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PRESERVICE SECONDARY MATHEMATICS TEACHERS’ REACTIONS TO
CONFLICTING OUTPUT BETWEEN MATHEMATICS TECHNOLOGIES

A Thesis in
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by
Sheri N. Stayton

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The thesis of Sheri N. Stayton was reviewed and approved* by the following:

Rose Mary Zbiek  
Professor of Education (Mathematics Education)  
Department Head, Curriculum and Instruction  
Thesis Advisor

E. Frances Arbaugh  
Associate Professor of Education (Mathematics Education)

William S. Carlsen  
Professor of Education (Science Education)  
Director of Undergraduate and Graduate Studies, Curriculum and Instruction

*Signatures are on file in the Graduate School
ABSTRACT

This study used task-based interviews and on-line questionnaires to investigate how six preservice secondary mathematics teachers at a large public university mathematically resolved output conflicts between mathematics technologies, their perspectives on using mathematics technologies and conflicting technological output in their future classrooms after experiencing conflicts themselves, and how their reactions related to their prior experiences and beliefs. Participants’ mathematical resolution methods involved common elements of recognizing salient similarities and differences, recalling facts and procedures seen as relevant to the conflicts, and using facts and procedures or technological evidence to determine the equivalence or validity of outputs. Participants used a considerable variety of mathematics and technological features to resolve the conflicts. Participants saw conflicting outputs as potentially useful teaching tools, but most seemed comfortable using conflicting output in their future teaching only if they could control how and when students encountered the conflicts. While beliefs and prior experiences seemed connected to participants’ reactions to conflicting output in some cases, it was not possible to connect experiences and beliefs to reactions across all participants or across all tasks. Results suggest that conflicting technological output could provoke rich discussions in teacher education courses that could broaden content knowledge and conceptions of technology use.
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Chapter 1

Introduction

Calls for the appropriate use of technology are prevalent in recent efforts for improving mathematics education. According to the National Council of Teachers of Mathematics, “Technology is essential in the teaching and learning of mathematics” (NCTM, 2000, p. 24), and classroom technology use should lead students to develop “a deeper understanding of mathematical concepts” (NCTM, 2009, p. 14). The Common Core State Standards for Mathematics (CCSSM) call for students to “use appropriate tools strategically,” which includes deciding when “tools might be helpful, recognizing both the insight to be gained and their limitations” (NGA Center & CCSSO, 2010, p. 7). Teaching students to use technology strategically will require more than using technology for demonstration purposes or answer checking. It will also require helping students learn to interpret and use technological output. Interpreting technological output is particularly difficult when results differ from what one expects to see. Output from technology can go against expectations if it differs from conventional paper-and-pencil results or if it differs between technologies.

As more mathematics technologies become readily available, such as various graphing calculator models and computer software that is free via the Internet, using different technologies for the same task and encountering different output from technologies has the potential to occur frequently. Therefore, it is important to understand what experiences and skills students and teachers will need in order to understand and utilize the output given by technology, particularly when different technologies or seemingly equivalent input within the same technology produce conflicting output. In this paper, conflicting output refers to outputs for comparable commands.
that are noticeably different between technologies. When I describe outputs as conflicting, this
does not necessarily mean that the outputs are not mathematically equivalent.

The goals of this study are to learn how preservice secondary mathematics teachers react
to conflicting output from mathematics technologies and how these reactions might relate to their
prior experiences and beliefs. Specifically, this study attempts to answer the following research
questions:

1. How do preservice secondary mathematics teachers mathematically resolve output
   conflicts between mathematics technologies?
2. What perspectives do preservice secondary mathematics teachers have on using
   mathematics technologies and conflicting technological output in their future classrooms
   after they have experienced technological conflicts themselves?
3. How are preservice secondary mathematics teachers’ mathematical resolution methods
   and perspectives on using mathematics technologies and conflicting output in their future
   classrooms related to their prior experiences as learners and their beliefs about
   technology, teaching, and learning?

I chose to work with preservice secondary mathematics teachers for this study because
they are an accessible group and because they represent a bridge between mathematics student
and mathematics teacher. They can provide valuable information about the attitudes and
 technological experience with which new teachers enter the field. Because preservice teachers are
still in the mindset of being a student who is learning mathematics, they may be better at
articulating their own processes when resolving technological conflicts rather than the processes
that they expect from students.

To study preservice teachers’ reactions to conflicting output between mathematics
technologies, I chose to use computer algebra systems (CAS), which can be implemented on
calculators or computers, because CAS are specifically appropriate for secondary mathematics
classrooms and can be used to explore various mathematical topics. When working with CAS,
knowledge of mathematics content and technology becomes important. Also, CAS are
particularly useful for studying algebra and function, a major focus of secondary mathematics
content, because of their capability to manipulate equations and expressions symbolically as well as display graphs and tables for functions.

In order to answer the research questions stated above, two data sources are used. First, preservice secondary mathematics teachers completed on-line questionnaires that asked about their prior experience with mathematics technology and their beliefs about mathematics technology, teaching, and learning. Second, those that completed the questionnaires were invited to participate in task-based interviews where they encountered and reacted to conflicting output between mathematics technologies. Encountering the conflicting output put preservice secondary mathematics teachers in problematic situations related to mathematics technology use and mathematics content knowledge in order to see how they mathematically resolve the conflicts and how experiencing the conflicts shapes their opinions about using technology or technological conflicts in their future teaching. I chose conflicts that are likely to arise in high school mathematics classrooms that use more than one technology with CAS capabilities.

As various advanced technologies become standard tools for doing mathematics, answers to these research questions can inform teachers, teacher educators, curriculum developers, educational researchers, and mathematics technology developers. Such knowledge will be important to mathematics teachers, mathematics teacher educators, and curriculum developers in designing activities to increase students’ abilities to use technological tools strategically and understand mathematics more deeply. Understanding how beliefs about technology, teaching, and learning relate to preservice teachers’ reactions to conflicting technological output will further inform teacher educators and researchers who are looking for ways to influence teachers’ beliefs in order to increase appropriate technology use in mathematics classrooms. Furthermore, knowing more about preservice teachers’ reactions to conflicting output can inform mathematics technology developers about how teachers might use their products.
Chapter 2

Review of the Literature

In this chapter, I review literature in several areas of educational research and describe how this study can add to the existing body of knowledge in the field of mathematics education. Areas of educational literature that I have identified as being relevant to this study include writings on surprising machine results and their implications for teaching and learning, writings on how mathematics technologies are used in the classroom, and writings about teachers’ and preservice teachers’ beliefs as they relate to technology, pedagogy, or both.

Surprising machine results

Soon after calculators became available for classroom use, it was evident that they could be used both functionally and pedagogically. Before the advent of graphing calculators and computer algebra systems (CAS), Etlinger (1974) discussed functional and pedagogical views of the electronic, four-function calculator. He explained that one who has a pure-functional view of the calculator sees it only as a tool that saves time, and one that has a pure-pedagogical view believes that the calculator must be used to facilitate, not replace, learning. Etlinger provided this example of unexpected machine results from an early model of an electronic, four-function calculator: $(1 ÷ 3) \times 3 = 0.9999999$, and he pointed out, “in many cases it is the limitations of the calculator which are directly responsible for the pedagogical value of the machine” (p. 44).

As calculators became more sophisticated, it became possible to produce graphic and symbolic results as well as numeric results. Thus, the potential for surprising machine results of various types has increased. Surprising machine results are often encountered when using CAS.
CAS are software that enhance “numeric and graphic operations with tools for formal manipulation of symbolic expressions” (Fey, Cuoco, Kieran, McMullin, & Zbiek, 2003, p. 1). In other words, in addition to producing tables of values and graphs for functions, CAS are capable of performing many of the symbolic operations learned in secondary school (grades 7-12) mathematics, such as solving equations, factoring or expanding polynomial expressions, and differentiating functions.

Student reactions to surprising symbolic output from CAS have been documented in the literature. For example, studies have addressed student reactions to conflicts between CAS output and their own paper-and-pencil results (e.g., Drijvers, 2002; Guzmán, Kieran, & Martinez, 2010; Lagrange, 1999; Tonisson, 2013). Additionally, Drijvers (2002) pointed out that the obstacles that students face when encountering surprising CAS results can be learning opportunities because “working on overcoming an obstacle often also means working on the conceptual development of the mathematics involved” (p. 228). However, in Lagrange (1999), the presence of CAS and surprising results was not sufficient for students to engage in rational reflection about mathematics. In Tonisson (2013), even students who could solve trigonometric equations quickly and correctly on paper found it difficult to compare their answers with unexpected CAS results (p. 226). On the other hand, Guzmán et al. (2010) offered evidence that surprising results can provoke student reflection. In their pilot study, students worked in pairs to compare CAS results to their paper-and-pencil work, and conflicts between students’ incorrect answers and correct CAS answers did provoke students to reflect on the mathematics involved. While these students’ reflections allowed them to find ways to correct their errors, the students missed important mathematical connections, which led Guzmán et al. to conclude, “the importance of teacher intervention is inescapable” (p. 1503).

The studies cited above described how students reacted to surprising machine results, but no study had students use different technologies for the same task. Tonisson (2013) suggested the
comparison of results from different CAS as a possible task for students, but he only studied students’ reactions to CAS results that differed from their paper-and-pencil results. Additionally, Tonisson (2015) investigated and categorized different outputs given by the solve commands of eight different CAS when solving “school equations,” but he did not study students’ or teachers’ reactions to such conflicts. Furthermore, while all of the studies cited here focused on surprising symbolic output from CAS, none of them investigated student reactions to surprising graphic or tabular displays. Studying reactions to conflicting output between different technologies is important because seeing conflicting output between different technologies when the user believes that the inputs are equivalent may create even greater surprise and deeper reflection on the mathematics involved than surprising output from a single technology that differs from paper-and-pencil results. An individual’s reaction to conflicting output may depend on how much they trust technological results.

The studies discussed above imply that students will require teacher guidance in order to become effective CAS users. Yet, no study looked at teachers’ or preservice teachers’ reactions to surprising CAS results. In order to help students become effective users of CAS and other mathematics technologies, teachers must possess the technical and pedagogical knowledge required to guide students toward this goal. Knowing more about preservice teachers’ reactions to surprising machine results may provide information about what needs to be included in teacher education courses that encourage the use of mathematics technologies in the classroom. Additionally, studying preservice teachers’ reactions to output conflicts will provide more information about what mathematical and technological knowledge is required to resolve such conflicts. This information is essential for determining what mathematical and technological knowledge is needed to use mathematics technologies effectively and strategically. Such information may also be used to help students develop the fifth mathematical practice outlined in
the Common Core State Standards for Mathematics (CCSSM), which calls for students to “use appropriate tools strategically” (NGA Center & CCSSO, 2010).

Technology use in the classroom

In order for appropriate use of technology in mathematics classrooms to occur, “the teacher must decide if, when, and how technology will be used” (NCTM, 2000, p. 26). The factors that influence such decisions are complex and depend on much more than the availability of technology in schools. Garofalo, Drier, Harper, and Timmerman (2000) stressed, in order to use technology appropriately, educational activities with technology “should extend beyond or significantly enhance what could be done without technology” (p. 71). If technology is used as Garofalo et al. recommends, then using technology appropriately requires a change in what mathematics is taught and how it is taught. Yet, research in the past has shown that even when technology was widely accessible, teaching practices were resistant to change across multiple subject areas (Cuban, 2001). Additionally, Simmt (1997) found that the introduction of graphing calculators did not dramatically change mathematics teachers’ practice. The teachers in Simmt’s study “used graphing technology as an extension to the way they always taught” (p. 287). These results suggest that increasing appropriate technology use is much more complicated than simply providing teachers access to tools. Just because “technology can be used to advance the goals of reasoning and sense making” (NCTM, 2009, p. 14), access alone will not guarantee that it will be used to do so.

Even when mathematics technology is used well and efforts are coordinated, teachers will use technology differently. For example, Kendal and Stacey (1999) found that teachers who planned and taught the same lesson with the TI-92, a graphing calculator with CAS capabilities, “privileged,” or emphasized, different features of the technology. The teachers’ privileging of
different capabilities of the technology, such as graphing versus symbolic manipulation, affected what students learned. When used well, technologies enable more approaches to teaching, so greater variations in teaching and learning are to be expected (Kendal & Stacey, 2001). This is something that teachers must be aware of in order to make informed choices about appropriate technology use.

Researchers have also noted that enthusiastic teachers who want to use technology to enhance student learning in innovative ways experience difficulties incorporating technology in their classrooms (e.g., Almekbel, 2000; Lumb, Monaghan, & Mulligan, 2000). Almekbel (2000) studied 7th through 12th grade teachers in Ohio that participated in a teacher enhancement program designed to help them implement National Council of Teachers of Mathematics (NCTM) principles. Teachers in Almekbel’s study cited short class periods and a lack confidence in their TI-92 skills as obstacles to using technology in their classrooms. Lumb et al. (2000) found that two teachers who chose to incorporate CAS into their upper-level mathematics courses in England spent a considerable amount of extra time planning because they had to learn to use the software, select topics from the syllabus that were appropriate for using the software, and write their own worksheets for classroom use because the textbook exercises were not structured in a way that the teachers’ deemed appropriate for software use. Considerable time was also required for students to learn to use DERIVE (the CAS software used in Lumb et al.’s study), and the teachers often introduced additional software to their students that did not have as many capabilities as DERIVE but were easier to use for particular purposes, such as creating graphs or spreadsheets.

The teachers in Lumb et al.’s (2000) study also felt pressure to prepare students for national assessments that did not allow software use. Lumb et al. explained that these pressures added to concerns about time because teachers felt pressure to ensure that lesson time spent using computer algebra needed “to be directed toward assessable outcomes” (p. 236). Teachers in
Almekbel (2000) expressed similar concerns about the Proficiency Test required in Ohio when deciding whether to use calculators and “non-traditional activities” with their students. Since these studies were conducted, it has become easier to access a variety of mathematics software that is appropriate for students at various levels. However, this means that teachers are presented with even more choices about when, how, and which software to use in their classrooms.

In order to provide more information about how powerful mathematics technologies give teachers many options for structuring mathematics lessons, Pierce and Stacey (2010) mapped pedagogical opportunities provided by mathematics analysis software (MAS). Pierce and Stacey described MAS as “adaptable software, where the user specifies what it will do” (p. 3). Additionally, the user’s ability “to use the software to perform selected mathematical algorithms with the user’s own input” was identified as a key feature of MAS (p. 3). I see the term MAS, used by Pierce and Stacey, as synonymous with the term “mathematics technologies,” but I use the latter term in this study because I thought it would be more familiar to the preservice teachers that I interviewed.

In Pierce and Stacey’s (2010) study, five successful teachers were observed using MAS in a variety of ways in their classrooms. Pierce and Stacey pointed out that using technology in more ways does not necessarily indicate better practice, but “a teacher who is aware of more pedagogical opportunities has stronger skills for meeting a variety of teaching needs” (p. 18). It is notable that all five of the teachers in Pierce and Stacey’s study were observed taking advantage of the functional opportunities of MAS, such as performing algorithms quickly. However, each pedagogical opportunity included in the map was only taken by at most four of the five teachers during observations. It is also interesting to note which pedagogical opportunities more teachers took and which pedagogical opportunities fewer teachers took. For example, four of the five teachers were observed using MAS to link representations and to re-balance emphasis on skills, concepts, and applications. Yet, only one of the five teachers was observed using MAS to
“exploit contrast of ideal and machine mathematics.” Having students explore and react to conflicts between mathematics technologies is one way in which teachers could exploit this contrast.

Additionally, Wachira, Keengwe, and Onchwari (2008) provided evidence that preservice teachers are more aware of the functional opportunities provided by mathematics technologies than the pedagogical opportunities. While many of the preservice teachers in their study saw technology as a good tool for computation, visualization, and answer checking, no preservice teacher in the study indicated that technology could be used to explore patterns or investigate mathematical relationships. Furthermore, Wachira et al. noted that many of the preservice teachers in their study had limited experiences with mathematics technologies as learners and believed that students should not use technology until they had mastered basic skills.

I ask preservice teachers about how technology was used in their prior mathematics courses in my study to see if time and advances in technology have produced more preservice teachers with meaningful technological experiences and whether differences in experiences lead to different reactions to conflicting technological output. Preservice teachers’ reactions to output conflicts between mathematics technologies can provide information about new teachers’ abilities to take advantage of the pedagogical opportunities provided by surprising machine results. I also ask preservice teachers whether or not they would consider presenting conflicts between mathematics technologies to students in their future classrooms. Their answers can shed light on whether the pedagogical opportunity of exploiting the contrast between machine results and ideal mathematics (Pierce & Stacey, 2010) would be used by teachers who are aware of it or if there are other factors that might motivate teachers to avoid student encounters with conflicting technological output.
Beliefs, practice, and technology use

The previous section discussed how technology provides opportunities for teachers to change their practice and the fact that such changes do not occur consistently across teachers. In order to understand teachers’ decisions about technology use in the classroom, it is important to consider teachers’ beliefs and attitudes. Providing working definitions to clarify these terms, which are often used without definition, Philipp (2007) defined attitudes as “manner of acting, feeling, or thinking that show one’s disposition or opinion” and beliefs as “psychologically held understandings, premises, or propositions about the world that are thought to be true.” To further distinguish the two terms, Philipp said, “Beliefs are more cognitive, are felt less intensely, and are harder to change than attitudes. Beliefs might be thought of as lenses that affect one’s view of some aspect of the world or as dispositions toward action” (p. 259). This latter sentence is included here because Philipp linked the term “belief” with the term “dispositions” and also described beliefs as affecting “one’s view.” The terms “dispositions” and “views” are also common in research on teachers’ beliefs.

I use the terms beliefs, attitudes, and views throughout this paper because I am using descriptions that already exist in the literature. These terms frequently appear in instruments for assessing belief and in descriptions of teachers’ and preservice teachers’ characteristics. I chose to use language consistent with previous literature without making careful distinctions between these terms because I do not think that the subtle distinctions between beliefs, attitudes, and views discussed by Philipp (2007) affect the interpretation of the results of my study. This is particularly true because many authors cited in this paper, whose language I have adopted, used the terms beliefs, attitudes, and views without clearly defining them. Thus, it is difficult to know if their definitions align with the definitions of others. For the purposes of this study, I think of beliefs,
attitudes, and views as opinions and perspectives of individuals that can influence their actions in various situations.

Even though the terms discussed above are not always clearly defined, it is not difficult to find research on teachers’ and preservice teachers’ beliefs. Some researchers have studied how teachers’ or preservice teachers’ beliefs, attitudes, or dispositions affect whether and how technology will be incorporated into classrooms (e.g., Anderson, Groulx, & Maninger, 2011; Cullen & Greene, 2011; Honey & Graham, 2003; Tharp, Fitzsimmons, & Brown Ayers, 1997; Vannatta & Fordham, 2004; Wachira et al., 2008). Others have investigated relationships between teachers’ beliefs and practice without considering technology use (e.g., Barkatsas, 2008; Judson, 2006; Rakes, Fields, & Cox, 2006; Raymond, 1997). While there is no question that teachers’ beliefs and attitudes affect their practice, the task of linking specific beliefs and attitudes to specific practices is difficult. This section reviews findings of studies that investigate teachers’ and preservice teachers’ beliefs and their implications for practice. However, before talking specifically about the results of previous studies, it is necessary to define and elaborate on some common terms that many studies use to categorize teachers’ beliefs and practice.

**Common categories of beliefs and practices in mathematics education**

It is common for literature that focuses on teachers’ beliefs and practice to study teachers’ beliefs about teaching and learning. For example, researchers sometimes investigate beliefs and practices based on constructivist views of learning. Often, researchers associate constructivist views of learning with student-centered, as opposed to teacher-centered, practices (e.g., Judson, 2006; Kaufman & Moss, 2010). Furthermore, constructivist views of learning and technology use are commonly associated with educational reform. Educational reform documents specific to mathematics within the past 20 years have combined these ideas with the importance of problem
solving and learning mathematics with understanding (e.g., NCTM, 2000; NGA Center & CCSSO, 2010). This brings up questions about what teachers believe about the nature of mathematics, mathematics learning, and mathematical understanding. In this section, I elaborate on categories used by researchers when studying mathematics teachers’ beliefs. These categories include constructivist versus traditional views of teaching and learning, different views of mathematics, and different views of mathematical understanding.

In the literature, mathematics instruction based on constructivist views of learning has been contrasted with traditional mathematics instruction. Goldsmith and Shifter (1997) explained:

Traditional mathematics instruction is grounded in the belief that students learn by receiving clear, comprehensible, and correct information about mathematical procedures and by having the opportunity to consolidate, automatize, and generalize the information they have received by practicing the demonstrated procedures (p. 22-23).

Although the term “traditional” has been used frequently to describe instruction in the literature, this term is problematic because tradition is not constant across culture, time, or place. In my view, an important feature of instruction described as “traditional” is that the teacher designs instruction with the goal of transmitting knowledge to students. Thus, when I picture a “traditional” mathematics classroom, or a mathematics classroom designed to transmit knowledge to students, I picture a classroom where the teacher tells students about mathematical concepts and demonstrates how to correctly perform mathematical tasks and procedures by working through examples. The students are then expected to master the material by practicing problems similar to what they have been shown by the teacher. In a mathematics classroom where the teacher’s goal is to transmit knowledge to students, I also expect to see the teacher speaking more frequently and doing more mathematics than the students during a typical class period.

In contrast, constructivists believe that knowledge is constructed within the individual rather than imposed from the outside world, and constructivists assume that teachers “should structure situations such that learners become actively involved with the content through
manipulation of materials and social interaction” (Schunk, 2008, p. 237). Well-known mathematics education reform documents promote the use of technology as a way to encourage students to manipulate mathematical objects and socially interact around mathematical concepts (e.g., NCTM, 2000; NGA Center & CCSSO, 2010). When I think of mathematics instruction that is based on constructivist views of learning, I picture a classroom where students learn by working through problems with which they do not necessarily have prior experience that are designed to guide them toward understanding mathematical concepts. I also expect to see students speaking more and doing more mathematics than the teacher during a typical class period while the teacher acts as a guide and motivator. Whether a mathematics teacher’s practice can be described as “traditional” or in line with constructivist principles is likely related to that teacher’s beliefs about pedagogy, learning, and mathematics.

Ernest (1988) described three views of mathematics. Those with an instrumentalist view see mathematics as “an accumulation of facts, rules, and skills to be used in the pursuance of some external end . . . a set of unrelated but utilitarian rules and facts.” Those with a Platonist view see mathematics as “a static but unified body of certain knowledge,” and those with a problem solving view see mathematics as “a dynamic, continually expanding field of human creation and invention, a cultural product” (p. 2). Tharp et al. (1997) described views of mathematical learning as rule-based or non-rule-based. They described a rule-based view as “the view that mathematics learning is mostly oriented toward processes that involve the manipulation of symbols and the memorization of facts” and a non-rule-based view as “the view that mathematics learning is based on reasoning about relationships and patterns” (p. 555).

These terms are important to this study because teachers’ and preservice teachers’ beliefs about teaching, learning, or mathematics may influence their choices about technology use in the classroom and their perspectives on how and when to use technology with students. For example, a teacher with a rule-based view of mathematics learning may fear that introducing technology
too early would prevent students from memorizing important mathematical facts or that exposing students to conflicting technological output would prevent students from learning conventional ways to write and manipulate mathematical expressions. On the other hand, a teacher with a constructivist view of teaching and learning may see technology or conflicts between technological outputs as a way for students to explore mathematical relationships. Therefore, the views of teaching, learning, and mathematics that participants in this study possess may influence how they react to conflicting technological output in the interviews and are considered in the analysis.

It is also important to note that even when terminology differs, researchers have assigned similar characteristics to teachers’ views of mathematics and beliefs about learning. For example, when Raymond (1997) categorized elementary teachers’ views about the nature of mathematics as traditional or nontraditional, her criteria for traditional views were similar to Ernest’s instrumentalist view, her criteria for nontraditional views were similar to Ernest’s problem solving view, and her categories between these extremes included ideas similar to Ernest’s Platonist view. In addition, Raymond’s categorizations of traditional and nontraditional beliefs about learning mathematics were similar to traditional and constructivist views of learning, respectively.

Furthermore, teachers’ views of mathematics and mathematics learning are related to how a teacher thinks about understanding mathematics. For example, an instrumentalist view of mathematics can be associated with a belief that instrumental understanding, as described by Skemp (1976/2006), is the goal of mathematics teaching. Skemp (1976/2006) described the differences between relational understanding: “knowing what to do and why” and instrumental understanding: “rules without reasons.” Elaborating on instrumental understanding, Skemp noted, “for many pupils and their teachers the possession of such a rule, and the ability to use it, was what they meant by ‘understanding’” (p. 89). The fact that people define mathematical
understanding differently is likely related to their different views of mathematics and mathematics learning. It makes sense that those with an instrumentalist view of mathematics, as described by Ernest, or a rule-based view of mathematics learning, as described by Tharp et al., would claim to understand certain mathematics, even if they only had an instrumental understanding, as described by Skemp. This is because one with an instrumentalist view of mathematics or a rule-based view of mathematics learning may not realize that there is more to know about a mathematical topic than performing procedures that lead to correct answers, and they may equate learning and performing correct procedures with learning and doing mathematics. For example, one with an instrumental view of mathematics might say that they “understand” how to solve inequalities because they know that you must remember to reverse the inequality sign when multiplying or dividing both sides of an inequality by a negative number without understanding why this is necessary. In this case, I would describe their “understanding” of solving inequalities as instrumental understanding.

Different views of mathematics, mathematics learning, and mathematical understanding have implications for teacher practice because those with different views of mathematics have different instructional goals. Because “CAS do, with no effort, what we previously thought we wanted the students to do!” (Goldenberg, 2003, p. 13), incorporating advanced mathematics technologies into the classroom will require a shift in beliefs about what it means to learn and do mathematics. This belief shift may be more difficult for some teachers than for others based on their beliefs about technology, teaching, learning, and mathematics. Knowing more about how preservice teachers’ reactions to conflicting output between mathematics technologies relate to their beliefs could help teacher educators provide preservice teachers with opportunities to address, reflect on, and possibly change beliefs that might prevent them from using mathematics technologies effectively in their future teaching. It is important to provide preservice teachers with opportunities to reflect on their beliefs because “teachers’ ability to be reflective is
unquestionably linked to teacher change” (Wilson & Cooney, 2002, p. 142). As mathematics technologies become essential tools for teaching and learning mathematics, it is particularly important to address preservice teachers’ beliefs about mathematics technologies.

**Findings from studies on beliefs, practice, and technology use**

With the intention of improving pedagogical knowledge about technology, many teacher education programs have added courses to help preservice teachers learn about technology and how to incorporate it appropriately into their teaching. Although these courses are not enough to ensure appropriate technology use in all future classrooms, there is some evidence that they have a positive impact. For example, Bai and Ertmer (2008) found that taking a technology course significantly improved preservice teachers’ attitudes about the educational benefits of technology.

It is unclear, however, if changes in attitudes will transfer to practice. Raymond (1997) studied relationships between elementary teachers’ practices and their mathematics beliefs, which included beliefs about the nature of mathematics along with beliefs about mathematics teaching and learning. She found that teacher education programs were more likely to influence mathematics beliefs than mathematics teaching practices and that the effect of teacher education programs on either beliefs or practices was limited when compared to other factors, such as past school experiences and social teaching norms.

Some teacher education programs have attempted to influence preservice teachers’ beliefs about technology and encourage technology use in their future classrooms by providing preservice teachers with opportunities to explore mathematics concepts using technology. According to Garofalo et al. (2000), “teachers who learn to use technology while exploring relevant mathematical topics are more likely to see its potential benefits and use it in their subsequent teaching” (p. 68). While this position makes sense, Walen, Williams, and Garner
Walen et al. (2003) suggested that it may be naïve. Walen et al. hoped that providing prospective elementary teachers the opportunity to learn more about mathematics content using calculators would change their beliefs about using calculators in their future teaching. They concluded:

We have found that our prospective teachers, although quite capable of learning mathematics in the new, reformed classrooms provided in this study, fall back on their cultural views of teaching mathematics when their attention is shifted from their own learning or doing of mathematics to their (prospective) teaching of mathematics (p. 460).

This “fall back on their cultural views” may happen because preservice teachers enter teacher education programs with beliefs that are strongly related to their prior experiences as students, and these beliefs affect what they will learn from teacher education programs (Richardson, 2003). Thus, while providing preservice teachers with the opportunity to learn mathematics more deeply with technology in teacher education programs is commendable, it may not be enough to encourage them to provide similar learning opportunities for their future students.

Teacher education programs are just one of many factors that can influence teachers’ beliefs, and many researchers have worked to better understand the relationships between teachers’ beliefs and practice by considering other factors. Using large samples of teachers or preservice teachers from various content areas, some studies have linked beliefs and other behaviors and attitudes with classroom technology use or intentions of future classroom technology use (e.g., Anderson et al., 2011; Cullen & Greene, 2011; Rakes et al., 2006; Tharp et al., 1997; Vannatta & Fordham, 2004). These studies relied on self-reporting of individuals on questionnaires, surveys, or in journal entries. Anderson et al. (2011) found value beliefs, or how much technology was valued for teaching and learning, to be the most significant predictor of preservice teachers’ intentions to use a variety of software and to use it frequently in their future classrooms. Cullen and Greene (2011) found preservice teachers’ positive attitudes toward technology use to be the best predictor of intrinsic and extrinsic motivation to use technology in their future classrooms, while amotivation was best predicted by negative attitudes toward
technology use and negative social norms. Rakes et al. (2006) found significant, positive
correlations between teachers’ constructivist instructional practices and both personal computer
use and levels of classroom technology use. Tharp et al. (1997) found that mathematics and
science teachers of grades 6-12 that had a rule-based view of mathematics learning were more
likely to feel that graphing calculators may hinder, rather than enhance, student learning. After
attempting inquiry approaches to learning, teachers with rule-based views also tended to “quickly
return to carefully controlling the amount and type of use of calculators in their classrooms” (pp.
566-567). In the classrooms of the teachers with non-rule-based views, “students were freer to
use the calculator as they wished” (p. 567). Vannatta and Fordham (2004) found that the
combination of factors that best predicted classroom technology use were amount of technology
training, time spent beyond the contractual work week, and openness to change. It is important to
note that with the exception of the study by Tharp et al., which focused on graphing calculator
use, the studies mentioned in this paragraph defined technology use very broadly. Thus,
technology use could mean that students and teachers were using technology for communication,
exploration, or a variety of other purposes.

While the studies mentioned in the previous paragraph support the notion that particular
beliefs and experiences are connected to particular practices, other studies that used observational
data and that had fewer participants found that observed practice and reported beliefs are not
always consistent (e.g., Barkatsas, 2008; Honey & Graham, 2003; Judson, 2006; Raymond,
1997). In Barkatsas (2008), experienced Greek secondary mathematics teachers’ reported beliefs
about mathematics teaching and learning were less traditional than their teaching practices that
were observed by the researcher (p. 213). Judson (2006), who observed the practices of 32
teachers representing primary and secondary levels, found that constructivist teaching
philosophies and attitudes toward technology were not significantly related to teachers’ practices
of technology integration. Honey and Graham (2003) found that three student teachers working
toward their Post-Graduate Certificate in Education in England, who each reported different beliefs and attitudes about graphing calculator use, were all reluctant to use graphing calculators to teach secondary mathematics lessons. Raymond (1997) found that an elementary teacher’s traditional beliefs about the nature of mathematics were more related to her practice than her nontraditional beliefs about mathematics pedagogy. The overall picture painted by these studies is that beliefs and attitudes have an effect on teachers’ intentions, but other factors, such as the “immediate classroom situation” (Raymond, 1997), may interact with these beliefs in ways that suggest to observers that beliefs and practice are inconsistent.

There may be other reasons why teachers’ beliefs and practice seem inconsistent to some researchers. In a review of the literature on beliefs about mathematics and mathematics teaching, Forgasz and Leder (2008) said, “teachers . . . are quite likely to have adopted reform oriented rhetoric without altering their instructional practice” (p. 180). Interestingly, Tharp et al. (1997) attempted to adjust for this phenomenon in their study. They noted that teachers were hesitant to indicate that they favored a rule-based view of mathematics learning on the questionnaires used in their study because it was not in line with NCTM recommendations for reform. “Hence teachers who indicated any tendency toward disagreement with a strong non-rule-based stance were classified as rule-based” (p. 560). Teachers’ reluctance to disagree with reform rhetoric could explain differences between the way teachers answer questions about their beliefs on questionnaires and the choices they make in the classroom.

However, the issue of inconsistencies between beliefs and practice is likely more complicated. Other scholars have warned that finding inconsistencies between teachers’ beliefs and actions indicates that there are other factors contributing to the relationships between beliefs and practice. For example, in a review of the literature on teachers’ beliefs and affect, Philipp (2007) pointed out that most of the researchers that he cited found that inconsistencies between teachers’ beliefs and actions “ceased to exist after they better understood the teachers’ thinking
about some aspects of their contexts” (p. 276). Similarly, Beswick (2005) said, “It is unreasonable to expect consistency between broad collections of beliefs that are not closely linked with a specific context, and practice that is not described in equally broad, contextually independent terms” (p. 42). Thus, the contexts in which teacher practices are observed must be considered when trying to link beliefs and practice or when beliefs and practice appear to be inconsistent.

Additionally, the contexts of the questions that researchers ask teachers or preservice teachers must also be taken into consideration when beliefs and intentions seem inconsistent. Using open ended questionnaires with student teachers, Kaufman and Moss (2010) found that preservice teachers espoused progressive beliefs in line with current educational reforms when asked questions that were framed to elicit general responses. However, when questions asked preservice teachers to specifically contemplate their future practice, “any notion of progressive, constructivist, or learner-centered approaches and environments almost completely vanished in favor of an even greater focus on behavior and control” (p. 128). My study uses questionnaire items that ask participants to indicate a level of agreement with general statements about teaching, learning, and mathematics technology. In the interviews, I ask participants to contemplate their future practice regarding mathematics technology after experiencing technological conflicts. Thus, the context in which questions are asked is considered in the analysis. Raymond (1997) indicated in her revised model of relationships between mathematics beliefs and practice that mathematics beliefs and the immediate classroom situation have strong influences on teacher practice. Kaufman and Moss’ (2010) results indicate that preservice teachers’ anticipation of the need to control student behavior may influence their beliefs and intentions in the classroom even before they become teachers.

Classroom management and planning concerns can also affect teachers’ decisions about using technology in the classroom. In Honey and Graham (2003), a student teacher with a positive attitude toward graphing calculators did not use graphing calculators in any of her
secondary mathematics lessons. She had been disappointed by the outcome of one of her mentor 
teacher’s lessons that had attempted to incorporate graphing calculators because the mentor 
teacher’s calculator knowledge was limited making it difficult to help students. She said, “I don’t 
think I’m confident enough to use them with a class unless I know that if anything cropped up, I 
could deal with it.” (p. 91). Another student teacher in the same study had a neutral attitude 
toward graphing calculators. This student teacher had a positive experience using graphing 
calculators while teaching one lesson. However, she remained concerned about the time required 
to plan such a lesson. As stated earlier, time and confidence were also obstacles to technology 
incorporation for secondary teachers in Almekbel (2000). These results indicate that even when 
teachers believe technology is valuable or have positive experiences teaching with technology, 
lack of confidence with the technology and concerns over management and time can still prevent 
them from incorporating technology into their teaching.

There is evidence in the literature that teachers should be concerned about classroom 
management as it relates to technology integration at lower grade levels. Means (2010) studied 
the technology implementation practices of 13 schools that were using the same reading software 
in third and fourth grade and the same mathematics software in sixth grade. Means found that the 
greatest differences between the schools with above-average student learning gains and schools 
with below-average student learning gains were effective classroom management, the use of 
software generated reports to inform instruction, and teacher collaboration around software use. 
Additionally, when teachers in Means’ study were asked what advice they would give to other 
teachers who were using the software for the first time, they “were more likely to provide 
recommendations on classroom management than on any other topic” (p. 293). It seems clear that 
teachers have management concerns regarding technology use at all grade levels. However, 
specific management concerns and how they are best addressed to increase student achievement 
may be different for secondary classrooms than for the third, fourth, and sixth grade classrooms
in Means’ study. For example, when secondary students encounter conflicting output between mathematics technologies, their teachers may be more concerned with managing reactions, questions, and time spent on discussion, while student attention and routines for technology use may be of greater concern with younger students.

Given that effective classroom management is important for effective technology integration in classrooms, it is important that teachers find management techniques that keep the class running smoothly but do not limit students’ learning opportunities. If teachers become preoccupied with getting work accomplished rather than student achievement, their management concerns can limit students’ learning opportunities even when student engagement is high (Doyle, 1986, p. 418). In a review of classroom management research, V. Jones (1996) indicated that management systems are most successful when they support instructional systems (p. 511). However, teachers may have little help in creating management systems that support technology integration in ways that are in line with currently recommended teaching practices. Martin (2004) said, “knowledge of classroom management has not developed with changing ideas of more active and socially interactive teaching and learning” (p. 406). McCaslin and Good (1992) argued that conceptions of management are incompatible with the promotion of a problem-solving curriculum. They warned, “We cannot expect that students will profit from the incongruous messages we send when we manage for obedience and teach for exploration and risk taking” (p. 12). Although this statement was made over 20 years ago, Kaufman and Moss’ (2010) study, in which student teachers espoused beliefs in line with student-centered teaching practices when asked general questions about pedagogy but changed their focus to controlling student behavior when asked to contemplate their own practice, suggests that teachers are still likely to struggle with mismatches between management systems and instructional systems.

Overall, the literature suggests that beliefs about teaching and learning are connected to teaching practice, and beliefs about technology are connected to technology use in the classroom.
However, it is not easy to predict teachers’ actions based on their beliefs because there are so many factors that influence teachers’ decisions, and it is difficult, if not impossible, to attend to all of these factors when studying the relationships between beliefs and practice through research. However, each study can add a piece to the larger puzzle of how beliefs and practice are related. According to Cobb, Wood, and Yackel (1990), “Beliefs are expressed in practice, and problems or surprises encountered in practice give rise to opportunities to reorganize beliefs” (p. 145). The technological conflicts that I chose for this study are designed to surprise preservice teachers. Studying how preservice secondary mathematics teachers react to surprising technological output and how that might affect how they intend to use mathematics technologies in their future classroom will further inform both the relationship between beliefs and mathematical knowledge used to resolve technological conflicts and the relationship between beliefs and future pedagogical intentions regarding mathematics technologies.

**Conclusion**

Previous studies have documented student reactions to technological output that conflicted with their paper-and-pencil results (e.g., Drijvers, 2002; Guzmán et al., 2010; Lagrange, 1999; Tonisson, 2013) and how different CAS results can conflict with each other (Tonisson, 2015). However, no study has investigated the reactions of students or teachers to conflicting results between mathematics technologies. Preservice teachers’ reactions to conflicting technological output can begin to provide the information needed to help students and teachers become strategic users of mathematics technologies and effective interpreters of technological output.

Additionally, technology provides more options for teaching, including various functional and pedagogical opportunities (Pierce & Stacey, 2010), and teachers will choose to use
technology differently, which can affect student learning (Kendal & Stacey, 1999, 2001).

Teachers’ choices about when and how to use technology in the classroom can be influenced by their attitudes and beliefs about pedagogy, learning, technology, and mathematics, which are influenced by their experiences. Yet, results of studies that have attempted to link specific beliefs and experiences to specific practices have been mixed. Some studies that relied on teachers’ or pre-service teachers’ self-reporting linked beliefs, attitudes, and experiences to intentions or practices related to technology (e.g., Anderson et al., 2011; Cullen & Greene, 2011; Rakes et al., 2006; Tharp et al., 1997; Vannatta & Fordham, 2004), but other studies that relied on observational data found inconsistencies between beliefs and practice (e.g., Barkatsas, 2008; Honey & Graham, 2003; Judson, 2006; Raymond, 1997). Furthermore, Kaufman & Moss (2010) found that the beliefs that preservice teachers expressed depended on the contexts in which questions were posed to them.

Therefore, more information is needed about the complex relationships between beliefs and practice. Learning about preservice teachers’ perspectives on using conflicts in their future classrooms can provide insight into their ideas and concerns about using technology for teaching and learning. Furthermore, understanding more about how preservice teachers’ reactions to output conflicts between mathematics technologies are related to their prior experiences and beliefs can further inform teacher educators and researchers who hope to influence preservice teachers’ beliefs in order to increase effective technology use in future mathematics classrooms. Such knowledge can also provide information to mathematics technology developers about how future teachers might choose to use their products. The next chapters discuss the methods, results, and implications of this study.
Chapter 3

Methods

In this chapter, I describe the participants and the materials involved in the data collection for this study and the methods used to analyze the data. Data for this study came from on-line questionnaires and individual task-based interviews. Interviews were audio-recorded and analyzed using GarageBand. Portions of the interviews were selected and transcribed for further analysis. I use data from the interviews to answer the first two research questions, and I use data from the questionnaires and the interviews to answer the third research question.

Participants

All participants were preservice secondary mathematics teachers at a large public university. Participants were recruited from a pool of 18 students enrolled in mathematics education courses during the Spring 2015 semester. Recruitment materials are available in Appendix A. Nine students completed on-line questionnaires that assessed their prior experience with mathematics technology, their beliefs about mathematics technology, and their beliefs about teaching and learning. These nine students were then invited to participate in task-based interviews where they would react to conflicting output between mathematics technologies. Six of these students completed the interview. Throughout this paper, I refer to participants by pseudonyms.

The mathematics education courses from which participants were recruited were 400-level courses. Students enrolled in these courses were upperclassmen, who had completed introductory courses for mathematics majors and had been accepted into a program designed to
prepare them for certification to teach secondary mathematics. These courses were chosen for recruitment to ensure that participants had strong mathematics backgrounds and had begun to study mathematics pedagogy. This ensured that participants were likely to be able to work through the technological conflicts mathematically and express opinions about their use in their future teaching.

**Technology**

The technologies used in this study were a TI-89 Titanium graphing calculator (Texas Instruments, 2005), GeoGebra version 5.0.42.0-3D (International GeoGebra Institute, 2015), and Core Math Tools (Keller, 2012). These technologies were selected because they are intended for use in secondary mathematics classrooms, and they all have CAS capabilities. That is, they can manipulate algebraic expressions and equations symbolically as well as produce graphs and tables for functions. The TI-89 Titanium calculator, while not the newest model, remains widely used in upper level high school mathematics classrooms and has been used for many years. Furthermore, its output is consistent with newer TI calculator models, such as the TI-Nspire CAS. Since the advent of the TI-89, free computer programs with CAS capabilities have also become available. Examples are GeoGebra and Core Math Tools. As more students have access to computers, this free software has the potential for widespread use along side of or in place of graphing calculators in the classroom.

**Questionnaires**

The purpose of the on-line questionnaires was to gather information about preservice secondary mathematics teachers’ prior experience with mathematics technologies as learners and
their attitudes and beliefs about teaching, learning, and using mathematics technology. A complete list of questionnaire items is included in Appendix B. Questionnaire data was collected through SurveyMonkey, a website that allows one to design questionnaires and securely collect responses. Beliefs and attitudes about mathematics technology, teaching, and learning were assessed using 22 items. Twenty of these items were adapted from questionnaires and surveys used in previous studies (Anderson et al., 2011; Christensen & Knezek, 2009; Vannatta & Fordham, 2004), and two were self-developed in order to assess whether participants valued mathematics technology for its functional opportunities, its pedagogical opportunities, or both. See Pierce and Stacey (2010) for further discussion of functional and pedagogical opportunities provided by mathematics analysis software (MAS), which includes CAS and other mathematics technologies. Additionally, the questionnaires included 13 self-developed items designed to assess prior experience with mathematics technologies.

The 20 items from questionnaires and surveys used in previous studies were selected to assess participants’ anxiety toward mathematics technology use, enthusiasm for mathematics technology use, value of mathematics technology for teaching and learning, and traditional views of teaching and learning. I chose items designed to assess anxiety, enthusiasm, and value specifically because they were connected to preservice teachers’ intentions to use technology in their future classrooms in previous studies. Positive attitudes (enthusiasm) toward technology use were found to be the best predictors of preservice teachers’ intrinsic and extrinsic motivation to use technology in their future classrooms, and preservice teachers’ amotivation to use technology in their future classrooms was best predicted by negative attitudes (anxiety) toward technology and negative social norms (Cullen & Greene, 2011). Value beliefs were found to be the most significant predictor of preservice teachers’ intentions to use a variety of software and to use it frequently in their future classrooms (Anderson et al., 2011). I attempted to assess participants’ traditional views of teaching and learning because teaching mathematics in ways aligned with
mathematics education reform often requires teachers to shift from a traditional mode of teaching “to a mode based on a constructivist view of learning” (Nelson, 1997, p. 5). I think of this shift as one where there are fewer instances of the teacher demonstrating procedures for students to practice and more instances of the teacher acting as a guide while students work on types of problems that they have not necessarily seen before designed to help them construct mathematical knowledge.

To assess participants’ traditional views of teaching and learning, I selected five of 14 items (see items 21-24, and 33 in Appendix B) designed to assess teaching philosophy directly from the Teacher Attribute Survey (Vannatta & Fordham, 2004). According to Vannatta and Fordham, the items that I selected originated from Becker and Anderson’s (1998) Teaching, Learning, and Computing Survey. While I was able to find information about the Teaching, Learning, and Computing Survey, I was unable to locate the original items because the website that Vannatta and Fordham referenced was no longer available. Thus, Vannatta and Fordham’s versions of these items were used. Because these items ask participants to select a level of agreement with statements about general views of teaching and learning without any references to technology or mathematics, I did not modify them.

To assess positive and negative attitudes about mathematics technology use, I adapted ten items (see items 6-15 in Appendix B) from the questionnaire titled Teachers’ Attitudes Toward Computers (TAC version 6) designed to assess enthusiasm (positive attitude) and anxiety (negative attitude) toward general computer use (Christensen & Knezek, 2009). The adaptation involved changing the word “computer” to “mathematics technologies.” I also changed the possible answer choices from the typical Likert Scale (strongly disagree, disagree, undecided, agree, strongly agree) to the following: strongly disagree, moderately disagree, slightly disagree, slightly agree, moderately agree, and strongly agree. This change was made so that these items matched the structure of the items taken directly from Vannatta and Fordham's (2004) Teacher
Attribute Survey. I also wanted to be able to determine whether participants’ beliefs were aligned with or in opposition to particular statements, and this change eliminated the undecided choice. The items that I used from the TAC version 6 were similar to TAC questionnaire items that Cullen and Green (2011) linked to preservice teachers’ motivation and amotivation to use technology in their future classrooms. The TAC version 6, deemed to be a refined, reliable version of the TAC by Christensen and Knezek (2009), used slightly fewer items to assess enthusiasm and anxiety than Cullen and Green.

To assess participants’ value of mathematics technology for teaching and learning, I adapted five items (see items 16-20 in Appendix B) designed to assess “beliefs about the value of classroom technology integration” (Anderson et al., 2011, p. 327). The adaptation involved changing the word “technology” to “mathematics technologies” and switching from five answer choices to six answer choices as described above. Since I was assessing participants’ value of mathematics technology for teaching and learning, I thought it was important to include questionnaire items that distinguished between valuing technology for its functional opportunities and valuing technology for its pedagogical opportunities, as those terms are described by Pierce and Stacey (2010). In other words, I wanted to know whether my participants valued mathematics technology because they believed it saved time or because they believed that it could be used to help students learn and understand mathematical concepts. Therefore, two self-developed items (see items 34 and 35 in Appendix B) were added.

Of the 13 items that I developed to assess participants’ prior experience with mathematics technologies, two items (see items 1-2 in Appendix B) asked participants about the frequency of technology use in their secondary school (grades 7-12) and college mathematics courses. Three items (see items 3-5 in Appendix B) asked participants to list courses in which they had used mathematics technologies to graph functions, create tables of values for functions, and symbolically manipulate algebraic expressions and equations. The other eight items (see items
25-32 in Appendix B) addressed whether participants had used mathematics technologies in their previous courses in ways that Wachira, Keengwe, and Onchwari (2008) identified as being recommended by the National Council of Teachers of Mathematics (NCTM) and others and in ways that preservice teachers in the study conducted by Wachira et al. believed technology should be used (p. 301).

**Interviews**

Semi-structured, task-based interviews were used for this study. The interview protocol is included in Appendix C. In the interview, students were presented with three types of conflicts: symbolic, tabular, and graphic. Since prior experience with the technologies was not assumed, participants were instructed on how to use each command that yielded the conflicting results by doing a sample task that produced similar output between each of the technologies directly before they were asked to use the same command for a task that would produce conflicting output. Besides providing instruction for the use of particular commands, seeing a similar example for which all technologies agreed before seeing conflicts metaphorically balanced the scale before each task so that participants did not suspect that one particular technology was more correct or conventional in its displays than others. During the interview, participants used each of the technologies to factor polynomials symbolically, create tables of values for functions, create graphs for functions, and differentiate trigonometric functions symbolically.

Lists of common derivative rules and trigonometric identities were available for reference during the interview (see Appendix D). This reference was provided because the last conflict in the interview involved differentiating a trigonometric function, and I did not want participants’ choices in resolving this conflict to depend on whether they had memorized derivative rules or trigonometric identities. I expected that some participants would use paper-and-pencil methods
and that others would try to resolve this conflict by using technology to test the equivalence of expressions symbolically, graphically, or by creating tables of values. The paper-and-pencil resolution method that I anticipated required trigonometric identities that participants may not have used extensively. The presence of printed trigonometric identities provided participants who preferred to rely on paper-and-pencil methods and available facts, rather than the capabilities of the available technology, access to the tools they needed to do so. However, including relevant trigonometric identities with irrelevant ones along with derivative rules in a format typical of calculus textbooks, provided participants with these tools without strongly influencing their resolution path. So that participants who preferred to use technology to resolve conflicts would feel comfortable doing so, participants were told that they could ask me how to perform any operation with the technologies that they wished if they did not already know how to do so.

Part 1 of the interview asked participants to use factor commands to factor trinomials. Using factor commands to factor \( x^2 - 5x + 4 \) and \( x^2 - 3x - 4 \) produced symbolic conflicts between Core Math Tools and the other two technologies. Table 3-1 shows the symbolic output that participants saw during the interview.

Part 2 asked participants to create tables of values for functions. Creating tables of values for the function \( f(x) = \frac{1}{x} \) produced a conflict between the TI-89 and the other two technologies when \( f(0) \) was evaluated. Table 3-2 shows the tabular output that participants saw during the interview.

Part 3 asked participants to graph functions. Graphing the functions \( f(x) = x^{1/5} \) and \( g(x) = x^{0.2} \) produced conflicts between the TI-89 and the other two technologies. Table 3-3 shows the graphical output that participants saw during the interview. Participants also were informed and asked to react to the fact that previous versions of GeoGebra had graphed \( f(x) = x^{1/5} \) and \( g(x) = x^{0.2} \) differently. While the graph \( g(x) = x^{0.2} \) has remained consistent for different versions of
GeoGebra, graphing only positive input values, other versions of GeoGebra graph both negative and positive input values for \( f(x) = x^{1/5} \) as the TI-89 does.

Part 4 asked participants to use derivative commands. Taking the derivative of the function \( \cos(3x - \pi/6) \) with respect to \( x \) produced symbolic conflicts between all three technologies. Table 3-1 shows this symbolic output. This problem was used by Lagrange (1999) as an example of output from the TI-92 that did not match the answer usually accepted in the paper-and-pencil context. In this case, the TI-92 and the TI-89 produce the same output, and GeoGebra’s result is most similar to the paper-and-pencil result one would get when applying the chain rule.

After producing outputs for all three technologies for a particular task, participants were asked to react to the output. Interviews were semi-structured to ensure that participants encountered the same conflicts and answered the same questions pertaining to the output while allowing me the flexibility to ask different follow-up questions depending on participants’ responses and comments. All participants saw the outputs shown in Tables 3-1, 3-2, and 3-3, and all participants were asked the following questions after each conflict:

1. Take a look at these three outputs; how are they similar or different?
2. Do you think these outputs are mathematically equivalent?
3. How do you convince yourself that these outputs are equivalent (or not equivalent)?
4. Can you think of another way that you could convince someone that these outputs are equivalent (or not equivalent) other than what you have already explained?

Other output seen by participants or questions asked depended upon participants’ responses and resolution methods.

After seeing all of the conflicts and responding to the above questions for each, participants were told about other possible strategies that one might use to resolve conflicts between technological output that they had not already offered during the interview and asked to react to each. Finally, in part 5 of the interview, participants were asked to react to the overall
experience. They were then asked whether they would consider presenting conflicts such as these to students in their future classrooms and why.

<table>
<thead>
<tr>
<th>TI-89 Titanium</th>
<th>GeoGebra</th>
<th>Core Math Tools</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Sample 1</strong></td>
<td><em>factor(x^2 + 5x + 4)</em>&lt;br&gt;(x + 1)(x + 4)</td>
<td><em>factor(x^2 + 5x + 4)</em>&lt;br&gt;(x + 1)^*(x + 4)</td>
</tr>
<tr>
<td><strong>Conflict 1a</strong></td>
<td><em>factor(x^2 - 5x + 4)</em>&lt;br&gt;(x - 4)(x - 1)</td>
<td><em>factor(x^2 - 5x + 4)</em>&lt;br&gt;(1-x)^*(4-x)</td>
</tr>
<tr>
<td><strong>Conflict 1b</strong></td>
<td><em>factor(x^2 - 3x - 4)</em>&lt;br&gt;(x - 4)(x + 1)</td>
<td><em>factor(x^2 - 3x - 4)</em>&lt;br&gt;-1*(4-x)^*(x+1)</td>
</tr>
<tr>
<td><strong>Sample 4</strong></td>
<td>Derivative[cos(3x)]&lt;br&gt;-3 sin(3x)</td>
<td><em>der(cos(3x),x)</em>&lt;br&gt;-1*(3<em>sin(3</em>x))</td>
</tr>
<tr>
<td><strong>Conflict 4</strong></td>
<td>Derivative[cos(3x-pi/6)]&lt;br&gt;-3 sin(-1/6 pi + 3x)</td>
<td><em>der(cos(3x-pi/6),x)</em>&lt;br&gt;-1*&lt;br&gt;[-3 \sin \left(3^* x - \frac{\pi}{6}\right)]</td>
</tr>
</tbody>
</table>

Table 3-1: Symbolic output seen by participants during the interview.
<table>
<thead>
<tr>
<th>TI-89 Titanium</th>
<th>GeoGebra</th>
<th>Core Math Tools</th>
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<td>Sample 2</td>
<td><img src="image1.png" alt="Image" /></td>
<td><img src="image2.png" alt="Image" /></td>
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<td></td>
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<td>B</td>
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<td>5</td>
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<tr>
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<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2</td>
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<td></td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>10</td>
</tr>
</tbody>
</table>

| Conflict 2     | ![Image](image3.png) | ![Image](image4.png) |
|                | A | B              |                           |
|                | -2 | -0.5           |                           |
|                | -1 | -1             |                           |
|                | 0  | ∞               |                           |
|                | 1  | 1               |                           |
|                | 2  | 0.5             |                           |
|                | 3  | 0.333           |                           |

Note: User may scroll up and down to see other values.

Note: Table output is not automatic in GeoGebra. Output varies based on what is entered into column A.

Note: User may scroll down to see other values.

Table 3-2: Tabular output seen by participants during the interview.
Table 3-3: Graphic output seen by participants during the interview. The GeoGebra output above is from version 5.0.42.0-3D, which was the most current version of GeoGebra available at the time of the interviews. While the graph of \( g(x) = x^{0.2} \) is displayed consistently across different versions of GeoGebra, the graph of \( f(x) = x^{1/5} \) is not. The graph of \( f(x) = x^{1/5} \) is displayed with both positive and negative input values in other versions of GeoGebra.

Analysis

Questionnaire responses were transferred from SurveyMonkey to tables designed to compare individual responses by item and by category. Specifically, I created tables that listed participants’ pseudonyms under each possible response for each item, and I organized those
tables by belief category. Using these tables, I could see patterns in the responses of particular participants and how their responses compared to the responses of other participants. After interviews were audio-recorded, GarageBand was used to split up and arrange participants’ responses so that they could be listened to and transcribed by question. Participants’ responses were then transcribed for further analysis. Memo writing as described in Corbin and Strauss (2008) was employed before and after each interview and early in the analysis to record my initial perspectives and observations and to look for common elements in participants’ responses. After these initial steps were completed, further analysis was done for each research question.

My first research question asked how preservice secondary mathematics teachers mathematically resolve output conflicts between mathematics technologies. For this question, I used participants’ transcribed responses as they tried to resolve each conflict, and I looked for common elements across tasks and across participants. Unique strategies were also noted. I then coded common elements within the participants’ discussions of conflicts, such as the recognition of salient differences between technological outputs, the recall of facts and procedures, and the use of technological evidence to resolve the conflicts.

My second research question asked what perspectives preservice secondary mathematics teachers have on using mathematics technologies and conflicting technological output in their future classrooms after experiencing technological conflicts themselves. For this question, I used participants’ transcribed responses to the final interview questions, which explored what they “took away” from the experience and whether and why they would consider presenting technological conflicts to students in their future classrooms. I looked for common elements between responses, and I noted unique perspectives.

The third research question asked how preservice secondary mathematics teachers’ mathematical resolution methods and perspectives on using mathematics technologies and conflicting output in their future classrooms related to their prior experiences as learners and their
beliefs about technology, teaching, and learning. For this research question, I attempted to connect participants’ questionnaire responses to participants’ interview responses. Questionnaire data, interview transcripts, and memos written by the researcher were analyzed. Similar responses were grouped and unique responses were noted. When similar responses during the interview did not appear to be connected to questionnaire responses in expected ways or across participants, further research and analysis was done in order to find possible reasons for the inconsistencies. For example, common elements of control and consistency in the classroom arose in participants’ responses to the question, “What do you take away from this experience?” These elements were common to most of the participants regardless of their questionnaire responses. Thus, more research was done on preservice teachers’ conceptions of classroom management and their desire to maintain classroom control in order to make sense of this result.

**Conclusion**

This chapter has provided details about the preservice secondary mathematics teachers that participated in this study, the technology that was used during the interviews, the design of the questionnaires and interview protocol used to collect the data, and the data analysis. Data from both the interviews and the questionnaires can inform teacher educators and researchers about the perspectives with which new teachers enter the field and how these perspectives relate to prior experiences and beliefs.

Because this study uses mathematical output that is likely to arise in secondary school mathematics classrooms from technologies designed for educational use (TI-89, GeoGebra, and Core Math Tools), preservice teachers’ reactions to the conflicts in this study can provide information to teachers, researchers, teacher educators, and curriculum developers about the mathematical and technical knowledge needed to understand and utilize the output given by
mathematics technologies designed for secondary mathematics education. Such information could be used to develop activities to help students and teachers become effective mathematics technology users. Additionally, preservice teachers’ reactions to conflicting technological output and their perspectives on how to use conflicts in their future classrooms can inform mathematics technology developers about how their products may be used by future teachers. The results of this study are presented in the next chapter. The final chapter discusses implications and limitations of the results as well as ideas for future studies.
Chapter 4

Results

In this chapter, I report answers to the research questions. The three research questions addressed in this study are as follows:

1. How do preservice secondary mathematics teachers mathematically resolve output conflicts between mathematics technologies?
2. What perspectives do preservice secondary mathematics teachers have on using mathematics technologies and conflicting technological output in their future classrooms after they have experienced technological conflicts themselves?
3. How are preservice secondary mathematics teachers’ mathematical resolution methods and perspectives on using mathematics technologies and conflicting output in their future classrooms related to their prior experiences as learners and their beliefs about technology, teaching, and learning?

Answers to the first two research questions are supported by examples from the interviews. Answers to the third research question are supported by examples from the interviews and the questionnaire responses. I refer to individual participants by pseudonyms throughout this paper.

How participants mathematically resolved output conflicts (question 1)

For the preservice secondary mathematics teachers interviewed in this study, resolving output conflicts between mathematics technologies involved several common elements. These elements included recognizing salient similarities and differences, attempting to make sense of unexpected output, and determining the equivalence or validity of outputs by applying facts and procedures or by generating technological evidence. This section describes similarities and differences in how participants mathematically resolved output conflicts in detail.
Recognizing salient similarities and differences

One element common to all participants was the recognition of salient similarities and differences between the outputs. For example, factoring \(x^2 - 5x + 4\) with the TI-89, GeoGebra, and Core Math Tools produced the outputs \((x - 4)(x - 1)\), \((x - 4)(x - 1)\), and \((1 - x)(4 - x)\), respectively. All of the participants recognized and mentioned the salient difference that Core Math Tools ordered the terms within the factors in an unconventional way. However, only one participant, Walter, recognized and mentioned, “all three of them [the technologies] have a different way of symbolizing multiplication.”

Similarly, for the tabular conflict, all participants quickly noticed the salient difference between “\(\infty\)” and “undef” as the value of \(f(x) = 1/x\) when \(x = 0\), but only three of the six participants (Lily, Richard, and Walter) recognized and commented on the less noticeable discrepancy that each technology displayed different numbers of significant digits for decimal output (see Table 3-2). Only Walter commented on display differences between the tables that were less mathematically salient. For example, referring to the way in which Core Math Tools displays the \(x\)- and \(y\)-values in a table, he said, “the input and output have different colors.”

For the graphic conflicts, all participants recognized the same salient difference between displays; the TI-89’s graph displayed negative and positive input values for both \(f(x) = x^{1/5}\) and \(g(x) = x^{0.2}\), while GeoGebra and Core Math Tools displayed only positive input values (see Table 3-3). Only Walter pointed out other differences between the graphical displays that were less mathematically salient. For example, he said, “the \(x\)-axis on the Core Math Tools is on the bottom, it’s labeled from negative ten to ten.” He contrasted this with GeoGebra placing the axes in the middle of the screen rather than on the borders (see Table 3-3). Walter also pointed out the lack of numeric labels on the TI-89’s display saying, “On the calculator, there’s no labeling of tick marks, but there are tick marks.”
Making sense of unexpected output

Another commonality was an attempt to make sense of unexpected output. As participants attempted to make sense of unexpected output, they often suggested ways in which they thought that technology might have operated on the input. For example, as Richard talked about the conflicting graphical outputs for \( f(x) = x^{1/5} \), he said that GeoGebra and Core Math Tools “just decide that anytime you have a power between zero and one, they only want to show the positive domain.” Mickey was the only participant that did not comment on how the technologies might have operated on the input. All of the others did this for at least two of the conflicts that they encountered during the interviews.

In addition, all participants mentioned mathematical facts and procedures that they associated with each conflict as they attempted to make sense of the conflicts. Although participants recalled facts and procedures as they attempted to make sense of conflicting output, these facts and procedures were not always used in their final arguments about the validity or equivalence of outputs. For example, as participants looked at the tabular conflict between “undef” and “\( \infty \),” Lily, Richard, Katrina, and Walter mentioned facts and procedures about left and right hand limits of a function at a point. Lily, Richard, and Katrina indicated that, if left and right hand limits do not match, then the limit of the function at that point “does not exist.” However, limits and the procedure described were only relevant to Lily’s and Walter’s arguments for preferring “undef” to “\( \infty \)” for the table values of \( f(x) = 1/x \) when \( x = 0 \). Lily and Walter each indicated that “\( \infty \)” would have been appropriate if the left and right hand limits of \( f(x) \) as \( x \) approached 0 were both positive infinity. Although every participant preferred “undef” to “\( \infty \),” rather than using facts and procedures related to limits to make their argument, Richard, Katrina, Liam, and Mickey argued that “\( \infty \)” was inappropriate because tables should display function
values, and $f(x) = 1/x$ has no value at $x = 0$. For example, Richard said, “It’s not asking for the limit. It’s asking for the value, and that value is undefined.”

**Using facts and procedures to resolve conflicts**

When facts or procedures could be applied mentally and with certainty, it was common for participants to use them to resolve output conflicts. A procedure commonly used by participants involved the application of the distributive property to express the product of two binomials as a sum. All of the participants either used or suggested this procedure with certainty as a way to confirm the equivalence of the output generated by factor commands. Lily, Richard, Katrina, and Walter specifically referred to this procedure as “FOIL,” a commonly taught acronym that stands for First, Outer, Inner, Last, which is intended to help students remember to multiply each term in the first binomial by each term in the second. Only Lily carried out this procedure using pencil and paper; the others did so mentally.

A fact commonly applied by participants during the interviews was $x^{1/n} = \sqrt[n]{x}$. Participants used this as a fact without discussing the possibility that $x^{1/n} = \sqrt[n]{x}$ may not apply for any value of $x$ or $n$. All of the participants, except Walter, stated that $x^{1/5} = \sqrt[5]{x}$ in order to make sense of the function $f(x) = x^{1/5}$ and argued that it is possible to take the fifth root of negative numbers, which provided evidence that the graph of $f(x) = x^{1/5}$ should display negative input values. Walter made a similar argument, but he argued directly that $f(x) = x^{1/5}$ was defined for negative input values without mentioning radicals as the others had done. He said, “If you plug in a negative value for $x$, the function is defined and has an output value.”

The statements about exponents and radicals described above were used as facts by the participants, and these facts were enough to quickly convince Richard, Katrina, and Walter that the graph of $f(x) = x^{1/5}$ should display negative input values. At first, Mickey suggested that the
graph of \( f(x) = x^{1/5} \) should not include negative input values because she thought it made sense for its graph to be similar to the graph of \( f(x) = x^{1/2} \), as these functions have similar algebraic structures. She said, “I know what the function \( x \) to the one half looks like, and they [the graphs of \( f(x) = x^{1/5} \) and \( g(x) = x^{0.2} \)] would kind of be similar to that I guess.” However, Mickey eventually changed her mind and made an argument similar to that of Richard, Katrina, and Walter. She said, “for the fifth root it could . . . have negative numbers, so \( x \) can be negative.”

**Searching for additional evidence when uncertain about facts**

Unlike the other participants, Lily and Liam seemed uncertain about the truth of the facts they had recalled about exponents and radicals, and they searched for additional evidence. Lily tried to remember and apply other facts and procedures, and Liam tried to find additional technological evidence to support the validity of one graph of \( f(x) = x^{1/5} \) over that of the other. These alternative approaches did not lead these two participants to a conclusion about the validity of the graphs because their approaches led them to contradictions that they were unable to resolve during the interview. Lily’s and Liam’s attempt to resolve the graphic conflict are described in more detail in the next three paragraphs.

As Lily tried to recall facts and procedures, she said, “we’ve always learned that you can’t take the . . . root of a negative number,” which to her seemed contradictory to the fact that multiplying -1 by itself five times would result in -1. In contrast to Lily’s attempt to remember facts, Liam chose to consult technology. He decided to find an arbitrary \( x \)-value using the trace feature on the TI-89. He chose \( x = -3.417722 \) from the TI-89 screen to substitute into the expression \( x^{1/5} \) using GeoGebra. He wanted to see if the point would fall on the graph generated by GeoGebra, but he saw that it would not because GeoGebra did not graph any points with negative input values, and the tables generated by GeoGebra were consistent with the graphs. He
decided to input \((-3.417722)^{1/5}\) into GeoGebra’s CAS window and saw that the result matched the TI-89’s value of \(y\) stating, “Yeah, they are the same.”

Liam had just generated output in the TI-89 and GeoGebra that agreed, but he had not yet provided a definitive conclusion about the validity of the displayed graphs. I knew that the output he had just generated would not have agreed if he had chosen to enter \((-3.417722)^{0.2}\) instead of \((-3.417722)^{(1/5)}\), and I wanted to see his reaction to this difference. Because I knew that it would produce an imaginary number, I encouraged Liam to enter \((-3.417722)^{0.2}\) in GeoGebra’s CAS window. His response to the output was, “So the graph matches the table, which neither matches the algebra system.” At this time he decided that the graph should not include negative input values making a majority rules argument. He said, “It shouldn’t graph the bottom . . . because just about every other mapping of points we have done does not include that bottom section.” At this point Liam had seen that Core Math Tools and GeoGebra did not graph negative values for \(f(x) = x^{1/5}\) or \(g(x) = x^{0.2}\) and that the TI-89 did, that GeoGebra did not compute output values for negative input values in the table for either \(f(x) = x^{1/5}\) or \(g(x) = x^{0.2}\), and that GeoGebra’s CAS window produced an imaginary result when a negative value was substituted into the expression \(x^{0.2}\) and a real number consistent with the TI-89’s trace feature when a negative value was substituted into the expression \(x^{1/5}\). See Figure 4-1 for conflicting symbolic and tabular output from GeoGebra similar to what Liam referred to when he indicated that the majority of the outputs did “not include the bottom section.”

Next, I told Liam that prior versions of GeoGebra displayed different graphs for \(f(x) = x^{1/5}\) and \(g(x) = x^{0.2}\). I also had him look at the table of values produced by the TI-89, which produced a real number output value for each negative input value for both \(f(x) = x^{1/5}\) and \(g(x) = x^{0.2}\). I did this because Liam had previously argued that the graphs should not include negative input values because the majority of output he had seen did not, and I wanted to know whether he would change his argument after seeing more output that did include negative input values. After seeing
the TI-89’s tables and learning of GeoGebra’s update, Liam remained uncertain. He said, “My gut is telling me that there should be something down there [in quadrant III], but . . . they [GeoGebra] updated their software, and I would expect their update knows something.” Though he could find negative values that seemed to work in the function mentally (e.g., -1, -32), the majority of the technological output in front of him suggested otherwise, and he was reluctant to believe that a software update would be incorrect.

Figure 4-1: GeoGebra output similar to what Liam generated as he tried to decide whether the function \( f(x) = x^{1/5} \) would allow negative input values. Column B of the spreadsheet is programmed to calculate \( f(x) \) (column A), with \( f(x) = x^{1/5} \). The same values are displayed in the table for \( g(x) = x^{0.2} \) when using version 5.0.42.0-3D of GeoGebra. Table results in other versions of GeoGebra differ.

**Using technological evidence to resolve conflicts**

When comparing symbolic outputs involving the derivative of a trigonometric function, fewer participants applied facts and procedures in order to resolve output conflicts, even though trigonometric identities and derivative rules were available during the interview (see Appendix D). Instead, more participants used technological evidence to draw conclusions, which seemed to be a shift from their previous resolution methods. For the earlier symbolic conflicts between outputs generated by factor commands, all of the participants applied facts and procedures to determine that the outputs were equivalent. With the exception of Liam, the use of facts and
procedures continued as participants tried to determine the validity of the graphs generated for
\( f(x) = x^{1/5} \). Additionally, participants referred to facts when talking about the tabular conflicts
between “undef” and “\( \infty \).” However, Lily, Katrina, Mickey, and Walter also referred to the graph
of \( f(x) = 1/x \) as they argued about their preference for “undef.” The graph of \( f(x) = 1/x \) was
automatically generated by GeoGebra and visible to all participants, except Liam, as they worked
through this conflict. It was not visible to Liam because he had closed the graphics window
before entering \( f(x) = 1/x \), and he did not reopen it as he worked through this conflict.

In order to resolve the final conflict, five of the six participants convinced themselves that
\(-3\sin(-1/6\pi + 3x)\) was equivalent to \(3\cos(3x + \pi/3)\) by graphing the corresponding functions with
technology to see if the graphs matched. This technological evidence was very convincing to
participants. Before generating these graphs, Richard, Liam, and Walter suspected that the
outputs were equivalent, but Lily and Katrina indicated that the two outputs were not equivalent
because the latter differed from the conventional derivative rule \( d/dx \cos x = -\sin x \). Referring to
the TI-89’s output, Lily said, “they’re not taking the derivative of the cosine,” and Katrina said,
“the TI-89 didn’t do the derivative.” For all five of these participants, seeing that the graphs
matched led them to announce that \(-3\sin(-1/6\pi + 3x)\) was equivalent to \(3\cos(3x + \pi/3)\). Only
Mickey did not use graphs to determine the equivalence of these expressions. Instead, she used
trigonometric identities, and she did so by looking at the trigonometric identities provided and
talking through the proof without using pencil and paper. She did not seek technological evidence
to support her conclusion.

**Combining facts and technological evidence**

Two participants offered suggestions for convincing others of the validity of outputs that
combined both mathematical facts and technological evidence. Walter suggested that generating
and comparing the graph of $f(x) = x^5$ to the conflicting graphs of $f(x) = x^{1/5}$ would convince someone that the latter function should display negative input values because when compared to the former, “they both have the s-shape that an odd function has.” He was the only participant to talk about graphic relationships between $f(x) = x^{1/5}$ and $f(x) = x^5$.

Another participant, Liam, suggested that one could prove $-3\sin(-1/6\pi + 3x)$ was equivalent to $3\cos(3x + \pi/3)$ by dividing the two expressions using CAS to see if the resulting output was 1. He said, “you could take the one function and divide it by the other, and you should get 1 if they are equivalent.” Of the technology used in the interview, only the TI-89 simplifies such an expression to 1; the output that each technology generates can be seen in Table 4-1. Liam only generated the output seen in Table 4-1 in GeoGebra. Seeing that GeoGebra did not simplify the expression, he tried the TI-89 next. However, a typo that was not noticed by Liam or myself during the interview, in which the $x$ was missing from one of the expressions, produced output that appeared to us that the TI-89 did not simplify the expression. Believing that the technologies would not simplify the output, Liam then decided to graph the functions that corresponded to the conflicting output to test for equivalence as other participants had done. Liam’s division of these expressions was the only case in any of the interviews where a participant used the symbolic manipulation capabilities of CAS to try to produce technological evidence in order to confirm the equivalence of outputs. Later in the interview, Liam indicated that he thought of this strategy as an attempted proof. When asked, “After seeing different outputs from different technologies, some people may have tried to algebraically prove that the two expressions are equivalent, what do you think of that strategy?” Liam replied, “I think I tried it, and the computer couldn't do it.”
<table>
<thead>
<tr>
<th>TI-89 Titanium</th>
<th>GeoGebra</th>
<th>Core Math Tools</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-3sin(-1/6π(3x)) / (3cos(3x + π/6)))</td>
<td>sin((\frac{1}{6} (\pi - 18x))) / cos((\frac{1}{6} (\pi + 18x)))</td>
<td>(-\frac{3*\sin\left(\frac{3\times x + \pi}{3}\right)}{3\cos\left(\frac{3\times x + \pi}{3}\right)})</td>
</tr>
</tbody>
</table>

Table 4-1: Output given when dividing -3sin(-1/6 π + 3x) by 3cos(3x + π/3); only the TI-89 automatically simplifies the quotient to 1.

**Question 1 summary**

Although there was variation in the ways in which participants resolved output conflicts between mathematics technologies, the resolution work of all participants involved the common elements of recognizing salient similarities and differences; attempting to make sense of unexpected output, which included mentioning facts or procedures that might be relevant; and using facts and procedures, technological evidence, or a combination of both to try to determine the equivalence or validity of outputs. Participants tended to apply facts and procedures when they could do so mentally and when they were certain about the facts, as seen when participants resolved conflicts between graphic results by applying the fact \(x^{1/n} = \sqrt[n]{x}\) and when participants resolved conflicts produced by factor commands by applying the distributive property. If they were uncertain about facts and procedures, participants looked for additional evidence. As it became more difficult to mentally apply facts and procedures to determine the equivalence or validity of outputs, fewer participants did so and most chose instead to look for technological evidence to confirm the equivalence or validity of outputs. Additionally, when participants used technological evidence to resolve conflicts, they chose to reference or generate graphs more frequently than symbolic or numerical output. It is notable that considerable variation was observed in the mathematics and technological features used by participants to resolve conflicts within a small sample size.
Participants’ perspectives on using mathematics technologies and conflicting output in their future classrooms (question 2)

For most of the participants, the experience of encountering conflicting output between mathematics technologies focused their attention on concerns about consistency in the classroom and the teacher’s role in providing that consistency. However, this desire to provide consistency in the classroom was not associated with a desire to avoid student encounters with technological conflicts under all circumstances. Rather, participants seemed to be expressing a desire to maintain control over what students encountered at particular times. In fact, when asked, all of the participants said that they would consider presenting conflicts between mathematics technologies to their future students and saw the conflicts as potentially useful teaching tools in various ways.

Consistency and control

While their reasons for doing so varied slightly, five participants expressed a desire to control the technological output that students see at particular times in order to maintain consistency in the classroom. Lily, Richard, and Katrina focused on the teacher’s knowledge of technology. Specifically, they mentioned that it would be important for teachers to be aware of specific technological output before using it with students and to choose technology such that students see output that is aligned with what the teacher wants them to learn. These participants seemed to be expressing a desire to provide consistency between the goals of the lesson and the output given by technology during that lesson. They would choose technology based on whether it gave output that they deemed helpful for students learning particular topics. The following comment by Katrina exemplified this reaction: “Before you’re going to use the technology in the classroom, make sure you test it and see that the outputs are going to give you what you want the students to see.”
Liam and Mickey said that they would choose to use only one kind of calculator or computer software in the classroom so that students would not get inconsistent results from technology. Mickey worried that it would be difficult to grade student assignments if their answers were incorrect because of differences in technological output. She said:

If you had the option of they [students] could use whichever one of these three systems that they want on a test or homework assignment, you might be getting a whole bunch of different responses from each student, and you might not consider all those responses correct.

Liam indicated that keeping technological displays consistent would benefit students who were working together during a lesson. He said, “If you can have kids helping kids and getting the same display, it will be a bigger help for everyone to learn.”

Additionally, most of the participants that focused on controlling the output that students see seemed to be making the assumption that students would only use the technologies that the teacher provided or recommended, which suggests that participants believe that the teacher will be able to maintain this control. Only one participant brought up the possibility that students might use different software at home than at school. Richard mentioned that it would be beneficial for students to know about conflicting output if they used free software at home when they did not have access to a graphing calculator. He said it would be important to show students conflicts between technologies, “just so they can see how they differ.” He elaborated:

Say they don’t have TI’s that they can take home, and so they want to use one of these [Core Math Tools or GeoGebra]; they can. They know these differences are going to come up when they are doing it on their own, and they don’t think they’re doing it wrong.

Richard was also the only participant to comment that access to technology at home might be different than access to technology at school.

Only one participant, Walter, did not express a desire to make sure that the technological output that students see during a lesson is consistent or to choose the technology that students use in class. Initially, he indicated that if an answer he obtains by one method does not match
technological output, the answers “might both be correct . . . so it might be good to solve
problems . . . in more than one way to verify that you’re results are what you’re looking for.”
When asked whether he would consider presenting conflicts to students, he said, “Yeah, I think it
would be important. I think it [conflicting output] would probably come up naturally.” He added
that it would be important to discuss conflicts that arose with students and find out their opinions
about the conflicting output.

**How participants would use conflicts in their future classrooms**

Although most participants expressed a desire to provide consistency by controlling the
technological output that students see, this desire did not lead participants to conclude that all
student encounters with conflicting technological output should be avoided. Instead, every
participant said that they would consider presenting technological conflicts to students, and they
saw various ways in which conflicts could be useful teaching tools. Some of the participants
implied that conflicts would help students learn about the limitations of technology. For example,
Lily said that the conflicts would help teach students, “they can’t just blindly use technology,”
while Walter said that students might think that technology will “solve all of their problems for
them . . . and they need to understand that that’s not how it is.” Richard and Katrina mentioned
that the conflicts could encourage students to think critically or think about the mathematics
involved more deeply. Mickey suggested that conflicts could be used to show that results that
look different can be equivalent. She said, “it would help students learn more about identities,”
and “it would help students see that there is different ways of actually factoring something.” Liam
said that he would like to show conflicts to students because he personally found them interesting,
but he qualified this response saying that due to time concerns he would probably only show
conflicts that are easier to resolve because “you’ve only got so many hours in a day.”
It is notable that no participant talked about ways in which conflicts could be used in their teaching when responding to what they “took away” from the experience. Not until they were asked if they would “consider presenting conflicts like these to their future students” did they offer the potential uses of technological conflicts for teaching described in the previous paragraph. Prior to the prompt about “presenting conflicts,” their focus was on consistency and control, with the exception of Walter. Perhaps the phrase “presenting conflicts” in the prompt directed the participants to think about using the conflicts in ways that allowed the teacher to be in control of how students encountered them, thereby reducing initial concerns that technological conflicts would create undesirable inconsistencies and allowing participants to consider benefits of student encounters with the conflicts.

**Question 2 summary**

It is not surprising that most of the preservice teachers initially reacted to the conflicts by talking about their own preparation and their desire to maintain control over what students see because it is common for student teachers to be concerned with their own actions and their ability to maintain control over their classrooms. Elaborating on Doyle's (1986) descriptions of teachers’ concern for order and control, M. G. Jones and Vesilind (1995) stated, “Student teachers’ desire to control their students and classroom events is interwoven with their attempts to establish and maintain classroom management. This desire for control emerges as student teachers attempt to reduce the complexity of their environment” (p. 313). It seemed the participants in this study had already developed a desire to reduce the complexity of their future teaching environment and saw technological conflicts as something that would complicate this environment if the teacher did not plan them. Richard, the only participant with student teaching experience, said, “You don’t want to be surprised.” He explained that this was undesirable because when technological output that
the teacher does not expect appears in class, students might ask questions that distract from the lesson goals. It is also notable that most participants in this study did not seem to consider that students might use different technologies regardless of what the teacher uses or recommends. Participants in this study saw conflicting outputs as potentially useful teaching tools, but with the exception of Walter, they only seemed comfortable using conflicting output in their future teaching if they could control how and when students encountered the conflicts.

How participants’ reactions related to prior experiences and beliefs (question 3)

Although I am not able to connect participants’ reactions to conflicting output in the interviews to prior experiences and beliefs across all participants or across all tasks, I am able to report interesting connections for some individuals between reactions and questionnaire responses or between reactions and interview comments. Because the research question that I address in this section attempts to link reactions in the interviews to prior experiences and beliefs, I only report on the questionnaire responses of the six participants who completed the interview. I begin by discussing connections between participants’ experiences with mathematics technologies in their prior mathematics courses and the ways in which they mathematically resolved output conflicts in the interviews. Then, I discuss connections between the beliefs and attitudes of individuals and the ways in which they mathematically resolved output conflicts. Finally, I discuss participants’ perspectives on whether to use mathematics technologies or technological conflicts in their future classrooms and some examples of how the perspectives of individuals were connected to their prior experiences, beliefs, and attitudes.
How prior experience related to mathematical resolution methods

When participants used technology to resolve conflicts, most seemed to use the technologies and the technological features with which they reported having had more past experience. For example, all participants, except Mickey, convinced themselves of the equivalence of \(-3\sin(-1/6\pi + 3x)\) and \(3\cos(3x + \pi/3)\) by using technology to generate the graphs of the corresponding functions. Before generating graphs, Liam attempted to use the symbolic manipulation capabilities of technology while trying to resolve this conflict, and he was the only participant that attempted to use the symbolic manipulation capabilities of technology to resolve any conflict during the interviews. This seemed connected to participants’ reported prior experiences because participants indicated that they used technology more for graphing functions than for symbolic manipulation in their prior mathematics courses, and Liam was the only participant who had used the symbolic manipulation capabilities of the TI-89 before the interview. In the remainder of this section, I elaborate on how the frequent use of technologically generated graphs and the infrequent use of the symbolic manipulation capabilities of CAS to resolve conflicts relates to the prior experiences that participants reported on the questionnaires and in the interviews.

The first two questionnaire items (see Appendix B) asked participants to rate the frequency with which they had used mathematics technologies in secondary school (grades 7-12) and college mathematics courses. The next three questionnaire items asked participants to list courses in which they had used mathematics technologies to graph functions, create tables of values for functions, and symbolically manipulate equations and expressions. All interviewed participants, except Liam, reported using mathematics technologies more frequently in their secondary mathematics courses than in their college mathematics courses. Additionally, while all participants reported using technology to generate graphs in multiple subject areas, this was not
the case for using technology to generate tables of values or to symbolically manipulate equations and expressions. The secondary school subject areas for which interviewed participants reported using technology to graph functions, generate tables of values for functions, and symbolically manipulate equations or expressions are shown in Table 4-2. An “X” in the table indicates that a participant listed at least one prior mathematics course in a particular subject area in which they used mathematics technologies for the indicated purpose. Table 4-2 does not provide any information about how frequently technology was used in a particular subject area for a particular purpose. Thus, one should not assume that participants spent similar amounts of time using technology to graph functions or to symbolically manipulate equations or expressions simply because Table 4-2 shows that they reported using technology for those purposes in a particular subject area.

Only subject areas that are taught in grades 7-12 are included in Table 4-2 because participants’ reports of technological experience in subjects beyond this level were difficult to interpret, as participants interpreted the questions differently. It was evident that the questions about technology use in college courses were unclear to participants because some participants included experience in mathematics education courses while others did not, and some participants included experience in courses in which they were currently enrolled while others did not. Furthermore, the conflicts that participants encountered during the interviews did not include content beyond introductory calculus because I chose conflicts that were likely to arise in secondary classrooms and in subject areas that the participants were likely to teach in the future. Thus, it makes sense to attend to participants’ past experience with technology in secondary school subject areas.
Table 4-2: Secondary mathematics subject areas in which participants reported using mathematics technologies to graph functions, create tables of values for functions, and symbolically manipulate equations or expressions in past mathematics courses on the questionnaires. Trig/PC refers to trigonometry and/or precalculus. *Liam did not report using mathematics technologies to symbolically manipulate equations or expressions on the questionnaires, but he did say during the interview that he used the TI-89 on his own to check integrals in his college calculus courses.

Participants’ responses during the interviews provided further evidence of a disparity between participants’ experience using technology to generate graphs and participants’ experience using CAS for symbolic manipulation. For example, during the interviews, five of the six participants specifically mentioned that in high school they had used the TI-84, which can create graphs and tables of values for functions but does not have CAS capabilities. All participants were aware of CAS, as they had recently been introduced to the TI-Nspire CAS in mathematics education courses that they were currently taking. However, only Liam had used the TI-89 prior to the interviews. While Liam did not report using mathematics technologies to symbolically manipulate equations or expressions in his prior mathematics courses on the questionnaires (see Table 4-2), he said in the interview that one of his friends owned a TI-89, and he and his friend had used the CAS capabilities of the TI-89 to check integrals in college calculus.
Liam also said, “I normally use the TI-84 Silver,” indicating that he usually used a calculator that did not have CAS capabilities.

Furthermore, all of the participants had used GeoGebra, but most of them reported using it in their college Geometry course, and they were not familiar with GeoGebra’s CAS features. Only Richard had used Core Math Tools prior to the interviews. Although Richard reported using CAS in several subject areas on the questionnaires (see Table 4-2), when I asked him how he had used CAS in the past, he said, “When they showed us how to use them, we maybe practiced some of the commands,” which indicated that his CAS experience was not extensive. This comment also provides further evidence that the frequency with which participants used technology in particular subject areas cannot be inferred from Table 4-2. Additionally, no participant used CAS to directly test the equivalence of two expressions during the interview, and four of the participants said that they were not aware that this was possible when asked about this strategy, which suggests that participants did not draw on what CAS experience they did have when resolving conflicts.

The disparity between participants’ experiences with using technology to generate graphs and using CAS for symbolic manipulation seemed connected to the technological strategies that they chose to resolve conflicts. For example, Liam, the only participant that had prior experience with the TI-89 and its CAS commands, was also the only participant to attempt to resolve a conflict using the TI-89’s symbolic manipulation capabilities. In contrast, strategies involving graphs were much more common. As mentioned earlier, five of the six participants used graphs to decide whether $3\cos(3x + \pi/3)$ was equivalent to $-3\sin(-1/6 \pi + 3x)$, and four participants specifically mentioned the graph of $f(x) = 1/x$ when describing their preference for “undef” over “$\infty$” as the best output for the value of the function when $x = 0$.

It makes sense that participants would choose to use features of technology or the specific tool with which they have had the most past experience. However, it is unclear whether
participants chose to refer to or generate graphs more often than tables or symbolic output in the interview because of their past experiences with technology or because of their past experiences learning the mathematical content associated with the conflicts. For example, it could be that more participants chose to generate graphs to confirm the equivalence of trigonometric expressions because they associated trigonometric functions more strongly with graphs than with symbolic manipulation. Similarly, participants may have associated factoring trinomials more strongly with mental or paper-and-pencil symbolic manipulation than with graphic or other representations generated by technology. Only one participant, Richard, suggested using graphs or tables as possible ways to check for equivalence between the outputs generated by factor commands. The other participants were unable to think of ways to test the equivalence of factored output during the interview other than factoring or multiplying to expand outputs mentally or by hand. Therefore, it is feasible that participants’ past experiences as learners had an effect on the ways in which they attempted to resolve conflicts between mathematics technologies, but it is uncertain whether experiences with specific mathematical content, experiences with specific technological features and tools, or some combination of both influenced participants’ strategies.

**How beliefs and attitudes related to mathematical resolution methods**

Although it was not possible to connect questionnaire responses about beliefs and attitudes to mathematical resolution methods across all participants or across all tasks, some relationships between resolution methods and beliefs and attitudes were found for individual participants. In this section, I discuss beliefs and attitudes that the questionnaires attempted to assess. Additionally, I discuss how the beliefs and attitudes assessed by the questionnaires related to Liam’s mathematical resolution methods during the interview and how questionnaire responses did not seem to relate to the resolution methods of others. I also discuss how views of
mathematics, mathematics learning, and mathematical understanding, not assessed by the questionnaires but indicated by comments made during the interviews, may have contributed to Katrina’s and Lily’s mathematical resolution methods. I did not assess views of mathematics on the questionnaires because I focused primarily on beliefs and attitudes that previous researchers had linked to technology use in the classroom (e.g., Anderson et al., 2011; Cullen & Greene, 2011; Rakes et al., 2006). The connection between mathematical resolution methods and views of mathematics, mathematics learning, and mathematical understanding was made after analyzing comments that Lily and Katrina made during the interviews.

Beliefs and attitudes assessed by the questionnaires included the following categories: anxiety toward mathematics technology use, enthusiasm for mathematics technology use, value of mathematics technology for teaching and learning, and traditional views of teaching and learning. All questionnaire items are available in Appendix B. Twenty questionnaire items with five items per category adapted from previous studies (Anderson et al., 2011; Christensen & Knezek, 2009; Vannatta & Fordham, 2004) assessed these beliefs and attitudes. Table 4-3 shows the questionnaire responses of the interviewed participants for items adapted from previous studies. Numerical values were assigned to participants’ responses for greater readability and to make it easier to look for patterns among the responses.

One can see in Table 4-3 that Liam reported strongly valuing mathematics technologies for teaching and learning on the questionnaires. He also reported that he was very enthusiastic and not at all anxious about using mathematics technologies. Previous studies showed that attitudes and beliefs similar to Liam’s significantly predicted preservice teachers’ intentions to use technology in their future classrooms (Anderson et al., 2011; Cullen & Greene, 2011). These positive attitudes and beliefs about mathematics technologies also seemed related to Liam’s strategies for resolving conflicts during the interview, as Liam attempted to use technology more often than the others to resolve conflicts.
Anxiety toward mathematics technology use
Enthusiasm for mathematics technology use
Value of mathematics technology for teaching and learning
Traditional views of teaching and learning

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Table 4-3: Interviewed participants’ responses to questionnaire items that assessed beliefs and attitudes by category. Items 6-24 had six possible responses, and item 33 had five possible responses. For items 6-24, numbers in the table correspond to participants’ responses as follows: strongly disagree = 1, moderately disagree = 2, slightly disagree = 3, slightly agree = 4, moderately agree = 5, strongly agree = 6. *Item 33 presented two opposing statements; statement 1 was aligned with constructivist views of teaching and learning, and statement 2 was aligned with traditional views of teaching and learning. For item 33, numbers in the table correspond to participants’ responses as follows: strongly agree with statement 1 and disagree with statement 2 = 1, agree more with statement 1 than with statement 2 = 2, unsure which statement I agree with more = 3, agree more with statement 2 than with statement 1 = 4, strongly agree with statement 2 and disagree with statement 1 = 5.

However, as described earlier in this chapter, Liam also encountered more conflicts when he tried to look for technological evidence to confirm whether the graph of \( f(x) = x^{1/5} \) should include both positive and negative input values. He made the statements, “I don’t like this” and “that’s just confusing me now” when he encountered additional conflicts. Lily also reported high enthusiasm and low anxiety toward using mathematics technologies on the questionnaires (see Table 4-3). Interestingly, Liam and Lily were the only participants that did not reach a definitive conclusion about the graphic conflict during the interview. As described earlier, while Lily did not reach a definitive conclusion because she could not remember rules about roots, Liam did not reach a definitive conclusion because when he looked for other ways to confirm the correct graph using technology, he encountered more conflicts. He was also reluctant to believe that a software
update from GeoGebra could have been incorrect saying “I would expect their update knows something.” He explained that even though he suspected that the domain of the function \( f(x) = x^{1/5} \) should include some negative values, he was uncertain because the most updated version of GeoGebra’s graphs and tables indicated otherwise. Before conducting the interviews, I would not have predicted that the participants that expressed high enthusiasm and low anxiety toward technology use would have struggled to resolve some of the conflicts. However, it could be that technology enthusiasts struggle with conflicting output because they are confident in technological results, and thus find inconsistencies disconcerting.

Liam also made more comments than others about software inconsistencies, and he was the only participant to comment on software coding. In fact, the first thing he said when asked what he “took away” from the overall experience was:

> I take away that you have different technologies; they will be coded different, so they should give different answers depending on how their coding works. But the main thing I take away is that GeoGebra is not consistent in all their output because their CAS is not exactly matching their graphics or spreadsheet.

It is also interesting that Liam seemed highly enthusiastic about mathematics technologies despite not having used them frequently in his past mathematics courses according to his questionnaire responses. He was the only participant to report using mathematics technologies rarely in his secondary school courses on the questionnaires, with all other participants indicating a higher frequency of mathematics technology use in grades 7-12. However, Liam’s comments about coding during the interview indicated that he might have had other technological experiences and interests such as computer science. I do not know this for sure, as I did not follow up on Liam’s other technical interests because I did not regard his comments as unique until interviews were completed and transcripts were analyzed.

In addition to beliefs about the value of technology and attitudes about technology use, the questionnaires also contained items that attempted to assess participants’ traditional views of
teaching and learning. In Table 4-3, one can see that Katrina’s questionnaire responses indicated that her beliefs were aligned with traditional views of teaching and learning and that she was anxious about using mathematics technologies. Lily’s responses, on the other hand, indicated that she was not anxious about mathematics technology use and that her beliefs were less aligned with traditional views of teaching and learning (see Table 4-3). Lily also reported using mathematics technologies more often and in more ways than Katrina in her prior mathematics courses on the questionnaires. Yet, Katrina and Lily seemed to approach conflicts in similar ways and make similar comments during the interviews. Specifically, both Katrina and Lily referred to rules and routine procedures that they had previously learned more frequently than others, and they were the only participants to spontaneously express the opinion that students should master mathematical procedures by hand before using technology. These comments indicated that Lily and Katrina valued paper-and-pencil skills and that they may have had similar views of mathematics and mathematics learning.

Although most of their responses shown in Table 4-3 differ, Katrina and Lily were the only participants interviewed to agree with questionnaire item 21 which states, “Students are not ready for meaningful learning until they have acquired basic reading and math skills.” However, their beliefs about teaching and learning seem to be in opposition according to their responses to questionnaire item 33, which asked participants to indicate which of two opposing statements they agreed with more. Lily strongly agreed with the first statement, which said, “I mainly see my role as a facilitator. I try to provide opportunities and resources for my students to discover or construct concepts for themselves.” Katrina agreed more with second statement: “That’s all nice but students really won’t learn the subject matter unless you go over the material in a structured way. It’s my job to explain, to show the students how to do the work and to assign specific practice.” The former statement is in line with constructivist views of teaching and learning, as it focuses on students constructing their own knowledge and the teacher acting as a guide. The latter
statement is in line with traditional views of teaching and learning, as it focuses on the teacher showing students how to do the work and the students practicing what the teacher has explained. Thus, it does not seem that the similarities between Katrina’s and Lily’s approaches to conflicts can be easily explained by their reported beliefs and attitudes on the questionnaires. However, it is possible that their similarities may have had less to do with their views of teaching and learning or their attitudes about technology use and more to do with their views of mathematics, mathematics learning, and mathematical understanding described by Ernest (1988), Tharp et al. (1997), and Skemp (1976/2006), respectively. Next, I discuss how Katrina’s and Lily’s comments during the interviews indicated similar mathematical views and how these views related to their frequent instrumental approaches to resolving conflicts.

Both Katrina and Lily indicated that they valued paper-and-pencil symbolic manipulation, and they seemed to equate it with doing and understanding mathematics in their statements. Katrina said, “I’m a big believer that [one should] do math by hand, but I guess that is because I’ve always done it by hand.” Lily said that her college calculus courses that did not allow calculators, “really drilled me on knowing how to understand it . . . we actually had to understand the concepts behind it and do it by hand.” In this excerpt, Lily seemed to be using the word “understand” to mean that she learned the rules and was able to use them. This seems related to Skemp’s (1976/2006) description of instrumental understanding, which said, “for many pupils and their teachers the possession of such a rule, and the ability to use it, was what they meant by ‘understanding’” (p. 89)

Additionally, the importance that Lily and Katrina placed on knowing how to do symbolic manipulation by hand indicated that their views might be in line with a rule-based view of mathematics learning, which Tharp et al. (1997) described as “the view that mathematics learning is mostly oriented toward processes which involve the manipulation of symbols and memorization of facts as opposed to the view that mathematics is based on reasoning about
relationships and patterns” (p. 555). Although Ernest’s instrumental view of mathematics (1988), Skemp’s instrumental understanding of mathematics (1976/2006), and Tharp et al.’s (1997) rule-based view of mathematics learning have a similar focus on the importance of mathematical rules and procedures, I think that describing Katrina’s and Lily’s views of mathematics learning as rule-based is more appropriate than describing them as having an instrumental view of mathematics or an instrumental understanding of mathematics. While I think that Lily and Katrina saw learning rules and facts as important, and perhaps necessary, for learning and understanding mathematics, their comments did not indicate that they viewed mathematics simply as a set of “unrelated” facts (Ernest, 1988, p. 2) or of mathematical understanding as “rules without reasons” (Skemp, 1976/2006, p. 89). Lily’s and Katrina’s comments did indicate that they placed a priority on the knowledge of facts, rules, and paper-and-pencil procedures for teaching and learning mathematics. This emphasis on facts, rules, and paper-and-pencil procedures seemed to influence the ways in which Katrina and Lily mathematically resolved output conflicts between mathematics technologies more than their dissimilar beliefs and attitudes about technology, teaching, and learning that the questionnaires attempted to assess.

For example, when they encountered the outputs $3\cos(3x + \pi/3)$ and $-3\sin(-1/6 \pi + 3x)$, Lily and Katrina were the only participants that initially thought that the former output from the TI-89 was incorrect because it did not follow the conventional derivative rule $d/dx \cos x = -\sin x$. However, both Lily and Katrina changed their minds when they saw that the graphs of the corresponding functions matched. After graphs persuaded her that the expressions were equivalent, Katrina commented, “I think if you are using that for the purpose of teaching the derivative, it would not be good to use the calculator [the TI-89] ‘cause the kids might not recognize that the derivative of cosine is negative sine.” Here, Katrina stresses the importance of teaching and learning the derivative rule in its conventional form and indicates that making sure
students know the conventional derivative rule is an important goal when “teaching the derivative.”

Additionally, while working through the conflicts, Lily showed evidence of trying to take an instrumental approach. For example, Lily tried to work through the graphic conflict by recalling rules as described earlier. Having recalled rules that seemed contradictory, she remained unsure whether the domain for \( f(x) = x^{1/5} \) contained negative values. I asked her how she could find out, and she replied, “that’s the problem, I don’t remember these rules.” Lily seemed unable to think of another way to proceed with this problem because she could not recall the rules she had previously been taught.

In contrast, Katrina did not experience difficulties with this conflict. Her approach was similar to Lily’s, but she remembered the rules, and she had insights into this conflict that other participants did not. She was the only participant to explain that \( x^{0.2} \) could be interpreted differently than \( x^{1/5} \) when asked. Katrina said, “it could be interpreted with doing the tenth root first and then squaring the whole thing, and then [with] that you can’t do negative numbers because you can’t do the tenth root of a negative.” She also mentioned that if squaring were done before taking the tenth root, then negative numbers would be allowed for \( x \). She wrote down this series of expressions on paper: \( x^{0.2}, x^{2/10}, \frac{10}{\sqrt{x}^2}, \left(\frac{10}{\sqrt{x}}\right)^2 \). In this case, Katrina’s procedural fluency with rational exponents and radicals facilitated unique insights into this conflict, but her approach was clearly instrumental. She arrived at this conclusion through the application of routine algebraic procedures. Katrina’s and Lily’s different levels of success with resolving this conflict illustrate how purely instrumental approaches depend on recalling appropriate facts and procedures.

In summary, while beliefs and attitudes about mathematics, technology, teaching, and learning seemed to be related to the ways in which participants mathematically resolved output conflicts between mathematics technologies, it was difficult to make detailed connections
between their resolution methods and the beliefs and attitudes assessed by questionnaires. Liam’s questionnaire responses seemed connected to his resolution methods, but Lily who had similar attitudes about technology use did not have similar resolution methods. Instead, her resolution methods seemed similar to Katrina’s, who had reported different attitudes toward technology and different beliefs about teaching and learning on the questionnaires. Additionally, questionnaire items designed to assess participants’ views of mathematics, mathematics learning, or mathematical understanding, which my questionnaires did not include, may have been more helpful in relating participants’ resolution methods to beliefs than questionnaire items that attempted to assess participants’ attitudes toward technology use and views of teaching and learning. On the other hand, questionnaire items may not accurately reveal teachers’ or preservice teachers’ mathematical views because respondents may answer in ways that they believe are “correct.” For example, Tharp et al. (1997) noted that teachers were reluctant to indicate agreement with a rule-based viewpoint in their questionnaire responses because a rule-based view of mathematics learning was not in line with National Council of Teacher of Mathematics (NCTM) recommendations.

Furthermore, it is likely that some of the difficulty in using the questionnaire responses to link beliefs and attitudes to mathematical resolution methods for technological conflicts resulted because the questionnaires asked participants to agree or disagree with general statements about technology, teaching, and learning. Questionnaire items did not ask about beliefs and attitudes in the context of technological conflicts because I wanted the conflicts to be a surprise to participants during the interview. However, as other researchers have stated, decontextualized beliefs often seem inconsistent with practices in particular contexts (Beswick, 2005; Kaufman & Moss, 2010; Philipp, 2007). Thus, while my questionnaire items may predict whether participants would be likely to use technology in their future teaching as suggested by Anderson et al. (2011)
and Cullen and Greene (2011), they did not seem to consistently predict how my participants would mathematically resolve technological conflicts.

**How perspectives on using mathematics technologies and conflicts for teaching related to beliefs, attitudes, and prior experience**

As was the case with connecting beliefs and attitudes to mathematical resolution methods in the previous section, it was difficult to connect beliefs and prior experiences assessed by the questionnaires to participants’ perspectives on whether to present conflicts to students in their future classrooms. Only one participant, Walter, reported views of teaching and learning on the questionnaires that seemed connected to his perspectives on using conflicts in his future teaching. Otherwise, specific prior experiences and beliefs that participants shared during the interviews, rather than questionnaire responses, seemed connected to participants’ perspectives on whether to present conflicts to students in their future classrooms. This section discusses these connections.

As described earlier, Walter was the only participant that did not express any desire to control the conflicts that students encountered, and he indicated that he would want to discuss students’ opinions about technological conflicts that might come up in class. Walter’s perspective of allowing students to encounter and discuss conflicts that arose “naturally” was in line with his questionnaire responses, which indicated that he disagreed with traditional views of teaching and learning (see Table 4-3). For example, Walter was the only participant to strongly disagree with questionnaire items 22 and 23 which stated respectively: “Instruction should be built around problems with clear, correct answers, and around ideas that most students can grasp quickly,” and “Student projects often result in students learning all sorts of wrong knowledge.” While Walter’s strong disagreement with these statements and his willingness to discuss conflicts with students as they arise are aligned with constructivist views of teaching and learning, it is difficult to argue
that Walter’s questionnaire responses about teaching and learning were considerably different from the responses of other participants. With the exception of Katrina, the other participants also expressed disagreement with these items, and it is difficult to say whether strong disagreement in Walter’s case is considerably different than the moderate or slight disagreement with these statements reported by others. Thus, while it makes sense that disagreement with traditional views of teaching and learning would make a teacher less likely to want to control the technological output that students see, it is difficult to argue that responses to the questionnaire items were connected to this perspective across participants.

Connections between reported views of teaching and learning on the questionnaires and perspectives on presenting conflicts to students may not align well because the questionnaire items were not stated in the context of technology use. It is possible that participants have similar views of teaching and learning in general but not when it comes to using technology for teaching and learning. Their views of teaching and learning could be different when considering technology use because their conceptions of mathematics technology use may be limited by their prior experiences with technology as learners. In Wachira et al. (2008), 20 preservice middle school teachers’ common conceptions of appropriate technology use included visualization, speedy computation, and answer checking. They also worried that students would over-rely on technology at the expense of learning basic facts. Wachira et al. characterized these preservice teachers’ conceptions of technology use as limited because they did not conceive of using technology in ways that NCTM and others suggested, such as for the in depth exploration of patterns and concepts. Wachira et al. attributed these preservice teachers’ limited conceptions of technology use to a lack of meaningful experiences with technology in their own mathematics learning, as only five of the 20 preservice teachers in the study said they had used calculator technology in their schooling, and 13 of 20 reported that computers were mainly used for presentations, drill and practice exercises, or searching for information.
While all of the participants in my study reported using mathematics technologies in their prior mathematics courses, they most commonly reported using them in ways that aligned with the limited conceptions of technology use observed in Wachira et al. (2008). According to responses on the questionnaires, the only way in which all of the interviewed participants reported using mathematics technologies sometimes or often in their secondary school mathematics courses was for checking answers. Five of the six interviewed participants also reported that their teachers used mathematics technologies for demonstration purposes and that students used mathematics technologies for speedy and convenient computation sometimes or often in their secondary school mathematics courses. In contrast, none of the interviewed participants reported using mathematics technologies often for dynamic graphing or drawing or for the in depth exploration of patterns and concepts in their secondary school mathematics courses, and most of them reported that they used mathematics technologies rarely or never for these purposes. Thus, due to a lack of experience using mathematics technologies for the meaningful learning of secondary mathematics content, it may be difficult for participants who view learning as constructivist activity in general to see mathematics technologies as tools that students can use to construct mathematical knowledge.

Views of mathematics learning, which influenced the ways in which Katrina and Lily mathematically resolved output conflicts as described earlier, also seemed to influence Katrina’s and Lily’s perspectives on how they would use technology in their future classrooms. Katrina and Lily were the only participants to spontaneously express the opinion that students should learn to do mathematics by hand before using technology. For example, Katrina said, “I think I will definitely teach by hand first, and then after the students understand the concept then you can put in the calculator.” Since Lily and Katrina were the only participants to bring this up, I asked them whether they would ever consider teaching students to use technology before paper-and-pencil. Lily emphatically said no because:
They’re just going to do that [use technology] all the time. You give them homework, say solve by hand, who’s to take away their calculator or watch them and make sure they are not using technology to solve it . . . I think you have to understand it before you can use technology.

It seemed to me that Lily was equating understanding with the ability to do paper-and-pencil mathematics, so I asked her if she felt that being able to do mathematics by hand was the same as understanding it, and she said:

Um, yes and no. I mean I feel like as a teacher, you are definitely giving your students more tools that way [by teaching them to do it by hand]. You are trying to get them to understand concepts, but then that’s up to the student if they want to grasp those concepts or not.

In this excerpt, Lily seemed to admit that students might do mathematics by hand without understanding it, but the goal of teaching students to perform mathematical procedures by hand was to give them more tools to “understand concepts.” These statements further suggested that Lily believed that knowing how to do paper-and-pencil symbolic manipulation was necessary for mathematical understanding, and they also seemed related to Lily’s perspective about how conflicts could be used in the classroom, to teach students, “They can’t just use technology blindly.”

In contrast, Katrina laid out particular instances when she might use technology before teaching students a paper-and-pencil method. She said, “I think like with doing the derivative and factoring, things that are very computational, that I would do by hand, but maybe to show graphing . . . you need to see what the thing looks like.” She went on to say that technology would help students see the general shape of a particular type of graph that would be difficult to do by plotting a few points. While she saw the value of technology for visualizing graphs, she did not see CAS as valuable for learning symbolic manipulation. Katrina reported having experience with CAS in fewer subject areas than she did with using technology for graphing and creating tables of values (see Table 4-2). She also said in the interview that in high school, graphing and checking answers was “all I really used technology for.” Therefore, it may be difficult for her to
see that just as technology can help students see the general shape of graphs, CAS can be used to help students look for patterns in algebraic structure and perhaps develop procedures from these patterns before learning the procedures by hand. Like Lily, Katrina seemed to think that using technology for symbolic manipulation before mastering it by hand would prevent students from learning paper-and-pencil methods. She said, “When you teach factoring, you shouldn’t just skip right to the computer because students won’t actually know how to factor by hand then.”

It is also possible that both Lily’s and Katrina’s perspectives on making sure students learned symbolic procedures by hand before using technology were related to rule-based views of mathematics learning (Tharp et al., 1997). Interestingly, Tharp et al. (1997) found that teachers with rule-based views were more likely than teachers with non-rule-based views to carefully control the amount and type of use of calculators in their classes, which seemed to be true for Lily and Katrina. My findings also indicate that all but one of my participants expressed a desire to control the technological output that students see at particular times regardless of whether their comments in the interviews indicated a rule-based view of mathematics learning.

Tharp et al. (1997) also found that teachers with a rule-based view of mathematics learning “were more wary than non-rule based teachers of the effects of the calculator on their students and were more likely to feel that graphing calculators might be a hindrance to their students’ mathematics and science learning” (pp. 566-567). Katrina was the only interviewed participant to express a concern that mathematics technologies might hinder learning on the questionnaires. For questionnaire item 34 (see Appendix B), participants were asked to determine with which of two opposing statements they agreed more. Katrina, as well as three participants that were not interviewed, agreed more with the statement, “Mathematics technology can be detrimental to student learning because it does too much of the math for them and should be used sparingly in the classroom.” All other interviewed participants agreed more with the opposing statement which said, “Mathematics technology is very valuable in helping students learn and
understand mathematics concepts.” Lily, Richard, and Walter responded that they strongly agreed
with this latter statement and disagreed with the former. Lily’s response to this item is interesting
because she seemed concerned that technology use might prevent student learning in the
interviews. However, upon closer inspection of her comments, it seemed that her main concern
was that technology would interfere with students’ learning of paper-and-pencil procedures if
they used technology to do them before learning them by hand. Otherwise, she expressed the
opinion that it was important for students to learn both paper-and-pencil and technological skills.

In addition to mathematical views of learning, specific past experiences also seemed to
influence participants’ perspectives on how they would use mathematics technology and
technological conflicts in their future classrooms. Sometimes participants spontaneously shared a
past classroom experience as they talked about their perspectives. For example, Liam was the
only participant that expressed concern about the instructional time it would take for students to
work through technological conflicts, particularly the one that arose between the TI-89 and
GeoGebra when taking the derivative of \( \cos(3x - \pi/6) \) with respect to \( x \). To show \( 3\cos(3x + \pi/3) \)
and \( -3\sin(-1/6 \pi + 3x) \) were equivalent Liam explained, “that could be one class if you are just
writing it all out by hand, and I mean my high school only had 35 minute class periods.” In this
statement, Liam links personal experience with concerns about class time commonly expressed
by other teachers in the literature that have tried to incorporate technology and inquiry learning
into their classrooms (e.g., Almekbel, 2000; Lumb et al., 2000). In contrast, Liam said he would
be more likely to show students the conflict between GeoGebra and Core Math Tools when
taking the derivative of \( \cos(3x - \pi/6) \) with respect to \( x \). Even though Core Math Tools’ output
looks very complex (see Figure 4-2), this would require less time because one only needs to do
some basic simplification and realize that the derivative of \( \pi \) with respect to \( x \) is 0 to get Core
Math Tools’ output to match GeoGebra’s output of \( -3\sin(-1/6 \pi + 3x) \). In this case, all of the
participants confirmed the validity of Core Math Tools’ output by performing this simplification
mentally during the interview, and there was little variation in their methods. Thus, Liam is probably correct in thinking that reconciling Core Math Tools’ and GeoGebra’s output in this case would take little class time. However, this particular conflict may not provide students with as many mathematical learning opportunities as the conflict between the TI-89 and GeoGebra.

\[
\text{\texttt{der}}(\cos(3x-\pi/6),x) = -1 \left( 3 - \frac{d(\pi)}{dx} \right) \sin\left( 3x - 1 \times \frac{\pi}{6} \right)
\]

Figure 4-2: Output from Core Math Tools for the derivative of \(\cos(3x - \pi/6)\) with respect to \(x\).

Lily provided another example of a participant relating a specific past classroom experience to their perspectives on using technological conflicts in their future teaching. When asked if she would consider presenting conflicts to her future students, Lily said yes and then shared that she had a high school mathematics teacher that would show students incorrect or surprising technological results. She explained, “but she didn’t trust technology, so that was her logic behind that.” When asked what she would want her students to learn from encountering conflicts, Lily said, “I would want them to know what technology they were using, and that they can’t just blindly use technology.” Since she mentioned her previous teacher’s goals before describing her own, it seemed that this experience as a student influenced her perspectives on using technological conflicts as a teaching tool.

Additionally, Richard was the only participant who suggested that exposure to technological conflicts would be important for students because those who did not have access to graphing calculators at home might use free software at home and graphing calculators at school, and results might differ between technologies. Although he did not cite any past experiences while sharing this perspective, it is interesting that he was the only participant who had student
teaching experience and the only participant to mention issues of student access to technology at home. Perhaps more experience with students leads preservice teachers to consider issues of technology use more broadly, just as more experience resulted in student teachers considering issues of classroom management more broadly in M. G. Jones and Vesilind (1995).

It was also evident that participants’ particular experiences with the conflicts during the interview influenced their perspectives on whether to use those conflicts with their future students. When participants were able to resolve a conflict with certainty while also gaining insights by making new mathematical connections, they tended to cite that example later in the interview as something that could be used effectively with students. For example, Katrina was the only participant who stated that the expression \( x^{0.2} \) might be interpreted in different ways that could change the graph of the corresponding function. At the end of the interview, she cited the conflict between the graphs of \( f(x) = x^{1/5} \) as an example of a conflict that could “get students thinking more deeply why the math is working that way.” Mickey was the only participant who resolved a conflict by proving two expressions equivalent using trigonometric identities. She did not know whether they were equivalent before using the available trigonometric identities to talk through the proof. She later cited this conflict as a way to help students learn about trigonometric identities. This suggests that participants’ particular experiences with the conflicts influenced their perspectives on whether to present technological conflicts to students. It seemed that if a participant was able to learn something new by resolving a particular conflict, they tended to cite it as helpful for students. Moreover, when participants were unable to resolve a particular conflict with certainty, they did not cite it as helpful for students.
**Question 3 Summary**

These results indicate that beliefs, prior experiences, and current learning experiences can influence preservice teachers’ perspectives on whether and how to present technological conflicts to students. Participants cited benefits of having students encounter technological conflicts, but the benefits they cited varied considerably within this small sample. Thus, it seems that it would be beneficial for preservice teachers to share and discuss their perspectives on presenting conflicts to students. Such discussions could provide preservice teachers with opportunities to examine their beliefs about the pedagogical uses of technology in the classroom and broaden their perspectives on incorporating mathematics technologies in their future teaching.

**Conclusion**

Participants’ mathematical resolution methods contained common elements of recognizing salient similarities and differences, attempting to make sense of questionable output by recalling related facts and procedures, and using facts and procedures or technological evidence to determine the equivalence or validity of outputs. Most of the participants focused on providing consistency in the classroom by controlling the technological output that students see at particular times after experiencing the conflicts, and all said they would consider presenting conflicting output to their future students when asked. Beliefs, attitudes, and prior experiences seemed connected to participants’ resolution methods and their perspectives on presenting conflicts to their future students in some cases, but it was not possible to make connections that hold across all participants or across all tasks.

Because many unique ideas about mathematics, technology use, and pedagogy arose in the comments of only six interviewees, output conflicts between mathematics technologies may
have the potential to promote rich discussions among preservice teachers in teacher education courses. Such discussions can expand preservice teachers’ content knowledge and “conceptions of appropriate technology use” (Wachira et al., 2008). These and other implications are discussed in greater detail in the following chapter.
Chapter 5

Discussion

In this chapter, I summarize the results of this study and discuss their implications. Implications for teacher education, professional development, curriculum materials, and technology development are addressed. I also discuss the limitations of this study and suggest ideas for future studies.

Results and implications

My first research question asked how preservice secondary mathematics teachers mathematically resolved output conflicts between mathematics technologies. I found that participants’ strategies had common elements of recognizing salient similarities and differences, attempting to make sense of unexpected output by recalling and mentioning facts and procedures that they saw as relevant to the conflicts, and using facts and procedures or technological evidence to determine the equivalence or validity of outputs. Though there were elements common to participants’ resolution strategies, considerable variation appeared in the mathematics and the technological features used to resolve conflicts within a small sample. It seemed that each participant had at least one perspective on output conflicts that was unique within the group of six interviewees. For example, only Katrina used routine algebraic procedures to explain that multiple interpretations of the expression $x^{0.2}$ were possible, and only Liam attempted to test the equivalence of two expressions by dividing them using computer algebra systems (CAS) to see if the output was 1.
It is notable that many different mathematical and technological ideas surfaced during the interviews within a small number of participants enrolled in upper level courses of the same teacher education program. This finding indicates that having preservice teachers work through conflicts on their own and share their strategies and conclusions with their peers could promote rich discussions in teacher education courses. Such discussions could help preservice teachers develop important content knowledge and broaden their perspectives on classroom technology use. For example, learning about different ways in which their peers resolved conflicts between outputs could lead to discussion of the mathematics involved and awareness of pedagogical opportunities provided by mathematics technologies, such as exploiting the contrast between machine results and ideal mathematics described by Pierce and Stacey (2010).

Furthermore, the same types of discussions could be helpful for practicing teachers. Since practicing teachers tend to be more diverse in their levels of education and experience than preservice teachers, there is the potential for more perspectives on mathematics content and technology use to surface in their discussions of conflicting output. Therefore, it could be useful to incorporate discussions of output conflicts between mathematics technologies into professional development activities about the use of mathematics technologies. Professional development providers may need to carefully consider how to guide discussions involving practicing teachers. While their greater diversity may generate more ideas about using technology, it may be more difficult to keep the conversation focused on creative ways to integrate technology that would enhance students’ mathematical understanding. Discussing the potential for surprising or conflicting results from mathematics technologies and their pedagogical opportunities in the professional development stage could provide teachers with ideas for using surprising or conflicting results for pedagogical purposes and lower the likelihood that teachers would avoid using mathematics technology simply because they do not know how to address unexpected output.
My second research question asked what perspectives preservice secondary mathematics teachers have on using mathematics technologies and conflicting technological output in their future classrooms after experiencing technological conflicts themselves. I found that five of the six participants discussed ways in which they would control the technological output that students see when asked what they “took away” from the experience. Katrina, Lily, and Richard talked about the need for teachers to make sure that technological output was consistent with what they wanted their students to learn in a particular lesson. Mickey and Liam said that they would choose one technology for primary use in the classroom to avoid inconsistencies. Although all participants said that they would consider “presenting conflicts” to students when asked, they did not mention using conflicts in their classroom until prompted. The phrase “presenting” may have allowed them to think about using conflicts pedagogically while still maintaining control as the teacher.

Participants’ desire to provide consistency by controlling the technological output that students see could be related to a desire to reduce the complexity of the teaching environment, as described in Doyle (1986) and M. G. Jones and Vesilind (1995). Since teachers already simultaneously focus on presenting content clearly, guiding student discussions, and maintaining student attention and appropriate behavior, surprising technological results can further complicate this juggling act. Concerns about undesirable complexity surfaced in some participants’ comments at the end of the interviews. For example, after Richard commented that the teacher should make sure that the technological output that students see is consistent with the goals of the lesson, he said, “You don’t want to be surprised.” He explained that being surprised was undesirable because if an unexpected result that the teacher is unprepared for appeared in class, the teacher would potentially have to field questions from students that did not pertain to the goals of their lesson. Another participant, Mickey, worried that it would be difficult to grade student assignments if their answers were incorrect because of differences in technological
output. These responses seemed indicative of participants seeing conflicting output that they did not control as a complication to avoid. Only Walter seemed unconcerned about surprising technological conflicts. He said, “If we were using a tool and a discrepancy comes up . . . I wouldn’t want to just buzz past it.” He went on to say that he would want to get students’ opinions about the conflicting output and to ask students why they think technologies give answers in different forms.

If teacher educators and curriculum developers want teachers to use technology in ways that enhance students’ learning of mathematics and software developers want their technology to be used, teachers’ concerns about managing technology use, including how to deal with surprising or conflicting results, need to be addressed. Software and curriculum developers might provide more guidance about how to manage software use in the classroom and provide examples in which the output from software might differ from conventional paper-and-pencil results that teachers could use to prompt student discussion. Curriculum developers might include activities or questions designed to help students make sense of surprising technological results. This does not necessarily mean that examples or activities would have to focus on specific software. Questions could be posed that asked students to investigate the output of the tools available to them. For example, students might be asked to investigate whether or how available CAS automatically simplify symbolic expressions or how available commands rewrite expressions in different forms. When CAS in not available, students might be asked to investigate how their tools display graphs for functions such as $f(x) = x^{1/5}$ and $g(x) = x^{0.2}$. Alternatively, curriculum materials could state that some technologies display positive and negative input values when graphing these functions while others only display positive input values and ask students to discuss how functions of this type should be graphed and why.

Additionally, teacher educators and professional development providers might challenge teachers and preservice teachers to think about how their learning might be similar to or different
from their future students’ learning and how they might best manage technology integration in a classroom setting to benefit their students. Examples of rich questions about technological output and examples of how to manage technology use from teacher educators, software developers, or curriculum materials could help teachers see technological differences, which are inevitable with the increased availability of different mathematics technologies, as pedagogical opportunities that can enhance students’ mathematical understanding and not as reasons to avoid technology or limit students’ technology use. This is not to say that limiting students’ technology is not sometimes appropriate. However, teachers and preservice teachers should be encouraged to ask themselves how placing limits on technology use is productive or counterproductive to the goal of helping students to learn mathematics or learn technological skills that will be important for learning and doing mathematics in the future.

My third research question asked how participants’ reactions to conflicting technological output were related to their prior experiences and beliefs. The following characteristics of participants were assessed by questionnaires: prior experience with mathematics technologies as learners, anxiety toward mathematics technology use, enthusiasm for mathematics technology use, value of mathematics technology for teaching and learning, and traditional views of teaching and learning. The questionnaire responses seemed connected to preservice teachers’ reactions in some cases. For example, Liam was very enthusiastic about technology use and valued it highly for teaching and learning mathematics according to his questionnaire responses, and he attempted to use technology more often than other participants to resolve output conflicts during the interview. However, it was difficult to connect questionnaire responses to participants’ reactions in the interview across all participants or across all tasks.

Comments that participants made during the interviews suggested that beliefs about mathematics that were not assessed by the questionnaires and individuals’ specific experiences might have had more influence on participants’ reactions than the beliefs, attitudes, and
experiences assessed by the questionnaires. For example, Lily’s and Katrina’s comments about valuing by-hand procedures seemed connected to their instrumental approaches to resolving the conflicts and their perspective of wanting to teach students paper-and-pencil procedures before using technology for similar tasks. Additionally, some participants connected their perspectives to specific prior experiences by mentioning a past experience when sharing their opinions. For example, Liam indicated that he would likely only present conflicts to students that they could resolve quickly because he was concerned about the instructional time required for discussion, and he also shared that his high school class periods were only 35 minutes.

At times, participants’ experiences with the specific conflicts seemed to influence their perspectives on using them with students. For example, Katrina said that the conflicts between the graphs of $g(x) = x^{0.2}$ could help students think about mathematics more deeply after seeing that there was more than one way to interpret the expression $x^{0.2}$ and that different interpretations could lead to different graphs of the corresponding function. Additionally, Mickey was the only participant to determine that $3\cos(3x + \pi/3)$ and $-3\sin(-1/6 \pi + 3x)$ were equivalent by manipulating trigonometric identities during the interview. She later cited this conflict as one that she might use with students because “it helped show that you had to use identities to show that they were equivalent. It would help students learn more about identities and why they necessarily are there.” These last two examples suggest that when resolving conflicts led to learning something new about mathematical relationships, participants tended to cite that conflict as helpful for students. Such comments by participants may also suggest participants were thinking about specific examples of conflicts by the end of the interviews but not yet reasoning abstractly about how technological conflicts might be used for teaching and learning. Abstract reasoning of this nature would require more time and perhaps discussion with others. If preservice teachers share what they learn by resolving conflicts on their own, their peers may also learn new mathematical connections that affect their perspectives on using technology with students.
Although it was difficult to connect general categories of attitudes and beliefs directly to participants’ reactions to technological conflicts, conversations with preservice teachers, as described in the previous two paragraphs, indicated that some prior experiences and beliefs had some influence on their mathematical resolution strategies and their perspectives on whether to present technological conflicts to students. This observation suggests that preservice teachers might benefit from opportunities to share and reflect on prior experiences that have shaped their beliefs. Furthermore, it is important for teacher educators to address preservice teachers’ beliefs and assumptions about technology use as they relate to learning mathematics. Richardson (2003) pointed out that preservice teachers’ prior experiences and beliefs affect what they will learn from teacher education programs. In order to change counterproductive beliefs about using technology for teaching and learning mathematics, teacher educators might further emphasize to preservice teachers the importance of learning to reflect on beliefs and how past experiences may have shaped those beliefs, as the ability to be reflective has been linked to teacher change (Wilson & Cooney, 2002). Preservice teachers’ reflections in teacher education courses could continue to be a rich source of data for further research on beliefs or belief change.

Limitations

Only six preservice teachers from the same teacher preparation program were interviewed in this study, which limits the generalizability of the results. I found that participants tended to use facts and procedures to resolve conflicts when they could do so mentally and with certainty and that when participants used technology to mathematically resolve conflicts they tended to generate or refer to graphs more frequently than they generated or referred to symbolic or tabular output. I found that all but one participant focused on controlling technological output that students see in order to maintain consistency when asked what they “took away” from the
experience. A larger and better chosen sample is needed to know if these reactions are typical of the broader population of preservice secondary mathematics teachers or if they are specific to the participants in my sample and if other reactions, not observed in this study, are common. I also found that all participants said that they would consider presenting conflicts to their future students when asked, but since they did not suggest that they would do this until prompted, it is possible that the nature of this study suggested to them that an affirmative response to this question was the “correct” response.

Additionally, links made between participants’ reactions to conflicting output and their prior experiences and beliefs were often made using comments that participants offered spontaneously during the interviews. I cannot claim that participants who reacted differently do not share the same opinions or experiences simply because they did not spontaneously offer them. For example, Liam’s concerns about time seemed connected to his experience of having 35 minute classes in high school, but I do not know that others that did not mention concerns about time did not also have short class periods in high school. Furthermore, Katrina and Lily spontaneously mentioned their value of paper-and-pencil procedures several times, which seemed to indicate a rule-based view of mathematics learning (Tharp et al., 1997). These views seemed to affect the ways in which they resolved conflicts. However, I do not know that others did not share their opinions; I only know that others did not mention their opinions on these matters.

While designing this study, I expected to be able to link responses to questionnaire items designed to assess participants beliefs, attitudes, and prior experience to participants’ reactions to conflicting output. However, I was not able to do this across participants or across tasks for self-developed or for previously developed and tested questionnaire items. This suggests that there may be a problem with the questionnaire items themselves or in the way I was attempting to the use the questionnaire items. Perhaps reaction to conflicting technological output is a phenomenon that is too specific to link to general attitudes and beliefs about technology, teaching, and
learning. Teachers’ beliefs and practice can seem inconsistent when contexts are not considered (Beswick, 2005; Philipp, 2007), and preservice teachers’ responses to questionnaires may vary based on the context in which a question is posed to them (Kaufman & Moss, 2010).

Further limitations of the questionnaires were evident. For example, asking students to list courses in which they had used technology to graph functions did not give me any information about how frequently they used technology for this purpose. For example, I did not know whether participants used the technology for this purpose regularly or if they were just shown commands once during a course. More detailed questions about how technology was used to graph functions may have been more helpful. Furthermore, the items that asked participants to rate their level of agreement with particular statements proved difficult to analyze. One problem is that it is not possible to know whether answers such as “strongly disagree” and “moderately disagree” differ in the same way as “slightly disagree” and “slightly agree” from item to item or from participant to participant. It is also impossible to know whether each participant has similar criteria for strongly agreeing with a particular statement and whether these criteria change based on the content of the item. Similar problems existed for items that asked participants to rate the frequency with which technology was used for particular purposes. “Often” or “sometimes” might mean different things to different participants, and one participant’s criteria for “often” might change based on what they perceive as typical. For example, if a participant used technology to generate graphs daily in high school algebra, they may report that they used technology to generate graphs rarely in college when they only did so monthly. On the other hand, if they never used technology to symbolically manipulate expressions in high school algebra, they may rate a calculus class that used technology for this purpose monthly as using technology to perform symbolic manipulation often. Since many assumptions must be made to compare questionnaire responses of this nature, findings based on questionnaire results are limited.
Ideas for future research

One reason I investigated preservice secondary mathematics teachers’ reactions to conflicting output between mathematics technologies was because I wanted to know what mathematical and technical knowledge they would use to interpret unexpected output and resolve output conflicts. In order to better understand how to teach students to use and interpret technological output, it would be helpful to study secondary students’ reactions to similar output conflicts. Prior studies have investigated student reactions to technological output that conflicted with their paper-and-pencil results (e.g., Guzmán et al., 2010; Lagrange, 1999; Tonisson, 2013), but students’ reactions may be different when the outputs of technologies conflict, particularly if students trust technological output. Thus, conflicts between technological outputs may provide different learning opportunities than surprising results from one technology. It would also be interesting to know whether conflicting output between mathematics technologies would promote productive discussions among secondary school students or if they would simply ignore or dismiss conflicts as technological glitches. Such research may also reveal more insights about the characteristics that prevent or allow students to successfully resolve conflicts between mathematics technologies.

Encountering conflicting technological output in the classroom may also provide opportunities for students to work with and learn about multiple mathematics technologies and to make informed choices about which technologies to use. Often, schools adopt one particular brand of calculator, or curriculum materials structure technological activities such that they provide instructions for one particular calculator or software. It would be interesting to compare students who were limited to one mathematics technology for each type of task throughout high school to those who used multiple mathematics technologies throughout high school for the same tasks to see if one group had an easier time than the other utilizing available technologies or
transitioning to the use of new technologies later. Results of studies of this nature could help provide information about whether and how technology choices affect students’ abilities to “use appropriate tools strategically,” one of the mathematical practices that the Common Core State Standards for Mathematics (CCSSM) deem necessary for mathematical proficiency (NGA Center & CCSSO, 2010). This could inform teachers and school districts as they make decisions about the technology that is available to their students in the classroom.

Furthermore, teachers’ concerns and beliefs must be addressed if teacher educators and technology developers want teachers to use technology productively to help students learn mathematics. More information about why teachers choose to use technology and why they choose to avoid technology is needed beyond linking responses to prewritten questionnaire items to self-reported classroom activities or intentions. Rather than trying to connect responses to pre-written questionnaire items to teachers’ actions or intentions, it might be helpful for researchers to directly ask teachers what experiences they attribute to their beliefs about technology and their decisions about how they use it in their classrooms. Preservice teachers’ written reflections on topics such as how their beliefs about technology relate to their prior experiences can also offer rich sources of data. When using data of this nature, it is still possible that influential experiences and beliefs would not be reported, but influential experiences and strong beliefs would not be missed because they were not assessed.

It would also be interesting to know whether the experiences teachers attribute to their beliefs about technology use are similar to the characteristics assessed on widely used instruments, such as the Teachers’ Attitudes Toward Computers (TAC version 6) questionnaire (Christensen & Knezek, 2009). Such information could help researchers develop better instruments for assessing teachers’ beliefs or help them better understand the limitations of questionnaires and surveys. This could lead to better instruments, which could lead to more accurate results from research. This information could also help curriculum and technology
developers design better tools and activities that address teachers’ concerns, thereby reducing the likelihood that teachers will avoid or limit students’ technology use in mathematics classrooms in ways that limit students’ learning opportunities.

Summary

This study provides information about preservice secondary mathematics teachers’ reactions to output conflicts between mathematics technologies. While resolving conflicts, participants in this study recognized salient similarities and differences between outputs, attempted to make sense of unexpected output by recalling mathematical facts and procedures that they saw as relevant to the conflicts, and used facts and procedures or technological evidence to determine the equivalence or validity of outputs. A variety of mathematical and technological strategies arose within these common elements.

Experiencing these conflicts focused most participants on controlling the technological output that students see in order to provide consistency in the classroom, although all participants said they would consider presenting conflicting output to their future students when asked. While prior experiences and beliefs seemed connected to participants’ reactions to conflicts in some cases, it was not possible to connect prior experiences and beliefs to reactions across all participants or across all tasks.

These results suggest that reactions within a small sample of preservice teachers vary enough to provide opportunities for rich discussions in teacher education courses or professional development activities that can deepen content knowledge and broaden perspectives on technology use. The results also suggest that the questionnaire items used in this study may not be helpful in predicting preservice teachers’ reactions to specific situations regarding technology use. Additionally, the results raise questions about how students might react to conflicting output
between mathematics technologies, whether students would react differently to output conflicts between mathematics technologies than to output that surprises them because it differs from conventional forms or paper-and-pencil results, and how learning to use multiple technologies would affect students’ mathematics and technical skills or their ability to learn to use new technologies in the future. As more mathematics technologies become available and mathematics technologies become standard tools for doing and learning mathematics, interpreting unexpected output and being able to resolve output conflicts between mathematics technologies will be important. Carefully chosen examples of conflicting output could provide new ways of helping students and teachers understand mathematical concepts at a deeper level while becoming more effective and strategic users of technology.
References


Appendix A

Recruitment Materials

Recruitment email sent to mathematics education majors

Subject line: opportunity to participate in math education research

Dear future secondary math teacher,

My name is Sheri Stayton and I am a graduate student in mathematics education. I am looking for volunteers who are planning to teach secondary mathematics to participate in my thesis research.

Summary:
I am trying to understand how preservice secondary mathematics teachers think about output from different mathematics technologies like graphing calculators or GeoGebra and how this relates to their prior experiences and beliefs.

Participation Requirements:
To participate you must be at least 18 years old and have completed or be enrolled in a 400-level mathematics education course.

What’s Involved?
Participating will involve filling out an on-line consent form and two short on-line questionnaires that will ask about your past experiences with and beliefs about mathematics technology. This is estimated to take between 12 and 20 minutes. Additionally, I will be inviting some of those who participate in the questionnaire to do a 90-minute task-based interview where they will use and react to some different mathematics technologies. Prior knowledge of the technology is not required, and the mathematics involved will be drawn from high school content. Participating in the questionnaire does not obligate you to participate in the interview. Only I will be able to connect your real name with your responses.

Benefits to Participants:
Filling out the questionnaires will give you the opportunity to reflect on your past experiences with and beliefs about mathematics technologies, which may help you think about how mathematics technologies fit into your philosophy of education and your future teaching. If you are asked to participate in the interview, you will have the opportunity to learn some new things about how different mathematics technologies work and display output differently. Thinking and talking about the mathematics content and the technology used in the interview should be helpful and relevant to your future teaching experiences. Because the interview will take considerably more time than the questionnaires, those who are invited and who complete the interview will receive a $10 Amazon gift card.
How to Participate:
If you are currently enrolled in a 400-level mathematics education course at [Penn State], I will be visiting these classes to provide you with more information and instructions to proceed if you wish to participate. If you are not enrolled in a 400-level mathematics education course this semester, but have already completed at least one of these courses and are interested in participating, please email me at sns180@psu.edu for further instructions.

Your participation is voluntary and would be very much appreciated. Your decision to participate will have no effect on your course grades.

Sincerely,
Sheri Stayton

Recruitment script read to 400-level mathematics education classes

My name is Sheri Stayton and I am a graduate student in mathematics education. I am here today to ask you for help with my research for my master’s thesis. I am trying to understand how preservice secondary mathematics teachers think about output from different mathematics technologies such as graphing calculators or GeoGebra and how this is related to their prior experiences and beliefs. I am looking for volunteers to fill out two short on-line questionnaires about their experiences and beliefs about mathematics technology. This is expected to take between 12 and 20 minutes.

Within two weeks after the questionnaires are completed, I will be inviting 9 of those who completed the questionnaires to do an audio recorded, 90-minute task-based interview where they will use and react to output from different mathematics technologies. All of the math involved will be drawn from high school content. Prior experience with the technology is not required. Agreeing to do the on-line questionnaires does not guarantee that you will be asked to participate in the interviews, nor does it obligate you to do so if asked. In fact, you may withdraw at any time.

Filling out the questionnaires will give you the opportunity to reflect on your past experiences with and beliefs about mathematics technologies, which may help you think about how mathematics technologies fit into your philosophy of education and your future teaching. If you are asked to participate in the interview, you will have the opportunity to learn some new things about how different mathematics technologies work and display output differently. Thinking and talking about the mathematics content and the technology used in the interview should be helpful and relevant to your future teaching experiences. Because the interview will take considerably more time than the questionnaires, those who are invited and who complete the interview will receive a $10 Amazon gift card.

It is important that I have participants that have various opinions about and experience with technology. So whether you love, hate, or are indifferent toward mathematics technologies, your participation will be very helpful to me.
If you are interested in participating, you simply need to use the link on the card that I will be handing out, read and complete the on-line consent form, and then follow the links on the consent form to the questionnaires. You can complete the questionnaires at any time that is convenient for you within the next week. You will be providing a pseudonym on the consent form that you will use on the questionnaires. I will be the only one that will be able to connect your responses to your real name. Your real names will not be used in any reporting of my results.

My contact information will also be on the card I will be handing out. If you have any questions, feel free to ask me now or contact me later.

Information card handed out during classroom visits

Title of Project: Conflicting Output Between Mathematics Technologies: How Preservice Secondary Mathematics Teachers' Reactions Relate to Their Past Learning Experiences and Beliefs

Principal Investigator: Sheri Stayton
Telephone Number: 570-768-0165
Email: sns180@psu.edu

To begin the process, go to this link to read and complete the on-line consent form: https://www.surveymonkey.com/s/7SPYZLT

Then follow the instructions on the consent form to get to and complete the questionnaires.
Appendix B

Questionnaire Items and Directions

When answering this questionnaire, please keep in mind that mathematics technology specifically refers to technology that is used to perform mathematical tasks. This includes but is not limited to calculators, spreadsheets, dynamic geometry software, and software or calculators with graphing and/or symbolic manipulation capabilities. Mathematics technology does not refer to technology like word processors, presentation software, videos, or Internet search engines.

1. As a student, which of the following best describes your experience with mathematics technologies (calculators/mathematics software) in your college mathematics courses up to this point?
   - Mathematics technologies were rarely used as a learning tool in my college mathematics courses.
   - Mathematics technologies were sometimes used as a learning tool in my college mathematics courses.
   - Mathematics technologies were used as a learning tool about half of the time in my college mathematics courses.
   - Mathematics technologies were used as a learning tool most of the time in my college mathematics courses.

2. As a student, which of the following best describes your experience with mathematics technologies (calculators/mathematics software) in your secondary (grades 7-12) mathematics courses up to this point?
   - Mathematics technologies were rarely used as a learning tool in my secondary (grades 7-12) mathematics courses.
   - Mathematics technologies were sometimes used as a learning tool in my secondary (grades 7-12) mathematics courses.
   - Mathematics technologies were used as a learning tool about half of the time in my secondary (grades 7-12) mathematics courses.
   - Mathematics technologies were used as a learning tool most of the time in my secondary (grades 7-12) mathematics courses.

3. As a student, in which of your mathematics courses did you use mathematics technologies to graph functions? In the appropriate blanks below, list the courses in which this use occurred to the best of your recollection (e.g., Algebra 1, Calculus, Statistics). Write “none” if you have not used technology to graph functions in one or both of these settings.

   Grades 7-12
   College
4. As a student, in which of your mathematics courses did you use mathematics technologies to generate a table of values for a function? In the appropriate blanks below, list the courses in which this use occurred to the best of your recollection (e.g., Algebra 1, Calculus, Statistics). Write “none” if you have not used technology to generate a table of values for a function in one or both of these settings.

Grades 7-12_____________________________________________________________
College________________________________________

5. As a student, in which of your mathematics courses did you use mathematics technologies to symbolically manipulate equations or expressions? An example would be using a factor command and getting output in algebraic form. In the appropriate blanks below, list the courses in which this use occurred to the best of your recollection (e.g., Algebra 1, Calculus, Statistics). Write “none” if you have not used technology to symbolically manipulate equations or expressions in one or both of these settings.

Grades 7-12_____________________________________________________________
College________________________________________

Indicate how much you agree or disagree with the following statements by selecting one level of agreement (SD = strongly disagree, MD = moderately disagree, SLD = slightly disagree, SLA = slightly agree, MA = moderately agree, SA = strongly agree).

6. I get a sinking feeling when I think of trying to use mathematics technologies.
7. Working with mathematics technologies makes me feel tense and uncomfortable.
8. Working with mathematics technologies makes me nervous.
9. Mathematics technologies intimidate me.
10. Using mathematics technologies is very frustrating.
11. I think that working with mathematics technologies would be enjoyable and stimulating.
12. I want to learn a lot about mathematics technologies.
13. The challenge of learning about mathematics technologies is exciting.
15. I can learn many things when I use mathematics technologies.

The previous items asked about your experiences and opinions as a student. These items ask about your beliefs as a future educator.

Indicate how much you agree or disagree with the following statements by selecting one level of agreement (SD = strongly disagree, MD = moderately disagree, SLD = slightly disagree, SLA = slightly agree, MA = moderately agree, SA = strongly agree).

16. I believe that I will be a better educator when I use mathematics technology for my work.
17. I believe that using mathematics technology in class activities will be time well spent.
18. I believe that students will be more motivated when they use mathematics technology for assignments.
19. I believe that when students use mathematics technology they will create products that show high levels of learning.
20. I believe that the positive effects of mathematics technology on my students will outweigh any negative effects such use might have.
21. Students are not ready for meaningful learning until they have acquired basic reading and math skills.
22. Student projects often result in students learning all sorts of wrong knowledge.
23. Instruction should be built around problems with clear, correct answers, and around ideas that most students can grasp quickly.

24. How much students learn depends on how much background knowledge they have—that is why the teaching of facts is so necessary.

The following items ask about different ways that *mathematics technologies* might be used in a mathematics class. Select the frequency that best matches your experiences in your college and then secondary math courses as a student.

25. Did your teachers use *mathematics technologies* for demonstration purposes in your math courses?

<table>
<thead>
<tr>
<th>College:</th>
<th>Never</th>
<th>Rarely</th>
<th>Sometimes</th>
<th>Often</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grades 7 -12:</td>
<td>Never</td>
<td>Rarely</td>
<td>Sometimes</td>
<td>Often</td>
</tr>
</tbody>
</table>

26. Were *mathematics technologies* used to check answers in your math courses?

<table>
<thead>
<tr>
<th>College:</th>
<th>Never</th>
<th>Rarely</th>
<th>Sometimes</th>
<th>Often</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grades 7 -12:</td>
<td>Never</td>
<td>Rarely</td>
<td>Sometimes</td>
<td>Often</td>
</tr>
</tbody>
</table>

27. Were *mathematics technologies* used to explore multiple representations of a concept in your math courses?

<table>
<thead>
<tr>
<th>College:</th>
<th>Never</th>
<th>Rarely</th>
<th>Sometimes</th>
<th>Often</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grades 7 -12:</td>
<td>Never</td>
<td>Rarely</td>
<td>Sometimes</td>
<td>Often</td>
</tr>
</tbody>
</table>

28. Were *mathematics technologies* used to provide dynamic graphing or drawing (with dragging or sliders) in your math courses?

<table>
<thead>
<tr>
<th>College:</th>
<th>Never</th>
<th>Rarely</th>
<th>Sometimes</th>
<th>Often</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grades 7 -12:</td>
<td>Never</td>
<td>Rarely</td>
<td>Sometimes</td>
<td>Often</td>
</tr>
</tbody>
</table>

29. Were *mathematics technologies* used by students for speedy and convenient computation in your math courses?

<table>
<thead>
<tr>
<th>College:</th>
<th>Never</th>
<th>Rarely</th>
<th>Sometimes</th>
<th>Often</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grades 7 -12:</td>
<td>Never</td>
<td>Rarely</td>
<td>Sometimes</td>
<td>Often</td>
</tr>
</tbody>
</table>

30. Were *mathematics technologies* used to aid in solving problems or modeling math in real-life contexts in your math courses?

<table>
<thead>
<tr>
<th>College:</th>
<th>Never</th>
<th>Rarely</th>
<th>Sometimes</th>
<th>Often</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grades 7 -12:</td>
<td>Never</td>
<td>Rarely</td>
<td>Sometimes</td>
<td>Often</td>
</tr>
</tbody>
</table>

31. Were *mathematics technologies* used to make or test conjectures in your math courses?

<table>
<thead>
<tr>
<th>College:</th>
<th>Never</th>
<th>Rarely</th>
<th>Sometimes</th>
<th>Often</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grades 7 -12:</td>
<td>Never</td>
<td>Rarely</td>
<td>Sometimes</td>
<td>Often</td>
</tr>
</tbody>
</table>
32. Were mathematics technologies used for in depth exploration of patterns and concepts in your math courses?

<table>
<thead>
<tr>
<th>College</th>
<th>Never</th>
<th>Rarely</th>
<th>Sometimes</th>
<th>Often</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grades 7 -12</td>
<td>Never</td>
<td>Rarely</td>
<td>Sometimes</td>
<td>Often</td>
</tr>
</tbody>
</table>

The previous items asked about your experiences as a student. These last items ask about your beliefs as a future educator.

There are three pairs of opposing viewpoints below. Read both statements for each pair before responding. Select the answer that indicates how close your beliefs are to a particular statement. (SAS1 = Strongly agree with statement 1 and disagree with statement 2, AS1 = Agree more with statement 1 than statement 2, U = Unsure which statement I agree with more, AS2 = Agree more with statement 2 than with statement 1, SAS2 = I strongly agree with statement 2 and disagree with statement 1).

33. Statement 1: I mainly see my role as a facilitator. I try to provide opportunities and resources for my students to discover or construct concepts for themselves.

Statement 2: That’s all nice but students really won’t learn the subject matter unless you go over the material in a structured way. It’s my job to explain, to show the students how to do the work and to assign specific practice.

34. Statement 1: Mathematics technology is very valuable in helping students learn and understand mathematics concepts.

Statement 2: Mathematics technology can be detrimental to student learning because it does too much of the math for them and should be used sparingly in the classroom.

35. Statement 1: Using mathematics technology in the classroom saves time by helping students do parts of problems quickly and check their answers.

Statement 2: Using mathematics technology in the classroom is often not worth the instructional time it takes for students and teachers to become fluent users.
Appendix C

Interview Protocol

Greeting and introduction

Good morning (or afternoon). I want to thank you for your participation in my study. Is it okay with you if I record the audio for this interview?

My goal for this study is to find out how preservice secondary mathematics teachers interpret the output from different mathematics technologies. This interview will be audio recorded so that I can better analyze your responses, but only my advisor and I will listen to these recordings. I will not use your real name when reporting my results. Since I am trying to study your thinking and interpretation of technological output, it would be helpful if you could describe what you are thinking and doing as much as possible. I may frequently ask you to clarify a statement or tell me more about what you are thinking. If I do this, it does not indicate that you are right or wrong. I am simply trying to make sure I understand your thinking.

Though you were asked to participate in this study in your MTHED _______ course, your responses or participation in this interview will not have any effect on your grades.

Do you have any questions before we begin?

Today we will be working with the TI-89, GeoGebra, and Core Math Tools. Have you had experience with any of these technologies?

If they say they have had experience ask:
In what ways have you used these technologies in the past?
Would you say that you are a beginning, intermediate or advanced user?

I am going to have you use these three technologies for different types of tasks. We will be using tables of values, graphs, and commands for factoring and taking the derivative symbolically. Before each type, I will show you a similar example so that you can see how to use the necessary commands for each of the technologies. I am not assuming that you are familiar with all of these technologies. If you have any questions about using the technology or if you would like to do something with the technology that you do not know how to do during any part of this interview, feel free to ask me. I am also not assuming that you have derivative rules or trigonometric identities memorized. I have a list of these available here, and you should feel free to consult them.
Part 1: Symbolic output for factoring

All of these technologies have Computer Algebra System capabilities, often called CAS. We are going to start by looking at the output from the factor command. Have you used CAS for symbolic manipulation in any of these technologies before?

If they say yes, ask:
   In what ways have you used CAS before?

I am going to have you factor $x^2 + 5x + 4$ to show you how the factor commands work.

Explain how to find and/or type the factor command for each of the technologies if they do not know how to do so.

Output will be:

<table>
<thead>
<tr>
<th>TI-89 Titanium</th>
<th>GeoGebra</th>
<th>Core Math Tools</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Factor" /></td>
<td>Factor[$x^2+5x+4$] $\rightarrow (x+1)(x+4)$</td>
<td>$\nabla$ factor($x^2+5x+4$) $(x+1)^*(x+4)$</td>
</tr>
</tbody>
</table>

Do you have any comments or questions about how this command works for each of the technologies?

Now I would like you to use each of the technologies to factor $x^2 - 5x + 4$.

Output will be:

<table>
<thead>
<tr>
<th>TI-89 Titanium</th>
<th>GeoGebra</th>
<th>Core Math Tools</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image2.png" alt="Factor" /></td>
<td>Factor[$x^2-5x+4$] $\rightarrow (x-4)(x-1)$</td>
<td>$\nabla$ factor($x^2-5x+4$) $(1-x)^*(4-x)$</td>
</tr>
</tbody>
</table>

Look at the three outputs. How are they similar or different?

If the interviewee does not say anything about the equivalence of the outputs, ask:
   Do you think these outputs are mathematically equivalent?

After the interviewee’s opinion about the equivalence of the outputs is established, ask the following referring specifically to what they identified as equivalent or not equivalent:
   How would you convince yourself that _______ are equivalent (or not equivalent)?

Next, I would like you to factor $x^2 - 3x - 4$ using each of the technologies.
Look at the three outputs. How are they similar or different?

If the interviewee does not say anything about the equivalence of the output, ask:

Do you think these outputs are mathematically equivalent?

After the interviewee’s opinion about the equivalence of the outputs is established, ask the following referring specifically to what they identified as equivalent or not equivalent:

How would you convince yourself that _______ are equivalent (or not equivalent)?

Can you think of another way that you could convince someone that these outputs from either factoring example are equivalent (or not equivalent) other than what you have already explained?
Part 2: Tabular output

Next we are going to generate a table of values for a function with each of these technologies.

First, I am going to have you generate a table of values for the function \( f(x) = x^2 + 1 \) as an example.

Explain how to bring up the table of values on the TI-89 and Core Math Tools and how to create a spreadsheet in GeoGebra that will display the \( x \) and \( y \)-values of a function if they do not know how to do so.

Output will be:

<table>
<thead>
<tr>
<th>TI-89 Titanium</th>
<th>GeoGebra</th>
<th>Core Math Tools</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>5</td>
</tr>
<tr>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
</tr>
</tbody>
</table>

Do you have any comments or questions about how to generate a table of values for each of the technologies?
Next, I would like you to generate a table of values for the function \( f(x) = \frac{1}{x} \).

Output will be:

<table>
<thead>
<tr>
<th>TI-89 Titanium</th>
<th>GeoGebra</th>
<th>Core Math Tools</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="TI-89 Titanium" /></td>
<td><img src="image2.png" alt="GeoGebra" /></td>
<td><img src="image3.png" alt="Core Math Tools" /></td>
</tr>
</tbody>
</table>

Look at the three outputs. How are they similar or different?

If the interviewee does not talk about the “undef” versus “\( \infty \)” output, ask:
What do you notice here? (Point to the rows in the tables where \( x = 0 \).)

If the interviewee only points out the differences in the displays without offering any opinion about the meanings of undefined and infinity, ask:
Do you think infinity and undefined are mathematically equivalent?

After the interviewee’s opinion about the equivalence of the output is established, ask:
How do you know that they are equivalent (or not equivalent)?

Can you think of another way that you could convince someone that these outputs are equivalent (or not equivalent) other than what you have already explained?
Part 3: Graphic output

Next, we are going to look at some graphs of functions. First, I am going to have you generate the graph of the function $f(x) = x^{1/2}$ as an example.

Explain how to generate these graphs if they do not already know how to do so.

Output will be:

<table>
<thead>
<tr>
<th>TI-89 Titanium</th>
<th>GeoGebra</th>
<th>Core Math Tools</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="TI-89 Titanium Graph" /></td>
<td><img src="image2.png" alt="GeoGebra Graph" /></td>
<td><img src="image3.png" alt="Core Math Tools Graph" /></td>
</tr>
</tbody>
</table>

Do you have any comments or questions about how to generate a graph for each of the technologies?

Next, I would like you to generate the graph for the function $f(x) = x^{1/5}$ using each of the technologies.

Output will be:

<table>
<thead>
<tr>
<th>TI-89 Titanium</th>
<th>GeoGebra</th>
<th>Core Math Tools</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image4.png" alt="TI-89 Titanium Graph" /></td>
<td><img src="image5.png" alt="GeoGebra Graph" /></td>
<td><img src="image6.png" alt="Core Math Tools Graph" /></td>
</tr>
</tbody>
</table>

Look at the three outputs. How are they similar or different?

Now, I would like you to generate the graph of $g(x) = x^{0.2}$ using each of the technologies. If both this function and the previous are entered, each of these technologies allows you to select or deselect the graph that you want to see.
(Explain how to select or deselect which graphs to display and/or how to show the path of the 2nd graph on the TI-89 if they do not know how to do so).

Now, I would like you to adjust the settings to view only the outputs for \( g(x) = x^{0.2} \).

Output will be:

<table>
<thead>
<tr>
<th>TI-89 Titanium</th>
<th>GeoGebra</th>
<th>Core Math Tools</th>
</tr>
</thead>
<tbody>
<tr>
<td>![TI-89 Titanium Image]</td>
<td>![GeoGebra Image]</td>
<td>![Core Math Tools Image]</td>
</tr>
</tbody>
</table>

Look at the three outputs. How are they similar or different?

How do they compare to the graphs of \( f(x) = x^{1/5} \)?

If the interviewee does not say anything about the equivalence of the output, ask:
Do you think that these graphs correctly represent the functions that you entered? Why or why not?

After the interviewee’s opinion about the correctness of the graphs are established, ask:
How would you convince yourself that you are correct?
Can you think of another way that you could convince someone that these graphs are (or are not) correct, other than what you have already explained?

Previous versions of GeoGebra graphed \( f(x) = x^{1/5} \) like the TI-89, and \( g(x) = x^{0.2} \) the same way it does now. With a recent update, GeoGebra now graphs these as you see here. What do you think about that update?

How do you interpret the expression \( x^{0.2} \)?

Can you think of another way that someone might interpret the expression \( x^{0.2} \)?
Part 4: Symbolic output for derivatives

We will now return to generating symbolic output using CAS. I am going to have you use the
derivative commands to find the derivative of the function \( f(x) = \cos(3x) \) with respect to \( x \) so that
you can see how to use the derivative command.

Explain where to find and/or type the derivative command for each of the technologies if they do
not already know how to do so.

Output will be:

<table>
<thead>
<tr>
<th>TI-89 Titanium</th>
<th>GeoGebra</th>
<th>Core Math Tools</th>
</tr>
</thead>
</table>
| ![TI-89 screenshot](image1.png) | \( \text{Derivative}[\cos(3x)] \)  
\( \rightarrow -3 \sin(3x) \) | \( \text{der} (\cos(3x),x) \)  
\( -1^* (3^* \sin(3^* x)) \) |

Do you have any comments or questions about how this command works for each of the
technologies?

Now I would like you to use each of the technologies to find the derivative of the function \( f(x) = \cos(3x - \pi/6) \).

Output will be:

<table>
<thead>
<tr>
<th>TI-89 Titanium</th>
<th>GeoGebra</th>
<th>Core Math Tools</th>
</tr>
</thead>
</table>
| ![TI-89 screenshot](image2.png) | \( \text{Derivative}[\cos(3x-\pi/6)] \)  
\( \rightarrow -3 \sin \left( -\frac{1}{6} + 3 \right) \) | \( \text{der} (\cos(3x-\pi/6),x) \)  
\( -1^* \left( \frac{d(x)}{dx} \left[ \frac{6^* \left( \sin(x) \right)}{3^* - 3^*} \right] \sin(3^* x - 1^* \frac{\pi}{6}) \right) \) |

Look at the three outputs. How are they similar or different?

If the interviewee does not say anything about the equivalence of the output, ask:

Do you think these outputs are mathematically equivalent?

After the interviewee’s opinion about the equivalence of the output is established, ask:

How would you convince yourself that they are equivalent (or not equivalent)? Can you think of
another way that you could convince someone that these outputs are equivalent (or not
equivalent) other than what you have already explained?

Now that you have seen several examples of conflicts between outputs, have you thought of any
other ways in which someone could figure out whether or not outputs that look different are
equivalent?
If the interviewee has not mentioned some common ways of checking for equivalence using tables, graphs, or algebra, ask about any of the following that they have not mentioned as time permits:

After seeing different outputs from different technologies, some people may have tried to algebraically prove that the two expressions were equivalent. What do you think about that strategy?

After seeing different outputs from different technologies, some people may have entered the two expressions into the computer or calculator as two functions to see if their graphs matched. What do you think about that strategy?

After seeing different outputs from different technologies some people may have entered the two expressions into the computer or calculator as two functions to see if their tables of values matched. What do you think about that strategy?

Were you aware that the TI-89 allows you to test the equivalence of two symbolic expressions? What do you think about using the calculator to test for equivalence?

If you believed that two expressions were equivalent algebraically but their graphs or tables did not match, what would you do next to try to understand the discrepancy?

**Part 5: Pedagogical questions**

Today you worked with three different mathematics technologies and saw that they do not always give the same output. What do you take away from that experience?

If students do not mention whether or not they would show the conflicts to their future students, ask:

Would you consider presenting conflicts like these to students in your future classrooms? Why or why not?

Again, I would like to thank you for your participation. These interviews are central to my master’s thesis, and you have helped me a lot. Since I will be doing this interview with others, I would appreciate it if you would not discuss what you did in this interview with other math education majors until all interviews are completed.
Appendix D

Formula Sheets Provided

Derivative Rules

\( \text{power rule: } \frac{d}{dx} (x^n) = nx^{n-1} \)

\( \text{product rule: } \frac{d}{dx} (f(x) \cdot g(x)) = f'(x) \cdot g(x) + f(x) \cdot g'(x) \)

\( \text{quotient rule: } \frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{(g(x))^2} \)

\( \text{chain rule: } \frac{d}{dx} (f(g(x))) = f'(g(x)) \cdot g'(x) \)

\( \frac{d}{dx} (e^x) = e^x \quad \frac{d}{dx} \ln x = \frac{1}{x} \quad \frac{d}{dx} (a^x) = a^x \cdot \ln a \)

\( \frac{d}{dx} \sin x = \cos x \quad \frac{d}{dx} \cos x = -\sin x \quad \frac{d}{dx} \tan x = \sec^2 x \)

\( \frac{d}{dx} \csc x = -\csc x \cdot \cot x \quad \frac{d}{dx} \sec x = \sec x \cdot \tan x \quad \frac{d}{dx} \cot x = -\csc^2 x \)

\( \frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}} \quad \frac{d}{dx} \arccos x = \frac{1}{x\sqrt{x^2-1}} \quad \frac{d}{dx} \arctan x = \frac{1}{1+x^2} \)
Trigonometric Identities

Reciprocal Identities:
\[
\begin{align*}
\sin \theta &= \frac{1}{\csc \theta} & \cos \theta &= \frac{1}{\sec \theta} & \tan \theta &= \frac{1}{\cot \theta} \\
csc \theta &= \frac{1}{\sin \theta} & \sec \theta &= \frac{1}{\cos \theta} & \cot \theta &= \frac{1}{\tan \theta}
\end{align*}
\]

Quotient Identities:
\[
\begin{align*}
\tan \theta &= \frac{\sin \theta}{\cos \theta} & \cot \theta &= \frac{\cos \theta}{\sin \theta}
\end{align*}
\]

Opposite Angle Identities:
\[
\begin{align*}
\sin(-\theta) &= -\sin \theta & \cos(-\theta) &= \cos \theta & \tan(-\theta) &= -\tan \theta
\end{align*}
\]

Cofunction Identities:
\[
\begin{align*}
\sin \left( \frac{\pi}{2} - \theta \right) &= \cos \theta & \cos \left( \frac{\pi}{2} - \theta \right) &= \sin \theta & \tan \left( \frac{\pi}{2} - \theta \right) &= \cot \theta \\
csc \left( \frac{\pi}{2} - \theta \right) &= \sec \theta & \sec \left( \frac{\pi}{2} - \theta \right) &= \csc \theta & \cot \left( \frac{\pi}{2} - \theta \right) &= \tan \theta
\end{align*}
\]

Pythagorean Identities:
\[
\begin{align*}
\sin^2 \theta + \cos^2 \theta &= 1 & 1 + \tan^2 \theta &= \sec^2 \theta & 1 + \cot^2 \theta &= \csc^2 \theta
\end{align*}
\]

Double Angle Identities:
\[
\begin{align*}
\sin 2\theta &= 2 \sin \theta \cdot \cos \theta & \cos 2\theta &= \cos^2 \theta - \sin^2 \theta & \tan 2\theta &= \frac{2\tan \theta}{1 - \tan^2 \theta} \\
\cos 2\theta &= 2 \cos^2 \theta - 1 & \cos 2\theta &= 1 - 2\sin^2 \theta
\end{align*}
\]

Half Angle Identities:
\[
\begin{align*}
\sin \left( \frac{\theta}{2} \right) &= \pm \sqrt{\frac{1 - \cos \theta}{2}} & \cos \left( \frac{\theta}{2} \right) &= \pm \sqrt{\frac{1 + \cos \theta}{2}} \\
\tan \left( \frac{\theta}{2} \right) &= \pm \frac{1 - \cos \theta}{1 + \cos \theta}, \cos \theta \neq -1
\end{align*}
\]

Sum and Difference Identities:
\[
\begin{align*}
\sin(\alpha \pm \beta) &= \sin \alpha \cdot \cos \beta \pm \cos \alpha \cdot \sin \beta & \cos(\alpha \pm \beta) &= \cos \alpha \cdot \cos \beta \mp \sin \alpha \cdot \sin \beta \\
\tan (\alpha \pm \beta) &= \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \cdot \tan \beta}
\end{align*}
\]