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DYNAMIC PRICING, COMPETITION AND UNCERTAINTY

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Changhyun Kwon

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The dissertation of Changhyun Kwon was reviewed and approved* by the following:

Terry L. Friesz
Harold and Inge Marcus Chaired Professor of Industrial Engineering
Director of the Center for Service and Enterprise Engineering
Dissertation Advisor, Chair of Committee

Tom M. Cavalier
Professor of Industrial Engineering

Tao Yao
Assistant Professor of Industrial Engineering

Robert D. Weaver
Professor of Agricultural Economics

Richard J. Koubek
Professor of Industrial Engineering
Head of the Harold and Inge Marcus Department of Industrial and Manufacturing Engineering

*Signatures are on file in the Graduate School.

Abstract

In this thesis, we study various dynamic pricing problems in the form of infinite-dimensional mathematical programming, i.e., optimal control problems and differential variational inequalities (DVIs), in support of accurate and efficient algorithms. In addition, we describe a method for handling uncertainty in optimal control problems via a robust optimization approach. We provide a formal method for handling uncertainty in the objective function, which may be nonlinear in states, controls, and uncertain parameters.

To develop an algorithm for DVIs, we consider a gap function for DVIs and study an equivalent optimal control problem. In particular, we employ a differential gap function and its gradient to form a descent method for DVIs. To show the descent method is effective, we investigate an application of DVIs to differential Nash games. In particular, we solve an abstract linear-quadratic differential Nash game using our proposed descent method.

We study dynamic pricing problems of three different classes: (1) infrastructure pricing, (2) service pricing, and (3) manufactured good pricing. First, we present a theory of dynamic congestion pricing for vehicular traffic networks; we consider day-to-day as well as within-day time scales in the formulation of a dynamic optimal toll problem with equilibrium constraints. Second, we study a non-cooperative differential game between service providers using a demand learning mechanism. Third, we provide a robust optimal control formulation of a dynamic pricing and inventory control problem in the presence of demand uncertainty.

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To my little angel,
Bomin

Chapter 1

Introduction

The optimal setting of prices for goods has historically been a very important economic issue. The objectives of optimal pricing include maximizing one's profit, minimizing one's cost, producing an efficient and effective economic system, and maximizing social welfare. The strategy of dynamic pricing has emerged during the last few decades, and is associated with the information technology revolution. With current information technology, we are able to observe immediate changes in our economic system in real-time. With such data, we can modify pricing strategies on the basis of computational decision-making systems supported by large-capacity, low-cost computing hardware. The objective of this thesis is to develop computationally attractive dynamic pricing strategies for various applications that reflect the current dynamic and immediate decision-making environment. To this end, we are interested in infinite-dimensional mathematical programming with differential equations; we provide their mathematical backgrounds and offer related new developments.

As the world economy has grown and economic systems have changed, pricing has gone beyond manufactured and agricultural goods. The subject of pricing now includes financial derivatives, infrastructure systems, services, and many other non-traditional products. In this thesis, we are interested in solving dynamic pricing problems to answer the following questions:

- How may toll prices be set dynamically to maximize the social welfare of vehicular network users? (Chapter 4)

- What will be the market equilibrium of service prices resulting from competition between revenue maximizers using demand learning mechanisms? (Chapter 5)
- How can we, with proper dynamic pricing, avoid the failure of inventory systems while pursuing maximum profit? (Chapter 6)

Each of these questions is answered by a chapter in this thesis. The following sections provide introductions for these chapters.

1.1 Theory and Algorithms of Infinite-Dimensional Mathematical Programming

The aim of Chapter 2 is to provide mathematical backgrounds for the study of dynamic pricing problems presented in this thesis. In particular, we define optimal control problems, robust optimal control problems and differential variational inequalities (DVIs) and study their mathematical properties. We also describe how those infinite-dimensional mathematical programming problems may be solved by a descent method in Hilbert spaces and a fixed point algorithm.

Since the development of the Maximum (Minimum) Principle by Pontryagin et al. (1962), the theory of optimal control has been applied to many economic problems. We refer Bryson and Ho (1975) for the general theory and Bensoussan et al. (1974) and Sethi and Thompson (2000) for the applications in problems of economics, management and decision making.

To extend the philosophy of robust optimization to uncertain optimal control problems, we further define robust optimal control problems. Robust optimization has become a popular technique for representing optimization problems in which the uncertainty of parameters cannot be ignored. Application areas include business decision making, optimal design in engineering, transportation planning, signal processing and more; see Ben-Tal and Nemirovski Ben-Tal and Nemirovski (2008) for a brief survey of robust optimization applications. In such problems, it is impossible to obtain precise and exact values of design parameters. A robust optimization approach generally assumes that the decision maker has a good estimate of parameter values and knowledge of the set of uncertainty from which the parameters are drawn. That is, for a one-dimensional uncertain parameter, the decision maker recognizes the nominal, minimal and maximal values of the parameter. For a larger number of uncertain parameters, a set of uncertainty may be modeled as an interval, a box constrained set, an ellipsoid, etc.

The theory of robust optimization has been developed by many scholars. To name a few, we refer to Ben-Tal and Nemirovski (2002), Ben-Tal and Nemirovski (2008), Bertsimas and Sim (2004), Bertsimas and Sim (2006), El Ghaoui et al. (1998), Zhang (2007) and references therein. We note that Zhang (2007) provides an approximate robust optimization approach to general nonlinear programming problems, while other research is limited to the problems with linear constraints. The method of Zhang (2007) is intensively tested for various optimization problems in Hale and Zhang Hale and Zhang (2007). In addition, the approach in Zhang (2007) is readily applicable to continuous-time optimal control problems with uncertainty; however, this approach gives a formulation which is only locally valid around nominal values, hence it may be called *locally robust.* In Chapter 2, we adopt and improve the approach of Zhang (2007) to properly accommodate uncertainty in a nonlinear criterion functional.

DVIs are infinite-dimensional variational inequalities constrained by ordinary differential equations, called state dynamics in optimal control theory. DVIs arise in mechanics, mathematical economics, transportation research, and many other complex engineering systems. Pang and Stewart (2007) defined DVIs formally, and present illustrative applications to differential Nash games. Moreover, Friesz et al. (2006) used a DVI to model shippers' oligopolistic competition on networks. The DVI of interest here should not be confused with the differential variational inequality of Aubin and Cellina (1984), which may be called a variational inequality of evolution, a name suggested by Pang and Stewart (2007).

1.2 Descent Method for Differential Variational Inequalities

In Chapter 3, we develop a descent method for solving DVIs. Analysis and algorithms for variational inequalities, especially for finite-dimensional problems, have been studied in depth over the last two decades. See, in particular, Harker and Pang (1990) and Facchinei and Pang (2003). Among the algorithms to solve variational inequality problems, descent methods with gap functions allows a variational inequality problem to be converted to an equivalent optimization problem whose optimal objective value is zero if and only if the optimal solution solves the original variational inequality problem. A number of algorithms in this class have been developed for finite-dimensional variational inequalities including Zhu and Marcotte (1994), Yamashita et al. (1997), Patriksson (1997), and Peng (1997). For infinite-dimensional problems, Zhu and Marcotte (1998) and Konnov et al. (2002) developed descent methods based on gap functions in Banach spaces and Hilbert spaces, respectively. Moreover, Konnov and Kum (2001) provided a descent method for mixed variational inequalities in Hilbert spaces. In Chapter 3, we propose a descent method for DVIs involving explicit control variables and explicit state dynamics based on proper gap functions for DVIs. In particular, we show that the framework of gap functions considered by Konnov et al. (2002) and Konnov and Kum (2001) for problems *without* explicit state dynamics and explicit controls can be extended to the DVI setting. Due to the presence of the state dynamics, the gap functions will form an optimal control problem, whose optimum also solves the original DVI when the optimal objective is zero.

1.3 Dynamic Pricing of Infrastructure

In Chapter 4, we present a theory of dynamic congestion pricing for the day-to-day as well as the within-day time scales. The advent of new commitments by municipal, state and federal governments to construct and operate roadways whose tolls may be set dynamically has brought into sharp focus the need for a computable theory of dynamic tolls. Moreover, it is clear from the policy debates that surround the issue of dynamic tolls that pure economic efficiency is not the sole or even the most prominent objective of any dynamic toll mechanism that will be implemented. Rather, equity considerations as well as preferential treatment for certain categories of commuters must be addressed by such a mechanism. Accordingly, we introduce in this chapter the dynamic user equilibrium optimal toll problem and discuss two plausible algorithms for its solution; we also provide detailed numerical results that document the performance of the two algorithms.

The dynamic user equilibrium optimal toll problem should not be considered a simple dynamic extension of the traditional congestion pricing paradigm associated with static user equilibrium and usually accredited to Beckmann et al. (1956). Rather, the dynamic user equilibrium optimal toll problem is most closely related to the equilibrium network design problem which is now widely recognized to be a specific instance of a mathematical program with equilibrium constraints (MPEC). In fact it will be convenient to refer to the dynamic user equilibrium optimal toll problem as the dynamic optimal toll problem with equilibrium constraints or DOTPEC, where it is understood that the equilibrium of interest is a dynamic user equilibrium.

The relevant background literature for the DOTPEC includes a paper by Friesz et al. (2002) who discuss a version of the DOTPEC but for the day-to-day time scale rather than the dual (within-day as well as day-to-day) time scale formulation emphasized. Also pertinent are the paper by Friesz et al. (1996) which discusses dynamic disequilibrium network design and the review by Liu (2004) which considers multi-period efficient tolls. Although the DOTPEC is not the same as the problem of determining efficient tolls including the latter's multiperiod generalization, the exact nature of the differences and similarities is not known and has never been studied. To study the DOTPEC, it is necessary to employ some form of dynamic user equilibrium model. We elect the formulation due to Friesz et al. (2001), Friesz and Mookherjee (2006) and its varieties analyzed by Ban et al. (2005) and others. The dynamic efficient toll formulation will be constructed by direct analogy to the static efficient toll problem formulation of Hearn and Yildirim (2002). We describe how an infnite dimensional mathematical programming perspective may be employed to create an algorithm for the DOTPEC. A numerical example is provided.

1.4 Dynamic Pricing of Service

We study a non-cooperative differential game between service providers with demand learning mechanism in Chapter 5. In the rapidly growing literature on revenue management – see McGill and van Ryzin (1999) and Talluri and van Ryzin

(2004) for comprehensive studies and a survey – one of the most important issues is how to model service provider demand learning. Demand is usually represented as a function of price, explicitly and/or implicitly, and the root tactic upon which revenue management is based is to change prices dynamically to maximize immediate or short-run revenue. In this sense, the more accurate the model of demand employed in revenue optimization, the more revenue we can generate. Although demand may be viewed theoretically as the result of utility maximization, an actual demand curve and its parameters are generally unobservable in most markets. In this chapter, we first describe the dynamics of demand as a differential equation based on an evolutionary game theory perspective and then observe the actual sales data to obtain estimates of parameters that govern the evolution of demand. A dynamic non-zero sum evolutionary game among service providers is expressed, in Chapter 5, as a differential variational inequality. The providers also have fixed upper bounds on output as each faces capacity constraints on available resources.

Among many others, the demand learning approach proposed in Chapter 5 is based on the Kalman Filter. As a state-space estimation method, Kalman filtering has gained attention in economics and revenue management as one of the most successful forecasting methods. The Kalman filter, developed by Kalman (1960) and Kalman and Bucy (1961), originally to filter out system and sensor noise in electrical/mechanical systems, is based on a system of linear state dynamics, observation equations and normally distributed noise terms with mean zero. The Kalman filter provides an *a prior* estimate for the model parameter, which is adjusted to *a posteriori* estimate by combining the observations. With this new model parameter, one repeatedly solves the dynamic pricing problem to maximize revenue in the next time period. For a detailed discussion and derivations of Kalman filter dynamics, see Bryson and Ho (1975), or, for an introduction suitable for revenue management researchers, see Talluri and van Ryzin (2004). We provide numerical examples of the revenue management model along with a parameter estimation mechanism based on Kalman filtering.

1.5 Dynamic Pricing of Manufactured Goods

In Chapter 6, we provide a robust optimal control formulation of a dynamic pricing and inventory control problem in the presence of demand uncertainty. Some recent literature on dynamic pricing and/or inventory control problems in the robust optimization framework includes the works of Adida and Perakis (2006), Ben-Tal et al. (2005), Bertsimas and Thiele (2006) and Perakis and Sood (2006). In this chapter, given its focus on pricing and inventory management, we are particularly interested in the model studied by Adida and Perakis (2006). To the best of our knowledge, Adida and Perakis (2006) is the first application paper to model a dynamic pricing and inventory control problem as a robust optimal control problem in continuous time. Since the nominal dynamic pricing and inventory control problem considered in this chapter is very similar to that of Adida and Perakis (2006), it is interesting to note some of the similarities and differences of the robust formulations of problem. The robust formulation herein is locally valid around the nominal parameter values and is thus appropriate for small to modest ranges of uncertainty, while the model of Adida and Perakis is globally valid. Both modeling approaches allow the decision maker to decide the level of uncertainty and provides protection from the uncertainty in inequality constraints. Adida and Perakis note that nonlinearity in both control variables and uncertain parameters of the criterion functional is hard to accommodate in a global robust optimization perspective; therefore, they provide a robust formulation with a partially uncertain criterion functional. However, our model allows nonlinear inventory holding costs and is general enough to handle nonlinear functions of the uncertain parameters. The robust formulation minimizes the impact of uncertainty on the criterion over the time horizon. The notion of maximum allowable magnitude of uncertainty is defined and analyzed and leads to some qualitative pricing and inventory management insights. The robust optimal control formulation of pricing and inventory management is shown to be numerically tractable using a numerical example that is solved via off-the-shelf software.

Chapter 2

Preliminaries

In this chapter, we provide mathematical backgrounds for studying dynamic pricing problems. To define infinite-dimensional mathematical programming problems and study their properties, we will need some results of functional analysis. A very useful discussion on functional analysis and infinite-dimensional mathematical programming is found in Minoux (1986).

Definition 1 (G-differentiability) *A functional J is Gateaux differentiable or G-differentiable at $v \in V$ in the direction $\varphi \in V$, if the limit*

$$\lim_{\theta \to 0} \frac{J(v + \theta\varphi) - J(v)}{\theta} \tag{2.1}$$

exists. This limit is denoted by $\delta J(v, \varphi)$.

Theorem 2 (Riesz) *Let V be a Hilbert space and $L \in V^*$ a continuous linear form on V. Then there exists a unique element $u_L \in V$ such that*

$$\forall v \in V : \qquad L(v) = \langle u_L, v \rangle$$

and

$$\| L \|_{V^*} = \| u_L \|_V .$$

Conversely, we can associate with each $u \in V$ the continuous linear form L_u defined by

$$\forall v \in V : \qquad L_u(v) = \langle u, v \rangle .$$

Definition 3 (Gradient) *Let V be a Hilbert space with the scalar product $\langle \cdot, \cdot \rangle$. If J is G-differentiable at $v \in V$, and if $\delta J(v, \varphi)$ is a* continuous linear form *with respect to φ, then, by Theorem of 2, there exists an element $\frac{\partial J}{\partial v} \in V$ such that*

$$\forall \varphi \in V : \qquad \delta J(v, \varphi) = \left\langle \frac{\partial J}{\partial v}, \varphi \right\rangle .$$

$\frac{\partial J}{\partial v}$ *is called the* gradient of J at v.

2.1 Optimal Control Problems Defined

We consider the following optimal control problem:

$$\min_u J = K\left(x\left(t_f\right), t_f\right) + \int_{t_0}^{t_f} F\left(x\left(t\right), u\left(t\right), t\right) dt \tag{2.2}$$

subject to

$$\frac{dx\left(t\right)}{dt} = f\left(x\left(t\right), u\left(t\right), t\right) \quad \forall t \in \left[t_0, t_f\right] \tag{2.3}$$

$$x\left(t_0\right) = x_0 \tag{2.4}$$

$$u\left(t\right) \in U \tag{2.5}$$

where

$$\begin{aligned}
u &\in U \subseteq \mathbb{R}^m \\
x &\in \left(\mathcal{H}^1\left[t_0, t_f\right]\right)^n \\
x^0 &\in \mathbb{R}^n \\
F &: \left(\mathcal{H}^1\left[t_0, t_f\right]\right)^n \times \left(L^2\left[t_0, t_f\right]\right)^m \times \mathbb{R}^1_+ \longrightarrow L^2\left[t_0, t_f\right] \\
f &: \left(\mathcal{H}^1\left[t_0, t_f\right]\right)^n \times \left(L^2\left[t_0, t_f\right]\right)^m \times \mathbb{R}^1_+ \longrightarrow \left(L^2\left[t_0, t_f\right]\right)^n \\
K &: \left(\mathcal{H}^1\left[t_0, t_f\right]\right)^n \times \mathbb{R}^1_+ \longrightarrow \mathbb{R}^1
\end{aligned}$$

We denote x by state variables, u by control variables, t by time, and ε by uncertain parameters. In the remainder of this thesis, we suppress the time argument t when the meaning is obvious. Note that $\left(L^2\left[t_0, t_f\right]\right)^m$ is the m-fold product of the space of square-integrable functions $L^2[t_0, t_f]$ and the inner product defined by

$$\langle x(t), y(t)\rangle \equiv \int_{t_0}^{t_f} [x(t)]^T y(t)\, dt \tag{2.6}$$

while $(\mathcal{H}^1[t_0, t_f])^n$ is the n-fold product of the Sobolev space $\mathcal{H}^1[t_0, t_f]$. We assume this parametric optimal control problem satisfies the regularity conditions defined as follows:

Definition 4 (Regularity) *The parametric optimal control problem (2.2)-(2.5) is regular if the following conditions hold:*

1. $F(x, u, t; \varepsilon)$ *is convex and continuously differentiable with respect to* x,u *and* ε;
2. $f(x, u, t; \varepsilon)$ *is convex and continuously differentiable with respect to* x *and* u;
3. $f(x, u, t; \varepsilon)$ *is continuously differentiable with respect to* ε;
4. $f(x, u, t; \varepsilon)$ *and* $\partial f(x, u, t; \varepsilon)/\partial x$ *are bounded so that the operator*

   $$x(u, t) : \left(L^2[t_0, t_f]\right)^m \times \mathbb{R}^1_+ \longrightarrow \left(\mathcal{H}^1[t_0, t_f]\right)^n$$

   exists and is continuous and Gateaux-differentiable with respect to u;
5. $K(x, t)$ *is continuously differentiable with respect to* x;
6. U *is non-empty, convex and compact; and*
7. $x^0 \in \mathbb{R}^n$ *is known and fixed.*

When the regularity conditions hold, the operator $x(u, t)$ exists which is implicitly defined by the state dynamics $\frac{dx}{dt} = f(x, u, t)$. Further, the operator $x(u, t)$ is continuous and Gateaux-differentiable by the following two theorems, which are reproduced from Theorem 4.2 and 4.4 of Bressan and Piccoli (2007), respectively.

Theorem 5 *Suppose the regularity conditions in Definition (4) hold. Then, for every* $t_f > t_0$, $u \in U$, *the Cauchy problem*

$$\frac{dx(t)}{dt} = f(x(t), u(t), t) \quad \forall t \in [t_0, t_f]; \qquad x(t_0) = x_0 \tag{2.7}$$

has a unique solution $x(u,t)$ defined for all $t \in [t_0, t_f]$. The input-output map $u(t) \longmapsto x(u,t)$ is continuous.

Theorem 6 *Suppose the regularity conditions in Definition (4) hold. Let $u(t) \in U$ be a control whose corresponding solution $x(u,t)$ of (2.7) is defined on $[t_0, t_f]$. Then, for every bounded measurable $\Delta u(t)$ and every $t \in [t_0, t_f]$, the map $\delta \longmapsto x(u + \delta \Delta u, t)$ is differentiable, hence the operator $x(u,t)$ is Gateaux-differentiable.*

Now we give the gradient of the optimal control problem (2.2)-(2.5).

Theorem 7 *The objective functional J in (2.2)-(2.5) is continuously differentiable in the sense of Gateaux, and*

$$\nabla J(u) = \frac{\partial}{\partial u} H(x, u, \lambda, t)$$

where H is the Hamiltonian

$$H \equiv F(x, u, t) + \lambda^T f(x, u, t)$$

and the adjoint variable λ is a solution of the following final value problem:

$$\begin{aligned} -\frac{d\lambda}{dt} &= \left(\frac{\partial f}{\partial x}\right)^T \lambda + \left(\frac{\partial g}{\partial x}\right)^T \\ \lambda(t_f) &= \frac{\partial K[x(t_f), t_f]}{\partial x(t_f)} \end{aligned}$$

Proof. The derivative of J in the direction ρ becomes, by Lemma 17,

$$\delta J(u; \rho) = \frac{\partial K(x(t_f), t_f)}{\partial x(t_f)} y(t_f) + \int_{t_0}^{t_f} \left\{ \frac{\partial F}{\partial x} y + \frac{\partial F}{\partial u} \rho \right\} dt \tag{2.8}$$

where $y \equiv \delta x$ is a variation in x which implicitly depends on ρ. Furthermore, by definition

$$x(t) = x_0 + \int_{t_0}^{t} f(x, u, s)\, ds$$

and therefore

$$y \equiv \delta x = \int_{t_0}^{t} \left[\frac{\partial f}{\partial x} y + \frac{\partial f}{\partial u} \rho \right] ds$$

We introduce adjoint variables λ defined by the final value problem

$$-\frac{d\lambda}{dt} = \left(\frac{\partial F}{\partial x}\right)^T + \left(\frac{\partial f}{\partial x}\right)^T \lambda \tag{2.9}$$

$$\lambda\left(t_f\right) = \frac{\partial K\left[x\left(t_f\right), t_f\right]}{\partial x\left(t_f\right)} \tag{2.10}$$

so that (2.8) becomes

$$\delta J\left(u;\rho\right) = \lambda\left(t_f\right) y\left(t_f\right) + \int_{t_0}^{t_f} \left\{\left[-\left(\frac{d\lambda}{dt}\right)^T - \lambda^T \frac{\partial f}{\partial x}\right] y + \frac{\partial F}{\partial u}\rho\right\} dt$$

The integration by parts yields

$$\begin{aligned}\int_{t_0}^{t_f} -\left(\frac{d\lambda}{dt}\right)^T y dt &= -\lambda\left(t_f\right) y\left(t_f\right) + \int_{t_0}^{t_f} \lambda^T \frac{dy}{dt} dt \\ &= -\lambda\left(t_f\right) y\left(t_f\right) + \int_{t_0}^{t_f} \lambda^T \left[\frac{\partial f}{\partial x} y + \frac{\partial f}{\partial u}\rho\right] dt\end{aligned}$$

It follows that

$$\begin{aligned}\delta J\left(u;\rho\right) &= \lambda\left(t_f\right) y\left(t_f\right) - \lambda\left(t_f\right) y\left(t_f\right) + \int_{t_0}^{t_f} \left\{\lambda^T \left[\frac{\partial f}{\partial x} y + \frac{\partial f}{\partial u}\rho\right] - \lambda^T \frac{\partial f}{\partial x} y + \frac{\partial F}{\partial u}\rho\right\} dt \\ &= \int_{t_0}^{t_f} \left\{\lambda^T \frac{\partial f}{\partial u} + \frac{\partial F}{\partial u}\right\} \rho dt \\ &= \left\langle \lambda^T \frac{\partial f}{\partial u} + \frac{\partial F}{\partial u}, \rho \right\rangle\end{aligned}$$

Therefore, by Theorem 3, the gradient of $J\left(u\right)$ becomes

$$\begin{aligned}\nabla J\left(u\right) &= \lambda^T \frac{\partial f}{\partial u} + \frac{\partial F}{\partial u} \\ &= \frac{\partial H}{\partial u}\end{aligned}$$

This completes the proof. ■

Now we are ready to state the necessary conditions for the optimal control problem (2.2)-(2.5).

Theorem 8 *A necessary condition for $u^* \in U$ to be a minimum solution of (2.2)-*

(2.5) is

$$\left\langle \frac{\partial H\left(x^{*},u^{*},\lambda^{*},t\right)}{\partial u}, u-u^{*}\right\rangle \geq 0 \text{ for all } u\in U \tag{2.11}$$

Proof. Let $u \in U$ be arbitrary. Since U is convex, $u^{*} \in U$ implies

$$u^{*}+\theta\left(u-u^{*}\right)\in U \quad \forall\theta\in[0,1]$$

Hence for u^{*} to be a minimum of J on U, it is necessary that $\forall u \in U$

$$\left[\frac{d}{d\theta}J\left(u^{*}+\theta\left(u-u^{*}\right)\right)\right]_{\theta=0} = \delta J(u^{*},u-u^{*}) \geq 0.$$

Since J is G-differentiable at u and δJ is well-defined by Theorem 3, we have

$$\begin{aligned}\delta J(u^{*},u-u^{*}) &= \left\langle \lambda^{T}\frac{\partial f\left(x^{*},u^{*},t\right)}{\partial u}+\frac{\partial F\left(x^{*},u^{*},t\right)}{\partial u}, u-u^{*}\right\rangle \\ &= \left\langle \frac{\partial H\left(x^{*},u^{*},\lambda^{*},t\right)}{\partial u}, u-u^{*}\right\rangle \geq 0 \qquad \forall u\in U\end{aligned}$$

Hence proof. ■

In fact, (2.11) is a variational inequality, which will be introduced later in this chapter.

2.2 Robust Optimal Control Problem Defined

Recently, Zhang (2007) introduced a robust-optimization approach for nonlinear programming which may also be applied to optimal control problems. The method is applicable when a decision maker has reasonable estimates for uncertain model parameters and the magnitude of uncertainty is moderate. Zhang (2007) uses a local linear approximation based on a Taylor series expansion to obtain a locally robust formulation. While the robust optimization method developed by Bertsimas and Sim (2004) and Bertsimas and Sim (2006) uses duality theory to derive a globally robust formulation, their method is limited to convex conic programming. On the other hand, while Zhang (2007) gave a robust counterpart formulation appropriate for general nonlinear constraint sets, uncertainty in the objective function was not considered. In this section, we succeed and improve the methodology of

Zhang (2007) to accommodate this weakness.

We consider an optimal control problem with uncertainty, which is of the form of a parametric optimal control problem:

$$\min_{u} J = \int_{t_0}^{t_f} F\left(x\left(t\right), u\left(t\right), t; \varepsilon\left(t\right)\right) dt \tag{2.12}$$

subject to

$$\frac{dx\left(t\right)}{dt} = f\left(x\left(t\right), u\left(t\right), t; \varepsilon\left(t\right)\right) \quad \forall t \in \left[t_0, t_f\right] \tag{2.13}$$

$$x\left(t_0\right) = x_0 \tag{2.14}$$

$$g\left(x\left(t\right), u\left(t\right), t; \varepsilon\left(t\right)\right) \leq 0 \tag{2.15}$$

$$h\left(x\left(t_f\right), t_f; \varepsilon\left(t_f\right)\right) \leq 0 \tag{2.16}$$

$$u\left(t\right) \in U \tag{2.17}$$

where

$$\begin{aligned}
&u \in U \subseteq \mathbb{R}^m \\
&x \in \left(\mathcal{H}^1\left[t_0, t_f\right]\right)^n \\
&\varepsilon \in \left(L^2\left[t_0, t_f\right]\right)^s \\
&x^0 \in \mathbb{R}^n \\
&F : \left(\mathcal{H}^1\left[t_0, t_f\right]\right)^n \times \left(L^2\left[t_0, t_f\right]\right)^m \times \mathbb{R}^1_+ \times \left(L^2\left[t_0, t_f\right]\right)^s \longrightarrow L^2\left[t_0, t_f\right] \\
&f : \left(\mathcal{H}^1\left[t_0, t_f\right]\right)^n \times \left(L^2\left[t_0, t_f\right]\right)^m \times \mathbb{R}^1_+ \times \left(L^2\left[t_0, t_f\right]\right)^s \longrightarrow \left(L^2\left[t_0, t_f\right]\right)^n \\
&g : \left(\mathcal{H}^1\left[t_0, t_f\right]\right)^n \times \left(L^2\left[t_0, t_f\right]\right)^m \times \mathbb{R}^1_+ \times \left(L^2\left[t_0, t_f\right]\right)^s \longrightarrow \left(L^2\left[t_0, t_f\right]\right)^l \\
&h : \left(\mathcal{H}^1\left[t_0, t_f\right]\right)^n \times \mathbb{R}^1_+ \times \left(L^2\left[t_0, t_f\right]\right)^s \longrightarrow \left(L^2\left[t_0, t_f\right]\right)^l
\end{aligned}$$

We denote x by state variables, u by control variables, t by time, and ε by uncertain parameters. In the remainder of this thesis, we suppress the time argument t when the meaning is obvious. Note that $\left(L^2\left[t_0, t_f\right]\right)^m$ is the m-fold product of the space of square-integrable functions $L^2\left[t_0, t_f\right]$ and the inner product is again defined by

$$\left\langle x\left(t\right), y\left(t\right)\right\rangle \equiv \int_{t_0}^{t_f} \left[x\left(t\right)\right]^T y\left(t\right) dt \tag{2.18}$$

while $\left(\mathcal{H}^1\left[t_0, t_f\right]\right)^n$ is the n-fold product of the Sobolev space $\mathcal{H}^1\left[t_0, t_f\right]$. We assume this parametric optimal control problem satisfies the regularity conditions defined as follows:

Definition 9 (Regularity) *The parametric optimal control problem (2.12)-(2.17) is regular if the following conditions hold:*

1. $F(x, u, t; \varepsilon)$ *is convex and continuously differentiable with respect to* x,u *and* ε*;*
2. $f\left(x, u, t; \varepsilon\right)$ *is convex and continuously differentiable with respect to* x *and* u*;*
3. $f\left(x, u, t; \varepsilon\right)$ *is continuously differentiable with respect to* ε*;*
4. $f\left(x, u, t; \varepsilon\right)$ *and* $\partial f\left(x, u, t; \varepsilon\right)/\partial x$ *are bounded;*
5. $g\left(x, u, t; \varepsilon\right)$ *is continuously differentiable with respect to* x, u *and* ε*;*
6. $h\left(x\left(t_f\right), t_f; \varepsilon\left(t_f\right)\right)$ *is continuously differentiable with respect to* $x\left(t_f\right)$ *and* $\varepsilon\left(t_f\right)$*;*
7. U *is non-empty, convex and compact; and*
8. $x^0 \in \mathbb{R}^n$ *is known and fixed.*

When the regularity conditions hold, the operator $x\left(u, t; \varepsilon\right)$ exists which is implicitly defined by the state dynamics $\frac{dx}{dt} = f\left(x, u, t; \varepsilon\right)$. Further, the operator $x\left(u, t; \varepsilon\right)$ is continuous and Gateaux-differentiable by Theorems 5 and 6, respectively. In addition, we assume that the uncertain parameter ε is drawn from the set of uncertainty:

$$\varepsilon \in E \equiv \{\hat{\varepsilon} + \tau D\delta : \|\delta\| \leq 1\} \subset \left(L^2\left[t_0, t_f\right]\right)^s \tag{2.19}$$

where $D \subset \mathbb{R}^{s\times s}$ is the uncertainty-parameter incidence matrix, $\tau \in \mathbb{R}^1_{++}$ is the magnitude of uncertainty, and $\hat{\varepsilon}$ is the decision maker's best guess of the value of the uncertain parameters. The norm in (2.19) is the L^2 norm, $\|\cdot\|_{L^2}$, corresponding to the inner product defined by (2.18).

Before we proceed, we reformulate above the parametric problem (2.12) - (2.17) by introducing a new state variable y:

$$\frac{dy}{dt} = F\left(x, u, t; \varepsilon\right), \;\; y\left(t_0\right) = 0 \tag{2.20}$$

that is

$$y\left(t\right) = \int_{t_0}^{t} F\left(x, u, t; \varepsilon\right) dt \tag{2.21}$$

Then the equivalent reformulation is

$$\min_u J = y\left(t_f; \varepsilon\right) \tag{2.22}$$

subject to

$$\begin{aligned} \frac{dy}{dt} &= F\left(x, u, t; \varepsilon\right), \quad y\left(t_0\right) = 0 \\ \frac{dx}{dt} &= f\left(x, u, t; \varepsilon\right), \quad x\left(t_0\right) = x_0 \\ g\left(x, u, t; \varepsilon\right) &\leq 0 \\ h\left(x\left(t_f\right), t_f; \varepsilon\left(t_f\right)\right) &\leq 0 \\ u &\in U \end{aligned}$$

The robust optimal control problem of our interest to minimize the *worst-case* criterion functional is

$$\min_u R \tag{2.23}$$

subject to

$$R = \max_{\varepsilon \in E} y\left(t_f; \varepsilon\right) \tag{2.24}$$

$$\frac{dy}{dt} = F\left(x, u, t; \varepsilon\right), \quad y\left(t_0\right) = 0 \tag{2.25}$$

$$\frac{dx}{dt} = f\left(x, u, t; \varepsilon\right), \quad x\left(t_0\right) = x_0 \tag{2.26}$$

$$g\left(x, u, t; \varepsilon\right) \leq 0 \qquad \forall \varepsilon \in E \tag{2.27}$$

$$h\left(x\left(t_f\right), t_f; \varepsilon\left(t_f\right)\right) \leq 0 \tag{2.28}$$

$$u \in U \tag{2.29}$$

where we call R the *robust index*. In what follows, we investigate the effect of uncertainty on the problem (2.23) - (2.29). We improve the work of Zhang (2007) to provide the robust counterpart formulation that includes analysis of the effect of uncertainty to the criterion functional.

2.2.1 Effect of Uncertainty on State Dynamics

In addition to the original state dynamics, we have an additional state dynamics induced by the introduction of a new state variable y. Hence, we have two state dynamics:

$$\frac{dy}{dt} = F(x, u, t; \varepsilon), \quad y(t_0) = 0 \tag{2.30}$$

$$\frac{dx}{dt} = f(x, u, t; \varepsilon), \quad x(t_0) = x_0 \tag{2.31}$$

Differentiating both sides of equations w.r.t. ε under the regularity conditions 1 and 2 in Definition 9, we obtain

$$\frac{d}{dt}\frac{\partial y}{\partial \varepsilon} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial \varepsilon} + \frac{\partial F}{\partial \varepsilon} \tag{2.32}$$

$$\frac{d}{dt}\frac{\partial x}{\partial \varepsilon} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \varepsilon} + \frac{\partial f}{\partial \varepsilon} \tag{2.33}$$

We note that (2.32) and (2.33) are matrix equations with appropriate Jacobian matrices. There is no effect of uncertainty on the initial conditions because they are explicitly known and not subject to uncertainty. Hence,

$$\frac{\partial y}{\partial \varepsilon}(t_0) = 0, \quad \frac{\partial x}{\partial \varepsilon}(t_0) = 0$$

We define

$$\alpha(t; \varepsilon) \equiv \frac{\partial x}{\partial \varepsilon}(t; \varepsilon)$$

$$\beta(t; \varepsilon) \equiv \frac{\partial y}{\partial \varepsilon}(t; \varepsilon)$$

to obtain simplified dynamics

$$\frac{d\beta}{dt} = \frac{\partial F}{\partial x}\alpha + \frac{\partial F}{\partial \varepsilon}, \quad \beta(t_0) = 0 \tag{2.34}$$

$$\frac{d\alpha}{dt} = \frac{\partial f}{\partial x}\alpha + \frac{\partial f}{\partial \varepsilon}, \quad \alpha(t_0) = 0 \tag{2.35}$$

In the robust formulation, we have (2.34) and (2.35) in addition to (2.30) and (2.30). Obviously, this leads to a robust formulation with increased dimensionality.

2.2.2 Effect of Uncertainty on Inequality Constraints

We have set of inequality constraints

$$g(x, u, t; \varepsilon) \leq 0 \quad \forall \varepsilon \in E \tag{2.36}$$

which may contain pure control, pure state, and mixed state-control inequalities. To create the robust counterpart, constraints (2.36) are replaced by

$$\max_{\varepsilon \in E} g_i(x, u, t; \varepsilon) \leq 0 \quad \forall i \ = 1, 2, ..., l$$

Recall that the set of uncertainty is defined

$$\varepsilon \in E \equiv \{\hat{\varepsilon} + \tau D\delta : \|\delta\| \leq 1\} \subset \left(L^2[t_0, t_f]\right)^s$$

Hence, using a first-order Taylor expansion we obtain

$$\max_{\varepsilon \in E} g_i(x, u, t; \varepsilon) \approx g_i(x, u, t; \hat{\varepsilon}) + \tau \max_{\|\delta\| \leq 1} \left\langle D^T \nabla_\varepsilon g_i(x, u, t; \hat{\varepsilon}), \delta \right\rangle \tag{2.37}$$

$$= g_i(x, u, t; \hat{\varepsilon}) + \tau \left\| D^T \nabla_\varepsilon g_i(x, u, t; \hat{\varepsilon}) \right\| \tag{2.38}$$

where (2.38) is obtained from (2.37) by using the Hölder inequality as seen in Zhang (2007). That is

$$\left\langle D^T \nabla_\varepsilon g_i(x, u, t; \hat{\varepsilon}), \delta \right\rangle \leq \left\| D^T \nabla_\varepsilon g_i(x, u, t; \hat{\varepsilon}) \right\| \|\delta\|$$

therefore,

$$\max_{\|\delta\|\leq 1} \left\langle D^T \nabla_\varepsilon g_i \left(x, u, t; \hat{\varepsilon}\right), \delta \right\rangle = \left\| D^T \nabla_\varepsilon g_i \left(x, u, t; \hat{\varepsilon}\right) \right\| \|\delta\|$$

Hence we obtain robust constraints

$$g_i \left(x, u, t; \hat{\varepsilon}\right) + \tau \left\| D^T \nabla_\varepsilon g_i \left(x, u, t; \hat{\varepsilon}\right) \right\| \leq 0 \quad \forall i \quad = 1, 2, ..., l$$

where

$$\begin{aligned} \nabla_\varepsilon g_i \left(x, u, t; \hat{\varepsilon}\right) &= \frac{\partial g_i \left(x, u, t; \hat{\varepsilon}\right)}{\partial x} \frac{\partial x \left(t, \hat{\varepsilon}\right)}{\partial \varepsilon} + \frac{\partial g_i \left(x, u, t; \hat{\varepsilon}\right)}{\partial \varepsilon} \\ &= \frac{\partial g_i \left(x, u, t; \hat{\varepsilon}\right)}{\partial x} \alpha \left(t; \hat{\varepsilon}\right) + \frac{\partial g_i \left(x, u, t; \hat{\varepsilon}\right)}{\partial \varepsilon} \end{aligned} \tag{2.39}$$

In this thesis, we do not consider equality constraints for simplicity. Effects on equality constraints may be studied following the analysis for the state dynamics, as state dynamics are special cases of equality constraints. Consult Zhang (2007) for a treatment of the effect on such equality constraints.

2.2.3 Effect of Uncertainty on Terminal Time State Constraints

We have terminal time state constraints:

$$h \left(x \left(t_f\right), t_f; \varepsilon \left(t_f\right)\right) \leq 0 \quad \forall \varepsilon \in E$$

These constraints are a special case of the inequality constraints discussed in the previous section except that they only hold at the terminal time. Performing a similar analysis, we obtain their replacement:

$$h_i \left(x \left(t_f\right), t_f; \varepsilon \left(t_f\right)\right) + \tau \left\| D^T \nabla_\varepsilon h_i \left(x \left(t_f\right), t_f; \hat{\varepsilon} \left(t_f\right)\right) \right\| \leq 0 \quad \forall i \ = 1, 2, ..., l$$

where

$$\nabla_{\varepsilon} h_i\left(x\left(t_f\right), t_f; \varepsilon\left(t_f\right)\right) = \left[\frac{\partial h_i\left(x, t; \hat{\varepsilon}\right)}{\partial x}\right]_{t=t_f} \left[\frac{\partial x\left(t, \hat{\varepsilon}\right)}{\partial \varepsilon}\right]_{t=t_f} + \left[\frac{\partial h_i\left(x, t; \hat{\varepsilon}\right)}{\partial \varepsilon}\right]_{t=t_f}$$
$$= \left[\frac{\partial h_i\left(x, t; \hat{\varepsilon}\right)}{\partial x}\right]_{t=t_f} \alpha\left(t_f; \hat{\varepsilon}\right) + \left[\frac{\partial g_i\left(x, t; \hat{\varepsilon}\right)}{\partial \varepsilon}\right]_{t=t_f} \tag{2.40}$$

and $[\cdot]_{t=t_f}$ indicates the evaluation at $t = t_f$.

2.2.4 Effect of Uncertainty on the Robust Index

We have reviewed the effect of uncertainty on the state dynamics and the inequality constraints as studied in Zhang (2007). In this section, we introduce a new technique to consider the effect of uncertainty on the objective function within the same framework. This new method induces a new state variable y which replaces the criterion as seen in (2.20) - (2.22). The objective function is now in the terms of the terminal time and is defined as the robust index. We have the robust index:

$$R = \max_{\varepsilon \in E} y\left(t_f; \varepsilon\right) \tag{2.41}$$

Taking a first-order Taylor expansion and using Hölder's inequality as seen in Section 2.2.2 and Zhang (2007) on the RHS of (2.41), we observe that

$$\max_{\varepsilon \in E} y\left(t; \varepsilon\right) \approx y\left(t; \hat{\varepsilon}\right) + \tau \max_{\|\delta\| \leq 1} \left\langle D^T \frac{\partial y\left(t; \hat{\varepsilon}\right)}{\partial \varepsilon}, \delta \right\rangle$$
$$= y\left(t; \hat{\varepsilon}\right) + \tau \left\| D^T \frac{\partial y\left(t; \hat{\varepsilon}\right)}{\partial \varepsilon} \right\| \qquad \forall t \in \left[t_0, t_f\right]$$

Hence

$$R = \max_{\varepsilon \in E} y\left(t_f; \varepsilon\right)$$
$$\approx y\left(t_f; \hat{\varepsilon}\right) + \tau \left\| D^T \frac{\partial y\left(t; \hat{\varepsilon}\right)}{\partial \varepsilon} \right\|_{t=t_f}$$
$$= y\left(t_f; \hat{\varepsilon}\right) + \tau \left\| D^T \beta\left(t; \hat{\varepsilon}\right) \right\|_{t=t_f} \tag{2.42}$$

where the subscript $t = t_f$ means the norm is evaluated at the terminal time and we defined

$$\beta(t;\varepsilon) \equiv \frac{\partial y}{\partial \varepsilon}(t;\varepsilon)$$

Note that $\|\cdot\|$ is defined corresponding to (2.18) is not a function of time, and as such, the evaluation at $t = t_f$ is not necessary. From (2.21), we note that

$$\begin{aligned}\beta(t;\varepsilon) &\equiv \frac{\partial y}{\partial \varepsilon}(t;\varepsilon) \\ &= \frac{\partial}{\partial \varepsilon}\int_{t_0}^{t} F(x,u,t;\varepsilon)\, dt\end{aligned}$$

We note that $y(t;\varepsilon)$ defined in (2.21) is the *cumulative contribution* to the criterion. We interpret $\beta(t;\varepsilon)$ as the effect of uncertainty on the cumulative contribution to the criterion. Hence when the robust index R in (2.42) is minimized, the effect of the uncertainty on the robust index is also minimized throughout the entire time horizon, $[t_0, t_f]$.

2.2.5 A Locally Robust Formulation

We have, from (2.30), (2.31), (2.34), (2.35), (2.39), (2.40) and (2.42), the final form of the robust counterpart:

$$\min_u y(t_f;\hat{\varepsilon}) + \tau \left\| D^T \beta(t_f;\hat{\varepsilon}) \right\| \tag{2.43}$$

subject to

$$\frac{dy}{dt} = F(x,u,t;\varepsilon), \quad y(t_0) = 0 \tag{2.44}$$

$$\frac{dx}{dt} = f(x,u,t;\varepsilon), \quad x(t_0) = x_0 \tag{2.45}$$

$$\frac{d\beta}{dt} = \frac{\partial F}{\partial x}\alpha + \frac{\partial F}{\partial \varepsilon}, \quad \beta(t_0) = 0 \tag{2.46}$$

$$\frac{d\alpha}{dt} = \frac{\partial f}{\partial x}\alpha + \frac{\partial f}{\partial \varepsilon}, \quad \alpha(t_0) = 0 \tag{2.47}$$

$$g_i\left(x,u,t;\hat{\varepsilon}\right)+\tau\left\|D^T\left[\frac{\partial g_i\left(x,u,t;\hat{\varepsilon}\right)}{\partial x}\alpha\left(t;\varepsilon\right)+\frac{\partial g_i\left(x,u,t;\hat{\varepsilon}\right)}{\partial \varepsilon}\right]\right\|\leq 0 \quad \forall i=0,1,...,r \tag{2.48}$$

$$h_i\left(x\left(t_f\right),t_f;\varepsilon\left(t_f\right)\right)+\tau\left\|D^T\nabla_\varepsilon h_i\left(x\left(t_f\right),t_f;\hat{\varepsilon}\left(t_f\right)\right)\right\|\leq 0 \quad \forall i=0,1,...,r \tag{2.49}$$

$$u\in U \tag{2.50}$$

We call this a *locally* robust optimal control problem, as we use a *local* linear approximation based on a Taylor series expansion. However, if every explicit and implicit function of the uncertainty ε is linear in ε, first-order Taylor expansion around the nominal value $\hat{\varepsilon}$ becomes exact and, hence, the robust optimal control problem is global.

2.3 Descent Method in Hilbert Spaces for Optimal Control Problems

Note that

$$H\left(x,u,\lambda,t\right)=F(x,u,t)+\lambda^T f\left(x,u,t\right)$$

will be the Hamiltonian for the optimal control problem (2.2)-(2.5). Now we state the algorithm.

Descent Method in Hilbert Spaces

$\boxed{\textit{Step 0}}$: *Initialization.* Set $k=0$ and pick $u^0\left(t\right)\in\left(L^2\left[t_0,t_f\right]\right)^m$.

$\boxed{\textit{Step 1}}$: *Find State Trajectory.* Using $u^k\left(t\right)$ solve the state initial value problem

$$\frac{dx}{dt}=f\left(x,u^0,t\right)$$
$$x\left(t_0\right)=x^0$$

and call the solution $x^k\left(t\right)$.

$\boxed{\textit{Step 2}}$: *Find Adjoint Trajectory.* Using $u^k(t)$ and $x^k(t)$ solve the adjoint final value problem

$$(-1)\frac{d\lambda}{dt} = \frac{\partial H\left(x^k, u^k, \lambda, t\right)}{\partial x}$$
$$\lambda(t_f) = \frac{\partial K\left[x(t_f), t_f\right]}{\partial x}$$

and call the solution $\lambda^k(t)$.

$\boxed{\textit{Step 3}}$: *Find Gradient.* Using $u^k(t)$, $x^k(t)$ and $\lambda^k(t)$ calculate

$$\nabla_u J(u^k) = \frac{\partial H\left(x^k, u^k, \lambda, t\right)}{\partial u}$$
$$= \frac{\partial F(x^k, u^k, t)}{\partial u} + \left(\lambda^k\right)^T \frac{\partial f\left(x^k, u^k, t\right)}{\partial u}$$

$\boxed{\textit{Step 4}}$: *Update and Apply Stopping Test.* For a suitably small step size θ_k, update according to

$$u^{k+1} = P_U\left[u^k - \theta_k \nabla J\left(u^k\right)\right]$$

where $P_U[\cdot]$ denotes the minimum norm projection onto U. If an appropriate stopping test is satisfied, declare

$$u^*(t) \approx u^{k+1}(t)$$

Otherwise set $k = k + 1$ and go to Step1.

This descent method in Hilbert space has known convergence properties. See Mookherjee (2006).

2.4 Differential Variational Inequalities Defined

A finite-dimensional variational inequality problem is, for a closed convex set $U \subseteq \mathbb{R}^n$ and a vector function F, to find $u \in U$ such that

$$\langle F(u), v - u\rangle \geq 0 \qquad \forall v \in U$$

where $\langle \cdot, \cdot \rangle$ denotes the corresponding inner product. Let us give this problem a symbolic name, $VI(F, U)$. This can be easily extended to an infinite-dimensional setting, for example, for $U \subseteq \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space. It is well-known that a VIP is closely related to an optimization problem. When a variable of an optimization problem has a representation of an ordinary differential equation, we call the problem an optimal control problem, which is closely related to a differential variational inequality problem we will introduce. We begin by letting

$$\begin{aligned} & u \in \left(L^2 [t_0, t_f]\right)^m \\ & x(u,t) = \arg\left\{\frac{dy}{dt} = f(y,u,t),\ y(t_0) = y, \Gamma\left[y(t_f), t_f\right] = 0\right\} \in \left(\mathcal{H}^1 [t_0, t_f]\right)^n \end{aligned} \tag{2.51}$$

The entity $x(u,t)$ is to be interpreted as an operator that tells us the state variable x for each vector u and each time $t \in [t_0, t_f] \subset \mathbb{R}^1_+$; constraints on u are enforced separately. This notation $x(u,t)$ is found in Minoux (1986), Bressan and Piccoli (2007) and Zhang (2007). In each case it denotes an operator that expresses the state vector for any control vector u and instant of time t. It should be noted that expression (2.51) does not assume that a solution to DVI is known *a priori*, but rather that for the given initial and terminal time values of the state a solution of the state dynamics exists for any feasible control; that is, there is an implicit reachability assumption. Later in this section, we will provide the regularity conditions for the existence, continuity and differentiability of the operator $x(u,t)$. We assume that every control vector is constrained to lie in a set U, where U is defined so as to ensure the terminal conditions may be reached from the initial conditions intrinsic to (2.51). In light of the operator (2.51), the variational inequality of interest takes the form:

$$\begin{aligned} & \text{find } u^* \in U \text{ such that} \\ & \langle F(x(u^*,t), u^*, t), u - u^* \rangle \geq 0 \text{ for all } u \in U \end{aligned} \tag{2.52}$$

where

$$U \subseteq \left(L^2\left[t_0, t_f\right]\right)^m \tag{2.53}$$

$$x^0 \in \mathbb{R}^n \tag{2.54}$$

$$F : \left(\mathcal{H}^1\left[t_0, t_f\right]\right)^n \times \left(L^2\left[t_0, t_f\right]\right)^m \times \mathbb{R}^1_+ \longrightarrow \left(L^2\left[t_0, t_f\right]\right)^m \tag{2.55}$$

$$f : \left(\mathcal{H}^1\left[t_0, t_f\right]\right)^n \times \left(L^2\left[t_0, t_f\right]\right)^m \times \mathbb{R}^1_+ \longrightarrow \left(L^2\left[t_0, t_f\right]\right)^n \tag{2.56}$$

$$\Gamma : \left(\mathcal{H}^1\left[t_0, t_f\right]\right)^n \times \mathbb{R}^1_+ \longrightarrow \left(\mathcal{H}^1\left[t_0, t_f\right]\right)^r \tag{2.57}$$

Note that $(L^2\left[t_0, t_f\right])^m$ is the m-fold product of the space of square-integrable functions $L^2\left[t_0, t_f\right]$ and the inner product in (2.52) is defined by

$$\langle F\left(x\left(u^*, t\right), u^*, t\right), u - u^*\rangle \equiv \int_{t_0}^{t_f} \left[F\left(x\left(u^*, t\right), u^*, t\right)\right]^T \left(u - u^*\right) \geq 0$$

while $(\mathcal{H}^1\left[t_0, t_f\right])^n$is the n-fold product of the Sobolev space $\mathcal{H}^1\left[t_0, t_f\right]$. We refer to (2.52) as a differential variational inequality and give it the symbolic name $DVI(F, f, U)$.

To analyze (2.52) we will rely on the following notion of regularity:

Definition 10 (Regularity) *We call $DVI(F, f, U)$ regular if:*

1. $\Gamma\left(x, t\right)$: *is continuously differentiable with respect to x;*

2. *$F(x, u, t)$ is continuous with respect to x and u;*

3. *$f\left(x, u, t\right)$ is convex and continuously differentiable with respect to x and u;*

4. *$f\left(x, u, t\right)$ and $\partial f\left(x, u, t\right)/\partial x$ are bounded so that the operator*

$$x\left(u, t\right) : \left(L^2\left[t_0, t_f\right]\right)^m \times \mathbb{R}^1_+ \longrightarrow \left(\mathcal{H}^1\left[t_0, t_f\right]\right)^n$$

exists and is continuous and Gateaux-differentiable with respect to u;

5. *U is convex and compact; and*

6. *$x^0 \in \mathbb{R}^n$ is known and fixed.*

The existence of the operator $x(u,t)$ and its continuity and differentiability with respect to u is provided by Theorems 5 and 6. The motivation for this definition of regularity is to parallel as closely as possible those assumptions needed to analyze traditional optimal control problems from the point of view of infinite dimensional mathematical programming: see Definition 4.

We next note that (2.52) may be restated as the following optimal control problem

$$\min \gamma^T \Gamma\left[x\left(t_f\right), t_f\right] + \int_{t_0}^{t_f} \left[F\left(x^*, u^*, t\right)\right]^T u dt \tag{2.58}$$

subject to

$$\frac{dx}{dt} = f(x, u, t) \tag{2.59}$$

$$u \in U \tag{2.60}$$

$$x(t_0) = x^0 \tag{2.61}$$

where $x^* = x(u^*)$ is the optimal state vector. We point out that this optimal control problem is a mathematical abstraction and of no use for computation, since its criterion depends on knowledge of the variational inequality solution u^*. In what follows we will need the Hamiltonian for (2.58) through (2.61), namely

$$H(x, u, \lambda, t) = \left[F\left(x^*, u^*, t\right)\right]^T u + \lambda^T f(x, u, t) \tag{2.62}$$

where $\lambda(t)$ is the adjoint vector that solves the adjoint equations and transversality conditions for the given state and control variables. Note that for a given state vector and a given instant in time (2.62) is convex in u when $DVI(F, f, U)$ is regular. Furthermore, there is a fixed point form of $DVI(F, f, U)$. In particular we state the following result:

Theorem 11 (Fixed Point Problem) *When regularity in the sense of Definition 10 holds, $DVI(F, f, U)$ is equivalent to the following fixed point problem:*

$$u = P_U\left[u - \alpha F(x(u), u, t)\right]$$

where $P_U[.]$ is the minimum norm projection onto $U \subseteq \left(L^2[t_0, \tau]\right)^m$ and $\alpha \in \mathbb{R}^1_{++}$.

The minimum norm projection requires that

$$u = \arg\min_{v} \left\{ \frac{1}{2} \left\| u - \alpha F\left(x(u), u, t\right) - v \right\|^2 : v \in U \right\} \tag{2.63}$$

where $\alpha \in \mathbb{R}^1_{++}$ *is any strictly positive real number.*

We now state and prove the following existence result:

Theorem 12 (Existence) *When the regularity in the sense of Definition 10 holds,* $DVI(F, f, U)$ *has a solution.*

Proof. By the assumption of regularity $x(u, t)$ is well defined and continuous. So $F(x(u, t), u, t)$ is continuous in u. Also by regularity we know U is convex and compact. Consequently, by Theorem 2 of Browder (1968), $DVI(F, f, U)$ has a solution. ■

2.5 Fixed Point Algorithm for Differential Variational Inequalities

Naturally there is an associated fixed point algorithm based on the iterative scheme

$$u^{k+1} = P_U\left[u^k - \alpha F\left(x\left(u^k, t\right), u^k, t\right)\right]$$

The detailed structure of the fixed point algorithm is:

Fixed Point Algorithm

$\boxed{\textit{Step 0}}$. Initialization: identify an initial feasible solution $u^0 \in U$ and set $k = 0$.
$\boxed{\textit{Step 1}}$. Solve optimal control problem: call the solution of the following optimal control problem u^{k+1}.

$$\min_{v} J^k(v) = \gamma^T \Gamma\left[x(t_f), t_f\right] + \int_{t_0}^{t_f} \frac{1}{2}\left[u^k - \alpha F\left(x^k, u^k, t\right) - v\right]^2 dt \tag{2.64}$$

$$\text{subject to } \frac{dx}{dt} = f(x, v, t);\ x(t_0) = x^0 \tag{2.65}$$

$$v \in U \tag{2.66}$$

$\boxed{\textit{Step 2}}$. Stopping test: if $\left\|u^{k+1}-u^{k}\right\| \leq \varepsilon$ where $\varepsilon \in \mathbb{R}_{++}^{1}$ is a preset tolerance, stop and declare $u^{*} \approx u^{k+1}$. Otherwise set $k=k+1$ and and go to Step 1.
The convergence of this algorithm is guaranteed by the following result:

Theorem 13 (Convergence of Fixed Point Algorithm) *When $DVI(F, f, U)$ is regular in the sense of Definition 10 and $f(x, u, t) : \left(\mathcal{H}^{1}\left[t_{0}, t_{f}\right]\right)^{n} \times\left(L^{2}\left[t_{0}, \tau\right]\right)^{m} \times \mathbb{R}_{+}^{1} \longrightarrow\left(L^{2}\left[t_{0}, t_{f}\right]\right)^{n}$ is convex, while additionally $F(x, u, t)$ is strongly monotonic for $u \in U$, the fixed point algorithm presented above converges.*

It is important to realize that the fixed point algorithm can be carried out in continuous time provided we employ a continuous time representation of the solution of each subproblem (2.64)-(2.66) from Step 1 of the fixed point algorithm. This may be done using a continuous time gradient projection method. For our present circumstances, that algorithm may be stated as

Descent Algorithm in Hilbert Space for the Projection Sub-Problems

$\boxed{\textit{Step 0}}$. *Initialization.* Pick $v^{k,0}(t) \in U$ and set $j=0$.
$\boxed{\textit{Step 1}}$. *Finding state variables.* Solve the state dynamics

$$\frac{dx}{dt}=f\left(x, v^{k,j}, t\right) \tag{2.67}$$

$$x\left(t_{0}\right)=x^{0} \tag{2.68}$$

Call the solution $x^{k,j}(t)$ .In the event a discrete time method is used to solve the state dynamics (2.67) and (2.68), curve fitting is used to obtain the continuous time state vector $x^{k,j}(t)$.
$\boxed{\textit{Step 2}}$. *Finding adjoint variables.* Solve the adjoint dynamics

$$(-1) \frac{d\lambda}{dt}=\nabla_{x} H^{k}\left.\right|_{x=x^{k,j}} ; \ \lambda\left(t_{f}\right)=\frac{\partial \Gamma\left[x^{k,j}\left(t_{f}\right), t_{f}\right]}{\partial x\left(t_{f}\right)} \tag{2.69}$$

where

$$H^{k}=\frac{1}{2}\left[u^{k}-\alpha F\left(x^{k}, u^{k}, t\right)-v\right]^{2}+\lambda^{T} f\left(x, v^{k,j}, t\right)$$

Call the solution $\lambda^{k,j}(t)$.In the event a discrete time method is used to solve the adjoint dynamics (2.69), curve fitting is used to obtain the continuous time adjoint vector $\lambda^{k,j}(t)$.

$\boxed{\textit{Step 3}}$. *Finding the gradient.* Determine

$$\nabla_v J^{k,j}(t) = \nabla_v H^k$$

$\boxed{\textit{Step 4}}$. *Stopping test.* For a fixed and suitably small fixed step size

$$\theta_k \in \mathbb{R}^1_{++}$$

determine

$$v^{k,j+1}(t) = P_U\left[v^{k,j}(t) - \theta_k \nabla_v J^{k,j}\right] \tag{2.70}$$

In the event a discrete time method is used to solve the above projection subproblem, curve fitting is used to obtain the continuous time control vector (2.70).
$\boxed{\textit{Step 5}}$. *Stopping test.* For $\varepsilon_2 \in \mathbb{R}^1_{++}$, a pre-set tolerance, stop if $\left\|v^{k,j+1} - v^{k,j}\right\| < \varepsilon_1$and declare $v^{k*} \approx v^{k,j+1}$. Otherwise set $j = j + 1$ and go to Step 1.

2.6 Differential Games

Recently, Pang and Stewart (2007) introduced an application of DVIs in differential Nash games. In a differential game, the state of a dynamical system at each time instant is described by a set of state variables, whose evolution over time is usually modelled by differential equations. Here, by summarizing the Pang and Stewart (2007) result, we provide an instance of a differential Nash game where each player solves their own optimal control problem. We will show that the game necessary conditions for the equilibrium constitute a DVI.

Let us consider a differential Nash game with n players where player i's problem

(PR_i) is stated as:

$$(PR_i) \quad \min \ J_i = \Gamma_i \left[x\left(T\right), T\right] + \int_0^T \Psi_i \left(x\left(t\right), u\left(t\right), t\right) dt \tag{2.71}$$

subject to

$$\frac{dx_i}{dt} = g_i \left(x_i \left(t\right), u_i \left(t\right), t\right) \tag{2.72}$$

$$x_i \left(0\right) = x_i^0 \tag{2.73}$$

$$u_i \left(t\right) \in U_i \tag{2.74}$$

where

$$x\left(t\right) = \left(x_i \left(t\right)\right)_{i=1}^n$$
$$u\left(t\right) = \left(u_i \left(t\right)\right)_{i=1}^n$$

Let us denote the adjoint variable of the problem PR_i by p_i. The corresponding Hamiltonian becomes

$$H_i \left(x, u, p, t\right) = \Psi_i \left(x, u, t\right) + p_i^T g_i \left(x_i, u_i, t\right)$$

The first-order necessary conditions for u_i to be a solution of the problem PR_i are

$$\int_0^T \sum_{i=1}^n \frac{\partial H_i \left(x_i, u_i, p_i, t\right)}{\partial u_i} \left(v_i - u_i\right) dt \geq 0 \qquad \forall v_i \in U_i$$

$$\frac{dx_i}{dt} = g_i \left(x_i, u_i, t\right) \qquad x_i \left(0\right) = x_i^0$$
$$\frac{dp_i}{dt} = -\frac{\partial H_i \left(x, u, p, t\right)}{\partial x_i} \qquad p_i \left(T\right) = \frac{\partial \Gamma_i \left[x\left(T\right), T\right]}{\partial x\left(T\right)}$$

Combining all the players' problems, we have a DVI of the form:

$$\int_0^T \left[F\left(x, u, p, t\right)\right]^T \left(v - u\right) dt \geq 0 \qquad \forall v \in U \tag{2.75}$$

$$\int_0^T [F(x,u,p,t)]^T (v-u)\, dt \geq 0 \qquad \forall v \in U$$

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dp}{dt} \end{pmatrix} = \begin{pmatrix} g(x,u,t) \\ h(x,u,p,t) \end{pmatrix} \tag{2.76}$$

$$x(0) = x^0 \tag{2.77}$$

$$p(T) = \Theta \tag{2.78}$$

where

$$\begin{aligned} p(t) &= (p_i(t))_{i=1}^n \\ F(x,u,p,t) &= \left(\frac{\partial H_i(x_i,u_i,p_i,t)}{\partial u_i}\right)_{i=1}^n \\ g(x,u,p,t) &= (f_i(x_i,u_i,t))_{i=1}^n \\ h(x,u,p,t) &= \left(-\frac{\partial H_i(x,u,p,t)}{\partial x_i}\right)_{i=1}^n \\ x^0 &= \left(x_i^0\right)_{i=1}^n \\ \Theta &= \left(\frac{\partial \Gamma_i[x(T),T]}{\partial x(T)}\right)_{i=1}^n \\ U &= \prod_{i=1}^n U_i \end{aligned}$$

Note that the adjoint variables of the players' optimal control problems become state variables in the DVI form (2.75)-(2.78).

Chapter 3

A Descent Method for Differential Variational Inequalities

We propose a descent method for differential variational inequalities (DVIs) involving explicit control variables and explicit state dynamics. This chapter is organized as follows. In Section 3.1, we summarize previously studied gap functions for variational inequalities *without* dynamics. Section 3.2 discusses the gap function considered by Konnov et al. (2002) within the DVI framework. Using the gap functions studied, we formulate an equivalent optimal control problem for a DVI with controls and state equations and derive its gradient using the theory of calculus of variation in Section 3.3. For solving the optimal control problem, we suggest a descent method with the convergence result in Section 3.4, and test it for a simple problem in Section 3.5. One important application of DVIs with controls and states, differential Nash games are discussed and we apply the gap function and associated descent method for a linear-quadratic game and an oligopolistic competition problem in Section 3.6. Finally we conclude this chapter in Section 3.7.

3.1 Background

In this section, we review the results developed in finite dimensions. To recall, a finite-dimensional variational inequality problem is, for a closed convex set $U \subseteq \mathbb{R}^n$

and a vector function F, to find $u \in U$ such that

$$\langle F(u), v-u \rangle \geq 0 \qquad \forall v \in U \tag{3.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the corresponding inner product. Let us give this problem a symbolic name, $VI(F,U)$. This can be easily extended to an infinite-dimensional setting, for example, for $U \subseteq \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space.

Gap function

Let us now introduce gap functions, whose definition is followed by:

Definition 14 (Gap Function) *A function $\varphi : U \longrightarrow \mathbb{R}_+$ is called a gap function for $VI(F,U)$ when the following statements hold:*

1. $\varphi(u) \geq 0$ *for all* $u \in U$
2. $\varphi(u) = 0$ *if and only if* u *is the solution of* $VI(F,U)$

An example function of this class given by Auslender (1976) is

$$\varphi(u) = \max_{v \in U} \langle F(u), u-v \rangle \tag{3.2}$$

Since the RHS of (3.2) has the first-order necessary and sufficient condition

$$F(u)(v'-v) \geq 0 \qquad \forall v' \in U$$

it is easily verified that $\varphi(u)$ is a non-negative function of u. The gap function (3.2) allows us to re-formulate $VI(F,U)$ as an optimization problem:

$$\min_{u \in U} \varphi(u) \tag{3.3}$$

whose optimal solution also solves $VI(F,U)$. An intuitive interpretation of (3.3) is to find u such that (3.1) holds; that is, we consider the worst possible case by minimizing $\langle F(u), v \rangle$ and see if $\langle F(u), v \rangle - \langle F(u), u \rangle \geq 0$ still holds. It is more obvious if we rewrite (3.3) as

$$\min_{u \in U} \varphi(u) = \min_{u \in U} \max_{v \in U} \langle F(u), u-v \rangle = \min_{u \in U} \left\{ \langle F(u), u \rangle - \min_{v \in U} \langle F(u), v \rangle \right\} \tag{3.4}$$

However, $\varphi(u)$ is not differentiable, in general, even if F is differentiable. It is because we can not guarantee the one-to-one mapping of $u \longmapsto \varphi(u)$ for general cases.

Regularized Gap Function

Fukushima (1992) and Auchmuty (1989) independently suggested a class of differentiable gap functions, that is

$$\varphi_\alpha(u) = \max_{v \in U} \left\{ \langle F(u), u - v \rangle - \frac{\alpha}{2} \|u - v\|^2 \right\} \tag{3.5}$$

for $\alpha > 0$. This is differentiable whenever F is differentiable, since the inner objective function is strongly convex in v therefore there is a unique maximizer. In particular, its gradient is given by

$$\nabla \varphi_\alpha(u) = F(u) + \langle \nabla F(u), u - v_\alpha \rangle - \alpha \|u - v\|$$

where v_α denotes the unique maximizer of (3.5). The RHS of (3.5) is strongly convex in v resulting in a unique maximizer v_α. In turn, this assures the differentiability of (3.5). Based on this idea, a generalization can be made.

A Generalization of the Regularized Gap Function

Wu et al. (1993) considered the following generalized class of regularized gap functions

$$\varphi_\alpha(u) = \max_{v \in U} \{ \langle F(u), u - v \rangle - \alpha \phi(u, v) \} \tag{3.6}$$

where we assume ϕ is a function which satisfies the following assumptions: (1) ϕ is continuously differentiable on $\mathbb{R}^{2n}$; (2) ϕ is non-negative on $\mathbb{R}^{2n}$; (3) $\phi(u, \cdot)$ is strongly convex for any $u \in \mathbb{R}^n$; and (4) $\phi(u, v) = 0$ if and only if $u = v$. Note that this generalization of the regularized gap function (3.6) allows us to have an equivalent *constrained* optimization problem for a variational inequality problem (VIP) of interest, which is

$$\min_{u \in U} \varphi_\alpha(u)$$

D-gap Function

Furthermore, we may form an equivalent *unconstrained* optimization problem to a VIP. Peng (1997) considered a so-called D-gap function, and Yamashita et al. (1997) generalized it. A D-gap function is the difference between two gap functions, in particular for $0 < \alpha < \beta$,

$$\begin{aligned}\psi_{\alpha\beta}(u) &= \varphi_\alpha(u) - \varphi_\beta(u) \\ &= \max_{v\in U}\{\langle F(u), u-v\rangle - \alpha\phi(u,v)\} - \max_{v\in U}\{\langle F(u), u-v\rangle - \beta\phi(u,v)\}\end{aligned}$$

The unconstrained optimization problem which is equivalent to the VIP is

$$\min_u \psi_{\alpha\beta}(u)$$

3.2 The Gap Functions for Differential Variational Inequalities

We are interested with the following structure:

$$\begin{aligned} u &\in \left(L^2[t_0,t_f]\right)^m \\ x(u,t) &= \arg\left\{\frac{dy}{dt} = f(y,u,t);\ y_i(t_0) = y_i^0\ \forall i \in \mathcal{I}_1; \Gamma[y(t_f), t_f] = 0\right\} \\ &\in \left(\mathcal{H}^1[t_0,t_f]\right)^n \end{aligned} \tag{3.7}$$

where we define two sets of indices

$$\begin{aligned}\mathcal{I}_1 &= \{i : x_i(t_0) \text{ is known }\} \\ \mathcal{I}_2 &= \{i : x_i(t_0) \text{ is unknown}\}\end{aligned}$$

The special structure of (3.7) is relevant to differential Nash games.

In this section, we extend the gap function considered by Konnov et al. (2002) to the DVI setting. When the regularity conditions given in Definition 10 hold, $DVI(F,f,U)$ belongs to the class of variational inequalities in a Hilbert space $\mathcal{H}$ considered by Konnov et al. (2002); i.e. U is a non-empty closed and convex subset of a Hilbert space, and F is a continuously differentiable mapping of u. This allows us to analyze $DVI(F,f,U)$ by considering gap functions, which are defined by

Definition 15 (Gap Function) *A function $\varphi : U \longrightarrow \mathbb{R}_+$ is called a gap function for $DVI\,(F,f,U)$ when the following statements hold:*

1. $\varphi\,(u) \geq 0$ *for all* $u \in U$
2. $\varphi\,(u) = 0$ *if and only if* u *is the solution of* $DVI\,(F,f,U)$

Let us consider a function

$$\varphi_\alpha\,(u) = \max_{v \in U} \Phi_\alpha\,(u,v) \tag{3.8}$$

where

$$\begin{aligned} \Phi_\alpha\,(u,v) &= \langle F\,(x,u,t)\,, u - v\rangle - \alpha\phi\,(u,v) \\ x\,(u,t) &= \arg\left\{ \frac{dy}{dt} = f\,(y,u,t)\,,\ y\,(t_0) = y^0, \Gamma\,[y\,(t_f)\,, t_f] = 0 \right\} \in \left(\mathcal{H}^1\,[t_0, t_f]\right)^n \\ U &\subseteq \left(L^2\,[t_0, t_f]\right)^m \end{aligned}$$

and it is assumed that ϕ is a function which satisfies following assumptions: (1) ϕ is continuously differentiable on $(L^2\,[t_0,t_f])^{2m}$; (2) ϕ is non-negative on $(L^2\,[t_0,t_f])^{2m}$; (3) $\phi\,(u,\cdot)$ is strongly convex with modulus $c > 0$ for any $u \in (L^2\,[t_0,t_f])^m$; and (4) $\phi\,(u,v) = 0$ if and only if $u = v$. Yamashita *et al.* discovered the following functions which satisfy assumptions (1) through (4) in finite-dimensional spaces:

- $\phi_1\,(u,v) = \tau_1\,(u-v)$, where τ_1 is non-negative, continuously differentiable, strongly convex, and $\tau_1\,(0) = 0$.
- $\phi_2\,(u,v) = \tau_2\,(v) - \tau_2\,(u) - \langle \nabla\tau_2\,(u)\,, u - v\rangle$, where τ_2 is twice continuously differentiable, and strongly convex.
- $\phi_3\,(u,v) = \langle u - v, G\,(u)\,(u-v)\rangle$, where $G\,(u)$ is a continuously differentiable, symmetric, and uniformly positive-definite matrix.

In Hilbert spaces, Konnov and Kum (2001) and Konnov et al. (2002) considered ϕ_2 and ϕ_3, respectively.

The maximization problem (3.8) has a unique solution since $\Phi_\alpha(u,v)$ is strongly convex in v and U is convex. Let the solution of (3.8) be $v_\alpha(u)$ such that $\varphi_\alpha(u) = \Phi_\alpha(u, v_\alpha(u))$. The following lemma can be proved in a very similar way to Konnov et al. (2002):

Lemma 16 *The function $\varphi_\alpha(u)$ defined by (3.8) is a gap function for $DVI(F,f,U)$, and u is the solution to $DVI(F,f,U)$ if and only if $u = v_\alpha(u)$.*

Proof. The optimality condition for (3.8) is

$$\left\langle \frac{\partial \Phi_\alpha(u, v_\alpha)}{\partial v}, v - v_\alpha \right\rangle \leq 0 \qquad \forall v \in U$$

That is

$$\langle -F(x,u,t) - \alpha \nabla_v \phi(u, v_\alpha), v - v_\alpha \rangle \leq 0 \qquad \forall v \in U \tag{3.9}$$

Substituting u for v in (3.9), we obtain

$$\langle F(x,u,t), u - v_\alpha \rangle \geq -\alpha \langle \nabla_v \phi(u, v_\alpha), u - v_\alpha \rangle$$

By the definition and the strong convexity of $\phi(u)$,

$$\begin{aligned} \varphi_\alpha(u) &= \Phi_\alpha(u, v_\alpha) \\ &= \langle F(x,u,t), u - v_\alpha \rangle - \alpha \phi(u, v_\alpha) \\ &\geq -\alpha \langle \nabla_v \phi(u, v_\alpha), u - v_\alpha \rangle - \alpha \phi(u, v_\alpha) \\ &= \alpha [\phi(u,u) - \phi(u, v_\alpha) - \langle \nabla_v \phi(u, v_\alpha), u - v_\alpha \rangle] \\ &\geq \frac{\alpha c}{2} \|v_\alpha - u\|^2 \end{aligned} \tag{3.10}$$

where the property $\phi(u,u) = 0$ is used. Therefore, $\varphi_\alpha(u) \geq 0$ for all $u \in U$. Moreover, if $\varphi_\alpha(u) = 0$, then, by (3.10), $u = v_\alpha(u)$, and from (3.9), we obtain $DVI(F,f,U)$. Suppose now that u is a solution of $DVI(F,f,U)$. Then

$$\langle F(x,u,t), v - u \rangle \geq 0 \qquad \forall v \in U$$

and

$$\Phi_\alpha(u,v) = \langle F(x,u,t), u-v\rangle - \alpha\phi(u,v)$$
$$\leq -\alpha\phi(u,v)$$

for all $v \in U$. By definition,

$$\varphi_\alpha(u) = \max_{v\in U}\Phi_\alpha(u,v) \leq -\alpha\phi(u,v_\alpha)$$

which conflicts the nonnegativity property of $\varphi_\alpha(u)$, unless $\varphi_\alpha(u) = 0$ and $u = v_\alpha(u)$. We completes the proof. ■

While $\varphi_\alpha(u)$ is not differentiable in general, $\psi_{\alpha\beta}(u)$ is Gateaux-differentiable, which will be shown in the next section. Recall that the D-gap function is defined as the difference between two gap functions

$$\psi_{\alpha\beta}(u) = \varphi_\alpha(u) - \varphi_\beta(u)$$

for $0 < \alpha < \beta$. To show that $\psi_{\alpha\beta}(u)$ is a gap function, we only need to show the nonnegativity property holds. By the definition and the strong convexity of $\phi(u)$,

$$\begin{aligned}
\psi_{\alpha\beta}(u) &= \varphi_\alpha(u) - \varphi_\beta(u)\\
&= \Phi_\alpha(u,v_\alpha) - \Phi_\beta(u,v_\beta)\\
&\geq \Phi_\alpha(u,v_\beta) - \Phi_\beta(u,v_\beta)\\
&= \langle F(x,u,t), u-v_\beta\rangle - \alpha\phi(u,v_\beta) - \langle F(x,u,t), u-v_\beta\rangle + \beta\phi(u,v_\beta)\\
&= (\beta-\alpha)\phi(u,v_\beta)
\end{aligned}$$

This proves $\psi_{\alpha\beta}(u) \geq 0$, and, of course, $\varphi_\alpha(u) \geq \varphi_\beta(u)$ for all $u \in U$.

3.3 An Equivalent Optimal Control Problem

Fukushima (1992) considered a regularized gap function, a special case of the gap function, and Konnov et al. (2002) extended it to Hilbert spaces; in particular,

$$\phi(u,v) = \frac{1}{2}\|v-u\|^2 \tag{3.11}$$

which satisfies the assumption on $\phi(\cdot)$ of strong convexity with modulus $c > 0$. Adopting Fukushima's regularized gap function, we obtain

$$\Phi_\alpha(u,v) = \langle F(x,u,t), u-v\rangle - \frac{\alpha}{2}\|v-u\|^2$$

The corresponding D-gap function becomes

$$\psi_{\alpha\beta}(u) = \varphi_\alpha(u) - \varphi_\beta(u) = \max_{v\in U}\Phi_\alpha(u,v) - \max_{v\in U}\Phi_\beta(u,v)$$

or, in detail,

$$\psi_{\alpha\beta}(u) = \langle F(x,u,t), v_\beta(u) - v_\alpha(u)\rangle - \frac{\alpha}{2}\|v_\alpha(u)-u\|^2 + \frac{\beta}{2}\|v_\beta(u)-u\|^2 \tag{3.12}$$

where we denote

$$v_\alpha(u) = \arg\max_{v\in U}\Phi_\alpha(u,v) \tag{3.13}$$

$$v_\beta(u) = \arg\max_{v\in U}\Phi_\beta(u,v) \tag{3.14}$$

It should be noted that, for a fixed $u \in U$, the maximization problem (3.13) is equivalent to the following problem:

$$v_\alpha(u) = \arg\min_{v\in U}\left\|v - \left(u - \frac{1}{\alpha}F(x,u,t)\right)\right\|^2$$

which may be rewritten in the form of a projection operator as

$$v_\alpha(u) = P_U\left[u - \frac{1}{\alpha}F(x,u,t)\right]$$

which can be easily shown as Fukushima (1992) in finite dimension and Konnov et al. (2002) in infinite dimension. This result with Lemma 16 is equivalent to the fixed point theorem for DVIs (Theorem 11).

With the D-gap function (3.12), $DVI(F, f, U)$ is equivalent to the following optimal control problem $OCP\left(\psi_{\alpha\beta}, f, U\right)$:

$$\begin{aligned}
\min\ & \psi_{\alpha\beta}\left(u\right) \\
&= \left\langle F\left(x,u,t\right), v_{\beta}\left(u\right) - v_{\alpha}\left(u\right)\right\rangle - \frac{\alpha}{2}\left\|v_{\alpha}\left(u\right) - u\right\|^{2} + \frac{\beta}{2}\left\|v_{\beta}\left(u\right) - u\right\|^{2} \\
&= \int_{t_0}^{t_f}\left\{F\left(x,u,t\right)\left[v_{\beta}\left(u\right) - v_{\alpha}\left(u\right)\right] - \frac{\alpha}{2}\left[v_{\alpha}\left(u\right) - u\right]^{2} + \frac{\beta}{2}\left[v_{\beta}\left(u\right) - u\right]^{2}\right\} dt
\end{aligned} \tag{3.15}$$

subject to

$$\frac{dx}{dt} = f\left(x,u,t\right) \tag{3.16}$$

$$x_i\left(0\right) = x_i^0 \qquad \forall i \in \mathcal{I}_1 \tag{3.17}$$

$$\Gamma\left[x\left(t_f\right), t_f\right] = 0 \tag{3.18}$$

This is a Bolza form of the standard optimal control problem, except the objective functional involves the maximizers of subproblems defined by (3.13) and (3.14), $v_{\alpha}\left(u\right)$ and $v_{\beta}\left(u\right)$.

Now we are interested in the gradient of the objective functional $\psi_{\alpha\beta}\left(u\right)$, which is, in fact, equivalent to the gradient of the corresponding Hamiltonian function in the theory of optimal control. Let us define the Hamiltonian function for $OCP\left(\psi_{\alpha\beta}, f, U\right)$

$$\begin{aligned}
H\left(x,u,\lambda,t\right) = {} & F\left(x,u,t\right)\left[v_{\beta}\left(u\right) - v_{\alpha}\left(u\right)\right] \\
& - \frac{\alpha}{2}\left[v_{\alpha}\left(u\right) - u\right]^{2} + \frac{\beta}{2}\left[v_{\beta}\left(u\right) - u\right]^{2} + \lambda f\left(x,u,t\right)
\end{aligned} \tag{3.19}$$

To obtain the gradient of $\psi_{\alpha\beta}\left(u\right)$, we need to consider $v_{\alpha}\left(\cdot\right)$ and $v_{\beta}\left(\cdot\right)$ which are maximizers defined by (3.13) and (3.14). Note that $v_{\alpha}\left(\cdot\right)$ and $v_{\beta}\left(\cdot\right)$ are unique by the strong concavity of $\Phi_{\alpha}\left(u,v\right)$ and $\Phi_{\beta}\left(u,v\right)$ in v and the convexity of the set U. To treat the maximizers, we need the following lemma from Pshenichnyi (1971).

Lemma 17 *Let $h : \mathcal{H} \times U \to \mathbb{R}$ be a function such that $\nabla_u h\left(u,v\right)$ exists, and is* *continuous on $\mathcal{H} \times U$. Define two functions as follows:*

$$w(u) = \max_{v \in U} h(u, v)$$
$$z(u) = \{v \in U : w(u) = h(u, v)\}$$

Then the derivative in the direction ρ becomes

$$\delta w(u; \rho) = \max_{v \in z(u)} \delta h(u, v; \rho)$$

Furthermore, if $z(u)$ is a singleton for all $u \in \mathcal{H}$, and z is a continuous function on $\mathcal{H}$, then w is continuously differentiable and the gradient becomes

$$\nabla w(u) = \nabla \max_{v \in U} h(u, v) = \nabla_u h(u, z(u)).$$

This result is now used to obtain the gradient of the objective functional $\psi_{\alpha\beta}(u)$:

Theorem 18 *Suppose $F(x, u, t)$ is Lipschitz continuous on every bounded subset of $(L^2[t_0, t_f])^m$. Then $\psi_{\alpha\beta}(u)$ is continuously differentiable in the sense of Gateaux, and*

$$\begin{aligned}
\nabla \psi_{\alpha\beta}(u) &= \frac{\partial}{\partial u} H(x, u, \lambda, t) \\
&= \frac{\partial F(x, u, t)}{\partial u} [v_\beta(u) - v_\alpha(u)] \\
&\quad + \alpha [v_\alpha(u) - u] - \beta [v_\beta(u) - u] + \lambda \frac{\partial f(x, u, t)}{\partial u}
\end{aligned}$$

where the adjoint variable λ is a solution of the following final value problem:

$$\begin{aligned}
-\frac{d\lambda}{dt} &= \left(\frac{\partial f}{\partial x}\right)^T \lambda + \left(\frac{\partial g}{\partial x}\right)^T \\
\lambda(t_f) &= \gamma^T \frac{\partial \Gamma[x(t_f), t_f]}{\partial x(t_f)} \\
\lambda_i(t_0) &= 0 \quad \forall i \in \mathcal{I}_2
\end{aligned}$$

$$g = \left\{F\left(x,u,t\right)\left[u - v_\alpha\left(u\right)\right] - \frac{\alpha}{2}\left[v_\alpha\left(u\right) - u\right]^2\right\}$$
$$- \left\{F\left(x,u,t\right)\left[u - v_\beta\left(u\right)\right] - \frac{\beta}{2}\left[v_\beta\left(u\right) - u\right]^2\right\}$$

Proof. Augmenting the terminal time state space constraint $\Gamma\left[x\left(t_f\right), t_f\right] = 0$ with a Lagrangian dual variable γ, rewrite the objective functional as

$$\begin{aligned}
\psi_{\alpha\beta}\left(u\right) &= \varphi_\alpha\left(u\right) - \varphi_\beta\left(u\right) + \gamma^T\Gamma\left[x\left(t_f\right), t_f\right] \\
&= \max_{v\in U}\left\{\int_{t_0}^{t_f} F\left(x,u,t\right)\left[u - v\right] - \frac{\alpha}{2}\left[v - u\right]^2 dt\right\} \\
&\quad - \max_{v\in U}\left\{\int_{t_0}^{t_f} F\left(x,u,t\right)\left[u - v\right] - \frac{\beta}{2}\left[v - u\right]^2 dt\right\} + \gamma^T\Gamma\left[x\left(t_f\right), t_f\right]
\end{aligned}$$

Let us define

$$g_\alpha\left(x,u,v\right) = F\left(x,u,t\right)\left[u - v\right] - \frac{\alpha}{2}\left[v - u\right]^2$$
$$g_\beta\left(x,u,v\right) = F\left(x,u,t\right)\left[u - v\right] - \frac{\beta}{2}\left[v - u\right]^2$$

Then the derivative in the direction $\rho \equiv \delta u$ becomes, by Lemma 17,

$$\delta\varphi_\alpha\left(u;\rho\right) = \int_{t_0}^{t_f}\left\{\frac{\partial g_\alpha\left(x,u,v_\alpha\right)}{\partial x}y + \frac{\partial g_\alpha\left(u,v_\alpha\right)}{\partial u}\rho\right\}dt$$
$$\delta\varphi_\beta\left(u;\rho\right) = \int_{t_0}^{t_f}\left\{\frac{\partial g_\beta\left(x,u,v_\beta\right)}{\partial x}y + \frac{\partial g_\beta\left(u,v_\beta\right)}{\partial u}\rho\right\}dt$$

so that

$$\delta\psi_{\alpha\beta}\left(u,\rho\right) = \int_{t_0}^{t_f}\left\{\frac{\partial g}{\partial x}y + \frac{\partial g}{\partial u}\rho\right\}dt + \gamma^T\frac{\partial\Gamma\left[x\left(t_f\right), t_f\right]}{\partial x\left(t_f\right)}y\left(t_f\right) \tag{3.20}$$

where we write $g\left(x,u\right) = g_\alpha\left(x,u,v_\alpha\right) - g_\beta\left(x,u,v_\beta\right)$ for simplicity and $y \equiv \delta x$ is a variation in x which implicitly depends on ρ. Furthermore, by definition

$$x\left(t\right) = x\left(t_0\right) + \int_{t_0}^{t} f\left(x,u,y\right)ds$$

and therefore

$$y = \delta x = y(t_0) + \int_{t_0}^{t} \left[\frac{\partial f}{\partial x} y + \frac{\partial f}{\partial u} \rho \right] ds$$

Since $x_i(t_0) = x_i^0$ fixed $\forall i \in \mathcal{I}_1$, we have

$$y_i(t_0) = 0 \quad \forall i \in \mathcal{I}_1$$

We introduce adjoint variables λ defined by the final value problem

$$\begin{aligned} -\frac{d\lambda}{dt} &= \left(\frac{\partial f}{\partial x}\right)^T \lambda + \left(\frac{\partial g}{\partial x}\right)^T \\ \lambda(t_f) &= \gamma^T \frac{\partial \Gamma[x(t_f), t_f]}{\partial x(t_f)} \\ \lambda_i(t_0) &= 0 \;\; \forall i \in \mathcal{I}_2 \end{aligned}$$

so that (3.20) becomes

$$\delta\psi_{\alpha\beta}(u;\rho) = \int_{t_0}^{t_f} \left\{ \left[-\left(\frac{d\lambda}{dt}\right)^T - \lambda^T \frac{\partial f}{\partial x} \right] y + \frac{\partial g}{\partial u} \rho \right\} dt + \lambda(t_f)\, y(t_f)$$

The integration by parts yields

$$\begin{aligned} \int_{t_0}^{t_f} -\left(\frac{d\lambda}{dt}\right)^T y dt &= \lambda(t_0)\, y(t_0) - \lambda(t_f)\, y(t_f) + \int_{t_0}^{t_f} \lambda^T \frac{dy}{dt} dt \\ &= \lambda(t_0)\, y(t_0) - \lambda(t_f)\, y(t_f) + \int_{t_0}^{t_f} \lambda^T \left[\frac{\partial f}{\partial x} y + \frac{\partial f}{\partial u} \rho \right] dt \end{aligned}$$

We note that

$$\begin{aligned} y_i(t_0) &= 0 \;\; \forall i \in \mathcal{I}_1 \\ \lambda_i(t_0) &= 0 \;\; \forall i \in \mathcal{I}_2 \end{aligned}$$

hence $\lambda(t_0)\, y(t_0) = 0$. It follows that

$$
\begin{aligned}
\delta\psi_{\alpha\beta}(u;\rho) &= -\lambda(t_f)\, y(t_f) + \int_{t_0}^{t_f} \left\{ \lambda^T \left[\frac{\partial f}{\partial x} y + \frac{\partial f}{\partial u} \rho \right] - \lambda^T \frac{\partial f}{\partial x} y + \frac{\partial g}{\partial u} \rho \right\} dt + \lambda(t_f)\, y(t_f) \\
&= \int_{t_0}^{t_f} \left\{ \lambda^T \frac{\partial f}{\partial u} + \frac{\partial g}{\partial u} \right\} \rho dt \\
&= \left\langle \lambda^T \frac{\partial f}{\partial u} + \frac{\partial g}{\partial u}, \rho \right\rangle
\end{aligned}
$$

Therefore, the gradient of $\psi_{\alpha\beta}(u)$ becomes

$$
\begin{aligned}
\nabla\psi_{\alpha\beta}(u) &= \lambda^T \frac{\partial f}{\partial u} + \frac{\partial g}{\partial u} \\
&= \nabla_u H(x, u, \lambda, t)
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
\nabla\psi_{\alpha\beta}(u) &= \nabla_u H(x, u, \lambda, t) \\
&= \lambda \frac{\partial f(x, u, t)}{\partial u} + \frac{\partial F(x, u, t)}{\partial u} \left[v_\beta(u) - v_\alpha(u) \right] \\
&\quad + \alpha \left[v_\alpha(u) - u \right] - \beta \left[v_\beta(u) - u \right]
\end{aligned}
$$

and, also

$$
-\frac{d\lambda}{dt} = \nabla_x H(x, u, \lambda, t)
$$

This completes the proof. ■

Using the gradient, we propose a descent method for solving $OCP(\psi_{\alpha\beta}, f, U)$ in the following section.

3.4 A Descent Method for $OCP(\psi_{\alpha\beta}, f, U)$

Numerous computational methods have been developed for continuous-time optimal control problems, for example see Polak (1973). Here, we propose the following descent algorithm for $OCP(\psi_{\alpha\beta}, f, U)$, in which the main philosophy remains the same as in usual descent algorithms for optimal control problems (see Minoux 1986 for such methods); however, we need to solve the state dynamics, the adjoint dynamics and two sub-maximization problems for v_α and v_β to obtain the current information of the descent direction. We define $\mathcal{I} \equiv \mathcal{I}_1 \cup \mathcal{I}_2$.

Descent Method for Differential Variational Inequalities

$\boxed{\textit{Step 0.}}$ *Initialization.* Choose $0 < \alpha < \beta$. Pick $u^k(t) \in U$ and set $k = 0$.
$\boxed{\textit{Step 1}}$. *Finding state variables.* Solve the state dynamics

$$\begin{aligned} \frac{dx_i}{dt} &= f_i\left(x, u^k, t\right) \qquad \forall i \in \mathcal{I} \\ x_i\left(t_0\right) &= x_i^0 \quad \forall i \in \mathcal{I}_1 \\ \Gamma\left[x\left(t_f\right), t\right] &= 0 \end{aligned}$$

and call the solution $x^k(t)$.
$\boxed{\textit{Step 2.}}$ *Solving subproblems.* Solve the projection subproblmes using current values of $u^k(t)$ and $x^k(t)$.

$$\begin{aligned} v_\alpha\left(u^k\right) &= P_U\left[u^k - \frac{1}{\alpha}F\left(x^k, u^k, t\right)\right] \\ v_\beta\left(u^k\right) &= P_U\left[u^k - \frac{1}{\beta}F\left(x^k, u^k, t\right)\right] \end{aligned}$$

$\boxed{\textit{Step 3.}}$ *Finding adjoint variables.* Solve the adjoint dynamics using current values of $u^k(t)$ and $x^k(t)$

$$\begin{aligned} -\frac{d\lambda_i}{dt} &= \left[\frac{\partial F\left(x^k, u^k, t\right)}{\partial x_i}\right]^T \left[v_\beta\left(u^k\right) - v_\alpha\left(u^k\right)\right] + \left[\frac{\partial f\left(x^k, u^k, t\right)}{\partial x_i}\right]^T \lambda \\ \lambda_i\left(t_f\right) &= \gamma^T \frac{\partial \Gamma\left[x^k\left(t_f\right), t_f\right]}{\partial x_i\left(t_f\right)} \quad \forall i \in \mathcal{I}_1 \\ \lambda_i\left(t_0\right) &= 0 \quad \forall i \in \mathcal{I}_2 \end{aligned}$$

and call the solution $\lambda^k(t)$.

$\boxed{\textit{Step 4.}}$ *Finding the gradient.* Determine

$$
\begin{aligned}
\nabla\psi^k_{\alpha\beta}(t) &= \nabla_u H\left(x^k, u^k, \lambda^k, t\right) \\
&= \frac{\partial F\left(x^k, u^k, t\right)}{\partial u}\left[v_\beta\left(u^k\right) - v_\alpha\left(u^k\right)\right] \\
&+ \alpha\left[v_\alpha\left(u^k\right) - u^k\right] - \beta\left[v_\beta\left(u^k\right) - u^k\right] + \lambda\frac{\partial f\left(x^k, u^k, t\right)}{\partial u}
\end{aligned}
$$

$\boxed{\textit{Step 5.}}$ *Updating the current control.* For a suitably small step size

$$\theta_k \in \mathbb{R}^1_{++}$$

determine

$$u^{k+1}(t) = P_U\left[u^k(t) - \theta_k \nabla\psi^k_{\alpha\beta}(t)\right]$$

$\boxed{\textit{Step 6.}}$ *Stopping Test.* For $\epsilon \in \mathbb{R}^1_{++}$, a preset tolerance, stop if

$$||\psi_{\alpha\beta}\left(u^{k+1}\right)|| < \epsilon$$

and declare

$$u^* \approx u^{k+1}$$

Otherwise set $k = k+1$ and go to Step1.

Note that Step 2 requires us to solve a two-point-boundary-value problem which is generally very difficult to solve. However, in many differential game applications of DVI form, the initial-value problem can be separated. That is, we may have

$$\frac{dx_i}{dt} = f_i\left(\left(x_i\right)_{i\in\mathcal{I}_1}, u^k, t\right), \quad x_i(t_0) = x_i^0$$

for all $i \in \mathcal{I}_1$ and, in sequence,

$$\frac{dx_i}{dt} = f_i\left(x, u^k, t\right), \quad x_i(t_f) = \gamma\left[\left(x_i\left(t_f\right)\right)_{i\in\mathcal{I}_1}, t_f\right]$$

for all $i \in \mathcal{I}_2$ where γ may be obtained from Γ. Similarly, we can separate the two-point-boundary-value problem in Step 3. Since

$$\frac{\partial f_j\left(x^k, u^k, t\right)}{\partial x_i} = 0 \quad \forall i \in \mathcal{I}_2\ , j \in \mathcal{I}_1$$

we have a initial-value problem

$$\begin{aligned} -\frac{d\lambda_i}{dt} &= \sum_{j\in\mathcal{I}} \left[\frac{\partial F_j\left(x^k, u^k, t\right)}{\partial x_i}\right]\left[v_{\beta,j}\left(u^k\right) - v_{\alpha,j}\left(u^k\right)\right] + \sum_{j\in\mathcal{I}_2}\left[\frac{\partial f_j\left(x^k, u^k, t\right)}{\partial x_i}\right]\lambda_j \\ \lambda_i\left(t_0\right) &= 0 \end{aligned}$$

for all $i \in \mathcal{I}_2$, in sequence,

$$\begin{aligned} -\frac{d\lambda_i}{dt} &= \sum_{j\in\mathcal{I}} \left[\frac{\partial F_j\left(x^k, u^k, t\right)}{\partial x_i}\right]\left[v_{\beta,j}\left(u^k\right) - v_{\alpha,j}\left(u^k\right)\right] + \sum_{j\in\mathcal{I}}\left[\frac{\partial f_j\left(x^k, u^k, t\right)}{\partial x_i}\right]\lambda_j \\ \lambda_i\left(t_f\right) &= \gamma^T \frac{\partial \Gamma\left[x^k\left(t_f\right), t_f\right]}{\partial x_i\left(t_f\right)} \end{aligned}$$

for all $i \in \mathcal{I}_1$.

We provide the convergence property of the proposed algorithm as follows:

Theorem 19 (Convergence) *Suppose the functional $\psi_{\alpha\beta} : U \longrightarrow \mathbb{R}_+$ is strongly convex with modulus $\rho > 0$ and $\nabla\psi_{\alpha\beta}(u)$ is defined and satisfies the Lipschitz condition*

$$\left\|\nabla\psi_{\alpha\beta}\left(u_1\right) - \nabla\psi_{\alpha\beta}\left(u_2\right)\right\| \leq \delta\left\|u_1 - u_2\right\| \tag{3.21}$$

for all $u_1, u_2 \in U$. Then the decent algorithm converges to the minimum u^* of $\psi_{\alpha\beta}$ on U for step size choices

$$\theta \in \left(0, \frac{2\rho}{\delta^2}\right) \tag{3.22}$$

Proof. We see from Theorem 11 that u^* must be the projection of $u^* - \theta\nabla\psi_{\alpha\beta}\left(u^*\right)$; that is

$$u^* = P_U\left[u^* - \theta\nabla\psi_{\alpha\beta}\left(u^*\right)\right] \tag{3.23}$$

From this observation and the algorithm itself, it is an easy matter to construct the difference

$$u^{k+1} - u^* = P_U\left[u^k - \theta\nabla\psi_{\alpha\beta}\left(u^k\right)\right] - P_U\left[u^* - \theta\nabla\psi_{\alpha\beta}\left(u^*\right)\right] \tag{3.24}$$

Since the projection mapping is a contraction, we know

$$\left\|u^{k+1} - u^*\right\| \leq \left\|\left(u^k - \theta\nabla\psi_{\alpha\beta}\left(u^k\right)\right) - \left(u^* - \theta\nabla\psi_{\alpha\beta}\left(u^*\right)\right)\right\| \tag{3.25}$$

from which we obtain

$$\left\|u^{k+1} - u^*\right\|^2 \leq \left\|\left(u^k - \theta\nabla\psi_{\alpha\beta}\left(u^k\right)\right) - \left(u^* - \theta\nabla\psi_{\alpha\beta}\left(u^*\right)\right)\right\|^2 \tag{3.26}$$

The right hand side (RHS) of (3.26) can be restated using the given Lipschitz condition and strong convexity. In particular, we have

$$\begin{aligned} \text{RHS} = {} & \left\|u^k - u^*\right\|^2 - 2\theta\left\langle\nabla\psi_{\alpha\beta}\left(u^k\right) - \nabla\psi_{\alpha\beta}\left(u^*\right), u^k - u^*\right\rangle \\ & + \theta^2\left\|\nabla\psi_{\alpha\beta}\left(u^k\right) - \nabla\psi_{\alpha\beta}\left(u^*\right)\right\|^2 \\ & \leq \left\|u^k - u^*\right\|^2 - 2\rho\theta\left\|u^k - u^*\right\|^2 + \delta^2\theta^2\left\|u^k - u^*\right\|^2 \\ & = \left(1 - 2\alpha\theta + \delta^2\theta^2\right)\left\|u^k - u^*\right\|^2 \end{aligned} \tag{3.27}$$

Results (3.26) and (3.27) tell us that

$$\begin{aligned} & \left\|u^{k+1} - u^*\right\|^2 \leq \left(1 - 2\rho\theta + \beta^2\theta^2\right)\left\|u^k - u^*\right\|^2 \\ & \qquad \Longrightarrow \left\|u^{k+1} - u^*\right\| \leq \left(1 - 2\rho\theta + \delta^2\theta^2\right)^{\frac{1}{2}}\left\|u^k - u^*\right\| \end{aligned} \tag{3.28}$$

Inequality (3.28) will establish convergence if

$$\begin{aligned} & \left(1 - 2\rho\theta + \delta^2\theta^2\right) < 1 \\ & \qquad \Longrightarrow \delta^2\theta^2 < 2\rho\theta \\ & \qquad \Longrightarrow \theta < \frac{2\rho}{\delta^2} \end{aligned} \tag{3.29}$$

Since $\theta > 0$, the desired result follows. ■

Now we are ready to proceed to examples and applications.

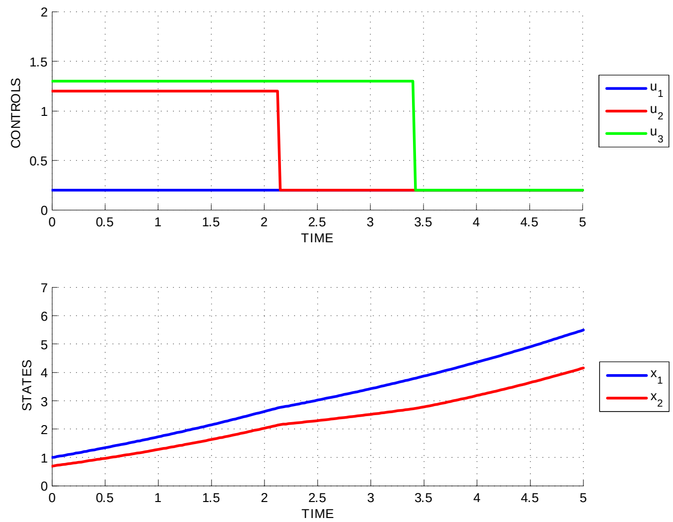

Figure 3.1. Result by Gap Function (gap$< 10^{-10}$, $\alpha = 0.5, \beta = 2$)

3.5 An Abstract Numerical Example

Let us consider an example DVI, originally considered by Friesz and Mookherjee (2005), including 3 controls and 2 states:

$$u \in \left(L^2\left[0,1\right]\right)^3;\ x \in \left(\mathcal{H}\left[0,1\right]\right)^2;\ x\left(t_0\right) = \begin{pmatrix} 1 \\ 0.7 \end{pmatrix}; \left[t_0, t_f\right] = \left[0,5\right]$$

$$F(x,u) = \begin{pmatrix} F_1(x,u) \\ F_2(x,u) \\ F_3(x,u) \end{pmatrix} = \begin{pmatrix} x_1^2 - u_1\left(t\right) + u_2\left(t\right) \\ x_2 - u_2^2\left(t\right) - u_3\left(t\right) \\ \frac{1}{10}x_2^2 - u_3^2\left(t\right) \end{pmatrix}$$

$$f\left(x,u\right) = \begin{pmatrix} f_1(x,u) \\ f_2(x,u) \end{pmatrix} = \begin{pmatrix} \frac{1}{5}x_1\left(t\right) + \frac{1}{2}u_1\left(t\right) + \frac{3}{10}u_2\left(t\right) \\ \frac{1}{4}x_2\left(t\right) + \frac{1}{2}u_2\left(t\right) - \frac{1}{5}u_3\left(t\right) \end{pmatrix}$$

$$U = \left\{u : 0.2 \leq u_1 \leq 1;\ 0.2 \leq u_2 \leq 1.2; 0.2 \leq u_3 \leq 1.3\right\}$$

The computation results are shown in Figures 3.1 and 3.2. In 11 iterations, the algorithm converges with a gap less than 10^{-10}.

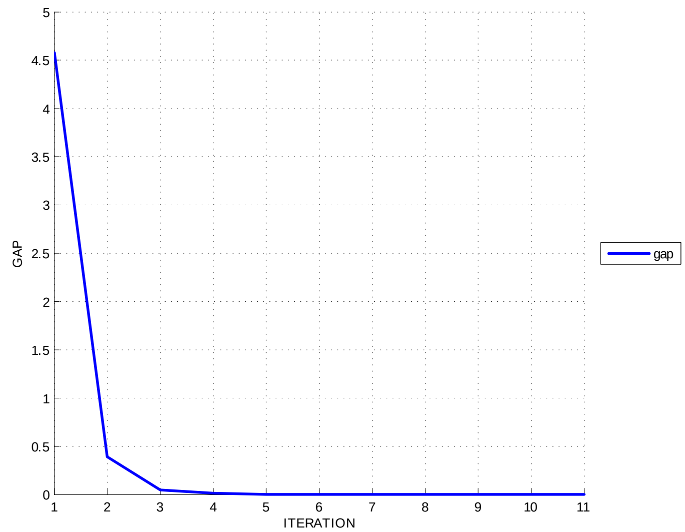

Figure 3.2. The convergence of the descent algorithm, which is terminated with a gap less than 10^{-10}

3.6 A Linear-Quadratic Differential Game Example

When Ψ_i in (2.71) is a quadratic function and g_i in (2.72)is a linear function of x and u, the corresponding differential game is called linear-quadratic. Dockner et al. (2000) obtained the exact solution of a two-person linear-quadratic differential Nash game in which Player 1 minimizes the cost function

$$J_1 = \frac{1}{2}\int_0^T e^{-rt}\left\{g_1 x^2(t) + g_2 [u_1(t)]^2\right\} dt$$

and Player 2 minimizes

$$J_2 = \frac{1}{2}\int_0^T e^{-rt}\left\{m_1 x^2(t) + m_2 [u_2(t)]^2\right\} dt$$

where x is the state variable, and u_1 and u_2 are controls of Player 1 and Player 2, respectively. The state equation is given by

$$\frac{dx}{dt} = ax(t) + bu_1(t) + cu_2(t), \qquad x(0) = x_0$$

The system of optimal control problems involves two controls and one state. Assume that g_1, g_2, m_1 and m_2 are positive constants, b and c are non-zero, and a is in most cases negative, but not necessarily, so that the uncontrolled system is stable. In this case, Dockner et al. (2000) gave the open-loop Nash equilibrium of the game as following: the optimal trajectories are

$$u_1^*(t) = x_0 \left(\frac{b}{g_2}\right)\left(\frac{g_1}{a - s_2}\right)\exp(s_1 t) \tag{3.30}$$

$$u_2^*(t) = x_0 \left(\frac{c}{m_2}\right)\left(\frac{m_1}{a - s_2}\right)\exp(s_1 t) \tag{3.31}$$

where

$$s_1 = \frac{r}{2} - \sqrt{\frac{r^2}{4} - a(r-a) + c^2\frac{m_1}{m_2} + b^2\frac{g_1}{g_2}}$$

$$s_2 = \frac{r}{2} + \sqrt{\frac{r^2}{4} - a(r-a) + c^2\frac{m_1}{m_2} + b^2\frac{g_1}{g_2}}$$

$$s_3 = r - a$$

For a detailed derivation of the solutions, see Dockner et al. (2000).

Using the necessary conditions, which are also sufficient in this case because the problem is linear-quadratic, we obtain the corresponding differential variational inequality, $DVI(F, f, U)$, where

$$F(x,u,p,t) = \begin{pmatrix} e^{-rt}g_2u_1 + bp_1 \\ e^{-rt}m_2u_2 + cp_2 \end{pmatrix}, \qquad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$f(x,u,p,t) = \begin{pmatrix} ax(t) + bu_1(t) + cu_2(t) \\ e^{-rt}\{g_1x\} + ap_1 \\ e^{-rt}\{m_1x\} + ap_2 \end{pmatrix}, \qquad \begin{array}{c} x(0) = x_0 \\ p_1(T) = 0 \\ p_2(T) = 0 \end{array}$$

$$U = \mathbb{R}^2$$

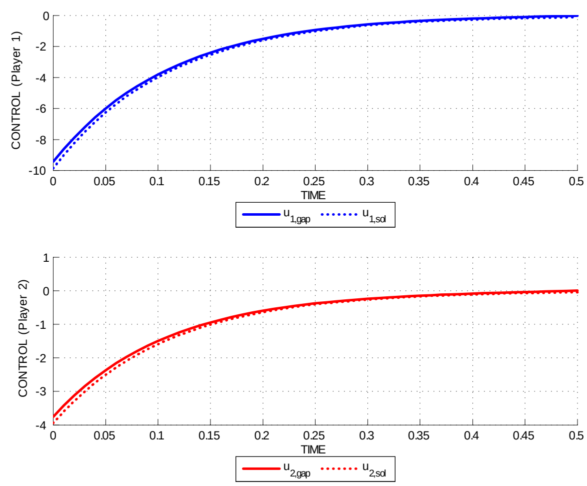

Figure 3.3. Analytical solutions (dashed lines) and numerical results (solid lines) for a differential linear-quadratic Nash game.

As an example, consider the set of parameters

$$\begin{aligned} a &= -1, b = 5, c = 8 \\ g_1 &= 2, g_2 = 1 \\ m_1 &= 1, m_2 = 2 \\ x_0 &= 10 \end{aligned}$$

In Figure 3.3, the exact solution given by (3.30) and (3.31), and the numerical results of the descent algorithm proposed are plotted in dashed lines and solid lines, respectively. In addition, the convergence property is shown in Figure 3.4. It is clear that the numerical results are very close to the analytic solution provided by Dockner et al. (2000). The descent algorithm is terminated in 42 iterations converging to the solution with a gap less than 10^{-6}.

3.7 Concluding Remarks

In this chapter, we have extended the D-gap functions considered by Konnov and Kum (2001) for VIPs in Hilbert spaces to DVIs with control and state variables. We have studied the equivalent optimal control problem for such DVIs and proposed

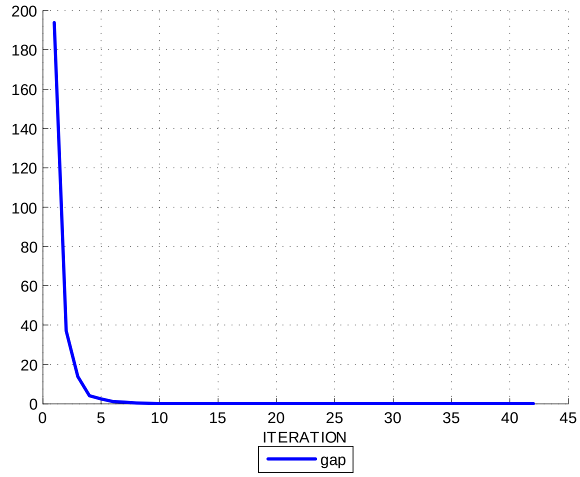

Figure 3.4. The convergence of the descent algorithm, which is terminated with the gap less than 10^{-6}

a descent method to solve it. As an application, we solved a linear-quadratic differential Nash game example by the descent method proposed, which concluded that the numerical solution is very close to the analytic solution given by Dockner et al. (2000).

The equivalent optimal control problem for a DVI is in fact a bi-level mathematical program. It should be noted that we can apply infinite-dimensional Karush-Kuhn-Tucker (KKT) conditions for the sub-problems of v_α and v_β, which will convert the bi-level problem to a single-level optimal control problem with complementarity problems in the constraint set since the sub-problems are concave maximization programs. Combined with discretizations, the KKT approach leads to a finite-dimensional nonlinear program which may be solved by various nonlinear programming algorithms.

Chapter 4

A Computable Theory of Dynamic Congestion Pricing

The main focus of this chapter is the formulation and solution of the dynamic optimal toll problem with equilibrium constraints, or DOTPEC. To this end, using the DUE formulation reported in Friesz et al. (2001) and Friesz and Mookherjee (2006), we will form a Stackelberg game that envisions a central authority minimizing social costs through its control of link tolls subject to DUE constraints with the potential for additional side constraints for equity and other policy considerations. Also, since we will allow multiple target arrival times of the users, the within-day scale model, we show how to easily extend the formulation to include the day-to-day evolution of demand. Of course there are several ways such a model may be formulated. The dual-time scale formulation we shall emphasize is based on our prior work on differential variational inequalities and equilibrium network design and follows the qualitative theory conjectured (but not analyzed) by Friesz et al. (1996).

Central to the study of the DOTPEC in this chapter is the dynamic generalization of a result due to Tan et al. (1979) and reprised by Friesz and Shah (2001) showing that a system of inequalities expressing the relationship of average effective delay to minimum delay is equivalent to a static user equilibrium. This system of inequalities allows one to state the equilibrium network design problem as a single level mathematical program. Extension of this result to a dynamic setting allows us in this chapter to state the DOTPEC as an equivalent, non-hierarchical optimal control problem. We consider two principal methods for solving this optimal control problem: (1) descent in Hilbert space without time discretization, and (2) a finite dimensional approximation solved as a nonlinear program. In both approaches we employ an implicit fixed point scheme like that in Friesz and Mookherjee (2006) for dealing with time shifts in differential variational inequalities. In an example provided near the end of this chapter, we numerically study a small network and determine its optimal dynamic tolls.

4.1 Notation and Model Formulation

In this section we purposely repeat key portions of the time-lagged DUE formulation given in Friesz et al. (2001), because of its key role in this manuscript. The reader familiar with the notation and time-shifted DUE model presented in Friesz et al. (2001) may skip this section.

4.1.1 Dynamics, Delay Operators and Constraints

The network of interest will form a directed graph $G(\mathcal{N}, \mathcal{A})$, where $\mathcal{N}$ denotes the set of nodes and $\mathcal{A}$ denotes the set of arcs; the respective cardinalities of these sets are $|\mathcal{N}|$ and $|\mathcal{A}|$. An arbitrary path $p \in \mathcal{P}$ of the network is

$$p \equiv \left\{a_1, a_2, ..., a_i, ..., a_{m(p)}\right\}$$

where $\mathcal{P}$ is the set of all paths and $m(p)$ is the number of arcs of p. We also let t_e denote the time at which flow exists an arc, while t_d is the time of departure from the origin of the same flow. The exit time function $\tau^p_{a_i}$ therefore obeys

$$t_e = \tau^p_{a_i}(t_d)$$

The relevant arc dynamics are

$$\frac{dx^p_{a_i}(t)}{dt} = g^p_{a_{i-1}}(t) - g^p_{a_i}(t) \qquad \forall p \in \mathcal{P}, \quad i \in \{1, 2, ..., m(p)\} \tag{4.1}$$

$$x^p_{a_i}(t) = x^p_{a_{i,0}} \qquad \forall p \in \mathcal{P}, \quad i \in \{1, 2, ..., m(p)\} \tag{4.2}$$

where $x_{a_i}^p$ is the traffic volume of arc a_i contributed by path p, $g_{a_i}^p$ is flow exiting arc a_i and $g_{a_{i-1}}^p$ is flow entering arc a_i of path $p \in \mathcal{P}$. Also, g_{a0}^p is the flow exiting the origin of path p; by convention we call this the flow of path p and use the symbolic name

$$h_p = g_{a_0}^p$$

Furthermore

$$\delta_{a_i p} = \begin{cases} 1 & \text{if } a_i \in p \\ 0 & \text{if } a_i \notin p \end{cases}$$

so that

$$x_a(t) = \sum_{p \in \mathcal{P}} \delta_{ap} x_a^p(t) \qquad \forall a \in \mathcal{A}$$

is the total arc volume.

Arc unit delay is $D_a(x_a)$ for each arc $a \in \mathcal{A}$. That is, arc delay depends on the number of vehicles in front of a vehicle as it enters an arc. Of course total path traversal time is

$$D_p(t) = \sum_{i=1}^{m(p)} \left[\tau_{a_i}^p(t) - \tau_{a_{i-1}}^p(t)\right] = \tau_{a_{m(p)}}^p(t) - t \qquad \forall p \in \mathcal{P}$$

It is expedient to introduce the following recursive relationships that must hold in light of the above development:

$$\tau_{a_1}^p(t) = t + D_{a_1}\left[x_{a_1}(t)\right] \quad \forall p \in \mathcal{P}$$
$$\tau_{a_i}^p(t) = \tau_{a_{i-1}}^p(t) + D_{a_i}\left[x_{a_i}(\tau_{a_{i-1}}^p(t))\right] \quad \forall p \in \mathcal{P}, \quad i \in \{2, 3, ..., m(p)\}$$

from which we have the nested path delay operators first proposed by Friesz et al. (1993):

$$D_p(t, x) \equiv \sum_{i=1}^{m(p)} \delta_{a_i p} \Phi_{a_i}(t, x) \quad \forall p \in \mathcal{P},$$

where

$$x = (x_{a_i}^p : p \in \mathcal{P}, i \in \{1, 2, ..., m(p)\}$$

and

$$\begin{aligned}
\Phi_{a_1}(t,x) &= D_{a_1}(x_{a_1}(t)) \\
\Phi_{a_2}(t,x) &= D_{a_2}(x_{a_2}(t+\Phi_{a_1})) \\
\Phi_{a_3}(t,x) &= D_{a_3}(x_{a_3}(t+\Phi_{a_1}+\Phi_{a_2})) \\
&\vdots \\
\Phi_{a_i}(t,x) &= D_{a_i}(x_{a_i}(t+\Phi_{a_1}+\cdots+\Phi_{a_{i-1}})) \\
&= D_{a_i}\left(x_{a_i}\left(t+\sum_{j=1}^{i-1}\Phi_{a_j}\right)\right).
\end{aligned}$$

To ensure realistic behavior, we employ asymmetric early/late arrival penalties

$$F\left[t+D_p\left(t,x\right)-t_A\right]$$

where t_A is the desired arrival time and

$$\begin{aligned}
t+D_p(t,x) > t_A &\Longrightarrow \quad F(t+D_p(t,x)-t_A) = \chi^L(x,t) > 0 \\
t+D_p(t,x) < t_A &\Longrightarrow \quad F(t+D_p(t,x)-t_A) = \chi^E(x,t) > 0 \\
t+D_p(t,x) = t_A &\Longrightarrow \quad F(t+D_p(t,x)-t_A) = 0
\end{aligned}$$

while

$$\chi^L(t,x) > \chi^E(t,x)$$

Let us further denote arc tolls by y_a for each arc $a \in \mathcal{A}$. We assume that users pay any toll imposed on an arc at the entrance of the arc. Then the path tolls y_p for each path $p \in \mathcal{P}$ are

$$y_p\left(t\right) = \sum_{i=1}^{m(p)} \delta_{a_i p} y_{a_i}\left(t+\sum_{j=1}^{i-1}\Phi_{a_j}(t,x)\right) \quad \forall p \in \mathcal{P}$$

where $\Phi_{a_0}(t,x) = 0$. If the tolls are paid when users exit arcs, then the path toll becomes

$$y_p\left(t\right) = \sum_{i=1}^{m(p)} \delta_{a_i p} y_{a_i}\left(t+\sum_{j=1}^{i}\Phi_{a_j}(t,x)\right) \quad \forall p \in \mathcal{P}$$

We now combine the actual path delays and arrival penalties to obtain the *effective delay operators*

$$\Psi_p(t,x) = D_p(t,x) + F\left(t + D_p(x,t) - T_A\right) \qquad \forall p \in \mathcal{P} \tag{4.3}$$

Since the volume which enters and exits an arc should conserve flow, we must have

$$\int_0^t g^p_{a_{i-1}}(t)\,dt = \int_{D_{a_i}(x_{a_i}(0))}^{t+D_{a_i}(x_{a_i}(t))} g^p_{a_i}(t)\,dt \quad \forall p \in \mathcal{P}, i \in [1, m(p)] \tag{4.4}$$

where $g^p_{a_0}(t) = h_p(t)$. Differentiating both sides of (4.4) with respect to time t and using the chain rule, we have

$$h_p(t) = g^p_{a_1}(t + D_{a_1}(x_{a_1}(t)))(1 + D'_{a_1}(x_{a_1}(t))\dot{x}_{a_1}) \qquad \forall p \in \mathcal{P} \tag{4.5}$$

$$g^p_{a_{i-1}}(t) = g^p_{a_i}(t+D_{a_i}(x_{a_i}(t)))(1+D'_{a_i}(x_{a_i}(t))\dot{x}_{a_i}) \qquad \forall p \in \mathcal{P}, \quad i \in [2, m(p)] \tag{4.6}$$

These are *proper flow progression constraints* derived in a fashion that makes them completely *consistent with the chosen dynamics and point queue model of arc delay.* These constraints involve a state-dependent time lag $D_{a_i}(x_{a_i}(t))$ but make no explicit reference to the exit time functions. These flow propagation constraints describe the expansion and contraction of vehicle platoons; they were presented by Friesz et al. (1995). Astarita (1995,1996) independently proposed flow propagation constraints that may be readily placed in the above form.

The final constraints to consider are those of flow conservation and non-negativity:

$$\sum_{p\in\mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p(t)dt = Q_{ij} \quad \forall\,(i,j) \in \mathcal{W} \tag{4.7}$$

$$h_p \geq 0 \quad \forall\,(i,j) \in \mathcal{P}_{ij} \tag{4.8}$$

$$g^p_{a_i} \geq 0 \quad \forall p \in \mathcal{P}, i \in [1, m(p)] \tag{4.9}$$

$$g^p_{a_i} \geq 0 \quad \forall p \in \mathcal{P}, i \in [1, m(p)] \tag{4.10}$$

where $\mathcal{W}$ is the set of origin-destination pairs, $\mathcal{P}_{ij}$ is the set of paths connecting origin-destination pair (i,j), $t_f > t_0$, and $t_f - t_0$ defines the planning horizon.

Furthermore, Q_{ij} is the travel demand (a volume) for the period $[t_0.t_f]$. In what follows h will denote the vector of all path flows, g the vector of all arc exit flows. Finally, we denote the set of all feasible exit flow vectors (h, g) by Ω; that is

$$\Omega \equiv \{(h, g) : (4.1), (4.2), (4.5), (4.6), (4.7), (4.8), (4.9), (4.10) \text{ are satisfied}\} \tag{4.11}$$

4.1.2 Dynamic User Equilibrium

Given the effective unit travel delay Ψ_p for path p, the infinite dimensional variational inequality formulation for dynamic network user equilibrium itself is: find $(g^*, h^*) \in \Omega$ such that

$$\langle \Psi (t, x (h^*, g^*)), (h - h^*) \rangle = \sum_{p \in \mathcal{P}} \int_{t_0}^{t_f} \Psi_p [t, x (h^*, g^*)] \cdot \left[h_p (t) - h_p^* (t) \right] dt \geq 0 \tag{4.12}$$

for all $(h, g) \in \Omega$, where Ψ denotes the vector of effective path delay operators. Friesz et al. (2001) show all solutions of (4.12) are dynamic user equilibria[1]. In particular the solutions of (4.12) obey

$$\Psi_p (t, x (g^*, h^*)) > \mu_{ij} \Longrightarrow h_p^* (t) = 0 \tag{4.13}$$

$$h_p^* (t) > 0 \Longrightarrow \Psi_p (t, x (g^*, h^*)) = \mu_{ij} \tag{4.14}$$

for $p \in \mathcal{P}_{ij}$ where μ_{ij} is the lower bound on achievable costs for any ij-traveler, given by

$$\mu_p = \operatorname{ess\,inf} \{\Theta_p (t, x) : t \in [t_0, t_f]\} \geq 0$$

and

$$\mu_{ij} = \min \{\mu_p : p \in \mathcal{P}_{ij}\} \geq 0$$

We call a flow pattern satisfying (4.13) and (4.14) a *dynamic user equilibrium.* The behavior described by (4.13) and (4.14) is readily recognized to be a type

[1] Although we have purposely supressed the functional analysis subtleties of the formulation, it should be noted that (4.12) involves an inner product in a Hilbert space, namely $\left(L^2 [0, T]\right)^{|\mathcal{P}|}$.

of Cournot-Nash non-cooperative equilibrium. It is important to note that these conditions do not describe a stationary state, but rather a time varying flow pattern that is a Cournot-Nash equilibrium (or user equilibrium) at each instant of time.

4.2 The Dynamic Efficient Toll Problem (DETP)

Hearn and Yildirim (2002) studied the efficient toll in the static setting with the traveling cost which is linear in the traffic flow. The objective of the efficient toll is to make the user equilibrium traffic flow equivalent to the system optimum by appropriate congestion pricing. To study the dynamic efficient toll problem (DETP), we introduce the notion of a *tolled effective delay operator*:

$$\Theta_p(t,x,y_p) = D_p(t,x) + F\left\{t + D_p(x,t) - T_A\right\} + y_p\left(t\right) \qquad \forall p \in \mathcal{P}$$

where y_p denotes the toll for path p. Of course we have the relationship

$$\Theta_p\left(t,x,y_p\right) = \Psi_p\left(t,x\right) + y_p\left(t\right) \tag{4.15}$$

4.2.1 Analysis of the System Optimum

The dynamic system optimum (DSO) is achieved by solving

$$\min \; J_1 = \int_{t_0}^{t_f} \sum_{p\in\mathcal{P}} e^{-rt}\Psi_p\left(t,x\right)h_p\left(t\right)dt$$

subject to

$$\frac{dx_{a_i}^p\left(t\right)}{dt} = g_{a_{i-1}}^p\left(t\right) - g_{a_i}^p\left(t\right) \qquad \forall p \in \mathcal{P}, \quad i \in \left\{1,2,...,m\left(p\right)\right\} \tag{4.16}$$

$$x_{a_i}^p\left(t\right) = x_{a_{i,0}}^p \qquad \forall p \in \mathcal{P}, \quad i \in \left\{1,2,...,m\left(p\right)\right\}$$

$$g_{a_{i-1}}^p\left(t\right) = g_{a_i}^p(t + D_{a_i}(x_{a_i}(t)))(1 + D'_{a_i}(x_{a_i}(t))\dot{x}_{a_i}) \qquad \forall p \in \mathcal{P}, \quad i \in [1,m(p)] \tag{4.17}$$

$$\sum_{p\in\mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p\left(t\right)dt = Q_{ij} \quad \forall\left(i,j\right) \in \mathcal{W} \tag{4.18}$$

$$x \geq 0 \qquad g \geq 0 \qquad h \geq 0 \tag{4.19}$$

where we have used the convention

$$g_{a_0}^p = h_p$$

It will be convenient to employ the following shorthand for shifted variables:

$$\bar{g}_{a_i}^p \mathrel{\mathring{=}} g_{a_i}^p(t + D_{a_i}(x_{a_i}(t))) \quad \forall p \in \mathcal{P}, \quad i \in [0, m(p)]$$

Penaltizing (4.17) we obtain

$$J_1 = \int_{t_0}^{t_f} \left\{ \sum_{p \in \mathcal{P}} e^{-rt} \Psi_p(t, x) \, h_p(t) \right. \\ \left. + \sum_{p \in \mathcal{P}} \sum_{i=1}^{m(p)} \frac{\mu_{a_i}^p}{2} \left[g_{a_{i-1}}^p(t) - \bar{g}_{a_i}^p(t) \left(1 + D'_{a_i}(x_{a_i}(t)) \dot{x}_{a_i}\right) \right]^2 \right\} dt \tag{4.20}$$

where $\mu_{a_i}^p$ is the penalty coefficient. Let us then define the set of feasible controls

$$\Lambda \equiv \left\{ (h, g) : \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p(t) \, dt = Q_{ij} \ \ \forall (i, j) \in \mathcal{W}, h \geq 0, \quad g \geq 0 \right\} \tag{4.21}$$

Optimal control problem (4.20) and (4.21) is an instance of the time-shifted optimal control problem analyzed in Friesz et al. (2001). We also employ the following notation for the state vector and control vector, respectively:

$$x = \left(x_{a_i}^p\right)_{p \in \mathcal{P}, i \in [1, m(p)]}$$

$$g = \left(g_{a_i}^p\right)_{p \in \mathcal{P}, i \in [0, m(p)]}$$

The DSO Hamiltonian is

$$H_1(t,x,h,g,\lambda;\mu) \equiv \sum_{p\in\mathcal{P}} e^{-rt}\Psi_p(t,x)\,h_p(t)$$
$$+\sum_{p\in\mathcal{P}}\sum_{i=1}^{m(p)} \frac{\mu^p_{a_i}}{2}\left\{g^p_{a_{i-1}}(t) - \bar{g}^p_{a_i}(t)(1+D'_{a_i}(x_{a_i}(t))\dot{x}_{a_i})\right\}^2$$
$$+\sum_{p\in\mathcal{P}}\sum_{i=1}^{m(p)} \lambda^p_{a_i}\left(g^p_{a_{i-1}}(t) - g^p_{a_i}(t)\right)$$

Let us introduce the vector

$$F(t,x,h,g,\lambda;\mu) = \left(F^p_{a_i}(t,x,h,g,\lambda;\mu)\right)_{p\in\mathcal{P},i\in[0,m(p)]}$$

where

$$F^p_{a_0}(t,x,h,g,\lambda;\mu) = \frac{\partial H_1(t,x,h,g,\lambda;\mu)}{\partial h_p} \quad \forall p\in\mathcal{P} \tag{4.22}$$

$$F^p_{a_i}(t,x,h,g,\lambda;\mu) = \begin{cases} \dfrac{\partial H_1(t,x,h,g,\lambda;\mu)}{\partial g^p_{a_i}} & \text{if } t\in[t_0, D_{a_i}(x(t_0))] \\[2ex] \dfrac{\partial H_1(t,x,h,g,\lambda;\mu)}{\partial g^p_{a_i}} + \left[\dfrac{\partial H_1(t,x,h,g,\lambda;\mu)}{\partial \bar{g}^p_{a_i}}\dfrac{1}{1+D'_{a_i}(x_{a_i}(t))\dot{x}_{a_i}}\right]_{s_{a_i}(t)} & \text{if } t\in[D_{a_i}(x(t_0)), t_f + D_{a_i}(x(t_f))] \end{cases}$$
$$\forall p\in\mathcal{P},\quad i\in[1,m(p)] \tag{4.23}$$

and each $s_{a_i}(t)$ is a solution of the fixed point problem

$$s_{a_i}(t) = \arg\left[s = t - D_{a_i}(x(s))\right]$$

We may write (4.22) and (4.23) in detail as

$$F^p_{a_0}(t,x,h,g,\lambda;\mu) = e^{-rt}\left[\Psi_p(t,x) + \frac{\partial \Psi_p(t,x)}{\partial h_p} h_p\right] + \mu^p_{a_1}\left[g^p_{a_0}(t) - \bar{g}^p_{a_1}(t)(1 + D'_{a_1}(x_{a_1}(t))\dot{x}_{a_1})\right] + \lambda^p_{a_1} \quad \forall p \in \mathcal{P} \tag{4.24}$$

$$F^p_{a_i}(t,x,h,g,\lambda;\mu) = \begin{cases} \mu^p_{a_{i+1}}\left\{g^p_{a_i}(t) - \bar{g}^p_{a_{i+1}}(t)(1 + D'_{a_{i+1}}(x_{a_{i+1}}(t))\dot{x}_{a_{i+1}})\right\} - \lambda^p_{a_i} + \lambda^p_{a_{i+1}} \\ \qquad \text{if } t \in [t_0, D_{a_i}(x(t_0))] \\ \mu^p_{a_{i+1}}\left\{g^p_{a_i}(t) - \bar{g}^p_{a_{i+1}}(t)(1 + D'_{a_{i+1}}(x_{a_{i+1}}(t))\dot{x}_{a_{i+1}})\right\} - \lambda^p_{a_i} \\ +\lambda^p_{a_{i+1}} - \left[\mu^p_{a_i}\left\{g^p_{a_{i-1}}(t) - \bar{g}^p_{a_i}(t)(1 + D'_{a_i}(x_{a_i}(t))\dot{x}_{a_i})\right\}\right]_{s_{a_i}(t)} \\ \qquad \text{if } t \in [D_{a_i}(x(t_0)), t_f + D_{a_i}(x(t_f))] \end{cases}$$

$$\forall p \in \mathcal{P}, \quad i \in [1, m(p)-1] \tag{4.25}$$

$$F^p_{a_i}(t,x,h,g,\lambda;\mu) = \begin{cases} -\lambda^p_{a_i} \\ \qquad \text{if } t \in [t_0, D_{a_i}(x(t_0))] \\ -\lambda^p_{a_i} - \left[\mu^p_{a_i}\left\{g^p_{a_{i-1}}(t) - \bar{g}^p_{a_i}(t)(1 + D'_{a_i}(x_{a_i}(t))\dot{x}_{a_i})\right\}\right]_{s_{a_i}(t)} \\ \qquad \text{if } t \in [D_{a_i}(x(t_0)), t_f + D_{a_i}(x(t_f))] \end{cases}$$

$$\forall p \in \mathcal{P}, \quad i = m(p) \tag{4.26}$$

Then a necessary condition for $\left(h^S, g^S\right) \in \Lambda$ to be the system optimum is

$$0 \leq \sum_{p\in\mathcal{P}} \sum_{i=0}^{m(p)} F_{a_i}^p \left(t, x^S, h^S, g^S, \lambda^S; \mu\right) \left(g_{a_i}^p - g_{a_i}^{pS}\right) \qquad \forall \left(h, g\right) \in \Lambda \tag{4.27}$$

for each time instant $t \in \left[t_{0,} \sup_{a_i \in A} \left\{t_f + D_{a_i}\left(x\left(t_f\right)\right)\right\}\right]$, together with the state dynamics (4.16) and the following adjoint equations and boundary conditions

$$-\frac{d\lambda_{a_i}^{p,S}}{dt} = \frac{\partial H_1^S}{\partial x_{a_i}^p} = e^{-rt} \frac{\partial \Psi_p \left(t, x^S\right)}{\partial x_{a_i}^p} \qquad \forall p \in \mathcal{P}, \quad i \in [1, m(p)]$$

$$\lambda_{a_i}^{p,S}\left(t_f\right) = 0 \qquad \forall p \in \mathcal{P}, \quad i \in [1, m(p)]$$

where the superscript S denotes a trajectory corresponding to a system optimum.

4.2.2 Analysis of the User Equilibrium in the Presence of Tolls

However, a dynamic tolled user equilibrium must obey

$$\sum_{p\in\mathcal{P}} \int_{t_0}^{t_f} e^{-rt} \left\{\Theta_p \left[t, x\left(h^U\right), y_p^U\right]\right\} \left[h_p\left(t\right) - h_p^U\left(t\right)\right] dt \geq 0 \quad \text{for all } \left(h, g\right) \in \Lambda \tag{4.28}$$

where the state dynamics as well as all other state and control constraints are identical to those introduced above for DSO. In particular, the set of feasible controls Λ referred to in (4.28) remains unchanged. We formulate an optimal control problem[2] from the above dynamic user equilibrium variational inequality problem; its objective is

$$\min \; J_2 = \sum_{p\in\mathcal{P}} \int_{t_0}^{t_f} e^{-rt} \Theta_p \left[t, x\left(h^U\right), y_p^U\right] h_p\left(t\right) dt$$

[2]may not be used for numerical computation as its statement depends on knowledge of the dynamic user equilibrium being sought. However, it may be employed for qualitative analyses like those which follow.

with the same constraints introduced previously. As previously done for the system optimum problem, we penaltize the flow propagation constraints to obtain the modified criterion

$$J_2 = \sum_{p\in\mathcal{P}} \int_{t_0}^{t_f} \Bigg\{ e^{-rt}\Theta_p \left[t, x\left(h^U\right), y_p^U\right] h_p\left(t\right) \\ + \sum_{p\in\mathcal{P}} \sum_{i=1}^{m(p)} \frac{\mu_{a_i}^p}{2} \left[g_{a_{i-1}}^p\left(t\right) - \bar{g}_{a_i}^p(t)(1 + D'_{a_i}(x_{a_i}(t))\dot{x}_{a_i})\right]^2 \Bigg\} dt \quad (4.29)$$

Then we have another standard form time-shifted optimal control problem, although it is subtely but importantly different than that for DSO. In particular, the Hamiltonian now becomes

$$H_2\left(t, x, h, g, \lambda; \mu\right) \equiv \sum_{p\in\mathcal{P}} e^{-rt}\Theta_p \left[t, x\left(h^U\right), y_p^U\right] h_p\left(t\right) \\ + \sum_{p\in\mathcal{P}} \sum_{i=1}^{m(p)} \frac{\mu_{a_i}^p}{2} \left\{g_{a_{i-1}}^p\left(t\right) - \bar{g}_{a_i}^p(t)(1 + D'_{a_i}(x_{a_i}(t))\dot{x}_{a_i})\right\}^2 \\ + \sum_{p\in\mathcal{P}} \sum_{i=1}^{m(p)} \lambda_{a_i}^p \left(g_{a_{i-1}}^p\left(t\right) - g_{a_i}^p\left(t\right)\right)$$

An analysis of necessary conditions similar to that for DSO is now possible. The key difference is that the counterpart of (4.24) must in the user equilibrium case be written as follows:

$$G_{a_0}^p\left(t, x, h, g, \lambda; \mu\right) = e^{-rt}\Theta_p \left[t, x\left(h^U\right), y_p^U\right] \\ + \mu_{a_1}^p \left[g_{a_0}^p\left(t\right) - \bar{g}_{a_1}^p(t)(1 + D'_{a_1}(x_{a_1}(t))\dot{x}_{a_1})\right] + \lambda_{a_1}^p \quad \forall p \in \mathcal{P} \quad (4.30)$$

$$G_{a_i}^p\left(t, x, h, g, \lambda; \mu\right) = F_{a_i}^p\left(t, x, h, g, \lambda; \mu\right) \qquad \forall p \in \mathcal{P}, \quad i \in [1, m(p)] \quad (4.31)$$

Then a necessary condition for $\left(h^S, g^S\right) \in \Lambda$ to be a dynamic user equilibrium (DUE) is

$$0 \leq \sum_{p\in\mathcal{P}} \sum_{i=0}^{m(p)} G_{a_i}^p\left(t, x^U, h^U, g^U, \lambda^U; \mu\right) \left(g_{a_i}^p - g_{a_i}^{pU}\right) \qquad g \in \Lambda \qquad (4.32)$$

for each time instant $t \in \left[t_{0,} \sup_{a_i \in A} \{t_f + D_{a_i}(x(t_f))\}\right]$, together with the state dynamics (4.16) and the following adjoint equations and boundary conditions:

$$-\frac{d\lambda_{a_i}^{p,U}}{dt} = \frac{\partial H_2^U}{\partial x_{a_i}^p} = e^{-rt} \frac{\partial \Theta_p \left[t, x\left(h^U\right), y_p^U\right]}{\partial x_{a_i}^p} \qquad \forall p \in \mathcal{P}, \quad i \in [1, m(p)]$$

$$\lambda_{a_i}^{p,U}(t_f) = 0 \qquad \forall p \in \mathcal{P}, \quad i \in [1, m(p)]$$

where the superscript U denotes a trajectory corresponding to a dynamic user equilibrium in the presence of tolls.

4.2.3 Characterizing Efficient Tolls

It is the purpose of efficient tolls to make the criteria J_1 amd J_2 identical along solution trajectories for which flow propagation and other constraints are satisfied, for then the system optimal total costs are identical to the tolled user optimal total costs. Furthermore, the vectors of path flows (departure rates) obey

$$h^U(t) = h^S(t) \tag{4.33}$$

There are as well identical arc exit flows and identical arc volumes. Therefore, along solution trajectories

$$\lambda_{a_1}^{p,S} = \frac{\partial J_1}{\partial x_{a_1}^{p,S}} = \frac{\partial J_2}{\partial x_{a_1}^{p,U}} = \lambda_{a_1}^{p,U} \tag{4.34}$$

With (4.34) in mind and upon comparing (4.27) and (4.32), we find

$$\begin{aligned} e^{-rt}\left\{\Psi_p\left(t, x^S\right) + \frac{\partial \Psi_p\left(t, x^S\right)}{\partial h_p} h_p^S\right\} &= e^{-rt}\left\{\Theta_p\left[t, x\left(h^U\right), y_p^U\right]\right\} \\ &= e^{-rt}\left\{\Psi_p\left(t, x^U\right) + y_p^U(t)\right\} \end{aligned}$$

which may be immediately re-stated as the following decision rule:

$$y_p^U(t) = \frac{\partial \Psi_p\left(t, x^S\right)}{\partial h_p} h_p^S \qquad \forall t \in [t_0, t_f] \tag{4.35}$$

This result is completely analogous to that for an efficiently tolled static user equilibrium.

4.3 The Dynamic Optimal Toll Problem with Equilibrium Constraints (DOTPEC)

We now introduce the dynamic optimal toll problem with equilibrium constraints (DOTPEC). The DOTPEC is a type of dynamic network design problem for which a central authority seeks to minimize congestion in a transport network, whose flows obey a dynamic network user equilibrium, by dynamically adjusting tolls. In particular the central authority seeks to solve the optimal control problem

$$\min \ J = \int_{t_0}^{t_f} \sum_{p \in \mathcal{P}} \Psi_p \left(t, x \right) h_p \left(t \right) dt \tag{4.36}$$

subject to

$$\sum_{p \in \mathcal{P}} \int_{t_0}^{t_f} \Theta_p \left[t, x \left(h, g \right), y_p \right] \left(w_p - h_p \right) dt \geq 0 \quad \forall \left(w, g \right) \in \Lambda \tag{4.37}$$

$$\frac{dx_{a_i}^p \left(t \right)}{dt} = g_{a_{i-1}}^p \left(t \right) - g_{a_i}^p \left(t \right) \qquad \forall p \in \mathcal{P}, i \in \left\{ 1, 2, ..., m \left(p \right) \right\} \tag{4.38}$$

$$x_{a_i}^p \left(t \right) = x_{a_{i,0}}^p \qquad \forall p \in \mathcal{P}, \quad i \in \left\{ 1, 2, ..., m \left(p \right) \right\} \tag{4.39}$$

$$h_p \left(t \right) = g_{a_1}^p (t + D_{a_1}(x_{a_1}(t)))(1 + D'_{a_1}(x_{a_1}(t))\dot{x}_{a_1}) \quad \forall p \in \mathcal{P} \tag{4.40}$$

$$g_{a_{i-1}}^p \left(t \right) = g_{a_i}^p (t + D_{a_i}(x_{a_i}(t)))(1 + D'_{a_i}(x_{a_i}(t))\dot{x}_{a_i}) \quad \forall p \in \mathcal{P}, i \in [2, m(p)] \tag{4.41}$$

$$\sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p \left(t \right) dt = Q_{ij} \quad \forall \left(i, j \right) \in \mathcal{W} \tag{4.42}$$

$$x_{a_i}^p \geq 0 \qquad g_{a_i}^p \geq 0 \qquad h_p \geq 0 \qquad \forall p \in \mathcal{P}, i \in \left\{ 1, 2, ..., m \left(p \right) \right\} \tag{4.43}$$

where Λ is the set of feasible controls (exit flows) defined previously. In the DUE constraints (4.37), we have introduced the notion of an *effective delay operator in* *the presence of tolls*, by which is meant

$$\Theta_p(t, x, y_p) = D_p(t, x) + F\{t + D_p(x, t) - T_A\} + y_p(t) \qquad \forall p \in \mathcal{P}$$

where y_p denotes the toll for path p. Of course we have the relationship

$$\Theta_p(t, x, y_p) = \Psi_p(t, x) + y_p(t) \tag{4.44}$$

where we recall from Friesz et al. (2001) that

$$y_p(t) = \sum_{i=1}^{m(p)} \delta_{a_i p} y_{a_i}\left(t + \Phi_{a_{i-1}}(t, x)\right) \quad \forall p \in \mathcal{P}$$

The variational-inequality constrained optimization problem (4.36) through (4.43) is a bi-level problem that is intrinsically difficult to solve. Note in particular that, even for a single instant of time, the number of constraints of the type (4.37) is uncountable.

In this chapter, to numerically solve specific instances of (4.36)-(4.43), we may exploit the following alternative to expressing the underlying DUE problem as an infinite dimensional variation inequality:

Theorem 20 *Given that the effective travel delay for path p is $\Theta_p[t, x(t), y_p(t)]$, a nonnegative path flow vector $h \geq 0$ is a user equilibrium if and only if the conditions*

$$\Theta_p \geq \frac{\sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} \Theta_p[t, x(t), y_p(t)]\, h_p(t)\, dt}{\sum_{p \in P_{ij}} \int_{t_0}^{t_f} h_p(t)\, dt} = \mu_{ij} \qquad \forall p \in \mathcal{P}_{ij}, \quad (i, j) \in \mathcal{W} \tag{4.45}$$

are satisfied

Proof: The analysis of necessary conditions for dynamic user equilibrium provides a definition of such equilibria for so-called open paths $p \in \mathcal{P}_{ij}$ for all $(i, j) \in \mathcal{W}$ that takes the following form in the presence of tolls:

$$h_p > 0 \Longrightarrow \Theta_p(t, x, y_p) = \mu_{ij}$$
$$\mu_{ij} > \Theta_p(t, x, y_p) \Longrightarrow h_p = 0$$

These conditions are equivalent to the following complementarity subproblem:

$$[\Theta_p(t,x,y_p)-\mu_{ij}]\,h_p(t)=0 \qquad \forall\,(i,j)\in\mathcal{W},\,p\in\mathcal{P}_{ij},t\in[t_0,t_f] \tag{4.46}$$

$$\Theta_p(t,x,y_p)-\mu_{ij}\geq 0 \qquad \forall\,(i,j)\in\mathcal{W},\,p\in\mathcal{P}_{ij},t\in[t_0,t_f] \tag{4.47}$$

$$h_p(t)\geq 0 \qquad \forall\,(i,j)\in\mathcal{W},\,p\in\mathcal{P}_{ij},t\in[t_0,t_f] \tag{4.48}$$

To show necessity we integrate the complementarity condition in (4.46) over the time horizon and sum over all paths to obtain

$$\sum_{p\in\mathcal{P}_{ij}}\int_{t_0}^{t_f}[\Theta_p(t,x,y_p)-\mu_{ij}]\,h_p dt=0 \qquad \forall\,(i,j)\in\mathcal{W}$$

or

$$\sum_{p\in\mathcal{P}_{ij}}\int_{t_0}^{t_f}\Theta_p(t,x,y_p)\,h_p dt=\mu_{ij}\sum_{p\in\mathcal{P}_{ij}}\int_{t_0}^{t_f}h_p dt \qquad \forall\,(i,j)\in\mathcal{W} \tag{4.49}$$

which is a particular case of (4.45). To show sufficiency begin with

$$\mu_{ij}=\frac{\sum_{q\in\mathcal{P}_{ij}}\int_{t_0}^{t_f}\Theta_q(t,x,y_q)\,h_q dt}{\sum_{q\in\mathcal{P}_{ij}}\int_{t_0}^{t_f}h_q dt}\leq\Theta_p(t,x,y_p) \qquad \forall\,(i,j)\in\mathcal{W},\,p\in\mathcal{P}_{ij} \tag{4.50}$$

We note that

$$[\Theta_p(t,x,y_p)-\mu_{ij}]\,h_p\geq 0 \qquad \forall\,(i,j)\in\mathcal{W},\,p\in\mathcal{P}_{ij}$$

due to cost positivity, demand positivity, flow conservation and flow non-negativity.

Substitution based on (4.50) gives

$$\begin{aligned}
&\sum_{p\in\mathcal{P}_{ij}} \int_{t_0}^{t_f} \left[\Theta_p\left(t,x,y_p\right)-\mu_{ij}\right] h_p dt \\
&= \sum_{p\in\mathcal{P}_{ij}} \int_{t_0}^{t_f} \left[\Theta_p\left(\xi,x,y_p\right) - \frac{\sum_{q\in\mathcal{P}_{ij}} \int_{t_0}^{t_f} \Theta_q\left(t,x,y_q\right) h_q dt}{\sum_{q\in\mathcal{P}_{ij}} \int_{t_0}^{t_f} h_q dt}\right] h_p d\xi \\
&= \sum_{p\in\mathcal{P}_{ij}} \int_{t_0}^{t_f} \Theta_p\left(\xi,x,y_p\right) h_p d\xi - \frac{\sum_{q\in\mathcal{P}_{ij}} \int_{t_0}^{t_f} \Theta_q\left(t,x,y_q\right) h_q dt}{\sum_{q\in\mathcal{P}_{ij}} \int_{t_0}^{t_f} h_q dt} \sum_{p\in\mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p d\xi \\
&= \sum_{p\in\mathcal{P}_{ij}} \int_{t_0}^{t_f} \Theta_p\left(\xi,x,y_p\right) h_p d\xi - \sum_{q\in\mathcal{P}_{ij}} \int_{t_0}^{t_f} \Theta_q\left(t,x,y_q\right) h_q dt = 0
\end{aligned}$$

from which we conclude that (4.46) holds. This completes the proof. ■

By virtue of Theorem 20, we may replace the DUE constraint (4.37) by the equality and inequality constraints (4.45) to obtain the following equivalent form of the DOTPEC:

$$\min\ J = \int_{t_0}^{t_f} \sum_{p\in\mathcal{P}} \Psi_p\left(t,x\right) h_p\left(t\right) dt \tag{4.51}$$

subject to

$$\mu_{ij} = \frac{\sum_{p\in\mathcal{P}_{ij}} \int_{t_0}^{t_f} \Theta_p\left[t,x\left(t\right),y_p\left(t\right)\right] h_p\left(t\right) dt}{\sum_{p\in P_{ij}} \int_{t_0}^{t_f} h_p\left(t\right) dt} \qquad \forall\left(i,j\right)\in\mathcal{W} \tag{4.52}$$

$$\Theta_p \geq \mu_{ij} \qquad \forall p\in\mathcal{P}_{ij}, \quad (i,j)\in\mathcal{W} \tag{4.53}$$

$$\frac{dx_{a_i}^p\left(t\right)}{dt} = g_{a_{i-1}}^p\left(t\right) - g_{a_i}^p\left(t\right) \qquad \forall p\in\mathcal{P}, i\in\{1,2,...,m\left(p\right)\} \tag{4.54}$$

$$x_{a_i}^p\left(t\right) = x_{a_{i,0}}^p \qquad \forall p\in\mathcal{P}, \quad i\in\{1,2,...,m\left(p\right)\} \tag{4.55}$$

$$h_p\left(t\right) = g_{a_1}^p(t+D_{a_1}(x_{a_1}(t)))(1+D_{a_1}'(x_{a_1}(t))\dot{x}_{a_1}) \quad \forall p\in\mathcal{P} \tag{4.56}$$

$$g_{a_{i-1}}^p\left(t\right) = g_{a_i}^p(t+D_{a_i}(x_{a_i}(t)))(1+D_{a_i}'(x_{a_i}(t))\dot{x}_{a_i}) \quad \forall p\in\mathcal{P}, i\in[2,m(p)] \tag{4.57}$$

$$\sum_{p\in\mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p\left(t\right) dt = Q_{ij} \quad \forall\left(i,j\right)\in\mathcal{W} \tag{4.58}$$

$$x_{a_i}^p \geq 0 \qquad g_{a_i}^p \geq 0 \qquad h_p \geq 0 \qquad \forall p\in\mathcal{P}, i\in\{1,2,...,m\left(p\right)\} \tag{4.59}$$

Note that the above formulation is an infinite dimensional mathematical program with inequality and equality constraints in standard form, and that the number of constraints for any given instant of time is countable.

4.4 Multiple Time Scales

We have investigated the within-day behavior of road network users so far. In this section we describe a day-to-day adjust process that sets daily travel demand. Our perspective is very simple: if today, commuters experience a level of congestion above a threshold representing the budget or tolerance for congestion of the typical commuter, travel demand will be less tomorrow and more workers will elect to stay at home (telecommute). To operationalize this idea, we take the perspective of evolutionary game theory to describe the day-to-day demand learning process in terms of the *moving average* of congestion and difference equations.

Let $\tau \in \Upsilon \equiv \{1, 2, ..., L\}$ be one typical discrete day within the planning horizon, and take the length of each day to be Δ, while the continuous clock time t within each day is presented by $t \in [(\tau - 1)\,\Delta, \tau\Delta]$ for all $\tau \in \{1, 2, ..., L\}$. The entire planning horizon spans L consecutive days. As noted above, we assume the travel demand for each day changes based on the moving average of congestion experienced over previous days. In fact we postulate that the travel demands Q_{ij}^{τ} for day τ between a given OD pair $(i, j) \in \mathcal{W}$ are determined by the following system of difference equations:

$$Q_{ij}^{\tau+1} = \left[Q_{ij}^{\tau} - s_{ij}^{\tau} \left\{ \frac{\displaystyle\sum_{p \in \mathcal{P}_{ij}} \sum_{j=0}^{\tau-1} \int_{j \cdot \Delta}^{(j+1)\cdot\Delta} \Psi_p \left[t, x\left(h^*, g^*\right)\right] dt}{|\mathcal{P}_{ij}| \cdot \tau \cdot \Delta} - \chi_{ij} \right\} \right]^{+}$$

$$\forall \tau \in \{1, 2, ...L - 1\} \quad (4.60)$$

with boundary condition

$$Q_{ij}^{1} = \tilde{Q}_{ij} \quad (4.61)$$

where $\tilde{Q}_{ij} \in \mathbb{R}_{+}$ is the fixed travel demand for the OD pair $(i, j) \in \mathcal{W}$ for the first day and χ_{ij} is the representative threshold. The operator $[x]^+$ is shorthand from $\max[0,x]$. The parameter s_{ij}^{τ} is related to the rate of change of inter-day travel demand.

4.5 Algorithms for Solving the DOTPEC

In this section, we provide two different algorithms for solving the DOTPEC: (1) descent in Hilbert space without time discretization, and (2) a finite dimensional discrete time approximation solved as a nonlinear program.

4.5.1 The Implicit Fixed Point Perspective

In both approaches, state-dependent time shifts must and can be accommodated using an implicit fixed point perspective, as innovated for the dynamic user equilibrium by Friesz and Mookherjee (2006). More specifically, in such an approach, one employs control and state information from a previous iteration to approximate current time shifted functions. This perspective may be summarized as follows:

1. Articulate the current approximate states (volumes) and controls (arc exit rates) by spline or other curve fitting techniques as continuous functions of time.

2. Using the aforementioned continuous functions of time, express time shifted controls as pure functions of time, while leaving unshifted controls as decision functions to be updated within the current iteration.

3. Update the states and controls, then repeat Step 2 and Step 3 until the control controls converge to a suitable approximate solution.

4.5.2 Descent in Hilbert Space

To articulate what is meant by descent in Hilbert space, it is much easier to study an abstract problem rather than the DOTPEC because of the notational complexity of the underlying DUE problem. To that end, let us consider an abstract optimal control problem with mixed state-control constraints involving state-dependent time shifts from the point of view of infinite dimensional mathematical programming:

$$\min \; J = \int_{t_0}^{t_f} F(x, u, u_D, t)dt \tag{4.62}$$

subject to

$$\begin{aligned} x(u, u_D, t) &\in \Lambda \\ &= \left\{ x : \frac{dx}{dt} = f(x, u, u_D, t), x(0) = 0, G(x, u, u_D, t) = 0, x \geq 0 \right\} \\ &\in (\mathcal{H}^1[t_0, t_f])^n \end{aligned}$$

where

$$\begin{aligned} u &\in U \subseteq (L^2[t_0, t_f])^m \\ u_D \equiv u(t + D(x)) &: (\mathcal{H}^1[t_0, t_f])^n \times \mathbb{R}^1_+ \longrightarrow (L^2[t_0, t_f])^m \\ f &: (\mathcal{H}^1[t_0, t_f])^n \times (L^2[t_0, t_f])^{2m} \times \mathbb{R}^1_+ \longrightarrow (L^2[t_0, t_f])^m \\ F &: (\mathcal{H}^1[t_0, t_f])^n \times (L^2[t_0, t_f])^{2m} \times \mathbb{R}^1_+ \longrightarrow (L^2[t_0, t_f])^m \\ G &: (\mathcal{H}^1[t_0, t_f])^n \times (L^2[t_0, t_f])^{2m} \times \mathbb{R}^1_+ \longrightarrow (L^2[t_0, t_f])^m \end{aligned}$$

In the above, $(L^2[t_0, t_f])^m$ is the m-fold product of the space of square integrable functions $L^2[t_0, t_f]$ and $(\mathcal{H}^1[t_0, t_f])^n$ is the n-fold product of the Sobolev space $\mathcal{H}^1[t_0, t_f]$ for the real interval $[t_0, t_f] \subset \mathbb{R}^1_+$. In applying descent in Hilbert space to this problem, it is convenient to use quadratic-loss penalty functions and a logarithmic barrier function to create the unconstrained program:

$$\begin{aligned} \min \; J_1 = \int_{t_0}^{t_f} F(x, u, u_D, t)dt + \frac{1}{2} \int_{t_0}^{t_f} \sum_i \eta_i (G_i(x, u, u_D, t))^2 dt \\ + \frac{1}{2} \int_{t_0}^{t_f} \sum_i \rho_i \min(0, x_i)^2 dt \end{aligned} \tag{4.63}$$

where it is understood that x denotes the operator

$$x(u, u_D, t) \in \Lambda_1 = \left\{ x : \frac{dx}{dt} = f(x, u, u_D, t), x(0) = x_0 \right\} \in (\mathcal{H}^1[t_0, t_f])^n,$$

and η_i and ρ_i are penalty and barrier multipliers to be adjusted from iteration to iteration. The resulting problem can be solved using a continuous time steepest descent method. For the penalized criterion (4.63), the algorithm can be stated as following:

Descent Method in Hilbert Spaces for the Penaltized Problem with Time Shifts

$\boxed{\textit{Step 0}}$. *Initialization.* Pick $u^0(t) \in U$ and set $k = 1$.
$\boxed{\textit{Step 1}}$. *Finding state variables.* Solve the state dynamics

$$\begin{aligned} \frac{dx}{dt} &= f(x, u^{k-1}, u_D^{k-1}, t) \\ x(0) &= x_0 \end{aligned}$$

and call the solution $x^k(t)$, using curve fitting to create an approximation to $x^k(t)$ when necessary.
$\boxed{\textit{Step 2}}$. *Finding adjoint variables.* Solve the adjoint dynamics

$$\begin{aligned} -\frac{d\lambda}{dt} &= \left[\nabla_x H(x, u^{k-1}, u_D^{k-1}, \lambda, t)\right]_{x=x^k} \\ \lambda(t_f) &= 0 \end{aligned}$$

where the Hamiltonian is given by

$$\begin{aligned} H(x, u, u_D, \lambda, t) = F(x, u, u_D, t) &+ \frac{1}{2} \sum_i \rho_i \min(0, x_i)^2 \\ &+ \frac{1}{2} \sum_i \eta_i (G_i(x, u, u_D, t))^2 + \lambda^T f(x, u, u_D, t) \end{aligned}$$

Call the solution $\lambda^k(t)$, using curve fitting to create an approximation to $\lambda^k(t)$ when necessary..

$\boxed{\textit{Step 3}}$. *Finding the gradient.* Determine

$$\nabla_u J^k \equiv \left[\nabla_u H(x^k, u, u_D^{k-1}, \lambda^k, t)\right]_{u=u^k}$$

$\boxed{\textit{Step 4}}$. *Updating the current control.* For a suitably small step size

$$\theta_k \in \mathbb{R}^1_{++}$$

determine

$$u^k(t) = u^{k-1}(t) - \theta_k \nabla_u J^k$$

$\boxed{\textit{Step 5}}$. *Stopping Test.* For $\epsilon \in \mathbb{R}^1_{++}$, a preset tolerance, stop if

$$||u^{k+1} - u^k|| < \epsilon$$

and declare

$$u^* \approx u^{k+1}$$

Otherwise set $k = k + 1$ and go to Step1.

4.5.3 Discrete-time Approximation of DOTPEC

The optimal control problem (4.51)-(4.59) may be given the following discrete time approximation:

$$\min \ J = \sum_{k=0}^{N} \sum_{p \in \mathcal{P}} \phi\left(k\right) \Psi_p \left[t_k, x\left(t_k\right)\right] h_p\left(t_k\right) \Delta$$

subject to

$$\mu_{ij} = \frac{\sum_{p\in\mathcal{P}_{ij}}\sum_{k=0}^{N}\phi\left(k\right)\Theta_p\left[t_k, x\left(t_k\right), y_p\left(t_k\right)\right]h_p\left(t_k\right)\Delta}{\sum_{p\in P_{ij}}\sum_{k=0}^{N}\phi\left(k\right)h_p\left(t_k\right)\Delta} \qquad \forall\left(i,j\right)\in\mathcal{W}$$

$$\Theta_p\left(t_k\right) \geq \mu_{ij} \qquad \forall k\in\left[0,N\right], \quad p\in\mathcal{P}_{ij}, \quad \left(i,j\right)\in\mathcal{W}$$

$$x^p_{a_i}\left(t_{k+1}\right) = x^p_{a_i}\left(t_k\right) + \Delta\left[g^p_{a_{i-1}}\left(t_k\right) - g^p_{a_i}\left(t_k\right)\right]$$

$$\forall k\in\left[0,N-1\right], \quad p\in\mathcal{P}, \quad i\in\left[1,m(p)\right]$$

$$x^p_{a_i}\left(t_0\right) = x^p_{a_{i,0}} \qquad \forall p\in\mathcal{P}, \quad i\in\left\{1,2,...,m\left(p\right)\right\}$$

$$x\left(t_k\right)\geq 0 \qquad \forall k\in\left[0,N\right]$$

$$h_p\left(t_k\right) = g^p_{a_1}(t_k + D_{a_1}(x_{a_1}(t_k)))(1 + D'_{a_1}(x_{a_1}(t_k))\dot{x}_{a_1}) \quad \forall k\in\left[0,N\right], p\in\mathcal{P}$$

$$g^p_{a_{i-1}}\left(t_k\right) = g^p_{a_i}(t_k + D_{a_i}(x_{a_i}(t_k)))(1 + D'_{a_i}(x_{a_i}(t_k))\dot{x}_{a_i})$$

$$\forall k\in\left[0,N\right], p\in\mathcal{P}, i\in\left[2,m(p)\right]$$

$$\sum_{p\in\mathcal{P}_{ij}}\sum_{k=0}^{N}\phi\left(k\right)h_p\left(t_k\right)\Delta = Q_{ij} \quad \forall\left(i,j\right)\in\mathcal{W}$$

$$y_a\left(t_k\right)\geq 0 \qquad \forall a\in\mathcal{A}, \quad k\in\left[0,N\right]$$

$$x\left(t_k\right)\geq 0 \qquad g\left(t_k\right)\geq 0 \qquad h\left(t_k\right)\geq 0 \qquad \forall k\in\left[0,N\right]$$

where k takes non-negative integer values, Δ is the discrete time step that divides the time interval $[t_0, t_f]$ into N equal segments, $\phi\left(k\right)$ is the coefficient which arises from a trapezoidal approximation of integrals, that is

$$\phi\left(k\right) = \begin{cases} 0.5 & \text{if } k=0 \text{ and } N \\ 1 & \text{otherwise} \end{cases}$$

and

$$t_k = k\Delta$$

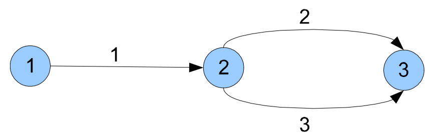

Figure 4.1. 3-Arc 3-Node Traffic Network

One advantage of time discretization is that we can now completely eliminate state variables (arc volumes) from the problem by noting that

$$x_{a_i}^p (t_{k+1}) = x_{a_{i,0}}^p + \sum_{r=0}^{k} \Delta \left[g_{a_{i-1}}^p (t_r) - g_{a_i}^p (t_r) \right]$$

$$\forall k \in [0, N-1], \quad p \in \mathcal{P}, \quad i \in \{1, 2, ..., m(p)\}$$

As a consequence, one obtains a finite dimensional mathematical program, which may be solved by conventional algorithms developed for such problems. We employ GAMS/MINOS for the numerical example of Section 4.6.1.

4.6 Numerical Example

In what follows, we consider a 3-arc, 3-node network shown in Figure 4.1. The arc labels and arc delay functions for this network are summarized in the following table:

Arc name	From node	To node	Arc delay, $D_a(x_a(t))$
a_1	1	2	$2 + (x_{a_1}/200)$
a_2	2	3	$1 + (x_{a_2}/150)$
a_3	2	3	$3 + (x_{a_3}/100)$

There are 2 paths connecting the single OD pair formed by nodes 1 and 3, namely:

$$\mathcal{P}_{13} = \{p_1, p_2\}, \qquad p_1 = \{a_1, a_2\}, \qquad p_2 = \{a_1, a_3\}$$

The controls (path flows and arc exit flows) and states (path-specific arc traffic volumes) associated with the network are:

Path	Path Flow	Arc Exit Flow	Traffic Volume of Arc
p_1	h_{p_1}	$g_{a_1}^{p_1}, g_{a_2}^{p_1}$	$x_{a_1}^{p_1}, x_{a_2}^{p_1}$
p_2	h_{p_2}	$g_{a_1}^{p_2}, g_{a_3}^{p_2}$	$x_{a_1}^{p_2}, x_{a_3}^{p_2}$

We consider three-day toll planning in which each day is 24 hours, hence, $\Delta = 24$ and $L = 14$ (two weeks). We assume there is the initial travel demand $\tilde{Q} = 150$ units from node 1 (origin) to node 3 (destination). The threshold for travel cost is $\chi = 20000$ and the inter-day rate of change in travel demand is $s_{13} = 0.7$. The desired arrival time for each day is $t_A = 12$, and we employ the symmetric early/late arrival penalty

$$F\left[t + D_p(x,t) - t_A\right] = 5\left[t + D_p(x,t) - t_A\right]^2$$

Further, without any loss of generality, we take

$$x_{a_i}^p(0) = 0 \quad \forall i \in [1, m(p)], p \in \mathcal{P}$$

In what follows we forgo the detailed symbolic statement of this example, and, instead, provide numerical results in graphical form.

4.6.1 DOTPEC Computation Based on Time Discretization and GAMS/ MINOS

Path flows and arc exit flows for paths p_1 and p_2 are presented in Figures 4.2 and 4.3, while path flows and tolls for each arc are given in Figures 4.4, 4.5 and 4.6, for three days from the computed fourteen-day results. We see that tolls tend to be proportional to the path flows. When, for path p_1, we compare the effective path delays (including tolls) with path flows (origin departure rates) by plotting both for the same time scale, Figure 4.7 is obtained. This figure shows that departure rate peaks when the associated effective path delay achieves a local minimum, thereby demonstrating that a dynamic user equilibrium has been found. Similar

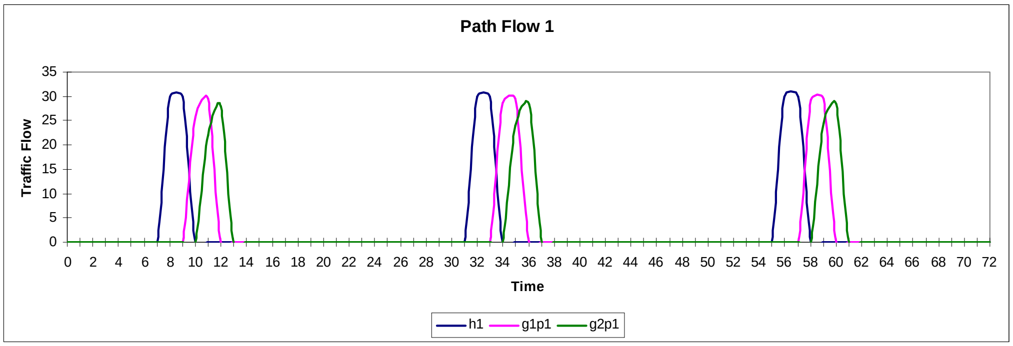

Figure 4.2. Path and arc exit flows for path p_1.

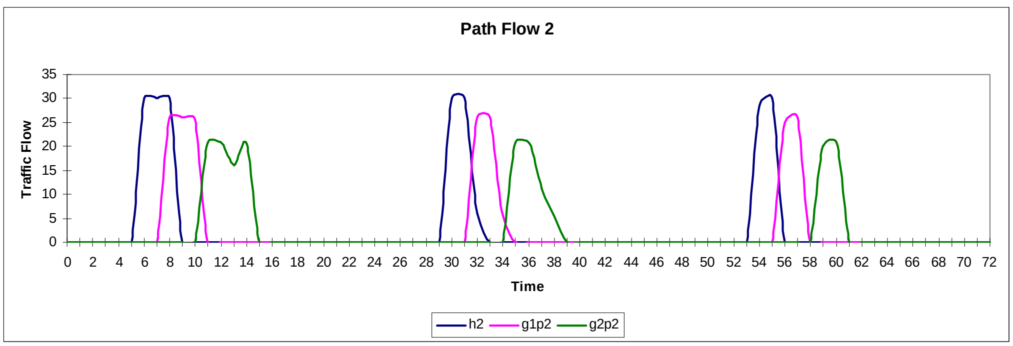

Figure 4.3. Path and arc exit flows for path p_2.

comparisons are made for paths p_2 in Figure 4.8. The daily changes of travel demand from the origin to destination according to the difference equation (4.60) are given in Figure 4.9.

4.6.2 DOTPEC Computation based on Descent in Hilbert Space

The same numerical example was also solved by descent in Hilbert space, a continuous-time numerical scheme described in Section 4.5.2. While employing the implicit fixed point approach, we penalize the flow propagation constraints, the travel demand constraint, and the DUE conditions which are converted to a set of inequality constriants. We present the path tolls in Figures 4.10 and 4.11. As in the previous section we again show the resulting flows are a dynamic user equilibrium by plotting the travel cost and departure flow on the same time axis in Figures 4.12

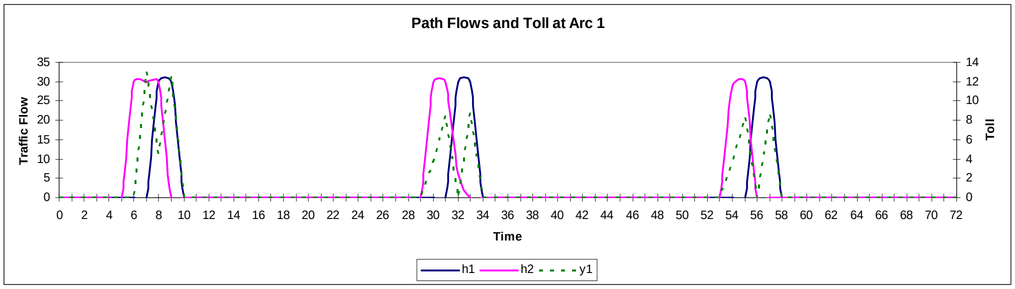

Figure 4.4. Path flows and toll at arc a_1.

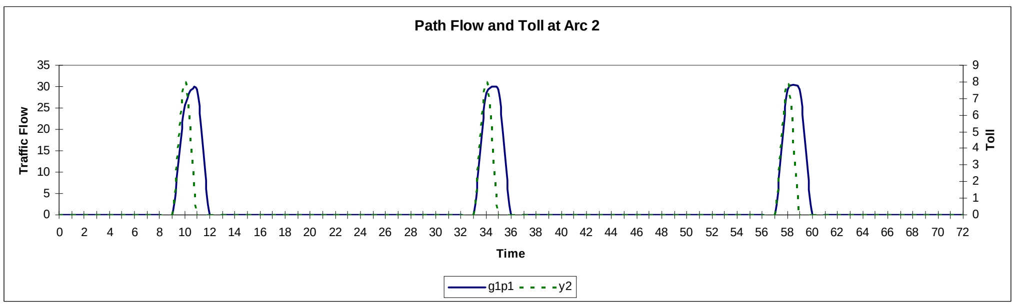

Figure 4.5. Path flow and toll at arc a_2.

and 4.13.

4.6.3 Comparison of Tolls

To compare, the tolls by DETP and DOTPEC with two algorithms of choice, we suggest a computational scheme for DETP. Recall that the decision rule for the

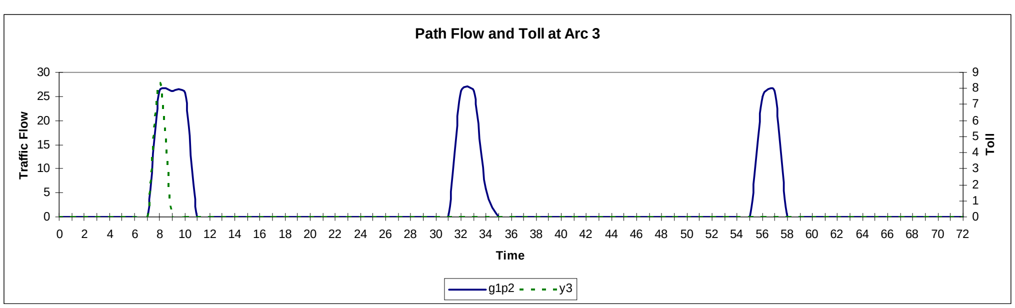

Figure 4.6. Path flow and toll at arc a_3.

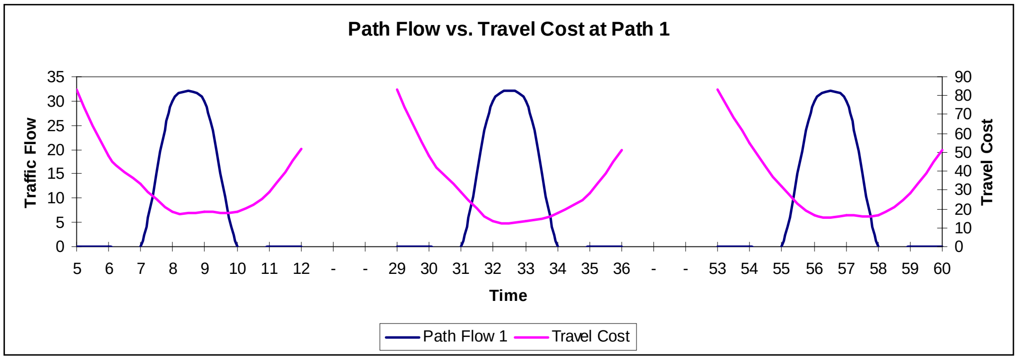

Figure 4.7. Comparison of path flow and associated unit travel costs for path p_1.

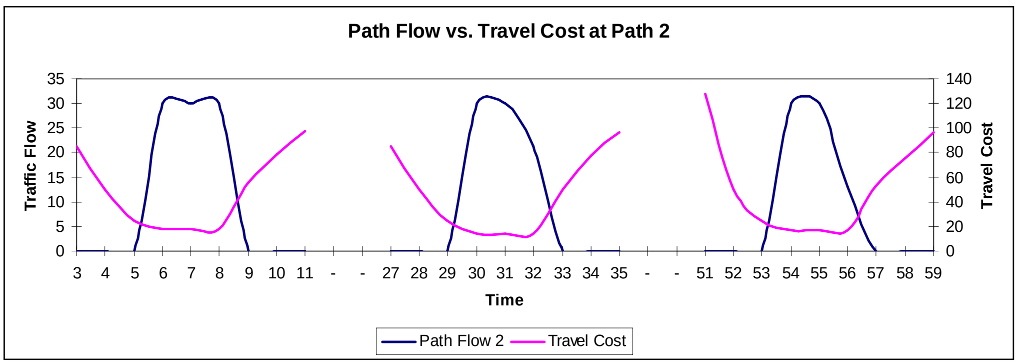

Figure 4.8. Comparison of path flow and associated unit travel costs for path p_2.

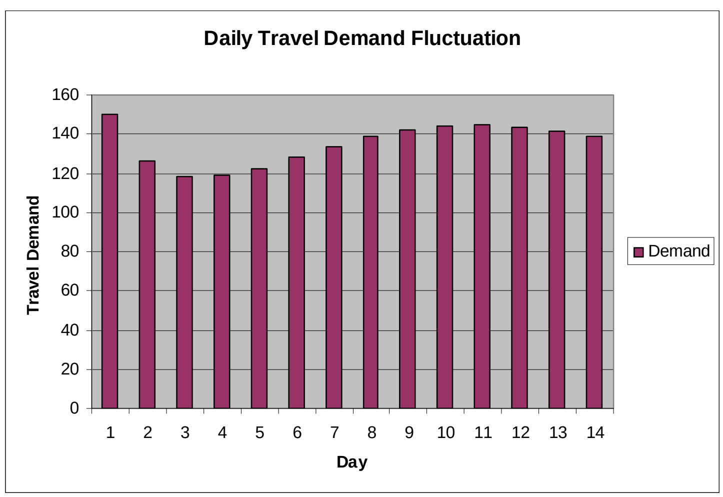

Figure 4.9. Daily changes of travel demand from the origin (node 1) to the destination (node 3)

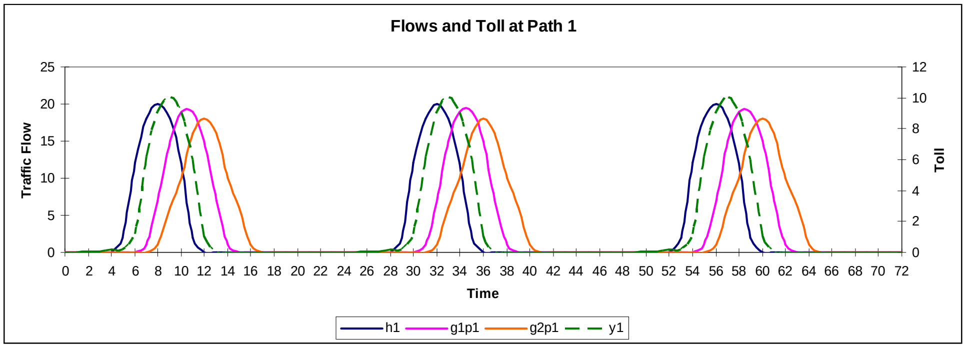

Figure 4.10. Path flows and toll at path p_1.

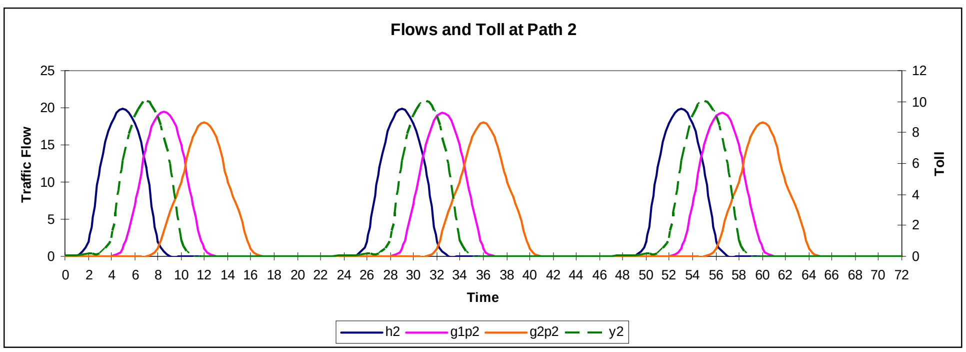

Figure 4.11. Path flows and toll at path p_2.

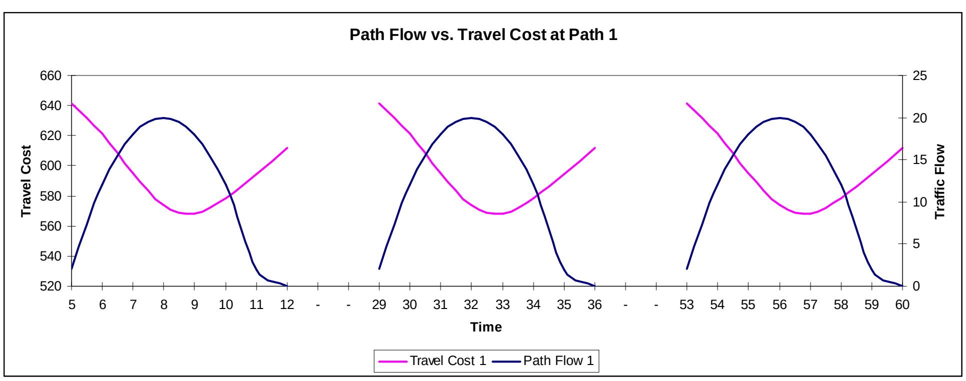

Figure 4.12. Comparison of path flow and associated unit travel costs for path p_1.

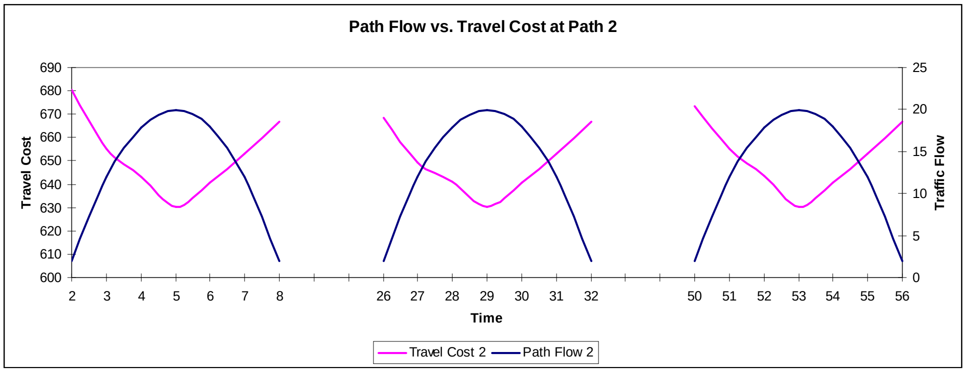

Figure 4.13. Comparison of path flow and associated unit travel costs for path p_2.

dynamic efficient toll is:

$$y_p^U(t) = \frac{\partial \Psi_p\left(t, x^S\right)}{\partial h_p} h_p^S \qquad \forall t \in [t_0, t_f]$$

Note that the partial derivative of $\Psi_p\left(t, x^S\right)$ with respect to the path flow h_p is not zero, since the state variable x is an implicit function of the control h_p as the relationship is expressed in the state dynamics. Further we cannot calculate the derivative directly due to the nested delay operator appears in $\Psi_p(\cdot, \cdot)$. However, from the numerical study of the dynamic system optimum traffic assignment, it is known that the controls are zero or singular. When the departure rate is nonzero, it as well as the states obtained from it are smooth and the delay operator is differentiable, although the derivative $\frac{\partial \Psi_p\left(t, x^S\right)}{\partial h_p}$ does not exist at the time moments where there are kinks in the controls. The derivative is numerically approximated as:

$$\frac{\partial \Psi_p\left[t, x\left(h^*, g^*\right)\right]}{\partial h_p} \cong \frac{\Psi_p\left[t, x\left(h+\delta, g\right)\right] - \Psi_p\left[t, x\left(h, g\right)\right]}{\delta}$$

A numerical comparison of the tolls found from the DETP with those from the DOTPEC is given in Figures 4.14 and 4.15. We see that the efficient toll has a more spike-like behavior than that for the DOTPEC. It is also interesting to note that the total congestion cost for the DETP is $(26.43, 38.85)$ while the total congestion cost for the DOTPEC is $(38.30, 46.85)$ by discrete approximation and $(43.09, 45.13)$ by descent in Hilbert spaces for paths (p_1, p_2).

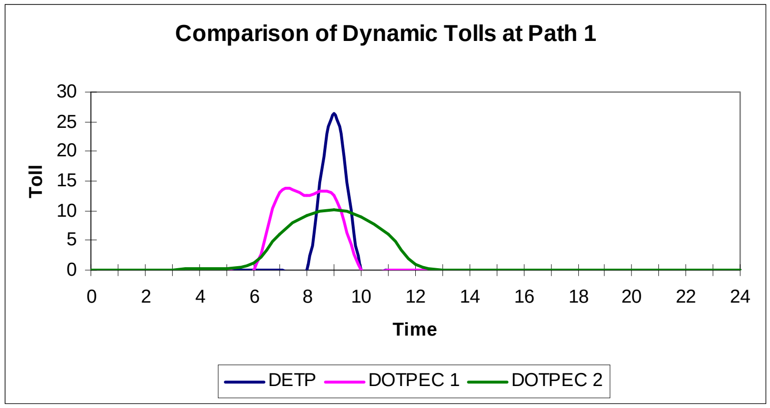

Figure 4.14. Comparison of Dynamic Tolls by DETP, DOTPEC solved by discrete time approximation (DOTPEC 1), and DOTPEC solved by descent in Hilbert spaces (DOTPEC 2) for path p_1

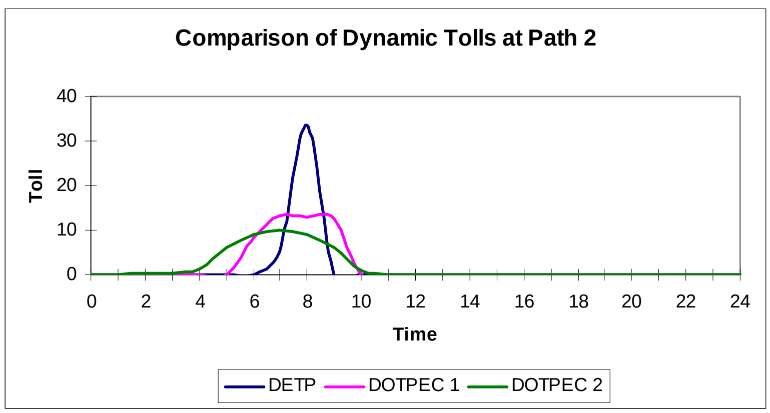

Figure 4.15. Comparison of Dynamic Tolls by DETP, DOTPEC solved by discrete time approximation (DOTPEC 1), and DOTPEC solved by descent in Hilbert spaces (DOTPEC 2) for path p_2

4.7 Concluding Remarks

We have presented a mathematical formulation of the DOTPEC and have shown how it may be directly solved using the notion of descent in Hilbert space for a small illustrative problem. We have also computed solutions using the more familiar approach of time discretization combined with off-the-shelf nonlinear programming software. Clearly, in-depth testing and comparison of these solution methods is required before one can be recommended over the other.

We have not explored in this manuscript the difficult theoretical questions of algorithm convergence, existence of solutions to the dynamic efficient toll and the DOTPEC problems, the Braess paradox and the price of anarchy. These topics are being addressed in a separate manuscript still in preparation. Given that serious efforts are already under way to implement versions of the optimal dynamic toll problem in the U.S. and elsewhere, our initial focus on computation seems fully justified.

We close by commenting that analytical DUE models – in our opinion – are far and away the best starting point for studies of the theoretical aspects of dynamic efficient tolls and dynamic congestion pricing. In particular, we have shown in this chapter that an intuitive generalization to a dynamic setting of the efficient static toll rule is correct – something that could not be established in such a definitive way with a simulation model.

Chapter 5

Non-cooperative Competition Among Revenue Maximizing Service Providers with Demand Learning

The service providers of interest in this chapter are in oligopolistic game theoretic competition according to a learning process that is similar to evolutionary game-theoretic dynamics and for which price changes are proportional to their signed excursion from a market clearing price. We stress that in this model firms are setting prices for their services while simultaneously determining the levels of demand they will serve. This is unusual in that, typically, firms in oligopolistic competition are modeled as setting either prices or output flows. The joint adjustment of prices and outputs is modeled here by comparing the current price to the price that would have cleared the market for the demand that has most recently been served. However, the service providers are unable to make this comparison until the current round of play is completed as knowledge of the total demand served by all competitors is required.

Kachani et al. (2004) put forward a revenue management model for service providers to address such joint pricing and demand learning in an oligopolistic setting with fixed capacity constraints. The model they consider assumes the demand faced by a service provider is a linear function of its price and other competitors' prices; each company learns to set their parameters over time, although the impact of a change in price on demand in one period does not automatically propagate to latter time periods. In our work we allow this impact to propagate to all subsequent time periods.

In this chapter, we will only be considering a single class of customers, so-called bargain-hunting buyers searching for personal or, to a limited extent, business services or products at the most competitive prices; these buyers are willing to sacrifice some convenience for the sake of a lower price. Because the services and products are assumed to be homogeneous, if two sellers offer the same price, the tie is broken randomly. In other words, the consumer has no concept of brand preference.

Forecasting demand is crucial in pricing and planning for any firm in that the forecasts have huge impacts on the revenues. In revenue management, demand is usually modeled as an exogenous stochastic process with a known distribution (Gallego and van Ryzin 1994; Feng and Gallego 1995). Such models are restrictive because (1) they depend largely on complete knowledge of demand before pricing starts; (2) they do not incorporate any learning mechanisms which will improve the demand estimation as more information becomes available.

In this chapter, demands for each firm's services are governed by dynamics controlled by prices. However, the parameters in the demand dynamics are unknown. Demand forecasting is often flexible in the sense that it is able to handle incomplete information about the demand function. Furthermore, demand is learned over time and each firm can update demand functions as new information becomes available. By using a learning mechanism, the firm can better estimate demand functions, thereby improving its profitability.

Demand learning has been studied extensively in many research areas. A typical approach to model learning is Bayesian learning. In Bayesian learning, demand at any time is a stochastic process following a known distribution with unknown parameters. Observed demand data are used to modify beliefs about unknown parameters based on Bayes' rule. In this approach, uncertainty in the parameter is resolved as more observations become available, and the distribution of any demand will approach its true distribution (Murray Jr and Silver 1966; Eppen and Iyer 1997; Bitran and Wadhwa 1996). Recently, researchers have developed other learning mechanisms to resolve demand uncertainty. Bertsimas and Perakis (2006) develop a demand learning mechanism which estimates the parameters for the linear demand function via a least square method. Yelland and Lee (2003) use class II mixture models to capture both the uncertainty in model specification and demand. They demonstrate that class II mixture models are more efficient in forecasting demand. Lin (2006) proposes to use real time demand data to improve the estimation of customer arrival rates which in turn can be used to better predict future demand distributions and develops a variable-rate policy which is immune to changes in the customer arrival rate.

The demand learning approach proposed in this chapter is based on the Kalman Filter. In the economics and revenue management literature, Balvers and Cosimano (1990) study a stochastic linear demand model in a dynamic pricing problem and obtain a dynamic programming formulation to maximize revenue. They use the Kalman filter to estimate the exact value of the intercept and elasticity in the stochastic linear demand model. Closely related, Carvalho and Puterman (2008) consider a log-linear demand model whose parameters are also stochastic, and test the model using Monte Carlo simulation. In addition, Xie et al. (1997) find an application of the Kalman filter in estimation of new product diffusion models. They concluded the Kalman filter approach gives superior predictions compared to other estimation methods in such an environment.

The remainder of the chapter is organized as follows. We give a detailed exposition of our revenue management model of dynamic competition in Section 5.1. Kalman filtering is described for the revenue management model in Section 5.2. We provide a scenario for a firm and a numerical method to solve its optimal control problem without considering the game in Section 5.3. Section 5.4 shows how dynamic revenue management competition may be expressed as a DVI. Section 5.5 provides a detailed numerical example considering competition. Section 5.6 summarizes our findings and describes future research.

5.1 Revenue Management Model

A set of service providers are competing in an oligopolistic setting, each with the objective of maximizing their revenue. These service providers have very high fixed costs compared to their relatively low variable or operating costs. Therefore, the providers focus only on maximizing their own revenue.

Each company provides a set of services for which each service type is assumed to be homogeneous. For example, the difference between an economy class seat on Southwest and an economy class seat on Jet Blue is indiscernible by customers; the only differences that the customers perceive are the prices charged by the different service providers.

All service providers can set the price for each of their services. The price that they charge for each service in one time period will affect the demand that they receive for that service in the next time period. The price that the service provider charges is compared to the rolling average price of their competitors. The amount of service that each company can provide has an upper bound. The providers must therefore choose prices that create an amount of demand for their services that will maximize their revenue while ensuring that the demands do not exceed their capacities.

5.1.1 Basic Notation

We denote the set of revenue managing firms as $\mathcal{F}$, each of whom is providing a set of services $\mathcal{S}$. Continuous time is denoted by the scalar $t \in \mathbb{R}^1_+$, while t_0 is the finite initial time and $t_f \in \mathbb{R}^1_{++}$ the finite terminal time so that $t \in [t_0, t_f] \subset \mathbb{R}^1_+$. Each firm $f \in \mathcal{F}$ controls prices

$$\pi_i^f \in L^2 [t_0, t_f]$$

corresponding to each service type $i \in \mathcal{S}$. The control vector of each firm $f \in \mathcal{F}$ is

$$\pi^f \in \left(L^2 [t_0, t_f]\right)^{|\mathcal{S}|}$$

which is the concatenation of

$$\pi \in \left(L^2 [t_0, t_f]\right)^{|\mathcal{S}| \times |\mathcal{F}|},$$

the complete vector of controls. We also let

$$D_i^f (\pi, t) : \left(L^2 [t_0, t_f]\right)^{|\mathcal{S}| \times |\mathcal{F}|} \times \mathbb{R}^1_+ \longrightarrow \mathcal{H}^1 [t_0, t_f]$$

denote the demand for service $i \in \mathcal{S}$ of firm $f \in \mathcal{F}$ and define the vector of all such demands for firm f to be

$$D^f \in \left(\mathcal{H}^1\left[t_0, t_f\right]\right)^{|\mathcal{S}|}$$

All such demands for service $i \in \mathcal{S}$ of firm $f \in \mathcal{F}$ are denoted by

$$D \in \left(\mathcal{H}^1\left[t_0, t_f\right]\right)^{|\mathcal{S}| \times |\mathcal{F}|}$$

We will use the notation

$$D^{-f} = \left(D_i^g : i \in \mathcal{S}, g \in \mathcal{F} - \{f\}\right)$$

for the vector of all service levels provided by the competitors of firm $f \in \mathcal{F}$.

5.1.2 Demand Dynamics

In evolutionary game theory the notion of comparing a moving average to the current state is used to develop ordinary differential equations describing learning processes; see Fudenberg and Levine (1999). To proceed, we first assume the qualities of services provided by different agents are homogeneous, hence the customers' decisions depend only on their prices.

The demand for the service offerings of firm $f \in \mathcal{F}$ evolve according to the following evolutionary game-theoretic dynamics:

$$\frac{dD_i^f}{dt} = \eta_i^f \cdot \left(\tilde{\pi}_i - \pi_i^f\right) \qquad \forall i \in \mathcal{S}, \;\; f \in \mathcal{F} \tag{5.1}$$

$$D_i^f\left(t_0\right) = K_{i,0}^f \qquad \forall i \in \mathcal{S}, \;\; f \in \mathcal{F} \tag{5.2}$$

where $\tilde{\pi}_i$ is the moving average price for service $i \in \mathcal{S}$ given by

$$\tilde{\pi}_i\left(t\right) = \frac{1}{|\mathcal{F}|\left(t - t_0\right)} \int_{t_0}^{t} \sum_{g \in \mathcal{F}} \pi_i^g(\tau) d\tau \quad \forall i \in \mathcal{S}$$

while $K_{i,0}^f \in \mathbb{R}_{++}^1$ and $\eta_i^f \in \mathbb{R}_{++}^1$ are exogenous parameters for each $i \in \mathcal{S}$ and $f \in \mathcal{F}$. The firms set the parameter η_i^f by analyzing the past demand data and the sensitivity of the demand with respect to price. The demand for service type

i of a firm f changes over time in accordance with the excess between the firm's price and the moving average of all agents' prices for the particular service. The coefficient η_i^f controls how quickly demand reacts to price changes for each firm f and service type i. Some providers may specialize in certain services and may be able to adjust more quickly than their competitors.

In the literature, it is commonly assumed that observed demands in periods t and $t+1$ are independent and demand at time t is influenced by price at time t which is set at time $t-1$ prior to observing the demand for period t. This is a stronger assumption and we know in reality there are some learning that takes place in terms of consumer's expectations and price-anticipation. In our model we capture this "learning" behavior in a naive way: customers have some "reference price" of a differentiated commodity which gets updates at the end of every period as they learn about the market condition and demand for service/goods is proportional to the price differential (tatonnement dynamics). This way demand at time t not only depends on the price set at time t, but on the complete price trajectories $[0, t]$. In this sense, the dynamic equation (5.1) based on "moving average price" of demand learning is more realistic. The 'replicator dynamics' (see Hofbauer and Sigmund 1998) , of which equation (5.1) is an instance, reflects a widely respect theoretical view or hypothesis that learning is the notion of comparison to the moving average.

These dynamics represent a learning mechanism for the firms. As stated here, the dynamics are reminiscent of replicator dynamics which are used in evolutionary games. The rate of growth of demand, can be viewed as the rate of growth of the firm f with respect to service type i. This growth follows the "basic tenet of Darwinism" and may be interpreted as the difference between the fitness (price) of the firm for the service and the rolling average fitness of all the agents for that service.

5.1.3 Constraints

There are positive upper and lower bounds, based on market regulations or knowledge of customer behavior, on service prices charged by firms. Thus we write

$$\pi^f_{\min,i} \leq \pi^f_i \leq \pi^f_{\max,i} \quad \forall i \in \mathcal{S}, \;\; f \in \mathcal{F}$$

where the $\pi^f_{\min,i} \in \mathbb{R}^1_{++}$ and $\pi^f_{\max,i} \in \mathbb{R}^1_{++}$ are known constants. Similarly, there will be an upper bound on the demand for services of each type by each firm as negative demand levels are meaningless; that is

$$D^f_i \geq 0 \quad \forall i \in \mathcal{S}, \quad f \in \mathcal{F}$$

Let $\mathcal{R}$ be the set of resources that the firms can utilize to provide the services, while $|\mathcal{R}|$ is the cardinality of $\mathcal{R}$. Define the incidence matrix (Talluri and van Ryzin 2004) $A = (a_{lm})$ by

$$a_{lm} = \begin{cases} 1 & \text{if resource } l \text{ is used by the service type } m \\ 0 & \text{otherwise} \end{cases}$$

Joint resource-constraints for firm f are

$$0 \leq AD^f_i \leq C^f_i \quad \forall i \in \mathcal{S}, \quad f \in \mathcal{F} \tag{5.3}$$

where C^f_i is firm f's capacity of resources used for service i.

5.1.4 The Firm's Optimal Control Problem

Since revenue management firms have negligible variable costs and high fixed costs, each firm's objective is to maximize revenue which in turn ensures the maximum profit. We further note that each firm $f \in \mathcal{F}$ faces the following problem: with the π^{-f} as exogenous inputs, solve the following optimal control problem:

$$\max_{\pi^f} J_f(\pi^f, \pi^{-f}, t) = \int_{t_0}^{t_f} e^{-\rho t} \left(\sum_{i \in \mathcal{S}} \pi^f_i \cdot D^f_i \right) dt - e^{-\rho t_0} \Psi^f_0 \tag{5.4}$$

$$\text{s.t.} \quad \frac{dD^f_i}{dt} = \eta^f_i \cdot \left(\tilde{\pi}_i - \pi^f_i \right) \quad \forall i \in \mathcal{S}, \quad f \in \mathcal{F} \tag{5.5}$$

$$D^f_i(t_0) = K^f_{i,0} \quad \forall i \in \mathcal{S} \tag{5.6}$$

$$\pi^f_{\min,i} \leq \pi^f_i \leq \pi^f_{\max,i} \quad \forall i \in \mathcal{S} \tag{5.7}$$

$$0 \leq AD^f_i \leq C^f_i \quad \forall i \in \mathcal{S} \tag{5.8}$$

where Ψ_0^f is the fixed cost of production for firm f which can be dropped from the problem later, ρ is the nominal discount rate compounded continuously, and $\int_{t_0}^{t_f} e^{-\rho t}\left(\sum_{i\in\mathcal{S}} \pi_i^f \cdot D_i^f\right) dt$ is the net present value (NPV) of revenue. From familiarity with these dynamics, we may restate them for all $f \in \mathcal{F}$ as

$$\frac{dD_i^f}{dt} = \eta_i^f \cdot \left(\frac{y_i}{|\mathcal{F}|\,(t-t_0)} - \pi_i^f\right) \quad \forall i \in \mathcal{S} \tag{5.9}$$

$$\frac{dy_i}{dt} = \sum_{g\in\mathcal{F}} \pi_i^g \quad \forall i \in \mathcal{S} \tag{5.10}$$

$$N_i^f\,(t_0) = K_{i,0}^f \quad \forall i \in \mathcal{S} \tag{5.11}$$

$$y_i\,(t_0) = 0 \quad \forall i \in \mathcal{S} \tag{5.12}$$

As a consequence we may rewrite the optimal control problem of firm $f \in \mathcal{F}$ as

$$\max_{\pi^f} J_f(\pi^f, \pi^{-f}, t) \; = \; \int_{t_0}^{t_f} e^{-\rho t}\left(\sum_{i\in\mathcal{S}} \pi_i^f \cdot D_i^f\right) dt \tag{5.13}$$

$$\text{s.t.} \qquad \frac{dD_i^f}{dt} = \eta_i^f \cdot \left(\frac{y_i}{|\mathcal{F}|\,(t-t_0)} - \pi_i^f\right) \quad \forall i \in \mathcal{S} \tag{5.14}$$

$$\frac{dy_i}{dt} = \sum_{g\in\mathcal{F}} \pi_i^g \quad \forall i \in \mathcal{S} \tag{5.15}$$

$$D_i^f\,(t_0) = K_{i,0}^f \quad \forall i \in \mathcal{S} \tag{5.16}$$

$$y_i\,(t_0) = 0 \quad \forall i \in \mathcal{S} \tag{5.17}$$

$$\pi_{\min,i}^f \le \pi_i^f \le \pi_{\max,i}^f \quad \forall i \in \mathcal{S} \tag{5.18}$$

$$0 \le AD_i^f \le C_i^f \quad \forall i \in \mathcal{S} \tag{5.19}$$

Consequently,

$$D(\pi) = \arg\left\{ \begin{aligned} \frac{dD_i^f}{dt} &= \eta_i^f \cdot \left(\frac{y_i}{|\mathcal{F}|(t-t_0)} - \pi_i^f \right) \\ \frac{dy_i}{dt} &= \sum_{g\in\mathcal{F}} \pi_i^g; \quad D_i^f(t_0) = K_{i,0}^f \\ 0 &\leq AD_i^f \leq C_i^f \quad \forall f \in \mathcal{F}, i \in \mathcal{S} \end{aligned} \right\} \tag{5.20}$$

where we implicitly assume that the dynamics have solutions for all feasible controls. In compact notation this problem can be expressed as: with the π^{-f} as exogenous inputs, compute π^f in order to solve the following optimal control problem:

$$\begin{aligned} \max \quad & J_f(\pi^f, \pi^{-f}, t) \\ \text{s.t.} \quad & \pi^f \in \Lambda_f \end{aligned} \tag{5.21}$$

for all $f \in \mathcal{F}$ where

$$\Lambda_f = \left\{ \pi^f : (5.14) - (5.19) \text{ hold} \right\}$$

5.2 Estimation of Model Parameters

So far, we have introduced a revenue management model for service providers which is deterministic. However, in reality, model parameters are usually unknown to the modeler and the firm and follow probability distributions. Let us suppose the sales season or the total planning horizon is $[T_0, T_F]$, and we want to update model parameters L times within the season. Hence, the time horizon $[T_0, T_F]$ is divided into the following L sub-intervals as

$$[T_0, T_0 + \Delta T], \ [T_0 + \Delta T, T_0 + 2\Delta T], \ \ldots, \ [T_0 + (L-1)\Delta T, T_F]$$

where $\Delta T = (T_F - T_0)/L$. At time moment at $t = T_0 + l\Delta T$, each firm updates model parameters based on observations from the interval $[T_0 + (l-1)\Delta T, T_0 + l\Delta T]$, for $l = 1, 2, ..., L-1$.

Assuming that modeling errors and observation errors follow normal distributions, we may employ Kalman filtering to estimate parameters, η_i^f. For Kalman filtering we minimize the squared estimation error. *A posterior* estimates are updated as long as the new observations are available. For most other methods, the estimation is conducted only once and ignores the rich information contained in new observations. Also, the Kalman filter is robust in the sense that it can handle substantially inaccurate observations. Other methods usually assume that the observations are essentially accurate.

For brevity, we drop the superscript f for each firm. The dynamics for the demand are then

$$\frac{dD_i(t)}{dt} = \eta_i \left(\frac{\int_{t_0}^{t} \sum_{g \in \mathcal{F}} \pi_i^g(\tau) d\tau}{t - t_0} - \pi(t) \right), \forall i \in \mathcal{I} \tag{5.22}$$

After one planning period is completed, we update the model parameter, η_i, to have a more precise pricing policy for the next planning horizon. For this purpose, we collect data during the previous planning horizon. Although we decide the pricing policy in continuous time, most data collecting activities typically occur in discrete time in practice. Let us employ the discrete time index k and assume we collect data K times within each planning period. Using the vector notation $\eta = (\eta_i : i \in \mathcal{S})$, the dynamics of η may be expressed as

$$\eta(k+1) = \eta(k) + \xi(k)$$

where $\xi(k)$ is a random noise from a normal distribution $N(0, Q)$. The matrix Q is called the process-noise covariance matrix.

The value of the parameter η cannot be observed directly but rather only by the change in realized demand may be observed, which can be defined as

$$\begin{aligned} z(k) &\equiv \Delta D(k) \\ &= \eta(k) \left(\frac{\sum_{j=0}^{k} \sum_{g \in \mathcal{F}} \pi_i^g(j)}{k \Delta k} - \pi(k) \right) \Delta k + \omega(k) \end{aligned} \tag{5.23}$$

where

$$
\begin{aligned}
\Delta D(k) &= D(k+1) - D(k) \\
D &= (D_i : i \in \mathcal{S}) \\
\Delta k &\equiv \frac{t_f - t_0}{K}
\end{aligned}
$$

and $\omega(k)$ is a random observation noise from a normal distribution $N(0, R)$. The matrix R is called the measurement noise covariance matrix. Expression (5.23) is obtained by discretizing the state dynamics (5.22) of D.

According to Section 12.6 of Bryson and Ho (1975), the Kalman filter dynamics are

$$
\begin{aligned}
\hat{\eta}(k) &= \bar{\eta}(k) + V(k)\left[z(k) - H(k)\,\bar{\eta}(k)\right] \\
\bar{\eta}(k+1) &= \hat{\eta}(k) \\
P(k) &= \left[M(k)^{-1} + H(k)^T R^{-1} H(k)\right]^{-1} \\
M(k+1) &= P(k) + Q
\end{aligned}
$$

where we defined

$$
\begin{aligned}
V(k) &\equiv P(k)\,H(k)\,R^{-1} \\
H(k) &\equiv \left(\frac{\sum_{j=0}^{k}\sum_{g\in\mathcal{F}} \pi_i^g(j)}{k\Delta k} - \pi(k)\right)\Delta k
\end{aligned}
$$

where $\bar{\eta}(k)$ is an *a priori* estimate of $\eta(k)$ before observation and $\hat{\eta}(k)$ is an *a posteriori* estimate after observation. Hence, the estimation process is:

1. From the previous planning period obtain sequences of exercised prices $\{\pi(k)\}$ and observed changes in demand $\{z(k)\}$.

2. Let the time index $k = 0$ and assume the initial values $P(0) = 1$ and $\hat{\eta}(0) = \eta_{\text{previous}}$.

3. For index k, forecast according to

$$\bar{\eta}(k+1) = \hat{\eta}(k)$$
$$M(k+1) = P(k) + Q$$

4. Based on the observation z_{k+1}, update according to

$$\hat{\eta}(k+1) = \bar{\eta}(k+1) + V(k+1)\left[z(k+1) - H(k+1)\bar{\eta}(k+1)\right]$$

where

$$H(k+1) \equiv \left(\frac{\sum_{j=0}^{k+1} \sum_{g \in \mathcal{F}} \pi_i^g(j)}{(k+1)\Delta k} - \pi(k+1) \right) \Delta k$$
$$P(k+1) = \left[M(k+1)^{-1} + H(k+1)^T R^{-1} H(k+1) \right]^{-1}$$
$$V(k+1) \equiv P(k+1) H(k+1) R^{-1}$$

5. If $k = K$ stop; otherwise set $k = k+1$ and go to step 3.

When we finish the estimation process, we have $\bar{\eta}(K)$ at hand, which is the value of η which will be used in the next planning period.

5.3 A Numerical Example of the Single Firm's Problem

Before we proceed to the game-theoretic model for competition among service providers, we first describe the so-called gradient projection algorithm, for the case when only one firm exists. We obtain an optimal control problem, in which the criterion is nonlinear, the state dynamics are linear, and the control set is convex and compact with a state-space constraint (5.3). Although the gradient projection algorithm is very popular among optimal control researchers, it is not easy to find a written statement of the algorithm in the revenue management literature. More information, such as convergence and varieties of the method, is found in Minoux (1986), Polak (1971) and Bryson (1999).

To continue, we penalize the state-space constraint so that the criteria becomes

$$\begin{aligned}\max_{\pi^f} J_f(\pi^f, \pi^{-f}, t) = & \int_{t_0}^{t_1} \left[e^{-\rho t} \left(\sum_{i \in \mathcal{S}} \pi_i^f \cdot D_i^f \right) \right. \\ & \left. - \frac{\mu}{2} \left\{ \max\left(0, AD^f - C^f\right)^2 + \min\left(0, D^f\right)^2 \right\} \right] dt \quad (5.24)\end{aligned}$$

where μ is the penalty coefficient. This problem can be solved by the descent method in Hilbert spaces presented in Section 2.3.

Let us consider a market where a single service provider f is offering four services. The planning horizon for this problem is a month or 30 days. Each firm updates the model parameter once based on daily demand observations. The firm wants to maximize revenue using the following parameters:

$$\eta = \begin{bmatrix} 0.1 \\ 0.08 \\ 0.12 \\ 0.09 \end{bmatrix}$$

$$\pi_{\min} = \begin{bmatrix} 30 \\ 40 \\ 60 \\ 130 \end{bmatrix}, \pi_{\max} = \begin{bmatrix} 85 \\ 135 \\ 180 \\ 205 \end{bmatrix},$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 300 \\ 210 \\ 150 \\ 60 \\ 255 \end{bmatrix}$$

Once they have the optimal pricing policy, the firms exercise it and observe what really happens in the market. The observation of the actual realized demand occurs every week so that the firm has 52 observations for the year. With this data, the firm learns the model parameter η with the process-noise covariance and

	Revenue
a priori	$\$119,990$
observed	$\$145,180$
a posteriori	$\$154,110$

Table 5.1. A priori, observed and a posteriori revenue of the firm

the measurement noise covariance matrices

$$Q = \begin{bmatrix} 0.3 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.5 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.2 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.3 \end{bmatrix}, R = \begin{bmatrix} 0.2 & 0.2 & 0.1 & 0.1 \\ 0.1 & 0.3 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.2 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.3 \end{bmatrix}$$

According to the results presented in Table 5.1, the observed revenue is higher than the *a priori* revenue estimation which is obtained by solving the model with the initial η. This means that the actual speed of change in demand dynamics is slower than the value used in planning. Hence, the firm could increase price of service without worrying about demand decreases. With the Kalman filter dynamics, we adjust the parameter η

$$\eta_{\text{adjusted}} = \begin{bmatrix} 0.0403 \\ 0.0436 \\ 0.0775 \\ 0.0471 \end{bmatrix}$$

whose values are less than the original values. With this new forecast, we perform another dynamic price planning for the next year.

The demand and price for each service is presented in Figures 5.1 and 5.2, respectively. After estimation, service prices are set at higher levels, generating more revenue. We should note that after estimation one does *not* follow the optimal pricing rule associated with observed demand, but rather one obtains a more accurate pricing rule based on updated parameters.

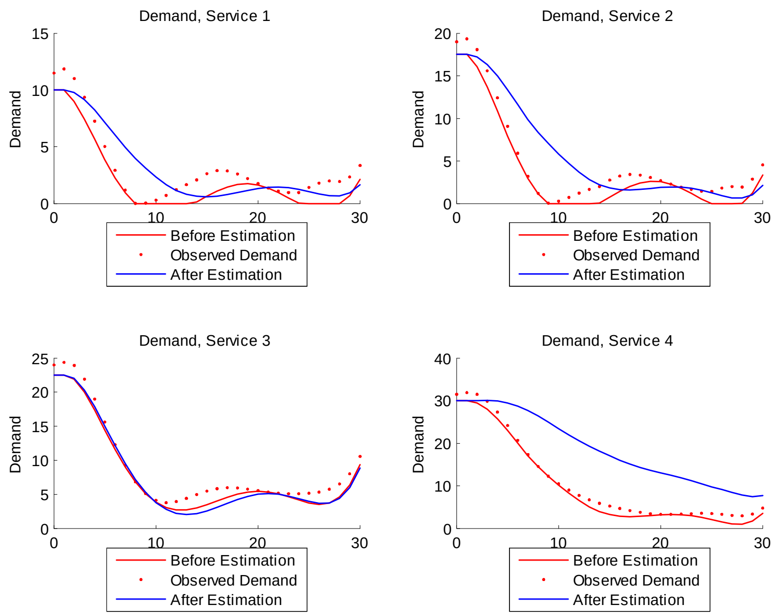

Figure 5.1. *A priori* demand, observed demand and *a posteriori* demand for each service

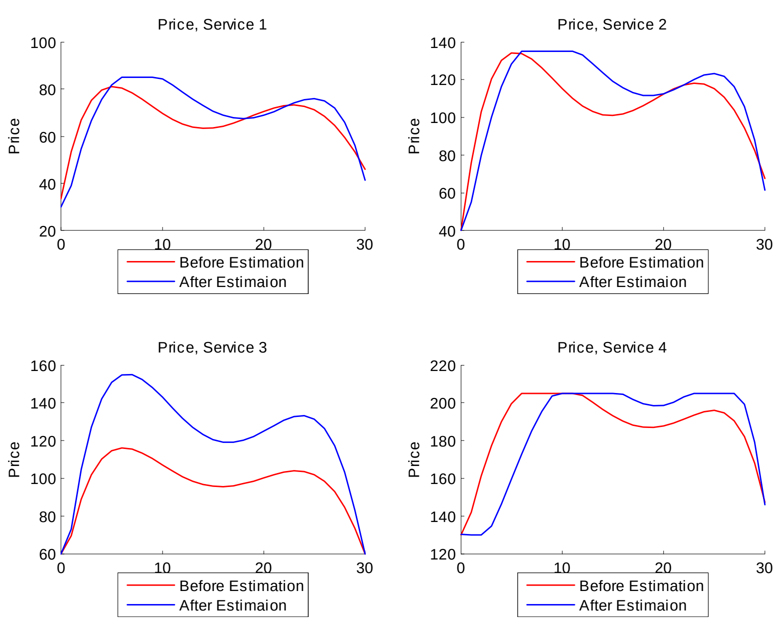

Figure 5.2. *A priori* optimal price and *a posteriori* optimal price for each service

5.4 DVI Formulation of the Competition

Each service provider is a Cournot-Nash agent that knows and employs the current instantaneous values of the decision variables of other firms to make its own non-cooperative decisions. Therefore (5.21) defines a set of coupled optimal control problems, one for each firm $f \in \mathcal{F}$. It is useful to note that (5.21) is an optimal control problem with fixed terminal time and fixed terminal state. Its Hamiltonian is

$$\begin{aligned} &H_f\left(\pi^f; D^f; \lambda^f; \sigma, \alpha^f, \beta^f; \pi^{-f}; t\right) \\ &\qquad \equiv e^{-\rho t}\left(\sum_{i\in\mathcal{S}} \pi_i^f \cdot D_i^f\right) + \Phi_f\left(\pi^f; D^f; \lambda^f; \sigma^f; \alpha^f; \beta^f; \pi^{-f}\right) \end{aligned} \quad (5.25)$$

where

$$\begin{aligned} &\Phi_f\left(\pi^f; D^f; \lambda^f; \sigma, \alpha^f, \beta^f; \pi^{-f}\right) \\ &\qquad = \sum_{i\in\mathcal{S}} \lambda_i^f \left[\eta_i^f \cdot \left(\frac{y_i}{|\mathcal{F}|\left(t-t_0\right)} - \pi_i^f\right)\right] \\ &\qquad + \sum_{i\in\mathcal{S}} \sigma_i \left(\pi_i^f + \sum_{g\in\mathcal{F}\setminus f} \pi_i^g\right) + \sum_{i\in\mathcal{S}} \alpha_i^f\left(-D_i^f\right) \\ &\qquad + \sum_{i\in\mathcal{S}} \beta_i^f\left(AD_i^f - C_i^f\right) \end{aligned} \quad (5.26)$$

while $\lambda_i^f \in \mathcal{H}^1\left[t_0, t_f\right]$ is the adjoint variable for the dynamics associated with the firm f with service type i while $\lambda \in \left(\mathcal{H}^1\left[t_0, t_f\right]\right)^{|\mathcal{F}|\times|\mathcal{S}|}$, $\sigma_i \in \mathbb{R}_+^1, \alpha_i^f \in \mathbb{R}_+^1$ and $\beta_i^f \in \mathbb{R}_+^1$ are the dual variables arising from the auxiliary state variables $(y_i, i \in \mathcal{S})$ and state space constraints. The instantaneous revenue for firm f is $\sum_{i\in\mathcal{S}} \pi_i^f \cdot D_i^f$. We assume in the balance of this chapter that the game (5.21) is regular in the sense of Definition 10. Therefore, the maximum principle (Bryson and Ho 1975) tells us that an optimal solution to (5.21) is a sextuplet $\left\{\pi^{f*}(t), D^{f*}(t); \lambda^{f*}(t), \sigma^*, \alpha^{*f}, \beta^{*f}\right\}$ requiring that the nonlinear program

$$\max H_f \quad \text{s.t.} \quad \pi_{\min}^f \leq \pi^f \leq \pi_{\max}^f$$

be solved by every firm $f \in \mathcal{F}$ for every instant of time $t \in [t_0, t_f]$ where

$$\pi^f_{\text{min}} = \left\{\pi^f_i \, : i \in \mathcal{S}\right\}$$
$$\pi^f_{\text{max}} = \left\{\pi^f_i \, : i \in \mathcal{S}\right\}$$

Consequently, any optimal solution must satisfy at each time $t \in [t_0, t_f]$

$$\pi^{f*} = \arg\left\{\max_{\pi^f_{\text{min}} \leq \pi^f \leq \pi^f_{\text{max}}} H_f\left(\pi^f; D^f; \lambda^f; \sigma, \alpha^f, \beta^f; \pi^{-*f}; t\right)\right\} \tag{5.27}$$

which in turn, by virtue of regularity, is equivalent to

$$\left[\nabla_{\pi^{*f}} H^*_f\right]^T \left(\pi^f - \pi^{*f}\right) \leq 0$$
$$\text{for all} \begin{pmatrix} \pi^f_{\text{min}} \\ \pi^f_{\text{min}} \end{pmatrix} \leq \begin{pmatrix} \pi^{*f} \\ \pi^f \end{pmatrix} \leq \begin{pmatrix} \pi^f_{\text{max}} \\ \pi^f_{\text{max}} \end{pmatrix} \tag{5.28}$$

where

$$H^*_f \equiv e^{-\rho t}\left(\sum_{i\in\mathcal{S}} \pi^{*f}_i \cdot D^{*f}_i\right) + \Phi^*_f \tag{5.29}$$

and

$$\Phi^*_f = \Phi_f\left(\pi^{*f}; D^{*f}; \lambda^{*f}; \sigma^{*f}; \alpha^{*f}; \beta^{*f}; \pi^{-f}\right) \tag{5.30}$$

From (5.25)

$$\left[\nabla_{\pi^f}\left[e^{-\rho t}\left(\sum_{i\in\mathcal{S}} \pi^{*f}_i \cdot D^{*f}_i\right)\right] + \nabla_{\pi^f}\Phi^*_f\right]^T \left(\pi^f - \pi^{*f}\right) \leq 0$$
$$\text{for all} \begin{pmatrix} \pi^f_{\text{min}} \\ \pi^f_{\text{min}} \end{pmatrix} \leq \begin{pmatrix} \pi^{*f} \\ \pi^f \end{pmatrix} \leq \begin{pmatrix} \pi^f_{\text{max}} \\ \pi^f_{\text{max}} \end{pmatrix}$$

Furthermore, adjoint dynamics and state dynamics are

$$\frac{\partial H_f}{\partial D^f_i} = (-1)\,\frac{d\lambda^{f*}_i}{dt} \tag{5.31}$$
$$\frac{\partial H_f}{\partial \lambda^f_i} = \frac{dD^{f*}_i}{dt} \tag{5.32}$$

Due to absence of a terminal time constraint, the transversality condition is

$$\lambda^{f*}(t_f) = \gamma^T \frac{\partial \Gamma \left[D^{f*}(t_f), t_f \right]}{\partial D^{f*}(t_f)} = 0$$

which gives rise to a two-point boundary value problem.

With this preceding background, we are now in a position to create a variational inequality for non-cooperative competition among the firms. We consider the following DVI which has solutions that are Cournot-Nash equilibria for the game described above in which individual firms maximize their revenue in light of current information about their competitors: find $\pi^* \in \Omega$ such that

$$\int_{t_0}^{t_f} \left(\sum_{s\in\mathcal{S}} \sum_{f\in\mathcal{F}} \frac{\partial H_f^*}{\partial \pi_i^f} \left(\pi_i^f - \pi_i^{f*} \right) \right) dt \leq 0 \text{ for all } \pi \in \Omega = \prod_{f\in\mathcal{F}} \Omega_f \tag{5.33}$$

where H_f^* is defined in (5.29)-(5.30) and

$$\Omega_f = \left\{ \pi^f : \pi^f_{\min} \leq \pi^f \leq \pi^f_{\max} \right\}$$

This DVI is regular in the sense of Definition 10 and a convenient way of expressing the Cournot-Nash game that is our present interest.

5.5 A Numerical Example for the Competition with Model Parameter Update

Consider a market where two service providers $f = 1, 2$ are offering four services. The planning horizon for this problem is a month or 30 days. Each firm updates parameters once daily based on demand observations. The speed of change in demand η takes on the values below

	Service type, i	1	2	3	4
η :	Firm 1	0.10	0.08	0.12	0.09
	Firm 2	0.11	0.07	0.10	0.15

Each firm's initial demand at time t_0 for services K_0 is shown below.

	Service type, i	1	2	3	4
K_0 :	Firm 1	10.0	17.5	22.5	30.0
	Firm 2	9.5	16.5	20.0	31.0

The resource upper bound for each firm C is given below

	Resource type, j	1	2	3	4	5
C :	Firm 1	300	210	150	60	255
	Firm 2	180	150	120	75	210

Assume the bounds on prices $\pi_{\max}$ and $\pi_{\min}$ are those given below

	Service type, i	1	2	3	4
$\pi_{\max}$:	Firm 1	85	135	180	205
	Firm 2	75	108	185	210

	Service type, i	1	2	3	4
$\pi_{\min}$:	Firm 1	30	40	60	130
	Firm 2	45	50	65	115

Once the firms have determined on optimal pricing policy, they exercise it and observe what really happens in the market. The observation of the actual realized demand occurs every week so that the firm has 52 observations for the year. With this data, the firm learns the model parameter η when the process-noise covariance and the measurement noise covariance matrices are the following:

$$Q = \begin{bmatrix} 0.3 & 0.1 & 0.2 & 0.2 \\ 0.1 & 0.5 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.2 & 0.3 \end{bmatrix}, R = \begin{bmatrix} 0.2 & 0.2 & 0.1 & 0.1 \\ 0.1 & 0.3 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.2 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.3 \end{bmatrix}$$

	Revenue, Firm 1	Revenue, Firm 2
before estimation	$65,060	$48,720
observed	$90,100	$71,870
after estimation	$396,900	$440,010

Table 5.2. Before-estimation, observed and after-estimation revenue of the firm

As the results are presented in Table 5.2, the observed revenue is higher than the projected *a priori* best revenue estimation. This means that the actual speed of change in demand is slower than used in planning. The firm may increase price without worrying that demand will decrease. With the Kalman filter dynamics, we adjust the parameter η according to

$$\eta_{\text{adjusted}} = \begin{bmatrix} 0.0403 \\ 0.0436 \\ 0.0775 \\ 0.0471 \end{bmatrix}$$

whose values are less than the original value. With this new forecast, we perform an analysis for the next period.

The demand trajectories for each service are presented in Figures 5.3 and 5.4 for firms 1 and 2, respectively. We observe that demand after estimation is less sensitive to the price than the demand before estimation, as the parameter value η is smaller. The trajectories of the price charged for each service are presented in Figures 5.5 and 5.6 for firms 1 and 2, respectively. The prices are relatively higher for the simulation after estimation than before estimation.

We note that the price will not reach a stationary state, in general. The optimal price trajectories we would obtain in each planning period are up-to-date reflections of customers' choice behavior; hence the price trajectories will depend on the current market state which is uncertain. However, if there is no shock to the market and we have a long enough time horizon, the price trajectory must reach a stationary state.

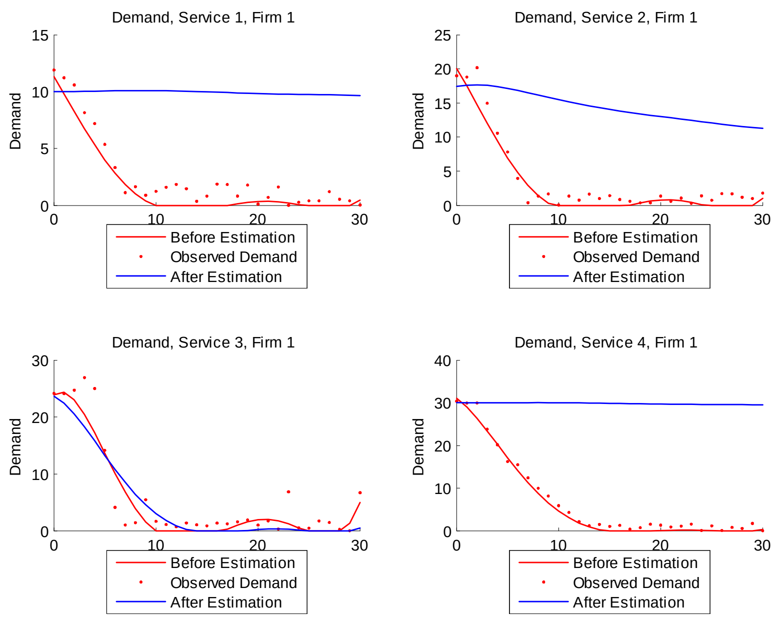

Figure 5.3. Firm 1's demand trajectories (before estimation, observed, after estimation) for different services

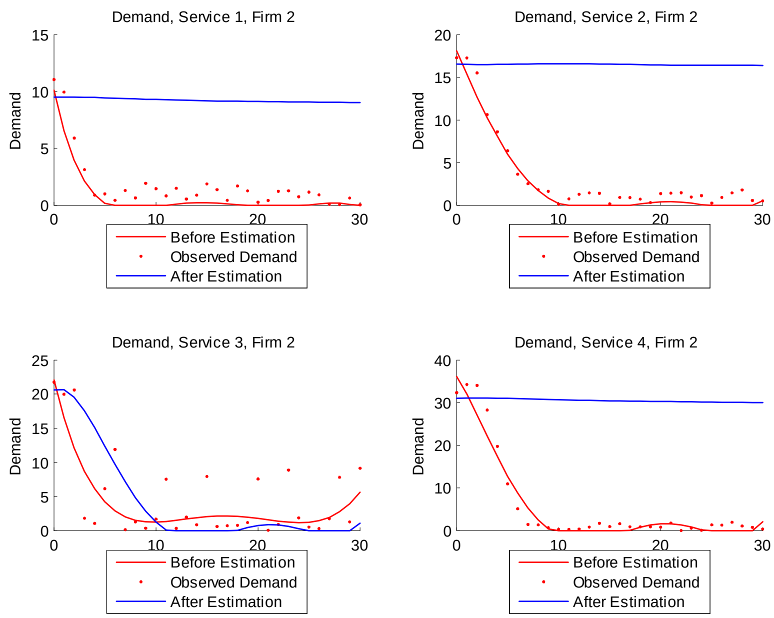

Figure 5.4. Firm 2's demand trajectories (before estimation, observed, after estimation) for different services

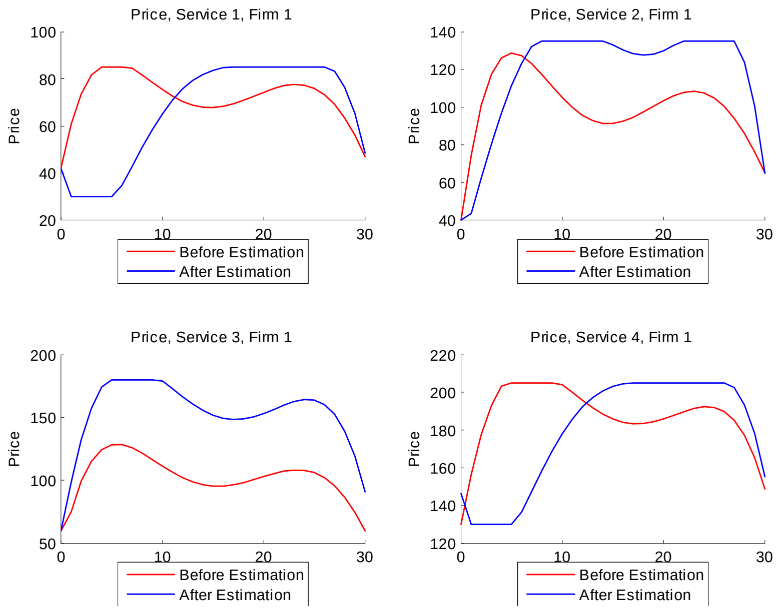

Figure 5.5. Firm 1's price trajectories (before estimation, after estimation) for different services

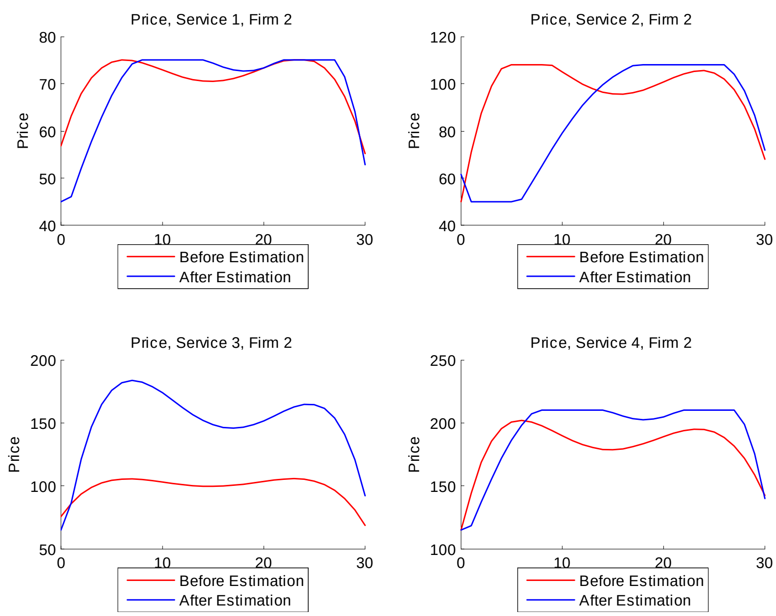

Figure 5.6. Firm 2's price trajectories (before estimation, after estimation) for different services

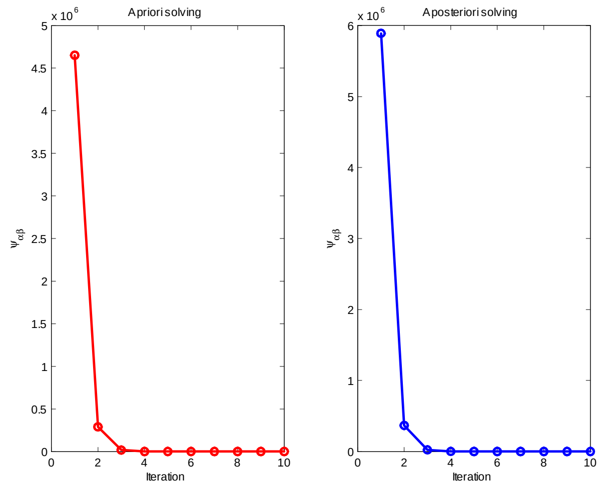

Figure 5.7. Change in gap function values

5.5.1 Computational Performance

The descent method described earlier converged after 10 iterations for this numerical example with the preset tolerance $\varepsilon = 10^{-4}$ for the gap function. In Figure 5.7, the objective values, which are values of the gap function, are presented for each iteration for the *a priori* and *a posteriori* problems. Even though the sizes of the gap function are very large at the first iterations for both problems, they decrease very rapidly toward zero, which is the solution. The run time for this example is less than 10 seconds using a desktop computer with an Intel Xeon processor and 4 GB RAM. The computer code for the descent method is written in MATLAB 7.0.

5.6 Concluding Remarks

We have shown that non-cooperative competition among service firms who engage in demand learning may be modeled as a differential variational inequality involving Kalman filtering. We have also shown that such a model may be effectively solved using the notion of a gap function together with descent in Hilbert space. Indeed the performance of the gap function/descent algorithm we propose suggests that revenue optimization/pricing calculations for real markets involving oligopolistic competition among service providers who are Nash agents engaged in demand learning may be carried out and used for decision support. The most obvious extension of the modeling framework proposed herein is to create a differential Stackelberg leader-follower game wherein the followers are described by a differential variational inequality like the one we have presented. The leader of such a game will be a powerful service firm that seeks to enter the market and who is described by an appropriate optimal control problem for revenue maximization.

Chapter 6

Locally Robust Dynamic Pricing and Inventory Control

In most industries, decision makers face uncertainty when making decisions regarding pricing policies or inventory control. Examples of uncertainty include production errors, supply-side disruptions, unexpected changes in costs, and demand. The objective of this chapter is to provide a proper method for solving a dynamic pricing and inventory control problem under demand uncertainty. This is accomplished through the articulation of a robust counterpart as shown in Chapter 2 for an abstract optimal control problem.

This chapter is organized as follows. We propose a dynamic pricing and inventory control problem and derive the robust formulation in Section 6.1. In Section 6.2 the mathematical and qualitative properties of the robust formulation are studied and analyzed, and the definition of the parameter set of controllable uncertainty is given. In Section 6.3 we propose a proper numerical solution approach for the model and examine a numerical example in Section 6.4. We conclude this chapter by stating the key findings in Section 6.5.

6.1 Robust Dynamic Pricing and Inventory Control

In this section, we consider a robust dynamic pricing and inventory control problem with nonlinear production and inventory holding costs, linear inventory dynamics, and uncertainty in demands. The uncertain demand function is assumed to be a function of price. We assume upper and lower bounds on the inventory levels which allows the modeler to specify whether back-orders are allowed and to what extent.

6.1.1 Notation

Throughout the rest of this chapter, we use the following notation and definitions:

$$\begin{aligned}
i = 1, 2, ...n &: \text{ product} \\
r = 1, 2, ..., m &: \text{resource} \\
p_i(t) &: \text{price} \\
q_i(t) &: \text{production rate} \\
I_i(t) &: \text{inventory level} \\
[L_i, U_i] &: \text{inventory capacity} \\
d_i(p_i(t); \varepsilon_i(t)) &: \text{uncertain demand function} \\
V_i(q_i(t)) &: \text{production cost} \\
W_i(I_i(t)) &: \text{inventory holding cost} \\
k_{ir} &: \text{resource requirement of } r \text{ to produce product } i \\
K_r(t) &: \text{available resource of } r \text{ at time } t
\end{aligned}$$

In this problem, the control variables of the firm are price $p_i(t)$ and $q_i(t)$, and the state variables are inventory level $I_i(t)$. In this problem we do not make any assumptions on back-orders; we simply restrict the inventory levels by lower and upper bounds. In the case that one wants to allow back-orders for a certain product i, the modeler may set L_i to a negative number so that $|L_i|$ denotes the upper-limit of back-orders which the firm can take. If the firms is to allow no back-orders for a certain product i, the modeler may simply set $L_i = 0$. However, to make the model meaningful, we constrain the terminal time inventory level to be non-negative; this indicates that the firm must fulfill all back-orders by the terminal time. We do not employ an explicit penalty for back-orders, while it may be included in a carefully defined inventory holding cost function, e.g., quadratic function of inventory level. Further, we introduce the following assumption:

Assumption 21 *We assume the following condtions hold for all $i = 1, 2, ..., n$:*

1. $d_i\left(p_i\left(t\right); \varepsilon_i\left(t\right)\right)$ *is bounded;*
2. $p_i\left(t\right) d_i\left(p_i\left(t\right); \varepsilon_i\left(t\right)\right)$ *is a continuously differentiable and concave function of* $p_i\left(t\right)$*;*
3. $V_i\left(q_i\left(t\right)\right)$ *is a continously differentiable and convex function of* $q_i\left(t\right)$*;*
4. $W_i\left(I_i\left(t\right)\right)$ *is a continously differentiable and monotonically increasing convex function of* $I_i\left(t\right)$*;*
5. $\dfrac{\partial W_i\left(I_i\left(t\right)\right)}{\partial I_i}$ *is a continuous differentiable and convex function of* $I_i\left(t\right)$*; and*
6. *the demand uncertainty is additive and linear.*

The condition 6 in Assumption 21 is that we have

$$d_i\left(p_i\left(t\right); \varepsilon_i\left(t\right)\right) = \hat{d}_i\left(p_i\left(t\right)\right) + \varepsilon_i\left(t\right)$$

where $\hat{d}_i\left(p_i\left(t\right)\right)$ is a mean demand function and the uncertainty $\varepsilon_i\left(t\right)$ shifts the demand around the mean demand.

6.1.2 The Model Formulation

Under demand uncertainty, the firm of interest controls production rates and prices to maximize the worst-case profit, which means the firm's problem is

$$\max_{p,q} \min_{\varepsilon \in E} \int_{t_0}^{t_f} \sum_{i=1}^{N} \left\{p_i\left(t\right) d_i\left(p_i\left(t\right); \varepsilon_i\left(t\right)\right) - V_i\left(q_i\left(t\right)\right) - W_i\left(I_i\left(t\right)\right)\right\} dt \tag{6.1}$$

subject to

$$\frac{dI_i}{dt} = q_i(t) - d_i(p_i(t); \varepsilon_i(t)) \qquad \forall i = 1, 2, ..., n \tag{6.2}$$

$$I_i(t_0) = I_{i,0} \qquad \forall i = 1, 2, ..., n \tag{6.3}$$

$$I_i(t_f) \geq 0 \qquad \forall i = 1, 2, ..., n \tag{6.4}$$

$$L_i \leq I_i(t) \leq U_i \qquad \forall i = 1, 2, ..., n \tag{6.5}$$

$$d_i(p_i(t); \varepsilon_i(t)) \geq 0 \tag{6.6}$$

$$p_i(t) \geq 0 \qquad \forall i = 1, 2, ..., n \tag{6.7}$$

$$q_i(t) \geq 0 \qquad \forall i = 1, 2, ..., n \tag{6.8}$$

$$\sum_{i=1}^{N} k_{ir} q_i(t) \leq K_r(t) \qquad \forall r = 1, 2, ..., m \tag{6.9}$$

The inventory dynamics are represented as ordinary differential equations in (6.2), and their initial conditions and terminal-time constraints are given in (6.3) and (6.4), respectively. The constraints (6.5) represent the lower and upper bounds of the inventory levels. To make any demand function meaningful, we enforce the non-negativity in constraints (6.6). The constraints (6.7) and (6.8) are non-negativity constraints for the control variables p_i's and q_i's. Finally, we have the resource capacity constraints in (6.9). The uncertainty set is

$$E \equiv \{\hat{\varepsilon} + \tau D\delta : \|\delta\| \leq 1\} \tag{6.10}$$

Following the method developed in Section 2.2, we introduce a new state variable:

$$y(t) \equiv \int_{t_0}^{t} \{p_i(t)\, d_i(p_i(t); \varepsilon_i(t)) - V_i(q_i(t)) - W_i(I_i(t))\}\, dt$$

so that

$$\frac{dy}{dt} = \sum_{i=1}^{N} \{p_i(t)\, d_i(p_i(t); \varepsilon_i(t)) - V_i(q_i(t)) - W_i(I_i(t))\}$$

$$y(t_0) = 0$$

We call $y(t)$ the *cumulative profit function*. We also define

$$\begin{aligned} R &\equiv \min_{\varepsilon \in E} y(t_f) \\ &= -\max_{\varepsilon \in E} \left[-y(t_f)\right] \end{aligned}$$

which is the robust index. Then, the above problem (6.1) - (6.9) is equivalent to

$$\max_{p,q} R \tag{6.11}$$

subject to

$$R = -\max_{\varepsilon \in E} \left[-y(t_f)\right] \tag{6.12}$$

$$\frac{dy}{dt} = \sum_{i=1}^{N} \left\{p_i(t)\, d_i(p_i(t); \varepsilon_i(t)) - V_i(q_i(t)) - W_i(I_i(t))\right\} \tag{6.13}$$

$$y(t_0) = 0 \tag{6.14}$$

$$\frac{dI_i}{dt} = q_i(t) - d_i(p_i(t); \varepsilon_i(t)) \qquad \forall i = 1, 2, ..., n \tag{6.15}$$

$$I_i(t_0) = I_{i,0} \qquad \forall i = 1, 2, ..., n \tag{6.16}$$

$$I_i(t_f) \geq 0 \qquad \forall i = 1, 2, ..., n \tag{6.17}$$

$$L_i \leq I_i(t) \leq U_i \qquad \forall i = 1, 2, ..., n \tag{6.18}$$

$$d_i(p_i(t); \varepsilon_i(t)) \geq 0 \qquad \forall i = 1, 2, ..., n \tag{6.19}$$

$$q_i(t) \geq 0 \qquad \forall i = 1, 2, ..., n \tag{6.20}$$

$$p_i(t) \geq 0 \qquad \forall i = 1, 2, ..., n \tag{6.21}$$

$$\sum_{i=1}^{N} k_{ir} q_i(t) \leq K_r(t) \qquad \forall r = 1, 2, ..., m \tag{6.22}$$

Our objective in this section is to develop a robust counterpart of the parametric optimal control problem (6.11)-(6.22) under uncertainty following the discussion in the previous section. In what follows, we investigate the effects of uncertainty on each component of the model in sequence.

The effect of uncertainty on the inventory dynamics, (6.15) and (6.16), is

$$\frac{d}{dt}\frac{\partial I_i}{\partial \varepsilon_j} = \begin{cases} 0 & \text{if } i \neq j \\ -\dfrac{\partial d_i}{\partial \varepsilon_j} & \text{if } i = j \end{cases}$$
$$\frac{\partial I_i(t_0)}{\partial \varepsilon_j} = 0 \quad \forall i = 1, 2, ..., n, \; j = 1, 2, ..., n$$

Hence

$$\frac{\partial I_i}{\partial \varepsilon_j} = \begin{cases} 0 & \text{if } i \neq j \\ -\displaystyle\int_{t_0}^{t} \frac{\partial d_i}{\partial \varepsilon_j} d\xi & \text{if } i = j \end{cases} \tag{6.23}$$

where ξ is a dummy variable for the time argument t. The effect of uncertainty on the new state dynamics, (6.13) and (6.14), is

$$\frac{d}{dt}\frac{\partial y}{\partial \varepsilon_j} = p_j \frac{\partial d_j}{\partial \varepsilon_j} - \sum_{i=1}^{N} \frac{\partial W_i(I_i)}{\partial I_i}\frac{\partial I_i}{\partial \varepsilon_j} = p_j \frac{\partial d_j}{\partial \varepsilon_j} + \frac{\partial W_j(I_j)}{\partial I_j} \int_{t_0}^{t} \frac{\partial d_j}{\partial \varepsilon_j} d\xi \quad \forall\, j = 1, 2, ..., n \tag{6.24}$$

$$\frac{\partial y(t_0)}{\partial \varepsilon_j} = 0 \quad \forall\, j = 1, 2, ..., n \tag{6.25}$$

where the simplification is obtained by using (6.23). We define

$$\beta_j \equiv \frac{\partial y}{\partial \varepsilon_j}, \quad \beta = (\beta_j : j = 1, 2, ..., n)$$

so that (6.24) and (6.25) becomes

$$\frac{d\beta_j}{dt} = p_j \frac{\partial d_j}{\partial \varepsilon_j} + \frac{\partial W_j(I_j)}{\partial I_j} \int_{t_0}^{t} \frac{\partial d_j}{\partial \varepsilon_j} d\xi \quad \forall\, j = 1, 2, ..., n \tag{6.26}$$
$$\beta_j(t_0) = 0 \quad \forall\, j = 1, ..., n \tag{6.27}$$

The effect of uncertainty on the lower bound of the inventory in (6.18) is:

$$I_i(t; \hat{\varepsilon}) - L_i - \tau \left\| D^T \frac{\partial I_i(t; \hat{\varepsilon})}{\partial \varepsilon} \right\| \geq 0 \quad \forall\, i = 1, 2, ..., n \tag{6.28}$$

Element-wise, the j-th component of $D^T \frac{\partial I_i(t;\hat{\varepsilon})}{\partial \varepsilon}$ for all $i = 1, 2, ..., n$ is

$$\begin{aligned}\left(D^T \frac{\partial I_i(t;\hat{\varepsilon})}{\partial \varepsilon}\right)_j &= \sum_{k=1}^{n} D_{kj} \frac{\partial I_i(t;\hat{\varepsilon})}{\partial \varepsilon_k} \\ &= D_{ij} \frac{\partial I_i(t;\hat{\varepsilon})}{\partial \varepsilon_i} \\ &= -D_{ij} \int_{t_0}^{t} \left[\frac{\partial d_i}{\partial \varepsilon_i}\right]_{\varepsilon_i=\hat{\varepsilon}_i} d\xi\end{aligned}$$

by (6.23). Further

$$\begin{aligned}\left\| D^T \frac{\partial I_i(t;\hat{\varepsilon})}{\partial \varepsilon} \right\| &= \sqrt{\int_{t_0}^{t_f} \sum_{j=1}^{n} \left(-D_{ij} \int_{t_0}^{\zeta} \left[\frac{\partial d_i}{\partial \varepsilon_i}\right]_{\varepsilon_i=\hat{\varepsilon}_i} d\xi \right)^2 d\zeta} \\ &= \sqrt{\int_{t_0}^{t_f} \left(\int_{t_0}^{\zeta} \left[\frac{\partial d_i}{\partial \varepsilon_i}\right]_{\varepsilon_i=\hat{\varepsilon}_i} d\xi \right)^2 \left(\sum_{j=1}^{n} (D_{ij})^2 \right) d\zeta}\end{aligned}$$

where ζ is another dummy variable for the time argument t. Hence, we replace (6.28) by the following:

$$I_i(t;\hat{\varepsilon}) - L_i - \tau \sqrt{\int_{t_0}^{t_f} \left(\int_{t_0}^{\zeta} \left[\frac{\partial d_i}{\partial \varepsilon_i}\right]_{\varepsilon_i=\hat{\varepsilon}_i} d\xi \right)^2 \left(\sum_{j=1}^{n} (D_{ij})^2 \right) d\zeta} \geq 0 \quad \forall\, i = 1, 2, ..., n \tag{6.29}$$

where D_{ij} is (i, j)-th element of matrix D. Similarly, we obtain the effect on the upper bound of the inventory in (6.18):

$$I_i(t;\hat{\varepsilon}) - U_i + \tau \sqrt{\int_{t_0}^{t_f} \left(\int_{t_0}^{t} \left[\frac{\partial d_i}{\partial \varepsilon_i}\right]_{\varepsilon_i=\hat{\varepsilon}_i} d\xi \right)^2 \left(\sum_{j=1}^{n} (D_{ij})^2 \right) dt} \geq 0 \quad \forall\, i = 1, 2, ..., n \tag{6.30}$$

The effect of uncertainty on the terminal time inventory level constraint (6.17) is:

$$I_i(t_f;\hat{\varepsilon}) - \tau \left\| D^T \frac{\partial I_i(t_f;\hat{\varepsilon})}{\partial \varepsilon} \right\| \geq 0 \quad \forall\, i = 1, 2, ..., n$$

or

$$I_i\left(t_f;\hat{\varepsilon}\right)-\tau\sqrt{\int_{t_0}^{t_f}\left(\int_{t_0}^{t_f}\left[\frac{\partial d_i}{\partial \varepsilon_i}\right]_{\varepsilon_i=\hat{\varepsilon}_i}d\xi\right)^2\left(\sum_{j=1}^{n}\left(D_{ij}\right)^2\right)d\zeta}\geq 0 \;\; \forall\; i=1,2,...,n \tag{6.31}$$

The integrand of the outer integral in (6.31) is not a function of time t, hence we have

$$I_i\left(t_f;\hat{\varepsilon}\right)-\tau\sqrt{\left(t_f-t_0\right)\left(\int_{t_0}^{t_f}\left[\frac{\partial d_i}{\partial \varepsilon_i}\right]_{\varepsilon_i=\hat{\varepsilon}_i}d\xi\right)^2\left(\sum_{j=1}^{n}\left(D_{ij}\right)^2\right)}\geq 0 \;\; \forall\; i=1,2,...,n \tag{6.32}$$

The effect of uncertainty on the non-negative demand constraint (6.19) is

$$d_i\left(p_i\left(t\right);\hat{\varepsilon}_i\left(t\right)\right)-\tau\left\|D^T\left[\frac{\partial d_i}{\partial \varepsilon}\right]_{\varepsilon=\hat{\varepsilon}}\right\|\geq 0 \;\; \forall\; i=1,2,...,n$$

or

$$d_i\left(p_i\left(t\right);\hat{\varepsilon}_i\left(t\right)\right)-\tau\sqrt{\int_{t_0}^{t_f}\left(\left[\frac{\partial d_i}{\partial \varepsilon_i}\right]_{\varepsilon_i=\hat{\varepsilon}_i}\right)^2\sum_{j=1}^{n}\left(D_{ij}\right)^2 dt}\geq 0 \;\; \forall\; i=1,2,...,n \tag{6.33}$$

Finally, the robust index (6.11) becomes

$$\begin{aligned}R &= -\max_{\varepsilon\in E}\left[-y\left(t_f\right)\right]\\ &= -\left[-y\left(t_f;\hat{\varepsilon}\right)+\tau\left\|-D^T\frac{\partial y\left(t,\hat{\varepsilon}\right)}{\partial \varepsilon}\right\|_{t=t_f}\right]\\ &= -\left[-y\left(t_f;\hat{\varepsilon}\right)+\tau\left\|-D^T\beta\left(t,\hat{\varepsilon}\right)\right\|_{t=t_f}\right]\\ &= y\left(t_f;\hat{\varepsilon}\right)-\tau\left\|D^T\beta\left(t,\hat{\varepsilon}\right)\right\|\\ &= y\left(t_f;\hat{\varepsilon}\right)-\tau\sqrt{\int_{t_0}^{t_f}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}D_{ij}\beta_j\left(t,\hat{\varepsilon}\right)\right)^2 dt}\end{aligned} \tag{6.34}$$

We note that the new state variable y only appears in (6.13), (6.14) and (6.34), so that we can replace (6.34) by the following:

$$R = \int_{t_0}^{t_f} \sum_{i=1}^{N} \{p_i(t)\, d_i(p_i(t); \hat{\varepsilon}_i(t)) - V_i(q_i(t)) - W_i(I_i(t))\}\, dt - \tau \sqrt{\int_{t_0}^{t_f} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} D_{ij} \beta_j(t, \hat{\varepsilon}) \right)^2 dt} \tag{6.35}$$

Consequently, from (6.15), (6.16), (6.20), (6.21), (6.22), (6.24), (6.25), (6.26), (6.27), (6.29), (6.30), (6.32), (6.33) and (6.35), we obtain the following robust counterpart:

Theorem 22 *The robust counterpart of uncertain optimal control problem (6.1)-(6.9) for dynamic pricing and inventory control is:*

$$\max_{p,q} R = \int_{t_0}^{t_f} \sum_{i=1}^{N} \{p_i(t) \quad - V_i(q_i(t)) - W_i(I_i(t))\}\, dt - \tau \sqrt{\int_{t_0}^{t_f} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} D_{ij} \beta_j(t, \hat{\varepsilon}) \right)^2 dt} \tag{6.36}$$

subject to

$$\frac{dI_i}{dt} = q_i(t) - d_i(p_i(t); \hat{\varepsilon}_i(t)) \qquad \forall i = 1, 2, ..., n \tag{6.37}$$

$$I_i(t_0) = I_{i,0} \qquad \forall i = 1, 2, ..., n \tag{6.38}$$

$$I_i(t_f; \hat{\varepsilon}) - \tau \sqrt{(t_f - t_0) \left(\int_{t_0}^{t_f} \left[\frac{\partial d_i}{\partial \varepsilon_i} \right]_{\varepsilon_i = \hat{\varepsilon}_i} d\xi \right)^2 \left(\sum_{j=1}^{n} (D_{ij})^2 \right)} \geq 0 \qquad \forall i = 1, 2, ..., n \tag{6.39}$$

$$I_i(t; \hat{\varepsilon}) - L_i - \tau \sqrt{\int_{t_0}^{t_f} \left(\int_{t_0}^{\zeta} \left[\frac{\partial d_i}{\partial \varepsilon_i} \right]_{\varepsilon_i = \hat{\varepsilon}_i} d\xi \right)^2 \left(\sum_{j=1}^{n} (D_{ij})^2 \right) d\zeta} \geq 0 \qquad \forall i = 1, 2, ..., n \tag{6.40}$$

$$I_i(t; \hat{\varepsilon}) - U_i + \tau \sqrt{\int_{t_0}^{t_f} \left(\int_{t_0}^{\zeta} \left[\frac{\partial d_i}{\partial \varepsilon_i} \right]_{\varepsilon_i = \hat{\varepsilon}_i} d\xi \right)^2 \left(\sum_{j=1}^{n} (D_{ij})^2 \right) d\zeta} \geq 0 \qquad \forall i = 1, 2, ..., n \tag{6.41}$$

$$\frac{d\beta_i}{dt} = p_i \frac{\partial d_i}{\partial \varepsilon_i} + \frac{\partial W_i(I_i)}{\partial I_i} \int_{t_0}^{t} \frac{\partial d_i}{\partial \varepsilon_i} d\xi \quad \forall\ i = 1, 2, ..., n \tag{6.42}$$

$$\beta_i(t_0) = 0 \quad \forall\ i = 1, 2, ..., n \tag{6.43}$$

$$d_i(p_i(t); \hat{\varepsilon}_i(t)) - \tau \sqrt{\int_{t_0}^{t_f} \left(\left[\frac{\partial d_i}{\partial \varepsilon_i} \right]_{\varepsilon_i = \hat{\varepsilon}_i} \right)^2 \sum_{j=1}^{n} (D_{ij})^2 \, d\zeta} \geq 0 \ \ \forall\ i = 1, 2, ..., n \tag{6.44}$$

$$p_i(t), q_i(t) \geq 0 \qquad \forall i = 1, 2, ..., n \tag{6.45}$$

$$\sum_{i=1}^{N} k_{ir} q_i(t) \leq K_r(t) \qquad \forall r = 1, 2, ..., m \tag{6.46}$$

where the state variables are $I_i(t)$ and $\beta_i(t)$, and the control variables are $p_i(t)$ and $q_i(t)$.

While we have developed the robust counterpart (6.36)-(6.46) which is general enough to include any type of uncertainty, we have in fact a simpler form by the condition 6 in Assumption 21. If the uncertainty is additive and linear

$$\frac{\partial d_i}{\partial \varepsilon_i} = 1$$

$$\int_{t_0}^{\zeta} \left[\frac{\partial d_i}{\partial \varepsilon_i} \right]_{\varepsilon_i = \hat{\varepsilon}_i} d\xi = \zeta - t_0$$

for all $i = 1, 2, ..., n$. We can replace (6.39), (6.40), (6.41) and (6.44) by

$$I_i(t_f; \hat{\varepsilon}) - \tau \sqrt{(t_f - t_0)^3 \left(\sum_{j=1}^{n} (D_{ij})^2 \right)} \geq 0 \qquad \forall i = 1, 2, ..., n$$

$$I_i(t; \hat{\varepsilon}) - L_i - \tau \sqrt{\frac{(t_f - t_0)^3}{3} \left(\sum_{j=1}^{n} (D_{ij})^2 \right)} \geq 0 \qquad \forall i = 1, 2, ..., n$$

$$I_i(t; \hat{\varepsilon}) - U_i + \tau \sqrt{\frac{(t_f - t_0)^3}{3} \left(\sum_{j=1}^{n} (D_{ij})^2 \right)} \geq 0 \qquad \forall i = 1, 2, ..., n$$

$$d_i\left(p_i\left(t\right);\hat{\varepsilon}_i\left(t\right)\right)-\tau\sqrt{\left(t_f-t_0\right)\sum_{j=1}^{n}\left(D_{ij}\right)^2}\geq 0 \quad \forall\, i=1,2,...,n$$

respectively.

6.2 Properties of the Robust Formulation

The robust optimal control problem (6.36)-(6.46) has the following mathematical properties:

1. *criterion functional*: Under Assumption 21, the first integral in the criterion functional (6.36) is concave. It can be easily shown that the second component in the criterion functional (6.36) is also concave. Any norm $\|\cdot\|$ is convex, since $\|\rho x+(1-\rho)\,y\|\leq\rho\,\|x\|+(1-\rho)\,\|y\|$ for $0<\rho<1$ by the triangle inequality. Because the norm used herein is a monotonically increasing function, for any convex function $g\left(\cdot\right)$,

$$\begin{aligned}\|g\left(\rho x+(1-\rho)\,y\right)\| &\leq \|\rho g\left(x\right)+(1-\rho)\,g\left(y\right)\|\\ &\leq \rho\,\|g\left(x\right)\|+(1-\rho)\,\|g\left(y\right)\|\end{aligned}$$

   for $0<\rho<1$. Hence, the function $\|g\left(\cdot\right)\|$ is convex. Therefore, the second component (including '-' sign) is concave.

2. *dynamics*: Robust inventory dynamics (6.37) keeps the properties of uncertain inventory dynamics (6.2). The robust counter part of the new dynamics (6.42) is generally nonlinear and depends on the time argument t. However, when $\dfrac{\partial W_i\left(I_i\right)}{\partial I_i}$ is linear in I_i and the uncertainty in demand is additive and linear as in Assumption 21, the dynamics (6.42) is also linear.

3. *state- and control-space constraints*: Zhang (2007) notes that, in general, the robust formulation induces non-convex constraints; however, in this particular dynamic pricing and inventory control problem all state and control space constraints remain convex as observed in (6.39), (6.40), (6.41), (6.44), (6.45) and (6.46) under Assumption 21.

The parameters defining the set of uncertainty in (6.10) are the magnitude of uncertainty τ and the uncertainty-parameter incidence matrix D. The choice of D allows us to deal with different scales of uncertainty in different parameters. However, the magnitude of uncertainty τ cannot be made arbitrarily large for two reasons. First, it is not generally desirable for the decision maker to consider a very large uncertainty which gives a less-efficient solution for pricing and inventory control. The other reason is that the problem may become infeasible for arbitrarily large values of τ. We note there are constraints (6.38), (6.40), (6.41), (6.44) and (6.45) where τ plays a role. From (6.38) and (6.40) at $t = t_0$, we have

$$\begin{aligned} I_i(t_0) &= I_{i,0} \\ &\geq L_i + \tau \sqrt{\int_{t_0}^{t_f} \left(\int_{t_0}^{\zeta} \left[\frac{\partial d_i}{\partial \varepsilon_i} \right]_{\varepsilon_i = \hat{\varepsilon}_i} d\xi \right)^2 \left(\sum_{j=1}^{n} (D_{ij})^2 \right) d\zeta} \\ &= L_i + \tau \sqrt{\int_{t_0}^{t_f} (\zeta - t_0)^2 \left(\sum_{j=1}^{n} (D_{ij})^2 \right) d\zeta} \\ &= L_i + \tau \sqrt{\frac{(t_f - t_0)^3}{3} \left(\sum_{j=1}^{n} (D_{ij})^2 \right)} \end{aligned} \tag{6.47}$$

Similary, we obtain

$$I_i(t_0) = I_{i,0} \leq U_i - \tau \sqrt{\frac{(t_f - t_0)^3}{3} \left(\sum_{j=1}^{n} (D_{ij})^2 \right)} \tag{6.48}$$

To avoid infeasibility of the problem, we must have from (6.47) and (6.48),

$$\tau \leq \min_{i=1,2,\ldots,n} \frac{(I_{i,0} - L_i)}{\sqrt{\sum_{j=1}^{n} (D_{ij})^2}} \sqrt{\frac{3}{(t_f - t_0)^3}} \tag{6.49}$$

$$\tau \leq \min_{i=1,2,\ldots,n} \frac{(U_i - I_{i,0})}{\sqrt{\sum_{j=1}^{n} (D_{ij})^2}} \sqrt{\frac{3}{(t_f - t_0)^3}} \tag{6.50}$$

We now turn our attention to (6.44) and (6.45). If the magnitude of uncertainty τ goes very large, there is a chance that (6.44) violates (6.45). To observe that, we consider, for example, the case when the demand function is linear, i.e.,

$$d_i\left(p_i\left(t\right), \varepsilon_i\left(t\right)\right) = a_i\left(t\right) - b_i\left(t\right) p_i\left(t\right) + \varepsilon_i\left(t\right) \quad \forall i = 1, 2, ..., n \tag{6.51}$$

where

$$a_i\left(t\right), b_i\left(t\right) > 0 \qquad \forall t \in \left[t_0, t_f\right]$$

With the linear demand function (6.51), the constraints (6.44) become

$$a_i\left(t\right) - b_i\left(t\right) p_i\left(t\right) + \hat{\varepsilon}_i\left(t\right) - \tau \sqrt{\sum_{j=1}^{n} \left(D_{ij}\right)^2 \left(t_f - t_0\right)} \geq 0 \ \ \forall \ i = 1, 2, ..., n$$

Hence the price trajectories $p_i\left(t\right)$ must satisfy

$$0 \leq p_i\left(t\right) \leq \frac{a_i\left(t\right) + \hat{\varepsilon}_i\left(t\right)}{b_i\left(t\right)} - \frac{\tau}{b_i\left(t\right)} \sqrt{\sum_{j=1}^{n} \left(D_{ij}\right)^2 \left(t_f - t_0\right)} \quad \forall t \in \left[t_0, t_f\right], \forall i = 1, ..., n$$

To avoid the infeasibilty, the parameters τ and D must satisfy the following inequalities:

$$\tau \leq \min_{i=1,2,...,n} \inf_{t \in \left[t_0, t_f\right]} \frac{1}{\sqrt{\sum_{j=1}^{n} \left(D_{ij}\right)^2}} \frac{a_i\left(t\right) + \hat{\varepsilon}_i\left(t\right)}{\sqrt{t_f - t_0}} \tag{6.52}$$

Observing that (6.49), (6.50) and (6.52) provides the feasible intervals of magnitude of uncertainty τ, we define the set C by

$$C \equiv \left\{ \begin{aligned} \tau : \tau &\leq \min_{i=1,2,...,n} \frac{\left(I_{i,0} - L_i\right)}{\sqrt{\sum_{j=1}^{n} \left(D_{ij}\right)^2}} \sqrt{\frac{3}{\left(t_f - t_0\right)^3}}, \\ \tau &\leq \min_{i=1,2,...,n} \frac{\left(U_i - I_{i,0}\right)}{\sqrt{\sum_{j=1}^{n} \left(D_{ij}\right)^2}} \sqrt{\frac{3}{\left(t_f - t_0\right)^3}}, \\ \tau &\leq \min_{i=1,2,...,n} \inf_{t \in \left[t_0, t_f\right]} \frac{1}{\sqrt{\sum_{j=1}^{n} \left(D_{ij}\right)^2}} \frac{a_i\left(t\right) + \hat{\varepsilon}_i\left(t\right)}{\sqrt{t_f - t_0}} \end{aligned} \right\} \tag{6.53}$$

We call the value $\bar{\tau} \equiv \sup_{\tau \in C} \tau$ the *maximum allowable magnitude of uncertainty.* This value gives a decision maker very important information. Usually, the scale matrix D is known to the decision maker, while the magnitude τ may or may not. The set (6.53) informs the decision maker on how much uncertainty may be controlled by proper decision making. If the magnitude τ is known to the decision maker and τ exceeds the value $\bar{\tau}$, the decision maker must take some other action. For example, let us suppose inequality (6.49) is violated for some product i. That could be the case when the firm of interest receives a very large order for product i at time $t = t_0 + 0$. To make $\bar{\tau}$ larger, the decision maker could try to increase the quantity $(I_{i,0} - L_i)\sqrt{\dfrac{3}{(t_f - t_0)^3}}$ in (6.49). To achieve this, the decision maker should prepare more initial stocks in inventory (larger $I_{i,0}$), or take a shorter planning horizon to avoid failure (smaller $t_f - t_0$). If τ is not known, the decision maker may determine τ satisfying (6.53) upon his preference.

We have considered a dynamic pricing and inventory control problem with upper and lower bounds on inventory levels; such bounds are critical in robust planning. With this in mind, consider (6.36), (6.39), and (6.42). We observe:

1. The robust lower bounds of inventory levels (6.39) indicate that the firm must keep extra stocks in inventory to accommodate possible extreme orders. The larger the magnitude of uncertainty and the longer the planning time horizon, the larger the safety stock should be for every time instant.

2. In the objective functional (6.36), we minimize the magnitude of accumulated β_i's, which is the effect of uncertainty in the cumulative contribution to the profit of the firm. The dynamic equations of β_i (6.42) are dependent on the price trajectories p_i and the marginal inventory costs $\dfrac{\partial W_i(I_i)}{\partial I_i}$ of product i.

6.3 A Solution Method

In solving a continuous-time model numerically, one must discretize at a certain point. There are two classes of numerical methods for solving a continuous time model: (1) *a priori* discretization and (2) *a posteriori* discretization, see Dontchev (1996). In the *a priori* discretization method, one discretizes the continuous model and then develops a solution method for the discretized model. In the *a posteriori* discretization method, one develops a solution method first for the continuous model and then uses discrete approximations only for solving sub-problems and/or obtaining computational information such as objective function values and gradient values.

In the robust optimal control problem formulation for dynamic pricing and inventory control developed in previous sections, we have a problematic component, the state-space path constraints (6.39). Such constraints make a problem very difficult to solve by *a posteriori* discretization, while other path constraints in the control space such as (6.44), (6.45) and (6.46) may be treated by simple projections. One such method is to introduce penalty functions in the objective function using 'max' operators. See Feehery and Barton (1998) and Kameswaran and Biegler (2006). However, *a priori* discretization has an advantage in solving state-space path constrained problems, since nonlinear programming based methods have the flexibility to deal with path constraints. In this chapter, we take a *a priori* discretization approach.

There are two issues in discrete-time approximation, integration and differentiation. We approximate integrations and differentiations by Riemann-type approximations and finite difference approximations, respectively. That is,

$$\int_{t_0}^{t_f} F(x,u,t)\,dt \approx \sum_{\gamma=0}^{\Gamma} F(x_\gamma, u_\gamma, t_\gamma)\,\Delta t$$

$$\frac{dx}{dt} \approx \frac{x_{\gamma+1} - x_\gamma}{\Delta t}$$

where $\Delta t = \frac{t_f - t_0}{\Gamma}$, and $t_\gamma = t_0 + \gamma \Delta t$. Using the above discretization method, we obtain the discretized statement of the robust counterpart:

$$\max_{p,q,I,\beta} J = \sum_{\gamma=0}^{\Gamma} \sum_{i=1}^{N} \{p_{i,\gamma} d_i (p_{i,\gamma}; \hat{\varepsilon}_i(t_\gamma)) - V_i(q_{i,\gamma}) - W_i(I_{i,\gamma})\}\,\Delta t \tag{6.54}$$

$$- \tau \sqrt{\sum_{\gamma=0}^{\Gamma} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} D_{ij} \beta_{j,\gamma} \right)^2 \Delta t} \tag{6.55}$$

subject to

$$I_{i,\gamma+1} = I_{i,\gamma} + \Delta t \left[q_{i,\gamma} - d_i \left(p_{i,\gamma}; \hat{\varepsilon}_i \left(t_\gamma \right) \right) \right] \qquad \forall i = 1, 2, ..., n \tag{6.56}$$

$$I_{i,0} = \text{given} \qquad \forall i = 1, 2, ..., n \tag{6.57}$$

$$I_{i,\Gamma} \geq \tau \sqrt{\left(t_f - t_0 \right)^3 \left(\sum_{j=1}^{n} \left(D_{ij} \right)^2 \right)} \qquad \forall i = 1, 2, ..., n \tag{6.58}$$

$$I_{i,\gamma} \geq L_i + \tau \sqrt{\frac{t_f^3 - t_0^3}{3} \sum_{j=1}^{n} \left(D_{ij} \right)^2} \qquad \forall i = 1, ..., n \tag{6.59}$$

$$I_{i,\gamma} \leq U_i - \tau \sqrt{\frac{t_f^3 - t_0^3}{3} \sum_{j=1}^{n} \left(D_{ij} \right)^2} \qquad \forall i = 1, ..., n \tag{6.60}$$

$$\beta_{i,\gamma+1} = \beta_{i,\gamma} + \Delta t \left[p_{i,\gamma} + \left(t_\gamma - t_0 \right) \frac{\partial W_i \left(I_{i,\gamma} \right)}{\partial I_i} \right] \qquad \forall\ i = 1, ..., n \tag{6.61}$$

$$\beta_{i,0} = 0 \quad \forall\ i = 1, ..., n \tag{6.62}$$

$$d_i \left(p_{i,\gamma}; \hat{\varepsilon}_i \left(t_\gamma \right) \right) - \tau \sqrt{\left(t_f - t_0 \right)^3 \sum_{j=1}^{n} \left(D_{ij} \right)^2} \geq 0 \quad \forall\ i = 1, 2, ..., n \tag{6.63}$$

$$p_{i,\gamma}, q_{i,\gamma} \geq 0 \qquad \forall i = 1, 2, ..., n \tag{6.64}$$

$$\sum_{i=1}^{N} k_{ir} q_{i,\gamma} \leq K_r \left(t_\gamma \right) \qquad \forall r = 1, 2, ..., m \tag{6.65}$$

An important advantage of the discrete-time approximation is that the state variables may be completely removed from the model. Using (6.56) and (6.57), we obtain

$$I_{i,\gamma} = I_{i,0} + \Delta t \sum_{\rho=0}^{\gamma-1} \left[q_{i,\rho} - d_i \left(p_{i,\rho}; \hat{\varepsilon}_i \left(t_\rho \right) \right) \right]$$

and using (6.61) and (6.62), we obtain

$$\beta_{i,\gamma} = \Delta t \sum_{\rho=0}^{\gamma-1} \left[p_{i,\rho} + \left(t_\rho - t_0 \right) \frac{\partial W_i \left(I_{i,\rho} \right)}{\partial I_i} \right]$$

Product i	$V_i(q_i(t))$			$W_i(I_i(t))$		
	$\eta_{0,i}$	$\eta_{1,i}$	$\eta_{2,i}$	$\mu_{0,i}$	$\mu_{1,i}$	$\mu_{2,i}$
1	1	2	0.1	0	0.2	0.1
2	3	2	0.1	0	0.3	0.1
3	1	2	0.1	0	0.4	0.1

Table 6.1. Parameters for production cost and inventory holding cost functions.

Under Assumption 21, the entire problem becomes a finite-dimensional maximization problem with a nonlinear concave objective function and linear constraints. Hence this problem should be easily solved by an existing NLP solver.

6.4 A Numerical Example

In this section, we study an example for the robust dynamic pricing and inventory control problem and solve it numerically. We suppose there are $n = 3$ products and $m = 4$ resources. We consider a planning time horizon of $t \in [t_0, t_f] = [0, 5]$. We employ production cost functions and inventory holding cost functions of the following forms:

$$V_i(q_i(t)) = \eta_{0,i} + \eta_{1,i} q_i(t) + \frac{1}{2}\eta_{2,i}(q_i(t))^2 \quad \forall i = 1, 2, 3$$
$$W_i(I_i(t)) = \mu_{0,i} + \mu_{1,i} I_i(t) + \frac{1}{2}\mu_{2,i}(I_i(t))^2 \quad \forall i = 1, 2, 3$$

which are quadratic. We will also need

$$\frac{\partial W_i(I_i(t))}{\partial I_i(t)} = \mu_{1,i} + \mu_{2,i} I_i(t) \quad \forall i = 1, 2, 3$$

The parameters we consider for the production and inventory holding cost functions are given in Table 6.1. In this numerical example, we consider a time-invariant linear demand function defined by

$$d_i(p_i(t), \varepsilon_i(t)) = a_i(t) - b_i(t)\, p_i(t) + \varepsilon_i(t) \quad \forall i = 1, 2, 3 \tag{6.66}$$

Product i	$a_i(t)$	$b_i(t)$
1	95	2
2	100	3
3	105	2

Table 6.2. Parameters for demand function.

and parameters are given in Table 6.2. The initial inventories are:

$$I_{1,0} = 10, \quad I_{2,0} = 7, \quad I_{3,0} = 5$$

The inventory capacity vectors are given as:

$$L = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, U = \begin{bmatrix} 20 \\ 40 \\ 30 \end{bmatrix}$$

where we assume no back-orders for any product. The resource-product incidence matrix (k_{ir}) and the resource capacity vector K are:

$$(k_{ir}) = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}, K = \begin{bmatrix} 100 \\ 100 \\ 100 \\ 100 \end{bmatrix}$$

The uncertainty set is given as

$$E \equiv \{\hat{\varepsilon} + \tau D\delta : \|\delta\| \leq 1\}$$

where

$$\hat{\varepsilon} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tau = 0.2, D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6.67}$$

Product i	Price	Production Rate	Inventory Level	Demand
1	+	+	+	–
2	+	–	+	–
3	+	+	+	–

Table 6.3. Changes of optimal strategies when the magnitude of uncertainty increases

In this setting, the maximum allowable magnitude of uncertainty defined by (6.53) is $\bar{\tau} = 0.7746$. Hence, we must have

$$\tau \leq 0.7746$$

This indicates that the magnitude of uncertainty, $\tau = 0.2$, given in (6.67) will not make the problem infeasible.

With $\Gamma = 50$ number of discretization steps, the problem was solved in GAMS with the MINOS NLP solver. We perform several scenario tests with different values of $\tau = 0, 0.2, 0.4, 0.6$. Of course $\tau = 0$ represents the nominal problem when there is no uncertainty. The optimal price, production rate, inventory level, and demand trajectories of product 1, 2 and 3 are plotted in Figures 6.1–6.4, Figures 6.5–6.8, and Figures 6.9–6.12, respectively.

The key observations are summarized in Table 6.3, which shows how optimal strategies change when the magnitude of uncertainty increases, i.e., τ increases. We denote '+' signs for increase and '–' signs for decrease. For example, the observation for product 1 in Table 6.3 is that, as the magnitude of uncertainty increases, the firm should set a higher price, produce more, keep more inventory in stock, and expect lower demand. However, the change in the optimal production rate of product 2 is different than that of other products. To interpret this result, we observe that the production rates and the demands tend to balance later in the planning horizon. We also note that we assumed a higher price sensitivity in the demand of product 2 as shown in Table 6.2. By definition, the demand decreases as prices increase. As the demand decreases, we also expect the production rate to decrease.

We compare the objective function values (the robust index), the profit and the uncertainty effect for different values of τ in Figures 6.13, 6.14 and 6.15, re-

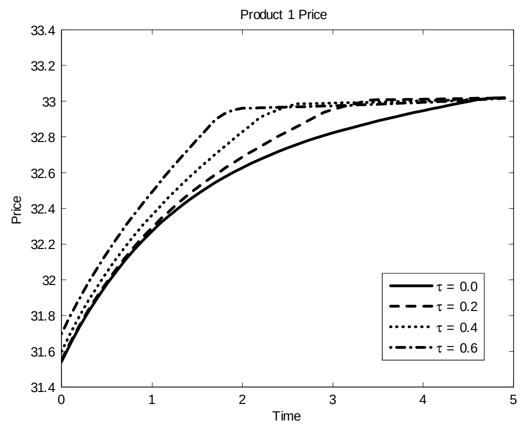

Figure 6.1. Price trajectories of product 1

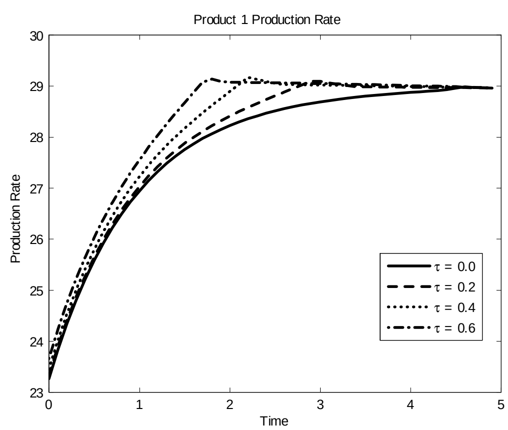

Figure 6.2. Production rate trajectories of product 1

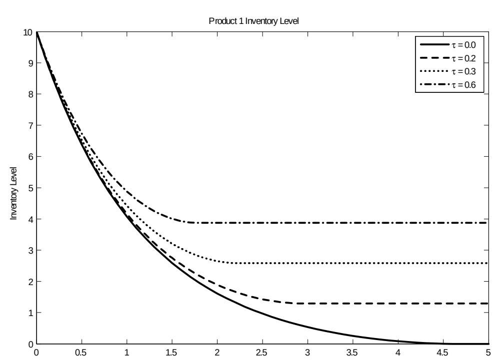

Figure 6.3. Inventory level trajectories of product 1

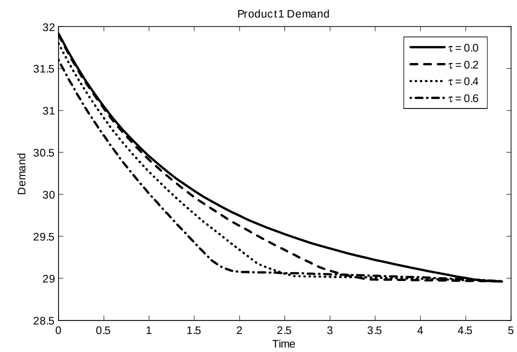

Figure 6.4. Demand trajectories of product 1

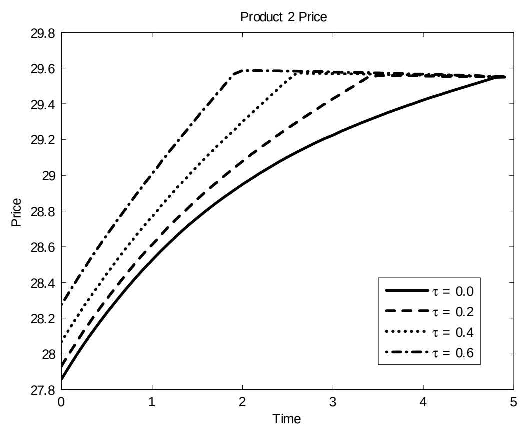

Figure 6.5. Price trajectories of product 2

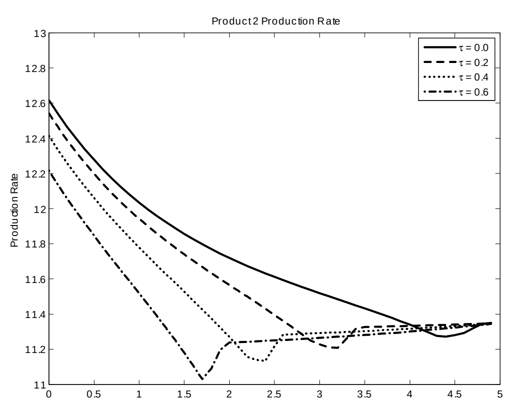

Figure 6.6. Production rate trajectories of product 2

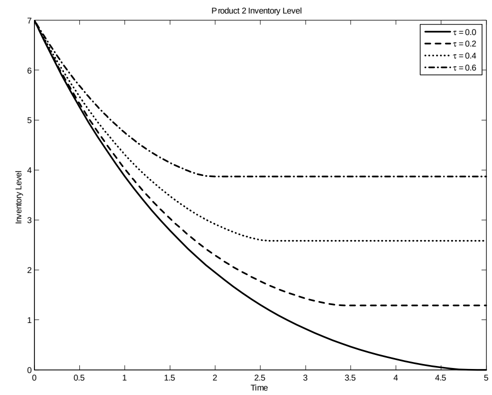

Figure 6.7. Inventory level trajectories of product 2

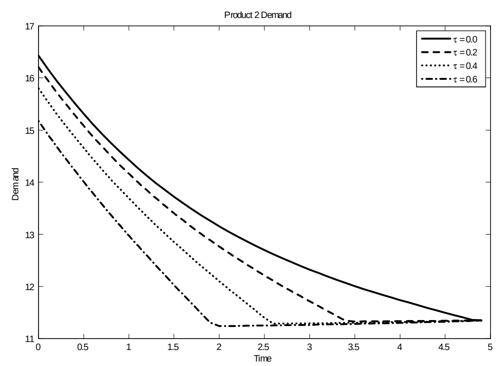

Figure 6.8. Demand trajectories of product 2

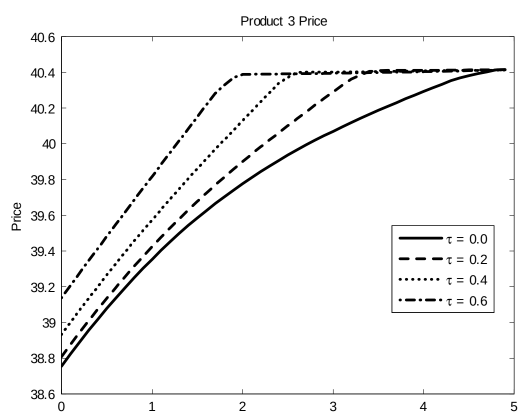

Figure 6.9. Price trajectories of product 3

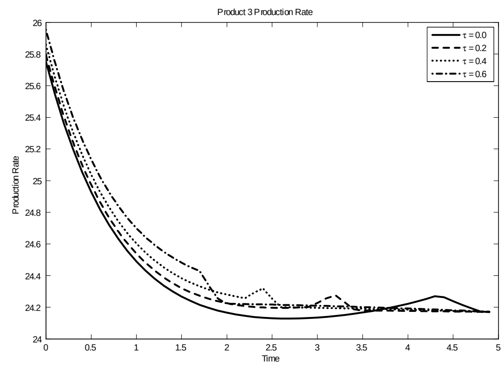

Figure 6.10. Production rate trajectories of product 3

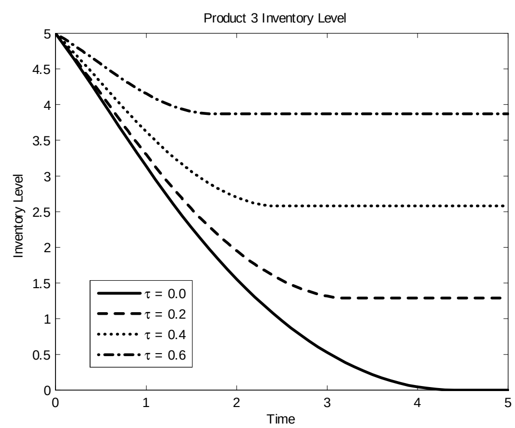

Figure 6.11. Inventory level trajectories of product 3

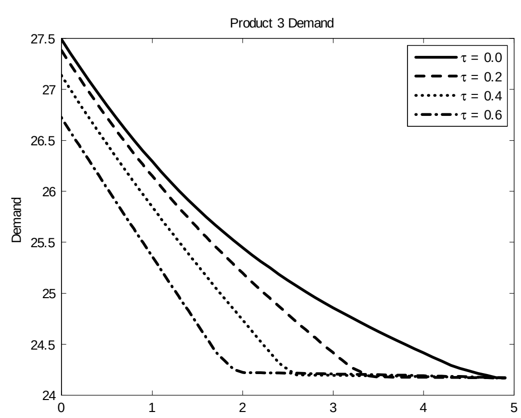

Figure 6.12. Demand trajectories of product 3

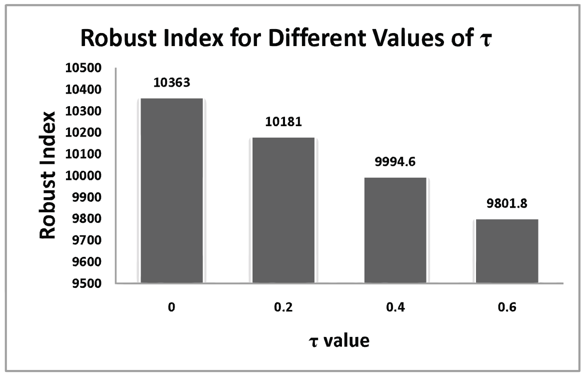

Figure 6.13. Robust index values

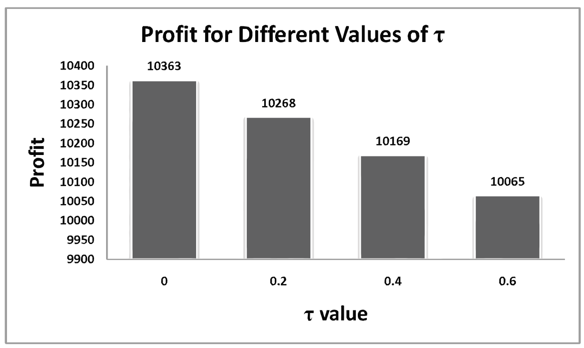

Figure 6.14. Profit values

spectively. We observe that the profit decreases and the size of the uncertainty effect increases when τ increases.

We previously discussed the similarities and differences between the robust formulation herein and that of Adida and Perakis (2006). It was noted that both approaches allow the decision maker to choose the level of uncertainty, though through different approaches; Adida and Perakis use the notion of a budget of uncertainty, while the magnitude of uncertainty τ and the uncertainty-parameter incidence matrix D are given herein. Although the set of uncertainty is defined in a different way and parameters for the numerical experiments are different, we can compare some qualitative properties of our solution trajectories with the results of Adida and Perakis (2006). Table 6.4 shows the observed changes in the solution trajectories for both models as the level of uncertainty is increased. While the production rate does not have uni-directional changes (marked by '?'), the

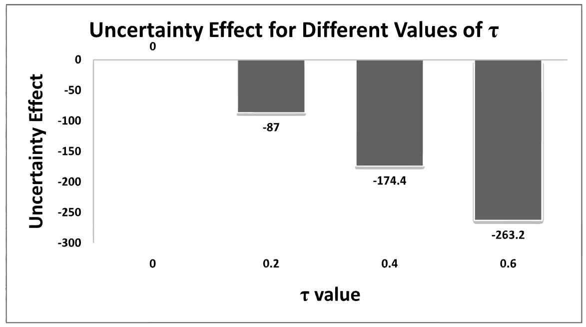

Figure 6.15. Uncertainty effect

Model	Price	Production Rate	Inventory Level	Demand	Profit
This Chapter	+	?	+	−	−
Adida and Perakis (2006)	+	?	+	−	−

Table 6.4. Comparison of changes of optimal solutions when the magnitude of uncertainty increases

changes in price, inventory level, demand and profit have uni-directional tendencies (marked by '+' or '-') for both models.

6.5 Concluding Remarks

This chapter has provided a locally robust formulation of optimal control problems with uncertainty and a robust dynamic pricing and inventory control model with bounded inventory level assumptions. From the uncertain continuous-time optimal control problem formulation of a dynamic pricing and inventory control problem, we reformulated it as a locally robust optimal control problem assuming demand uncertainty. This approach does not require any probabilistic distributions on uncertain parameters. We have discovered the following:

1. Nonlinear production cost functions and inventory holding cost functions can be studied in the framework of robust optimization developed in this chapter.

2. The robust dynamic pricing and inventory control model developed in this

chapter has a concave maximization objective, linear dynamics, and convex constraints.

3. Under mild assumptions, the discrete-time approximation to the model has a nonlinear concave maximization objective and linear constraints.

4. The robust model minimizes the magnitude of the uncertainty effect to the cumulative contribution to the profit of the firm throughout the planning horizon.

5. To avoid infeasibility of the model, the magnitude of uncertainty must be smaller than the maximum allowable magnitude of uncertainty defined in this chapter.

6. To accommodate large demand uncertainty, a firm needs more initial stocks in inventory and/or a shorter planning horizon.

To solve the problem, we have given a discrete-time approximation of the continuous-time model for a 'discretize-then-optimize' solution approach. For a numerical example, we have solved a problem involving 3 different products using 4 different resources.

Chapter 7

Conclusion

From optimal control problems to differential variational inequalities, we have shown how continuous-time mathematical programming theories may be used to solve various pricing issues. By combining the theories with proper numerical algorithms, we have provided a mathematically rigorous framework for solving dynamic pricing problems for competition and/or uncertainty. Pricing problems for infrastructure, services and traditional products are formulated in a function space, and solved with the corresponding numerical algorithm. In this chapter, we conclude this dissertation by summarizing the key contributions and suggest future research topics.

In this dissertation, we have:

- provided a robust optimal control problem formulation, permitting general nonlinear functions of state, control and uncertain parameters;
- developed a numerical algorithm effective for solving differential variational inequalities using gap functions;
- demonstrated the analogy between dynamic and static efficient toll problems;
- formulated the dynamic optimal toll problem as a continuous-time mathematical program with equilibrium constraints;
- solved the dynamic optimal toll problem using a heuristic algorithm;

- provided a differential variational inequality formulation that describes a differential Nash game between non-cooperative service providers;
- shown that the Kalman filter may be used for the demand learning process in a game setting;
- formulated a robust dynamic pricing and inventory control problem;
- analyzed the dynamic pricing and inventory control problem to obtain quantitative and qualitative insights; and
- shown that, for most cost functions, the robust problem can be solved efficiently with optimization software.

In the future, an extension of the theories, formulations and algorithms developed in this dissertation are recommended to accommodate more complicated problems. Potential research topics that would build upon the theories in this dissertation include:

- a proof that gap functions for differential variational inequalities provide error bounds. In static variational inequality problems, gap functions may be used in a special way. For stochastic variational inequality problems, we may formulate an optimization problem that minimizes the expected residual. This method is often called the expected residual minimization (ERM). A similar method for differential variational inequalities may be investigated in the future.
- stochastic dynamic optimal toll problems. We are interested in how optimal toll prices change in uncertain environments. The gap functions as an error bound for differential variational inequalities that may impact this research topic.
- comparison of the performance of algorithms in large-scale problems. There exists several algorithms for solving differential variational inequalities: the fixed point algorithm, the descent method based on gap functions, the time-stepping method, algorithms with complementarity problem reformulations and other finite-dimensional approximations. However, no studies compare

the performance of these algorithms. In addition, in large-scale problems, comparing performance would help to increase the understanding.

- dynamic pricing games with robust players. In a competition under uncertainty, robust optimization techniques may be used to describe the characteristics of Nash players and the uncertainty induced by incomplete and imperfect information. We are interested in how uncertainty factors change the dynamic pricing games of services and manufactured goods.

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Vita

Changhyun Kwon is a Ph.D. candidate in the Department of Industrial and Manufacturing Engineering at the Pennsylvania State University, where he received his M.S. degree in Industrial Engineering and Opeartions Research in 2005. He received his B.S. degree in Mechanical Engineering from the Korea Advanced Institute of Science and Technology, Daejeon, Republic of Korea in 2000. His research interests are dynamic pricing, production and distribution, congestion pricing and various equilibrium problems. He is a member of INFORMS, IIE, and SIAM.