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# ESSAYS ON MONETARY ECONOMIES WITH HETEROGENEOUS-AGENTS

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by

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# Abstract

This dissertation consists of three essays in monetary economics. Although the topic of each chapter differs, the approach is shared: I extend a random matching model of money by augmenting the set of money holdings, and compute socially desirable allocations in the spirit of mechanism design analysis. The augmentation is not just technically improving the model, but making the model rich enough to think about the economic problem that each chapter delves into. I document some interesting properties of the desirable allocations, and highlight the differences generated by the extension.

Chapter 1. "A beneficial role of government bonds"
I study a random matching model of money to show that the existence of bonds can be beneficial to a society, compared to having only money. In the model, anonymous agents randomly meet in pairs to produce and consume, hence money becomes essential. I compare two identical economies except the availability of bonds, in the sense that people can use any available assets as payments. Following the mechanism design approach, I define implementable allocations and the optimum. Under the notion of the implementability, social planner can devise trading mechanisms that induce people to hold both assets without exogenously given advantages of money as means of payment. I find that having both bonds and money in the economy can improve social welfare over having only money. This role of bonds is associated with a beneficial effect of inflation produced by lump-sum transfers, and it is achieved differently from the previously documented mechanism.

Chapter 2. "Optimal intervention in a random-matching model of money" (joint with Wataru Nozawa)
Wallace [2014] conjectures that there generically exists an inflation-financed transfer scheme that improves welfare over no intervention in pure-currency economies. We investigate this conjecture in the Shi-Trejos-Wright model with different upper bounds on money holdings. The choice of an upper bound affects the results as some potentially beneficial transfer schemes cannot be studied under small upper bounds. Numerical optima are computed

for different degrees of discounting rate and risk aversion. As the upper bound on money holdings increases, optima are more likely to have positive money creation (and inflation), and this result is in line with the conjecture.

Chapter 3. "Optimal inflation in a model of inside money: A further result" (joint with Wataru Nozawa)
We extend the Deviatov and Wallace [2014] model of inside money in which they find some examples where inflation is beneficial. Their model is restrictive in that it cannot address policies that provide interests on cash (Friedman rule). With a higher upper bound on money holdings than what they use, such policies can be engineered without inflation and resulting allocations are potentially better than what they find, in which case positive inflation is not a property of good allocation. We investigate this possibility and confirm their results in a more generalized setting for some parameters. At optima for the examples, interest on cash is not provided and positive inflation arises in a similar manner to their work. Welfare at optimum increases monotonically with respect to discount factor and public monitoring capacity of a society, but other variables change in a more complex way.

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# Chapter 1

# A Beneficial Role of Government Bonds

## 1.1 Introduction

In the well-known Hicks [1935], he asks why an individual chooses to hold money rather than interest bearing assets, and asserts that this is "the central issue in the pure theory of money". Putting it differently, if there are two types of risk-free assets that can potentially be used as payments for trades, and one of them earns more interest than the other, why does one hold an asset with lower interest? Once we deal with this issue, it naturally leads us to ask: Is it desirable for a society to have both money and higher return assets? In particular, given that nominal risk-free bonds do not provide additional risk-sharing over money, what additional benefit can such bonds provide over money?

The goal of this paper is to examine this question in a monetary economy in which coexistence of money and interest bearing assets (Hicks question) is achieved endogenously. I study a monetary economy in which people cannot borrow and lend, and cannot be taxed, due to frictions such as complete anonymity. Even under this extreme circumstance, social planner can circulate government bonds, whose return is higher than money, by financing the interest through money creation. The model builds on Deviatov [2006], where he studies a model related to Shi [1995] and Trejos and Wright [1995] with an augmented set of money holdings. Following the mechanism design approach in Monetary Economics (Wallace [2010]), he considers the set of implementable allocations and finds numerical examples that some degree of money creation (and inflation) is welfare improving. In this paper, I add an interest bearing asset (inflation-indexed bond) to that economy as a representative of government bonds, and show that having this asset can enhance social welfare.

The coexistence is achieved in the spirit of Zhu and Wallace [2007], where they show that there exists a trading mechanism under which 1) trade outcomes lie in pairwise core and 2) both money and bonds are held by people. Specifically, the trading mechanism divides gains from trade depending on the proportion of money in consumers' portfolios, using the multiplicity of pairwise core allocations. Hence, agents hold some amount of money and forgo interest from bonds for the sake of the gains from trade. Under the information structure and thet notion of defection that I assume, Zhu and Wallace [2007] trading mechanism is included to what social planner can choose. Consequently, the coexistence can be achieved without relying on any assumed advantages of money as a payment.

After laying out the environment and definition of implementability, I study the numerical examples that are adopted from Deviatov [2006], where he considers a same model with only money. I consider a richer set of transfer scheme than what he considers. Computation result shows that there are better allocations which are implementable with money and bonds, compared to the optima with only money. Compared to the optima with only money, transfer and inflation rate are higher in the better allocations. The result indicates that the beneficial effect of inflation produced by lump-sum transfers, as in Deviatov [2006], Green and Zhou [2005], Kehoe et al. [1992], Molico [2006], can be achieved more effectively by the existence of bonds.

Although this paper primarily aims to discover a role of government bonds, it also sheds some light on the welfare implication of currency substitution. When the inflation rate of a country is positive, the currency of that country loses its value over time. Suppose that agents in that country have an access to foreign currency, which is issued in a foreign country where inflation rate is zero. In that case, foreign currency is equivalent to bonds considered in this model. If social planner cannot force people to use only domestic currency, agents may hold and use foreign currency (currency substitution). Hence, the result can also be viewed as an implication of currency substitution.

### 1.1.1 Related Literature

In this subsection, I review two papers in which the coexistence is necessary to achieve desirable allocations. For the literature that attempt to rationalize the coexistence, I refer readers to Lagos [2013] and Hu and Rocheteau [2013].

Kocherlakota [2003] is the first normative analysis of the coexistence of money and higher return assets. He argues that the existence of illiquid bonds (illiquid because they cannot be used as payment for consumption goods) can improve social welfare, while liquid bonds cannot. The coexistence of money and bonds is achieved by assuming cash-in-advance constraint. Agents are subject to idiosyncratic one period marginal utility shock, and trade takes place in a competitive market. As people can adjust their portfolios after the shock realizes, agents with higher marginal utility can effectively borrow from agents with lower marginal utility by transferring purchasing power from the former to the latter using illiquid bonds. This implicit borrowing and lending enabled by illiquid bonds improves welfare in Kocherlakota [2003][1]. If bonds can be readily used as a payment as money, people will hold both assets only when there are no interest on bonds, so bonds are equivalent to money. There are

[1] Andolfatto [2011], Boel and Camera [2006], Shi [2008] show that the result persists in steady state. See also Kim and Lee [2009].

two main differences that distinguishes mine from Kocherlakota [2003]. Although there is a idiosyncratic shock (producer and consumer status), which is similar to the preference shock in Kocherlakota [2003], I assume that people can adjust their asset holdings only before the shock realizes. More importantly, while it is assumed that bonds can be made illiquid in Kocherlakota [2003], it may not be a desirable assumption. If bonds are not illiquid per se, people have incentive to deviate from using only money when they trade. Even when social planner can force people to use only money for payment, it may not be optimal to do so. I address this issue by considering a class of trading mechanism, under which people can jointly defect in pairwise trade.

Hu and Rocheteau [2013] uses an economy that builds on Lagos and Wright [2005], where they extend the setting by adding physical capital. Physical capital plays double duty in that economy, as an input for production in one subperiod and means of payment in another subperiod. Hence, physical capital can be accumulated too much to supplement money as a medium of exchange. Like mine, they also follow the mechanism design approach and consider a class of trading mechanisms, and characterize the optimal one. Under the optimal trading mechanism, physical capital have higher rate of return than money to prevent the overaccumulation of capital. So, coexistence of money and assets with higher return (physical capital) is necessary to achieve welfare improvement. As I consider intrinsically useless bonds rather than physical capital as higher return assets, such inefficiency does not arise here. I investigate a distinctive channel through which higher return assets improve social welfare.

## 1.2 The Model

The background setting is a variant of Shi [1995] and Trejos and Wright [1995]. Time is discrete and infinite. There are nonatomic measure of people, who live forever and maximize expected lifetime utility with discount factor $\beta \in (0, 1)$. People are anonymous, so their histories are private information and they cannot commit to future actions. In each period, agents sequentially enter to portfolio choice stage and pairwise trade stage.

All production and consumption occur in pairwise trade stage. In this stage, people randomly meet in pairs and the period status, a consumer, a producer, or inactive, is determined randomly according to the parameter $\mu \in \left(0, \frac{1}{2}\right)$. One becomes a producer or a consumer with probability $\mu$ respectively, and inactive with probability $1 - 2\mu$. Underlying structure of this specification is in Shi [1995] and Trejos and Wright [1995] (see also Williamson and Wright [2010]). Period utility function of an agent is $u(y) - x$, where $y,\ x \in \mathbb{R}_+$ are amount of consumption and production respectively. As one cannot produce and consume in a same period, at most one of $y$ and $x$ is strictly greater than zero in a period for an agent. $u(y)$ is strictly increasing, strictly concave, differentiable, and satisfies $u(0) = 0$. Also, $\hat{y} \equiv \max_{y \geq 0} [u(y) - y]$ is strictly positive. All produced goods are perishable. As no record-keeping is feasible in this economy, one cannot use any form of credit for trades. As a result, it is not possible to achieve any production and consumption without a medium of exchange.

There are two kinds of intrinsically useless and indivisible assets which can be used as a medium of exchange. I will call the first asset "currency" and the second asset "bonds"[2]. These assets are fully recognizable and cannot be counterfeited. An agent can hold assets up to 2 units in sum at any point of time. The set of feasible individual portfolio is

$$\mathbb{Z} \equiv \left\{(M_C, M_B) \in \{0,1,2\}^2 \mid M_C + M_B \leq 2\right\} = \{(0,0), (0,1), (0,2), (1,0), (1,1), (2,0)\}$$

In portfolio choice stage, agents can redeem their bonds for currency at one to one rate, and they can purchase bonds using currency also at one to one rate through a window that social planner operates. While this window can potentially be more sophisticated, by differentiating

[2]I call it "currency" rather than "money" as "bonds" can be used as medium-of-exchange here.

**Figure 1.2.1.** Sequence of actions.


exchange rate depending on the asset holdings, one virtue of this window is that it can be implemented by a competitive market. I assume that bonds mature in one period and are redeemed only in the following period[3]. One crucial (and the only meaningful) difference between two assets is that currency may disintegrate at the end of a period while bonds do not, in the following sense.

At the end of a period, people who are not at the upper bound of assets holding[4] receive an additional unit of currency with probability $\epsilon(z) \in [0,1]$, where the rate can depend on the asset holdings. After the transfer, each unit of currency disintegrates with probability $\tau \in [0,1]$. Hence, an agent with $j$ unit of currency will lose all of her currency with probability $\tau^j$, $j-1$ unit of currency with probability $(1-\tau)\,\tau^{j-1}$ and so on. This process is our stand-in for currency transfer and inflation, and they are modeled in this way to cope with the upper bound on assets holdings. This technique was also used in Li [1995], Deviatov and Wallace [2001] and Deviatov [2006]. Bonds do not suffer from this disintegration process, and this difference makes bonds a higher return asset. For simplicity, I only consider such inflation-indexed bonds. At least conceptually, it is easy to accommodate bonds with different interest rates in this setup. Lastly, two rates are chosen by social planner as policy variables.

[3]It is equivalent to assume that bonds can be redeemed at any periods and the date of issuance is disregarded.

[4]For example, if an agent has one unit of each asset, he cannot get the transfer since he is at the upper bound already. This assumption is made due to the upper bound on the asset holdings and for simplicity. In this sense, the scheme cannot perfectly resemble lump-sum transfer.

**Table 1.1.** Information structure (column) and notion of defection (row).

Trade stage

| defection \ info. | sym | asym |
| --- | --- | --- |
| individual | | |
| group | X | |

Transfer stage

| defection \ info. | sym | asym |
| --- | --- | --- |
| individual | | X |
| group | | |

## 1.3 Implementable Allocations

Before defining the implementability, I describe information structure and the notion of defection. Table 1 shows the possible specifications in each stage, where a cell is a combination of the allowed defection (row) and the information structure (column). The specification that I use is marked with X. I assume that people in a pair can defect jointly (group defection), in the sense that they can reject the designated trade by the planner if it is not in the pairwise core. This assumption puts more discipline on social planner, as the trading mechanism must exploit all the gains from trade. There is no private information in trade stage (symmetric information), hence the pairwise core is well-defined in that stage.

People cannot jointly defect in transfer stage (individual defection). An implication of this assumption is that they cannot pool their assets to exploit the transfer from the planner. While they cannot overstate their asset holdings (it can be verified simply by asking to show their assets), they can understate (asymmetric information). Hence, transfer policy is incentive compatible only when it is an increasing function of asset holdings.

I focus only to stationary allocations due to tractability. Hence, all time subscripts are omitted. Briefly speaking, implementable allocations satisfy three conditions in addition to stationarity: (1) *optimal portfolio choice*, (2) *coalition-proofness*, (3) *incentive compatibility of transfer policy*.

I will denote the expected discounted utility for an agent with portfolio $z$ who enters the pairwise trade meeting stage by $v(z)$. The wealth (the sum of currency and bonds) distribution at the portfolio choice stage is $\{\pi_k\}_{k\in\{0,1,2\}}$. As we don't need to distinguish

**Table 1.2.** Notations.

| Symbol | Description |
|---|---|
| $\epsilon(z) \in [0,1]$ | Transfer rate |
| $\tau \in [0,1]$ | Disintegration rate |
| $t^{(1)}: \{0,1,2\} \to \Delta(\mathbb{Z})$ | Transition Probability (Portfolio Choice Stage) |
| $t^{(2)}: \mathbb{Z} \to \Delta(\mathbb{Z})$ | Transition Probability (Trade Meeting Stage) |
| $t^{(3)}: \mathbb{Z} \to \Delta(\{0,1,2\})$ | Transition Probability (Transfer and Disintegration) |
| $v: \mathbb{Z} \to \mathbb{R}_+$ | Expected Discounted Utility (Pre Trade Stage) |
| $w: \mathbb{Z} \to \mathbb{R}_+$ | Expected Discounted Utility (Post Trade Stage) |
| $p^{(1)} \in \Delta(\mathbb{Z})$ | Portfolio Holding Dist. (Pre Trade Stage) |
| $p^{(2)} \in \Delta(\mathbb{Z})$ | Portfolio Holding Dist. (Post Trade Stage) |
| $\pi \in \Delta(\{0,1,2\})$ | Wealth Distribution (Portfolio Choice Stage) |
| $y: \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}_+$ | Production Level (Trade Stage) |
| $\lambda_S: \mathbb{Z} \times \mathbb{Z} \to \Delta(\mathbb{Z})$ | producer's Portfolio Dist. (Post Trade Meeting) |
| $\lambda_B: \mathbb{Z} \times \mathbb{Z} \to \Delta(\mathbb{Z})$ | consumer's Portfolio Dist. (Post Trade Meeting) |
| $\gamma: \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}_+$ | producer's Surplus (Trade Meeting) |

currency and bonds when people are entering this stage, it suffices to track the wealth distribution here. Transition probability from wealth level $k$ to portfolio $z$ at the portfolio choice stage is $t^{(1)}_{zk}$. All variables are listed in the table 2.

In portfolio choice stage, each agent chooses a portfolio that maximizes his expected utility. *Optimal portfolio choice condition* is satisfied when

$$z' \in \underset{\{x \in \mathbb{Z} | x_C + x_B \leq k\}}{\arg\max} \; v(x) \;\; \text{, if } t^{(1)}_{z'k} > 0 \tag{1.3.1}$$

For given $t^{(1)}$and $\pi_k$, the measure of people with portfolio $z$ at trade stage $p^{(1)}_z$ is

$$p^{(1)}_z = \sum_{k \in \{0,1,2\}} t^{(1)}_{zk} \pi_k$$

Given that there are 6 possible portfolios that consumers and producers can hold, we have 36 types of different meetings. Each type of meeting is a combination of the portfolios of the producer and the consumer $(z_p,\ z_c)$. Denote the production level in each meeting by $y(z_p, z_c)$, and let $\lambda_c(z'_c; z_p, z_c)$, $\lambda_p\left(z'_p; z_p, z_c\right)$ be the distribution of the portfolios of the consumer and the producer, as a result of the trade meeting. Payments of currency and bonds are embedded in this representation, since those distributions can be derived from a given payment of currency and bonds, and vice-versa. In sum, transition probability from portfolio $z$ to portfolio $z'$ at the trade stage $t^{(2)}_{z'z}$ is characterized by the probability of entering to each type of meeting and having a certain portfolio after that meeting.

$$t^{(2)}_{z'z} = \begin{cases} \mu \sum_x p^{(1)}_x \left[\lambda_S\left(z'; z, x\right) + \lambda_B\left(z'; x, z\right)\right] & \text{if } z' \neq z \\ 1 - \sum_{z' \neq z} t^{(2)}_{z'z} & \text{if } z' = z \end{cases}$$

The distribution of agents over portfolios after the trade meeting stage, $p^{(2)}$, is

$$p^{(2)}_{z'} = \sum_{z \in \mathbb{Z}} t^{(2)}_{z'z} p^{(1)}_z$$

The currency transfer rate $\epsilon_z \in [0, 1]$ and the disintegration rate $\tau \in [0, 1]$, which are chosen by the social planner, determine the transition probability from portfolio $z$ to wealth level $m$ at transfer and disintegration stage $t^{(3)}_{mz}$. For example, if an agent holds one unit of currency, she will end up with one unit of currency with probability $t^{(3)}_{(10,10)} = \epsilon_{10}\tau + \left(1 - \epsilon_{10}\right)\left(1 - \tau\right)$, where the first term represents the probability of receiving one unit of currency and losing it, and the second term represents the probability of not receiving nor losing. Note that agents are allowed to hide their asset holdings. So, they will misrepresent their asset holdings if the transfer scheme is not incentive compatible. The transfer policy is *incentive compatible* if and only if

$$\epsilon_z \geq \epsilon_{z'} \text{ whenever } z \geq z' \tag{1.3.2}$$

The wealth (the sum of currency and bonds) distribution at the portfolio choice stage in the next period satisfies

$$\pi'_k = \sum_{z \in \mathbb{Z}} t^{(3)}_{kz} p^{(2)}_z \text{ for } k \in \{0, 1, 2\}$$

Two value functions, before and after trade meeting stage, are

$$\begin{aligned}
v(z) &= \mu \sum_{x\in\mathbb{Z}} p_x^{(1)} \left\{ u\left[y(x,z)\right] + E_{\lambda_C(x,z)} w - y(z,x) + E_{\lambda_P(z,x)} w \right\} \\
&\quad + (1-2\mu)\, w(z) \\
w(z) &= \beta \sum_{x\in\mathbb{Z}} \sum_{m\in\{0,1,2\}} t_{mz}^{(3)} t_{xm}^{(1)} v'(x)
\end{aligned}$$

for all $z \in \mathbb{Z}$, where

$$\begin{aligned}
E_{\lambda_C(x,z)} w &\equiv \sum_{z'_C} \lambda_C\left(z'_C; x, z\right) w\left(z'_C\right) \\
E_{\lambda_P(x,z)} w &\equiv \sum_{z'_P} \lambda_P\left(z'_P; x, z\right) w\left(z'_P\right)
\end{aligned}$$

Lastly, stationarity requires

$$\pi'_k = \pi_k \text{ for all } k \in \{0,1,2\} \tag{1.3.3}$$

$$v'(z) = v(z) \text{ for all } z \in \mathbb{Z} \tag{1.3.4}$$

While many works using this class of models adopt some kind of bargaining solution to determine production levels and payments in trade meeting, I adopt mechanism design approach as described in Wallace [2010], which considers all implementable allocations. Using this approach has two important implications. First, it enables to achieve coexistence of currency and bonds in the spirit of Zhu and Wallace [2007]. As two assets have no differences in physical properties (i.e. recognizability, storage costs among many others) other than their labels and vulnerability to disintegration, agents will not hold currency if trade outcomes are determined by a bargaining solution under which trade outcomes depend only on the feasible allocations and termination values. Second, it eliminates the loss attributable to sub-optimal trading rule. I aim to show that bonds can still improve welfare even when the welfare loss due to sub-optimal trading rule disappears.

Production and payment are required to maximize consumers' utility subject to making producers at least better than no trade. $\gamma(z_P, z_C)$ is the surplus of producers in each type of meeting. Formally, for any given $w$, $\{y(\cdot,\cdot), \lambda_C(\cdot;\cdot,\cdot), \ \lambda_P(\cdot;\cdot,\cdot)\}$ are *coalition-proof* if for every $z_P$ and $z_C$, there exists $\gamma(z_P, z_C) \geq 0$ such that $\{y(z_P, z_C), \lambda_C(z_P, z_C), \lambda_P(z_P, z_C)\}$ is a solution to the following problem:

$$\begin{aligned}
\max_{x\in\mathbb{R}_+,\eta_C\in\Delta(\mathbb{Z}),\eta_P\in\Delta(\mathbb{Z})} \quad & u(x) + E_{\eta_C} w - w(z_C) \\
\text{subject to} \quad & -x + E_{\eta_P} w - w(z_P) \geq \gamma(z_P, z_C) \\
& u(x) + E_{\eta_C} w - w(z_C) \geq 0 \\
& \eta_C(z_P + z_C - z') = \eta_P(z') \text{ for } z' \in \{(m,b) \,|\, 0 \leq (m,b) \leq z_P + z_C\} \\
& \eta_P(z') = \eta_C(z') = 0 \text{ for } z' \notin \{(m,b) \,|\, 0 \leq (m,b) \leq z_P + z_C\}
\end{aligned}$$

Now, we can define implementable allocations.

**Definition 1.** An allocation $p^{(1)} \in \Delta(\mathbb{Z})$, $y: \ \mathbb{Z}\times\mathbb{Z} \to \mathbb{R}_+$, and $\lambda: \ \mathbb{Z}\times\mathbb{Z} \to \Delta(\mathbb{Z})\times\Delta(\mathbb{Z})$ is *implementable* if there exist $\epsilon$, $\tau$, $v$, $\pi$, $t^{(1)}$ that satisfy *optimal portfolio choice condition (3.1), incentive compatibility of transfer policy (3.2), stationarity (3.3), (3.4) and coalition-proofness.*

As standard in the literature, I use the ex-ante utility level, where the initial portfolios of agents are randomly assigned according to the stationary distribution, as the welfare criterion.

$$W \equiv \sum_z p_z^{(1)} v(z)$$

Maximizing it is equivalent to maximizing one period total surplus

$$\sum_{z}\sum_{z'} p_z^{(1)} p_{z'}^{(1)} \left[u\left(y_{zz'}\right) - y_{zz'}\right]$$

An optimum achieves the highest welfare among all the implementable allocations.

Under this notion of the implementability, the set of implementable allocations in an economy with only currency (henceforth, currency economy) is nested to the set in an economy with currency and bonds (henceforth, bond economy). In other words, adding bonds to an economy is always weakly better. Pick an arbitrary implementable allocation in currency economy. To implement the same allocation in bond economy, it suffices to restrain agents from choosing bonds in portfolio choice stage, and that can be done by the following trading mechanism. Suppose that agents are asked to produce and pay as before in any meetings in which people hold only currency. In meetings where only one person holds bonds, the bond holder gets no gains from trade and the meeting partner gets all gains. If this is the case, given that no one else holds bonds, an agent will not choose to hold bonds at the portfolio choice stage. Since production and payment are the same as before, this trading mechanism implements a same allocation in bond economy. This nesting argument relies on two assumptions that I made. First, if the bond earns higher interest, agents may sacrifice the period surplus for the sake of the interest. In this case, social planner is not able to prevent people from holding bonds as above. Second, this argument is not possible if agents' portfolios are not visible in trade stage.

*Remark.* An implementable allocation in currency economy is also implementable in bond economy.

Before moving on to the next section, I want to make and justify an assumption on the meetings where producers already have two units of assets. When a producer is already at the upper bound of asset holdings with some currency (1 unit of each asset, or 2 units of currency), he may be willing to produce to earn bonds by swapping assets. Suppose that a producer with two units of currency meets a consumer with one unit of bond. Even though the producer has two units of assets already, he may be willing to produce to exchange his currency to a bond. In the following, I assume that this kind of trade cannot happen.

**Assumption.** *If a producer has two units of any assets, he cannot alter his portfolio by exchanging assets in trade stage.*

By making this assumption, I want to shut down one channel through which bonds can improve the welfare in this class of model, a reminiscent of Aiyagari et al. [1996]. Hence, welfare improvement after making this assumption will be attributed to a different channel. In the Appendix A, I explain the rationale of the assumption using a same economy with a one-unit upper bound instead of a two-unit upper bound on asset holdings. I construct a numerical example in which adding bonds improves welfare by increasing the number of trade meetings, similar to Aiyagari et al. [1996], and then prove that this improvement disappears once asset swapping is not allowed. Hence, adding another asset in a two-unit upper bound economy is not a mere extension of the one in a one-unit upper bound economy. With richer set of asset holdings, we can see a different channel of welfare improvement from adding bonds.

## 1.4 Numerical Examples

In this section, I use numerical examples to learn about a beneficial role of government bonds. I use same utility function and parameters in Deviatov [2006], which are arbitrary except for the discount factor $\beta$. Define the first-best as people produce and consume $y^* \equiv \max_{y \geq 0} \left[u\left(y\right) - y\right]$ in every trade meeting. If perfect monitoring is possible, the first-best allocation is implementable whenever $\beta$ satisfies

$$\frac{u\left(y^{*}\right)}{y^{*}} \geq \frac{1-\beta}{\beta\mu}+1$$

The discount factors that I used satisfy above inequality, so that the first-best allocation would be implementable under perfect monitoring.

I first compute the optimum in currency economy. The optimum in this economy provides the upper bound of welfare that the planner can achieve using only currency. The only difference between this computation and Deviatov [2006] is that transfer can now depend on asset holdings. Then, I formulate the maximization problem of social planner in bond economy, and find a better allocation. For simplicity, I assume that transfer rate is the same regardless of asset holdings. This assumption is innocuous for my purpose as it restricts what government can do in bond economy. I use GAMS (General Algebraic Modeling System) and BARON (Branch-And-Reduce Optimization Navigator) solver to compute these problems. GAMS is a modeling system for mathematical programming and optimization, and BARON solver can be used in this system. BARON solver uses deterministic global optimization algorithms of the branch-and-bound type, and it is one of the most robust global solver according to Neumaier et al. [2005]. While the program did not complete the computation for the bond economy[5], I could find a better allocation than the optimum in currency economy. In the course of finding the global solution, the solver constantly updates and records its candidate solution, and the candidate is an implementable allocation. By comparing that allocation to the optimum in currency economy, we can learn how government can use bonds to improve welfare.

I use $u\left(y\right)=y^{0.2}$, the coincidence parameter $\mu=\frac{1}{3}$, and two discount factors $\beta \in \left\{\frac{1}{2}, \frac{2}{3}\right\}$. Under this specification, the first-best production level is $y^{*} \approx 0.1337$. In Deviatov [2006],

[5]It is known that global solvers are generally slower than local solvers, but other alternatives are not more suitable for my purposes.

he finds positive transfer and inflation optimal for the lower discount factor ($\beta = \frac{1}{2}$), while no transfer and zero inflation optimal for the higher one ($\beta = \frac{2}{3}$). It turns out that allowing different transfer rate for different asset holdings does not change the optimum for these examples, as the optimum in currency economy shows same transfer rate for different asset holdings. The computed optimum is same as the optimum in Deviatov [2006][6].

Table 3 shows welfare and aggregates of the optimum in currency economy and a better allocation in bond economy, when discount factor $\beta$ is $\frac{1}{2}$. The welfare level, relative to the first-best one, is in the first row. The second and third row show transfer and disintegration rate in each allocation. Transfer rate is only one number as optimal transfer rate is same for different asset holdings in currency economy, and it is assumed to be same in bond economy. They are strictly positive in both allocations, but transfer rate and disintegration rate are higher in the bond allocation. This change is consistent with the change in distribution ($\pi$), which is in the last row. Each number in the cell of that row shows the proportion of people with 0,1,2 units of assets respectively, in the beginning of each period. In the bond allocation, agents with 2 units of assets choose one of each asset at portfolio choice stage. Agents with 1 unit of asset choose to hold currency. Hence, there are 4 types of trade meeting that happen on equilibrium in each economy, and we can compare those meetings across the economies.

Each cell in table 4 contains production and payment in a meeting in which the producer and the consumer have portfolios corresponding to the row and the column. For the portfolios in bond economy, I use the first digit for currency and the second digit for bonds. The first number in each cell shows the production level relative to first-best, and the number(s) in the right is the amount of assets paid from the consumer to the producer. Note that it is not optimal to randomize over production as the utility function is concave and cost

[6] While it is out of scope of this paper, an example showed that incentive compatibility of transfer policy is binding. When asset holdings are not private information at transfer stage, we can disregard the incentive compatibility of transfer policy. In this case, the optimum of currency economy did not show same rate for different asset holdings, and welfare was strictly higher than the one with equal transfer rate.

**Table 1.3.** Welfare and aggregates ($\beta = 0.5$).

| | Currency Optimum | A Better Bond Allocation |
|---|---|---|
| $\frac{W}{W^*}$ | 0.419 | 0.426 |
| $\epsilon(\%)$ | 0.250 | 0.472 |
| $\tau(\%)$ | 0.176 | 0.346 |
| $\pi$ | 0.369 / 0.397 / 0.234 | 0.346 / 0.418 / 0.237 |

**Table 1.4.** Output (relative to the first best level) and payment in trade stage.

Currency Optimum

| $P \backslash C$ | 1 | 2 |
|---|---|---|
| 0 | 0.435 / 1 | 0.435 / 1 |
| 1 | 0.120 / 1 | 0.120 / 1 |

A Better Bond Allocation

| $P \backslash C$ | 1,0 | 1,1 |
|---|---|---|
| 0,0 | 0.282 / 1,0 | 0.431 / 0,1 |
| 1,0 | 0.103 / 1,0 | 0.206 / 0,1 |

function is linear. It must be deterministic to satisfy coalition-proofness constraint. Although lotteries for payments are potentially useful, they are not used in both allocations. In all of meetings that occur on equilibrium, consumers get all the surplus of trade and producers become indifferent to no trade. Hence, the trading mechanism is equivalent to the one where consumers make take-it-or-leave-it offers in those meetings. Needless to say, that is not the case for the meetings that do not occur on equilibrium. Production level is higher in (10,11) meeting, where the first element indicates the producer's portfolio, than the comparable (1,2) meeting in currency economy. However, production levels in all other meetings are lower. In a sense, the production is smoothed over different types of meetings. In all of meetings in both economies, one unit of asset is transferred with certainty (numbers in the parentheses). When a consumer has bonds, one unit of bond is paid. Otherwise, he pays with one unit of currency.

One may wonder why the output level of (00,11) meeting in bond economy is lower than the output of (0,2) meeting in currency economy (first row, second column). The producer in bond economy gets a bond, which does not suffer from disintegration, while his counterpart in currency economy gets a unit of currency. One reason is that currency has lower value in the bond allocation. Because of the higher transfer and inflation in the bond allocation, the return on currency decreases. As the value of bonds is tied to the value of currency, it can result in lower output level. In contrast, the output level in (10,11) meeting is higher than the one in (1,2) meeting. That can be explained by the concavity of value functions. If we compare the two value functions in currency optimum and the bond allocation ($v^C(\cdot)$ and $v^B(\cdot)$),

$$\begin{aligned} v^C(1) - v^C(0) &\geq v^B(10) - v^B(00) \\ v^B(11) - v^B(10) &\geq v^C(2) - v^C(1) \end{aligned}$$

For an agent who already has a unit of currency, attaining an additional asset is more worthwhile in the bond allocation. As a result, the output level in (10,11) meeting is higher than the one in (1,2) meeting.

I find a similar result with a higher discount factor, $\beta = \frac{2}{3}$. In this example, the optimal transfer and inflation rate is zero in currency optimum. But, a better allocation can be implemented with bonds, positive transfer and inflation. Now, agents with one unit of asset choose a bond, while agents with two units of asset still choose one unit of each asset. The pattern of aggregates is similar to the previous one: higher transfer, disintegration, and more people in the center of the distribution. One thing to note is that the magnitude of transfer rate is smaller than the disintegration rate, while it was the opposite before. As more people hold bonds now, higher inflation follows even with small amount of currency creation.

As before, consumers get all the surplus of trade in meetings that occur on equilibrium. Production level decreases in all meetings that actually happen. So, the improvement on welfare is mostly attributable to the improved extensive margin. When consumers have only one asset, they use lottery for payment in both economies. In addition, people in the bond allocation pay different assets from previous example. Consumers with two units of assets pay one unit of currency now, while consumers with one unit of asset pay a bond.

**Table 1.5.** Welfare and aggregates ($\beta = \frac{2}{3}$).

| | Currency Optimum | A Better Bond Allocation |
|---|---|---|
| $\frac{W}{W^*}$ | 0.478 | 0.487 |
| $\epsilon(\%)$ | 0 | 0.057 |
| $\tau(\%)$ | 0 | 0.176 |
| $\pi$ | 0.323 / 0.430 / 0.247 | 0.316 / 0.477 / 0.207 |

**Table 1.6.** Output (relative to the first best level) and payment in trade stage.

Currency Optimum

| $P\backslash C$ | 1 | 2 |
|---|---|---|
| 0 | 1 / 0.587 | 1.703 / 1 |
| 1 | 0.212 / 0.431 | 0.493 / 1 |

A Better Bond Allocation

| $P\backslash C$ | 0,1 | 1,1 |
|---|---|---|
| 0,0 | 1 / 0, 0.656 | 1.127 / 1,0 |
| 0,1 | 0.139 / 0, 0.442 | 0.257 / 1,0 |

So how do bonds improve the welfare in this economy? The optimal rate of transfer and disintegration balance the trade-off between *beneficial effect* on the extensive margin (number of trade meeting) and *harmful effect* on the producer's incentive to produce for getting currency. Lump-sum transfer and inflation is beneficial on the extensive margin, as they alter the asset holdings distribution in a way that makes use of more trade opportunities. If a producer has two units of assets, or a consumer has none, that trade opportunity will be wasted. Some lump-sum transfer and inflation are helpful to prevent such a loss. However, they also decrease the return on currency, so that a producer would be willing to produce less for getting currency. If the *harmful effect* on producers' incentive can be mitigated, higher welfare can be achieved through higher rate of currency transfer. Note that wealthy agents loses more when $\tau$ increases. Hence, by inducing wealthy agents to hold one unit of bond, social planner can mitigate the incentive effect in more sophisticated way. In consequence, the bond allocation results in higher welfare by having higher transfer rate (and disintegration rate), while the presence of bonds mitigate the loss to wealthy agents resulting from this increase in transfer and inflation rate. The change in distribution is in line with this argument.

Comparison of $v$ provides a different account of the welfare gain. In the first example, people with 0 wealth level and 2 wealth level are better off, while agents with 1 unit of wealth

**Table 1.7.** Expected utility (relative to first-best level) for each wealth level.

$\beta = \frac{1}{2}$

| | Currency optimum | The Bond allocation |
|---|---|---|
| 0 | 0.0987 | 0.1273 |
| 1 | 0.5776 | 0.5395 |
| 2 | 0.653 | 0.6637 |

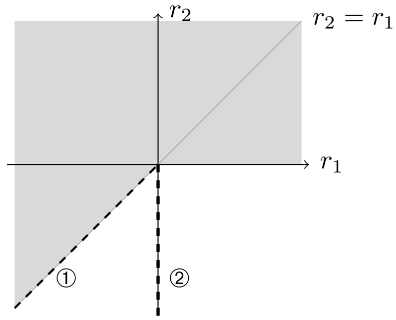
$\beta = \frac{1}{2}$

| Currency optimum | The Bond allocation |
|---|---|
| 0 | 0.0553 |
| 0.6387 | 0.6492 |
| 0.8234 | 0.7731 |

are not in the bond allocation. Note that 0 wealth agents can only be a producer, who earns no trade surplus in trade stage in both allocations. However, he has a higher chance of having 1 wealth, due to higher rate of currency creation. 2 wealth agents are better off since they can induce more production from 1 wealth agents, and the loss from disintegration is mitigated by holding bonds. On the contrary, high currency creation rate and disintegration rate is not beneficial to 1 wealth agents since the gain from transfer (becoming 2 wealth agents by earning free currency) is not compensating the loss (becoming 0 wealth agents due to disintegration). Also, he consumes less due to lower value of currency in this economy. The second example shows a different pattern. While people with 0 wealth still get better in the bond allocation, changes in expected utility for people with 1 and 2 wealth are in contrast to the first example. People with 1 wealth get better due to the chance of earning free currency and the changes in distribution, while disintegration does not decrease the value of their payment (bonds). Even though 2 wealth agents in the bond allocation hold a unit of bond, he (possibly) loses currency due to disintegration, while they don't in currency optimum as the disintegration rate is zero. In addition, they consume less as their payment has lower value. While they could use a bond for payment, they choose to retain it for not putting themselves at risk of having no asset due to disintegration.

Based on this explanation, it is reasonable to conjecture that the beneficial role of bonds will disappear for some parameters. In Deviatov [2006], positive transfer and inflation is not beneficial for a range of parameters (high discount factor, low risk aversion), as the harmful effect dominates the beneficial effect for small transfer and inflation. As the role of bonds is

**Figure 1.4.1.** Real rates of return on assets.




associated with the beneficial effect of transfer and inflation, I expect that the optimum will not make use of bonds under high discount factor and low risk aversion. Verification of this conjecture requires finding the optimum with two assets, which is not feasible as of now due to computational limitations. However, the program did not find any better allocations than the currency optimum even after considerable computation time for a high discount factor ($\frac{1}{1.05}$) case, and this is consistent with the conjecture.

Figure 4.1 illustrates the region that Deviatov [2006] and this paper investigated. Consider an economy where people are anonymous and cannot be taxed as in this paper. Each axis represents the (real) rate of return of a fiat asset, and think of a point as an economy with two fiat assets with returns corresponding to the point. Since fiat assets are intrinsically worthless, it is impossible to support any points in first quadrant including the axes in that region without tax. Also, all points above 45 degree line are mirror-image of points in the region below the line. So the gray area is not of interest. Deviatov [2006] considers the region on the 45 degree line, currency only economy, excluding the aforementioned regions, and the area is ①. This paper expands the region to include negative y-axis, so ① and ②. Points between those two lines can be studied by making bonds partially indexed to inflation rate. Some points in the fourth quadrant may be achievable by paying higher interest rate than inflation rate, but not all of them are. For example, if we think of a point $(x, y)$ in the fourth quadrant with high $x$ and $y$ close to zero, inflation rate should be close to zero while the real interest rate on bonds is very high. Since they are intrinsically worthless assets, it is infeasible to support such a high return without causing high inflation. In general, if we can define a feasible region, the planner would not choose a point on the lines that are studied here. I leave for future work the study of the optimal rate of returns.

## 1.5 Concluding Remarks

Should two forms of government liabilities - currency and government bonds - exist? Friedman [1948] argues that government should get rid of interest bearing government liabilities and finance its expenditures through money creation and tax. Although there already exist a number of papers on societal benefits of interest bearing assets following Kocherlakota [2003], this paper differentiates itself from the previous ones as following. First, coexistence is achieved without relying on any assumed advantages of currency over bonds used. Second, the interest bearing asset still improves welfare after eliminating the loss from sub-optimal trading mechanism. Lastly, the role exists only when money creation is potentially useful[7].

As I have presented only two examples, one may wonder if the result holds more generally. It is limited in the sense that assets are indivisible and there is a somewhat small upper bound on assets holding. Applying other parameters can be easily done, but it does not seem to add much as the contribution of this paper is the discovery of new channel through which bonds improve social welfare. I suspect that the result would persist with divisible and unbounded asset holdings as well. Suppose there are currency and inflation-indexed bond, and social planner is able to choose the transfer (in a lump-sum fashion) rate of currency. Social planner will still have some power, though limited, to induce wealthy agents to choose a portfolio with more of indexed bond, by appropriately dividing trade surplus contingent on portfolios. This will create another policy dimension, and the presence of inflation-indexed

[7] Wallace [2014] conjectures that there generically exists beneficial government money creation in a class of monetary economies

bond can implement an allocation which was not implementable before. If money to be essential, we ought to put ourselves in a world where first-best is not achievable (See Wallace [2010]). The additional fiat asset with higher return helps to get closer to the first-best.

Other crucial assumptions are the complete information of portfolios in trade meetings and the size of pairwise meeting. When the portfolios of a consumer and a producer is not common knowledge, we need to use a different notion of pairwise core, core under incomplete information. It is expected that the choice of the notion would affect the result. Regarding the size of meetings, consider the other extreme case in which people meet altogether. In this case, the only core allocation is the competitive equilibrium allocation. Since social planner uses multiplicity of core to induce agents to hold a particular portfolio, what the planner can do will be more limited.

Features of the optimum is also of interest. Due to computational limitations, I find a welfare superior allocation instead of an optimum. Optimal transfer, inflation rate and trading mechanism may give some further insights to understand the beneficial role of government bonds.

# Chapter 2

# Optimal Intervention in a Random-matching Model of Money

## 2.1 Introduction

Wallace [2014] conjectures that in a class of economies in which all trade must involve money and there is no explicit taxation, there exist beneficial inflation-financed transfer schemes. A simple class of such schemes that he discusses involves a transfer to a person with $m$ amount of money equal to $\max\{0,\ a + bm\}$, where $b \geq 0$[1]. The conjecture applies to economies in which trades and policies affect both current-period payoffs and future states of the economy, the typical situation in heterogeneous-agent economies. Wallace discusses two examples, the alternating endowment economy with random "switches" (see Levine [1991] and Kehoe et al. [1992]) and a random-matching model due to Shi [1995] and Trejos and Wright [1995], but with a rich set of individual money holdings. However, he presents results only for the former. Here, we present numerical results for a version of the latter model.

In order to do that, we are forced to study a version with a relatively small set of individual

[1]When $a = 0$, such transfers are neutral; when $a > 0$, they are equivalent to lump-sum transfers; and when $a < 0$, they pay interest on money to those with holdings that exceed $-a/b$.

money holdings, $\{1, 2, \ldots, B\}$ with $B$ small. Both the discreteness and the bound force us to adapt the policies and the way we model the inflation that results from the transfers. The bound limits transfers to those at the bound. The discreteness forces both transfers and inflation to be probabilistic, where inflation is modeled as a probabilistic version of a proportional tax on money holdings—a tax which is nothing but a normalization when money is divisible. Our main results are for $B = 3$, which, as described below, is mainly dictated by computational feasibility. This magnitude of $B$ is interesting because it is the smallest $B$ that gives potential scope to policies with $a > 0$ and to policies with $a < 0$. Deviatov [2006] studies optima under $B = 2$ and we repeat that case here, but that case does not give scope to policies with $a < 0$. Policies with $a < 0$ require a positive holding with transfers given only to those with higher holdings. That positive holding must be at least one. With $B = 2$, all holdings that exceed one are at the bound and, therefore, ineligible for a transfer. With $B = 3$, those with two units of money can receive a transfer, a transfer which gives an additional incentive to those with one unit to produce and acquire additional money.

As in Deviatov [2006], we study alternative steady states in which the planner is choosing the steady-state distribution of money holdings, the trades in meetings subject to those trades being in the pairwise core in each meeting, and the above policies in order to maximize ex ante representative-agent utility. We present results for various combinations of two aspects of preferences: the discount factor and the finite marginal utility of consumption at zero. Consistent with Deviatov [2006], when $B = 2$ there are few cases in which intervention helps and those interventions have $a > 0$, are lump-sum transfers. When $B = 3$, more cases have desirable intervention; some have $a > 0$ and others have $a < 0$. We also made attempts to study optima for $B = 4$, but we could not get reliable results for all the parameter combinations. Nevertheless, the findings are broadly consistent with the surmise that the set of parameters for which no-intervention is optimal shrinks as $B$ gets larger.

## 2.2 Environment

The environment is borrowed from a random matching model in Shi [1995] and Trejos and Wright [1995]. Time is discrete and the horizon is infinite. There are a nonatomic measure of infinitely-lived agents. In each period, pairwise meetings for production and consumption occur in the following way. An agent becomes a producer (who meets a random consumer) with probability $\frac{1}{K}$, becomes a consumer (who meets a random producer) with probability $\frac{1}{K}$, or becomes inactive and enters no meeting with probability $1 - \frac{2}{K}$. In a meeting, the producer can produce $q$ units of a consumption good for the consumer in the meeting at the cost of disutility $c(q)$, where $c$ is strictly increasing, convex, and differentiable and $c(0) = 0$. The consumer obtains period utility $u(q)$, where $u$ is strictly increasing, strictly concave, differentiable function on $\mathbb{R}_+$ and satisfies $u(0) = 0$. The consumption good is perishable: it must be consumed in a meeting or discarded. Agents maximize the expected sum of discounted period utilities with discount factor $\beta \in (0, 1)$.

Individual money holdings are restricted to be in $\{0, 1, ..., B\}$. The state of the economy entering a date is a distribution over that set. Then there are pairwise meetings at random at which lottery trades occur: in single coincidence meetings some amount of output goes from the producer to the consumer and there is a lottery that determines the amount of money that consumers gives the producer. Next, there are transfers. We let $\tau_i \geq 0$ be the transfer to a person who ends trade with $i$ units and impose only that $\tau_i$ is weakly increasing in $i$ for $i \in \{0, 1, ..., B - 1\}$ and that $\tau_B = 0$. Finally, inflation occurs via probabilistic distintegration of money. Each unit of money held disappears with probability $\delta$.

We assume that people cannot commit to future actions and that there is no public monitoring in the sense that histories of agents are private. However, we assume that money holdings and consumer-producer status are known within meetings, but that money holdings are private at the transfer stage which is why we assume that $\tau_i$ is weakly increasing in $i$ for

**Table 2.1.** Variables constitute an allocation

| | |
|---|---|
| $\pi_k$ | fraction with $k$ units of money before meetings |
| $q(k,k')$ | production in $(k,k')$ meeting |
| $\lambda_p^{k,k'}(i)$ | probability that producer has $i$ money after $(k,k')$ meeting |
| $\lambda_c^{k,k'}(i)$ | probability that consumer has $i$ money after $(k,k')$ meeting |
| $\tau_k$ | transfer rate for agents with $k$ units of money |
| $\delta$ | probability that money disintegrates after meetings |

$i \in \{0,1,...,B-1\}$.

All our computations are for $K=3$, $c(q)=q$, and $u(q)=1-e^{-\kappa q}$, which implies that $u'(0)=\kappa$. We study optima for a subset of

$$(\beta,\kappa) \in \{0.15, 0.2, 0.25, 0.3, 0.35, 0.4, 0.5, 0.6, 0.7, 0.8\} \times \{2,3,4,5,6,8,10,12,15,20\},$$

a subset that satisfies

$$\kappa > 1 + \frac{K(1-\beta)}{\beta}. \tag{2.2.1}$$

This condition is necessary and sufficient for the production of constant positive output in a version of the model with perfect monitoring[2]. It is necessary for existence of a monetary equilibrium. When it is imposed the subset contains 55 elements.

## 2.3 The planner's problem

We study allocations that are stationary and symmetric meaning that agents in the same situation (money holdings, producer-consumer status) take the same action. Therefore, productions and monetary payments are constant over all meetings in which a producer has $k$ units of money and a consumer has $k'$ units of money, a $(k,k')$ meeting. A stationary and symmetric allocation consists of choices for the variables listed in Table 2.1:

[2]The inequality is derived from $\frac{d}{dq}_{q=0}\left(-c(q) + \frac{\beta}{K(1-\beta)}\left[u(q)-c(q)\right]\right) > 0$.

The planner chooses production and payments in meetings, disintegration and transfer rates to maximize ex-ante expected utility before money are assigned. It can be easily shown that ex ante expected utility is proportional to

$$\sum_{0\leq k\leq B}\sum_{0\leq k'\leq B}\pi_k\pi_{k'}\left[u(q(k,k'))-q(k,k')\right], \tag{2.3.1}$$

the expected gains from trade in meetings. The choice is subject to the following constraints.

**Physical feasibility and stationarity** First, money holdings that result from meetings must be feasible: in $(k,k')$ meeting, if the consumer has $i$ units, then the producer must have $k+k'-i$ units. Also, money holdings cannot be negative or exceed the total amount brought into the meeting.

$$\lambda_c^{k,k'}(i)=\lambda_p^{k,k'}(k+k'-i) \text{ if } 0\leq i\leq k+k' \tag{2.3.2}$$

$$\lambda_c^{k,k'}(i)=\lambda_p^{k,k'}(i)=0 \text{ if } i<0 \text{ or } k+k'<i \tag{2.3.3}$$

Let $\Lambda(k,k')$ denote the set of pairs of probabilities $(\lambda_c^{k,k'},\lambda_p^{k,k'})$ that satisfy the above constraints.

The money holding distribution is required to be stationary and consistent with transition probability specified by monetary payments in meetings and disintegration and transfers. Given a money holding distribution $\{\pi_k\}_{k\in\{0,1,\ldots,B\}}$ and a money holding transition probability$\{\lambda_p^{k,k'}(i),\lambda_c^{k,k'}(i)\}_{(k,k',i)\in\{0,1,\ldots,B\}^3}$, the transition probability that an agent with $k$ units of money before meeting has $k'$ units of money after meeting is determined as

$$t^{(1)}(k,k')=\frac{1}{K}\sum_{i\in\{0,\ldots,B\}}\pi_i\left[\lambda_p^{k,i}(k')+\lambda_c^{i,k}(k')\right]+\frac{K-2}{K}1_{k=k'}.$$

The transition caused by a transfer is expressed by

$$t^{(2)}(k,k') = \begin{cases} 1-\tau_k & \text{if } k < B \text{ and } k' = k, \\ \tau_k & \text{if } k < B \text{ and } k' = k+1, \\ 1 & \text{if } k' = k = B, \\ 0 & \text{otherwise,} \end{cases}$$

and the transition caused by disintegration is expressed by

$$t^{(3)}(k,k') = \begin{cases} \begin{pmatrix} k \\ k' \end{pmatrix} \delta^{k-k'}(1-\delta)^{k'} & \text{if } k \geq k', \\ 0 & \text{otherwise.} \end{cases}$$

Denote $T^{(i)}$ the matrix whose $(n, n')$ elements are $t^{(i)}\left(n-1, n'-1\right)$. Specifically,

$$T^{(i)} \equiv \begin{bmatrix} t^{(i)}\left(0,0\right) & t^{(i)}\left(0,1\right) & \ldots \\ t^{(i)}\left(1,0\right) & t^{(i)}\left(1,1\right) & \ldots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

The stationarity constraint can be stated as

$$\pi = \pi T^{(1)} T^{(2)} T^{(3)}, \tag{2.3.4}$$

where

$$\pi = \begin{bmatrix} \pi_0 & \cdots & \pi_B \end{bmatrix}.$$

**Incentive compatibility** We assume that agents can deviate from trades chosen by the planner individually and also cooperatively in the meeting stage. Individual deviations lead to no trade, and profitable cooperative deviations lead to a Pareto-improving alternative trade. To state incentive compatibility constraints induced by such deviations, it is convenient to introduce discounted expected utility. Discounted utility for an agent with $k$ units of money before pairwise meeting is denoted by $v(k)$, and that after the meeting stage but before transfer and disintegration is denoted by $w(k)$. They are defined for each symmetric and stationary allocation in the standard way. Specifically, for each $k \in \{0, 1, ..., B\}$, they satisfy

$$\begin{aligned} v(k) =& \frac{1}{K} \sum_{k' \in \{0,\dots,B\}} \pi_{k'} \left[ u(q(k',k)) + \beta \sum_{0 \le i \le k+k'} \lambda_c^{k',k}(i) w(i) \right] \\ &+ \frac{1}{K} \sum_{k' \in \{0,\dots,B\}} \pi_{k'} \left[ -q(k,k') + \beta \sum_{0 \le i \le k+k'} \lambda_p^{k,k'}(i) w(i) \right] \\ &+ \frac{K-2}{K} \beta w(k) \end{aligned} \tag{2.3.5}$$

$$w(k) = \sum_{k' \in \{0,\dots,B\}} T^{(2)}(k,k') \sum_{i \in \{0,\dots,B\}} T^{(3)}(k',i) v(i) \tag{2.3.6}$$

Trades are immune to both individual and cooperative deviations if post-trade allocations are in the pairwise core, and we call this pairwise core constraint. To state the constraint for $(k, k')$ meeting, let $\vartheta(k, k')$ denote a surplus (over no-trade) for a producer in the meeting. The constraint can be stated as follows: $q(k, k')$, $\lambda_p^{k,k'}$, and $\lambda_c^{k,k'}$ solve

$$\begin{aligned} \max_{q \ge 0,\ (\lambda_p, \lambda_c) \in \Lambda(k,k')} & u(q) + \beta \sum_{0 \le i \le k+k'} \lambda_c(i) w(i) \\ \text{s.t. } & -q + \beta \sum_{0 \le i \le k+k'} \lambda_p(i) w(i) = \beta w(k) + \vartheta(k,k') \\ & u(q) + \beta \sum_{0 \le i \le k+k'} \lambda_c(i) w(i) \ge \beta w(k') \end{aligned} \tag{2.3.7}$$

for some $\vartheta(k, k') \geq 0$.[3] The Karush-Kuhn-Tucker condition is necessary and sufficient for the optimality, and we can derive a set of equations and inequalities from the condition. See Appendix B.4 for the detail.

The planner maximizes the ex-ante expected utility (2.3.1), subject to the physical feasibility conditions, the stationarity conditions, and the pairwise core constraints.

## 2.4 Computational Procedure

We compute solutions for the planner's problem using two solvers that are compatible with the GAMS interface, KNITRO and BARON. KNITRO is a local solver for large scale optimization problems. For a given initial point, it quickly converges to a local solution (or shows that it cannot reach one), but it does not guarantee global optimality. This issue is usually dealt with by using a large number of initial values. The solver automatically feeds in different initial values as we change an option that controls the number of initial values. In contrast, BARON (Branch-And-Reduce Optimization Navigator) is a global solver for nonconvex optimization problems. It continues to update an upper bound and a lower bound on the objective by evaluating the values of variables satisfying the constraints, and stops when the difference between the two bounds becomes smaller than a threshold. It guarantees global optimality under mild conditions, but it generally takes much longer time to converge than local solvers. Even before it converges, we can terminate it and see its candidate solution. When Baron did not finish in a reasonable time span, we stopped it and checked the candidate solution

[3] Solving the problem is necessary for trades being in the pairwise core. That is also sufficient if the utility function of the producer and the consumer are strictly monotone in consumption goods and money holdings (see, for example, Mas-Collel et al. [1995]). Here, the utility function may not be strictly increasing in money holdings; Some additional units of money may not be valued in some allocations, and hence the value function $w$, which specifies the preference for money holdings in trade meetings, may be non-strictly increasing in a part of the domain. In effect, we are solving a relaxed problem using this formulation. For example, if we find that a numerical solution has non-strictly increasing $w$, it may not be an optimum as solving above problem is not a sufficient condition in that case. It is verified that numerical solutions have strictly increasing $w$, and thus it is assured that the solutions solve the problem of our interest.

with the solution from KNITRO.

For $B = 2$, BARON under its default criterion usually finished in about an hour, and it reproduces the solution that KNITRO finds using 250 different initial points. For $B = 3$, BARON did not finish in 200 hours. In all cases that we tried, the candidate solution was not updated after roughly 20 hours. (The remaining time was being used to verify that other feasible allocations are not better than the candidate solution.) We ran KNITRO with 1000 initial points and found that its solution coincides with the intermediate output from BARON, which is the best lower bound. Also, to check whether the computation is sensitive to the number of initial points we use, we ran KNITRO with 8000 initial points and made sure that the results are the same. We also tried the same approach with $B = 4$, but we could not find robust results for some examples. We report them in appendix B.3.

## 2.5 Results

For $B = 2$, we found that intervention is optimal for only 2 parameter combinations, $(\kappa, \beta) = (15, 0.7)$ and $(20, 0.6)$. Those interventions were lump-sum in the sense that they satisfy $\tau_0 = \tau_1$, the transfer rate was 2.8% and 1.4%. For $B = 3$, interventions were optimal in 21 of the 55 parameter combinations. Moreover, all the interventions were either lump-sum, $\tau_0 = \tau_1 = \tau_2 > 0$, or had $\tau_0 = \tau_1 = 0$ and $\tau_2 > 0$. In other words, they turned out to fit the class discussed in the introduction. Because all the solutions have $\tau_0 = \tau_1$, it is convenient to represent them in a table as the two numbers, $x/y$ , where $x$ is the common magnitude of $\tau_0$ and $\tau_1$ and $y$ is the magnitude of $\tau_2$.

The type of transfer is related to the value of $\beta$ and $\kappa$. Among cases in which some intervention is optimal, the optimal transfer tends to be lump-sum when $\beta$ and $\kappa$ are both high, and non-lump-sum when either of $\beta$ and $\kappa$ is low. To understand this result, it is helpful to explain the benefits and the costs of two types of transfers. The cost that is common for

**Table 2.2.** Transfer (%), $B = 3$

| | $\kappa \backslash \beta$ | 0.15 | 0.2 | 0.25 | 0.3 | 0.35 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\begin{pmatrix} \tau_0 = \tau_1 \\ \tau_2 \end{pmatrix}$ | 20 | 0<br>0 | 0<br>72.3 | 0<br>73.4 | 0<br>73.7 | 0<br>0 | 35.3<br>35.3 | 31.2<br>31.2 | 13.0<br>13.0 | 11.4<br>11.4 | 0<br>0 |
| | 15 | - | 0<br>0 | 0<br>65.4 | 0<br>66.8 | 0<br>67.1 | 0<br>0 | 0<br>0 | 53.1<br>53.1 | 2.8<br>2.8 | 0<br>0 |
| | 12 | - | - | 0<br>0 | 0<br>59.0 | 0<br>61.0 | 0<br>0 | 0<br>0 | 2.1<br>2.1 | 0<br>0 | 0<br>0 |
| | 10 | - | - | - | 0<br>0 | 0<br>54.6 | 0<br>55.7 | 0<br>0 | 0<br>0 | 0<br>0 | 0<br>0 |
| | 8 | - | - | - | - | 0<br>0 | 0<br>47.2 | 0<br>0 | 0<br>0 | 0<br>0 | 0<br>0 |
| | 6 | - | - | - | - | - | 0<br>0 | 0<br>38.4 | 0<br>0 | 0<br>0 | 0<br>0 |
| | 5 | - | - | - | - | - | - | 0<br>0 | 0<br>0 | 0<br>0 | 0<br>0 |
| | 4 | - | - | - | - | - | - | - | 0<br>0.6 | 0<br>0 | 0<br>0 |
| | 3 | - | - | - | - | - | - | - | - | 0<br>0.9 | 0<br>0 |
| | 2 | - | - | - | - | - | - | - | - | - | 0<br>0 |

any transfer is inflation, as it is lowering the producer's incentive. The benefit of lump-sum transfer is the risk-sharing. If an agent without money becomes a consumer, he will forego the opportunities to consume and this is wasteful from a society's point of view. The transfer is helpful for reducing such loss. However, as people get an additional money regardless of their money holdings, this type of transfer further lowers the producers' incentive to earn money. In other words, it tightens producers' participation constraint. Lastly, the benefit of non-lump-sum transfer used in this example, transfer rate is strictly increasing in an interval, is that it enhances the producer's incentive, particularly those who already owns some money. Such transfer is helpful to relax the participation constraint of producers who already own some money.

The pattern of optimal transfer seems to be intuitive in light of the explanation on the benefit and the cost of each transfer type: when people have high incentive to work for future consumption and averse risk more (that is, when $\beta$ and $\kappa$ are both high), the optimal transfer tends to provide risk-sharing. If people have low incentive to work for future consumption but do not averse risk much, (that is, when $\beta$ and $\kappa$ are both low), the optimal transfer tends to enhance the incentive of producers. Although there still are several examples where no intervention is optimal with the upper bound of three, we believe that the region where no-intervention is optimal will shrink as the upper bound increases comparing the optimal transfer rates for $B = 2$ and $B = 3$. We present results with the upper bound of four, which is consistent with this conjecture.

How much do these interventions improve? We measure the improvement over no intervention in terms of consumption good, calculating $x$ that satisfies

$$\sum_{k,k'} \pi_k \pi_{k'} \left[ u\left( \frac{100 - x}{100} q^*(k, k') \right) - c(q^*(k, k')) \right] = \sum_{k,k'} \pi_k \pi_{k'} \left[ u(q^0(k, k')) - c(q^0(k, k')) \right],$$

where $q^*$ is the optimal production with intervention and $q^0$ is the optimal production with no intervention. Table 2.5 shows the result. The gain varies from 0.12 percentage point to 6.50 percentage point. The largest gain found from an example with relatively high discount factor and risk aversion, with which the optimal transfer is lump-sum.

We report some features of optima in the following, and contain the related results in the appendix. As expected, welfare at the optimum is increasing with discount factor. The welfare at optimum appears to increase with $B$ as well. A result in Zhu [2003] shows that the set of implementable allocations for lower $n$ is a subset of the set for larger $n$ when $B$ increases as in $B_n = m^n$ for any integer $m > 1$. Hence, welfare is weakly increasing function for $n$ in that case. However, it is not straightforward whether the result will extend to $B_n = n$ for $n = 1, 2, \ldots$. At least to our knowledge, numerical results that compared welfare at the optimum for different $B$ (greater than 1) do not exist. Our numerical results are in line with

**Table 2.3.** Gains from intervention

| | $\kappa \backslash \beta$ | 0.15 | 0.2 | 0.25 | 0.3 | 0.35 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $B = 2$ | 20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.40 | 0 | 0 |
| | 15 | - | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.08 | 0 |
| $B = 3$ | 20 | 0 | 3.52 | 4.92 | 1.87 | 0 | 0.68 | 2.04 | 6.50 | 2.42 | 0 |
| | 15 | - | 0 | 2.70 | 3.71 | 0.37 | 0 | 0 | 1.40 | 0.48 | 0 |
| | 12 | - | - | 0 | 2.48 | 2.44 | 0 | 0 | 0.21 | 0 | 0 |
| | 10 | - | - | - | 0 | 2.52 | 1.05 | 0 | 0 | 0 | 0 |
| | 8 | - | - | - | - | 0 | 1.18 | 0 | 0 | 0 | 0 |
| | 6 | - | - | - | - | - | 0 | 0.58 | 0 | 0 | 0 |
| | 5 | - | - | - | - | - | - | 0 | 0 | 0 | 0 |
| | 4 | - | - | - | - | - | - | - | 0.12 | 0 | 0 |
| | 3 | - | - | - | - | - | - | - | - | 0.18 | 0 |
| | 2 | - | - | - | - | - | - | - | - | - | 0 |

the hypothesis of increasing welfare with respect to $B$.

A measure of money supply is defined as the mean money holdings of people. Optimal money supply monotonically increses with discount factor and risk aversion. Comparing across two upper bounds, optimal money supply is less volatile with higher upper bound. This is intuitive as the wealth of an agent can be more finely recorded with richer money holdings. It does not exceeds the half of $B$ in any cases.

The lottery is potentially useful as money is indivisible and money holding is limited (due to the concavity of utility function, it is not helpful to use lottery for deciding production level). One measure we constructed to see the use of lottery is the percentage of trade meetings where the lottery is used to any degree. It tends to be used more with high discount rate and high risk aversion, but is not monotonically increasing. As the value of money at optimum increases with the discount rate, the lottery gets used more to overcome the indivisibility.

We construct a similar measure for the trade meetings in which consumers making take-it-or-leave-it offer. As some works using this class of model assume such trading mechanism for simplicity, it is interesting to see whether such trading mechanism is close to the optimal mechanism. Our measure does not capture the exact welfare loss from imposing take-it-or-leave-it mechanism, but it is indicative. In an example, almost 50% of meeting is departing from take-it-or-leave-it mechanism. Again here, it happens when both parameters are high. It may be instructive to consider the optimum with $B = 1$ to understand this result. When the upper bound is one and discount rate is sufficiently high, the optimum is achievable with money held by the half of population and producers producing the first-best level of production. Importantly, consumers don't get all the surplus in that case, as it induces to produce over the first-best level. The result with higher $B$ seems to show the similar pattern, consumers don't get all the surplus when discount rate is high.

## 2.6 Concluding remarks

We attempted to validate Wallace's conjecture against the background of Shi-Trejos-Wright model in a weak sense. We constricted our attention only to stationary allocations, while the conjecture was posed without such a restriction. Also, one might say that exploring only up to $B = 3$ is premature to conclude. We believe that increasing $B$ will not overturn the main result that positive money creation and inflation becomes optimal more generically as $B$ increases. Increasing from $B = 2$ to $B = 3$ is significant as it opens up a new role for monetary transfer. If we are to explore all implementable allocations and find optima, increasing $B$ increases computational load exponentially. An alternative to our modeling choice is Molico [2006], a similar model with unrestricted $B$ and buyers make take-it-or-leave-it offers to sellers. However, assuming such trading mechanism is not innocuous to welfare, as it may not be an optimal trading mechanism as our examples show.

Despite that, there are at least three reasons that one might be interested in our results. First, this is an example that is consistent with the conjecture. While Wallace [2014] discusses some examples, our work is the first that explores the conjecture within a search model of money, which is a workhorse model for monetary theory these days. Second, it shows the delicacy of the upper bound in this model. The type of optimal transfer changes dramatically as the upper bound changes. One may want to be cautious generalizing result from the model with $B = 1$ or 2, given that it might not generalize as seen here. Lastly, we present numerical optimum for different upper bounds, and getting optimum with an upper bound greater than 2 is of interest itself. For example, we could observe that welfare at optimum appears to increase with $B$.

# Chapter 3

# Optimal Inflation in a Model of Inside Money: A Further Result

## 3.1 Introduction

Is inflation a necessary evil? While Friedman rule, setting opportunity cost of holding money equals to production cost of money, has been a cornerstone of monetary theory, several papers have shown that inflation is a feature of desirable allocations (Kehoe et al. [1992], Molico [2006], Green and Zhou [2005], and Deviatov [2006] to name a few). The optimality of inflation attained within pure-currency economies (Lucas [1980], Wallace [2014]) rely on two properties of pure-currency economy with idiosyncratic shock: 1) lump-sum transfer is beneficial as they provide risk-sharing, 2) there is no available tax instruments other than inflation tax. Wallace [2014] takes stock of this literature and conjectures that positive inflation is generically optimal in pure-currency economies.

If we go beyond pure-currency economy, which is extreme, necessity of inflation is less clear as government transfers can be financed through some other taxes. We argue that inflation may still be optimal but for a different reason. We build on Deviatov and Wallace

[2014] where they study a model of inside money where there are two types of people in the model, who are subject to public monitoring and can issue inside money (monitored agents), and who are anonymous and cannot issue inside money (non-monitored agents). They find that optima may have positive inflation as it enables monitored agents can spend more than what they earn from non-monitored agents[1]. However, their model is restrictive in that some potentially beneficial policies that are not inflationary policies are physically ruled out due to small upper bound on money holdings. While some taxes are feasible, the small upper bound severely limits the ways to use tax revenue. In particular, their setting does not give scope to Friedman rule-like policies (providing interest on cash holdings). Hence, one may wonder if positive inflation remains optimal when such policies were feasible.

We extend their model by adopting the smallest upper bound that makes Friedman rule-like policies become feasible, which is 3. This extension provides a rich setting in which we can study whether Friedman rule is desirable and whether inflation is necessary. We define a mechanism design problem and study optima by computing those for a range of parameters on the discount factor, the finite marginal utility of consumption at zero, and the fraction of monitored agents in population. This is an additional contribution compared to Deviatov and Wallace [2014], who concentrate on one example.

In all examples we study, positive inflation ariese at optimum. While direct monetary transfer from government is very rarely used (which is in contrast to optima in pure-currency economies) inflation arises due to net positive creation of inside money from monitored agents. Hence, giving scope to Friedman rule do not necessarily invalidate the result of Deviatov and Wallace [2014]. Optima in economies without monitored agents (outside money economy) and economies with monitored agents (inside money economy) show many differences.

[1] See Antinolfi et al. [2007] for a related study.

## 3.2 Environment

The environment is borrowed from Deviatov and Wallace [2014], which uses a random matching model in Shi [1995], Trejos and Wright [1995] as the background. If you are familiar with Cavalcanti and Wallace [1999], this model is identical to it except money holding and inflation scheme. Time is discrete and the horizon is infinite. There are a nonatomic measure of infinitely-lived agents.

In each period, pairwise meetings for production and consumption occur in the following way. An agent becomes a producer and meets a random consumer with probability $1/K$, becomes a consumer and meets a random producer with probability $1/K$, or becomes inactive and enters no meeting with probability $1-2/K$. In a meeting, the producer can produce $q$ units of a consumption good for the consumer in the meeting at the cost of disutility $c\,(q)$, where $c$ is strictly increasing, convex, and differentiable and $c(0)=0$. The consumer obtains period utility $u(q)$, where $u$ is strictly increasing, strictly concave, differentiable on $\mathbb{R}_+$ and satisfies $u(0)=0$. Also, $q^*=\arg\max_{q\in\mathbb{R}_+}[u(q)-c(q)]$ exists and is strictly positive. The consumption good is perishable: it must be consumed in a meeting or discarded. Agents maximize the expected sum of discounted period utility with discount factor $\beta\in(0,1)$.[2]

Each agent and the planner have printing presses that can produce an intrinsically useless and indivisible asset which can be used as a medium of exchange. We call the asset money. As monitoring is imperfect as described soon, money can potentially help the economy to achieve a good allocation. Money produced by any agent is distinguishable from that produced by other agents. Individual money holdings are restricted to be in $\{0,1,2,3\}$. While we can consider an economy where the only medium of exchange is outside money when monitored agents exit, we only focus on inside money economy as it is welfare superior. (Deviatov and Wallace [2014])

[2]This exposition is a simplified version of Shi [1995], Trejos and Wright [1995].

Monitoring is imperfect in the language of Cavalcanti and Wallace [1999]. A fraction of agents are permanently monitored and called $m$-agents, and the rest are permanently not monitored and called $n$-agents. Histories of $m$-agents, who they meet and what they did, are common knowledge, while those of $n$-agents are private. Fraction of $m$-agents is a parameter of the environment and denoted by $\alpha$. Also, agents cannot commit to any future action.

Sequence of actions are as following. First, pairwise meeting takes place and people realize their type as a producer or a consumer or become inactive. Depending on their shock realization, people can produce or consume, and period utility is determined here. All information, regarding their type and money holding, are common knowledge between them. Following the mechanism design approach, we will consider broader trading protocol than using a bargaining solution. Given a trading protocol, we assume that people can individually defect and the pair in any meeting can jointly defect to it. When $n$-agents defect, government cannot punish them consistent to the definition of $n$-agents. When $m$-agents defect, they can be treated as $n$-agents as a punishment. This is a loss as they lose their ability to issue valued money. We also allow that $m$-agents can become $n$-agents whenever they want to.

After pairwise meeting, government make monetary transfer and tax. At this point, people can only defect individually. Money holdings of $n$-agents are not visible, and government can ask them to differ the transfer rate depending on their money holdings. Let $\tau_i \in [0, 1]$ be the probability that an agent with $i$ units of money receives another unit for each $i$. In response, people can report any units less than what they have and that impoes that $\tau_i$ must be weakly increasing in $i$. Also, we must assume $\tau_B = 0$. For $m$-agents, all money holdings of them are returned to government and eliminated at this point. After transfer and tax, each unit of money disintegrates with probability $\delta \in [0, 1]$, independent to their money holdings. This is a device to resemble inflation in this class of model, to make agents' money holdings stay in the given set of possible money holdings (Li [1995]).

All our compuations are for $K = 3$; $c(q) = q$, and $u(q) = 1 - e^{-\kappa q}$, which implies that

$u'(0) = \kappa$. We study optima for a subset of

$$(\beta, \kappa) \in \{0.1, 0.2, \ldots, 0.8\} \times \{10, 20, 40\}$$

a subset that satisfies

$$\kappa > 1 + \frac{K(1-\beta)}{\beta}$$

This condition is necessary and sufficient for the production of constant positive output with $\alpha = 1$ (perfect monitoring), and it leaves us 21 pairs of $(\beta, \kappa)$. Regarding $\alpha$, we study $\alpha \in \{0, 0.25, 0.5, 0.75\}$. When $\alpha = 0$, this model becomes a version of Shi [1995], Trejos and Wright [1995], with a richer money holding. (See Nozawa and Yang for a study on optimal intervention in that version.)

## 3.3 Implementable allocations

To preserve tractability, we restrict our attention to stationary and symmetric allocations[3]: Money holding distribution is invariant to time and people act identically given that they have same monitoring type, money holding and shock realization. Therefore, productions and monetary payments are constant over all meetings in which a producer has $k$ units of money and a consumer has $k'$ units of money, a $(k, k')$ meeting. A stationary and symmetric allocation consists of choices for the variables listed in Table 3.1:

The planner chooses production and payments in meetings, disintegration and transfer rates to maximize ex-ante expected utility before money are assigned. It can be easily shown

[3] This may not be innocuous. In a similar model, Bertolai et al. [2012] finds that there can be ex-ante pareto superior nonstationary allocations to the stationary optima.

**Table 3.1.** Variables constitute an allocation

| | |
|---|---|
| $\pi_k$ | fraction with $k$ units of money before meetings |
| $q(k, k')$ | production in $(k, k')$ meeting |
| $\lambda_p^{k,k'}(i)$ | probability that producer has $i$ money after $(k, k')$ meeting |
| $\lambda_c^{k,k'}(i)$ | probability that consumer has $i$ money after $(k, k')$ meeting |
| $\tau_k$ | transfer rate for agents with $k$ units of money |
| $\delta$ | probability that money disintegrates after meetings |

that ex-ante expected utility is proportional to

$$\sum_{0 \le k \le B} \sum_{0 \le k' \le B} \pi_k \pi_{k'} \left[ u(q(k, k')) - q(k, k') \right], \tag{3.3.1}$$

the expected gains from trade in meetings. The choice is subject to the following constraints.

We define the first-best allocation as production and consumption equal to $q^*$ in every single-coincidence meeting. Accordingly, the first-best welfare level is $u(q^*) - q^*$.

**Physical feasibility and stationarity** First, money holdings that result from meetings must be feasible: in $(k, k')$ meeting, if the consumer has $i$ units, then the producer must have $k + k' - i$ units. Also, money holdings cannot be negative or exceed the total amount brought into the meeting.

$$\lambda_c^{k,k'}(i) = \lambda_p^{k,k'}(k + k' - i) \text{ if } 0 \le i \le k + k' \tag{3.3.2}$$

$$\lambda_c^{k,k'}(i) = \lambda_p^{k,k'}(i) = 0 \text{ if } i < 0 \text{ or } k + k' < i \tag{3.3.3}$$

Let $\Lambda(k, k')$ denote the set of pairs of probabilities $(\lambda_c^{k,k'}, \lambda_p^{k,k'})$ that satisfy the above constraints.

Given this, the transition probability that a person in state $k$ transits to state $k'$ during pairwise trade meeting stage is

$$t^{(1)}(k,k') = \begin{cases} \frac{1}{K}\sum_{k_0\in\{0,\ldots,B\}\cup\{m\}} \pi_{k_0}\left[\lambda_p^{k,k_0}(k') + \lambda_c^{k_0,k}(k')\right] + \frac{K-2}{K}1_{k=k'} & \text{if } 0\le k,k'\le 3 \\ 1 & \text{if } k=m \text{ and } k'=m \\ 0 & \text{otherwise} \end{cases}$$

The transition probability that a person in state $k$ transits to state $k'$ caused by transfer is denoted by

$$t^{(2)}(k,k') = \begin{cases} 1-\tau_k & \text{if } k\in\{0,1,2\} \text{ and } k'=k, \\ \tau_k & \text{if } k\in\{0,1,2\} \text{ and } k'=k+1, \\ 1 & \text{if } k'=k=3 \text{ or } k'=k=m, \\ 0 & \text{otherwise,} \end{cases}$$

and the transition caused by disintegration is expressed by

$$t^{(3)}(k,k') = \begin{cases} \begin{pmatrix} k \\ k' \end{pmatrix} \delta^{k-k'}(1-\delta)^{k'} & \text{if } k\in\{0,1,2,3\} \text{ and } k\ge k', \\ 1 & \text{if } k=k'=m, \\ 0 & \text{otherwise.} \end{cases}$$

Denote $T^{(i)}$ the following matrix

$$T^{(i)} \equiv \begin{bmatrix} t^{(i)}(0,0) & t^{(i)}(0,1) & t^{(i)}(0,2) & t^{(i)}(0,3) & t^{(i)}(0,m) \\ t^{(i)}(1,0) & t^{(i)}(1,1) & t^{(i)}(1,2) & t^{(i)}(1,3) & t^{(i)}(1,m) \\ t^{(i)}(2,0) & t^{(i)}(2,1) & t^{(i)}(2,2) & t^{(i)}(2,3) & t^{(i)}(2,m) \\ t^{(i)}(3,0) & t^{(i)}(3,1) & t^{(i)}(3,2) & t^{(i)}(3,3) & t^{(i)}(3,m) \\ t^{(i)}(m,0) & t^{(i)}(m,1) & t^{(i)}(m,2) & t^{(i)}(m,3) & t^{(i)}(m,m) \end{bmatrix}.$$

The stationarity constraint can be stated as

$$\pi = \pi T^{(1)} T^{(2)} T^{(3)}, \tag{3.3.4}$$

where

$$\pi = \begin{bmatrix} \pi_0 & \cdots & \pi_B, \pi_m \end{bmatrix}$$

and $\pi_m = \alpha$.

**Incentive compatibility** Incentive compatibility is defined by underlying information and defection assumptions in two stages for transfer and pairwise meeting. Information assumption is on the history of agents and their money holdings, and defection assumptions are whether they can defect jointly or not. As we mentioned already, history of agents are determined by their monitoring status, hence information specification only determines the visibility of money holdings. We assume that (1) in transfer stage, money holdings are private information and people can only defect individually, (2) in pairwise meeting stage, money holdings are visible within each meeting and people can jointly defect within a pair. Although money holdings are private information in transfer stage, they can only hide not overstate, as it can be verified. Hence, the truth-telling constraint on transfer rates is

$$\tau_{i-1} \leq \tau_i$$

for $i \in \{0, 1, 2\}$.

This is one possible specification out of 16, which is the number of combinations of information and defection specifications in each stage. Some of other specifications can be studied in a similar manner. We think it is desirable to allow joint defection within a meeting and hide money holdings in transfer stage, as it seems natural in this pairwise meeting and anonymous environment. If we want to allow joint defection in transfer stage (with visible money holdings), transfer rate must be restricted to linear in money holdings, and this can be easily incorporated. Other than that, our choice of specification is not more justified than others.

Two small differences from Deviatov and Wallace [2014] are notable. In Deviatov and Wallace [2014], $n$-agents can hide their money in a meeting, and this potentially matters only when $n$ consumers meet $m$ producers. In ours, allowing $n$-agents to hide their money holdings create more complications. Hence, we choose different specification on that. Next, we allow $m$ agents can issue money to $n$ agents even when $m$ producer meets $n$ consumer. This is potentially useful for government to engineer transfer, as information in pairwise meeting is visible while it is not in transfer stage. This is not beneficial in Deviatov and Wallace [2014] for the same reason why transfer is not used at optimum. However, in our settings, it potentially is.

Due to stationarity, we omit all time scripts. Following definitions are made under stationarity. Before defining incentive compatibility, it is convenient to define discounted expected utility before meetings. For each money holdings $k \in \{0, \ldots, B\}$,

$$\begin{aligned} v(k) = {} & \frac{1}{K} \sum_{k' \in \{0,\ldots,B\} \cup \{m\}} \pi_{k'} \left[ u(q(k',k)) + \beta \sum_{k_0} \lambda_c^{k',k}(k_0) w(k_0) \right] \\ & + \frac{1}{K} \sum_{k' \in \{0,\ldots,B\} \cup \{m\}} \pi_{k'} \left[ -q(k,k') + \beta \sum_{k_0} \lambda_p^{k,k'}(k_0) w(k_0) \right] \\ & + \frac{K-2}{K} \beta w(k), \end{aligned}$$

and, for $k = m$,

$$\begin{aligned} v(m) = {} & \frac{1}{K} \sum_{k' \in \{0,\ldots,B\} \cup \{m\}} \pi_{k'} \left[ u(q(k',m)) + \beta w(m) \right] \\ & + \frac{1}{K} \sum_{k' \in \{0,\ldots,B\} \cup \{m\}} \pi_{k'} \left[ -q(m,k') + \beta w(m) \right] \\ & + \frac{K-2}{K} \beta w(m). \end{aligned}$$

where $w(\cdot)$ is discounted expected utility before transfer. Under stationarity,

$$w(k) = \sum_{k' \in \{0,\ldots,B\}} t^{(2)}(k,k') \sum_{i \in \{0,\ldots,B\}} t^{(3)}(k',i) v(i)$$

for $k \in \{0, \ldots, B\}$. And for $k = m$,

$$w(m) = v(m)$$

Note that agents with 3 units of money cannot earn more. We also require that monitored agents prefer to stay monitored, rather than becoming non-monitored agents with money they earned in pairwise meeting.

$$w(m) \geq w\left( \max_{k \in \{0,\ldots,B\}} \{x\{m,k\}\} \right) \tag{3.3.5}$$

where $x(m,k)$ is the amount that $m$ producer receives from a meeting with $n$ consumer with $k$ units of money.

As discussed in Deviatov and Wallace [2014], monitored agents start new period with no money at optimum. Money they earned in previous periods are entirely taxed, and they issue money under their name when they need to pay. It is easy to see why it creates a tighter incentive compatibility than 3.3.5 if they don't, as $m$ agents can obtain more money than one can get in one meeting.

We also require that trade results in pairwise core to immune to individual defection and joint defection in a pair. If $n$ agents defect to a given trade, it results in no trade in the current period and expecting $w(k)$ next period. If $m$ agents defect, they lose their status and become $n$ agents. To state the constraint for $(k,k')$ meeting for , let $\vartheta(k,k')$ denote a surplus (over no-trade) for a producer in the meeting. The constraint can be stated as follows: $q(k,k')$, $\lambda_p^{k,k'}$, and $\lambda_c^{k,k'}$ solve

$$\begin{aligned} &\max_{q\geq 0,\ (\lambda_p,\lambda_c)\in\Lambda(k,k')} u(q)+\beta\sum_{0\leq i\leq k+k'}\lambda_c(i)w(i) \\ &\text{s.t.}\ \ -q+\beta\sum_{0\leq i\leq k+k'}\lambda_p(i)w(i)=\beta w(k)+\vartheta(k,k') \\ &\qquad u(q)+\beta\sum_{0\leq i\leq k+k'}\lambda_c(i)w(i)\geq\beta w(k') \end{aligned} \tag{3.3.6}$$

or some $\vartheta(k,k')\geq 0$.[4] The Karush-Kuhn-Tucker condition is necessary and sufficient for the optimality, and we can derive a set of equations and inequalities from the condition. If either

[4]Solving the problem is necessary for trades being in the pairwise core. That is also sufficient if the utility function of the producer and the consumer are strictly monotone in consumption goods and money holdings (see, for example, Mas-Collel et al. [1995]). Here, the utility function may not be strictly increasing in money holdings; Some additional units of money may not be valued in some allocations, and hence the value function $w$, which specifies the preference for money holdings in trade meetings, may be non-strictly increasing in a part of the domain. In effect, we are solving a relaxed problem using this formulation. For example, if we find that a numerical solution has non-strictly increasing $w$, it may not be an optimum as solving above problem is not a sufficient condition in that case. It is verified that numerical solutions have strictly increasing $w$, and thus it is assured that the solutions solve the problem of our interest.

a consumer or a producer is $m$ agent, individual rationality suffices

$$u\left(q(k',k)\right)+\beta\sum_{k_0}\lambda_c^{k',k}(k_0)w(k_0)\geq\beta w\left(k\right),\ \text{for}\ k\in\{0,1,2,3\}\cup\{m\}\,,\ k'=m \tag{3.3.7}$$

$$u\left(q(k',m)\right)+\beta w\left(m\right)\geq\beta w\left(0\right),\ \text{for}\ k'\in\{0,1,2,3\}\cup\{m\} \tag{3.3.8}$$

$$-q(k,k')+\beta\sum_{k_0}\lambda_p^{k,k'}(k_0)w\left(k_0\right)\geq\beta w\left(k\right),\ \text{for}\ k\in\{0,1,2,3\}\cup\{m\}\,,\ k'=m \tag{3.3.9}$$

$$-q(m,k')+\beta w\left(m\right)\geq\beta w\left(0\right),\ \text{for}\ k'\in\{0,1,2,3\}\cup\{m\} \tag{3.3.10}$$

The planner maximizes the ex-ante expected utility (3.3.1), subject to the physical feasibility conditions, the stationarity conditions, the pairwise core constraints, and the individual rationarity constraints.

## 3.4 Computational procedure

We compute solutions for the planner's problem using two solvers that are compatible with the GAMS interface, KNITRO and BARON. KNITRO is a local solver for large scale optimization problems. For a given initial point, it quickly converges to a local solution (or shows that it cannot reach one), but it does not guarantee global optimality. This issue is usually dealt with by using a large number of initial values. The solver automatically feeds in different initial values as we change an option that controls the number of initial values. In contrast, BARON (Branch-And-Reduce Optimization Navigator) is a global solver for nonconvex optimization problems. It continues to update an upper bound and a lower bound on the objective by evaluating the values of variables satisfying the constraints, and stops when the difference between the two bounds becomes smaller than a threshold. It guarantees global optimality under mild conditions, but it generally takes much longer time to converge than local solvers. Even before it converges, we can terminate it and see its candidate solution. When Baron did not finish in a reasonable time span, we stopped it and checked the candidate solution with the solution from KNITRO.

Due to the complexity of the problem, BARON did not complete its computation after spending a day, which is the limit of time we can use for a single routine. In all examples that we tried, the candidate solution was not updated after roughly 12 hours. The remaining time was being used to verify that other feasible allocations are not better than the candidate solution. The intermediate result earned after one day run is our first candidate of the solution. We ran KNITRO with 1000 initial points and found that its solution coincides with the intermediate output from BARON, which is the best lower bound. With KNITRO, we used 3000 initial starting points and checked if the results from using 5000 initial points are the same. This becomes our second candidate when they coincide. We compare two candidates, the candidates from two solvers, and determine that results are consistent when maximized values and aggregate variables are same within tolerance level, which is 0.001. The following results meet this criteria.

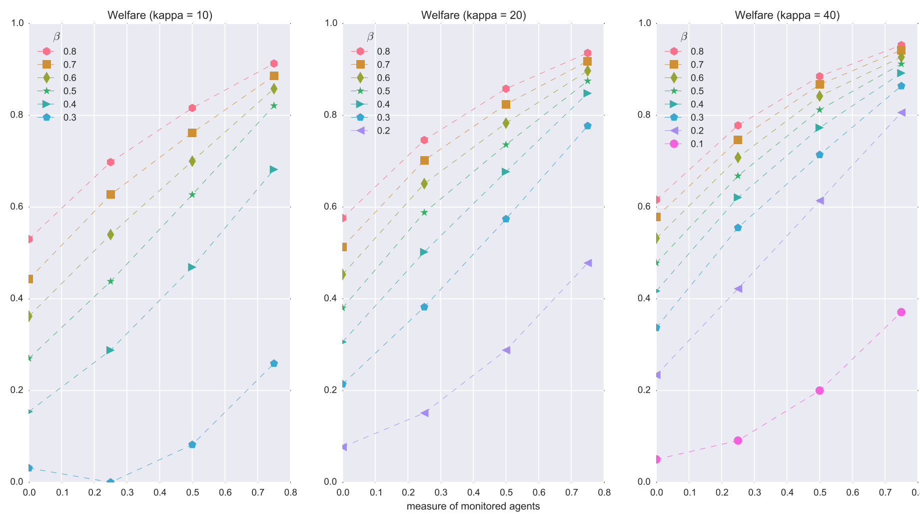
## 3.5 Results

The first two figures show the welfare (relative to first-best level) and money supply at optima for different values of $\beta, \kappa, \alpha$. The money supply is defined as

$$\frac{\pi_1 + 2\pi_2 + 3\pi_3}{1-\alpha}$$

The welfare increases with $\beta$ as patient producers can endure more production, increases with $\alpha$ as the economy has the higher monitoring capacity. One exception is $(\beta, \kappa) = (0.3, 10)$, in the first panel where the welfare decreases as $\alpha$ increases from 0 to 0.25. Moreover, the welfare is close to zero in that example, suggesting that implementable allocations are very

**Figure 3.5.1.** Welfare

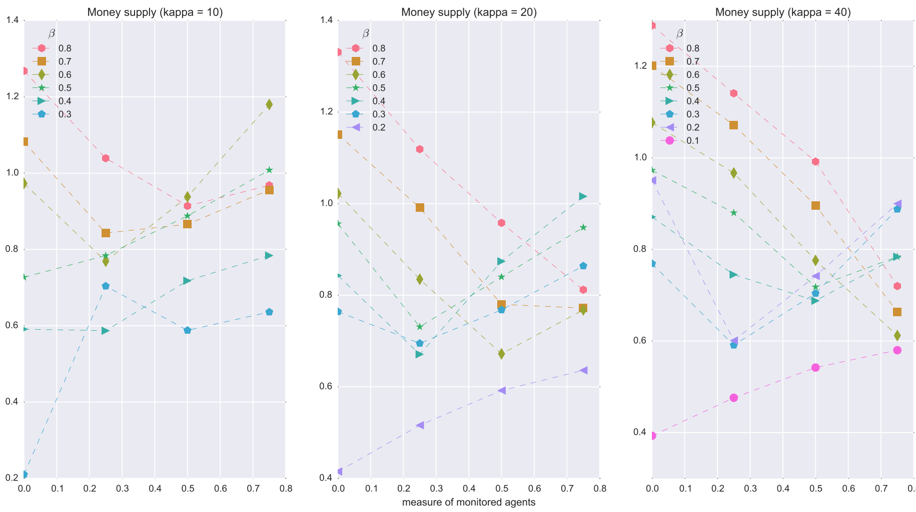



close to an allocation with no production. This is due to the incentive compatibility we imposed. While Wallace [2010] shows that welfare is weakly increasing in $\alpha$, the proof of this claim is relying on social planner treating some monitored agents as non-monitored agents. For simplicity, we require the expected utility of all monitored agents to be equal, and higher than that of any non-monitored agents'. In this case, welfare is not guaranteed to be weakly increasing in $\alpha$, as we can see in this result. This example shows that social planner may actually want to treat some monitored agents as non-monitored agents, and it might be better to use less than full monitoring capacity.

Broadly, money supply decreases with $\alpha$ when $\beta$ is high, and it increases $\alpha$ when $\beta$ is low. For intermediate values of $\beta$, money supply decreases with initially and then increases with $\alpha$. In sum, money supply does not change monotonically with $\alpha$, but it is less than $B/2$ in all examples.

Figure 3 shows the inflation rate at optima. It is strictly positive in all examples with $\alpha > 0$, while transfer rate is positive only in one of them. In this model, both social planner and monitored agents can create money, and that will increase money supply and lead to

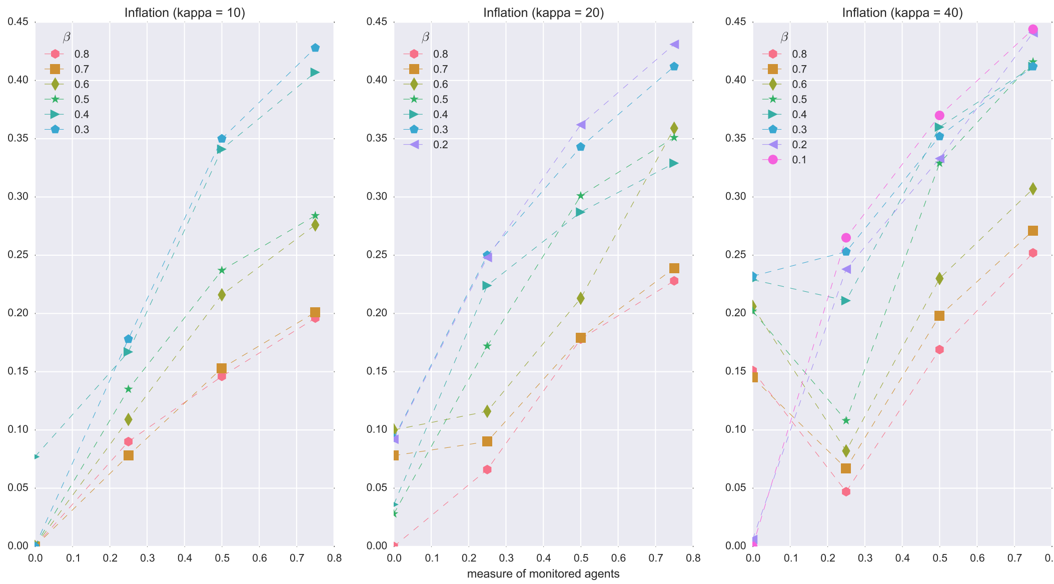
**Figure 3.5.2.** Money supply




inflation. In examples where inflation rate is positive and transfer rates are zero, inflation is caused by net money creation of monitored agents, which is the amount of money spent by monitored to non-monitored agents net of the amount of money earned by monitored from non-monitored agents. Hence, the result is consistent with Deviatov and Wallace [2014]. Even though we extend their model to incorporate policies that are potentially beneficial but not inflationary, it is still not beneficial to tax monitored agents. Even in the example with positive transfer rates, inflation may not be solely caused by government transfer. It can be both government transfer and net money creation of monitored agents that cause inflation.

To understand this result, it is helpful to discuss the limit on the role of policies, generated by the one-unit upper bound on money holdings in Deviatov and Wallace [2014]. Taxing monitored agents - by making monitored agents spend less money to non-monitored agents than what they earn from non-monitored agents - is feasible even under the one-unit upper bound (given that it is incentive compatible). However, it cannot be beneficial for welfare with the one-unit upper bound, as transfer scheme is too limited. With the three-unit upper bounds as in here, taxing monitored agents can be beneficial. Social planner can improve

**Figure 3.5.3.** Inflation rate




risk-sharing through lump-sum transfers, relax participation constraints of some producers through policies resembling Friedman rule. The cost of inflation resulting from such transfers can be reduced by taxing monitored agents. However, it will make monitored agents to consume less and decrease their utility. At optima, net issue of money from monitored to non-monitored agents is positive, as in the example of Deviatov and Wallace [2014].

Compared with the counterpart pure-currency economies ($\alpha = 0$), transfer is used ($\tau_i > 0$ for any $i$) much infrequently. According to the conjecture in Wallace [2014], there generically exists some government transfer schemes that improve welfare over no transfer in pure-currency economies. We find that government transfer is used in 2/3 of examples with $\alpha = 0$ (See Chapter 2 for a discussion on the upper bound and the conjecture). There are two reasons why transfer is not used as frequently as in pure-currency economies. First, transfer is done through monitored agents. We allow not only that monitored agents can produce for non-monitored agents with no money, but also can give money to them. In 1/2 examples with $\alpha > 0$, monitored producers both produce and give money to non-monitored consumers with no money. For the social planner, this is a more efficient way to give money to agents with no money than the direct transfer. While transfer is subject to incentive compatibility constraint $(\tau_0 \leq \tau_1 \leq \tau_2)$ , we assume perfect information in pairwise meetings. Hence, risk-sharing can be achieved more efficiently through monitored agents in pairwise meetings, than the direct transfer. Second, as some inflation is already caused by net money creation of monitored agents, transfer can be too costly as it will further increase the inflation rate.

## 3.6 Concluding remarks

Let us quote Deviatov and Wallace [2014] to highlight the message of this study: this is a counter-example to the view that "inside-money economies should be regulated so as to avoid inflation", and that "inflation is harmful in the absence of nominal rigidities". Inflation can be optimal in a model with inside money as it enables those who can issue money spend more than they earn. However, their model does not give scope to policies resembling Friedman rule because of the small upper bound on money holdings, and the result can potentially be overturned if such policies financed through tax on monitored (those who can issue money) were possible and optimal. We extend their model by adopting higher upper bound, and find that their result is robust. Inflation is optimal for all the examples we considered, and it arises from the net positive money creation by monitored agents, not from the government transfer.

The upper bound of three, which is what we used, is not large but not small either. It is the smallest $B$ that gives potential scope to policies resembling Friedman rule, and we cannot think of a reason why adopting a higher $B$ will overturn the result.

# Chapter A

# Chapter 1: An example

In this appendix, I change the set of feasible asset holdings to have a one-unit upper bound while keeping environment and solution concepts the same.

$$\mathbb{Z} \equiv \left\{ (M_C, M_B) \in \{0,1\}^2 \mid M_C + M_B \leq 1 \right\} = \{(0,0), (0,1), (1,0)\}$$

**Example.** This example is about two implementable allocations in an economy with a unit upper bound. By constructing these examples, I want to show the necessity of the restriction on trades. One example is the optimum of currency economy, and the other example is a welfare superior allocation in bond economy, compared to the optimum.

The optimum is depending on the following inequality in the economy with $\{0,1\}$ money holding.

$$\frac{u(y)}{y} \geq \frac{1-\beta}{\mu\beta} \frac{1}{1-m} + 1$$

If $y = \hat{y}$, where $\hat{y} \equiv \max_{y \geq 0} [u(y) - y]$, and $m = \frac{1}{2}$ satisfies the inequality, it is the optimum. If not, optimal production $y^o$ and amount of money $m^o$ holds above inequality with equality and $y^o < \hat{y}$, $m^o < \frac{1}{2}$. Using the equality, social planner's maximization problem

$$\max_{m \in [0,1], y \geq 0} W^{(1)} \equiv m(1-m)(u(y) - y)$$

becomes

$$\begin{aligned} \max_{y\geq 0} \quad & \left(\tfrac{1-\beta}{\beta\mu}y\right)\left(1-\tfrac{1-\beta}{\beta\mu}\tfrac{y}{u(y)-y}\right) \\ \text{subject to} \quad & m = 1-\tfrac{1-\beta}{\beta\mu}\tfrac{y}{u(y)-y} \in [0,1] \\ & 1-m = \tfrac{1-\beta}{\beta\mu}\tfrac{y}{u(y)-y} \in [0,1] \end{aligned}$$

Using the first parameter in section 4, we get $W^{(1)} = 0.1319$, $y = 0.0909$, $m = 0.4839$.

Now, construct an allocation in the bond economy as follows. Assume that $m_C = m_B = \frac{1}{4}$, disintegration rate $\tau = 0.01$ (the transfer rate that preserves stationarity is $\epsilon = \frac{0.01*0.25}{0.99*0.5}$) and buyers make take-it-or-leave-it offers in (0,C), (C,B) meeting, where two elements are the asset holdings of a producer and a consumer respectively. Also, coexistence requires that agents are indifferent to choose currency or bonds at the portfolio choice stage. Following expressions are the value functions of agents holding each asset before and after trade meeting stage.

$$\begin{aligned} V_C &= \mu\left(1-m_C-m_B\right)\left(u\left(y_C\right)+\beta W_0\right)+\mu\left(m_B\right)\left(-y_{CB}+\beta W_B\right)+\left(1-\mu\left(1-m_C\right)\right)\beta W_C \\ V_B &= \frac{1-m_C-m_B}{N}\left(u\left(y_B\right)+\beta W_0\right)+\frac{m_C}{N}\left(u\left(y_{CB}\right)+\beta W_C\right)+\left(1-\frac{1-m_B}{N}\right)\beta W_B \\ V_0 &= \mu m_C\left(-y_C+\beta W_C\right)+\mu m_B\left(-y_B+\beta W_B\right)+\left(1-\mu\left(m_B+m_C\right)\right)\beta W_0 \\ W_C &= \left(1-\tau\right)V_C+\tau V_0 \\ W_B &= V_B \\ W_0 &= \epsilon\left(1-\tau\right)V_C+\left(1-\epsilon+\epsilon\tau\right)V_0 \end{aligned}$$

As buyers make take-it-or-leave-it offers in (0,C), (C,B) meeting,

$$-y_C+\beta\left(W_C-W_0\right) = 0$$

$$-y_{CB} + \beta\left(W_B - W_C\right) = 0$$

and these are equivalent to

$$\begin{aligned} y_C &= \beta\left(1-\epsilon\right)\left(1-\tau\right)\left(V_C - V_0\right) \\ y_{CB} &= \beta\tau\left(V_C - V_0\right) \end{aligned}$$

using $V_C = V_B$ (coexistence condition) . After some algebra, $V_C = V_B$ implies

$$\left(1-m_C-m_B\right)\left(u\left(y_C\right)-u\left(y_B\right)\right) = \frac{m_C+m_B}{2}\left(u\left(y_{CB}\right)+y_{CB}\right)+\left(\frac{1}{\mu}-1\right)y_{CB}$$

Lastly, $V_C - V_0$ can be expressed as

$$\begin{aligned} \left(V_C - V_0\right) &= \mu\left(1-m_C-m_B\right)u\left(y_C\right)+\mu m_C y_C + m_B\left(y_B - y_{CB}\right) \\ &\quad +\beta\left(1-\mu\right)\left(1-\epsilon\right)\left(1-\tau\right)\left(V_C - V_0\right) \end{aligned}$$

One solution that satisfies above four equations is $y_C = 0.0802$, $y_B = 0.0253$, $y_{CB} = 0.000814$ . The welfare level of this allocation

$$W^{(2)} \equiv \sum_{i=B,C} m_i\left(1-m_B-m_C\right)\left(u\left(y_i\right)-y_i\right)+m_B m_C\left(u\left(y_{CB}\right)-y_{CB}\right)$$

is $W^{(2)} = 0.1372 > 0.1319 = W^{(1)}$.

Hence, this allocation is welfare superior to the optimum in currency economy.

This example is a reminiscent of Aiyagari et al. [1996], where they show that there exists a better equilibrium with two different color of money, compared to the unique equilibrium with only one money. Even though two monies differ only in their appearances, one of them is more valuable in the better equilibrium. By valuing two colors of money differently, it endogenously generates a denomination structure, effectively a richer set of money holdings, and leads to welfare improvement. While the argument does not straightforwardly extend to the economy we consider, due to many differences between Aiyagari et al. [1996] and mine, above example shows that a similar result arises.

Next proposition proves that with a one-unit upper bound asset holdings, the welfare improvement disappears when asset swapping is not possible.

**Proposition.** *In a one-unit upper bound economy, bonds do not improve welfare when asset swapping is not possible.*

*Proof.* I show that for any given $m, \epsilon, \tau \in (0, 1)$, where $m = m_C + m_B$, the set of implementable $y_C$ in bond economy is a subset of the set of implementable $y_C$ in currency economy. Since $y_C > y_B$ is necessary to achieve the coexistence whenever transfer and disintegration rate is strictly positive, this proves the claim. Consider following social planner's problem:

Given $m, \epsilon, \tau$

$$\begin{aligned} \max_{m_C \in [0,m], y_C, y_B} \quad & (1-m)\, m_C \left(u\left(y_C\right) - y_C\right) + (1-m)\left(m - m_C\right)\left(u\left(y_B\right) - y_B\right) \\ \text{subject to} \quad & -y_C + \beta W_C \geq \beta W_0 \\ & u\left(y_C\right) + \beta W_0 \geq \beta W_c \\ & -y_B + \beta W_B \geq \beta W_0 \\ & u\left(y_B\right) + \beta W_0 \geq \beta W_B \end{aligned}$$

where

$$\begin{aligned} V_C &= \mu\left(1-m\right)\left(u\left(y_C\right) + \beta W_0\right) + \left(1 - \mu\left(1-m\right)\right)\beta W_C \\ V_B &= \mu\left(1-m\right)\left(u\left(y_B\right) + \beta W_0\right) + \left(1 - \mu\left(1-m\right)\right)\beta W_B \\ V_0 &= \mu m_C\left(-y_C + \beta W_C\right) + \mu\left(m - m_C\right)\left(-y_B + \beta W_B\right) + \left(1 - \mu m\right)\beta W_0 \end{aligned}$$

$$
\begin{aligned}
W_C &= (1-\tau) V_C + \tau V_0 \\
W_B &= V_B \\
W_0 &= \epsilon (1-\tau) V_C + (1-\epsilon+\epsilon\tau) V_0
\end{aligned}
$$

By subtracting $V_0$ from $V_C$, we get

$$
\begin{aligned}
\mu (1-m) u (y_C) + \mu (m_C y_C + (m-m_C) y_B) &= \Delta (1-\beta (1-\epsilon)(1-\tau)) \\
&\quad + \beta\mu\Delta ((1-\epsilon)(1-\tau) + \tau (m-m_C))
\end{aligned}
$$

where $\Delta \equiv V_C - V_0$. Using this, we can write the participation constraint of a producer who meets a consumer with currency as

$$
\Omega (\mu (1-m) u (y_C) + \mu (m_C y_C + (m-m_C) y_B)) \geq y_C
$$

where $\Omega = \frac{\beta(1-\epsilon)(1-\tau)}{1-\beta(1-\epsilon)(1-\tau)+\mu\beta((1-\epsilon)(1-\tau)+\tau(m-m_C))}$. This is equivalent to

$$
(1-m) \frac{u (y_C)}{y_C} + m_C + (m-m_C) \frac{y_B}{y_C} \geq \mu \left( \frac{1}{\beta (1-\epsilon)(1-\tau)} - 1 \right) + \frac{\tau}{1-\tau} (1-\epsilon)(m-m_C)
$$

The largest set of $y_C$ that satisfies this necessary condition can be attained when $m_C = m$, and it is a subset of the set of implementable allocations in currency economy. $\square$

Given the access to the window in every period, the only way to achieve the coexistence, in the sense that people are indifferent to hold currency or bonds, is by making currency buys more goods in trade stage than bonds, as bonds have an advantage in the disintegration process. While the extensive margin (number of trade meeting) does not increase by adding bonds as we forbid asset swapping, the intensive margin (amount of production and consumption) worsens in order to achieve the coexistence.

Welfare improvements in the example is associated with the upper bound and indivisibility on the feasible set of asset holding, while those assumptions are made solely to make the model tractable. As I do not want to focus on that issue, restrictions on such trades are in order. While above results are attained in an economy with a one-unit upper bound, it hints that we are turning ourselves away from the result in Aiyagari et al. [1996] by not allowing asset swapping.

One crucial difference between a one-unit upper and a two-unit upper bound is that inflation created by lump-sum transfer can be beneficial in a two-unit upper bound economy. The beneficial extensive margin effect can present only when people can hold more than one unit of asset. This observation helps to understand the difference between the proposition and the results in the section 4.

# Chapter B

# Chapter 2

## B.1 Money supply

**Table B.1.** Money supply ($\frac{\text{Average money holdings}}{B}$), $B = 2$

| $\kappa \backslash \beta$ | 0.15 | 0.2 | 0.25 | 0.3 | 0.35 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 |
|---|---|---|---|---|---|---|---|---|---|---|
| 20 | 0.0285 | 0.1145 | 0.1810 | 0.2330 | 0.2765 | 0.3190 | 0.3950 | 0.4305 | 0.4505 | 0.4655 |
| 15 | - | 0.0390 | 0.1120 | 0.1715 | 0.2225 | 0.2660 | 0.3445 | 0.4130 | 0.4530 | 0.4675 |
| 12 | - | - | 0.0510 | 0.1160 | 0.1715 | 0.2200 | 0.3065 | 0.3815 | 0.4485 | 0.4660 |
| 10 | - | - | - | 0.0645 | 0.1240 | 0.1765 | 0.2700 | 0.3540 | 0.4290 | 0.4615 |
| 8 | - | - | - | - | 0.0595 | 0.1160 | 0.2185 | 0.3125 | 0.4005 | 0.4565 |
| 6 | - | - | - | - | - | 0.0275 | 0.1365 | 0.2440 | 0.3515 | 0.4495 |
| 5 | - | - | - | - | - | - | 0.0770 | 0.1895 | 0.3065 | 0.4300 |
| 4 | - | - | - | - | - | - | - | 0.1125 | 0.2400 | 0.3825 |
| 3 | - | - | - | - | - | - | - | - | 0.127 | 0.290 |
| 2 | - | - | - | - | - | - | - | - | - | 0.0835 |

**Table B.2.** Money supply ($\frac{\text{Average money holdings}}{B}$), $B = 3$

| $\kappa\backslash\beta$ | 0.15 | 0.2 | 0.25 | 0.3 | 0.35 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 |
|---|---|---|---|---|---|---|---|---|---|---|
| 20 | 0.029 | 0.138 | 0.2063 | 0.2547 | 0.258 | 0.281 | 0.3187 | 0.3407 | 0.3837 | 0.4437 |
| 15 | - | 0.041 | 0.133 | 0.195 | 0.2433 | 0.2413 | 0.2923 | 0.335 | 0.3933 | 0.436 |
| 12 | - | - | 0.054 | 0.136 | 0.1937 | 0.194 | 0.27 | 0.329 | 0.3837 | 0.4293 |
| 10 | - | - | - | 0.07 | 0.143 | 0.197 | 0.2423 | 0.3243 | 0.3607 | 0.4227 |
| 8 | - | - | - | - | 0.064 | 0.1333 | 0.194 | 0.282 | 0.336 | 0.4123 |
| 6 | - | - | - | - | - | 0.031 | 0.154 | 0.2147 | 0.302 | 0.392 |
| 5 | - | - | - | - | - | - | 0.087 | 0.1653 | 0.2753 | 0.362 |
| 4 | - | - | - | - | - | - | - | 0.1283 | 0.2077 | 0.3167 |
| 3 | - | - | - | - | - | - | - | - | 0.147 | 0.2557 |
| 2 | - | - | - | - | - | - | - | - | - | 0.101 |

| $\kappa\backslash\beta$ | 0.15 | 0.2 | 0.25 | 0.3 | 0.35 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 |
|---|---|---|---|---|---|---|---|---|---|---|
| 20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 26.01 | 53.36 |
| 15 | - | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 51.67 |
| 12 | - | - | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 28.1 |
| 10 | - | - | - | 0 | 0 | 0 | 0 | 0 | 0 | 27.13 |
| 8 | - | - | - | - | 0 | 0 | 0 | 0 | 0 | 26.01 |
| 6 | - | - | - | - | - | 0 | 0 | 0 | 0 | 0 |
| 5 | - | - | - | - | - | - | 0 | 0 | 0 | 30.54 |
| 4 | - | - | - | - | - | - | - | 0 | 0 | 0 |
| 3 | - | - | - | - | - | - | - | - | 0 | 0 |
| 2 | - | - | - | - | - | - | - | - | - | 0 |

**Table B.3.** Lottery usage (% of all trade meeting), $B = 2$

## B.2 Some features of the optima

First three tables document the use of lottery in meetings. The first and second table show the fraction of trade meetings where the lottery is used in each example. The third table shows the fraction of examples where the lottery is used in each meeting.

Secondly, the following tables illustrate that optimum is sometimes not achieved through consumers making take-it-or-leave-it offer (approximately one third of the examples). The trade surplus is defined as the difference between the expected utility from following the designated trade (production, consumption, monetary payments) and running away from the

| $\kappa \backslash \beta$ | 0.15 | 0.2 | 0.25 | 0.3 | 0.35 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 |
|---|---|---|---|---|---|---|---|---|---|---|
| 20 | 0 | 2.05 | 3.05 | 3.68 | 0 | 0 | 10.22 | 0 | 7.25 | 42.72 |
| 15 | - | 0 | 2.09 | 3.04 | 3.71 | 0 | 0 | 9.61 | 0 | 38.67 |
| 12 | - | - | 0 | 2.3 | 3.16 | 0 | 6.67 | 11.04 | 0 | 26.16 |
| 10 | - | - | - | 0 | 2.5 | 3.35 | 5.95 | 12.32 | 7.55 | 25.62 |
| 8 | - | - | - | - | 0 | 2.52 | 0 | 0 | 16.80 | 25.21 |
| 6 | - | - | - | - | - | 0 | 3.22 | 5.94 | 19.01 | 43.99 |
| 5 | - | - | - | - | - | - | 0 | 4.64 | 0 | 37.99 |
| 4 | - | - | - | - | - | - | - | 0 | 0 | 24.05 |
| 3 | - | - | - | - | - | - | - | - | 4.62 | 7.9 |
| 2 | - | - | - | - | - | - | - | - | - | 3.68 |

**Table B.4.** Lottery usage (% of trade meetings), $B = 3$

| (Producer, Consumer) | (0,1) | (0,2) | (0,3) | (1,1) | (1,2) | (1,3) | (2,1) | (2,2) | (2,3) |
|---|---|---|---|---|---|---|---|---|---|
| B=2 | 1/55 | 0/55 | - | 6/55 | 2/55 | - | - | - | - |
| B=3 | 3/55 | 0/55 | 6/55 | 6/55 | 7/55 | 5/55 | 23/55 | 0/55 | 0/55 |

**Table B.5.** Lottery usage (proportion of examples for each meeting)

trade. The first and second table show the fraction of trade meetings where the consumers all the trade surplus in each example. The third table shows the fraction of examples where the consumers take all the trade surplus (hence the producers' participation constraint is binding) in each meeting.

| $\kappa \backslash \beta$ | 0.15 | 0.2 | 0.25 | 0.3 | 0.35 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 |
|---|---|---|---|---|---|---|---|---|---|---|
| 20 | 100 | 100 | 100 | 100 | 100 | 59.13 | 66.43 | 69.42 | 47.36 | 53.36 |
| 15 | - | 100 | 100 | 100 | 100 | 100 | 61.70 | 68.00 | 46.99 | 51.67 |
| 12 | - | - | 100 | 100 | 100 | 100 | 100 | 65.21 | 46.16 | 50.42 |
| 10 | - | - | - | 100 | 100 | 100 | 100 | 62.62 | 69.37 | 49.23 |
| 8 | - | - | - | - | 100 | 100 | 100 | 100 | 66.92 | 47.83 |
| 6 | - | - | - | - | - | 100 | 100 | 100 | 100 | 71.06 |
| 5 | - | - | - | - | - | - | 100 | 100 | 100 | 69.46 |
| 4 | - | - | - | - | - | - | - | 100 | 100 | 100 |
| 3 | - | - | - | - | - | - | - | - | 100 | 100 |
| 2 | - | - | - | - | - | - | - | - | - | 100 |

**Table B.6.** Percentage of meetings where consumers take all the surplus, $B = 2$

| $\kappa\backslash\beta$ | 0.15 | 0.2 | 0.25 | 0.3 | 0.35 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 |
|---|---|---|---|---|---|---|---|---|---|---|
| 20 | 100 | 100 | 100 | 100 | 66.69 | 69.79 | 65.32 | 64.79 | 69.03 | 54.46 |
| 15 | - | 100 | 100 | 100 | 100 | 100 | 63.27 | 63.27 | 68.76 | 52.72 |
| 12 | - | - | 97.80 | 100 | 100 | 100 | 100 | 62.19 | 67.40 | 62.87 |
| 10 | - | - | - | 97.34 | 100 | 100 | 100 | 61.51 | 79.20 | 61.66 |
| 8 | - | - | - | - | 97.50 | 100 | 100 | 100 | 75.75 | 70.57 |
| 6 | - | - | - | - | - | 100 | 100 | 100 | 100 | 100 |
| 5 | - | - | - | - | - | - | 96.83 | 100 | 100 | 100 |
| 4 | - | - | - | - | - | - | - | 95.89 | 100 | 100 |
| 3 | - | - | - | - | - | - | - | - | 95.38 | 100 |
| 2 | - | - | - | - | - | - | - | - | - | 96.32 |

**Table B.7.** Percentage of meetings where consumers take all the surplus, $B = 3$

| (Producer, Consumer) | (0,1) | (0,2) | (0,3) | (1,1) | (1,2) | (1,3) | (2,1) | (2,2) | (2,3) |
|---|---|---|---|---|---|---|---|---|---|
| B=2 | 36/55 | 47/55 | - | 55/55 | 55/55 | - | - | - | - |
| B=3 | 37/55 | 43/55 | 51/55 | 55/55 | 53/55 | 55/55 | 46/55 | 55/55 | 55/55 |

**Table B.8.** Fraction of examples where consumers take all the surplus

## B.3 Some results for $B = 4$

We report numerical results for $B = 4$ in this subsection. For this case, the risk aversion parameter $\kappa$ is varied over $\{2, 3, 4, 5, 6, 8, 10, 12, 15, 20\}$ and the discount factor parameter $\beta$ is varied over $\{0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8\}$. Optimum is computed for each of 46 pairs of $\kappa$ and $\beta$, under which the condition (2.2.1) is satisfied, out of 70 possible pairs, using (i) KNITRO with 8000 initial points, (ii) KNITRO with 16000 initial points, or (iii) BARON.

We found the numerical results for $B = 4$ are less reliable than those for $B = 2$ or 3: We found optima do not coincide for 6 pairs of $\kappa$ and $\beta$ between (i) and (ii) and for 9 pairs between (i) and (iii). So the results for $B = 4$ are reported not in the body, but here.

| | $\kappa \backslash \beta$ | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 |
|---|---|---|---|---|---|---|---|---|
| $\tau_0 = \tau_1 = \tau_2$ | 20 | 0 | 0 | 5.0 | 15.9 | 14.5 | 20.7 | 16.1 |
| | 15 | 0 | 0 | 0 | 0 | 3.9 | 14.4 | 11.7 |
| | 12 | - | 0 | 0 | 0.7 | 0 | 9.3 | 7.7 |
| | 10 | - | 0 | 0 | 0 | 0 | 4.8 | 0 |
| | 8 | - | - | 0 | 0 | 0 | 0 | 0 |
| | 6 | - | - | 0 | 0 | 0 | 0 | 0 |
| | 5 | - | - | - | 0 | 0 | 0 | 0 |
| | 4 | - | - | - | - | 0 | 0 | 0 |
| | 3 | - | - | - | - | - | 0 | 0 |
| | 2 | - | - | - | - | - | - | 0 |
| $\tau_3$ | 20 | 100.0 | 76.4 | 30.5 | 56.8 | 14.5 | 20.7 | 16.1 |
| | 15 | 100.0 | 70.2 | 14.8 | 5.0 | 3.9 | 14.4 | 11.7 |
| | 12 | - | 64.3 | 8.1 | 0.7 | 0 | 9.3 | 7.7 |
| | 10 | - | 58.1 | 61.7 | 0 | 0 | 4.8 | 0 |
| | 8 | - | - | 55.0 | 0 | 0 | 0 | 0 |
| | 6 | - | - | 100.0 | 47.3 | 0 | 0 | 0 |
| | 5 | - | - | - | 100.0 | 0 | 0 | 0 |
| | 4 | - | - | - | - | 0 | 0 | 0 |
| | 3 | - | - | - | - | - | 0 | 0 |
| | 2 | - | - | - | - | - | - | 0 |
| $\delta$ | 20 | 8.7 | 4.81 | 6.69 | 15.27 | 10.03 | 12.24 | 8.61 |
| | 15 | 7.06 | 5.62 | 1.24 | 0 | 2.79 | 8.94 | 6.4 |
| | 12 | - | 5.16 | 0.61 | 0.66 | 0 | 6.03 | 4.3 |
| | 10 | - | 4.42 | 5.13 | 0 | 0 | 3.13 | 0 |
| | 8 | - | - | 4.63 | 0 | 0 | 0 | 0 |
| | 6 | - | - | 6.76 | 4.22 | 0 | 0 | 0 |
| | 5 | - | - | - | 8.35 | 0 | 0 | 0 |
| | 4 | - | - | - | - | 0 | 0 | 0 |
| | 3 | - | - | - | - | - | 0 | 0 |
| | 2 | - | - | - | - | - | - | 0 |

**Table B.9.** Transfer and Disintegration (%), $B = 4$

## B.4 Pairwise core constraint

The problem (3.3.6) can be equivalently rewritten as

$$\max_{\lambda_c} u \left( \beta \sum_{0 \le i \le k+k'} \lambda_c(i) w\left(k + k' - i\right) - \beta w(k) - \vartheta(k, k') \right) + \beta \sum_{0 \le i \le k+k'} \lambda_c(i) w(i)$$

$$
\begin{aligned}
\text{s.t. } & [\mu_q^{k,k'}] - \left( \beta \sum_{0 \le i \le k+k'} \lambda_c(i) w\left(k + k' - i\right) - \beta w(k) - \vartheta(k,k') \right) \le 0 \\
& [\mu_0^{k,k'}(i)] - \lambda_c(i) \le 0 \text{ for all } i \\
& [\mu_{\text{sum}}^{k,k'}] 1 - \sum_{0 \le i \le k+k'} \lambda_c(i) = 0
\end{aligned}
$$

where $\mu$'s (the variables in square brackets before the constraints) denote multipliers. Since all the constraints are linear, the constraint qualification is satisfied. As the objective is concave, the Karush-Kuhn-Tucker (KKT) condition is necessary and sufficient for the optimality. The KKT condition can be stated as follows:

$$
0 = \left[u'\left(q(k,k')\right) + \mu_q^{k,k'}\right] \beta w\left(k + k' - i\right) + \beta w\left(i\right) + \mu_0^{k,k'}(i) + \mu_{\text{sum}}^{k,k'} \text{ for all } i, \tag{B.4.1}
$$

$$
0 \ge -q(k,k'), \tag{B.4.2}
$$

$$
0 \ge -\lambda_c^{k,k'}(i) \text{ for all } i, \tag{B.4.3}
$$

$$
0 = 1 - \sum_{0 \le i \le k+k'} \lambda_c^{k,k'}(i), \tag{B.4.4}
$$

$$
0 = \mu_q^{k,k'} q(k,k'), \tag{B.4.5}
$$

$$
0 = \mu_0^{k,k'}(i) \lambda_c^{k,k'}(i) \text{ for all } i, \tag{B.4.6}
$$

and

$$
q(k,k') = \beta \sum_{0 \le i \le k+k'} \lambda_c^{k,k'}(i) w(k + k' - i) - \beta w(k) - \vartheta(k,k').
$$

In the following, we show that the last condition and the surplus variable $\vartheta(k,k')$ are redundant.

The original planner's problem (Problem O), which is with $\vartheta$, can be stated as

$$
\max_{\vartheta,q,\lambda,\delta,\tau,\pi,v,w} \quad \frac{\sum_{0 \le k \le B} \sum_{0 \le k' \le B} \pi_k \pi_{k'} \left[u\left(q(k,k')\right) - q(k,k')\right]}{K\left(1 - \beta\right)}
$$

$$\begin{array}{lll} \text{s.t.} & \{\text{IRs}\} & -q(k,k') + \beta \sum_{0\leq i\leq k+k'} \lambda_c^{k,k'}(i) w(k+k'-i) \geq \beta w(k) \\ & & \text{(Other IRs)} \\ & & \vartheta(k,k') \geq 0 \\ & \{\text{PCs}\} & -q(k,k') + \beta \sum_{0\leq i\leq k+k'} \lambda_c^{k,k'}(i) w(k+k'-i) = \beta w(k) + \vartheta(k,k') \\ & & \text{(Other PCs)} \\ & & \text{(All the other constraints)} \end{array}$$

Note that $\vartheta$ appears only in the pairwise core constraints explicitly written above. The reduced problem (Problem R) is

$$\max_{q,\lambda,\delta,\tau,\pi,v,w} \quad \frac{\sum_{0\leq k\leq B} \sum_{0\leq k'\leq B} \pi_k \pi_{k'} \left[u\left(q(k,k')\right) - q(k,k')\right]}{K\left(1-\beta\right)}$$

$$\begin{array}{lll} \text{s.t.} & \{\text{IRs}\} & -q(k,k') + \beta \sum_{0\leq i\leq k+k'} \lambda_c^{k,k'}(i) w(k+k'-i) \geq \beta w(k) \\ & & \text{(Other IRs)} \\ & \{\text{PCs}\} & \text{(Other PCs)} \\ & & \text{(All the other constraints)} \end{array}$$

Let $S_O$ and $S_R$ denote the set of solutions for Problem O and that for Problem R, respectively. Define $\tilde{S}_O = \{(q,\lambda,\delta,\tau,\pi,v,w) | \exists\vartheta, (\vartheta,q,\lambda,\delta,\tau,\pi,v,w) \in S_O\}$. The precise claim we want to show is the following.

*Remark.* $\tilde{S}_O = S_R$.

*Proof.* We prove two inclusion relationships, $\tilde{S}_O \subset S_R$ and $S_R \subset \tilde{S}_O$.

- ($\tilde{S}_O \subset S_R$) Let $(q,\lambda,\delta,\tau,\pi,v,w) \in \tilde{S}_O$. By definition, there exists $\vartheta$ such that $(\vartheta,q,\lambda,\delta,\tau,\pi,v,w) \in S_O$. The feasibility of $(\vartheta,q,\lambda,\delta,\tau,\pi,v,w)$ in Problem O clearly implies the feasibility of $(q,\lambda,\delta,\tau,\pi,v,w)$ in Problem R. Suppose, by way of contradiction, $(q,\lambda,\delta,\tau,\pi,v,w) \notin S_R$: there exists $(\tilde{q},\tilde{\lambda},\tilde{\delta},\tilde{\tau},\tilde{\pi},\tilde{v},\tilde{w})$ such that $(\tilde{q},\tilde{\lambda},\tilde{\delta},\tilde{\tau},\tilde{\pi},\tilde{v},\tilde{w})$

is feasible and better than $(q,\lambda,\delta,\tau,\pi,v,w)$ in Problem R. Define $\tilde{\vartheta}$, using

$$\tilde{q}(k,k') \;=\; \beta \sum_{0\le i\le k+k'} \tilde{\lambda}_c^{k,k'}(i)\tilde{w}(k+k'-i) - \beta\tilde{w}(k) - \tilde{\vartheta}(k,k').$$

Then, the IR implies $\tilde{\vartheta}(k,k') \ge 0$, so $(\tilde{\vartheta},\tilde{q},\tilde{\lambda},\tilde{\delta},\tilde{\tau},\tilde{\pi},\tilde{v},\tilde{w})$ satisfies all the constraints in Problem O. Also, the supposition that $(\tilde{q},\tilde{\lambda},\tilde{\delta},\tilde{\tau},\tilde{\pi},\tilde{v},\tilde{w})$ is strictly better than $(q,\lambda,\delta,\tau,\pi,v,w)$ in Problem R implies that $(\tilde{\vartheta},\tilde{q},\tilde{\lambda},\tilde{\delta},\tilde{\tau},\tilde{\pi},\tilde{v},\tilde{w})$ is strictly better than $(q,\lambda,\delta,\tau,\pi,v,w)$ in Problem O. That contradicts the assumption $(\vartheta,q,\lambda,\delta,\tau,\pi,v,w) \in S_O$, so we have $(q,\lambda,\delta,\tau,\pi,v,w) \in S_R$.

- $(S_R \subset \tilde{S}_O)$ Let $(q,\lambda,\delta,\tau,\pi,v,w) \in S_R$. Define $\vartheta$, using

$$q(k,k') \;=\; \beta \sum_{0\le i\le k+k'} \lambda_c^{k,k'}(i)w(k+k'-i) - \beta w(k) - \vartheta(k,k').$$

Then, the IR implies $\vartheta(k,k') \ge 0$, so $(\vartheta,q,\lambda,\delta,\tau,\pi,v,w)$ is feasible in Problem O. The optimality of $(q,\lambda,\delta,\tau,\pi,v,w)$ in Problem R implies the optimality of $(\vartheta,q,\lambda,\delta,\tau,\pi,v,w)$ in Problem O in the way similar to the above. So we have $(q,\lambda,\delta,\tau,\pi,v,w) \in \tilde{S}_R$.

$\square$

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