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NONPARAMETRIC IDENTIFICATION AND ESTIMATION OF K-DOUBLE AUCTIONS

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Abstract

This dissertation consists of two chapters on nonparametrically identifying and estimating the sealed-bid k-double auction models between single buyer and single seller.

Chapter 1: Nonparametric Identification and Estimation of k-Double Auctions Using Bid Data

This chapter studies the nonparametric identification and estimation of double auctions with one buyer and one seller. This model assumes that both bidders submit their own sealed bids, and the transaction price is determined by a weighted average between the submitted bids when the buyer's offer is higher than the seller's ask. It captures the bargaining process between two parties. Working within this double auction model, we first establish the nonparametric identification of both the buyer's and the seller's private value distributions in two bid data scenarios; from the ideal situation in which all bids are available, to a more realistic setting in which only the transacted bids are available. Specifically, we can identify both private value distributions when all of the bids are observed. However, we can only partially identify the private value distributions on the support with positive (conditional) probability of trade when only the transacted bids are available in the data. Second, we estimate double auctions with bargaining using a two-step procedure that incorporates bias correction. We then show that our value density estimator achieves the same uniform convergence rate as Guerre, Perrigne, and Vuong (2000) for one-sided auctions. Monte Carlo experiments show that, in finite samples, our estimation procedure works well on the whole support and significantly reduces the large bias of the standard estimator without bias correction in both interior and boundary regions.

Chapter 2: Nonparametric Identification of k-Double Auctions Using Price Data

This chapter studies the model identification problem of k-double auctions between one buyer and one seller when the transaction price, rather than the traders' bids, can be observed. Given that only the price data is available, I explore an identification strategy that utilizes the double auctions with extreme pricing weight ($k = 1$ or 0) and exclusive covariates that shift only one trader's value distribution to identify both the buyer's and the seller's value distributions nonparametrically.

First, as each exclusive covariate can take at least two values, the buyer's and the seller's value distributions are partially identified from the price distribution for $k = 1$ or $k = 0$. The identified set is sharp and can be easily computed. I provide a set of sufficient conditions under which the traders' value distributions are point identified. Second, when the exclusive covariates are continuous, it is shown that the buyer's and the seller's value distributions will be uniquely determined by a partial differential equation that only depends on the price distribution, provided that the value distributions are known for at least one value of the exclusive covariates.

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Chapter 1

Nonparametric Identification and Estimation of k-Double Auctions Using Bid Data

1.1 Introduction

For more than 100 years, trade in the most important field markets for homogeneous goods has been governed primarily by double auction rules (see Friedman, 1993). With one buyer and one seller, a double auction model captures the nature of bargaining under incomplete information. Applications of such a model range from the settlement of a claim out of court, to union-management negotiations,[1] to the purchase and sale of a used automobile (see Chatterjee and Samuelson, 1983).

While the theoretical properties of double auctions with bargaining are well established, the theory of identification and estimation in these double auctions is presently very sparse. On the theoretical side, the double auction model with one buyer and one seller has been extensively studied by Chatterjee and Samuelson (1983), Myerson and Satterthwaite (1983), Leininger, Linhart, and Radner (1989), Satterthwaite and Williams (1989), and Kadan (2007), among others. In addition, there is literature which examines the theoretical properties of double auctions with one buyer and one seller in an experimental setting; see, e.g., Radner and Schotter (1989) and Rapoport and Fuller (1995), among others. Nevertheless, there have been few studies of the identification and estimation of double auctions with bargaining from field data. This constrains the corresponding empirical analysis.

Motivated by this gap in the literature, we study the nonparametric identification and estimation

[1] Treble (1987, 1990) obtained documentation of offers and asks in union-management wage negotiations for most UK coalfields over 1893-1914.

of the buyer's and the seller's value distributions in double auctions with bargaining, and obtain the following results: First, in addition to characterizing all the restrictions on the observables (i.e. bid distributions) imposed by the theoretical double auction model with bargaining, we establish point identification of model primitives (i.e. value distributions) from the observables in the case where all bids are observed. In the case when only transacted bids are observed,[2] we provide a sharp identified set of bidders' value distributions.[3] We show that, in the latter case, the conditional distributions of bidders' valuations given positive (conditional) probability of trade are point identified. Second, we propose a (boundary and interior) bias corrected two-step estimator of the buyer's and the seller's value distributions. In a double auction setting, we show that our estimator achieves the same uniform convergence rate as the one-sided auctions provided by Guerre, Perrigne, and Vuong (2000). Third, using Monte Carlo experiments, we show that it is important to implement the bias correction (especially bias correction in the interior of the support) in the two-step estimation of value distributions. In particular, we show that, without bias correction, the statistical inference is almost infeasible, not only on the boundaries, but also in a significant part of the interior.

Our research builds upon a large body of work which examines nonparametric identification and estimation of one-sided auctions. This work was pioneered by Guerre, Perrigne, and Vuong (2000) for identification and estimation of first-price auctions, and has been followed by many other papers. Examples include Li, Perrigne, and Vuong (2000, 2002), Athey and Haile (2002), Haile, Hong, and Shum (2003), Haile and Tamer (2003), Hendricks, Pinkse, and Porter (2003), Li and Zheng (2009), An, Hu, and Shum (2010), Athey, Levin, and Seira (2011), Krasnokutskaya (2011), Tang (2011), Hu, McAdams, and Shum (2013), Gentry and Li (2014), among others. For a comprehensive survey, see Athey and Haile (2007). Among these, the identification part of this chapter is similar to Haile and Tamer (2003), who obtained bounds on the distribution of valuations by placing two simple assumptions on the relation between valuations and bids without a full characterization of bidding behavior in ascending auctions. It compliments McAdams (2008), who provided upper and lower bounds on the distribution of bidder values in multi-unit auctions, as well as Tang (2011), who bounded counterfactual revenue distributions in auctions with affiliated values. The identification part is also analogous to Gentry and Li (2014), who obtained constructive bounds on model fundamentals which collapse to point identification when available entry variation is continuous in auctions with selective entry. Compared to this research line, however, we consider a different auction setting (namely, double auctions with bargaining) which introduces not only

[2]In a transaction, a buyer's bid (or offer) must be no lower than seller's bid (or ask).

[3]This result parallels the typical finding that limitations on data observation (such as interval valued data) induce partial identification in nonparametric mean regression and semi-parametric binary regression; see, e.g., Manski and Tamer (2002), Magnac and Maurin (2008), Wan and Xu (2015), among others.

asymmetric information but also asymmetric bidding strategies.[4]

Our work is also closely related to the structural analysis of non-cooperative bargaining models. Many papers in this literature recover the model primitives by exploiting the data on the timing and terms of reaching an agreement after sequential bargaining. Complete information examples include Merlo (1997), Diermeier, Eraslan, and Merlo (2003), Eraslan (2008), Merlo and Tang (2012, 2015), and Simcoe (2012), while Sieg (2000), Watanabe (2006), Merlo, Ortalo-Magne, and Rust (2015), are a set of examples which highlight the role of asymmetric information. This chapter, however, uses the data on offers and asks at the beginning of the bargaining process to estimate the initial valuation distributions of both participating parties. Consequently, our work can be viewed as complimentary to this growing literature.

More broadly, we contribute to a third literature on kernel density estimation with boundary correction. In this line of research, the boundary bias can be corrected by several different methods such as the reflection method (e.g. Silverman, 1986), the boundary kernel approach (e.g. Gasser and Müller, 1979), the transformation method (e.g. Wand, Marron, and Ruppert, 1991), the local linear method (e.g. Cheng, 1997, Cheng, Fan, and Marron, 1997, Zhang and Karunamuni, 1998), the nearest internal point approach (e.g. Imbens and Ridder, 2009), and the reflection of transformed data approach (e.g. Karunamuni and Zhang, 2008, Zhang, Karunamuni, and Jones, 1999). Among these, Zhang, Karunamuni, and Jones (1999) proposed a generalized reflection method, which involves a reflection of transformed data, and established the pointwise consistency of their estimator. This approach was later improved by Karunamuni and Zhang (2008). In (one-sided) first-price auctions, Hickman and Hubbard (2014) applied their method to correct the boundary bias of the two-step value density estimator which was first proposed by Guerre, Perrigne, and Vuong (2000). We also adopt the bias correction ideas of Zhang, Karunamuni, and Jones (1999) and Karunamuni and Zhang (2008) to estimate both bid and private value densities in double auctions with bargaining. Relative to these two papers, however, we generalize their density estimation approach so that it can correct both boundary and interior biases which exist in the equilibrium outcome of our double auction model. Furthermore, we establish the uniform consistency of our generalized density estimator on the whole support.

The rest of this chapter is organized as follows. In Section 1.2, we present the sealed-bid double auction model with bargaining and characterize its equilibrium. Section 1.3 then studies the identification of private value distributions in two different scenarios. In the first scenario, all of the submitted bids can be observed. In contrast, only those bids with successful transactions can be observed in the second scenario. In Section 1.4, we estimate both the bid and the value densities with bias correction and establish its uniform consistency. Section 1.5 uses Monte Carlo

[4]The asymmetry of bidding strategies arises from the fact that the buyer and the seller have different roles in our double auction model.

experiments to illustrate the finite sample performance of our estimator. We briefly discuss the extension of our estimation approach to the case with auction-specific heterogeneity and/or higher order bias reduction in Section 1.6. Supplementary results and proofs are collected in Appendix A.

1.2 The *k*-Double Auction Model

We consider a k-double auction where a single and indivisible object is auctioned between a buyer and a seller. Each of them simultaneously submits a bid. If the buyer's offer is no lower than the seller's ask, a transaction is made at a price of their weighted average, i.e. at a price $p(B,S) = kB + (1-k)S$ where k is a constant in $[0,1]$, B is the buyer's offer, and S is the seller's ask. Otherwise, there is no transaction. The buyer has a value V for the auctioned object, and the seller has a reservation value C. Consequently, the buyer's payoff is $V - p(B,S)$ and the seller's payoff is $p(B,S) - C$ if a trade occurs; their payoffs are both zero otherwise. Each of them does not know her opponent's valuation but only knows that it is drawn from a distribution F_j $(j = C, V)$. The distributions F_V, F_C, and the pricing rule are all common knowledge between buyer and seller.

Restricted to single buyer and single seller, the k-double auction considered can be viewed as a static bargaining game between two parties, while the mathematical expression of the specified pricing rule indeed shares some similarity with the asymmetric Nash bargaining solution. However, it should be clarified that we will model the k-double auction as non-cooperative game and focus on the traders' equilibrium strategies. So such a pricing rule is completely exogenous rather than an endogenous outcome. At the same time, differing from the bargaining models which often concentrate on the two-player scenario, k-double auctions essentially do allow there to be multiple buyers trading lots of goods with multiple sellers. Considering only one buyer and one seller is mainly due to the limitation that the game theoretic properties of this simple case have been more well-understood. The k-double auctions with a large number of traders will be an interesting and important extension of our current model in the future.

We first impose the following assumption on the private values and their distributions.

Assumption A. *(i) V and C are independent. (ii)* F_V *is absolutely continuous on the support* $[\underline{v}, \overline{v}] \subset \mathbb{R}_+$ *with density* f_V*.* F_C *is absolutely continuous on the support* $[\underline{c}, \overline{c}] \subset \mathbb{R}_+$ *with density* f_C*.*

Under Assumption A, the seller's private value is independent of the buyer's, and the value distributions are absolutely continuous on bounded supports. Such an assumption has been adopted by most theoretical papers on double auctions with bargaining; see, e.g., Chatterjee and Samuelson (1983), Myerson and Satterthwaite (1983), Leininger, Linhart, and Radner (1989) and Satterthwaite and Williams (1989).

We also impose the following restriction on the supports of F_V and F_C.

Assumption B. *The supports of F_V and F_C satisfy $\underline{c} < \overline{v}$.*

This assumption requires that the buyer's maximum value must be higher than the seller's minimum cost. It rules out the trivial case of $\overline{v} \leqslant \underline{c}$ in which there is zero probability of trade in any equilibrium. The special cases of such a support condition have been commonly adopted by the theoretical double auction literature; e.g., Myerson and Satterthwaite (1983), Leininger, Linhart, and Radner (1989), and Satterthwaite and Williams (1989).

Denote by $\beta_B : [\underline{v}, \overline{v}] \to \mathbb{R}_+$ and $\beta_S : [\underline{c}, \overline{c}] \to \mathbb{R}_+$ the buyer's and the seller's strategies, respectively. Let $b = \beta_B(v)$ denote the bid of a buyer with realized private value v under strategy β_B. Then, the expected profit of the buyer given the seller's strategy is

$$\pi_B(b, v) = \begin{cases} \int_{\underline{s}}^{b} [v - p(b, s)] \, \mathrm{d}G_S(s) = \int_{\underline{s}}^{b} [v - kb - (1-k)s] \, \mathrm{d}G_S(s), & \text{if } b \geqslant \underline{s}, \\ 0, & \text{if } b < \underline{s}, \end{cases} \tag{1.1}$$

where G_S is the distribution function of the seller's bid and $\underline{s}$ is the lower endpoint of its support. Similarly, let $s = \beta_S(c)$ denote the ask of a seller with realized private reservation value c under strategy β_S. Then, the expected profit of the seller given the buyer's strategy is

$$\pi_S(s, c) = \begin{cases} \int_{s}^{\overline{b}} [p(b, s) - c] \, \mathrm{d}G_B(b) = \int_{s}^{\overline{b}} [kb + (1-k)s - c] \, \mathrm{d}G_B(b), & \text{if } s \leqslant \overline{b}, \\ 0, & \text{if } s > \overline{b}, \end{cases} \tag{1.2}$$

where G_B is the distribution function of the buyer's bid and $\overline{b}$ is the upper endpoint of its support.

We adopt the Bayesian Nash equilibrium (BNE) concept throughout.

Definition (Best response). *A buyer's strategy β_B is a best response to β_S if for any buyer's strategy $\tilde{\beta}_B : [\underline{v}, \overline{v}] \to \mathbb{R}_+$ and each value $v \in [\underline{v}, \overline{v}]$, $\pi_B(\beta_B(v), v) \geqslant \pi_B(\tilde{\beta}_B(v), v)$. The seller's best response is defined in an analogous way.*

Definition (Bayesian Nash equilibrium). *A strategy profile (β_B, β_S) constitutes a Bayesian Nash equilibrium if β_B and β_S are best responses to each other.*

We exclude some irregular equilibria and focus on those which are well-behaved as described in Chatterjee and Samuelson (1983). Precisely, we impose the following restrictions on the equilibrium:

Assumption C (Regular equilibrium). *The equilibrium strategy profile (β_B, β_S) satisfies*

A1. β_B and β_S are continuous and strictly increasing on their whole domains;

A2. β_B is continuously differentiable on $[\underline{s}, \overline{v}]$ if $\underline{s} < \overline{v}$; β_S is continuously differentiable on $[\underline{c}, \overline{b}]$ if $\underline{c} < \overline{b}$;

A3. $\beta_B(v) = v$ if $v \leqslant \underline{s}$; $\beta_S(c) = c$ if $c \geqslant \overline{b}$.

We say that an equilibrium satisfying Assumption C is regular. Assumption C basically restricts us to strictly monotone and (piecewise) differentiable strategy equilibria which are quite intuitive in bilateral k-double auctions. As demonstrated by Satterthwaite and Williams (1989, Theorem 3.2), there exist a continuum of regular equilibria when $k \in (0,1)$ and $[\underline{v}, \overline{v}] = [\underline{c}, \overline{c}] = [0,1]$. Following most of the empirical game literature, we adopt the following equilibrium selection mechanism when multiple regular equilibria exist:

Assumption D. *In all observed auctions, the buyers and the sellers play the same regular equilibrium.*

Notice that Assumption D is not restrictive when there is a unique regular equilibrium.

The following lemma characterizes some basic properties of the equilibrium strategy profile.

Lemma 1.1. *For any equilibrium* (β_B, β_S),

(i) when $v > \underline{s}$, $\beta_B(v) \leqslant v$ *with strict inequality if* $k > 0$;

(ii) when $c < \overline{b}$, $\beta_S(c) \geqslant c$ *with strict inequality if* $k < 1$.

Proof. See Appendix A.2. □

Note that the conclusion of Lemma 1.1 holds for any BNE (i.e., not only for regular BNE). With condition A3 of Assumption C, it implies that, in regular equilibrium, the buyer will never bid higher than her private value and the seller will never bid lower than her private value. Under the special case of $k = 1/2$, Leininger, Linhart, and Radner (1989) constructed a lemma similar to our Lemma 1.1.

1.3 Nonparametric Identification

In this section, we study the nonparametric identification of private value distributions in two cases which differ in the degree of available data. In the first case, econometricians can observe both the transacted bids and the bids where no transaction takes place.[5] In the second case, econometricians can only observe the transacted bids.

In both cases, we assume that the pricing weight k in the pricing rule is known to econometricians. Such an assumption is not restrictive because the value of k can be recovered by using some additional information about the transaction price, given that the transacted bids are observed. For example, when the mean transaction price is observed, the parameter k is determined by $k = \frac{\mathbb{E}(P) - \mathbb{E}(S^*)}{\mathbb{E}(B^*) - \mathbb{E}(S^*)}$ since $\mathbb{E}(P) = k\mathbb{E}(B^*) + (1-k)\mathbb{E}(S^*)$ where (B^*, S^*) are the transacted bids. Alternatively, when we observe some quantile of the transaction price, k can be identified by exploiting the property that the price distribution function is continuous and monotone in k (see Appendix A.1 for detailed discussion).

[5] We say that a pair of bids (B, S) is transacted if $B \geqslant S$.

1.3.1 Identification with Perfect Observability of Bid Distribution

We first consider the nonparametric identification of the k-double auction model with bargaining when researchers observe both the parameter k and the distribution of all submitted bids (including the bids that are not transacted).[6]

As shown in Chatterjee and Samuelson (1983) and Satterthwaite and Williams (1989), a regular equilibrium (β_B, β_S) in a k-double auction with bargaining can be characterized by the following two differential equations for $v \geqslant \underline{s}$ and $c \leqslant \overline{b}$,

$$\beta_B^{-1}(\beta_S(c)) = \beta_S(c) + k\beta_S'(c)\frac{F_C(c)}{f_C(c)}, \tag{1.3}$$

$$\beta_S^{-1}(\beta_B(v)) = \beta_B(v) - (1-k)\beta_B'(v)\frac{1-F_V(v)}{f_V(v)}, \tag{1.4}$$

where $\beta_B^{-1}(\cdot)$ and $\beta_S^{-1}(\cdot)$ are the inverse bidding strategies.[7] For buyer with value $v \geqslant \underline{s}$, the equilibrium bid under strategy β_B is $b = \beta_B(v)$. Let $\tilde{c} = \beta_S^{-1}(b)$. Since strategy β_S is strictly increasing, $G_S(b) = F_C(\beta_S^{-1}(b)) = F_C(\tilde{c})$. Noting that

$$g_S(b) = \frac{f_C(\beta_S^{-1}(b))}{\beta_S'(\beta_S^{-1}(b))} = \frac{f_C(\tilde{c})}{\beta_S'(\tilde{c})}, \quad v = \beta_B^{-1}(b) = \beta_B^{-1}(\beta_S(\tilde{c})),$$

by (1.3), we have

$$v = b + k\frac{G_S(b)}{g_S(b)}. \tag{1.5}$$

Similarly, for seller with value $c \leqslant \overline{b}$, we have the following condition by (1.4)

$$c = s - (1-k)\frac{1-G_B(s)}{g_B(s)}. \tag{1.6}$$

Note that (1.5) and (1.6) only hold for $v \geqslant \underline{s}$ and $c \leqslant \overline{b}$. In such a case, we have $\Pr(\beta_B(V) \geqslant \beta_S(C) \,|\, V = v) > 0$ when $v > \underline{s}$ and $\Pr(\beta_B(V) \geqslant \beta_S(C) \,|\, C = c) > 0$ when $c < \overline{b}$. In other words, given the private values, both the buyer and the seller expect that trade occurs with positive probability.[8] For the buyer with value $v < \underline{s}$ or the seller with value $c > \overline{b}$, there will be no transaction under strategy profile (β_B, β_S). We define functions $\xi(b, G_S)$ and $\eta(s, G_B)$ as the right-hand sides of (1.5) and (1.6), respectively. That is,

$$\xi(b, G_S) \equiv b + k\frac{G_S(b)}{g_S(b)}, \quad \underline{s} \leqslant b \leqslant \overline{s}, \tag{1.7}$$

[6]This observational environment is theoretically interesting and empirically relevant.
[7]When $c = \underline{c}$, (1.3) implies that $\beta_B^{-1}(\underline{s}) = \underline{s}$. Similarly, (1.4) implies that $\beta_S^{-1}(\overline{b}) = \overline{b}$ when $v = \overline{v}$.
[8]The transaction occurs when $\beta_B(V) \geqslant \beta_S(C)$.

$$\eta(s, G_B) \equiv s - (1-k)\frac{1 - G_B(s)}{g_B(s)}, \quad \underline{b} \leqslant s \leqslant \overline{b}. \tag{1.8}$$

By definition, it is straightforward that $\xi(\underline{s}, G_S) = \underline{s}$ and $\eta(\overline{b}, G_B) = \overline{b}$.

We define $\mathcal{P}_{\mathscr{A}}$ as the collection of absolutely continuous probability distributions with support $\mathscr{A}$. Let G denote the joint distribution of (B, S). Here, we restrict ourselves to the regular equilibrium strategies which are strictly increasing and (piecewise) differentiable.

Theorem 1.1. *Under Assumptions C and D, if $G \in \mathcal{P}_{\mathscr{D}}$ is the joint distribution of regular equilibrium bids (B, S) in a sealed-bid k-double auction with some (F_V, F_C) satisfying Assumptions A and B, then*

C1. The support $\mathscr{D} = [\underline{b}, \overline{b}] \times [\underline{s}, \overline{s}]$ with $\underline{b} \leqslant \underline{s} < \overline{b} \leqslant \overline{s}$;

C2. $G(b, s) = G_B(b) \cdot G_S(s)$ and $G_B \in \mathcal{P}_{[\underline{b}, \overline{b}]}$, $G_S \in \mathcal{P}_{[\underline{s}, \overline{s}]}$;

C3. The function $\xi(\,\cdot\,, G_S)$ defined in (1.7) is strictly increasing on $[\underline{s}, \overline{b}]$ and its inverse is differentiable on $[\xi(\underline{s}, G_S), \xi(\overline{b}, G_S)]$;

C4. The function $\eta(\,\cdot\,, G_B)$ defined in (1.8) is strictly increasing on $[\underline{s}, \overline{b}]$ and its inverse is differentiable on $[\eta(\underline{s}, G_B), \eta(\overline{b}, G_B)]$;

C5. For any $\overline{b} \leqslant b' \leqslant \overline{s}$ and for any $b \leqslant \overline{b}$ such that $\xi(b, G_S) \geqslant b'$,

$$[\xi(b, G_S) - b']G_S(b') - [\xi(b, G_S) - b]G_S(b) + (1-k)\int_b^{b'} G_S(s)\,\mathrm{d}s \leqslant 0; \tag{1.9}$$

C6. For any $\underline{b} \leqslant s' \leqslant \underline{s}$ and for any $s \geqslant \underline{s}$ such that $\eta(s, G_B) \leqslant s'$,

$$[s' - \eta(s, G_B)][1 - G_B(s')] - [s - \eta(s, G_B)][1 - G_B(s)] + k\int_{s'}^{s} [1 - G_B(b)]\,\mathrm{d}b \leqslant 0. \tag{1.10}$$

Proof. See Appendix A.3. □

Theorem 1.1 shows that the theoretical model of a k-double auction with bargaining does impose some restrictions on the joint distribution of observed bids. Together with Theorem 1.2 which will be shown immediately, these restrictions can be used to establish a formal test of the theory of k-double auction with bargaining. Specifically, condition C1 of Theorem 1.1 shows that the buyer's minimum (or maximum) bid is not higher than the seller's minimum (or maximum) bid, and the intersection between the buyer's and the seller's bid supports has a non-empty interior. The latter is mainly due to Assumption B about the supports of private value distributions, which implies that there is always positive probability of trade in any regular equilibrium. Condition C2 shows that the buyer's bid is independent of seller's. This independence result is intuitive given that the buyer's value is independent of the seller's. Conditions C3 and C4 say that the functions $\xi(\,\cdot\,, G_S)$ and $\eta(\,\cdot\,, G_B)$, which can be regarded as the inverse bidding strategies, are strictly increasing and differentiable on the interval where there is a positive probability of trade. The strict monotonicity property of inverse bidding strategies comes from the fact that the equilibrium strategies are strictly increasing. Conditions C5 and C6 restrict the bid distributions to have small enough probability in the cases where buyer offers less than minimum ask $\underline{s}$ or seller asks more than maximum offer $\overline{b}$.[9]

The following theorem shows that, under Assumptions C and D, the converse of Theorem 1.1 is also true.

Theorem 1.2. *Under Assumptions C and D, if $G \in \mathcal{P}_{\mathscr{D}}$ satisfies C1–C6, then there exists a unique pair of (F_V, F_C) satisfying Assumptions A and B such that G is the joint distribution of some regular equilibrium bids (B, S) in a sealed-bid k-double auction with (F_V, F_C).*

Proof. See Appendix A.4. □

Theorem 1.2 is important for several reasons. First, it shows that conditions C1–C6 on the bid distribution G are sufficient to prove the existence of model structure (F_V, F_C) which satisfies Assumptions A and B. Second, suppose that the buyer and the seller behave as predicted by the theoretical model of k-double auction with bargaining, Theorem 1.2 then shows that the private value distributions (F_V, F_C) under which regular equilibrium exists are identified from the joint distribution of observed bids. Third, the inverse bidding strategies, which are equal to $\xi(\cdot, G_S)$ and $\eta(\cdot, G_B)$ on the trading interval $[\underline{s}, \overline{b}]$, only rely on the knowledge of distribution G as well as the parameter k. Thus, we can avoid solving the linked differential equations (1.3) and (1.4) in order to determine the equilibrium strategy profile (β_B, β_S). It is worth pointing out that, to identify (F_V, F_C), Theorem 1.2 needs all observables to come from the same equilibrium because of the existence of multiple regular equilibria.

Conditions C5 and C6 are less intuitive, and could be difficult to check in some applications. It will be helpful to provide their sufficient conditions which are easy to verify. We are going to show in the following theorem, which can be viewed as a corollary of Theorem 1.2, that condition C5 will be automatically satisfied if function $\xi(\cdot, G_S)$ is strictly increasing not only on interval $[\underline{s}, \overline{b}]$ but also on $[\overline{b}, \overline{s}]$ whenever $\overline{s} > \overline{b}$, and that condition C6 will hold if function $\eta(\cdot, G_B)$ is strictly increasing on the entire domain $[\underline{b}, \overline{b}]$.

Theorem 1.3. *The conclusion of Theorem 1.2 holds if $G \in \mathcal{P}_{\mathscr{D}}$ satisfies C1–C2 and*

C7. The function $\xi(\cdot, G_S)$ defined in (1.7) *is strictly increasing on $[\underline{s}, \overline{s}]$ and its inverse is differentiable on $[\xi(\underline{s}, G_S), \xi(\overline{b}, G_S)]$;*

C8. The function $\eta(\cdot, G_B)$ defined in (1.8) *is strictly increasing on $[\underline{b}, \overline{b}]$ and its inverse is differentiable on $[\eta(\underline{s}, G_B), \eta(\overline{b}, G_B)]$.*

Proof. See Appendix A.5. □

[9]Otherwise, the buyer with very high private value or the seller with very low reservation value will have incentive to deviate from the given equilibrium strategy.

1.3.2 Identification with Limited Observability of the Bid Distribution

We now discuss the nonparametric identification of the k-double auction model with less data. In particular, we consider the case where econometricians only observe the weight parameter k and the distribution of transacted bids, but never observe the non-transacted bids. Such a case is more empirically realistic than the first one because the bids without transaction are usually not documented in many data sets.

Our identification strategy consists of two steps. In the first step, we identify the bid distribution in an area, namely $[\underline{s}, \overline{b}]^2$, from the distribution of the transacted bids. In the second step, we then find the inverse bidding strategies for the bids in that area so that we can recover the corresponding private values for the buyer and the seller.

Suppose Assumptions A to D hold. Let G_1 denote the joint distribution of the bids located in $[\underline{s}, \overline{b}]^2$ and let G_2 denote the joint distribution of the transacted bids.[10] Notice that G_2 is known by assumption. In the next paragraph, we will show that G_1 can be identified from G_2.

By the definition of conditional density, the corresponding densities of G_1 and G_2, namely g_1 and g_2, are proportional to the density of G, namely g, in their respective supports. Specifically,

$$g_1(b,s) = \frac{g(b,s)}{m}, \quad g_2(b,s) = \frac{g(b,s)}{m'}, \tag{1.11}$$

where $m = \Pr(\underline{s} \leqslant S \leqslant \overline{b}, \underline{s} \leqslant B \leqslant \overline{b})$ and $m' = \Pr(\underline{s} \leqslant S \leqslant B \leqslant \overline{b})$. By Theorem 1.1, bids B and S are independent, i.e. $g(b,s) = g_B(b) \cdot g_S(s)$ for any $(b,s) \in [\underline{b}, \overline{b}] \times [\underline{s}, \overline{s}]$. Consequently, the conditional density g_1 can be expressed in terms of g_2 as follows:

$$g_1(b,s) = \begin{cases} \dfrac{m'}{m} \cdot g_2(b,s) & \text{if } \underline{s} \leqslant s \leqslant b \leqslant \overline{b} \\ \dfrac{m'}{m} \cdot \dfrac{g_2(b',s) g_2(b,s')}{g_2(b',s')} & \text{if } \underline{s} \leqslant b < s \leqslant \overline{b} \end{cases}$$

where in the latter case b' and s' are chosen such that $\underline{s} \leqslant s' < b$ and $s < b' \leqslant \overline{b}$ (see Figure 1.1).[11] For example, we can choose $b' = (\overline{b} + s)/2$ and $s' = (b + \underline{s})/2$. Since $g_2(\cdot, \cdot)$ is identified directly from the observables, the ratio m'/m is then identified by the fact that $\int_{[\underline{s}, \overline{b}]^2} g_1(b,s)\, \mathrm{d}b\, \mathrm{d}s = 1$ as

$$\frac{m'}{m} = \left[1 + \int_{\underline{s} \leqslant b < s \leqslant \overline{b}} \frac{g_2(b',s) g_2(b,s')}{g_2(b',s')} \, \mathrm{d}b\, \mathrm{d}s \right]^{-1},$$

where b' and s' are chosen for each (b,s) in region II of Figure 1.1 so that (b',s), (b,s') and (b',s')

[10] Precisely, $G_1(b,s) = \Pr(B \leqslant b, S \leqslant s \mid (B,S) \in [\underline{s}, \overline{b}]^2)$ and $G_2(b,s) = \Pr(B \leqslant b, S \leqslant s \mid \underline{s} \leqslant S \leqslant B \leqslant \overline{b})$.

[11] In the latter case, we use the independence property of the joint density g such that $g(b,s) = g_B(b) \cdot g_S(s)$, $g(b',s) = g_B(b') \cdot g_S(s)$, $g(b,s') = g_B(b) \cdot g_S(s')$ and $g(b',s') = g_B(b') \cdot g_S(s')$.

all locate in region I. Consequently, the conditional density $g_1(\cdot,\cdot)$ is identified on the support of $[\underline{s},\overline{b}]^2$.

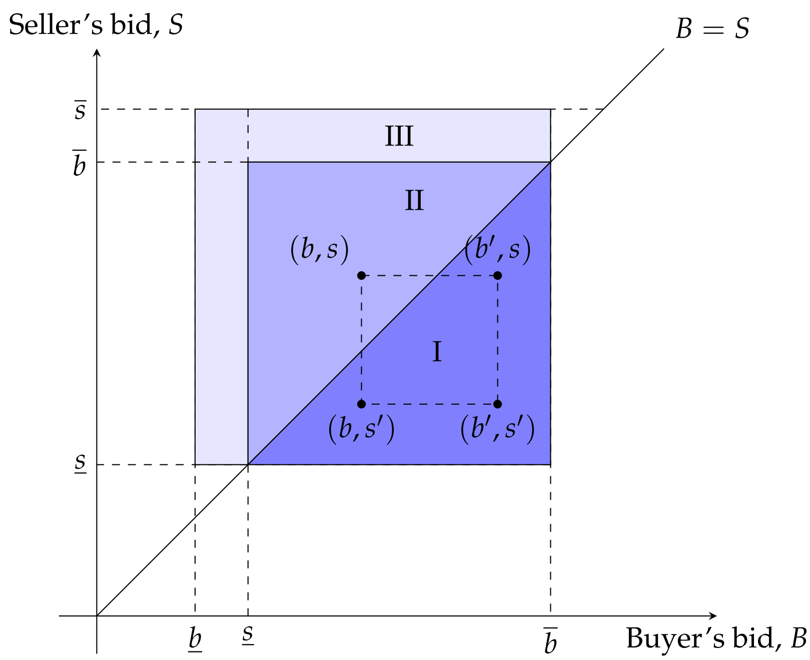


Figure 1.1: Recover G_1 in region II. Here, $\underline{s} \leqslant s' < b$ and $s < b' \leqslant \overline{b}$.

Next, we recover the inverse bidding strategies for the bids in the area $[\underline{s},\overline{b}]^2$, i.e. regions I and II of Figure 1.1. Let G_{1B} and G_{1S} denote the buyer's and the seller's marginal bid distributions, respectively, of the joint distribution G_1, and let g_{1B} and g_{1S} be their densities. In addition, for any $b, s \in [\underline{s},\overline{b}]$, define

$$\tilde{\xi}(b, G_{1S}) \equiv b + k\frac{G_{1S}(b)}{g_{1S}(b)}, \tag{1.12}$$

$$\tilde{\eta}(s, G_{1B}) \equiv s - (1-k)\frac{1-G_{1B}(s)}{g_{1B}(s)}. \tag{1.13}$$

The following lemma shows how to recover the inverse bidding strategy for bids in regions I and II from the identified G_1:

Lemma 1.2. *If $G \in \mathcal{P}_{\mathscr{D}}$ satisfies C1 and C2, then for all $b, s \in [\underline{s},\overline{b}]$,*

$$\xi(b, G_S) = \tilde{\xi}(b, G_{1S}), \tag{1.14}$$

$$\eta(s, G_B) = \tilde{\eta}(s, G_{1B}). \tag{1.15}$$

Proof. See Appendix A.6. □

Lemma 1.2 shows that both the buyer's and the seller's inverse bidding strategies are identified in regions I and II of Figure 1.1, since the conditional distributions of bids in those regions, i.e. G_{1B} and G_{1S}, have been identified in our previous step. Based on this result, we can recover the conditional joint distribution (and hence its marginal distributions) of private values under which the equilibrium bids locate in regions I and II. The specific expressions of those conditional marginal distributions are given by (1.16) of Theorem 1.4.

The following theorem summarizes the above discussion.

Theorem 1.4. *Under Assumptions C and D:*

(i) If $G_2 \in \mathcal{P}_{\mathscr{D}'}$ is the joint distribution of transacted bids under some regular equilibrium in a sealed-bid k-double auction with (F_V, F_C) satisfying Assumptions A and B, then

D1. The support $\mathscr{D}' = \left\{(b,s) \mid \underline{s} \leqslant s \leqslant b \leqslant \overline{b}\right\}$ with $\underline{s} < \overline{b}$;

D2. For any $\underline{s} \leqslant s' \leqslant s \leqslant b \leqslant b' \leqslant \overline{b}$, the density of G_2 satisfies $g_2(b,s) \cdot g_2(b',s') = g_2(b,s') \cdot g_2(b',s)$;

D3. The function $\tilde{\xi}(\,\cdot\,, G_{1S})$ defined in (1.12) is strictly increasing on $[\underline{s}, \overline{b}]$ and its inverse is differentiable on $[\tilde{\xi}(\underline{s}, G_{1S}), \tilde{\xi}(\overline{b}, G_{1S})]$;

D4. The function $\tilde{\eta}(\,\cdot\,, G_{1B})$ defined in (1.13) is strictly increasing on $[\underline{s}, \overline{b}]$ and its inverse is differentiable on $[\tilde{\eta}(\underline{s}, G_{1B}), \tilde{\eta}(\overline{b}, G_{1B})]$.

(ii) Suppose that Assumptions A and B also hold, and $G_2 \in \mathcal{P}_{\mathscr{D}'}$ satisfies D1–D4, then G_2 is the joint distribution of transacted bids under some regular equilibrium in a sealed-bid k-double auction with (F_V, F_C) if and only if (F_V, F_C) satisfies

E1. $\underline{c} \leqslant \underline{s}$, $\overline{v} \geqslant \overline{b}$;

E2. For all $(v,c) \in [\underline{s}, \tilde{\xi}(\overline{b}, G_{1S})] \times [\tilde{\eta}(\underline{s}, G_{1B}), \overline{b}]$,[12]

$$\Pr\left(V \leqslant v \mid V \geqslant \underline{s}\right) = G_{1B}(\tilde{\xi}^{-1}(v, G_{1S})), \quad \Pr\left(C \leqslant c \mid C \leqslant \overline{b}\right) = G_{1S}(\tilde{\eta}^{-1}(c, G_{1B})) \tag{1.16}$$

where $\Pr\left(V \leqslant v \mid V \geqslant \underline{s}\right) = \dfrac{F_V(v) - F_V(\underline{s})}{1 - F_V(\underline{s})}$ *and* $\Pr\left(C \leqslant c \mid C \leqslant \overline{b}\right) = \dfrac{F_C(c)}{F_C(\overline{b})}$*;*

E3. For any $b' \geqslant \overline{b}$ and for any $b \leqslant \overline{b}$ such that $\tilde{\xi}(b, G_{1S}) \geqslant b'$,

$$\begin{aligned}
&[\tilde{\xi}(b, G_{1S}) - b']F_C(b') - [\tilde{\xi}(b, G_{1S}) - b]F_C(\tilde{\eta}(b, G_{1B})) \\
&\qquad + (1-k)\left[\int_b^{\overline{b}} F_C(\tilde{\eta}(s, G_{1B}))\,\mathrm{d}s + \int_{\overline{b}}^{b'} F_C(s)\,\mathrm{d}s\right] \leqslant 0;
\end{aligned} \tag{1.17}$$

For any $s' \leqslant \underline{s}$ and for any $s \geqslant \underline{s}$ such that $\tilde{\eta}(s, G_{1B}) \leqslant s'$,

[12] Notice that we have $\tilde{\xi}(\underline{s}, G_{1S}) = \underline{s}$ and $\tilde{\eta}(\overline{b}, G_{1B}) = \overline{b}$ by the definitions of functions $\tilde{\xi}$ and $\tilde{\eta}$.

$$[s' - \tilde{\eta}(s, G_{1B})][1 - F_V(s)] - [s - \tilde{\eta}(s, G_{1B})][1 - F_V(\tilde{\xi}(s, G_{1S}))] \\ + k \left\{ \int_{s'}^{\underline{s}} [1 - F_V(b)] \,\mathrm{d}b + \int_{\underline{s}}^{s} [1 - F_V(\tilde{\xi}(b, G_{1S}))] \,\mathrm{d}b \right\} \leqslant 0. \quad (1.18)$$

Proof. See Appendix A.7. □

Part (i) of Theorem 1.4 shows that the conclusion of Theorem 1.1 carries over to the transacted bids area, i.e. region I of Figure 1.1, although some (non-transacted) bids cannot be observed now. Specifically, condition D1 says that the support of the distribution of observed (transacted) bids is a triangle in which the buyer's bid is no less than the seller's. Condition D2 means that the multiplication of conditional densities evaluated at (b, s) and (b', s') is the same as the multiplication of conditional densities evaluated at (b, s') and (b', s) as long as these four points are located in the transacted bids area, i.e. region I. Such a condition arises mainly due to the independence of private values. By Lemma 1.2, conditions D3 and D4 state that both the buyer's and the seller's inverse bidding strategies are strictly increasing and differentiable on the interval of all possible transacted bids values, namely $[\underline{s}, \overline{b}]$.

Part (ii) of Theorem 1.4 gives the conditions under which the private value distributions rationalize a given distribution of transacted bids. It mainly requires, in the private value interval with positive probability of trade, that both the buyer's and the seller's conditional private value distributions have to generate the corresponding bid distributions when we treat functions $\tilde{\xi}^{-1}$ and $\tilde{\eta}^{-1}$ as buyer's and seller's bidding strategies, respectively. Moreover, part (ii) states that any private value distributions satisfying conditions E1–E3 can rationalize the given distribution of transacted bids, and hence they are observationally equivalent. Although there can be many private value distributions which explain a given distribution of transacted bids, (1.16) shows that the buyer's and the seller's conditional private value distributions are point identified on their value intervals where there is a positive probability of trade.

1.4 Estimation

Based on the identification strategy, we provide a nonparametric estimation procedure when all bids can be observed by the researchers, i.e. in the case of Section 1.3.1. We further assume that all of the observed k-double auctions are homogeneous.

Our estimation procedure extends the two-step estimator proposed by Guerre, Perrigne, and Vuong (2000) for the estimation of sealed-bid first-price auctions: In the first step, a sample of buyers' and sellers' "pseudo private values" is constructed by (1.5) and (1.6), where G_S and G_B are estimated by their empirical distribution functions, and g_S and g_B are estimated by their kernel density estimators with boundary (and interior) bias correction. In the second step, this sample of pseudo private values is used to nonparametrically estimate the densities of buyers' and sellers' private values with boundary bias correction. Notice that, due to the regular equilibrium assumption, a bidder's private value is equal to her bid in the first step if the bidder is a buyer offering less than $\underline{s}$ or if the bidder is a seller asking more than $\overline{b}$.

It is worth pointing out that a boundary correction is implemented in all kernel density estimators of our two-step procedure.[13] This is motivated by the fact that boundary bias is worse in double auctions than in first-price auctions. Specifically, as pointed out by Guerre, Perrigne, and Vuong (2000), the estimators of bid density and private value density suffer from boundary bias in the two-step estimation of first-price auctions, since the supports of these two densities are finite. This issue carries over to the double auction setup, and is made worse by the discontinuity of bid densities in the interior of their supports. The interior discontinuity of bid densities occurs because of the strategic asymmetry between the buyer and the seller in double auctions; in a regular equilibrium, the buyer's (or seller's) pseudo private value is recovered via the distribution of her opponent's bid instead of her own by (1.5) (or (1.6)) when her bid is within $[\underline{s}, \overline{b}]$, and is equal to her bid otherwise. This results in the discontinuity of buyer's (or seller's) bid density at interior point $\underline{s}$ (or at interior point $\overline{b}$). Consequently, the two-step estimator of private value density with boundary and interior bias correction will have much better performance than the one without bias correction (e.g. the one with sample trimming instead) in finite samples. This is confirmed by our Monte Carlo experiments in Section 1.5.

1.4.1 Definition of the Estimator

To clarify our idea, we consider n homogeneous k-double auctions. In each auction $i = 1, 2, \ldots, n$, there is one buyer with private value V_i and one seller with private value C_i. We observe a sample that consists of all of the buyers' bids $\{B_1, B_2, \ldots, B_n\}$ and all of sellers' bids $\{S_1, S_2, \ldots, S_n\}$. Let $\hat{\underline{b}}$ and $\hat{\overline{b}}$ ($\hat{\underline{s}}$ and $\hat{\overline{s}}$) be the minimum and maximum of the buyers' (sellers') n observed bids.

We first give the general definition of our boundary corrected kernel density estimator. For a random sample $\{X_1, \ldots, X_n\}$ that is drawn from distribution F with density f and support $[\underline{x}, \overline{x}]$,[14] the boundary corrected kernel density estimator of f *on interval* $[a_1, a_2] \subseteq [\underline{x}, \overline{x}]$ is defined as[15]

$$\hat{f}(x) = \frac{1}{nh}\sum_{i=1}^{n} \mathbb{1}(a_1 \leqslant X_i \leqslant a_2)\Big[K\left(\frac{x - X_i}{h}\right) + K\left(\frac{x - a_1 + \hat{\gamma}_1(X_i - a_1)}{h}\right) + K\left(\frac{a_2 - x + \hat{\gamma}_2(a_2 - X_i)}{h}\right)\Big], \quad (1.19)$$

[13] We also need to implement interior bias correction in the estimation of bid densities, i.e. in the first step.

[14] The support $[\underline{x}, \overline{x}]$ is not necessarily bounded.

[15] We adapt the boundary correction technique proposed by Zhang, Karunamuni, and Jones (1999) and Karunamuni and Zhang (2008) to our double auction setup.

where

$$\hat{\gamma}_1(u) = u + \hat{d}_1 u^2 + A\hat{d}_1^2 u^3, \quad \hat{\gamma}_2(u) = u + \hat{d}_2 u^2 + A\hat{d}_2^2 u^3,$$

with

$$\begin{aligned}
\hat{d}_1 = \frac{1}{h'} \Bigg\{ \log\left[\frac{1}{nh'} \sum_{i=1}^{n} \mathbb{1}(a_1 \leqslant X_i \leqslant a_2) K\left(\frac{h' - X_i + a_1}{h'}\right) + \frac{1}{n^2} \right] \\
- \log\left[\max\left(\frac{1}{nh'_0} \sum_{i=1}^{n} \mathbb{1}(a_1 \leqslant X_i \leqslant a_2) K_0\left(\frac{a_1 - X_i}{h'_0}\right), \frac{1}{n^2} \right) \right] \Bigg\}, \\
\hat{d}_2 = \frac{1}{h'} \Bigg\{ \log\left[\frac{1}{nh'} \sum_{i=1}^{n} \mathbb{1}(a_1 \leqslant X_i \leqslant a_2) K\left(\frac{h' + X_i - a_2}{h'}\right) + \frac{1}{n^2} \right] \\
- \log\left[\max\left(\frac{1}{nh'_0} \sum_{i=1}^{n} \mathbb{1}(a_1 \leqslant X_i \leqslant a_2) K_0\left(\frac{X_i - a_2}{h'_0}\right), \frac{1}{n^2} \right) \right] \Bigg\},
\end{aligned}$$

$K_0(u) = (6 + 18u + 12u^2) \cdot \mathbb{1}(-1 \leqslant u \leqslant 0)$ and

$$h'_0 = \left[\frac{\left(\int_{-1}^{1} u^2 K(u)\,\mathrm{d}u\right)^2 \left(\int_{-1}^{0} K_0^2(u)\,\mathrm{d}u\right)}{\left(\int_{-1}^{0} u^2 K_0(u)\,\mathrm{d}u\right)^2 \left(\int_{-1}^{1} K^2(u)\,\mathrm{d}u\right)} \right]^{1/5} \cdot h'.$$

Our estimation proceeds as follows: In the first step, we use the observed sample of all bids to estimate the distribution and density functions of the buyers' and sellers' bids by their empirical distribution functions and boundary and interior corrected kernel density estimators, respectively, i.e. by

$$\hat{G}_B(b) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(B_i \leqslant b), \quad \hat{G}_S(s) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(S_i \leqslant s),$$

and kernel density estimators $\hat{g}_B$ and $\hat{g}_S$ which are estimated on $[\hat{\underline{s}}, \hat{\overline{b}}]$ as shown in (1.19). Specifically, the estimator of the buyers' bid density $\hat{g}_B$ uses kernel function K_B, primary bandwidth h_B, secondary bandwidth h'_B and coefficient $A = A_B$, while the estimator of the sellers' bid density $\hat{g}_S$ uses K_S, h_S, h'_S and $A = A_S$. We then define the buyer's pseudo private value $\hat{V}_i$ corresponding to B_i and the seller's pseudo private value $\hat{C}_i$ corresponding to S_i, respectively, as

$$\hat{V}_i = \begin{cases} B_i + k\dfrac{\hat{G}_S(B_i)}{\hat{g}_S(B_i)} & \text{if } B_i \geqslant \hat{\underline{s}}, \\ B_i & \text{otherwise,} \end{cases} \qquad \hat{C}_i = \begin{cases} S_i - (1-k)\dfrac{1 - \hat{G}_B(S_i)}{\hat{g}_B(S_i)} & \text{if } S_i \leqslant \hat{\overline{b}}, \\ S_i & \text{otherwise,} \end{cases} \tag{1.20}$$

where $\hat{G}_B(\cdot)$, $\hat{G}_S(\cdot)$, $\hat{g}_B(\cdot)$, and $\hat{g}_S(\cdot)$ are the empirical distribution functions and bias-corrected kernel density estimators defined earlier.

In the second step, we use the pseudo private value samples, $\{\hat{V}_1, \dots, \hat{V}_n\}$ and $\{\hat{C}_1, \dots, \hat{C}_n\}$, to estimate the buyers' and sellers' respective value densities. Specifically, the estimator of the buyers' value density $\hat{f}_V$ is obtained by applying the bias correction approach in (1.19) to the sample of the buyers' pseudo private values on $[\hat{\underline{v}}, \hat{\overline{v}}]$, where $\hat{\underline{v}}$ and $\hat{\overline{v}}$ are respectively the minimum and maximum of the buyers' pseudo private values, with kernel function K_V, primary bandwidth h_V, secondary bandwidth h'_V, and coefficient $A = A_V$. Similarly, we get the estimator of the sellers' value density $\hat{f}_C$ on interval $[\hat{\underline{c}}, \hat{\overline{c}}]$ by the sample of the sellers' pseudo private values with kernel function K_C, primary bandwidth h_C, secondary bandwidth h'_C, and coefficient $A = A_C$.

1.4.2 Asymptotic Properties

To show the asymptotic properties of our estimator, for technical convenience, we limit the regular equilibria to those equilibrium strategy profiles (β_B, β_S) that satisfy $\beta'_B(v) > 0$ for all $v \in [\underline{s}, \overline{v}]$ and $\beta'_S(c) > 0$ for all $c \in [\underline{c}, \overline{b}]$. The next assumption concerns the generating process of buyers' and sellers' private values $(V_i, C_i), i = 1, \dots, n$.

Assumption E. *V_i, $i = 1, 2, \dots, n$, are independently and identically distributed as F_V with density f_V; C_i, $i = 1, 2, \dots, n$, are independently and identically distributed as F_C with density f_C.*

This assumes that the bidders' private values are independent across auctions. In addition, we impose a smoothness condition on the latent value distributions as follows:

Assumption F. *F_V and F_C admit up to $R+1$ continuous bounded derivatives on $[\underline{v}, \overline{v}]$ and $[\underline{c}, \overline{c}]$, respectively, with $R \geqslant 1$. In addition, $f_V(v) \geqslant \alpha_V > 0$ for all $v \in [\underline{v}, \overline{v}]$; $f_C(c) \geqslant \alpha_C > 0$ for all $c \in [\underline{c}, \overline{c}]$.*

Assumption F requires that, on compact supports, the latent value distributions have $R+1$ continuous derivatives and their density functions are bounded away from zero. As shown in the following lemma, this assumption implies that the generated equilibrium bid distributions will also satisfy a similar smoothness condition.

Lemma 1.3. *Given Assumption F, the distributions of regular equilibrium bids G_B and G_S satisfy:*

(i) for any $b \in [\underline{b}, \overline{b}]$ and any $s \in [\underline{s}, \overline{s}]$, $g_B(b) \geqslant \alpha_B > 0$, $g_S(s) \geqslant \alpha_S > 0$;

(ii) G_B and G_S admit up to $R+1$ continuous bounded derivatives on $[\underline{s}, \overline{b}]$;

(iii) g_B and g_S admit up to $R+1$ continuous bounded derivatives on $[\underline{s}, \overline{b}]$.

Proof. See Appendix A.8. □

The striking feature of Lemma 1.3 is part (iii). It shows that the bid densities are smoother than their corresponding latent value densities. A similar result is obtained by Guerre, Perrigne, and Vuong (2000) in first-price auctions.

We turn to the choice of kernels, bandwidths and other tuning parameters which define our estimator.

Assumption G. *(i) The kernels K_B, K_S, K_V, K_C are symmetric with support $[-1, 1]$ and have twice continuous bounded derivatives. (ii) K_B, K_S, K_V and K_C are of order $R+1$, $R+1$, R, and R, with $R \geqslant 1$.*

Assumption G is standard. The orders of kernels are chosen according to the smoothness of the estimated densities. Specifically, the kernels for bid densities are of order $R+1$, since the bid densities admit up to $R+1$ continuous bounded derivatives by Lemma 1.3. And the kernels for the private value densities are of order R because by Assumption F, the private value densities admit up to R continuous bounded derivatives.

We then give two parallel assumptions which mainly concern the choice of bandwidths.

Assumption H1. *The bandwidths h_B, h_S, h_V, h_C are of the form:*

$$h_j = \lambda_j \left(\frac{\log n}{n} \right)^{\frac{1}{2R+3}}, \quad j = B, S, V, C,$$

where the λ_j's are positive constants.

Assumption H2. *The bandwidths h_B, h_S, h_V, h_C are of the form:*

$$h_j = \lambda_j \left(\frac{\log n}{n} \right)^{\frac{1}{5}}, \quad j = B, S, V, C,$$

where the λ_j's are positive constants. The parameters $A_B, A_S, A_V, A_C > 1/3$ and the secondary bandwidths are of the form:

$$h'_j = \tau_j n^{-\frac{1}{5}}, \quad h'_S = \tau_S n^{-\frac{1}{5}}, \quad j = B, S, V, C,$$

where the τ_j's are positive constants.

Assumption H1 chooses the primary bandwidths for both the bid and private value densities of order $(\log n/n)^{1/(2R+3)}$. To implement the bias correction technique, we adopt Assumption H2 to choose all primary bandwidths of order $(\log n/n)^{1/5}$ and the secondary bandwidths h'_B, h'_S, h'_V, and h'_C of order $n^{-1/5}$.[16] To extend Assumption H2 for higher order bias reduction, a brief discussion can be found in Section 1.6.2.

Our main estimation results establish the uniform consistency of the two-step estimator with its rate of convergence. They are built on the following two important lemmas: The first lemma shows the uniform consistency (with rates of convergence) of the first-step nonparametric estimators of

[16] Such choices of secondary bandwidths minimize the mean squared errors of estimating d's in the transform functions for bias correction.

the bid densities. The second lemma gives the rate at which the pseudo private values $\hat{V}_i$ and $\hat{C}_i$ converge uniformly to the true private values.

Lemma 1.4. *(i) Under Assumptions E to G and Assumption H2,*

$$\sup_{b\in[\underline{b},\overline{b}]} |\hat{g}_B(b) - g_B(b)| = O_p\left(\left(\frac{\log n}{n}\right)^{\frac{2}{5}}\right), \quad \sup_{s\in[\underline{s},\overline{s}]} |\hat{g}_S(s) - g_S(s)| = O_p\left(\left(\frac{\log n}{n}\right)^{\frac{2}{5}}\right).$$

(ii) Under Assumptions E to G and Assumption H1, for any (fixed) closed inner subset $\mathscr{C}_g$ of $[\underline{s},\overline{b}]$,[17]

$$\sup_{b\in\mathscr{C}_g} |\hat{g}_B(b) - g_B(b)| = O_p\left(\left(\frac{\log n}{n}\right)^{\frac{R+1}{2R+3}}\right), \quad \sup_{s\in\mathscr{C}_g} |\hat{g}_S(s) - g_S(s)| = O_p\left(\left(\frac{\log n}{n}\right)^{\frac{R+1}{2R+3}}\right).$$

Proof. See Appendix A.9. □

Lemma 1.5. *(i) Under Assumptions E to G and Assumption H2,*

$$\sup_i |\hat{V}_i - V_i| = O_p\left(\left(\frac{\log n}{n}\right)^{\frac{2}{5}}\right), \quad \sup_i |\hat{C}_i - C_i| = O_p\left(\left(\frac{\log n}{n}\right)^{\frac{2}{5}}\right).$$

(ii) Under Assumptions E to G and Assumption H1, for any (fixed) closed inner subsets $\mathscr{C}_V$ of $[\underline{s},\overline{v}]$ and $\mathscr{C}_C$ of $[\underline{c},\overline{b}]$,

$$\sup_i \mathbb{1}(V_i \in \mathscr{C}_V)|\hat{V}_i - V_i| = O_p\left(\left(\frac{\log n}{n}\right)^{\frac{R+1}{2R+3}}\right), \quad \sup_i \mathbb{1}(C_i \in \mathscr{C}_C)|\hat{C}_i - C_i| = O_p\left(\left(\frac{\log n}{n}\right)^{\frac{R+1}{2R+3}}\right).$$

Proof. See Appendix A.10. □

Part (i) of Lemma 1.4 shows that, after bias correction with bandwidth choice outlined in Assumption H2, the kernel density estimators of the bid distributions will uniformly converge in probability to the true densities at a rate of $(\log n/n)^{2/5}$ on their entire supports. Part (i) of Lemma 1.5 then shows that, after the bias correction, all pseudo private values converge uniformly in probability to the true private values at a rate of $(\log n/n)^{2/5}$ under Assumption H2.

Furthermore, part (ii) of Lemmas 1.4 and 1.5 show that, if the primary bandwidths h_B and h_S are of order $(\log n/n)^{1/(2R+3)}$ according to Assumption H1, the bid density estimators and the pseudo private values can have a faster rate of uniform convergence. However, this rate can only be achieved by the bid density estimators on the subsets of the bid interval with positive probability of trade which are strictly bounded away from the support boundaries of bid distributions, and by the pseudo private values corresponding to the observed bids inside these subsets. The rate of

[17]We call closed set $\mathscr{A}' \subseteq \mathscr{A}$ a closed inner subset of $\mathscr{A}$ if $\mathscr{A}'$ is also a subset of the interior of $\mathscr{A}$.

uniform convergence in this case, $(\log n/n)^{(R+1)/(2R+3)}$, is the same as the optimal rate obtained by Guerre, Perrigne, and Vuong (2000) for the first-price auctions.

We now give the first main result of the estimation section.

Theorem 1.5. *Under Assumptions E to G and Assumption H1, for any (fixed) closed inner subsets $\mathscr{C}_V$ of $[\underline{v}, \overline{v}]\backslash\{\underline{s}\}$ and $\mathscr{C}_C$ of $[\underline{c}, \overline{c}]\backslash\{\overline{b}\}$,*

$$\sup_{v\in\mathscr{C}_V} |\hat{f}_V(v) - f_V(v)| = O_p\left(\left(\frac{\log n}{n}\right)^{\frac{R}{2R+3}}\right), \quad \sup_{c\in\mathscr{C}_C} |\hat{f}_C(c) - f_C(c)| = O_p\left(\left(\frac{\log n}{n}\right)^{\frac{R}{2R+3}}\right).$$

Proof. See Appendix A.11. □

Theorem 1.5 establishes the uniform consistency of our two-step estimator of the bidders' private value densities. The rate of convergence coincides with the result of Guerre, Perrigne, and Vuong (2000) for the first-price auctions. It is worth pointing out that the convergence rate of the buyers' value density estimator $\hat{f}_V(\cdot)$ can be improved to $(\log n/n)^{R/(2R+1)}$ on the closed inner subsets of $[\underline{v}, \underline{s}]$ when the bandwidth h_V has an order of $(\log n/n)^{1/(2R+1)}$ rather than $(\log n/n)^{1/(2R+3)}$ under Assumption H1. This is due to the fact that the buyer will bid her true value in a regular equilibrium if it is in $[\underline{v}, \underline{s}]$ when $\underline{v} < \underline{s}$ (i.e. we can observe directly her value in this case) . A similar conclusion holds for the sellers' value density estimator $\hat{f}_C(\cdot)$ on closed inner subsets of $[\overline{b}, \overline{c}]$.

Theorem 1.5, however, does not provide the uniform convergence rate of the buyers' (or sellers') value density estimator on a closed inner subset containing $\underline{s}$ (or $\overline{b}$), although the value density is continuous at this interior point $\underline{s}$ (or $\overline{b}$). This is caused by the existence of bias in the buyers' (or sellers') value density estimator close to $\underline{s}$ (or $\overline{b}$). Our next main estimation result addresses this issue by adopting bias correction.

Theorem 1.6. *Under Assumptions E to G and Assumption H2, for any (fixed) closed inner subsets $\overline{\mathscr{C}}_V$ of $[\underline{v}, \overline{v}]$ and $\overline{\mathscr{C}}_C$ of $[\underline{c}, \overline{c}]$,*

$$\sup_{v\in\overline{\mathscr{C}}_V} |\hat{f}_V(v) - f_V(v)| = O_p\left(\left(\frac{\log n}{n}\right)^{\frac{1}{5}}\right), \quad \sup_{c\in\overline{\mathscr{C}}_C} |\hat{f}_C(c) - f_C(c)| = O_p\left(\left(\frac{\log n}{n}\right)^{\frac{1}{5}}\right).$$

Proof. See Appendix A.12. □

Theorem 1.6 establishes the uniform convergence rate of the buyers' (or sellers') value density estimator in any closed inner subsets containing $\underline{s}$ (or $\overline{b}$). Consequently, we expect that, in comparison to the two-step estimator without bias correction, the one with bias correction will have better finite sample performance close to $\underline{s}$ for the buyers' value density estimator and close to $\overline{b}$ for the sellers'. This is confirmed by our Monte Carlo experiments in the next section. Nevertheless, Theorem 1.6 does not say anything about the uniform convergence rate on the entire support. The main difficulty comes from the low accuracy in estimation of the boundary points $\underline{v}$, $\overline{v}$, $\underline{c}$ and $\overline{c}$, since they are estimated from the pseudo private values which converge to their true values at a nonparametric rate.

1.5 Monte Carlo Experiments

To study the finite sample performance of our two-step estimation procedure, we conduct Monte Carlo experiments. We consider two cases of buyers' and sellers' true value distributions and pricing weights. In the first case, both buyers' and sellers' private values are uniformly distributed on $[0, 1]$. The bidding strategies of the buyer and the seller are given by

$$\beta_B(v) = \begin{cases} \dfrac{v}{1+k} + \dfrac{k(1-k)}{2(1+k)}, & \text{if } \dfrac{1-k}{2} \leqslant v \leqslant 1, \\ v, & \text{if } 0 \leqslant v < \dfrac{1-k}{2}; \end{cases} \quad \beta_S(c) = \begin{cases} \dfrac{c}{2-k} + \dfrac{1-k}{2}, & \text{if } 0 \leqslant c \leqslant \dfrac{2-k}{2}, \\ c, & \text{if } \dfrac{2-k}{2} < c \leqslant 1, \end{cases}$$

where k is the pricing weight. Moreover, we set the pricing weight $k = 1/2$ so that the buyer and the seller have equal bargaining power in determining the transaction price. This case has been frequently studied in the theoretical literature (e.g., Chatterjee and Samuelson, 1983). In the second case, we allow asymmetry between buyers' and sellers' value distributions, and asymmetry between their pricing weights. Specifically, we set the pricing weight to $k = 3/4$, and the true densities of buyers' and sellers' private value distributions to be:

$$f_V(v) = \frac{(8v+12)\sqrt{16v^2 - 128v + 553} - 32v^2 + 80v - 105}{(7\sqrt{553} - 31)\sqrt{16v^2 - 128v + 553}},$$

$$f_C(c) = \frac{1}{511 + \sqrt{73} - 1076e^{-3/4}} \left[4 - \frac{8c}{9} + \frac{9+16c}{\sqrt{81+16c^2}} - \frac{2}{9}\sqrt{81+16c^2} + \mathbb{1}(c \geqslant 3)\frac{(c-3)^3}{3} e^{\frac{3-c}{4}} \right],$$

with identical supports, $[\underline{v}, \overline{v}] = [\underline{c}, \overline{c}] = [0, 6]$.[18] In this case, it can be verified that the buyer's and the seller's bidding strategies given by[19]

$$\beta_B(v) = \begin{cases} v, & \text{if } 0 \leqslant v < 1, \\ \dfrac{4v + 28 - \sqrt{16v^2 - 128v + 553}}{11}, & \text{if } 1 \leqslant v \leqslant 6; \end{cases}$$

$$\beta_S(c) = \begin{cases} \dfrac{4c + \sqrt{16c^2 + 81}}{9}, & \text{if } 0 \leqslant c \leqslant 3, \\ c, & \text{if } 3 < c \leqslant 6, \end{cases}$$

form a regular equilibrium. Figure 1.2 plots the true value densities, the equilibrium bidding strategies, and the induced bid densities in the second case.

Our Monte Carlo experiment consists of 5000 replications for each case. In each replication, we first randomly generate n buyers' and n sellers' private values from their true value distributions. We then compute the corresponding bids according to the true bidding strategies. Next, we apply our bias-corrected two-step estimation procedure to the generated sample of bids for each replication. In the first step, we estimate the distribution functions and densities of buyers' and sellers' bids using the empirical distribution functions and bias-corrected kernel density estimators, respectively. We then use (1.20) to obtain the buyers' and the sellers' pseudo private values. In the second step, we use the sample of buyers' and sellers' pseudo private values to estimate buyers' and sellers' value densities by their bias-corrected kernel density estimators.

To satisfy Assumption G on the kernels,[20] we choose the triweight kernel for all of $K_B(\cdot)$, $K_S(\cdot)$, $K_V(\cdot)$, and $K_C(\cdot)$, i.e. $K_B(u) = K_S(u) = K_V(u) = K_C(u) = (35/32)(1-u^2)^3 \cdot \mathbb{1}(-1 \leqslant u \leqslant 1)$. We then choose the primary bandwidths h_B, h_S, h_V and h_C according to the rule of optimal global bandwidth (see Silverman, 1986) as

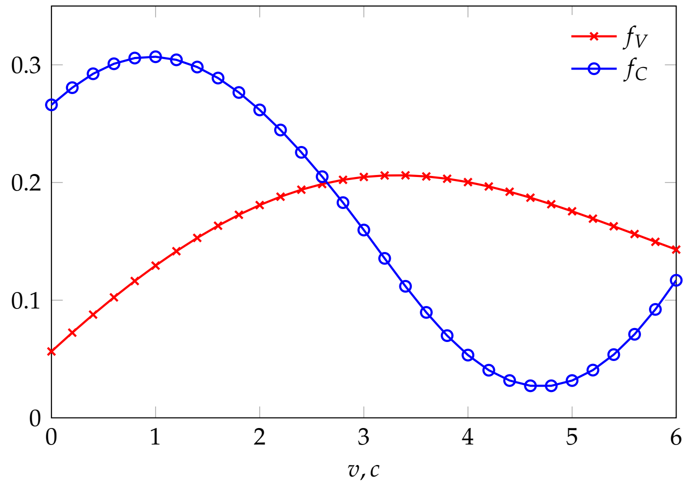
$$h_j = \min\left(n^{-\frac{1}{5}} \hat{\sigma}_j \left[\frac{8\sqrt{\pi} \int_{-1}^{1} K_j^2(u)\, \mathrm{d}u}{3 \left(\int_{-1}^{1} u^2 K_j(u)\, \mathrm{d}u \right)^2} \right]^{\frac{1}{5}}, \frac{\hat{r}_j}{2} \right), \quad j = B, S, V, C,$$

[18] As a matter of fact, we also add some curvature to the true value densities $f_V(\cdot)$ and $f_C(\cdot)$ in this case.

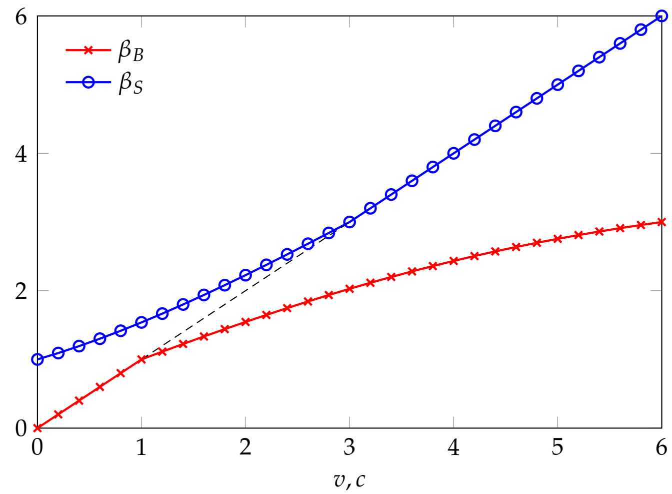
[19] It can also be verified that the corresponding bid densities are

$$g_B(b) = \begin{cases} f_V(b), & \text{if } 0 \leqslant b < 1, \\ \dfrac{121b}{28\sqrt{553} - 124}, & \text{if } 1 \leqslant b \leqslant 3, \\ 0, & \text{otherwise;} \end{cases} \qquad g_S(s) = \begin{cases} \dfrac{36 - 9s}{2044 + 4\sqrt{73} - 4304e^{-3/4}}, & \text{if } 1 \leqslant s \leqslant 3, \\ f_C(s), & \text{if } 3 < s \leqslant 6, \\ 0, & \text{otherwise.} \end{cases}$$

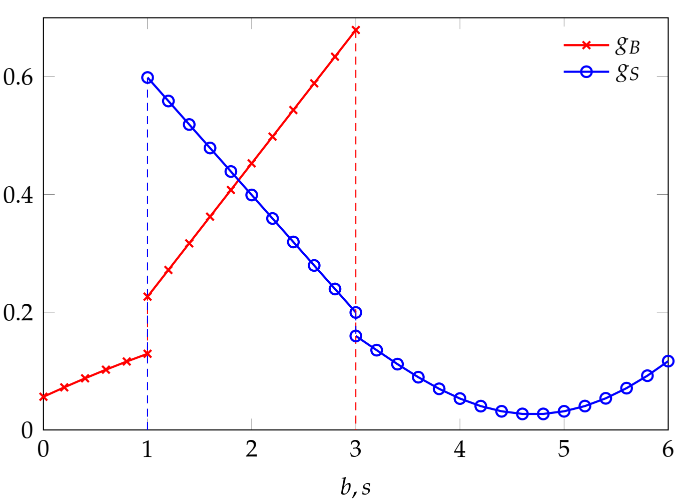
[20] Notice that, in both cases, the private value densities $f_V(\cdot)$ and $f_C(\cdot)$ are continuously twice differentiable on the entire support.



(a) Buyers' and sellers' private value densities



(b) Equilibrium bidding strategies



(c) Density functions of induced bids

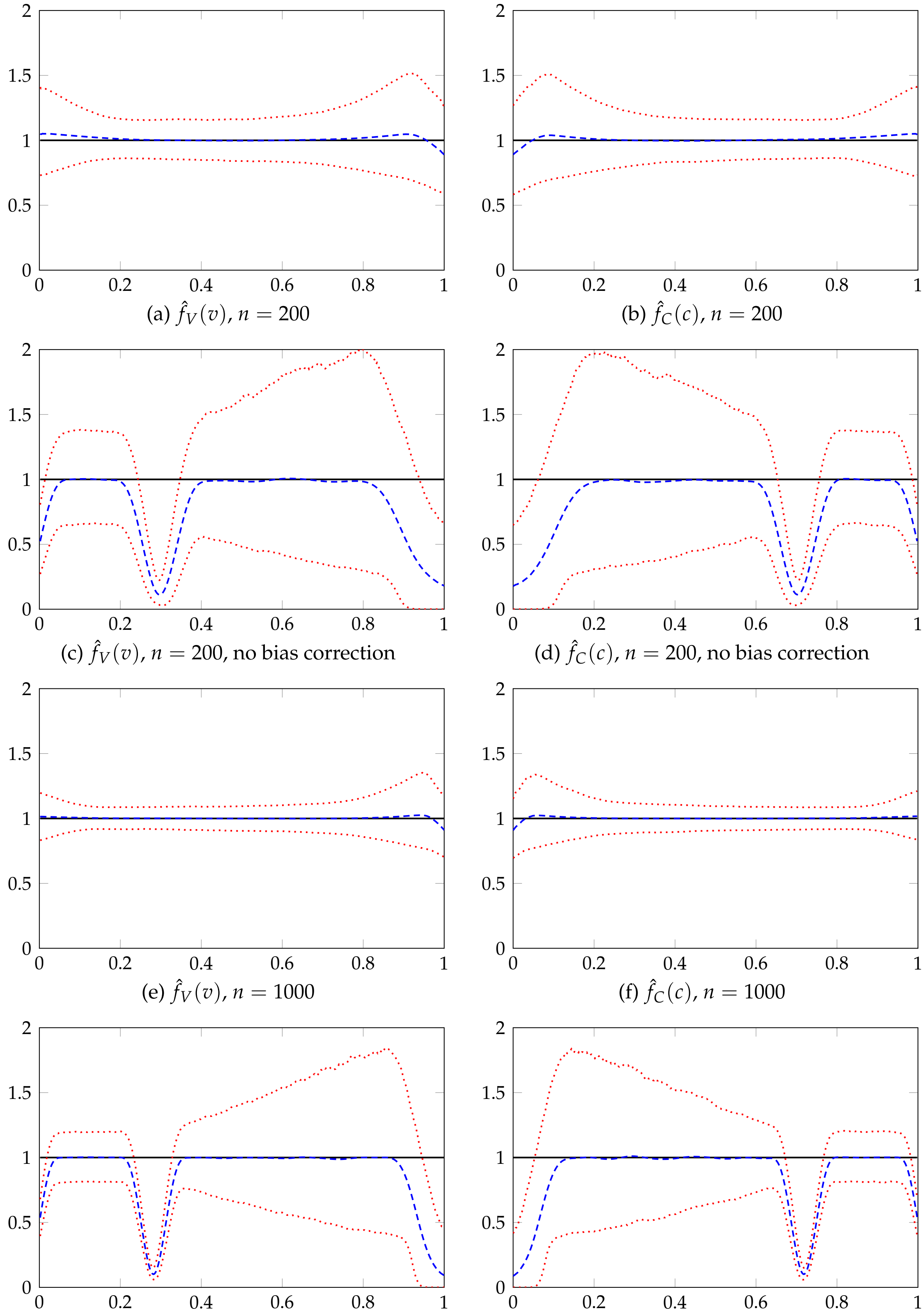
Figure 1.2: True private value densities, equilibrium bidding strategies and bid densities in Monte Carlo experiment 2

where n is the sample size of the observed bids, $\hat{\sigma}_j$ is the estimated standard deviation of observed bids for $j = B, S$ or pseudo private values for $j = V, C$, $K_j(\cdot)$ is the kernel function, and $\hat{r}_j$ is the length of the interval on which the corresponding bid or value density is estimated. In addition, the parameters of bias correction are chosen as follows: all of the coefficients A_B, A_S, A_V and A_C are set at 0.65; each of the secondary bandwidths is equal to its counterpart among the primary bandwidths,[21] i.e. $h'_j = h_j$ for $j = B, S, V, C$.

Our Monte Carlo results for the first case are summarized in Figure 1.3. It shows the two-step estimates of value densities with and without bias correction under the sample sizes of $n = 200$ and $n = 1000$, when both buyers' and sellers' private values are uniformly distributed on $[0, 1]$. The true value densities are displayed in solid lines. For each value of $v \in [0, 1]$ (or $c \in [0, 1]$), we plot the mean of the estimates with a dashed line, and the 5th and 95th percentiles with dotted lines. The latter gives the (pointwise) 90% confidence interval for $f_V(v)$ (or $f_C(c)$). Figure 1.3 shows that our bias-corrected two-step density estimates behave well. First, the true curves fall within their corresponding confidence bands. Second, the mean of the estimates for each density closely matches the true curve. Third, as sample size increases, both the bias and variance of the estimates decrease. Figure 1.3 also shows that bias correction plays an important role in estimating the value densities in double auctions with bargaining. As shown by Figures 1.3c, 1.3d, 1.3g and 1.3h, the standard kernel density estimator (without bias correction) has large bias not only at the boundaries but also in an interior area. When the sample size n increases, this bias will not diminish, although the variance will shrink. The appearance of bias in the interior shows that bias correction is necessary to estimate value densities in double auctions with bargaining.

Figure 1.4 reports the simulation results of the second case under the sample sizes of $n = 200$ and $n = 1000$. Similarly, the true densities, means, and 5th/95th percentiles are respectively displayed in solid lines, dashed lines, and dotted lines. It shows that, with some curvature in the value densities and asymmetry between buyers and sellers, the conclusions in Figure 1.3 still hold; that is, (i) the bias-corrected two-step density estimates perform well, and (ii) bias correction plays an important role for estimating the value densities in our double auction model.

[21] We tried other values of coefficients A_j and secondary bandwidths h'_j, $j = B, S, V, C$, in our experiments, but found that, as long as Assumption H2 holds, the estimates of both buyers' and sellers' value densities are almost the same for different values of A_j and h'_j.



(a) $\hat{f}_V(v)$, $n = 200$

(b) $\hat{f}_C(c)$, $n = 200$

(c) $\hat{f}_V(v)$, $n = 200$, no bias correction

(d) $\hat{f}_C(c)$, $n = 200$, no bias correction

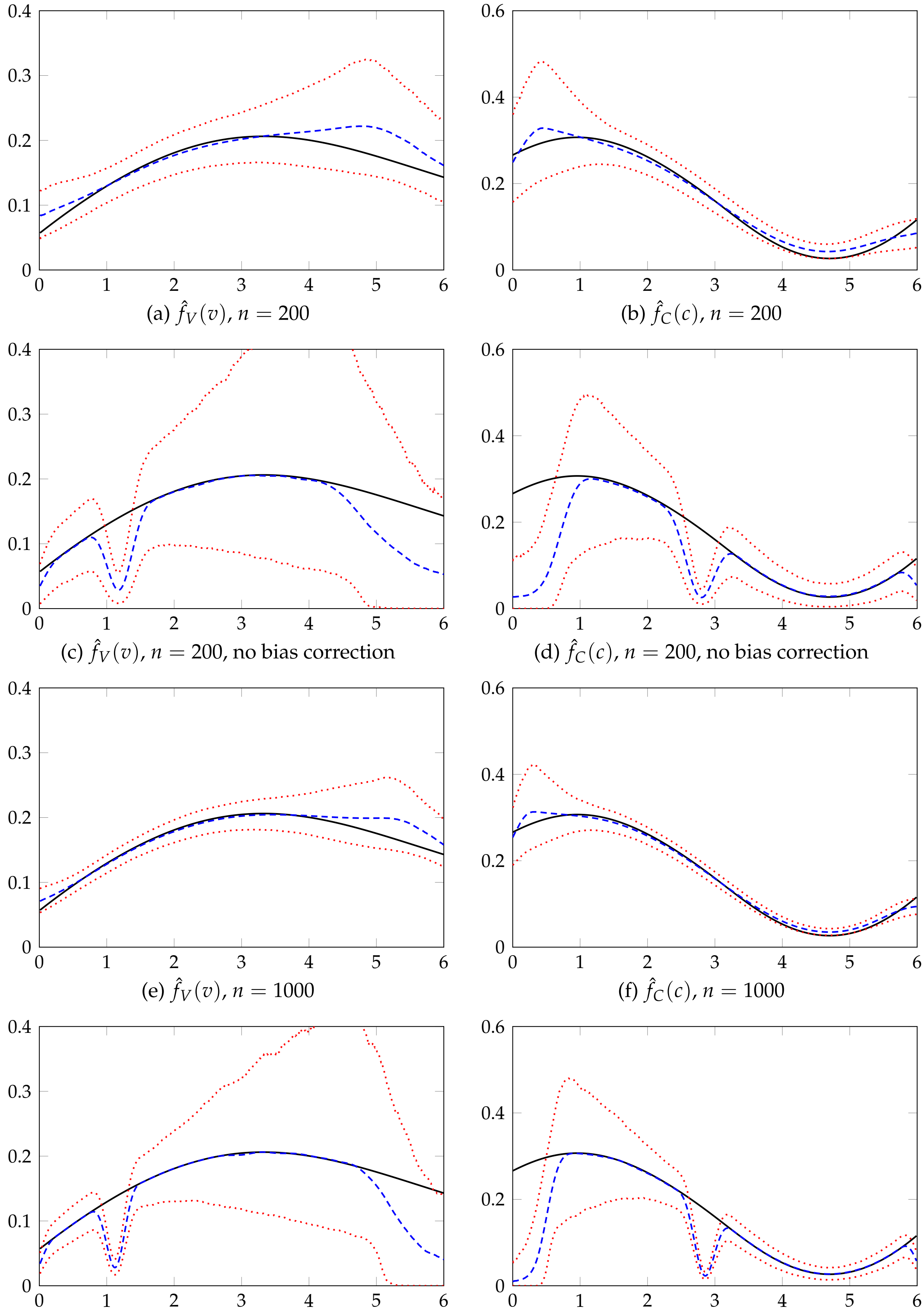
(e) $\hat{f}_V(v)$, $n = 1000$

(f) $\hat{f}_C(c)$, $n = 1000$

(g) $\hat{f}_V(v)$, $n = 1000$, no bias correction

(h) $\hat{f}_C(c)$, $n = 1000$, no bias correction

Figure 1.3: True and estimated densities of private values. $V_i \sim U[0, 1]$, $C_i \sim U[0, 1]$.



(a) $\hat{f}_V(v)$, $n = 200$

(b) $\hat{f}_C(c)$, $n = 200$

(c) $\hat{f}_V(v)$, $n = 200$, no bias correction

(d) $\hat{f}_C(c)$, $n = 200$, no bias correction

(e) $\hat{f}_V(v)$, $n = 1000$

(f) $\hat{f}_C(c)$, $n = 1000$

(g) $\hat{f}_V(v)$, $n = 1000$, no bias correction

(h) $\hat{f}_C(c)$, $n = 1000$, no bias correction

Figure 1.4: True and estimated densities of private values under asymmetry.

1.6 Discussion

1.6.1 Auction-Specific Heterogeneity

We now briefly discuss how to generalize our identification and estimation approach to allow for auction-specific heterogeneity.[22] Let $X \in \mathbb{R}^d$ be a random vector that characterizes the heterogeneity of auctions. For auctions with $X = x$, let $F_{V|X}(\cdot|x)$ and $F_{C|X}(\cdot|x)$ be the buyers' and sellers' private value distributions, and $G_{B|X}(\cdot|x)$ and $G_{S|X}(\cdot|x)$ be their respective bid distribution functions with densities $g_{B|X}(\cdot|x)$ and $g_{S|X}(\cdot|x)$. Let all of our previous assumptions hold for every x in the support of X wherever it applies. The buyer's and the seller's inverse bidding functions in an auction with characteristic $X = x$ are, respectively,

$$v = \begin{cases} b + \dfrac{G_{S|X}(b\,|\,x)}{g_{S|X}(b\,|\,x)}, & \text{if } b \geqslant \underline{s}(x), \\ b, & \text{otherwise,} \end{cases} \qquad c = \begin{cases} s - \dfrac{1 - G_{B|X}(s\,|\,x)}{g_{B|X}(s\,|\,x)}, & \text{if } s \leqslant \overline{b}(x), \\ s, & \text{otherwise,} \end{cases} \tag{1.21}$$

where $\underline{s}(x)$ is the lower bound of the support of $G_{S|X}(\cdot\,|\,x)$, $\overline{b}(x)$ is the upper bound of the support of $G_{B|X}(\cdot\,|\,x)$.

We can then generalize most of our identification and estimation results to auctions with heterogeneity. Specifically, our identification and model restrictions results (Theorems 1.1 to 1.4) still hold as long as the value and bid distributions are simply replaced by the corresponding conditional distributions given X and all relevant conditions hold for every realization of X.

For estimation, our two-step procedure can be generalized to incorporate auction-specific heterogeneity. In the first step, for each auction, we use (1.21) to recover both the buyers' and the sellers' pseudo private values. Notice that, in (1.21), the estimation of conditional bid densities $g_{S|X}$ and $g_{B|X}$ needs to first recover the joint densities g_{SX} and g_{BX} of the bids and the covariates (as well as the marginal density f_X of the covariates), since $g_{S|X}(s\,|\,x) = g_{SX}(s,x)/f_X(x)$ and $g_{B|X}(b\,|\,x) = g_{BX}(b,x)/f_X(x)$. In the second step, we use the covariate data $\{X_1, \ldots, X_n\}$ and pseudo private values recovered previously to estimate the conditional value densities $f_{C|X}$ and $f_{V|X}$. Again, this needs the estimation of joint densities of valuation and covariates f_{CX} and f_{VX}. It is then possible to extend our estimation results in Section 1.4 to this new two-step estimator. However, the new estimator will suffer the "curse of dimensionality" with the introduction of auction-specific heterogeneity $X \in \mathbb{R}^d$. Moreover, for $d \geqslant 1$, the (interior and boundary) bias correction in kernel estimation of bid densities g_{SX} and g_{BX} will be an issue in a multi-dimensional scenario.[23] This

[22] The existence of auction-specific heterogeneity allows for correlation between the buyer's and the seller's private values. Such correlation, however, exists only through the auction-specific heterogeneity.

[23] Notice that the supports of S and B are finite. In addition, the bid densities can have discontinuity points in the interior of the supports (see Figure 1.2c).

issue is challenging, in that, to our knowledge, little is known in the existing literature regarding the boundary bias correction of kernel density estimators in a multi-dimensional setting.

1.6.2 Higher Order Bias Reduction

We can also have higher order boundary (and interior) bias reduction at the cost of more tedious calculations. Due to space limitations, we only illustrate the idea of achieving higher order bias reduction here.

To achieve higher order boundary (and interior) bias reduction, we need to specify both a higher order kernel and a proper functional form for the data transformation. For demonstration purposes, suppose that $\{X_1, X_2, \dots, X_n\}$ is a random sample drawn from a distribution with a density function $f(\cdot)$ and support $[0, \overline{x}]$. To simplify the analysis, we further assume that the density $f(\cdot)$ has a discontinuity point only at 0, i.e. we assume $\lim_{x \to \overline{x}^-} f(x) = 0$. Denote the transformation function by $\gamma(\cdot)$.[24] The (boundary-corrected) kernel density estimator of $f(\cdot)$ with a generalized reflection is given by

$$\hat{f}(x) = \frac{1}{nh} \sum_{i=1}^{n} \left[K\left(\frac{x - X_i}{h}\right) + K\left(\frac{x + \gamma(X_i)}{h}\right) \right],$$

where $K(\cdot)$ is a kernel function on support $[-1, 1]$, and h is a bandwidth parameter. Suppose the underlying density $f(\cdot)$ admits up to $R + 1$ continuous bounded derivatives. Let $w(\cdot) = f(\gamma^{-1}(\cdot))/\gamma'(\gamma^{-1}(\cdot))$ with $\gamma(\cdot)$ being strictly increasing on $[0, +\infty)$ and $(R + 1)$-times continuously differentiable. Then, for $x = \rho h$ with $0 \leqslant \rho \leqslant 1$, the bias of $\hat{f}$ at x can be obtained as

$$E\hat{f}(x) - f(x) = [w(0) - f(0)] \int_{\rho}^{1} K(t) \, \mathrm{d}t + \sum_{j=1}^{R} \frac{W_j}{j!} h^j + O\left(h^{R+1}\right), \tag{1.22}$$

where

$$W_j = f^{(j)}(0) \left[\sum_{l=1}^{j} \binom{j}{l} (-1)^l \rho^{j-l} \int_{-1}^{1} t^l K(t) \, \mathrm{d}t \right] + \left[w^{(j)}(0) - (-1)^j f^{(j)}(0) \right] \int_{\rho}^{1} (t - \rho)^j K(t) \, \mathrm{d}t.$$

Consequently, if we choose a kernel $K(\cdot)$ of order $(R + 1)$ and a transformation function $\gamma(\cdot)$ such that (i) $w(0) = f(0)$, (ii) $w^{(j)}(0) = (-1)^j f^{(j)}(0)$ for all $j = 1, 2, \dots, R$, (iii) $\gamma'(\cdot) > 0$ on $[0, +\infty)$, and (iv) $(R + 1)$-th derivative of $\gamma(\cdot)$ exists,[25] then the boundary bias $E\hat{f}(x) - f(x) = O(h^{R+1})$ for any $x = \rho h$ with $0 \leqslant \rho \leqslant 1$. To see this, condition (i) eliminates the first term on the right-hand

[24] In Section 1.4, we follow Zhang, Karunamuni, and Jones (1999) and Karunamuni and Zhang (2008) and employ a cubic transformation function of $\gamma(u) = u + d \cdot u^2 + A \cdot d^2 \cdot u^3$ where d is the derivative of log-density at the boundary point.

[25] As a matter of fact, conditions (iii) and (iv) are not essential for the higher order bias reduction.

side of (1.22), and condition (ii) together with $(R+1)$-th order kernel $K(\cdot)$ implies $W_j = 0$ for all $j = 1, \ldots, R$ which makes the second term on the right-hand side of (1.22) zero. With the bias of order $O(h^{R+1})$ on the boundary, the kernel density estimator $\hat{f}(\cdot)$ with a generalized reflection then converges uniformly to the true density function $f(\cdot)$ at a rate of $O_p\left(h^{R+1} + \sqrt{\log n/(nh)}\right)$ on the entire support $[0, \bar{x}]$.

Chapter 2

Nonparametric Identification of *k*-Double Auctions Using Price Data

2.1 Introduction

Double auctions are one of the most common exchange institutions. They permit both offers to buy and offers to sell and usually set the transaction price according to traders' offers from both sides. They are extensively used in many field markets such as stock markets and commodity markets.

Recent theoretical studies on double auctions within a game theoretic framework, e.g. Chatterjee and Samuelson (1983), Myerson and Satterthwaite (1983), Leininger, Linhart, and Radner (1989), Satterthwaite and Williams (1989, 2002), Kadan (2007), provide researchers and practitioners with insights about the behavioral theory of double auction markets. Meanwhile, there has been a large body of work which examines identification and estimation of one-sided auctions. Examples includes Laffont, Ossard, and Vuong (1995), Laffont and Vuong (1996), Donald and Paarsch (1996), Guerre, Perrigne, and Vuong (2000, 2009), Li, Perrigne, and Vuong (2000, 2002), Athey and Haile (2002), Haile, Hong, and Shum (2003), Haile and Tamer (2003), Hendricks, Pinkse, and Porter (2003), McAdams (2008), Li and Zheng (2009), An, Hu, and Shum (2010), Krasnokutskaya (2011), Tang (2011), Marmer and Shneyerov (2012), Hubbard, Li, and Paarsch (2012), Hu, McAdams, and Shum (2013), Gentry and Li (2014). However, in contrast to the intensive investigation of one-sided auctions, the empirical analysis of double auction models is still in its infancy.

Motivated by the lack of identification and estimation results for double auction model, I study a simple yet important type of double auctions called k-double auctions with a single buyer and a single seller, which employs a pricing rule that takes the weighted average of the two traders' offers as the transaction price. Such a double auction model is closely related to the structural analysis of noncooperative bargaining models with incomplete information (see, e.g. Sieg, 2000; Watanabe, 2006; Merlo, Ortalo-Magne, and Rust, 2015) and has a wide range of applications including negotiations, dispute settlements, arbitration for sport,[1], and real estate sales. In previous chapter, within the independent private value paradigm, I obtained theoretical results for nonparametric identification and estimation of the buyer's and the seller's value distributions if their bids are observed, both in the case in which bids are observed for all double auctions and when they are observed only in double auctions where transactions take place. In This chapter, I pursue the problem of nonparametrically identifying both traders' value distributions if I only observe the transaction price in each double auction with a transaction. It is common in many applications that the researchers can only access limited sets of observables for reasons such as the design of the trading mechanism. For markets governed by the double auction institution, the transaction prices rather than the traders' offers are usually more readily available. Limiting the observables to the transaction price makes identification more appealing but also more difficult. A similar problem has been addressed for one-sided auctions, see, e.g. Athey and Haile (2002), Adams (2007). However, the transaction price in double auctions depends on both the buyer's and the seller's strategic offers at the same time. This creates greater challenges to identify the traders' value distributions from the price data alone.

As part of my research, I focus on the case in which the pricing weight $k = 1$ or $k = 0$, noting that my results can be readily extended to the situations in which k is observed and equals one or zero some of the time. Based on the distribution of the transaction price in double auctions with $k = 1$ or $k = 0$, under mild assumptions, I give the sharp bound for the identified value distributions of the buyer and the seller on part of their supports, provided that there exist exogenous value distribution shifters Y for the buyer and Z for the seller, while point identification can be reached under stronger conditions. I further establish identification of the two value distributions when the value distribution shifters are continuously distributed.

The rest of this chapter proceeds as follows. Section 2.2 presents the k-double auction model. In Section 2.3, I exploit exclusion restrictions to achieve nonparametric identification of the buyer's and the seller's private value distributions, mainly in the case where the pricing weight on the buyer's bid is equal to 1. Section 2.4 concludes with a discussion of possible estimation approaches. The supplementary results and the proofs are collected in the appendix.

2.2 The Sealed-Bid *k*-Double Auction Model

Consider a sealed-bid k-double auction where a single indivisible good is traded between a buyer and a seller. The value of the good to the buyer is V and the reservation value to the seller is C.

[1] Examples of arbitration similar to the k-double auction mechanism under review include the final offer arbitration employed by Major League Baseball.

Both traders are risk neutral expected utility maximizers. In the auction, both the buyer and the seller simultaneously submit sealed bids B and S, respectively. If $B \geqslant S$, the transaction is struck at price $P = kB + (1-k)S$ where $0 \leqslant k \leqslant 1$. The seller's utility is $P - C$ and the buyer's utility is $V - P$. If $B < S$, there is no transaction occurring and each gets zero utility. Each trader knows his own private value and observes some auction-specific covariates X. However, he only knows his adversary's value is drawn from a certain distribution. The joint distribution of these random variables and the pricing rule (including the pricing weight k) are all common knowledge between the buyer and the seller.

I impose the following assumption on the traders' value distributions.

Assumption I (Independent Private Value)**.**

(i) *V and C are independent conditional on X;*

(ii) *The conditional distributions of V and C given $X = x$, $F_V(\cdot \mid x)$ and $F_C(\cdot \mid x)$, are absolutely continuous with densities $f_V(\cdot \mid x)$ and $f_C(\cdot \mid x)$ on the same support $[\underline{c}(x), \overline{v}(x)] \subset \mathbb{R}_+$.*

Assumption I requires that both traders' values are conditionally independent and drawn from absolutely continuous distributions which share the same bounded supports. It allows unconditional correlation between V and C as long as the auction-specific covariates X account for all the dependence structure.

Denote by $\beta_B(\cdot, x) : [\underline{c}(x), \overline{v}(x)] \to \mathbb{R}_+$ and $\beta_S(\cdot, x) : [\underline{c}(x), \overline{v}(x)] \to \mathbb{R}_+$ the respective strategies of the buyer and the seller. The Bayesian Nash equilibrium (BNE) concept is adopted throughout. However, Leininger, Linhart, and Radner (1989) and Satterthwaite and Williams (1989) showed that there can exist multiple BNE's in a given k-double auction. To exclude some irregular cases and focus on a certain class of equilibria which are well-behaved as described in Chatterjee and Samuelson (1983), the following restrictions are imposed on the equilibrium under consideration here:

Assumption J (Regular equilibrium)**.** *The equilibrium strategy profile (β_B, β_S) satisfies, for any x,*

(i) *$\beta_B(\cdot, x)$ and $\beta_S(\cdot, x)$ are continuous and strictly increasing;*

(ii) *for traders who have positive probability of trade under the strategy profile, $\beta_B(\cdot, x)$ and $\beta_S(\cdot, x)$ are continuously differentiable;*

(iii) *for traders who have zero probability of trade under the strategy profile, $\beta_B(v, x) = v$ and $\beta_S(c, x) = c$.*

An equilibrium is called "regular" if it satisfies Assumption J. Assumption J restricts me to the equilibria with strictly monotonic and (piecewise) differentiable strategies. Here the equilibrium strategies can depend on covariates X in two ways. First, the equilibrium strategies will change as the model primitives such as the value distributions or the pricing weight k vary. Second, when the k-double auction with given model primitives has multiple regular equilibria, the equilibrium strategies can differ as the covariates X affect the equilibrium selection.

Let $G_{Bk}(\cdot \mid x)$ and $G_{Sk}(\cdot \mid x)$ denote the respective distributions of buyer's and seller's equilibrium bids conditional on $X = x$, which are induced by the value distributions and some equilibrium strategy profile (β_{Bk}, β_{Sk}). The k in the subscript is used to indicate the dependence of these functions on the pricing weight k. Since the regular equilibrium strategies are strictly increasing, it follows that $F_V(v \mid x) = G_{Bk}(\beta_{Bk}(v, x) \mid x)$, $F_C(c \mid x) = G_{Sk}(\beta_{Sk}(c, x) \mid x)$, and the respective supports of G_{Bk} and G_{Sk} are given by $[\underline{b}_k(x), \overline{b}_k(x)] = [\beta_{Bk}(\underline{c}(x), x), \beta_{Bk}(\overline{v}(x), x)]$ and $[\underline{s}_k(x), \overline{s}_k(x)] = [\beta_{Sk}(\underline{c}(x), x), \beta_{Sk}(\overline{v}(x), x)]$. As shown in Chapter 1, the regular equilibrium bids of the buyer and of the seller are independent conditional on X and their supports should satisfy $\underline{b}_k(x) \leqslant \underline{s}_k(x) < \overline{b}_k(x) \leqslant \overline{s}_k(x)$. Because the transaction price P is defined only when the buyer's bid is greater than the seller's bid, the support of P is $[\underline{s}_k(x), \overline{b}_k(x)]$. Therefore, by the conditional independence of B and S, the density function of the transaction price is

$$h_k(p \mid x) = a_k(x) \int_0^{T_k(p,x)} g_{Bk}(p + (1-k)t \mid x) g_{Sk}(p - kt \mid x) \, \mathrm{d}t, \tag{2.1}$$

where $g_{Bk}(\cdot \mid x)$ and $g_{Sk}(\cdot \mid x)$ are the corresponding densities of $G_{Bk}(\cdot \mid x)$ and $G_{Bk}(\cdot \mid x)$, $a_k(x)$ is a constant which makes $\int_{\underline{s}_k(x)}^{\overline{b}_k(x)} h_k(p \mid x) \, \mathrm{d}p = 1$, and the upper limit of the integral is $T_k(p, x) = \min\left(\frac{\overline{b}_k(x) - p}{1-k}, \frac{p - \underline{s}_k(x)}{k}\right)$.

2.3 Nonparametric Identification

In this section, I study the identification of the buyer's and the seller's conditional private value distributions, $F_V(\cdot \mid x)$ and $F_C(\cdot \mid x)$. In contrast to Chapter 1, here I assume the econometricians have less information about the traders' behavior—rather than the bids of the buyer and the seller, they can only observe the final transaction price P, auction-specific covariates X and the pricing rule parameter k.

Chapter 1 showed that given that the traders' private values are independent, the value distributions are nonparametrically identified as long as the econometricians can observe the bids of the buyer and the seller (at least those bids with a successful transaction). So one way of identifying the value distributions is to (2.1) to recover the bid distributions from the price distribution then apply the conclusion of Chapter 1. However, this approach faces several challenges.

First, the difficulty comes from the fact that the price distribution is obtained by projecting the joint bid distribution in a certain direction so the price distribution compresses the information of both the buyer's and the seller's bid distributions. It is usually impossible to recover a two-dimensional bid distribution from a one-dimensional price distribution, even if both traders' bids are independent (in this case, it aims to recover two one-dimensional bid distributions).

Second, although the price distribution can be treated as a weighted mixture of the two bid distributions because $P = kB + (1-k)S$, the methods that are typically used to identify the component distributions in finite mixture models will not be applicable due to special features of double auction models. In double auctions, the transaction price is defined only when the buyer's bid is greater than the seller's bid. This means that the price distribution is obtained from a truncated bid distribution. The original independence between the buyer's and the seller's bids breaks down because of the truncation. Therefore, the deconvolution method that is usually used to decompose the mixture distribution does no longer work without the independence condition. In addition, the parameter k does not only play a role as a weight used to calculate the price in double auction transaction and thereby a mixture weight, but it also directly determines the buyer's and the seller's equilibrium bidding strategies and therefore the equilibrium bid distributions. So any changes in the value of k will inevitably change the two bid distributions. As a result, the method employed in many studies about finite mixture models, which rely on the existence of a variable that shifts the mixture weight without affecting the component distributions, cannot be used in the double auction case.

In view of these difficulties, I explore a different identification strategy based on extreme values of pricing weight and exclusion restrictions to identify the buyer's and the seller's conditional private value distributions nonparametrically.

To start, I posit the following assumption on the observed pricing rule parameter k.

Assumption K. *There exist double auctions with pricing weight $k = 0$ or $k = 1$.*

This assumption requires that the econometricians are able to observe the double auctions in which one of the two traders has full bargaining power to unilaterally set the final transaction price once a successful transaction takes place. This condition is satisfied in many applications, for example, either the buyer or the seller can propose a take-it-or-leave-it offer.

The pricing weight k taking the extreme values brings several benefits. As shown by Satterthwaite and Williams (1989), when $k = 0$ or $k = 1$, the k-double auction game has a unique regular equilibrium, so specifying an equilibrium selection mechanism can be avoided. Meanwhile, in these two extreme cases, the strategies in that equilibrium have closed-form expressions, which allows me to interpret the corresponding equilibrium bid distributions as well as the distribution of transaction price in terms of the value distributions and then establish a direct connection between the observables and the model primitives of interest. Precisely, according to Satterthwaite and Williams (1989): When $k = 1$, the seller will choose the weakly dominant strategy of bidding his private value truthfully, therefore the seller's equilibrium inverse bidding function is $\beta_{S1}^{-1}(s, x) = s$

and the buyer's equilibrium inverse bidding function is given by

$$\beta_{B1}^{-1}(b,x) = b + \mathbb{1}(b \geqslant \underline{s}_1(x)) \cdot \lambda(b,x),$$

where $\lambda(\cdot,x) = F_C(\cdot \mid x)/f_C(\cdot \mid x)$. Then by (2.1), the density function of the transaction price when $k = 1$ is

$$\begin{aligned} h_1(p \mid x) &= a_1(x) \int_0^{p-\underline{s}_1(x)} g_{B1}(p \mid x) g_{S1}(p-t \mid x)\, \mathrm{d}t \\ &= a_1(x) g_{B1}(p \mid x) G_{S1}(p \mid x) \\ &= a_1(x) F_C(p \mid x) f_V(p + \lambda(p,x) \mid x)\, [1 + \partial_1 \lambda(p,x)] \end{aligned} \tag{2.2}$$

for $\underline{s}_1(x) \leqslant p \leqslant \overline{b}_1(x)$, where $\partial_1\lambda$ denotes the partial derivative of λ with respect to the first argument. When $k = 0$, the buyer plays the truth-telling strategy in the unique regular equilibrium, so the buyer's equilibrium inverse bidding function is $\beta_{B0}^{-1}(b,x) = b$ and the seller's equilibrium inverse bidding function is

$$\beta_{S0}^{-1}(s,x) = s - \mathbb{1}(s \leqslant \overline{b}_0(x)) \cdot \delta(s,x),$$

where $\delta(\cdot,x) = [1 - F_V(\cdot \mid x)]/f_C(\cdot \mid x)$. As a result, the price density when $k = 0$ is

$$\begin{aligned} h_0(p \mid x) &= a_0(x) g_{S0}(p \mid x)\, [1 - G_{B0}(p \mid x)] \\ &= a_0(x)[1 - F_V(p \mid x)] f_C(p - \delta(p,x) \mid x)\, [1 - \partial_1 \delta(p,x)] \end{aligned} \tag{2.3}$$

for $\underline{s}_0(x) \leqslant p \leqslant \overline{b}_0(x)$. Here $\partial_1\delta$ is the partial derivative of δ with respect to the first argument.

Another key identification restriction is a source of variation in one trader's value distribution that leaves the other trader's value distribution unchanged. Suppose that the observed covariates can be partitioned into three parts $X = (Y, Z, W)$ where $Y \in \mathbb{R}^{d_y}$, $Z \in \mathbb{R}^{d_z}$ and $W \in \mathbb{R}^{d_w}$, and suppose Y and Z only affect the value distribution of one trader. I assume the following exclusion restriction holds.

Assumption L (Exclusion Restriction)**.** *For any realization $x = (y, z, w)$:*

(i) $\underline{c}(y,z,w) = \underline{c}(w)$, $\overline{v}(y,z,w) = \overline{v}(w)$;

(ii) $F_V(v \mid y,z,w) = F_V(v \mid y,w)$ *and* $F_C(c \mid y,z,w) = F_C(c \mid z,w)$ *for any* $v, c \in [\underline{c}(w), \overline{v}(w)]$.

According to Assumption L, Y only affects the buyer's value distribution while Z only affects the seller's. However, the exclusive covariates Y and Z only change the shape but not the supports of the private value distributions.

For simplicity, w is dropped from the notation unless specifically stated otherwise; all quantities considered are implicitly functions of w. To illustrate the idea of the identification strategy, I will focus on identification for the $k = 1$ case. Symmetric results for the $k = 0$ case can be obtained by using similar assumptions and arguments (see Appendix C.1).

2.3.1 Identification with Binary-valued Z

Let $\mathscr{Y}$ and $\mathscr{Z}$ denote the respective supports of Y and Z. To guarantee the existence of a regular equilibrium when $k = 1$, i.e. to ensure the buyer has a strictly increasing and differentiable bidding strategy, I assume that the seller's conditional value distribution $F_C(\cdot \mid z)$ satisfies the following assumption.

Assumption M. *For any $z \in \mathscr{Z}$, $\lambda(\underline{c}, z) = 0$, and $\lambda(\cdot, z)$ is continuously differentiable with $0 < \partial_1 \lambda(c, z) < \infty$ for all $c \in [\underline{c}, \overline{v}]$.*

Such an admissibility condition, which requires that the seller's conditional value distribution admits a continuously differentiable and strictly decreasing reverse hazard rate, is usually imposed in the literature about one-sided auctions and double auctions (see Satterthwaite and Williams, 1989). Then it can be shown first that:

Lemma 2.1. *Under Assumptions I to M, for any $y \in \mathscr{Y}$ and any $z \in \mathscr{Z}$,*

(i) $\underline{s}_1(y, z) = \underline{c}$, and $h_1(\underline{c} \mid y, z) = 0$;

(ii) $\overline{b}_1(y, z) = \overline{b}_1(z)$ which solves $\overline{b}_1(z) + \lambda(\overline{b}_1(z), z) = \overline{v}$.

Proof. See Appendix C.2. □

According to this lemma, when $k = 1$, all price distributions have identical lower endpoints of their support at $\underline{c}$, where the price distribution has zero density. It is also implied that the upper endpoint of the price distribution support only depends on the value of Z which affects the seller's value distribution. These properties are mainly attributed to the exclusion restrictions assumed in Assumption L.

Given this result, for $y^*, y^{**} \in \mathscr{Y}$ and $z \in \mathscr{Z}$, if I define[2]

$$\Gamma_1(p, z) \equiv \lim_{q \to \underline{c}} \left[\frac{h_1(p \mid y^*, z)}{h_1(q \mid y^*, z)} \bigg/ \frac{h_1(p \mid y^{**}, z)}{h_1(q \mid y^{**}, z)} \right] \tag{2.4}$$

for $\underline{c} < p < \overline{b}_1(z)$ and let $\Gamma_1(\underline{c}, z) \equiv \lim_{p \to \underline{c}} \Gamma_1(p, z)$, $\Gamma_1(\overline{b}_1(z), z) \equiv \lim_{p \to \overline{b}_1(z)} \Gamma_1(p, z)$, then it follows from (2.2) that

$$\Gamma_1(p, z) = \frac{f_V(p + \lambda(p, z) \mid y^*)}{f_V(p + \lambda(p, z) \mid y^{**})} \cdot \frac{f_V(\underline{c} \mid y^{**})}{f_V(\underline{c} \mid y^*)}. \tag{2.5}$$

[2] I take the limit as $q \to \underline{c}$ rather than directly take $q = \underline{c}$ because the price has zero density at $\underline{c}$ if the price is generated from the k-double auction specified in Section 2.2. Here y^*, y^{**} are regarded as fixed in order to simplify the notation.

Because the function Γ_1 only depends on the price density h_1 by definition, (2.5) means that the likelihood ratio of the buyer's value distributions in the k-double auction model as specified in Section 2.2 is identified up to scale. Note that the buyer's unobservable private value can be inferred from the identified $\Gamma_1(\cdot, z)$ function if it is invertible. So in order to have invertibility, I assume that for some values of Y, the buyer's conditional value distribution possesses the monotone likelihood ratio property. Specifically,

Assumption N. *There exist $y^* \neq y^{**}$ in $\mathscr{Y}$ such that $f_V(\cdot \mid y^*)/f_V(\cdot \mid y^{**})$ is continuously differentiable with negative derivative on $[\underline{c}, \overline{v}]$.*

Therefore, for the values y^* and y^{**} of covariate Y such that Assumption N holds, I have:

Lemma 2.2. *Under Assumptions I to N,*

(i) $\Gamma_1(\cdot, z)$ is continuously differentiable with negative derivative on $[\underline{c}, \overline{b}_1(z)]$ for any $z \in \mathscr{Z}$;

(ii) $\Gamma_1(\underline{c}, z^*) = \Gamma_1(\underline{c}, z^{**}) = 1$ and $\Gamma_1(\overline{b}_1(z^*), z^*) = \Gamma_1(\overline{b}_1(z^{**}), z^{**})$ for any $z^*, z^{**} \in \mathscr{Z}$.

Proof. See Appendix C.3. □

Lemma 2.2 shows some restrictions that the specified nonparametric double auction model imposes on the observable conditional distribution of the transaction price in terms of Γ_1. First, condition (i) mainly points out that the model implies a strictly decreasing $\Gamma_1(\cdot, z)$. This is mainly due to the monotonicity of the reverse hazard rate of the seller's value distribution and the monotonicity of the likelihood ratio of the buyer's value distribution. Together with the monotonicity, condition (ii) implies that the range of $\Gamma_1(\cdot, z)$, whose domain coincides with the support of transaction price, should keep constant as the value of the exclusive covariate for seller's value distribution changes.

I will start the discussion about identifying the traders' value distributions by showing a few important properties of the traders' conditional value distributions that can rationalize a given price data by the k-double auction model.

For $y^* \neq y^{**}$ in $\mathscr{Y}$ and $z^* \neq z^{**}$ in $\mathscr{Z}$, since the functions $\Gamma_1(\cdot, z^*)$ and $\Gamma_1(\cdot, z^{**})$ induced by the price densities $h_1(\cdot \mid y, z)$ with $y \in \{y^*, y^{**}\}$ and $z \in \{z^*, z^{**}\}$, satisfy the conditions of Lemma 2.2, that is, both $\Gamma_1(\cdot, z^*)$ and $\Gamma_1(\cdot, z^{**})$ are strictly decreasing and have the same range, so for any p in $\Gamma_1(\cdot, z^*)$'s domain $[\underline{c}, \overline{b}_1(z)]$, there is a unique $\psi(p) \in [\underline{c}, \overline{b}_1(z^{**})]$ such that[3]

$$\Gamma_1(p, z^*) = \Gamma_1(\psi(p), z^{**}). \tag{2.6}$$

By the implicit function theorem, $\psi(\cdot)$ defined above is continuously differentiable and strictly increasing and it satisfies $\psi(\underline{c}) = \underline{c}$. Then under Assumptions I to M, by (2.5), equation (2.6) is

[3]In fact, $\psi(p) = \Gamma_1^{-1}(\Gamma_1(p, z^*), z^{**})$ where $\Gamma_1^{-1}(\cdot, z^{**})$ is the inverse function of $\Gamma_1(\cdot, z^{**})$ provided that $\Gamma_1(\cdot, z^{**})$ is strictly monotone.

equivalent to

$$p + \lambda(p, z^*) = \psi(p) + \lambda(\psi(p), z^{**}). \tag{2.7}$$

Because in a k-double auction with $k = 1$, when the seller has private value distribution $F_V(\cdot \mid z)$, the equilibrium inverse bidding function for the buyer is $b + \lambda(b, z)$ where b is the bid, (2.7) means that a buyer who bids $\psi(p)$ when facing a seller with value distribution $F_C(\cdot \mid z^{**})$, will have the same private value as a buyer who bids p when the seller's value distribution is $F_C(\cdot \mid z^*)$.

Moreover, (2.7) directly implies

$$f_V(p + \lambda(p, z^*) \mid y^*) = f_V(\psi(p) + \lambda(\psi(p), z^{**}) \mid y^*),$$

and differentiating both sides of (2.7) with respect to p yields

$$1 + \partial_1 \lambda(p, z^*) = [1 + \partial_1 \lambda(\psi(p), z^{**})] \cdot \psi'(p).$$

Then, by combining these two equations and (2.2), I have for any $p \in [\underline{c}, \overline{b}_1(z^*)]$,

$$\frac{F_C(\psi(p) \mid z^{**})}{F_C(p \mid z^*)} = \frac{a_1(y^*, z^*)}{a_1(y^*, z^{**})} \cdot \frac{h_1(\psi(p) \mid y^*, z^{**})}{h_1(p \mid y^*, z^*)} \psi'(p) = \frac{a_1(y^*, z^*)}{a_1(y^*, z^{**})} m(p), \tag{2.8}$$

where

$$m(p) \equiv \frac{h_1(\psi(p) \mid y^*, z^{**})}{h_1(p \mid y^*, z^*)} \psi'(p).$$

Equation (2.8) requires the ratio $F_C(\psi(\cdot) \mid z^{**}) / F_C(\cdot \mid z^*)$ to be proportional to the function $m(\cdot)$, which is determined by the price distributions, on the interval $(\underline{c}, \overline{b}_1(z^*)]$. It indeed imposes another restriction besides (2.7) on the seller's conditional value distribution that can rationalizes the observed price distribution. This is because by the exclusion restrictions, the same conditional value distribution for the buyer generates price densities $h_1(\cdot \mid y^*, z^*)$ and $h_1(\cdot \mid y^*, z^{**})$ with the seller's conditional value distributions $F_C(\cdot \mid z^*)$ and $F_C(\cdot \mid z^{**})$, respectively. However, it follows from (2.2) that

$$F_V(v \mid y) = \frac{1}{a_1(y, z)} \int_{\underline{c}}^{b} \frac{h_1(u \mid y, z)}{F_C(u \mid z)} \, \mathrm{d}u,$$

where b solves $b + \lambda(b, z) = v$. It suggests that any conditional value distribution for the seller that satisfies Assumption M will automatically induce a conditional value distribution for the buyer,[4] namely

$$\tilde{F}_V(v \mid y, z) = \left[\int_{\underline{c}}^{\overline{b}_1(z)} \frac{h_1(u \mid y, z)}{F_C(u \mid z)} \, \mathrm{d}u \right]^{-1} \int_{\underline{c}}^{b} \frac{h_1(u \mid y, z)}{F_C(u \mid z)} \, \mathrm{d}u, \quad v \in [\underline{c}, \overline{v}], \tag{2.9}$$

[4]By definition, such a conditional value distribution for the buyer which is induced by the price density and a given seller's conditional value distribution will depend not only the covariate Y but also the covariate Z.

and $\tilde{F}_V(v \mid y, z)$ rationalizes a given price density by a k-double auction with $k = 1$. So the implication of condition (2.8) is to make sure that, the buyer's conditional value distribution induced by $F_C(\cdot \mid z^*)$ and $h_1(\cdot \mid y^*, z^*)$ is the same as the one induced by $F_C(\cdot \mid z^{**})$ and $h_1(\cdot \mid y^*, z^{**})$, and therefore it does not depend on Z. Meanwhile, because any given seller's conditional value distribution induces an associated conditional value distribution for the buyer in the above way, when I try to identify both traders' value distributions, it suffices to consider only the one for the seller.

These restrictions are summarized by the following theorem.

Theorem 2.1. *Under Assumptions I to N, $h_1(\cdot \mid y, z)$ can be rationalized by sealed-bid k-double auction with $k = 1$ for some $F_V(\cdot \mid y)$, $y \in \{y^*, y^{**}\}$ and $F_C(\cdot \mid z)$, $z \in \{z^*, z^{**}\}$ if and only if $F_C(\cdot \mid z^*)$ and $F_C(\cdot \mid z^{**})$ satisfy equations* (2.7) *and* (2.8).

Proof. See Appendix C.4. □

Theorem 2.1 is important because it shows that equations (2.7) and (2.8) fully characterize the identified set of the model primitives. On one hand, by showing that only those seller's conditional value distributions satisfying (2.7) and (2.8) are consistent with price data, it gives necessary conditions for the private value distributions that can rationalize the given distribution of transaction price. On the other hand, it also shows that, under those model assumptions, any seller's conditional value distributions that satisfy (2.7) and (2.8) can generate the given price distribution. This indicates that such an identified set is actually sharp.

The identified set characterized by (2.7) and (2.8) can be easily computed. To see that, first rewrite (2.8) as

$$F_C(\psi(p) \mid z^{**}) = \frac{a_1(y^*, z^*)}{a_1(y^*, z^{**})} \cdot F_C(p \mid z^*) m(p),$$

and then taking the derivative with respect to p yields

$$f_C(\psi(p) \mid z^{**}) = \frac{a_1(y^*, z^*)}{a_1(y^*, z^{**})} \cdot \frac{f_C(p \mid z^*) m(p) + F_C(p \mid z^*) m'(p)}{\psi'(p)}.$$

Plugging these two into (2.7) will give

$$m'(p) \left[\lambda(p, z^*)\right]^2 + \left[(p - \psi(p)) m(p)\right]' \lambda(p, z^*) + (p - \psi(p)) m(p) = 0 \tag{2.10}$$

for $p \in [\underline{c}, \overline{b}_1(z^*)]$. By the construction of (2.10), $F_C(\cdot \mid z^*)$ and $F_C(\cdot \mid z^{**})$ satisfy (2.7) and (2.8) if and only if $F_C(\cdot \mid z^*)$ satisfies (2.10) (and induces $F_C(\cdot \mid z^{**})$ according to (2.7) or (2.8)). This allows me to obtain the identified set by looking for the solution to (2.10). Since $\psi(\cdot)$ and $m(\cdot)$ are identified from the price distribution, (2.10) only serves to determine $\lambda(\cdot, z^*)$. Interestingly, for any fixed

$p \in (\underline{c}, \overline{b}_1(z^*)]$, (2.10) is a quadratic equation whenever $m'(p) \neq 0$,[5] so it will have two real solutions

$$\lambda(p,z^*) = \frac{-[(p-\psi(p))m(p)]' \pm \sqrt{\{[(p-\psi(p))m(p)]'\}^2 - 4(p-\psi(p))m(p)m'(p)}}{2m'(p)} \tag{2.11}$$

provided that $\Delta_1(p) \equiv \{[(p-\psi(p))m(p)]'\}^2 - 4(p-\psi(p))m(p)m'(p) \geqslant 0$. Then, the identified set for $\lambda(\,\cdot\,,z^*)$, or equivalently for $F_C(\,\cdot\mid z^*)$, can be obtained by collecting all functions that conform to the form of (2.11) among those specified by the model assumptions.

It should be noticed that the condition $\Delta_1(p) \geqslant 0$ for all $p \in (\underline{c}, \overline{b}_1]$ is indeed another empirical implication which can be potentially used to test the model. This is because if the observed prices come from the regular equilibrium of a k-double auction with $k=1$ and conditional value distributions determined by Assumptions I, L, M and N, then $\lambda(\,\cdot\,,z^*)$ corresponding to the true conditional value distribution for the seller given $Z = z^*$ will be a solution to equation (2.10), so it must be true that $\Delta_1(p) \geqslant 0$.[6]

Furthermore, utilizing the quadratic feature of (2.10), I can find additional conditions on the model structure, under which the identified set for the seller's conditional value distributions (on the part of support that permits positive probability of trade) will collapse to a singleton so that the model is point identified. The following assumption and theorem provide an example.

Assumption O. *Let $\xi(p,z) \equiv p + \lambda(p,z)$. $F_C(\,\cdot\mid z^*)$ and $F_C(\,\cdot\mid z^{**})$ satisfy $\partial_1 \xi^{-1}(v,z^*) > \partial_1 \xi^{-1}(v,z^{**})$ for any $v \in (\underline{c}, \overline{v}]$, where $\xi^{-1}(\,\cdot\,,z)$ is the inverse function of $\xi(\,\cdot\,,z)$.*

Recall that $\xi(\,\cdot\,,z)$ defined above is the buyer's inverse bidding function in the regular equilibrium of a k-double auction with $k=1$ when the seller's value distribution is $F_C(\,\cdot\mid z)$, so $\xi^{-1}(v,z)$ gives the equilibrium bid of a buyer with private value v. Therefore, condition $\partial_1 \xi^{-1}(v,z^*) > \partial_1 \xi^{-1}(v,z^{**})$ for all $v > \underline{c}$, means that the seller's conditional value distributions should be such that the buyer will choose a steeper bidding strategy when $Z = z^*$ than when $Z = z^{**}$.

This assumption is testable. It can be done by comparing the value of the derivative of $\psi(\cdot)$ with unity. To see that, for any $p \in (\underline{c}, \overline{b}_1(z^*)]$, let $v \in (\underline{c}, \overline{v}]$ be the buyer's private value such that $v = \xi(p,z^*) = p + \lambda(p,z^*)$. Therefore, $p = \xi^{-1}(v,z^*)$ and it follows from (2.7) that $\psi(p) = \xi^{-1}(v,z^{**})$. Then, by (2.7),

$$\psi'(p) = \frac{\partial_1 \xi(p,z^*)}{\partial_1 \xi(\psi(p),z^{**})} = \frac{\partial_1 \xi(\xi^{-1}(v,z^*),z^*)}{\partial_1 \xi(\xi^{-1}(v,z^{**}),z^{**})} = \frac{\partial_1 \xi^{-1}(v,z^{**})}{\partial_1 \xi^{-1}(v,z^*)}.$$

[5]If $m'(p) = 0$ but $[(p-\psi(p))m(p)]' \neq 0$, (2.10) will become a linear equation and has a unique solution $\lambda(p,z^*) = -(p-\psi(p))m(p)/[(p-\psi(p))m(p)]'$ which coincides with the limit of original quadratic equation's solution $\left\{-[(p-\psi(p))m(p)]' + \operatorname{sgn}\left([(p-\psi(p))m(p)]'\right)\sqrt{\Delta_1(p)}\right\}/[2m'(p)]$ as p approaches the zeros of $m'(\cdot)$.

[6]This can also be seen by alternatively showing that $\Delta_1(p) = \left\{m(p)\left[\frac{\lambda(p,z^*)}{\lambda(\psi(p),z^{**})}\psi'(p) - \frac{\lambda(\psi(p),z^{**})}{\lambda(p,z^*)}\right]\right\}^2 \geqslant 0$.

Since the buyer's equilibrium bidding strategy $\xi^{-1}(\cdot, z)$ is strictly increasing, so I have the condition of Assumption O is satisfied if and only if $\psi'(p) < 1$ for all $p \in (\underline{c}, \overline{b}_1(z^*)]$.

Assumption O in fact imposes some restrictions on the coefficients of equation (2.10) so that under all the model assumptions there exists only one possible $\lambda(\cdot, z^*)$ that satisfies this quadratic equation. With the rest assumptions, now it can be shown that:

Theorem 2.2. *Under Assumptions I to O,* $F_C(\cdot \mid z^*)$ *is identified on* $[\underline{c}, \overline{b}_1(z^*)]$ *and* $F_C(\cdot \mid z^{**})$ *is identified on* $[\underline{c}, \overline{b}_1(z^{**})]$*.*

Proof. See Appendix C.5. □

The conclusion of Theorem 2.2 holds if Assumption O is replaced by a weaker one which calls for $F_C(\cdot \mid z^*)$ and $F_C(\cdot \mid z^{**})$ that satisfy (i) $\xi^{-1}(v, z^*) > \xi^{-1}(v, z^{**})$ for all $v \in (\underline{c}, \overline{v}]$, and (ii) $F_C(\xi^{-1}(\cdot, z^{**}) \mid z^{**}) / F_C(\xi^{-1}(\cdot, z^*) \mid z^*)$ is strictly decreasing on $(\underline{c}, \overline{v}]$.[7] Condition (i) suggests that any buyer will always bid more aggressively when $Z = z^*$ than when $Z = z^{**}$ by offering a higher price. This is the case if and only if $\lambda(c, z^{**}) > \lambda(c, z^*)$ for all $c \in (\underline{c}, \overline{b}_1(z^*)]$, or equivalently, $F_C(\cdot \mid z^{**}) / F_C(\cdot \mid z^*)$ is strictly decreasing on $(\underline{c}, \overline{b}_1(z^*)]$.[8] Meanwhile, for condition (ii), because the seller will bid his true private value in regular equilibrium in a k-double auction with $k = 1$ and because the transaction takes place only when the buyer's bid is no less than the seller's bid, $F_C(\xi^{-1}(v, z^*) \mid z^*)$ and $F_C(\xi^{-1}(v, z^{**}) \mid z^{**})$ represent the respective trade probabilities of a buyer with value v in the cases of $Z = z^*$ and $Z = z^{**}$, and therefore, this condition requires that the ratio of these two trade probabilities must be decreasing in the buyer's valuation for the good. Similarly, for any $p \in (\underline{c}, \overline{b}_1(z^*)]$, if I let $v = \xi(p, z^*) = p + \lambda(p, z^*)$ be the buyer's private value, then condition (i) is equivalent to $p > \psi(p)$ as $p = \xi^{-1}(v, z^*)$ and (2.7) implies $\psi(p) = \xi^{-1}(v, z^{**})$, and condition (ii) is equivalent to $m'(p) < 0$ because it follows from (2.8) that $m(p) \propto F_C(\psi(p) \mid z^{**}) / F_C(p \mid z^*) = F_C(\xi^{-1}(v, z^{**}) \mid z^{**}) / F_C(\xi^{-1}(v, z^*) \mid z^*)$. So these two conditions can be tested by examining the corresponding properties of functions $\psi(\cdot)$ and $m(\cdot)$.

There are a few points regarding Theorem 2.1 and Theorem 2.2 that need further clarification. First, the conclusions of there two theorems only need two different values for the exclusive covariate Z. This is useful in establishing nonparametric identification no matter whether the covariate Z is discrete or continuous. Second, these two theorems aim to recover the buyer's and the seller's conditional value distributions from the price densities $h_1(\cdot \mid y, z)$ only for $y \in \{y^*, y^{**}\}$ and $z \in \{z^*, z^{**}\}$. When the supports of the covariates, espcially $\mathcal{Z}$, are richer and allow for more variation in the covariates, it is possible to shrink the identified set that Theorem 2.1 gives by choosing other values for Y and Z. Finally, it should be pointed out that Assumption O is sufficient

[7] These two conditions are implied by Assumption O (see the proof in Appendix C.5) but not vise versa.

[8] Refer to Shaked and Shanthikumar (2007) about the latter equivalence.

but not necessary for point identifying the seller's conditional value distributions (see the example in Appendix C.7 where the condition is violated but $\lambda(\cdot, z^*)$ is still point identified).

2.3.2 Identification with Continuous Z

When the covariate Z that exclusively shifts the seller's value distribution is continuous, it is feasible in theory but practically inefficient to establish the identification of the seller's value distribution, by repeatedly applying the results from the previous subsection to investigate all pairs of Z's values in its support. But it will be shown next that the property of Z varying continuously provides a shortcut to identify the seller's conditional value distribution for all $z \in \mathscr{Z}$.

For ease of discussion, I will assume that Z is scalar (i.e. $d_z = 1$) and the support $\mathscr{Z}$ is an interval $[\underline{z}, \overline{z}]$ in $\mathbb{R}$ for the time being.

First, as a supplement to Assumption M, I assume that:

Assumption P. *When Z is continuous, $\lambda(c, z)$ is continuously differentiable in z.*

With this assumption, Lemma 2.2 is augmented to include the following conclusion.

Lemma 2.3. *When Z is continuous, under Assumptions I to N and P, $\Gamma_1(p, z)$ is continuously differentiable in z.*

Now suppose the function $\Gamma_1(p, z)$ defined for $z \in \mathscr{Z}$ and $p \in [\underline{c}, \overline{b}_1(z)]$ satisfies the conditions in Lemmas 2.2 and 2.3, then define[9]

$$\ell_1(p, z) = \frac{\partial_2 \Gamma_1(p, z)}{\partial_1 \Gamma_1(p, z)}, \quad z \in \mathscr{Z}, \quad p \in [\underline{c}, \overline{b}_1(z)], \tag{2.12}$$

where $\partial_1 \Gamma_1$ and $\partial_2 \Gamma_1$ are the partial derivatives of Γ_1 with respect to the first and the second arguments, respectively. By (2.5), I have

$$\ell_1(p, z) = \frac{\partial_2 \lambda(p, z)}{1 + \partial_1 \lambda(p, z)}$$

which can be written as the following form

$$\partial_2 \lambda(p, z) - \ell_1(p, z) \cdot \partial_1 \lambda(p, z) = \ell_1(p, z), \tag{2.13}$$

where $\partial_2 \lambda$ denotes the partial derivative of λ with respect to the second argument. Equation (2.13) turns out to be a first-order linear inhomogeneous partial differential equation about λ, which should be satisfied by all the conditional value distributions for the seller that can rationalize the

[9] $\ell_1(p, z)$ is always well-defined because $\partial_1 \Gamma_1(p, z) < 0$ by Lemma 2.2.

price distribution. Thus, applying the theory of partial differential equations gives the following identification result when Z is continuous.

Theorem 2.3. *Under Assumptions I to N and P, if $\lambda(\cdot, z^*)$ is known for some $z^* \in \mathscr{Z}$, then $\lambda(\cdot,\cdot)$ is identified on the set $\{(p,z) : z \in \mathscr{Z},\ p \in [\underline{c}, \overline{b}_1(z)]\}$.*

Proof. See Appendix C.6. □

According to Theorem 2.3, I can pin down the value of $\lambda(\cdot, z)$ for all other values of z in the support $\mathscr{Z}$ by solving the partial differential equation (2.13), as long as $\lambda(\cdot, z)$ or $F_C(\cdot \mid z)$ is identified for just one realization of covariate Z, which can be done by applying Theorem 2.1 or Theorem 2.2. As a result, the seller's conditional value distribution is identified as

$$\frac{F_C(c \mid z)}{F_C(\overline{b}_1(z) \mid z)} = \exp\left(- \int_c^{\overline{b}_1(z)} \frac{1}{\lambda(u,z)} \, \mathrm{d}u \right), \quad z \in \mathscr{Z}, \quad c \in [\underline{c}, \overline{b}_1(z)],$$

and the buyer's conditional value distribution is given by (2.9) for $y \in \{y^*, y^{**}\}$.

Theorem 2.3 holds for the case where covariate Z is vector-valued (i.e. $d_z > 1$) and the support $\mathscr{Z}$ takes the form of $[\underline{z}_1, \overline{z}_1] \times \cdots \times [\underline{z}_{d_z}, \overline{z}_{d_z}] \subset \mathbb{R}^{d_z}$. This is because, by taking the partial derivatives of $\Gamma_1(\cdot,\cdot)$ and $\lambda(\cdot,\cdot)$ with respect to each component of Z instead, I can define a series of functions similar to (2.12) and construct a system of partial differential equations similar to (2.13), and then it can be shown that the solution to the system of partial differential equations is still uniquely determined by a boundary value condition such as a known $\lambda(\cdot, z^*)$.

2.4 Conclusion

This chapter addresses the problem of nonparametric identification of the buyer's and the seller's value distributions in k-double auctions given only the transaction price is observed. I use exclusion restrictions which take the form of two exogenous covariates that respectively shift the buyer's and the seller's value distributions. I show that in the k-double auctions with either $k = 1$ or $k = 0$, both traders' value distributions can be partially identified in general from the distribution of transaction price, as long as both exclusive value distribution shifters can take at least two distinct values. Besides showing my bound for the identified set is sharp, I provide some sufficient conditions under which the traders' value distributions are point identified. When the value distribution shifters are continuous, I also show that the traders' value distributions can be recovered by solving a partial differential equation that only depends on the observed price distribution.

A nonparametric estimation method for the buyer's and the seller's conditional value distributions, $(F_V(\cdot \mid y), F_C(\cdot \mid z))$, can be developed given they are point identified. A strategy could rely on the previous identification strategy. Specifically, for example, when $k=1$, first let $\hat{\Gamma}_1(p,z)$ be the estimate of $\Gamma_1(p,z)$ in (2.4), where $h_1(p\,|\,y,z)$ is replaced by its nonparametric estimate $\hat{h}_1(p\,|\,y,z)$. Then define the estimates for $\psi(p)$ and $m(p)$ by

$$\hat{\Gamma}_1(p,z^*) = \hat{\Gamma}_1(\hat{\psi}(p),z^{**}) \quad \text{and} \quad \hat{m}(p) = \frac{\hat{h}_1(\hat{\psi}(p),z^{**})}{\hat{h}_1(p,z^*)}\hat{\psi}'(p).$$

Finally, an estimator for $\lambda(\,\cdot\,,z^*)$ is obtained by solving the counterpart of equation (2.10), that is,

$$\hat{m}'(p)\left[\hat{\lambda}(p,z^*)\right]^2 + \left[(p-\hat{\psi}(p))\hat{m}(p)\right]'\hat{\lambda}(p,z^*) + (p-\hat{\psi}(p))\hat{m}(p)\hat{m}'(p) = 0.$$

However, because it requires estimating the functions $\Gamma_1(\,\cdot\,,\,\cdot\,)$, $\psi(\,\cdot\,)$ and $m(\,\cdot\,)$ as intermediate steps which involves operations such as taking limits or derivatives, this strategy could be computationally demanding and complicated in implementation.

Another possible strategy relies on the feature that when $k=1$ or 0, the conditional price density $h_1(\,\cdot\,|\,y,z)$ or $h_0(\,\cdot\,|\,y,z)$ can be explicitly expressed as a function of the traders' conditional value distributions. This feature allows the conditional value distributions to be estimated by directly search the parameter space for $F_V(\,\cdot\,|\,y)$ and $F_C(\,\cdot\,|\,z)$ to match the observed and the predicted distributions of transaction price. Consider a simple example of n double auctions with $k=1$, for each of which the observables consist of the transaction price P_i and the associated covariates (Y_i,Z_i). First, estimate $\underline{c}$ by the lowest observed price as $\hat{\underline{c}} = \min_i P_i$ and assume the rest unknown parameters $\theta = (\overline{v}, F_V(\,\cdot\,|\,\cdot\,), F_C(\,\cdot\,|\,\cdot\,)) \in \Theta = \mathcal{V}\times\mathcal{F}_V\times\mathcal{F}_C$, where $\mathcal{V}$ is a compact subset of $\mathbb{R}_+$ and $\mathcal{F}_V$, $\mathcal{F}_C$ are the respective sets of the buyer's and the seller's conditional value distributions that satisfy all the relevant assumptions on traders' value distributions. Next, take $\Theta_n = \mathcal{V}\times\mathcal{F}_{V,n}\times\mathcal{F}_{C,n}$ as the sieve approximation of Θ such that the sieve preserves the shape and smoothness restrictions on the unknown functions, and then an estimator for θ is given by the minimizer of a criterion function Q_n, i.e. $\hat{\theta} = \arg\min_{\theta_n=(\overline{v}_n,F_{Vn},F_{Cn})\in\Theta_n} Q_n(\theta_n)$. A candidate for the criterion function is the negative log-likelihood function which, by (2.2), takes the form of

$$Q_n(\theta_n) = -\frac{1}{n}\sum_{i=1}^{n}\log\left[\frac{H_{1n}(P_i,Y_i,Z_i)}{\int_{\hat{\underline{c}}}^{\overline{b}_{1n}(Z_i)} H_{1n}(u,Y_i,Z_i)\,\mathrm{d}u}\right],$$

where

$$H_{1n}(p,y,z) = F'_{Vn}\left(p + \frac{F_{Cn}(p\,|\,z)}{F'_{Cn}(p\,|\,z)}\,\middle|\, y\right) F_{Cn}(p\,|\,z)\left\{2 - \frac{F_{Cn}(p\,|\,z)F''_{Cn}(p\,|\,z)}{[F'_{Cn}(p\,|\,z)]^2}\right\}$$

and $\overline{b}_{1n}(Z_i)$ is determined by $\overline{b}_{1n}(Z_i) + F_{Cn}(\overline{b}_{1n}(Z_i)\,|\,Z_i)/F'_{Cn}(\overline{b}_{1n}(Z_i)\,|\,Z_i) = \overline{v}_n$. Alternatively, inspired by Bierens and Song (2012, 2014), the criterion function can be chosen as

$$Q_n(\theta_n) = \int_{[-t,t]^{d_y+d_z+1}} \left| \frac{1}{n} \sum_{i=1}^{n} \exp\left[\mathrm{i} \cdot (P_i, Y_i, Z_i)\tau\right] - \frac{1}{n} \sum_{i=1}^{n} \exp\left[\mathrm{i} \cdot (\tilde{P}_i(\theta_n), Y_i, Z_i)\tau\right] \right|^2 \mathrm{d}\tau$$

with some $t > 0$, where $\tau \in \mathbb{R}^{d_y+d_z+1}$, $\mathrm{i} = \sqrt{-1}$, and $\tilde{P}_i(\theta_n)$ is the simulated transaction price in a double auction with $k = 1$ for the traders' value distributions specified by $F_{Vn}(\cdot \mid Y_i)$ and $F_{Cn}(\cdot \mid Z_i)$.

Appendix A

Supplementary Results and Proofs to Chapter 1

A.1 Identification of pricing weight k from quantiles of transaction price

Let $\Psi_k(p) \equiv \Pr(P \leqslant p)$ be the distribution function of transaction price, where the subscript k indicates the value of this function could also depend on the pricing weight k. Since $\Psi_k(p) = \Pr(kB + (1-k)S \leq p \,|\, \underline{s} \leq S \leq B \leq \overline{b})$, for $0 < k < 1$, we have

$$\Psi_k(p) = \begin{cases} \displaystyle\int_{\underline{s}}^{p} \int_{s}^{\frac{p-(1-k)s}{k}} g_2(b,s)\,\mathrm{d}b\,\mathrm{d}s, & \text{if } p \leq k\overline{b} + (1-k)\underline{s}, \\ \displaystyle 1 - \int_{p}^{\overline{b}} \int_{\frac{p-kb}{1-k}}^{b} g_2(b,s)\,\mathrm{d}s\,\mathrm{d}b, & \text{if } p > k\overline{b} + (1-k)\underline{s}, \end{cases} \tag{A.1}$$

where density function $g_2(b,s)$ is defined by (1.11). When $k = 0$, since $P = S$,

$$\Psi_0(p) = \int_{\underline{s}}^{p} \int_{s}^{\overline{b}} g_2(b,s)\,\mathrm{d}b\,\mathrm{d}s, \tag{A.2}$$

and similarly, when $k = 1$,

$$\Psi_1(p) = \int_{\underline{s}}^{p} \int_{\underline{s}}^{b} g_2(b,s)\,\mathrm{d}s\,\mathrm{d}b = \int_{\underline{s}}^{p} \int_{s}^{p} g_2(b,s)\,\mathrm{d}b\,\mathrm{d}s. \tag{A.3}$$

In order to establish the conditions on recovering k from the distributions of bids and price, we firstly show the following lemma.

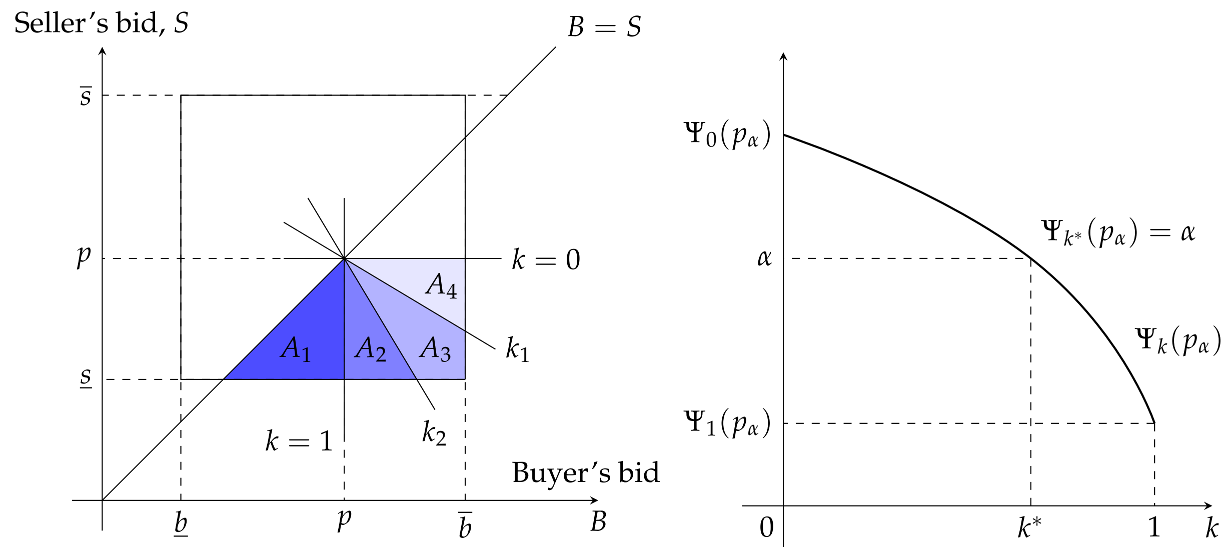
Lemma A.1. *For any fixed $p \in (\underline{s}, \overline{b})$, $\Psi_k(p)$ is continuous and strictly decreasing in $k \in [0,1]$.*

Proof. See Appendix A.13. □

The intuition behind Lemma A.1 is given in Figure A.1a. This lemma implies that the distribution function (and hence the quantile function) of transaction price is continuous and strictly monotonic in k. If we know some αth-quantile of the transaction price P, say p_α, such that $\underline{s} < p_\alpha < \overline{b}$ and $\Psi_1(p_\alpha) \leq \alpha \leq \Psi_0(p_\alpha)$, then by Lemma A.1, there exists a unique $k^* \in [0,1]$ such that

$$\Psi_{k^*}(p_\alpha) = \alpha. \tag{A.4}$$

Thus, the value of k can be obtained by solving equation (A.4) for k^*.[1] Such an idea is shown by Figure A.1b.



(a) Intuition of Lemma A.1. Here $0 < k_1 < k_2 < 1$, then $\Psi_{k_1}(p) = \iint_{A_1 \cup A_2 \cup A_3} g_2(b,s)\,\mathrm{d}b\,\mathrm{d}s$, $\Psi_{k_2}(p) = \iint_{A_1 \cup A_2} g_2(b,s)\,\mathrm{d}b\,\mathrm{d}s$.

(b) Recovering k from a price quantile p_α.

Figure A.1: Identification of pricing weight k from quantiles of transaction price

A.2 Proof of Lemma 1.1

First, we prove that $v > \underline{s}$ implies $\beta_B(v) \leqslant v$.

When $k = 0$, that is, the transaction price is completely determined by the seller's bid, a buyer with private value $v \geqslant \underline{s}$ will get

$$\pi_B(b,v) = \int_{\underline{s}}^{b} (v - s)\,\mathrm{d}G_S(s)$$

[1] Notice that, for fixed k and p, $\Psi_k(p)$ is identified from the distribution of transacted bids by (A.1).

from bidding b. Note that the integrand, $v - s$, is strictly decreasing in s, thus

$$\int_{\underline{s}}^{b} (v-s)\,\mathrm{d}G_S(s) \leqslant \int_{\underline{s}}^{+\infty} \max\{v-s, 0\}\,\mathrm{d}G_S(s). \tag{A.5}$$

Since $v > \underline{s}$, the equality in (A.5) holds if $b = v$, and the equality holds for all G_S only if $b = v$. This implies that, when $k = 0$, the truthful strategy $\beta_B(v) = v$ is the unique (weakly) dominant strategy for the buyer.

When $k \in (0, 1]$, we shall show that it is better for the buyer with value $v > \underline{s}$ to bid her value v than any bid $b > v$. Since $\underline{s}$ is the lower bound of the support of G_S, $G_S(\underline{s}) = 0$ and $G_S(v) > 0$, then

$$\begin{aligned}
\pi_B(v,v) - \pi_B(b,v) &= \int_{\underline{s}}^{v} [v - kv - (1-k)s]\,\mathrm{d}G_S(s) - \int_{\underline{s}}^{b} [v - kb - (1-k)s]\,\mathrm{d}G_S(s) \\
&= \int_{\underline{s}}^{v} [v - kv - (1-k)s]\,\mathrm{d}G_S(s) - \int_{\underline{s}}^{v} [v - kb - (1-k)s]\,\mathrm{d}G_S(s) \\
&\qquad - \int_{v}^{b} [v - kb - (1-k)s]\,\mathrm{d}G_S(s) \\
&= \int_{\underline{s}}^{v} k(b-v)\,\mathrm{d}G_S(s) - \int_{v}^{b} [v - kb - (1-k)s]\,\mathrm{d}G_S(s) \\
&= k(b-v)G_S(v) + \int_{v}^{b} [kb + (1-k)s - v]\,\mathrm{d}G_S(s).
\end{aligned}$$

Since $b > v$ and $G_S(v) > 0$, the first term is positive and the second term

$$\int_{v}^{b} [kb + (1-k)s - v]\,\mathrm{d}G_S(s) \geqslant \int_{v}^{b} [kb + (1-k)v - v]\,\mathrm{d}G_S(s) = k(b-v)[G_S(b) - G_S(v)] \geqslant 0.$$

This completes the proof of $\beta_B(v) \leqslant v$.

To see that $\beta_B(v) < v$ for $v > \underline{s}$ if $k > 0$, note that by (1.1),

$$\left.\frac{\partial \pi_B(b,v)}{\partial b}\right|_{b=v} = -kG_S(v) < 0.$$

It implies that there exists $\Delta > 0$ small enough such that $\pi_B(v-\Delta, v) > \pi_B(v,v)$, therefore, bidding the true value for the buyer with private value v is no longer optimal, i.e. $\beta_B(v) \neq v$. Since we have already shown that $\beta_B(v) \leqslant v$, the desired result follows.

In an analogous way, the second conclusion can be proved by showing that truthful bidding strategy is dominant when $k = 1$, and is dominated by some $\tilde{\beta}_S(c) > c$ when $k \in [0, 1)$ and $c < \overline{b}$. □

A.3 Proof of Theorem 1.1

Let $\beta_B(\cdot)$ and $\beta_S(\cdot)$ be the respective regular equilibrium bidding strategies of the buyer and the seller that induce the bid distribution G.

By condition A1 of Assumption C, strictly increasing and continuous bidding strategies imply the support of bid distribution is a rectangular region, namely $[\underline{b}, \overline{b}] \times [\underline{s}, \overline{s}]$ with $\underline{b} = \beta_B(\underline{v})$, $\overline{b} = \beta_B(\overline{v})$, $\underline{s} = \beta_S(\underline{c})$ and $\overline{s} = \beta_S(\overline{c})$. To show that $\overline{b} \leqslant \overline{s}$ and $\underline{b} \leqslant \underline{s}$, firstly suppose $\overline{b} > \overline{s}$, then any buyer bidding $b > \overline{s}$ will be strictly inferior to just bidding $\overline{s}$. Because this doesn't make the buyer lose any trades but the expected profit on each trade will increase by lowering the transaction price. This deviation is contradicted by the assumption that (β_B, β_S) is an equilibrium. Applying similar argument to the seller bidding $s < \underline{b}$, we can prove the second conclusion $\underline{s} \geqslant \underline{b}$. Then we show that $\underline{s} < \overline{b}$. Suppose not, then: (i) If $\overline{b} \leqslant \underline{s} < \overline{v}$, the buyer with value $\overline{v}$ will have incentive to bid $\frac{\underline{s}+\overline{v}}{2}$ instead of $\overline{b}$, because by bidding $\frac{\underline{s}+\overline{v}}{2}$ he can get

$$\pi\left(\frac{\underline{s}+\overline{v}}{2}, \overline{v}\right) = \int_{\underline{s}}^{\frac{\underline{s}+\overline{v}}{2}} \left[\overline{v} - k\frac{\underline{s}+\overline{v}}{2} - (1-k)s\right] \mathrm{d}G_S(s) = \frac{k}{2}(\overline{v} - \underline{s}) + (1-k)\int_{\underline{s}}^{\frac{\underline{s}+\overline{v}}{2}} (\overline{v} - s)\, \mathrm{d}G_S(s) > 0$$

while bidding $\overline{b} \leqslant \underline{s}$ gives him zero expected profit. This contradicts the equilibrium requirement. (ii) If $\underline{c} < \overline{b} \leqslant \underline{s}$, then analogous argument can show that bidding $\frac{\overline{b}+\underline{c}}{2}$ is a profitable deviation for the seller with value $\underline{c}$, which presents a contradiction to the equilibrium condition, too. (iii) If $\overline{b} \leqslant \underline{c} < \overline{v} \leqslant \underline{s}$, then condition A3 of Assumption C is contradicted because it requires that $\underline{s} = \underline{c} < \overline{v} = \overline{b}$. From the above, C1 hold.

Because V and C are independent and because $\beta_B(\cdot)$ and $\beta_S(\cdot)$ are deterministic functions, it follows that the bids, $B = \beta_B(V)$ and $S = \beta_S(C)$, are also independent. More precisely, since $\beta_B(\cdot)$ and $\beta_S(\cdot)$ are continuous and strictly increasing, so there exist inverse functions, $\beta_B^{-1}(\cdot)$ and $\beta_S^{-1}(\cdot)$, which are also continuous and strictly increasing. Thus

$$\begin{aligned} G(b,s) &= \Pr(\beta_B(V) \leqslant b, \beta_S(C) \leqslant s) \\ &= \Pr(V \leqslant \beta_B^{-1}(b), C \leqslant \beta_S^{-1}(s)) \\ &= \Pr(V \leqslant \beta_B^{-1}(b)) \Pr(C \leqslant \beta_S^{-1}(s)) = F_V(\beta_B^{-1}(b)) F_C(\beta_S^{-1}(s)). \end{aligned}$$

Define

$$G_B(b) = F_V(\beta_B^{-1}(b)) \tag{A.6}$$

$$G_S(s) = F_C(\beta_S^{-1}(s)) \tag{A.7}$$

for every $b \in [\underline{b}, \overline{b}]$ and $s \in [\underline{s}, \overline{s}]$. Since $\beta_B^{-1}(\cdot)$ is continuous and strictly increasing on $[\underline{b}, \overline{b}] =$

$[\beta_B(\underline{v}), \beta_B(\overline{v})]$, we have $G_B \in \mathcal{P}_{[\underline{b},\overline{b}]}$ by (A.6) and the assumption $F_V \in \mathcal{P}_{[\underline{v},\overline{v}]}$. Similar argument can be applied to show $G_S \in \mathcal{P}_{[\underline{s},\overline{s}]}$. Now we get C2.

In order to show C3 and C4, note that $G_B(\cdot)$ and $G_S(\cdot)$ defined in (A.6) and (A.7) must be the distributions of observed (equilibrium) bids of the buyer and the seller, respectively. Now, $\beta_B(\cdot)$ and $\beta_S(\cdot)$ must solve the set of first-order differential equations (1.3) and (1.4). Since (1.5) and (1.6) follow from (1.3) and (1.4), then $\beta_B(\cdot)$ and $\beta_S(\cdot)$ must satisfy

$$\xi(\beta_B(v), G_S) = v, \quad \eta(\beta_S(c), G_B) = c$$

for all $v \geqslant \underline{s}$ and all $c \leqslant \overline{b}$. Noting that $\underline{s} = \beta_S(\underline{c})$ and $\overline{b} = \beta_B(\overline{v})$ and making the change of variable $v = \beta_B^{-1}(b)$ and $c = \beta_S^{-1}(s)$, we obtain

$$\xi(b, G_S) = \beta_B^{-1}(b) \tag{A.8}$$

$$\eta(s, G_B) = \beta_S^{-1}(s) \tag{A.9}$$

for all $b, s \in [\underline{s}, \overline{b}]$. By condition A1 of Assumption C, both $\beta_B^{-1}(\cdot)$ and $\beta_S^{-1}(\cdot)$ are strictly increasing, and by condition A3 ofAssumption C, $\beta_B(\cdot)$ is differentiable on $[\underline{s}, \overline{v}]$ and so is $\beta_S(\cdot)$ on $[\underline{c}, \overline{b}]$. Thus C3 and C4 follow from the fact that $\xi(\underline{s}, G_S) = \underline{s}$ by (1.5), $\eta(\overline{b}, G_B) = \overline{b}$ by (1.6), and $\overline{v} = \beta_B^{-1}(\overline{b}) = \xi(\overline{b}, G_S)$, $\underline{c} = \beta_S^{-1}(\underline{s}) = \eta(\underline{s}, G_B)$.

It is remained to show C5 and C6. Given $b \leqslant \overline{b}$, for buyer with private value v such that $\beta_B(v) = b$, bidding any $b' \in [\overline{b}, \overline{s}]$ should not give him greater profit than bidding b because β_B is the equilibrium bidding strategy for the buyer. That is,

$$\begin{aligned}
0 \geqslant \pi_B(b', v) - \pi_B(b, v) &= \int_{\underline{s}}^{b'} [v - kb' - (1-k)s] \, \mathrm{d}G_S(s) - \int_{\underline{s}}^{b} [v - kb - (1-k)s] \, \mathrm{d}G_S(s) \\
&= v[G_S(b') - G_S(b)] - kb'G_S(b') + kbG_S(b) - (1-k) \int_b^{b'} s \, \mathrm{d}G_S(s) \\
&= k(v - b')G_S(b') - k(v-b)G_S(b) \\
&\quad + (1-k) \left[(v - b')G_S(b') - (v-b)G_S(b) + \int_b^{b'} G_S(s) \, \mathrm{d}s \right] \\
&= (v - b')G_S(b') - (v-b)G_S(b) + (1-k) \int_b^{b'} G_S(s) \, \mathrm{d}s.
\end{aligned}$$

Because $v = \beta_B^{-1}(b) = \xi(b, G_S)$ by (A.8), replacing v by $\xi(b, G_S)$ in the above inequality will yield (1.9). Similarly, for seller with private value c such that $\beta_S(c) = s \geqslant \underline{s}$, using the argument that any deviation of bidding $s' \in [\underline{b}, \underline{s}]$ would not be profitable, we can show that (1.10) must hold. This completes the proof of C6 and the theorem. $\square$

A.4 Proof of Theorem 1.2

To show the sufficiency of C1–C4, define

$$F_V(v) = \begin{cases} G_B(v) & \text{if } v < \underline{s} \\ G_B(\xi^{-1}(v, G_S)) & \text{if } \underline{s} \leqslant v \leqslant \xi(\overline{b}, G_S) \\ 1 & \text{if } v > \xi(\overline{b}, G_S) \end{cases} \tag{A.10}$$

$$F_C(c) = \begin{cases} 0 & \text{if } c < \eta(\underline{s}, G_B) \\ G_S(\eta^{-1}(c, G_B)) & \text{if } \eta(\underline{s}, G_B) \leqslant c \leqslant \overline{b} \\ G_S(c) & \text{if } c > \overline{b} \end{cases} \tag{A.11}$$

and

$$\underline{v} = \underline{b}, \quad \overline{v} = \xi(\overline{b}, G_S), \quad \underline{c} = \eta(\underline{s}, G_B), \quad \overline{c} = \overline{s}.$$

Condition C1 guarantees the functions $\xi(\,\cdot\,, G_S)$ in (1.5) and $\eta(\,\cdot\,, G_S)$ in (1.6) are well-defined. Since $\underline{b}$ is the lower endpoint of the support of G_B, so for all $v \leqslant \underline{v} = \underline{b}$, $F_V(v) = 0$, and by definition, $F_V(v) = 1$ for all $v > \overline{v} = \xi(\overline{b}, G_S)$. Moreover, because $F_V(\overline{v}) = G_B(\xi^{-1}(\xi(\overline{b}, G_S), G_S)) = G_B(\overline{b}) = 1$, $F_V(\underline{s}) = G_B(\xi^{-1}(\xi(\underline{s}, G_S), G_S)) = G_B(\underline{s})$, G_B is continuous and strictly increasing on $[\underline{b}, \overline{b}]$ by C2, and $\xi^{-1}(\,\cdot\,, G_S)$ is continuous and strictly increasing on $[\xi(\underline{s}, G_S), \xi(\overline{b}, G_S)]$ by C3. Then $F_V(\,\cdot\,)$ defined by (A.10) is continuous and strictly increasing on $[\underline{b}, \xi(\overline{b}, G_S)] = [\underline{v}, \overline{v}]$. Therefore F_V is a valid absolutely continuous distribution with support $[\underline{v}, \overline{v}]$, i.e. $F_V \in \mathcal{P}_{[\underline{v}, \overline{v}]}$ as required. We can also show $F_C \in \mathcal{P}_{[\underline{c}, \overline{c}]}$ in similar way.

We shall show that the distributions F_V and F_C of buyer's and seller's respective private values can rationalize G in a sealed-bid k-double auction, i.e. $G_B(b) = F_V(\beta_B^{-1}(b))$ on $[\underline{b}, \overline{b}]$ and $G_S(s) = F_C(\beta_S^{-1}(s))$ on $[\underline{s}, \overline{s}]$ for some regular equilibrium profile (β_B, β_S). By construction of F_V and F_C, we have

$$\begin{aligned} G_B(b) &= F_V(b)\mathbb{1}(\underline{b} \leqslant b < \underline{s}) + F_V(\xi(b, G_S))\mathbb{1}(\underline{s} \leqslant b \leqslant \overline{b}) \\ &= F_V\left(b\mathbb{1}(\underline{b} \leqslant b < \underline{s}) + \xi(b, G_S)\mathbb{1}(\underline{s} \leqslant b \leqslant \overline{b})\right) \end{aligned}$$

for $b \in [\underline{b}, \overline{b}]$ and

$$\begin{aligned} G_S(s) &= F_C(\eta(s, G_B))\mathbb{1}(\underline{s} \leqslant s \leqslant \overline{b}) + F_C(s)\mathbb{1}(\overline{b} < s \leqslant \overline{s}) \\ &= F_C\left(\eta(s, G_B)\mathbb{1}(\underline{s} \leqslant s \leqslant \overline{b}) + s\mathbb{1}(\overline{b} < s \leqslant \overline{s})\right) \end{aligned}$$

for $s \in [\underline{s}, \overline{s}]$, where $\mathbb{1}(\cdot)$ is the indicator function. Define

$$\xi_*(b, G_S) \equiv b\mathbb{1}(\underline{b} \leqslant b < \underline{s}) + \xi(b, G_S)\mathbb{1}(\underline{s} \leqslant b \leqslant \overline{b}),$$
$$\eta_*(s, G_B) \equiv \eta(s, G_B)\mathbb{1}(\underline{s} \leqslant s \leqslant \overline{b}) + s\mathbb{1}(\overline{b} < s \leqslant \overline{s}),$$

then by C3 and C4, $\xi_*(\cdot, G_S)$ is continuous and strictly increasing on $[\underline{b}, \overline{b}]$ and so is $\eta_*(\cdot, G_B)$ on $[\underline{s}, \overline{s}]$. Define bidding strategies

$$\beta_B(v) = \begin{cases} v & \text{if } \underline{v} \leqslant v \leqslant \underline{s} \\ \xi^{-1}(v, G_S) & \text{if } \underline{s} < v \leqslant \overline{v} \end{cases} \tag{A.12}$$

$$\beta_S(c) = \begin{cases} \eta^{-1}(c, G_B) & \text{if } \underline{c} \leqslant c < \overline{b} \\ c & \text{if } \overline{b} \leqslant c \leqslant \overline{c} \end{cases} \tag{A.13}$$

so that $\beta_B(\cdot) = \xi_*^{-1}(\cdot, G_S)$ and $\beta_S(\cdot) = \eta_*^{-1}(\cdot, G_B)$. By construction of these strategies, A1–A3 in Assumption C are satisfied, and also, $G_B(b) = F_V(\beta_B^{-1}(b))$ and $G_S(s) = F_C(\beta_S^{-1}(s))$ so that G is the induced bid distribution for (F_V, F_C) defined in (A.10) and (A.11) by the strategy profile (β_B, β_S) defined above. Thus it is remained to show (β_B, β_S) is indeed an equilibrium. We show that the optimal bid for the buyer with private value v is $\beta_B(v)$. A similar argument shows that β_S is optimal for the seller.

Obviously, if $v \leqslant \underline{s}$, then the buyer cannot make an advantageous trade and bidding $\beta_B(v) = v$ achieves zero as her greatest possible expected profit. Suppose $v > \underline{s}$, since G_S is the induced seller's bid distribution, then for bid $b \in [\underline{s}, \overline{b}]$, by (1.1) we obtain

$$\begin{aligned} \frac{\partial \pi_B(b,s)}{\partial b} &= -kG_S(b) + (v - kb)g_S(b) - (1-k)bg_S(b) \\ &= g_S(b)\left[v - \left(b + k\frac{G_S(b)}{g_S(b)}\right)\right] = g_S(b)\left[v - \xi(b, G_S)\right]. \end{aligned}$$

Because $g_S(b)$ is positive, the monotonicity of $\xi(\cdot, G_S)$ by C3 implies that $\partial\pi_B(b,v)/\partial b > 0$ for all $b < \xi^{-1}(v, G_S)$ and $\partial\pi_B(b,v)/\partial b < 0$ for all $b > \xi^{-1}(v, G_S)$. Therefore, $b = \xi^{-1}(v, G_S) = \beta_B(v)$ is the unique maximizer of the buyer's expected profit in $[\underline{s}, \overline{b}]$. Now we show that the buyer would not want to choose bid within $[\overline{b}, \overline{s}]$, either. Recall that we have already shown that C5 is equivalent to $\pi_B(b', v) \leqslant \pi_B(b, v)$ for any $v \geqslant \overline{b}$ and any $b' \in [\overline{b}, \overline{s}]$ when $b = \xi^{-1}(v, G_S) = \beta_B(v)$ in the proof of Theorem 1.1, this is established straightforwardly because choosing a bid within $[\overline{b}, \overline{s}]$ is profitable only for the buyer with private value $v \geqslant \overline{b}$. Finally, given $\overline{s}$ is the highest seller's bid, any buyer's bid greater than $\overline{s}$ will be dominated by $\overline{s}$. This completes the proof of sufficiency.

From the proof of Theorem 1.1, we know that $\xi(\cdot, G_S) = \beta_B^{-1}(\cdot)$ and $\eta(\cdot, G_B) = \beta_S^{-1}(\cdot)$

on $[\underline{s}, \overline{b}]$ when $F_V(\cdot)$ and $F_C(\cdot)$ exist. Since $F_V(\cdot) = G_B(\beta_B(\cdot))$ and $F_C(\cdot) = G_S(\beta_S(\cdot))$, then $F_V(\cdot) = G_B(\xi_*^{-1}(\cdot, G_S))$ and $F_C(\cdot) = G_S(\eta_*^{-1}(\cdot, G_B))$. Because $\xi(\cdot, G_S)$ is uniquely determined by $G_S(\cdot)$ and $\eta(\cdot, G_B)$ is uniquely determined by $G_B(\cdot)$, it follows that $\xi_*(\cdot, G_S)$ and $\eta_*(\cdot, G_B)$ are uniquely determined by G. Hence, the private value distribution (F_V, F_C) that rationalizes G is unique. □

A.5 Proof of Theorem 1.3

Given Theorem 1.2, it suffices to show that C5 and C6 are implied by C7 and C8.

We shall only show C7, more precisely, the monotonicity of $\xi(\cdot, G_S)$, implies C5. A similar argument can show that C8 implies C6. For buyer with private value v, since

$$\frac{\partial \pi_B(b, v)}{\partial b} = g_S(b) \left[v - \xi(b, G_S)\right],$$

then strictly increasing $\xi(\cdot, G_S)$ on $[\underline{s}, \overline{s}]$ ensures that for any $b \in \left(\xi^{-1}(v, G_S), \overline{s}\right]$, $\partial \pi_B(b, v)/\partial b < 0$, therefore, the expected profit of the buyer $\pi_B(b, v)$ is strictly decreasing in the buyer's bid. For $b' \in [\overline{b}, \overline{s}]$ and $b \leqslant \overline{b}$ such that $\xi(b, G_S) \geqslant b'$, let $v = \xi(b, G_S)$, then it follows from the above conclusion that

$$b' \geqslant \overline{b} \geqslant b = \xi^{-1}(v, G_S) \quad \Rightarrow \quad \pi_B(b', v) \leqslant \pi_B(b, v),$$

which is equivalent to C5 as shown in the proof of Theorem 1.1. □

A.6 Proof of Lemma 1.2

By definition, G_{1B} and G_{1S} are the conditional marginal distributions of B and S given $(B, S) \in [\underline{s}, \overline{b}]^2$. So

$$G_{1B}(b) \equiv \Pr\left(B \leqslant b \mid (B, S) \in [\underline{s}, \overline{b}]^2\right) = \int_{\underline{s}}^{b} \int_{\underline{s}}^{\overline{b}} g_1(x, y)\, \mathrm{d}y\, \mathrm{d}x,$$
$$G_{1S}(s) \equiv \Pr\left(S \leqslant s \mid (B, S) \in [\underline{s}, \overline{b}]^2\right) = \int_{\underline{s}}^{\overline{b}} \int_{\underline{s}}^{s} g_1(x, y)\, \mathrm{d}y\, \mathrm{d}x,$$
$$g_{1B}(b) \equiv \int_{\underline{s}}^{\overline{b}} g_1(b, y)\, \mathrm{d}y, \quad g_{1S}(s) \equiv \int_{\underline{s}}^{\overline{b}} g_1(x, s)\, \mathrm{d}x.$$

By independence between buyer's offer and seller's ask, namely $g(b, s) = g_B(b) g_S(s)$, equation (1.11) implies that

$$1 - G_{1B}(b) = \int_{b}^{\overline{b}} \int_{\underline{s}}^{\overline{b}} \frac{1}{m} g_B(x) g_S(y)\, \mathrm{d}y\, \mathrm{d}x = \frac{1}{m} G_S(\overline{b}) \left[1 - G_B(b)\right], \tag{A.14}$$

$$G_{1S}(s) = \int_{\underline{s}}^{s} \int_{\underline{s}}^{\overline{b}} \frac{1}{m} g_B(x) g_S(y)\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{m}\left[1 - G_B(\underline{s})\right] G_S(s), \tag{A.15}$$

$$g_{1B}(b) = \int_{\underline{s}}^{\overline{b}} \frac{1}{m} g_B(b) g_S(y)\,\mathrm{d}y = \frac{1}{m} g_B(b) G_S(\overline{b}), \tag{A.16}$$

$$g_{1S}(s) = \int_{\underline{s}}^{\overline{b}} \frac{1}{m} g_B(x) g_S(s)\,\mathrm{d}x = \frac{1}{m} g_S(s)\left[1 - G_B(\underline{s})\right]. \tag{A.17}$$

Thus, by (A.14)–(A.17),

$$\frac{G_S(b)}{g_S(b)} = \frac{G_{1S}(b)}{g_{1S}(b)}, \quad \frac{1 - G_B(s)}{g_B(s)} = \frac{1 - G_{1B}(s)}{g_{1B}(s)}.$$

So we finally get

$$\tilde{\xi}(b, G_{1S}) = b + k\frac{G_{1S}(b)}{g_{1S}(b)} = b + k\frac{G_S(b)}{g_S(b)} = \xi(b, G_S),$$
$$\tilde{\eta}(s, G_{1B}) = s - (1-k)\frac{1 - G_{1B}(s)}{g_{1B}(s)} = s - (1-k)\frac{1 - G_B(s)}{g_B(s)} = \eta(s, G_B)$$

for all $b, s \in [\underline{s}, \overline{b}]$. □

A.7 Proof of Theorem 1.4

First, we show part (i). By Theorem 1.1, C1–C4 hold. By definition of G_2, D1 is the direct corollary of C1. Using $g_2(b,s) = g(b,s)/m'$ and $g(b,s) = g_B(b)g_S(s)$ by C2, we have

$$g_2(b,s)g_2(b',s') = g_2(b,s')g_2(b',s) = \frac{g_B(b)g_B(b')g_S(s)g_S(s')}{m'^2},$$

so D2 holds. D3 is implied by C3 and (1.14); D4 is implied by C4 and (1.15).

Next, let us prove the conclusion of part (ii). Notice that, by D3 and D4, (1.16) is equivalent to

$$\frac{F_V(\tilde{\xi}(b, G_{1S})) - F_V(\underline{s})}{1 - F_V(\underline{s})} = G_{1B}(b), \quad \frac{F_C(\tilde{\eta}(s, G_{1B}))}{F_C(\overline{b})} = G_{1S}(s) \tag{A.18}$$

for $(b,s) \in [\underline{s}, \overline{b}]^2$. For any (F_V, F_C) satisfying E1–E3, by Assumption B and D1, $\underline{c} \leqslant \underline{s} < \overline{b} \leqslant \overline{v}$. Consider the following strategy profile

$$\tilde{\beta}_B(v) = \begin{cases} v & \text{if } v \leqslant \underline{s}, \\ \tilde{\xi}^{-1}(v, G_{1S}) & \text{otherwise}, \end{cases} \tag{A.19}$$

$$\tilde{\beta}_S(c) = \begin{cases} c & \text{if } c \geqslant \bar{b}, \\ \tilde{\eta}^{-1}(c, G_{1B}) & \text{otherwise,} \end{cases} \tag{A.20}$$

where $\tilde{\xi}^{-1}(\cdot, G_{1S})$ and $\tilde{\eta}^{-1}(\cdot, G_{1B})$ are respective inverse functions of $\tilde{\xi}(\cdot, G_{1S})$ and $\tilde{\eta}(\cdot, G_{1B})$. D3 and D4 ensure that both $\tilde{\xi}^{-1}(\cdot, G_{1S})$ and $\tilde{\eta}^{-1}(\cdot, G_{1B})$ are well-defined, strictly increasing, and differentiable on $[\tilde{\xi}(\underline{s}, G_{1S}), \tilde{\xi}(\bar{b}, G_{1S})]$ and $[\tilde{\eta}(\underline{s}, G_{1B}), \tilde{\eta}(\bar{b}, G_{1B})]$, respectively.

We firstly show that $(\tilde{\beta}_B, \tilde{\beta}_S)$ will induce the same distribution of transacted bids as G_2.

Since $\tilde{\xi}(\underline{s}, G_{1S}) = \underline{s}$ and $\tilde{\eta}(\bar{b}, G_{1B}) = \bar{b}$, so $\tilde{\xi}^{-1}(\underline{s}, G_{1S}) = \underline{s}$ and $\tilde{\eta}^{-1}(\bar{b}, G_{1B}) = \bar{b}$, then both $\tilde{\beta}_B$ and $\tilde{\beta}_S$ defined above are continuous and strictly increasing. Moreover, $\bar{b}$ is the upper bound of G_{1B}'s support, so by (A.18),

$$\frac{F_V(\tilde{\xi}(\bar{b}, G_{1S})) - F_V(\underline{s})}{1 - F_V(\underline{s})} = G_{1B}(\bar{b}) = 1 \quad \Rightarrow \quad F_V(\tilde{\xi}(\bar{b}, G_{1S})) = 1;$$

and for any $b < \bar{b}$,

$$\frac{F_V(\tilde{\xi}(b, G_{1S})) - F_V(\underline{s})}{1 - F_V(\underline{s})} = G_{1B}(b) < 1 \quad \Rightarrow \quad F_V(\tilde{\xi}(b, G_{1S})) < 1.$$

This means $\xi(\bar{b}, G_{1S})$ should equal to the upper bound of F_V's support, i.e. $\bar{v} = \tilde{\xi}(\bar{b}, G_{1S})$. Similar argument can show that $\underline{c} = \tilde{\eta}(\underline{s}, G_{1B})$.

Define the induced bids $\tilde{B} = \tilde{\beta}_B(V)$ and $\tilde{S} = \tilde{\beta}_S(C)$. Then by the continuity and monotonicity of these strategies, we have the support of $\tilde{B}$ is $[\tilde{\beta}_B(\underline{v}), \tilde{\beta}_B(\bar{v})] = [\underline{v}, \tilde{\xi}^{-1}(\tilde{\xi}(\bar{b}, G_{1S}), G_{1S})] = [\underline{v}, \bar{b}]$ and the support of $\tilde{S}$ is $[\tilde{\beta}_S(\underline{c}), \tilde{\beta}_S(\bar{c})] = [\tilde{\eta}^{-1}(\tilde{\eta}(\underline{s}, G_{1B}), G_{1B}), \bar{c}] = [\underline{s}, \bar{c}]$. Since V and C are independent, so $\tilde{B}$ and $\tilde{S}$ are also independent. Thus for all $b, s \in [\underline{s}, \bar{b}]$,

$$\begin{aligned}
\tilde{G}_{1B}(b) &\equiv \Pr\left(\tilde{B} \leqslant b \,|\, (\tilde{B}, \tilde{S}) \in [\underline{s}, \bar{b}]^2\right) = \Pr(\tilde{B} \leqslant b \,|\, \underline{s} \leqslant \tilde{B} \leqslant \bar{b}) \\
&= \Pr(\tilde{\beta}_B(V) \leqslant b \,|\, \underline{s} \leqslant \tilde{\beta}_B(V) \leqslant \bar{b}) = \Pr(V \leqslant \tilde{\xi}(b, G_{1S}) \,|\, \tilde{\xi}(\underline{s}, G_{1S}) \leqslant V \leqslant \tilde{\xi}(\bar{b}, G_{1S})) \\
&= \frac{\Pr(\tilde{\xi}(\underline{s}, G_{1S}) \leqslant V \leqslant \tilde{\xi}(b, G_{1S}))}{\Pr(\tilde{\xi}(\underline{s}, G_{1S}) \leqslant V \leqslant \tilde{\xi}(\bar{b}, G_{1S}))} = \frac{\Pr(\underline{s} \leqslant V \leqslant \tilde{\xi}(b, G_{1S}))}{\Pr(\underline{s} \leqslant V \leqslant \bar{v})} \\
&= \frac{F_V(\tilde{\xi}(b, G_{1S})) - F_V(\underline{s})}{1 - F_V(\underline{s})} = G_{1B}(b), \\
\tilde{G}_{1S}(s) &\equiv \Pr\left(\tilde{S} \leqslant s \,|\, (\tilde{B}, \tilde{S}) \in [\underline{s}, \bar{b}]^2\right) = \Pr(\tilde{S} \leqslant s \,|\, \underline{s} \leqslant \tilde{S} \leqslant \bar{b}) \\
&= \Pr(\tilde{\beta}_S(C) \leqslant s \,|\, \underline{s} \leqslant \tilde{\beta}_S(C) \leqslant \bar{b}) = \Pr(C \leqslant \tilde{\eta}(s, G_{1B}) \,|\, \tilde{\eta}(\underline{s}, G_{1B}) \leqslant C \leqslant \tilde{\eta}(\bar{b}, G_{1B})) \\
&= \frac{\Pr(\tilde{\eta}(\underline{c}, G_{1B}) \leqslant C \leqslant \tilde{\eta}(s, G_{1B}))}{\Pr(\tilde{\eta}(\underline{s}, G_{1B}) \leqslant C \leqslant \tilde{\eta}(\bar{b}, G_{1B}))} = \frac{\Pr(\underline{c} \leqslant C \leqslant \tilde{\eta}(s, G_{1B}))}{\Pr(\underline{c} \leqslant C \leqslant \bar{b})} \\
&= \frac{F_C(\tilde{\eta}(s, G_{1B}))}{F_C(\bar{b})} = G_{1S}(s).
\end{aligned}$$

Consequently, the corresponding conditional marginal density $\tilde{g}_{1B}(b) = g_{1B}(b)$ and $\tilde{g}_{1S}(s) = g_{1S}(s)$ for all $b, s \in [\underline{s}, \overline{b}]$.

By D2 and the definition of g_1, we know that g_1 has the following property

$$\forall\, (b,s),(b',s') \in [\underline{s}, \overline{b}]^2 : \quad g_1(b,s)g_1(b',s') = g_1(b,s')g_1(b',s).$$

Then

$$\forall\, (b,s),(b',s') \in [\underline{s}, \overline{b}]^2 : \quad \frac{g_1(b',s')}{g_1(b,s')} = \frac{g_1(b',s)}{g_1(b,s)}.$$

Integrating both sides with respect to b', we have

$$\frac{g_{1S}(s')}{g_1(b,s')} = \int_{\underline{s}}^{\overline{b}} \frac{g_1(b',s')}{g_1(b,s')}\,\mathrm{d}b' = \int_{\underline{s}}^{\overline{b}} \frac{g_1(b',s)}{g_1(b,s)}\,\mathrm{d}b' = \frac{g_{1S}(s)}{g_1(b,s)} \;\Rightarrow\; g_{1S}(s')g_1(b,s) = g_{1S}(s)g_1(b,s').$$

Integrating both sides again with respect to s', we get

$$g_1(b,s)\int_{\underline{s}}^{\overline{b}} g_{1S}(s')\,\mathrm{d}s' = g_1(b,s) = g_{1S}(s)\int_{\underline{s}}^{\overline{b}} g_1(b,s')\,\mathrm{d}s' = g_{1B}(b)g_{1S}(s),$$

which holds for all $(b,s) \in [\underline{s}, \overline{b}]^2$. Thus, by the independence between $\tilde{B}$ and $\tilde{S}$, the conditional joint density of $(\tilde{B}, \tilde{S})$ given $(\tilde{B}, \tilde{S}) \in [\underline{s}, \overline{b}]^2$, namely $\tilde{g}_1(b,s)$, satisfies

$$\tilde{g}_1(b,s) = \tilde{g}_{1B}(b)\tilde{g}_{1S}(s) = g_{1B}(b)g_{1S}(s) = g_1(b,s)$$

for all $(b,s) \in [\underline{s}, \overline{b}]^2$. Since

$$g_2(b,s) = \frac{g_1(b,s)}{\int_{\underline{s}}^{\overline{b}}\int_{\underline{s}}^{s} g_1(b,s)\,\mathrm{d}b\,\mathrm{d}s}, \quad \tilde{g}_2(b,s) = \frac{\tilde{g}_1(b,s)}{\int_{\underline{s}}^{\overline{b}}\int_{\underline{s}}^{s} \tilde{g}_1(b,s)\,\mathrm{d}b\,\mathrm{d}s},$$

thus the conditional density of the induced transacted bids $(\tilde{B}, \tilde{S})$ must be $\tilde{g}_2(b,s) = g_2(b,s)$ for all $(b,s) \in \mathscr{D}'$.

Secondly, we shall show that $(\tilde{\beta}_B, \tilde{\beta}_S)$ is a regular equilibrium for such (F_V, F_C). Since A1–A3 in Assumption C are all satisfied by definitions of $\tilde{\beta}_B$ and $\tilde{\beta}_S$, it suffices to verify that $(\tilde{\beta}_B, \tilde{\beta}_S)$ maximizes the expected profit for buyer with $v > \underline{s}$ and seller with $c < \overline{b}$.

Let $\tilde{G}_B$ and $\tilde{G}_S$ denote the distributions of the induced bids $\tilde{B} = \tilde{\beta}_B(V)$ and $\tilde{S} = \tilde{\beta}_S(C)$, respectively. Then, for $v > \underline{s}$ and $b \in [\underline{s}, \overline{b}]$, by (1.1) we have

$$\frac{\partial \pi_B(b,v)}{\partial b} = -k\tilde{G}_S(b) + (v-b)\tilde{g}_S(b) = -kF_C(\tilde{\beta}_S^{-1}(b)) + (v-b)\frac{f_C(\tilde{\beta}_S^{-1}(b))}{\tilde{\beta}_S'(\tilde{\beta}_S^{-1}(b))}.$$

Since $\tilde{\beta}_S^{-1}(b) \leqslant \overline{b}$, then by (A.20) and (1.16),

$$F_C(\tilde{\beta}_S^{-1}(b)) = F_C(\overline{b})G_{1S}(b), \quad f_C(\tilde{\beta}_S^{-1}(b)) = F_C(\overline{b})g_{1S}(b)\tilde{\beta}_S'(\tilde{\beta}_S^{-1}(b)).$$

Hence,

$$\begin{aligned}\frac{\partial \pi_B(b,v)}{\partial b} &= -kF_C(\overline{b})G_{1S}(b) + (v-b)F_C(\overline{b})g_{1S}(b) \\ &= F_C(\overline{b})g_{1S}(b)\left[v - \left(b + k\frac{G_{1S}(b)}{g_{1S}(b)}\right)\right] = F_C(\overline{b})g_{1S}(b)\left[v - \tilde{\xi}(b, G_{1S})\right].\end{aligned}$$

Because $F_C(\overline{b})g_{1S}(b) > 0$, the monotonicity of $\tilde{\xi}(\cdot, G_{1S})$ by D3 implies that $b = \tilde{\xi}^{-1}(v, G_{1S}) = \tilde{\beta}_B(v)$ is the unique maximizer of the buyer's expected profit in $[\underline{s}, \overline{b}]$. For $b' \geqslant \overline{b}$ and $v \geqslant b'$, let $b = \tilde{\beta}_B(v)$, then by construction of $(\tilde{\beta}_B, \tilde{\beta}_S)$, we have $\tilde{G}_S(b') = F_C(b')$, $\tilde{G}_S(b) = F_C(\tilde{\beta}_S^{-1}(b)) = F_C(\tilde{\eta}(b, G_{1B}))$ and

$$\int_b^{b'} \tilde{G}_S(s)\,\mathrm{d}s = \int_b^{\overline{b}} F_C(\tilde{\eta}(s, G_{1B}))\,\mathrm{d}s + \int_{\overline{b}}^{b'} F_C(s)\,\mathrm{d}s.$$

Note that by definition

$$\xi(b, \tilde{G}_S) = b + k\frac{\tilde{G}_S(b)}{\tilde{g}_S(b)} = b + k\frac{F_C(\tilde{\beta}_S^{-1}(b))}{f_C(\tilde{\beta}_S^{-1}(b))/\tilde{\beta}_S'(\tilde{\beta}_S^{-1}(b))} = b + k\frac{G_{1S}(b)}{g_{1S}(b)} = \tilde{\xi}(b, G_{1S}),$$

therefore, (1.17) implies

$$\left[\xi(b, \tilde{G}_S) - b'\right]\tilde{G}_S(b') - \left[\xi(b, \tilde{G}_S) - b\right]\tilde{G}_S(b) + (1-k)\int_b^{b'} \tilde{G}_S(s)\,\mathrm{d}s \leqslant 0,$$

and it follows that $\pi_B(b', v) \leqslant \pi_B(b, v)$ under $\tilde{G}_S$. Hence, we show that the buyer would not bid any $b' \geqslant \overline{b}$ and $\tilde{\beta}_B$ gives the buyer the maximal expected profit. Similar argument can show the optimality of $\tilde{\beta}_S$ and it completes the proof of sufficiency of E1–E3.

It remains to be shown that only those (F_V, F_C) satisfying E1–E3 can rationalize the same distribution of transacted bids as G_2.

By Lemma 1.1, we have already known that in a regular equilibrium the buyer will never bid higher than her private value and the seller will never bid lower than her private value, so the conditions $\underline{c} \leqslant \underline{s}$ and $\overline{v} \geqslant \overline{b}$ are straightforward. For any distribution of regular equilibrium bids, $G \in \mathcal{P}_{[\underline{b},\overline{b}]\times[\underline{s},\overline{s}]}$, such that G_2 is the corresponding distribution of transacted bids, the bids B and S are independent by Theorem 1.1, so G_1 that is uniquely derived from G_2 must be the corresponding conditional distribution of (B, S) given $(B, S) \in [\underline{s}, \overline{b}]^2$. Since it has already be shown that $G_1(b, s) = G_{1B}(b)G_{1S}(s)$ because $g_1(b, s) = g_{1B}(b)g_{1S}(s)$, then G_{1B} is the conditional distribution of B given $B \geqslant \underline{s}$ and G_{1S} is the conditional distribution of S given $S \leqslant \overline{b}$, in other words,

$$G_{1B}(b) = \frac{G_B(b) - G_B(\underline{s})}{1 - G_B(\underline{s})}, \quad G_{1S}(s) = \frac{G_S(s)}{G_S(\overline{b})}. \tag{A.21}$$

According to the proof of Theorem 1.2, G can only be rationalized by (F_V, F_C) defined in (A.10) and (A.11) which imply

$$F_V(\xi(b, G_S)) = G_B(b), \quad F_C(\eta(s, G_B)) = G_S(s) \tag{A.22}$$

for $\underline{s} = \xi^{-1}(\underline{s}, G_S) \leqslant b \leqslant \overline{b}$ and $\underline{s} \leqslant s \leqslant \eta^{-1}(\overline{b}, G_B) = \overline{b}$. By (1.14), (1.15), (A.21), (A.22) and using $\tilde{\xi}(\underline{s}, G_{1S}) = \underline{s}$, $\tilde{\eta}(\overline{b}, G_{1B}) = \overline{b}$, we have condition (A.18) should hold for all (F_V, F_C) that can rationalize G_2. In addition, according to Theorem 1.1, G is rationalizable only if G satisfies conditions C5 and C6. Given the equilibrium strategies are regular, we have $G_S(s) = F_C(s)$ for all $s > \overline{b}$ and $G_S(s) = F_C(\eta(s, G_B)) = F_C(\tilde{\eta}(s, G_{1B}))$ for all $s \leqslant \overline{b}$ by Lemma 1.2, therefore, (1.17) immediately follows from (1.9). A similar argument can show (1.18) follows from (1.10), too. The assertion of part (ii) is then established, which completes the proof. □

A.8 Proof of Lemma 1.3

First, we will establish the following two properties on bidding strategies: (M1) under Assumption F, any regular equilibrium strategies β_B and β_S admit up to $R+1$ continuous and bounded derivatives on $[\underline{s}, \overline{v}]$ and $[\underline{c}, \overline{b}]$, respectively; (M2) for any $v \in [\underline{s}, \overline{v}]$ and any $c \in [\underline{c}, \overline{b}]$, $\beta_B'(v) \geqslant \epsilon_B > 0$ and $\beta_S'(c) \geqslant \epsilon_S > 0$. To show (M1), we need to rewrite (1.3) and (1.4) as follows:

$$\beta_S'(c) = \frac{f_C(c)\left[\beta_B^{-1}(\beta_S(c)) - \beta_S(c)\right]}{k \cdot F_C(c)}, \tag{A.23}$$

$$\beta_B'(v) = \frac{f_V(v)\left[\beta_B(v) - \beta_S^{-1}(\beta_B(v))\right]}{(1-k) \cdot [1 - F_V(v)]}. \tag{A.24}$$

By definition, any pair of regular equilibrium strategies β_B and β_S is continuously differentiable on $[\underline{s}, \overline{v}]$ and $[\underline{c}, \overline{b}]$, respectively (see Assumption C). Consequently, under Assumption F, (A.23) and (A.24) imply that $\beta_S'(\cdot)$ and $\beta_B'(\cdot)$ are continuously differentiable on $[\underline{c}, \overline{b}]$ and $[\underline{s}, \overline{v}]$, respectively. This further implies that β_S and β_B are twice continuously differentiable on $[\underline{c}, \overline{b}]$ and $[\underline{s}, \overline{v}]$. Again, under Assumption F, (A.23) and (A.24) imply that $\beta_S'(\cdot)$ and $\beta_B'(\cdot)$ are twice continuously differentiable, and hence β_S and β_B admit up to third continuous bounded derivatives on $[\underline{c}, \overline{b}]$ and $[\underline{s}, \overline{v}]$, respectively. This argument can go on until we conclude that β_S and β_B admit up to $R+1$ continuous bounded derivatives, respectively, on $[\underline{c}, \overline{b}]$ and $[\underline{s}, \overline{v}]$. This completes the proof of (M1).

Now we establish (M2). By definition of regular equilibrium, the seller's and buyer's bidding strategies are continuously differentiable with positive derivative on $[\underline{c}, \overline{b}]$ and $[\underline{s}, \overline{v}]$, respectively (see condition A2 of Assumption C), i.e., $\beta_S'(\cdot)$ and $\beta_B'(\cdot)$ are continuous and positive on $[\underline{c}, \overline{b}]$ and $[\underline{s}, \overline{v}]$. By extreme value theorem, $\beta_S'(\cdot)$ and $\beta_B'(\cdot)$ have positive minimum and maximum on $[\underline{c}, \overline{b}]$ and $[\underline{s}, \overline{v}]$, respectively. The conclusion of (M2) therefore follows.

It was shown earlier that $\xi(\cdot, G_S)$ and $\eta(\cdot, G_B)$ solve

$$\forall\, b, s \in [\underline{s}, \overline{b}]: \quad \beta_B(\xi(b, G_S)) = b, \quad \beta_S(\eta(s, G_B)) = s,$$

it follows from (M1), (M2) and Lemma C1 of Guerre, Perrigne, and Vuong (2000) that both $\xi(\cdot, G_S)$ and $\eta(\cdot, G_B)$ admit up to $R+1$ continuous and bounded derivatives on $[\underline{s}, \overline{b}]$. Note that

$$g_B(b) = \frac{f_V(\beta_B^{-1}(b))}{\beta_B'(\beta_B^{-1}(b))}, \quad g_S(s) = \frac{f_C(\beta_S^{-1}(s))}{\beta_S'(\beta_S^{-1}(s))}.$$

In addition, f_V and f_C are bounded away from 0 by Assumption F, and β_B' and β_S' are bounded by (M2). The conclusion of part (i) then follows. Because $G_B(b) = F_V(\beta_B^{-1}(b)) = F_V(\xi(b, G_S))$ for $b \in [\underline{s}, \overline{b}]$, the result about G_B in part (ii) follows from that both $F_V(\cdot)$ and $\xi(\cdot, G_S)$ have $R+1$ continuous and bounded derivatives on $[\underline{s}, \overline{b}]$. The result about G_S in part (ii) can be proven similarly. Lastly, to prove part (iii), we note that (1.5) and (1.6) give

$$g_S(s) = k\frac{G_S(s)}{\xi(s, G_S) - s}, \quad g_B(b) = (1-k)\frac{1 - G_B(b)}{b - \eta(b, G_B)}.$$

Since every term on the right-hand side admits up to $R+1$ continuous and bounded derivatives, the desired result follows. □

A.9 Proof of Lemma 1.4

We will first show part (ii), and then show part (i). For part (ii), we shall show the convergence rate of $\sup_{b \in \mathscr{C}_g} |\hat{g}_B(b) - g_B(b)|$, and the other conclusion can be proven analogously.

Note $\hat{\underline{s}} \geqslant \underline{s}$, $\hat{\overline{b}} \leqslant \overline{b}$ and as $n \to \infty$, $\hat{\underline{s}} \xrightarrow{p} \underline{s}$ and $\hat{\overline{b}} \xrightarrow{p} \overline{b}$. Given $\lim_{n\to\infty} h_B = 0$ by Assumption H1, for sufficiently large n, $\mathscr{C}_g \subset [\hat{\underline{s}} + h_B, \hat{\overline{b}} - h_B]$ and therefore the boundary-corrected kernel density estimator $\hat{g}_B$ will be numerically identical to the standard kernel density estimator $\tilde{g}_B$. Thus, using the existing results for the standard kernel density estimator (see Li and Racine (2006), page 31, Theorem 1.4), we have under Assumptions E to G and Assumption H1,

$$\sup_{b \in \mathscr{C}_g} |\hat{g}_B(b) - g_B(b)| = O_p\left(h_B^{R+1} + \sqrt{\frac{\log n}{nh_B}}\right) = O_p\left(\left(\frac{\log n}{n}\right)^{\frac{R+1}{2R+3}}\right).$$

For part (i), since $\hat{\underline{s}} \geqslant \underline{s}$, $\hat{\overline{b}} \leqslant \overline{b}$ and $|\hat{\underline{s}} - \underline{s}| = O_p(1/n)$, $|\hat{\overline{b}} - \overline{b}| = O_p(1/n)$, the estimation error of $\hat{\underline{s}}$ and $\hat{\overline{b}}$ is negligible. Therefore the uniform consistency result on $[\underline{s}, \overline{b}]$ directly follows from the following lemma about the uniform convergence rate of our boundary-corrected kernel density estimator.[2]

Lemma. *Suppose*

(i) $X_1, \dots, X_n$ *are independently and identically distributed as F with density f and support* $[\underline{x}, \overline{x}]$*;*

(ii) f *has r-th continuous bounded derivative on* $[a,b] \subseteq [\underline{x}, \overline{x}]$ $(r = 1, 2)$*;* $f(x) \geqslant c_0 > 0$ *for all* $x \in [a,b]$*;*

(iii) The kernel K is symmetric with support $[-1,1]$ *and has twice continuous bounded derivative on* $\mathbb{R}$*, and K is of order 2, i.e.* $\int_{-\infty}^{\infty} K(u)\,\mathrm{d}u = 1$, $\int_{-\infty}^{\infty} uK(u)\,\mathrm{d}u = 0$, $\int_{-\infty}^{\infty} u^2 K(u)\,\mathrm{d}u = \kappa < \infty$*;*

(iv) h *satisfies* $0 < h < (b-a)/2$, $h \to 0$ *and* $nh/\log n \to \infty$ *as* $n \to \infty$*;*

(v) h' *satisfies* $h' = O(h)$ *and* $1/\sqrt{nh'^3} = O(h)$ *as* $n \to \infty$*;* $A > 1/3$.

Let $\hat{f}$ *be the boundary-corrected kernel density estimator on interval* $[a,b] \subseteq [\underline{x}, \overline{x}]$ *as defined in* (1.19)*, then*

$$\sup_{x \in [a,b]} \left| \hat{f}(x) - f(x) \right| = O_p\left(h^r + \sqrt{\frac{\log n}{nh}} \right).$$

Although g_B (or g_S) is discontinuous at $\underline{s}$ (or $\overline{b}$), we can similarly use the boundary-corrected density kernel estimator to estimated g_B (or g_S) on interval $[\underline{b}, \underline{s}]$ (or interval $[\overline{b}, \overline{s}]$) and with the same argument we can get that $\hat{g}_B$ (or $\hat{g}_S$) converges to the true density at the same rate as on interval $[\underline{s}, \overline{b}]$, then the desired uniform consistency results on the whole support of g_B or g_S follow. □

A.10 Proof of Lemma 1.5

We first show part (ii). For part (ii), we shall show the convergence rate of $\sup_i \mathbb{1}(V_i \in \mathscr{C}_V)|\hat{V}_i - V_i|$. The other conclusion can be proven analogously.

Define $\mathscr{C}_B = \{b \in [\underline{s}, \overline{b}] \,|\, \xi(b, G_S) \in \mathscr{C}_V\}$. Because $\xi(\cdot, G_S)$ is a strictly increasing continuous function and $\mathscr{C}_V$ is a closed inner subset of $[\underline{s}, \overline{v}]$, then $\mathscr{C}_B$ is also a (fixed) closed inner subset of $[\underline{s}, \overline{b}]$. Hence, it follows from the definition of $\xi(b, G_S)$ and (1.20) that

$$\mathbb{1}(V_i \in \mathscr{C}_V)|\hat{V}_i - V_i| = \mathbb{1}(B_i \in \mathscr{C}_B) \cdot k \left| \frac{\hat{G}_S(B_i)}{\hat{g}_S(B_i)} - \frac{G_S(B_i)}{g_S(B_i)} \right|$$

[2] The proof of this lemma for the case of $r = 2$ is in Appendix B. The case of $r = 1$ can be shown similarly.

$$
\begin{aligned}
&= \mathbb{1}(B_i \in \mathscr{C}_B) k \left| \frac{\hat{G}_S(B_i) - G_S(B_i)}{g_S(B_i)} - \frac{G_S(B_i)}{g_S(B_i)^2} \left[\hat{g}_S(B_i) - g_S(B_i) \right] \right. \\
&\qquad \left. + o\left(\hat{G}_S(B_i) - G_S(B_i)\right) + o\left(\hat{g}_S(B_i) - g_S(B_i)\right) \right| \\
&\leqslant \mathbb{1}(B_i \in \mathscr{C}_B) \left\{ \frac{|\hat{G}_S(B_i) - G_S(B_i)|}{g_S(B_i)} + \frac{G_S(B_i)}{g_S(B_i)^2} |\hat{g}_S(B_i) - g_S(B_i)| \right. \\
&\qquad \left. + o\left(|\hat{G}_S(B_i) - G_S(B_i)|\right) + o\left(|\hat{g}_S(B_i) - g_S(B_i)|\right) \right\} \\
&\leqslant \sup_{B_i \in \mathscr{C}_B} \left\{ \frac{|\hat{G}_S(B_i) - G_S(B_i)|}{g_S(B_i)} + \frac{G_S(B_i)}{g_S(B_i)^2} |\hat{g}_S(B_i) - g_S(B_i)| \right. \\
&\qquad \left. + o\left(|\hat{G}_S(B_i) - G_S(B_i)|\right) + o\left(|\hat{g}_S(B_i) - g_S(B_i)|\right) \right\} \\
&\leqslant \frac{\sup_{b \in \mathscr{C}_B} |\hat{G}_S(b) - G_S(b)|}{\alpha_S} + \frac{1}{\alpha_S^2} \sup_{b \in \mathscr{C}_B} |\hat{g}_S(b) - g_S(b)| \\
&\qquad + o\left(\sup_{b \in \mathscr{C}_B} |\hat{G}_S(b) - G_S(b)| \right) + o\left(\sup_{b \in \mathscr{C}_B} |\hat{g}_S(b) - g_S(b)| \right).
\end{aligned}
$$

where the last inequality holds since, for any b, $g_S(b) \geqslant \alpha_S$ and $G_S(b) \leqslant 1$. Then,

$$
\begin{aligned}
\sup_i \mathbb{1}(V_i \in \mathscr{C}_V) |\hat{V}_i - V_i| &\leqslant \frac{\sup_{b \in \mathscr{C}_B} |\hat{G}_S(b) - G_S(b)|}{\alpha_S} + \frac{1}{\alpha_S^2} \sup_{b \in \mathscr{C}_B} |\hat{g}_S(b) - g_S(b)| \\
&\qquad + o\left(\sup_{b \in \mathscr{C}_B} |\hat{G}_S(b) - G_S(b)| \right) + o\left(\sup_{b \in \mathscr{C}_B} |\hat{g}_S(b) - g_S(b)| \right).
\end{aligned}
$$

Since $\sup_{b \in \mathscr{C}_B} |\hat{G}_S(b) - G_S(b)| \leqslant \sup_{b \in \mathbb{R}} |\hat{G}_S(b) - G_S(b)| = O_p(1/\sqrt{n})$, the desired result follows from Lemma 1.4-(ii) and $O_p\left(\max\left(1/\sqrt{n}, (\log n/n)^{(R+1)/(2R+3)}\right)\right) = O_p\left((\log n/n)^{(R+1)/(2R+3)}\right)$.

For part (i), by similar argument, we have

$$
\begin{aligned}
\sup_i \mathbb{1}(V_i \in [\underline{s}, \overline{v}]) |\hat{V}_i - V_i| &\leqslant \frac{\sup_{b \in [\underline{s}, \overline{b}]} |\hat{G}_S(b) - G_S(b)|}{\alpha_S} + \frac{1}{\alpha_S^2} \sup_{b \in [\underline{s}, \overline{b}]} |\hat{g}_S(b) - g_S(b)| \\
&\qquad + o\left(\sup_{b \in [\underline{s}, \overline{b}]} |\hat{G}_S(b) - G_S(b)| \right) + o\left(\sup_{b \in [\underline{s}, \overline{b}]} |\hat{g}_S(b) - g_S(b)| \right)
\end{aligned}
$$

Then it follows from Lemma 1.4-(i) that

$$
\sup_i \mathbb{1}(V_i \in [\underline{s}, \overline{v}]) |\hat{V}_i - V_i| = O_p\left(\left(\frac{\log n}{n} \right)^{\frac{2}{5}} \right). \tag{A.25}
$$

Since by regular equilibrium assumption, the buyer with private value $v < \underline{s}$ will bid $b = v$ and hence $\hat{V}_i = B_i = V_i$. Then we can extend the result in (A.25) to all $V_i \in [\underline{v}, \overline{v}]$ so that

$$\sup_i |\hat{V}_i - V_i| = \sup_i \mathbb{1}(V_i \in [\underline{s}, \overline{v}])|\hat{V}_i - V_i| = O_p\left(\left(\frac{\log n}{n}\right)^{\frac{2}{5}}\right).$$

The result for $\sup_i |\hat{C}_i - C_i|$ can be shown analogously. □

A.11 Proof of Theorem 1.5

We shall show the uniform consistency of $|\hat{f}_V(v) - f_V(v)|$, the other conclusion can be proven analogously.

First, we consider the case that $\mathscr{C}_V$ is a closed inner subset of $[\underline{s}, \overline{v}]$. Let $\tilde{f}_V(v)$ define the (infeasible) one-step boundary-corrected kernel density estimator which uses the unobserved true private values V_i instead of $\hat{V}_i$. Applying similar argument to show Lemma 1.4, we can show that $\sup_{v \in \mathscr{C}_V} |\tilde{f}_V(v) - f_V(v)| = O_p\left((\log n/n)^{R/(2R+3)}\right)$ given non-optimal bandwidth $h_V = \lambda_V (\log n/n)^{1/(2R+3)}$. Since $\hat{f}_V(v) - f_V(v) = [\hat{f}_V(v) - \tilde{f}_V(v)] + [\tilde{f}_V(v) - f_V(v)]$, we are left with the first term.

Let $\mathscr{C}'_V = \bigcup_{v \in \mathscr{C}_V} [v - \Delta, v + \Delta]$ and $\mathscr{C}''_V = \bigcup_{v \in \mathscr{C}'_V} [v - \Delta, v + \Delta]$ for some $\Delta > 0$. By construction, $\mathscr{C}'_V$ and $\mathscr{C}''_V$ are also closed, and $\mathscr{C}_V \subset \mathscr{C}'_V \subset \mathscr{C}''_V$. Since $\mathscr{C}_V$ is a closed inner subset of $[\underline{s}, \overline{v}]$, Δ can be chosen small enough such that $\mathscr{C}''_V \subset [\underline{s}, \overline{v}]$. Now by Lemma 1.5, for $v \in \mathscr{C}_V$ and n large enough, $\hat{f}_V(v)$ uses at most observations $\hat{V}_i$ in $\mathscr{C}'_V$ and for which V_i is in $\mathscr{C}''_V$. Because for any $v \in \mathscr{C}_V$, $\tilde{f}_V(v)$ uses at most V_i in $\mathscr{C}''_V$ and both $\hat{f}_V(v)$ and $\tilde{f}_V(v)$ are numerically identical to the standard kernel density estimator, we obtain

$$\hat{f}_V(v) - \tilde{f}_V(v) = \frac{1}{nh_V} \sum_{i=1}^{n} \mathbb{1}(V_i \in \mathscr{C}''_V) \left[K_V\left(\frac{v - \hat{V}_i}{h_V}\right) - K_V\left(\frac{v - V_i}{h_V}\right)\right].$$

A second-order Taylor expansion gives

$$\left|\hat{f}_V(v) - \tilde{f}_V(v)\right| = \left|\frac{1}{nh_V} \sum_{i=1}^{n} \left[\mathbb{1}(V_i \in \mathscr{C}''_V)(\hat{V}_i - V_i) \cdot \frac{1}{h_V} K'_V\left(\frac{v - V_i}{h_V}\right)\right] + \frac{1}{2nh_V} \sum_{i=1}^{n} \left[\mathbb{1}(V_i \in \mathscr{C}''_V)(\hat{V}_i - V_i)^2 \cdot \frac{1}{h_V^2} K''_V\left(\frac{v - \tilde{V}_i}{h_V}\right)\right]\right|$$

where $\tilde{V}_i$ is some point between $\hat{V}_i$ and V_i. By triangular inequality,

$$\left|\hat{f}_V(v)-\tilde{f}_V(v)\right| \leqslant \frac{1}{nh_V^2}\sum_{i=1}^n \mathbb{1}(V_i\in\mathscr{C}_V'')\left|\hat{V}_i-V_i\right|\cdot\left|K_V'\left(\frac{v-V_i}{h_V}\right)\right| + \frac{1}{2nh_V^3}\sum_{i=1}^n \mathbb{1}(V_i\in\mathscr{C}_V'')\left(\hat{V}_i-V_i\right)^2\cdot\left|K_V''\left(\frac{v-\tilde{V}_i}{h_V}\right)\right|. \quad \text{(A.26)}$$

Because $\left|K_V''\left(\frac{v-\tilde{V}_i}{h_V}\right)\right| \leqslant \sup_u |K_V''(u)|$, then the right-hand side of (A.26) is bounded by

$$\frac{1}{h_V}\sup_i \mathbb{1}(V_i\in\mathscr{C}_V'')\left|\hat{V}_i-V_i\right|\cdot\frac{1}{nh_V}\sum_{i=1}^n\left|K_V'\left(\frac{v-V_i}{h_V}\right)\right| + \frac{1}{2h_V^3}\sup_i \mathbb{1}(V_i\in\mathscr{C}_V'')\left|\hat{V}_i-V_i\right|^2\cdot\sup_u|K_V''(u)|.$$

By Lemma 1.5-(ii) and Assumption H1,

$$\left|\hat{f}_V(v)-\tilde{f}_V(v)\right| \leqslant O_p\left(\left(\frac{\log n}{n}\right)^{\frac{R}{2R+3}}\right)\cdot\frac{1}{nh_V}\sum_{i=1}^n\left|K_V'\left(\frac{v-V_i}{h_V}\right)\right| + O_p\left(\left(\frac{\log n}{n}\right)^{\frac{2R-1}{2R+3}}\right)\cdot\sup_u|K_V''(u)|. \quad \text{(A.27)}$$

It can be shown that $\frac{1}{nh_V}\sum_{i=1}^n\left|K_V'\left(\frac{v-V_i}{h_V}\right)\right|$ converges uniformly to $f_V(v)\int_{-\infty}^{\infty}|K_V'(u)|\,\mathrm{d}u$ thus it is bounded uniformly. Moreover, $\sup_u|K_V''(u)|<\infty$ by Assumption G. Since $R\geqslant 1$ implies $\frac{2R-1}{2R+3}\geqslant\frac{R}{2R+3}$, it follows that $\sup_{v\in\mathscr{C}_V}|\hat{f}_V(v)-\tilde{f}_V(v)| = O_p\left((\log n/n)^{R/(2R+3)}\right)$ and therefore $\sup_{v\in\mathscr{C}_V}|\hat{f}_V(v)-f_V(v)| = O_p\left((\log n/n)^{R/(2R+3)}\right)$.

Now we consider the other case that $\mathscr{C}_V$ is a closed inner subset of $[\underline{v},\underline{s}]$ when $\underline{s}>\underline{v}$. By regular equilibrium assumption, the buyer with private value $v<\underline{s}$ will bid $b=v$, thus we have $\hat{V}_i=B_i=V_i$. Thus $\hat{f}_V$ is in fact the one-step boundary-corrected kernel estimator for f_V on $[\underline{v},\underline{s}]$. Same argument gives $\sup_{v\in\mathscr{C}_V}|\hat{f}_V(v)-f_V(v)| = O_p\left((\log n/n)^{R/(2R+3)}\right)$.

Since any given closed inner subset $\mathscr{C}_V\subseteq[\underline{v},\overline{v}]\backslash\{\underline{s}\}$ is a union of at most two closed inner subsets respectively belonging to the two cases above, the final conclusion is proven. □

A.12 Proof of Lemma 1.6

This conclusion is obtained by applying a similar argument to that in Theorem 1.5 where we show the uniform convergence rate of $\hat{f}_V(\cdot)$ (or $\hat{f}_C(\cdot)$) in the closed inner subset of $[\underline{s},\overline{v}]$ (or $[\underline{c},\overline{b}]$). However, we use part (i) (instead of part (ii)) of Lemma 1.5 here. □

A.13 Proof of Lemma A.1

First, note that when $k\in(0,1]$, we can rewrite (A.1) and (A.3) together as

$$\Psi_k(p) = \int_{\underline{s}}^{p}\int_{s}^{\min\left(\frac{p-(1-k)s}{k},\overline{b}\right)} g_2(b,s)\,\mathrm{d}b\,\mathrm{d}s. \quad \text{(A.28)}$$

Keep $p \in (\underline{s}, \overline{b})$ fixed and define a function φ as the inner integral in (A.28), i.e.

$$\varphi(k,s) = \int_{s}^{\min\left(\frac{p-(1-k)s}{k}, \overline{b}\right)} g_2(b,s)\,\mathrm{d}b, \quad k \in (0,1],\ s \in [\underline{s}, p]. \tag{A.29}$$

Since $g_2(b,s)$ is integrable, so φ is continuous in the upper limit of integral. And since the upper limit, $\min\left(\frac{p-(1-k)s}{k}, \overline{b}\right)$, is continuous in k, so φ is continuous in k. Note that $g_2(b,s) > 0$ because the interval of integration is in the support of G, and note that $\min\left(\frac{p-(1-k)s}{k}, \overline{b}\right) \leq \overline{b}$, thus for any $k \in (0,1]$,

$$0 \leq \varphi(k,s) \leq \int_{s}^{\overline{b}} g_2(b,s)\,\mathrm{d}b \equiv \bar{\varphi}(s), \quad \forall s \in [\underline{s}, p].$$

Therefore, for any $k \in (0,1]$, for any sequence $\{k_n\}$ in $(0,1]$ such that $k_n \to k$ as $n \to \infty$, by continuity of φ in k, we have $\tilde{\varphi}_n(s) \equiv \varphi(k_n, s)$ converges pointwise to $\tilde{\varphi}(s) \equiv \varphi(k,s)$ on $[\underline{s}, p]$. Since $\bar{\varphi}(s)$ is integrable, by dominated convergence theorem, as $n \to \infty$,

$$\int_{\underline{s}}^{p} \tilde{\varphi}_n(s)\,\mathrm{d}s \to \int_{\underline{s}}^{p} \tilde{\varphi}(s)\,\mathrm{d}s,$$

hence, $\Psi_{k_n}(p) \to \Psi_k(p)$.

To see the (right) continuity at $k = 0$, we just need to rewrite (A.1) and (A.2) as

$$\Psi_k(p) = 1 - \int_{p}^{\overline{b}} \int_{\frac{p-kb}{1-k}}^{b} g_2(b,s)\,\mathrm{d}s\,\mathrm{d}b, \quad 0 \leq k < \frac{p-\underline{s}}{\overline{b}-\underline{s}}$$

and define

$$\psi(k,b) = -\int_{\frac{p-kb}{1-k}}^{b} g(b,s)\,\mathrm{d}s, \quad k \in \left[0, \frac{p-\underline{s}}{\overline{b}-\underline{s}}\right),\ b \in [p, \overline{b}].$$

Then applying analogous argument, we have ψ is continuous in k so that for sequence $\{k_n\}$ in $\left[0, \frac{p-\underline{s}}{\overline{b}-\underline{s}}\right)$ such that $k_n \to 0$, the sequence $\{\tilde{\psi}_n(b) \equiv \psi(k_n, b)\}$ converges pointwise to $\tilde{\psi}(b) \equiv \psi(0,b)$. Since $\{\tilde{\psi}_n(b)\}$ is dominated by $\bar{\psi}(b) \equiv \int_{\underline{s}}^{b} g(b,s)\,\mathrm{d}s$, we finally can get $\Psi_{k_n}(p) \to \Psi_0(p)$.

It remains to show the monotonicity of $\Psi_k(p)$ in k. Suppose $0 \leq k_1 < k_2 \leq 1$, then by (A.1), (A.2), and (A.3):

(i) If $k_2 < \frac{p-\underline{s}}{\overline{b}-\underline{s}}$, then

$$\Psi_{k_1}(p) - \Psi_{k_2}(p) = \int_{p}^{\overline{b}} \int_{b-\frac{b-p}{1-k_2}}^{b-\frac{b-p}{1-k_1}} g_2(b,s)\,\mathrm{d}s\,\mathrm{d}b > 0$$

due to $\frac{b-p}{1-k_2} > \frac{b-p}{1-k_1}$.

(ii) If $k_1 \geq \frac{p-\underline{s}}{\overline{b}-\underline{s}}$, then

$$\Psi_{k_1}(p) - \Psi_{k_2}(p) = \int_{\underline{s}}^{p} \int_{s+\frac{p-s}{k_2}}^{s+\frac{p-s}{k_1}} g_2(b,s)\,\mathrm{d}b\,\mathrm{d}s > 0$$

due to $\frac{p-s}{k_2} < \frac{p-s}{k_1}$.

(iii) If $k_1 < \frac{p-\underline{s}}{\overline{b}-\underline{s}} \leq k_2$, then

$$\Psi_{k_1}(p) - \Psi_{k_2}(p) = \int_{\underline{s}}^{p} \int_{s+\frac{p-s}{k_2}}^{s+\frac{(p-s)(\overline{b}-\underline{s})}{p-\underline{s}}} g_2(b,s)\,\mathrm{d}b\,\mathrm{d}s + \int_{p}^{\overline{b}} \int_{b-\frac{(b-p)(\overline{b}-\underline{s})}{\overline{b}-p}}^{b-\frac{b-p}{1-k_1}} g_2(b,s)\,\mathrm{d}s\,\mathrm{d}b > 0,$$

where the first term is non-negative and the second one is positive. □

Appendix B

Uniform Consistency of Boundary Corrected Kernel Density Estimator

Let f denote a probability density function with support $\mathscr{X}$, and consider a random sample $X_1, \ldots, X_n \in \mathbb{R}$ drawn from f. The standard kernel estimator of f at $x \in \mathscr{X}$ is given by

$$f_n(x) = \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{x - X_i}{h}\right), \tag{B.1}$$

where h is the bandwidth and K is a kernel function that satisfies $\int_{-\infty}^{\infty} K(u)\,\mathrm{d}u = 1$, $\int_{-\infty}^{\infty} uK(u)\,\mathrm{d}u = 0$ and $\int_{-\infty}^{\infty} u^2 K(u)\,\mathrm{d}u < \infty$. When the support $\mathscr{X}$ is bounded then the standard kernel density estimator (B.1) is not necessarily consistent at those points near the boundary of the support unless $f(x) = 0$. This phenomenon is known as the "boundary effect" of the standard kernel estimator. If the density f is discontinuous at some points in the interior of the support, similar problem can also happen around these points of discontinuity and the consistency of standard kernel estimator will break down, also.

It has already been extensively studied in literature how to correct this boundary effect. Some well-known methods include the simple reflection method (Cline and Hart, 1991, Schuster, 1985), the boundary kernel method (Gasser and Müller, 1979, Gasser, Müller, and Mammitzsch, 1985, Müller, 1991), the local linear method (Cheng, 1994, Cheng, Fan, and Marron, 1997), the transformation method (Marron and Ruppert, 1994, Wand, Marron, and Ruppert, 1991), and the pseudodata method (Cowling and Hall, 1996). Zhang, Karunamuni, and Jones (1999, hereforth ZKJ for short) proposed an improved boundary corrected density estimator, which is a combination of the pseudodata, transformation, and reflection methods. Their method consists of three basic steps: first, use some function g to transform the original data $X_1, \ldots, X_n$ to a new set of observations while keeping the original data; second, reflect those transformed data around the boundary of the support; finally, estimate the density function in the same way as standard kernel estimator (B.1) but based on the enlarged data sample including the original data as well as the transformed and reflected data. I will focus on the ZKJ method because this method has several advantages compared with other boundary correction approaches. Whereas the simple reflection has bad bias and those approaches involving only kernel modifications such as the boundary kernel-related methods are usually associated with larger variance, the ZKJ estimator both controls the bias and keeps the variance relatively small. This makes the estimator present better performance for various shapes of densities. Meanwhile, unlike the boundary kernel method and the local linear method, the ZKJ method ensures that the estimated density function is nonnegative everywhere. Moreover, the ZKJ method only corrects the boundary bias in the region where the standard kernel estimator loses consistency and as a result the estimator reduces to the standard kernel estimator numerically in the interior region which is at least one bandwidth distance away from the boundary points. These features allow the ZKJ estimator to inherit all the good properties of the standard kernel estimator in the interior region and make computation easy.

Zhang, Karunamuni, and Jones (1999) and Karunamuni and Zhang (2008) give the pointwise asymptotic properties of the ZKJ estimator. However, in many applications such as estimating economic models, researchers are more interested in knowing the whole density function and this requires that the estimator converges to the true density uniformly over the inference region. To address this problem, this appendix chapter will establishes the uniform consistency of the ZKJ estimator by showing the uniform rate of convergence.

B.1 Boundary-corrected kernel density estimator

For a random sample $X_1, \ldots, X_n \in \mathbb{R}$ from an unknown density f with support $\mathscr{X}$, following Zhang, Karunamuni, and Jones (1999), I define the *boundary-corrected kernel density estimator on compact interval* $[a, b] \subseteq \mathscr{X}$ as

$$\tilde{f}(x) = \frac{1}{nh} \sum_{i=1}^{n} \mathbb{1}(a \leqslant X_i \leqslant b) \left[K\left(\frac{x - X_i}{h}\right) + K\left(\frac{x - a + g_1(X_i - a)}{h}\right) + K\left(\frac{b - x + g_2(b - X_i)}{h}\right) \right], \tag{B.2}$$

where

$$g_1(u) = u + d_1 u^2 + A_1 d_1^2 u^3, \quad g_2(u) = u + d_2 u^2 + A_2 d_2^2 u^3 \tag{B.3}$$

are transformation functions with

$$d_1 = \frac{f'(a)}{f(a)}, \quad d_2 = -\frac{f'(b)}{f(b)}.$$

Here the kernel function K, bandwidth h, and the constants $A_1, A_2 > \frac{1}{3}$ are all tuning parameters.

The density estimator (B.2) is slightly different from the one in Zhang, Karunamuni, and Jones (1999). My estimator essentially estimate the density on an interval by using only those observations lying in that interval and the boundary correction is implemented in form at both endpoints. I define the estimator in the form for the purpose of versatility. My estimator can handle, not only the standard problem of correcting boundary effect at the boundaries of support $\mathscr{X}$ by simply setting $[a, b] = \mathscr{X}$, but also the inconsistency issue raised by the discontinuity of density function. The latter is achieved by following similar approach suggested by Schuster (1985) and Cline and Hart (1991), in which I let one or both of a, b be the discontinuous point(s) of f so that f is continuous inside interval $[a, b]$.[1]

Note that g_1 and g_2 depend on the unknown density function then the kernel density estimator $\tilde{f}$ given in (B.2) is usually infeasible unless d_1 and d_2 are known. Thus, I define the feasible counterpart of $\tilde{f}$ as

$$\hat{f}(x) = \frac{1}{nh}\sum_{i=1}^{n}\mathbb{1}(a \leqslant X_i \leqslant b)\left[K\left(\frac{x - X_i}{h}\right) + K\left(\frac{x - a + \hat{g}_1(X_i - a)}{h}\right) + K\left(\frac{b - x + \hat{g}_2(b - X_i)}{h}\right)\right], \tag{B.4}$$

where $\hat{g}_1$ and $\hat{g}_2$ are obtained by replacing d_1 and d_2 in (B.3) with the following estimators

$$\begin{aligned}
\hat{d}_1 = \frac{1}{h_1}\Bigg\{\log\left[\frac{1}{nh_1}\sum_{i=1}^{n}\mathbb{1}(a \leqslant X_i \leqslant b)K\left(\frac{h_1 - X_i + a}{h_1}\right) + \frac{1}{n^2}\right] \\
- \log\left[\max\left(\frac{1}{nh_0}\sum_{i=1}^{n}\mathbb{1}(a \leqslant X_i \leqslant b)K_0\left(\frac{a - X_i}{h_0}\right), \frac{1}{n^2}\right)\right]\Bigg\}, \quad \text{(B.5)} \\
\hat{d}_2 = \frac{1}{h_1}\Bigg\{\log\left[\frac{1}{nh_1}\sum_{i=1}^{n}\mathbb{1}(a \leqslant X_i \leqslant b)K\left(\frac{h_1 + X_i - b}{h_1}\right) + \frac{1}{n^2}\right] \\
- \log\left[\max\left(\frac{1}{nh_0}\sum_{i=1}^{n}\mathbb{1}(a \leqslant X_i \leqslant b)K_0\left(\frac{X_i - b}{h_0}\right), \frac{1}{n^2}\right)\right]\Bigg\}, \quad \text{(B.6)}
\end{aligned}$$

where $K_0(u)$ is a so-called endpoint kernel satisfying

$$\int_{-1}^{0} K_0(u)\,\mathrm{d}u = 1, \quad \int_{-1}^{0} uK_0(u)\,\mathrm{d}u = 0, \quad \int_{-1}^{0} u^2K_0(u)\,\mathrm{d}u \neq 0,$$

and

$$h_0 = h_1 \cdot \left[\frac{\left(\int_{-1}^{1} u^2K(u)\,\mathrm{d}u\right)^2\left(\int_{-1}^{0} K_0^2(u)\,\mathrm{d}u\right)}{\left(\int_{-1}^{0} u^2K_0(u)\,\mathrm{d}u\right)^2\left(\int_{-1}^{1} K^2(u)\,\mathrm{d}u\right)}\right]^{1/5}. \tag{B.7}$$

[1]In fact, the results of this part keep valid even if f is smooth at a and b but the boundary correction is forcibly implemented. However, this is not recommended since it unnecessarily causes poor performance of the density estimates.

Specifically, I can set $K_0(u) = (6 + 18u + 12u^2) \cdot \mathbb{1}(-1 \leqslant u \leqslant 0)$ which is the optimal endpoint kernel in the sense of minimizing the MSE at the boundary point among all the kernel functions that change sign only once on the support $[-\infty, 0]$ and satisfy $\int_{-\infty}^{0} K(u)\,\mathrm{d}u = 1$ as well as $\int_{-\infty}^{0} uK(u)\,\mathrm{d}u = 0$ (see Zhang and Karunamuni, 1998). Following Karunamuni and Zhang (2008), I allow the bandwidth h_1 (secondary bandwidth) used in estimating d_1, d_2 to differ from the bandwidth h (primary bandwidth) used in estimating f. This will bring potential improvement of the estimator's performance when the uniform consistency is of interest.

B.2 Uniform consistency of the infeasible estimator

I start with a series of assumptions about the underlying data generating process and the choice of tuning parameter values.

Assumption Q. *X_i, $i = 1, 2, \ldots, n$ are independently and identically distributed as f.*

Assumption R. *f is twice continuously differentiable with bounded f'' on $[a, b]$. $f(x) \geqslant c_0 > 0$ for all $x \in [a, b]$.*

Assumption S. *The kernel K is symmetric with support $[-1, 1]$ and has twice continuous bounded derivative on $\mathbb{R}$. K is of order 2, i.e. $\int_{-\infty}^{\infty} K(u)\,\mathrm{d}u = 1$, $\int_{-\infty}^{\infty} uK(u)\,\mathrm{d}u = 0$, $\int_{-\infty}^{\infty} u^2K(u)\,\mathrm{d}u = \kappa < \infty$.*

Assumption T. *h satisfies $0 < h < (b-a)/2$. $h \to 0$ and $nh/\log n \to \infty$ as $n \to \infty$.*

Now I show the uniform consistency of the infeasible estimator (B.2) on $[a, b]$.

Lemma B.1. *Under Assumptions Q, R, S and T, when n is sufficiently large,*

$$\sup_{x \in [a,b]} \left| E\tilde{f}(x) - f(x) \right| = O(h^2). \tag{B.8}$$

Proof. See Appendix B.5.1. □

Lemma B.2. *Under Assumptions Q, R, S and T, when n is sufficiently large,*

$$\sup_{x \in [a,b]} \mathrm{Var}\left(\tilde{f}(x)\right) = O\left(\frac{1}{nh}\right). \tag{B.9}$$

Proof. See Appendix B.5.2. □

Theorem B.1. *Under Assumptions Q, R, S and T, as $n \to \infty$,*

$$\sup_{x \in [a,b]} |\tilde{f}(x) - f(x)| = O\left(h^2 + \sqrt{\frac{\log n}{nh}}\right) \quad \textit{almost surely.} \tag{B.10}$$

Proof. Lemmas 1 and 2 give the uniform convergence rates of the bias and the variance of estimator (B.2). Given those, the proof is similar to showing the uniform rate of convergence for standard kernel estimator on a compact interior subset[2] of the support (see, e.g. Li and Racine, 2006, section 1.12) and is therefore omitted here. □

B.3 Uniform consistency of the feasible estimator

Now I turn to discuss the uniform convergence rate of the feasible estimator (B.4). I denote the corresponding terms in (B.5) and (B.6) by

$$f_1^*(a+h_1) \equiv \frac{1}{nh_1}\sum_{i=1}^{n}\mathbb{1}(a \leqslant X_i \leqslant b)K\left(\frac{h_1 - X_i + a}{h_1}\right), \quad f_1^*(a) \equiv \frac{1}{nh_0}\sum_{i=1}^{n}\mathbb{1}(a \leqslant X_i \leqslant b)K_0\left(\frac{a - X_i}{h_0}\right),$$
$$f_2^*(b-h_1) \equiv \frac{1}{nh_1}\sum_{i=1}^{n}\mathbb{1}(a \leqslant X_i \leqslant b)K\left(\frac{h_1 + X_i - b}{h_1}\right), \quad f_2^*(b) \equiv \frac{1}{nh_0}\sum_{i=1}^{n}\mathbb{1}(a \leqslant X_i \leqslant b)K_0\left(\frac{X_i - b}{h_0}\right),$$

and

$$f_1(a+h_1) \equiv f_1^*(a+h_1) + \frac{1}{n^2}, \qquad f_1(a) \equiv \max\left(f_1^*(a), \frac{1}{n^2}\right),$$
$$f_2(b-h_1) \equiv f_1^*(b-h_1) + \frac{1}{n^2}, \qquad f_2(b) \equiv \max\left(f_2^*(b), \frac{1}{n^2}\right).$$

Here $f_1(\cdot)$, $f_1^*(\cdot)$, $f_2(\cdot)$ and $f_2^*(\cdot)$ can be seen as some kind of estimators of f at the corresponding points, and then $\hat{d}_1$ and $\hat{d}_2$ can be written as

$$\hat{d}_1 = \frac{\log f_1(a+h_1) - \log f_1(a)}{h_1}, \qquad \hat{d}_2 = \frac{\log f_2(b-h_1) - \log f_2(b)}{h_1}. \tag{B.11}$$

I first show some consistency results for f_1^*, f_2^*.

Lemma B.3. *Under Assumptions Q, R and S, if $h_1 \to 0$ and $nh_1 \to \infty$ as $n \to \infty$, then for sufficiently large n and any $1 \leqslant p \leqslant n$,*

$$E\left\{[f_1^*(a+h_1) - f(a+h_1)]^2 \mid X_p\right\} = O\left({h_1}^4 + \frac{1}{nh_1}\right), \tag{B.12}$$
$$E\left\{[f_1^*(a) - f(a)]^2 \mid X_p\right\} = O\left({h_1}^4 + \frac{1}{n^2h_1}\right), \tag{B.13}$$
$$E\left\{[f_2^*(b-h_1) - f(b-h_1)]^2 \mid X_p\right\} = O\left({h_1}^4 + \frac{1}{nh_1}\right), \tag{B.14}$$
$$E\left\{[f_2^*(b) - f(b)]^2 \mid X_p\right\} = O\left({h_1}^4 + \frac{1}{nh_1}\right). \tag{B.15}$$

[2]A set A is called an interior subset of set B if it is a subset of the interior of B.

Proof. See Appendix B.5.3. □

Because the factor $1/n^2$ in (B.5) and (B.6) is used to make f_1 and f_2 bounded away from zero, in fact, it does not affect the statistical properties of f_1^* and f_2^*. Thus, the following lemma directly follows Lemma B.3.

Lemma B.4. *Under Assumptions Q, R and S, if $h_1 \to 0$ and $nh_1 \to \infty$ as $n \to \infty$, then for sufficiently large n and any $1 \leqslant p \leqslant n$,*

$$E\left\{[f_1(a+h_1) - f(a+h_1)]^2 \mid X_p\right\} = O\left({h_1}^4 + \frac{1}{nh_1}\right), \tag{B.16}$$

$$E\left\{[f_1(a) - f(a)]^2 \mid X_p\right\} = O\left({h_1}^4 + \frac{1}{nh_1}\right), \tag{B.17}$$

$$E\left\{[f_2(b-h_1) - f(b-h_1)]^2 \mid X_p\right\} = O\left({h_1}^4 + \frac{1}{nh_1}\right), \tag{B.18}$$

$$E\left\{[f_2(b) - f(b)]^2 \mid X_p\right\} = O\left({h_1}^4 + \frac{1}{nh_1}\right). \tag{B.19}$$

Proof. See Appendix B.5.4. □

The main intermediate step in proving the uniform convergence rate of the feasible estimator is to establish the statistical properties of the estimator for d_1 and d_2.

Lemma B.5. *Under Assumptions Q, R and S, if $h_1 \to 0$ and $n{h_1}^3 \to \infty$ as $n \to \infty$, then for sufficiently large n and any $1 \leqslant p \leqslant n$,*

$$E\left[(\hat{d}_1 - d_1)^2 \,\middle|\, X_p\right] = O\left({h_1}^2 + \frac{1}{n{h_1}^3}\right), \quad E\left[(\hat{d}_2 - d_2)^2 \,\middle|\, X_p\right] = O\left({h_1}^2 + \frac{1}{n{h_1}^3}\right). \tag{B.20}$$

Proof. See Appendix B.5.5. □

Lemma B.5 implies the "optimal" secondary bandwidth h_1 in the sense of making the convergence rates given by Lemma B.5 as fast as possible, is of the form $h_1 = \lambda' n^{-1/5}$. Given that, $E[(\hat{d}_1 - d_1)^2 \mid X_p] = O(n^{-2/5})$ and $E[(\hat{d}_2 - d_2)^2 \mid X_p] = O(n^{-2/5})$. With the following additional assumption about the convergence rate of h_1 relative to the primary bandwidth h, I can show the feasible density estimator $\hat{f}$ has the same uniform convergence rate (in probability) as the infeasible estimator $\tilde{f}$.

Assumption U. *h_1 satisfies $h_1 = O(h)$ and $1/\sqrt{n{h_1}^3} = O(h)$ as $n \to \infty$.*

And the main result is the following theorem.

Theorem B.2. *Under Assumptions Q, R, S, T and U, when n is sufficiently large,*

$$\sup_{x\in[a,b]} \left|\hat{f}(x) - f(x)\right| = O_p\left(h^2 + \sqrt{\frac{\log n}{nh}}\right). \tag{B.21}$$

Proof. See Appendix B.5.6. □

Assumption U requires that as $n \to \infty$, the secondary bandwidth h_1 should converge to zero not slower than the primary bandwidth h but, meanwhile, it should not converge too fast compared to h. When h takes the form of $\lambda n^{-1/5}$, the only possible convergence rate for h_1 that satisfies Assumption U is $h_1 = \lambda' n^{-1/5}$, that is, both h and h_1 have the same convergence rate to zero. And if I take $h = o(n^{-1/5})$, then any sequence of h_1 will violate the condition of Assumption U. Therefore, to make the result in Theorem B.2 hold, Assumption U implicitly imposes some restrictions on the primary bandwidth that can be chosen.

It is worth to note that when the support $\mathscr{X}$ is a compact interval and $[a,b] = \mathscr{X}$, Theorem B.2 shows that after boundary correction, my estimator can achieve the same uniform rate of convergence on the entire support as what the standard kernel estimator gets on fixed compact interior subset of the support. Provided the assumption that $h = \lambda(\log n/n)^{-1/5}$, my estimator has the optimal uniform convergence rate of order $O_p\left((\log n/n)^{-2/5}\right)$.

B.4 Higher order bias correction

My results show that under almost the same assumptions, the boundary corrected kernel estimator can extend the existing uniform consistency result of the standard kernel estimator on a fixed compact interior subset of the support to the entire support. More specifically, I show that my boundary corrected estimator uniformly converges to the true density at the rate of $O_p(h^2 + \sqrt{\log n/n})$, and the optimal uniform rate of convergence is $O_p\left((\log n/n)^{-2/5}\right)$ achieved when the bandwidth is chosen in the form of $h = \lambda(\log n/n)^{-1/5}$.

In respect to removing the boundary bias that causes inconsistency, like many other boundary correction methods in literature, the ZKJ method aims to reduce the bias of the proposed estimator to the order of h^2, which yields a uniform rate of convergence in the order of $O_p(h^2 + \sqrt{\log n/n})$. It is appealing to explore whether the bias can be further reduced, for example, to the order of h^3 or even h^4, by using a similar boundary correction technique, if the underlying density function is smooth enough and it is allowed to choose kernel function freely. For ZKJ method, the key step of boundary correction is done by using a transformation function g to generated transformed data. So following the same idea, it is expected that the higher order bias can be removed if the transformation function is designed appropriately.

For simplicity, let me assume the support $\mathscr{X} = [0, \infty)$ so 0 is the only boundary point, and define the (infeasible) density estimator for the density function $f(x)$, which now is assumed to have up to r continuous bounded derivatives, as

$$\tilde{f}(x) = \frac{1}{nh} \sum_{i=1}^{n} \left[K\left(\frac{x - X_i}{h}\right) + K\left(\frac{x + g(X_i)}{h}\right) \right],$$

where $g(\cdot)$ is the transformation function. Then

$$E\tilde{f}(x) = \frac{1}{h} E\left[K\left(\frac{x - X_1}{h}\right) + K\left(\frac{x + g(X_1)}{h}\right) \right] = \frac{1}{h} EK\left(\frac{x - X_1}{h}\right) + \frac{1}{h} EK\left(\frac{x + g(X_1)}{h}\right).$$

For $x = \rho h$ with $\rho \in [0, 1]$: by changing variable and Taylor expansion, I have the first term

$$\begin{aligned} J_1 \equiv \frac{1}{h} EK\left(\frac{x - X_1}{h}\right) &= \frac{1}{h} \int_0^{+\infty} K\left(\frac{x - z}{h}\right) f(z)\,\mathrm{d}z = \int_{-\infty}^{\rho} K(t) f(x - th)\,\mathrm{d}t \\ &= \int_{-\infty}^{\rho} K(t) \left[f(0) + \sum_{j=1}^{r} \frac{(-h)^j (t - \rho)^j}{j!} f^{(j)}(0) + o\left(h^r (\rho - t)^r\right) \right] \mathrm{d}t \\ &= f(0) \int_{-\infty}^{\rho} K(t)\,\mathrm{d}t + \sum_{j=1}^{r} (-1)^j \frac{h^j}{j!} f^{(j)}(0) \int_{-\infty}^{\rho} (t - \rho)^j K(t)\,\mathrm{d}t + o(h^r); \end{aligned}$$

and provided that g is strictly increasing, I have the second term

$$J_2 \equiv \frac{1}{h} EK\left(\frac{x + g(X_1)}{h}\right) = \frac{1}{h} \int_0^{+\infty} K\left(\frac{x + g(z)}{h}\right) f(z)\,\mathrm{d}z = \int_{\rho}^{+\infty} K(t) \frac{f(g^{-1}(th - x))}{g'(g^{-1}(th - x))}\,\mathrm{d}t.$$

Define $w(\cdot) = f(g^{-1}(\cdot))/g'(g^{-1}(\cdot))$. Then by changing variable and Taylor expansion,

$$\begin{aligned} J_2 &= \int_{\rho}^{+\infty} K(t) \left[w(0) + \sum_{j=1}^{r} \frac{h^j (t - \rho)^j}{j!} w^{(j)}(0) + o\left(h^r (t - \rho)^r\right) \right] \mathrm{d}t \\ &= w(0) \int_{\rho}^{+\infty} K(t)\,\mathrm{d}t + \sum_{j=1}^{r} \frac{h^j}{j!} w^{(j)}(0) \int_{\rho}^{+\infty} (t - \rho)^j K(t)\,\mathrm{d}t + o(h^r). \end{aligned}$$

Note that

$$\begin{aligned} J_1 = f(0) &\left[\int_{-\infty}^{+\infty} K(t)\,\mathrm{d}t - \int_{\rho}^{+\infty} K(t)\,\mathrm{d}t \right] \\ &+ \sum_{j=1}^{r} (-1)^j \frac{h^j}{j!} f^{(j)}(0) \left[\int_{-\infty}^{+\infty} (t - \rho)^j K(t)\,\mathrm{d}t - \int_{\rho}^{+\infty} (t - \rho)^j K(t)\,\mathrm{d}t \right] + o(h^r) \end{aligned}$$

and $\int_{-\infty}^{+\infty} K(t)\,\mathrm{d}t = 1$, combining J_1 and J_2 yields

$$E\tilde{f}(x) = J_1 + J_2 = f(0) + \sum_{j=1}^{r}(-1)^j \frac{h^j}{j!} f^{(j)}(0) \int_{-\infty}^{+\infty} (t-\rho)^j K(t)\,\mathrm{d}t + [w(0) - f(0)] \int_{\rho}^{+\infty} K(t)\,\mathrm{d}t$$
$$+ \sum_{j=1}^{r} \frac{h^j}{j!} \left[w^{(j)}(0) - (-1)^j f^{(j)}(0) \right] \int_{\rho}^{+\infty} (t-\rho)^j K(t)\,\mathrm{d}t + o(h^r).$$

Using the Taylor expansion

$$f(x) = f(0) + \sum_{j=1}^{r} \frac{x^j}{j!} f^{(j)}(0) + o(x^r)$$
$$\Rightarrow f(0) = f(x) - \sum_{j=1}^{r} \frac{x^j}{j!} f^{(j)}(0) - o(x^r) = f(x) - \sum_{j=1}^{r} \frac{\rho^j h^j}{j!} f^{(j)}(0) + o(h^r),$$

I can get

$$\begin{aligned} E\tilde{f}(x) &= f(x) + \sum_{j=1}^{r} \frac{h^j}{j!} f^{(j)}(0) \left[\int_{-\infty}^{+\infty} (\rho - t)^j K(t)\,\mathrm{d}t - \rho^j \right] \\ &\quad + [w(0) - f(0)] \int_{\rho}^{+\infty} K(t)\,\mathrm{d}t + \sum_{j=1}^{r} \frac{h^j}{j!} \left[w^{(j)}(0) - (-1)^j f^{(j)}(0) \right] \int_{\rho}^{+\infty} (t-\rho)^j K(t)\,\mathrm{d}t + o(h^r) \\ &= f(x) + \sum_{j=1}^{r} \frac{h^j}{j!} f^{(j)}(0) \left[\sum_{l=1}^{j} \binom{j}{l} (-1)^l \rho^{j-l} \int_{-\infty}^{+\infty} t^l K(t)\,\mathrm{d}t + \rho^j \int_{-\infty}^{+\infty} K(t)\,\mathrm{d}t - \rho^j \right] \\ &\quad + [w(0) - f(0)] \int_{\rho}^{+\infty} K(t)\,\mathrm{d}t + \sum_{j=1}^{r} \frac{h^j}{j!} \left[w^{(j)}(0) - (-1)^j f^{(j)}(0) \right] \int_{\rho}^{+\infty} (t-\rho)^j K(t)\,\mathrm{d}t + o(h^r) \\ &= f(x) + [w(0) - f(0)] \int_{\rho}^{+\infty} K(t)\,\mathrm{d}t + \sum_{j=1}^{r} \frac{h^j}{j!} \left\{ f^{(j)}(0) \left[\sum_{l=1}^{j} \binom{j}{l} (-1)^l \rho^{j-l} \int_{-\infty}^{+\infty} t^l K(t)\,\mathrm{d}t \right] \right. \\ &\quad + \left. \left[w^{(j)}(0) - (-1)^j f^{(j)}(0) \right] \int_{\rho}^{+\infty} (t-\rho)^j K(t)\,\mathrm{d}t \right\} + o(h^r). \end{aligned}$$

Thus, the bias of the ZKJ estimator is, as (1.22), $E\tilde{f}(x) - f(x) = [w(0) - f(0)] \int_{\rho}^{+\infty} K(t)\,\mathrm{d}t + \sum_{j=1}^{r} \frac{W_j}{j!} h^j + o(h^r)$ where

$$W_j = f^{(j)}(0) \left[\sum_{l=1}^{j} \binom{j}{l} (-1)^l \rho^{j-l} \int_{-\infty}^{+\infty} t^l K(t)\,\mathrm{d}t \right] + \left[w^{(j)}(0) - (-1)^j f^{(j)}(0) \right] \int_{\rho}^{+\infty} (t-\rho)^j K(t)\,\mathrm{d}t.$$

According to this expression, the bias $E\tilde{f}(x) - f(x) = o(h^r)$ if $K(\cdot)$ is of order $(r+1)$ and one chooses function $g(\cdot)$ such that $w(0) = f(0)$ and $w^{(j)} = (-1)^j f^{(j)}(0)$ for $j = 1, 2, \ldots, r$.

B.5 Proofs

B.5.1 Proof of Lemma B.1

Since X_i, $i = 1, 2, \ldots, n$ are i.i.d. draws from f by Assumption Q, then

$$E\tilde{f}(x) = \frac{1}{h}\int_a^b K\left(\frac{x-z}{h}\right) f(z)\,\mathrm{d}z + \frac{1}{h}\int_a^b K\left(\frac{x-a+g_1(z-a)}{h}\right) f(z)\,\mathrm{d}z + \frac{1}{h}\int_a^b K\left(\frac{b-x+g_2(b-z)}{h}\right) f(z)\,\mathrm{d}z. \quad \text{(B.22)}$$

By changing variable and taking Taylor expansion with Lagrangian remainder, I have

$$\begin{aligned}\frac{1}{h}\int_a^b K\left(\frac{x-z}{h}\right) f(z)\,\mathrm{d}z &= \int_{(x-b)/h}^{(x-a)/h} K(t) f(x-ht)\,\mathrm{d}t \\ &= f(x)\int_{(x-b)/h}^{(x-a)/h} K(t)\,\mathrm{d}t - hf'(x)\int_{(x-b)/h}^{(x-a)/h} tK(t)\,\mathrm{d}t \\ &\quad + \frac{h^2}{2}\int_{(x-b)/h}^{(x-a)/h} t^2 K(t) f''(x-\lambda_{0t}ht)\,\mathrm{d}t, \end{aligned} \quad \text{(B.23)}$$

where $\lambda_{0t} \in [0,1]$ may depend on t. Using the properties that $g_1(0) = 0$, $g_1'(0) = 1$ and $g_1''(0) = 2f'(a)/f(a)$,[3] I have

$$\begin{aligned}\frac{1}{h}\int_a^b K\left(\frac{x-a+g_1(z-a)}{h}\right) f(z)\,\mathrm{d}z &= \int_{(x-a)/h}^{[x-a+g_1(b-a)]/h} K(t)\frac{f(a+g_1^{-1}(ht-x+a))}{g_1'(g_1^{-1}(ht-x+a))}\,\mathrm{d}t \\ = \int_{(x-a)/h}^{[x-a+g_1(b-a)]/h} K(t) &\left\{\frac{f(a+g_1^{-1}(0))}{g_1'(g_1^{-1}(0))}\right. \\ &+ h\left(t-\frac{x-a}{h}\right)\frac{f'(a+g_1^{-1}(0))g_1'(g_1^{-1}(0)) - f(a+g_1^{-1}(0))g_1''(g_1^{-1}(0))}{[g_1'(g_1^{-1}(0))]^3} \\ &\left. + \frac{h^2}{2}\left(t-\frac{x-a}{h}\right)^2 \mathcal{G}_1(\lambda_{1t}(ht-x+a))\right\}\mathrm{d}t \\ = [f(a)+(x-a)f'(a)]\int_{(x-a)/h}^{[x-a+g_1(b-a)]/h} K(t)\,\mathrm{d}t &- hf'(a)\int_{(x-a)/h}^{[x-a+g_1(b-a)]/h} tK(t)\,\mathrm{d}t \\ &+ \frac{h^2}{2}\int_{(x-a)/h}^{[x-a+g_1(b-a)]/h}\left(t-\frac{x-a}{h}\right)^2 K(t)\mathcal{G}_1(\lambda_{1t}(ht-x+a))\,\mathrm{d}t \end{aligned} \quad \text{(B.24)}$$

where $\lambda_{1t} \in [0,1]$ may depend on t and

$$\mathcal{G}_1(u) = \frac{f''(a+g_1^{-1}(u))g_1'(g_1^{-1}(u)) - f(a+g_1^{-1}(u))g_1'''(g_1^{-1}(u))}{[g_1'(g_1^{-1}(u))]^4}$$

[3] By construction, g_1 and g_2 are strictly increasing continuous functions satisfying $g_1(0) = g_2(0) = 0$, $g_1'(0) = g_2'(0) = 1$, $g_1''(0) = 2f'(a)/f(a)$ and $g_2''(0) = -2f'(b)/f(b)$.

$$- \frac{3g_1''(g_1^{-1}(u))[f'(a+g_1^{-1}(u))g_1'(g_1^{-1}(u)) - f(a+g_1^{-1}(u))g_1''(g_1^{-1}(u))]}{[g_1'(g_1^{-1}(u))]^5}.$$

Similarly, using the symmetry of K and the properties that $g_2(0) = 0$, $g_2'(0) = 1$ and $g_2''(0) = -2f'(b)/f(b)$, I have

$$\frac{1}{h}\int_a^b K\left(\frac{b-x+g_2(b-z)}{h}\right) f(z)\,\mathrm{d}z = [f(b)+(x-b)f'(b)]\int_{[x-b-g_2(b-a)]/h}^{(x-b)/h} K(t)\,\mathrm{d}t$$
$$- hf'(b)\int_{[x-b-g_2(b-a)]/h}^{(x-b)/h} tK(t)\,\mathrm{d}t + \frac{h^2}{2}\int_{[x-b-g_2(b-a)]/h}^{(x-b)/h}\left(t-\frac{x-b}{h}\right)^2 K(t)\mathcal{G}_2(\lambda_{2t}(-ht+x-b))\,\mathrm{d}t \tag{B.25}$$

where $\lambda_{2t} \in [0,1]$ may depend on t and

$$\mathcal{G}_2(u) = \frac{f''(b-g_2^{-1}(u))g_2'(g_2^{-1}(u)) - f(b-g_2^{-1}(u))g_2'''(g_2^{-1}(u))}{[g_2'(g_2^{-1}(u))]^4}$$
$$- \frac{3g_2''(g_2^{-1}(u))[f'(b-g_2^{-1}(u))g_2'(g_2^{-1}(u)) + f(b-g_2^{-1}(u))g_2''(g_2^{-1}(u))]}{[g_2'(g_2^{-1}(u))]^5}.$$

Using the fact that

$$\begin{aligned} f(x) &= f(a)+(x-a)f'(a)+(x-a)^2 f''(a+\gamma_1(x-a)) \\ &= f(b)+(x-b)f'(b)+(x-b)^2 f''(b+\gamma_2(x-b)), \\ f'(x) &= f'(a)+(x-a)f''(a+\theta_1(x-a)) \\ &= f'(b)+(x-b)f''(b+\theta_2(x-b)) \end{aligned}$$

for some $\gamma_1, \gamma_2, \theta_1, \theta_2 \in [0,1]$ and then combining (B.23), (B.24) and (B.25), I get

$$\begin{aligned} E\tilde{f}(x) = f(x)\int_{[x-b-g_2(b-a)]/h}^{[x-a+g_1(b-a)]/h} K(t)\,\mathrm{d}t - h\int_{[x-b-g_2(b-a)]/h}^{[x-a+g_1(b-a)]/h} tK(t)\,\mathrm{d}t \\ + \frac{h^2}{2}\left[\int_{(x-b)/h}^{(x-a)/h} t^2K(t)f''(x-\lambda_{0t}ht)\,\mathrm{d}t + \int_{(x-a)/h}^{[x-a+g_1(b-a)]/h} K(t)\mathcal{R}_1(x,t)\,\mathrm{d}t \right. \\ \left. + \int_{[x-b-g_2(b-a)]/h}^{(x-b)/h} K(t)\mathcal{R}_2(x,t)\,\mathrm{d}t\right], \end{aligned}$$

where

$$\begin{aligned} \mathcal{R}_1(x,t) = \left(t-\frac{x-a}{h}\right)^2 \mathcal{G}_1(\lambda_{1t}(ht-x+a)) \\ + \frac{2(x-a)t}{h}f''(a+\theta_1(x-a)) - 2\left(\frac{x-a}{h}\right)^2 f''(a+\gamma_1(x-a)), \end{aligned}$$

$$\mathcal{R}_2(x,t) = \left(t - \frac{x-b}{h}\right)^2 \mathcal{G}_2(\lambda_{2t}(-ht + x - b)) + \frac{2(x-b)t}{h} f''(b + \theta_2(x-b)) - 2\left(\frac{x-b}{h}\right)^2 f''(b + \gamma_2(x-b)).$$

By Assumption T, $h \to 0$ as $n \to \infty$, so when n is sufficiently large, $h \leqslant \min(g_1(b-a), g_2(b-a))$ then for any $x \in [a,b]$,

$$\int_{[x-b-g_2(b-a)]/h}^{[x-a+g_1(b-a)]/h} K(t)\,\mathrm{d}t = \int_{-1}^{1} K(t)\,\mathrm{d}t = 1, \quad \int_{[x-b-g_2(b-a)]/h}^{[x-a+g_1(b-a)]/h} tK(t)\,\mathrm{d}t = \int_{-1}^{1} tK(t) = 0,$$

since $x - a + g_1(b-a) \geqslant g_1(b-a) \geqslant h$ and $x - b - g_2(b-a) \leqslant -g_2(b-a) \leqslant -h$. Hence,

$$\begin{aligned}
\left|E\tilde{f}(x) - f(x)\right| &= \frac{h^2}{2}\left|\int_{(x-b)/h}^{(x-a)/h} t^2 K(t) f''(x - \lambda_{0t}ht)\,\mathrm{d}t \right. \\
&\qquad \left. + \int_{(x-a)/h}^{[x-a+g_1(b-a)]/h} K(t)\mathcal{R}_1(x,t)\,\mathrm{d}t + \int_{[x-b-g_2(b-a)]/h}^{(x-b)/h} K(t)\mathcal{R}_2(x,t)\,\mathrm{d}t\right| \\
&\leqslant \frac{h^2}{2}\left|\int_{(x-b)/h}^{(x-a)/h} t^2 K(t) f''(x - \lambda_{0t}ht)\,\mathrm{d}t\right| + \left|\frac{h^2}{2}\int_{(x-a)/h}^{[x-a+g_1(b-a)]/h} K(t)\mathcal{R}_1(x,t)\,\mathrm{d}t\right| \\
&\qquad + \left|\frac{h^2}{2}\int_{[x-b-g_2(b-a)]/h}^{(x-b)/h} K(t)\mathcal{R}_2(x,t)\,\mathrm{d}t\right|.
\end{aligned} \tag{B.26}$$

Now I shall show the three terms on the right-hand side of inequality (B.26) are uniformly bounded for all $x \in [a,b]$.

First, by Assumption R,

$$\overline{f} \equiv \sup_{z\in[a,b]} f(z) < \infty, \quad \overline{f'} \equiv \sup_{z\in[a,b]} |f'(z)| < \infty, \quad \overline{f''} \equiv \sup_{z\in[a,b]} |f''(z)| < \infty. \tag{B.27}$$

Thus

$$\left|\int_{(x-b)/h}^{(x-a)/h} t^2 K(t) f''(x - \lambda_{0t}ht)\,\mathrm{d}t\right| \leqslant \overline{f''}\int_{(x-b)/h}^{(x-a)/h} t^2 K(t)\,\mathrm{d}t \leqslant \overline{f''}\int_{-\infty}^{\infty} t^2 K(t)\,\mathrm{d}t = \kappa\overline{f''}. \tag{B.28}$$

Second, in the integral of (B.24), $\frac{x-a}{h} \leqslant t \leqslant \frac{x-a+g_1(b-a)}{h}$ and $0 \leqslant \lambda_{1t} \leqslant 1$, so $0 \leqslant \lambda_{1t}(ht - x + a) \leqslant g_1(b-a)$ and then $0 \leqslant g_1^{-1}(\lambda_{1t}(ht - x + a)) \leqslant b - a$ by the monotonicity of g_1^{-1}. Since $g_1(u) = u + d_1u^2 + A_1d_1^2u^3$, it is easy to see that

(i) $g_1'(u) \leqslant \max(1, 1 + 2d_1(b-a) + 3A_1d_1^2(b-a)^2)$ and

$$g_1'(u) = 1 + 2d_1u + 3A_1d_1^2u^2 \geqslant \frac{3A_1 - 1}{3A_1} > 0 \tag{B.29}$$

for any $0 \leqslant u \leqslant b - a$;

(ii) $|g_1''(u)| = |2d_1 + 6A_1 d_1^2 u| \leqslant |2d_1| + 6A_1 d_1^2 (b-a)$ for any $0 \leqslant u \leqslant b-a$; and

(iii) $g_1'''(u) = 6A_1 d_1^2 > 0$ is constant.

Define

$$\overline{g_1'} = \max(1, 1 + 2d_1(b-a) + 3A_1 d_1^2 (b-a)^2), \quad \overline{g_1''} = |2d_1| + 6A_1 d_1^2 (b-a), \quad \overline{g_1'''} = 6A_1 d_1^2,$$

then

$$\begin{aligned}
\sup_{0 \leqslant u \leqslant g_1(b-a)} |\mathcal{G}_1(u)| &\leqslant \sup_{0 \leqslant u \leqslant b-a} \left| \frac{f''(a+u) g_1'(u)}{[g_1'(u)]^4} \right| + \sup_{0 \leqslant u \leqslant b-a} \left| \frac{f(a+u) g_1'''(u)}{[g_1'(u)]^4} \right| \\
&\quad + 3 \sup_{0 \leqslant u \leqslant b-a} \left| \frac{f'(a+u) g_1'(u) g''(u)}{[g_1'(u)]^5} \right| + 3 \sup_{0 \leqslant u \leqslant b-a} \left| \frac{f(a+u) g_1''(u)^2}{[g_1'(u)]^5} \right| \\
&\leqslant \frac{\overline{f''} \cdot \overline{g_1'} + \overline{f} \cdot \overline{g_1'''}}{[1 - 1/(3A_1)]^4} + \frac{3\overline{f'} \cdot \overline{g_1'} \cdot \overline{g_1''} + 3\overline{f} \left(\overline{g_1''}\right)^2}{[1 - 1/(3A_1)]^5} \equiv \overline{\mathcal{G}_1} < \infty.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left| \frac{h^2}{2} \int_{(x-a)/h}^{[x-a+g_1(b-a)]/h} K(t) \mathcal{R}_1(x,t) \, \mathrm{d}t \right| = \left| \frac{h^2}{2} \int_{(x-a)/h}^{[x-a+g_1(b-a)]/h} t^2 K(t) \mathcal{G}_1(\lambda_{1t}(ht - x + a)) \, \mathrm{d}t \right. \\
&\qquad + \frac{h^2}{2} \int_{(x-a)/h}^{[x-a+g_1(b-a)]/h} \frac{2(x-a)}{h} t K(t) [f''(a + \theta_1(x-a)) - \mathcal{G}_1(\lambda_{1t}(ht - x + a))] \, \mathrm{d}t \\
&\qquad \left. + \frac{h^2}{2} \int_{(x-a)/h}^{[x-a+g_1(b-a)]/h} 2 \left(\frac{x-a}{h} \right)^2 K(t) [\mathcal{G}_1(\lambda_{1t}(ht - x + a)) - f''(a + \gamma_1(x-a))] \, \mathrm{d}t \right| \\
&\quad \leqslant \frac{h^2}{2} \int_{(x-a)/h}^{[x-a+g_1(b-a)]/h} t^2 K(t) \left| \mathcal{G}_1(\lambda_{1t}(ht - x + a)) \right| \, \mathrm{d}t \\
&\qquad + h \int_{(x-a)/h}^{[x-a+g_1(b-a)]/h} (x-a)|t| K(t) \left[\left| f''(a + \theta_1(x-a)) \right| + \left| \mathcal{G}_1(\lambda_{1t}(ht - x + a)) \right| \right] \, \mathrm{d}t \\
&\qquad + \int_{(x-a)/h}^{[x-a+g_1(b-a)]/h} (x-a)^2 K(t) \left[\left| \mathcal{G}_1(\lambda_{1t}(ht - x + a)) \right| + \left| f''(a + \gamma_1(x-a)) \right| \right] \, \mathrm{d}t \\
&\quad \leqslant \frac{h^2}{2} \overline{\mathcal{G}_1} \int_{-\infty}^{\infty} t^2 K(t) \, \mathrm{d}t + h(\overline{\mathcal{G}_1} + \overline{f''}) \int_{(x-a)/h}^{[x-a+g_1(b-a)]/h} (x-a)|t| K(t) \, \mathrm{d}t \\
&\qquad + (\overline{\mathcal{G}_1} + \overline{f''}) \int_{(x-a)/h}^{[x-a+g_1(b-a)]/h} (x-a)^2 K(t) \, \mathrm{d}t \\
&\quad = \frac{h^2}{2} \kappa \overline{\mathcal{G}_1} + h(\overline{\mathcal{G}_1} + \overline{f''}) \int_{(x-a)/h}^{[x-a+g_1(b-a)]/h} (x-a)|t| K(t) \, \mathrm{d}t + (\overline{\mathcal{G}_1} + \overline{f''}) \int_{(x-a)/h}^{[x-a+g_1(b-a)]/h} (x-a)^2 K(t) \, \mathrm{d}t
\end{aligned}$$

When $x > a + h$, $(x-a)/h > 1$ so $K(t) = 0$ on the entire interval of integration. Thus

$$\int_{(x-a)/h}^{[x-a+g_1(b-a)]/h} (x-a)|t| K(t) \, \mathrm{d}t = \int_{(x-a)/h}^{[x-a+g_1(b-a)]/h} (x-a)^2 K(t) \, \mathrm{d}t = 0.$$

When $a \leqslant x \leqslant a+h$,

$$\int_{(x-a)/h}^{[x-a+g_1(b-a)]/h} (x-a)|t|K(t)\,\mathrm{d}t \leqslant h\int_{-\infty}^{\infty} |t|K(t)\,\mathrm{d}t \leqslant h\left[\int_{-\infty}^{\infty} |t|^2 K(t)\,\mathrm{d}t\right]^{1/2} = h\sqrt{\kappa},$$

where the last inequality follows from the fact that $K(t)$ is indeed a probability density function and the Liapounov's inequality, and,

$$\int_{(x-a)/h}^{[x-a+g_1(b-a)]/h} (x-a)^2 K(t)\,\mathrm{d}t \leqslant h^2 \int_{-\infty}^{\infty} K(t)\,\mathrm{d}t = h^2.$$

Combining these two cases, I have

$$\left|\frac{h^2}{2}\int_{(x-a)/h}^{[x-a+g_1(b-a)]/h} K(t)\mathcal{R}_1(x,t)\,\mathrm{d}t\right| \leqslant h^2\left[\frac{1}{2}\kappa\overline{\mathcal{G}_1} + \left(\overline{\mathcal{G}_1} + \overline{f''}\right)\left(1+\sqrt{\kappa}\right)\right]. \tag{B.30}$$

Third, similarly, I have

$$\left|\frac{h^2}{2}\int_{[x-b-g_2(b-a)]/h}^{(x-b)/h} K(t)\mathcal{R}_2(x,t)\,\mathrm{d}t\right| \leqslant h^2\left[\frac{1}{2}\kappa\overline{\mathcal{G}_2} + \left(\overline{\mathcal{G}_2} + \overline{f''}\right)\left(1+\sqrt{\kappa}\right)\right]. \tag{B.31}$$

where

$$\overline{\mathcal{G}_2} \equiv \frac{\overline{f''}\cdot\overline{g_2'} + \overline{f}\cdot\overline{g_2'''}}{[1-1/(3A_2)]^4} + \frac{3\overline{f'}\cdot\overline{g_2'}\cdot\overline{g_2''} + 3\overline{f}\left(\overline{g_2''}\right)^2}{[1-1/(3A_2)]^5} < \infty$$

with

$$\overline{g_2'} = \max(1, 1+2d_2(b-a)+3A_2d_2^2(b-a)^2), \quad \overline{g_2''} = |2d_2| + 6A_2d_2^2(b-a), \quad \overline{g_2'''} = 6A_2d_2^2.$$

Finally, plugging (B.28), (B.30) and (B.31) into (B.26), I get

$$\left|E\tilde{f}(x) - f(x)\right| \leqslant h^2\left[\frac{\kappa}{2}\left(\overline{f''} + \overline{\mathcal{G}_1} + \overline{\mathcal{G}_2}\right) + (1+\sqrt{\kappa})\left(2\overline{f''} + \overline{\mathcal{G}_1} + \overline{\mathcal{G}_2}\right)\right]. \tag{B.32}$$

Note that the coefficient of h^2 on the right-hand side doesn't depend on x (but depends on a and b) and hence the right-hand side is a uniform upper bound, so the desired result follows. □

B.5.2 Proof of Lemma B.2

Since X_i, $i = 1, 2, \ldots, n$ are i.i.d. draws from f by Assumption Q, then

$$\begin{aligned}\mathrm{Var}\left(\tilde{f}(x)\right) = \frac{1}{n}\mathrm{Var}\Bigg\{&\frac{\mathbb{1}(a \leqslant X_1 \leqslant b)}{h}\left[K\left(\frac{x-X_1}{h}\right)\right.\\ &\left.+K\left(\frac{x-a+g_1(X_1-a)}{h}\right) + K\left(\frac{b-x+g_2(b-X_1)}{h}\right)\right]\Bigg\}\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{1}{n} E\left\{\frac{\mathbb{1}(a \leqslant X_1 \leqslant b)}{h^2}\left[K\left(\frac{x-X_1}{h}\right)\right.\right. \\
&\qquad\left.\left. +K\left(\frac{x-a+g_1(X_1-a)}{h}\right)+K\left(\frac{b-x+g_2(b-X_1)}{h}\right)\right]^2\right\} \\
&= \frac{1}{nh^2}\int_a^b K\left(\frac{x-z}{h}\right)^2 f(z)\,\mathrm{d}z + \frac{1}{nh^2}\int_a^b K\left(\frac{x-a+g_1(z-a)}{h}\right)^2 f(z)\,\mathrm{d}z \\
&\quad + \frac{1}{nh^2}\int_a^b K\left(\frac{b-x+g_2(b-z)}{h}\right)^2 f(z)\,\mathrm{d}z \\
&\quad + \frac{2}{nh^2}\int_a^b K\left(\frac{x-z}{h}\right) K\left(\frac{x-a+g_1(z-a)}{h}\right) f(z)\,\mathrm{d}z \\
&\quad + \frac{2}{nh^2}\int_a^b K\left(\frac{x-z}{h}\right) K\left(\frac{b-x+g_2(b-z)}{h}\right) f(z)\,\mathrm{d}z \\
&\quad + \frac{2}{nh^2}\int_a^b K\left(\frac{x-a+g_1(z-a)}{h}\right) K\left(\frac{b-x+g_2(b-z)}{h}\right) f(z)\,\mathrm{d}z, \qquad \text{(B.33)}
\end{aligned}
$$

where the inequality follows from the fact that $\mathrm{Var}(Z) = E(Z^2) - (EZ)^2 \leqslant E(Z^2)$ for any random variable Z.

Let $R_K \equiv \int_{-1}^{1} K(u)^2\,\mathrm{d}u$. By Assumption S, K is bounded and has finite support, so $R_K < \infty$. In (B.33),

$$
\frac{1}{nh^2}\int_a^b K\left(\frac{x-z}{h}\right)^2 f(z)\,\mathrm{d}z = \frac{1}{nh}\int_{(x-b)/h}^{(x-a)/h} K(t)^2 f(x-ht)\,\mathrm{d}t \leqslant \frac{\overline{f}}{nh}\int_{-\infty}^{\infty} K(t)^2\,\mathrm{d}t = \frac{R_K\overline{f}}{nh}. \qquad \text{(B.34)}
$$

where $\overline{f} = \sup_{z\in[\underline{x},\overline{x}]} f(z) < \infty$ as defined in (B.27). And

$$
\frac{1}{nh^2}\int_a^b K\left(\frac{x-a+g_1(z-a)}{h}\right)^2 f(z)\,\mathrm{d}z = \frac{1}{nh}\int_{(x-a)/h}^{[x-a+g_1(b-a)]/h} K(t)^2 \frac{f(a+g_1^{-1}(ht-x+a))}{g_1'(g_1^{-1}(ht-x+a))}\,\mathrm{d}t.
$$

Since $\frac{x-a}{h} \leqslant t \leqslant \frac{x-a+g_1(b-a)}{h}$, so $0 \leqslant ht-x+a \leqslant g_1(b-a)$ and then $0 \leqslant g_1^{-1}(ht-x+a) \leqslant b-a$ by monotonicity of g_1^{-1}. Then, using (B.29), I have

$$
\begin{aligned}
\frac{1}{nh^2}\int_a^b K\left(\frac{x-a+g_1(z-a)}{h}\right)^2 f(z)\,\mathrm{d}z &\leqslant \frac{1}{nh}\frac{\overline{f}}{1-1/(3A_1)}\int_{(x-a)/h}^{[x-a+g_1(b-a)]/h} K(t)^2\,\mathrm{d}t \\
&\leqslant \frac{1}{nh}\frac{\overline{f}}{1-1/(3A_1)}\int_{-\infty}^{\infty} K(t)^2\,\mathrm{d}t = \frac{1}{nh}\frac{R_K\overline{f}}{1-1/(3A_1)},
\end{aligned}
\qquad \text{(B.35)}
$$

Similarly,

$$
\frac{1}{nh^2}\int_a^b K\left(\frac{b-x+g_2(b-z)}{h}\right)^2 f(z)\,\mathrm{d}z \leqslant \frac{1}{nh}\frac{R_K\overline{f}}{1-1/(3A_2)}. \qquad \text{(B.36)}
$$

Let $\overline{K} \equiv \sup_{u\in[-1,1]} K(u)$ then by Assumption S I have $\overline{K} < \infty$. By similar changing variable procedure, I can get

$$\begin{aligned}\frac{2}{nh^2}\int_a^b K\left(\frac{x-z}{h}\right)K\left(\frac{x-a+g_1(z-a)}{h}\right)f(z)\,\mathrm{d}z \\ &= \frac{2}{nh}\int_{(x-b)/h}^{(x-a)/h} K(t)K\left(\frac{x-a+g_1(x-a-ht)}{h}\right)f(x-ht)\,\mathrm{d}t \\ &\leqslant \frac{2\overline{K}\cdot\overline{f}}{nh}\int_{-\infty}^{\infty}K(t)\,\mathrm{d}t = \frac{2\overline{K}\cdot\overline{f}}{nh},\end{aligned} \tag{B.37}$$

and similarly,

$$\frac{2}{nh^2}\int_a^b K\left(\frac{x-z}{h}\right)K\left(\frac{b-x+g_2(b-z)}{h}\right)f(z)\,\mathrm{d}z \leqslant \frac{2\overline{K}\cdot\overline{f}}{nh}. \tag{B.38}$$

By changing variable and applying similar argument to show (B.35), I can also get

$$\begin{aligned}&\frac{2}{nh^2}\int_a^b K\left(\frac{x-a+g_1(z-a)}{h}\right)K\left(\frac{b-x+g_2(b-z)}{h}\right)f(z)\,\mathrm{d}z \\ &= \frac{2}{nh}\int_{(x-a)/h}^{[x-a+g_1(b-a)]/h} K(t)K\left(\frac{b-x+g_2(b-a-g_1^{-1}(ht-x+a))}{h}\right)\frac{f(a+g_1^{-1}(ht-x+a))}{g_1'(g_1^{-1}(ht-x+a))}\,\mathrm{d}t \\ &\leqslant \frac{2}{nh}\frac{\overline{K}\cdot\overline{f}}{1-1/(3A_1)}\int_{-\infty}^{\infty}K(t)\,\mathrm{d}t = \frac{2}{nh}\frac{\overline{K}\cdot\overline{f}}{1-1/(3A_1)}.\end{aligned} \tag{B.39}$$

Plugging (B.34), (B.35), (B.36), (B.37), (B.38) and (B.39) into (B.33), I finally get

$$\operatorname{Var}\left(\tilde{f}(x)\right) \leqslant \frac{1}{nh}\left[R_K\overline{f}\left(1+\frac{3A_1}{3A_1-1}+\frac{3A_2}{3A_2-1}\right)+2\overline{K}\cdot\overline{f}\frac{9A_1-2}{3A_1-1}\right]. \tag{B.40}$$

Note that the coefficient of $(nh)^{-1}$ on the right-hand side doesn't depend on x (and even doesn't depend on a or b). Therefore the right-hand side is a uniform upper bound for the variance of $\tilde{f}(x)$ and then the desired result follows. □

B.5.3 Proof of Lemma B.3

I prove (B.12) first. Using the c_r inequality,

$$\begin{aligned}E\{[f_1^*(a+h_1)-f(a+h_1)]^2\mid X_p\} &= E\left\{[f_1^*(a+h_1)-Ef_1^*(a+h_1)+Ef_1^*(a+h_1)-f(a+h_1)]^2\mid X_p\right\} \\ &\leqslant 2E\left\{[f_1^*(a+h_1)-Ef_1^*(a+h_1)]^2\mid X_p\right\}+2[Ef_1^*(a+h_1)-f(a+h_1)]^2.\end{aligned} \tag{B.41}$$

Denote $U_i = \mathbb{1}(a\leqslant X_i\leqslant b)K\left(\frac{h_1-X_i+a}{h_1}\right)$, $i=1,2,\ldots,n$. Since X_i's are i.i.d., then so are U_i's. By definition of $f_1^*(a+h_1)$,

$$E\{[f_1^*(a+h_1)-Ef_1^*(a+h_1)]^2\mid X_p\}$$

$$
\begin{aligned}
&= \frac{1}{n^2{h_1}^2} E\left\{\left[\sum_{i=1}^{n} \mathbb{1}(a \leqslant X_i \leqslant b) K\left(\frac{h_1 - X_i + a}{h_1}\right) - \sum_{i=1}^{n} E\left(\mathbb{1}(a \leqslant X_i \leqslant b) K\left(\frac{h_1 - X_i + a}{h_1}\right)\right)\right]^2 \middle| X_p\right\} \\
&= \frac{1}{n^2{h_1}^2} E\left\{\left[(U_p - EU_p) + \left(\sum_{1\leqslant i\leqslant n,\ i\neq p} U_i - E\left(\sum_{1\leqslant i\leqslant n,\ i\neq p} U_i\right)\right)\right]^2 \middle| X_p\right\} \\
&\leqslant \frac{2}{n^2{h_1}^2} E\left[(U_p - EU_p)^2 \mid X_p\right] + \frac{2}{n^2{h_1}^2} E\left\{\left[\sum_{1\leqslant i\leqslant n,\ i\neq p} U_i - E\left(\sum_{1\leqslant i\leqslant n,\ i\neq p} U_i\right)\right]^2 \middle| X_p\right\} \\
&= \frac{2}{n^2{h_1}^2}(U_p - EU_p)^2 + \frac{2}{n^2{h_1}^2} E\left[\sum_{1\leqslant i\leqslant n,\ i\neq p} U_i - E\left(\sum_{1\leqslant i\leqslant n,\ i\neq p} U_i\right)\right]^2 \\
&= \frac{2}{n^2{h_1}^2}(U_p - EU_p)^2 + \frac{2}{n^2{h_1}^2} \operatorname{Var}\left(\sum_{1\leqslant i\leqslant n,\ i\neq p} U_i\right) \\
&= \frac{2}{n^2{h_1}^2}(U_p - EU_p)^2 + \frac{2(n-1)}{n^2{h_1}^2} \operatorname{Var}(U_1) \\
&\leqslant \frac{2}{n^2{h_1}^2}(U_p - EU_p)^2 + \frac{2}{n{h_1}^2} EU_1^2, \qquad \text{(B.42)}
\end{aligned}
$$

where the first inequality follows from the c_r inequality and the second inequality is due to the fact that $\operatorname{Var}(U_1) = EU_1^2 - (EU_1)^2 \leqslant EU_1^2$. Since for all $i = 1, 2, \ldots, n$, $0 \leqslant U_i = \mathbb{1}(a \leqslant X_i \leqslant b) K\left(\frac{h_1 - X_i + a}{h_1}\right) \leqslant \overline{K}$ where $\overline{K} = \sup_{u\in\mathbb{R}} K(u) < \infty$, then $0 \leqslant |EU_i| = EU_i \leqslant \overline{K}$ and

$$
\frac{1}{n^2{h_1}^2}(U_p - EU_p)^2 \leqslant \frac{(|U_p| + |EU_p|)^2}{n^2{h_1}^2} \leqslant \frac{(2\overline{K})^2}{n^2{h_1}^2} = O\left(\frac{1}{n^2{h_1}^2}\right).
$$

And because

$$
EU_1^2 = \int_a^b K\left(\frac{h_1 - z + a}{h_1}\right)^2 f(z)\,\mathrm{d}z = h_1 \int_{1-(b-a)/h_1}^{1} K(t)^2 f(a + (1-t)h_1)\,\mathrm{d}t \leqslant h_1 \overline{f} \int_{-\infty}^{\infty} K(t)^2\,\mathrm{d}t,
$$

where $\overline{f} = \sup_{z\in[a,b]} f(z)$. By Assumption S, $R_K \equiv \int_{-\infty}^{\infty} K(t)^2\,\mathrm{d}t < \infty$ and hence $EU_1^2/(n{h_1}^2) \leqslant h_1 R_K \overline{f}/(n{h_1}^2) = O\left(\frac{1}{nh_1}\right)$. Provided that $nh_1 \to \infty$ as $n \to \infty$, I get

$$
E\left\{[f_1^*(a+h_1) - Ef_1^*(a+h_1)]^2 \mid X_p\right\} \leqslant O\left(\frac{1}{n^2{h_1}^2}\right) + O\left(\frac{1}{nh_1}\right) = O\left(\frac{1}{nh_1}\right). \qquad \text{(B.43)}
$$

Also, X_i's are i.i.d. by Assumption Q, so I have

$$
\begin{aligned}
Ef_1^*(a+h_1) &= \frac{1}{h_1} E\left[\mathbb{1}(a \leqslant X_1 \leqslant b) K\left(\frac{h_1 - X_1 + a}{h_1}\right)\right] \\
&= \frac{1}{h_1} \int_a^b K\left(\frac{h_1 - z + a}{h_1}\right) f(z)\,\mathrm{d}z = \int_{1-(b-a)/h_1}^{1} K(t) f(a + h_1 - th_1)\,\mathrm{d}t
\end{aligned}
$$

$$= f(a+h_1)\int_{1-(b-a)/h_1}^{1} K(t)\,\mathrm{d}t - h_1 f'(a+h_1)\int_{1-(b-a)/h_1}^{1} tK(t)\,\mathrm{d}t + {h_1}^2 \int_{1-(b-a)/h_1}^{1} t^2 K(t) f''(a+h-\lambda t h_1)\,\mathrm{d}t,$$

where $\lambda \in [0,1]$. Since $h_1 \to 0$ as $n \to \infty$, so $h_1 < (b-a)/2$ when n is sufficiently large, then using $\int_{1-(b-a)/h_1}^{1} K(t)\,\mathrm{d}t = \int_{-1}^{1} K(t)\,\mathrm{d}t = 1$ and $\int_{1-(b-a)/h_1}^{1} tK(t)\,\mathrm{d}t = \int_{-1}^{1} tK(t)\,\mathrm{d}t = 0$, I have

$$Ef_1^*(a+h_1) - f(a+h_1) = {h_1}^2 \int_{-1}^{1} t^2 K(t) f''(a+h-\lambda t h_1)\,\mathrm{d}t \leqslant {h_1}^2 \overline{f''} \int_{-1}^{1} t^2 K(t)\,\mathrm{d}t = {h_1}^2 \overline{f''}\kappa = O({h_1}^2),$$

where $\overline{f''} = \sup_{z \in [a,b]} f''(z)$. Hence,

$$[Ef_1^*(a+h_1) - f(a+h_1)]^2 = O({h_1}^4). \tag{B.44}$$

Now (B.12) is proven by combining (B.41), (B.43) and (B.44).

By letting $U_i = \mathbb{1}(a \leqslant X_i \leqslant b) K\left(\frac{h_1 + X_i - b}{h_1}\right)$, I can show (B.14) with analogous argument. To show (B.13) and (B.15), I first can use similar argument to show

$$E\left\{[f_1^*(a) - f(a)]^2 \mid X_p\right\} = O\left({h_0}^4 + \frac{1}{nh_0}\right), \quad E\left\{[f_2^*(b) - f(b)]^2 \mid X_p\right\} = O\left({h_0}^4 + \frac{1}{nh_0}\right)$$

Then the desired results follow from (B.7). □

B.5.4 Proof of Lemma B.4

I only show (B.16) and (B.17) here. (B.18) and (B.19) can be proven analogously.

By definition of $f_1(a+h_1)$, using the c_r inequality, I have

$$\begin{aligned} E\left\{[f_1(a+h_1) - f(a+h_1)]^2 \mid X_p\right\} &= E\left\{\left[f_1^*(a+h_1) - f(a+h_1) + \frac{1}{n^2}\right]^2 \,\middle|\, X_p\right\} \\ &\leqslant 2E\left\{[f_1^*(a+h_1) - f(a+h_1)]^2 \mid X_p\right\} + 2E\left[\left(\frac{1}{n^2}\right)^2 \,\middle|\, X_p\right] \\ &= O\left({h_1}^4 + \frac{1}{nh_1}\right) + \frac{2}{n^4}. \end{aligned}$$

Recall that I have known the fastest convergence rate of the first term in the right-hand side is of $O(n^{-4/5})$, which is slower than the second term, whence (B.16) follows.

To see (B.17), by definition of $f_1(a)$,

$$E\left\{[f_1(a) - f(a)]^2 \mid X_p\right\} = E\left\{\left[\max\left(f_1^*(a), 1/n^2\right) - f(a)\right]^2 \,\middle|\, X_p\right\}.$$

Because $f_1^*(a) \leqslant \max\left(f_1^*(a), 1/n^2\right) \leqslant f_1^*(a) + 1/n^2$,

$$\left|\max\left(f_1^*(a), \frac{1}{n^2}\right) - f(a)\right| \leqslant |f_1^*(a) - f(a)| + \left|f_1^*(a) + \frac{1}{n^2} - f(a)\right|,$$

and then I have

$$\begin{aligned}
E\left\{[f_1(a) - f(a)]^2 \mid X_p\right\} &\leqslant E\left\{\left[|f_1^*(a) - f(a)| + \left|f_1^*(a) - f(a) + \frac{1}{n^2}\right|\right]^2 \middle| X_p\right\} \\
&\leqslant 2E\left\{[f_1^*(a) - f(a)]^2 \mid X_p\right\} + 2E\left\{\left[f_1^*(a) - f(a) + \frac{1}{n^2}\right]^2 \middle| X_p\right\} \\
&\leqslant 2E\left\{[f_1^*(a) - f(a)]^2 \mid X_p\right\} + 4E\left\{[f_1^*(a) - f(a)]^2 \mid X_p\right\} + \frac{4}{n^4} \\
&= O\left({h_1}^4 + \frac{1}{nh_1}\right) + O\left(\frac{1}{n^4}\right).
\end{aligned}$$

where the second and the third inequalities follow from the c_r inequality. Then by the same argument to show (B.16), I have (B.17) proven. □

B.5.5 Proof of Lemma B.5

I only show the conclusion for $E[(\hat{d}_1 - d_1)^2 \mid X_p]$ here. By repeatedly applying the c_r inequality, I can get

$$\begin{aligned}
E\left[(\hat{d}_1 - d_1)^2 \middle| X_p\right] &= E\left\{\left[\left(\hat{d}_1 - \frac{\log f(a+h_1) - \log f(a)}{h_1}\right) + \left(\frac{\log f(a+h_1) - \log f(a)}{h_1} - d\right)\right]^2 \middle| X_p\right\} \\
&\leqslant 2E\left[\left(\hat{d}_1 - \frac{\log f(a+h_1) - \log f(a)}{h_1}\right)^2 \middle| X_p\right] \\
&\qquad + 2E\left[\left(\frac{\log f(a+h_1) - \log f(a)}{h_1} - d\right)^2 \middle| X_p\right] \\
&= 2E\left[\left(\frac{\log f_1(a+h_1) - \log f(a+h_1)}{h_1} - \frac{\log f_1(a) - \log f(a)}{h_1}\right)^2 \middle| X_p\right] \\
&\qquad + 2\left(\frac{\log f(a+h_1) - \log f(a)}{h_1} - d\right)^2 \\
&\leqslant \frac{4}{{h_1}^2} E\left\{[\log f_1(a+h_1) - \log f(a+h_1)]^2 \mid X_p\right\} \\
&\qquad + \frac{4}{{h_1}^2} E\left\{[\log f_1(a) - \log f(a)]^2 \mid X_p\right\} + 2\left(\frac{\log f(a+h_1) - \log f(a)}{h_1} - d\right)^2 \\
&= \frac{4}{{h_1}^2} J_1 + \frac{4}{{h_1}^2} J_2 + 2J_3, \qquad \text{(B.45)}
\end{aligned}$$

where

$$J_1 = E\left\{[\log f_1(a+h_1) - \log f(a+h_1)]^2 \mid X_p\right\},$$
$$J_2 = E\left\{[\log f_1(a) - \log f(a)]^2 \mid X_p\right\},$$
$$J_3 = \left(\frac{\log f(a+h_1) - \log f(a)}{h_1} - d\right)^2.$$

First, by the mean value theorem, I have

$$\begin{aligned}
J_1 &= E\left\{\left[\frac{f_1(a+h_1) - f(a+h_1)}{\lambda f_1(a+h_1) + (1-\lambda) f(a+h_1)}\right]^2 \middle| X_p\right\} \\
&\leqslant E\left\{\left[\frac{f_1(a+h_1) - f(a+h_1)}{(1-\lambda) f(a+h_1)}\right]^2 \middle| X_p\right\} = \frac{E\left\{[f_1(a+h_1) - f(a+h_1)]^2 \mid X_p\right\}}{[(1-\lambda) f(a+h_1)]^2} \\
&\leqslant \frac{1}{(1-\lambda)^2 c_0^2} E\left\{[f_1(a+h_1) - f(a+h_1)]^2 \mid X_p\right\} \\
&= O\left({h_1}^4 + \frac{1}{nh_1}\right),
\end{aligned} \tag{B.46}$$

where $\lambda \in (0,1)$ is a constant, the first inequality is because the estimator $f_1(a+h_1) > 0$ by definition, the second inequality is because $f(a+h_1) \geqslant c_0 > 0$ by Assumption R, and the last equality is obtained by using Lemma B.4.

Similarly, by using the fact that $f_1(a) > 0$ and $f(a) \geqslant c_0 > 0$, I have

$$J_2 \leqslant \frac{1}{(1-\lambda')^2 c_0^2} E\left\{[f_1(a) - f(a)]^2 \mid X_p\right\} = O\left({h_1}^4 + \frac{1}{nh_1}\right), \tag{B.47}$$

where $\lambda' \in (0,1)$ is a constant.

For J_3, by second-order Taylor expansion of $\log f(\cdot)$, I have

$$\begin{aligned}
J_3 &= \left\{\frac{1}{h_1}\left[h_1 \frac{f'(a)}{f(a)} + \frac{{h_1}^2}{2} \cdot \frac{f''(a+\lambda'' h_1) f(a+\lambda'' h_1) - [f'(a+\lambda'' h_1)]^2}{[f(a+\lambda'' h_1)]^2}\right] - \frac{f'(a)}{f(a)}\right\}^2 \\
&= \frac{{h_1}^2}{4}\left\{\frac{f''(a+\lambda'' h_1) f(a+\lambda'' h_1) - [f'(a+\lambda'' h_1)]^2}{[f(a+\lambda'' h_1)]^2}\right\}^2 \\
&\leqslant \frac{{h_1}^2}{4} \cdot \frac{\left[\overline{f''} \cdot \overline{f} + (\overline{f'})^2\right]^2}{c_0^4} \\
&= O({h_1}^2),
\end{aligned} \tag{B.48}$$

where $\lambda'' \in [0,1]$ is a constant. Here $\overline{f''} = \sup_{z \in [a,b]} |f''(z)| < \infty$, $\overline{f'} = \sup_{z \in [a,b]} |f'(z)| < \infty$ and

$\overline{f} = \sup_{z\in[a,b]} f(z) < \infty$. Then combining (B.45), (B.46), (B.47) and (B.48), I get

$$E\left[(\hat{d}_1 - d_1)^2 \,\middle|\, X_p\right] = O\left({h_1}^2 + \frac{1}{n{h_1}^3}\right) + O\left({h_1}^2 + \frac{1}{n{h_1}^3}\right) + O({h_1}^2) = O\left({h_1}^2 + \frac{1}{n{h_1}^3}\right).$$

The other conclusion can be proven analogously. □

B.5.6 Proof of Theorem B.2

By triangular inequality,

$$\sup_{x\in[a,b]} \left|\hat{f}(x) - f(x)\right| = \sup_{x\in[a,b]} \left|\hat{f}(x) - \tilde{f}(x) + \tilde{f}(x) - f(x)\right| \leqslant \sup_{x\in[a,b]} \left|\hat{f}(x) - \tilde{f}(x)\right| + \sup_{x\in[a,b]} \left|\tilde{f}(x) - f(x)\right|.$$

Since I have already know that the second term above is of order $O\left(h^2 + \sqrt{\log n/(nh)}\right)$ almost surely and hence $O_p\left(h^2 + \sqrt{\log n/(nh)}\right)$ by Theorem B.1, I am left with the first term.

By definition of $\hat{f}$ and $\tilde{f}$, I have for any $x \in [a, b]$,

$$\begin{aligned}
\left|\hat{f}(x) - \tilde{f}(x)\right| &= \left|\frac{1}{nh}\sum_{i=1}^{n} \mathbb{1}(a \leqslant X_i \leqslant b)\left[K\left(\frac{x-a+\hat{g}_1(X_i-a)}{h}\right) - K\left(\frac{x-a+g_1(X_i-a)}{h}\right)\right.\right. \\
&\qquad \left.\left. +K\left(\frac{b-x+\hat{g}_2(b-X_i)}{h}\right) - K\left(\frac{b-x+g_2(b-X_i)}{h}\right)\right]\right| \\
&\leqslant \frac{1}{nh}\sum_{i=1}^{n}\left|\mathbb{1}(a \leqslant X_i \leqslant b)\left[K\left(\frac{x-a+\hat{g}_1(X_i-a)}{h}\right) - K\left(\frac{x-a+g_1(X_i-a)}{h}\right)\right]\right| \\
&\qquad + \frac{1}{nh}\sum_{i=1}^{n}\left|\mathbb{1}(a \leqslant X_i \leqslant b)\left[K\left(\frac{b-x+\hat{g}_2(b-X_i)}{h}\right) - K\left(\frac{b-x+g_2(b-X_i)}{h}\right)\right]\right| \\
&= J_1 + J_2, \qquad\qquad \text{(B.49)}
\end{aligned}$$

where

$$J_1 = \frac{1}{nh}\sum_{i=1}^{n}\left|\mathbb{1}(a \leqslant X_i \leqslant b)\left[K\left(\frac{x-a+\hat{g}_1(X_i-a)}{h}\right) - K\left(\frac{x-a+g_1(X_i-a)}{h}\right)\right]\right|,$$
$$J_2 = \frac{1}{nh}\sum_{i=1}^{n}\left|\mathbb{1}(a \leqslant X_i \leqslant b)\left[K\left(\frac{b-x+\hat{g}_2(b-X_i)}{h}\right) - K\left(\frac{b-x+g_2(b-X_i)}{h}\right)\right]\right|.$$

Consider J_1 first. Because for any $d \in \mathbb{R}$, $u \geqslant 0$ and $A > 1/3$,

$$u + du^2 + Ad^2u^3 = (1 + du + Ad^2u^2)u = \left[\left(\sqrt{A}du + \frac{1}{2\sqrt{A}}\right)^2 + \frac{4A-1}{4A}\right]u \geqslant \frac{4A-1}{4A}u,$$

it is easy to see that $g_1(u) > h$, $\hat{g}_1(u) > h$ for $u > \rho h$, where $\rho = 4A_1/(4A_1 - 1)$. Hence, for

$x \in [a,b]$, when $X_i - a > \rho h$,

$$K\left(\frac{x-a+\hat{g}_1(X_i-a)}{h}\right) = K\left(\frac{x-a+g_1(X_i-a)}{h}\right) = 0,$$

since $K(u) = 0$ for $|u| > 1$ by Assumption S. By Assumption T, $h \to 0$ as $n \to \infty$, thus by applying first-order Taylor expansion, I have for sufficiently large n such that $h < (b-a)/\rho$,

$$\begin{aligned}
J_1 &= \frac{1}{nh}\sum_{i=1}^{n}\Bigg|\mathbb{1}(a \leqslant X_i \leqslant a+\rho h)\Bigg[\frac{\hat{g}_1(X_i-a)-g_1(X_i-a)}{h} \\
&\qquad\qquad \times K'\left(\frac{x-a+\lambda_i\hat{g}_1(X_i-a)+(1-\lambda_i)g_1(X_i-a)}{h}\right)\Bigg]\Bigg| \\
&\leqslant \frac{\overline{K'}}{nh^2}\sum_{i=1}^{n}\left|\mathbb{1}(a \leqslant X_i \leqslant a+\rho h)\left[\hat{g}_1(X_i-a)-g_1(X_i-a)\right]\right| \\
&= \frac{\overline{K'}}{nh^2}\sum_{i=1}^{n}\left|\mathbb{1}(a \leqslant X_i \leqslant a+\rho h)\left[(\hat{d}_1-d_1)(X_i-a)^2 + A_1(\hat{d}_1^2-d_1^2)(X_i-a)^3\right]\right| \\
&\leqslant \frac{\overline{K'}}{nh^2}\sum_{i=1}^{n}\left|\mathbb{1}(a \leqslant X_i \leqslant a+\rho h)(X_i-a)^2(\hat{d}_1-d_1)\right| \\
&\qquad\qquad + \frac{A_1\overline{K'}}{nh^2}\sum_{i=1}^{n}\left|\mathbb{1}(a \leqslant X_i \leqslant a+\rho h)(X_i-a)^3(\hat{d}_1^2-d_1^2)\right|, \\
&\leqslant \frac{\overline{K'}}{nh^2}\sum_{i=1}^{n}\rho^2h^2\left|\mathbb{1}(a \leqslant X_i \leqslant a+\rho h)(\hat{d}_1-d_1)\right| + \frac{A_1\overline{K'}}{nh^2}\sum_{i=1}^{n}\rho^3h^3\left|\mathbb{1}(a \leqslant X_i \leqslant a+\rho h)(\hat{d}_1^2-d_1^2)\right|,
\end{aligned}$$

where $\lambda_i \in [0,1]$ and $\overline{K'} = \sup_{u\in\mathbb{R}}|K'(u)| < \infty$. The last inequality is obtained by using the fact that $0 \leqslant X_i - a \leqslant \rho h$ when $a \leqslant X_i \leqslant a + \rho h$. Because $|\hat{d}_1^2 - d_1^2| = |(\hat{d}_1-d_1)^2 + 2d_1(\hat{d}_1-d_1)| \leqslant (\hat{d}_1-d_1)^2 + 2|d_1||\hat{d}_1-d_1|$, then by triangular inequality, I have

$$\begin{aligned}
J_1 &\leqslant \left(\rho^2h^2\overline{K'} + 2A_1\rho^3h^3|d_1|\overline{K'}\right)\cdot\frac{1}{nh^2}\sum_{i=1}^{n}\mathbb{1}(a \leqslant X_i \leqslant a+\rho h)|\hat{d}_1-d_1| \\
&\qquad + 2A_1\rho^3h^4|d_1|\overline{K'}\cdot\frac{1}{nh^3}\sum_{i=1}^{n}\mathbb{1}(a \leqslant X_i \leqslant a+\rho h)(\hat{d}_1-d_1)^2. \quad \text{(B.50)}
\end{aligned}$$

Under Assumptions T and U, $h_1 \to 0$ and $n{h_1}^3 \to \infty$ as $n \to \infty$, and Lemma B.5 gives $E[(\hat{d}_1 - d_1)^2 \mid X_i] = O\left({h_1}^2 + \frac{1}{n{h_1}^3}\right) = O\left(\max\left({h_1}^2, \frac{1}{n{h_1}^3}\right)\right)$ for all $i = 1, 2, \ldots, n$. Thus by the Hölder's inequality,

$$E\left[|\hat{d}_1 - d_1| \,\middle|\, X_i\right] \leqslant \left\{E\left[(\hat{d}_1-d_1)^2 \mid X_i\right]\right\}^{1/2} = O\left(\max\left(h_1, \frac{1}{\sqrt{n{h_1}^3}}\right)\right).$$

It means that for sufficiently large n,

$$E\left[|\hat{d}_1 - d_1| \,\middle|\, X_i\right] \leqslant C_1 \max\left(h_1, \frac{1}{\sqrt{n{h_1}^3}}\right), \quad E\left[(\hat{d}_1 - d_1)^2 \,\middle|\, X_i\right] \leqslant C_2 \max\left({h_1}^2, \frac{1}{n{h_1}^3}\right),$$

for some constants $C_1, C_2 > 0$. Then, for any $M > 0$, by Markov's inequality,

$$\begin{aligned}
\Pr\left(\frac{1}{nh^2}\sum_{i=1}^n \mathbb{1}(a \leqslant X_i \leqslant a+\rho h)|\hat{d}_1 - d_1| \geqslant M\right) &\leqslant \frac{1}{M}E\left[\frac{1}{nh^2}\sum_{i=1}^n \mathbb{1}(a \leqslant X_i \leqslant a+\rho h)|\hat{d}_1 - d_1|\right] \\
&= \frac{1}{nh^2M}\sum_{i=1}^n E\left[\mathbb{1}(a \leqslant X_i \leqslant a+\rho h)E\left(|\hat{d}_1 - d_1| \mid X_i\right)\right] \\
&\leqslant \frac{1}{nh^2M}\sum_{i=1}^n C_1 \max\left(h_1, \frac{1}{\sqrt{n{h_1}^3}}\right) \cdot E[\mathbb{1}(a \leqslant X_i \leqslant a+\rho h)] \\
&= \frac{C_1}{h^2M}\max\left(h_1, \frac{1}{\sqrt{n{h_1}^3}}\right)\int_a^{a+\rho h} f(z)\,\mathrm{d}z \\
&\leqslant \frac{C_1}{h^2M}\max\left(h_1, \frac{1}{\sqrt{n{h_1}^3}}\right)\overline{f}\rho h = \frac{C_1\overline{f}\rho}{M}\cdot\frac{\max\left(h_1, \frac{1}{\sqrt{n{h_1}^3}}\right)}{h}
\end{aligned}$$

and similarly

$$\begin{aligned}
\Pr\left(\frac{1}{nh^3}\sum_{i=1}^n \mathbb{1}(a \leqslant X_i \leqslant a+\rho h)(\hat{d}_1 - d_1)^2 \geqslant M\right) &\leqslant \frac{1}{M}E\left[\frac{1}{nh^3}\sum_{i=1}^n \mathbb{1}(a \leqslant X_i \leqslant a+\rho h)(\hat{d}_1 - d_1)^2\right] \\
&= \frac{1}{nh^3M}\sum_{i=1}^n E\left\{\mathbb{1}(a \leqslant X_i \leqslant a+\rho h)E\left[(\hat{d}_1 - d_1)^2 \mid X_i\right]\right\} \\
&\leqslant \frac{1}{nh^3M}\sum_{i=1}^n C_2 \max\left({h_1}^2, \frac{1}{n{h_1}^3}\right) \cdot E[\mathbb{1}(a \leqslant X_i \leqslant a+\rho h)] \\
&\leqslant \frac{C_2\overline{f}\rho}{M}\cdot\frac{\max\left({h_1}^2, \frac{1}{n{h_1}^3}\right)}{h^2},
\end{aligned}$$

where $\overline{f} = \sup_{z\in[a,b]} f(z)$. Since $h_1 = O(h)$ and $1/\sqrt{n{h_1}^3} = O(h)$, then there exists a constant $C_3 > 0$ such that for sufficiently large n,

$$\frac{\max\left(h_1, \frac{1}{\sqrt{n{h_1}^3}}\right)}{h} \leqslant C_3, \quad \frac{\max\left({h_1}^2, \frac{1}{n{h_1}^3}\right)}{h^2} \leqslant C_3,$$

which implies that for any $\epsilon > 0$, I can pick M sufficiently large, e.g. $M = C_3\overline{f}\rho\max(C_1, C_2)/\epsilon$, such that

$$\Pr\left(\frac{1}{nh^2}\sum_{i=1}^{n}\mathbb{1}(a \leqslant X_i \leqslant a+\rho h)|\hat{d}_1 - d_1| \geqslant M\right) \leqslant \epsilon,$$
$$\Pr\left(\frac{1}{nh^3}\sum_{i=1}^{n}\mathbb{1}(a \leqslant X_i \leqslant a+\rho h)(\hat{d}_1 - d_1)^2 \geqslant M\right) \leqslant \epsilon.$$

Therefore, it follows from (B.50) that

$$J_1 \leqslant (\rho^2 h^2 \overline{K'} + 2A_1\rho^3 h^3 |d_1| \overline{K'}) \cdot O_p(1) + 2A_1 \rho^3 h^4 |d_1| \overline{K'} \cdot O_p(1) = O_p(h^2) + O_p(h^3) + O_p(h^4) = O_p(h^2). \tag{B.51}$$

Note that the upper bound for J_1 given in (B.50) doesn't depend on x, so the order of J_1 given in (B.51) is uniform for all $x \in [a, b]$. Similarly, I have $J_2 = O_p(h^2)$ uniformly for all $x \in [a, b]$. Hence,

$$\sup_{x\in[a,b]}\left|\hat{f}(x) - \tilde{f}(x)\right| \leqslant \sup_{x\in[a,b]} J_1 + \sup_{x\in[a,b]} J_2 = O_p(h^2),$$

which completes the proof of the theorem. □

Appendix C

Supplementary Results and Proofs to Chapter 2

C.1 Results for the $k = 0$ case

The identification results based on the price distributions of k-double auctions with $k = 0$ can be established in a similar way by using symmetric assumptions. Here I will list the relevant assumptions and provide the statements of the conclusions without proof.

When $k = 0$, I assume that the buyer's conditional value distribution $F_V(\cdot \mid y)$ satisfies:

Assumption V. *For any $y \in \mathcal{Y}$, $\delta(\overline{v}, y) = 0$, and $\delta(\cdot, y)$ is continuously differentiable with $-\infty < \partial_1\delta(v, z) < 0$ for all $v \in [\underline{c}, \overline{v}]$.*

Just a reminder, here $\delta(\cdot, y) = [1 - F_V(\cdot \mid y)]/f_V(\cdot \mid y)$ and $\partial_1\delta(v, y)$ denotes the partial derivative of $\delta(\cdot, \cdot)$ with respect to the first argument evaluated at (v, y). I also assumed that the seller's conditional value distribution $F_C(\cdot \mid z)$ satisfies:

Assumption W. *There exist $z^* \neq z^{**}$ in $\mathcal{Z}$ such that $f_C(\cdot \mid z^*)/f_C(\cdot \mid z**)$ is continuously differentiable with positive derivative on $[\underline{c}, \overline{v}]$.*

Assumption V and Assumption W are parallel to Assumptions M and N, respectively.

With these assumptions, I can show the following parallel results for the $k = 0$ case.

Lemma C.1. *Under Assumptions I to L and Assumption V, for any $y \in \mathcal{Y}$ and any $z \in \mathcal{Z}$,*

(i) $\overline{b}_0(y, z) = \overline{v}$, and $h_0(\overline{v} \mid y, z) = 0$;

(ii) $\underline{s}_0(y, z) = \underline{s}_0(y)$ which solves $\underline{s}_0(y) - \delta(\underline{s}_0(y), y) = \underline{c}$.

By defining

$$\Gamma_0(p, y) \equiv \lim_{q \to \overline{v}} \left[\frac{h_0(p \mid y, z^*)}{h_0(q \mid y, z^*)} \middle/ \frac{h_0(p \mid y, z^{**})}{h_0(q \mid y, z^{**})} \right]$$

for $\underline{s}_0(y) < p < \overline{v}$ and $\Gamma_0(\underline{s}_0(y), y) \equiv \lim_{p \to \underline{s}_0(y)} \Gamma_0(p, y)$, $\Gamma_0(\overline{v}, y) \equiv \lim_{p \to \overline{v}} \Gamma_0(p, y)$, it follows that

Lemma C.2. *Under Assumptions I to L and Assumptions V and W,*

(i) $\Gamma_0(\cdot, y)$ *is continuously differentiable with positive derivative on* $[\underline{s}_0(y), \overline{v}]$ *for any* $y \in \mathcal{Y}$*;*

(ii) $\Gamma_0(\overline{v}, y^*) = \Gamma_0(\overline{v}, y^{**}) = 1$ *and* $\Gamma_0(\underline{s}_0(y^*), y^*) = \Gamma_0(\underline{s}_0(y^{**}), y^{**})$ *for any* $y^*, y^{**} \in \mathcal{Y}$*.*

For $y^* \neq y^{**}$ and for $p \in [\underline{s}_0(y^*), \overline{v}]$, define $\varphi(p)$ be such that

$$\Gamma_0(p, y^*) = \Gamma_0(\varphi(p), y^{**}),$$

and let

$$r(p) \equiv \frac{h_0(\varphi(p) \,|\, y^{**}, z^*)}{h_0(p \,|\, y^*, z^*)} \varphi'(p),$$

then

Theorem C.1. *Under Assumptions I to L and Assumptions V and W,* $h_0(\cdot \,|\, y, z)$ *can be rationalized by sealed-bid k-double auction with* $k = 0$ *for some* $F_V(\cdot \,|\, y)$*,* $y \in \{y^*, y^{**}\}$ *and* $F_C(\cdot \,|\, z)$*,* $z \in \{z^*, z^{**}\}$ *if and only if* $F_V(\cdot \,|\, y^*)$ *and* $F_V(\cdot \,|\, y^{**})$ *satisfy*

$$p - \delta(p, y^*) = \varphi(p) - \delta(\varphi(p), y^{**}),$$

and

$$\frac{1 - F_V(\varphi(p) \,|\, y^{**})}{1 - F_V(p \,|\, y^*)} = \frac{a_0(y^*, z^*)}{a_0(y^{**}, z^*)} \cdot r(p),$$

for all $p \in [\underline{s}_0(y^*), \overline{v}]$*.*

For the point identification result, assume additionally:

Assumption X. *Let* $\eta(p, y) = p - \delta(p, y)$*.* $F_V(\cdot \,|\, y^*)$ *and* $F_V(\cdot \,|\, y^{**})$ *satisfy* $\partial_1 \eta^{-1}(c, y^*) < \partial_1 \eta^{-1}(c, y^{**})$ *for any* $c \in [\underline{c}, \overline{v})$*, where* $\eta^{-1}(\cdot, y)$ *is the inverse function of* $\eta(\cdot, y)$*.*

Theorem C.2. *Under Assumptions I to L and Assumptions V to X,* $1 - F_V(\cdot \,|\, y^*)$ *is identified on* $[\underline{s}_0(y^*), \overline{v}]$ *and* $1 - F_V(\cdot \,|\, y^*)$ *is identified on* $[\underline{s}_0(y^{**}), \overline{v}]$*.*

Similarly, it suffices to use conditions weaker than Assumption X: (i) $\eta^{-1}(c, y^*) > \eta^{-1}(c, y^{**})$ for all $c \in [\underline{c}, \overline{v})$, and (ii) $[1 - F_V(\eta^{-1}(\cdot, y^{**}) \,|\, y^{**})] / [1 - F_V(\eta^{-1}(\cdot, y^*) \,|\, y^*)]$ is strictly decreasing on $[\underline{c}, \overline{v})$.

When the covariate Y is continuous random variable and the support $\mathcal{Y} = [\underline{y}, \overline{y}] \subset \mathbb{R}$, I assume in addition that:

Assumption Y. *When* Y *is continuous,* $\delta(v, y)$ *is continuously differentiable in* y*.*

Then,

Lemma C.3. *Under Assumptions I to L, V, W, and Y, $\Gamma_0(p, y)$ is continuously differentiable in y.*

Therefore, similarly define $\ell_0(p, y) = \partial_1 \Gamma_0(p, y)/\partial_2 \Gamma_0(p, y)$ for $y \in \mathcal{Y}$ and $p \in [\underline{s}_0(y), \overline{v}]$, and the following theorem parallel to Theorem 2.3 holds.

Theorem C.3. *Under Assumptions I to L, V, W, and Y, if $\delta(\cdot, y^*)$ is known for some $y^* \in \mathcal{Y}$, then $\delta(\cdot,\cdot)$ is identified on the set $\{(p, y) : y \in \mathcal{Y},\ p \in [\underline{s}_0(y), \overline{v}]\}$ through the following partial differential equation*

$$\ell_0(p, y) \cdot \partial_1 \delta(p, y) - \partial_2 \delta(p, y) = \ell_0(p, y).$$

C.2 Proof of Lemma 2.1

By Assumption L, the lowest possible private values of the buyer and the seller are $\underline{c}$. When $k = 1$, because the equilibrium bidding strategy for the seller is submitting his true valuation, the seller with the lowest private value will bid $\underline{c}$. Also, the buyer with the lowest private value will also bid $\underline{c}$ because $\underline{c} + \lambda(\underline{c}, z) = \underline{c} + 0 = \underline{c}$. Therefore, the lower endpoint of the price support is $\underline{s}_1(y, z) = \underline{c}$, which is independent from Y and Z; and it follows from (2.2) directly that $h_1(\underline{c} \mid y, z) = 0$ because $F_C(\underline{c} \mid z) = 0$.

For conclusion (ii), because the seller's value distribution does not depend on covariate Y by condition (i) of Assumption L, the buyer's equilibrium bidding strategy, whose inverse function is given by $b + \lambda(b, z)$, does not depend on Y, either. Also, because the highest possible private value of the buyer, $\overline{v}$, does not depend on Y or Z, the buyer's highest bid $\overline{b}_1(y, z)$ will just depend on covariate Z. Since the buyer's inverse bidding function is strictly increasing as $\lambda(\cdot, z)$ is assumed to be strictly increasing, the buyer's highest bid, which is also the upper endpoint of the price support, is the unique solution to equation $\overline{b}_1(z) + \lambda(\overline{b}_1(z), z) = \overline{v}$. □

C.3 Proof of Lemma 2.2

For (i), given the differentiability of $f_V(\cdot \mid y^*)/f_V(\cdot \mid y^{**})$ and $\lambda(\cdot, z)$ assumed in Assumptions M and N, differentiating $\Gamma_1(p, z)$ as (2.5) with respect to p yields

$$\partial_1 \Gamma_1(p, z) = \frac{f_V(\underline{c} \mid y^{**})}{f_V(\underline{c} \mid y^*)} \cdot \frac{\mathrm{d}}{\mathrm{d}u} \left[\frac{f_V(u \mid y^*)}{f_V(u \mid y^{**})} \right] \Bigg|_{u = p + \lambda(p,z)} \cdot \left[1 + \partial_1 \lambda(p, z)\right].$$

Since $\frac{\mathrm{d}}{\mathrm{d}u}\left[f_V(u \mid y^*)/f_V(u \mid y^{**})\right]$ is continuous and both $p + \lambda(p, z)$ and $1 + \partial_1 \lambda(p, z)$ are continuous in p, $\partial_1 \Gamma_1(p, z)$ is continuous in p and therefore $\Gamma_1(\cdot, z)$ is continuously differentiable. In the mean time, $\partial_1\Gamma_1(p,z) < 0$ because $\frac{\mathrm{d}}{\mathrm{d}u}\left[f_V(u \mid y^*)/f_V(u \mid y^{**})\right] < 0$ by Assumption N and $1+\partial_1\lambda(p,z) > 1$ by Assumption M.

To show (ii), first note that by Assumption M, $\lambda(\underline{c},z^*) = \lambda(\underline{c},z^{**}) = 0$, so by (2.5),

$$\Gamma_1(\underline{c},z^*) = \frac{f_V(\underline{c}+\lambda(\underline{c},z^*) \mid y^*)}{f_V(\underline{c}+\lambda(\underline{c},z^*) \mid y^{**})} \cdot \frac{f_V(\underline{c} \mid y^{**})}{f_V(\underline{c} \mid y^*)} = \frac{f_V(\underline{c} \mid y^*)}{f_V(\underline{c} \mid y^{**})} \cdot \frac{f_V(\underline{c} \mid y^{**})}{f_V(\underline{c} \mid y^*)} = 1$$

and

$$\Gamma_1(\underline{c},z^{**}) = \frac{f_V(\underline{c}+\lambda(\underline{c},z^{**}) \mid y^*)}{f_V(\underline{c}+\lambda(\underline{c},z^{**}) \mid y^{**})} \cdot \frac{f_V(\underline{c} \mid y^{**})}{f_V(\underline{c} \mid y^*)} = \frac{f_V(\underline{c} \mid y^*)}{f_V(\underline{c} \mid y^{**})} \cdot \frac{f_V(\underline{c} \mid y^{**})}{f_V(\underline{c} \mid y^*)} = 1.$$

Next, it follows from conclusion (ii) of Lemma 2.1 that $\overline{b}_1(z^*)$ and $\overline{b}_1(z^{**})$ satisfy that $\overline{b}_1(z^*) + \lambda(\overline{b}_1(z^*),z^*) = \overline{b}_1(z^{**}) + \lambda(\overline{b}_1(z^{**}),z^{**}) = \overline{v}$. Therefore,

$$\Gamma_1(\overline{b}_1(z^*),z^*) = \frac{f_V(\overline{b}_1(z^*)+\lambda(\overline{b}_1(z^*),z^*) \mid y^*)}{f_V(\overline{b}_1(z^*)+\lambda(\overline{b}_1(z^*),z^*) \mid y^{**})} \cdot \frac{f_V(\underline{c} \mid y^{**})}{f_V(\underline{c} \mid y^*)} = \frac{f_V(\overline{v} \mid y^*)}{f_V(\overline{v} \mid y^{**})} \cdot \frac{f_V(\underline{c} \mid y^{**})}{f_V(\underline{c} \mid y^*)},$$
$$\Gamma_1(\overline{b}_1(z^{**}),z^{**}) = \frac{f_V(\overline{b}_1(z^{**})+\lambda(\overline{b}_1(z^{**}),z^{**}) \mid y^*)}{f_V(\overline{b}_1(z^{**})+\lambda(\overline{b}_1(z^{**}),z^{**}) \mid y^{**})} \cdot \frac{f_V(\underline{c} \mid y^{**})}{f_V(\underline{c} \mid y^*)} = \frac{f_V(\overline{v} \mid y^*)}{f_V(\overline{v} \mid y^{**})} \cdot \frac{f_V(\underline{c} \mid y^{**})}{f_V(\underline{c} \mid y^*)},$$

and then $\Gamma_1(\overline{b}_1(z^*),z^*) = \Gamma_1(\overline{b}_1(z^{**}),z^{**})$. □

C.4 Proof of Theorem 2.1

First, for the "only if" part, I shall show that under Assumptions I to N, for any buyer's conditional value distributions $F_V(\cdot \mid y)$, $y \in \{y^*,y^{**}\}$ and seller's conditional value distributions $F_C(\cdot \mid z)$, $z \in \{z^*,z^{**}\}$ that can rationalize the price distributions with densities $h_1(\cdot \mid y,z)$, $y \in \{y^*,y^{**}\}$, $z \in \{z^*,z^{**}\}$, the seller's conditional value distributions $F_C(\cdot \mid z^*)$ and $F_C(\cdot \mid z^{**})$ must satisfy (2.7) and (2.8). This is straightforward based on the derivation of (2.7) and (2.8).

Since $F_V(\cdot \mid y)$, $y \in \{y^*,y^{**}\}$ and $F_C(\cdot \mid z)$, $z \in \{z^*,z^{**}\}$ can rationalize the given price distribution, then by (2.2) it must be true that for $y \in \{y^*,y^{**}\}$ and $z \in \{z^*,z^{**}\}$, the price density can be written as

$$h_1(p \mid y,z) = a_1(y,z)F_C(p \mid z)f_V(p+\lambda(p,z) \mid y)\left[1+\partial_1\lambda(p,z)\right], \quad p \in [\underline{c},\overline{b}_1(z)],$$

for some $a_1(y,z) > 0$. Then by the definition of $\Gamma_1(\cdot,z)$ (i.e. (2.4)), it follows that for $z \in \{z^*,z^{**}\}$,

$$\Gamma_1(p,z) = \lim_{q\to\underline{c}}\left[\frac{h_1(p \mid y^*,z)}{h_1(q \mid y^*,z)} \Big/ \frac{h_1(p \mid y^{**},z)}{h_1(q \mid y^{**},z)}\right]$$
$$= \lim_{q\to\underline{c}}\left[\frac{f_V(p+\lambda(p,z) \mid y^*)}{f_V(p+\lambda(p,z) \mid y^{**})} \cdot \frac{f_V(q+\lambda(q,z) \mid y^{**})}{f_V(q+\lambda(q,z) \mid y^*)}\right]$$

$$= \frac{f_V(p+\lambda(p,z)\,|\,y^*)}{f_V(p+\lambda(p,z)\,|\,y^{**})} \cdot \frac{f_V(\underline{c}\,|\,y^{**})}{f_V(\underline{c}\,|\,y^*)},$$

which is equation (2.5). The last equality is due to $\lambda(\underline{c}, z) = 0$ and the continuity of $\lambda(\cdot, z)$ by Assumption M as well as the continuity of $f_V(\cdot\,|\,y^{**})/f_V(\cdot\,|\,y^*)$ by Assumption N. Because the function $\psi(\cdot)$ is defined as such that $\Gamma_1(p, z^*) = \Gamma_1(\psi(p), z^{**})$ for $p \in [\underline{c}, \overline{b}_1(z^*)]$, and because $f_V(\cdot\,|\,y^*)/f_V(\cdot\,|\,y^{**})$ is strictly monotone by Assumption N, $F_C(\cdot\,|\,z^*)$ and $F_C(\cdot\,|\,z^{**})$, or $\lambda(\cdot\,|\,z^*)$ and $\lambda(\cdot\,|\,z^{**})$, must satisfy

$$\frac{f_V(p+\lambda(p,z^*)\,|\,y^*)}{f_V(p+\lambda(p,z^*)\,|\,y^{**})} = \frac{f_V(\psi(p)+\lambda(\psi(p),z^{**})\,|\,y^*)}{f_V(\psi(p)+\lambda(\psi(p),z^{**})\,|\,y^{**})} \;\Rightarrow\; p+\lambda(p,z^*) = \psi(p)+\lambda(\psi(p),z^{**}),$$

which is equation (2.7).

Given $F_C(\cdot\,|\,z^*)$ and $F_C(\cdot\,|\,z^{**})$ satisfy (2.7), differentiating both sides of (2.7) yields

$$1+\partial_1\lambda(p,z^*) = \psi'(p)+\partial_1\lambda(\psi(p),z^{**})\cdot\psi'(p) = \psi'(p)\left[1+\partial_1\lambda(\psi(p),z^{**})\right].$$

Since for $y \in \{y^*, y^{**}\}$ and $p \in [\underline{c}, \overline{b}_1(z^*)]$,

$$h_1(p\,|\,y,z^*) = a_1(y,z^*)F_C(p\,|\,z^*)f_V(p+\lambda(p,z^*)\,|\,y)\left[1+\partial_1\lambda(p,z^*)\right],$$
$$h_1(\psi(p)\,|\,y,z^{**}) = a_1(y,z^{**})F_C(\psi(p)\,|\,z^{**})f_V(\psi(p)+\lambda(\psi(p),z^{**})\,|\,y)\left[1+\partial_1\lambda(\psi(p),z^{**})\right],$$

so taking the ratio of these two and using $f_V(p+\lambda(p,z^*)\,|\,y) = f_V(\psi(p)+\lambda(\psi(p),z^{**})\,|\,y)$ implied by (2.7), I have that $F_C(\cdot\,|\,z^*)$ and $F_C(\cdot\,|\,z^{**})$ should also satisfy

$$\frac{h_1(p\,|\,y,z^*)}{h_1(\psi(p)\,|\,y,z^{**})} = \frac{a_1(y,z^*)}{a_1(y,z^{**})}\cdot\frac{F_C(p\,|\,z^*)}{F_C(\psi(p)\,|\,z^{**})}\psi'(p),$$

which can be written into the form of (2.8).

Now I shall show the "if" part, i.e. under Assumptions I to L, for any seller's conditional value distributions $F_C(\cdot\,|\,z^*)$ and $F_C(\cdot\,|\,z^{**})$ that satisfy Assumption M, (2.7) and (2.8), there exist buyer's conditional value distributions $F_V(\cdot\,|\,y^*)$ and $F_V(\cdot\,|\,y^{**})$ satisfying Assumption N such that $h_1(\cdot\,|\,y,z)$, $y \in \{y^*, y^{**}\}$, $z \in \{z^*, z^{**}\}$ are the price densities generated from the regular equilibrium of a k-double auction with $k = 1$.

I will start off by claiming that given a price density $h_1(\cdot\,|\,y,z)$ and a seller's conditional value distribution $F_C(\cdot\,|\,z)$ that satisfies Assumption M, a distribution given by (2.9) with $\overline{v} = \overline{b}_1(z) + \lambda(\overline{b}_1(z), z)$ is the conditional value distribution for the buyer with which the seller's conditional value distribution rationalizes the price distribution. To show this, first, because b solves $b +$

$\lambda(b,z) = v$, it follows that

$$\frac{\mathrm{d}b}{\mathrm{d}v} + \partial_1\lambda(b,z)\cdot\frac{\mathrm{d}b}{\mathrm{d}v} = 1 \;\Rightarrow\; \frac{\mathrm{d}b}{\mathrm{d}v} = \frac{1}{1+\partial_1\lambda(b,z)}.$$

Note that $\left\{\int_{\underline{c}}^{\overline{b}_1(z)}[h_1(u\,|\,y,z)/F_C(u\,|\,z)]\,\mathrm{d}u\right\}^{-1}$ in (2.9) is a constant only depending on y and z so denote it by $\sigma_1(y,z)$. Therefore, differentiating (2.9) with respect to v gives the corresponding density of $\tilde{F}_V(\cdot\,|\,y,z)$ as

$$\tilde{f}_V(v\,|\,y,z) = \sigma_1(y,z)\cdot\frac{h_1(b\,|\,y,z)}{F_C(b\,|\,z)}\cdot\frac{\mathrm{d}b}{\mathrm{d}v} = \sigma_1(y,z)\cdot\frac{h_1(b\,|\,y,z)}{F_C(b\,|\,z)}\cdot\frac{1}{1+\partial_1\lambda(b,z)}. \tag{C.1}$$

Since $v = b + \lambda(b,z)$, so (C.1) can be written as

$$\tilde{f}_V(b+\lambda(b,z)\,|\,y,z) = \sigma_1(y,z)\cdot\frac{h_1(b\,|\,y,z)}{F_C(b\,|\,z)}\cdot\frac{1}{1+\partial_1\lambda(b,z)} \tag{C.2}$$

for $b \in [\underline{c},\overline{b}_1(z)]$. Then by (2.2), the density of the price distribution generated by $F_C(\cdot\,|\,z)$ and $\tilde{F}_V(\cdot\,|\,y,z)$ is

$$\tilde{h}_1(p\,|\,y,z) = \tilde{a}_1(y,z)F_C(p\,|\,z)\tilde{f}_V(p+\lambda(p,z)\,|\,y,z)\left[1+\partial_1\lambda(p,z)\right],$$

where $\tilde{a}_1(y,z)$ is the normalization constant. It immediately follows from (C.2) that

$$\tilde{h}_1(p\,|\,y,z) = \tilde{a}_1(y,z)\sigma_1(y,z)h_1(p\,|\,y,z).$$

Since both $\tilde{h}_1(\cdot\,|\,y,z)$ and $h_1(\cdot\,|\,y,z)$ are density functions on support $[\underline{c},\overline{b}_1(z)]$,

$$1 = \int_{\underline{c}}^{\overline{b}_1(z)}\tilde{h}_1(p\,|\,y,z)\,\mathrm{d}p = \tilde{a}_1(y,z)\sigma_1(y,z)\int_{\underline{c}}^{\overline{b}_1(z)}h_1(p\,|\,y,z)\,\mathrm{d}p = \tilde{a}_1(y,z)\sigma_1(y,z)$$

and hence $\tilde{h}_1(p\,|\,y,z) = h_1(p\,|\,y,z)$ for all $p \in [\underline{c},\overline{b}_1(z)]$.

Let $\tilde{F}_V(\cdot\,|\,y,z)$, $y \in \{y^*,y^{**}\}$, $z \in \{z^*,z^{**}\}$ be the respective buyer's conditional value distributions induced by the corresponding $h_1(\cdot\,|\,y,z)$ and the seller's conditional value distributions $F_C(\cdot\,|\,z^*)$, $F_C(\cdot\,|\,z^{**})$ that satisfy (2.7) and (2.8). Next I will show that for $y \in \{y^*,y^{**}\}$, $\tilde{F}_V(\cdot\,|\,y,z^*)$ and $\tilde{F}_V(\cdot\,|\,y,z^{**})$ are in fact the same distribution.

First, by condition (ii) of Lemma 2.2, I have $\psi(\underline{c}) = \underline{c}$ and $\psi(\overline{b}_1(z^*)) = \overline{b}_1(z^{**})$, then (2.7) implies

$$\overline{b}_1(z^*) + \lambda(\overline{b}_1(z^*),z^*) = \psi(\overline{b}_1(z^*)) + \lambda(\psi(\overline{b}_1(z^*)),z^{**}) = \overline{b}_1(z^{**}) + \lambda(\overline{b}_1(z^{**}),z^{**}).$$

Since the induce buyer's conditional value distribution has support $[\underline{c},\overline{v}]$ with $\overline{v} = \overline{b}_1(z) +$

$\lambda(\overline{b}_1(z),z)$, this means that both $F_V(\cdot\,|\,y,z^*)$ and $F_V(\cdot\,|\,y,z^{**})$ have the same support. Second, for any $p \in [\underline{c},\overline{b}_1(z^*))$, let $v = p + \lambda(p,z^*)$. Given $F_C(\cdot\,|\,z^*)$ and $F_C(\cdot\,|\,z^{**})$ satisfy (2.7), $v = \psi(p) + \lambda(\psi(p),z^{**})$, too. So by taking $b = p$ and $b = \psi(p)$ in (C.2) for $z = z^*$ and $z = z^{**}$ respectively, I get

$$\tilde{f}_V(v\,|\,y,z^*) = \sigma_1(y,z^*)\cdot\frac{h_1(p\,|\,y,z^*)}{F_C(p\,|\,z^*)}\cdot\frac{1}{1+\partial_1\lambda(p,z^*)},$$
$$\tilde{f}_V(v\,|\,y,z^{**}) = \sigma_1(y,z^{**})\cdot\frac{h_1(\psi(p)\,|\,y,z^{**})}{F_C(\psi(p)\,|\,z^{**})}\cdot\frac{1}{1+\partial_1\lambda(\psi(p),z^{**})}.$$

Then by (2.8),

$$\begin{aligned}\tilde{f}_V(v\,|\,y,z^{**}) &= \sigma_1(y,z^{**})\frac{a_1(y,z^{**})}{a_1(y,z^*)}\cdot\frac{h_1(p\,|\,y,z^*)}{F_C(p\,|\,z^*)}\cdot\frac{1}{[1+\partial_1\lambda(\psi(p),z^{**})]\,\psi'(p)}\\ &= \sigma_1(y,z^{**})\frac{a_1(y,z^{**})}{a_1(y,z^*)}\left[\frac{1+\partial_1\lambda(p,z^*)}{\sigma_1(y,z^*)}\cdot\tilde{f}_V(v\,|\,y,z^*)\right]\frac{1}{[1+\partial_1\lambda(\psi(p),z^{**})]\,\psi'(p)}\\ &= \frac{\sigma_1(y,z^{**})a_1(y,z^{**})}{\sigma_1(y,z^*)a_1(y,z^*)}\cdot\frac{1+\partial_1\lambda(p,z^*)}{[1+\partial_1\lambda(\psi(p),z^{**})]\,\psi'(p)}\cdot\tilde{f}_V(v\,|\,y,z^*).\end{aligned}$$

Equation (2.7) implies $1+\partial_1\lambda(p,z^*) = \psi'(p)+\partial_1\lambda(\psi(p),z^{**})\psi'(p)$, so $\frac{1+\partial_1\lambda(p,z^*)}{[1+\partial_1\lambda(\psi(p),z^{**})]\psi'(p)} = 1$. Consequently, $\tilde{f}_V(v\,|\,y,z^*) = \tilde{f}_V(v\,|\,y,z^{**})$ because

$$\frac{\sigma_1(y,z^{**})a_1(y,z^{**})}{\sigma_1(y,z^*)a_1(y,z^*)} = \frac{\sigma_1(y,z^{**})a_1(y,z^{**})}{\sigma_1(y,z^*)a_1(y,z^*)}\int_{\underline{c}}^{\overline{v}}\tilde{f}_V(v\,|\,y,z^*)\,\mathrm{d}v = \int_{\underline{c}}^{\overline{v}}\tilde{f}_V(v\,|\,y,z^{**})\,\mathrm{d}v = 1.$$

Thus, the induced $\tilde{F}_V(\cdot\,|\,y,z)$, $y\in\{y^*,y^{**}\}$ do not depend on covariate Z, so $\tilde{F}_V(\cdot\,|\,y,z) = \tilde{F}_V(\cdot\,|\,y)$ and they are valid conditional value distributions for the buyer that satisfies Assumption L.

As a final point, it is remained to show that $\tilde{F}_V(\cdot\,|\,y^*)$ and $\tilde{F}_V(\cdot\,|\,y^{**})$ satisfy Assumption N. Since $\tilde{F}_V(\cdot\,|\,y^*)$ and $\tilde{F}_V(\cdot\,|\,y^{**})$ can rationalize the observed price distributions, straightforwardly, for either $z = z^*$ or $z = z^{**}$,

$$\Gamma_1(p,z) = \frac{\tilde{f}_V(p+\lambda(p,z)\,|\,y^*)}{\tilde{f}_V(p+\lambda(p,z)\,|\,y^{**})}\cdot\frac{\tilde{f}_V(\underline{c}\,|\,y^{**})}{\tilde{f}_V(\underline{c}\,|\,y^*)},\quad p\in[\underline{c},\overline{b}_1(z)].$$

Given that $\lambda(\cdot,z)$ is continuously differentiable and strictly increasing by Assumption M, the likelihood ratio $\tilde{f}_V(\cdot\,|\,y^*)/\tilde{f}_V(\cdot\,|\,y^{**})$ is continuously differentiable and strictly decreasing as $\Gamma_1(\cdot,z)$ is continuously differentiable and strictly decreasing due to Lemma 2.2. □

C.5 Proof of Theorem 2.2

First, I shall show that under Assumptions I to N, it is implied by Assumption O that $\psi(p) < p$ and $m'(p) < 0$ for all $p \in (\underline{c}, \overline{b}_1(z^*)]$. Because $\xi(\cdot, z)$ is the buyer's equilibrium inverse bidding function when $Z = z$, let $v \in (\underline{c}, \overline{v}]$ be such that $v = \xi(p, z^*) = p + \lambda(p, z^*)$ then $v = \psi(p) + \lambda(\psi(p), z^{**}) = \xi(\psi(p), z^{**})$ by (2.7), which means $\psi(p) = \xi^{-1}(v, z^{**})$ while $p = \xi^{-1}(v, z^*)$. Since differentiating (2.7) with respect to p yields $1 + \partial_1 \lambda(p, z^*) = \psi'(p)\, [1 + \partial_1 \lambda(\psi(p), z^{**})]$, so $\partial_1 \xi^{-1}(v, z^*) > \partial_1 \xi^{-1}(v, z^{**})$ implies

$$\psi'(p) = \frac{1 + \partial_1 \lambda(p, z^*)}{1 + \partial_1 \lambda(\psi(p), z^{**})} = \frac{\partial_1 \xi(p, z^*)}{\partial_1 \xi(\psi(p), z^{**})} = \frac{\partial_1 \xi(\xi^{-1}(v, z^*), z^*)}{\partial_1 \xi(\xi^{-1}(v, z^{**}), z^{**})} = \frac{\partial_1 \xi^{-1}(v, z^{**})}{\partial_1 \xi^{-1}(v, z^*)} < 1.$$

Since $\psi(\underline{c}) = \underline{c}$, it is straightforward that

$$\psi(p) = \psi(\underline{c}) + \int_{\underline{c}}^{p} \psi'(u)\, \mathrm{d}u < \underline{c} + \int_{\underline{c}}^{p} 1\, \mathrm{d}u = p.$$

To see $m'(p) < 0$, note that by (2.8), $m(p) = \frac{a_1(y^*, z^{**})}{a_1(y^*, z^*)} \cdot \frac{F_C(\psi(p) \mid z^{**})}{F_C(p \mid z^*)}$ and then

$$\begin{aligned} m'(p) &= \frac{a_1(y^*, z^{**})}{a_1(y^*, z^*)} \left[\frac{f_C(\psi(p) \mid z^{**}) \psi'(p)}{F_C(p \mid z^*)} - \frac{F_C(\psi(p) \mid z^{**}) f_C(p \mid z^*)}{F_C(p \mid z^*)^2} \right] \\ &= \frac{a_1(y^*, z^{**})}{a_1(y^*, z^*)} \cdot \frac{F_C(\psi(p) \mid z^{**})}{F_C(p \mid z^*)} \left[\frac{f_C(\psi(p) \mid z^{**})}{F_C(\psi(p) \mid z^{**})} \psi'(p) - \frac{f_C(p \mid z^*)}{F_C(p \mid z^*)} \right] \\ &= m(p) \left[\frac{\psi'(p)}{\lambda(\psi(p), z^{**})} - \frac{1}{\lambda(p, z^*)} \right]. \end{aligned}$$

Because $\psi(p) < p$, (2.7) implies $\lambda(p, z^*) < \lambda(\psi(p), z^{**})$; then it follows from $\psi'(p) < 1$ that

$$\lambda(p, z^*) \psi'(p) < \lambda(p, z^*) < \lambda(\psi(p), z^{**}) \Rightarrow \frac{\psi'(p)}{\lambda(\psi(p), z^{**})} - \frac{1}{\lambda(p, z^*)} < 0,$$

and therefore $m'(p) < 0$ for $m(p) > 0$.

Because for any $p \in (\underline{c}, \overline{b}_1(z^*)]$, $m'(p) < 0$ and $\Delta_1(p) \geqslant 0$, (2.10) has two real solutions, namely $\lambda_{(1)}(p, z^*)$ and $\lambda_{(2)}(p, z^*)$.

Since $m(p) > 0$, so by Vieta's formulas, when $\psi(p) < p$ and $m'(p) < 0$, the product of these two solutions

$$\lambda_{(1)}(p, z^*) \cdot \lambda_{(2)}(p, z^*) = \frac{(p - \psi(p)) m(p)}{m'(p)} < 0.$$

It implies that either one of the two solutions is positive and the other is negative. Because it is assumed that $\lambda(p, z^*) > 0$ for all $p \in (\underline{c}, \overline{b}_1(z^*)]$, the negative solution should be ruled out.

Therefore, $\lambda(\cdot, z^*)$ is identified as the positive solution, i.e.

$$\lambda(p, z^*) = \frac{-[(p-\psi(p))m(p)]' + \operatorname{sgn}(p-\psi(p))\sqrt{\{[(p-\psi(p))m(p)]'\}^2 - 4(p-\psi(p))m(p)m'(p)}}{2m'(p)},$$

where $\operatorname{sgn}(\cdot)$ is the sign function.

Then by (2.7), $\lambda(\cdot, z^{**})$ will be identified as

$$\lambda(p, z^{**}) = \psi^{-1}(p) + \lambda(\psi^{-1}(p), z^*) - p, \quad p \in [\underline{c}, \overline{b}_1(z^{**})],$$

where $\psi^{-1}(\cdot)$ is the inverse function of $\psi(\cdot)$. It follows from the definition of $\lambda(\cdot,\cdot)$ that the corresponding seller's conditional value distributions are identified as

$$\frac{F_C(c \mid z)}{F_C(\overline{b}_1(z) \mid z)} = \exp\left(-\int_c^{\overline{b}_1(z)} \frac{1}{\lambda(u,z)} \, \mathrm{d}u\right), \quad c \in [\underline{c}, \overline{b}_1(z)],$$

for $z \in \{z^*, z^{**}\}$. □

C.6 Proof of Theorem 2.3

By applying the method of characteristics (see Rhee, Aris, and Amundson, 1986 or Vvedensky, 1993), the first-order linear differential equation in the form of (2.13) can be solved by solving its system of characteristic differential equations

$$\frac{\mathrm{d}z}{1} = \frac{\mathrm{d}p}{-\ell_1(p,z)} = \frac{\mathrm{d}\lambda}{\ell_1(p,z)} \tag{C.3}$$

with the boundary condition $\lambda(\cdot, z^*) = \Lambda(\cdot)$ where z^* is arbitrary point in $\mathcal{Z}$ and $\Lambda(\cdot)$ is a continuously differentiable function satisfying $\Lambda(\underline{c}) = 0$ and $\Lambda'(\cdot) > 0$.

Rewrite the boundary condition curve into its parametric form:

$$z(u) = z^*, \quad p(u) = u, \quad \lambda(u) = \Lambda(u), \quad u \in [\underline{c}, \overline{b}_1(z^*)].$$

Because

$$1 \cdot [-p'(u)] - [-\ell_1(p(u), z(u))] \cdot z'(u) = -1 \neq 0$$

for all $t \in [\underline{c}, \overline{b}_1(z^*)]$, the boundary condition prescribe above is non-characteristic; that is, geometrically, the projection of the boundary condition curve onto the (p,z)-plane does not coincide with the projections of any integral curves that satisfy (C.3). Therefore, the solution to the partial differential equation (2.13) subject to the boundary condition $\lambda(\cdot, z^*) = \Lambda(\cdot)$ is unique and the desired result follows. □

Remark. As a matter of fact, given the partial differential equation under review is in fact strictly linear, it can be shown that how the solution is uniquely obtained. First, it follows from (C.3) that

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\ell_1(p,z).$$

Since $\ell_1(\cdot,\cdot)$ is continuous by the fact that $\Gamma_1(p,z)$ is continuously differentiable in both p and z, this ordinary differential equation can be solved and denote the solution by $p = \rho(z,A)$ where A is a constant of integration which is determined by boundary condition $\rho(z^*,A) = p^*$ where $p^* \in [\underline{c}, \overline{b}_2(z^*)]$. Next, it also follows from (C.3) that

$$\frac{\mathrm{d}\lambda}{\mathrm{d}z} = \ell_1(p,z).$$

Then for any value of A, since $\lambda(\rho(z^*,A),z^*) = \lambda(p^*,z^*) = \Lambda(p^*)$ by the boundary condition specified, then taking the integral of the above ordinary differential equation along the curve $\{(p,z) : z \in \mathcal{Z},\ p = \rho(z,A)\}$, I can get for any $z \in \mathcal{Z}$,

$$\lambda(\rho(z,A),z) = \Lambda(\rho(z^*,A)) + \int_{z^*}^{z} \ell_1(\rho(t,A),t)\,\mathrm{d}t,$$

which is the desired solution of the partial differential equation (2.13).

C.7 Example that violates Assumption O but has $\lambda(\cdot,z^*)$ identified

It is implied by Assumption O that $p > \psi(p)$ and $m'(p) < 0$ for all $p \in (\underline{c}, \overline{b}_1(z^*)]$. Here is an example in which $m(p)$ is not monotone and $p - \psi(p)$ changes sign as p varies within $[\underline{c}, \overline{b}_1(z^*)]$ so that Assumption O is not satisfied. However, $\lambda(p,z^*)$ is still point identified in this example.

Suppose the private value support is $[\underline{c}, \overline{v}] = [0,1]$. The (true) seller's conditional value distributions are specified as such that

$$\lambda(p,z^*) = \frac{F_C(p \mid z^*)}{f_C(p \mid z^*)} = p, \quad 0 \leqslant p \leqslant \frac{1}{2}$$

and

$$\lambda(p,z^{**}) = \frac{F_C(p \mid z^{**})}{f_C(p \mid z^{**})} = \frac{2\sqrt{9p+1}-2}{3} - p, \quad 0 \leqslant p \leqslant \frac{7}{12}.$$

Let the (true) buyer's conditional value densities be $f_V(v \mid y^*) = 1$ and $f_V(v \mid y^{**}) = 2v$. Given such specification, the price densities $h_1(\cdot \mid y^*, z^*)$ and $h_1(\cdot \mid y^{**}, z^*)$ will have support $[\underline{c}, \overline{b}_1(z^*)] = \left[0, \frac{1}{2}\right]$;

the price densities $h_1(\cdot \mid y^*, z^{**})$ and $h_1(\cdot \mid y^{**}, z^{**})$ will have support $[\underline{c}, \overline{b}_1(z^{**})] = \big[0, \frac{7}{12}\big]$.

It can be verified that the price distributions imply

$$\psi(p) = p^2 + \frac{2p}{3}, \quad 0 \leqslant p \leqslant \frac{1}{2},$$

which features $\psi(p) < p$ if $0 < p < \frac{1}{3}$ and $\psi(p) > p$ if $\frac{1}{3} < p \leqslant \frac{1}{2}$. Meanwhile, it can also be verified that with some $A > 0$,

$$m(p) = A \cdot (4-3p)^{-\frac{5}{2}} p^{-\frac{1}{2}}, \quad m'(p) = A \cdot (9p-2)(4-3p)^{-\frac{7}{2}} p^{-\frac{3}{2}}, \quad 0 \leqslant p \leqslant \frac{1}{2}.$$

So $m(p)$ is strictly decreasing when $0 \leqslant p \leqslant \frac{2}{9}$ and strictly increasing when $\frac{2}{9} \leqslant p \leqslant \frac{1}{2}$. These mean that the conditions in Theorem 2.2 are not satisfied, except

$$\Delta_1(p) = A^2 \cdot \frac{\left(9p^2 - 42p + 10\right)^2}{9p(4-3p)^7} \geqslant 0$$

for all $0 \leqslant p \leqslant \frac{1}{2}$.

However, note that for $p \in \big[0, \frac{1}{2}\big] \setminus \big\{\frac{2}{9}\big\}$, $m'(p) \neq 0$, so by the quadratic formula, the two real solutions are

$$\lambda_{(1)}(p, z^*) = \frac{-\left[(p - \psi(p))m(p)\right]' + \sqrt{\Delta_1(p)}}{2m'(p)} = \begin{cases} \dfrac{9p^3 - 15p^2 + 4p}{27p - 6} & \text{if } p < \dfrac{7 - \sqrt{39}}{3}, \\ p & \text{otherwise;} \end{cases}$$

$$\lambda_{(2)}(p, z^*) = \frac{-\left[(p - \psi(p))m(p)\right]' + \sqrt{\Delta_1(p)}}{2m'(p)} = \begin{cases} p & \text{if } p < \dfrac{7 - \sqrt{39}}{3}, \\ \dfrac{9p^3 - 15p^2 + 4p}{27p - 6} & \text{otherwise.} \end{cases}$$

When $p = \frac{2}{9}$, (2.10) has only one real solution

$$-\frac{(p - \psi(p))m(p)}{\left[(p - \psi(p))m(p)\right]'}\bigg|_{p = \frac{2}{9}} = \frac{2}{9}.$$

The definition of solution $\lambda_{(2)}(p, z^*)$ above can be modified to accommodate this case.

However, there is only one solution of (2.10) satisfying Assumption M, i.e. $\lambda(p, z^*)$ is continuously differentiable and strictly increasing with $\lambda(\underline{c}, z^*) = 0$ (see Figure C.1):

$$\lambda(p, z^*) = \mathbb{1}\left(0 \leqslant p < \frac{7 - \sqrt{39}}{3}\right) \cdot \lambda_{(2)}(p, z^*) + \mathbb{1}\left(\frac{7 - \sqrt{39}}{3} \leqslant p \leqslant \frac{1}{2}\right) \cdot \lambda_{(1)}(p, z^*) = p.$$

Therefore, the seller's conditional value distribution $F_C(\cdot \mid z^*)$ on interval $[\underline{c}, \overline{b}_1(z^*)] = \big[0, \frac{1}{2}\big]$ is still identified, though conditions (ii) and (iii) required by Theorem 2.2 are not fulfilled.

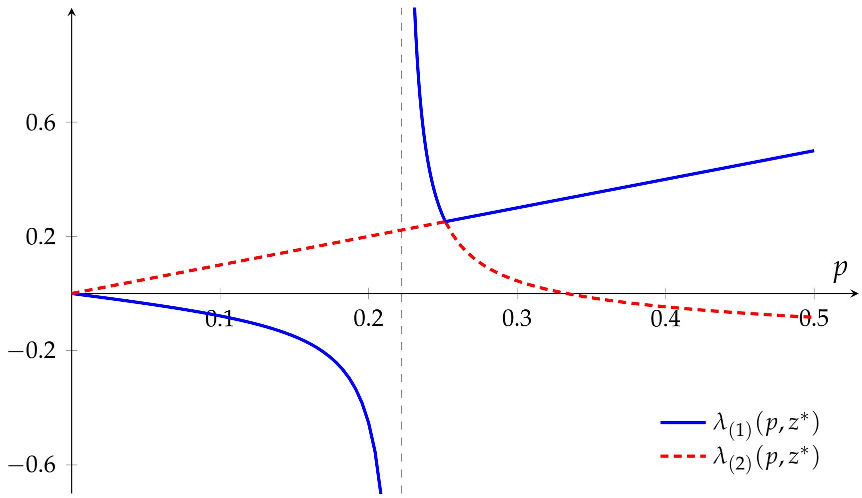


Figure C.1: Examine the solutions of equation (2.10)

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"Nonparametric Identification and Estimation of Double Auctions with Bargaining," with Nianqing Liu, 2015.
"Uniform Consistency of a Boundary Corrected Kernel Density Estimator," 2014.

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