline{\pi}(\tau, \varphi)$.

Definition 3.2.9. *(Vogan, [Vog85, 5.3]* Let $\mathscr{C} \subseteq K$ be an associate class of representations of K, $A = A(\mathscr{C})$ a corresponding special vector subgroup of G, and $\delta = \delta(\mathscr{C})$ the representation of $M(\mathscr{C})$ associated to $\mathscr{C}$ by Proposition 3.2.3. Define

$$N(\mathscr{C}) = \text{the normalizer of the triple } (A, M, \delta) \text{ in } K,$$

and the *Weyl group of* $\mathscr{C}$,

$$W(\mathscr{C}) = N(\mathscr{C})/(M \cap K).$$

Remark. $W(\mathscr{C})$ is finite and acts on both A and $\widehat{A}$ (as well as on $\mathfrak{a}$ and $\widehat{\mathfrak{a}}$).

Using the above, Vogan details the equivalences between these representations as follows.

Proposition 3.2.10. *(Vogan, [Vog85, Proposition 5.4]) The representations* $\overline{\pi}(\tau, \varphi)$ *and* $\overline{\pi}(\tau, \varphi')$ *are equivalent if and only if* φ_1 *and* φ_2 *lie in the same* $W(\mathscr{C})$ *orbit.*

Propositions 3.2.8 and 3.2.10 give us a fairly explicit description of those representations of G with a specified minimal K-type. We must now describe the precise set of minimal K-types for each of the $\overline{\pi}(\tau, \varphi)$ in order to complete the classification of $\widehat{G}$. This brings us back to the R group of Proposition 3.1.7.

Definition 3.2.11. *(Vogan, [Vog85, 5.6]* Let $\mathscr{C} \subseteq \widehat{K}$ be an associate class of representations of K, and $W(\mathscr{C})$ as in Definition 3.2.9. Define $R(\mathscr{C})$ to be the group of homomorphisms from $W(\mathscr{C})$ into the set $\{\pm 1\}$. In addition, set

$$W(\mathscr{C}, \varphi) = \{w \in W(\mathscr{C}) \,|\, w \cdot \varphi = \varphi\},$$

$$R(\mathscr{C}, \varphi) = \{\rho \in R(\mathscr{C}) \,|\, \forall\, w \in W(\mathscr{C}, \varphi)\ \rho(w) = 1\}.$$

It turns out that for most φ, $W(\mathscr{C},\varphi) = 1$ and $R(\mathscr{C},\varphi) = R(\mathscr{C})$.

As discussed in Proposition 3.1.7, there is a group $R_{\overline{\pi}(\tau,\varphi)}$ such that the set of minimal K-types of $\overline{\pi}(\tau,\varphi)$ is precisely the orbit of τ under $R_{\overline{\pi}(\tau,\varphi)}$. The following result allows us to find $R_{\overline{\pi}(\tau,\varphi)}$.

Proposition 3.2.12. *(Vogan, [Vog85, Proposition 5.9]) The set of minimal K-types of $\overline{\pi}(\tau,\varphi)$ is*

$$R_{\overline{\pi}(\tau,\varphi)} \cdot \tau \subseteq \mathscr{C}(\tau).$$

Now, the results of this section may be used to give the following classification of $\widehat{G}$ by minimal K-types.

Theorem 3.2.13. *(Vogan, [Vog85, Theorem 5.12]) To each associate class $\mathscr{C} \subseteq \widehat{K}$ of representations of K, attach a special vector subgroup $A(\mathscr{C})$, and a series of representations $\{\pi(\varphi) \,|\, \varphi \in \widehat{A}(\mathscr{C})\}$ (Propositions 3.2.1, 3.2.6). Then the correspondence which associates to each pair $(\mathscr{C},\varphi)$, where $\mathscr{C} \subseteq \widehat{K}$ is an associate class of representations of K and $\varphi \in \widehat{A}(\mathscr{C})/W(\mathscr{C})$, the set*

$$\{\overline{\pi}(\tau,\varphi) \,|\, \tau \in \mathscr{C}\} \subseteq \widehat{G},$$

is a one-to-finite correspondence onto $\widehat{G}$. The set $\{\overline{\pi}(\tau,\varphi) \,|\, \tau \in \mathscr{C}\}$ has

$$|R(\mathscr{C})/R(\mathscr{C},\varphi)|$$

elements, each of which has a single orbit of $R(\mathscr{C}, \varphi)$ *in* $\mathscr{C}$ *as its set of minimal* K*-types.*

Now we see that the representations of G with a given minimal K-type are (essentially) parametrized by the associated $\widehat{A}/W$. As a result, identifying the special vector subgroup attached to each associate class is the first step in this classification procedure. As discussed above, there is an explicit way to construct the special vector subgroup, but it is quite involved, and will not be discussed here. However, the following gives us an alternative method for characterizing $A(\mathscr{C})$.

Theorem 3.2.14. *(Vogan, [Vog85, Proposition 4.12]) The correspondence which sends an associate class* $\mathscr{C} \subseteq \widehat{K}$ *of representations of* K *to the* K*-conjugacy class of the triple* $(A(\mathscr{C}), M(\mathscr{C}), \delta(\mathscr{C}))$ *is a bijection.*

Thus, for a given associate class $\mathscr{C} \subseteq \widehat{K}$, the special vector subgroup $A(\mathscr{C})$ is uniquely determined by the requirements of Proposition 3.2.3. This Theorem is the main result which will allow us to construct all representations of a given minimal K-type in what follows.

Chapter 4

Classifying $\widehat{G}_0$ by minimal K-types

Let G be a connected semisimple Lie group with finite center and maximal compact subgroup K. Let $G_0 = K \ltimes \mathfrak{g}/\mathfrak{k}$ the associated semidirect product group. The goal of this dissertation is to construct a reasonably natural bijection between $\widehat{G}$ and $\widehat{G}_0$. On the face of it, there does not seem to be an obvious way to compare the standard picture of $\widehat{G}$ (obtained via parabolic induction) and that of $\widehat{G}_0$ (obtained via the Mackey machine). In order to make this comparison, we would like to be able to describe each dual using similar techniques.

Having summarized Vogan's approach to classifying $\widehat{G}$ by minimal K-types in Chapter 3, in this chapter we begin our own contributions by suggesting an analogous method for classifying $\widehat{G}_0$ by minimal K-types. While many of the following statements will remain conjectural for general semisimple Lie groups, we will be able to prove them for the group $\mathrm{SL}(n, \mathbb{R})$, as well as for complex semisimple Lie groups.

The main organizational theme of this dissertation is that if G is a connected semisimple Lie group with finite there should be a very natural bijection between $\widehat{G}$ and $\widehat{G}_0$, and this bijection is made precise in Conjecture 4.0.7.

Before stating this conjecture, we must establish some notation. We begin by defining the set which will parametrize representations of a given minimal K-type, much like the special vector subgroup A did in Section 3.2.

Definition 4.0.1. Let $\mathscr{C} \subseteq K$ be an associate class of representations of K, and $A = A(\mathscr{C})$ a corresponding special vector subgroup of G. Define

$$\mathfrak{a}(\mathscr{C}) = Lie(A(\mathscr{C})),$$

be the corresponding special subalgebra of $\mathfrak{g}$.

Note that Vogan's explicit construction [Vog81, p.270] will not be used here either. Instead, we will be able to find $\mathfrak{a}$ through the characterization of Theorem 3.2.14.

Definition 4.0.2. Let $\mathscr{C} \subseteq \widehat{K}$ be an associate class of representations of K, and $M(\mathscr{C}) \subseteq G$ the group associated to $\mathscr{C}$ by Definition 3.2.2. Set

$$K(\mathscr{C}) = M(\mathscr{C}) \cap K.$$

We now turn our attention to constructing representations of G_0 with a specified minimal K-type. To do so, we will need a preliminary conjecture, which we will prove for $\mathrm{SL}(n, \mathbb{R})$.

Conjecture 4.0.3. *Let $\mathscr{C} \subseteq \widehat{K}$ be an associate class of representations of K, and let $K(\mathscr{C})$ denote the group constructed in Definition 4.0.2. Then there is a $\delta_0 = \delta_0(\mathscr{C}) \in \widehat{K}(\mathscr{C})$ such that $\mathscr{C}$ is the set of a minimal K-types of*

$$\mathrm{Ind}_{K(\mathscr{C})}^{K} \delta_0.$$

The representation δ_0 is determined up to conjugation under the normalizer of A in K.

Now, we will define a family of representations of G_0 in analogy with Definition 3.2.4.

Definition 4.0.4. Let $\mathscr{C} \subseteq \widehat{K}$ be an associate class of representations of K, and let $\mathfrak{a}(\mathscr{C})$ and $K(\mathscr{C})$ be as in Definitions 4.0.1 and 4.0.2. In addition, let $\delta_0 \in \widehat{K}(\mathscr{C})$ be as in Proposition 4.0.3, and $\varphi \in \widehat{\mathfrak{a}}(\mathscr{C})$, viewed as a balanced character of $\mathfrak{g}/\mathfrak{k}$. Then define

$$\pi_0(\delta_0, \varphi) = \mathrm{Ind}_{K(\mathscr{C}) \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0} (\delta_0 \otimes \varphi).$$

In analogy with Proposition 3.2.6, we obtain a continuous family of representations, with prescribed minimal K-types, once we establish the validity of Conjecture 4.0.3.

Conjecture 4.0.5. *Let $\mathscr{C} \subseteq \widehat{K}$ be an associate class of representations of K, $\mathfrak{a} = \mathfrak{a}(\mathscr{C})$ a corresponding special subalgebra of $\mathfrak{g}$, and $\delta_0 = \delta_0(\mathscr{C})$ the representation of $K(\mathscr{C})$ associated to $\mathscr{C}$ by Conjecture 4.0.3. Then there is a continuous family*

$$\{\pi_0(\varphi) = \pi_0(\delta_0 \otimes \varphi) \mid \varphi \in \widehat{\mathfrak{a}}\}$$

of (possibly reducible) representations of G_0, each with $\mathscr{C}$ as its set of minimal K-types, and each of these minimal K-types occurs with multiplicity one.

Then, as in Definition 3.2.7, we can isolate the irreducible constituent of these representations containing a minimal K-type.

Definition 4.0.6. Fix $\tau \in \widehat{K}$, and let $\mathscr{C} = \mathscr{C}(\tau)$. Let $\{\pi_0(\varphi)\}$ be the family of representations built from $\mathscr{C}$ by Conjecture 4.0.5. Define $\overline{\pi}_0(\tau, \varphi)$ to be the irreducible subquotient of $\pi_0(\tau, \varphi)$ containing τ as a minimal K-type (with multiplicity one), following the construction from Definition 3.2.7.

This brings us to our main conjecture, that there should be a very natural bijection between $\widehat{G}$ and $\widehat{G}_0$, and we make this bijection explicit.

Conjecture 4.0.7. *Let G be a connected semisimple Lie group with finite center and maximal compact subgroup K. Let $G_0 = K \ltimes \mathfrak{g}/\mathfrak{k}$ be the associated semidirect product group. Let $\tau \in \widehat{K}$, and let $\overline{\pi}(\tau, \varphi)$ and $\overline{\pi}_0(\tau, \varphi)$ be the representations of G and G_0 from Definitions 3.2.7 and 4.0.6, respectively. The correspondence which* *associates to the representation* $\overline{\pi}(\tau, \varphi) \in \widehat{G}$ *the representation* $\overline{\pi}_0(\tau, \varphi) \in \widehat{G}_0$ *is a well-defined bijection preserving minimal* K*-types.*

Note that, by construction, for each K-type τ, the bijection is actually a homeomorphism from the set of those representations in $\widehat{G}$ with minimal K-types in $\mathscr{C}(\tau)$ to those representations in $\widehat{G}_0$ with minimal K-types in $\mathscr{C}(\tau)$.

The main result of this dissertation is Theorem 4.0.11, which establishes a this conjecture for $G = \mathrm{SL}(n, \mathbb{R})$ (or a connected complex semisimple Lie group, for that matter). With this in mind, let us take G to be either a complex semisimple Lie group, or $\mathrm{SL}(n, \mathbb{R})$ for the remainder of this chapter. We now briefly outline the proof which will be presented in the following chapters.

In order to find the δ_0 from Conjecture 4.0.3, we must look ahead to Chapter 6. Let $\tau \in \widehat{K}$. In the case of a connected complex semisimple Lie group G, we will see in Section 6.1 that the subgroup M associated to τ is just the maximal torus of K, and the corresponding δ is simply the highest weight of τ (see Lemma 6.1.2). On the other hand, from definition 4.0.2, above, we see that $K(\tau) = M$ and the corresponding $\delta_0 = \delta$, as well.

In the case of $G = \mathrm{SL}(n, \mathbb{R})$, the construction of δ_0 is a little more involved, but once again, there is a straightforward way to build δ_0 once we have δ in hand. In this case, the subgroup $M(\tau)$ associated to the K-type τ is of the form (see 7.1.1)

$$M(\tau) \cong [\mathrm{SL}^{\pm}(2, \mathbb{R})]^p \times (\mathbb{Z}/2\mathbb{Z})^q.$$

where $\mathrm{SL}^{\pm}(2,\mathbb{R})$ is the group of matrices with determinant ± 1. The situation here is complicated by the disconnectedness of M. Vogan defines a representation $\delta^{\#}$ on

$$M(\tau)^{\#} \cong [\mathrm{SL}(2,\mathbb{R})]^{p} \times (\mathbb{Z}/2\mathbb{Z})^{q},$$

to be essentially the discrete series representation of $[\mathrm{SL}(2,\mathbb{R})]^{p}$ with minimal highest weight equal to the highest weight of τ (the precise construction from [Vog85, p.282] will be summarized in Section 7.1). He then takes δ to be the representation induced from $\delta^{\#}$ up to $M(\tau)$.

We will construct $\delta_0^{\#}$ in a similar manner. Here, we take $\delta_0^{\#}$ essentially to be the representation on

$$K(\tau)^{\#} \cong [\mathrm{SO}(2)]^{p} \times (\mathbb{Z}/2\mathbb{Z})^{q},$$

to be essentially the representation of $[\mathrm{SO}(2)]^{p}$ with highest weight equal to the highest weight of τ (the precise construction from will be summarized in Section 7.1. We then take δ to be the representation induced from $\delta^{\#}$ up to $K(\tau)$.

Thus, we can explicitly construct δ_0 in a very similar manner to that of δ. When M is compact, they are, in fact, equal. However, when M is not compact, δ_0 is constructed in such a way as to be a compact analog of δ.

Just as in the case of semisimple groups, we would like to be able to realize all unitary representations in a certain way inside these representations.

Theorem 4.0.8. *Let π be an irreducible representation of G_0 containing τ as a* *minimal K-type. Then there is a $\varphi \in \widehat{\mathfrak{a}}$ such that π is equivalent to $\overline{\pi_0}(\tau, \varphi)$.*

We now consider the equivalences among these representations.

Theorem 4.0.9. *Let $W(\mathscr{C})$ be as in Definition 3.2.9. Then the representations $\overline{\pi}_0(\tau, \varphi_1)$ and $\overline{\pi}_0(\tau, \varphi_2)$ are equivalent if and only if φ_1 and φ_2 lie in the same $W(\mathscr{C})$ orbit.*

Finally, if we let $R(\mathscr{C})$ and $R(\mathscr{C}, \varphi)$ be as in Definition 3.2.11, the classification of $\widehat{G}_0$ by minimal K-types would be completed with the following.

Theorem 4.0.10. *To each associate class $\mathscr{C} \subseteq \widehat{K}$ of representations of K, attach a special subalgebra $\mathfrak{a}(\mathscr{C})$, and a series of representations $\{\pi_0(\varphi) \mid \varphi \in \widehat{\mathfrak{a}}(\mathscr{C})\}$ (Propositions 4.0.1, 4.0.5). Then the correspondence which associates to each pair $(\mathscr{C}, \varphi)$, where $\varphi \in \widehat{\mathfrak{a}}(\mathscr{C})/W(\mathscr{C})$, the set*

$$\{\overline{\pi}_0(\tau, \varphi) \mid \tau \in \mathscr{C}\} \subseteq \widehat{G}_0,$$

is a one-to-finite correspondence onto $\widehat{G}_0$. The set $\{\overline{\pi}_0(\tau, \varphi) \mid \tau \in \mathscr{C}\}$ has

$$|R(\mathscr{C})/R(\mathscr{C}, \varphi)|$$

elements, each of which has a single orbit of $R(\mathscr{C}, \varphi)$ in $\mathscr{C}$ as its set of minimal K-types.

Comparing Theorem 3.2.13 and Theorem 4.0.10, we see that identifying $\widehat{A}(\mathscr{C})$

with $\widehat{\mathfrak{a}}(\mathscr{C})$ gives us the desired bijection of $\widehat{G}$ and $\widehat{G}_0$.

Theorem 4.0.11. *The correspondence which associates to the representation* $\overline{\pi}(\tau, \varphi) \in \widehat{G}$ *the representation* $\overline{\pi}_0(\tau, \varphi) \in \widehat{G}_0$ *is a well-defined bijection preserving minimal* K*-types.*

Chapter 6 is devoted to proving Theorem 4.0.11 for connected complex semisimple Lie groups and for the groups $\mathrm{SL}(2, \mathbb{R})$ and $\mathrm{SL}(3, \mathbb{R})$. In chapter 7 we build on those results, and prove the theorem in the more general setting of $\mathrm{SL}(n, \mathbb{R})$.

It should be noted that the way in which we approach Theorem 4.0.11 is to describe the representations of G_0 via those of G, which in some sense runs counter to Mackey's approach. It was his goal to develop an understanding of the representations of G by using his own machinery for G_0. However, there is one interesting consequence of our work on $\mathrm{SL}(n, \mathbb{R})$ which does fall more in line with the spirit of Mackey's idea. One of the more difficult aspects of understanding $\widehat{G}$ are the ideas surrounding the R-group and reducibility of various induced representations. Theorem 4.0.11 allows us to understand these issues purely in terms of the representation theory of G_0, where the representations of concern are all irreducible. Most importantly, the calculations in the following provide evidence to support the broader claim of Conjecture 4.0.7.

Chapter 5

Branching Rules

In this chapter we summarize the various branching rules and related results necessary to classify representations of the semidirect product groups associated to $\mathrm{SL}(n,\mathbb{R})$ by minimal K-types. We begin by examining the group $\mathrm{SO}(n)$, and we will then extend the results to $\mathrm{O}(n)$.

5.1 The Group $\mathrm{SO}(n)$

Since $\mathrm{SO}(n)$ is a compact group, its irreducible representations are determined by their highest weights. The first step in describing $\widehat{\mathrm{SO}(n)}$, therefore, is to choose a maximal torus T inside $\mathrm{SO}(n)$. Let R_θ be the 2×2 matrix of the rotation of the plane by an angle θ:

$$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Then, the groups

$$T_{2k} = \left\{ \begin{pmatrix} R_{\theta_1} & & \\ & \ddots & \\ & & R_{\theta_k} \end{pmatrix} \;\middle|\; \theta_1, \ldots, \theta_k \in \mathbb{R} \right\},$$

and,

$$T_{2k+1} = \left\{ \begin{pmatrix} R_{\theta_1} & & & \\ & \ddots & & \\ & & R_{\theta_k} & \\ & & & 1 \end{pmatrix} \;\middle|\; \theta_1, \ldots, \theta_k \in \mathbb{R} \right\},$$

are maximal tori for $\mathrm{SO}(2k)$ and $\mathrm{SO}(2k+1)$, respectively (see [BtD85, p.185], for example). As a result, the dual of $\mathrm{SO}(n)$ has a slightly different characterization depending on whether n is even or odd. In fact (see [BtD85, p.272], for example),

$$\widehat{\mathrm{SO}(2k)} \cong \{(a_1, \ldots, a_k) \in \mathbb{Z}^k \mid a_1 \geq a_2 \geq \cdots \geq a_{k-1} \geq |a_k|\},$$

whereas,

$$\widehat{\mathrm{SO}(2k+1)} \cong \{(b_1, \ldots, b_k) \in \mathbb{Z}^k \mid b_1 \geq b_2 \geq \cdots \geq b_{k-1} \geq b_k \geq 0\}.$$

Let $\tau_{(a_1,\ldots,a_k)}$ denote the representation of $\mathrm{SO}(2k)$ whose highest weight acts on T_{2k} by

$$\begin{pmatrix} R_{\theta_1} & & \\ & \ddots & \\ & & R_{\theta_k} \end{pmatrix} \longmapsto \prod_{j=1}^{k} e^{ia_j\theta_j},$$

and similarly, the the highest weight of the representation $\tau_{(b_1,\ldots,b_k)}$ acts on T_{2k+1} by

$$\begin{pmatrix} R_{\theta_1} & & & \\ & \ddots & & \\ & & R_{\theta_k} & \\ & & & 1 \end{pmatrix} \longmapsto \prod_{j=1}^{k} e^{ib_j\theta_j}$$

Now, consider $\mathrm{SO}(m)$ as a subgroup of $\mathrm{SO}(n)$, for $m < n$, in the usual way:

$$SO(m) \hookrightarrow \left\{ \begin{pmatrix} T & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \in \mathrm{SO}(n) \;\middle|\; T \in \mathrm{SO}(m) \right\}.$$

The way in which representations restrict from $\mathrm{SO}(n)$ to $\mathrm{SO}(m)$ for $m < n$ can be computed using the following branching rules [Kna02, p.570].

Proposition 5.1.1. *Let* $\tau_{(a_1,\ldots,a_k)} \in \widehat{\mathrm{SO}(2k)}$*. Then,*

$$\tau_{(a_1,\ldots,a_k)}|_{\mathrm{SO}(2k-1)} \cong \bigoplus \tau_{(b_1,\ldots,b_{k-1})},$$

where the direct sum is over all $\tau_{(b_1,\ldots,b_{k-1})} \in \widehat{\mathrm{SO}(2k-1)}$ *such that*

$$a_1 \geq b_1 \geq a_2 \geq b_2 \geq \cdots \geq a_{k-1} \geq b_{k-1} \geq |a_k|.$$

Proposition 5.1.2. *Let* $\tau_{(b_1,\ldots,b_k)} \in \widehat{\mathrm{SO}(2k+1)}$*. Then,*

$$\tau_{(b_1,\ldots,b_k)}|_{\mathrm{SO}(2k)} \cong \bigoplus \tau(a_1,\ldots,a_k),$$

where the direct sum is over all $\tau_{(a_1,\ldots,a_k)} \in \widehat{\mathrm{SO}(2k)}$ *such that*

$$b_1 \geq a_1 \geq b_2 \geq a_2 \geq \cdots \geq b_k \geq |a_k|.$$

With these branching rules established, we may now compute the minimal K-types of induced representations as a direct consequence of Frobenius reciprocity.

Theorem 5.1.3. *The minimal* K*-type of*

$$\mathrm{Ind}_{\mathrm{SO}(2k-1)}^{\mathrm{SO}(2k)} \tau_{(b_1,\ldots,b_{k-1})}$$

is $\tau_{(b_1,\ldots,b_{k-1},0)}$. ■

Theorem 5.1.4. *The minimal K-type of*

$$\operatorname{Ind}_{\mathrm{SO}(2k)}^{SO(2k+1)} \tau_{(a_1,\ldots,a_k)}$$

is $\tau_{(a_1,\ldots,a_k,0)}$. ■

5.2 The Group $\mathrm{O}(n)$

Since $\mathrm{SO}(n)$ is the semi-direct product of $\mathrm{SO}(n)$ and a two-element group, $\widehat{\mathrm{O}(n)}$ may be determined from $\widehat{\mathrm{SO}(n)}$. As in the preceding, we will need to treat the cases of even and odd dimensions separately.

To begin, note that in the case of $\mathrm{O}(2k+1)$, we actually have [BtD85, p.292]

$$\mathrm{O}(2k+1) \cong \mathrm{SO}(2k+1) \times \mathbb{Z}_2.$$

As a result,

$$\mathrm{O}\widehat{(2k+1)} \cong \mathrm{SO}\widehat{(2k+1)} \times \mathbb{Z}_2 \cong \{(b_1,\ldots,b_k,\epsilon) \mid b_1 \geq b_2 \geq \cdots \geq b_k \geq 0,\, \epsilon = \pm 1\}$$

Here the action of $\tau_{(b_1,\ldots,b_k,\epsilon)}$ is identical to that of $\tau_{(b_1,\ldots,b_k)} \in \mathrm{SO}\widehat{(2k+1)}$ when $\epsilon = 1$. When $\epsilon = -1$, the action of $\tau_{(b_1,\ldots,b_k)} \in \mathrm{SO}\widehat{(2k+1)}$ is followed by the determinant action.

On the other hand, the (non-trivial) action of $\mathbb{Z}_2$ on $\mathrm{SO}(2k)$ is given by [BtD85, p.296]

$$\tau_{(a_1,\ldots,a_k)} \mapsto \tau_{(a_1,\ldots,-a_k)}.$$

As a result, only those representations of SO$(2k)$ of the form $\tau_{(a_1,\ldots,a_{k-1},0)}$ have non-trivial isotropy, and the representations induced from $\tau_{(a_1,\ldots,a_k)}$ and $\tau_{(a_1,\ldots,-a_k)}$ in $\widehat{\mathrm{SO}(2k)}$ must be equivalent. Thus, the dual of O$(2k)$ can be described as follows,

$$\widehat{\mathrm{O}(2k)} \cong \left\{ (a_1,\ldots,a_k,\epsilon) \mid a_1 \geq a_2 \geq \cdots \geq a_k \geq 0,\ \epsilon = \left\{ \begin{array}{ll} \pm 1 & \text{if } a_k = 0 \\ 1 & \text{if } a_k \neq 0 \end{array} \right\} \right. .$$

As above, consider O(m) as a subgroup of O(n), for $m < n$, in the following way:

$$\mathrm{O}(m) \hookrightarrow \left\{ \begin{pmatrix} T & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \in \mathrm{O}(n) \;\middle|\; T \in O(m) \right\}.$$

The branching rules for the orthogonal groups are similar to those for the special orthogonal groups, with the presence of the sign ϵ being the only complication. The following results are due to King [Kin75, pp.438-441].

Proposition 5.2.1. *Let* $\tau_{(a_1,\ldots,a_k,\epsilon)} \in \widehat{\mathrm{O}(2k)}$. *Then,*

$$\tau_{(a_1,\ldots,a_k,\epsilon)}\big|_{O(2k-1)} \cong \bigoplus \tau_{(b_1,\ldots,b_{k-1},\epsilon')},$$

where the direct sum is over all $\tau_{(b_1,\ldots,b_{k-1},\epsilon')}$ *such that*

$$a_1 \geq b_1 \geq a_2 \geq \cdots \geq a_{k-1} \geq b_{k-1} \geq a_k,$$

$$\text{and } \epsilon' = \begin{cases} \epsilon & \text{if } a_k = 0, \\ \pm 1 & \text{if } a_k \neq 0. \end{cases}$$

Proposition 5.2.2. *Let* $\tau_{(a_1,\ldots,a_k,\epsilon)} \in \widehat{\mathrm{O}(2k+1)}$. *Then,*

$$\tau_{(a_1,\ldots,a_k,\epsilon)}|_{\mathrm{O}(2k)} \cong \bigoplus \tau_{(b_1,\ldots,b_k,\epsilon')},$$

where the direct sum is over all $\tau_{(b_1,\ldots,b_k,\epsilon')}$ *such that*

$$a_1 \geq b_1 \geq a_2 \geq \cdots \geq a_k \geq b_k \geq 0,$$

$$\text{and } \epsilon' = \begin{cases} \epsilon & \text{if } b_k = 0, \\ 1 & \text{if } b_k \neq 0. \end{cases}$$

With these branching rules established, we may now use Frobenius reciprocity, as in Section 5.1, to determine the minimal K-types of induced representations.

Theorem 5.2.3. *The minimal* K*-type of*

$$\mathrm{Ind}_{\mathrm{O}(2k-1)}^{\mathrm{O}(2k)} \tau_{(b_1,\ldots,b_{k-1},\epsilon)}$$

is $\tau_{(a_1,\ldots,a_{k-1},0,\epsilon)}$. ■

Theorem 5.2.4. *The minimal K-type of*

$$\mathrm{Ind}_{\mathrm{O}(2k)}^{\mathrm{O}(2k+1)} \tau_{(a_1,\ldots,a_k,\epsilon)},$$

is $\begin{cases} \tau_{(a_1,\ldots,a_k,\pm1)} & \text{if } a_k \neq 0, \\ \tau_{(a_1,\ldots,a_k,\epsilon)} & \text{if } a_k = 0. \end{cases}$ ■

Finally, one may iterate this process to obtain a method for determining the minimal K-types of further induced representations.

Theorem 5.2.5. *Let $m, n \in \mathbb{N}$, with $m < n-1$. If m is even, the minimal K-type of*

$$\mathrm{Ind}_{\mathrm{O}(m)}^{\mathrm{O}(n)} \tau_{(a_1,\ldots,a_k,\epsilon)}$$

is $\begin{cases} \tau_{(a_1,\ldots,a_k,0,\ldots,0,\pm1)} & \text{if } a_k \neq 0, \\ \tau_{(a_1,\ldots,a_k,0,\ldots,0,\epsilon)} & \text{if } a_k = 0. \end{cases}$

If m is odd, the minimal K-type of

$$\mathrm{Ind}_{\mathrm{O}(m)}^{\mathrm{O}(n)} \tau_{(b_1,\ldots,b_k,\epsilon)}$$

is $\tau_{(b_1,\ldots,b_k,0,\ldots,0,\epsilon)}$. ■

As a consequence of this theorem, we also have the following.

Theorem 5.2.6. *The minimal K-type of*

$$\operatorname{Ind}_{\mathrm{O}(2)^r}^{\mathrm{O}(n)} \tau_{(a_1,\ldots,a_r,\epsilon)}$$

is $\tau_{(a_1,\ldots,a_r,0,\ldots,0,\epsilon)}$. ■

Chapter 6

Motivating Examples Using the K-type Approach

In this chapter, we will use Vogan's approach to classifying the dual of a semisimple Lie group, as outlined in Chapter 3, to reexamine the Mackey analogy for complex semisimple Lie groups. In addition, we will study the groups $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(3,\mathbb{R})$ from this perspective. We prove our main theorem (Theorem 4.0.11) for these groups, and our calculations motivate the treatment of $\mathrm{SL}(n,\mathbb{R})$ which will follow in Chapter 7.

6.1 Complex Semisimple Lie Groups

We begin our application of Vogan's K-type approach with the case of complex semisimple Lie groups. Let G be a connected complex semisimple Lie group with maximal compact subgroup K, and let T be a maximal torus of K. Let $G_0 = K \ltimes \mathfrak{g}/\mathfrak{k}$ be the associated semidirect product group. The goal in this section is to organize both $\widehat{G}$ and $\widehat{G}_0$ by minimal K-types, in order to recover the bijection of Theorem 2.5.3.

We will use the group $G = \mathrm{SL}(n, \mathbb{C})$ as a running example in this section. Here we take $K = \mathrm{SU}(n)$, the maximal compact subgroup of $G = \mathrm{SL}(n, \mathbb{C})$, and

$$T = \left\{ \begin{pmatrix} e^{i\theta_1} & & & \\ & e^{i\theta_2} & & \\ & & \ddots & \\ & & & e^{i\theta_n} \end{pmatrix} \middle| \prod_{j=1}^{n} e^{i\theta_j} = 1 \right\}, \tag{6.1.1}$$

is a maximal torus of K.

Since K is connected for complex semisimple Lie groups, we can use Theorem 2.4.9, to give us

$$\widehat{K} \cong \widehat{T}/W,$$

where $W = N_K(T)/T$ is the Weyl group of K.

Then in our example, where $K = \mathrm{SU}(n)$, we have (see [BtD85, p.275], for example)

$$\widehat{K} = \{(a_1, a_2, \dots, a_{n-1}) \in \mathbb{Z}^{n-1} \mid a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq 0\},$$

where the highest weight of the representation $\tau_{(a_1,a_2,\ldots,a_{n-1})}$ acts on T by

$$\begin{pmatrix} e^{i\theta_1} & & & \\ & e^{i\theta_2} & & \\ & & \ddots & \\ & & & e^{i\theta_n} \end{pmatrix} \longmapsto \prod_{j=1}^{n} e^{ia_j\theta_j}.$$

Now, let $P = MAN$ be the minimal, and only cuspidal, parabolic subgroup of G, and note that $M = T$ is the maximal torus of K. In the case of $G = \mathrm{SL}(n, \mathbb{C})$, we have

$$A = \left\{ \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_n \end{pmatrix} \,\middle|\, \alpha_j \in \mathbb{R}, \alpha_j > 0 \text{ and } \prod \alpha_j = 1,\ j = 1, \ldots, n \right\},$$

and N the group of unipotent upper triangular matrices.

Returning to the general case, since P is the only cuspidal parabolic of G, we see that for every K-type $\tau \in \widehat{K}$, the series of representations $\overline{\pi}(\tau, \varphi)$ from Definition 3.2.8 must be induced from P. As a result, for each $\tau \in \widehat{K}$, we know that $M(\tau) = M$ and $A(\tau) = A$, but we need to find the appropriate $\delta(\tau) \in \widehat{M}$.

Lemma 6.1.2. *Let $\tau \in \widehat{K}$ with highest weight $\delta \in \widehat{M}$. Then τ is the unique* *minimal K-type of* $\operatorname{Ind}_M^K \delta$.

Proof. Since $\delta \in \tau|_T$, by Frobenius reciprocity $\tau \in \operatorname{Ind}_M^K \delta$. However, if $\mu \in \widehat{K}$ with $\|\mu\| < \|\tau\|$, then (by definition) the highest weight of μ is less than δ. Thus $\delta \notin \mu|_M$, and by Frobenius reciprocity again, we must have $\mu \notin \operatorname{Ind}_M^K \delta$. Uniqueness follows from Proposition 2.4.10, which tells us that if $\|\mu\| = \|\tau\|$, then $\mu \cong \tau$. ■

Thus, if $\tau \in \widehat{K}$, with highest weight δ, we will take $\delta(\tau) = \delta$. We can then create the family of representations $\{\pi(\tau, \varphi) \; = \operatorname{Ind}_{MAN}^G (\delta \otimes \varphi \otimes 1) \,|\, \varphi \in \widehat{A}\}$, as in Proposition 3.2.6.

Note that each of these representations is actually a unitary principal series representation of G, and as a result, irreducible (see Theorem 2.2.2). As a result, we have the following.

Lemma 6.1.3. *Fix $\tau \in \widehat{K}$ with highest weight $\delta \in \widehat{M}$, and $\varphi \in \widehat{A}$. Then the representation $\overline{\pi}(\tau, \varphi)$, the irreducible subquotient of $\pi(\tau, \varphi)$ with minimal K-type τ, from Definition 3.2.7, is just the principal series representation with parameter (δ, φ):*

$$\overline{\pi}(\tau, \varphi) = \pi_{(\delta,\varphi)} = \operatorname{Ind}_{MAN}^G (\delta \otimes \varphi \otimes 1). \quad ■$$

Theorem 6.1.4. *Let G be a connected complex semisimple Lie group with maximal compact subgroup K, minimal parabolic subgroup $P = MAN$, and Weyl group W. Fix $\tau \in \widehat{K}$ with highest weight $\delta \in \widehat{M}/W$. Then the set of representations in $\widehat{G}$ containing τ as a minimal K-type is the collection of all principal series* *representations with δ as the discrete parameter:*

$$\{\pi_{(\delta,\varphi)} \,|\, \varphi \in \widehat{A}/W(\tau)\} \cong \widehat{A}/W(\tau),$$

where $W(\tau)$ is the isotropy subgroup of δ in the Weyl group of G.

Thus, we have the following description of $\widehat{G}$:

$$\widehat{G} \cong \bigsqcup_{\delta \in \widehat{M}/W} \widehat{A}/W(\tau). \quad \blacksquare$$

In our example of $G = \mathrm{SL}(n, \mathbb{C})$, we have $\widehat{A} \cong \mathbb{R}^{n-1}$. In addition, for a generic dominant weight $\delta \in \widehat{M}/W$, W_δ will be trivial, but if $\delta = (a_1, a_2, \ldots, a_{n-1})$ with $a_j = a_{j+1}$ say, then W_δ will be a two element group permuting the j^{th} and $j+1^{\text{st}}$ parameters. Thus the space of representations of $G = \mathrm{SL}(n, \mathbb{C})$ with lowest K-type τ just a quotient of $\mathbb{R}^{n-1}$, and $\widehat{G}$ is just the disjoint union over all K-types (or highest weights) of these quotients.

6.1.1 The Associated Semidirect Product Group

With the description of $\widehat{G}$ in terms of minimal K-types given by Theorem 6.1.4, we now turn our attention to understanding $\widehat{G}_0$ in a similar manner, as outlined in Section 4.

To begin, since G has only the one parabolic subgroup $P = MAN$, we see that for any $\tau \in \widehat{K}$, we have

$$\mathfrak{a}(\tau) = \mathfrak{a} = \mathrm{Lie}(A),$$

the Lie algebra of A, and

$$K(\tau) = M \cap K = M = T,$$

the maximal torus of K.

In our example where $G = \mathrm{SL}(n, \mathbb{C})$, then

$$\mathfrak{a} = \left\{ \begin{pmatrix} \beta_1 & & & \\ & \beta_2 & & \\ & & \ddots & \\ & & & \beta_n \end{pmatrix} \,\middle|\, \beta_j \in \mathbb{R}, \text{ and } \sum_{j=1}^{n} \beta_j = 0 \right\},$$

and $K(\tau) = T$ as in equation 6.1.1.

In the general case, if $\tau \in \widehat{K}$ with highest weight $\delta \in \widehat{M} = \widehat{K}(\tau)$, construct the family of representations $\{\pi_0(\tau, \varphi) \,|\, \varphi \in \widehat{\mathfrak{a}} = \widehat{\mathfrak{a}}(\tau)\}$, where

$$\pi_0(\tau, \varphi) = \mathrm{Ind}_{M \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0} (\delta \otimes \varphi)$$

as in Definition 4.0.4.

We are now in a position to prove Conjecture 4.0.5 in this case.

Proposition 6.1.5. *Fix $\tau \in \widehat{K}$ with highest weight $\delta \in \widehat{M} = \widehat{K}(\tau)$. Then each member of the family of representations $\{\pi_0(\tau,\varphi) \,|\, \varphi \in \widehat{\mathfrak{a}}\}$ of G_0 has τ as its minimal K-type.*

Proof. First, note that

$$\pi_0(\tau,\varphi)|_K = \operatorname{Ind}_M^K \delta = \pi(\tau,\varphi)|_K,$$

and as we saw above, τ is the unique minimal K-type of $\pi(\tau,\varphi)$. ■

Next we would like to make sure that we exhaust all representations of G_0 as we vary the K-type in these families. First, let $\overline{\pi}_0(\tau,\varphi)$ be the irreducible subquotient of $\pi_0(\tau,\varphi)$ with minimal K-type τ, as in Definition 4.0.6. Then, we need to show that each Mackey datum occurs as some $\overline{\pi}_0(\tau,\varphi)$ in order to prove Theorem 4.0.8.

Theorem 6.1.6. *Let $\pi \in \widehat{G}_0$ with minimal K-type $\tau \in \widehat{K}$. Then there is a $\varphi \in \widehat{a}$ such that π is equivalent to $\overline{\pi}_0(\tau,\varphi)$.*

Proof. Proposition 2.5.2 tells us that there is a $\sigma \in \widehat{M}$ and $\varphi \in \widehat{a}$ (viewed as a balanced character of $\mathfrak{g}/\mathfrak{k}$) such that

$$\pi \cong \operatorname{Ind}_{K_\varphi \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0} (\tau_\sigma \otimes \varphi),$$

where $\tau_\sigma \in \widehat{K}_\varphi$ has highest weight σ. Now, let $\delta \in \widehat{M}$ be the highest weight of τ.

Then, if τ is the minimal K-type of π, we must have $\sigma = \delta$. As a result,

$$\tau_\sigma \in \operatorname{Ind}_M^{K_\varphi} \delta$$

and so,

$$\pi \in \operatorname{Ind}_{M \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0} (\delta \otimes \varphi)$$

Thus, since $\overline{\pi}_0(\tau, \varphi)$ is the irreducible subquotient of $\operatorname{Ind}_{M \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0} (\delta \otimes \varphi)$ with minimal K-type τ, it follows that $\pi \cong \overline{\pi}_0(\tau, \varphi)$. ∎

Of course, every $\overline{\pi}_0(\tau, \varphi)$ is equivalent to some Mackey datum As a result, the equivalences among the various representations can be determined via the Mackey machine, proving Theorem 4.0.9.

Theorem 6.1.7. *The representations $\overline{\pi}_0(\tau, \varphi_1)$ and $\overline{\pi}_0(\tau, \varphi_2)$ are equivalent if and only if $\varphi_1 = w \cdot \varphi_2$ for some $w \in W(\tau)$.*

Proof. We have $\overline{\pi}_0(\tau, \varphi_1) = \operatorname{Ind}_{M \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0}(\delta \otimes \varphi_1)$, and $\overline{\pi}_0(\tau, \varphi_2) = \operatorname{Ind}_{M \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0}(\delta \otimes \varphi_2)$. The Mackey machine tells us that these two representations are equivalent if and only if the pair (δ, φ_1) is conjugate to the pair (δ, φ_2) under the action of the Weyl group. Thus, we must have φ_1 and φ_2 are conjugate under the action of W_δ, the isotropy subgroup of δ in W. But, by definition, $W_\delta = W(\tau)$. ∎

As a result, we now have the following description of $\widehat{G}_0$ in terms of minimal K-types, proving Theorem 4.0.10.

Theorem 6.1.8. *To each* $\tau \in \widehat{K}$*, attach a special subalgebra* $\mathfrak{a}(\tau)$*, and a series of representations* $\{\pi_0(\varphi) \,|\, \varphi \in \widehat{\mathfrak{a}}(\mathscr{C})\}$*. Then the correspondence which associates to each pair* (τ, φ)*, where* $\varphi \in \widehat{\mathfrak{a}}/W(\tau)$*, the representation* $\overline{\pi}_0(\tau, \varphi) \in \widehat{G}_0$*, is a bijection. The representation* $\overline{\pi}_0(\tau, \varphi)$ *has minimal* K*-type* τ*.* ■

In our example of $G = \mathrm{SL}(n, \mathbb{C})$, we have $\widehat{\mathfrak{a}} \cong \widehat{A} \cong \mathbb{R}^{n-1}$. Thus the space of representations of G_0 with lowest K-type τ just a quotient of $\mathbb{R}^{n-1}$, and $\widehat{G}_0$ is just the disjoint union over all K-types (or highest weights) of these quotients, just as with $\widehat{G}$.

Now that we have described both $\widehat{G}$ and $\widehat{G}_0$, we have establish Theorem 4.0.11 for connected complex semisimple Lie groups.

Theorem 6.1.9. *The correspondence which associates to the representation* $\overline{\pi}(\tau, \varphi) \in \widehat{G}$ *the representation* $\overline{\pi}_0(\tau, \varphi) \in \widehat{G}_0$ *is a well-defined bijection preserving minimal* K*-types.* ■

So, we have

$$\widehat{G} \cong \bigsqcup_{\delta \in \widehat{M}/W} \widehat{A}/W(\tau) \cong \bigsqcup_{\delta \in \widehat{M}/W} \widehat{\mathfrak{a}}/W(\tau) \cong \widehat{G}_0,$$

and as mentioned in Chapter 4, this bijection is, in fact, a homeomorphism at the level of K-types.

6.2 The Group $\mathrm{SL}(2,\mathbb{R})$

We now turn our focus away from complex semisimple Lie groups, and begin our study of $\mathrm{SL}(n,\mathbb{R})$ by looking at the specific example of $G = \mathrm{SL}(2,\mathbb{R})$. As it turns out, this example contains virtually all of the complications inherent in the general case of $\mathrm{SL}(n,\mathbb{R})$, and our work here will be indicative of what is to follow in Chapter 7.

We may take $K = \mathrm{SO}(2)$ to be the maximal compact subgroup of $G = \mathrm{SL}(2,\mathbb{R})$. Then every element of K is a matrix of the form

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

from which it is easy to see that K is abelian. As a result K is its own maximal torus. Then, the representations of K are simply indexed by integers, where the representation τ_n acts on K by

$$\tau_n \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = e^{in\theta}.$$

Since the Weyl group of K is trivial in this case ($W = N_K(T)/T$), there are no equivalences among these representations, and so $\widehat{K} \cong \mathbb{Z}$.

Now, as discussed earlier, real semisimple Lie groups may have more than one parabolic subgroup, and this is the case with $G = \mathrm{SL}(2,\mathbb{R})$ (see [Vog85, p.281], for example). The minimal parabolic of G is $P_1 = M_1A_1N_1$, where

$$M_1 = \left\{ \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \,\middle|\, \epsilon = \pm 1, \right\},$$

and

$$A_1 = \left\{ \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \,\middle|\, t_j > 0, \text{ and, } t_1 \cdot t_2 = 1 \right\}.$$

There is one additional parabolic subgroup, as well. Let $P_2 = M_2A_2N_2$ denote this parabolic subgroup, where

$$M_2 = G = \mathrm{SL}(2, \mathbb{R}),$$

and

$$A_2 = \{I\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Thus, in order to classify $\widehat{G}$ by minimal K-types, we must now find for each associate class $\mathscr{C} \subseteq \widehat{K}$ of representations of K a δ in the discrete series of either M_1 or M_2 such that $\mathrm{Ind}_{M_j \cap K}^{K} \left(\delta|_{M_j \cap K}\right)$ has $\mathscr{C}$ as its set of minimal K-types.

We will begin by examining the representations of M_1. Up to equivalence, their are only two representations of M_1. Let χ_0 denote the trivial representation of M_1, and let χ_1 denote the representation of M_1 given by

$$\chi_1 \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} = \epsilon.$$

Then by Frobenius reciprocity, the minimal K-type of

$$\mathrm{Ind}_{M_1 \cap K}^{K} (\chi_0|_{M_1 \cap K}) = \mathrm{Ind}_{M_1}^{K} \chi_0$$

is the trivial representation τ_0 of K. However, the induced representation

$$\mathrm{Ind}_{M_1 \cap K}^{K} (\chi_1|_{M_1 \cap K}) = \mathrm{Ind}_{M_1}^{K} \chi_1$$

contains both τ_1 and τ_{-1} as minimal K-types, by Frobenius reciprocity again.

Now we will examine the representations of M_2. Proposition 3.2.3 requires us to look only at the discrete series of M_2. The discrete series of $\mathrm{SL}(2, \mathbb{R})$ may be parametrized by their minimal K-type, or Blattner parameter. Let D_n denote the discrete series representation of $\mathrm{SL}(2, \mathbb{R})$ which has τ_n as its minimal K-type. These discrete series representations exist only for $|n| \geq 2$ (see [Bla87, Section 3], for example). Thus, the minimal K-type of

$$\mathrm{Ind}_{M_2 \cap K}^{K} (D_n|_{M_2 \cap K})$$

is just τ_n.

Thus, the only associate K-types are τ_1 and τ_{-1}, and we see that for $n \neq \pm 1$, $\mathscr{C}(\tau_n) = \{\tau_n\}$, while $\mathscr{C}(\tau_1) = \mathscr{C}(\tau_{-1}) = \{\tau_1, \tau_{-1}\}$. From the discussion above, we also know that $\delta(\tau_0) = \chi_0 \in \widehat{M_1}$, $\delta(\mathscr{C}(\tau_{\pm 1})) = \chi_1 \in \widehat{M_1}$, and $\delta(\tau_n) = D_n \in E_2 M_2$ for $|n| > 1$. In addition, $A(\tau_0) = A(\mathscr{C}(\tau_{\pm 1})) \cong \mathbb{R}$, while $A(\tau_n) \cong \{0\}$ for $|n| > 1$.

Now for each associate class $\mathscr{C} \subseteq \widehat{K}$ of representations of K, we can construct the family of irreducible representations $\{\overline{\pi}(\delta(\mathscr{C}), \varphi) \,|\, \varphi \in \widehat{A}(\mathscr{C})\}$ with minimal K-types in $\mathscr{C}$, as in Proposition 3.2.6 and Definition 3.2.7. We would now like to describe the equivalences among these representations.

The normalizer in K of M_1 and A_1 is just the two-element group permuting the diagonal entries. Evidently, this group also fixes both χ_0 and χ_1. Thus, following Definition 3.2.9, we have

$$W(\tau_0) \cong W(\mathscr{C}(\tau_{\pm 1})) \cong \mathbb{Z}/2\mathbb{Z}.$$

On the other hand, the normalizer in K of M_2 is trivial. Thus $W(\tau_n)$ is trivial as well, for $|n| > 1$. As a result, we have

$$\widehat{A}(\tau_0)/W(\tau_0) \cong \widehat{A}(\mathscr{C}(\tau_{\pm 1}))/W(\mathscr{C}(\tau_{\pm 1})) \cong \mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \,|\, x \geq 0\},$$

and,

$$\widehat{A}(\tau_n)/W(\tau_n) \cong \{0\}, \text{ for } |n| > 1.$$

At this point, for $n \neq \pm 1$, since $\mathscr{C}(\tau_n) = \{\tau_n\}$, the above gives us a complete picture of the set of (equivalence classes of) representations with minimal K-type τ_n. However, since $\mathscr{C}(\tau_{\pm 1})$ has two elements, to determine all the representations with minimal K-types in $\mathscr{C}(\tau_{\pm 1})$ we must now come to grips with the R-groups of Definition 3.2.11.

Since $W(\mathscr{C}(\tau_{\pm 1})) \cong \mathbb{Z}/2\mathbb{Z}$, we have

$$R(\mathscr{C}(\tau_{\pm 1})) \cong \mathbb{Z}/2\mathbb{Z},$$

as well. We also have

$$W(\mathscr{C}(\tau_{\pm 1}), \varphi) \cong \mathbb{Z}/2\mathbb{Z},$$

if $\varphi = 0$, and $W(\mathscr{C}(\tau_{\pm 1}), \varphi)$ is trivial if $\varphi \neq 0$. Recall from Theorem 3.2.13 that the number of elements in the set

$$\{\overline{\pi}(\tau, \varphi) \mid \tau \in \mathscr{C}(\tau_{\pm 1})\},$$

is given by

$$|R(\mathscr{C}(\tau_{\pm 1}))/R(\mathscr{C}(\tau_{\pm 1}), \varphi)| = \begin{cases} 2 & \text{if } \varphi = 0, \\ 1 & \text{if } \varphi \neq 0. \end{cases}$$

So, we now have a clear description of the tempered dual of $G = \mathrm{SL}(2, \mathbb{R})$, organized by minimal K-types. For each $|n| > 1$ there is a single representation of

G with minimal K-type τ_n. The set of representations in $\widehat{G}$ with τ_0 as the minimal K-type can be parametrized by the non-negative real numbers. And finally, the representations with both τ_1 and τ_{-1} as minimal K-types can be parametrized by the positive real numbers, while there is a single representation with minimal K-type τ_1, and a single representation with minimal K-type τ_{-1}.

Figure 6.1, on the following page, gives us a picture of $\widehat{G}$, organized by minimal K-types.

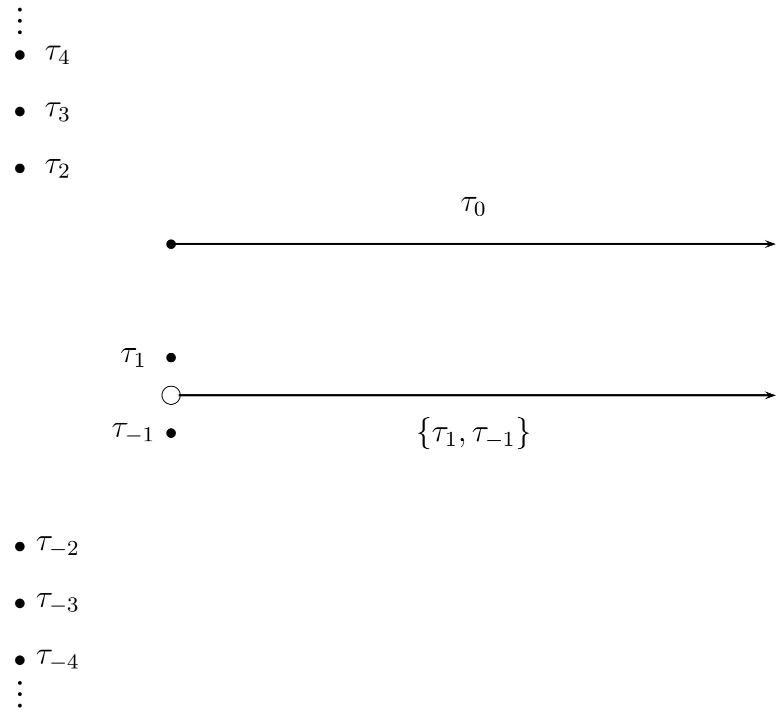

Figure 6.1. The tempered dual of $G = \mathrm{SL}(2, \mathbb{R})$ organized by minimal K-types.

6.2.1 The Associated Semidirect Product Group

As in the previous section, let $G = \mathrm{SL}(2,\mathbb{R})$ and $K = \mathrm{SO}(2)$ the maximal compact subgroup of G. Now that we have classified all tempered representations of G by minimal K-types, we will now focus on the associated semidirect product group $G_0 = K \ltimes \mathfrak{g}/\mathfrak{k}$. We will now prove the theorems of Chapter 4 for $G = \mathrm{SL}(2,\mathbb{R})$, by classifying all unitary representations of G_0 by minimal K-types.

To begin, just as in the previous section, we have $K = \mathrm{SO}(2)$, and thus $\widehat{K} \cong \mathbb{Z}$. We will again use the notation τ_n to denote the representation of K with highest weight n. Then in analogy with the previous section, we will define

$$K_1 = M_1 \cap K = M_1 = \left\{ \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \middle| \epsilon = \pm 1, \right\},$$

and

$$\mathfrak{a}_1 = \mathrm{Lie}(A_1) = \left\{ \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \middle| t_1 + t_2 = 0 \right\}.$$

We will also define

$$K_2 = M_2 \cap K = K = \mathrm{SO}(2),$$

and

$$\mathfrak{a}_2 = \{0\} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

Now, the Mackey machine tells us that for every $\pi \in \widehat{G}_0$ there is a $\varphi \in \mathfrak{a}_1$ (viewed as a balanced character of $\mathfrak{g}/\mathfrak{k}$) and $\sigma \in \widehat{K}_\varphi$ such that

$$\pi \cong \mathrm{Ind}_{K_\varphi \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0}(\sigma \otimes \varphi).$$

However, we would like to classify $\widehat{G}_0$ using the alternate approach outlined in Section 4. We begin by looking at each K-type individually. Following Definitions 4.0.1 and 4.0.2, for the trivial K-type we have

$$\mathfrak{a}(\tau_0) = \mathfrak{a}_1,$$

and

$$K(\tau_0) = K_1$$

Next, for $\tau_{\pm 1}$ we have

$$\mathfrak{a}(\tau_{\pm 1}) = \mathfrak{a}_1,$$

and

$$K(\tau_{\pm 1}) = K_1$$

And finally, for $|n| > 1$,

$$\mathfrak{a}(\tau_n) = \mathfrak{a}_2,$$

and

$$K(\tau_n) = K_2.$$

We must now find the appropriate δ_0 associated to each K-type. To do so, we examine the representations of K_1 and K_2. Let χ_0 and χ_1 be the representations of $K_1 = M_1$ as in the previous section. Then just as above, the minimal K-type of

$$\operatorname{Ind}_{K_1}^{K} \chi_0$$

is the trivial representation (τ_0) of K. And, the induced representation

$$\operatorname{Ind}_{K_1}^{K} \chi_1$$

contains both τ_1 and τ_{-1} as minimal K-types.

Seeing that $K_2 = K = \mathrm{SO}(2)$, the representations of K_2 are just the representations τ_n and clearly

$$\operatorname{Ind}_{K_2}^{K} \tau_n = \tau_n.$$

Once again, we see that the only associate K-types are τ_1 and τ_{-1}, and for $n \neq \pm 1$, $\mathscr{C}(\tau_n) = \{\tau_n\}$, while $\mathscr{C}(\tau_1) = \mathscr{C}(\tau_{-1}) = \{\tau_1, \tau_{-1}\}$. From the discussion above, we have established Conjecture 4.0.3.

Theorem 6.2.1. *Let $\mathscr{C} \subseteq \widehat{K}$ be an associate class of representations of K, and let $K(\mathscr{C})$ denote the group constructed in Definition 4.0.2. Then there is a* $\delta_0 =$

$\delta_0(\mathscr{C}) \in \widehat{K}(\mathscr{C})$ such that $\mathscr{C}$ is the set of a minimal K-types of

$$\mathrm{Ind}_{K(\mathscr{C})}^{K}(\delta_0).$$

The representation δ_0 is determined up to conjugation under the normalizer of A in K.

Proof. From the discussion above, we take $\delta_0(\tau_0) = \chi_0 \in \widehat{K}_1$, $\delta_0(\mathscr{C}(\tau_{\pm 1})) = \chi_1 \in \widehat{K}_1$, and $\delta_0(\tau_n) = \tau_n \in \widehat{K}_2$ for $|n| > 1$. ■

Now, we have $\mathfrak{a}(\tau_0) = \mathfrak{a}(\mathscr{C}(\tau_{\pm 1})) \cong \mathbb{R}$, while $\mathfrak{a}(\tau_n) \cong \{0\}$ for $|n| > 1$. So for each $\tau \in \widehat{K}$, we can construct the family of irreducible representations $\{\overline{\pi}_0(\tau, \varphi) \,|\, \varphi \in \widehat{\mathfrak{a}}(\mathscr{C})\}$ with minimal K-types in $\mathscr{C}$, following Conjecture 4.0.5 and Definition 4.0.6. The following lemmas will characterize these representations.

Lemma 6.2.2. *Let $\overline{\pi}_0(\tau_0, \varphi)$ be as in Definition 4.0.6. Then, we have*

$$\overline{\pi}_0(\tau_0, \varphi) \cong \begin{cases} \tau_0 \otimes \varphi & \text{if } K_\varphi = K, \\ \mathrm{Ind}_{K_1 \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0}(\chi_0 \otimes \varphi) & \text{if } K_\varphi \cong K_1. \end{cases}$$

Proof. By construction,

$$\pi_0(\tau_0, \varphi) = \mathrm{Ind}_{K_1 \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0}(\chi_0 \otimes \varphi).$$

If $K_\varphi = K$, then since $\tau_0 \in \operatorname{Ind}_{K_1}^{K} \chi_0$, we also have

$$\tau_0 \otimes \varphi \in \pi_0(\tau_0, \varphi).$$

The Mackey machine tells us that $\tau_0 \otimes \varphi$ is irreducible, and since $\tau_0 \otimes \varphi$ has minimal K-type τ_0, we have

$$\overline{\pi}_0(\tau_0, \varphi) \cong \tau_0 \otimes \varphi.$$

Now, if $K_\varphi \cong K_1$, we still have

$$\pi_0(\tau_0, \varphi) = \operatorname{Ind}_{K_1 \ltimes \mathfrak{a}_1}^{G_0} (\chi_0 \otimes \varphi),$$

which, by the Mackey machine, is irreducible. Hence

$$\overline{\pi}_0(\tau_0, \varphi) = \operatorname{Ind}_{K_1 \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0} (\chi_0 \otimes \varphi),$$

since τ_0 is the minimal K-type of $\operatorname{Ind}_{K_1 \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0} (\chi_0 \otimes \varphi)$. ∎

Lemma 6.2.3. *Let $\overline{\pi}_0(\tau_{\pm 1}, \varphi)$ be as in Definition 4.0.6. Then, we have*

$$\overline{\pi}_0(\tau_{\pm 1}, \varphi) \cong \begin{cases} \tau_{\pm 1} \otimes \varphi & \text{if } K_\varphi = K, \\ \operatorname{Ind}_{K_1 \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0} (\chi_1 \otimes \varphi) & \text{if } K_\varphi \cong K_1. \end{cases}$$

Proof. By construction,

$$\pi_0(\tau_{\pm 1}, \varphi) = \operatorname{Ind}_{K_1 \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0}(\chi_1 \otimes \varphi).$$

If $K_\varphi = K$, then since $\tau_{\pm 1} \in \operatorname{Ind}_{K_1}^{K} \chi_1$, we also have

$$\tau_{\pm 1} \otimes \varphi \in \pi_0(\tau_{\pm 1}, \varphi).$$

The Mackey machine tells us that $\tau_{\pm 1} \otimes \varphi$ is irreducible, and since $\tau_{\pm 1} \otimes \varphi$ has minimal K-type $\tau_{\pm 1}$, we have

$$\overline{\pi}_0(\tau_{\pm 1}, \varphi) \cong \tau_{\pm 1} \otimes \varphi.$$

Now, if $K_\varphi \cong K_1$, we still have

$$\pi_0(\tau_{\pm 1}, \varphi) = \operatorname{Ind}_{K_1 \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0}(\chi_1 \otimes \varphi),$$

which, by the Mackey machine, is irreducible. Hence

$$\overline{\pi}_0(\tau_{\pm 1}, \varphi) = \operatorname{Ind}_{K_1 \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0}(\chi_1 \otimes \varphi),$$

since τ_1 *and* τ_{-1} are the minimal K-types of $\operatorname{Ind}_{K_1 \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0}(\chi_1 \otimes \varphi)$. ∎

Lemma 6.2.4. *Let $|n| > 1$, and let $\overline{\pi}_0(\tau_n, \varphi)$ be as in Definition 4.0.6. Then, we* *have*

$$\overline{\pi}_0(\tau_n, \varphi) \cong \tau_n \otimes \varphi,$$

where φ is trivial.

Proof. By construction,

$$\pi_0(\tau_0, \varphi) = \tau_n \otimes \varphi.$$

The Mackey machine tells us that $\tau_n \otimes \varphi$ is irreducible, and since $\tau_n \otimes 0$ has minimal K-type τ_n, the result follows. ∎

Since the above lemmas exhaust all possible Mackey data for G_0, we have established Theorem 4.0.8.

Theorem 6.2.5. *Let π be an irreducible representation of G_0 containing τ as a minimal K-type. Then there is a $\varphi \in \widehat{\mathfrak{a}}(\tau)$ such that π is equivalent to $\overline{\pi_0}(\tau, \varphi)$.* ∎

Once again, since each $\overline{\pi_0}(\tau, \varphi)$ is equivalent to some Mackey datum, we may use the Mackey machine, as above, to determine the equivalences between these representations, proving Theorem 4.0.9.

Proposition 6.2.6. *The representations $\overline{\pi}_0(\tau, \varphi_1)$ and $\overline{\pi}_0(\tau, \varphi_2)$ are equivalent if and only if $\varphi_1 = w \cdot \varphi_2$ for some $w \in W(\tau)$.*

Proof. We have $\overline{\pi}_0(\tau, \varphi_1) = \mathrm{Ind}_{M \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0}(\delta \otimes \varphi_1)$, and $\overline{\pi}_0(\tau, \varphi_2) = \mathrm{Ind}_{M \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0}(\delta \otimes \varphi_2)$. The Mackey machine tells us that these two representations are equivalent if and only if the pair (δ, φ_1) is conjugate to the pair (δ, φ_2) under the action of the Weyl group. Thus, we must have φ_1 and φ_2 are conjugate under the action of W_δ, the isotropy subgroup of δ in W. But, by definition, $W_\delta = W(\tau)$. ■

Thus, for $n \neq \pm 1$ we have a complete description of the set of (equivalence classes of) irreducible unitary representations with minimal K-type τ_n – they can be parametrized by the set $\widehat{\mathfrak{a}}(\tau_n)/W(\tau_n)$. Just as above, we have

$$W(\tau_0) \cong W(\mathscr{C}(\tau_{\pm 1})) \cong \mathbb{Z}/2\mathbb{Z},$$

while $W(\tau_n)$ is trivial for $|n| > 1$. As a result, we have

$$\widehat{\mathfrak{a}}(\tau_0)/W(\tau_0) \cong \widehat{\mathfrak{a}}(\mathscr{C}(\tau_{\pm 1}))/W(\mathscr{C}(\tau_{\pm 1})) \cong \mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \,|\, x \geq 0\},$$

and,

$$\widehat{\mathfrak{a}}(\tau_n)/W(\tau_n) \cong \{\text{pt.}\}, \text{ for } |n| > 1.$$

Now, if we examine the set of representations in $\widehat{G}_0$ with minimal K-types in $\mathscr{C}(\tau_{\pm 1})$, we see that if $\varphi = 0$, then

$$\overline{\pi}_0(\tau_1, \varphi) = \tau_1 \otimes \varphi,$$

whereas

$$\overline{\pi}_0(\tau_{-1}, \varphi) = \tau_{-1} \otimes \varphi,$$

and these representations are *not* equivalent. On the other hand, if $\varphi \neq 0$, both $\overline{\pi}_0(\tau_1, \varphi)$ and $\overline{\pi}_0(\tau_{-1}, \varphi)$ are given by

$$\mathrm{Ind}_{K_1 \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0}(\chi_1 \otimes \varphi),$$

and hence are equivalent.

Thus we have established the following.

Lemma 6.2.7. *Let $\varphi \in \widehat{\mathfrak{a}}_1 \cong \widehat{A}_1$. Then the number of elements in the set*

$$\{\pi_0(\tau, \varphi) \,|\, \tau \in \mathscr{C}(\tau_{\pm 1})\} = \{\pi(\tau, \varphi) \,|\, \tau \in \mathscr{C}(\tau_{\pm 1})\}. \quad \blacksquare$$

Putting these results together proves Theorem 4.0.10.

Theorem 6.2.8. *To each associate class $\mathscr{C} \subseteq \widehat{K}$ of representations of K, attach a special subalgebra $\mathfrak{a}(\mathscr{C})$, and a series of representations $\{\pi_0(\varphi) \,|\, \varphi \in \widehat{\mathfrak{a}}(\mathscr{C})\}$ (Propositions 4.0.1, 4.0.5). Then the correspondence which associates to each pair $(\mathscr{C}, \varphi)$, where $\varphi \in \widehat{\mathfrak{a}}(\mathscr{C})/W(\mathscr{C})$, the set*

$$\{\overline{\pi}_0(\tau, \varphi) \,|\, \tau \in \mathscr{C}\} \subseteq \widehat{G}_0,$$

is a one-to-finite correspondence onto $\widehat{G}_0$. The set $\{\overline{\pi}_0(\tau, \varphi) \,|\, \tau \in \mathscr{C}\}$ has

$$|R(\mathscr{C})/R(\mathscr{C}, \varphi)|$$

elements, each of which has a single orbit of $R(\mathscr{C}, \varphi)$ in $\mathscr{C}$ as its set of minimal K-types. ∎

So, we now have a clear description of the unitary dual of G_0, organized by minimal K-types, and it is precisely the same as that for $\widehat{G}$. For each $|n| > 1$ there is a single representation of G with minimal K-type τ_n. The set of representations in $\widehat{G}$ with τ_0 as the minimal K-type can be parametrized by the non-negative real numbers. And finally, the representations with both τ_1 and τ_{-1} as minimal K-types can be parametrized by the positive real numbers, while there is a single representation with minimal K-type τ_1, and a single representation with minimal K-type τ_{-1}. Thus, we have established the bijection between $\widehat{G}$ and $\widehat{G}_0$, proving Theorem 4.0.11.

Theorem 6.2.9. *The correspondence which associates to the representation $\overline{\pi}(\tau, \varphi) \in \widehat{G}$ the representation $\overline{\pi}_0(\tau, \varphi) \in \widehat{G}_0$ is a well-defined bijection preserving minimal K-types.* ∎

And, as we can see, for each K-type τ, the bijection is a homeomorphism from the set of those representations in $\widehat{G}$ with minimal K-types in $\mathscr{C}(\tau)$ to those representations in $\widehat{G}_0$ with minimal K-types in $\mathscr{C}(\tau)$.

6.3 The Group $\mathrm{SL}(3,\mathbb{R})$

As we will see in the following chapter, the representation theory of $\mathrm{SL}(n,\mathbb{R})$ takes on different characteristics depending on whether n is even or odd. So before we look at the general case of $\mathrm{SL}(n,\mathbb{R})$, we will examine the group $\mathrm{SL}(3,\mathbb{R})$ from the minimal K-type perspective, using results from [Val85] to tailor Vogan's approach to this specific group.

Let $G = \mathrm{SL}(3,\mathbb{R})$, and we will take $K = \mathrm{SO}(3)$ to be the maximal compact subgroup of G. In addition, we have

$$T = \left\{ \begin{pmatrix} X & \\ & 1 \end{pmatrix} \middle| X \in \mathrm{SO}(2) \right\} \cong \mathrm{SO}(2),$$

a maximal torus of K.

So, much like in the previous section, we will let τ_n denote the irreducible representation of $K = \mathrm{SO}(3)$ whose highest weight acts on T by

$$\begin{pmatrix} \cos\theta & \sin\theta & \\ -\sin\theta & \cos\theta & \\ & & 1 \end{pmatrix} \longmapsto e^{in\theta}.$$

However, the Weyl group of K is nontrivial, and maps τ_n to τ_{-n}. So, we have (see [BtD85, p.272] for example)

$$\widehat{K} \cong \{0\} \cup \mathbb{N}.$$

Once again G has two cuspidal parabolic subgroups (see [Val85, p.540], for example). Let $P_1 = M_1A_1N_1$, where

$$M_1 = \left\{ \begin{pmatrix} \epsilon_1 & & \\ & \epsilon_2 & \\ & & \epsilon_3 \end{pmatrix} \middle| \, \epsilon_j = \pm 1, \text{ and } \epsilon_1\epsilon_2\epsilon_3 = 1 \right\},$$

and

$$A_1 = \left\{ \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix} \middle| \, t_j > 0, \text{ and } t_1t_2t_3 = 1 \right\}.$$

We also have the parabolic subgroup $P_2 = M_2A_2N_2$, where

$$M_2 = \left\{ \begin{pmatrix} X & \\ & \epsilon \end{pmatrix} \middle| \, S \in \mathrm{SL}^{\pm}(2, \mathbb{R}), \epsilon = \pm 1, \text{ and } \det(S)\epsilon = 1 \right\},$$

where $\mathrm{SL}^{\pm}(2, \mathbb{R}) = \{X \in \mathrm{GL}(2, \mathbb{R}) \,|\, \det(X) = \pm 1\}$, and

$$A_1 = \left\{ \begin{pmatrix} s & & \\ & s & \\ & & t \end{pmatrix} \middle| \, s, t > 0, \text{ and } s^2t = 1 \right\}.$$

Thus, in order to classify $\widehat{G}$ by minimal K-types, we must now find for each associate class $\mathscr{C} \subseteq \widehat{K}$ of representations of K a δ in the discrete series of either

M_1 or M_2 such that $\operatorname{Ind}_{M_j\cap K}^{K}(\delta|_{M_j\cap K})$ has $\mathscr{C}$ as its set of minimal K-types.

We will begin by examining the representations of M_1. Up to equivalence, there are only two representations of M_1. Let χ_0 denote the trivial representation of M_1, and let χ_1 denote the representation of M_1 given by

$$\chi_1\begin{pmatrix}\epsilon_1 & & \\ & \epsilon_2 & \\ & & \epsilon_3\end{pmatrix} = \epsilon_1.$$

Then by Frobenius reciprocity, the minimal K-type of

$$\operatorname{Ind}_{M_1\cap K}^{K}(\chi_0|_{M_1\cap K}) = \operatorname{Ind}_{M_1}^{K}\chi_0$$

is the trivial representation (τ_0) of K. However, the induced representation

$$\operatorname{Ind}_{M_1\cap K}^{K}(\chi_1|_{M_1\cap K}) = \operatorname{Ind}_{M_1}^{K}\chi_1$$

contains τ_1 as its minimal K-type (see [Vog85, p.282]).

Now we will examine the representations of M_2. Proposition 3.2.3 requires us to look only at the discrete series of M_2. The discrete series of $M_2 \cong \mathrm{SL}^{\pm}(2,\mathbb{R})$ is just the set $\{D_n \in E_2\mathrm{SL}(2,\mathbb{R}) \,|\, n > 1\}$ (see [Val85, p.541], for example). As above, the minimal K-type of

$$\operatorname{Ind}_{M_2\cap K}^{K}(D_n|_{M_2\cap K})$$

is just τ_n.

Thus, for each $\tau_n \in \widehat{K}$, we have $\mathscr{C}(\tau_n) = \{\tau_n\}$. From the discussion above, we also know that $\delta(\tau_0) = \chi_0 \in \widehat{M_1}$, $\delta(\tau_1) = \chi_1 \in \widehat{M_1}$, and $\delta(\tau_n) = D_n \in E_2M_2$ for $n > 1$. In addition, $A(\tau_0) = A(\tau_1) \cong \mathbb{R}^2$, while $A(\tau_n) \cong \mathbb{R}$ for $n > 1$.

Now for each associate class $\tau_n \in \widehat{K}$, we can construct the family of irreducible representations $\{\overline{\pi}(\delta(\tau_n), \varphi) \,|\, \varphi \in \widehat{A}(\mathscr{C})\}$ with minimal K-type τ_n, as in Proposition 3.2.6 and Definition 3.2.7. We would now like to describe the equivalences among these representations.

The normalizer in K of M_1 and A_1 is isomorphic to S_3, the group of permutations on three letters. Evidently, this group also normalizes both χ_0. Thus, following Definition 3.2.9, we have

$$W(\tau_0) \cong S_3.$$

On the other hand, the normalizer of χ_1 in S_3 is isomorphic to S_2, and so is $W(\tau_1)$, then. Finally, the normalizer in K of M_2 is trivial. Thus $W(\tau_n)$ is trivial as well, for $n > 1$. As a result, we have

$$\widehat{A}(\tau_0)/W(\tau_0) \cong \{(x, y) \in \mathbb{R}^2 \,|\, x \geq |y|\},$$

$$\widehat{A}(\tau_1)/W(\tau_1) \cong \{(x, y) \in \mathbb{R}^2 \,|\, x \geq 0\},$$

and

$$\widehat{A}(\tau_n)/W(\tau_n) \cong \mathbb{R}, \text{ for } n > 1.$$

So, we now have a clear description of the tempered dual of $G = \mathrm{SL}(3, \mathbb{R})$, organized by minimal K-types. For each $n > 1$ there is a family of representations of G with minimal K-type τ_n, parametrized by $\mathbb{R}$. The set of representations in $\widehat{G}$ with τ_1 as the minimal K-type can be parametrized by the right half-plane in $\mathbb{R}^2$. And finally, the representations with τ_0 as the minimal K-type can be parametrized by the cone $x \geq y$ in R^2.

Figure 6.2, on the following page, gives us a picture of $\widehat{G}$, organized by minimal K-types.

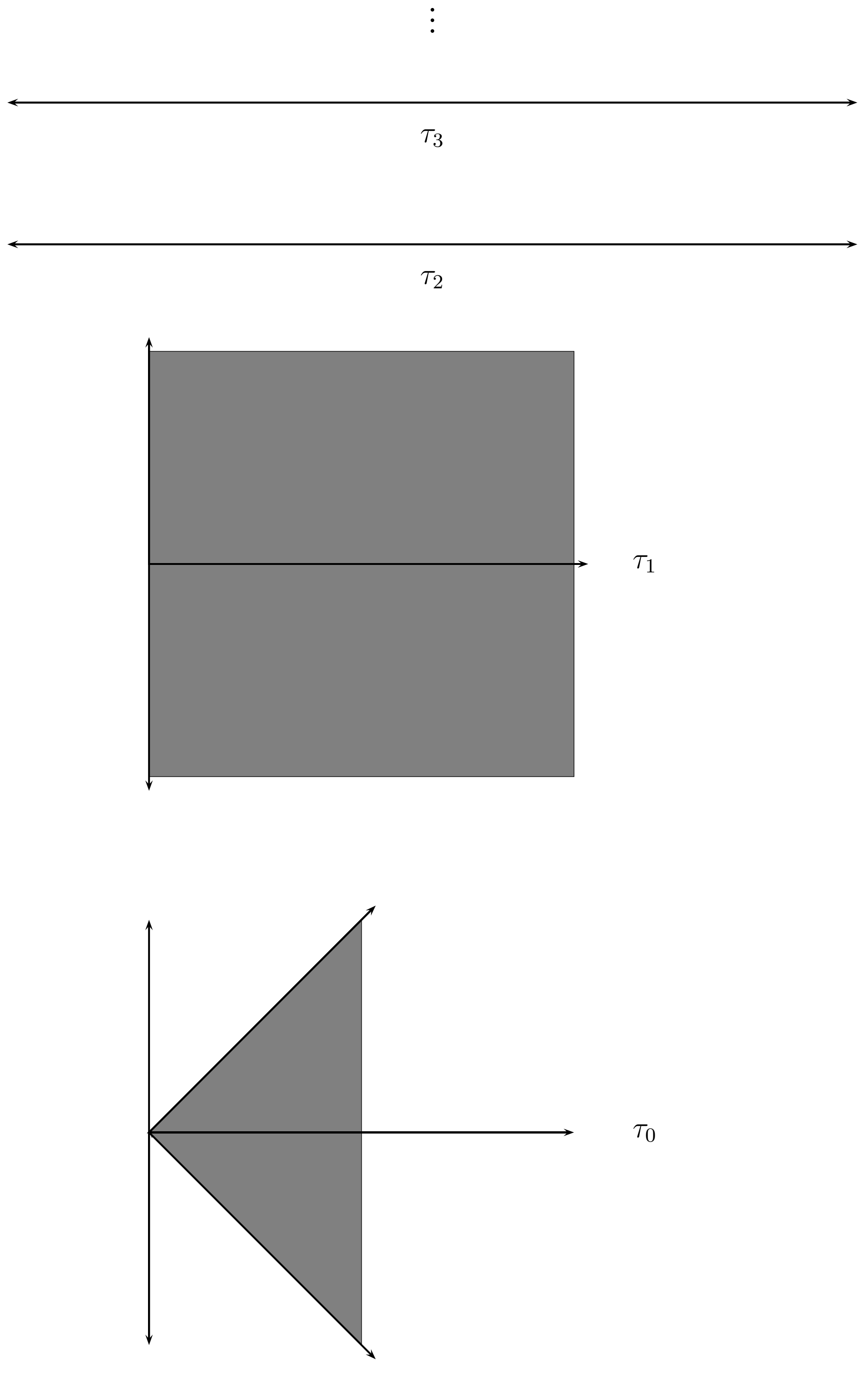

Figure 6.2. The tempered dual of $G = \mathrm{SL}(3, \mathbb{R})$ organized by minimal K-types.

6.3.1 The Associated Semidirect Product Group

As in the previous section, let $G = \mathrm{SL}(3, \mathbb{R})$ and let $K = \mathrm{SO}(3)$, which is a maximal compact subgroup of G. Now that we have classified all tempered representations of G by minimal K-types, we will now focus on the associated semidirect product group $G_0 = K \ltimes \mathfrak{g}/\mathfrak{k}$. Following the method outlined in Chapter 4 we will now classify all unitary representations of G_0 by minimal K-types, and then build a bijection between $\widehat{G}$ and $\widehat{G}_0$ which preserves minimal K-types, just as we did for the group $\mathrm{SL}(2, \mathbb{R})$.

To begin, just as in the previous section, we have $K = \mathrm{SO}(3)$, and thus $\widehat{K} \cong \{0\} \cup \mathbb{N}$. We will again use the notation τ_n to denote the representation of K with highest weight n. Then in analogy with the previous section, we will define

$$K_1 = M_1 \cap K = M_1 = \left\{ \begin{pmatrix} \epsilon_1 & & \\ & \epsilon_2 & \\ & & \epsilon_3 \end{pmatrix} \middle| \epsilon_j = \pm 1, \right\},$$

and

$$\mathfrak{a}_1 = \mathrm{Lie}(A_1) = \left\{ \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix} \middle| t_1 + t_2 + t_3 = 0 \right\}.$$

We will also define

$$K_2 = M_2 \cap K = \left\{ \begin{pmatrix} X & \\ & \epsilon \end{pmatrix} \middle| X \in \mathrm{O}(2),\, \epsilon = \pm 1, \text{ and } \det(X)\epsilon = 1 \right\} \cong \mathrm{O}(2),$$

and

$$\mathfrak{a}_2 = \left\{ \begin{pmatrix} s & & \\ & s & \\ & & t \end{pmatrix} \middle| 2s + t = 0 \right\}.$$

Now, the Mackey machine tells us that for every $\pi \in \widehat{G}_0$ there is a $\varphi \in \mathfrak{a}_1$ (viewed as a balanced character of $\mathfrak{g}/\mathfrak{k}$) and $\sigma \in \widehat{K}_\varphi$ such that

$$\pi \cong \mathrm{Ind}_{K_\varphi \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0} (\sigma \otimes \varphi).$$

However, we would like to classify $\widehat{G}_0$ using the alternate approach outlined in Section 4. We begin by looking at each K-type individually. Following Definitions 4.0.1 and 4.0.2, for the trivial K-type we have

$$\mathfrak{a}(\tau_0) = \mathfrak{a}_1,$$

and

$$K(\tau_0) = K_1$$

Next, for τ_1 we have

$$\mathfrak{a}(\tau_1) = \mathfrak{a}_1,$$

and

$$K(\tau_1) = K_1$$

And finally, for $n > 1$,

$$\mathfrak{a}(\tau_n) = \mathfrak{a}_2,$$

and

$$K(\tau_n) = K_2.$$

We must now find the appropriate δ_0 associated to each K-type. To do so, we examine the representations of K_1 and K_2. Let χ_0 and χ_1 be the representations of $K_1 = M_1$ as in the previous section. Then just as above, the minimal K-type of

$$\operatorname{Ind}_{K_1}^{K} \chi_0$$

is the trivial representation (τ_0) of K. And, the induced representation

$$\operatorname{Ind}_{K_1}^{K} \chi_1$$

has τ_1 as its minimal K-type.

Next, we examine $K_2 \cong \mathrm{O}(2)$. To begin, note that $\mathrm{O}(2)$ is the semidirect product of SO(2) with the group generated by

$$\omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since τ_0 is the only element of $\widehat{\mathrm{SO}(2)}$ fixed by ω, by the Mackey machine, we have (see Section 5.2 for the bijection)

$$\widehat{\mathrm{O}(2)} \cong \left\{ (n, \epsilon) \,|\, n \in \mathbb{Z} \text{ and } \epsilon = \begin{cases} \pm 1 & \text{if } n = 0 \\ 1 & \text{if } n \neq 0 \end{cases} \right\}.$$

We now need to understand the minimal K-types of representations induced from O(2). To begin, Theorem 5.2.4 tell us that the minimal K-type of

$$\mathrm{Ind}_{\mathrm{O}(2)}^{\mathrm{O}(3)} \tau_{(n,\epsilon)},$$

is $\begin{cases} \tau_{(n,\pm 1)} & \text{if } n \neq 0, \\ \tau_{(n,\epsilon)} & \text{if } n = 0. \end{cases}$

In addition, since $\mathrm{O}(3) \cong \mathrm{SO}(3) \times Z/2$, the only representation τ of SO(3) for which $\tau_{(n,\epsilon)} \in \widehat{\mathrm{O}(3)}$ is a minimal K-type of

$$\mathrm{Ind}_{\mathrm{SO}(3)}^{\mathrm{O}(3)} \tau,$$

is $\tau = \tau_n$.

Proposition 6.3.1. *Let* $\tau_{(n,1)} \in \widehat{K}_2$*, with* $|n| > 1$*. Then the minimal* K*-type of*

$$\operatorname{Ind}_{K_2}^{K} \tau_{(n,\epsilon)},$$

is $\tau_n \in \widehat{K}$*.*

Proof. By Mackey's induction in stages (see [Mac52, Theorem 4.1], for example), for every $\tau \in \widehat{K}_2$, we have

$$\operatorname{Ind}_{K}^{\mathrm{O}(3)} \operatorname{Ind}_{K_2}^{K} \tau \cong \operatorname{Ind}_{K_2}^{\mathrm{O}(3)} \tau.$$

Since the minimal K-types of

$$\operatorname{Ind}_{K_2}^{\mathrm{O}(3)} \tau_{(n,1)},$$

are $\tau_{(n,\pm 1)} \in \widehat{\mathrm{O}(3)}$, and since the only irreducible representation τ of SO(3) such that

$$\operatorname{Ind}_{\mathrm{SO}(3)}^{\mathrm{O}(3)} \tau,$$

has minimal K-types $\tau_{(n,\pm 1)}$ is $\tau = \tau_n$, we see that the minimal K-type of

$$\operatorname{Ind}_{K_2}^{\mathrm{SO}(3)} \tau_(n, 1),$$

must be $\tau_n \in \widehat{\mathrm{SO}(3)} = \widehat{K}$. ∎

Just as above, we see that for each $\tau_n \in \widehat{\mathrm{SO}(3)}$ we have $\mathscr{C}(\tau_n) = \{\tau_n\}$, and we can now prove Conjecture 4.0.3.

Theorem 6.3.2. *Let $\tau \in \widehat{K}$, and let $K(\tau)$ denote the group constructed in Definition 4.0.2. Then there is a $\delta_0 = \delta_0(\tau) \in \widehat{K}(\tau)$ such that τ is the set of a minimal K-type of*

$$\mathrm{Ind}_{K(\tau)}^{K}(\delta_0).$$

The representation δ_0 is determined up to conjugation under the normalizer of A in K.

Proof. From the discussion above, we can take $\delta_0(\tau_0) = \chi_0 \in \widehat{K}_1$, $\delta_0(\tau_1) = \chi_1 \in \widehat{K}_1$, and $\delta_0(\tau_n) = \tau_n \in \widehat{K}_2$ for $n > 1$. ∎

Now, we have $\mathfrak{a}(\tau_0) = \mathfrak{a}(\tau_1) \cong \mathbb{R}^2$, while $\mathfrak{a}(\tau_n) \cong \mathbb{R}$ for $n > 1$. As we did for the group $\mathrm{SL}(2,\mathbb{R})$, for each $\tau \in \widehat{K}$, we can construct the family of irreducible representations $\{\overline{\pi}_0(\tau,\varphi) \,|\, \varphi \in \widehat{\mathfrak{a}}(\tau)\}$ with minimal K-type τ, following Conjecture 4.0.5 and Definition 4.0.6. The following lemmas will characterize these representations.

Lemma 6.3.3. *Let $\pi = \mathrm{Ind}_{K_\varphi \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0}(\sigma \otimes \varphi) \in \widehat{G}_0$ with minimal K-type τ_0. Then,*

$$\pi \cong \overline{\pi}_0(\tau_0, \varphi).$$

Proof. Since the minimal K-type of

$$\mathrm{Ind}_{K_\varphi}^{K} \sigma,$$

is $\tau_0 \in \widehat{\mathrm{SO}(3)}$, σ must be the trivial representation of K_φ. Thus,

$$\sigma \in \operatorname{Ind}_{K_1}^{K_\varphi} \chi_0,$$

and so

$$\pi = \operatorname{Ind}_{K_\varphi \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0} (\sigma \otimes \varphi) \in \pi_0(\tau_0, \varphi).$$

Since π is irreducible, the result follows. ∎

Lemma 6.3.4. *Let* $\pi = \operatorname{Ind}_{K_\varphi \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0} (\sigma \otimes \varphi) \in \widehat{G}_0$ *with minimal* K*-type* τ_1*. Then,*

$$\pi \cong \overline{\pi}_0(\tau_1, \varphi).$$

Proof. If $K_\varphi = K_1$, then (up to equivalence) $\sigma = \chi_1$ is the only irreducible representation of K_1 such that

$$\operatorname{Ind}_{K_1}^{K} \sigma,$$

has minimal K-type τ_1. Thus

$$\pi \cong \overline{\pi}_0(\tau_1, \varphi).$$

However, if $K_\varphi = K_2 \cong \mathrm{O}(2)$, then either $\sigma \cong \tau_{(0,-1)}$ or $\sigma \cong \tau_{(1,1)}$, both of which are contained in

$$\operatorname{Ind}_{K_1}^{K_2} \chi_1,$$

combining 5.2.1 and Frobenius reciprocity. Thus, we have

$$\pi \cong \overline{\pi}_0(\tau_1, \varphi).$$

Finally, if $K_\varphi = K$, we know that τ_1 is the minimal K-type of

$$\operatorname{Ind}_{K_1}^{K} \chi_1,$$

hence

$$\pi \cong \overline{\pi}_0(\tau_1, \varphi),$$

and the result follows. ■

Lemma 6.3.5. *Let $n > 1$, and let $\pi = \operatorname{Ind}_{K_\varphi \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0}(\sigma \otimes \varphi) \in \widehat{G}_0$ with minimal K-type τ_n. Then,*

$$\pi \cong \overline{\pi}_0(\tau_n, \varphi).$$

Proof. First note that if $\sigma \in K_1$, then the minimal K-type of

$$\operatorname{Ind}_{K_1}^{K} \sigma,$$

is either τ_0 or τ_1. Thus, we must have $K_\varphi = K_2$ or $K_\varphi = K$. If $K_\varphi = K$, then

$\pi = \tau_n \otimes \varphi = \overline{\pi}_0(\tau_n, \varphi)$. If $K_\varphi = K_2$, then

$$\pi = \mathrm{Ind}_{K_2 \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0}(\tau_n, \varphi) = \overline{\pi}_0(\tau_n, \varphi).$$

Once again, the result follows. ■

Combining these three lemmas gives us the following proves Theorem 4.0.8.

Theorem 6.3.6. *Let $\pi \in \widehat{G}_0$ with minimal K-type τ. Then $\pi \cong \overline{\pi}_0(\tau, \varphi)$ for some $\varphi \in \widehat{\mathfrak{a}}_1$.* ■

Now, since each $\overline{\pi}_0(\tau_n, \varphi)$ is equivalent to some Mackey datum, we can use the Mackey machine, just as above, to describe the equivalences among these representations, proving Theorem 4.0.9.

Theorem 6.3.7. *The representations $\overline{\pi}_0(\tau, \varphi_1)$ and $\overline{\pi}_0(\tau, \varphi_2)$ are equivalent if and only if $\varphi_1 = w \cdot \varphi_2$ for some $w \in W(\tau)$.*

Proof. We have $\overline{\pi}_0(\tau, \varphi_1) = \mathrm{Ind}_{M \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0}(\delta \otimes \varphi_1)$, and $\overline{\pi}_0(\tau, \varphi_2) = \mathrm{Ind}_{M \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0}(\delta \otimes \varphi_2)$. The Mackey machine tells us that these two representations are equivalent if and only if the pair (δ, φ_1) is conjugate to the pair (δ, φ_2) under the action of the Weyl group. Thus, we must have φ_1 and φ_2 are conjugate under the action of W_δ, the isotropy subgroup of δ in W. But, by definition, $W_\delta = W(\tau)$. ■

We must now calculate $W(\tau)$ for each K-type τ. First, we have the normalizer of K_1 and $\mathfrak{a}_1$ is isomorphic to S_3, and normalizes χ_0, as well. And, the isotropy subgroup of χ_1 is isomorphic to S_2. Thus, we have

$$W(\tau_0) \cong S_3,$$

and

$$W(\tau_1) \cong S_2.$$

Next, the normalizer of K_2 and $\mathfrak{a}_2$ is trivial, so $W(\tau_n)$ is trivial for $n > 1$. Thus, we have a complete description of the set of (equivalence classes of) irreducible unitary representations with minimal K-type τ_n – they can be parametrized by the set $\widehat{\mathfrak{a}}(\tau_n)/W(\tau_n)$. Just as above, we have

$$\widehat{\mathfrak{a}}(\tau_0)/W(\tau_0) \cong \mathbb{R}^2,$$

$$\widehat{\mathfrak{a}}(\tau_1)/W(\tau_1) \cong \{(x,y) \in \mathbb{R}^2 \mid x \geq 0\},$$

and

$$\widehat{\mathfrak{a}}(\tau_n)/W(\tau_n) \cong \mathbb{R}, \text{ for } n > 1.$$

Thus we have established Theorem 4.0.10.

Theorem 6.3.8. *To each $\tau \in \widehat{K}$, attach a special subalgebra $\mathfrak{a}(\tau)$, and a series of representations $\{\pi_0(\varphi) \mid \varphi \in \widehat{\mathfrak{a}}(\tau)\}$. Then the correspondence which associates to* *each pair* (τ, φ)*, where* $\varphi \in \widehat{\mathfrak{a}}(\tau)/W(\tau)$*, the representation*

$$\overline{\pi}_0(\tau, \varphi) \in \widehat{G}_0,$$

is a one-to-one correspondence onto $\widehat{G}_0$*. The minimal* K*-type of* $\overline{\pi}_0(\tau, \varphi)$ *is* τ*.*

So, we now have a clear description of the unitary dual of G_0, organized by minimal K-types, and it is precisely the same as that for $\widehat{G}$. For each $n > 1$ there is a family of representations of G_0 with minimal K-type τ_n, parametrized by $\mathbb{R}$. The set of representations in $\widehat{G}_0$ with τ_1 as the minimal K-type can be parametrized by the right half-plane in $\mathbb{R}^2$. And finally, the representations with τ_0 as the minimal K-type can be parametrized by the cone $x \geq y$ in R^2. Once again, we have established the bijection between $\widehat{G}$ and $\widehat{G}_0$, proving Theorem 4.0.11.

Theorem 6.3.9. *The correspondence which associates to the representation* $\overline{\pi}(\tau, \varphi) \in \widehat{G}$ *the representation* $\overline{\pi}_0(\tau, \varphi) \in \widehat{G}_0$ *is a well-defined bijection preserving minimal* K*-types.*

Also, just as with the previous examples, we see that the bijection is a homeomorphism at the level of K-types.

Chapter 7

The Mackey Analogy for $SL(n, \mathbb{R})$

In this chapter, we use the results from Chapter 3 to classify the representations of $G = \mathrm{SL}(n, \mathbb{R})$ by minimal K-types, and we will do the same for the associated semidirect product group G_0, following Chapter 4. Just as we saw in Chapter 6, this will give rise to a very natural bijection between $\widehat{G}$ and $\widehat{G}_0$. There are slight differences in the details in even and odd dimension, so they will be dealt with separately. Combining the key results in these sections, Theorems 7.1.10 and 7.2.8, establishes the bijection between $\widehat{G}$, and $\widehat{G}_0$, thus proving Theorem 4.0.11.

Theorem 7.0.1. *Let $G = \mathrm{SL}(n, \mathbb{R})$. The correspondence which associates to the representation $\overline{\pi}(\tau, \varphi) \in \widehat{G}$ the representation $\overline{\pi}_0(\tau, \varphi) \in \widehat{G}_0$ is a well-defined bijection preserving minimal K-types.*

7.1 The Group $\mathrm{SL}(2k, \mathbb{R})$

We begin our study of the group $\mathrm{SL}(n, \mathbb{R})$ with the even-dimensional case. Let $G = \mathrm{SL}(2k, \mathbb{R})$, and take $K = \mathrm{SO}(2k)$ to be the maximal compact subgroup of G. The description of $\widehat{G}$ very closely follows that of [Vog85]. As discussed in Section 5.1, if we let

$$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

the 2×2 matrix of the rotation of the plane by an angle θ, then the group

$$T_{2k} = \left\{ \begin{pmatrix} R_{\theta_1} & & \\ & \ddots & \\ & & R_{\theta_k} \end{pmatrix} \;\middle|\; \theta_1, \ldots, \theta_k \in \mathbb{R} \right\},$$

is a maximal torus of $\mathrm{SO}(2k)$. Let $\tau_{(a_1,\ldots,a_k)}$ denote the representation of $\mathrm{SO}(2k)$ whose highest weight acts on T_{2k} by

$$\begin{pmatrix} R_{\theta_1} & & \\ & \ddots & \\ & & R_{\theta_k} \end{pmatrix} \longmapsto \prod_{j=1}^{k} e^{ia_j\theta_j}.$$

Then, we have (see [BtD85, p.272], for example)

$$\widehat{\mathrm{SO}(2k)} \cong \{(a_1, \ldots, a_k) \in \mathbb{Z}^k \mid a_1 \geq a_2 \geq \cdots \geq a_{k-1} \geq |a_k|\},$$

by the Cartan-Weyl theory of highest weights.

Now, as in the case of $\mathrm{SL}(2,\mathbb{R})$, G has multiple cuspidal parabolic subgroups. For $0 \leq r \leq k$, the cuspidal parabolic subgroups of G are of the form $P_r = M_r A_r N_r$, where

$$M_r = \left\{ S = \begin{pmatrix} X_1 & & & & & \\ & \ddots & & & & \\ & & X_{k-r} & & & \\ & & & \epsilon_1 & & \\ & & & & \ddots & \\ & & & & & \epsilon_{2r} \end{pmatrix} \;\middle|\; \det X_i = \pm 1, \epsilon_i = \pm 1, \text{ and } \det(S) = 1 \right\},$$

$$\cong [\mathrm{SL}^{\pm}(2,\mathbb{R})]^{k-r} \times (\mathbb{Z}/2\mathbb{Z})^{2r-1}$$

and,

$$A_r = \left\{ S = \begin{pmatrix} s_1 & & & & & & & \\ & s_1 & & & & & & \\ & & \ddots & & & & & \\ & & & s_{k-r} & & & & \\ & & & & s_{k-r} & & & \\ & & & & & t_1 & & \\ & & & & & & \ddots & \\ & & & & & & & t_{2r} \end{pmatrix} \middle| s_j, t_j > 0 \text{ and } \det(S) = 1 \right\},$$

since $\mathrm{SL}(m, \mathbb{R})$ has discrete series representations only when $n = 1$ or 2.

Let $\tau_{(a_1,\ldots,a_k)}$ be the representation of $\mathrm{SO}(2k)$ with highest weight $(a_1, \ldots, a_k)$. Recall that this is defined only if $a_1 \geq \cdots \geq a_{k-1} \geq |a_k|$. Let r be the number of a_j's that are equal to 0 or ± 1.

Thus, in order to classify $\widehat{G}$ by minimal K-types, we must now find for each associate class $\mathscr{C} \subseteq \widehat{K}$ of representations of K a δ in the discrete series of some M_r such that $\mathrm{Ind}_{M_r \cap K}^{K} (\delta|_{M_r \cap K})$ has $\mathscr{C}$ as its set of minimal K-types. We begin with an examination of the representations of each M_r.

We are only interested in the discrete series of M_r, and since

$$M_r \cong [\mathrm{SL}^{\pm}(2, \mathbb{R})]^{k-r} \times (\mathbb{Z}/2\mathbb{Z})^{2r-1}, \tag{7.1.1}$$

we have

$$E_2M_r \cong \{(a_1, \ldots, a_{k-r}, s) \in \mathbb{Z}^{k-r+1} \mid a_1 \geq a_2 \geq \cdots \geq a_{k-r} \geq 2, \text{ and } 0 \leq s \leq r\},$$

where the representation corresponding to $(a_1, \ldots, a_{k-r}, s)$ is

$$D_{a_1} \otimes \cdots \otimes D_{a_{k-r}} \otimes \chi_s,$$

where $D_{a_j} \in E_2(\mathrm{SL}^{\pm}(2, \mathbb{R}))$ (as discussed in Section 6.3) and

$$\chi_s \begin{pmatrix} \epsilon_1 & & & \\ & \epsilon_2 & & \\ & & \ddots & \\ & & & \epsilon_{2r} \end{pmatrix} = \prod_{j=1}^{s} \epsilon_j.$$

Note that the parametrization of E_2M is still valid even when $r = 0$, and hence $M_r \cong [\mathrm{SL}^{\pm}(2, \mathbb{R})]^{k-1} \times (\mathrm{SL}(2, \mathbb{R}))$.

Let $\delta = D_{a_1} \otimes \cdots \otimes D_{a_{k-r}} \otimes \chi_s$. Then the set of minimal K-types of

$$Ind_{M_r \cap K}^{K} (\delta|_{M_r \cap K})$$

is [Vog85, p.282]

$$\begin{cases} \{\tau_{(a_1,\ldots,a_{k-r},\underbrace{1,\ldots,1}_{\text{s-times}},0,\ldots,0)}\} & \text{if } s < r, \\ \{\tau_{(a_1,\ldots,a_{k-r},1,\ldots,1,\pm 1)}\} & \text{if } s = r. \end{cases}$$

Thus, we see that

$$\mathscr{C}(\tau_{(a_1,\ldots,a_k)}) = \begin{cases} \{\tau_{(a_1,\ldots,a_k)}\} & \text{if } a_k \neq \pm 1 \\ \{\tau_{(a_1,\ldots,\pm 1)}\} & \text{if } a_k = \pm 1 \end{cases}.$$

From the above discussion we now see that if $\tau = \tau_{(a_1,\ldots,a_k)} \in \widehat{K}$ with r of the a_j's equal to 0 or ± 1, and s of the a_j's equal to ± 1,

$$\delta(\mathscr{C}(\tau)) = D_{a_1} \otimes \cdots \otimes D_{a_{k-r}} \otimes \chi_s \in \widehat{M_r}.$$

And so we have $M(\mathscr{C}(\tau)) = M_r$ and $A(\mathscr{C}(\tau)) = A_r$.

Now for each associate class $\mathscr{C} \subseteq \widehat{K}$ of representations of K, we can construct the family of irreducible representations $\{\overline{\pi}(\delta(\mathscr{C}), \varphi) \,|\, \varphi \in \widehat{A}(\mathscr{C})\}$ with minimal K-types in $\mathscr{C}$, as in Proposition 3.2.6 and Definition 3.2.7. We would like to describe the equivalences among these representations, and to do so we need to find $W(\mathscr{C})$, as described in Definition 3.2.9.

To describe $W(\mathscr{C})$, let $\varphi = (\alpha_1, \ldots, \alpha_{k-r}, \beta_1, \ldots, \beta_{2r}) \in \widehat{A}(\mathscr{C})$, where $\alpha_i, \beta_j \in$

$\mathbb{C}$, be given by

$$\varphi\begin{pmatrix} s_1 & & & & & & & \\ & s_1 & & & & & & \\ & & \ddots & & & & & \\ & & & s_{k-r} & & & & \\ & & & & s_{k-r} & & & \\ & & & & & t_1 & & \\ & & & & & & \ddots & \\ & & & & & & & t_{2r} \end{pmatrix} = \left(\prod s_i^{2\alpha_i}\right)\left(\prod t_j^{\beta_j}\right),$$

and

$$2\sum \alpha_i + \sum \beta_j = 0.$$

Then the action of $W(\mathscr{C})$ permutes the α's and β's. Let $w = (\eta, \nu) \in W(\mathscr{C})$ be given by

$$\begin{aligned} w \cdot \varphi &= w(\alpha_1, \ldots, \alpha_{k-r}, \beta_1, \ldots \beta_{2r}) \\ &= (\alpha_{\eta(1)}, \ldots, \alpha_{\eta(k-r)}, \beta_{\nu(1)}, \ldots, \beta_{\nu(2r)}) \end{aligned}$$

Then the permutations η and ν must satisfy [Vog85, p.286]

$$ka_{\eta(i)} = a_i \text{ for } 1 \leq i \leq k-r,$$

and

$$\nu\{1,\dots,s\} = \{1,\dots,s\} \text{ or } \{s+1,\dots,2r\}.$$

To complete the description of $\widehat{G}$, we must determine the complete set of minimal K-types of each $\overline{\pi}(\delta(\mathscr{C}),\varphi)$, which is governed by the R-groups of Definition 3.2.11. Here, we have $R(\mathscr{C}(\tau_{(a_1,\dots,a_k)}))$ is nontrivial if and only if $a_k = \pm 1$ [Vog85, p.286], and in this case $R(\mathscr{C}(\tau_{(a_1,\dots,a_k)})) = \{1,\rho\}$ where

$$\rho(w) = \begin{cases} 1 & \text{if } \nu\{1,\dots,s\} = \{1,\dots,s\}, \\ -1 & \text{if } \nu\{1,\dots,s\} = \{s+1,\dots,2r\}. \end{cases} \tag{7.1.2}$$

Thus, following Definition 3.2.11 and Theorem 3.2.13, we have the set of minimal K-types of $\overline{\pi}(\tau_{(a_1,\dots,a_k)},\varphi)$, where $a_k = \pm 1$, is

$$\begin{cases} \{\tau_{(a_1,\dots,\pm a_k)}\} & \text{if } \{\beta_1,\dots,\beta_s\} = \{\beta_{s+1},\dots,\beta_{2r}\} \\ \{\tau_{(a_1,\dots,a_k)}\} & \text{if } \{\beta_1,\dots,\beta_s\} = \{\beta_1,\dots,\beta_s\}. \end{cases}$$

Theorem 3.2.13 then gives us a complete description of $\widehat{G}$.

Let $\tau = \tau_{(a_1,\dots a_k)} \in \widehat{K}$ with $a_k \neq \pm 1$. Then the set of representations of in $\widehat{G}$ with minimal K-type τ can be parametrized by $\widehat{A}(\tau)/W(\tau)$.

Similarly, if $\tau = \tau_{(a_1,\dots a_k)} \in \widehat{K}$ with $a_k = \pm 1$, then the set of representations of in $\widehat{G}$ with minimal K-types in $\mathscr{C}(\tau) = \{\tau_{(a_1,\cdots \pm a_k)}\}$ can be parametrized by $\widehat{A}(\tau)/W(\tau)$. However, for each $\varphi \in \widehat{A}(\tau)$ with $\{\beta_1,\dots,\beta_s\} = \{\beta_{s+1},\dots \beta_{2r}\}$, we have two representations, one with $\tau_{(a_1,\dots a_k)}$ as its minimal K-type, and one with $\tau_{(a_1,\dots -a_k)}$ as its minimal K-type.

7.1.1 The Associated Semidirect Product Group

As in the previous section, let $G = \mathrm{SL}(2k,\mathbb{R})$ and let $K = \mathrm{SO}(2k)$, a maximal compact subgroup of G. Now that we have classified all tempered representations of G by minimal K-types, we will focus on the associated semidirect product group $G_0 = K \ltimes \mathfrak{g}/\mathfrak{k}$. Following the method outlined in Chapter 4 we will now classify all unitary representations of G_0 by minimal K-types. Just as in the previous chapter, we will then use this description of $\widehat{G}_0$ to prove Theorem 4.0.11 for G.

To begin, just as in the previous section, we have $K = \mathrm{SO}(2k)$, and thus

$$\widehat{\mathrm{SO}(2k)} \cong \{(a_1,\dots,a_k) \in \mathbb{Z}^k \mid a_1 \geq a_2 \geq \cdots \geq a_{k-1} \geq |a_k|\},$$

via the highest weight correspondence from Section 5.1 We will again use the notation $\tau_{(a_1,\dots,a_k)}$ to denote the representation of K with highest weight $(a_1,\dots,a_k)$. In analogy with the previous section, we will define

$$K_r = M_r \cap K$$

$$= \left\{ S = \begin{pmatrix} X_1 & & & & & \\ & \ddots & & & & \\ & & X_{k-r} & & & \\ & & & \epsilon_1 & & \\ & & & & \ddots & \\ & & & & & \epsilon_{2r} \end{pmatrix} \middle| X_i \in \mathrm{O}(2), \epsilon_i = \pm 1, \text{ and } \det(S) = 1 \right\},$$

$$\cong [\mathrm{O}(2)]^{k-r} \times (\mathbb{Z}/2\mathbb{Z})^{2r-1},$$

and

$$\mathfrak{a}_r = \mathrm{Lie}(A_r)$$

$$= \left\{ S = \begin{pmatrix} s_1 & & & & & & & \\ & s_1 & & & & & & \\ & & \ddots & & & & & \\ & & & s_{k-r} & & & & \\ & & & & s_{k-r} & & & \\ & & & & & t_1 & & \\ & & & & & & \ddots & \\ & & & & & & & t_{2r} \end{pmatrix} \middle| \mathrm{tr}(S) = 0 \right\}.$$

The Mackey machine tells us that for every $\pi \in \widehat{G}_0$ there is a $\varphi \in \mathfrak{a}_k$ (viewed as a balanced character of $\mathfrak{g}/\mathfrak{k}$) and $\sigma \in \widehat{K}_\varphi$ such that

$$\pi \cong \mathrm{Ind}_{K_\varphi \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0}(\sigma \otimes \varphi).$$

However, as before, we would like to classify $\widehat{G}_0$ using the alternate approach outlined in Chapter 4. We must now find the appropriate δ_0 associated to each K-type. To do so, we examine the representations of K_r.

Let $\chi_m \in \widehat{K}_k$ be defined by

$$\chi_m \begin{pmatrix} \epsilon_1 & & & \\ & \epsilon_2 & & \\ & & \ddots & \\ & & & \epsilon_{2k} \end{pmatrix} = \prod_{j=1}^{m} \epsilon_j.$$

Then the discussion on p.282 of [Vog85] gives us the following result.

Lemma 7.1.3. *The set of minimal K-types of* $\mathrm{Ind}_{K_k}^{K} \chi_m$, *is*

$$\begin{cases} \{\tau_{(\underbrace{1,\ldots,1}_{m\text{-}times},0,\ldots,0)}\} & \text{if } m < k \\ \{\tau_{(1,\ldots,1,\pm 1)}\} & \text{if } m = k. \quad \blacksquare \end{cases}$$

Now, if $r < k$, let $\tau = \tau_{(a_1,\ldots,a_{k-r})} \otimes \chi_m \in \widehat{K}_r$ with each $a_j > 0$, then the results from Section 5.2 govern how $\tau_{(a_1,\ldots,a_{k-r})}$ induces up to K, while the discussion on p.282 of [Vog85] (and above) governs how χ_m induces up to K. Combining these establishes the following.

Lemma 7.1.4. *Let* $\tau = \tau_{(a_1,\dots,a_{k-r})} \otimes \chi_m \in \widehat{K}_r$. *The set of minimal* K*-types of* $\mathrm{Ind}_{K_r}^{K} \tau$ *is*

$$\begin{cases} \{\tau_{(a_1,\dots,a_{k-r},\underbrace{1,\dots,1}_{m\text{-}times},0,\dots,0)}\} & \text{if } m < r, \\ \{\tau_{(a_1,\dots,a_{k-r}1,\dots,1,\pm 1)}\} & \text{if } m = r. \end{cases}$$

Proof. From Section 5.2, we know that when we induce $\tau_{(a_1,\dots,a_{k-r})}$ up to K, the minimal K-type is $\tau_{(a_1,\dots,a_{k-r},0,\dots,0)}$. Tensoring $\tau_{(a_1,\dots,a_{k-r})}$ with χ_m has the effect of inserting m ± 1's after the a_j's (see [Vog85, p.282]). ■

Thus, we see that if $\tau = \tau_{(a_1,\dots,a_k)} \in \widehat{K}$ with r of the a_j's equal to 0 or ± 1, and s of the a_j's equal to ± 1, we can construct δ_0 of Conjecture 4.0.3 in an analogous way to δ. Let

$$\delta_0(\tau) = \tau_{(a_1,\dots,a_{k-r})} \otimes \chi_s \in \widehat{K}_r. \tag{7.1.5}$$

Theorem 7.1.6. *Let* $\tau \in \widehat{K}$*, and let* $K(\mathscr{C}(\tau))$ *denote the group constructed in Definition 4.0.2. Then there is a* $\delta_0 = \delta_0(\mathscr{C}(\tau)) \in \widehat{K}(\mathscr{C}(\tau))$ *such that* $\mathscr{C}(\tau)$ *is the set of a minimal* K*-type of*

$$\mathrm{Ind}_{K(\tau)}^{K}(\delta_0).$$

The representation δ_0 *is determined up to conjugation under the normalizer of* A *in* K.

Proof. From the preceding lemma, we that the choice of δ_0 in (7.1.5) has the desired properties. ■

Note that we now see, just as with the group $G = \mathrm{SL}(2,\mathbb{R})$,

$$\mathscr{C}(\tau_{(a_1,\dots,a_k)}) = \begin{cases} \{\tau_{(a_1,\dots,a_k)}\} & \text{if } a_k \neq \pm 1 \\ \\ \{\tau_{(a_1,\dots,\pm 1)}\} & \text{if } a_k = \pm 1 \end{cases}.$$

As a result, we have,

$$\mathfrak{a}(\mathscr{C}(\tau)) = \mathfrak{a}_r,$$

and for each associate class $\mathscr{C}(\tau) \in \widehat{K}$ of representations of K, we can construct the family of irreducible representations $\{\overline{\pi}_0(\tau,\varphi) \,|\, \varphi \in \widehat{\mathfrak{a}}(\mathscr{C})\}$ with minimal K-types in $\mathscr{C}(\tau)$, following Conjecture 4.0.5 and Definition 4.0.6. Then, we can prove Theorem 4.0.8.

Theorem 7.1.7. *Suppose* $\pi = \mathrm{Ind}_{K_\varphi \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0}(\sigma \otimes \varphi) \in \widehat{G}_0$ *contains* $\tau = \tau_{(a_1,\dots,a_k)}$ *as a minimal* K*-type. Then, we have*

$$\pi \cong \overline{\pi}_0(\tau,\varphi).$$

Proof. If $K_\varphi = K_r$, then (up to equivalence) $\tau_{(a_1,\dots,a_{k-r})} \otimes \chi_s$ is the only representation in $\widehat{K}_r$ which has minimal K-type τ when induced up to K. Thus we must have $\sigma \cong \tau_{(a_1,\dots,a_{k-r})} \otimes \chi_s$.

Now, if $K_\varphi \neq K_r$, there is a subgroup of K_φ conjugate to K_r, and so we may assume $K_r \subseteq K_\varphi$, without loss of generality. Then, combining the branching rules of Section 5.2 with Frobenius reciprocity, we must have

$$\sigma \in \operatorname{Ind}_{K_r}^{K_\varphi} \delta(\tau),$$

and since π is irreducible with minimal K-type τ, the result follows. ∎

Once again, since each $\overline{\pi_0}(\tau, \varphi)$ is equivalent to some Mackey datum, we may use the Mackey machine, as in previous examples, to determine the equivalences between these representations, proving Theorem 4.0.9.

Theorem 7.1.8. *The representations $\overline{\pi}_0(\tau, \varphi_1)$ and $\overline{\pi}_0(\tau, \varphi_2)$ are equivalent if and only if $\varphi_1 = w \cdot \varphi_2$ for some $w \in W(\mathscr{C}(\tau))$.*

Proof. We have $\overline{\pi}_0(\tau, \varphi_1) = \operatorname{Ind}_{M \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0}(\delta \otimes \varphi_1)$, and $\overline{\pi}_0(\tau, \varphi_2) = \operatorname{Ind}_{M \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0}(\delta \otimes \varphi_2)$. The Mackey machine tells us that these two representations are equivalent if and only if the pair (δ, φ_1) is conjugate to the pair (δ, φ_2) under the action of the Weyl group. Thus, we must have φ_1 and φ_2 are conjugate under the action of W_δ, the isotropy subgroup of δ in W. But, by definition, $W_\delta = W(\tau)$. ∎

Thus, for $a_k \neq \pm 1$ we have a complete description of the set of (equivalence classes of) irreducible unitary representations with minimal K-type $\tau = \tau_{(a_1,\ldots,a_k)}$ – they can be parametrized by the set $\widehat{\mathfrak{a}}(\tau)/W(\tau)$, in analogy with Section 7.1.

Similarly, if $\tau = \tau_{(a_1,\ldots a_k)} \in \widehat{K}$ with $a_k = \pm 1$, then the set of representations of in $\widehat{G}_0$ with minimal K-types in $\mathscr{C}(\tau) = \{\tau_{(a_1,\cdots \pm a_k)}\}$ can be parametrized by $\widehat{\mathfrak{a}}(\tau)/W(\tau)$. However, for each $\varphi = (\alpha_1, \ldots, \alpha_{k-r}, \beta_1, \ldots, \beta_r) \in \widehat{\mathfrak{a}}(\tau)$ with $\{\beta_1, \ldots, \beta_s\} = \{\beta_{s+1}, \ldots \beta_{2r}\}$, note that $\varphi \cong \varphi'$ where

$$\varphi' = (\alpha_1, \ldots, \alpha_{k-r}, \beta_1, \beta_1 \ldots, \beta_r, \beta_r).$$

Thus

$$K_\varphi \cong \left\{ S = \begin{pmatrix} X_1 & & & & & \\ & \ddots & & & & \\ & & X_p & & & \\ & & & Y_1 & & \\ & & & & \ddots & \\ & & & & & Y_q \end{pmatrix} \middle| X_i \in \mathrm{O}(m_j), Y_j \in \mathrm{O}(2), \text{ and } \det(S) = 1 \right\}$$

In this case, if $a_k = \pm 1$, the minimal K-type of

$$\mathrm{Ind}_{K_\varphi \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0} (\tau_{(a_1, \ldots a_k)} \otimes \varphi)$$

is just $\tau_{(a_1, \ldots a_k)}$. As a result, if $a_k = \pm 1$ and φ is such that $\{\beta_1, \ldots, \beta_s\} = \{\beta_{s+1}, \ldots \beta_{2r}\}$, we have two representations corresponding to φ with minimal K-type in $\mathscr{C}(\tau)$, one with $\tau_{(a_1, \ldots a_k)}$ as its minimal K-type, and one with $\tau_{(a_1, \cdots -a_k)}$ as its minimal K-type, which proves Theorem 4.0.10 for G.

Theorem 7.1.9. *To each associate class $\mathscr{C} \subseteq \widehat{K}$ of representations of K, attach a* *special subalgebra* $\mathfrak{a}(\mathscr{C})$*, and a series of representations* $\{\pi_0(\varphi)\,|\,\varphi \in \widehat{\mathfrak{a}}(\mathscr{C})\}$ *(Propositions 4.0.1, 4.0.5). Then the correspondence which associates to each pair* $(\mathscr{C}, \varphi)$*, where* $\varphi \in \widehat{\mathfrak{a}}(\mathscr{C})/W(\mathscr{C})$*, the set*

$$\{\overline{\pi}_0(\tau, \varphi)\,|\,\tau \in \mathscr{C}\} \subseteq \widehat{G}_0,$$

is a one-to-finite correspondence onto $\widehat{G}_0$*. The set* $\{\overline{\pi}_0(\tau, \varphi)\,|\,\tau \in \mathscr{C}\}$ *has*

$$|R(\mathscr{C})/R(\mathscr{C}, \varphi)|$$

elements, each of which has a single orbit of $R(\mathscr{C}, \varphi)$ *in* $\mathscr{C}$ *as its set of minimal* K*-types.*

So, we now have a clear description of the unitary dual of G_0, organized by minimal K-types, and once again, it is precisely the same as that for $\widehat{G}$. Thus, we have established Theorem 4.0.11 for $G = \mathrm{SL}(2k, \mathbb{R})$.

Theorem 7.1.10. *The correspondence which associates to the representation* $\overline{\pi}(\tau, \varphi) \in \widehat{G}$ *the representation* $\overline{\pi}_0(\tau, \varphi) \in \widehat{G}_0$ *is a well-defined bijection preserving minimal* K*-types.* ■

Note, that the bijection, just as with previous examples, is a homeomorphism at the level of minimal K-types.

7.2 The Group $\mathrm{SL}(2k+1,\mathbb{R})$

We conclude our study of the group $\mathrm{SL}(n,\mathbb{R})$ with the odd-dimensional-case. The arguments here are very similar to those in Section 7.1, however the triviality of the R-groups in this case simplifies matters considerably.

Let $G = \mathrm{SL}(2k+1,\mathbb{R})$ and let $K = \mathrm{SO}(2k+1)$, a maximal compact subgroup of G. As discussed in Section 5.1, if we let

$$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

the 2×2 matrix of the rotation of the plane by an angle θ, then the group

$$T_{2k+1} = \left\{ \begin{pmatrix} R_{\theta_1} & & & \\ & \ddots & & \\ & & R_{\theta_k} & \\ & & & 1 \end{pmatrix} \;\middle|\; \theta_1,\ldots,\theta_k \in \mathbb{R} \right\},$$

is a maximal toris for $\mathrm{SO}(2k+1)$. Let $\tau_{(a_1,\ldots,a_k)}$ denote the representation of

SO$(2k+1)$ whose highest weight acts on T_{2k+1} by

$$\begin{pmatrix} R_{\theta_1} & & & \\ & \ddots & & \\ & & R_{\theta_k} & \\ & & & 1 \end{pmatrix} \longmapsto \prod_{j=1}^{k} e^{ia_j\theta_j}.$$

Then, we have (see [BtD85, p.272], for example)

$$\widehat{\mathrm{SO}(2k+1)} \cong \{(a_1,\ldots,a_k)\in\mathbb{Z}^k \mid a_1\geq a_2\geq\cdots\geq a_{k-1}\geq a_k>0\},$$

again by the Cartan-Weyl theory of highest weights.

Now, as in the case of SL$(3,\mathbb{R})$, G has multiple cuspidal parabolic subgroups. For $0\leq r\leq k$, the cuspidal parabolic subgroups of G are of the form $P_r=M_rA_rN_r$, where

$$M_r = \left\{ S=\begin{pmatrix} X_1 & & & & & \\ & \ddots & & & & \\ & & X_{k-r} & & & \\ & & & \epsilon_1 & & \\ & & & & \ddots & \\ & & & & & \epsilon_{2r+1} \end{pmatrix} \middle| \det X_i=\pm1, \epsilon_i=\pm1, \text{ and } \det(S)=1 \right\},$$

$$\cong [\mathrm{SL}^{\pm}(2,\mathbb{R})]^{k-r}\times(\mathbb{Z}/2\mathbb{Z})^{2r},$$

much like above, since $\mathrm{SL}(m,\mathbb{R})$ has discrete series representations only when $n = 1$ or 2), and

$$A_r = \left\{ S = \begin{pmatrix} s_1 & & & & & & & \\ & s_1 & & & & & & \\ & & \ddots & & & & & \\ & & & s_{k-r} & & & & \\ & & & & s_{k-r} & & & \\ & & & & & t_1 & & \\ & & & & & & \ddots & \\ & & & & & & & t_{2r+1} \end{pmatrix} \middle| \, s_j, t_j > 0 \text{ and } \det(S) = 1 \right\}.$$

Now, let $\tau_{(a_1,\ldots,a_k)}$ be the representation of $\mathrm{SO}(2k+1)$ with highest weight $(a_1,\ldots,a_k)$. Recall that this is defined only if $a_1 \geq \cdots \geq a_{k-1} \geq a_k \geq 0$. Let r be the number of a_j's that are equal to 0 or 1.

Thus, in order to classify $\widehat{G}$ by minimal K-types, we must now find for each associate class $\mathscr{C} \subseteq \widehat{K}$ of representations of K a δ in the discrete series of some M_r such that $\mathrm{Ind}_{M_r \cap K}^{K}(\delta|_{M_r \cap K})$ has $\mathscr{C}$ as its set of minimal K-types. We begin with an examination of the representations of each M_r.

We are only interested in the discrete series of M_r, and since

$$M_r \cong [\mathrm{SL}^{\pm}(2,\mathbb{R})]^{k-r} \times (\mathbb{Z}/2\mathbb{Z})^{2r},$$

we have

$$E_2M_r \cong \{(a_1, \dots, a_{k-r}, s) \in \mathbb{Z}^{k-r+1} \mid a_1 \geq a_2 \geq \cdots \geq a_{k-r} \geq 2, \text{ and } 0 \leq s \leq r\},$$

where the representation corresponding to $(a_1, \dots, a_{k-r}, s)$ is

$$D_{a_1} \otimes \cdots \otimes D_{a_{k-r}} \otimes \chi_s,$$

where $D_{a_j} \in E_2(\mathrm{SL}^{\pm}(2, \mathbb{R}))$ (as discussed in Section 6.3) and

$$\chi_s \begin{pmatrix} \epsilon_1 & & & \\ & \epsilon_2 & & \\ & & \ddots & \\ & & & \epsilon_{2r+1} \end{pmatrix} = \prod_{j=1}^{s} \epsilon_j.$$

Let $\delta = D_{a_1} \otimes \cdots \otimes D_{a_{k-r}} \otimes \chi_s$. Then the set of minimal K-types of

$$\mathrm{Ind}_{M_r \cap K}^{K} \left(\delta|_{M_r \cap K}\right)$$

is [Vog85, p.282]

$$\begin{cases} \{\tau_{(a_1, \dots, a_{k-r}, \underbrace{1, \dots, 1}_{\text{s-times}}, 0, \dots, 0)}\} & \text{if } s < r, \\ \{\tau_{(a_1, \dots, a_{k-r}, 1, \dots, 1)}\} & \text{if } s = r. \end{cases}$$

Thus, we see that

$$\mathscr{C}(\tau_{(a_1,\dots,a_k)}) = \{\tau_{(a_1,\dots,a_k)}\}.$$

From the above discussion we now see that if $\tau = \tau_{(a_1,\dots,a_k)} \in \widehat{K}$ with r of the a_j's equal to 0 or 1, and s of the a_j's equal to 1,

$$\delta(\tau) = D_{a_1} \otimes \cdots \otimes D_{a_{k-r}} \otimes \chi_s \in \widehat{M_r}.$$

And so we have $M(\tau) = M_r$ and $A(\tau) = A_r$.

Now for each $\tau \in \widehat{K}$, we can construct the family of irreducible representations $\{\overline{\pi}(\delta(\tau), \varphi) \,|\, \varphi \in \widehat{A}(\tau)\}$ with minimal K-type τ, as in Proposition 3.2.6 and Definition 3.2.7. We would like to describe the equivalences among these representations, and to do so we need to find $W(\tau)$, as described in Definition 3.2.9.

To describe $W(\tau)$, let $\varphi = (\alpha_1, \dots, \alpha_{k-r}, \beta_1, \dots, \beta_{2r+1}) \in \widehat{A}(\tau)$, where $\alpha_i, \beta_j \in$

$\mathbb{C}$, be given by

$$\varphi \begin{pmatrix} s_1 & & & & & & & \\ & s_1 & & & & & & \\ & & \ddots & & & & & \\ & & & s_{k-r} & & & & \\ & & & & s_{k-r} & & & \\ & & & & & t_1 & & \\ & & & & & & \ddots & \\ & & & & & & & t_{2r+1} \end{pmatrix} = \left(\prod s_i^{2\alpha_i}\right)\left(\prod t_j^{\beta_j}\right),$$

and

$$2\sum \alpha_i + \sum \beta_j = 0.$$

Then the action of $W(\tau)$ permutes the α's and β's. Let $w = (\eta, \nu) \in W(\tau)$ be given by

$$\begin{aligned} w \cdot \varphi &= w(\alpha_1, \ldots, \alpha_{k-r}, \beta_1, \ldots \beta_{2r+1}) \\ &= (\alpha_{\eta(1)}, \ldots, \alpha_{\eta(k-r)}, \beta_{\nu(1)}, \ldots, \beta_{\nu(2r+1)}) \end{aligned}$$

Then the permutations η and ν must satisfy [Vog85, p.286]

$$ka_{\eta(i)} = a_i \text{ for } 1 \leq i \leq k - r,$$

and

$$\nu\{1, \ldots, s\} = \{1, \ldots, s\}.$$

Note that in this case, the condition on ν is slightly different than that for $\mathrm{SL}(2k, \mathbb{R})$, due to the fact there are an odd number of β's now. Also, since for each $\tau \in \widehat{K}$ we have $\mathscr{C}(\tau) = \{\tau\}$, the R-groups for $G = \mathrm{SL}(2k+1, \mathbb{R})$ must be trivial, in contrast to the situation for $\mathrm{SL}(2k, \mathbb{R})$.

Theorem 3.2.13 then gives us a complete description of $\widehat{G}$. The set of representations of in $\widehat{G}$ with minimal K-type τ can be parametrized by $\widehat{A}(\tau)/W(\tau)$.

7.2.1 The Associated Semidirect Product Group

As in the previous section, let $G = \mathrm{SL}(2k+1, \mathbb{R})$ and let $K = \mathrm{SO}(2k+1)$, a maximal compact subgroup of G. Now that we have classified all tempered representations of G by minimal K-types, we will focus on the associated semidirect product group $G_0 = K \ltimes \mathfrak{g}/\mathfrak{k}$. Just as for $\mathrm{SL}(2k, \mathbb{R})$, we will follow the method outlined in Chapter 4 to classify all unitary representations of G_0 by minimal K-types. Just as in the previous chapter, we will then use this description of $\widehat{G}_0$ to prove Theorem 4.0.11 for G.

To begin, just as in the previous section, we have $K = \mathrm{SO}(2k+1)$, and thus

$$\widehat{\mathrm{SO}(2k+1)} \cong \{(a_1, \ldots, a_k) \in \mathbb{Z}^k \mid a_1 \geq a_2 \geq \cdots \geq a_{k-1} \geq a_k \geq 0\}.$$

We will again use the notation $\tau_{(a_1,\ldots,a_k)}$ to denote the representation of K with highest weight $(a_1,\ldots,a_k)$. Then in analogy with the previous section, we will define

$$\begin{aligned}
K_r &= M_r \cap K \\
&= \left\{ S = \begin{pmatrix} X_1 & & & & & \\ & \ddots & & & & \\ & & X_{k-r} & & & \\ & & & \epsilon_1 & & \\ & & & & \ddots & \\ & & & & & \epsilon_{2r+1} \end{pmatrix} \;\middle|\; X_i \in \mathrm{O}(2), \epsilon_i = \pm 1, \text{ and } \det(S) = 1 \right\}, \\
&\cong [\mathrm{O}(2)]^{k-r} \times (\mathbb{Z}/2\mathbb{Z})^{2r},
\end{aligned}$$

and

$$\mathfrak{a}_r = \mathrm{Lie}(A_r)$$

$$= \left\{ S = \begin{pmatrix} s_1 & & & & & & & \\ & s_1 & & & & & & \\ & & \ddots & & & & & \\ & & & s_{k-r} & & & & \\ & & & & s_{k-r} & & & \\ & & & & & t_1 & & \\ & & & & & & \ddots & \\ & & & & & & & t_{2r+1} \end{pmatrix} \middle| \operatorname{tr}(S) = 0 \right\}.$$

Now, the Mackey machine tells us that for every $\pi \in \widehat{G}_0$ there is a $\varphi \in \mathfrak{a}_k$ (viewed as a balanced character of $\mathfrak{g}/\mathfrak{k}$) and $\sigma \in \widehat{K}_\varphi$ such that

$$\pi \cong \operatorname{Ind}_{K_\varphi \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0} (\sigma \otimes \varphi).$$

However, we would like to classify $\widehat{G}_0$ using the alternate approach outlined in Section 4. We must now find the appropriate δ_0 associated to each K-type. To do so, we examine the representations of K_r.

Let $\chi_m \in \widehat{K}_k$ be defined by

$$\chi_r \begin{pmatrix} \epsilon_1 & & & \\ & \epsilon_2 & & \\ & & \ddots & \\ & & & \epsilon_{2k+1} \end{pmatrix} = \prod_{j=1}^{m} \epsilon_j .$$

Then combining the branching rules of Section 5.2 and the discussion on p.282 of [Vog85] gives us the following result.

Lemma 7.2.1. *Then the minimal* K*-type of* $\operatorname{Ind}_{K_k}^{K} \chi_m$, *is*

$$\begin{cases} \tau_{(\underbrace{1,\ldots,1}_{m\text{-}times},0,\ldots,0)} & \text{if } m < k \\ \tau_{(1,\ldots,1)} & \text{if } m = k. \quad \blacksquare \end{cases}$$

Now, if $r < k$, let $\tau = \tau_{(a_1,\ldots,a_{k-r})} \otimes \chi_m \in \widehat{K}_r$ with each $a_j > 0$, then the results from Section 5.2 govern how $\tau_{(a_1,\ldots,a_{k-r})}$ induces up to K, while the discussion on p.282 of [Vog85] (and above) governs how χ_m induces up to K. Combining these establishes the following, just as for $\mathrm{SL}(2k, \mathbb{R})$.

Lemma 7.2.2. *Let* $\tau = \tau_{(a_1,\ldots,a_{k-r})} \otimes \chi_m \in \widehat{K}_r$. *The minimal* K*-type of* $\operatorname{Ind}_{K_r}^{K} \tau$ *is*

$$\begin{cases} \tau_{(a_1,\ldots,a_{k-r},\underbrace{1,\ldots,1}_{m\text{-}times},0,\ldots,0)} & \text{if } m < r, \\ \tau_{(a_1,\ldots,a_{k-r}1,\ldots,1)} & \text{if } m = r. \end{cases}$$

Proof. From Section 5.2, we know that when we induce $\tau_{(a_1,\ldots,a_{k-r})}$ up to K, the minimal K-type is $\tau_{(a_1,\ldots,a_{k-r},0,\ldots,0)}$. Tensoring $\tau_{(a_1,\ldots,a_{k-r})}$ with χ_m has the effect of inserting m ± 1's after the a_j's (see [Vog85, p.282]). ■

Thus, we see that if $\tau = \tau_{(a_1,\ldots,a_k)} \in \widehat{K}$ with r of the a_j's equal to 0 or 1, and s of the a_j's equal to 1, we can once again define

$$\delta_0(\tau) = \tau_{(a_1,\ldots,a_{k-r})} \otimes \chi_s \in \widehat{K}_r. \tag{7.2.3}$$

Theorem 7.2.4. *Let $\tau \in \widehat{K}$, and let $K(\mathscr{C}(\tau))$ denote the group constructed in Definition 4.0.2. Then there is a $\delta_0 = \delta_0(\mathscr{C}(\tau)) \in \widehat{K}(\mathscr{C}(\tau))$ such that $\mathscr{C}(\tau)$ is the set of a minimal K-type of*

$$\mathrm{Ind}_{K(\tau)}^{K}(\delta_0).$$

The representation δ_0 is determined up to conjugation under the normalizer of A in K.

Proof. From the preceding lemma, we that the choice of δ_0 in (7.2.3) has the desired properties. ■

As a result, we have,

$$\mathfrak{a}(\tau) = \mathfrak{a}_r,$$

and for each $\tau \in \widehat{K}$, we can construct the family of irreducible representations $\{\overline{\pi}_0(\tau,\varphi) \,|\, \varphi \in \widehat{\mathfrak{a}}(\tau)\}$ with minimal K-type τ, following Conjecture 4.0.5 and Definition 4.0.6. Then, we can prove Theorem 4.0.8.

Theorem 7.2.5. *Suppose* $\pi = \mathrm{Ind}_{K_\varphi \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0}(\sigma \otimes \varphi) \in \widehat{G}_0$ *contains* $\tau = \tau_{(a_1,\ldots,a_k)}$ *as a minimal* K*-type. Then, we have*

$$\pi \cong \overline{\pi}_0(\tau, \varphi).$$

Proof. If $K_\varphi = K_r$, then (up to equivalence) $\tau_{(a_1,\ldots,a_{k-r})} \otimes \chi_s$ is the only representation in $\widehat{K}_r$ which has minimal K-type τ when induced up to K. Thus we must have $\sigma \cong \tau_{(a_1,\ldots,a_{k-r})} \otimes \chi_s$.

Now, if $K_\varphi \neq K_r$, there is a subgroup of K_φ conjugate to K_r, and so we may assume $K_r \subseteq K_\varphi$, without loss of generality. Then, combining the branching rules of Section 5.2 with Frobenius reciprocity, we must have

$$\sigma \in \mathrm{Ind}_{K_r}^{K_\varphi} \delta(\tau),$$

and since π is irreducible with minimal K-type τ, the result follows. ∎

Once again, since each $\overline{\pi_0}(\tau, \varphi)$ is equivalent to some Mackey datum, we may use the Mackey machine, as in previous examples, to determine the equivalences between these representations, proving Theorem 4.0.9.

Theorem 7.2.6. *The representations* $\overline{\pi}_0(\tau, \varphi_1)$ *and* $\overline{\pi}_0(\tau, \varphi_2)$ *are equivalent if and only if* $\varphi_1 = w \cdot \varphi_2$ *for some* $w \in W(\mathscr{C}(\tau))$.

Proof. We have $\overline{\pi}_0(\tau, \varphi_1) = \mathrm{Ind}_{M \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0}(\delta \otimes \varphi_1)$, and $\overline{\pi}_0(\tau, \varphi_2) = \mathrm{Ind}_{M \ltimes \mathfrak{g}/\mathfrak{k}}^{G_0}(\delta \otimes \varphi_2)$. The Mackey machine tells us that these two representations are equivalent if and only if the pair (δ, φ_1) is conjugate to the pair (δ, φ_2) under the action of the Weyl group. Thus, we must have φ_1 and φ_2 are conjugate under the action of W_δ, the isotropy subgroup of δ in W. But, by definition, $W_\delta = W(\tau)$. ∎

Thus, for each $\tau \in \widehat{K}$ we have a complete description of the set of (equivalence classes of) irreducible unitary representations with minimal K-type τ – they can be parametrized by the set $\widehat{\mathfrak{a}}(\tau)/W(\tau)$, in analogy with Section 7.2, which proves Theorem 4.0.10.

Theorem 7.2.7. *To each $\tau \in \widehat{K}$, attach a special subalgebra $\mathfrak{a}(\tau)$, and a series of representations $\{\pi_0(\varphi) \,|\, \varphi \in \widehat{\mathfrak{a}}(\tau)\}$. Then the correspondence which associates to each pair (τ, φ), where $\varphi \in \widehat{\mathfrak{a}}(\tau)/W(\tau)$, the representation*

$$\overline{\pi}_0(\tau, \varphi) \in \widehat{G}_0,$$

is a one-to-one correspondence onto $\widehat{G}_0$. The minimal K-type of $\overline{\pi}_0(\tau, \varphi)$ is τ.

So, we now have a clear description of the unitary dual of G_0, organized by minimal K-types, and once again, it is precisely the same as that for $\widehat{G}$. Thus, we have established Theorem 4.0.11 for $G = \mathrm{SL}(2k+1, \mathbb{R})$.

Theorem 7.2.8. *The correspondence which associates to the representation $\overline{\pi}(\tau, \varphi) \in \widehat{G}$ the representation $\overline{\pi}_0(\tau, \varphi) \in \widehat{G}_0$ is a well-defined bijection preserving* *minimal K-types.* ■

Note, that the bijection, just as with previous examples, is a homeomorphism at the level of minimal K-types.

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