CONTROL MODELS OF LEHMAN'S 7TH LAW OF SOFTWARE ENGINEERING IN SINGLE & DUOPOLISTIC MARKETS

A Thesis in Mathematics
by
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Abstract

This thesis proposes an optimal control model and a differential game model of a firm’s costly effort towards product maintenance in the digital software market. Both models analyze a digital vendor’s profit maximization problem, where dedication to maintenance is expensive but also increases market share. Through a rumor-spreading state equation explained with a Bass-like model, digital vendors can capture market share by focusing efforts on post-launch product maintenance. We first derive theoretical results expressing market share, the state variable, in terms of effort put towards maintenance, the control variable. We then use numerical examples to illustrate optimal control paths for vendors facing various market conditions. Both models demonstrate that optimal product maintenance always declines over a product’s life cycle, consistent with Lehman’s 7th law of software evolution. Lehman’s 7th law states that “the quality of an E-type system will appear to be declining unless it is rigorously maintained and adapted to operational environment changes.” Furthermore, we see that initial market conditions can heavily influence the optimal path to pursue and affect final market share.
# Table of Contents

List of Figures                                      v  
List of Tables                                       vi 
Acknowledgments                                      vii 

Chapter 1  
Introduction                                       1 

Chapter 2  
Literature Review  
  2.1 Optimal Control Theory                          3  
  2.2 Differential Games Theory                       5  
  2.3 Motivation in Related Disciplines              7 

Chapter 3  
Optimal Control Model for Product Maintenance of Mobile Software Applications  
  3.1 Optimal Control Model                           9  
  3.2 Numerical Examples                             15 
  3.3 Discussion and Conclusions                     19 

Chapter 4  
Differential Game Model of Maintenance-Based Competition for Mobile Software  
  4.1 Differential Game Model of Digital Distribution 21 
  4.2 Numerical Examples                             26 
  4.3 Discussion                                     28 

Chapter 5  
Conclusions                                        29 

Bibliography                                        30
# List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>Market conditions with $x_0 = 0.5$</td>
<td>18</td>
</tr>
<tr>
<td>3.2</td>
<td>Market conditions with $x_0 = 0.3$</td>
<td>18</td>
</tr>
<tr>
<td>3.3</td>
<td>Market conditions with $x_0 = 0.4$</td>
<td>19</td>
</tr>
<tr>
<td>4.1</td>
<td>Symmetric System: $x(0) = 0.5; T = 3; \alpha_1 = \alpha_2 = 1; P_1 = P_2 = 1; \beta = 0.5; C_1 = C_2 = 1; A = 0.05$</td>
<td>27</td>
</tr>
<tr>
<td>4.2</td>
<td>Incumbent Advantage vs. Competitive Advantage: $x(0) = 0.4; T = 3; \alpha_1 = \alpha_2 = 1; P_1 = 1; P_2 = 0.5; \beta = 0.5; C_1 = C_2 = 0.5; A = 0.05$</td>
<td>27</td>
</tr>
<tr>
<td>4.3</td>
<td>Ratio of Cost and Quality Parameters: $x(0) = 0.5; T = 3; \alpha_1 = \alpha_2 = 1; P_1 = 1; P_2 = 0.5; \beta = 0.5; C_1 = 1; C_2 = 0.5; A = 0.05$</td>
<td>28</td>
</tr>
</tbody>
</table>
List of Tables

3.1 Parameters and Variables ................................................. 11
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Chapter 1

Introduction

This thesis is motivated by an interest in the digital distribution of software products and how firms respond to the unique benefits and challenges of digital markets. Digital distribution platforms for mobile applications (apps) represent a relatively new environment for product distribution. These platforms give rise to digital markets that significantly decrease the time and startup costs required to sell software products, thus facilitating the distribution of apps to consumers anywhere in the world. However, large distributors such as Apple and Google must ensure only high-quality products enter the digital market, and the most common implementation of quality-assurance is user-generated reviews. User reviews allow consumers to interact with one another and influence the reputation and desirability of a given mobile app. In addition, these reviews foster rumor-spreading app diffusion within their platforms, heavily influencing the product’s market share. As such, firms look to maximize market share through effort towards product maintenance. A strong focus on maintenance is recognized; however, maintaining this dedication is expensive and faces decreasing returns over time. Consequently, developers must balance current and future effort costs to maximize revenue over a product’s life cycle.

We believe the problem faced by a digital vendor is especially relevant now. Digital distribution platforms (e.g., GooglePlay, iTunes, Steam) have revolutionized the way consumers purchase digital products like music, apps, and video games. These platforms also represent a significant venue of profit; for example, Apple’s App Store, recorded over $10 billion in sales during 2013 alone [1]. The paradigm shift in product distribution is most prominent in software applications for mobile apps, which is the focus of our study. These products include games, social media, and utility applications for all manners of mobile devices; smart phones, tablets, smart watches, and even eyewear in the form of Google Glasses can run mobile apps. A unique feature of mobile apps is that they are almost exclusively distributed digitally, enhancing the effects of rumor-spreading through user-generated reviews. Firms looking to maximize potential revenue from mobile apps must first understand the digital market and its particular challenges.

In this thesis, we model the dynamic decision-making process that app developers face, first
as a monopoly and then under duopolistic competition. In both settings, a firm controls its level of effort towards product maintenance over a product life cycle to capture market share and maximize total revenue stream for a digital good. In the monopolistic setting, the firm must balance current and future investment towards maintenance effort and market share over the product’s life cycle. We introduce an optimal control model of a single vendor’s path for maintenance and resulting market share in Chapter 3. From the model, we prove analytical results about its structure and establish the basic interaction between market share and product investment in a dynamic setting. Then, we illustrate the model’s applications using numerical examples to highlight insights for firms.

In the duopolistic setting, we model the decision-making process as a differential game between two firms in Chapter 4. Each firm controls its product maintenance level over a product life cycle to capture market share and maximize total revenue stream for its respective digital good. From the model, we again prove analytical results about its structure and establish basic interactions between relative market share, competitive advantages, and perceived quality in a dynamic setting. Then, we illustrate the model’s applications using numerical examples to highlight insights for firms. Our theoretical and numerical findings demonstrate that, in the face of quadratic costs on product investment, firms should steadily lower their investment effort over any product’s life cycle in both monopolistic and duopolistic settings. These results are supported by established literature on software evolution, as both of our models provide an analytic underpinning for Lehman’s 7th law of software evolution, which states that "the quality of an E-type system will appear to be declining unless it is rigorously maintained and adapted to operational environment changes" [2, 3, 4, 5].

The thesis is organized as follows: In Chapter 2 we review the relevant literature. In Chapter 3, we present an optimal control model where a single firm optimizes their product maintenance level to capture market share. In Chapter 4, we present a differential game model of two firms competing for digital market share through maintenance effort. In Chapter 5, we discuss our findings.
Chapter 2

Literature Review

2.1 Optimal Control Theory

In Chapters 3 and 4, we model a firm’s focus on product maintenance as an optimal control problem\(^1\). For tractability of the model due to the nonlinear nature of the equation of motion, we define a single state variable \(x_t\) as the proportion of users who have adopted the vendor’s product at time \(t\), i.e., the market share at time \(t\). We define \(u_t : \mathbb{R}_+ \to [0, \infty)\) as a single valued function of time as the control input that captures all efforts related to maintaining product post launch. The equation of motion, or state dynamic, is the first order differential equation that dictates the path of the control variables. \(^2\) The goal of the optimal control problem is to find an optimal path for the control variable \(u_t\) that minimizes (or maximizes) the objective. Our objective function fits the form of a standard objective function, given as follows:

\[
\begin{align*}
\min \psi(x(T)) + \int_0^T f(x_t, u_t) \, dt \\
\text{s.t.} \quad \dot{x} = g(x_t, u_t)
\end{align*}
\]

(2.1)

where \(\psi(x(T))\) is the endpoint cost (or salvage value in a maximization problem). This optimal control problem is of the Bolza type, and the standard approach is to minimize the cost function \(f(x_t, u_t)\). Note that in Chapters 3 and 4, we seek to maximize the objective function, as a firm would seek to maximize total profit over a product’s lifecycle. Therefore, we will state the necessary theorems as maximization principles. The Hamiltonian is defined as:

\[
H(x_t^*, u_t^*, \lambda) = f(x_t, u_t) + \lambda^T g(x_t, u_t),
\]

where \(f\) is the function defined in Equation 2.1, \(g\) is right hand side of the state equation, and \(\lambda\) is a vector of the co-state multipliers. Note that in our formulation, \(\lambda\) is a single co-state equation.

\(^1\)For an introduction to optimal control problems, see [6].
\(^2\)Two equations of motion are introduced in the next two chapters.
in Chapter 3 and two co-state equations in Chapter 4. This leads us to the following standard theorem, proven in [6] as well as most standard references on optimal control:

**Theorem 2.1.1. (Necessary Conditions for Optimal Control).** Consider the Bolza problem given in Equation 2.1. If \( u^* \) is an optimal control, then

\[
H(x^*_t, u^*_t, \lambda^*(t)) \geq H(x^*_t, u_t, \lambda^*(t))
\]

for all \( t \in [0, T] \) and for all admissible inputs \( u \in U \), and must satisfy the following conditions:

1. **Pontryagin’s Maximum Principle:** \( \dot{u} = \frac{\partial H}{\partial u} = 0 \) and \( \frac{\partial^2 H}{\partial u^2} \) is negative definite,

2. **Co-state Dynamics:** \( \dot{\lambda} = -\frac{\partial H}{\partial x} = -\lambda^T \frac{\partial g(x,u)}{\partial x} + \frac{\partial f(x,u)}{\partial x} \),

3. **State Dynamics:** \( \dot{x} = -\frac{\partial H}{\partial \lambda} = g(x,u) \),

4. **Initial Condition:** \( x(0) = x_0 \), and

5. **Transversality Condition:** \( \lambda(T) = -\frac{\partial \psi}{\partial x}(x(T)) \).

The partial derivatives with respect to vectors, such as \( \frac{\partial H}{\partial u} \), denote the gradient restricted only to those variables to which we differentiate.

Next we note that any solution \( u^* \) in Equation 2.1 must satisfy both the necessary conditions set forth in Theorem 2.1.1 as well as the sufficiency conditions known as Mangaserian’s Theorem [7] and Arrow’s Theorem [8] to be an optimal control.

**Theorem 2.1.2. (Mangaserian’s Theorem).** Suppose the admissible pair \( (x^*, u^*) \) satisfies all of the relevant continuous-time optimal control problem necessary conditions for the optimal control problem, the Hamiltonian \( H \) is jointly convex in \( x \) and \( u \) for all admissible solutions, \( t_0 \) and \( t_f \) are fixed, \( x_0 \) is fixed, and there is no terminal time conditions \( \psi[x(T), T] = 0 \). Then any solution of the continuous-time optimal control necessary conditions is a global maximum.

**Theorem 2.1.3. (Arrow’s Theorem).** Let \( (x^*, u^*) \) be an admissible pair for the optimal control problem when the Hamiltonian \( H \) is jointly convex in \( x \) and \( u \) for all admissible solutions, \( t_0 \) and \( t_f \) are fixed, \( x_0 \) is fixed, and there is no terminal time conditions \( \psi[x(T), T] = 0 \). If there exists a continuous and piecewise continuously differentiable function \( \lambda = (\lambda_1, ..., \lambda_n)^T \) such that the following conditions are satisfied:

1. \( \dot{\lambda}_i = -\frac{\partial H}{\partial x_i} \), almost everywhere, \( i = 1, ..., n \),

2. \( H(x^*, u, \lambda, t) \geq H(x^*, u^*, \lambda^*, t) \) for all \( u \in U \) and all \( t \in [0, T] \),

3. \( \dot{H}(x, \lambda, t) = \min_{u \in U} H(x, u, \lambda, t) \) exists and is convex in \( x \) for all \( t \in [0, T] \),

then \( (x^*, u^*) \) solves the optimal control problem. If \( \dot{H}(x, \lambda, t) \) is strictly convex in \( x \) for all \( t \), then \( x^* \) is unique (but \( u^* \) is not necessarily unique).

In Chapter 3, we formulate our control model in accordance with the above theorems to ensure existence of an optimal control path.
2.2 Differential Games Theory

In Chapter 4, we model duopolistic competition between two firms looking to maximize revenue through market share competition as a differential game from an economic and operations research perspective. Again, for tractability of the model, we define a single state variable $x_t$ to be the market share of Firm 1 at time $t$, with $0 \leq x_t \leq 1$ for all $t$. Naturally, $1 - x_t$ is market share of Firm 2. Let $u_t : \mathbb{R}_+ \rightarrow [0, \infty]$ be a single valued function of time that captures all efforts related to maintaining a quality product for Firm 1. Likewise, let $v_t : \mathbb{R}_+ \rightarrow [0, \infty]$ be Firm 2’s product maintenance function. We employ a single state equation, given as:

$$
\dot{x} = \beta (x(1-x)) (P_1 u - P_2 v),
$$

This first order differential equation dictates the paths of the controls for both firms given separate objective functions. It can be generalized to a game with $n$ firms. In generality, the objective function for firm $i$ has the following standard form:

$$
\begin{align*}
\max \ & \psi_i(x(T)) + \int_0^T f_i(x_t, u_t) \, dt \\
\text{s.t.} \ & \dot{x} = g(x_t, u^i_t) \\
\end{align*}
$$

(2.2)

where $u = (u_1, ..., u_n)$ represents the vector of controls for $n$ firms. To solve this game, we can solve $n$ optimal control problems simultaneously for each firm. We now define a Markovian Nash equilibrium for the differential game.

**Definition 1.** The $N$-tuple $(\phi^1, \phi^2, ..., \phi^N)$ of functions $\phi^i : X \times [0, T) \rightarrow \mathbb{R}^{m_i}$, $i \in \{1, 2, ..., N\}$, is a Markovian Nash equilibrium if, for each $i \in \{1, 2, ..., N\}$, an optimal control path $u^i(\cdot)$ of the problem 2.2 exists and is given by the Markovian strategy $u^i_t = \phi^i(x(t), t)$.

Thus, finding a Markovian Nash equilibrium is the same as solving the system of $N$ independent optimal control problems. Next, we characterize an open-loop Nash equilibrium for differential games.

**Definition 2.** The $N$-tuple $(\phi^1, \phi^2, ..., \phi^N)$ of functions $\phi^i : X \times [0, T) \rightarrow \mathbb{R}^{m_i}$, $i \in \{1, 2, ..., N\}$, is an open-loop Nash equilibrium if, for each $i \in \{1, 2, ..., N\}$, an optimal control path $u^i(\cdot)$ of the problem 2.2 exists and is given by the open-loop strategy $u^i_t = \phi^i(x(t), t)$.

Note that every open-loop Nash equilibrium is also a Markovian Nash equilibrium, so we focus on Markovian Nash equilibriums. Next, we consider when the $N$-tuple $(\phi^1, \phi^2, ..., \phi^N)$ is a Markovian Nash equilibrium in the next two theorems. The proofs for both theorems can be found in [9] as well as most textbooks on differential games.

**Theorem 2.2.1.** Let $(\phi^1, \phi^2, ..., \phi^N)$ be a given $N$-tuple of functions $\phi^i : X \times [0, T) \rightarrow \mathbb{R}^{m_i}$, $i \in \{1, 2, ..., N\}$, with the following assumptions:
1. There exists a unique absolutely continuous solution \( x : [0, T) \mapsto X \) of the initial value problem:
   \[
   \dot{x} = g(x(t), \phi^1(x(t), t), \phi^2(x(t), t), \ldots, \phi^N(x(t), t)), x(0) = x_0.
   \]

2. For all \( i \in \{1, 2, \ldots, N\} \) there exists a continuously differentiable function \( V^i : X \times [0, T) \mapsto \mathbb{R} \) such that the Hamilton-Jacobi-Bellman equations
   \[
   r^i V^i(x, t) - V^i_t(x, t) = \max \left\{ f^i_{\phi^i}(x, u^i, t) + V^i_x(x, t) g^i_{\phi^i}(x, u^i, t) | u^i \in U^i_{\phi^i}(x, t) \right\}
   \]
   are satisfied for all \( (x, t) \in X \times [0, T) \).

3. If \( T < \infty \) then \( V^i(x, T) = \Psi^i(x) \) for all \( i \in \{1, 2, \ldots, N\} \) and for all \( x \in X \),

4. If \( T = \infty \) then for all \( i \in \{1, 2, \ldots, N\} \) either \( V^i \) is a bounded function and \( r^i > 0 \) or \( V^i \) is bounded below, \( r^i > 0 \), and \( \limsup_{t \to \infty} e^{-r^i t} V^i(x, t) \leq 0 \).

Denote by \( \Phi^i(x, t) \) the set of all \( u^i \in U^i \) which maximize the right-hand side of Equation 2.3. If \( \phi^i(x(t), t) \in \Phi^i(x(t), t) \) holds for all \( i \in \{1, 2, \ldots, N\} \) and almost all \( t \in [0, T) \) then \( (\phi^1, \phi^2, \ldots, \phi^N) \) is a Markovian Nash equilibrium. (If \( T = \infty \), optimality is understood in the sense of catching up optimality).

**Theorem 2.2.2.** Let an \( N \)-tuple \( (\phi^1, \phi^2, \ldots, \phi^N) \) of functions \( \phi^i : X \times [0, T) \mapsto \mathbb{R}^{m^i}, i \in \{1, 2, \ldots, N\} \) be given and let assumption (1) of Theorem 2.2.1 be satisfied. Define for all \( i \in \{1, 2, \ldots, N\} \) the Hamiltonians \( H^i_{\phi^i} : X \times \mathbb{R}^{m^i} \times \mathbb{R}^n \times [0, T) \mapsto \mathbb{R} \) by
   \[
   H^i_{\phi^i}(x, u^i, \lambda^i, t) = f^i_{\phi^i}(x, u^i, t) + \lambda^i g^i_{\phi^i}(x, u^i, t)
   \]
   and the maximized Hamiltonians \( H^{i*}_{\phi^i} : X \times \mathbb{R}^n \times [0, T) \mapsto \mathbb{R} \) by
   \[
   H^{i*}_{\phi^i}(x, \lambda^i, t) = \max \left\{ H^i_{\phi^i}(x, u^i, \lambda^i, t) | u^i \in U^i_{\phi^i}(x, t) \right\}
   \]
   Assume that the state space is convex, the scrap value functions \( \Psi^i \) are continuously differentiable and concave, and that there exist \( N \) absolutely continuous functions \( \lambda^i : [0, T) \mapsto \mathbb{R}^n \) such that

1. the maximum condition \( H^{i*}_{\phi^i}(x(t), \phi^i(x(t), t), \lambda^i(t), t) = H^{i*}_{\phi^i}(x(t), \lambda^i(t), t) \) holds for all \( i \in \{1, 2, \ldots, N\} \) and almost all \( t \in [0, T) \),

2. the adjoint equation \( \lambda^i(t) = r^i \lambda^i(t) - (\partial / \partial x) H^{i*}_{\phi^i}(x(t), \lambda^i(t), t) \) holds for all \( i \in \{1, 2, \ldots, N\} \) and almost all \( t \in [0, T) \),

3. if \( T < \infty \) then \( \lambda^i(T) = S^i_{\lambda^i}(x(T)) \) holds for all \( i \in \{1, 2, \ldots, N\} \),

4. if \( T = \infty \) then either \( \lim_{t \to \infty} e^{-r^i t} \lambda^i(t) \hat{x}(t) = 0 \) holds for all \( i \in \{1, 2, \ldots, N\} \) and all feasible state trajectories \( \hat{x}(\cdot) \), or there exists a vector \( a \in \mathbb{R}^n \) such that \( x \geq a \) for all \( x \in X, \lambda^i(T) \geq 0 \) for all \( i \in \{1, 2, \ldots, N\} \) and all sufficiently large \( t \), and \( \limsup_{t \to \infty} e^{-r^i t} \lambda^i(t)|x(t) - a| \leq 0 \).
5. the function \( x \mapsto H_{\phi^i}(x, \lambda^i, t) \) is continuously differentiable and concave for all \( i \in \{1, 2, ..., N\} \) and all \( t \in [0, T) \).

Then \( (\phi^1, \phi^2, ..., \phi^N) \) is a Markovian Nash equilibrium. (If \( T = \infty \), optimality is understood in the sense of catching up optimality).

In Chapter 4, we formulate our differential game model in accordance with the above theorems to ensure existence of a Markovian Nash equilibrium.

### 2.3 Motivation in Related Disciplines

Modeling the development of large software systems has been an on-going area of research since at least the 1970’s [10]. Models of reliability of software that include costs have also been investigated [11] in the literature since the early 90s. Even before the advent of the modern digital eco-systems, digital distribution has been a popular topic of research in software engineering and management science in the past two decades. This paper investigates the optimal policy for post-launch maintenance of a software product to maximize firm revenue; as such, our research sits at the intersection of software engineering and operations research. Therefore, in this section we highlight the two streams of literature pertinent to our paper, as well as our contribution to the literatures.

Within the software engineering literature, a lot of attention has been given to the quality of software systems over time. The seminal work by Lehman proposes eight laws of software evolution that have been the subject of rigorous empirical research; for us, the 7th law is especially relevant, which states that ”the quality of an E-type system will appear to be declining unless it is rigorously maintained and adapted to operational environment changes” [2]. For example, Munson notes that developing metrics for software measurement is a key element in software engineering [12]. Furthermore, they have direct implications for evaluating Lehman’s eight laws, though not all of the laws are directly measurable [4]. The author does propose a precise definition for software fault as a means of measuring declining quality in a software system and offers empirical support for the 7th law. Johari & Kaur studies the applicability of Lehman’s laws on Object Oriented Software Systems in their 2011 paper, which examined the change over versions of several software applications [13]. The authors find empirical support for the 2nd and 7th law, which are noted to be similar in nature. Kaur et. al. undertake a similar study by analyzing two software developed in C++, also using object oriented metrics to evaluate Lehman’s laws. The paper finds support for the laws of continuous change, growth, and increasing complexity. The authors note that the law of declining quality is reflected in the 2nd law of increasing complexity. Neamtiu et. al. conduct a longer study over 705 releases of nine open-source projects [14]. Using statistical hypothesis testing, the authors find evidence for the laws of continuing change and growth, though they could not find definitive evidence for the other laws.

Drouin & Badri also find support for the laws of continuing change and growth in their 2013 study [15]. In addition, the authors find support for three other laws, including the law of declin-
ing quality. Through use of a synthetic metric denoted as the Quality Assurance Indicator, the study applies the metric to two Java software systems and finds support for five of Lehman’s laws. Moazeni et. al. discusses Lehman’s laws and their implications for the concept of Incremental Development Productivity Decline [3]. The authors qualitatively argue that Lehman’s laws support the concept; moreover, the paper finds that continuing maintenance on a software system is necessary to avoid loss of quality. Yu and Mishra’s uses the accumulated defects-density metric to measure the focus on quality [16]. The authors apply this metric to two open source software programs and their bug reports, which are data mined. Their findings also support Lehman’s 7th law, noting that focus on quality for software does indeed decrease over the product’s lifespan in the way of increasing defect reports. Zhang et. al. conduct an exploratory study on the validity of Lehman’s laws in the mobile applications setting; in particular, they perform a case study on two mobile apps, VLC and ownCloud [17]. The study focuses on the laws of continuing change, increasing complexity, and declining quality, and the authors show that both the laws of continuing change and declining quality apply in this setting.

Within the operations management (OM) literature, attention has been given to the unique characteristics of digital goods, factors impacting software quality, and the influence of user-generated reviews on sales. Jones and Mendelson study differences in cost structure and competition between digital goods and industrial goods [18]. The authors find that markets for information goods lack the segmentation inherent in markets for traditional goods. A key result is that a monopolistic firm will offer only a single digital product. Competition can lead to highly concentrated information-good markets, though the leading firm is able to behave like a monopolist even with no barriers to entry. Lehmann-Grube models the strategic choice of quality as a two-firm, two-staged game and showed the importance of focusing on quality to profit [19]. In both simultaneous and sequential games, choosing the higher level of quality captures greater market share and profits. Other studies have highlighted the impact of online user reviews to sales and market share (see [20, 21]). The general consensus among marketing and information researchers is that these reviews have a considerable impact on product performance. We note that most studies in this literature analyze sales of physical products, though we do not expect the effect to be lessened for digital goods. We utilize these concepts to motivate our model for finding the optimal control path of quality for single & dual vendor settings.
Chapter 3

Optimal Control Model for Product Maintenance of Mobile Software Applications

In this chapter, we construct our optimal control model for a monopolist in a mobile apps market. Our goal is to model the behavior of vendors using digital distribution platforms. We begin with a single firm looking to maximize profits over a product’s post-launch life cycle by controlling its effort towards product maintenance.

3.1 Optimal Control Model

We define $u_t : \mathbb{R}_+ \rightarrow [0, \infty)$ as a single valued function of time that captures all efforts related to maintaining a quality product. These efforts include debugging, on-going customer support, and additional content creation. It is important to note that product maintenance for digital goods is costly, which is reflected in our profit function in Equation (3.1). In our model, $u_t$ is nonnegative, with 0 representing no effort towards product maintenance. Let $x_t \in [0, 1]$ denote the proportion of users who have adopted the vendor’s product at time $t$, i.e., the market share of the firm at time $t$. The firm aims to maximize total profits over the product’s life-cycle, which is equivalent to solving the following maximization problem:

$$\max \Psi(x(T)) + \int_0^T R(u) \cdot x - Cu^2 \, dt$$

This functional form states that the revenue stream of the firm depends on both $u_t$ and $x_t$, which are implicit functions of time; we often drop the subscripts $t$ for notational convenience. Dynamic optimization problems of this form are common in many analytic fields such as electrical engineering, economics, and operations research [6]. We use this functional form to show that
the decision of post-launch maintenance effort is a dynamic and continuous one. In this generic functional form, let \( R(u) \) represent the revenue stream as a function of \( u \), and let \( C \) be the cost coefficient to product maintenance. Because both revenue and costs depend on maintenance effort, the firm faces a dynamic optimization problem where it must balance revenue, costs, and market share.

The cost of product maintenance is quadratic, which is a standard modeling assumption in many economic models, as well as in the literature on digital goods [22]. Because we are restricting our analysis to the tradeoffs of product maintenance, costs in our model depend only on \( u_t \). The function \( \Psi(x(T)) \) is the salvage value of market share at time \( T \), which translates to how much value the firm places on end-of-life market share for the product. The salvage value determines the transversality condition, which is used to characterize the optimal control path of \( u_t \) [6]. In our specific model, we assume \( R(u) \) to be a linear function such that \( R(u) = \alpha u \). This simplifying assumption is made so that the model is tractable.

Next, we motivate our equation of motion, which describes how market share changes as a function of the control variable, \( u_t \). We assume a population of users who will choose to purchase the vendor’s product if the perceived quality of the product surpasses the utility gained from using the default, or next best, product. Users are assumed to be homogeneous and judge a product based on perceived quality. This is a simplifying assumption we make to keep the analytic results straightforward; assuming a heterogeneous population would not change our qualitative findings in this section, though it is a possible extension of the model. In our equation of motion, we wish to capture the dynamics of rumor spreading and its impact on market share. Online reviews have been shown to be influential to market share in the economics and marketing literature (see [21, 20]), and we believe this effect is magnified for market share of digitally distributed software. Therefore, we employ a Bass-like model of product diffusion [23] in the first part of the equation, as the model extends nicely to the digital distribution setting:

\[
\dot{x} = \frac{dx}{dt} = \beta x(1-x) (\pi(u) - \pi_0) \quad (3.2)
\]

The Bass equation, \( \dot{x} = \beta x(1-x) \), in our context is a logistic model of rumor spreading, and the intuition behind the model is straightforward. Changes in market share, denoted by \( \dot{x} \), occur via rumor spreading and are greatest when half of the population owns the product. These changes can be positive or negative. When market share is small (close to 0), rumors about the software and adoption rates change slowly because only a small proportion of the population owns the product. When market share is large, rumors and adoption rates change slowly because most of the population has already adopted the product.

In the second part of the equation, the relationship between \( \pi(u) \) and \( \pi_0 \) governs the sign of the equation of motion. The value function of \( u_t \) is given by \( \pi(u) \), which denotes the utility users derive from the product at time \( t \). \( \pi_0 \) is the value users derive from the next best product, assumed to be constant over a product’s life cycle. We can think of \( \pi_0 \) as the utility of the next best outside option. This option can be a competing firm’s digital product, or even a traditional
Table 3.1: Parameters and Variables

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho )</td>
<td>Linear salvage value of final market share</td>
</tr>
<tr>
<td>( x(t) )</td>
<td>Market share at time ( t ), normalized to be between 0 and 1</td>
</tr>
<tr>
<td>( R(u) = \alpha \cdot u )</td>
<td>Linear revenue generated by maintenance effort</td>
</tr>
<tr>
<td>( u(t) )</td>
<td>Firm’s effort towards product maintenance</td>
</tr>
<tr>
<td>( C )</td>
<td>Cost coefficient of product maintenance</td>
</tr>
<tr>
<td>( \dot{x} )</td>
<td>Change in market share at time ( t )</td>
</tr>
<tr>
<td>( \beta )</td>
<td>Impact coefficient of how much changes in ( x ) affect ( \dot{x} )</td>
</tr>
<tr>
<td>( \pi(u) = P \cdot u )</td>
<td>Consumer’s linear utility derived from maintenance effort</td>
</tr>
<tr>
<td>( \pi_0 )</td>
<td>Utility of the outside option</td>
</tr>
</tbody>
</table>

A product that offers sufficient functionality. For example, the Wall Street Journal application faces stiff competition from both the New York Times application as well as traditional print newspapers [24].

For simplicity, we can normalize \( \pi(u) \) and \( \pi_0 \) to lie between 0 and 1 without losing analytic power. When \( \pi(u) > \pi_0 \), indicating that the value derived from the firm’s digital product is greater than the value of the next best alternative, the sign of \( \dot{x} \) is positive, and market share increases. The reverse holds as well. Let \( \pi(u) = P \cdot u \), which is linear and monotonically increasing. We can think of \( P \) as the impact of maintenance effort on users’ perception of quality, or the effectiveness of maintenance on perceived quality; as \( P \) increases, focus on product maintenance garners higher returns for the firm.

Finally, we let \( \Psi(x(T)) = \rho x(T) \), with \( \rho \in [0, 1] \), so that our salvage value function is linear. This is done for simplicity, though using the generalized form \( \Psi(x(T)) \) for salvage value does not qualitatively change our analytic findings. Our optimal control problem becomes:

\[
\begin{align*}
\max & \quad \rho x(T) + \int_0^T \alpha x u - Cu^2 \ dt \\
\text{s.t.} & \quad \dot{x} = \beta \left( 1 - x \right) \left( Pu - \pi_0 \right) \\
& \quad x(0) = x_0, \ u \geq 0
\end{align*}
\]

(3.3)

Table 3.1 summarizes our model’s variables and parameters, in the order they appear in (3.3). From Equation (3.3), the standard optimal control approach [6] is to numerically solve the system using the two-point boundary-value problem, which we do in Section 3.2. However, we can first exploit the structure of the model to solve for \( \dot{u} \), allowing us to derive a relationship between \( x_t \) and \( u_t \) on the optimal control path\(^1\) as long as the Hamiltonian is concave in \( x_t \) and \( u_t \), which is discussed in Proposition 3.1.1. This non-standard technique is shown to work given the form of our objective function and state equation when \( u_t \) is not at the boundary, which we show in our working paper [25]. Applying this technique allows us to glean insight into the relationship between market share and maintenance over time, shown in Proposition 3.1.1\(^2\).

\(^1\)We take the notation \( \dot{u} \) to mean \( \frac{\partial u}{\partial t} \), the time derivative of \( u_t \).

\(^2\)A variation of these results are first presented in [25] and are considered as general control problems in [26].
Proposition 3.1.1. Assume the Hamiltonian is concave in $x_t$ and $u_t$. If $u_t \in [0, \infty)$ for all $t$, then $x_t$ and $u_t$ satisfy the following differential equations:

$$
\begin{cases}
\dot{x} = \beta x(1-x)(Pu - \pi_0) \\
\dot{u} = \frac{-\alpha\beta \pi_0}{2C} x(1-x)
\end{cases}
$$

(3.4)

Therefore, focus on maintenance and perceived quality declines over a product’s life cycle.

Proof of Proposition 3.1.1. We use Pontryagin’s maximum principle, a standard approach in optimal control, to derive the system of differential equations (see [6] for a comprehensive introduction). We note that the derivation of $\dot{u}$ is not a standard approach but gives insights into the model. Let $\lambda(t)$ be the co-state variable, where the co-state equation is the first order differential equation of the Hamiltonian with respect to the state variable. Accordingly, minimizing the Hamiltonian is a necessary condition for solving our optimal control model. The Hamiltonian is given as:

$$H(x, u, \lambda, t) = \alpha ux - Cu^2 + \lambda[\beta x(1-x)(Pu - \pi_0)]$$

(3.5)

Assuming concavity, maximizing Equation (3.5) with respect to $u$ yields:

$$u^* = -\frac{\beta P \lambda x^2 - \beta P \lambda x - \alpha x}{2C}$$

(3.6)

We substitute $u^*$ into the equation of motion for $\dot{x}$, and using the Pontryagin conditions, we solve for $\dot{\lambda} = -\frac{\partial H}{\partial x}$, which is omitted here due to length. From our state equation, $\dot{x}$ is already given, so we can now differentiate Equation (3.6) using the derived values for $\dot{\lambda}$ and $\dot{x}$. This gives us the expression above for $\dot{u}$ purely as a function of the state variable.

From Proposition 3.1.1, note that $\dot{u} < 0$. This signifies that at optimality, focus on product maintenance is a monotonically decreasing function. That is to say, firms should always devote the most effort towards product maintenance at the beginning of the product life cycle and continuously scale back maintenance effort over time. Similarly, perceived quality, which is a linear function of $u_t$, also declines. This finding is consistent with Lehman’s 7th law of software evolution, often dubbed the “declining quality” law, which again states that "the quality of E-type systems will appear to be declining unless they are rigorously maintained and adapted to operational environment changes [2]. Because focus on product maintenance is costly to maintain in our model, we see that as a product approaches the end of its life cycle, perceived quality should be declining through time. Consequently, Lehman’s law of declining quality is an emergent property of this model. Our model shows digital vendors that it is always optimal to devote the most resources to maintenance effort at the beginning of the product life cycle. From the system of differential equations given in Proposition 3.1.1, we can express $x_t$ solely as a function of $u_t$ to illustrate the cases in which app developers may find themselves while on the
optimal control path. We first derive an expression for $\partial x/\partial u$, which is given below:

$$\frac{\partial x}{\partial u} = -\frac{2C}{\alpha \pi_0} (Pu - \pi_0).$$

(3.7)

Furthermore, by integrating Equation (3.7), we have an expression for the state variable as a function of the control, given below.

$$x(u) = \tilde{C} - \frac{CP}{\alpha \pi_0} u^2 + \frac{2C}{\alpha} u,$$

(3.8)

where $\tilde{C}$ is the constant of integration. With this expression, we see that $x_t$ varies quadratically in $u_t$ while firms are on the optimal path. From Proposition 3.1.1, we see that $u(t)$ is a monotonically decreasing function of time. Hence, we can say that $u(T) - u(0) \leq 0$, and furthermore, we know that $x(u)$ is maximized when $u_t = \pi_0/P$. This gives us the following result:

**Proposition 3.1.2.** Exactly one of the following holds:

1. $u(0) > \pi_0/P$, $u(T) \geq \pi_0/P$ and thus $x(T) \geq x(0)$ and market share increases monotonically.

2. $u(0) > \pi_0/P$, $u(T) < \pi_0/P$ and market share increases and then decreases, and the final relationship between $x(T)$ and $x(0)$ is determined by the relationship between $u(T)$ and $u(0)$.

3. $u(0) \leq \pi_0/P$ and thus $x(T) \leq x(0)$ and market share decreases monotonically.

**Proof of Proposition 3.1.2.** From Equation (3.8), we see that $x(u)$ is clearly concave in $u$, and $x(u)$ is maximized at $u_t = \pi_0/P$. The cases from Proposition 3.1.2 follow immediately. □

Proposition 3.1.2 describes the inherent relationship between maintenance effort and market share in the model. These three market scenarios are the only possible cases a firm may face when behaving optimally. In Case 1, initial focus on product maintenance as well as terminal focus is sufficiently high such that the firm experiences increasing market share over the entire product life cycle. This is always the case if there are no market alternatives, such that $\pi_0 = 0$. In Case 2, initial focus on product maintenance is sufficiently high to increase market share, but it is too costly to maintain, and eventually the firm experiences decreasing market share until the end of the product life cycle. In Case 3, maintenance effort is never high enough to generate increasing market share, and firms only lose market share from product launch.

Depending on the specific market conditions, such as initial market share and impact of maintenance effort on user perception, a digital vendor may find itself in any of these scenarios. In all three cases, initial maintenance effort is greater than terminal maintenance effort, as $u_t$ is monotonically decreasing on the optimal path. In Section 3.2, we will explore the shape of the optimal path for market share and maintenance effort, along with the implications for firms. Our next proposition characterizes the situation when a firm would experience monotonically decreasing market share for its product’s life cycle.
Proposition 3.1.3. Market share decreases (i.e., $x(T) \leq x(0)$) if and only if:

$$2 - \frac{P}{\pi_0} (u(T) + u(0)) \geq 0,$$

$$\implies P \leq \frac{2\pi_0}{u(T) + u(0)}.$$

Proof of Proposition 3.1.3. From Equation 3.8, we may compute:

$$x(T) - x(0) = 2C \frac{\alpha}{\alpha} (u(T) - u(0)) - \frac{CP}{\alpha \pi_0} (u(T)^2 - u(0)^2)$$

$$= 2C \left[ \frac{\alpha}{\alpha} (u(T) - u(0)) - \frac{CP}{\alpha \pi_0} (u(T) + u(0)) (u(T) - u(0)) \right] (u(T) - u(0))$$

$$= \frac{C}{\alpha} \left[ 2 - \frac{P}{\pi_0} (u(T) + u(0)) \right] (u(T) - u(0)).$$

As noted, $u(T) - u(0) \leq 0$. Thus, $x(T) - x(0) \leq 0$ if and only if:

$$2 - \frac{P}{\pi_0} (u(T) + u(0)) \geq 0.$$

This proposition emphasizes the relationship between a firm’s ability to increase consumers’ perceived quality through product maintenance and the outside option. Since firms decide their maintenance effort for a given product, the relationship between $P$ and $\pi_0$ predicts whether a firm can profitably increase market share. If $P$ is less than the threshold value given in Proposition 3.1.3, firms are not incentivized to increase market share through costly maintenance effort. Therefore, firms should plan on experiencing decreasing market share throughout a product’s life cycle if maintenance effort has sufficiently small benefits to perceived quality.

Proposition 3.1.3 also provides an obvious check on the sensibility of the model: if a user derives less utility from the product than the alternative, market share must (necessarily) decrease as the next best option is adopted. This insight is especially relevant to firms that are able to estimate the benefits of focus on product maintenance and have a good understanding of the competition or outside option. For example, in market settings with a strong incumbent that produces a quality product, other firms may seek to capitalize on short-term profits at the expense of long-run market share.

Lastly we can relate $\tilde{C}$ directly to $x(T)$. Note we can compute $u(T) \in [0, \infty)$ by substituting $x(T)$ into Expression 3.6 and simplifying. This yields:

$$u(T) = \left[ \frac{\alpha x(T) + \beta P r x(T) (1 - x(T))}{2C} \right]_0.$$
From Equation 3.8 we can compute:
\[ \hat{C} = x(T) + \frac{CP}{\alpha \pi_0} u(T)^2 - \frac{2C}{\alpha} u(T). \]

From the expression for \( u(T) \), we can analyze the equation for the optimal \( u(T) \) in two parts, given as:
\[ \frac{\alpha x(T)}{2C} \quad \text{and} \quad \frac{\beta \rho x(T)(1 - x(T))}{2C}. \]
The first part represents the marginal profit for the revenue stream; the second part represents the marginal increase in market share. It is also worth noting that if the marginal salvage value, \( \Psi'(x(T)) = \rho \), is 0, then the only marginal value of maintenance effort appears in the revenue stream. We explore this in Section 3.2, and specifically in Equation 3.9.

We have the following result, which is a direct consequence of the expression for \( u(T) \).

**Proposition 3.1.4.** The final value \( u(T) \) is positive if and only if \( x(T) \) is positive.

**Proof of Proposition 3.1.4.** The expression:
\[ \frac{\alpha x(T) + \beta P x(T)(1 - x(T)) \rho}{2C} \]
is concave in \( x(T) \) and must be non-negative because of the signs of the parameters and the fact \( x(T) \in [0, 1] \). It has zeros in \( x(T) \) only when \( x(T) = 0 \) and when:
\[ x(T) = \frac{\alpha + \beta P \rho}{\beta P \rho} \]
Note, the previous expression is always greater than 1 for any \( \rho > 0 \) and this root does not exist when \( \rho = 0 \). Thus, \( u(T) > 0 \) if and only if \( x(T) > 0 \).

Thus, we see that the optimal control \( u^* \) is always positive just as long as \( x(0) > 0 \), since the dynamics prevent \( x(t) \) from ever reaching its fixed point 0 in finite time. Therefore, the adjoint dynamics are never required for analyzing a case when the controller may hit its lower bound zero. That is, analysis of Equation 4.3 is sufficient.

From the propositions given in this section, we provide an analytic underpinning to Lehman’s 7th law of software evolution. We also delineate the different optimal control paths of product maintenance that developers should follow. Because closed-form solutions for the optimal paths do not exist, we turn to numerical techniques. In Section 3.2, we extend our analysis by illustrating the three possible market scenarios a firm faces on the revenue-maximizing path in the presence of discounting.

### 3.2 Numerical Examples

We rely on standard techniques in optimal control to derive additional insight into the behavior of \( x_t \) and \( u_t \) through numerical evaluation. This is particularly relevant when we add a discounting
interest rate and a product innovation term to the model (see Expression 3.10). These elements are not handled in our theoretical analysis, but may be a part of a software firm’s market model. Let $r$ be the discount factor for future revenue, and let $A(1 - x)$ be the coefficient of product innovation. We first see the coefficient for product innovation as part of the Generalized Bass Model, where $A(1 - x)$ allows for growth when products have no market share [27].

For small discount factors (i.e., $r \ll 1$) and small product innovation terms (i.e., $A \ll 1$), qualitatively similar behavior will be observed since the discount and product innovation terms act as a small perturbations on the differential equations. If we set $r$ and $A$ to be equal to zero, we have the same model as above. We choose to include the discount factor and coefficient of product innovation as a robustness check for the model; unfortunately, including these variables in the analytic model would have rendered the optimization problem intractable. Numerically, we can account for these factors, which firms may face in practice. The qualitative findings are the same in both sections.

Because both market share and focus on maintenance are dynamic through time, solving for the optimal path of $u_t$ is crucial to understanding the qualitative behavior of the model. Our analysis uses the standard Euler-Lagrange two-point boundary value problem approach [6].

Our variation on the dynamics of Equation 4.1 includes the coefficient of product innovation, and is given by the following equation of motion:

$$\dot{x} = \beta (A(1 - x) + x(1 - x)) (\pi(u) - \pi_0)$$  \hspace{1cm} (3.9)

We further assume $\Psi(x(T)) = 0$, or that there is no salvage value of market share for products at the end of their life cycle. While this assumption does not always hold in practice, it simplifies the model’s insights and does not qualitatively change our results. Introducing a positive value for $\Psi(x(T))$ acts as an upward shift on the optimal path. If we assume a sufficiently high salvage value $\Psi(x(T))$, we can generate the case where a digital vendor focuses entirely on market share at the detriment of revenue. In this scenario, the optimal path for maintenance effort would be to set $u(t)$ as large as possible for all time. Our theoretical analysis did not consider a maximum value of $u$, to avoid the possibility of boundary conditions requiring the use of the adjoint equation. A software startup company may very well employ this strategy upon entering a new market to capture future market share for its products. The company will run short-term deficits to ensure higher initial market share for future products. Because initial market share is exogenous in our model, we do not directly account for this scenario in the firm’s objective function.

Assuming no salvage value for market share and a discount rate $r$ s.t. $0 < r \ll 1$, the firm
faces the following optimal control problem:

\[
\begin{aligned}
\max & \int_0^T e^{-rt} \left( \alpha u x - Cu^2 \right) \, dt \\
\text{s.t.} & \quad \dot{x} = \beta (A(1-x) + x(1-x)) (\pi(u) - \pi_0) \\
& \quad u_t > 0, \quad x \in [0,1], \quad x(0) = x_0
\end{aligned}
\]  

(3.10)

This optimization problem is similar to the one given in Equation 3.3, with the addition of the discount factor and coefficient of product innovation. Note that we are assuming continuous discounting in the form of \(e^{-rt}\), so that firms value future revenue less than present revenue. This is a standard economic assumption consistent with human behavior. Since our objective function and state equation satisfy the necessary conditions (see [6]) for a feasible control path, the Hamiltonian is given as:

\[
H(x, u, \lambda, t) = \alpha u x - C u^2 + \lambda \left[ \beta (A(1-x) + x(1-x)) (\pi(u) - \pi_0) \right]
\]

Again, we solve for \(u^*_t\) and substitute into \(\dot{x}\) and \(\dot{\lambda}\). Note that due to the discount rate, the costate is given as \(\dot{\lambda} = r \lambda - \frac{\partial H}{\partial x}\). Again, no convenient closed-form analytic solution for this system of differential equations exists. Even if one did exist, the solution also most likely would yield little intuition on the nature of the problem; therefore, we turn to numerical techniques to gain insight into the firm’s optimal path. We use Maple to determine a numerical solution with the following parameters:

\[
T = 4; \alpha = 2; \pi_0 = 0.5; P = 1; \beta = 0.5; r = 0.05; C = 1; A = 0.05;
\]

We divide the figures into the three cases presented in Proposition 3.1.2. Each case represents a different starting market share; we see that the optimal control path for the focus on product maintenance heavily depends on \(x_0\). Recognizing the initial market conditions is crucial to finding the revenue-maximizing control path. The first graph shows optimal market share behavior over time, and the second figure shows optimal maintenance effort. We also include optimal paths for when firms do not discount future revenue; i.e., \(r = 0\). This shows that small values of \(r\) act as minor perturbations on the system; positive values of \(r\) shift both market share and maintenance effort downwards.

In Figure 3.1, we see the optimal control path for focus on product maintenance and market share for the firm under Case 1 of Proposition 3.1.2. At this level of market share, it is profitable to maintain a relatively high level of maintenance to maximize revenue. Market share monotonically increases for the product life cycle, even though product maintenance is decreasing over time, consistent with Lehman’s 7th law. Case 1 represents a vendor with a large, established market share. With a strong dedication to maintenance, the firm is able to secure more market share

---

3 The parameter \(T\) has a small value to ensure convergence of the differential system. \(A\) has empirically been found to be small-valued (see [28]).
throughout the product life cycle.

In Figure 3.2, we see the optimal control path for focus on maintenance and market share for the firm under Case 2 of Proposition 3.1.2. When \( x_0 = 0.3 \), it is no longer profitable to rigorously maintain the product; as a result, market share monotonically decreases for the product life cycle. Case 2 represents a vendor with small market share who wishes to maximize revenue without incurring high costs. In our model, it is too costly for this firm to gain market share. Instead, the firm capitalizes on short-term profits before market share deteriorates. Maintenance effort is much lower at every point in time in Case 2 compared to Case 1.

In Figure 3.3, we see the optimal control path for focus on maintenance and market share for the firm under Case 3 of Proposition 3.1.2. Because numerical examples are sensitive to initial values, a decrease in \( x_0 \) of 10\% is sufficient to shift both the optimal control path and market share behavior. There is less focus on maintenance at every time \( t \). Market share first increases, peaks, then decreases. From Proposition 3.1.2, we know that the peak is reached when \( u = \frac{\pi_0}{\rho} \).
Unlike the first case, the vendor is not incentivized to sustain the necessary maintenance effort for constant market growth. In this case, focus on product maintenance lies between efforts in Case 1 and 2.

Note that we have set $\rho = 0$, which implies that we are studying firms that are not interested in market share at the end of the product life cycle. We can take the opposite view of a firm interested solely in maximizing market share at the expense of profit. Such firms exist when they are recently capitalized (i.e., by venture capital). In this case, the optimal control can only be $u(t) = u_{max}$ for all $t \in [0, T]$ and the result is a (rapid) increase in market share with a corresponding increase in revenue but a (potentially) sub-optimal profit function. As the company increases its market share and transitions to Case 1 (see Figure 3.1), profit becomes a larger motivating factor and market share may continue to increase as a result of viral propagation. Ideally (from the company’s point of view) this occurs before initial capital is completely expended.

3.3 Discussion and Conclusions

The current study provides an examination of optimal firm behavior in a digital market setting. Because conditions are unique for a given firm and market, market share for a digital good may follow any of the trajectories presented in Figures 3.1 through 3.2. If a vendor can identify its marginal benefit and cost of product maintenance, the firm can estimate its optimal path to maximize revenue. When firms have a good estimate for parameters such as $C$ and $x_0$, these numerical examples highlight how initial market share impacts a firm’s strategic choice of quality dedication.

One interpretation of these numerical results is that having an established reputation and preexisting market share encourages a firm to devote more resources towards product maintenance. By doing so, the firm can capture even greater market share throughout a product’s life.
cycle. On the other end of the spectrum, a vendor with small market share wants focus efforts on other aspects of its business. These products may be perceived as lower tiered digital goods, and indeed the firm is less focused on maintenance by comparison. This self-fulfilling prophecy could add to the discussion of reputation and perceived-quality. Given the assumptions of our functional form, the three cases present the only possible optimal control paths. The numerical figures are meant to highlight the cases in Proposition 3.1.2, which characterizes the entire range of scenarios a digital vendor may face.

In Figure 3.2, we see the optimal control path for focus on maintenance and market share for the firm under Case 2 of Proposition 3.1.2. When \( x_0 = 0.3 \), it is no longer profitable to rigorously maintain the product; as a result, market share monotonically decreases for the product life cycle. Case 2 represents a vendor with small market share who wishes to maximize revenue without incurring high costs. In our model, it is too costly for this firm to gain market share. Instead, the firm capitalizes on short-term profits before market share deteriorates. Maintenance effort is much lower at every point in time in Case 2 compared to Case 1.

In Figure 3.3, we see the optimal control path for focus on maintenance and market share for the firm under Case 3 of Proposition 3.1.2. Because numerical examples are sensitive to initial values, a decrease in \( x_0 \) of 10% is sufficient to shift both the optimal control path and market share behavior. There is less focus on maintenance at every time \( t \). Market share first increases, peaks, then decreases. From Proposition 3.1.2, we know that the peak is reached when \( u = \frac{\pi_0}{P} \). Unlike the first case, the vendor is not incentivized to sustain the necessary maintenance effort for constant market growth. In this case, focus on product maintenance lies between efforts in Case 1 and 2.

Note that we have set \( \rho = 0 \), which implies that we are studying firms that are not interested in market share at the end of the product life cycle. We can take the opposite view of a firm interested solely in maximizing market share at the expense of profit. Such firms exist when they are recently capitalized (i.e., by venture capital). In this case, the optimal control can only be \( u(t) = 1 \) (for all \( t \in [0, T] \)) and the result is a (rapid) increase in market share with a corresponding increase in revenue but a (potentially) sub-optimal profit function. As the company increases its market share and transitions to Case 1 (see Figure 3.1), profit becomes a larger motivating factor and market share may continue to increase as a result of viral propagation. Ideally (from the company’s point of view) this occurs before initial capital is completely expended.
Chapter 4

Differential Game Model of Maintenance-Based Competition for Mobile Software

In this chapter, we introduce competition to our model, which we frame as a differential game between two firms. Each firm controls its product maintenance level over a product life cycle, aiming to capture market share and maximize total revenue stream for its respective digital good.

4.1 Differential Game Model of Digital Distribution

In this section, we construct our differential game model for two vendors in the mobile apps market. Our aim is to model competition between two firms over digital market share for their respective mobile app. Both mobile apps are out of the development phase and on the market, and firm efforts are focused towards product maintenance to increase market share. Product maintenance incorporates processes such as troubleshooting, bug fixing, and content testing; maintenance effort is assumed to have increasing marginal cost in the model. We also assume both firms are price takers, so price is exogenous; this assumption allows us to implicitly model price without loss of generality.

Each firm seeks to maximize profits over its digital product’s post-launch life cycle. Without loss of generality, we let $x_t$ denote the market share of Firm 1 at time $t$, with $0 \leq x_t \leq 1$ for all $t$. Thus, Firm 2’s market share is $1 - x_t$. Let $u_t : \mathbb{R}_+ \rightarrow [0, \infty]$ be a single valued function of time that captures all efforts related to maintaining a quality product for Firm 1. Likewise, let $v_t : \mathbb{R}_+ \rightarrow [0, \infty]$ be Firm 2’s product maintenance function. We assume that higher levels of product maintenance result in higher perceived quality and higher marginal profits per sale, as firms can charge more for higher quality products. However, product maintenance is
expensive, and we model costs as quadratic in maintenance effort for both firms. Products at the end of their life cycle have some salvage value, which can be looked at as market influence for future products by the same firm. Firms control the level of effort put forth towards product maintenance, denoted \( u_t \) and \( v_t \), for Firm 1 and 2 respectively. Therefore, both firms look to solve their respective maximization problem:\(^1\)

\[
\max_u \Psi_1(x(T)) + \int_0^T \alpha_1 ux - C_1 u^2 \, dt,
\]

\[
\max_v \Psi_2(1-x(T)) + \int_0^T \alpha_2 v(1-x) - C_2 v^2 \, dt.
\]

These functional forms states that the revenue stream of both firms depend on focus on maintenance and market share. By assuming the revenue streams depends on \( u_t \) and \( v_t \), we are assuming stating that consumers recognize well-maintained, quality products and are willing to pay more for them, now and in the future. One example of a digital good generating a revenue stream is the Wall Street Journal mobile app, which is a subscription-based service for smart phones, though many mobile apps continuously generate revenue past the salespoint.

Costs in the profit function are quadratic and depend only on the respective controls. The functions \( \Psi_1(x(T)) \) and \( \Psi_2(1-x(T)) \) are the salvage values of market share at time \( T \) and affect the transversality conditions used to solve the differential game. To motivate our equation of motion, we assume a population of users choosing between Firm 1’s or Firm 2’s product; users will purchase the digital product with the highest perceived utility. These users are assumed to be homogeneous and judge a product based solely on utility.

Next, we model the state equation governing changes in market share between firm 1 and firm 2. Again, customers will choose to purchase a vendor’s product if the perceived quality of the product is higher than its competitor’s. In our model, perceived quality is a direct function of effort towards product maintenance, and the difference in perception drives changes in respective market shares. To model the rumor spreading dynamic of digital market share, we employ an equation similar to the generalized Bass model of product diffusion [23], as the model extends nicely to the digital distribution setting. Therefore, our single state equation is given as:

\[
\dot{x} = \beta (A(1-x) + x(1-x))(P_1u - P_2v),
\]

where \( \dot{x} \) is the derivative of \( x \) with respect to time. The equation first part of the equation, \( \dot{x} = \beta(A(1-x) + x(1-x)) \), represents a logistic model of rumor spreading, and the intuition behind the model is straightforward. Market share changes for a firm, whether positive or negative, occur via rumor spreading and are greatest when half of the market population uses the product. When market share is small (close to 0), rumors about the mobile app spread slowly because only a small proportion of the population own the product. When market share is large, rumors spread slowly because most of the population have already adopted the product.

\(^1x, u, \text{ and } v \text{ are implicit functions of } t; \text{ we drop the subscripts for ease of reading.}\)
The variable \( A \) represents the coefficient of product innovation\(^2\), which we attribute to Firm 1. Without loss of generality, we chose to drop this term for Firm 2 to simplify notation. Therefore, we can think of Firm 2 as the incumbent in the market for our model. The coefficient of product innovation is only consequential when Firm 1’s market share is close to 0.

The relationship between \( P_1u_t \) and \( P_2v_t \) governs the sign of the equation of motion. The coefficients \( P_1 \) and \( P_2 \) describe the impact of maintenance effort on perceived quality for the respective firms. \( P_1u_t \) is the perceived quality of Firm 1’s product, and \( P_2v_t \) is the perceived quality for Firm 2’s product; both are assumed to be linear functions of maintenance effort. When \( P_1u_t > P_2v_t \), the perceived quality of firm 1’s digital product is greater, and the sign of \( \dot{x} \) is positive. This leads to market share increases for Firm 1. When \( P_1u_t < P_2v_t \), Firm 2 experiences market growth at the Firm 1’s expense. By coupling the two controls for product maintenance in one state equation, we can directly analyze the impact of relative maintenance efforts on market share in a parsimonious model.

We let \( \Psi_1(x(T)) = \rho_1 \cdot x(T) \) and \( \Psi_2(1-x(T)) = \rho_2 \cdot (1-x(T)) \) with \( \rho \in [0, 1] \), so that our salvage value function is linear. Firm 1’s maximization problem becomes:

\[
\begin{align*}
\max & \quad \rho_1 x(T) + \int_0^T \alpha_1 xu - C_1 u^2 \ dt \\
\text{s.t.} & \quad \dot{x} = \beta(A(1-x) + x(1-x)) (P_1u - P_2v) \\
& \quad x(0) = x_0, \ u, v \in [0, \infty].
\end{align*}
\] (4.1)

Firm 2’s symmetric maximization problem is given as:

\[
\begin{align*}
\max & \quad \rho_2(1-x(T)) + \int_0^T \alpha_2 (1-x)v - C_2 v^2 \ dt \\
\text{s.t.} & \quad \dot{x} = \beta(A(1-x) + x(1-x)) (P_1u - P_2v) \\
& \quad x(0) = x_0, \ u, v \in [0, \infty].
\end{align*}
\] (4.2)

From Equations 4.1 and 4.2, we can solve this differential game for both firms simultaneously as two optimal control problems (see [9]). By solving both systems as Euler-Lagrange two-point boundary-value problems\(^3\), we can describe the equilibrium response paths for both firms under various market conditions and appropriate concavity assumptions. No closed-form solution exists to describe the entire equilibrium path for \( x_t, u_t, \) and \( v_t \), so we employ numerical techniques in Section 4.2.

In the remaining portion of this section, we analyze the structure of the model to develop closed-form equations for \( \dot{u}, \dot{v}, \) and the relationships between \( x_t, u_t, \) and \( v_t \) in equilibrium. These closed-form equations provide insight into the relationships between relative maintenance efforts and market share over time. Our first proposition describes equilibrium behavior

\(^2\)For more information, see [23], [29].

\(^3\)This is a common technique in differential game theory.
Proposition 4.1.1. Assuming the Hamiltonian is concave in $x$, $u$, and $v$, market share and respective focus on product maintenance satisfy the following differential equations in equilibrium:

$$\begin{align*}
\dot{x} &= \beta(A(1-x) + x(1-x))(P_1u - P_2v), \\
\dot{u} &= -\frac{\alpha_1\beta P_2v}{2C_1} \cdot (A + x)(1-x), \\
\dot{v} &= -\frac{\alpha_2\beta P_1u}{2C_2} \cdot (A + x)(1-x),
\end{align*}$$

(4.3)

Proof. Proof of Proposition 4.1.1: The state equation for $\dot{x}$ is given. Let $\lambda(t)$ and $\mu(t)$ be the co-states of (4.1) and (4.2). The Hamiltonians are given as

$$H_1(x, u, v, \lambda, t) = \alpha_1ux - C_1 \cdot u^2 + \lambda[\beta(A(1-x) + x(1-x))(P_1u - P_2v)],$$

(4.4)

$$H_2(x, u, v, \mu, t) = \alpha_2v(1-x) - C_2 \cdot v^2 + \mu[\beta(A(1-x) + x(1-x))(P_1u - P_2v)].$$

(4.5)

Solving for $\frac{\partial H_1}{\partial u}$ and $\frac{\partial H_2}{\partial v}$ yields $u^*(x, \lambda)$ and $v^*(x, \mu)$, which are omitted here; we substitute $u^*(x, \lambda)$ and $v^*(x, \mu)$ into the state equation. From the Pontryagin conditions, we solve $\dot{\lambda} = -\frac{\partial H_1}{\partial x}$ and $\dot{\mu} = -\frac{\partial H_2}{\partial x}$. This gives us expressions for $\dot{x}$, $\dot{\lambda}$, and $\dot{\mu}$ solely as functions of $x$, $\lambda$, and $\mu$. Differentiating $u^*(x, \lambda)$ and $v^*(x, \mu)$ w.r.t. $t$ yields Proposition 4.1.1.

From Proposition 4.1.1, we see that both $\dot{u}$ and $\dot{v}$ are strictly negative. This signifies that at optimality, focus on product maintenance is a monotonically decreasing function for both firms. That is to say, firms should always devote the most effort towards product maintenance at the beginning of the product life cycle and continuously scale back maintenance effort over time. Because focus on product maintenance is costly to maintain, we see that as a product approaches the end of its life cycle, maintenance effort and perceived quality is declining through time for both firms. This is true regardless of the initial market conditions, though, the rates of optimal maintenance decline depend on the specific parameters.

This finding is again consistent with Lehman’s 7th law of software evolution, which states that the quality of a software good or system will appear to be declining unless it is rigorously maintained and adapted to operational environment changes [2]. In the previous chapter, we have found Lehman’s law to hold in a monopolistic setting, and we see it extends to the competitive framework. Consequently, Lehman’s 7th law of software evolution is an emergent property of this model. This result tells digital vendors that it is always optimal to devote the most resources to maintenance effort at the beginning of the product life cycle, even in the presence of competition.

Following Proposition 4.1.1, we can describe the relationship between $\dot{u}$ and $\dot{v}$ in equilibrium. This equation expresses the two rates of change as a ratio of the cost, revenue, and perceived quality parameters.

\[^{4}\text{Note that } \lambda \text{ and } \mu \text{ are also implicit functions of } t.\]
Proposition 4.1.2. In equilibrium, the relationship between Firm 1’s and Firm 2’s maintenance effort is given as:

$$\frac{\partial u}{\partial v} = \alpha_1 \frac{C_2 P_2}{\alpha_2 C_1 P_1} \cdot \frac{v}{u}.$$

Proof. This follows immediately from Proposition 4.1.1 by taking \(\frac{\dot{u}}{\dot{v}}\) to yield \(\frac{\partial u}{\partial v}\).

Propositions 4.1.1 and 4.1.2 highlight the coupled nature of our differential equation system and allow us to characterize equilibrium behavior for both firms. From Proposition 4.1.2, we obtain three corollaries.

Corollary 4.1.3. The absolute value of \(\dot{u}\) is positively correlated with \(\frac{\alpha_1}{\alpha_2}\) with \(\dot{v}\).

This states that Firm 1’s rate of decline for focus on product maintenance is positively correlated to the ratio of the relative revenue parameters, because we know from Proposition 4.1.1 that \(\dot{u}\) and \(\dot{v}\) are always negative. The corollary shows that a firm’s maintenance effort actually declines more rapidly when the revenue ratio favors it. While this corollary may seem counter-intuitive, the implication is that in optimality, the higher a firm’s relative revenue advantage, the higher its initial focus on product maintenance. The faster decline in maintenance effort results from the firm’s ability to “rest on its laurels” in a sense; with a higher initial level of focus on product maintenance, the firm can afford to decrease effort more rapidly.

Corollary 4.1.4. The absolute value of \(\dot{u}\) is inversely correlated with \(\frac{C_1}{C_2}\) with \(\dot{v}\).

This states that Firm 1’s rate of decline for maintenance effort is inversely correlated with the ratio of the relative cost parameters. Therefore, as Firm 1’s cost of maintenance effort increases relative to Firm 2, Firm 1’s speed of decline in product maintenance slows. Similar to the logic in Corollary 4.1.3, this counter-intuitive corollary arises because as maintenance costs increase, Firm 1 would decrease initial investment in maintenance effort.

Corollary 4.1.5. The absolute value of \(\dot{u}\) is inversely correlated with \(\frac{P_1}{P_2}\) with \(\dot{v}\).

This states that Firm 1’s rate of change for maintenance effort is inversely correlated with the ratio of the relative perceived utility impact parameters. Consequently, when a firm’s maintenance effort has higher impact on perceived quality, it should actually start with a lower level of dedication to maintenance. This is because the firm is able to sustain the perception of high quality with lower effort.

It is important to note that the corollaries hold with the assumption of ceteris paribus, or that all other things are held constant. They do not describe what relative initial maintenance efforts will be between two firms; instead, they describe how initial maintenance effort for one firm changes when the ratios change. From Proposition 4.1.2 and it’s corollaries, we observe the some interesting results about the nature of duopolistic competition between digital firms. Without loss of generality, we can take Firm 1’s perspective. Our model states that the relative marketing advantages Firm 1 has over Firm 2 collectively dictate the rate the change of Firm 1’s product maintenance. The optimal rate of decline in maintenance effort is a function of relative
advantages in perceived quality, marginal revenue, and marginal cost. In Section 4.2 we turn to numerical methods to analyze the interactions between market share and relative marketing advantages.

### 4.2 Numerical Examples

We rely on standard techniques in optimal control and differential games to derive additional insight into the optimal paths for market share and focus on product maintenance. Because the structure of our state equation is non-linear, no closed-form solution for the optimal paths exist. Because both firm’s market shares and maintenance effort are dynamic through time, these numerical illustrations are crucial to understanding optimal behavior.

In this section, we solve both firm’s optimal control problems (Equations 4.1 and 4.2) simultaneously using the standard Euler-Lagrange two-point boundary value approach. The Hamiltonians are derived in Equations 4.4 and 4.5 and given below:

\[ H_1(x, u, v, \lambda, t) = \alpha_1 u x - C_1 \cdot u^2 + \lambda \beta (A(1-x) + x(1-x))(P_1 u - P_2 v), \]

\[ H_2(x, u, v, \mu, t) = \alpha_2 v (1-x) - C_2 \cdot v^2 + \mu \beta (A(1-x) + x(1-x))(P_1 u - P_2 v). \]

We solve for \( \frac{\partial H_1}{\partial u} \) and \( \frac{\partial H_2}{\partial v} \) to yield \( u^*(x, \lambda) \) and \( v^*(x, \mu) \), which are omitted. Next, we substitute \( u^*(x, \lambda) \) and \( v^*(x, \mu) \) into our state equation and utilize the Pontryagin conditions to solve \( \dot{\lambda} = -\frac{\partial H_1}{\partial x} \) and \( \dot{\mu} = -\frac{\partial H_2}{\partial x} \). This gives us expressions for \( \dot{x}, \dot{\lambda}, \) and \( \dot{\mu} \) solely as functions of \( x, \lambda, \) and \( \mu \). These expressions are used to numerically evaluate the optimal paths for market share and focus on product maintenance.

Again, due to the non-linearity of the equation of motion, no closed-form analytic solution for this system of differential equations exists. We turn to numerical techniques using Maple to better understand both firms’ optimal paths under various market conditions. We divide the figures into three interesting cases that highlight different aspects of the model. Note that we assume no salvage value in these numerical illustrations, implying that both firms place no value on the end of product life market share. Positive salvage value would simply shift the optimal paths for \( u_t \) and are unnecessary here.

In Figure 4.1, we have a sensibility check on the system. When both firms are symmetric in starting market share and relative parameters, equilibrium behavior is symmetric. While market share holds steady at 50% for each firm, optimal maintenance effort is decreasing for both firms. This result supports Proposition 4.1.1 and Lehman’s 7th Law of software evolution. Note that in our setting, the coefficient of product innovation is non-zero and affects Firm 1 only. However, because Firm 1 is well-established in the market and commands 50% of the market share, the coefficient is inconsequential to optimal behavior.

Figure 4.2 shows the optimal paths for market share and focus on product maintenance when Firm 2 is the incumbent and has a market share advantage. Firm 1 enjoys higher returns from
maintenance effort on perceived quality so that it is less costly to establish a product with better consumer reception. We see that despite the market share disadvantage, Firm 1 is incentivized to support higher product maintenance, eventually overtaking the majority of market share through a better perceived digital product. This is an example of how incumbent firms within a market can be dethroned by newcomers.

Figure 4.3 highlights the importance of the ratios between perceived quality and cost parameters. We see that when the ratio of \( \frac{P_1}{P_2} \) and \( \frac{C_2}{C_1} \) are equal so that product maintenance is two times as effective but also twice as costly, when all other factors are held equal, market share between the two firms remains equal. Firm 1’s optimal strategy is to minimize focus on product maintenance due to the prohibitive costs. At the same time, Firm 1 can leverage its perceived quality advantage to maintain constant market share despite the lower investment. Firm 2 takes the opposite approach because maintenance is relatively less costly. Both firms capture half the
market despite the differing strategies.

We use these numerical examples to emphasize the propositions and corollaries in Section 4.1. These examples give a graphical interpretation to Lehman's 7th Law in a competitive setting and to how optimal paths are governed by the ratios of model parameters. It is true that with specific numerical parameters, one could devise any optimal path behavior and equilibrium he or she wanted. However, our goal with the numerical examples is to reinforce our analytic findings; thus, we have attempted to use reasonable parameters that may be applicable in practice.

4.3 Discussion

This chapter investigates optimal behavior of digital vendors under competition in a dynamic setting. We use a differential game to model the strategic interactions between two competing firms that wish to maximize revenue over a product life cycle. Equilibrium results show that while the ratios of cost, revenue, and perceived quality parameters dictate market share behavior, focus on product maintenance should always be decreasing for both firms in optimality. In a duopolistic environment, our differential game model provides an analytic underpinning for Lehman’s 7th law of software evolution. Furthermore, we show that if a digital vendor understands its own competitive advantages and disadvantages, the firm should adjust its optimal path accordingly.
Chapter 5

Conclusions

In this thesis, we studied the optimal behavior for digital vendors post-launch in monopolistic and duopolistic settings. Because digital distribution of products has revolutionized the way consumers purchase digital goods like music, apps, and video games, we believe it is important to understand how focus on product maintenance can influence market share, revenue, and perceived quality of a product. We first model the single-firm setting as an optimal control problem. Next, we model the two-firm competitive setting as a differential game. We derive analytical and numerical results for both models.

We find that in both settings, firms should steadily lower their maintenance effort over any product’s life cycle. This result is supported by our analytic as well as numerical examples, providing evidence for Lehman’s 7th law of software evolution. In the future, we look to expand the model to accommodate entry and exiting of the market. Endogenizing peoples’ decision to adopt a product, switch to a competitor’s product, or leaving the market all-together introduces several complexities to the model, and a new differential equation model may be necessary. We may look to mathematical models of infectious diseases as potential starting points of research.
Bibliography


