The Pennsylvania State University
The Graduate School
Department of Mathematics

A NEW APPROACH TO BIVARIANT K-THEORY

A Thesis in
Mathematics
by
Constantin Dorin Dumitraşcu

© 2001 Constantin Dorin Dumitraşcu

Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Doctor of Philosophy

August 2001

We approve the thesis of Constantin Dorin Dumitraşcu.

Date of Signature

Nigel Higson
Distinguished Professor of Mathematics
Thesis Adviser
Chair of Committee

Paul Frank Baum
Evan Pugh Professor of Mathematics

John Roe
Professor of Mathematics

Shyamoli Chaudhuri Plassmann
Assistant Professor of Physics

Gary L. Mullen
Professor of Mathematics
Head of the Department of Mathematics

Abstract

We construct a new bivariant theory, that we call *KE-theory*, which is intermediate between the KK-theory of Gennadi Kasparov, and the E-theory of Alain Connes and Nigel Higson. It has an associative product, and there are natural transformations $KK^G \to KE^G$ and $KE^G \to E_G$ which preserve the product structures of the three theories. We obtain in this way an explicit description of the map between KK-theory and E-theory, abstractly known to exist due to their characterization using category theory. In an effort to further elucidate the relationship with the other two bivariant theories, we study some of the functoriality properties of the KE-theory groups and of the product. The thesis concludes with an example: we show that the new theory recovers ordinary K-theory. All the C^*-algebras that we consider are separable, graded, and admit an action of a locally compact σ-compact Hausdorff group.

The thesis adviser was Prof. Nigel Higson.

Contents

Acknowledgments

At Penn State University, I am indebted for everything that I have accomplished as a mathematician to my adviser Prof. Nigel Higson. I want to thank him for suggesting the subject of this thesis, and for many discussions, constructive criticism, and corrections related to the material within it. I also want to thank him for being such an enthusiastic teacher, and for his generous financial support over six summers at PSU. (Some money came from grants shared with P. Baum and J. Roe, and I want to thank them too.)

My appreciation goes to Prof. Paul Baum and Prof. John Roe for brightening so many days with their talks and courses. I want to thank Prof. Shyamoli Chaudhuri for kindly agreeing to be a member of my thesis committee. I also benefited from discussions with Prof. Victor Nistor. Last but not least, the Geometric Functional Analysis Seminar provided an excellent mathematical environment.

My friends Radu Popescu, Gabriel Prăjitură, and Heath Emerson kept my interest in math alive in moments of doubt. I cannot thank Radu enough for reading drafts of my thesis, and making suggestions that resulted in a more readable document. For everything else that I did not mention here, I know, they know, and God knows.

Mark Holowach proof-read the thesis and made pertinent style related observations. All the remaining inadvertencies, both mathematical and of language, are my very own responsibility.

Finally I am grateful to my son Radu Daniel, for the miracle that he really is, to my parents, for the trust they showed in me, and to my high-school math teacher, Liliana Niculescu.

Introduction

In the 1960's, the algebraic topology of manifolds produced one of the profound theorems of XX$^{\text{th}}$ century mathematics: the Index Theorem of Atiyah and Singer ([**AtSi63**], [**Pals**]). The conceptual proof given in [**AtSiI**] is based on the cohomology theory invented by M. Atiyah and F. Hirzebruch [**AtHr**] — namely K-theory. Using hints coming from various generalizations of the index theorem, Atiyah [**Atiy69**] also proposed a way of defining cycles of the *dual* theory — namely K-homology. The only thing left open by Atiyah was the definition of the equivalence relation that would make these cycles into a group. This issue was resolved by G. G. Kasparov [**Kas75**]. He succeeded in creating (see also [**Kas81**]) a *bivariant* theory — named KK-theory — which associates to any two C^*-algebras A and B a group $KK(A, B)$. His theory generalizes both K-theory for compact manifolds (obtained when $A = \mathbb{C}$, and $B =$ the continuous functions on the manifold) and K-homology (obtained when $A =$ the continuous functions on the manifold, and $B = \mathbb{C}$). For a very well written account of the origins of KK-theory see the introductory sections of [**Hg87a**] and [**Hg90a**].

Besides a wealth of functorial properties, the key feature of KK-theory is the existence for any separable C^*-algebras A, B, and D of an associative product map $KK(A, D) \otimes KK(D, B) \longrightarrow KK(A, B)$. Following an approach indicated by J. Cuntz ([**Cu83**], [**Cu84**]), N. Higson [**Hg87a**] gave the following description of KK-theory: it is the universal category with homotopy invariance, stability, and split-exactness. This category has separable C^*-algebras as objects, elements of KK-groups as morphisms, and the Kasparov product as the composition of morphisms.

In a subsequent paper [**Hg90b**] Higson described the universal category with homotopy invariance, stability, and *exactness*. The resulting new theory — named E-theory — has become important in C^*-algebra theory after A. Connes and N. Higson [**CoHg90**] described it concretely in terms of asymptotic morphisms. (An asymptotic morphism between two C^*-algebras is a family of maps between the two, indexed by $[1, \infty)$, which satisfies the conditions of a $*$-homomorphism in the limit at ∞.) The description of KK-theory and E-theory using category theory implies in a rather abstract and algebraic way the existence of a map $KK(A, B) \to E(A, B)$, for any two C^*-algebras A and B. This map is an isomorphism when A is nuclear [**Sk88**], and in general no explicit description at the level of bounded Kasparov modules has been given in the literature. A similar description of *equivariant* KK-theory [**Kas88**] and E-theory appears in [**Thms98**].

Equivariant KK-theory and E-theory have become essential tools in C^*-algebra theory because of their use in solving topological/geometrical problems, notably cases of the Novikov conjecture [**Kas88**], [**Rsn84**], and the Baum-Connes conjecture [**BC82**], [**BCH**].

In this thesis a new theory has been constructed, called *KE-theory*, which is intermediate between KK-theory and E-theory. It applies to C^*-algebras that are separable, graded, and admit an action of a locally compact, σ-compact Hausdorff group. For such a group G, and for any two such G-C^*-algebras A and B, the resulting abelian group is denoted by $KE^G(A, B)$. These groups satisfy some of the good functorial properties of the other two bivariant theories, and there exists an associative product $KE^G(A, D) \otimes KE^G(D, B) \to KE^G(A, B)$. We have also proved the existence of two natural transformations, $\Theta : KK^G(A, B) \to KE^G(A, B)$ and $\Xi : KE^G(A, B) \to E_G(A, B)$, which preserve the product structures. Their composition $\Xi \circ \Theta$ provides an *explicit* construction of the map $KK \to E$, abstractly known to exist because of the universality properties of the two theories (as we mentioned above). In the final part of the thesis we show that the new theory recovers ordinary K-theory of ungraded C^*-algebras. The idea of constructing a theory intermediate between KK-theory and E-theory was suggested some time ago by V. Lafforgue (private communication to N. Higson).

Intermediate theories between KK-theory and E-theory appear also in the work of J. Cuntz [**Cu97**], [**Cu98**]. Our construction is different in initial motivation, concrete realization, and final goal: we wanted to produce a solid framework for another proof to the Baum-Connes conjecture for a-T-menable groups [**HgKas97**], [**HgKas01**]. Details for this application will be given elsewhere.

The focus of **Chapter 1** is to briefly review the essential definitions, theorems and constructions related to KK-theory and E-theory. We also use it to set up notation. A short overview of KK-theory and E-theory, including their product is given in Section 1.4 and Section 1.5, respectively. We tried to keep this chapter as short as possible. This explains the abundance of references.

The standing assumptions for the entire thesis are: all C^*-algebras are *separable* and *graded*; all $*$-homomorphisms preserve the grading; all groups are locally compact, σ-compact, Hausdorff, and act continuously on C^*-algebras.

Chapter 2 constructs the new KE-theory. It contains its definition, the existence of the product between the KE-theory groups, and various functoriality properties of the groups and of the product.

In Section 2.1 we introduce and study the 'cycles' of the new theory, which we call *asymptotic Kasparov modules*. They are appropriate families of pairs, indexed by $[1, \infty)$. Each pair consists of a Hilbert module and an operator on it, that are put together in a field satisfying conditions that resemble those appearing in KK-theory. An example of such cycle, motivated by the K-homology class of the Dirac operator on a spin manifold, consists of a C^*-algebra A, a Hilbert space $\mathcal{H}$ (constant family), a $*$–homomorphism $\varphi : A \to \mathcal{B}(\mathcal{H})$, and a family $\{F_t\}_{t\in[1,\infty)}$ of bounded linear operators on $\mathcal{H}$ satisfying:

(aKm1) $F_t = F_t^*$, for all t;

(aKm2) $\| [F_t, \varphi(a)] \| \xrightarrow{t\to\infty} 0$, for all $a \in A$;

(aKm3) $\varphi(a)\,(F_t^2 - 1)\,\varphi(a)^* \geq 0$, modulo compact operators and operators which converge in norm to zero.

Such a family $\{(\mathcal{H}, F_t)\}_{t\in[1,\infty)}$ is an asymptotic Kasparov (A,$\mathbb{C}$)-module. Axiom (aKm2) encodes the pseudo-locality of first order elliptic differential operators, and axiom (aKm3) is supposed to encode the Fredholm property of elliptic operators on smooth manifolds. The definition can be also adapted to include a group action.

In Section 2.2 we define, for a locally compact group G and two graded separable G-C^*-algebras A and B, the group $KE^G(A,B)$ of homotopy equivalence classes of asymptotic Kasparov G-(A,B)-modules. Various functoriality properties of these groups are proved.

In Section 2.3 the product in KE-theory is constructed using the notions of 'two-dimensional' connection and quasi-central approximate unit. Let G be a locally compact group, and A_1, A_2, B_1, B_2, D be G-C^*-algebras. As in KK-theory, in its most general form, the product is a map

$$KE^G(A_1, B_1 \otimes D) \otimes KE^G(D \otimes A_2, B_2) \to KE^G(A_1 \otimes A_2, B_1 \otimes B_2), \quad (x,y) \mapsto x \sharp_D y\,.$$

Insight about the product in the new theory can be obtained by looking at the particular case when $B_1 = B_2 = D = \mathbb{C}$, which corresponds to the external product in K-homology. Consider two asymptotic Kasparov modules as described above: $\{(\mathcal{H}_1, F_{1,t})\}_t \in KE(A_1, \mathbb{C})$, and $\{(\mathcal{H}_2, F_{2,t})\}_t \in KE(A_2, \mathbb{C})$. Their product is $\{(\mathcal{H}_1 \otimes \mathcal{H}_2, F_{1,t} \otimes 1 + 1 \otimes F_{2,t})\}_t \in KE(A_1 \otimes A_2, \mathbb{C})$. The reader familiar with KK-theory will notice that no Kasparov Technical Theorem was used in our construction. The general case is more involved, but we hope that it is still simpler than in KK-theory. Our method is summarized in Overview 2.3.7.

Section 2.4 analyzes the algebra behind the product. We show that the product is associative and its various compatibilities with the functoriality of KE-groups are worked out. The stability of KE-theory is an easy consequence of the corresponding property of KK-theory.

Section 2.5 plays the role of an appendix of this chapter. It contains the proof of Theorem 2.3.9 used to construct the product.

Chapter 3 is a technical presentation that constructs the maps $\Theta : KK^G \to KE^G$ and $\Xi : KE^G \to E_G$. Both maps are proved to be functors, *i.e.* to preserve the product structures. The composition $\Xi \circ \Theta$ gives the explicit map between KK-theory and E-theory. The main results are Theorems 3.2.8 and 3.4.4.

Chapter 4 contains more observations related to the composition $\Xi \circ \Theta$, and proves that the KE-theory groups recover ordinary K-theory for trivially graded C^*-algebras, *i.e.* $KE(\mathbb{C}, A) = KK(\mathbb{C}, A) = K_0(A)$, for any C^*-algebra A.

Sections are numbered in each chapter in order of appearance. All results (theorems, propositions, corollaries, lemmas, important remarks and examples) are numbered in each section in order of appearance. This convention dictates that a reference of the type $m.n.p$ sends to the result p from section n of chapter m. All equations and diagrams are numbered after the chapter. The end of a proof is marked by $\blacksquare$. The notation used throughout this thesis is summarized at the beginning of the Index.

CHAPTER 1

Preliminaries: review and notation

The focus of this chapter is to briefly review the essential definitions, theorems and constructions related to KK-theory and E-theory. We also use it to set up notation. Among the covered topics we mention: tensor products of C^*-algebras, Hilbert modules and tensor products of Hilbert modules, group actions, approximate units, and Kasparov's Technical Theorem. A short overview of KK-theory and E-theory, including their product is given in Section 1.4 and Section 1.5, respectively.

The standing assumption for the entire thesis is: we shall work in the category $\mathbf{C^*}$**-alg**, whose objects are the *separable* and $(\mathbb{Z}_2$-)*graded* C^*-algebras, and whose morphisms are $*$-homomorphisms that preserve the grading.

1.1. C^*-algebras, tensor products and Hilbert modules

Given a separable graded C^*-algebra A, the *commutator* of two elements $a, b \in A$ is: $[a, b] = ab - (-1)^{\partial a\, \partial b} ba$. The C^*-algebra of complex numbers, $\mathbb{C}$, is trivially graded. As a general rule, given a locally compact space X, $C_0(X)$, the complex valued continuous functions on X vanishing at infinity, will be trivially graded. The major exception to this rule is $C_0(\mathbb{R})$, which when used in some constructions related to E-theory is graded by even and odd functions. We shall use $\mathcal{S}$ to designate this grading on $C_0(\mathbb{R})$.

All the tensor products are graded. The minimal C^*-algebra tensor product is denoted by $\otimes$, the maximal one by $\otimes_{max}$. For two C^*-algebras A_1 and A_2, there is a transposition isomorphism $A_1 \otimes A_2 \simeq A_2 \otimes A_1$, given on elementary tensors by $a_1 \otimes a_2 \mapsto (-1)^{\partial a_1\, \partial a_2} a_2 \otimes a_1$. We recall also two of the identities that hold true with graded tensor products: $(a_1 \otimes a_2)(b_1 \otimes b_2) = (-1)^{\partial a_2\, \partial b_1} a_1 b_1 \otimes a_2 b_2$, and $(a_1 \otimes a_2)^* = (-1)^{\partial a_1\, \partial a_2} a_1^* \otimes a_2^*$, for all $a_1, b_1 \in A_1$, and $a_2, b_2 \in A_2$.

Let $L = [1, \infty)$, and $LL = [1, \infty) \times [1, \infty)$. For any C^*-algebra B, let $BL = C_0(L, B)$, $BLL = C_0(LL, B)$, and $B[0, 1] = C([0, 1], B)$.

Given a Hilbert B-module $\mathcal{E}$, the C^*-algebra of *adjointable operators on* $\mathcal{E}$ (see [**Kas81**], [**Kas80**], [**Lan**]) is denoted by $\mathcal{B}(\mathcal{E})$. The closed ideal of *compact operators on* $\mathcal{E}$ is denoted by $\mathcal{K}(\mathcal{E})$. It is generated by the rank-one operators $\theta_{\xi,\eta}(\zeta) = \xi\langle\eta, \zeta\rangle$, for $\xi, \eta, \zeta \in \mathcal{E}$.

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be graded Hilbert modules over B_1 and B_2, respectively. The completion $\mathcal{E}_1 \otimes \mathcal{E}_2$ of the algebraic tensor product $\mathcal{E}_1 \odot \mathcal{E}_2$ with respect to the $B_1 \otimes B_2$-valued inner product $\langle\xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2\rangle = (-1)^{\partial\xi_2(\partial\xi_1+\partial\eta_1)}\langle\xi_1, \eta_1\rangle \otimes \langle\xi_2, \eta_2\rangle$ is a Hilbert $B_1 \otimes B_2$-module, called the *external tensor product of* $\mathcal{E}_1$ *and* $\mathcal{E}_2$. If $\varphi : B_1 \to \mathcal{B}(\mathcal{E}_2)$ is a $*$–homomorphism, we can also construct the *internal tensor product* $\mathcal{E}_1 \otimes_{B_1} \mathcal{E}_2$ *of* $\mathcal{E}_1$ *and* $\mathcal{E}_2$. (The notation $\mathcal{E}_1 \otimes_\varphi \mathcal{E}_2$ will also be used.) It is the Hilbert B_2-module obtained as completion of the algebraic tensor product $\mathcal{E}_1 \odot_{B_1} \mathcal{E}_2$ with respect to the B_2-valued inner product $\langle\xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2\rangle = \langle\xi_2, \varphi(\langle\xi_1, \eta_1\rangle)(\eta_2)\rangle$. In both cases the grading is $\partial(\xi_1 \otimes \xi_2) = \partial\xi_1 + \partial\xi_2$. It is not here the place to enter in details, but rather refer the reader to [**Kas80**], [**Lan**], or [**Blck**, Secs.13,14].

Given two Hilbert modules $\mathcal{E}_1$ and $\mathcal{E}_2$, there is an embedding $\mathcal{B}(\mathcal{E}_1) \otimes \mathcal{B}(\mathcal{E}_2) \to \mathcal{B}(\mathcal{E}_1 \otimes \mathcal{E}_2)$, given by $(F_1 \otimes F_2)(\xi_1 \otimes \xi_2) = (-1)^{\partial\xi_1\,\partial F_2} F_1(\xi_1) \otimes F_2(\xi_2)$. Its restriction to compact operators gives an isomorphism $\mathcal{K}(\mathcal{E}_1) \otimes \mathcal{K}(\mathcal{E}_2) \simeq \mathcal{K}(\mathcal{E}_1 \otimes \mathcal{E}_2)$. In the case of internal tensor product of Hilbert modules, we only have a natural graded $*$–homomorphism $\mathcal{B}(\mathcal{E}_1) \to \mathcal{B}(\mathcal{E}_1 \otimes_{B_1} \mathcal{E}_2)$, $F \mapsto F \otimes_{B_1} 1$, $(F \otimes_{B_1} 1)((\xi_1 \otimes \xi_2) = F(\xi_1) \otimes_{B_1} \xi_2$.

Given a Hilbert B-module $\mathcal{E}$ and a space X, $\mathcal{E}(X)$ is the Hilbert $B(X)$-module $C_0(X) \otimes \mathcal{E}$ (external tensor product of Hilbert modules). We shall use the notation: $\mathcal{E}L = C_0(L) \otimes \mathcal{E} = \{\mathcal{E}\}_t =$ constant family indexed by $[1, \infty)$, $\mathcal{E}LL = C_0(LL) \otimes \mathcal{E} =$ constant family indexed by $[1, \infty) \times [1, \infty)$.

The *multiplier algebra* $\mathcal{M}(A)$ of a C^*-algebra A is the largest C^*-algebra in which A embeds as an essential ideal. We recall the following two facts: $\mathcal{M}(\mathcal{K}(\mathcal{E})) \simeq \mathcal{B}(\mathcal{E})$, for any Hilbert B-module $\mathcal{E}$, and $\mathcal{M}\big(C_0([1, \infty), \mathcal{K}(\mathcal{E}))\big) \simeq C_b\big([1, \infty), \mathcal{B}(\mathcal{E})\big)$, with $\mathcal{B}(\mathcal{E})$ having the strict topology.

At some point in the proof of the Technical Theorem (Section 2.5) we shall need the next result. Its justification is similar with the corresponding description for Hilbert space operators.

LEMMA 1.1.1. *Let A be a C^*-algebra, then $T \in \mathcal{B}(A^{(n)})$ is positive if and only if $\langle T\xi, \xi \rangle \geq 0$, for every $\xi \in A^{(n)}$. ($A^{(n)}$ is the direct sum of n-copies of A.)*

1.2. Group actions

As reference for this section see [**Kas88**, Sec.1]. Besides being separable and graded, the C^*-algebras that we consider have an additional structure: the action of a group by automorphisms. A standing assumption for the entire thesis is the following: *all groups are supposed to be locally compact, σ-compact and Hausdorff.* Given such a group G and a C^*-algebra A, an *action of G on A* is a group homomorphism $G \to \mathrm{Aut}(A)$, where $\mathrm{Aut}(A)$ is the group of automorphisms of A, with no topology on it. An element $a \in A$ is called *G-continuous* if the map $G \to A$, $g \mapsto g(a)$ is continuous. We denote by **G-C*-alg**the category with objects the separable graded C^*-algebras equipped with G-action compatible with the grading and having all the elements G-continuous, and with morphisms the equivariant $*$-homomorphisms. The action of any group G on $\mathbb{C}$ is trivial.

DEFINITION 1.2.1. Given a group G, a G-C^*-algebra B, and a Hilbert B-module $\mathcal{E}$, an *action of G on $\mathcal{E}$*, or a *G-action*, is an action of G by grading preserving linear automorphisms such that: (i) $G \times \mathcal{E} \to \mathcal{E}$, $(g, \xi) \mapsto g(\xi)$, is continuous in the norm topology of $\mathcal{E}$; (ii) $g(\xi b) = g(\xi)g(b)$; and (iii) $\langle g(\xi), g(\eta) \rangle = g(\langle \xi, \eta \rangle)$, for all $\xi, \eta \in \mathcal{E}, b \in B, g \in G$.

Given an action of G on $\mathcal{E}$, there is an induced action of G on $\mathcal{B}(\mathcal{E})$ as follows: $g(T)(\xi) = g(T(g^{-1}\xi))$, for all $g \in G$, $T \in \mathcal{B}(\mathcal{E})$, and $\xi \in \mathcal{E}$. Let $\mathcal{E}_1$ be a Hilbert D-module, with a G-action, and $\mathcal{E}_2$ be a G-(D, B)-module. The action of G on the internal tensor product $\mathcal{E}_1 \otimes_D \mathcal{E}_2$ is given by $g(\xi \otimes_D \eta) = g(\xi) \otimes_D g(\eta)$, for all $\xi \in \mathcal{E}_1$, $\eta \in \mathcal{E}_2$. This implies, for $T \in \mathcal{B}(\mathcal{E}_1)$, that $g(\, T \otimes_D 1\,) = g(T) \otimes_D 1$.

The *standard Hilbert G-space* is $\mathcal{H}_G = L^2(G) \oplus L^2(G) \oplus \dots$, with infinitely many summands, graded alternately even and odd, and equipped with the left regular representation of G. Let $\mathcal{K} = \mathcal{K}(\mathcal{H}_G)$ be the compact operators on $\mathcal{H}_G$. For any G-C^*-algebra B, the *standard Hilbert G-B-module* is $\mathcal{H}_B = l^2 \otimes L^2(G) \otimes (B \otimes B^{\mathrm{op}})$. Let $\mathcal{K}(\mathcal{H}_B)$ be the compact operators on $\mathcal{H}_B$.

1.3. Quasi-central approximate units, and Kasparov's Technical Theorem

Recall that a C^*-algebra is called *σ-unital* if it has a countable approximate unit.

Definition 1.3.1. Let G be a group. Consider an inclusion $I \subset B \subset A$, where A is a G-C^*-algebra, B is a σ-unital G-C^*-subalgebra of A, and I is a σ-unital G-ideal of A. A *quasi-invariant quasi-central approximate unit for I in B* (abreviated q.i.q.c.a.u.) is a continuous family $\{u_t\}_{t\in[1,\infty)}$ of positive, increasing, even elements of I satisfying:

(a.u.) $\| xu_t - x \| \xrightarrow{t\to\infty} 0$, for all $x \in I$;

(q.c.) $\| yu_t - u_t y \| \xrightarrow{t\to\infty} 0$, for all $y \in B$; and

(q.i.) $\| g(u_t) - u_t \| \xrightarrow{t\to\infty} 0$, uniformly on compact subsets of G.

Proposition 1.3.2. *Let G be a group. A quasi-invariant quasi-central approximate unit exists for any closed G-invariant ideal I of a G-C^*-algebra A.*

For a proof see [**Kas88**, Lemma 1.4], or [**GHT**, 5.3]. Without a group action, the existence of quasi-central approximate units is proved in [**Pdrs**, 3.12.14], or [**Arvs**]. Such approximate units exist for any $I \lhd A$, but in this thesis we need a *countable* approximate unit $\{u_n\}_n$ (which by interpolation gives the family $\{u_t\}_t$), and this justifies the presence of the separable subalgebra B. It is usually clear from the context what B is (the biggest subalgebra that one needs in each particular application!), and we shall usually omit to mention it.

The following result in pure C^*-algebra theory is due to Kasaprov. His initial proof [**Kas81**, Sec.3] was complicated. N. Higson [**Hg87b**] gave an elegant proof in the non-equivariant case based on the notion of quasi-central approximate unit. The statement that follows is Theorem 1.5 of [**Kas88**]. We shall often abbreviate the result as KTT.

KASPAROV'S TECHNICAL THEOREM. *Let J be a σ-unital G-C^*-algebra. Assume that E_1 and E_2 are subalgebras of $\mathcal{M}(J)$, E_1 with G-action and having all the elements G-continuous, such that:*

(i) E_1, E_2 *are σ-unital,*

(ii) $E_1\, E_2 \subset J$.

Assume also that Δ is a subset of $\mathcal{M}(J)$, separable in the norm topology, consisting of G-continuous elements, and satisfying:

(iii) $[\Delta, E_1] \subset E_1$.

Further assume that $\phi : G \to \mathcal{M}(J)$ is a bounded function, such that:

(iv) $E_1\, \phi(G) \subset J$, $\phi(G)\, E_1 \subset J$,

(v) $g \mapsto a\phi(g)$, $g \mapsto \phi(g)a$ *are norm continuous on G, for any $a \in E_1 + J$.*

Then there exist G-continuous positive even elements $M, N \in \mathcal{M}(J)$ with the properties:

(1) $M + N = 1$;

(2) $M\, E_1 \subset J$, $N\, E_2 \subset J$;

(3) $[M, \Delta] \subset J$;

(4) $\big(g(M) - M\big) \in J$, *for all $g \in G$;*

(5) $N\, \phi(G) \subset J$, $\phi(G)\, N \subset J$; *and*

(6) $g \mapsto N\phi(g)$, $g \mapsto \phi(g)N$ *are norm continuous on G.*

The following observation of N. Higson in relation to KTT will be very useful when working with families of operators indexed by L. The point is contained in property ($2^\star$).

THEOREM 1.3.3. *With the same hypothesis as in Kasparov's Technical Theorem, there exist G-continuous positive even elements $\widehat{M} = \{M_t\}_{t\in[1,\infty)}$, $\widehat{N} = \{N_t\}_{t\in[1,\infty)} \in C_b(L, \mathcal{M}(J))$ with the properties:*

($1'$) $M_t + N_t = 1$, *for all t; $t \mapsto M_t$ is norm continuous;*

($2'$) $M_t\, E_1 \subset J$, $N_t\, E_2 \subset J$, *for all t;*

($2^\star$) $\|N_t\, T\| \xrightarrow{t\to\infty} 0$, *for all $T \in E_2$;*

(3′) $[M_t, \Delta] \subset J$, *for all* t, $\|[M_t, T]\| \xrightarrow{t\to\infty} 0$, *for all* $T \in \Delta$;

(4′) $\big(g(M_t) - M_t\big) \in J$, *for all* t, *and* $\|g(M_t) - M_t\| \xrightarrow{t\to\infty} 0$, *for all* $g \in G$; *and*

(5′) $N_t\,\phi(G) \subset J$, $\phi(G)\,N_t \subset J$, *for all* t, $\|N_t\,\phi(g)\|$, $\|\phi(g)\,N_t\| \xrightarrow{t\to\infty} 0$, *for all* $g \in G$.

Proof. Apply KTT to: $\widehat{J} = C_0(\mathbb{N}, J)$, $\widehat{E}_1 = C_0(\mathbb{N}, E_1)$ $\widehat{E}_2 = 1 \otimes E_2$ (here $1 \otimes E_2$ represents constant functions on $\mathbb{N}$ with values elements of E_2), $\widehat{\Delta} = 1 \otimes \Delta$, and $\widehat{\varphi} = 1 \otimes \varphi$. Interpolate the families $\{M_n\}_n$ and $\{N_n\}_n$ so obtained to get the desired families indexed by $t \in [1, \infty)$. ■

1.4. KK-theory

The KK-theory groups were introduced and studied in [**Kas75**], [**Kas81**], the equivariant ones under the action of a group in [**Kas95**], [**Kas88**], and under the action of a groupoid in [**leG99**]. We follow the equivariant presentation of Kasparov [**Kas88**].

Definition 1.4.1. Consider a group G, and two graded separable G-C^*-algebras A and B. A *Kasparov* G-(A, B)-*module* is a pair $(\mathcal{E}, F)$, where $\mathcal{E}$ is a Hilbert B-module, admitting a G-action and an action of A via a $*$-homomorphism $\varphi : A \to \mathcal{B}(\mathcal{E})$, and $F \in \mathcal{B}(\mathcal{E})$ is an odd G-continuous operator such that for every $a \in A$ and $g \in G$

$$(F - F^*)\varphi(a),\ [F, \varphi(a)],\ (F^2 - 1)\varphi(a),\ \text{and}\ (g(F) - F)\varphi(a)\ \text{belong all to}\ \mathcal{K}(\mathcal{E}). \tag{1.1}$$

The set of all Kasparov G-(A, B)-modules will be denoted by $kk^G(A, B)$. An Kasparov G-(A, B)-module $(\mathcal{E}, F)$ is said to be *degenerate* if for all $a \in A$ and $g \in G$: $(F - F^*)\varphi(a) = 0$, $[F, \varphi(a)] = 0$, $(F^2 - 1)\varphi(a) = 0$, and $(g(F) - F)\varphi(a) = 0$. Whenever there is no risk of confusion, we shall write a instead of $\varphi(a)$.

Definition 1.4.2. An element $(\mathcal{E}, F)$ of $kk^G(A, B[0, 1])$ gives by 'evaluation at s' a family $\{\,(\mathcal{E}_s, F_s) \in kk^G(A, B) \mid s \in [0, 1]\,\}$, with $\mathcal{E}_s = \mathcal{E} \otimes_{\mathrm{ev}_s} B$, $F_s = F \otimes_{\mathrm{ev}_s} 1$. Such an element $(\mathcal{E}, F)$ and the family that it generates are called a *homotopy* between $(\mathcal{E}_0, F_0)$ and $(\mathcal{E}_1, F_1)$.

DEFINITION 1.4.3. The set $KK^G(A,B)$ is defined as the quotient of $kk^G(A,B)$ by the equivalence relation given by homotopy. Given an element $x = (\mathcal{E}, F) \in kk^G(A,B)$, its class in $KK^G(A,B)$ will be denoted by $\boldsymbol{x}$. The *addition* of two Kasparov G-(A,B)-modules is given by the obvious notion of direct sum.

Under the above defined addition $KK^G(A,B)$ becomes an abelian group. The following elements play an important role in the theory: $1 = 1_{\mathbb{C}} \in KK^G(\mathbb{C},\mathbb{C})$, the class of the Kasparov module $(\mathbb{C}, 0)$, and $1_A \in KK^G(A,A)$, the class of the Kasparov module $(A, 0)$. Given A, B, and D, there is a map $\sigma_D : KK^G(A,B) \to KK^G(A \otimes D, B \otimes D)$, $(\mathcal{E}, F) \mapsto (\mathcal{E} \otimes D, F \otimes 1)$.

DEFINITION 1.4.4. ([**CoSk**, Def.A.1], [**Sk84**, Def.8]) Assume that the following elements are given: a Hilbert D-module $\mathcal{E}_1$, a Hilbert (D,B)-module $\mathcal{E}_2$, and $F_2 \in \mathcal{B}(\mathcal{E}_2)$. Let $\mathcal{E} = \mathcal{E}_1 \otimes_D \mathcal{E}_2$. An operator $\underline{F} \in \mathcal{B}(\mathcal{E})$ is called an F_2-*connection for* $\mathcal{E}_1$ if it has the same degree as F_2 and if it satisfies for every $\xi \in \mathcal{E}_1$:

$$\left(T_\xi \, F_2 - (-1)^{\partial\xi \cdot \partial F_2} \underline{F} \, T_\xi \right) \in \mathcal{K}(\mathcal{E}_2, \mathcal{E}), \text{ and } \left(F_2 \, T_\xi^* - (-1)^{\partial\xi \cdot \partial F_2} T_\xi \, \underline{F} \right) \in \mathcal{K}(\mathcal{E}, \mathcal{E}_2). \tag{1.2}$$

Here $T_\xi \in \mathcal{B}(\mathcal{E}_2, \mathcal{E})$ is defined by $T_\xi(\eta) = \xi \otimes_D \eta$, for $\eta \in \mathcal{E}_2$.

The properties of connections are listed in [**Sk84**, Prop.9]. Here is the one that interests us most (the equivariant part is contained in [**Kas88**, Lemma 2.6]):

LEMMA 1.4.5. *Consider the notation of the previous definition. If F_2 satisfies, for all $d \in D$, $[F_2, d] \in \mathcal{K}(\mathcal{E}_2)$, then an F_2-connection $\underline{F}$ exists for any countably generated $\mathcal{E}_1$. If $d\,F_2$ and $F_2\,d$ are G-continuous for any $d \in D$, then $\underline{F}\,(K \otimes_D 1)$ and $(K \otimes_D 1)\,\underline{F}$ are G-continuous, for any $K \in \mathcal{K}(\mathcal{E}_1)$.*

DEFINITION 1.4.6. ([**CoSk**, Thm.A.3], [**Sk84**, Def.10]) Let A, B, D be G-C^*-algebras, $x = (\mathcal{E}_1, F_1) \in kk^G(A,D)$, $y = (\mathcal{E}_2, F_2) \in kk^G(D,B)$, $\mathcal{E} = \mathcal{E}_1 \otimes_D \mathcal{E}_2$. Denote by $F_1 \sharp_D F_2$ the set of operators $F \in \mathcal{B}(\mathcal{E})$ satisfying:

(1) $(\mathcal{E}, F) \in kk^G(A,B)$;

(2) F is an F_2-connection for $\mathcal{E}_1$; and

(3) $a\,[F_1 \otimes_D 1, F]\,a^* \geq 0$, modulo $\mathcal{K}(\mathcal{E})$, for all $a \in A$.

For any $F \in F_1 \sharp_D F_2$, the pair $z = (\mathcal{E}, F)$ will be called *the product of x and y.* We shall use the notation $\boldsymbol{z} = \boldsymbol{x} \sharp_D \boldsymbol{y}$. The same notation $\sharp$ will be used to designate also the products in E-theory and in the new KE-theory. We hope that it will be clear from the context to what theory a certain product belongs to.

THEOREM 1.4.7. *Let G be a group, and A, B, and D be separable graded G-C^*-algebras. The product $\sharp_D$ exists, is unique up to homotopy, and defines a bilinear pairing:*

$$KK^G(A, D) \otimes KK^G(D, B) \xrightarrow{\sharp_D} KK^G(A, B),\ (\boldsymbol{x}, \boldsymbol{y}) \mapsto \boldsymbol{x} \sharp_D \boldsymbol{y}\,. \tag{1.3}$$

PROOF. ([**Kas88**, Thm.2.11], [**Sk84**, Thm.12]) As in the definition above, let $x = (\mathcal{E}_1, F_1) \in kk^G(A, D)$, $y = (\mathcal{E}_2, F_2) \in kk^G(D, B)$, $\mathcal{E} = \mathcal{E}_1 \otimes_D \mathcal{E}_2$. Let $\underline{F}$ be an F_2-connection for $\mathcal{E}_1$. Apply KTT for: $J = \mathcal{K}(\mathcal{E})$; $E_1 = \mathcal{K}(\mathcal{E}_1) \otimes_D 1 + \mathcal{K}(\mathcal{E})$; $E_2 =$ the algebra span of $[F_1 \otimes_D 1, \underline{F}]$, $(\underline{F} - \underline{F}^*)$, $(\underline{F}^2 - 1)$, and $[\underline{F}, A]$; $\Delta =$ the vector space span of $F_1 \otimes_D 1$, $\underline{F}$, and A; $\phi : G \to \mathcal{B}(\mathcal{E})$, $\phi(g) = g(\underline{F}) - \underline{F}$. With the elements M and N so obtained, define:

$$F = M^{\frac{1}{2}}(F_1 \otimes_D 1) + N^{\frac{1}{2}}\underline{F}. \tag{1.4}$$

Then F satisfies the conditions of Definition 1.4.6, and consequently the class $(\mathcal{E}, F)$ represents the product $\boldsymbol{x} \sharp_D \boldsymbol{y}$ of $\boldsymbol{x}$ and $\boldsymbol{y}$. ■

EXAMPLE 1.4.8 (External product in KK-theory). Let A_1, A_2, B_1, B_2 be G-C^*-algebras. The *external product* is the map:

$$KK^G(A_1, B_1) \otimes KK^G(A_2, B_2) \xrightarrow{\sharp_{\mathbb{C}}} KK^G(A_1 \otimes A_2, B_1 \otimes B_2). \tag{1.5}$$

Let $x = (\mathcal{E}_1, F_1) \in kk^G(A_1, B_1)$, $y = (\mathcal{E}_2, F_2) \in kk^G(A_2, B_2)$, $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$ (external product of Hilbert modules). We shall still apply Kasparov's Technical Theorem, as in the proof of the theorem above, but things are simpler because as F_2-connection we can choose $\underline{F} = 1 \otimes F_2$. Let: $J = \mathcal{K}(\mathcal{E}) = \mathcal{K}(\mathcal{E}_1) \otimes \mathcal{K}(\mathcal{E}_2)$; $E_1 = \mathcal{K}(\mathcal{E}_1) \otimes A_2 + J$; $E_2 = A_1 \otimes \mathcal{K}(\mathcal{E}_2)$;

$\Delta =$ the vector space span of $F_1 \otimes 1$, $1 \otimes F_2$, and $A_1 \otimes A_2$; and $\varphi \equiv 0$. With the elements given by KTT, define:

$$F = M^{\frac{1}{2}}(F_1 \otimes 1) + N^{\frac{1}{2}}(1 \otimes F_2). \tag{1.6}$$

Then F satisfies the conditions of Definition 1.4.6, and consequently the class $(\mathcal{E}, F)$ represents the product $\boldsymbol{x} \sharp_{\mathbb{C}} \boldsymbol{y}$ of $\boldsymbol{x}$ and $\boldsymbol{y}$.

1.5. E-theory

The E-theory groups were introduced and studied in [**CoHg**], [**CoHg90**], the equivariant ones under the action of a group in [**GHT**], [**Thms99**], and under the action of a groupoid in [**Pop**]. We shall use in the thesis the approach sketched in [**HgKas97**, Sec.2]. The theory is based on the notion of asymptotic family/asymptotic morphism.

Definition 1.5.1. Let G be a group, A and B be G-C^*-algebras. An *equivariant asymptotic family from A to B* is a family of functions $\{\varphi_t\}_{t\in[1,\infty)} : A \to B$ such that

- $t \mapsto \varphi_t(a)$ is bounded and norm continuous for all $a \in A$;
- $(g, a) \mapsto g(\varphi_t(a))$ is norm continuous in $g \in G$ and $a \in A$, uniformly in t; and

$$\lim_{t\to\infty} \left\{ \begin{array}{l} \varphi_t(a_1 + \alpha a_2) - \varphi_t(a_1) - \alpha\varphi_t(a_2) \\ \varphi_t(a_1 a_2) - \varphi_t(a_1)\varphi_t(a_2) \\ \varphi_t(a^*) - \varphi_t(a)^* \\ \varphi_t(g(a)) - g(\varphi_t(a)) \end{array} \right\} = 0, \text{ for all } a, a_1, a_2 \in A, \alpha \in \mathbb{C}, g \in G. \tag{1.7}$$

We shall use a broken arrow notation: $\{\varphi_t\} : A \dashrightarrow B$.

Two asymptotic families $\{\varphi_t\}_{t\in[1,\infty)}$, $\{\psi_t\}_{t\in[1,\infty)}$ from A to B are said to be *asymptotically equivalent* if $\lim_{t\to\infty}\big(\varphi_t(a) - \psi_t(a)\big) = 0$, for all $a \in A$. Asymptotic equivalence is an equivalence relation on the set of asymptotic families.

Definition 1.5.2. An *asymptotic morphism from A to B* is a $*$-homomorphism $\varphi : A \to C_b([1,\infty), B)/C_0([1,\infty), B)$.

LEMMA 1.5.3. *There is a bijective correspondence between asymptotic morphisms and equivalence classes of asymptotic families.*

A *homotopy* between two asymptotic families from A to B is an asymptotic family from A to $B[0,1]$. We denote by $[\![A, B]\!]$ the set of homotopy classes of asymptotic families from A to B, and by $[\![\varphi_t]\!]$ the homotopy class of an asymptotic family. Homotopy is an essential equivalence relation both for the definition of E-theory groups and for the construction of the composition of asymptotic families. Note that asymptotically equivalent asymptotic families are homotopic.

DEFINITION 1.5.4. ([**HgKas97**, Def.2.2]) We denote by $E_G(A, B)$ the set of all homotopy equivalence classes of asymptotic families from $\mathcal{S}A \otimes \mathcal{K}(\mathcal{H}_G) = \mathcal{S} \otimes A \otimes \mathcal{K}(\mathcal{H}_G)$ to $B \otimes \mathcal{K}(\mathcal{H}_G)$: $E_G(A, B) = [\![\, \mathcal{S}A \otimes \mathcal{K}(\mathcal{H}_G), B \otimes \mathcal{K}(\mathcal{H}_G)\,]\!]$.

A useful technique of constructing asymptotic morphisms is given in the following:

LEMMA 1.5.5. *Consider dense $*$-subalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ of the C^*-algebras A_1 and A_2, respectively. Let $\{\chi_t\}_{t\in[1,\infty)} : A_1 \dashrightarrow B$ and $\{\psi_t\}_{t\in[1,\infty)} : A_2 \dashrightarrow B$ be asymptotic families such that $\lim_{t\to\infty}\big[\chi_t(a_1), \psi_t(a_2)\big] = 0$, for all $a_1 \in \mathcal{A}_1$, $a_2 \in \mathcal{A}_2$. Then there exists an asymptotic family $\{\chi_t \otimes \psi_t\}_t : A_1 \otimes_{max} A_2 \dashrightarrow B$, such that (up to asymptotic equivalence) $(\chi_t \otimes \psi_t)(a_1 \otimes a_2) = \chi_t(a_1)\psi_t(a_2)$.*

The proof is nothing else but the universality property of the maximal tensor product $\otimes_{max}$. There is also a tensor product functor on the homotopy category of asymptotic families which associates to the asymptotic families $\{\chi_t\}_t : A_1 \dashrightarrow B_1$ and $\{\psi_t\}_t : A_2 \dashrightarrow B_2$ a tensor product $\{\chi_t \otimes \psi_t\}_t : A_1 \otimes_{max} A_2 \dashrightarrow B_1 \otimes_{max} B_2$.

For any C^*-algebra D, let $1_D : D \to D$ be the asymptotic family corresponding to identity. Given an asymptotic family $\{\varphi_t\}_t : \mathcal{S}A \otimes \mathcal{K} \dashrightarrow B \otimes \mathcal{K}$, apply the remark made in the previous paragraph to obtain an asymptotic family $\mathcal{S}(A \otimes_{max} D) \otimes \mathcal{K} \dashrightarrow (B \otimes_{max} D) \otimes \mathcal{K}$. This is the definition of the map $\sigma_D : E_G(A, B) \to E_G(A \otimes_{max} D, B \otimes_{max} D)$, $\sigma_D([\![\varphi_t]\!]) = [\![\varphi_t \otimes 1_D]\!]$.

Two $*$-homomorphisms will play an important role in our approach to E-theory. (To the best of our knowledge these were first introduced in [**HgKas97**, Sec.2]; see also [**HKT**, A.3,4], and [**BaaJlg**].) They are:

$$\Delta : \mathcal{S} \to \mathcal{S} \otimes \mathcal{S},\ f \mapsto f(X \otimes 1 + 1 \otimes X),\ \text{and}\ \epsilon : \mathcal{S} \to \mathbb{C}, f \mapsto f(0). \tag{1.8}$$

In the definition of Δ above (do not confuse with the vector space that appears in KTT!), X is the function $X(x) = x$ on $\mathbb{R}$, viewed as unbounded self-adjoint multiplier of $\mathcal{S}$. We then apply functional calculus ([**Co81**], [**BaaJlg**]) to the unbounded self-adjoint multiplier $(X \otimes 1 + 1 \otimes X)$ of $\mathcal{S} \otimes \mathcal{S}$. The following identities will be useful:

$$\Delta(e^{-x^2}) = e^{-x^2} \otimes e^{-x^2},\ \Delta(xe^{-x^2}) = xe^{-x^2} \otimes e^{-x^2} + e^{-x^2} \otimes xe^{-x^2}. \tag{1.9}$$

The pair of functions $\{e^{-x^2}, xe^{-x^2}\}$ is of interest in the context of E-theory because it represents a system of generators for $\mathcal{S}$, and because of the nice explicit formulas (1.9).

Definition 1.5.6. Given three G-C^*-algebras A, B, and D, the *product in E-theory* is the map:

$$E_G(A, D) \otimes E_G(D, B) \xrightarrow{\sharp_D} E_G(A, B),$$

$$\big(\{\varphi_t\}_t, \{\psi_t\}_t\big) \mapsto \text{ the composition } \mathcal{S}A \otimes \mathcal{K} \xrightarrow{\Delta} \mathcal{S}\mathcal{S}A \otimes \mathcal{K} \overset{1\otimes\varphi_t}{\dashrightarrow} \mathcal{S}D \otimes \mathcal{K} \overset{\psi_{h(t)}}{\dashrightarrow} B \otimes \mathcal{K}. \tag{1.10}$$

Note that a rescaling h of the parameter for $\{\psi_t\}_t$ is necessary in order to compose the two asymptotic families. We shall denote the product (1.10) by $[\![\varphi_t]\!] \sharp_D [\![\psi_t]\!]$.

Example 1.5.7 (External product in E-theory). The external product is the map:

$$E_G(A_1, B_1) \otimes E_G(A_2, B_2) \xrightarrow{\sharp_{\mathbb{C}}} E_G(A_1 \otimes_{max} A_2, B_1 \otimes_{max} B_2),$$

(1.11)

$$\big(\{\varphi_t\}_t, \{\psi_t\}_t\big) \mapsto \text{ the composition}$$

$$\big\{\mathcal{S}(A_1 \otimes_{max} A_2) \otimes \mathcal{K} \xrightarrow{\Delta} \mathcal{S}A_1 \otimes \mathcal{K} \otimes_{max} \mathcal{S}A_2 \otimes \mathcal{K} \overset{\varphi_t\otimes\psi_t}{\dashrightarrow} (B_1 \otimes_{max} B_2) \otimes \mathcal{K}\big\}_t.$$

No rescaling is necessary in this case.

CHAPTER 2

KE-theory: existence of the product and functorial properties

In this chapter we introduce the new bivariant theory. Its cycles are appropriate families of pairs, indexed by $[1,\infty)$. Each pair consists of a Hilbert module and an operator on it, and they are put together in a field satisfying conditions that resemble those appearing in KK-theory. Various functoriality properties of the theory are discussed. The product is defined and its associativity is proved.

2.1. Asymptotic Kasparov modules

DEFINITION 2.1.1. Consider a group G, and two G-C^*-algebras A and B. A G-(A,B)-*module* is a Hilbert B-module $\mathcal{E}$, admitting a G-action and a left action of A through an equivariant $*$–homomorphism $\varphi : A \to \mathcal{B}(\mathcal{E})$. If no group G is acting we shall use the terminology (A,B)-module. A *continuous field of* G-(A,B)-*modules* is a G-(A,BL)-module. (We recall the notation: $L=[1,\infty)$, $BL = C_0(L)\otimes B = C_0(L,B)$. Again, we omit G in the non-equivariant case.)

A continuous field $\mathcal{E}$ of G-(A,B)-modules can be thought of as a family $\{\mathcal{E}_t\}_{t\in[1,\infty)}$ of Hilbert B-modules, each acted on the left by A and G, satisfying certain continuity conditions for the left and right actions. Indeed, for any $t\in[1,\infty)$, let $ev_t : BL \to B$, $ev_t(f\otimes b) = f(t)b$, be the evaluation $*$-homomorphism at t. We obtain the Hilbert G-(A,B)-module $\mathcal{E}_t = \mathcal{E}\otimes_{ev_t} B$, with inner product $\langle \xi\otimes b, \xi'\otimes b'\rangle_t = b^*\, ev_t(\langle\xi,\xi'\rangle)\, b'$. The A-action an each $\mathcal{E}_t$ is $\varphi_t : A\to\mathcal{B}(\mathcal{E}_t)$, $\varphi_t(a) = \varphi(a)\otimes_{ev_t} 1$. Any operator $F\in\mathcal{B}(\mathcal{E})$ gives a family $\{F_t\}_{t\in[1,\infty)} = \{F\otimes_{ev_t} 1\}_{t\in[1,\infty)}$. When $\mathcal{E} = \mathcal{E}_\bullet L$, for a given Hilbert B-module $\mathcal{E}_\bullet$, the function $L\to\mathcal{B}(\mathcal{E}_\bullet)$, $t\mapsto F_t$, is 'bounded and $*$-strong continuous' [**Hg90a**, 3.16], *i.e.* the family $\{F_t\}_t$ is norm bounded, and for each $\xi \in \mathcal{E}_\bullet$ the functions $t \mapsto F_t(\xi)$ and $t \mapsto F_t^*(\xi)$ are norm continuous. On $\mathcal{B}(\mathcal{E}_\bullet)$ $*$-strong continuity is weaker than norm continuity.

Whenever there is no risk of confusion, we shall write a instead of $\varphi(a)$, and a_t instead of $\varphi_t(a)$.

For the remaining part of this section we assume no group action. Given any Hilbert BL-module $\mathcal{E}$, besides the *adjointable operators* $\mathcal{B}(\mathcal{E})$ on $\mathcal{E}$ and the *compact operators* $\mathcal{K}(\mathcal{E})$, two ideals will play an important role in the thesis.

Definition 2.1.2. The closed ideal of *compact-valued families of operators* is

$$\mathcal{C}(\mathcal{E}) = \{F \in \mathcal{B}(\mathcal{E}) \mid F\ f \in \mathcal{K}(\mathcal{E}),\ \text{for all } f \in C_0(L)\}. \tag{2.1}$$

(Here $(Ff)(\xi) \stackrel{\text{def}}{=} F(\xi f)$, for all $\xi \in \mathcal{E}$, and $\xi f = \lim_{n\to\infty} \xi(f \otimes b_n)$, with $\{b_n\}_n$ an approximate unit for B.) The closed ideal of *vanishing families of operators* is

$$\mathcal{J}(\mathcal{E}) = \{F \in \mathcal{B}(\mathcal{E}) \mid \lim_{t\to\infty} \|F_t\| = 0\}. \tag{2.2}$$

Lemma 2.1.3. $\mathcal{K}(\mathcal{E}) = \mathcal{C}(\mathcal{E}) \cap \mathcal{J}(\mathcal{E})$.

Proof. The inclusion $\mathcal{K}(\mathcal{E}) \subseteq \mathcal{C}(\mathcal{E}) \cap \mathcal{J}(\mathcal{E})$ is clear. Let $F \in \mathcal{C}(\mathcal{E}) \cap \mathcal{J}(\mathcal{E})$. From the fact that $F \in \mathcal{J}(\mathcal{E})$ it follows that for every positive integer n there exists t_n such that $\|F_t\| < 2^{-n}$, for all $t > t_n$. Consider a partition of unity for L, $\{\chi_0, \chi_1, \cdots, \chi_n, \cdots\}$, subordinated to the cover $\{[1, t_1+2^{-1})\} \cup \{(t_n, t_{n+1}+2^{-n-1}) | n = 1, 2, \dots\}$. Then $F = F\cdot 1 = \sum_{n=0}^{\infty} F\cdot\chi_n \in \mathcal{K}(\mathcal{E})$, due to the fact that each term $F \cdot \chi_n$ of the sum is compact ($F \in \mathcal{C}(\mathcal{E})$), and of norm less than 2^{-n} (for $n \geq 1$). ∎

Lemma 2.1.4. *If $\mathcal{E} = \mathcal{E}_\bullet L$ is a constant family of Hilbert B-modules, then any $F = \{F_t\}_t \in \mathcal{C}(\mathcal{E})$ is a norm-continuous family of operators in $\mathcal{K}(\mathcal{E}_\bullet)$.*

Proof. We first notice that the elements of $\mathcal{K}(\mathcal{E})$ generate norm-continuous families of operators. This is because any $\xi \in \mathcal{E}_\bullet L$ is a norm-continuous section vanishing at infinity in the constant field of Hilbert modules $\{\mathcal{E}_\bullet\}_t$. Consequently the generators $\theta_{\xi,\eta}$, $\xi, \eta \in \mathcal{E}_\bullet L$, of $\mathcal{K}(\mathcal{E})$ are norm-continuous. Now, given $F \in \mathcal{C}(\mathcal{E})$, the continuity of the family $\{F_t\}_t$ that it generates is a local property. For any t_0, choose$f \in C_c(L)$, $f \equiv 1$ in a neighborhood of t_0. The definition of $\mathcal{C}(\mathcal{E})$ says that $Ff \in \mathcal{K}(\mathcal{E})$, and consequently Ff is a norm-continuous family. This gives the norm-continuity of $\{F_t\}_t$ at t_0. ■

REMARK 2.1.5. $\mathcal{C}(\mathcal{E})$ does not coincide with $\{\, F \in \mathcal{B}(\mathcal{E}) \,|\, F_t \in \mathcal{K}(\mathcal{E}_t),$ for all $t \,\}$. Indeed, it is not difficult to construct a $*$-strongly continuous family $\{P_t\}_{t\in[1,\infty)}$ of rank-one projections on an infinite dimensional Hilbert space which is not norm continuous.

The relations between these various ideals are pictured in the following diagram:

$$\begin{array}{ccccc} & & \mathcal{B}(\mathcal{E}) & & \\ & \nearrow & & \nwarrow & \\ \mathcal{C}(\mathcal{E}) & & & & \mathcal{J}(\mathcal{E}) \\ & \nwarrow & & \nearrow & \\ & & \mathcal{K}(\mathcal{E}) & & \end{array} \tag{2.3}$$

DEFINITION 2.1.6. Let A and B be graded separable C^*-algebras (with no group action). An *asymptotic Kasparov* (A, B)*-module* is a pair $(\mathcal{E}, F)$, where $\mathcal{E}$ is a continuous field of (A,B)-modules, and $F \in \mathcal{B}(\mathcal{E})$ is odd and satisfies for any $a \in A$:

(aKm1) $(F - F^*)\varphi(a) \in \mathcal{J}(\mathcal{E})$;

(aKm2) $[F, \varphi(a)] \in \mathcal{J}(\mathcal{E})$; and

(aKm3) $\varphi(a)\,(F^2 - 1)\,\varphi(a)^* \geq 0$, modulo $\mathcal{C}(\mathcal{E}) + \mathcal{J}(\mathcal{E})$.

The set of all asymptotic Kasparov (A, B)-modules will be denoted by $ke(A, B)$.

REMARK 2.1.7. Compare these axioms with the ones that a Kasparov module $(\mathcal{E}, F)$ must satisfy (Definition 1.4.1, (1.1)). It is worth noticing that the third axiom of a Kasparov module, namely $(F^2 - 1)\varphi(a) \in \mathcal{K}(\mathcal{E})$, can be replaced by $\varphi(a)(F^2 - 1)\varphi(a)^* \geq 0$, modulo $\mathcal{K}(\mathcal{E})$, which looks more like our (aKm3).

REMARK 2.1.8. We introduce the following notation: given two operators $T, T' \in \mathcal{B}(\mathcal{E})$, then $T \sim T'$ if $(T - T') \in \mathcal{J}(\mathcal{E})$. With this convention (aKm1) reads $(F - F^*)\varphi(a) \sim 0$, and (aKm2) reads $[F, \varphi(a)] \sim 0$, for all $a \in A$.

REMARK 2.1.9. In terms of families we can rephrase the conditions of Definition 2.1.6 as follows: $\{\mathcal{E}_t\}_{t\in[1,\infty)}$ is a family of Hilbert (A,B)-modules, $\{F_t\}_{t\in[1,\infty)}$ is a bounded '$*$-strong continuous' family of odd operators, meaning that for each continuous section $\xi = \{\xi_t\}_t$ the maps $t \mapsto F_t(\xi_t)$ and $t \mapsto F_t^*(\xi_t)$ are continuous sections of the field $\{\mathcal{E}_t\}_{t\in[1,\infty)}$, and for any $a \in A$

(aKm1′) $\| (F_t - F_t^*)a_t \| \xrightarrow{t\to\infty} 0$;

(aKm2′) $\| [F_t, a_t] \| \xrightarrow{t\to\infty} 0$; and

(aKm3′) $a_t\,(F_t^2 - 1)\,a_t^* \geq K_t^a$, for a family $\{K_t^a\}_t \in \mathfrak{C}(\mathcal{E}) + \mathfrak{J}(\mathcal{E})$ depending on a.

(Here a_t denotes $\varphi_t(a) = \varphi(a) \otimes_{ev_t} 1$.)

EXAMPLE 2.1.10. Given a $*$-homomorphism $\psi : A \to B$, we form the asymptotic Kasparov (A,B)-module $(\mathcal{E}, F)$, where $\mathcal{E} = BL$ and $F = 0$. The representation of A is $\widehat{\psi} = 1 \otimes \psi$. Axioms (aKm1) and (aKm2) are trivially satisfied, and (aKm3) follows from the fact that $\widehat{\psi}(a)\,\widehat{\psi}(a)^* \in \mathfrak{C}(\mathcal{E})$. ($K_t^a = -\psi(a)\,\psi(a)^* \in B \simeq \mathcal{K}(B)$, for all $t \in [1,\infty)$ and all $a \in A$.) More generally, given a $*$-homomorphism $\psi : A \to \mathcal{K}(\mathcal{H}) \otimes B$, with $\mathcal{H}$ a countable generated Hilbert space, we form the asymptotic Kasparov (A,B)-module $(\mathcal{H}_B L, 0)$, with constant action of A on 'fibers' as above. In this situation $K_t^a = -\psi(a)\,\psi(a)^* \in \mathcal{K}(\mathcal{H}) \otimes B \simeq \mathcal{K}(\mathcal{H}_B)$. This simple but fundamental example implies the following principle: if a Kasparov module $(\mathcal{E}, F)$ with $F = 0$ exists, then an asymptotic Kasparov module can be constructed from it, namely $(\mathcal{E}L, 0)$.

EXAMPLE 2.1.11 (The K-homology class of the Dirac operator). Let M^{2n} be an even-dimensional, complete, spin^c-manifold, with spinor bundle $\mathbb{S} = \mathbb{S}_M$, and Dirac operator $D = D_M$. (D is essentially self-adjoint, and whenever functional calculus is used D actually denotes the closure $\overline{D} = D^*$.) The *fundamental asymptotic Kasparov* $(C_0(M), \mathbb{C})$-*module* is constructed as follows: $\mathcal{E} = \{L^2(M, \mathbb{S})\}_{t\in[1,\infty)}$, constant family; the action of $C_0(M)$ is the same on each 'fiber', by multiplication operators $\varphi_t(f) = M_f$; and $F = \{\chi(\frac{1}{t}D)\}_{t\in[1,\infty)}$, where χ is a *normalizing function* (*i.e.* $\chi : \mathbb{R} \to [-1,1]$ is odd, smooth, and $\lim_{x\to\pm\infty} \chi(x) = \pm 1$; for example we could take $\chi = x/(1+x^2)^{1/2}$). Let us show that this is an asymptotic Kasparov module. (For a thorough exposition of elliptic operators on manifolds see [**HgRoe**, Chaps.10,11]. This reference also explains the terminology that we use in this example.)

• $F \in \mathcal{B}(\mathcal{E})$. Indeed, this is implied by the norm continuity of $t \mapsto \chi(\frac{1}{t}D)$.

• *F satisfies* (aKm1). As noted above, when we write D we actually mean $\overline{D} = D^*$, which is self-adjoint, and the functional calculus gives $F = F^*$.

• *F satisfies* (aKm2). Let $f \in C_c^\infty(M)$. Then $\left[\,\frac{1}{t}D, f\,\right] = \frac{1}{t}\,(Df - fD) = \frac{1}{t}\,\nabla f \xrightarrow{t\to\infty} 0$, in norm ($\nabla f$ represents Clifford multiplication by the vector field ∇f). This gives ([**Hg91**]):

$$\begin{aligned} \left[(\tfrac{1}{t}D \pm i)^{-1}, f\right] &= (\tfrac{1}{t}D \pm i)^{-1} f - f(\tfrac{1}{t}D \pm i)^{-1} \\ &= (\tfrac{1}{t}D \pm i)^{-1}\{f(\tfrac{1}{t}D \pm i) - (\tfrac{1}{t}D \pm i)f\}(\tfrac{1}{t}D \pm i)^{-1} \\ &= (\tfrac{1}{t}D \pm i)^{-1}\,(\tfrac{1}{t}\,\nabla f)\,(\tfrac{1}{t}D \pm i)^{-1} \\ &\xrightarrow{t\to\infty} 0. \end{aligned} \tag{2.4}$$

It follows that we obtain norm convergence $[\,\phi(\frac{1}{t}D), f\,] \xrightarrow{t\to\infty} 0$, for all $\phi \in C_0(\mathbb{R}), f \in C_0(M)$. The significance is that the asymptotic Kasparov module that we construct will not depend on the normalizing function, any two such having difference in $C_0(\mathbb{R})$. Moreover it suffices now to prove (aKm2) for *one* particular normalizing function χ_0. We choose it such that its distributional Fourier transform $\widehat{\chi}_0$ is compactly supported, and $s \mapsto s\,\widehat{\chi}_0(s)$ is smooth (or in $L^1(\mathbb{R})$). (Such functions exist: see [**HgRoe**, 10.9.3].) We know, basically from Stone's theorem, that:

$$\langle\,[\chi_0(D), f]u, v\,\rangle = \int_{\mathbb{R}} \langle\,[e^{isD}, f]u, v\,\rangle\;\widehat{\chi_0}(s)\;ds, \text{ for all } u, v \in C_c^\infty(M, \mathbb{S}). \tag{2.5}$$

Assume now that $f \in C_c^\infty(M)$ takes actually values in $S^1 \subset \mathbb{C}$ (*i.e.* M_f is an unitary operator). We have:

$$\begin{aligned} [\chi_0(\tfrac{1}{t}D), f] &= \chi_0(\tfrac{1}{t}D)\,f - f\,\chi_0(\tfrac{1}{t}D) = f\left(f^{-1}\,\chi_0(\tfrac{1}{t}D)\,f - \chi_0(\tfrac{1}{t}D)\right) \\ &= f\left(\chi_0(\tfrac{1}{t}f^{-1}Df) - \chi_0(\tfrac{1}{t}D)\right). \end{aligned} \tag{2.6}$$

Putting together (2.5) and (2.6), we obtain:

$$\langle\,[\chi_0(\tfrac{1}{t}D), f]u, v\,\rangle = \int_{\mathbb{R}} \langle\,\left(e^{ist^{-1}f^{-1}Df} - e^{ist^{-1}D}\right)u, \overline{f}v\,\rangle\;\widehat{\chi_0}(s)\;ds. \tag{2.7}$$

By our first computation of this paragraph, $f^{-1}Df - D = f^{-1}[D, f] = f^{-1}\nabla f$ is a bounded operator. In accordance with [**HgRoe**, Lemma 10.3.6], applied to $T_1 = \frac{1}{t} f^{-1} D f$ and $T_2 = \frac{1}{t} D$, we have:

$$\|e^{isT_1} - e^{isT_2}\| \leq |s| \, \|T_1 - T_2\|, \quad \text{for all } s \in \mathbb{R}. \tag{2.8}$$

Because of (2.8), the inner product in the integral of (2.7) equals $|s|$ times a smooth function which is pointwise bounded by $\frac{1}{t} \|\nabla f\| \cdot \|u\| \cdot \|v\|$. The required norm asymptotic commutation now follows:

$$\| \, [\chi_0(\tfrac{1}{t} D), f] \, \| \leq \tfrac{1}{t} \, \|\nabla f\| \int_{\mathbb{R}} |s \, \widehat{\chi_0}(s)| \, ds.$$

The computation made in the last part of the argument above is [**HgRoe**, Prop. 10.3.7].

Finally, any arbitrary $f \in C_c^\infty(M)$ can be written as a linear combination of functions on M which are S^1-valued. Indeed, $f = \mathrm{Re}(f) + i \, \mathrm{Im}(f)$, and for a real valued f one writes:

$$f = (\|f\|/2) \left(\left(f/\|f\| + i\sqrt{1 - f^2/\|f\|^2} \right) + \left(f/\|f\| - i\sqrt{1 - f^2/\|f\|^2} \right) \right).$$

We are through.

• *F satisfies* (aKm3). The standard theory of elliptic first order differential operators shows that $f \, \left(\chi^2(\frac{1}{t} D) - 1 \right)$ is compact for $f \in C_0(M)$. It follows that $f \, (F^2 - 1) \, \overline{f} = 0$, modulo $\mathcal{C}$. (The norm continuity of $t \mapsto F_t$ was used again here.)

Remark 2.1.12. Given an asymptotic Kasparov (A, B)-module $(\mathcal{E}, F)$ then $(\mathcal{E}, (F + F^*)/2)$ is another such object. Indeed, the only axiom which is not obvious is (aKm3). It reduces to showing that $(F + F^*)^2/4 \geq (F^2 + (F^*)^2)/2$, which in turn is equivalent to $(F - F^*) \, (F - F^*)^* \geq 0$.

2.2. KE-theory

In this section we define the new bivariant theory and we study some of its functorial properties. *A group (locally compact, σ-compact, Hausdorff) is assumed to act continuously on all the objects under study.*

2.2.1. The KE-theory groups. We start with an extension of our previous Definition 2.1.6 to the equivariant context.

DEFINITION 2.2.1. Consider a group G, and two graded separable G-C^*-algebras A and B. An *asymptotic Kasparov G-(A,B)-module* is a pair $(\mathcal{E}, F)$, where $\mathcal{E}$ is a continuous field of G-(A,B)-modules (see Definition 2.1.1), and $F \in \mathcal{B}(\mathcal{E})$ is an odd G-continuous operator that satisfies **(aKm1)**, **(aKm2)**, **(aKm3)** of Definition 2.1.6, and the extra condition:
(aKm4) $\quad (\, g(F) - F\,)\, \varphi(a) \in \mathcal{J}(\mathcal{E})$, for all $g \in G$, $a \in A$.
In terms of families this reads:
(aKm4′) $\quad \|\, (g(F_t) - F_t)\, a_t \,\| \xrightarrow{t\to\infty} 0$, for all $g \in G$, $a \in A$, and with $a_t = \varphi_t(a)$.
The set of all asymptotic Kasparov G-(A,B)-modules is denoted by $ke^G(A,B)$.

DEFINITION 2.2.2. An element $(\mathcal{E}, F)$ of $ke^G(A, B[0,1])$ gives, by 'evaluation at s', a family $\{\, (\mathcal{E}_s, F_s) \in ke^G(A,B) \,|\, s \in [0,1]\,\}$, with $\mathcal{E}_s = \mathcal{E} \otimes_{\mathrm{ev}_s} BL$, $F_s = F \otimes_{\mathrm{ev}_s} 1$. Such an element $(\mathcal{E}, F)$ and the family that it generates are called a *homotopy* between $(\mathcal{E}_0, F_0)$ and $(\mathcal{E}_1, F_1)$. An *operator homotopy* is a homotopy $\{\, (\mathcal{E}, F_s) \,|\, s \in [0,1]\,\}$, with $s \mapsto F_s$ being norm continuous. Note that $\mathcal{E}$, and the action of A on it, are constant throughout an operator homotopy.

EXAMPLE 2.2.3. Given $(\mathcal{E}_0, F_0) = (\{\mathcal{E}_{0,t}\}_t, \{F_{0,t}\}_t) \in ke^G(A,B)$, it is homotopic to any of its 'translates' $(\{\mathcal{E}_{0,t+N}\}_t, \{F_{0,t+N}\}_t)$. It can also be 'stretched' by a homotopy to $(\mathcal{E}_1, F_1) = (\{\mathcal{E}_{0,h(t)}\}_t, \{F_{0,h(t)}\}_t)$, for any increasing bijective function $h : [1,\infty) \to [1,\infty)$.

DEFINITION 2.2.4. An asymptotic Kasparov G-(A,B)-module $(\mathcal{E}, F)$ is said to be *degenerate* if for all $a \in A$ and $g \in G$: $F = F^*$, $[F, \varphi(a)] = 0$, $(g(F) - F)\varphi(a) = 0$, and $\varphi(a)\,(F^2 - 1)\,\varphi(a)^* \geq 0$, modulo $\mathcal{J}(\mathcal{E})$. (Compare with Definition 1.4.1 for degenerate Kasparov modules.)

LEMMA 2.2.5. *If $(\mathcal{E}, F)$ is degenerate, then it is homotopic to the 0-module.*

PROOF. The pair $(C_0([0,1)) \otimes \mathcal{E}, 1 \otimes F)$, with A acting as $1 \otimes \varphi$, is a degenerate asymptotic Kasparov (A, BI)-module, which gives a homotopy between $(\mathcal{E}, F)$ and $(0,0)$. ∎

DEFINITION 2.2.6. $(\mathcal{E}, F') \in ke^G(A,B)$ is called a *'small perturbation'* of $(\mathcal{E}, F) \in ke^G(A,B)$ if $(F - F')\,\varphi(a) \in \mathfrak{J}(\mathcal{E})$, for all $a \in A$.

LEMMA 2.2.7. *Consider $(\mathcal{E}, F)$ in $ke^G(A,B)$, and F' a 'small perturbation' of F. Then $(\mathcal{E}, F)$ and $(\mathcal{E}, F')$ are operatorially homotopic.*

PROOF. Indeed, the straight line segment between F and F' is an operator homotopy: $\boldsymbol{F} = \{sF + (1-s)F'\}_{s\in[0,1]}$. We note that it is the same proof as in KK-theory for 'compact perturbations' [**Blck**, Def.17.2.4]. ∎

COROLLARY 2.2.8. *Any $(\mathcal{E}, F) \in ke^G(A,B)$ is homotopic to $(\mathcal{E}, (F + F^*)/2)$.*

PROOF. $(F + F^*)/2$ is a 'small perturbation' of F. ∎

From the corollary above it follows that (aKm1) can be strengthened: in Definitions 2.1.6 and 2.2.1 we could consider only self-adjoint operators F. Other changes are possible too.

A less trivial example of homotopy is provided by the next result (compare with [**Sk84**, Lemma 11]). Despite the simplicity of its proof, it will be very useful when we shall analyze in depth the product in KE-theory.

LEMMA 2.2.9. *Let $\mathcal{E}$ be a continuous field of G-(A,B)-modules. Consider two asymptotic Kasparov modules $(\mathcal{E}, F)$, $(\mathcal{E}, F') \in ke^G(A,B)$, such that $\varphi(a)\,[F, F']\,\varphi(a)^* \geq 0$, modulo $\mathcal{C}(\mathcal{E}) + \mathfrak{J}(\mathcal{E})$, for all $a \in A$. Then $(\mathcal{E}, F)$ and $(\mathcal{E}, F')$ are (operatorially) homotopic.*

PROOF. Put $F_s = \cos(s\,\pi/2)F + \sin(s\,\pi/2)F'$, for $s \in [0,1]$. Then the family $\{(\mathcal{E}, F_s)\}_s$ realizes the required homotopy. ∎

DEFINITION 2.2.10. The set $KE^G(A,B)$ is defined as the quotient of $ke^G(A,B)$ by the equivalence relation given by homotopy. (We shall omit G in the non-equivariant case.) Given $x = (\mathcal{E}, F) \in ke^G(A,B)$, its class in $KE^G(A,B)$ will be denoted by $[\![x]\!]$. The *addition* of two asymptotic Kasparov G-(A,B)-modules $(\mathcal{E}_1, F_1)$ and $(\mathcal{E}_2, F_2)$ is defined by $(\mathcal{E}_1, F_1) + (\mathcal{E}_2, F_2) = (\mathcal{E}_1 \oplus \mathcal{E}_2, F_1 \oplus F_2) \in ke^G(A,B)$.

THEOREM 2.2.11. *With the notation of the previous definition, $KE^G(A, B)$ is an abelian group.*

PROOF. The argument is similar to the one for KK-theory — see [**Sk84**, Prop.4]. ■

DEFINITION 2.2.12. For any group G, $1 = 1_{\mathbb{C}} \in KE^G(\mathbb{C}, \mathbb{C})$ is the class of the identity $*$-homomorphism $\psi = \mathrm{id} : \mathbb{C} \to \mathbb{C}$, *i.e.* the class of $(C_0(L), 0)$, with trivial action on $C_0(L)$. Note that 1 has nothing to do with the abelian group structure. More generally, given a G-C^*-algebra A, the element $1_A \in KE^G(A, A)$ is the class of the identity $*$-homomorphism $\psi = \mathrm{id} : A \to A$ (as in Example 2.1.10), *i.e.* the class of $(AL, 0)$. Given an equivariant $*$–homomorphism $\psi : A \to B$ or more generally $\psi : A \to \mathcal{K} \otimes B$, its class in $KE^G(A, B)$ is denoted by $[\![\psi]\!]$.

2.2.2. Functoriality properties. We discuss next some of the functoriality properties of the KE-groups. They are similar to the ones that the KK-theory groups satisfy.

(a) Given a $*$-homomorphism $\psi : A_1 \to A$, we obtain a map:

$$\psi^* : ke^G(A, B) \to ke^G(A_1, B)\,,\ (\mathcal{E}, F) \mapsto (\psi^*\mathcal{E}, F).$$

Here $\psi^*\mathcal{E}$ denotes the same Hilbert module $\mathcal{E}$, but with left action by A_1 given by the composition $\varphi \circ \psi : A_1 \to \mathcal{B}(\mathcal{E})$. We observe that ψ^* respects direct sums, and homotopy of asymptotic Kasparov modules. Consequently we get a well-defined map, denoted by the same symbol, at the level of groups: $\psi^* : KE^G(A, B) \to KE^G(A_1, B)$.

(b) Let $\psi : B \to B_1$ be a $*$-homomorphism. Using $1 \otimes \psi : BL \to B_1L$, we obtain a map:

$$\psi_* : ke^G(A, B) \to ke^G(A, B_1)\,,\ (\mathcal{E}, F) \mapsto (\mathcal{E} \otimes_{1\otimes\psi} B_1L, F \otimes_{1\otimes\psi} 1).$$

This map also respects direct sums, and homotopy of asymptotic Kasparov modules, and so gives a well-defined map: $\psi_* : KE^G(A, B) \to KE^G(A, B_1)$.

(c) For any G-C^*-algebra D there is a map:

$$\sigma_D : ke^G(A, B) \to ke^G(A \otimes D, B \otimes D),\ (\mathcal{E}, F) \mapsto (\mathcal{E} \otimes D, F \otimes 1). \tag{2.9}$$

It passes to quotients and gives a map $\sigma_D : KE^G(A,B) \to KE^G(A \otimes D, B \otimes D)$. Indeed, we verify first that the axioms for asymptotic Kasparov modules are satisfied.

• $F \otimes 1$ *satisfies* (aKm1). $(F \otimes 1 - (F \otimes 1)^*)\,(a \otimes d) = (F - F^*)a \otimes d \in \mathfrak{J}(\mathcal{E}) \otimes D \subseteq \mathfrak{J}(\mathcal{E} \otimes D)$.

• $F \otimes 1$ *satisfies* (aKm2). $(F \otimes 1)\,(a \otimes d) - (-1)^{\partial a + \partial d}(a \otimes d)\,(F \otimes 1) = [F,a] \otimes d \in \mathfrak{J}(\mathcal{E}) \otimes D \subseteq \mathfrak{J}(\mathcal{E} \otimes D)$.

• $F \otimes 1$ *satisfies* (aKm3). $(a \otimes d)\,(F^2 \otimes 1)(a^* \otimes d^*) = aF^2a^* \otimes dd^* \geq aa^* \otimes dd^*$, modulo $\mathcal{C}(\mathcal{E}) \otimes D + \mathfrak{J}(\mathcal{E} \otimes D) \subseteq \mathcal{C}(\mathcal{E} \otimes D) + \mathfrak{J}(\mathcal{E} \otimes D)$. The last inclusion follows from the isomorphism $\mathcal{K}(\mathcal{F} \otimes D) \simeq \mathcal{K}(\mathcal{F}) \otimes D$, where $\mathcal{F}$ is any Hilbert module.

• $F \otimes 1$ *satisfies* (aKm4). $(g(F \otimes 1) - F \otimes 1)\,(a \otimes d) = (g(F) - F)a \otimes d \in \mathfrak{J}(\mathcal{E} \otimes D)$.

Finally, σ_D sends homotopic Kasparov modules to homotopic asymptotic Kasparov modules, and this shows that σ_D is well defined at the level of groups.

PROPOSITION 2.2.13 (Homotopy invariance). *The bifunctor $KE^G(A,B)$ is homotopy invariant in both variables:*

(*a*) *let $\psi_0, \psi_1 : A_1 \to A$ be homotopic $*$-homomorphisms; then, for any B, $\psi_0^* = \psi_1^* : KE^G(A_1, B) \to KE^G(A,B)$;*

(*b*) *let $\psi_0, \psi_1 : B \to B_1$ be homotopic $*$-homomorphisms; then, for any A, $\psi_{0*} = \psi_{1*} : KE^G(A,B) \to KE^G(A, B_1)$.*

PROOF. Once again we may follow the same proof as in KK-theory.

(a) Let $\boldsymbol{\psi} : A_1 \to A[0,1]$ be a homotopy between ψ_0 and ψ_1. If $(\mathcal{E}, F) \in ke^G(A,B)$, then $\boldsymbol{\psi}^*\big(\sigma_{C([0,1])}((\mathcal{E},F))\big) \in ke^G(A_1, B[0,1])$ gives a homotopy between $\psi_0^*((\mathcal{E},F))$ and $\psi_1^*((\mathcal{E},F))$.

(b) Let $\boldsymbol{\psi} : B \to B_1[0,1]$ be a homotopy between ψ_0 and ψ_1. Because ev_0 and ev_1 are essential $*$-homomorphisms, it follows that $\psi_{i*} = ev_{i*} \circ \boldsymbol{\psi}_*$, for $i = 0,1$. Consequently, given $(\mathcal{E},F) \in ke^G(A,B)$, $\boldsymbol{\psi}_*((\mathcal{E},F))$ gives a homotopy between $\psi_{0*}((\mathcal{E},F))$ and $\psi_{1*}((\mathcal{E},F))$. ∎

2.2.3. Some technical results. We conclude this section with a technical result (namely Lemma 2.2.14), two definitions, and a 'diagonalization' process, that will be used in the definition of the product in Section 2.3. Recall that any self-adjoint element x of a C^*-algebra can be written as a difference of two positive elements $x = x_+ - x_-$, with $x_+ x_- = x_- x_+ = 0$. x_- is called the negative part of x.

LEMMA 2.2.14. *Let A and B be separable G-C^*-algebras. Given $(\mathcal{E}, F) \in ke^G(A, B)$, there exists a self-adjoint element $u \in \mathcal{C}(\mathcal{E})^{(0)}$ satisfying:*

(i) $[\,u, F\,] \in \mathcal{J}(\mathcal{E})$;

(ii) $[\,u, a\,] \in \mathcal{J}(\mathcal{E})$, *for all* $a \in A$;

(iii) $(1 - u^2)\,(a\,(\,F^2 - 1\,)\,a^*)_- \in \mathcal{J}(\mathcal{E})$, *for all* $a \in A$; *and*

(iv) $(\,g(u) - u\,) \in \mathcal{J}(\mathcal{E})$, *for all* $g \in G$.

PROOF. Consider a dense subset $\{a_n\}_{n=1}^\infty$ in A, and an appropriate (see below) cover of $[1, \infty)$ by closed intervals $\{I_n\}_{n=0}^\infty$, of the form $I_n = [t_n, t_{n+2}]$, with $t_0 = 1$, and $\{t_n\}_n$ being a strictly increasing sequence with $\lim_{n\to\infty} t_n = \infty$. Choose a partition of unity $\{\mu_n\}_{n=0}^\infty$ subordinated to this cover. For each positive integer n, let $r_n : DL \to D(I_n)$ be the restriction $*$-homomorphism, and use it to define the restriction of $\mathcal{E}$ and F to I_n: $\mathcal{E}|_{I_n} = (r_n)_*(\mathcal{E})$, $F|_{I_n} = (r_n)_*(F)$.

Let $u_{0,0}$ be an arbitrary even self-adjoint element of $\mathcal{K}(\mathcal{E}|_{I_0})$. For each $n \geq 1$, apply Proposition 1.3.2 to construct a quasi-invariant approximate unit $\{u_{n,k}\}_{k=1}^\infty$ for $\mathcal{K}(\mathcal{E}|_{I_n})$, which is quasi-central for $F|_{I_n}$, $A|_{I_n}$, and $\big\{\,(a\,(\,F^2 - 1\,)\,a^*)_-\,|_{I_n}\,\big|\,a \in A\big\}$. There exists an index k_n such that $\|\,[u_{n,k_n}, F]\,\| < 1/n$, $\|\,[u_{n,k_n}, a_m]\,\| < 1/n$, $\|\,(1-u_{n,k_n}^2)\,(a_m(F^2 - 1)a_m^*)_-\,\| < 1/n$, for $m = 1, 2, \ldots, n$. (For the third inequality, recall that (aKm3) implies $(\,a_m(F^2 - 1)a_m^*\,)_- \in \mathcal{C}(\mathcal{E}) + \mathcal{J}(\mathcal{E})$, with the $\mathcal{C}(\mathcal{E})$ part restricting to an element of $\mathcal{K}(\mathcal{E}|_{I_n})$, and the $\mathcal{J}(\mathcal{E})$ part having norm $< 1/2n$ by our initial choice of the partition $\{I_n\}_n$.) Define: $u = \sum_{n=0}^\infty \mu_n\, u_{n,k_n} \in \mathcal{C}(\mathcal{E})$. We observe that (i) is satisfied, and that (ii) and (iii) hold true for all the elements of the dense subset $\{a_n\}_n$ of A. A density argument finishes the proof. To have (iv) satisfied, one uses quasi-invariance, and a similar argument after choosing a dense subset $\{g_n\}_{n=1}^\infty$ of G. ∎

REMARK. The diagram (2.3) shows that the operators that appear in (i) and (ii) of the lemma above belong actually to $\mathcal{K}(\mathcal{E})$. If $\mathcal{E} = \mathcal{E}_\bullet L$, for a fixed Hilbert B-module $\mathcal{E}_\bullet$, then

$u = \{u_t\}_t$ is nothing but a quasi-invariant quasi-central approximate unit for $\mathcal{K}(\mathcal{E}_\bullet)$ in the smallest C^*-algebra containing $\mathcal{K}(\mathcal{E}_\bullet)$, A, F, and $(a\,(\,F^2 - 1\,)\,a^*)_-$, for all $a \in A$.

DEFINITION 2.2.15. A *section of* $[1,\infty) \times [1,\infty)$ is any increasing continuous function $h : [1,\infty) \to [1,\infty)$, with $h(1) = 1$, $\lim_{t\to\infty} h(t) = \infty$, differentiable on $[1,\infty)$, except maybe for a countable set of points where it has finite side derivatives. (Note that the differentiability assumption is just a convenience.)

LEMMA 2.2.16. *Given a countable family* $\{h_n\}_n$ *of sections of* $[1,\infty) \times [1,\infty)$, *one can find a suitable strict increasing sequence of numbers* $\{1 = x_0, x_1, x_2, \ldots, x_n, \ldots\}$, *with* $\lim_{n\to\infty} x_n = \infty$, *and a section* h *satisfying the following condition: for each* n, $h \geq h_i$, *for* $i = 1, 2, \ldots, n$, *over the closed interval* $[x_{n-1}, x_n]$.

PROOF. The definition of a section implies the existence, for each h_i, of a sequence $\{1 = x_0^i, x_1^i, \ldots, x_n^i, \ldots\}$, such that h_i is differentiable on (x_{n-1}^i, x_n^i), for each positive integer n, and has finite side derivatives at the end points. Let $h(1) = 1$, and for each integer $n \geq 1$ define:

$$x_n = \max_{1\leq i\leq n} \{x_n^i\},$$

$$m_n = \max_{1\leq i\leq n} \sup_{t\in[x_{n-1},x_n]} \{\ h_i'(t)\ \},\ \text{(side derivatives included)},$$

$$H_n = \max\left\{0, h_{n+1}(x_n) - (h(x_{n-1}) + m_n\,(x_n - x_{n-1}))\right\}.$$

Define h on $(x_{n-1}, x_n]$ by:

$$h(t) = h(x_{n-1}) + \left(m_n + \frac{H_n}{x_n - x_{n-1}}\right)\ (t - x_{n-1}).$$

■

DEFINITION 2.2.17. Consider a Hilbert BLL-module $\mathcal{E}$. Given a section h of $[1,\infty) \times [1,\infty)$ as in Definition 2.2.15, consider the restriction $*$-homomorphism: $\mathrm{Res}_h : BLL \to BL$, $f \mapsto f|_{\mathrm{graph}(h)}$. The *restriction of* $\mathcal{E}$ *to the graph of* h is the Hilbert BL-module $\mathcal{E}_h := (\mathrm{Res}_h)_*(\,\mathcal{E}\,) = \mathcal{E} \otimes_{\mathrm{Res}_h} BL$. Consider now any operator $F \in \mathcal{B}(\mathcal{E})$. The *restriction of* F *to the graph of* h is the operator $F_h := (\mathrm{Res}_h)_*(\,F\,) = F \otimes_{\mathrm{Res}_h} 1 \in \mathcal{B}(\mathcal{E}_h)$.

2.3. Construction of the product in KE-theory

Let G be a locally compact σ-compact Hausdorff group, and A_1, A_2, B_1, B_2, D be separable G-C^*-algebras. The aim of this section is to construct a certain bilinear map

$$KE^G(A_1, B_1 \otimes D) \otimes KE^G(D \otimes A_2, B_2) \to KE^G(A_1 \otimes A_2, B_1 \otimes B_2). \tag{2.10}$$

This is the *product in KE-theory* (compare with the product in KK-theory and in E-theory), and its construction is based on the particular case when $B_1 = A_2 = \mathbb{C}$. The intuition, based on examples coming from K-homology and K-theory, is that the product should have the form:

$$\big(\,(\mathcal{E}_1, F_1\,)\,,\,(\mathcal{E}_2, F_2\,)\,\big) \mapsto \big(\,\mathcal{E}_1 \boxtimes \mathcal{E}_2\,,\, F_1 \boxtimes 1 + 1 \boxtimes F_2\,\big), \tag{2.11}$$

where $\boxtimes$ is a certain 'tensor product.' Kasparov [**Kas75**], [**Kas81**] was the first mathematician to overcome the serious technical difficulties that arise in making sense of (2.11). We start our approach by providing a construction of the product (2.10) in the case when $D = \mathbb{C}$, known as *external product*. By doing so, we shall present a case when the formula (2.11) is actually correct. We shall also see the axioms (aKm1) - (aKm4) at work, and understand some of the difficulties involved in the general construction.

Example 2.3.1 (External product). Consider elements $(\mathcal{E}_1, F_1) \in ke^G(A_1, B_1)$ and $(\mathcal{E}_2, F_2) \in ke^G(A_2, B_2)$. Construct the $(A_1 \otimes A_2, B_1L \otimes B_2L)$-module $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$ (external tensor product of Hilbert modules), and $F = F_1 \otimes 1 + 1 \otimes F_2 \in \mathcal{B}(\mathcal{E})$. The claim is that the restriction $(\mathrm{Res}_h)_*(\,(\mathcal{E}, F)\,)$ to the graph of any section h satisfies (aKm1)—(aKm4). Indeed, due to the inclusions $\mathcal{J}(\mathcal{E}_1) \otimes \mathcal{B}(\mathcal{E}_2) \subset \mathcal{J}(\mathcal{E})$ and $\mathcal{B}(\mathcal{E}_1) \otimes \mathcal{J}(\mathcal{E}_2) \subset \mathcal{J}(\mathcal{E})$, it is easy to see that $(F - F^*)a$, $[F, a]$, $(g(F) - F)a \in \mathcal{J}(\mathcal{E})$, for all $a = a_1 \otimes a_2 \in A$. We also have:

$$
\begin{aligned}
&(a_1 \otimes a_2)\big(F^2 - 1\big)(a_1 \otimes a_2)^* \\
&\quad = (a_1 \otimes a_2)\big(F_1^2 \otimes 1 + 1 \otimes F_2^2 - 1\big)(a_1 \otimes a_2)^* \\
&\quad = \begin{cases} a_1(F_1^2 - 1)a_1^* \otimes a_2 a_2^* + a_1 a_1^* \otimes a_2 F_2^2 a_2^* \ \geq 0, & \text{modulo } J_1 = \big(\mathcal{C}(\mathcal{E}_1) + \mathcal{J}(\mathcal{E}_1)\big) \otimes \mathcal{B}(\mathcal{E}_2), \\ \text{ and} & \\ a_1 F_1^2 a_1^* \otimes a_2 a_2^* + a_1 a_1^* \otimes a_2(F_2^2 - 1)a_2^* \ \geq 0, & \text{modulo } J_2 = \mathcal{B}(\mathcal{E}_1) \otimes \big(\mathcal{C}(\mathcal{E}_2) + \mathcal{J}(\mathcal{E}_2)\big). \end{cases}
\end{aligned}
$$

Apply Lemma 2.3.2, with J_1, J_2 ideals in $\mathcal{B}(\mathcal{E}_1) \otimes \mathcal{B}(\mathcal{E}_2)$, to see that $(a_1 \otimes a_2)(F^2 - 1)(a_1 \otimes a_2)^* \geq 0$, modulo $J_1 J_2 \subseteq \mathcal{C}(\mathcal{E}) + \mathcal{J}(\mathcal{E})$. There is only one thing left: in order to obtain a right $(B_1 \otimes B_2)L$-module (and not an $(B_1 \otimes B_2)LL$-module as $\mathcal{E}$ is) we restrict $\mathcal{E}$ and F to the graph of $h(t) = t$. It is clear that F_h satisfies (aKm1)—(aKm4). The class of $(\mathrm{Res}_h)_*(\,(\mathcal{E}, F)\,)$ in $KE^G(A_1 \otimes A_2, B_1 \otimes B_2)$ is called the *external product* of $(\mathcal{E}_1, F_1)$ and $(\mathcal{E}_2, F_2)$. Compare with Example 1.4.8.

Conclusion. The external product of two asymptotic Kasparov G-modules $(\{\mathcal{E}_{1,t}\}_t, \{F_{1,t}\}_t)$ and $(\{\mathcal{E}_{2,t}\}_t,\ \{F_{2,t}\}_t)$, is an asymptotic Kasparov G-module as required, namely $(\{\mathcal{E}_{1,t} \otimes \mathcal{E}_{2,t}\}_t, \{F_{1,t} \otimes 1 + 1 \otimes F_{2,t}\}_t)$.

In the above example we used:

LEMMA 2.3.2. *Let J_1 and J_2 be closed ideals of the C^*-algebra A. If $a \geq 0$ mod J_1, and $a \geq 0$ mod J_2, then $a \geq 0$ mod $J_1 J_2 = J_1 \cap J_2$.*

PROOF. Given a C^*-subalgebra B and a closed ideal I of A, then $(B + I)$ is a C^*-subalgebra of A, and the map $(B + I)/I \to B/(B \cap I)$ is a $*$-isomorphism (see [**Pdrs**, 1.5.8]). Assume that $a \notin J_1$, otherwise interchange the roles of J_1 and J_2 in the argument below ($a \in J_1 \cap J_2$ being trivial). Consider B to be the C^*-subalgebra generated by J_2 and a, and $I = J_1$. We obtain the $*$-isomorphism: $(B + J_1)/J_1 \to B/(B \cap J_1) = B/(J_1 \cap J_2)$, $b + J_1 \mapsto b + J_1 J_2$. It sends the positive element $a + J_1$ to a positive element, namely $a + J_1 J_2$. ■

2.3.1. Two-dimensional connections. As in Kasparov's KK-theory, the general product will involve tensor products of Hilbert modules. Given a Hilbert DL-module $\mathcal{E}_1$ and a Hilbert BL-module $\mathcal{E}_2$, their tensor product (internal or external) will be a module over $[1,\infty)\times[1,\infty)$, (to be precise, it will be a module over the algebra BLL or $(D\otimes B)LL$). We shall call such modules over $[1,\infty)\times[1,\infty)$, and corresponding families of operators, 'two-dimensional.' The ones indexed by $[1,\infty)$ are 'one-dimensional.' Our construction of the product will be based on an appropriate notion of connection, which is going to be a 'two-dimensional' operator. The original definition of connection, on which ours is modelled, appears in [**CoSk**, Def.A.1] and [**Sk84**, Def.8] (see Definition 1.4.4).

Definition 2.3.3. Assume that the following elements are given: a Hilbert DL-module $\mathcal{E}_1$, a Hilbert (D,BL)-module $\mathcal{E}_2$, and $F_2\in\mathcal{B}(\mathcal{E}_2)$. Consider the Hilbert BLL-module $\mathcal{E}=\mathcal{E}_1\otimes_{DL}\mathcal{E}_2L$. An operator $\underline{F}\in\mathcal{B}(\mathcal{E})$ is called an *F_2-connection for $\mathcal{E}_1$* if it has the same degree as F_2 and if it satisfies, for every *compactly supported* ξ in $\mathcal{E}_1$,

$$\big(\, T_\xi\,(1\otimes F_2)-(-1)^{\partial\xi\cdot\partial F_2}\underline{F}\,T_\xi\,\big)\in\mathcal{J}(\mathcal{E}_2L,\mathcal{E}),$$

and

$$\big(\,(1\otimes F_2)\,T_\xi^*-(-1)^{\partial\xi\cdot\partial F_2}T_\xi^*\,\underline{F}\,\big)\in\mathcal{J}(\mathcal{E},\mathcal{E}_2L).$$

Here $T_\xi\in\mathcal{B}(\mathcal{E}_2L,\mathcal{E})$ is defined by $T_\xi(g\otimes\eta)=\xi\otimes_{DL}(g\otimes\eta)$, for $g\in C_0L$, and $\eta\in\mathcal{E}_2$. Moreover $\mathcal{J}(\mathcal{E}_2L,\mathcal{E})=\{T\in\mathcal{B}(\mathcal{E}_2L,\mathcal{E})\mid\lim_{(t_1,t_2)\to\infty}\|T_{t_1,t_2}\|=0\}$, and $\mathcal{J}(\mathcal{E},\mathcal{E}_2L)$ is defined similarly.

Remark. The above two conditions which a connection must satisfy are better remembered through the commutative modulo $\mathcal{J}$ diagrams

$$\begin{array}{ccc} \mathcal{E}_2L & \xrightarrow{F_2} & \mathcal{E}_2L \\ {\scriptstyle T_\xi}\downarrow & & \downarrow{\scriptstyle T_\xi} \\ \mathcal{E} & \xrightarrow[\underline{F}]{} & \mathcal{E} \end{array} \quad\text{and}\quad \begin{array}{ccc} \mathcal{E}_2L & \xrightarrow{F_2} & \mathcal{E}_2L \\ {\scriptstyle T_\xi^*}\uparrow & & \uparrow{\scriptstyle T_\xi^*} \\ \mathcal{E} & \xrightarrow[\underline{F}]{} & \mathcal{E} \end{array}\ . \tag{2.12}$$

The notation F_2 in relation with $\mathcal{E}_2L=C_0(L)\otimes\mathcal{E}_2$ means $1\otimes F_2$.

Proposition 2.3.4. *Consider the notation of the previous definition. If F_2 satisfies, for all $d \in D$, $[F_2, \varphi_2(d)] \in \mathcal{J}(\mathcal{E}_2)$, then an F_2-connection exists for any countably generated $\mathcal{E}_1$.*

Proof. According with the Stabilization Theorem [**Kas80**, Thm.2], there exists an element $V \in \mathcal{B}(\mathcal{E}_1, \mathcal{H}_{(DL)^\sim})$ of degree 0 such that $V^*V = 1$. (This follows from the isomorphism $\mathcal{E}_1 \oplus \mathcal{H}_{(DL)^\sim} \simeq \mathcal{H}_{(DL)^\sim}$.) Assume first that the unit of $(DL)^\sim$ acts as identity on $\mathcal{E}_2L$. There is then an obvious isomorphism $W : \mathcal{H}_{(DL)^\sim} \otimes_{(DL)^\sim} \mathcal{E}_2L \to \mathcal{H} \otimes \mathcal{E}_2L$, given on elementary tensors by $W((v \otimes f) \otimes_{(DL)^\sim} \eta) = v \otimes f\eta$, for $v \in \mathcal{H}$, $f \in (DL)^\sim$, $\eta \in \mathcal{E}_2L$. (In $\mathcal{H} \otimes \mathcal{E}_2L$ the tensor product is an external one.) The desired F_2-connection $\underline{F}$ is given by the composition of the operators depicted in the diagram below:

$$\begin{array}{ccc}
\mathcal{E}_1 \otimes_{DL} \mathcal{E}_2L & \xrightarrow{\underline{F}} & \mathcal{E}_1 \otimes_{DL} \mathcal{E}_2L \\
{\scriptstyle V \otimes_{(DL)^\sim} 1}\downarrow & & \uparrow{\scriptstyle V^* \otimes_{(DL)^\sim} 1} \\
\mathcal{H}_{(DL)^\sim} \otimes_{(DL)^\sim} \mathcal{E}_2L & & \mathcal{H}_{(DL)^\sim} \otimes_{(DL)^\sim} \mathcal{E}_2L \\
{\scriptstyle W}\downarrow & & \uparrow{\scriptstyle W^{-1}} \\
\mathcal{H} \otimes \mathcal{E}_2L & \xrightarrow[1 \otimes F_2]{} & \mathcal{H} \otimes \mathcal{E}_2L
\end{array},$$

i.e.

$$\underline{F} = (V^* \otimes 1)\, W^{-1}\, (1 \otimes F_2)\, W\, (V \otimes 1). \tag{2.13}$$

We shall verify only one of the conditions for an F_2-connection (the other one being similar). Let ξ be a compactly supported section of $\mathcal{E}_1$, and $V(\xi) = \sum_{i=1}^{\infty} e_i \otimes f_i$, where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis in $\mathcal{H}$, and $\sum_{i=1}^{\infty} f_i^* f_i < \infty$ in DL. We have of course $\partial\xi = \partial e_i + \partial f_i$, and $\operatorname{supp}(f_i) \subseteq \operatorname{supp}(\xi)$. A direct computation gives for any $\eta \in \mathcal{E}_2 L$:

$$\begin{aligned}
W\,(V \otimes 1)\,\big(\,T_\xi\,(1 \otimes F_2) - (-1)^{\partial\xi\cdot\partial F_2}\underline{F}\,T_\xi\,\big)(\eta) \\
&= W(\,V(\xi) \otimes_{(DL)^\sim} F_2(\eta)\,) - (-1)^{\partial\xi\cdot\partial F_2}\,(1 \otimes F_2)\,W\,(V(\xi) \otimes_{(DL)^\sim} \eta\,) \\
&= \sum_{i=1}^{\infty} e_i \otimes f_i F_2(\eta) - (-1)^{\partial\xi\cdot\partial F_2}(1 \otimes F_2)\big(\sum_{i=1}^{\infty} e_i \otimes f_i\eta\big) \\
&= \sum_{i=1}^{\infty} e_i \otimes f_i F_2(\eta) - (-1)^{\partial f_i\cdot\partial F_2}\sum_{i=1}^{\infty} e_i \otimes F_2(f_i\eta) \\
&= \sum_{i=1}^{\infty} e_i \otimes f_i F_2(\eta) - \sum_{i=1}^{\infty} e_i \otimes f_i F_2(\eta) - \sum_{i=1}^{\infty} e_i \otimes [F_2, f_i](\eta) \\
&= -\sum_{i=1}^{\infty} e_i \otimes [F_2, f_i](\eta).
\end{aligned}$$

Consequently, it remains to show the convergence of the last infinite sum and that it belongs to $\mathcal{J}(\mathcal{E}_2 L, \mathcal{E})$. This is accomplished by proving the convergence in *operator norm* of the partial sums $S_I = -\sum_{i=1}^{I} e_i \otimes [F_2, f_i]$, using the expression given after the third equal sign in the above computation. This, together with the fact that the partial sums belong to $\mathcal{J}(\mathcal{E}_2 L, \mathcal{E})$, will give the desired result.

Fix $\varepsilon > 0$, and consider $\eta \in \mathcal{E}_2 L$. We have:

$$\begin{aligned}
\big\|\big(S_{I+k} - S_I\big)(\eta)\big\| &= \Big\|\sum_{i=I+1}^{I+k} e_i \otimes f_i F_2(\eta) - (-1)^{\partial f_i\cdot\partial F_2}\sum_{i=I+1}^{I+k} e_i \otimes F_2 f_i(\eta)\Big\| \\
&\leq \underbrace{\Big\|\sum_{i=I+1}^{I+k} e_i \otimes f_i F_2(\eta)\Big\|}_{\alpha} + \underbrace{\Big\|\sum_{i=I+1}^{I+k} e_i \otimes F_2 f_i(\eta)\Big\|}_{\beta}.
\end{aligned}$$

Now:

$$\begin{aligned}
\alpha^2 &= \Big\|\,\Big\langle \sum_{i=I+1}^{I+k} e_i \otimes f_i F_2(\eta), \sum_{i=I+1}^{I+k} e_i \otimes f_i F_2(\eta)\Big\rangle\,\Big\| = \Big\|\,\Big\langle F_2(\eta), \Big(\sum_{i=I+1}^{I+k} f_i^* f_i\Big) F_2(\eta)\Big\rangle\,\Big\| \\
&\leq \Big\|\,\sum_{i=I+1}^{I+k} f_i^* f_i\,\Big\| \cdot \big\|F_2\big\|^2 \cdot \big\|\eta\big\|^2.
\end{aligned}$$

Choose I such that $\big\| \sum_{i\in\Omega} f_i^* f_i \big\| \leq \varepsilon^2/4\|F_2\|^2$, for every finite set Ω which does not intersect $\{1,2,\ldots,I\}$. Next:

$$\beta^2 = \big\| \langle \sum_{i=I+1}^{I+k} e_i \otimes F_2 f_i(\eta), \sum_{i=I+1}^{I+k} e_i \otimes F_2 f_i(\eta)\rangle \big\| = \big\| \langle \eta, \sum_{i=I+1}^{I+k} f_i^* \left(F_2^* F_2 \right) f_i(\eta)\rangle \big\|$$

$$\leq \big\|F_2^* F_2\big\| \cdot \big\| \sum_{i=I+1}^{I+k} f_i^* f_i \big\| \cdot \big\|\eta\big\|^2 \leq \big\|F_2\big\|^2 \cdot \big\| \sum_{i=I+1}^{I+k} f_i^* f_i \big\| \cdot \big\|\eta\big\|^2.$$

For the chosen I, we obtain: $\alpha + \beta \leq (\varepsilon/2 + \varepsilon/2) \big\|\eta\big\|$. Consequently, $\big\|S_{I+k} - S_I\big\| \leq \varepsilon$, for all positive integers k. This proves the norm convergence of the double sum, and the proposition in the case when the unit of $(DL)^\sim$ acts as identity on $\mathcal{E}_2 L$.

In the general case, the equation (2.13) needs replaced by:

$$\underline{F} = (V^* \otimes 1)\, W^{-1} \big(1 \otimes (1 \otimes F_2)|_{DL}\big)\, W\, (V \otimes 1),$$

where $W : \mathcal{H}_{DL} \otimes_{DL} \mathcal{E}_2 L \to \mathcal{H} \otimes (DL \cdot \mathcal{E}_2 L)$, and $(1 \otimes (1 \otimes F_2)|_{DL} : \mathcal{H} \otimes (DL \cdot \mathcal{E}_2 L) \to \mathcal{H} \otimes (DL \cdot \mathcal{E}_2 L)$. We recall the definition of the restriction operator $(1 \otimes F_2)\big|_{DL}$ of $1 \otimes F_2$ to the closed (but not necessarily complemented) subspace $DL \cdot \mathcal{E}_2 L$:

$$(1 \otimes F_2)\big|_{LD} = \sum_{n=1}^{\infty} (1 \otimes \varphi_2)(\delta_n^{1/2})\; (1 \otimes F_2)\; (1 \otimes \varphi_2)(\delta_n^{1/2}),$$

where $\{u_n\}_{n=1}^{\infty}$ is an approximate unit for DL, and $\delta_n = u_n - u_{n-1}$, $n = 1,2,\ldots$, and $u_0 = 0$. The computations are now longer, but there is no new idea involved in the proof. ∎

The next result gathers some useful properties of connections (compare with [**Sk84**, Prop.9]). The same notation as in Definition 2.3.3 is used.

PROPOSITION 2.3.5. *(i) Let $\underline{F}$ be an F_2-connection for $\mathcal{E}_1$, and $\underline{F}'$ be an F_2'-connection for $\mathcal{E}_1$. Then $(\underline{F}+\underline{F}')$ is an (F_2+F_2')-connection for $\mathcal{E}_1$, and $(\underline{F}\,\underline{F}')$ is an $(F_2\, F_2')$-connection for $\mathcal{E}_1$.*
(ii) The linear space of 0-connections for $\mathcal{E}_1$ is

$$\big\{\, \underline{F} \in \mathcal{B}(\mathcal{E}) \;\big|\; (K \otimes_{DL} 1)\underline{F},\ \underline{F}(K \otimes_{DL} 1) \in \mathcal{J}(\mathcal{E}),\ \ \textit{for all } K \in \mathcal{K}(\mathcal{E}_1) \,\big\}.$$

PROOF. Both (i) and (ii) follow immediately from the definition of connection. ∎

LEMMA 2.3.6. *Consider the notation of Definition 2.3.3 and assume that a separable set $K \subset \mathcal{C}(\mathcal{E}_1)$ is given. Then there exists a section h_{00} of $[1,\infty) \times [1,\infty)$ such that for any other section $h \geq h_{00}$ the following holds:*

$$(\mathrm{Res}_h)_*(\,[\,k \otimes_{DL} 1, \underline{F}\,]\,) \in \mathfrak{J}\,((\mathrm{Res}_h)_*(\,\mathcal{E}\,))\,, \text{for all } k \in K.$$

PROOF. Choose a dense subset $\{\,k_n\,\}_{n=1}^{\infty}$ of K. Assume that one is able to find for each k_n a section h_n such that $(\mathrm{Res}_h)_*(\,[k_n \otimes_{DL} 1, \underline{F}]\,) \in \mathfrak{J}((\mathrm{Res}_h)_*(\,\mathcal{E}\,))$, for any $h \geq h_n$. Apply the diagonalization process described in Lemma 2.2.16 to obtain a section h_{00} which makes the conclusion true for all k_n's. A density argument shows that the result holds for all $k \in K$.

Consequently it is enough to construct a section that works for a single element $k \in K$. As in the proof of (2.21) in the Technical Theorem (Section 2.5), one uses a partition of unity for L, an approximation of $k \otimes_{DL} 1$ by finite sums $\sum_i T_{\xi_i} T^*_{\eta_i}$, with ξ_i, $\eta_i \in \mathcal{E}_1$, and the properties of connections that $\underline{F}$ satisfies. ∎

2.3.2. Construction of the product. We are now ready to give the construction of the product (2.10) in the case when $B_1 = A_2 = \mathbb{C}$. Before stating the main theorem we present an overview of the proof.

OVERVIEW 2.3.7. Consider two asymptotic Kasparov modules $(\mathcal{E}_1, F_1) \in ke^G(A, D)$ and $(\mathcal{E}_2, F_2) \in ke^G(D, B)$. Their product, which is an element in $ke^G(A, B)$, is obtained by performing the following sequence of steps.

Step 1. Find a self-adjoint $u \in \mathcal{C}(\mathcal{E}_1)^{(0)}$ such that:

(1) $[\,u, F_1\,] \in \mathfrak{J}(\mathcal{E}_1)$,

(2) $[\,u, a\,] \in \mathfrak{J}(\mathcal{E}_1)$, for all $a \in A$, and

(3) $(1 - u^2)\,(a\,(\,F_1^2 - 1\,)\,a^*)_- \in \mathfrak{J}(\mathcal{E}_1)$, for all $a \in A$,

(4) $(g(u) - u) \in \mathfrak{J}(\mathcal{E}_1)$, for all $g \in G$.

Step 2. Define $\mathcal{E} = \mathcal{E}_1 \otimes_{DL} \mathcal{E}_2 L$. Find $\underline{F} = \underline{F}^*$ an F_2-connection for $\mathcal{E}_1$, and define $F = F_1 \otimes_{DL} 1 + (u \otimes_{DL} 1)\, \underline{F}$.

Step 3. Choose a section h_{00} of $[1,\infty) \times [1,\infty)$ such that the restrictions of the following operators to the graph of any other section $h \geq h_{00}$ are in $\mathcal{J}((\mathrm{Res}_h)_*(\,\mathcal{E}\,))$:

(5) $[\,u \otimes_{DL} 1, \underline{F}]$,

(6) $[\,u\, F_1 \otimes_{DL} 1, \underline{F}]$,

(7) $[\,u\, a \otimes_{DL} 1, \underline{F}]$, for all $a \in A$.

Step 4. Find $h_0 \geq h_{00}$ such that the restriction to the graph of any $h \geq h_0$ of:

(8) $(u \otimes_{DL} 1)\left(\underline{F}^2 - 1\right)(u \otimes_{DL} 1)$ is positive modulo $\mathcal{C}((\mathrm{Res}_h)_*(\,\mathcal{E}\,)) + \mathcal{J}((\mathrm{Res}_h)_*(\,\mathcal{E}\,))$ (see (2.21) in Section 2.5); and of

(9) $(u \otimes_{DL} 1)\big(g(\underline{F}) - \underline{F}\big)$ is in $\mathcal{J}((\mathrm{Res}_h)_*(\,\mathcal{E}\,))$, for all $g \in G$.

Once a triple $(u, \underline{F}, h_0)$ satisfying (1)–(9) is constructed, the conclusion is that the restriction of $(\mathcal{E}, F)$ to the graph of any $h \geq h_0$ gives an asymptotic Kasparov G-(A,B)-module $(\mathcal{E}_h, F_h)$:

$$\begin{aligned} \mathcal{E}_h &= (\mathrm{Res}_h)_*(\,\mathcal{E}\,) = (\mathrm{Res}_h)_*(\,\mathcal{E}_1 \otimes_{DL} \mathcal{E}_2 L\,), \\ F_h &= (\mathrm{Res}_h)_*(\,F\,) = (\mathrm{Res}_h)_*(\,F_1 \otimes_{DL} 1 + (u \otimes_{DL} 1)\,\underline{F}\,) = \widetilde{F_1 \otimes_D} 1 + 1 \widetilde{\otimes_D F_2}. \end{aligned} \tag{2.14}$$

The notation $\widetilde{F_1 \otimes_D} 1 = (\mathrm{Res}_h)_*(\,F_1 \otimes_{DL} 1\,)$, and $1 \widetilde{\otimes_D F_2} = (\mathrm{Res}_h)_*(\,(u \otimes_{DL} 1)\,\underline{F}\,)$ is suggested by the form of the product in the external product case, and will simplify the writing in the following sections. Note that in terms of families (2.14) reads:

$$\begin{aligned} \mathcal{E}_h &= \left\{\, \mathcal{E}_{1,t} \otimes_D \mathcal{E}_{2,h(t)} \,\right\}_{t \in [1,\infty)}, \\ F_h &= \left\{\, F_{1,t} \otimes_D 1 + (u_t \otimes_D 1)\,\underline{F}_{(t,h(t))} \,\right\}_{t \in [1,\infty)}. \end{aligned} \tag{2.15}$$

Remark 2.3.8. We do not have an axiomatic definition of the product as in [**Sk84**, Def.10], [**CoSk**, Thm.A.3] (see Definition 1.4.6), so the situation is more like in E-theory.

The following theorem guarantees that Steps 1-4 of Overview 2.3.7 can be performed. Its proof will be given in Section 2.5.

THEOREM 2.3.9 (Technical Theorem). *Let G be a locally compact σ-compact Hausdorff group, and let A, B, and D be separable graded G-C^*-algebras. Consider two asymptotic Kasparov modules $(\mathcal{E}_1, F_1) \in ke^G(A,D)$ and $(\mathcal{E}_2, F_2) \in ke^G(D,B)$. There exists a triple $(\,u, \underline{F}, h_0\,)$, with u a self-adjoint element of $\mathcal{C}^{(0)}(\mathcal{E}_1)$, $\underline{F}$ an F_2-connection for $\mathcal{E}_1$, and h_0 a section of $[1,\infty) \times [1,\infty)$, as in Overview 2.3.7, such that for any other section $h \geq h_0$*

$$\begin{aligned}(\mathcal{E}_h, F_h) &= \big(\,(\mathrm{Res}_h)_*(\mathcal{E}_1 \otimes_{DL} \mathcal{E}_2 L), \widetilde{F_1 \otimes_D 1 + 1 \otimes_D F_2}\,\big)\\ &= (\mathrm{Res}_h)_*\big(\,\mathcal{E}_1 \otimes_{DL} \mathcal{E}_2 L, F_1 \otimes_{DL} 1 + (u \otimes_{DL} 1)\,\underline{F}\,\big)\end{aligned}$$

is an asymptotic Kasparov G-(A,B)-module.

We can now give the definition of the *product map in KE-theory* in the form of:

THEOREM 2.3.10. *With the notation of the above theorem, the map $(\,(\mathcal{E}_1, F_1), (\mathcal{E}_2, F_2)\,) \mapsto (\mathcal{E}_{h_0}, F_{h_0})$ passes to quotients and defines the* product map*:*

$$KE^G(A,D) \otimes KE^G(D,B) \xrightarrow{\ \sharp_D\ } KE^G(A,B), \quad (x,y) \mapsto x \,\sharp_D\, y\,. \tag{2.16}$$

PROOF. The notation is that of Overview 2.3.7.
(I) Independence of the triple $(\,u, \underline{F}, h\,)$. (a) For any two $h_1, h_2 \geq h_0$ we have a homotopy between $(\mathcal{E}_{h_1}, F_{h_1})$ and $(\mathcal{E}_{h_2}, F_{h_2})$ given by the explicit formula:

$$\big\{(\mathrm{Res}_{sh_1+(1-s)h_2})_*(\,\widetilde{F_1 \otimes_D 1 + 1 \otimes_D F_2}\,)\big\}_{s\in[0,1]}.$$

Similarly, one can construct a homotopy between two asymptotic Kasparov modules corresponding to different h_0's in Step 4. This proves the independence of h. *(b)* In order to show independence of $\underline{F}$, consider two F_2-connections $\underline{F}$ and $\underline{F}'$, same u, and choose an h that works for both connections, *i.e.* both $F_h = F_1 \otimes_D 1 + (u \otimes_D 1)\,(\mathrm{Res}_h)_*(\,\underline{F}\,)$ and $F'_h = F_1 \otimes_D 1 + (u \otimes_D 1)\,(\mathrm{Res}_h)_*(\,\underline{F}'\,)$ give elements in $ke^G(A,B)$. Now $(\underline{F} - \underline{F}')$ is a 0-connection, and Proposition 2.3.5(ii) shows that $\big((u \otimes_D 1)\,(\mathrm{Res}_h)_*(\,\underline{F}\,) - (u \otimes_D 1)\,(\mathrm{Res}_h)_*(\,\underline{F}'\,)\big) \in \mathfrak{J}(\mathcal{E}_h)$. Lemma 2.2.7 applies and gives a homotopy between F_h and F'_h. *(c)* To show independence of u, choose two different such elements u and u', both satisfying the requirements of Step 1, same $\underline{F}$, and an h that works for both choices. We obtain a homotopy by the formula:

$$\big\{F_1 \otimes_D 1 + \big(s(u \otimes_D 1) + (1-s)(u' \otimes_D 1)\big)(\mathrm{Res}_h)_*(\,\underline{F}\,)\big\}_{s\in[0,1]}.$$

Combining (a), (b), and (c) above we get that the homotopy class of the element $(\mathcal{E}_h, F_h)$ constructed in Theorem 2.3.9 does not depend on the triple $(\,u, \underline{F}, h\,)$.

(II) Passage to quotients. Our goal is to show that the homotopy class of the product does not depend on the representatives in the class of $(\mathcal{E}_1, F_1)$ and $(\mathcal{E}_2, F_2)$. Consider $(\boldsymbol{\mathcal{E}_1}, \boldsymbol{F_1}) \in ke^G(A, B[0,1])$ a homotopy between $(\mathcal{E}_{1,0}, F_{1,0})$ and $(\mathcal{E}_{1,1}, F_{1,1})$. An F_2-connection for $\boldsymbol{\mathcal{E}_1}$ will be a 'three-dimensional' object, and the required homotopy between the product of $(\mathcal{E}_{1,0}, F_{1,0})$ and $(\mathcal{E}_2, F_2)$, and of $(\mathcal{E}_{1,1}, F_{1,1})$ and $(\mathcal{E}_2, F_2)$, respectively, will be given by the restriction to a 'surface' in this three-dimensional object. (Its existence follows from the existence of ordinary connections and the compactness of $[0,1]$.) The general case is not more complicated than this particular one, and requires one more restriction to a 'surface' in a three dimensional universe. We obtain that the map from the statement does pass to a well-defined map at the level of KE-theory groups. ■

Using Theorem 2.3.10 and the map σ, we are in position to construct the general product (2.10) mentioned at the very beginning of this section (compare with the definition in KK-theory [**Kas88**, Def.2.12]).

DEFINITION 2.3.11. Let G be a group, and let A_1, A_2, B_1, B_2, D be G-C^*-algebras. The *general product in KE-theory* is the map

$$KE^G(A_1, B_1 \otimes D) \otimes KE^G(D \otimes A_2, B_2) \to KE^G(A_1 \otimes A_2, B_1 \otimes B_2), \tag{2.10}$$

defined by:

$$x \sharp_D y \;=\; \sigma_{A_2}(x) \, \sharp_{B_1 \otimes D \otimes A_2} \, \sigma_{B_1}(y)\,. \tag{2.17}$$

The *external product* corresponds to $D = \mathbb{C}$.

This section is concluded by showing that, in the case of external product, the asymptotic Kasparov module constructed in Example 2.3.1 is homotopic with the one given by the general product of Definition 2.3.11. This will show that Example 2.3.1 really represents the construction of a product, and not merely of some other asymptotic Kasparov module. Let $x \in KE^G(A_1, B_1)$ be represented by $(\mathcal{E}_1, F_1)$, and $y \in KE^G(A_2, B_2)$ be represented by $(\mathcal{E}_2, F_2)$. According with Definition 2.3.11, $x \sharp_{\mathbb{C}} y = \sigma_{A_2}(x) \sharp_{B_1 \otimes A_2} \sigma_{B_1}(y)$. Now, $\sigma_{A_2}(x)$ is represented by $(\mathcal{E}_1 \otimes A_2, F_1 \otimes 1)$, and $\sigma_{B_1}(y)$ is represented by $(B_1 \otimes \mathcal{E}_2, 1 \otimes F_2)$. (Bear in mind the details related to the graded tensor product of Hilbert modules [**Blck**, 14.4.4].) To obtain a module that represents the product we follow the steps given in Overview 2.3.7. The q.i.q.c.a.u. of *Step 1* can be chosen of the form $\{\widetilde{u}_t \otimes \alpha_{h(t)}\}_t$, with $\{\widetilde{u}_t\}_t$ a q.i.q.c.a.u. for $\mathcal{K}(\mathcal{E}_1)$, $\{\alpha_t\}_t$ an a.u. for A_2, and h an arbitrary section. In *Step 2* we identify $\mathcal{E}$ with $\mathcal{E}_1 \cdot (B_1 L) \otimes A_2 \cdot \mathcal{E}_2$, which is a Hilbert $(B_1 L \otimes B_2 L)$-module, acted on the left by $A_1 \otimes A_2$. As two-dimensional connection we can take the constant field $\{1 \otimes F_{2,t_2}\}_{(t_1,t_2) \in LL}$. With the choices and identifications made so far, any section h_{00} will do in *Step 3*. In *Step 4* choose a section h that makes the restriction to its graph an asymptotic Kasparov module:

$$\big(\mathcal{E}_h, F_h\big) = \Big(\big\{\mathcal{E}_{1,t} \otimes \mathcal{E}_{2,h(t)}\big\}_t, \big\{F_{1,t} \otimes 1 + \widetilde{u}_t \otimes \alpha_{h(t)} F_{2,h(t)}\big\}_t \Big) \in ke^G(A_1 \otimes A_2, B_1 \otimes B_2).$$

Lemma 2.2.9 applies and gives a homotopy between $(\mathcal{E}_h, F_h)$ and

$$\big(\mathcal{E}'_h, F'_h\big) = \Big(\big\{\mathcal{E}_{1,t} \otimes \mathcal{E}_{2,h(t)}\big\}_t, \big\{F_{1,t} \otimes 1 + 1 \otimes F_{2,h(t)}\big\}_t \Big) \in ke^G(A_1 \otimes A_2, B_1 \otimes B_2).$$

Finally we notice that $(\{\mathcal{E}_{2,h(t)}\}_t, \{F_{2,h(t)}\}_t)$ is just another representative of y, obtained by 'stretching' (Example 2.2.3) the initial representative $(\mathcal{E}_2, F_2) = (\{\mathcal{E}_{2,t}\}_t, \{F_{2,t}\}_t)$. Consequently, using two homotopies, we succeeded to show that the product $\sharp_{\mathbb{C}}$ of Definition 2.3.11 is compatible with the external tensor product of Example 2.3.1.

2.4. Properties of the product

We study in this section some of the properties of the product in KE-theory. They are very similar with the ones that the Kasparov product satisfies in KK-theory. For our first result compare with [**Kas88**, Thm.2.14].

THEOREM 2.4.1. *The product $\sharp$ satisfies the following properties:*

(i) *it is bilinear;*

(ii) *it is contravariant in A,* i.e. $f^*(x)\,\sharp_D\, y \;=\; f^*(\,x\,\sharp_D\, y\,)$, *for any $*$-homorphism $f : A_1 \to A$, $x \in KE^G(A,D)$, and $y \in KE^G(D,B)$;*

(iii) *it is covariant in B,* i.e. $g_*(\,x\,\sharp_D\, y\,) = \; x\,\sharp_D\, g_*(y)$, *for any $*$-homorphism $g : B \to B_1$, $x \in KE^G(A,D)$, and $y \in KE^G(D,B)$;*

(iv) *it is functorial in D,* i.e. $f_*(x)\,\sharp_{D_2}\, y \;=\; x\,\sharp_{D_1}\, f^*(y)$, *for any $*$-homorphism $f : D_1 \to D_2$, $x \in KE^G(A,D_1)$, and $y \in KE^G(D_2,B)$;*

(v) $\sigma_{D_1}(\,x\,\sharp_D\, y\,) = \sigma_{D_1}(x)\,\sharp_{D\otimes D_1}\,\sigma_{D_1}(y)$, *for $x \in KE^G(A,D)$ and $y \in KE^G(D,B)$.*

PROOF. (i) Let $x = [\![(\mathcal{E}_1, F_1)]\!] \in KE^G(A,D)$, $y_1 = [\![(\mathcal{E}_2, F_2)]\!]$, $y_1 = [\![(\mathcal{E}_2', F_2')]\!] \in KE^G(D,B)$. Then: $x\,\sharp_D\, y_1 = [\![(\mathrm{Res}_{h_1})_*\big((\,\mathcal{E}_1 \otimes_{DL} \mathcal{E}_2 L, F_1 \otimes_{DL} 1 + (u_1 \otimes_{DL} 1)\underline{F}\,)\big)]\!]$, $x\,\sharp_D\, y_2 = [\![(\mathrm{Res}_{h_2})_*\big((\,\mathcal{E}_1 \otimes_{DL} \mathcal{E}_2' L, F_1 \otimes_{DL} 1 + (u_1 \otimes_{DL} 1)\underline{F}'\,)\big)]\!]$, $y_1 + y_2 = [\![(\mathcal{E}_2 \oplus \mathcal{E}_2', F_2 \oplus F_2')]\!]$. Let $h = \sup\{h_1, h_2\}$. Using $\mathcal{E}_1 \otimes_{DL} (\mathcal{E}_2 \oplus \mathcal{E}_2')L \simeq (\mathcal{E}_1 \otimes_{DL} \mathcal{E}_2 L) \oplus (\mathcal{E}_1 \otimes_{DL} \mathcal{E}_2' L)$, the definition of connection shows that $(\underline{F} \oplus \underline{F}')$ is an $(F_2 \oplus F_2')$-connection for $\mathcal{E}_1$. It is clear that:

$$
\begin{aligned}
x\,\sharp_D\, y_1 \;+\; & x\,\sharp_D\, y_2 \\
&= [\![(\mathrm{Res}_h)_*\big((\,\mathcal{E}_1 \otimes_{DL} (\mathcal{E}_2 \oplus \mathcal{E}_2')L, F_1 \otimes_{DL} (1 \oplus 1) + (u_1 \otimes_{DL} (1 \oplus 1))(\underline{F} \oplus \underline{F}'\,)\big)]\!] \\
&= \; x\,\sharp_D\,(\,y_1 + y_2).
\end{aligned}
$$

The linearity in the first variable is simpler.

(ii) Given $x = [\![(\mathcal{E}_1, F_1)]\!]$ and $y = [\![(\mathcal{E}_2, F_2)]\!]$, the result is clear once we recall that $f^*\mathcal{E}_1$ is the same Hilbert D-module $\mathcal{E}_1$, but with an A_1 action given by $\varphi_1 \circ f$. It follows that a triple $(u, \underline{F}, h_0)$ used to construct $x\,\sharp_D\, y$ can also be used to construct $f^*(x)\,\sharp_D\, y$.

(iii) Once a triple $(u, \underline{F}, h_0)$ has been obtained, one can choose $(u, \underline{F} \otimes_{1\otimes 1\otimes g} 1, h_0)$ as defining triple for $x\,\sharp_D\, g_*(y)$. We also use here the isomorphism $(\mathcal{E}_1 \otimes_{DL} \mathcal{E}_2 L) \otimes_{1\otimes 1\otimes g} B_1 LL \simeq \mathcal{E}_1 \otimes_{DL} (\mathcal{E}_2 \otimes_{1\otimes g} B_1 L)L$, and the identification: $(\mathrm{Res}_{h_0})_*(\,F_1 \otimes_{DL} 1 + (u_1 \otimes_{DL} 1)\underline{F}\,) \otimes_{1\otimes g} 1 = (\mathrm{Res}_{h_0})_*\big(F_1 \otimes_{DL} (1 \otimes 1) + (u_1 \otimes_{DL} (1 \otimes 1))(\underline{F} \otimes_{1\otimes 1\otimes g} 1\big)$.

(iv) Consider $x = [\![(\mathcal{E}_1, F_1)]\!]$ and $y = [\![(\mathcal{E}_2, F_2)]\!]$. Choose the same h for both products. The two-dimensional objects that represent $x\,\sharp_{D_1}\, f^*(y)$ and $f_*(x)\,\sharp_{D_2}\, y$ by restriction to the graph of h are:

$$(\mathcal{E}_{x(y)}, F_{x(y)}) = \big(\mathcal{E}_1 \otimes_{1\otimes(\varphi_2\circ f)} \mathcal{E}_2 L, F_1 \otimes_{1\otimes(\varphi_2\circ f)} 1 + (u \otimes_{1\otimes(\varphi_2\circ f)} 1)\underline{F}\big), \quad \text{and}$$

$$(\mathcal{E}_{(x)y}, F_{(x)y}) = \big((\mathcal{E}_1 \otimes_{1\otimes f} D_2 L) \otimes_{1\otimes\varphi_2} \mathcal{E}_2 L, (F_1 \otimes 1) \otimes_{1\otimes\varphi_2} 1 + (\widetilde{u} \otimes_{1\otimes\varphi_2} 1)\underline{F}'\big),$$

respectively. There is an isomorphism $\mathcal{E}_{(x)y} \to \mathcal{E}_{x(y)}$, $\big(\xi \otimes (k \otimes d_2)\big) \otimes_{1\otimes\varphi_2} (l \otimes \eta) \mapsto \xi \otimes_{1\otimes(\varphi_2\circ f)} \big(kl \otimes \varphi_2(d_2)\eta\big)$, where $\xi \in \mathcal{E}_1$, $\eta \in \mathcal{E}_2$, $k, l \in C_0(L)$. Under this isomorphism, $(F_1 \otimes 1) \otimes_{1\otimes\varphi_2} 1$ identifies with $F_1 \otimes_{1\otimes(\varphi_2\circ f)} 1$, and $\underline{F}'$ can be chosen to equal $\underline{F}$. By imposing an extra condition on $\widetilde{u}$, namely $[\widetilde{u}, u \otimes_{1\otimes f} 1] \in \mathfrak{J}(\mathcal{E}_1 \otimes_{1\otimes f} D_2 L)$, it follows that $a\,[\,(\mathrm{Res}_h)_*(\,F_{(x)y}\,), (\mathrm{Res}_h)_*(\,F_{x(y)}\,)\,]\,a^* \geq 0$, modulo $\mathfrak{J}((\mathrm{Res}_h)_*(\,\mathcal{E}_{(x)y}\,))$. Lemma 2.2.9 proves that the two representatives are homotopic.

(v) With $x = [\![(\mathcal{E}_1, F_1)]\!]$ and $y = [\![(\mathcal{E}_2, F_2)]\!]$, $\sigma_{D_1}(\,x \sharp_D y\,)$ is represented by the restriction of $\big((\mathcal{E}_1 \otimes_{DL} \mathcal{E}_2 L) \otimes D_1, (F_1 \otimes_{DL} 1) \otimes 1 + ((u \otimes_{DL} 1)\underline{F}) \otimes 1\big)$ to the graph of a section h. Let $\mathcal{E}_1' = \mathcal{E}_1 \otimes D_1$, $\mathcal{E}_2' = \mathcal{E}_2 \otimes D_1$, $D' = D \otimes D_1$. The product $\sigma_{D_1}(x) \sharp_{D\otimes D_1} \sigma_{D_1}(y)$ is represented by the restriction of $\big(\mathcal{E}_1' \otimes_{D'L} \mathcal{E}_2' L, (F_1 \otimes 1) \otimes_{D'L} 1 + (\widetilde{u} \otimes_{D'L} 1)\underline{F}'\big)$. Under the identification $\mathcal{E}_1' \otimes_{D'L} \mathcal{E}_2' L \simeq (\mathcal{E}_1 \otimes_{DL} \mathcal{E}_2 L) \otimes D_1$, we can take $\underline{F}' = \underline{F} \otimes 1$. Given any quasi-invariant approximate unit $\widetilde{d} = \{d_t\}_t$ for D_1, we can choose $\widetilde{u} = u \otimes \widetilde{d} \in \mathcal{C}^{(0)}(\mathcal{E}_1 \otimes D_1)$. Finally, after considering a common section for both products, Lemma 2.2.9 applies again and gives a homotopy between the two representatives. ■

Remark. In the proof of the next theorem the language of elementary calculus will be used again in order to 'visualize' the construction of a double product in KE-theory. A 3D-cartesian coordinate system is assumed, with LLL viewed as octant in this system. The quotations marks required by such imprecise, but suggestive, terminology will be dropped.

Definition 2.4.2. A *3D-section* is a function $h : L \to LL$, $t \mapsto (h_2(t), h_3(t))$, with h_2 and h_3 ordinary sections.

Theorem 2.4.3 (Associativity of the product). *Let A, B, D, and E be G-C^*-algebras. Then, for any $x_1 \in KE^G(A, D)$, $x_2 \in KE^G(D, E)$, and $x_3 \in KE^G(E, B)$,*

$$(\,x_1 \sharp_D x_2\,) \sharp_E x_3 \;=\; x_1 \sharp_D (\,x_2 \sharp_E x_3\,)\,.$$

Proof. Assume that x_1, x_2, x_3 are represented by $(\mathcal{E}_1, F_1) \in ke^G(A,D)$, $(\mathcal{E}_2, F_2) \in ke^G(D,E)$, $(\mathcal{E}_3, F_3) \in ke^G(E,B)$, respectively. We shall use the notation: $\mathcal{E}_{12} = \mathcal{E}_1 \otimes_{DL} \mathcal{E}_2 L$, $\mathcal{E}_{23} = \mathcal{E}_2 \otimes_{EL} \mathcal{E}_3 L$, $\mathcal{E} = \mathcal{E}_1 \otimes_{DL} \mathcal{E}_2 L \otimes_{ELL} \mathcal{E}_3 LL$, $x_{12,3} = (\, x_1 \sharp_D x_2 \,) \sharp_E x_3$, $x_{1,23} = x_1 \sharp_D (\, x_2 \sharp_E x_3 \,)$. An inner product $(\xi \otimes_{DL} \eta \otimes_{ELL} \zeta) \in \mathcal{E}$ is abreviated as $(\xi \otimes_D \eta \otimes_E \zeta)$, and similarly for operators on $\mathcal{E}$. In LLL, the first copy of L and the first coordinate t_1 correspond to $\mathcal{E}_1$, the second copy of L and the second coordinate t_2 correspond to $\mathcal{E}_2$, and the third copy of L and the third coordinate t_3 correspond to $\mathcal{E}_3$.

We first describe the product $x_{12,3}$. As explained in the previous section, $x_1 \sharp_D x_2$ is constructed from a triple $(u_1, \underline{F_{12}}, h_{12})$, with $u_1 \in \mathcal{C}(\mathcal{E}_1)$, $\underline{F_{12}}$ an F_2-connection for $\mathcal{E}_1$, and h_{12} a section in the (t_1, t_2)-plane. It is represented by $(\mathcal{E}_{12,h_{12}}, F_{12,h_{12}}) = (\mathrm{Res}_{h_{12}})_*(\,(\mathcal{E}_{12}, F_{12})\,)$, where

$$F_{12} = F_1 \otimes_{DL} 1 + (u_1 \otimes_{DL} 1)\underline{F_{12}}.$$

The product $x_{12,3}$ is constructed from a triple $(u_{12,h_{12}}, \underline{F_{12,3}}, h_3)$, with $u_{12,h_{12}} \in \mathcal{C}(\mathcal{E}_{12,h_{12}})$, $\underline{F_{12,3}}$ an F_3-connection for $\mathcal{E}_{12,h_{12}}$, and h_3 a section in the 'surface' $\Sigma_1 = \{(t_1,t_2,t_3) \in LLL \,|\, t_2 = h_{12}(t_1)\}$. It is represented by the restriction to the graph of h_3 of $\big(\mathcal{E}_{12,h_{12}} \otimes_{EL} \mathcal{E}_3 L, F_{12,h_{12}} \otimes_{EL} 1 + (u_{12,h_{12}} \otimes_{EL} 1)\underline{F_{12,3}}\big)$. There is a simpler way of describing a representative. Define the 3D-section $h(t) = (h_{12}(t), h_3(t))$. Consider the three-dimensional objects $\mathcal{E}$ and

$$F = F_1 \otimes_D 1 \otimes_E 1 + (u_1 \otimes_D 1 \otimes_E 1)(\underline{F_{12}} \otimes_E 1) + (u_{12} \otimes_E 1)\underline{F},$$

with u_1, $\underline{F_{12}}$ as before, $u_{12} \in \mathcal{C}(\mathcal{E}_{12})$, and $\underline{F}$ a three-dimensional F_3-connection for $\mathcal{E}_{12}$. (Such a three-dimensional connection is a straightforward generalization of our definition for two-dimensional connection. See (2.19) for one of the defining, commutative up to $\mathcal{J}$, diagrams.) The product is represented by the restriction of $(\mathcal{E}, F)$ to the graph of h.

Similarly, $x_2 \sharp_E x_3$ is constructed from a triple $(u_2, \underline{F_{23}}, h_{23})$, with $u_2 \in \mathcal{C}(\mathcal{E}_2)$, $\underline{F_{23}}$ an F_3-connection for $\mathcal{E}_2$, and h_{23} a section in the (t_2, t_3)-plane. It is represented by $(\mathcal{E}_{23,h_{23}}, F_{23,h_{23}}) = (\mathrm{Res}_{h_{23}})_*(\,(\mathcal{E}_{23}, F_{23})\,)$, where

$$F_{23} = F_2 \otimes_{EL} 1 + (u_2 \otimes_{EL} 1)\underline{F_{23}}.$$

The product $x_{1,23}$ is constructed from a triple $(u_1, \underline{F_{1,23}}, h'_3)$, with the same u_1 as before, $\underline{F_{1,23}}$ an $F_{23,h_{23}}$-connection for $\mathcal{E}_1$, and h'_3 a section in the 'surface' $\Sigma_2 = \{(t_1, t_2, t_3) \in LLL \,|\, t_3 = h_{23}(t_2)\}$. Let h' be the 3D-section whose graph is given by the graph of h'_3. We can describe a representative fo $x_{1,23}$ by the restriction to the graph of h' of

$$F' = F_1 \otimes_{DL} 1 + (u_1 \otimes_{DL} 1)\underline{F_{1,23}},$$

with $\underline{F_{1,23}}$ an F_{23}-connection for $\mathcal{E}_1$. The properties of connections given in Proposition 2.3.5 imply that we can take $\underline{F_{1,23}} = \underline{F_{12}} \otimes_E 1 + \underline{U_2}\,\underline{F'}$, where $\underline{U_2}$ is an $(u_2 \otimes_{EL} 1)$-connection for $\mathcal{E}_1$, and $\underline{F'}$ is an $\underline{F_{23}}$-connection for $\mathcal{E}_1$. The best way to see this choice for $\underline{F_{1,23}}$ is through the diagram below, which represents the first of the two diagrams (2.12) for the connections under discussion (the other one being constructed in a similar way):

$$\begin{array}{ccccc}
(\mathcal{E}_3 L)L & \xrightarrow{1\otimes(1\otimes F_3)} & (\mathcal{E}_3 L)L & & \\
\downarrow{\scriptstyle f_1\otimes T_\eta} & & \downarrow{\scriptstyle f_1\otimes T_\eta} & & \\
(\mathcal{E}_2 \otimes_{EL} \mathcal{E}_3 L)L & \xrightarrow{1\otimes\underline{F_{23}}} & (\mathcal{E}_2 \otimes_{EL} \mathcal{E}_3 L)L & \xrightarrow{1\otimes(u_2\otimes_{EL}1)} & (\mathcal{E}_2 \otimes_{EL} \mathcal{E}_3 L)L \\
\downarrow{\scriptstyle T_\xi} & & \downarrow{\scriptstyle T_\xi} & & \downarrow{\scriptstyle T_\xi} \\
\mathcal{E} & \xrightarrow[\underline{F'}]{} & \mathcal{E} & \xrightarrow[\underline{U_2}]{} & \mathcal{E}
\end{array} \tag{2.18}$$

(In the diagram: $f_1 \in C_0(L)$, $\eta \in \mathcal{E}_2$, $\xi \in \mathcal{E}_1$. We also have made the identification: $\mathcal{E}_1 \otimes_{DL} (\mathcal{E}_2 L \otimes_{ELL} \mathcal{E}_3 LL) \simeq \mathcal{E} \simeq (\mathcal{E}_1 \otimes_{DL} \mathcal{E}_2 L) \otimes_{ELL} \mathcal{E}_3 LL$.) The bottom squares of (2.18) show that $\underline{U_2}\,\underline{F'}$ is indeed a $(u_2 \otimes_{EL} 1)\underline{F_{23}}$-connection for $\mathcal{E}_1$. The left squares of (2.18) are nothing but an F_3-connection for $\mathcal{E}_{12}$:

$$\begin{array}{ccc}
(\mathcal{E}_3)LL & \xrightarrow{(1\otimes 1)\otimes F_3} & (\mathcal{E}_3)LL \\
\downarrow{\scriptstyle T_{\xi\otimes_{DL}(f_1\otimes\eta)}} & & \downarrow{\scriptstyle T_{\xi\otimes_{DL}(f_1\otimes\eta)}} \\
\mathcal{E} & \xrightarrow[\underline{F'}=\underline{F}]{} & \mathcal{E}
\end{array} \tag{2.19}$$

The outcome of all the above is the following: $x_{12,3}$ and $x_{1,23}$ can be represented by the restriction of 3D-pairs $(\mathcal{E}, F)$ and $(\mathcal{E}, F')$, where

$$\begin{aligned}
F &= F_1 \otimes_D 1 \otimes_E 1 + (u_1 \otimes_D 1 \otimes_E 1)(\underline{F_{12}} \otimes_E 1) + (u_{12} \otimes_E 1)\,\underline{F}, \\
F' &= F_1 \otimes_D 1 \otimes_E 1 + (u_1 \otimes_D 1 \otimes_E 1)(\underline{F_{12}} \otimes_E 1) + (u_1 \otimes_D 1 \otimes_E 1)\,\underline{U_2}\,\underline{F},
\end{aligned} \tag{2.20}$$

to the graphs of appropriate sections h and h', respectively. We complete the proof by showing that h and h' can be chosen the same, and that F and F' are homotopic.

The proof of Technical Theorem given in Section 2.5 (see also the remark that follows that proof) shows that, while the section h_0 that appears in the triple $(u, \underline{F}, h_0)$ used to define the product of two KE-modules is an important element, the 'right decay conditions' actually hold true on a two dimensional object, namely over $\cup_{n=0}^{\infty}[T_{1,n}, T_{1,n+1}] \times [T_{2,n}, \infty)$, or over $\{\, (t_1, t_2) \in LL \,|\, t_2 \geq h_0(t_1) \,\}$. (Notation as in the proof of Technical Theorem.) This implies that in the computation of a product the section is important only through the fact that it captures the behavior when both $t_1 \to \infty$ and $t_2 \to \infty$. This observation is summarized as:

LEMMA 2.4.4. *The products* $(\, x_1 \sharp_D x_2 \,) \sharp_E x_3$ *and* $x_1 \sharp_D (\, x_2 \sharp_E x_3 \,)$ *can be computed by restricting the operators of (2.20) to a* common *3D-section* h.

We need one more result:

LEMMA 2.4.5. *Define:* $\mathcal{J}_0(\mathcal{E}) = \{F \in \mathcal{B}(\mathcal{E}) \;\big|\; \lim_{(t_1,t_2,t_3)\to\infty} \|F_{(t_1,t_2,t_3)}\| = 0\}$. *(Here* $(t_1, t_2, t_3) \to \infty$ *means* $t_i \to \infty$, *for* $i = 1, 2, 3$.*) Then* $[\, u_1 \otimes_D 1 \otimes_E 1, \underline{U_2} \,] \in \mathcal{J}_0(\mathcal{E})$, *and* u_{12} *can be chosen such that* $[\, u_{12} \otimes_E 1, (u_1 \otimes_D 1 \otimes_E 1) \, \underline{U_2} \,] \in \mathcal{J}_0(\mathcal{E})$.

PROOF. Modulo an element in $\mathcal{J}(\mathcal{E}_1) \otimes_D 1 \otimes_E 1 \subset \mathcal{J}_0(\mathcal{E})$, $(u_1 \otimes_D 1 \otimes_E 1)$ can be approximated on compact intervals in t_1-variable by finite sums $\sum_i (T_{\xi_i} T^*_{\eta_i} \otimes_E 1)$, with $\xi_i, \eta_i \in \mathcal{E}_1$, compactly supported. (See the proof of Technical Theorem in Section 2.5.) This implies:

$$
\begin{aligned}
&[u_1 \otimes_D 1 \otimes_E 1, \underline{U_2}] \\
&\qquad = \sum_i \big((T_{\xi_i} T^*_{\eta_i} \otimes_E 1)\underline{U_2} - \underline{U_2}(T_{\xi_i} T^*_{\eta_i} \otimes_E 1) \big) \\
&\qquad \sim (-1)^{\partial \eta_i} \sum_i \big(T_{\xi_i}(1 \otimes (u_2 \otimes_{EL} 1))T^*_{\eta_i} - T_{\xi_i}(1 \otimes (u_2 \otimes_{EL} 1))T^*_{\eta_i} \big), \quad \text{modulo } \mathcal{J}(\mathcal{E}) \\
&\qquad = 0.
\end{aligned}
$$

This proves the first inclusion. For the second one, use the same approximation for $(u_1 \otimes_D 1 \otimes_E 1)$ as above to see that, modulo $\mathcal{J}_0(\mathcal{E})$, $(u_1 \otimes_D 1 \otimes_E 1) \, \underline{U_2}$ is an element of $\mathcal{B}(\mathcal{E}_{12}) \otimes_E 1$. The claimed asymptotic-commutativity follows by actually imposing it as an *extra requirement* for u_{12} (besides the conditions that appear in Step 1, Overview 2.3.7). ■

This last lemma implies that $a\,[F, F']\,a^* \geq 0$, modulo $\mathcal{J}(\mathcal{E}_h)$, for any section h, and consequently Lemma 2.2.9 gives the required homotopy. We have proved that $x_{12,3} = x_{1,23}$ in $KE^G(A, B)$. ■

REMARK. There is another way to see the homotopy between the operators from (2.20). It uses the following result, whose proof is left to the reader:

LEMMA 2.4.6. $(u_1 \otimes_D 1 \otimes_E 1)\,\underline{U_2}$ *satisfies the (properly modified) conditions of Step 1, Overview 2.3.7, that* $(u_{12} \otimes_E 1)$ *satisfies.*

Consequently, the straight line homotopy $\{\,(1-s)(u_{12} \otimes_E 1) + s(u_1 \otimes_D 1 \otimes_E 1)\,\underline{U_2}\,\}_{s\in[0,1]}$ can be used to give a homotopy between F and F'.

Recall from Definition 2.2.12 that $1 = 1_{\mathbb{C}} \in KE^G(\mathbb{C}, \mathbb{C})$ is the class of the identity homomorphism $\psi = \mathrm{id} : \mathbb{C} \to \mathbb{C}$. For the next result compare with [**Kas81**, Thm.4.5], [**Sk84**, Prop.17].

PROPOSITION 2.4.7. *Let A and B be separable G-C^*-algebras, then*

$$1_{\mathbb{C}} \sharp_{\mathbb{C}}\, x \;=\; x \,\sharp_{\mathbb{C}}\, 1_{\mathbb{C}} \;= x,\;\; \textit{for any } x \in KE^G(A, B).$$

PROOF. One equality is easy. We have: $x \,\sharp_{\mathbb{C}}\, 1_{\mathbb{C}} \overset{\text{def}}{=} x \,\sharp_B\, \sigma_B(1_{\mathbb{C}}) \;=\; x \,\sharp_B\, 1_B$. Let x be represented by $(\mathcal{E}, F)$ and 1_B be represented by $(BL, 0)$. As 0-connection for $\mathcal{E}$ we can take the 0 operator, and we can restrict to $h(t) = t$ in the construction of the product to obtain:

$$(\mathrm{Res}_h)_*(\,\mathcal{E} \otimes_{BL} BLL\,) \simeq \mathcal{E},\;\text{ via } (\mathrm{Res}_h)_*(\,\xi \otimes_{BL} (f \otimes g \otimes b)\,) \mapsto (\xi \cdot (fg \otimes b)),$$

$$(\mathrm{Res}_h)_*(\,F \otimes_{BL} 1\,) = F \text{ (under the previous isomorphism).}$$

Consequently $x \,\sharp_B\, 1_B = x$.

For the other equality, we start with: $1_{\mathbb{C}} \sharp_{\mathbb{C}}\, x \overset{\text{def}}{=} \sigma_A(1_{\mathbb{C}}) \,\sharp_A\, x \;=\; 1_A \,\sharp_A\, x$. Let 1_A be represented by $(AL, 0)$. Consider a quasi-invariant approximate unit $\{u_n\}_{n=1}^{\infty}$ for A, and construct an element $u = \{u_t\}_{t\in[1,\infty)} \in \mathcal{C}^{(0)}(AL)$ by interpolating the u_n's: $u_t = (1 -$

$\{t\})u_{[t]} + \{t\}u_{[t]+1}$, with $[t]$ denoting the gratest integer smaller that t, and $\{t\} = t - [t]$. We shall exhibit a homotopy between $(\mathcal{E}, F)$ and a representative of the product $1_A \sharp_A x$ constructed using u.

If A is unital, consider the projection $\varphi(1) = P \in \mathcal{B}(\mathcal{E})$. With the identification $AL \otimes_{AL} \mathcal{E}L \simeq (P\mathcal{E})L$, and after choosing $u \equiv 1$ and $h(t) = t$ in the definition of the product, we obtain as representative of $1_A \sharp_A x$ the asymptotic Kasparov module $(P\mathcal{E}, PF)$. There is an operator homotopy between $(\mathcal{E}, F)$ and $(\mathcal{E}, PF) = (P\mathcal{E}, PF) \oplus ((1-P)\mathcal{E}, 0)$, with the second summand being degenerate. This proves that $(\mathcal{E}, F)$ represents the product.

Assume now that A is not unital. Let $A^\sim$ be the unitization of A, with 1 acting as identity on $\mathcal{E}$. Following [**Sk84**, Prop.17], let $\widehat{A[0,1]}$ be the G-$(A, A^\sim[0,1])$-module:

$$\widehat{A[0,1]} = \{\, f : [0,1] \to A^\sim \mid f(1) \in A \,\} \subseteq A^\sim[0,1].$$

Notice that A acts as multiplication by constant functions. Let $\widetilde{\mathcal{E}} = \widehat{A[0,1]}L \otimes_{A^\sim[0,1]L} (\mathcal{E}[0,1])L$, and let $\widetilde{F}$ be an $(1 \otimes F)$-connection for $\widehat{A[0,1]}L$. Consider $\widetilde{u} = \{(1-s)1 + su\}_{s\in[0,1]} \in \mathcal{C}^{(0)}(\widehat{A[0,1]}L)$. Finally, let $\widetilde{h}$ be any section of LL that makes $(\widetilde{\mathcal{E}}_{\widetilde{h}}, \widetilde{F}_{\widetilde{h}}) = (\mathrm{Res}_{\widetilde{h}})_*(\,(\widetilde{\mathcal{E}}, \widetilde{F})\,)$ an asymptotic Kasparov G-$(A, B[0,1])$-module. Then $(\widetilde{\mathcal{E}}_{\widetilde{h},0}, \widetilde{F}_{\widetilde{h},0})$ is homotopic (via a 'stretching') with $(\mathcal{E}, F)$, and $(\widetilde{\mathcal{E}}_{\widetilde{h},1}, \widetilde{F}_{\widetilde{h},1})$ represents $1_A \sharp_A x$. ∎

Remark. Theorem 2.4.3 and Proposition 2.4.7 imply that, for any G-C^*-algebra A, $KE^G(A, A)$ is a ring with unit.

The following notion is important in further studying the properties of KE-theory and in applications. See [**Kas88**, 2.17], [**Blck**, 19.1] for the corresponding definition in KK-theory.

Definition 2.4.8. Let D_1 and D_2 be G-C^*-algebras. An element $\alpha \in KE^G(D_1, D_2)$ is called *KE-equivalence* (or *invertible*) if there exists an element $\beta \in KE^G(D_2, D_1)$ such that $\alpha \sharp_{D_2} \beta = 1_{D_1}$ and $\beta \sharp_{D_1} \alpha = 1_{D_2}$. If such an element α exists then D_1 and D_2 are called *KE-equivalent.*

Theorem 2.4.9 (Stability in KE-theory). *For any G-C^*-algebra A, A and $A \otimes \mathcal{K}(\mathcal{H}_G)$ are KE-equivalent.*

For another proof of this result we refer to the end of Section 3.2. It also follows from the next more general result, which is just an adaptation of [**Kas88**, 2.18].

THEOREM 2.4.10. *If* D_1 *and* D_2 *are linked by a* G*-imprimitivity bimodule* $\mathcal{I}$*, then there is an invertible element in* $KE^G(D_1, D_2)$*. Moreover, for any separable* G*-*C^**-algebras* A *and* B*,*

$$KE^G(A, B \otimes D_1) \simeq KE^G(A, B \otimes D_2), \quad \text{and} \quad KE^G(A \otimes D_1, B) \simeq KE^G(A \otimes D_2, B).$$

COROLLARY 2.4.11. *For any separable* G*-*C^**-algebras* A *and* B*, we have* $KE^G(A, B) \simeq KE^G(A, B \otimes \mathcal{K}(\mathcal{H}_G)) \simeq KE^G(A \otimes \mathcal{K}(\mathcal{H}_G), B) \simeq KE^G(A \otimes \mathcal{K}(\mathcal{H}_G), B \otimes \mathcal{K}(\mathcal{H}_G))$.

In the spirit of the other two bivariant theories [**Kas75**], [**Kas88**], [**GHT**], [**Thms98**], we can define *higher order* KE*-theory groups*. We shall be rather brief, because they will not be used in the thesis. Details, plus the study of issues like Bott periodicity, Thom isomorphism, long exact sequences etc. will form the substance of a future paper. We recall that $\mathcal{C}_{-n}$ is the Clifford algebra of $\mathbb{R}^n$, *i.e.* the universal algebra with odd generators $\{e_1, ..., e_n\}$ satisfying $e_i e_j + e_j e_i = -2\delta_{ij}$, for $1 \leq i, j \leq n$, ${e_i}^* = -e_i$, and $\|e_i\| = 1$. (The grading is the standard one, and the notation coincides with the one from [**Kas75**]. The adjoint and the norm refer to the fact that $\mathcal{C}_{-n}$ can be given the structure of a C^*-algebra.)

DEFINITION 2.4.12. $KE^G_n(A, B) = KE^G(A, B \otimes \mathcal{C}_{-n})$, for $n = 1, 2, \ldots$.

2.5. The proof of the technical theorem

In this section the following is proved:

TECHNICAL THEOREM (Theorem 2.3.9). *Let G be a locally compact σ-compact Hausdorff group, and let A, B, and D be separable graded G-C^*-algebras. Consider two asymptotic Kasparov modules $(\mathcal{E}_1, F_1) \in ke^G(A, D)$ and $(\mathcal{E}_2, F_2) \in ke^G(D, B)$. There exists a triple $(\, u, \underline{F}, h_0\,)$, with u a self-adjoint element of $\mathcal{C}^{(0)}(\mathcal{E}_1)$, $\underline{F}$ an F_2-connection for $\mathcal{E}_1$, and h_0 a section of $[1,\infty)\times[1,\infty)$, as in Overview 2.3.7, such that for any other section $h \geq h_0$*

$$(\mathcal{E}_h, F_h) = (\mathrm{Res}_h)_*\big(\, \mathcal{E}_1 \otimes_{DL} \mathcal{E}_2 L, F_1 \otimes_{DL} 1 + (u \otimes_{DL} 1)\, \underline{F}\,\big)$$

is an asymptotic Kasparov G-(A, B)-module.

PROOF. We shall justify Steps 1-4 of the Overview 2.3.7. *Step 1*, in which u is constructed, is nothing but Lemma 2.2.14 applied to $(\mathcal{E}_1, F_1)$. The existence of the connection $\underline{F} = \underline{F}^*$ in *Step 2* follows from Proposition 2.3.4 (and after choosing $F_2 = F_2^*$). So far we succeeded to create the pair of 'two-dimensional' objects $(\mathcal{E}, F) = (\mathcal{E}_1 \otimes_{DL} \mathcal{E}_2 L, F_1 \otimes_{DL} 1 + (u \otimes_{DL} 1)\, \underline{F})$. For *Step 3*, we obtain h_{00} by applying Lemma 2.3.6 for the set $K = \{u, uF_1, ua_1, ua_2, \ldots, ua_n, \ldots\}$, where $\{\, a_n\,\}_{n=1}^{\infty}$ is a dense subset of A. The essential *Step 4* is concerned with finding an appropriate section h_0 such that $(\mathcal{E}_{h_0}, F_{h_0}) = (\mathrm{Res}_{h_0})_*(\,(\mathcal{E}, F)\,)$ will be the asymptotic Kasparov G-(A, B)-module which represents the product. For this to happen, the axioms (aKm1)–(aKm4) must be satisfied. The tensor products that appear below are all inner (over DL), but the C^*-algebra will be omitted in order to simplify the writing.

• The simple computation: $(F - F^*)(a \otimes 1) = \big(F_1 \otimes 1 + (u \otimes 1)\, \underline{F} - F_1^* \otimes 1 - \underline{F}^*\, (u \otimes 1)\big)(a \otimes 1) = (F_1 - F_1^*)a \otimes 1 + [u \otimes 1, \underline{F}](a \otimes 1)$, shows that (aKm1) for F_h is satisfied for *any* $h \geq h_{00}$, due to (aKm1) for F_1, and (5) of Step 3.

• Next, given $a \in A$, we have $[F, a \otimes 1] = F_1 a \otimes 1 + (u \otimes 1)\, \underline{F}\, (a \otimes 1) - (-1)^{\partial a} a F_1 \otimes 1 - (-1)^{\partial a}(au \otimes 1)\, \underline{F} = [F_1, a] \otimes 1 - (-1)^{\partial a}[ua \otimes 1, \underline{F}] + (-1)^{\partial a}([u, a] \otimes 1)\, \underline{F} + [u \otimes 1, \underline{F}]\, (a \otimes 1)$.

Consequently (aKm2) for F_h is also satisfied for *any* $h \geq h_{00}$, because of (aKm2) for F_1, and (2), (5), and (7).

• For (aKm3), it is noted that:

$$
\begin{aligned}
& a\left(F^2-1\right)a^* \\
&\quad = (a\otimes 1)\Big(F_1^2\otimes 1 + (u\otimes 1)\underline{F}(F_1\otimes 1) + (F_1\otimes 1)(u\otimes 1)\underline{F} - (u\otimes 1)\underline{F}\,[\underline{F}, u\otimes 1] \\
&\qquad\qquad + (u\otimes 1)\underline{F}^2(u\otimes 1) - 1\Big)(a^*\otimes 1) \\
&\quad \sim \big((au)\,F_1^2\,(au)^*\big)\otimes 1 + (1-u^2)\big(a(F_1^2-1)a^*\big)\otimes 1 + (a\otimes 1)\big(\, [u\otimes 1, \underline{F}](F_1\otimes 1) \\
&\qquad\qquad + ([F_1, u]\otimes 1)\underline{F} + [uF_1\otimes 1, \underline{F}] - (u\otimes 1)\underline{F}\,[\underline{F}, u\otimes 1]\,\big)(a^*\otimes 1) \\
&\qquad\qquad + (a\otimes 1)\,(u\otimes 1)\,\big(\underline{F}^2-1\big)\,(u\otimes 1)\,(a\otimes 1)^*, \text{ modulo } \mathcal{J}(\mathcal{E}_1)\otimes_{DL} 1.
\end{aligned}
$$

(For the second equality $\sim$ above, we used (1) and (2) of Step 1, and the self-adjointness of u.) The restriction of the first six terms to *any* $h \geq h_{00}$ will give a positive element modulo $\mathcal{J}(\mathcal{E}_h)$, because of (3), (5) and (6). So we shall have (aKm3) satisfied provided that

$$
(u\otimes 1)\left(\underline{F}^2-1\right)(u\otimes 1) \text{ restricts to a positive element modulo } \mathcal{C}(\mathcal{E}_h)+\mathcal{J}(\mathcal{E}_h). \tag{2.21}
$$

Showing (2.21) is a critical point in the construction. Let $\{I_n\}_{n=0}^{\infty}$ be a cover of $[1,\infty)$ by closed intervals of the form $I_n = [t_n, t_{n+2}]$, with $t_0 = 1$, and $\{t_n\}_n$ being a strictly increasing sequence with $\lim_{n\to\infty} t_n = \infty$. Let $T_{1,n} = t_n$, for $n \geq 0$, and $T_{2,0} = 1$. If $\{\mu_n\}_{n=0}^{\infty}$ is a partition of unity subordinated to this cover, then $u\otimes 1 = \sum_{n=0}^{\infty}(\mu_n u\otimes 1)$. For each $n \geq 1$, we can approximate $(\mu_n u\otimes 1)$ by a *self-adjoint* finite rank operator

$$
K_n = \sum_{i=1}^{N_n} T_{\xi_i}\, T^*_{\eta_i} = \sum_{i=1}^{N_n} T_{\eta_i}\, T^*_{\xi_i}, \text{ with } \xi_i, \eta_i \in \mathcal{E}_1|_{I_n}, \text{ for } i = 1, 2, \ldots, N_n, \tag{2.22}
$$

and such that $\|(\mu_n u \otimes 1) - K_n\| < 1/(24n(\|F_2\|^2 + 1))$. Note that:

$$
\begin{aligned}
K_n\, \left(\underline{F}^2 - 1\right)\, K_n^* &= \sum_{i,j=1}^{N_n} T_{\xi_i}\, T_{\eta_i}^*\, \left(\underline{F}^2 - 1\right)\, T_{\eta_j}\, T_{\xi_j}^* \\
&\sim \sum_{i,j=1}^{N_n} T_{\xi_i}\, T_{\eta_i}^* T_{\eta_j}\, \left(F_2^2 - 1\right)\, T_{\xi_j}^*, \quad \text{modulo } \mathcal{J}(\mathcal{E}) \qquad (2.23)\\
&= \sum_{i,j=1}^{N_n} T_{\xi_i}\, \langle \eta_i, \eta_j\rangle\, \left(F_2^2 - 1\right)\, T_{\xi_j}^*.
\end{aligned}
$$

There exists $\tau_{n,1}$ such that $\left\|(\underline{F}^2 T_{\eta_j} - T_{\eta_j} F_2^2)_{(t_1,t_2)}\right\| < 1/(12nN_n^2)$, for all η_j, all $t_1 \in I_n$, and all $t_2 > \tau_{n,1}$. This implies that the error of the commutation that was used for the second line of equation (2.23) is smaller than $1/(12n)$, in norm and when restricted to the graph of any section h whose values on I_n are bigger than $\tau_{n,1}$. Using Lemma 1.1.1, we see that the matrix $P = (\langle \eta_i, \eta_j\rangle) \in M_{N_n}(DL)$ is positive, $P = QQ^*$, with $Q = (d_{ij})$, and we get:

$$
\begin{aligned}
\sum_{i,j=1}^{N_n} T_{\xi_i}\, \langle \eta_i, \eta_j\rangle\, \left(F_2^2 - 1\right)\, T_{\xi_j}^* &= \sum_{i,j=1}^{N_n} T_{\xi_i}\, \big(\sum_{k=1}^{N_n} d_{ik}\, d_{jk}^*\big)\, \left(F_2^2 - 1\right)\, T_{\xi_j}^* \qquad (2.24)\\
&\sim \sum_{k=1}^{N_n} \Big(\big(\sum_{i=1}^{N_n} T_{\xi_i}\, d_{ik}\big)\, \left(F_2^2 - 1\right)\, \big(\sum_{j=1}^{N_n} T_{\xi_j}\, d_{jk}\big)^*\Big), \qquad \text{modulo } \mathcal{J}(\mathcal{E}).
\end{aligned}
$$

There exists $\tau_{n,2}$ such that $\left\|[d_{jk}, F_2^2]_{(t_1,t_2)}\right\| < 1/(12nN_n^3)$, for all d_{jk}, all $t_1 \in I_n$, and all $t_2 > \tau_{n,2}$. This implies that the error due to asymptotic commutativity ((aKm2) for F_2, used to obtain the second line of equation (2.24)) is smaller than $1/(12n)$, in norm and when restricted to the graph of any section h whose values on I_n are bigger than $\tau_{n,1}$. Let $\{\delta_m\}_m$ be an approximate unit in D. Because of (aKm3) for F_2,

$$
\sum_{k=1}^{N_n} \Big(\big(\sum_{i=1}^{N_n} T_{\xi_i}\, d_{ik}\big)\, \delta_m\, \left(F_2^2 - 1\right)\, \delta_m\, \big(\sum_{j=1}^{N_n} T_{\xi_j}\, d_{jk}\big)^*\Big) \qquad (2.25)
$$

is positive modulo $\mathcal{C}(\mathcal{E}|_{I_n}) + \mathcal{J}(\mathcal{E}|_{I_n})$. Choose m_0 such that the entire sum from (2.25) approximates the one from the second line of (2.24) by $1/(12n)$.

Let $T_{2,n} = \max\{\tau_{n,1}, \tau_{n,2}, T_{2,(n-1)} + 1\}$. (To be precise, there is also an $\tau_{n,3}$ coming from (aKm4) to be taken into account, but we ignore it for the moment.) Once the sequence $\{T_{2,n}\}_n$ has been constructed, we define h_0 on $[T_{1,n}, T_{1,(n+1)}]$ as the linear function satisfying

$h(T_{1,n}) = T_{2,n}$ and $h_0(T_{1,(n+1)}) = T_{2,(n+1)}$. The estimates above show that the restriction to the graph of $h_0|_{I_n}$ of $(\mu_n u \otimes 1)\,(\underline{F}^2 - 1)\,(\mu_n u \otimes 1)^*$ is positive modulo $\mathcal{C}(\mathcal{E}_{h_0})$, with an error which is smaller than $1/(3n)$, in norm. At most three such terms are non-zero over I_n, this proves (2.21) for any $h \geq h_0$, and consequently F_h satisfies (aKm3).

• Finally, for any $g \in G$, we have:

$$\begin{aligned}\big(g(F) - F\big)(a \otimes 1) &= \big(g\,(F_1 \otimes 1) + g\,(u \otimes 1)\; g\,(\underline{F}) - (F_1 \otimes 1) - (u \otimes 1)\,\underline{F}\big)(a \otimes 1)\\ &= \big(\,g(F_1) - F_1\big)a \otimes 1 + \big((g(u) - u) \otimes 1\big)\; g(\underline{F})(a \otimes 1)\\ &\qquad + (u \otimes 1)\big(g(\underline{F}) - \underline{F}\big)(a \otimes 1).\end{aligned}$$

Due to (aKm4) for F_1 and (4) of Step 1, the first two terms put no extra constraints on h_0. For the third one, $u \otimes 1$ can be approximated, as in the proof of (aKm3), on each interval I_n, by a finite sum $\sum_i T_{\xi_i} T^*_{\eta_i}$. A simple computation shows that $g\,T^*_{\eta_i} = T^*_{g(\eta_i)}$. Consequently:

$$\begin{aligned}T_{\xi_i} T^*_{\eta_i}\big(g(\underline{F}) - \underline{F}\big) &= T_{\xi_i}\, g\big(g^{-1}(T^*_{\eta_i})\,\underline{F}\big) - T_{\xi_i} T^*_{\eta_i} \underline{F}\\ &\sim (-1)^{\partial\eta_i}\, T_{\xi_i}\, g\big(F_2 T^*_{g^{-1}(\eta_i)}\big) - (-1)^{\partial\eta_i}\, T_{\xi_i} F_2 T^*_{\eta_i}, \quad \text{modulo } \mathcal{J}(\mathcal{E})\\ &= (-1)^{\partial\eta_i}\, T_{\xi_i}\, \big(g(F_2) - F_2\big)\, T^*_{\eta_i}.\end{aligned}$$

Further modification (increase) of h_0, using (aKm4) for F_2, will make the above errors go to zero when restricted to the graph of h_0. (This is the place where the $\tau_{n,3}$ mentioned above makes its appearance.) This shows that (aKm4) holds for F_h, for any $h \geq$ (new h_0), and the proof of the Technical Theorem is complete. ∎

Remark. The only important fact that h_0 encodes in the construction of the product is a certain behavior that occurs when $t_1 \to \infty$ and $t_2 \to \infty$, with h_0 correlating t_1 and t_2. We have noticed that certain decay properties hold true on entire 'stripes' $[T_{1,n}, T_{1,n+1}] \times [T_{2,n}, \infty)$, and not only on the graph of h_0. This observation is used in the proof of the associativity of the product (see Lemma 2.4.4), where it allows us to focus on the analysis of the operators that appear in the construction rather than on the sections.

CHAPTER 3

KE-theory: relation with KK-theory and E-theory

Assume that a group G (locally compact, σ-compact, Hausdorff) is given. In this chapter we construct two functors: $\Theta : \mathbf{KK^G} \to \mathbf{KE^G}$, and $\Xi : \mathbf{KE^G} \to \mathbf{E_G}$. The three categories have all the same objects: the separable and graded G-C^*-algebras. The morphisms of $\mathbf{KK^G}$ ([**Hg87a**], [**Hg90b**], [**Thms98**]) are the KK-theory groups, with composition given by the Kasparov product (see Theorem 1.4.7). The morphisms of $\mathbf{KE^G}$ are the KE-theory groups, with composition given by the product defined in Section 2.3. The morphisms of $\mathbf{E_G}$ ([**GHT**], [**Thms98**]), are the E-theory groups, with the corresponding composition product (see Definition 1.5.6). Both functors are the identity on objects.

The action of Θ on morphisms is explained in Section 3.1, and in Section 3.2 it is proven that it preserves the composition rule. The similar properties for Ξ are contained in Section 3.3 and Section 3.4, respectively. We observe that the techniques used in the construction of Θ and Ξ are completely different: for Θ we use 'variations on Kasparov's Technical Theorem' (a key result in KK-theory), while for Ξ we use stabilization and stability (critical techniques in E-theory). These facts reinforce the point of view that KE-theory sits in a natural way 'in between' KK-theory and E-theory.

The outcome of the chapter is the construction of an *explicit* natural transformation, namely the composition $\Xi \circ \Theta$, between KK-theory and E-theory. This transformation *preserves the product structures* of the two theories. In detail, this connecting functor is:

(3.1)
$$\begin{array}{ccccc} KK^G(A,B) & \xrightarrow{\Theta} & KE^G(A,B) & \xrightarrow{\Xi} & E_G(A,B) \\ (\mathcal{E},F) & \mapsto & \big\{\big(\mathcal{E},(1-u_t)F(1-u_t)\big)\big\}_t & \mapsto & \big\{f\otimes a \overset{\varphi_t}{\mapsto} f\big((1-u_t)F(1-u_t)\big)\,a\big\}_t \end{array}$$

Here $(\mathcal{E}, F)$ is a Kasparov module, $\{u_t\}_t$ is a quasi-invariant quasi-central approximate unit for $\mathcal{K}(\mathcal{E})$, and $\{\varphi_t\}_t : C_0((-1,1)) \otimes A \dashrightarrow B \otimes \mathcal{K}$ is an asymptotic family.

3.1. The map $\Theta : KK^G \to KE^G$

Let G be a group, A and B be G-C^*-algebras. Consider $(\mathcal{E}, F) \in kk^G(A, B)$. This means that $\mathcal{E}$ is a graded Hilbert G-B-module, acted on by A, and $F \in \mathcal{B}(\mathcal{E})$ is an odd operator such that $(F - F^*)a$, $[F, a]$, $(F^2 - 1)a$, $(g(F) - F)a$ belong all to $\mathcal{K}(\mathcal{E})$, for all $a \in A$, $g \in G$. Denote by $C^*(\mathcal{K}(\mathcal{E}), A, F)$ the smallest C^*-subalgebra of $\mathcal{B}(\mathcal{E})$ that contains $\mathcal{K}(\mathcal{E})$, $\varphi(A)$, and F, and let $u = \{u_t\}_{t \in [1,\infty)}$ be a quasi-invariant quasi-central approximate unit of $\mathcal{K}(\mathcal{E}) \subset C^*(\mathcal{K}(\mathcal{E}), A, F) \subset \mathcal{B}(\mathcal{E})$. It will be convenient, at least for notational purposes, to regard u as an element of $\mathcal{C}(\mathcal{E}L)$. We make the notation: $\widehat{\mathcal{E}} = \mathcal{E}L$ (constant family of modules), and $\widehat{F} = \{(1-u_t)F(1-u_t)\}_t = (1-u)F(1-u)$.

Claim 3.1.1. $\big\{\big(\mathcal{E}, (1-u_t)F(1-u_t)\big)\big\}_t = \big(\widehat{\mathcal{E}}, \widehat{F}\big)$ is an asymptotic Kasparov G-(A,B)-module.

Proof. We shall use heavily the properties of the q.i.q.c.a.u. u.

- $\widehat{F} \in \mathcal{B}(\widehat{\mathcal{E}})$. This follows from the norm continuity of $\{F_t\}_t$, which in turn is based on the norm continuity of the familty $\{u_t\}_t$.
- $\widehat{F}$ *satisfies* (aKm1). For any $a \in A$, we have:

$$\begin{aligned}\left(\widehat{F} - \widehat{F}^*\right) a &= (1-u)F(1-u)a - (1-u)F^*(1-u)a \\ &\sim (1-u)\big((F-F^*)a\big)(1-u), \quad \text{modulo } \mathcal{J}(\widehat{\mathcal{E}}), \\ &\sim 0, \quad \text{modulo } \mathcal{J}(\widehat{\mathcal{E}}).\end{aligned}$$

We used for the second line of the formula above the quasi-centrality, $\lim_{t\to\infty} \|au_t - u_t a\| = 0$, which implies $[(1-u), a] \in \mathcal{J}(\widehat{\mathcal{E}})$. The last line is obtained from the fact that $\{u_t\}_t$ is an approximate unit for $\mathcal{K}(\mathcal{E})$: $\lim_{t\to\infty} \|(1-u_t)(F-F^*)a\| = 0$.

- $\widehat{F}$ *satisfies* (aKm2). Indeed, given $a \in A$: $[\widehat{F}, a] = (1-u)F(1-u)a - (-1)^{\partial a} a(1-u)F(1-u) \sim (1-u)[F,a](1-u) \sim 0$, modulo $\mathcal{J}(\widehat{\mathcal{E}})$. Exactly as for (aKm1), we used $\{u_t\}_t$ being quasi-central and approximate unit.

• $\widehat{F}$ *satisfies* (aKm3). For any $a \in A$, we have:

$$
\begin{aligned}
a\left(\widehat{F}^2 - 1\right) a^* &= a\left((1-u)F(1-u)^2F(1-u) - 1\right) a^* \\
&\sim a\left((1-u)^2F^2(1-u)^2 - (1-u)^4 + p(u)\right) a^* \quad \text{modulo } \mathcal{J}(\widehat{\mathcal{E}}) \\
&\sim (1-u)^2\, a(F^2-1)\,(1-u)^2 a^* + a\,p(u)\,a^* \qquad \text{modulo } \mathcal{J}(\widehat{\mathcal{E}}) \\
&\sim 0, \quad \text{modulo } \mathcal{C}(\widehat{\mathcal{E}}) + \mathcal{J}(\widehat{\mathcal{E}}).
\end{aligned}
\tag{3.2}
$$

Here $p(u) = u^4 - 4u^3 + 6u^2 - 4u$. We used: quasi-centrality of $\{u_t\}_t$ (for the second and third line of (3.2)), $a(F^2-1) \in \mathcal{K}(\mathcal{E})$, $\{u_t\}_t$ being an approximate unit for $\mathcal{K}(\mathcal{E})$, and the fact that $u \in \mathcal{C}(\widehat{\mathcal{E}})$.

• $\widehat{F}$ *satisfies* (aKm4). Indeed, given $a \in A$ and $g \in G$: $(g(\widehat{F}) - \widehat{F})\,a = \big(g((1-u)F(1-u)) - (1-u)F(1-u)\big)\,a \sim (1-u)\,\big((g(F)-F)a\big)\,(1-u) \sim 0$, modulo $\mathcal{J}(\widehat{\mathcal{E}})$. (The quasi-invariance of $\{u_t\}_t$, $\lim_{t\to\infty} \|g(u_t) - u_t\| = 0$, was also used above.) □

The connection between the KK-theory and KE-theory groups is given by the following:

THEOREM 3.1.2. *Let G be a group, A and B be G-C^*-algebras. Consider $(\mathcal{E}, F) \in kk^G(A,B)$, and let $u = \{u_t\}_{t\in[1,\infty)}$ be a quasi-invariant quasi-central approximate unit of $\mathcal{K}(\mathcal{E}) \subset C^*(\mathcal{K}(\mathcal{E}), A, F) \subset \mathcal{B}(\mathcal{E})$. The map*

$$
\Theta : kk^G(A,B) \to ke^G(A,B),\ \big(\mathcal{E}, F\big) \mapsto \big\{\, \big(\mathcal{E}, (1-u_t)F(1-u_t)\big) \,\big\}_{t\in[1,\infty)},
\tag{3.3}
$$

passes to quotients and gives a group homomorphism $\Theta : KK^G(A,B) \to KE^G(A,B)$.

PROOF. Knowing from the claim that $\Theta((\mathcal{E},F))$ is an asymptotic Kasparov module, we show next that its class in $KE^G(A,B)$ does not depend on the quasi-central approximate unit $\{u_t\}_t$ used in its definition. Indeed, consider another such q.i.q.c.a.u. $\{u'_t\}_t$. Let $\mathbf{u} = (1-s)u + su'$, $s \in [0,1]$. The computations above show that $\big(\mathcal{E}(L \times [0,1]), (1-\mathbf{u})F(1-\mathbf{u})\big) \in ke^G(A, B[0,1])$ is a homotopy between the constructions performed using u and u', respectively. In order to show that the map Θ is well defined at the level of homotopy, we consider a homotopy $(\boldsymbol{\mathcal{E}}, \mathbf{F}) \in kk^G(A, B[0,1])$ between $(\mathcal{E}, F)$ and $(\mathcal{E}', F')$. Then $\Theta((\boldsymbol{\mathcal{E}}, \mathbf{F})) \in ke^G(A, B[0,1])$, constructed using a q.i.q.c.a.u. $\mathbf{u}$ of $\mathcal{K}(\boldsymbol{\mathcal{E}})$, represents a homotopy between $\Theta((\mathcal{E}, F))$ and $\Theta((\mathcal{E}', F'))$, constructed using $u = \mathbf{u} \otimes_{\mathrm{ev}_0} 1$ and $u' = \mathbf{u} \otimes_{\mathrm{ev}_1} 1$, respectively. Consequently homotopic Kasparov modules are sent by Θ into homotopic asymptotic Kasparov modules, and this shows that Θ given by (3.3) is well defined. Finally, it is clear that Θ preserves the addition. ∎

REMARK 3.1.3. The proof of Claim 3.1.1 given above leads to an important observation. The map Θ will be well-defined for *any* approximate unit $u = \{u_t\}_t$ of $\mathcal{K}(\mathcal{E})$ satisfying the minimal requirements of being quasi-invariant and of asymptotically commuting with the elements of A and with F, *i.e.* $\lim_{t\to\infty} \|au_t - u_t a\| = 0$, for all $a \in A$, and $\lim_{t\to\infty} \|Fu_t - u_t F\| = 0$. Moreover the straight path homotopy between any two such approximate units will preserve these minimal requirements.

EXAMPLES 3.1.4. (a) $\Theta(1_A) = 1_A$, *i.e.* $\Theta((A, 0)) = (AL, 0)$, for any A.
(b) The KK-theory element that corresponds to a $*$-homomorphism $\varphi : A \to B \otimes \mathcal{K}$ is sent by Θ to the corresponding element in KE-theory (Example 2.1.10).
(c) We study next Example 2.1.11. The KK-theory class of the Dirac operator is given by $(L^2(M, \mathbb{S}), \chi(D))$, with χ a normalizing function. The characteristic of this example is that the *functional calculus* will provide a family of compact operators $\{u_t\}_t$, which depends on D but which satisfies the minimal requirements of Remark 3.1.3. Using this approximate unit in the definition of Θ we have exactly: $(1 - u_t)\,\chi(D)\,(1 - u_t) = \chi(\frac{1}{t}D)$.

We end this section by showing that the connecting map Θ commutes with the map σ:

PROPOSITION 3.1.5. *Let A, B, D be arbitrary G-C^*-algebras. The following diagram is commutative:*

$$\begin{array}{ccc} KK^G(A,B) & \xrightarrow{\Theta} & KE^G(A,B) \\ \sigma_D \big\downarrow & & \big\downarrow \sigma_D \\ KK^G(A\otimes D, B\otimes D) & \xrightarrow[\Theta]{} & KE^G(A\otimes D, B\otimes D). \end{array} \tag{3.4}$$

PROOF. The only problem is the choice of approximate units for the two horizontal maps. Let $u = \{u_t\}_t$ be a q.i.q.c.a.u. of $\mathcal{K}(\mathcal{E}) \subset C^*(\mathcal{K}(\mathcal{E}), A, F) \subset \mathcal{B}(\mathcal{E})$, let $\{d_t\}_t$ be a q.i.a.u. of D, and define $w = \{w_t = u_t \otimes d_t\}_t$. It is easy to check that $\{w_t\}_t$ is an approximate unit for $\mathcal{K}(\mathcal{E}) \otimes D \simeq \mathcal{K}(\mathcal{E} \otimes D) \lhd \mathcal{B}(\mathcal{E} \otimes D)$, which satisfies the minimal requirements discussed in Remark 3.1.3. Use $\{w_t\}_t$ to define the bottom map Θ. The asymptotic Kasparov modules $\sigma_D\big(\Theta((\mathcal{E}, F))\big)$ and $\Theta\big(\sigma_D((\mathcal{E}, F))\big)$ can be operator homotoped by Lemma 2.2.9 and we are through. $\square$

3.2. The map Θ is a functor

We show in this section that Θ preserves the product structures. As with the definition of the product in Section 2.3, the case of external product is studied first.

THEOREM 3.2.1. *Let G be a group, A_1, A_2, B_1, and B_2 be G-C^*-algebras. Consider $x = (\mathcal{E}_1, F_1) \in kk^G(A_1, B_1)$, and $y = (\mathcal{E}_2, F_2) \in kk^G(A_2, B_2)$. Then*

$$\Theta(\, \boldsymbol{x} \sharp_{\mathbb{C}} \boldsymbol{y} \,) \;=\; \Theta(\boldsymbol{x}) \sharp_{\mathbb{C}} \Theta(\boldsymbol{y})\,. \tag{3.5}$$

PROOF. Let $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$. We recall from Example 1.4.8 that the product $\boldsymbol{x} \sharp_{\mathbb{C}} \boldsymbol{y}$ is the class of $\big(\mathcal{E}, M^{\frac{1}{2}}(F_1 \otimes 1) + N^{\frac{1}{2}}(1 \otimes F_2)\big)$, with M and N obtained from Kasparov's Technical Theorem applied to: $J = \mathcal{K}(\mathcal{E}) = \mathcal{K}(\mathcal{E}_1) \otimes \mathcal{K}(\mathcal{E}_2)$, $E_1 = \mathcal{K}(\mathcal{E}_1) \otimes A_2$, $E_2 = A_1 \otimes \mathcal{K}(\mathcal{E}_2)$, and $\Delta =$ vector space span of $\{F_1 \otimes 1, 1 \otimes F_2, A_1 \otimes A_2\}$. Let $u = \{u_t\}_t$ be a q.i.q.c.a.u. for $\mathcal{K}(\mathcal{E}_1) \subset C^*(\mathcal{K}(\mathcal{E}_1), A_1, F_1) \subset \mathcal{B}(\mathcal{E}_1)$, $v = \{v_t\}_t$ be a q.i.q.c.a.u. for $\mathcal{K}(\mathcal{E}_2) \subset C^*(\mathcal{K}(\mathcal{E}_2), A_2, F_2) \subset \mathcal{B}(\mathcal{E}_2)$, and $w = \{w_t\}_t$ be a q.i.q.c.a.u. for $\mathcal{K}(\mathcal{E}) \subset C^*(\mathcal{K}(\mathcal{E}), E_1, E_2, \Delta) \subset \mathcal{B}(\mathcal{E})$. Let $u_t^{\perp} = (1 - u_t) \otimes 1$, $v_t^{\perp} = 1 \otimes (1 - v_t)$, and $w_t^{\perp} = 1 - w_t$.

The required result will be obtained by showing that the following operators determine homotopic asymptotic Kasparov modules:

$$\widehat{F} \;=\; \big\{\, w_t^{\perp} \big(M^{\frac{1}{2}}(F_1 \otimes 1) + N^{\frac{1}{2}}(1 \otimes F_2)\big)\, w_t^{\perp} \,\big\}_{t \in [1,\infty)}, \tag{3.6}$$

$$\widehat{F}'' \;=\; \big\{\, w_{s(t)}^{\perp} \big(M_t^{\frac{1}{2}}(F_1 \otimes 1) + N_t^{\frac{1}{2}}(1 \otimes F_2)\big)\, w_{s(t)}^{\perp} \,\big\}_{t \in [1,\infty)},$$

$$\widehat{F}''' \;=\; \big\{\, u_t^{\perp}(F_1 \otimes 1)u_t^{\perp} + w_{s(t)}^{\perp} N_t^{\frac{1}{2}}(1 \otimes F_2) w_{s(t)}^{\perp} \,\big\}_{t \in [1,\infty)},$$

$$\widehat{F}' \;=\; \big\{\, u_t^{\perp}(F_1 \otimes 1)u_t^{\perp} + v_t^{\perp}(1 \otimes F_2)v_t^{\perp} \,\big\}_{t \in [1,\infty)}. \tag{3.7}$$

Here: $\widehat{F}$ is the representative of $\Theta(\,\boldsymbol{x}\,\sharp_{\mathbb{C}}\,\boldsymbol{y}\,)$, and $\widehat{F}'$ is the representative of $\Theta(\boldsymbol{x})\,\sharp_{\mathbb{C}}\,\Theta(\boldsymbol{y})$ (as explained in Example 2.3.1, and using Theorem 3.1.2). The construction of the other two operators $\widehat{F}''$ and $\widehat{F}'''$, including the construction of s and of the families $\{M_t\}_t$ and $\{N_t\}_t$, and the justification of the homotopies $\widehat{F} \sim \widehat{F}'' \sim \widehat{F}''' \sim \widehat{F}'$ is relegated to the next six lemmas. Granting their validity, it follows that the representative $(\mathcal{E}L, \widehat{F})$ of $\Theta(\,\boldsymbol{x}\,\sharp_{\mathbb{C}}\,\boldsymbol{y}\,)$ and the representative $(\mathcal{E}L, \widehat{F}')$ of $\Theta(\boldsymbol{x})\,\sharp_{\mathbb{C}}\,\Theta(\boldsymbol{y})$ are (operatorially) homotopic, so their classes in $KE^G(A_1 \otimes A_2, B_1 \otimes B_2)$ are equal. This finishes the proof. ■

The next task is to justify the claims made in the proof above. *In the next six lemmas we shall use the notation introduced in the proof of Theorem 3.2.1.* We start with the construction of the families $\{M_t\}_t$ and $\{N_t\}_t$ which are used in defining $\widehat{F}''$.

LEMMA 3.2.2. *Let $\widehat{J} = C_0(L, J)$, $\widehat{E}_1 = C_0(L, E_1)$, $\widehat{E}_2 = 1 \otimes E_2$ (with $1 \otimes E_2$ representing constant functions on L with values elements of E_2), and $\widehat{\Delta}$ = vector space span of $1 \otimes \Delta \cup \{1 \otimes M, w, u \otimes 1, 1 \otimes v\}$. Then there exist two positive almost equivariant elements $\widehat{M} = \{M_t\}_t$ and $\widehat{N} = \{N_t\}_t$ in $\mathcal{M}(\widehat{J}) = C_b(L, \mathcal{B}(\mathcal{E}))$ such that*

(3.8)

(1) $\widehat{M} + \widehat{N} = 1$, (2) $\widehat{M}\widehat{E}_1 \subset \widehat{J}$, $\widehat{N}\widehat{E}_2 \subset \widehat{J}$, (3) $[\widehat{M}, \widehat{\Delta}] \subset \widehat{J}$, and $(\star)$ $\widehat{M}(1 \otimes E_1) \subset \mathcal{C}(\mathcal{E}L)$.

PROOF. This is a minor variation of Theorem 1.3.3. Note that $(\star)$ is a consequence of (2). ■

The next result defines $\widehat{F}''$.

LEMMA 3.2.3. *There exists a function $s : [1, \infty) \to [1, \infty)$ which is strictly increasing, bijective, and continuous, such that*

$$\| \, w_{s(t)}^{\perp} \, K_t \, \| \xrightarrow{t \to \infty} 0, \text{ for every } \{K_t\}_t \in \widehat{M}(1 \otimes E_1), \tag{3.9}$$

and

$$\| \, [w_{s(t)}, u_t \otimes 1] \, \|, \| \, [w_{s(t)}, 1 \otimes v_t] \, \| \xrightarrow{t \to \infty} 0. \tag{3.10}$$

For any s as above, let

$$\widehat{F}'' = \{ w^{\perp}_{s(t)} \big(M_t^{\frac{1}{2}}(F_1 \otimes 1) + N_t^{\frac{1}{2}}(1 \otimes F_2) \big) w^{\perp}_{s(t)} \}_{t \in [1,\infty)}. \tag{3.11}$$

Then $(\mathcal{E}L, \widehat{F}'')$ is an asymptotic Kasparov G-$(A_1 \otimes A_2, B_1 \otimes B_2)$-module.

Proof. We start by choosing a sequence $\{K_n = \{K_{n,t}\}_t\}$, with $K_1 = u \otimes 1$, $K_2 = 1 \otimes v$, and $\{ K_n \,|\, n = 3, 4, \ldots \}$ being a dense subset in $\widehat{M}(1 \otimes E_1)$ (E_1 is separable). For $n = 1, 2$, use the quasi-centrality of w to construct (possibly rapidly increasing) functions s_1, and s_2 that give the desired norm convergence to zero in (3.10). For $n \geq 3$, we recall that $(\star)$ of (3.8) gives $K_n \in \mathcal{C}(\mathcal{E}L)$, such that $K_{n,t} \in \mathcal{K}(\mathcal{E})$, for every t. Use next the fact that w is an approximate unit for $\mathcal{K}(\mathcal{E})$ to construct $s_n : [1, \infty) \to [1, \infty)$ such that $\| w^{\perp}_{s_n(t)} K_{n,t} \| \xrightarrow{t \to \infty} 0$. Finally, apply the diagonalization process described in Lemma 2.2.16 to find a function s which makes (3.9) and (3.10) hold true.

For the second part we check the axioms of an asymptotic Kasparov module.

- $\widehat{F}''$ *satisfies* (aKm1). For any $a_1 \otimes a_2 \in A_1 \otimes A_2$, we have:

$$\begin{aligned}
&\big(\widehat{F}'' - (\widehat{F}'')^*\big)\,(a_1 \otimes a_2) \\
&\qquad \sim w^{\perp}_{s(t)} M_t^{\frac{1}{2}} \big((F_1 - F_1^*) a_1 \otimes a_2\big) w^{\perp}_{s(t)} + w^{\perp}_{s(t)} [M_t^{\frac{1}{2}}, F_1^* \otimes 1] w^{\perp}_{s(t)} (a_1 \otimes a_2) \\
&\qquad\quad + (-1)^{\partial a_1} w^{\perp}_{s(t)} N_t^{\frac{1}{2}} \big(a_1 \otimes (F_2 - F_2^*) a_2\big) w^{\perp}_{s(t)} + w^{\perp}_{s(t)} [N_t^{\frac{1}{2}}, 1 \otimes F_2^*] w^{\perp}_{s(t)} (a_1 \otimes a_2), \\
&\qquad\quad \text{modulo } \mathcal{J}(\mathcal{E}L).
\end{aligned}$$

The first term above is in $\mathcal{J}(\mathcal{E}L)$ due to the property (3.9) of s. The commutators belong to $\widehat{J} \subset \mathcal{J}(\mathcal{E}L)$ because of property (3) of (3.8). The third term belongs to $\mathcal{J}(\mathcal{E}L)$ because of property (2) of (3.8). Consequently the entire sum is in $\mathcal{J}(\mathcal{E}L)$.

- $\widehat{F}''$ *satisfies* (aKm2). Indeed, given $a_1 \otimes a_2 \in A_1 \otimes A_2$:

$$\begin{aligned}
&[\,\widehat{F}'', a_1 \otimes a_2\,] \\
&\qquad \sim w^{\perp}_{s(t)} M_t^{\frac{1}{2}} \big([F_1, a_1] \otimes a_2\big) w^{\perp}_{s(t)} + (-1)^{\partial a_1} w^{\perp}_{s(t)} N_t^{\frac{1}{2}} \big(a_1 \otimes [F_2, a_2]\big) w^{\perp}_{s(t)}, \quad \text{modulo } \mathcal{J}(\mathcal{E}L).
\end{aligned}$$

The final inclusion in $\mathcal{J}(\mathcal{E}L)$ is obtained by using (3.9) and (2) of (3.8).

• $\widehat{F}''$ *satisfies* (aKm3). For any $a = a_1 \otimes a_2 \in A_1 \otimes A_2$, by writing $1 = M_t + N_t$, we have:

$$\begin{aligned}
& a\left((\widehat{F}'')^2 - 1 \right) a^* \\
& \quad \sim a \Big((w^{\perp}_{s(t)})^2 M_t ({F_1}^2 \otimes 1)(w^{\perp}_{s(t)})^2 - M_t \\
& \qquad\qquad + (w^{\perp}_{s(t)})^2 N_t (1 \otimes {F_2}^2)(w^{\perp}_{s(t)})^2 - N_t \Big)\, a^* \qquad \text{modulo } \mathfrak{C}(\mathcal{E}L) + \mathfrak{J}(\mathcal{E}L) \\
& \quad \sim (w^{\perp}_{s(t)})^2 M_t \big(a_1({F_1}^2 - 1) \otimes a_2\big)(w^{\perp}_{s(t)})^2 (a_1 \otimes a_2)^* \\
& \qquad\qquad + (w^{\perp}_{s(t)})^2 N_t \big(a_1 \otimes a_2({F_2}^2 - 1)\big)(w^{\perp}_{s(t)})^2 (a_1 \otimes a_2)^* \quad \text{modulo } \mathfrak{C}(\mathcal{E}L) + \mathfrak{J}(\mathcal{E}L) \\
& \quad \sim 0, \qquad\qquad \text{modulo } \mathfrak{C}(\mathcal{E}L) + \mathfrak{J}(\mathcal{E}L).
\end{aligned}$$

For the first $\sim$ sign we used (3) of (3.8), *i.e.* $[\widehat{M}, F_1 \otimes 1]$, $[\widehat{N}, 1 \otimes F_2] \in \mathfrak{J}(\mathcal{E}L)$, and $\{[M_t, w_{s(t)}]\}_t \in \mathfrak{C}(\mathcal{E}L) + \mathfrak{J}(\mathcal{E}L)$. For the second $\sim$ sign we used $w \in \mathfrak{C}(\mathcal{E}L)$. The first term there belongs to $\mathfrak{J}(\mathcal{E}L)$ because of (3.9), and the second one due to (2) of (3.8). This justifies the last step.

• $\widehat{F}''$ *satisfies* (aKm4). Indeed, given $a_1 \otimes a_2 \in A_1 \otimes A_2$:

$$\begin{aligned}
& \left(g(\widehat{F}'') - \widehat{F}'' \right)(a_1 \otimes a_2) \\
& \quad \sim w^{\perp}_{s(t)} M_t^{\frac{1}{2}} \big((g(F_1) - F_1) a_1 \otimes a_2\big) w^{\perp}_{s(t)} \\
& \qquad\qquad + (-1)^{\partial a_1} w^{\perp}_{s(t)} N_t^{\frac{1}{2}} \big(a_1 \otimes (g(F_2) - F_2) a_2\big) w^{\perp}_{s(t)}, \quad \text{modulo } \mathfrak{J}(\mathcal{E}L).
\end{aligned}$$

We used above the quasi-invariance of w, $\widehat{M}$, and $\widehat{N}$. The proof of the lemma is completed by applying (3.9) and (2) of (3.8). ■

Lemma 3.2.4. *Consider*

$$\widehat{F}''' = \left\{ u^{\perp}_t (F_1 \otimes 1) u^{\perp}_t + w^{\perp}_{s(t)} N_t^{\frac{1}{2}} (1 \otimes F_2) w^{\perp}_{s(t)} \right\}_{t \in [1,\infty)}. \tag{3.12}$$

Then $(\mathcal{E}L, \widehat{F}''')$ *is an asymptotic Kasparov* G-$(A_1 \otimes A_2, B_1 \otimes B_2)$-*module.*

PROOF. We check the axioms.

• $\widehat{F}'''$ *satisfies* (aKm1). For any $a_1 \otimes a_2 \in A_1 \otimes A_2$, we have:

$$\begin{aligned}
&\big(\widehat{F}''' - (\widehat{F}''')^*\big)\,(a_1 \otimes a_2) \\
&\quad \sim u_t^{\perp}\big((F_1 - F_1^*)a_1 \otimes a_2\big)u_t^{\perp} + (-1)^{\partial a_1} w_{s(t)}^{\perp} N_t^{\frac{1}{2}}\big(a_1 \otimes (F_2 - F_2^*)a_2\big)w_{s(t)}^{\perp}, \quad \text{modulo } \mathcal{J}(\mathcal{E}L) \\
&\quad \in \mathcal{J}(\mathcal{E}_1 L) \otimes \mathcal{B}(\mathcal{E}_2) + \mathcal{J}(\mathcal{E}L) \subseteq \mathcal{J}(\mathcal{E}L).
\end{aligned}$$

Compare this with the argument for (aKm1) given in the proof of Lemma 3.2.3.

• $\widehat{F}'''$ *satisfies* (aKm2) and (aKm4). See the computation above and the ones given for the corresponding axioms in the proof of Lemma 3.2.3.

• $\widehat{F}'''$ *satisfies* (aKm3). For any $a = a_1 \otimes a_2 \in A_1 \otimes A_2$, we have:

$$\begin{aligned}
&a\,\big((\widehat{F}''')^2 - 1\big)\,a^* \\
&\sim a\,\Big(\,(u_t^{\perp})^2({F_1}^2 \otimes 1)(u_t^{\perp})^2 + (w_{s(t)}^{\perp})^2 N_t (1 \otimes {F_2}^2)(w_{s(t)}^{\perp})^2 - 1\,\Big)\,a^*, \quad \text{modulo } \mathcal{J}(\mathcal{E}L) \\
&\sim (u_t^{\perp})^2 M_t\big(a_1({F_1}^2 - 1) \otimes a_2\big)(u_t^{\perp})^2 a^* + a(u_t^{\perp})^2 N_t^{\frac{1}{2}}({F_1}^2 \otimes 1)N_t^{\frac{1}{2}}(u_t^{\perp})^2 a^* \\
&\qquad + (w_{s(t)}^{\perp})^2 N_t\big(a_1 \otimes a_2({F_2}^2 - 1)\big)(w_{s(t)}^{\perp})^2 a^*, \quad \text{modulo } \mathcal{C}(\mathcal{E}L) + \mathcal{J}(\mathcal{E}L) \\
&\geq 0, \quad \text{modulo } \mathcal{C}(\mathcal{E}L) + \mathcal{J}(\mathcal{E}L).
\end{aligned}$$

At the first $\sim$ sign we used the asymptotic commutativity of $\{w_{s(t)}\}_t$ with $\{u_t \otimes 1\}_t$, given by (3.10), and the more subtler $[\widehat{N}, u \otimes 1] \in \mathcal{J}(\mathcal{E}L)$ contained in (3) of (3.8), to see that the cross terms belong to $\mathcal{J}(\mathcal{E}L)$. For the second $\sim$ sign we used $1 = M_t + N_t$, and the fact that $(p(u_t) \otimes 1)\, M_t\, (q(u_t) \otimes 1) \in \mathcal{C}(\mathcal{E}L)$, for any polynomials p and q, at least one of which vanish at 0. The first term there belongs to $\mathcal{J}(\mathcal{E}L)$ due to the fact that u is an approximate unit of $\mathcal{K}(\mathcal{E}_1)$, the second one is positive, and the third one is in $\mathcal{J}(\mathcal{E}L)$ due to (2) of (3.8). This justifies the last written inequality. ■

LEMMA 3.2.5. *$\widehat{F}$ and $\widehat{F}''$ are homotopic asymptotic Kasparov modules.*

PROOF. The family of operators

$$\left\{\, w_{s(t)}^{\perp} \Big((xM + (1-x)M_{s(t)})^{\frac{1}{2}}(F_1 \otimes 1) + (xN + (1-x)N_{s(t)})^{\frac{1}{2}}(1 \otimes F_2) \Big)\, w_{s(t)}^{\perp} \,\right\}_{x \in [0,1]} \tag{3.13}$$

gives a homotopy between $\widehat{F}''$ (at $x = 0$) and a representative of $\Theta(\boldsymbol{x} \sharp_{\mathbb{C}} \boldsymbol{y})$ constructed as in (3.6) but with $\{w_{s(t)}\}_t$ as approximate unit of $\mathcal{K}(\mathcal{E})$ insted of $\{w_t\}_t$ (at $x = 1$). The same argument as in the proof of Lemma 3.2.3 shows that for every $x \in [0, 1]$ we obtain an asymptotic Kasparov module. ■

LEMMA 3.2.6. *$a\,[\,\widehat{F}'', \widehat{F}'''\,]\,a^* \geq 0$, modulo $\mathcal{J}(\mathcal{E}L)$, for all $a \in A_1 \otimes A_2$.*

PROOF. Using the asymptotic commutativity of $\{w_{s(t)}\}_t$ and $\{u_t \otimes 1\}_t$ given by (3.10), and the subtler property $[\widehat{M}, u \otimes 1], [\widehat{N}, u \otimes 1] \in \mathcal{J}(\mathcal{E}L)$ contained in (3) of (3.8), we have:

$$\begin{aligned} a\,[\,\widehat{F}'', \widehat{F}'''\,]\,a^* &= a\,\big(\,\widehat{F}''\widehat{F}''' + \widehat{F}'''\widehat{F}''\,\big)\,a^* \\ &\sim a\,\Big(\; w^{\perp}_{s(t)} u^{\perp}_t M_t^{\frac14}(F_1^2 \otimes 1) M_t^{\frac14} u^{\perp}_t w^{\perp}_{s(t)} + (w^{\perp}_{s(t)})^2 M_t^{\frac12} N_t^{\frac12} (F_1 \otimes F_2)(w^{\perp}_{s(t)})^2 \\ &\qquad - w^{\perp}_{s(t)} u^{\perp}_t N_t^{\frac12}(F_1 \otimes F_2) u^{\perp}_t w^{\perp}_{s(t)} + (w^{\perp}_{s(t)})^2 N_t^{\frac12}(1 \otimes F_2^2) N_t^{\frac12}(w^{\perp}_{s(t)})^2 \\ &\qquad + w^{\perp}_{s(t)} u^{\perp}_t M_t^{\frac14}(F_1^2 \otimes 1) M_t^{\frac14} u^{\perp}_t w^{\perp}_{s(t)} + w^{\perp}_{s(t)} u^{\perp}_t N_t^{\frac12}(F_1 \otimes F_2) u^{\perp}_t w^{\perp}_{s(t)} \\ &\qquad - (w^{\perp}_{s(t)})^2 M_t^{\frac12} N_t^{\frac12}(F_1 \otimes F_2)(w^{\perp}_{s(t)})^2 + (w^{\perp}_{s(t)})^2 N_t^{\frac12}(1 \otimes F_2^2) N_t^{\frac12}(w^{\perp}_{s(t)})^2 \;\Big)\,a^*, \\ &\qquad\qquad\qquad\qquad \text{modulo } \mathcal{J}(\mathcal{E}L). \end{aligned}$$

Now: the 1st and 5th term are positive; the 2nd and 7th term cancel out; the 3rd and 6th term also cancel out; the 4th and 8th are equal and positive. This proves the lemma. ■

LEMMA 3.2.7. *$a\,[\,\widehat{F}''', \widehat{F}'\,]\,a^* \geq 0$, modulo $\mathcal{J}(\mathcal{E}L)$, for all $a \in A_1 \otimes A_2$.*

PROOF. The argument mimics the one given in the previous lemma and is simpler. For the positivity of the 4th and 8th term we use $[\widehat{N}, 1 \otimes v] \in \mathcal{J}(\mathcal{E}L)$, and this fact justifies the inclusion of $1 \otimes v$ in $\widehat{\Delta}$ in Lemma 3.2.2. ■

We have succeeded to prove all the claims that were made in the proof of Theorem 3.2.1. (Lemma 2.2.9 is used also in conjunction with Lemmas 3.2.6 and 3.2.7.) Looking back on our method, we want to make the following comment: the strengthened version of Kasparov's Technical Theorem in Lemma 3.2.2 is used to obtain the *critical* asymptotic commutativity of $\widehat{N}$ with $u \otimes 1$. All the other identities could have probably been obtained by an appropriate 'rescaling' of w, as exemplified by Lemma 3.2.3, without recourse to Lemma 3.2.2.

The main result of the section. Assume that a group G (locally compact, σ-compact, Hausdorff) is given. At the beginning of the chapter we introduced the following notation: $\mathbf{KK^G}$ and $\mathbf{KE^G}$ are the categories with objects te separable graded G-C^*-algebras, and with morphism sets the KK-theory and KE-theory groups, respectively. The composition of morphisms is given by the product $\sharp$.

THEOREM 3.2.8. *$\Theta : \mathbf{KK^G} \longrightarrow \mathbf{KE^G}$ is a functor.*

PROOF. The structure of the argument is similar with the one presented in the proof of Theorem 3.2.1, so we shall only provide the details that are significantly different. Let A, B, and D be separable graded G-C^*-algebras. Let $x = (\mathcal{E}_1, F_1) \in kk^G(A, D)$, $y = (\mathcal{E}_2, F_2) \in kk^G(D, B)$, $\mathcal{E} = \mathcal{E}_1 \otimes_D \mathcal{E}_2$. The statement of the theorem is equivalent to showing that $\Theta(\,\boldsymbol{x} \sharp_D \boldsymbol{y}\,) = \Theta(\boldsymbol{x}) \sharp_D \Theta(\boldsymbol{y})$, where the first product is in KK-theory and the second one is in KE-theory. We recall from the proof of Theorem 1.4.7 that the product $\boldsymbol{x} \sharp_D \boldsymbol{y}$ is the class of $\big(\mathcal{E}, M^{\frac{1}{2}}(F_1 \otimes_D 1) + N^{\frac{1}{2}}\underline{F}\big)$, with $\underline{F}$ an F_2-connection for $\mathcal{E}_1$, and M and N obtained from Kasparov's Technical Theorem. From the same proof recall also the definition of the elements J, E_1, E_2, Δ, and φ that appear in the hypothesis of KTT. Let $u = \{u_t\}_t$ be a q.i.q.c.a.u. for $\mathcal{K}(\mathcal{E}_1) \subset C^*(\mathcal{K}(\mathcal{E}_1), A, F_1) \subset \mathcal{B}(\mathcal{E}_1)$, satisfying also the conditions of 2.2.14, $v = \{v_t\}_t$ be a q.i.q.c.a.u. for $\mathcal{K}(\mathcal{E}_2) \subset C^*(\mathcal{K}(\mathcal{E}_2), D, F_2) \subset \mathcal{B}(\mathcal{E}_2)$, and $w = \{w_t\}_t$ be a q.i.q.c.a.u. for $\mathcal{K}(\mathcal{E}) \subset C^*(\mathcal{K}(\mathcal{E}), E_1, \Delta) \subset \mathcal{B}(\mathcal{E})$. Make the notation $u_t^\perp = (1 - u_t) \otimes_D 1$, $w_t^\perp = 1 - w_t$, $\widehat{F_1} = \{(1-u_t)F_1(1-u_t)\}_t$, and $\widehat{F_2} = \{(1-v_t)F_2(1-v_t)\}_t$. The goal is to show that the following operators over $\mathcal{E}L$ determine homotopic asymptotic Kasparov modules:

$$\widehat{F} = \Big\{\, w_t^\perp \,\big(M^{\frac{1}{2}}(F_1 \otimes_D 1) + N^{\frac{1}{2}}\underline{F}\big)\, w_t^\perp \,\Big\}_{t\in[1,\infty)}, \tag{3.14}$$

$$\widehat{F}'' = \Big\{\, w_{s(t)}^\perp \,\big(M_t^{\frac{1}{2}}(F_1 \otimes_D 1) + N_t^{\frac{1}{2}}\underline{F}\big)\, w_{s(t)}^\perp \,\Big\}_{t\in[1,\infty)},$$

$$\widehat{F}''' = \Big\{\, u_t^\perp(F_1 \otimes_D 1)u_t^\perp + w_{s(t)}^\perp N_t^{\frac{1}{2}}\underline{F} w_{s(t)}^\perp \,\Big\}_{t\in[1,\infty)},$$

$$\widehat{F}' = \Big\{\, u_t^\perp(F_1 \otimes_D 1)u_t^\perp + (\mathrm{Res}_h)_*\big((u \otimes_{DL} 1)\underline{\widehat{F}}\big) \,\Big\}_{t\in[1,\infty)}. \tag{3.15}$$

Here: $\widehat{F}$ is the representative of $\Theta(\boldsymbol{x} \sharp_D \boldsymbol{y})$, and $\widehat{F}' = \widetilde{\widehat{F_1} \otimes_D 1} + \widetilde{1 \otimes_D \widehat{F_2}}$ is the representative of $\Theta(\boldsymbol{x}) \sharp_D \Theta(\boldsymbol{y})$, using the two-dimensional $\widehat{F_2}$-connection $\underline{F}$ for $\mathcal{E}_1 L$, and the construction of the product from Section 2.3 (see Overview 2.3.7). The construction of the other two operators $\widehat{F}''$ and $\widehat{F}'''$, and the justification of the homotopies $\widehat{F} \sim \widehat{F}'' \sim \widehat{F}''' \sim \widehat{F}'$ will be given bellow. Granting that the homotopies are constructed, it follows that $(\mathcal{E}L, \widehat{F})$ and $(\mathcal{E}L, \widehat{F}')$ define the same class in $KE^G(A, B)$, and the poof of the theorem is completed. ■

For the remaining part of the section the notation is the one introduced in the proof of the theorem above. The justification of the claims needed to complete this proof occupies this last part of the section.

The families $\{M_t\}_t$ and $\{N_t\}_t$ which are used in defining $\widehat{F}''$ are constructed via the extended KTT 1.3.3. Again it is the case that $(\star)$ is a consequence of (2).

Lemma 3.2.9. *Let $\widehat{J} = C_0(L, J)$, $\widehat{E}_1 = C_0(L, E_1)$, $\widehat{E}_2 = 1 \otimes E_2$ (with $1 \otimes E_2$ representing constant functions on L with values elements of E_2), $\widehat{\Delta} =$ vector space span of $1 \otimes \Delta \cup \{1 \otimes M, w, u \otimes_D 1, \widetilde{1 \otimes_D \widehat{F_2}}\}$, and $\widehat{\varphi} = 1 \otimes \varphi$. Then there exist two positive elements $\widehat{M} = \{M_t\}_t$ and $\widehat{N} = \{N_t\}_t$ in $\mathcal{M}(\widehat{J}) = C_b(L, \mathcal{B}(\mathcal{E}))$ such that*

(3.16)

$$(1)\ \widehat{M} + \widehat{N} = 1,\ (2)\ \widehat{N}\,(1 \otimes E_2) \subset \mathcal{J}(\mathcal{E}L),\ (3)\ [\widehat{N}, \widehat{\Delta}] \subset \mathcal{J}(\mathcal{E}L),$$
$$(4)\ (g(\widehat{M}) - \widehat{M}) \in \mathcal{J}(\mathcal{E}L),\ (5)\ \widehat{N}\,(g(\underline{F}) - \underline{F}),\ (g(\underline{F}) - \underline{F})\,\widehat{N} \in \mathcal{J}(\mathcal{E}L),\ \text{ for all } g \in G,$$
$$(\star)\ \widehat{M}\,(\mathcal{K}(\mathcal{E}_1) \otimes_D 1) \subset \mathcal{C}(\mathcal{E}L).$$

The following result produces an 'accelerated' $\{w_{s(t)}\}_t$, which is used together with the families generated above to define $\widehat{F}''$. It is similar with Lemma 3.2.3, and its proof will be omitted.

Lemma 3.2.10. *There exists a function $s : [1, \infty) \to [1, \infty)$ which is strictly increasing, bijective, and continuous, such that*

$$\text{(3.17)}\qquad \begin{aligned} &\text{(a)}\quad \| w_{s(t)}^{\perp}\, K_t \| \xrightarrow{t\to\infty} 0,\ \text{for every } \{K_t\}_t \in \widehat{M}(\mathcal{K}(\mathcal{E}_1) \otimes_D 1),\\ &\text{(b)}\quad \| w_{s(t)}^{\perp}\, [u_t \otimes_D 1, \underline{F}] \| \xrightarrow{t\to\infty} 0,\ \text{and}\\ &\text{(c)}\quad \| [w_{s(t)}, u_t \otimes_D 1] \| \xrightarrow{t\to\infty} 0. \end{aligned}$$

For any s as above, let

$$\widehat{F}'' = \big\{ w^{\perp}_{s(t)} \big(M_t^{\frac{1}{2}}(F_1 \otimes_D 1) + N_t^{\frac{1}{2}}\underline{F}\big)\, w^{\perp}_{s(t)} \big\}_{t\in[1,\infty)}. \tag{3.18}$$

Then $(\mathcal{E}L, \widehat{F}'')$ is an asymptotic Kasparov G-(A,B)-module.

Remark. Behind the property (b) of (3.17) is the following claim: $[u_t \otimes_D 1, \underline{F}] \in \mathcal{C}(\mathcal{E}L) + \mathcal{J}(\mathcal{E}L)$. Indeed, for each integer n, on the interval $[n, n+2]$ we can approximate $u_t \otimes_D 1$ by a sum of the form $\sum_i T_{\xi_i} T^*_{\eta_i}$, with an error in operator norm less than $1/n$. But then: $[\sum_i T_{\xi_i} T^*_{\eta_i}, \underline{F}] = \sum_i \big(T_{\xi_i} T^*_{\eta_i} \underline{F} - \underline{F} T_{\xi_i} T^*_{\eta_i}\big) = \sum_i (-1)^{\partial \xi_i} \big(T_{\xi_i} F_2 T^*_{\eta_i} - T_{\xi_i} F_2 T^*_{\eta_i}\big) +$ error in $\mathcal{K}(\mathcal{E})$. This gives the claimed inclusion in $\mathcal{C}(\mathcal{E}L) + \mathcal{J}(\mathcal{E}L)$. Note that $\underline{F}$ is a Kasparov connection (Definition 1.4.4) and not a two-dimensional connection.

Lemma 3.2.11. *Consider*

$$\widehat{F}''' = \big\{ u_t^{\perp}(F_1 \otimes_D 1) u_t^{\perp} + w^{\perp}_{s(t)} N_t^{\frac{1}{2}} \underline{F} w^{\perp}_{s(t)} \big\}_{t\in[1,\infty)}. \tag{3.19}$$

Then $(\mathcal{E}L, \widehat{F}''')$ is an asymptotic Kasparov G-(A,B)-module.

Proof. We check the axioms.

• $\widehat{F}'''$ *satisfies* (aKm1). For any $a \in A$, we have:

$$\begin{aligned} &\big(\widehat{F}''' - (\widehat{F}''')^*\big)\,(a \otimes_D 1) \\ &\quad\sim u_t^{\perp}\big((F_1 - F_1^*)a \otimes_D 1\big)u_t^{\perp} + w^{\perp}_{s(t)} N_t^{\frac{1}{2}} \big(\underline{F} - \underline{F}^*\big) w^{\perp}_{s(t)}\,(a \otimes_D 1), \quad \text{modulo } \mathcal{J}(\mathcal{E}L) \\ &\quad\in \mathcal{J}(\mathcal{E}_1 L) \otimes_D 1 + \mathcal{J}(\mathcal{E}L) \subseteq \mathcal{J}(\mathcal{E}L). \end{aligned}$$

After the $\sim$ sign, we used for the first term the fact that u is an approximate unit for $\mathcal{K}(\mathcal{E}_1)$, and for the second term the relation (2) of (3.16) satisfied by $\widehat{N}$, keeping in mind that the constant family $(\underline{F} - \underline{F}^*)$ belongs to $\widehat{E}_2$. Compare with the argument for (aKm1) given in the proof of Lemma 3.2.3.

• $\widehat{F}'''$ *satisfies* (aKm2) and (aKm4). See the computation above and the ones given for the corresponding axioms in the proof of Lemma 3.2.3.

• $\widehat{F}'''$ *satisfies* (aKm3). It is probably time to give details for a computation that was used already in the proof of some of the previous lemmas, and which will be used below too.

We have:

$$
\begin{aligned}
&u_t^{\perp}(F_1\otimes_D 1)u_t^{\perp}w_{s(t)}^{\perp}N_t^{\frac12}\underline{F}w_{s(t)}^{\perp}\\
(3.20)\qquad &\sim\ w_{s(t)}^{\perp}u_t^{\perp}(F_1\otimes_D 1)u_t^{\perp}N_t^{\frac12}\underline{F}w_{s(t)}^{\perp}\quad \text{modulo } \mathcal{J}(\mathcal{E}L)\quad (3.17)(c), \text{ and q.c.}\\
&\sim\ w_{s(t)}^{\perp}u_t^{\perp}N_t^{\frac12}(F_1\otimes_D 1)u_t^{\perp}\underline{F}w_{s(t)}^{\perp}\quad \text{modulo } \mathcal{J}(\mathcal{E}L)\quad (3.16)(3)\\
&\sim\ w_{s(t)}^{\perp}u_t^{\perp}N_t^{\frac12}(F_1\otimes_D 1)\underline{F}u_t^{\perp}w_{s(t)}^{\perp}\quad \text{modulo } \mathcal{J}(\mathcal{E}L)\quad (3.17)(b)
\end{aligned}
$$

The point is that this computation, combined with the one for $w_{s(t)}^{\perp}N_t^{\frac12}\underline{F}w_{s(t)}^{\perp}u_t^{\perp}(F_1\otimes_D 1)u_t^{\perp}$, will make the sum of the two terms an element of $\mathcal{J}(\mathcal{E}L)$. Indeed, we use $[F_1\otimes_D 1,\underline{F}]\in E_2$, and (3.16)(2). This justifies the first $\sim$ sign below.

For any $a\in A$, with the notation $a=a\otimes_D 1$, we have:

$$
\begin{aligned}
&a\left((\widehat{F}''')^2-1\right)a^*\\
&\sim a\left((u_t^{\perp})^2({F_1}^2\otimes_D 1)(u_t^{\perp})^2+(w_{s(t)}^{\perp})^2N_t\underline{F}^2(w_{s(t)}^{\perp})^2-1\right)a^*,\quad \text{modulo } \mathcal{J}(\mathcal{E}L)\\
&\sim (u_t^{\perp})^2M_t\big(a({F_1}^2-1)\otimes_D 1\big)(u_t^{\perp})^2a^*+a(u_t^{\perp})^2N_t^{\frac12}({F_1}^2\otimes_D 1)N_t^{\frac12}(u_t^{\perp})^2a^*\\
&\qquad\qquad +a(w_{s(t)}^{\perp})^2N_t\big(\underline{F}^2-1)\big)(w_{s(t)}^{\perp})^2a^*,\quad \text{modulo } \mathcal{C}(\mathcal{E}L)+\mathcal{J}(\mathcal{E}L)\\
&\geq 0,\quad \text{modulo } \mathcal{C}(\mathcal{E}L)+\mathcal{J}(\mathcal{E}L).
\end{aligned}
$$

For the second $\sim$ sign we used $1=M_t+N_t$, and the fact that $(p(u_t)\otimes_D 1)\,M_t\,(q(u_t)\otimes_D 1)\in\mathcal{C}(\mathcal{E}L)$, for any polynomials p and q, at least one of which vanish at 0. The first term there belongs to $\mathcal{J}(\mathcal{E}L)$ due to the fact that u is an approximate unit of $\mathcal{K}(\mathcal{E}_1)$, the second one is positive, and the third one is in $\mathcal{J}(\mathcal{E}L)$ due to (2) of (3.16). This gives the inequality above and the proof of the lemma is now complete. ■

The following result has exactly the same proof as Lemma 3.2.5.

LEMMA 3.2.12. *$\widehat{F}$ and $\widehat{F}''$ are homotopic asymptotic Kasparov modules.*

The proof of the next lemma uses arguments similar with the ones that appear in the proof of Lemma 3.2.6 and of the axiom (aKm3) of Lemma 3.2.11.

LEMMA 3.2.13. *$a\,[\,\widehat{F}'',\widehat{F}'''\,]\,a^*\geq 0$, modulo $\mathcal{J}(\mathcal{E}L)$, for all $a\in A$.*

Finally we have:

LEMMA 3.2.14. *$a\,[\,\widehat{F}''' , \widehat{F}'\,]\,a^* \geq 0$, modulo $\mathcal{J}(\mathcal{E}L)$, for all $a \in A$.*

PROOF. The only problem here is the positivity of the terms $w_{s(t)}^{\perp} N_t^{\frac{1}{2}} \underline{F} w_{s(t)}^{\perp} \widetilde{(1 \otimes_D \widehat{F_2})}$ and $\widetilde{(1 \otimes_D \widehat{F_2})} w_{s(t)}^{\perp} N_t^{\frac{1}{2}} \underline{F} w_{s(t)}^{\perp}$. To see that that they are indeed positive, we notice that the asymptotic commutativity of $\widetilde{1 \otimes_D \widehat{F_2}}$ with $w_{s(t)}$ can be obtained by further 'accelerating' s, and that the inclusion of $\widetilde{1 \otimes_D \widehat{F_2}}$ in $\widehat{\Delta}$ and (3) of (3.16) gives the asymptotic commutativity of $\widetilde{1 \otimes_D \widehat{F_2}}$ with $\widehat{N}$. Everything is wrapped up by the following:

CLAIM. $\underline{F}\,\widetilde{(1 \otimes_D \widehat{F_2})}$ and $\widetilde{(1 \otimes_D \widehat{F_2})}\,\underline{F}$ are positive.

PROOF OF THE CLAIM. One can choose the representatives for the initial classes in KK-theory such that $\mathcal{E}_1 = \mathcal{H}_D$ and $D\mathcal{E}_2 = \mathcal{E}_2$. Then $\mathcal{E} = \mathcal{E}_1 \otimes_D \mathcal{E}_2 \simeq \mathcal{H} \otimes \mathcal{E}_2$, $\underline{F} = 1 \otimes F_2$, and $(\mathrm{Res}_h)_*(\,\underline{\widehat{F_2}}\,) = 1 \otimes \big((1 - v_{h(t)})F_2(1 - v_{h(t)})\big)$. The positivity of the two products is now obvious. □ ■

We have succeeded to justify the claims made in the proof of the main theorem. The last three lemmas give the homotopies $\widehat{F} \sim \widehat{F}'' \sim \widehat{F}''' \sim \widehat{F}'$, and because we also proved that they generate asymptotic Kasparov modules it follows that all operators generate the same class in $KE^G(A, B)$.

This section is concluded with the presentation of a second proof for the Stability Theorem 2.4.9, namely that A and $A \otimes \mathcal{K}(\mathcal{H}_G)$ are KE-theory equivalent. Indeed, we know that they are KK-theory equivalent, *i.e.* there exist elements $\boldsymbol{x} \in KK^G(A, A \otimes \mathcal{K})$ and $\boldsymbol{y} \in KK^G(A \otimes \mathcal{K}, A)$ such that $\boldsymbol{x} \sharp_{A \otimes \mathcal{K}} \boldsymbol{y} = 1_A$ and $\boldsymbol{y} \sharp_A \boldsymbol{x} = 1_{A \otimes \mathcal{K}}$. The functoriality of Θ now immediately shows that $\Theta(\boldsymbol{x}) \in KE^G(A, A \otimes \mathcal{K})$ and $\Theta(\boldsymbol{y}) \in KE^G(A \otimes \mathcal{K}, A)$ give the desired KE-equivalence.

The argument above illustrates the fact that KE-theory is a 'weaker' theory than KK-theory, in the sense that if products behave well in KK-theory (say, in applications to Baum-Connes conjecture or Novikov conjecture) then their images under Θ will behave well in KE-theory too. In other words, once a composition product is known to exist or is constructed in KK-theory, then the same thing will happen in KE-theory. The consequence is an extremely interesting (to us) research problem: to determine whether or not KE-theory is *equivalent* to KK-theory, either by showing that Θ is an isomorphism (equivalence of categories) or by constructing elements in KE-theory that do not come from KK-theory elements. In the first case, KE-theory would provide another 'picture' for KK-theory. It is in any case another tool for the operator algebraist interested also in applications to geometry.

3.3. The map $\Xi : KE^G \to E_G$

The main result of this section is Theorem 3.3.6, which constructs the connecting map Ξ between KE-theory and E-theory. This construction is performed via a description of the E-theory groups which involves $C_0((-1,1))$ instead of $\mathcal{S}$. (See the brief review of E-theory given in Section 1.5.) Such a modification seems more appropriate when working with bounded operators, and in this and the following section we shall also address the technical details necessary to express the product in this '$C_0((-1,1))$-picture' of E-theory. As with $\mathcal{S} = C_0(\mathbb{R})$, the C^*-algebra $C_0((-1,1))$ will be graded by even and odd functions.

Let G be a group, A and B be G-C^*-algebras. We consider first a particular case of asymptotic Kasparov (A,B)-modules: $(\mathcal{E}, F) = (\mathcal{E}_\bullet L, \{F_t\}_t) \in ke^G(A,B)$, where $\mathcal{E}_\bullet$ is a *fixed* Hilbert G-B-module acted upon by A through the $*$-homomorphism $\varphi : A \to \mathcal{B}(\mathcal{E}_\bullet)$ (or through a family of $*$-homomorphisms $\varphi_t : A \to \mathcal{B}(\mathcal{E}_\bullet)$, but the argument remains unchanged). This means that $F_t = F_t^* \in \mathcal{B}(\mathcal{E}_\bullet)$ is an odd self-adjoint operator, for every t, such that $[F_t, a]$, $(g(F_t) - F_t)a$ converge in norm to 0 as $t \to \infty$, for all $a \in A$, $g \in G$, and that $a(F_t^2 - 1)a^* \geq 0$, modulo compacts, with an error that converges in norm to 0 as $t \to \infty$.

Claim 3.3.1. The family of maps

(3.21)
$$\phi_F = \{\phi_{F,t}\}_{t\in[1,\infty)} : C_0((-1,1))\otimes A \to \mathcal{K}(\mathcal{E}_\bullet),\ f\otimes a \overset{\phi_{F,t}}{\mapsto} f(F_t)\, a,\ \text{for } f \in C_0((-1,1)), a \in A,$$

is an asymptotic family.

Proof. Lemma 1.5.5 will be applied to the varying family of $*$-homomorphisms $\{\chi_t\}_t : C_0((-1,1)) \to \mathcal{B}(\mathcal{E}_\bullet)$, $\chi_t : f \mapsto f(F_t)$, and to the (constant) family of $*$-homomorphisms $\{\varphi\}_t : A \to \mathcal{B}(\mathcal{E}_\bullet))$, in order to obtain the asymptotic family

$$\phi_F = \{\chi_t \otimes \varphi\}_t : C_0((-1,1)) \otimes A \to \mathcal{B}(\mathcal{E}_\bullet),\ f \otimes a \mapsto f(F_t)\, a. \tag{3.22}$$

This is almost what we have to prove, but there are three facts to be justified: (1) the construction of $\{\chi_t\}_t$; (2) the asymptotic commutativity $[\chi_t(f), \varphi(a)] = [f(F_t), a] \xrightarrow{t\to\infty} 0$

required in the hypothesis of Lemma 1.5.5; and (3) that the range of $\phi_{F,t}$'s is $\mathcal{K}(\mathcal{E}_\bullet)$ and not $\mathcal{B}(\mathcal{E}_\bullet)$ as given by (3.22).

The definition $\chi_t : f \mapsto f(F_t)$ is the functional calculus:

$$C_0((-1,1)) \rightharpoonup C(\text{spectrum}(F_t)) \xrightarrow{\text{functional calculus}} \mathcal{B}(\mathcal{E}_\bullet).$$

Here the algebras of functions are graded by even and odd functions, and $\rightharpoonup$ means either the inclusion $*$-homomorphism (by extension as 0 outside $(-1,1)$), or the restriction $*$-homomorphism (if $\text{spectrum}(F_t) \subset (-1,1)$).

The asymptotic commutativity follows from (aKm2) (and the mechanics behind the functional calculus). Finally, in order to compute the range of φ_t's we shall show that

$$a\, f(F) \in \mathcal{C}(\mathcal{E}) + \mathcal{J}(\mathcal{E}), \text{ for any } f \in C_c((-1,1)). \tag{3.23}$$

Granted this, (aKm2) implies that $f(F)\, a \in \mathcal{C}(\mathcal{E}) + \mathcal{J}(\mathcal{E})$, which makes ϕ_F asymptotically equivalent to a $\mathcal{K}(\mathcal{E}_\bullet)$-valued asymptotic family. A density argument shows that the desired inclusion holds for every $f \in C_0((-1,1))$.

To prove (3.23), let $a(F^2-1)a^* = p_+ - p_- \in \mathcal{B}(\mathcal{E})$ be the decomposition into positive and negative parts. The axiom (aKm3) implies that $p_- \in \mathcal{C}(\mathcal{E}) + \mathcal{J}(\mathcal{E})$. There exists $g \in C_c((-1,1))$ (concretely $g(x) = f(x)/\sqrt{1-x^2}$, for $x \in \text{supp}(f)$) such that $f(x)\overline{f}(x) = g(x)(1-x^2)\overline{g}(x)$. Then:

$$\begin{aligned}
0 &\leq a\, f(F)\, \overline{f}(F)\, a^* \\
&= a\, g(F)\, (1-F^2)\, \overline{g}(F)\, a^* \\
&= g(F)\, a\, (1-F^2)\, a^*\, \overline{g}(F) && (\text{modulo } \mathcal{J}(\mathcal{E})) \\
&= -g(F)\, a\, (F^2-1)\, a^*\, \overline{g}(F) \\
&= g(F)\, p_-\, \overline{g}(F) && (\text{because of initial positivity}) \\
&\in \mathcal{C}(\mathcal{E}) + \mathcal{J}(\mathcal{E}) && (\text{because of (aKm3)}).
\end{aligned}$$

Consequently, by polar decomposition, $a\, f(F) \in \mathcal{C}(\mathcal{E}) + \mathcal{J}(\mathcal{E})$. ■

The asymptotic family constructed above indicates that a '$C_0((-1,1))$-picture' of E-theory is in order. The next lemma is the first step towards such a characterization.

LEMMA 3.3.2. *Let A and D be G-C^*-algebras, and consider an equivariant asymptotic family $\phi_F = \{\phi_{F,t}\}_t : C_0((-1,1)) \otimes A \dashrightarrow D$. Then there exists a unique, up to homotopy, equivariant asymptotic family $\psi_F = \{\psi_{F,t}\}_t : \mathcal{S}A \dashrightarrow D$ such that the diagram*

$$\begin{array}{ccc} C_0((-1,1)) \otimes A & \xrightarrow{\phi_F} & D \\ {\scriptstyle \text{inclusion}} \big\downarrow {\scriptstyle \iota} & & \big\| \\ \mathcal{S}A & \xrightarrow[\psi_F]{} & D \end{array}$$

commutes up to homotopy.

PROOF. The inclusion ι from the statement is the extension by zero outside $(-1,1)$: $C_0((-1,1)) \hookrightarrow \mathcal{S}$, $f \mapsto \overset{\frown}{f}$. By choosing a homeomorphism between $(-1,1)$ and $\mathbb{R}$, say $x \mapsto \tan(\pi x/2)$, we can also construct a $*$-isomorphism: $\mathcal{S} \to C_0((-1,1)), f \mapsto \widehat{f}$, with $\widehat{f}(x) = f(\tan(\pi x/2))$. We define ψ_F as follows: $\psi_F(f \otimes a) = \phi_F(\widehat{f} \otimes a)$, for all $f \in \mathcal{S}$, and $a \in A$. A homotopy between ϕ_F and the composition $\psi_F \circ \iota$ is given by the formula $f \otimes a \mapsto \phi_F\big(\overset{\frown}{f} ((1-s)x + s\tan(\pi x/2)) \otimes a\big)$, $s \in [0,1]$. The uniqueness is clear. ■

Two more results are needed:

LEMMA 3.3.3 (Fell's trick). *Let $\mathcal{E}$ be a Hilbert G-D-module, and denote the action of G by π (as in Definition 1.2.1). There exists an isomorphism $U_{\mathcal{E}}$ intertwining $\pi \otimes \lambda$ and $1 \otimes \lambda$, where λ is the left regular representation.*

PROOF. Recall from representation theory that 'Fell's trick' refers to the statement: "The tensor product of any unitary representation of G with the regular representation is unitarily equivalent to a multiple of the regular representation." The left regular representation is the unitary representation $\lambda : G \to \mathcal{B}(L^2(G))$, $s \mapsto \lambda_s$, $\lambda_s(f)(t) = f(s^{-1}t)$, for all $f \in L^2(G)$, $s, t \in G$. We denote by $\mathcal{E}^0$ the Hilbert module $\mathcal{E}$ but with trivial G-action. Define $U_{\mathcal{E}} : \mathcal{E} \otimes L^2(G) \to \mathcal{E}^0 \otimes L^2(G)$, $\xi \otimes f \to U_{\mathcal{E}}(\xi \otimes f)$, $U_{\mathcal{E}}(\xi \otimes f)(t) = \pi(t^{-1})(\xi) f(t)$, for all $t \in G$. The commutativity of the diagram

$$\begin{array}{ccc} \mathcal{E} \otimes L^2(G) & \xrightarrow{U_{\mathcal{E}}} & \mathcal{E}^0 \otimes L^2(G) \\ {\scriptstyle \pi(s)\otimes\lambda_s}\downarrow & & \downarrow{\scriptstyle 1\otimes\lambda_s} \\ \mathcal{E} \otimes L^2(G) & \xrightarrow[U_{\mathcal{E}}]{} & \mathcal{E}^0 \otimes L^2(G) \end{array}$$

shows that $U_{\mathcal{E}}$ is G-equivariant. It is also clear that $U_{\mathcal{E}}$ is an isomorphism. ■

LEMMA 3.3.4. *Let $\mathcal{E}$ be a Hilbert G-D-module, with a* non-equivariant *isometry $V : \mathcal{E} \to \mathcal{H}_D$. Then there exists an* equivariant D-linear *isometry $W : \mathcal{E} \otimes L^2(G) \to \mathcal{H}_D \otimes L^2(G)$.*

PROOF. Use Lemma 3.3.3 to define $W = U_{\mathcal{H}_D}^{-1} \circ (V \otimes 1) \circ U_{\mathcal{E}}$, as pictured in the diagram below:

$$\mathcal{E} \otimes L^2(G) \xrightarrow[\text{Fell's trick}]{U_{\mathcal{E}}} \mathcal{E}^0 \otimes L^2(G) \xrightarrow{V\otimes 1} \mathcal{H}_D^0 \otimes L^2(G) \xrightarrow[\text{Fell's trick}]{U_{\mathcal{H}_D}^{-1}} \mathcal{H}_D \otimes L^2(G). \tag{3.24}$$

We have $W(\xi \otimes f)(t) = t\,(V\,\pi(t^{-1})\,\xi)\,f(t)$, for all $t \in G$, and with the action of t on D (or on $\mathcal{H}_D$) also denoted by t. D-linearity follows from the simple observation that the two actions of G that are present in the formula for W cancel one other out. ■

The point behind the previous two lemmas is a possible simplification in the definition of asymptotic Kasparov modules:

PROPOSITION 3.3.5. *Given two G-C^*-algebras A' and B', let $A = A' \otimes \mathcal{K}(\mathcal{H}_G)$ and $B = B' \otimes \mathcal{K}(\mathcal{H}_G)$. Then, in the definition of $KE^G(A, B)$ it is enough to consider modules of the form $(\mathcal{H}_{BL}, F)$.*

PROOF. Recall that the notation $\mathcal{K}(\mathcal{H}_G) = \mathcal{K}$ is used. Let $x = (\mathcal{E}, F)$ be an arbitrary module in $KE^G(A, B)$, and fix an isomorphism $\mathcal{K} \otimes \mathcal{K} \simeq \mathcal{K}$. This isomorphism transforms $(\mathcal{E}, F)$ in $(\mathcal{E} \otimes \mathcal{H}_G, F \otimes 1)$, with the obvious action of $A \otimes \mathcal{K}$ on $\mathcal{E} \otimes \mathcal{H}_G$. Next, in Lemma 3.3.4 let V be the isometry given by Stabilization Theorem ($\mathcal{E}^0 \oplus \mathcal{H}_{BL}^0 \simeq \mathcal{H}_{BL}^0$). The addition of the degenerate $(\mathcal{H}_{BL}^0 \otimes \mathcal{H}_G, 0) \in ke^G(A, B)$, with $A \otimes \mathcal{K}$ acting as zero operators, together with (3.24) show that the initial module can be represented by $\mathcal{H}_{BL}$ as required. ■

The proposition implies that the previous construction of the asymptotic morphism associated to an asymptotic Kasparov module with constant 'fibers' can be carried over the general case. Consider an arbitrary Kasparov module $(\mathcal{E}, F) \in ke^G(A, B)$. The formula (3.23) tells that we can construct, as in the proof of Claim 3.3.1, an asymptotic morphism $\phi : C_0((-1,1)) \otimes A \to \mathcal{C}(\mathcal{E})/\mathcal{K}(\mathcal{E})$. (Recall the equivalence provided by Lemma 1.5.3.) This in turn gives an asymptotic morphism:

$$\phi \otimes 1 : C_0((-1,1)) \otimes A \otimes \mathcal{K}(L^2(G)) \to \mathcal{C}(\mathcal{E} \otimes L^2(G))/\mathcal{K}(\mathcal{E} \otimes L^2(G)). \tag{3.25}$$

By ignoring the action of G, apply the Stabilization Theorem ([**Kas80**, Thm.2], with G=\{e\}) to get a non-equivariant isometry $V : \mathcal{E} \to \mathcal{H}_{BL}$. Apply next Lemma 3.3.4 to construct an equivariant BL-linear isometry $W : \mathcal{E} \otimes L^2(G) \to \mathcal{H}_{BL} \otimes L^2(G)$. Use it, and the fact that now we have a *constant field* $\mathcal{H}_{BL}$ of modules, to transform the asymptotic morphism $\phi \otimes 1$ of (3.25) into an asymptotic family:

$$\phi_F : C_0((-1,1)) \otimes A \otimes \mathcal{K}(L^2(G)) \dashrightarrow \mathcal{K}(\mathcal{H}_B) \otimes \mathcal{K}(L^2(G)). \tag{3.26}$$

After tensoring with $\mathcal{K}$, we can use Lemma 3.3.2 to obtain

$$\psi_F : \mathcal{S}A \otimes \mathcal{K} \dashrightarrow B \otimes \mathcal{K}. \tag{3.27}$$

The connection between KE-theory and E-theory is given by the following:

THEOREM 3.3.6. *For any group G, and any two G-C^*-algebras A and B, the map $\Xi : (\mathcal{E}, F) \mapsto \psi_F$, from asymptotic Kasparov G-(A, B)-modules to asymptotic families from $\mathcal{S}A \otimes \mathcal{K}$ to $B \otimes \mathcal{K}$, passes to quotients and gives a natural group homomorphism*

$$\Xi : KE^G(A, B) \to E_G(A, B),\ \Xi((\mathcal{E}, F)) = [\![\psi_F]\!]\,. \tag{3.28}$$

PROOF. The map was constructed in the paragraph above. It only remains to make the simple observation that all the steps used to obtain ψ_F transform homotopic elements into homotopic elements. This implies that Ξ is a well defined map at the level of groups. ∎

EXAMPLES 3.3.7. (a) $\Xi(1_A) = 1_A$, for any A.
(b) $\Xi(\llbracket\varphi\rrbracket) = \llbracket\{\varphi\}_t\rrbracket$, when $\varphi : A \to B \otimes \mathcal{K}$, *i.e.* the KE-theory element corresponding to the $*$-homomorphism φ (Example 2.1.10) is sent by Ξ to the corresponding element in E-theory.
(c) The construction of the asymptotic family which gives the K-homology class $\llbracket D\rrbracket$ of a Dirac operator D in E-theory is by now a standard fact. (See some of the early work of Connes and Higson [**CoHg**], the Ph.D. thesis of E. Guentner [**Gntn**], and others; in the graded context see also [**Dum**]). This construction is similar to (3.21), and we can even say that our formulation of (aKm2) was suggested by the asymptotic commutativity described in (2.4). Using the description of $\llbracket D\rrbracket$, it is immediate that $\Xi(\llbracket D\rrbracket) = \llbracket D\rrbracket$.

This section is concluded by studying the relation between Ξ and the map σ. We recall that a C^*-algebra A is called *nuclear* if for any other C^*-algebra B the algebraic tensor product $A \odot B$ admits a unique C^*-completion, in other words the minimal and maximal C^*-algebra tensor products coincide. For more comments on the restriction to nuclear C^*-algebras in the next statement see Section 4.1.

PROPOSITION 3.3.8. *Let A, B be arbitrary G-C^*-algebras, and D be an arbitrary* nuclear *G-C^*-algebra. The following diagram is commutative:*

$$\begin{array}{ccc} KE^G(A,B) & \xrightarrow{\Xi} & E_G(A,B) \\ {\scriptstyle\sigma_D}\big\downarrow & & \big\downarrow{\scriptstyle\sigma_D} \\ KE^G(A\otimes D, B\otimes D) & \xrightarrow[\Xi]{} & E_G(A\otimes D, B\otimes D). \end{array} \tag{3.29}$$

PROOF. In the case when $\mathcal{E} = \{\mathcal{E}_\bullet\}_t$, this is clear from the definitions. The general case follows after stabilization as before. □

3.4. The map Ξ is a functor

In this section it it shown that the map Ξ preserves the product structures of KE-theory and E-theory. Before stating and proving the results, we need to complete the '$C_0((-1,1))$-picture' of E-theory that was mentioned in the previous section. Lemma 3.3.2

was the first step in this direction. Our goal is to circumvent the lack of an explicit formula for the map $\widehat{\Delta}= \Delta|_{C_0((-1,1))}$ in the following commutative, up to homotopy, diagram:

$$
\begin{array}{ccc}
\mathcal{S} & \xrightarrow{\Delta} & \mathcal{S}\otimes\mathcal{S} \\
\simeq\uparrow & & \uparrow\simeq \\
C_0((-1,1)) & \xrightarrow[\widehat{\Delta}]{} & C_0((-1,1))\otimes C_0((-1,1))
\end{array}
\tag{3.30}
$$

The vertical $*$–isomorphisms come from a *fixed identification* of $C_0((-1,1))$ with $\mathcal{S}$ (say as in the proof of Lemma 3.3.2). To simplify the writing the interval $(-1,1)$ is denoted by I.

LEMMA 3.4.1. *The map $\Delta : \mathcal{S} \to \mathcal{S}\otimes\mathcal{S}$ described in (1.8) has a well-defined restriction $\widehat{\Delta}: C_0(I) \to C_0(I)\otimes C_0(I)$.*

PROOF. The important thing to bear in mind is that $C_0(I)$ is viewed here as *subalgebra of $\mathcal{S}$* via extension by zero outside $(-1,1)$. In Lemma 3.3.2 we used the notation $\iota : C_0(I) \hookrightarrow \mathcal{S}$, $f \mapsto \widehat{f}$, for this inclusion. Consequently, $\widehat{\Delta}\,(f) = \widehat{f}\,(X\otimes 1+1\otimes X)$. But $\widehat{f}\,(X\otimes 1+1\otimes X)$ is supported on the unit disk, and this observation shows that the range of $\widehat{\Delta}$ is indeed included in $C_0(I)\otimes C_0(I)$. ■

LEMMA 3.4.2. *Consider two equivariant asymptotic families $\phi_{F_1} : C_0(I)\otimes A_1\otimes\mathcal{K} \dashrightarrow B_1\otimes\mathcal{K}$ and $\phi_{F_2} : C_0(I)\otimes A_2\otimes\mathcal{K} \dashrightarrow B_2\otimes\mathcal{K}$. Let $\psi_{F_1} : \mathcal{S}A_1\otimes\mathcal{K} \dashrightarrow B_1\otimes\mathcal{K}$ and $\psi_{F_2} : \mathcal{S}A_2\otimes\mathcal{K} \dashrightarrow B_2\otimes\mathcal{K}$ be the corresponding asymptotic families constructed using Lemma 3.3.2 from ϕ_{F_1} and ϕ_{F_2}, respectively. In order to compute the product in E-theory $[\![\psi_{F_1}]\!]\ \sharp_{\mathbb{C}}\ [\![\psi_{F_2}]\!]$ it suffices to compute $[\![\phi]\!]$, with ϕ given by the composition:*

$$
\begin{aligned}
\phi : C_0(I)\otimes(A_1\otimes_{max}A_2)\otimes\mathcal{K} &\xrightarrow{\widehat{\Delta}} C_0(I)\otimes A_1\otimes\mathcal{K}\otimes_{max}C_0(I)\otimes A_2\otimes\mathcal{K} \\
&\overset{\phi_{F_1}\otimes\phi_{F_2}}{\dashrightarrow} (B_1\otimes_{max}B_2)\otimes\mathcal{K}.
\end{aligned}
$$

PROOF. Complete (3.30) to get a commutative, up to homotopy, diagram:

$$\begin{array}{ccc} \mathcal{S}(A_1 \otimes_{max} A_2) \otimes \mathcal{K} & \xrightarrow{\Delta} & \mathcal{S}A_1 \otimes \mathcal{K} \otimes_{max} \mathcal{S}A_2 \otimes \mathcal{K} \\ \simeq\uparrow & & \uparrow\simeq \\ C_0(I) \otimes (A_1 \otimes_{max} A_2) \otimes \mathcal{K} & \xrightarrow[\widehat{\Delta}]{} & C_0(I) \otimes A_1 \otimes \mathcal{K} \otimes_{max} C_0(I) \otimes A_2 \otimes \mathcal{K}. \end{array}$$

The required result is obtained by making the composition of the maps in this diagram with the ones from:

$$\begin{array}{ccc} \mathcal{S}A_1 \otimes \mathcal{K} \otimes_{max} \mathcal{S}A_2 \otimes \mathcal{K} & \xrightarrow{\psi_{F_1} \otimes \psi_{F_2}} & B_1 \otimes \mathcal{K} \otimes_{max} B_2 \otimes \mathcal{K} \\ \simeq\uparrow & & \| \\ C_0(I) \otimes A_1 \otimes \mathcal{K} \otimes_{max} C_0(I) \otimes A_2 \otimes \mathcal{K} & \xrightarrow[\phi_{F_1} \otimes \phi_{F_2}]{} & B_1 \otimes \mathcal{K} \otimes_{max} B_2 \otimes \mathcal{K} \end{array}$$

(This second diagram is also commutative up to homotopy because the ψ's are obtained from ϕ's as in Lemma 3.3.2.) Indeed, the composition from the bottom row represents ϕ, while the composition of the top row represents $[\![\psi_{F_1}]\!] \sharp_{\mathbb{C}} [\![\psi_{F_2}]\!]$. ■

The case of external product can now be settled.

THEOREM 3.4.3. *Let G be a group, A_1, A_2, B_1, and B_2 be G-C^*-algebras. Consider $x \in KE^G(A_1, B_1)$, and $y \in KE^G(A_2, B_2)$. Then $\Xi(x \sharp_{\mathbb{C}} y) = \Xi(x) \sharp_{\mathbb{C}} \Xi(y)$.*

PROOF. Let x be represented by $(\mathcal{E}_1, F_1)$ and y be represented by $(\mathcal{E}_2, F_2)$. We recall from Example 2.3.1 that the product $x \sharp_{\mathbb{C}} y$ is the class of $(\mathcal{E}, F) = \{(\mathcal{E}_{1,t} \otimes \mathcal{E}_{2,t}, F_{1,t} \otimes 1 + 1 \otimes F_{2,t})\}_t$. By definition $\Xi(x \sharp_{\mathbb{C}} y) = [\![\psi_F]\!]$. The product $\Xi(x) \sharp_{\mathbb{C}} \Xi(y) = [\![\psi_{F_1}]\!] \sharp_{\mathbb{C}} [\![\psi_{F_2}]\!]$ is represented by ϕ as described in Lemma 3.4.2. The theorem will be proven if it is shown that ϕ and ϕ_F represent asymptotically equivalent asymptotic families. In other words, one has to prove:

CLAIM.*For all $f \in C_0(I)$, $a_1 \in A_1$, $a_2 \in A_2$,*

$$\lim_{t \to \infty} \|\phi_t(f \otimes a_1 \otimes a_2) - f(F_{1,t} \otimes 1 + 1 \otimes F_{2,t})(a_1 \otimes a_2)\| = 0.$$

We denote $\phi_t(f \otimes a_1 \otimes a_2)$ by $\widehat{\Delta}\,(f)(F_{1,t}, F_{2,t})(a_1 \otimes a_2)$, where $\widehat{\Delta}\,(f)(F_{1,t}, F_{2,t})$ is the result of the composition of $\widehat{\Delta}\colon C_0(I) \to C_0(I) \otimes C_0(I)$ with $C_0(I) \otimes C_0(I) \to \mathcal{M}(\mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2})$, $f_1 \otimes f_2 \mapsto f_1(F_{1,t}) \otimes f_2(F_{2,t})$. Consequently it suffices to show that, for all $f \in C_0(I)$ and for all $t \in [1, \infty)$, the following is true:

$$\lim_{t \to \infty} \| \widehat{\Delta}\,(f)(F_{1,t}, F_{2,t}) - f(F_{1,t} \otimes 1 + 1 \otimes F_{2,t}) \| = 0.$$

In fact we show the above for *any* $f \in \mathcal{S}$. To do so we check on the generators $\{e^{-x^2}, xe^{-x^2}\}$ of $\mathcal{S}$. We have:

$$\begin{aligned} \Delta(e^{-x^2})\big(F_{1,t}, F_{2,t}\big) &= e^{-F_{1,t}^2} \otimes e^{-F_{2,t}^2} \\ &= (e^{-x^2})\big(F_{1,t} \otimes 1 + 1 \otimes F_{2,t}\big), \end{aligned}$$

and

$$\begin{aligned} \Delta(xe^{-x^2})\big(F_{1,t}, F_{2,t}\big) &= F_{1,t}\, e^{-F_{1,t}^2} \otimes e^{-F_{2,t}^2} + e^{-F_{1,t}^2} \otimes F_{2,t}\, e^{-F_{2,t}^2} \\ &= (xe^{-x^2})\big(F_{1,t} \otimes 1 + 1 \otimes F_{2,t}\big). \end{aligned}$$

Use a density argument to complete the proof. ■

The main result of the section. At the beginning of the chapter we introduced the following notation: $\mathbf{KE^G}$ and $\mathbf{E_G}$ are the categories with objects the separable graded G-C^*-algebras, and with morphisms the KE-theory and E-theory groups, respectively. The composition of morphisms is given by the product $\sharp$.

THEOREM 3.4.4. $\Xi : \mathbf{KE^G} \longrightarrow \mathbf{E_G}$ *is a functor.*

PROOF. Consider three G-C^*-algebras A, B, D, $x \in KE^G(A, D)$ and $y \in KE^G(D, B)$. The statement of the theorem is equivalent to showing that $\Xi(\, x \,\sharp_D\, y\,) \;=\; \Xi(x) \,\sharp_D\, \Xi(y)\,$, where the first product is in KE-theory and the second one is in E-theory. By making use of the stability property of KE-theory (Theorem 2.4.9), we can assume that $A = A' \otimes \mathcal{K}$, $B = B' \otimes \mathcal{K}$, $D = D' \otimes \mathcal{K}$, for some G-C^*-algebras A', B', D'. We can now use Proposition 3.3.5 to further assume that $x = (\mathcal{E}_1, F_1) = (\mathcal{H}_{DL}, F_1)$ and $y = (\mathcal{E}_2, F_2) = (\mathcal{H}_{BL}, F_2)$, with constant D action on the 'fibers' of $\mathcal{H}_{BL}$. Recall from Step 4, Overview 2.3.7, equation

(2.15), that the product $x \sharp_D y$ is represented by

$$\big(\mathcal{E}_h, F_h\big) = \big\{\big(\mathcal{E}_{1,t} \otimes_D \mathcal{E}_{2,h(t)}\ ,\ F_{1,t} \otimes_D 1 + (u_t \otimes_D 1)\,\underline{F}_{(t,h(t))}\big)\big\}_t,$$

and $\Xi(\, x \sharp_D y\,)$ is represented by ψ_{F_h}, as described in Theorem 3.3.6:

$$\mathcal{S}A \otimes \mathcal{K} \overset{\psi_{F_h}}{\dashrightarrow} B \otimes \mathcal{K}. \tag{3.31}$$

Note that $\mathcal{E}_h$ may be identified with $\mathcal{H} \otimes D\mathcal{H}_{BL}$. If necessary, the section that appears in the construction of $(\mathcal{E}_h, F_h)$ can be modified to coincide with the one used in (1.10) to represent the product of $[\![\psi_{F_1}]\!] \in E_G(A, D)$ and $[\![\psi_{F_2}]\!] \in E_G(D, B)$. Consequently, the product $\Xi(x) \sharp_D \Xi(y) = [\![\psi_{F_1}]\!] \sharp_D [\![\psi_{F_2}]\!]$ is represented by the composition:

$$\mathcal{S}A \otimes \mathcal{K} \xrightarrow{\Delta} \mathcal{S}\mathcal{S}A \otimes \mathcal{K} \overset{\psi_{F_{1,t}}}{\dashrightarrow} \mathcal{S}D \otimes \mathcal{K} \overset{\psi_{F_{2,h(t)}}}{\dashrightarrow} B \otimes \mathcal{K}. \tag{3.32}$$

Our goal is to show that the two asymptotic families given in (3.31) and (3.32), or rather their corresponding families in the $C_0(I)$-picture, are homotopic. As in the proof of Theorem 3.4.3, we denote by $\widehat{\Delta}\,(f)(F_{1,t}, F_{2,t})(a \otimes_D 1)$ the composition that corresponds to (3.32) in the $C_0(I)$-picture of E-theory. The problem is reduced to showing that, for all $f \in C_0(I)$ and all $a \in A$:

$$\lim_{t\to\infty} \|\, \widehat{\Delta}\,(f)(F_{1,t}, F_{2,t})(a \otimes_D 1) - f(F_h)(a \otimes_D 1)\| = 0. \tag{3.33}$$

(In this case we cannot simplify the proof by ignoring $(a \otimes_D 1)$, as it was done in the case of the external product.)

Claim 1. *We have:*

$$e^{-[F_1 \otimes_D 1 + (u \otimes_D 1)(\mathrm{Res}_h)_*(\,\underline{F}\,)]^2} = \big(e^{-F_1^2} \otimes_D 1\big)\, e^{-(u^2 \otimes_D 1)(\mathrm{Res}_h)_*(\,\underline{F}\,)^2}, \quad \textit{modulo } \mathfrak{J}(F_h), \tag{3.34}$$

together with a corresponding relation involving xe^{-x^2}.

Indeed, the properties that F_1, u, $\underline{F}$, and h satisfy, listed in Step 3, Overview 2.3.7, show that the problem reduces to the following one: 'Given operators $S, T \in \mathcal{B}(\mathcal{E}_h)$ such that $(ST + TS) \in \mathfrak{J}(F_h)$, then $\big(e^{-(S+T)^2} - e^{-S^2}e^{-T^2}\big) \in \mathfrak{J}(F_h)$.' This second statement follows easily from the fact that $\mathcal{J}(F_h)$ is a closed ideal of $\mathcal{B}(\mathcal{E}_h)$. Using the claim, the asymptotic family of $\Xi(\, x \,\sharp_D\, y\,)$ acts on generators as:

$$e^{-x^2} \otimes (a \otimes_D 1) \overset{\psi_{F_h}}{\mapsto} \big(e^{-F_1^2} a \otimes_D 1\big)\, e^{-(u^2 \otimes_D 1)(\mathrm{Res}_h)_*(\,\underline{F}\,)^2}, \tag{3.35}$$

with $(e^{-F_1^2} a \otimes_D 1) \in (\mathcal{K} \otimes D) \otimes_D 1$, and $e^{-F_1^2} a = \psi_{F_1}(e^{-x^2} \otimes a)$ representing exactly the second map in (3.32). There is a similar action on $xe^{-x^2} \otimes (a \otimes_D 1)$.

Claim 2. *Assume that $D\mathcal{H}_B = \mathcal{H}_B$. Under the identification $F_h = \mathcal{H} \otimes \mathcal{H}_{BL}$, $\underline{F} = 1 \otimes F_2$, and $u_t = \sum_i K_i^t \otimes d_i^t$, (3.33) holds true for $f(x) \in \{e^{-x^2}, xe^{-x^2}\}$.*

As in the proof of Proposition 2.3.4, the general case is more involved, but it can be also proved by a similar argument. ■

We conclude the chapter with some philosophical comments. The qualitative difference between the terms $\widetilde{F_1 \otimes_D} 1$ and $1 \widetilde{\otimes_D F_2}$ that appear in the product in KE-theory has correspondent in the fact that the product in E-theory treats differently the two asymptotic morphisms that are to be composed (the presence of h in (1.10)). This correspondence is incorporated in the way Ξ acts. Such a compatibility and the one reflected in the use of Kasparov's Technical Theorem when proving the functoriality of Θ show that KE-theory sits in natural way 'in between' KK-theory and E-theory. Meanwhile we also note that the techniques used in the construction of Θ and Ξ, respectively, are completely different. For more comments on Θ, Ξ, and their composition see next chapter.

CHAPTER 4

Final remarks

In this short chapter we further investigate the significance and consequences of (i) the two natural transformations Θ and Ξ that we have constructed in the previous chapter, and of (ii) the axioms (aKm1)–(aKm4) that lie at the foundation of KE-theory. A consequence of our discussion is that non-equivariant KE-theory groups recover the ordinary K-theory for trivially graded C^*-algebras.

4.1. On the composition $\Xi \circ \Theta$

Putting together the results of the previous chapter, it is clear that we have a composition of functors $KK^G \xrightarrow{\Theta} KE^G \xrightarrow{\Xi} E_G$:

(4.1)

$$\begin{array}{ccccc} KK^G(A,B) & \xrightarrow{\Theta} & KE^G(A,B) & \xrightarrow{\Xi} & E_G(A,B) \\ (\mathcal{E},F) & \mapsto & \big\{\big(\mathcal{E},(1-u_t)F(1-u_t)\big)\big\}_t & \mapsto & \big\{f\otimes a \overset{\varphi_t}{\mapsto} f\big((1-u_t)F(1-u_t)\big)\,a\big\}_t \end{array}$$

Here $(\mathcal{E},F)$ is a Kasaprov G-(A,B)-module, $\{u_t\}_t$ is a quasi-invariant quasi-central approximate unit for $\mathcal{K}(\mathcal{E})$, $f\in C_0((-1,1))$, and $a\in A$.

The interest in the composition $\Xi\circ\Theta$ is twofold. First, it explicitly provides a connecting map between the KK-theory groups and the E-theory groups, when the Kasparov modules are given in the bounded picture, *i.e.* when the operator F is bounded. (In the unbounded picture, such a map was sketched for a particular case in [**CoHg**, Sec.8] and possibly elsewhere.) Second, it is shown to preserve the essential feature of each of the three bivariant theories — the associative product.

The behavior of the natural transformations Θ and Ξ with respect to σ (Propositions 3.1.5 and 3.3.8), together with the general construction of the product (Definition 2.3.11, and the like in KK-theory and E-theory), show that more can be said about Θ and Ξ: they preserve the general product of the three bivariant theories, *under the assumption that we work only with K-nuclear C^*-algebras* [**Sk88**], or with *exact C^*-algebras* [**sWss**]. This restriction is due to the fact that E-theory does not behave well with respect to minimal tensor products, but see [**GHT**, Ch.4] for comments on this behavior under the presence of an exact C^*-algebra. Another way to obtain this extra-property is to redo all the definitions and constructions in KK-theory and KE-theory by replacing *all* the minimal tensor products $\otimes$ (at the level of C^*-algebras and Hilbert modules) by maximal ones $\otimes_{max}$ (see [**Sk88**]). We do not know how useful this second approach may be. The literature about KK-theory seems to give exclusive consideration to the minimal tensor product.

For arbitrary C^*-algebras, it seems to be a hard problem to decide whether or not KE-theory coincides with either of the other two theories. The first result that needs to be obtained in this direction is probably a characterization of the excision that KE-theory satisfies. In any case, in the non-equivariant context the bijectivity of the composition $\Xi \circ \Theta$ can be obtained in a rather easy and formal way for all K-nuclear C^*-algebras. The following result is Theorem 3.5 of [**Hg90b**].

THEOREM 4.1.1. *There is a unique (additive) functor* $\mathbf{KK} \to \mathbf{E}$ *such that the diagram*

$$\begin{array}{ccc} \mathbf{C^*\text{-alg}} & = & \mathbf{C^*\text{-alg}} \\ \downarrow & & \downarrow \\ \mathbf{KK} & \longrightarrow & \mathbf{E} \end{array}$$

commutes. If A is a K-nuclear C^*-algebra then for any B the homomorphism $KK(A,B) \to E(A,B)$ is an isomorphism.

By our construction of Θ and Ξ, these are additive functors, and consequently their composition $\Xi \circ \Theta$ *must be* the unique functor mentioned in the theorem. The second part shows that this composition is actually an isomorphism in the case when the C^*-algebra in the first variable is K-nuclear. It is not hard to conjecture at this point that there are isomorphisms $KK(A,B) \simeq KE(A,B) \simeq E(A,B)$, in the case when A is K-nuclear, but we were not able to prove it.

4.2. A non-equivariant example: K-theory

In this section the C^*-algebras are trivially graded (ungraded), and there is no group action. We show that the KE-theory groups recover, when the first C^*-algebra is $\mathbb{C}$, the ordinary K-theory for C^*-algebras.

PROPOSITION 4.2.1. *Let B be an ungraded C^*-algebra, then*

$$KE(\mathbb{C},B) = KK(\mathbb{C},B) = K_0(B).$$

PROOF. Our method consists in showing that the axioms of Kasparov modules (Definition 1.4.1, (1.1)) can be successively modified, in the case when $A = \mathbb{C}$, to give the axioms (aKm1–3) of asymptotic Kasparov modules (Definition 2.1.6). In this way we obtain intermediate abelian groups $\widetilde{KK}(\mathbb{C},B)$, $\widetilde{KE}(\mathbb{C},B)$, and group homomorphisms α, β, γ between the four groups under consideration, that can be depicted in the diagram:

$$KK(\mathbb{C},B) \xrightarrow{\alpha} \widetilde{KK}(\mathbb{C},B) \xrightarrow{\beta} \widetilde{KE}(\mathbb{C},B) \xrightarrow{\gamma} KE(\mathbb{C},B)\,. \tag{4.2}$$

(Note that $\widetilde{KK}(\mathbb{C},B)$ has nothing to do with the group denoted by same symbol in [**Sk84**, Def.2(8)], which is the quotient of $kk(\mathbb{C},B)$ by the equivalence relation generated by addition of degenerate elements and operatorial homotopy.) The claimed isomorphism between the KK-theory group and the KE-theory group follows from the fact that α, β, and γ will be proven to be isomorphisms.

$\widetilde{KK}(\mathbb{C},B)$ is the abelian group (under direct sum) of homotopy classes of pairs $(\mathcal{E},F)$, where $\mathcal{E}$ is a Hilbert B-module, admitting an action of $\mathbb{C}$ via a $*$-homomorphism $\varphi : \mathbb{C} \to \mathcal{B}(\mathcal{E})$, and $F \in \mathcal{B}(\mathcal{E})$ is an odd operator such that:

$$\varphi(1) = \text{id},\ F = F^*,\ \text{ and } (F^2 - 1/2) \geq 0,\ \text{ modulo } \mathcal{K}(\mathcal{E}). \tag{4.3}$$

(See Remark 2.1.7.) To construct the group homomorphism $\alpha : KK(\mathbb{C},B) \to \widetilde{KK}(\mathbb{C},B)$ we recall some of the standard simplifications of the axioms that a Kasparov module has to satisfy [**Blck**, 17.4]. Let $(\mathcal{E}, F) \in kk(\mathbb{C}, B)$ be an arbitrary Kasparov module. By replacing F with $F' = (F + F^*)/2$ we find a homotopic module $(\mathcal{E}, F')$ with the operator self-adjoint. Next, consider the projection $\varphi(1) = P \in \mathcal{B}(\mathcal{E})$. The pair $(\mathcal{E}, F')$ is operator homotopic to $(\mathcal{E}, PF'P) = (P\mathcal{E}, PF'P) + ((1-P)\mathcal{E}, 0)$, with the second summand being degenerate. Consequently, in the homotopy class of the initial Kasparov module we find a representative $(\widetilde{\mathcal{E}}, \widetilde{F}) = (P\mathcal{E}, PF'P)$, with $1 \in \mathbb{C}$ acting as identity, $\widetilde{F}$ self-adjoint, and $\widetilde{F}^2 = 1 \geq 1/2$, modulo $\mathcal{K}(\widetilde{\mathcal{E}})$. This defines the group homomorphism α (all the changes above preserve homotopies and direct sums):

$$\alpha : KK(\mathbb{C}, B) \to \widetilde{KK}(\mathbb{C}, B),\ (\mathcal{E}, F) \mapsto (\widetilde{\mathcal{E}}, \widetilde{F}). \tag{4.4}$$

For the inverse map, let $\psi : \mathbb{R} \to \mathbb{R}$, be $\psi(x) = -1$, for $x \leq -1/\sqrt{2}$, $\psi(x) = \sqrt{2}\,x$, for $x \in (-1/\sqrt{2}, 1/\sqrt{2})$, and $\psi(x) = 1$, for $x \geq 1/\sqrt{2}$. Define

$$\alpha' : (\widetilde{\mathcal{E}}, \widetilde{F}) \mapsto (\widetilde{\mathcal{E}}, \psi(\widetilde{F})).$$

The only non-trivial checking is $\psi(\widetilde{F})^2 - 1 = 2\widetilde{F}^2 - 1 \geq 0$, modulo $\mathcal{K}(\widetilde{\mathcal{E}})$. We observe that $[\psi(\widetilde{F}), \widetilde{F}] \geq 0$, and consequently both compositions $\alpha' \circ \alpha$ and $\alpha \circ \alpha'$ give results homotopic with the initial module. It follows that α is an isomorphism, with $\alpha^{-1} = \alpha'$.

Define $\widetilde{KE}(\mathbb{C}, B)$ to be the abelian group (under direct sum) of homotopy classes of asymptotic Kasparov $(\mathbb{C}, B)$-modules $(\widehat{\mathcal{E}}, \widehat{F})$ satisfying the *extra conditions*:

$$\varphi(1) = \mathrm{id},\ \widehat{F} = \widehat{F}^*,\ \text{ and } (\widehat{F}^2 - 1/2) \geq 0,\ \text{ modulo } \mathcal{C}(\widehat{\mathcal{E}}). \tag{4.5}$$

The map $\gamma : \widetilde{KE}(\mathbb{C}, B) \to KE(\mathbb{C}, B)$ is the forgetting map at the level of asymptotic Kasparov modules. To define the inverse γ', let $(\widehat{\mathcal{E}}, \widehat{F})$ be an arbitrary asymptotic Kasparov module. We can make the action of $\mathbb{C}$ unital as in KK-theory: there is a homotopy followed by a 'small perturbation' connecting $(\widehat{\mathcal{E}}, \widehat{F})$ with $(\widehat{\mathcal{E}}', \widehat{F}'') = (P\widehat{\mathcal{E}}, P\widehat{F}P)$, where $P = \varphi(1)$. As we have already observed in Corollary 2.2.8, there is a homotopy from this last pair to another one $(\widehat{\mathcal{E}}', \widehat{F}')$, with $\widehat{F}'$ self-adjoint. Finally, (aKm3) implies that $(\widehat{F}_t')^2 - 1 \geq U_t + V_t$, with $U = \{U_t\}_t \in \mathcal{C}(\widehat{\mathcal{E}}')$ and $V = \{V_t\}_t \in \mathcal{J}(\widehat{\mathcal{E}}')$. Let T be such that $\|V_t\| < 1/2$, for all $t > T$. It follows that $(\widehat{F}_t')^2 - 1/2 \geq U_t$, for $t > T$. We define γ' via a 'translation' (see

Example 2.2.3):

$$\gamma' : \{\, (\widehat{\mathcal{E}}_t, \widehat{F}_t)\,\}_t \mapsto \{\, (\widehat{\mathcal{E}}_{t+T}, \widehat{F}_{t+T})\,\}_t.$$

All the operations used to define γ' preserve homotopies and direct sums, and consequently both $\gamma' \circ \gamma$ and $\gamma \circ \gamma'$ are identity, and $\gamma^{-1} = \gamma'$.

Finally, define

$$\beta : \widetilde{KK}(\mathbb{C}, B) \to \widetilde{KE}(\mathbb{C}, B),\ (\widetilde{\mathcal{E}}, \widetilde{F}) \mapsto \{\, (\widetilde{\mathcal{E}}, \widetilde{F})\,\}_t \ \text{(constant family)}, \tag{4.6}$$

and

$$\beta' : \widetilde{KE}(\mathbb{C}, B) \to \widetilde{KK}(\mathbb{C}, B),\ (\widehat{\mathcal{E}}, \widehat{F}) = \{\, (\widehat{\mathcal{E}}_t, \widehat{F}_t)\,\}_t \mapsto (\widehat{\mathcal{E}_1}, \widehat{F_1}) \ \text{(the ‘fiber’ at } t = 1).$$

The composition $\beta' \circ \beta = \mathrm{id}$ is obvious. Let now $(\widehat{\mathcal{E}}, \widehat{F}) = \{\, (\widehat{\mathcal{E}}_t, \widehat{F}_t)\,\}_t$ be an element of $\widetilde{KE}(\mathbb{C}, B)$. There exists a homotopy $(\boldsymbol{\mathcal{E}}, \boldsymbol{F})$ between $(\widehat{\mathcal{E}}, \widehat{F})$ and $(\beta\circ\beta')((\widehat{\mathcal{E}}, \widehat{F})) = \{\, (\widehat{\mathcal{E}_1}, \widehat{F_1})\,\}_t$ given by explicit formulas:

$$\boldsymbol{\mathcal{E}}_{t,s} = \widehat{\mathcal{E}}_{s+(1-s)t},\ \boldsymbol{F}_{t,s} = \widehat{F}_{s+(1-s)t}, \quad \text{for } s \in [0,1], t \in [1, \infty).$$

This proves that β is also an isomorphism, with $\beta^{-1} = \beta'$.

The claimed isomorphism is $\gamma \circ \beta \circ \alpha : KK(\mathbb{C}, B) \to KE(\mathbb{C}, B)$. ■

Remark. The comment about the bijectivity of $\Xi \circ \Theta$ made at the end of the previous section shows that $\Xi \circ \Theta = \gamma \circ \beta \circ \alpha$, as group homomorphisms.

Bibliography

[Arvs] W. Arveson, *Notes on extensions of C^*-algebras*, Duke Math. J. **44** (1977), no. 2, 329–355; MR **55** #11056

[Atiy69] M. Atiyah, *Global theory of elliptic operators*, in *Proc. Internat. Conf. on Functional Analysis and Related Topics (Tokyo, 1969)*, 21–30, Univ. of Tokyo Press, Tokyo, 1970; MR **42** #1154

[AtHr] M. Atiyah and F. Hirzebruch, *Vector bundles and homogeneous spaces*, in *Proc. Sympos. Pure Math., Vol. III*, 7–38, Amer. Math. Soc., Providence, R.I., 1961; MR **25** #2617

[AtSi63] M. Atiyah and I. Singer, *The index of elliptic operators on compact manifolds*, Bull. Amer. Math. Soc. **69** (1963), 422–433; MR **28** #626

[AtSiI] M. Atiyah and I. Singer, *The index of elliptic operators. I*, Ann. of Math. (2) **87** (1968), 484–530; MR **38** #5243

[BaaJlg] S. Baaj and P. Julg, *Théorie bivariante de Kasparov et opérateurs non bornés dans les C^*-modules hilbertiens*, C. R. Acad. Sci. Paris Sér. I Math. **296** (1983), no. 21, 875–878; MR 84m:46091

[BC82] P. Baum and A. Connes, *Geometric K-theory for Lie groups and foliations*, Enseign. Math. (2) **46** (2000), no. 1-2, 3–42; CNO CMP 1 769 535

[BCH] P. Baum, A. Connes and N. Higson, *Classifying space for proper actions and K-theory of group C^*-algebras*, n *C^*-algebras: 1943–1993 (San Antonio, TX, 1993)*, 240–291, Contemp. Math., 167, Amer. Math. Soc., Providence, RI, 1994; MR 96c:46070

[Blck] B. Blackadar, *K-theory for operator algebras*, Second edition, Cambridge Univ. Press, Cambridge, 1998; MR 99g:46104

[Co81] A. Connes, *An analogue of the Thom isomorphism for crossed products of a C^*-algebra by an action of* $\mathbb{R}$, Adv. in Math. **39** (1981), no. 1, 31–55; MR 82j:46084

[CoHg] A. Connes and N. Higson, *Almost homomorphisms and KK-theory*, unpublished manuscript, `http://www.math.psu.edu/higson/research.html`

[CoHg90] A. Connes and N. Higson, *Déformations, morphismes asymptotiques et K-théorie bivariante*, C. R. Acad. Sci. Paris Sér. I Math. **311** (1990), no. 2, 101–106; MR 91m:46114

[CoSk] A. Connes and G. Skandalis, *The longitudinal index theorem for foliations*, Publ. Res. Inst. Math. Sci. **20** (1984), no. 6, 1139–1183; MR 87h:58209

[Cu83] J. Cuntz, *Generalized homomorphisms between C^*-algebras and KK-theory*, in *Dynamics and processes (Bielefeld, 1981)*, 31–45, Lecture Notes in Math., 1031, Springer, Berlin, 1983; MR 85j:46126

[Cu84] J. Cuntz, *K-theory and C^*-algebras*, in *Algebraic K-theory, number theory, geometry and analysis (Bielefeld, 1982)*, 55–79, Lecture Notes in Math., 1046, Springer, Berlin, 1984; MR 86d:46071

[Cu97] J. Cuntz, *Bivariante K-Theorie für lokalkonvexe Algebren und der Chern-Connes-Charakter*, Doc. Math. **2** (1997), 139–182 (electronic); MR 98h:19006

[Cu98] J. Cuntz, *A general construction of bivariant K-theories on the category of C^*-algebras*, in *Operator algebras and operator theory (Shanghai, 1997)*, 31–43, Contemp. Math., 228, Amer. Math. Soc., Providence, RI, 1998; MR 2000b:46125

[Dum] D. Dumitraşcu, *On asymptotic morphisms, K-homology, and Dirac operators*, submitted for publication, Pacific J. Math. (2000)

[Gntn] E. Guentner, *Relative E-theory, quantization and index theory*, Ph.D. thesis, Penn State University, 1994

[GHT] E. Guentner, N. Higson and J. Trout, *Equivariant E-theory for C^*-algebras*, Mem. Amer. Math. Soc. **148** (2000), no. 703, viii+86 pp.; MR 2001c:46124

[Hg87a] N. Higson, *A characterization of KK-theory*, Pacific J. Math. **126** (1987), no. 2, 253–276; MR 88a:46083

[Hg87b] N. Higson, *On a technical theorem of Kasparov*, J. Funct. Anal. **73** (1987), no. 1, 107–112; MR 88g:46064

[Hg90a] N. Higson, *A primer on KK-theory*, in *Operator theory: operator algebras and applications, Part 1 (Durham, NH, 1988)*, 239–283, Proc. Sympos. Pure Math., **51**, Amer. Math. Soc., Providence, RI, 1990; MR 92g:19005

[Hg90b] N. Higson, *Categories of fractions and excision in KK-theory*, J. Pure Appl. Algebra **65** (1990), no. 2, 119–138; MR 91i:19005

[Hg91] N. Higson, *A note on the cobordism invariance of the index*, Topology **30** (1991), no. 3, 439–443; MR 92f:58171

[HgKas97] N. Higson and G. G. Kasparov, *Operator K-theory for groups which act properly and isometrically on Hilbert space*, Electron. Res. Announc. Amer. Math. Soc. **3** (1997), 131–142 (electronic); MR 99e:46090

[HgKas01] N. Higson and G. G. Kasparov, *E-theory and KK-theory for groups which act properly and isometrically on Hilbert space*, Invent. Math. **144** (2001), no. 1, 23–74; CNO 1 821 144

[HgRoe] N. Higson and J. Roe, *Analytic K-homology*, Oxford Univ. Press, Oxford, 2000; CNO CMP 1 817 560

[HKT] N. Higson, G. Kasparov and J. Trout, *A Bott periodicity theorem for infinite dimensional Euclidean space*, Adv. Math. **135** (1998), no. 1, 1–40; MR 99j:19005

[Kas75] G. G. Kasparov, *Topological invariants of elliptic operators, I: K-homology*, Math. USSR-Izv. **9** (1975), no. 4, 751–792; translated from Izv. Akad. Nauk SSSR Ser. Mat. **39** (1975), no. 4, 796–838; MR **58** #7603

[Kas80] G. G. Kasparov, *Hilbert C^*-modules: theorems of Stinespring and Voiculescu*, J. Operator Theory **4** (1980), no. 1, 133–150; MR 82b:46074

[Kas81] G. G. Kasparov, *The operator K-functor and extensions of C^*-algebras*, Math. USSR Izvestiya **16** (1981), 513–572; translated from Izv. Akad. Nauk SSSR Ser. Mat. **44** (1980), no. 3, 571–636; MR 81m:58075

[Kas88] G. G. Kasparov, *Equivariant KK-theory and the Novikov conjecture*, Invent. Math. **91** (1988), no. 1, 147–201; MR 88j:58123

[Kas95] G. G. Kasparov, *K-theory, group C^*-algebras, and higher signatures (conspectus)*, in *Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993)*, 101–146, Cambridge Univ. Press, Cambridge, 1995; MR 97j:58153

[Lan] E. C. Lance, *Hilbert C^*-modules: a toolkit for operator algebraists*, Cambridge Univ. Press, Cambridge, 1995; MR 96k:46100

[leG99] P.-Y. Le Gall, *Théorie de Kasparov équivariante et groupoïdes, I*, *K*-Theory **16** (1999), no. 4, 361–390; MR 2000f:19006

[Pals] R. Palais, *Seminar on the Atiyah-Singer index theorem*, Princeton Univ. Press, Princeton, N.J., 1965; MR **33** #6649

[Pdrs] G. K. Pedersen, *C^*-algebras and their automorphism groups*, Academic Press, London, 1979; MR 81e:46037

[Pop] R. Popescu, *E-théorie équivariante et groupoïdes*, thése de doctorat, Université Claude Bernard - Lyon I, 2000

[Rsn84] J. Rosenberg, *Group C^*-algebras and topological invariants*, in *Operator algebras and group representations, Vol. II (Neptun, 1980)*, 95–115, Pitman, Boston, MA, 1984; MR 85g:22011

[Sk84] G. Skandalis, *Some remarks on Kasparov theory*, J. Funct. Anal. **56** (1984), no. 3, 337–347; MR 86c:46085

[Sk88] G. Skandalis, *Une notion de nuclearité en K-théorie (d'aprés J. Cuntz)*, K-Theory **1** (1988), no. 6, 549–573; MR 90b:46131

[Thms98] K. Thomsen, *The universal property of equivariant KK-theory*, J. Reine Angew. Math. **504** (1998), 55–71; MR 2000a:19008

[Thms99] K. Thomsen, *Asymptotic homomorphisms and equivariant KK-theory*, J. Funct. Anal. **163** (1999), no. 2, 324–343; MR 2000c:19008

[sWss] S. Wassermann, *Exact C^*-algebras and related topics*, Seoul Nat. Univ., Seoul, 1994; MR 95b:46081

Index

Vita

Constantin D. Dumitraşcu

Constantin Dorin Dumitraşcu was born in 1968, in Pleniţa, Dolj, Romania. He graduated from Nicolae Bălcescu High-School, Craiova, Romania, in June 1987. He received the Bachelor degree in Mathematics from the University of Craiova, in July 1993. Being awarded a European Union TEMPUS fellowship, he spent the academic year 1992-1993 at Centro de Matematica and Applicasoes, Faculdade de Ciências, Lisbon, Portugal. In August 1993 he was admitted as a graduate student at Virginia Tech. He received, while holding the position of teaching assistant, the Master of Science degree in Mathematics from Virginia Polytechnic Institute and State University, in July 1995. During the academic year 1995-1996 he was a visiting teaching assistant at University of Colorado, Boulder. He enrolled in the Doctoral program in Mathematics at the Pennsylvania State University in August 1996. During the duration of his studies, he held the position of teaching and research assistant within the Department of Mathematics.