NONLOCAL MODELS WITH CONVECTION EFFECTS

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by
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Abstract

This dissertation focuses on proposing and studying nonlocal models in the form of partial-integral-differential equations (PIDEs), as a generalization of local models, in the form of classical differential equations. It consists of two parts: the first part works on the volume-constraint problems associated with nonlocal convection-diffusion equations, the local counterpart part of which is the boundary-value problems associated with classical convection-diffusion differential equations. The second part deals with the initial-value problems of scalar nonlocal hyperbolic conservation laws.

In both parts, we show that the nonlocal models we propose are reasonable extensions of their local counterparts, by demonstrating the consistency of their most important properties. In particular, both nonlocal problems enjoy maximum principle, just as their local counterparts do. Additionally, in the first part, we identify the underlying stochastic jump processes for the nonlocal convection-diffusion problems with Dirichlet volume-constraint, Robin volume-constraint, or on the whole space. The local counterpart of these jump processes are the Brownian motions with reflective or censored behavior near the domain boundary, or Brownian motions in the free space. Both Monte Carlo simulations and Finite Difference Methods are performed to observe the nonlocal solutions. In the second part, for nonlocal hyperbolic conservation laws, we prove the uniqueness and existence of the nonlocal entropy solution, and establish a condition on the kernel function, under which the nonlocal solution develop no shocks from smooth initial condition. Numerically, we propose a monotone scheme, the solution of which, as the horizon \( \delta \) is fixed and \( \Delta x \to 0 \), converges to the entropy solution of the nonlocal conservation law, while as both \( \delta \) and \( \Delta x \) vanish, converges to the entropy solution of the local conservation law. Numerical experiments are performed to further study the solution behaviors.
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Dedication

To my parents, Jinfu Huang and Youxing Hu.
Chapter 1
Introduction

Increasing interests on 'nonlocal models' have been rising among mathematicians, physicists and engineers, and 'nonlocality' becomes one of the key words of mathematical modeling. Mathematically, nonlocality normally appears in the form of integrals. Beside integrals, nonlocal models may also contain derivatives of the unknown function, leading to the form of partial integral-differential equations (PIDE). While the 'local' models refer to the classical partial differential equations (PDE) involving only derivatives. From physical point of views, nonlocality often relates to the physical facts of nonlocal interactions. For example, in elastic materials, a particle in a continuum may not only be influenced by the particles immediately around it, but could also interact with particles of some distance—such models lead to the peridymanic theory developed by Silling [19] in 2000.

Besides the ability to describe nonlocal mechanisms, nonlocal models could also provide mathematical convenience compared to local models. Let’s take peridynamics as an example. For continuum deformations with discontinuities, PDEs do not directly apply since partial derivatives do not exist on crack surfaces and other singularities. While integral equations are free of such concern. Moreover, the nonlocal peridynamic models unites the mathematical modeling of continuous media, cracks, and particles within one single multi-scale framework, avoiding the need for the special techniques of fracture mechanics. More literature reviews of relevant nonlocal models will be introduced at the beginning of each chapter.

Our work focuses on nonlocal models with convection effects. It consists two chapters: the first chapter is on the volume-constraint problems associated with nonlocal convection-diffusion equations, corresponding to the boundary-value problems of classical convection-diffusion equations. In this part all problems are linear. The second part moves forward to nonlinear problems, namely, Cauchy problems of scalar nonlocal hyperbolic conservation laws.
In the first chapter, we introduce the Cauchy problem and time-dependent volume-constrained problems associated with a linear nonlocal convection-diffusion equation. These problems are shown to be well-posed and correspond to conventional convection-diffusion equations as the region of nonlocality vanishes. The problems also share a number of features such as the Maximum Principle, conservation and dispersion relations, all of which are consistent with their corresponding local counterparts. Moreover, these problems are the master equations for a class of finite activity Lévy-type processes with nonsymmetric Lévy measure. Monte Carlo simulations and Finite Difference schemes are applied to these nonlocal problems, to show the effects of time, kernel, nonlocality and different volume-constraints. This part of work forms a basis for studying the nonlinear problem. This part of work has been published [15].

In the second chapter, we propose a class of nonlocal conservation laws in such a way that it represents a reasonable generalization of the local conservation law. The nonlocal conservation law shares some important properties of the local conservation law, such as the conservation property and maximum principle, and nonlocal conservation law reduces to its local counterpart when we take special kernel. We prove the uniqueness and existence of the nonlocal entropy solution, and also establish a condition on the kernel function, under which the nonlocal solution develop no shocks from smooth initial condition. Numerically, we propose a monotone scheme, the solution of which, as the horizon $\delta$ is fixed and $\Delta x \to 0$, converges to the entropy solution of the nonlocal conservation law, while as both $\delta$ and $\Delta x$ vanish, converges to the entropy solution of the local conservation law. Results of numerical experiments are consistent with these theories.
Chapter 2
Nonlocal Linear Conservation Law

2.1 Introduction

The contribution of this part of work is to formulate and analyze problems associated with a linear nonlocal convection-diffusion equation introduced in §2.2 and the relationship to Markov jump processes of finite-range with a nonsymmetric jump-rate in §2.4. In addition to the free-space Cauchy problem, we also consider variations where the nonlocal equation is augmented with volume-constraints, the nonlocal analogue of boundary conditions. This allows us to consider Dirichlet and Robin volume-constrained problems and their corresponding stochastic processes. This work has been published [15].

Classical convection-diffusion equations as well as their boundary-value problems have often been used as representative mathematical models for problems in biology, engineering, physics and finance. The associated linear differential equations and differential operators are considered “local” models since the solution at a point determines whether the equation is satisfied. In our work, linear integral equations and operators replace their differential counterparts so that the resulting model is “nonlocal”, i.e., the solution at a point depends upon a continuum of nearby points. When the kernel of the integral operator is appropriately chosen, the nonlocal model can be related to the local model and so can be seen as a generalization. The interest in the nonlocal model arises because the regularity requirements associated with the local model may be significantly relaxed, or equivalently, the sample-path of the jump process contains discontinuities in contrast to the continuous process corresponding to Brownian motion.

Although there has been much research on nonlocal diffusion on a bounded domain, see, e.g., [12, 40] for citations to the literature, comparatively little is available for non-
local convection-diffusion problems with volume-constraints. One of the relative work is in [78], where Du et. al proposed and compared some nonlocal convection-diffusion volume-constrained problems and study their finite element approximations, to demonstrate the advantages of the upwind models from both physical and numerical perspectives.

We builds upon and extends the work in [74] where a one-dimensional nonlocal convection conservation law was considered. Unlike the formulation in [74], our treatment of the nonlocal advection term presented allows us to maintain the Maximum Principle and provide a probabilistic relationship. The paper [73] established a well-posed nonlocal equation with nonlinear convection and linear diffusion and asymptotic behavior of the solution for the Cauchy problem. We, on the other hand, consider a broad-class of volume-constrained problems and provide their stochastic interpretations. In our work we modify the nonlocal operator proposed in [12] to accommodate

• homogenous Dirichlet and Robin volume-constraints;
• a non-symmetric kernel that is not translationally invariant;

see §2.2. The corresponding nonlocal convection-diffusion problems are demonstrated to be well-posed in §2.3.1 whereas convergence to the local problem, Maximum Principle and a dispersion relation are given in §2.3. Discretization of the nonlocal equations and comparisons with Monte-Carlo simulations are given in §2.5, proceeded by the aforementioned §2.4 on the underlying Markov processes associated with the nonlocal problems. The immediate impact is to demonstrate that a general class of exit-time problems are well-posed and allow a deterministic interpretation; see §2.4. This extends the work introduced in the papers [26,27,76] that only considered symmetric kernels and a homogenous Dirichlet volume-constraint.

2.1.1 Notation

Throughout this chapter we use the following notations. Let $\Omega$ and $\Omega_d$ be open domains in $\mathbb{R}^n$. We let $\Omega = \Omega \cup \Omega_d$ be the domain containing both $\Omega$ and $\Omega_d$. The horizon parameter, denoted by $\varepsilon > 0$, represents the maximum distance within which nonlocal interactions can happen. When $\Omega$ is bounded, we denote its boundary by $\partial \Omega$ and define the inner and outer layers along $\partial \Omega$ with width $\varepsilon$ as, respectively,

$$\Omega_{in}^\varepsilon := \{ y \in \Omega : \text{dist}(y, \partial \Omega) \leq \varepsilon \}, \quad \Omega_{out}^\varepsilon := \{ y \in \mathbb{R}^n \setminus \Omega : \text{dist}(y, \partial \Omega) \leq \varepsilon \},$$

(2.1)
where “dist” represents Euclidean distance. Denote the indicator function

\[ 1_{\Omega}(x) = \begin{cases} 
1, & x \in \Omega, \\
0, & x \notin \Omega, 
\end{cases} \]

and \( B(0, \varepsilon) \) the closed ball in \( \mathbb{R}^n \) centered at the origin with radius \( \varepsilon \).

The paper [11] develops a nonlocal calculus, which we now briefly review for later formulation of the nonlocal operators and equations. Given an anti-symmetric vector mapping \( \alpha(x,y) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) with \( \alpha(y,x) = -\alpha(x,y) \), the action of the nonlocal divergence operator \( \mathcal{D} \) on a locally integrable mapping \( \nu = \nu(x,y) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is defined as

\[ \mathcal{D}(\nu)(x) := \int_{\mathbb{R}^n} (\nu(x,y) + \nu(y,x)) \cdot \alpha(x,y) dy, \quad x \in \mathbb{R}^n, \]  

(2.2)

where \( \mathcal{D}(\nu) : \mathbb{R}^n \rightarrow \mathbb{R} \) is a scalar-valued map. In general, the nonlocal operators are interpreted as operators in suitable function spaces so that (2.2) holds in the sense of distributions. Whenever appropriate, a point-wise definition of the integral in (2.2) should also be interpreted in the principal value sense [17]. In this work, however, we will mostly work with standard \( L^p \) function spaces so that the functions and integrals are well-defined almost everywhere.

Given a scalar mapping \( u : \mathbb{R}^n \rightarrow \mathbb{R} \), the adjoint operator \( \mathcal{D}^* \) corresponding to the nonlocal divergence operator \( \mathcal{D} \) is the operator whose action on \( u \) is given by

\[ \mathcal{D}^*(u)(x,y) = -(u(y) - u(x))\alpha(x,y), \quad \text{for } x,y \in \mathbb{R}^n, \]  

(2.3)

where \( \mathcal{D}^*(u) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \). With \( \mathcal{D}^* \) denoting the adjoint of the nonlocal divergence operator, one can view \(-\mathcal{D}^*\) as a nonlocal gradient.

In this part of work, we assume that the antisymmetric \( \alpha \) used to define \( \mathcal{D} \) and \( \mathcal{D}^* \) is translation invariant i.e., \( \alpha(x,y) = \alpha^\varepsilon(y-x) \), and has support in \( B(0,\varepsilon) \), a ball centered at the origin and of radius \( \varepsilon \), i.e., \( \alpha^\varepsilon(s) = 0 \) for \( s \notin B(0,\varepsilon) \) so that \( \alpha \) is also denoted as \( \alpha^\varepsilon \) to highlight the \( \varepsilon \)-dependence.

### 2.2 Nonlocal linear convection and diffusion

We introduce in this section the nonlocal convection-diffusion equation under consideration, and present formulations of the associated volume-constrained problems that are analogous
to the homogeneous Dirichlet and Robin boundary-value problems in the local case.

2.2.1 Nonlocal linear convection-diffusion equation

We first present a general time-dependent nonlocal convection-diffusion problem given by

\[
\begin{cases}
    u_t = D(1_{\Omega_w}(x)1_{\Omega_w}(y)\mu u) - D(1_{\Omega_w}(x)1_{\Omega_w}(y)\Theta D^* u), & x \in \Omega, \ t > 0, \\
    u(x, t) = 0, & x \in \Omega_d, \ t > 0, \\
    u(x, 0) = u_0(x), & x \in \Omega.
\end{cases}
\] (2.4)

Here, an integral-differential equation is imposed in \( \Omega \times (0, \infty) \) along with a homogeneous constraint of the Dirichlet type defined in \( \Omega_d \times (0, \infty) \). The two-point function \( \mu : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is symmetric and translational invariant, i.e., \( \mu(x, y) = \mu(y - x) = \mu(x - y) \). The two-point tensor \( \Theta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n} \) is symmetric with elements that are symmetric and translational invariant, i.e., \( \Theta(y - x) = \Theta^T(y - x) = \Theta(x - y) \). The classical model associated with (2.4) is

\[
    u_t = b \cdot \nabla u + \nabla \cdot (A \cdot \nabla u),
\] (2.5)

along with suitable initial and boundary conditions. Here \( A \in \mathbb{R}^{n \times n} \) is a symmetric, positive definite tensor with constant entries \( a_{ij} \), and \( b \in \mathbb{R}^n \) (see section 2.3.2 for more details).

The two terms on the right hand side of (2.4) represent nonlocal convection and diffusion, respectively. We interpret the first term as the nonlocal convection, because it introduces bias in the propagation of the substance, analogous to drift effect in the local case. Such nonlocal bias effect remains in the local limit in our scaling (see analysis in section 2.3.2 and numerical experiments in section 2.5.3).

We note that depending on the choices of \( \Omega \) and \( \Omega_d \), the problems can be classified as different types, such as nonlocal Cauchy problems and nonlocal initial and Dirichlet or Robin (or a mixed type) volume-constrained problems. A few examples of these special types of problems are discussed later.

2.2.2 The kernel

Define \( \gamma^\varepsilon : \mathbb{R}^n \rightarrow [0, \infty) \) as

\[
    \gamma^\varepsilon := 2\alpha^\varepsilon \cdot \Theta \alpha^\varepsilon - \mu \cdot \alpha^\varepsilon,
\]
such that $||\gamma^{\varepsilon}||_{L^1(\mathbb{R}^n)} < +\infty$. The assumption on the antisymmetric $\alpha^{\varepsilon}$ given at the end of §2.1.1 imply that $\gamma^{\varepsilon}$ is compactly supported on $B(0,\varepsilon)$ and that

$$\int_{\mathbb{R}^n} \gamma^{\varepsilon}(x) \, dx = \int_{\mathbb{R}^n} \gamma^{\varepsilon}(-x) \, dx \quad (2.6)$$

since $\mu$ in (2.4) is assumed symmetric.

Using the definition for the nonlocal divergence (2.2) and its adjoint (2.3), the nonlocal equation (2.4) can be rewritten as

$$u_t(x,t) = \int_{\Omega_w} \left( u(y,t)\gamma^{\varepsilon}(x-y) - u(x,t)\gamma^{\varepsilon}(y-x) \right) \, dy, \quad x \in \Omega, t > 0, \quad (2.7a)$$

where we identify

$$1_{\Omega_w}(x)1_{\Omega_w}(y)\gamma^{\varepsilon}(x-y) \quad (2.7b)$$

as the kernel accounting for the nonlocal interaction.

As an illustration, we give a one-dimensional example with $\Omega = (a,b)$ with the translationally invariant, piecewise constant kernel

$$\gamma^{\varepsilon}(x) = 2c_1\varepsilon^{-1}\varphi^{\varepsilon}(x) + 6c_2\varepsilon^{-2}\phi^{\varepsilon}(x), \quad (2.8)$$

where $c_1$ and $c_2$ are constants, and

$$\varphi^{\varepsilon}(x) = \frac{1}{\varepsilon}1_{[-\varepsilon,0]}(x), \quad \phi^{\varepsilon}(x) = \frac{1}{2\varepsilon}1_{[-\varepsilon,\varepsilon]}(x),$$

so that

$$||\gamma^{\varepsilon}||_{L^1(\mathbb{R})} = 2c_1\varepsilon^{-1} + 6c_2\varepsilon^{-2}.$$ 

In order that the kernel is nonnegative, we assume that

$$c_2 \geq 0 \quad \text{and} \quad 2c_1\varepsilon + 3c_2 \geq 0,$$

where the former constraint arises when $0 < x < \varepsilon^{-1}$. Defining $\gamma^{\varepsilon}_s$ and $\gamma^{\varepsilon}_a$ to be the symmetric and antisymmetric components of $\gamma^{\varepsilon}$, we have

$$\gamma^{\varepsilon}_s(x) = \frac{2(c_1\varepsilon + 3c_2)}{\varepsilon^2}\phi^{\varepsilon}(x) \quad \text{and} \quad \gamma^{\varepsilon}_a(x) = \frac{2c_1}{\varepsilon}\text{sgn}(x)\phi^{\varepsilon}(x),$$
where $\text{sgn}(x)$ extracts the sign of $x$. Because $\gamma^\varepsilon = \gamma^\varepsilon_h + \gamma^\varepsilon_a$, the identifications

$$\alpha^\varepsilon(x) = -\frac{1}{\varepsilon}\text{sgn}(x)\phi^\varepsilon(x), \quad \Theta = 4(c_1\varepsilon + 3c_2), \quad \text{and } \mu = 2c_1$$

then grant that $\gamma^\varepsilon = \left(\alpha^\varepsilon\right)^2\Theta - \mu\alpha^\varepsilon$.

### 2.2.3 Nonlocal Cauchy problem and volume-constrained problems

We may specify a number of special cases, depending on the definitions of $\Omega$ and $\Omega_d$. In the remaining part, we assume $u_0$ is a probability density function supported on $\Omega$, with

$$\int_{\Omega} u_0(x) dx = 1.$$  

First, let $\Omega = \mathbb{R}^n$ and $\Omega_d = \emptyset$ be the empty set, then $\Omega_w = \mathbb{R}^n$ and we have the following Cauchy problem

$$\begin{cases}
    u_t = D(\mu u) - D(\Theta D^* u), & x \in \mathbb{R}^n, \ t > 0, \\
    u(x, 0) = u_0(x), & x \in \mathbb{R}^n.
\end{cases} \quad (2.9)$$

For a bounded domain $\Omega$, we set $\Omega_d = \mathbb{R}^n\setminus\hat{\Omega}$, then we again have $\Omega_w = \mathbb{R}^n$ and $\mathbb{1}_{\Omega_w} \equiv 1$. A nonlocal homogeneous Dirichlet volume-constrained problem on $\Omega$ is then given by

$$\begin{cases}
    u_t = D(\mu u) - D(\Theta D^* u), & x \in \Omega, \ t > 0, \\
    u(x, t) = 0, & x \in \Omega_d, \ t > 0, \\
    u(x, 0) = u_0(x), & x \in \Omega.
\end{cases} \quad (2.10)$$

Similarly, for $\Omega$ bounded and $\Omega_d = \emptyset$, we have $\Omega_w = \Omega$ and the nonlocal homogeneous Robin volume-constrained problem is

$$\begin{cases}
    u_t = D(\mathbb{1}_{\Omega}(x)\mathbb{1}_{\Omega}(y)\mu u) - D(\mathbb{1}_{\Omega}(x)\mathbb{1}_{\Omega}(y)\Theta D^* u), & x \in \Omega, \ t > 0, \\
    u(x, 0) = u_0(x), & x \in \Omega.
\end{cases} \quad (2.11)$$

In addition, the formulation (2.4) also effectively provides us a mixed Dirichlet and Robin volume-constrained problem when $\Omega$ is taken to be a bounded domain, and $\Omega_d$ is a set that has a non-empty intersection with $\Omega \cup \Omega^\varepsilon_{\text{out}}$ but does not contain all of $\Omega^\varepsilon_{\text{in}} \cup \Omega^\varepsilon_{\text{out}}$. For instance, in the one dimension example given in (2.8) with $\Omega = (a, b)$ with a positive $c_1$, we may take $\Omega_d$ to be $(b, b + \varepsilon)$. 

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For convenience, we denote the nonlocal operator in (2.4) by $\mathcal{L}$, the nonlocal operators in (2.9)–(2.10) as $\tilde{\mathcal{L}}$ and the nonlocal operator in (2.11) as $\check{\mathcal{L}}$. More precisely, using the definitions and properties of the kernel functions, we have

$$\mathcal{L}u(x) := \mathcal{D}(1_{\Omega_w}(x)1_{\Omega_u}(y)\mu u)(x) - \mathcal{D}(1_{\Omega_w}(x)1_{\Omega_u}(y)\Theta D^*(u)(x,y))(x)$$

$$= \int_{\Omega_w} \left( u(y)\gamma^\varepsilon(x - y) - u(x)\gamma^\varepsilon(y - x) \right) dy,$$

$$(2.12a)$$

$$\tilde{\mathcal{L}}u(x) := \mathcal{D}(\mu u)(x) - \mathcal{D}(\Theta D^* u)(x)$$

$$= \int_{\mathbb{R}^n} \left( u(y) - u(x) \right) \gamma^\varepsilon(x - y) dy,$$  

$$(2.12b)$$

$$\check{\mathcal{L}}u(x) := \mathcal{D}(1_{\Omega}(x)1_{\Omega}(y)\mu u)(x) - \mathcal{D}(1_{\Omega}(x)1_{\Omega}(y)\Theta D^* u)(x)$$

$$= \int_{\Omega} \left( u(y)\gamma^\varepsilon(x - y) - u(x)\gamma^\varepsilon(y - x) \right) dy.$$  

$$(2.12c)$$

Under the assumption that $||\gamma^\varepsilon||_{L^1(\mathbb{R}^n)} < +\infty$, we can see that the above operators are bounded linear operators from $L^p(\Omega)$ to itself which leads to the well-posedness of the evolution equations (see discussions in section 3.1).

Recall, the solutions to the local boundary-value problems (2.5) are constrained on the surface $\partial\Omega$. In contrast, for the nonlocal case, the boundary condition is replaced by “volume constraint” in (2.10), where the constraint is imposed over a region in $\mathbb{R}^n\setminus\Omega$ with a nonzero volume.

**Remark 2.2.1.** Unlike the nonlocal Dirichlet problem, no additional equation is used to specify the volume-constraint for the nonlocal Robin problem (2.11). This departs slightly from the formulation given in [?], §5.3. In fact, the nonlocal Robin volume-constraint has been incorporated implicitly into the operator $\check{\mathcal{L}}$, which is obtained by modifying $\tilde{\mathcal{L}}$ on an inner layer $\Omega^\varepsilon$ of width $\varepsilon$ along the boundary of $\Omega$. To see this, let $x \in \Omega\setminus\Omega^\varepsilon$ so that $B(x,\varepsilon) \subset \Omega$, we have

$$\check{\mathcal{L}}u(x) = \int_{\Omega} \left( u(y)\gamma^\varepsilon(x - y) - u(x)\gamma^\varepsilon(y - x) \right) dy$$

$$= \int_{\Omega} (u(y)\gamma^\varepsilon(x - y)dy - \int_{\Omega} u(x)\gamma^\varepsilon(x - y)dy$$

$$= \int_{\Omega} (u(y) - u(x))\gamma^\varepsilon(x - y)dy = \check{\mathcal{L}}u(x),$$

where the second equality follows by (2.6). This implies that the nonlocal Robin operator $\check{\mathcal{L}}$ and the nonlocal Dirichlet operator $\tilde{\mathcal{L}}$ coincide over $\Omega\setminus\Omega^\varepsilon$. The two differ on $\Omega^\varepsilon$ where the Robin volume constraint is implicitly imposed.
2.3 Model properties

In this section we first establish the well-posedness of the time-dependent nonlocal problem (2.4). Then, we show that, as $\varepsilon \to 0$, the weak forms of the nonlocal Cauchy problem, homogeneous Dirichlet and Robin volume-constrained problems converge to those corresponding to local Cauchy problem, local homogeneous Dirichlet and Robin boundary-value problems. Moreover, our nonlocal problems preserve Maximum Principle and conservation law, just as their local counterparts do. The dispersion relation of the nonlocal equation also reduces to that of the local equation. In this sense, our nonlocal models are consistent with the local models.

2.3.1 Well-posed nonlocal problems

Let us define the constrained space by

$$L^p_c(\Omega) = \{ u \in L^p(\Omega_w) : u(x) \equiv 0 \text{ on } \Omega_d \}.$$

Note that if $\Omega_d = \emptyset$, then $L^p_c(\Omega) = L^p(\Omega)$.

The following result is needed to establish that the nonlocal problems are well-posed in the theorem that immediately follows.

**Lemma 2.3.1.** Let $u \in L^p_c(\Omega)$, $1 \leq p < \infty$, then there is a constant $C$ depending only on $p$ such that

$$||L^p u||_{L^p(\Omega)} \leq C||\gamma^\varepsilon||_{L^1(\mathbb{R}^n)}||u||_{L^p(\Omega)}.$$  \hspace{1cm} (2.13)

**Proof.** By definition, we have

$$||L^p u||_{L^p(\Omega)} = \left|\left| \int_{\Omega_w} \left( u(y) \gamma^\varepsilon(x-y) - u(x) \gamma^\varepsilon(y-x) \right) dy \right| \right|_{L^p(\Omega)}^p \leq 2^{p-1} \left\{ \int_{\Omega} \left| \int_{\Omega_w} u(y) \gamma^\varepsilon(x-y) dy \right|^p dx + \int_{\Omega} \left| u(x) \int_{\Omega_w} \gamma^\varepsilon(x-y) dy \right|^p dx \right\}.$$

Let $\tilde{u}$ be an extension of $u$ by zero outside $\Omega$. By Young’s inequality, we get

$$||L^p u||_{L^p(\Omega)} \leq C||\tilde{u} \ast \gamma^\varepsilon||_{L^p(\Omega)}^p + C||\gamma^\varepsilon||_{L^1(\mathbb{R}^n)}^p ||u||_{L^p(\Omega)}^p \leq C||\tilde{u}||_{L^p(\mathbb{R}^n)}^p ||\gamma^\varepsilon||_{L^1(\mathbb{R}^n)}^p + C||\gamma^\varepsilon||_{L^1(\mathbb{R}^n)}^p ||u||_{L^p(\Omega)}^p = C||u||_{L^p(\Omega)}^p ||\gamma^\varepsilon||_{L^1(\mathbb{R}^n)}^p + C||\gamma^\varepsilon||_{L^1(\mathbb{R}^n)}^p ||u||_{L^p(\Omega)}^p.$$
for a generic constant $C$ depending only on $p$. Here $\ast$ represents convolution.

**Theorem 2.3.2.** For any given $T > 0$, $1 \leq p < \infty$ and $u_0 \in L^p_c(\Omega)$, the nonlocal time-dependent problem (2.4) has a unique solution in $C^1(0,T; L^p_c(\Omega))$.

**Proof.** By equation (2.13) in Lemma 2.3.1, $\mathcal{L}$ is a bounded linear operator in $L^p_c(\Omega)$, and thus is Lipschitz continuous in $u$. By Picard-Lindelöf theorem, the nonlocal Cauchy problem (2.4) has a unique solution in $C^1(0,T; L^p_c(\Omega))$ for any given $T > 0$.

The above theorem implies the well-posedness of the nonlocal Cauchy problem, homogeneous Dirichlet and homogeneous Robin volume-constrained problems given by (2.9), (2.10), and (2.11) respectively.

### 2.3.2 Local limit

As the nonlocality vanishes, we further assume that

\begin{equation}
\int_{\mathbb{R}^n} z \gamma^\varepsilon(z) dz = -\int_{\mathbb{R}^n} z(\mu \cdot \alpha^\varepsilon)(z) dz \to -b,
\end{equation}

\begin{equation}
\int_{\mathbb{R}^n} z_i z_j \gamma^\varepsilon(z) dz = 2\int_{\mathbb{R}^n} z_i z_j \alpha^\varepsilon \cdot (\Theta \alpha^\varepsilon)(z) dz \to 2a_{ij}
\end{equation}

for $i, j = 1, \ldots, n$. Here $b = (b_1, \ldots, b_n)$ is a constant vector, and $a_{ij}$’s are the entries of a symmetric, positive definite constant matrix $A$. We refer to [73] for an analogous scaling of the nonlocal kernels that also yield (2.14) and thus allow the convergence of nonlocal convection-diffusion equations to a local convection-diffusion equation as the nonlocality vanishes.

#### 2.3.2.1 Weak form and local limit

For $v \in L^2_c(\Omega)$, we consider

\[
(Lu, v)_\Omega = (Lu, v)_{\Omega_w} = \int_{\Omega_w} \int_{\Omega_w} u(y) \gamma^\varepsilon(x - y) dv(x) dx - \int_{\Omega_w} \int_{\Omega_w} u(x) \gamma^\varepsilon(y - x) dv(x) dx
\]

\[
= 2 \int_{\Omega_w} \int_{\Omega_w} (u(y) - u(x)) \alpha^\varepsilon \cdot (\Theta \alpha^\varepsilon)(x - y) dv(x) dx
\]

\[
- \int_{\Omega_w} \int_{\Omega_w} (u(y) + u(x)) (\mu \cdot \alpha^\varepsilon)(x - y) dv(x) dx.
\]
Thus, by a change of variable, we have

\[
(Lu, v)_\Omega = -\int_{\Omega_w} \int_{\Omega_w} (u(y) - u(x))(v(y) - v(x)) \alpha^\varepsilon \cdot (\Theta \alpha^\varepsilon)(x - y) dy dx \\
- \int_{\Omega_w} \int_{\Omega_w} u(y)(\mu \cdot \alpha^\varepsilon)(x - y)(v(x) - v(y)) dy dx,
\]

which leads to a weak form for the time-dependent nonlocal problem: find \( u \in C^1(0, T; L^2_c(\Omega)) \) such that \( u(x, 0) = u_0 \) and for \( t \in (0, T) \),

\[
(u_t, v)_\Omega = \int_{\Omega_w} \int_{\Omega_w} (u(y) - u(x))(v(y) - v(x)) \alpha^\varepsilon \cdot (\Theta \alpha^\varepsilon)(x - y) dy dx \\
- \int_{\Omega_w} \int_{\Omega_w} u(y)(\mu \cdot \alpha^\varepsilon)(x - y)(v(x) - v(y)) dy dx, \quad \forall \, v \in L^2_c(\Omega).
\]

(2.15)

Our aim next is to demonstrate that as \( \varepsilon \to 0 \) the weak form of (2.4) converges to that of the equation in (2.5). If we assume that \( u \in H^1(\Omega_w) \) with \( u \equiv 0 \) in \( \Omega_d \) and suitably regular test functions, say \( v \in C^1(\mathbb{R}^n) \) with \( v \equiv 0 \) in \( \Omega_d \), then the assumptions (2.14) imply that as \( \varepsilon \to 0 \),

\[
(Lu, v)_\Omega \to -\int_{\Omega_w} \nabla u(x) \cdot A \cdot \nabla v(x) dx - \int_{\Omega_w} u(y)b \cdot \nabla v(y) dy \\
= -(A \cdot \nabla u, \nabla v)_{\Omega_w} - (bu, \nabla v)_{\Omega_w} \\
= -(A \cdot \nabla u, \nabla v)_{\Omega} - (bu, \nabla v)_{\Omega}.
\]

Thus the weak form of nonlocal problem (2.4) converges to that of the local problem (2.5) with the corresponding boundary condition depending on the solution and test function spaces. The next three subsections considers the three cases of interest.

2.3.2.2 Local limit of nonlocal Cauchy problem

For the nonlocal Cauchy problem (2.9), we see that the above derivation implies

\[
(Lu, v)_{\mathbb{R}^n} \to -(A \cdot \nabla u, \nabla v)_{\mathbb{R}^n} - (bu, \nabla v)_{\mathbb{R}^n}
\]

for \( u, v \in H^1(\mathbb{R}^n) \), thus the weak form of the local problem corresponds to precisely the local Cauchy problem:

\[
\begin{align*}
u_t &= b \cdot \nabla u + \nabla \cdot (A \cdot \nabla u), & x \in \Omega, t > 0, \\
u(x, 0) &= u_0(x), & x \in \Omega.
\end{align*}
\]

(2.17)
2.3.2.3 Local limit of nonlocal Dirichlet problem

For the nonlocal Dirichlet problem (2.10), for any \( u, v \in H^1_0(\Omega) \), by extending them to be zero outside \( \Omega \), we can again use the above derivation to get as \( \varepsilon \to 0 \),

\[
(\mathcal{L}u,v)_\Omega \to -(A \cdot \nabla u, \nabla v)_\Omega -(bu, \nabla v)_\Omega ,
\]

which is again the weak form corresponding to the local problem with a homogeneous Dirichlet boundary condition:

\[
\begin{cases}
  u_t = b \cdot \nabla u + \nabla \cdot (A \cdot \nabla u), & x \in \Omega, \ t > 0, \\
  u(x,t) = 0, & x \in \partial \Omega, \ t > 0, \\
  u(x,0) = u_0(x), & x \in \Omega.
\end{cases}
\]

(2.18)

2.3.2.4 Local limit of nonlocal Robin problem

Similar as the case of Dirichlet problems, since \( \Omega_d = \emptyset \), we do not need to impose any volume-constraint explicit on the solution and test functions, thus the earlier derivation gives: for any \( u, v \in H^1(\Omega) \),

\[
(\mathcal{L}u,v)_\Omega \to -(A \cdot \nabla u, \nabla v)_\Omega -(bu, \nabla v)_\Omega ,
\]

which implies the natural homogeneous Robin type boundary value problem for the local equation:

\[
\begin{cases}
  u_t = b \cdot \nabla u + \nabla \cdot (A \cdot \nabla u), & x \in \Omega, \ t > 0, \\
  A \frac{\partial u}{\partial n}(x,t) + b \cdot n u(x,t) = 0, & x \in \partial \Omega, \ t > 0, \\
  u(x,0) = u_0(x), & x \in \Omega.
\end{cases}
\]

(2.19)

2.3.3 Maximum Principle

Let \( T \) be a finite terminal time, \( U_T = \bar{\Omega} \times [0,T] \), and \( \Gamma_T = (\bar{\Omega}_w \setminus \Omega \times (0,T)) \cup (\Omega \times \{t = 0\}) \).

Lemma 2.3.3. Assume that \( \max_{U_T} u \) and \( \min_{U_T} u \) exist, and for any \( x \in \Omega \), \( u(x,t) \in C^1[0,T] \) and that

\[
\int_{\Omega_w} \gamma^\varepsilon(x_0 - y) \, dy = \int_{\Omega_w} \gamma^\varepsilon(y - x_0) \, dy , \quad \forall x_0 \in \Omega .
\]

(2.20)
If \( u_t \geq \mathcal{L}u \) on \( \Omega \times (0, T) \) then
\[
\min_{U_T} u = \min_{\bar{\Gamma}_T} u. \tag{2.21a}
\]

If \( u_t \leq \mathcal{L}u \) on \( \Omega \times (0, T) \) then
\[
\max_{U_T} u = \max_{\bar{\Gamma}_T} u. \tag{2.21b}
\]

Proof. We prove the results in three steps.

Step 1: We first establish the conclusion (3.43). Suppose that inequality is strictly “greater than”. Let \((x_0, t_0)\) be such that \( u(x_0, t_0) = \min_{U_T} u \). Suppose \((x_0, t_0) \in \Omega \times (0, T]\), then \( u_t(x_0, t_0) \leq 0 \). On the other hand, because \( \gamma^\varepsilon \) is nonnegative with support \( B(0, \varepsilon) \), one has
\[
\int_{\Omega_w} \left( u(y, t_0) \gamma^\varepsilon(x_0 - y) - u(x_0, t_0) \gamma^\varepsilon(y - x_0) \right) dy \\
= \int_{\Omega_w} \left( u(y, t_0) \gamma^\varepsilon(x_0 - y) - u(x_0, t_0) \gamma^\varepsilon(x_0 - y) \right) dy \geq 0,
\]
which leads to a contradiction. Hence \((x_0, t_0) \in \bar{\Gamma}_T\).

Step 2: Denote \( u^\delta(x, t) = u(x, t) + \delta t \) with \( \delta > 0 \); then \( u_t^\delta(x, t) = u_t(x, t) + \delta \), and
\[
u_t^\delta(x_0, t_0) > \int_{\Omega_w} \left( u(y, t_0) \gamma^\varepsilon(x_0 - y) - u(x_0, t_0) \gamma^\varepsilon(y - x_0) \right) dy.
\]
By the argument made in step 1, \( \min_{U_T} u^\delta = \min_{\bar{\Gamma}_T} u^\delta \). Letting \( \delta \to 0 \) leads to the desired conclusion (3.43).

Step 3: Replace \( u \) by \(-u\) in step 2 above to immediately obtain (3.45).

We remark that the condition (2.20) on \( \gamma^\varepsilon \) is a generalization of (2.6) to subdomains of \( \mathbb{R}^n \). We now show that the nonlocal problem (2.4) has a generalized maximum principle, i.e., the maximum or minimum value of \( u \) can be observed on \( \bar{\Gamma}_T \), that is, either at the initial time or on \( \Omega_w \backslash \Omega \).

Theorem 2.3.4. (Maximum principle for nonlocal volume-constrained problems)
Assume that \( u \) solves the nonlocal problem (2.4), \( u(x, t) \in C^1([0, T]) \) for any \( x \in \Omega \), and \( u \) obtains its maximum and minimum in \( U_T \). Assume in addition that (2.20) holds, then
\[
\max_{U_T} u = \max_{\bar{\Gamma}_T} u, \quad \min_{U_T} u = \min_{\bar{\Gamma}_T} u.
\]

Proof. The nonlocal problem (2.4) satisfies the two inequalities of Lemma 2.3.3, which leads to (3.43) and (3.45). 

2.3.4 Dispersion relation

We now look at the dispersion relation of the solution to the nonlocal Cauchy problem (2.9). Dispersion occurs when waves of different wavelengths have different propagation speeds. The wave speed, $v$, depends on the angular frequency $\varpi$ and nonnegative wavenumber $k = (k_1, \ldots, k_n)$ via $v \cdot k = \varpi$. A representative mode is described by $u(x, t) = e^{i(k \cdot x + \varpi t)}$ so that substitution into (2.9) yields

$$\varpi = \int_{\mathbb{R}^n} \sin(k \cdot z) \gamma^\varepsilon(-z) dz + 2i \int_{\mathbb{R}^n} \sin^2\left(\frac{k \cdot z}{2}\right) \gamma^\varepsilon(-z) dz.$$  

For the $\gamma^\varepsilon$ given in equation (2.8),

$$\varpi = \frac{c_1 \varepsilon k}{2} \left(\frac{\sin(k \varepsilon)}{k \varepsilon}\right)^2 + (c_1 \varepsilon^{-1} + c_2 \varepsilon^{-2}) \left(1 - \frac{\sin(k \varepsilon)}{k \varepsilon}\right) i.$$  

Dividing (3.106) by $k$ yields the dispersion relation

$$\frac{\varpi}{k} = \frac{c_1 \varepsilon}{2} \left(\frac{\sin(k \varepsilon)}{k \varepsilon}\right)^2 + \frac{c_1 \varepsilon^{-1} + c_2 \varepsilon^{-2}}{k} \left(1 - \frac{\sin(k \varepsilon)}{k \varepsilon}\right) i$$  

for the nonlocal equation (2.9)$_1$. One can easily verify that as $\varepsilon \to 0$, (2.23) converges to the dispersion relation of the local equation (2.24).

Moreover, by (2.2.2), we see that $\text{Im}(\varpi) \geq 0$ which yields

$$\left|e^{i \varpi t}\right| = e^{\text{Im}(\varpi) t} = e^{-\text{Im}(\varpi) t},$$

implying that the amplitude of each Fourier mode is non-increasing in time, a reflection of the dissipative nature of the underlying model.

2.3.5 Dissipation

The dispersion relation for the nonlocal Cauchy problem reveals the precise rate of dissipation of Fourier modes in the special one dimensional case. The following result demonstrates that dissipation can be associated with more general nonlocal convection diffusion problems in direct analogy to their local counterparts.
Lemma 2.3.5. Given \( u \) the solution of (2.4), if \( \Omega_w = \mathbb{R}^n \), then
\[
\frac{d}{dt} \int_{\Omega_w} u^2(x, t) dx \leq 0.
\]

Proof. We take \( v = u \) in the weak form (2.16) to get
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega_w} u^2(x, t) dx = \int_{\Omega_w} \int_{\Omega_w} \left( u(y) - u(x) \right)^2 \alpha^\varepsilon \cdot (\Theta \alpha^\varepsilon)(x - y) dy dx
\]
\[
- \int_{\Omega_w} \int_{\Omega_w} u(y)(\mu \cdot \alpha^\varepsilon)(x - y) \left( u(x) - u(y) \right) dx dy.
\]

By the fact that \( \mu \cdot \alpha^\varepsilon \) is antisymmetric, we have
\[
\int_{\Omega_w} \int_{\Omega_w} u(y)(\mu \cdot \alpha^\varepsilon)(x - y) u(x) dx dy = 0, \quad \int_{\mathbb{R}^n} (\mu \cdot \alpha^\varepsilon)(x - y) dx = 0,
\]

we get
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega_w} u^2(x, t) dx = - \int_{\Omega_w} \int_{\Omega_w} \left( u(y, t) - u(x, t) \right)^2 \alpha^\varepsilon \cdot (\Theta \cdot \alpha^\varepsilon)(x - y) dy dx \leq 0,
\]

which leads to the conclusion in the lemma. \( \square \)

According to our problem formulation, \( \Omega_w = \mathbb{R}^n \) is valid for both nonlocal Cauchy problems and homogeneous nonlocal Dirichlet volume constrained problems.

2.4 Markov jump process

The purpose of this section is to construct the Markov process associated with the nonlocal diffusion problem (2.4) with a volume constraint.

Let \( X_t \) be a jump process confined to remain in \( \Omega \) when conditioned on \( X_0 \in \Omega \) unless \( X_t \in \Omega_d \subset \mathbb{R}^n \setminus \Omega \), i.e., the process exits \( \Omega \), at which point the process is absorbed. This exit-time is given by the random variable
\[
\tau := \inf\{t > 0, X_t \in \Omega_d | X_0 \in \Omega\}.
\]

If we select the jump-rate equal to the kernel (2.7b) then the nonlocal system (2.4) evolves the probability density for the random variable \( \tau \) and represents the particles remaining in \( \Omega \). The first integrand of (2.7a) represents the rate \( \gamma^\varepsilon(x - y) dx \) to \( dx \) from \( y \) given the
probability \(u(y, t) \, dy\) while the second integrand of (2.7a) represents the rate \(\gamma^\varepsilon(y - x) \, dy\) to \(dy\) from \(x\) given the probability \(u(x, t) \, dx\). The difference in these two rates gives the rate of change of the probability \(u(x, t) \, dx\). The volume constraint \(u = 0\) over \(\Omega_d\) for \(t > 0\) for (2.4) ensures that the process is absorbed, i.e., does not reenter the domain \(\Omega\). At steady-state, the rates are equal and \(u_t \equiv 0\) over the domain \(\Omega\). We also see that the process \(X_t\) is Markov because \(X_{t+t'}\) for \(t' > 0\) only depends upon \(X_t\). Because the sample path transitions to a point at most a finite distance \(\varepsilon\) from \(X_t\) and the rate \(\gamma^\varepsilon\) is not assumed to be symmetric, the process sample path “jumps” are not symmetric and are discontinuous. We refer to such a process as a finite-range nonsymmetric jump process. This is in stark contrast to Brownian motion (without drift) where the sample path is continuous with equal probability transitions out of a point.

Two cases are of particular interest. The first case occurs when \(\Omega_d = \mathbb{R}^n \setminus \bar{\Omega}\) so that we have an absorbed process with master equation given by the nonlocal homogenous Dirichlet problem (2.10). The second case occurs when \(\Omega_d = \emptyset\) so that we have a censored process \([\Omega]\) with master equation given by the nonlocal homogeneous Robin volume-constrained problem (2.11). In general, however, the nonlocal system (2.4) is the master equation for a mixed absorbed/censored process.

We now investigate the conservation of probability over the domain \(\Omega\). Because

\[
\frac{d}{dt} \int_{\Omega} u(x, t) \, dx = \int_{\Omega} \int_{\Omega_d} (u(y, t)\gamma^\varepsilon(x - y) - u(x, t)\gamma^\varepsilon(y - x)) \, dy \, dx,
\]

\[
= \int_{\Omega} \int_{\Omega_d} (u(y, t)\gamma^\varepsilon(x - y) - u(x, t)\gamma^\varepsilon(y - x)) \, dy \, dx,
\]

\[
= - \int_{\Omega} \int_{\Omega_d} u(x, t)\gamma^\varepsilon(y - x) \, dy \, dx < 0,
\]

where the second equality follows by imposing the volume constraint, we then have

\[
\frac{d}{dt} \int_{\Omega} u(x, t) \, dx = - \int_{\Omega} \int_{\Omega_d} (u(y, t)\gamma^\varepsilon(x - y) - u(x, t)\gamma^\varepsilon(y - x)) \, dy \, dx = 0
\]

so that integrating in time grants that

\[
\int_{\Omega} u(x, t) \, dx = \int_{0}^{t} \int_{\Omega} \int_{\Omega_d} (u(y, t)\gamma^\varepsilon(x - y) - u(x, t)\gamma^\varepsilon(y - x)) \, dy \, dx \, ds = \int_{\Omega} u_0(x) \, dx = 1.
\]

In words, at any time \(t\), the particle is either located in \(\Omega\) or \(\Omega_d\) with probability given by
the first and second integrals above, respectively. In particular, the second integral is the
time-integrated flux into $\Omega_d$ out of $\Omega$. If $\int_{\Omega} u(x, t) \, dx \to 0$ as $t \to \infty$, then the probability of
the sample-path ending up in $\Omega_d$ tends to 1. The rate at which the probability escapes $\Omega$
decreases as the measure of $\Omega_d$ tends to zero and corresponds to our intuition that exiting is
more difficult when $\Omega_d$ is small. When $\Omega_d = \emptyset$ then a censored process results so that the
particle is relegated to remain in $\Omega$ for all time since the probability is conserved over $\Omega$.

2.5 Discretization and numerical experiments

In this section we present numerical solutions some one-dimensional nonlocal problems
associated with the previously discussed nonlocal Cauchy problem, Dirichlet and Robin
volume-constrained problems. In order to use a bounded domain to simulate the Cauchy
problem, we use a spatial periodic problem instead.

The solutions to the nonlocal problems are computed using both Monte Carlo simulations
and finite difference schemes. Experiments are conducted in groups to study the effects of
kernel, horizon and different volume constraints on the dynamics.

We consider the domain $\Omega = (0, 1)$ and the kernel $\gamma^\varepsilon$ as in (2.8), where $\varepsilon < 1$, $c_1$ and $c_2$
are taken as nonnegative constants, and referred to as the nonlocal convection and diffusion
coefficients, respectively. Indeed, for the $\gamma^\varepsilon$ under consideration, the corresponding local
equation, recovered under the limit $\varepsilon \to 0$, is

$$u_t = c_1 u_x + c_2 u_{xx}. \tag{2.24}$$

However, we caution the reader that nonlocal convection corresponds to nonsymmetric
diffusion for finite $\varepsilon$ and only becomes convection in the limit $\varepsilon \to 0$. Equivalently, non-
local convection is associated with a nonsymmetric jump process whereas convection is a
deterministic motion, or drift.

Two distinct initial data will be used, one is a linear function and the other is piecewise
constant:

$$u_0(x) = 2 - 2x, \quad x \in [0, 1], \tag{2.25}$$

$$u_0(x) = \begin{cases} 
0, & x \in [0, \frac{1}{2}), \\
2, & x \in [\frac{1}{2}, 1].
\end{cases} \tag{2.26}$$

To account for the stochastic interpretation of the nonlocal models, these initial condition
are taken as a probability density functions over $[0, 1]$.

To compare Monte Carlo simulation and finite difference schemes, we use the same spatial grid points for both methods. The grid points are denoted as $\{x_i\}_{i=0}^{N}$ with $x_i = \frac{i}{N}$, which partition $[0, 1]$ into $N$ bins $\{\Omega_i = (x_i, x_i + h)\}_{i=0}^{N-1}$ of equal size $h = 1/N$. We also consider some equally spaced grid points $\{x_i \in \bar{\Omega}_d\}_{i \in I_d}$ outside $\Omega$ where $I_d$ is an index set containing suitable indices that are either negative or larger than $N$. Let $I_w = \{i, 1 \leq i \leq N - 1\} \cup I_d$.

### 2.5.1 Monte Carlo methods

Monte Carlo simulations are applied to the nonlocal problem under consideration to approximate the solution $u$, which is a probability density function on $\mathbb{R}$. With $Y^j$ denoting the position of the $j$-th particle, the simulation is implemented as follows:

1. Input: the solution domain $\Omega = (0, 1)$, the domain $\Omega_w = \Omega \cup \Omega_d$, number of cells $N$ in $\Omega$, $h = 1/N$, grid points $\{x_i\}$ in $\bar{\Omega}_w$, cells $\{\Omega_i\}$, the terminal time $T$, time step size $\delta t$, the total sample size $M$, the horizon $\varepsilon$, jump-rate kernel (2.7b), the exponential wait-time density $\omega(t) := \frac{1}{\|\gamma\|_{L^1(\mathbb{R}^1)}}e^t/\|\gamma\|_{L^1(\mathbb{R}^1)}$ and initial data $u_0$ which is assumed to be a probability density function.

2. For $j$ from 1 to $M$: set the time $t = 0$, and sample initial position of the $j$-th path: $Y^j \sim u_0$.

   While $t < T$, and $Y^j \in \Omega$,
   - sample $\delta Y \sim \frac{\gamma(x)}{\|\gamma\|_{L^1(\mathbb{R}^1)}}$;
   - update $Y^j \leftarrow Y^j + \delta Y \mathbb{1}_{\Omega_w}(Y^j + \delta Y)$;
   - sample $\delta t \sim \omega(\cdot)$;
   - update $t \leftarrow t + \delta t$;

3. Output: $u(x_i, T) \approx \frac{1}{Mh} \sum_{j=1}^{M} \mathbb{1}_{\Omega_i}(Y^j)$, for $i = 0, \ldots, N - 1$.

Several comments are in order. First, the above three step algorithm simulates the nonlocal Cauchy problem, Dirichlet and Robin volume-constrained problems. For instance, for the Cauchy problem, since a spatial periodic formulation is used, $Y^j \leftarrow \text{mod}(Y^j, 1)$. For the third problem, $\Omega_w \equiv \Omega$ so that the particle remains in $\Omega$. Our second comment is that in Step (3), if the $j$-th path exits the domain, i.e., $Y^j \notin \Omega$, then the sample path is not located in any bin $\Omega_i$ and so $\mathbb{1}_{\Omega_i}(Y^j) = 0$ for all $i$. Hence this path does not contribute to
the summation in the statistics for the output.

2.5.2 Finite Difference scheme

Finite Difference schemes are applied to both nonlocal and local periodic, Dirichlet and Robin problems.

2.5.2.1 Finite Difference scheme for nonlocal problems

Composite trapezoidal rule is used to discretize the integral in the nonlocal problem (2.4). Denote the time-step as \( \Delta t \), and \( u^k_m \) as the finite difference solution at grid point \( x_m \) at time \( k\Delta t \). Then for the \( (k+1) \)-th time level, and \( 1 \leq m \leq N-1 \), we have

\[
\frac{u^{k+1}_m - u^k_m}{\Delta t} = h \sum_{m+i \in I_w} \omega_i \left( \gamma^\varepsilon (-ih)u^k_{m+i} - u^k_m \gamma^\varepsilon (ih) \right),
\]

(2.27)

where the coefficients \( \omega_i \) are equal to 1 except at the end points where \( w_i = 0.5 \). We let \( u^k_{m+1} = 0 \) for \( k \in I_d \). For the periodic problem, when \( m \pm i \notin \{0,1,\ldots,N\} \), we replace \( u^k_{m\pm1} \) by \( u^k_p \), with \( p = \text{mod}(m \pm i, N) \).

The summation in (2.27) are taken differently for the three types of nonlocal problems corresponding to the Cauchy problem

With suitable conditions on the volume constraints and the index set \( I_w \), one can derive a discrete Maximum Principle under the condition:

\[
\Delta t \leq \min_m \frac{1}{\sum_{m+i \in I_w} h \omega_i \gamma^\varepsilon (ih)} \sim \frac{1}{\| \gamma^\varepsilon \|_{L^1(\mathbb{R}^n)}}.
\]

2.5.2.2 Finite Difference scheme for the local problem

The corresponding local equation (2.24) is discretized via forward-in-time upwinding finite difference scheme:

\[
\frac{u^{k+1}_m - u^k_m}{\Delta t} = c_1 \frac{u^k_{m+1} - u^k_m}{h} + c_2 \frac{u^k_{m+1} - 2u^k_m + u^k_{m-1}}{h^2}
\]

(2.28)

or equivalently,

\[
u^{k+1}_m = \left( \frac{c_1 \Delta t}{h} + \frac{c_2 \Delta t}{h^2} \right) u^k_{m+1} + \left( 1 - \frac{2c_2 \Delta t}{h^2} - \frac{c_1 \Delta t}{\Delta x} \right) u^k_m + \frac{c_2 \Delta t}{h^2} u^k_{m-1}.
\]
The local periodic and homogeneous Dirichlet boundary conditions, \( u(x + 1) = u(x) \) and \( u(0) = u(1) = 0 \), can be embedded into scheme (2.28) in a straightforward way. The Robin boundary condition \( c_1 u(0) + c_2 u'(0) = c_1 u(1) + c_2 u'(1) = 0 \) is discretized as

\[
\begin{align*}
    u_{0}^{k+1} &= \left(1 - \frac{c_2 \Delta t}{\Delta x^2}\right) u_{0}^{k} + \left(\frac{c_1 \Delta t}{h} + \frac{c_2 \Delta t}{h^2}\right) u_{1}^{k}, \\
    u_{N}^{k+1} &= \left(\frac{c_2 \Delta t}{\Delta x^2}\right) u_{N-1}^{k} + \left(1 - \frac{c_1 \Delta t}{\Delta x} - \frac{c_2 \Delta t}{h^2}\right) u_{N}^{k}.
\end{align*}
\]

Under the stability condition,

\[2c_2 \frac{\Delta t}{h^2} + c_1 \frac{\Delta t}{h} \leq 1,
\]

the finite difference discretization (2.28) preserves Maximum Principle, conservation law, and also guarantees that \( u_{m}^{k} \geq 0 \) for all \( k \) and \( m \).

### 2.5.3 Numerical experiments

We first show that the numerical solutions to nonlocal problems by Monte Carlo method and the Finite Difference schemes are consistent. Figure 1 (a-c) demonstrate the time-evolutions of numerical solutions from both methods using linear initial data (2.25), for nonlocal periodic, Dirichlet and Robin problems respectively. The blue, cyan, green, magenta and red curves represent fluctuations correspond to the Monte Carlo solutions at time \( T = 0.01, 0.05, 0.09, 0.13, 0.17 \) respectively. The black lines provide the corresponding nonlocal finite difference solutions. Here the parameters are taken as \( \varepsilon = 0.04 \), \( c_1 = 4 \) and \( c_2 = 6 \). The Monte Carlo simulation uses \( N = 500 \) spatial cells on \([0, 1]\) and \( M = 10^6 \) sample paths while the Finite Difference scheme uses \( N = 1000 \) spatial cells and a time step-size of \( 10^{-4} \). Figure 1 (d-f) is from the same experiment except that the initial data is piece-wise constant given by (2.26).

Next we look at the effects of parameters \( c_1 \) and \( c_2 \) in the kernel of the nonlocal problems, using linear initial data (2.25) and Monte Carlo method. We fix the parameters \( T = 0.01, \varepsilon = 0.04 \), the number of spatial cells \( N = 500 \), and the number of paths \( M = 10^6 \). Figure 2 (a-c) show the solutions for \( c_1 = 0, 20, 40, 80, 160 \), in blue, cyan, green, magenta and red, respectively, with \( c_2 = 36 \). One can see that for any of the three nonlocal problems, larger \( c_1 \) gives faster nonlocal convection. Moreover, when \( c_1 \) is much larger than \( c_2 \), convection dominates diffusion, and as \( c_1 \) keeps increasing, the solution tends to build up a boundary
layer near the left boundary. This is because the nonlocal convection is to the left, as shown in the local limit (2.24). Figure 2 (d-f) show the nonlocal solutions for \(c_2 = 0, 6, 24, 96, 384\), in blue, cyan, green, magenta and red, respectively, with \(c_1 = 4\). One can see that for any of the three nonlocal problems, larger \(c_2\) gives faster diffusion. The case of \(c_2 = 0\) yields a strong convection effect and again, one sees the boundary layer built up. The boundary layer starts to be diffused more as \(c_2\) gets bigger.

Finally, we look at the behavior of the nonlocal solutions for decreasing horizon \(\varepsilon\). The computed results given in Fig 3 (a,c,e) and their respective zoomed-in pictures in Fig 3 (b,d,f) show that, for any of the three nonlocal problems, the nonlocal solution given by Monte Carlo simulations converge to the corresponding local solution computed by upwinding finite difference scheme, as predicted by the local limit analysis presented earlier. The blue, cyan, green, magenta and red curves correspond to nonlocal Monte Carlo solutions with \(\varepsilon = 0.2, 0.1, 0.05, 0.025\), respectively, and the black curve is the local finite difference solution. In this experiment we take the initial condition as linear function in (2.25), and fix as constants \(T = 0.01\), \(c_1 = 2\), \(c_2 = 6\), \(N = 500\) spatial cells and \(M = 10^7\) sample paths for Monte Carlo simulations, and \(N = 1000\) spatial cells and a time step-size of \(10^{-5}\) for the Finite Difference scheme.

### 2.6 Summary

In this work, we provide a general description of initial and initial volume-constrained problems associated with a linear nonlocal convection-diffusion equation. Some special cases include the nonlocal Cauchy problem and problems involving the nonlocal homogenous Dirichlet and Robin (or more general mixed type) volume-constrained conditions. We discuss results related to the well-posedness of these nonlocal problems, their local limits, the Maximum Principle, the conservation law and the dispersion relations. All these properties are consistent with those of the corresponding local problems as the horizon parameter vanishes. Moreover, the stochastic processes associated with these nonlocal problems are presented. Monte Carlo simulations and finite difference schemes are applied to solve the nonlocal problems, showing the effects of time, kernel, horizon and different volume constraints. Our results can be generalized to other similar cases such as the fractional differential convection-diffusion equations.
Fig. 2.1: Comparison of time evolutions of nonlocal solutions given by Monte Carlo method (fluctuating colored curves) and finite difference scheme (smooth black curves).
Fig. 2.2: Monte Carlo solutions showing effects of parameters $c_1$ and $c_2$. 
Fig. 2.3: Convergence of the nonlocal solution to the local solution when horizon $\varepsilon$ being reduced.
Chapter 3  
Numerical Studies of A Class of Nonlocal Conservation Laws

3.1 Introduction

3.1.1 Motivations

In Finite Difference the classical PDE theory, invicid Burger’s equation

\[ u_t + uu_x = 0, \quad x \in \mathbb{R}, t > 0. \]  \hspace{1cm} (3.1)

is one of the simplest and most important nonlinear conservation law. It has many important properties, such as non-uniqueness of the weak solutions, uniqueness of the entropy solution, and shock formation from smooth initial condition. A pivotal method for finding its entropy solution is the viscosity approach. It adds to the equation a viscosity term, such as \( \varepsilon u_{xx} \), to get the “viscous” Burger’s equation

\[ u_t + uu_x = \varepsilon u_{xx}, \quad x \in \mathbb{R}, t > 0, \]  \hspace{1cm} (3.2)

where \( \varepsilon > 0 \) is a small coefficient. By doing this, the pure convection equation (3.1) becomes a diffusion equation of \( u^\varepsilon \), whose unique solution is much easier to solve. Then one can prove that the viscosity solution \( u^\varepsilon \) converges to the unique entropy solution \( u \) of the invicid Burgers’ equation (3.1). One can also choose to add other viscous terms involving second-order or higher order partial derivatives, and a lot of work has been done in this direction.
Our idea is to propose a “nonlocal” version of the viscous Burger’s equation (3.2), where the nonlinear convection term $uu_x$ together with the viscous term $\varepsilon u_{xx}$ is replaced by one single integral of $u$, which involves some kernel function $\omega^\delta$ depending on a small parameter $\delta$. We want to propose it in such a way that, as the parameter $\delta \to 0$, its solution $u^\delta$ converges to the entropy solution of the inviscid Burger’s equation (3.1). An advantage of such “nonlocal” equation is that it reduces the regularity requirement on $u$, because it involves only an integral with respect to $x$, rather than partial derivatives on $x$ in classical differential equations.

The motivation for our work of nonlocal conservation laws also relates to the theory of peridynamics. Peridynamics is a nonlocal continuum theory that has been developed postulated by Silling [20] and successfully applied to critical phenomena such as material failure [19]. To illustrate the relevancy of our work, a linear peridynamic model for an infinite one-dimensional bar [21] can be written as

$$u_{tt}(x,t) = \int_R (u(y,t) - u(x,t))\phi^\delta(y-x)dy + b(x,t),$$

(3.3)

where $u$ denotes the displacement and $\phi = \phi^\delta$ is an even kernel function reflecting the material properties. In peridynamics, the kernel $\phi^\delta$ is taken to be a function supported on a spherical neighborhood within a radius $\delta$ (the peridynamic horizon) to account for nonlocal interactions. It has been shown that, when body force $b(x,t) = 0$, (3.3) can be rewritten into a system of two first order in time nonlocal equations —in direct analogy to the local case where second order linear wave equation can be split into a system of two first order convection equations [74]. Thus, instead of considering the second order in time peridynamic equation corresponding to elastic waves, we consider the first order in time, nonlocal nonlinear convection equation in our work, the “nonlocal nonlinear conservation laws”.

### 3.1.2 Previous work on nonlocal convections

Here we review some previous work on nonlocal models involving convection effects.

- Nonlocalization through fractional differential operators. In some studies, the local convection operators are maintained, and nonlocality is introduced through a fractional derivative operator that modifies the diffusive term in Burger’s equation; see, e.g.,
Dronio [56], Alibaud and coworkers [57,58]. Alternatively, others introduce nonlocality through and convection flux and retain or generalize the diffusive term of the viscous Burger’s equation; see, e.g., Ervin, Hewer and Roop [59], Biler and Woyczyński [60], Woyczyński [61], and Miśkinis [62].

- Standard, local flux, nonlocalized regularization. There is a significant literature for equations with nonlocal regularization term. For example, [63] consider the equation $u_t + uu_x = (\phi[u])_x$ where $\phi[u]$ is the nonlocal operator defined as $\phi[u] = \int G(x-y)u(y)dy$; this approach is generalized and analyzed in different ways by Liu [64], Chmaj [69], Duan,Feller and Zhu [65], Rohde [66], Kissling and Rohde [67], and Kissling, LeFloch and Rohde [68].

- Nonlocal convection through nonlocal wave speed. Logan [70] considers equations in the form of $u_t + (\int G(u)dy)u_x = 0$, where $G$ is a specified function.

- Nonlocalization through generalized flux, no regulation. Benzoni-Gavage [71] considers existence and stability for the generalize Burger’s equation $u_t + \mathcal{F}[u]_x$, where the Fourier transform of the operator $\mathcal{F}$ is given by $\hat{\mathcal{F}}[u](k) = \int \Gamma(k-l)\hat{u}(k-l)\hat{u}(l)dl$. Alì, Hunter and Parker [72] provide the motivation for and describe properties of the kernel $\Gamma$ for a generalized Burger’s equation of this form.

- Nonlocalized Convection and nonlocalized diffusion. Ignat and Rossi [73] analyze the nonlocal evolution equation

$$
\frac{\partial u}{\partial t}(x,t) = \int_{\mathbb{R}^n} (u(y,t) - u(x,t))J(y-x)dy \\
+ \int_{\mathbb{R}} (h(u)(y,t) - h(u)(x,t))K(y-x)dy, \quad (x,t) \in \mathbb{R}^d \times (0,\infty)
$$

where $J$ is even, $J$ and $K$ are density functions, and $h$ is nondecreasing, locally Lipschitz continuous with $h(0) = 0$.

Perhaps the the approach most closely related to our work is that considered by Du, Kamm, Lehoucq and Parks [74]. They analyze the nonlocal evolution equation

$$
\frac{\partial u}{\partial t}(x,t) = \int_{\mathbb{R}^n} \psi \left( \frac{u(y,t) + u(x,t)}{2} \right) \phi(y-x)dy = 0,
$$
where $\phi$ is an odd function, and $\psi$ is a generalized flux function. They identify the nonlocal flux, describe the connection to a nonlocal viscous regularization, which mimics the viscous Burger’s equation in an appropriate limit. They also demonstrate that, for integrable kernel, the nonlocal solutions do not develop a shock in finite time as long as the value of $u$ stays finite.

### 3.1.3 Our work on nonlocal conservation laws

The initial-value problems of a scalar hyperbolic conservation law

$$
\begin{align*}
&\begin{cases}
  u_t + \partial_x f(u) = 0, \\
  u(x, 0) = u_0(x),
\end{cases} \\
&\quad (x, t) \in \mathbb{R} \times [0, T], \\
&\quad x \in \mathbb{R},
\end{align*}
$$

have been thoroughly investigated, both analytically and numerically. It’s well known that even if the initial value $u_0$ is smooth, the solution typically develops discontinuities, or shock waves, as $t$ increases to some $t_0 > 0$. Thus the equation must be understood in a weak sense. However, the weak solutions are often non-unique, and normally people only concern about the physically relevant unique weak solution, the entropy solution. Numerically, a variety of algorithms have been developed to capture the entropy solution, among which the most basic and widely-used category is the Finite Difference (FD) method. Its simplest case is the three-point conservative scheme, taking the form

$$
u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x} [g(u_j^n, u_{j+1}^n) - g(u_{j-1}^n, u_j^n)],$$

where numerical flux $g$ is Lipschitz continuous, and is consistent with the flux function $f$ in that $g(u, u) = f(u)$. It’s well known that if $g$ is monotone, i.e., non-decreasing on the first argument, and non-increasing on the second argument, then the scheme (3.5) is monotone, and hence is Total-Variance Stable (TVS) and enjoys Maximum Principle in the discrete level [46,55]. Picking appropriate $g$ with such properties, the numerical solution of scheme (3.5) converges to the unique entropy solution of the local conservation law (3.4).

In this paper we propose a nonlocal generalization to the classical scalar conservation law (3.4), in the following form:

$$
u_t(x) + \int_0^\delta \left( \frac{g(u(x), u(x + h)) - g(u(x - h), u(x))}{h} \right) \omega^\delta(h) dh = 0,$$

(3.6)
where $g$ is consistent with some flux function $f$ and is monotone just like in the classical case (Here the dependence of $u$ on $t$ is omitted for convenience). Formally, equation (3.6) mimics the three-point conservative FD scheme (3.5), except that the pure difference approximation is replaced by an integral with a kernel function $\omega^\delta$, which depends on a parameter $\delta > 0$ called the nonlocal "horizon". Equation (3.6) is "nonlocal", in the sense that the interaction between two spatial points $x$ and $y$ is allowed as long as their distance is no larger than $\delta$. In contrast, the classical conservation law (3.4) is "local", since the derivative of flux $f(u)_x$ implies that interaction happens only within infinitesimal distance. By taking some special kernel $\omega^\delta$, the nonlocal conservation law (3.6) reduces to the local one (3.4) (see Section 3.4.2), so our nonlocal conservation law can be seen as an extension of its local counterpart. The choice of kernel $\omega^\delta$ greatly affects the solution properties, and in our work, we only consider nonnegative integrable kernel $\omega^\delta \in L^1(\mathbb{R})$, with the support of $\omega^\delta$ being in the interval $[0, \delta]$.

This paper studies on the initial-value problems of scalar nonlocal conservation law (3.6), in the following aspects:

1. Show the consistency of the nonlocal conservation law with the local one;
2. Find the condition under which the solution of nonlocal conservation law do not develop shocks from smooth initial condition $u_0$;
3. Provide a numerical scheme for the nonlocal conservation law, and prove the numerical convergence when $\delta$ is fixed and grid size $\Delta x \to 0$;
4. Prove that when $\delta$ and $\Delta x$ both go to zero, the entropy entropy solution of the nonlocal conservation law converges to the entropy solution of the local conservation law;
5. Existence and uniqueness of the entropy solution to the nonlocal conservation law;
6. Numerical experiments are performed to further explore the solution properties.

To show the consistency of nonlocal conservation law (3.6) and its local counterpart (3.4), we first identify the flux of nonlocal conservation law (Section 3.4.1), show that the nonlocal conservation law reduces to the local one under special kernel (Section 3.4.2), and prove the Maximum Principle (Section 3.4.4). We also show that, if $\omega(h) / h$ is integrable, then the nonlocal solutions develop no shock in finite time (Theorem 3.4.8).

Numerically, we propose a monotone scheme (3.57) with Maximum Principle and the Total-Varience-Stable property, and show that the "nonlocal Lax-Friedrich’s scheme" can be rewritten into the form of (3.57) with a new flux function $\tilde{g}$, just as the local case (Section
In the main convergence theorem of our work, Theorem 3.6.10, we show that, under the nonlocal CFL condition, when \( \delta \) is fixed and \( \Delta x \to 0 \), the numerical solution of the nonlocal scheme (3.57) converges to the entropy solution of the nonlocal conservation law; while as \( \delta \) and \( \Delta x \) both go to zero, the numerical solution converges to the entropy solution of the local conservation law. The proof of this theorem also provides the existence of the entropy solution for the nonlocal conservation law (Theorem 3.6.11), which, combining with the uniqueness Theorem 3.3.3, establishes the wellposedness of the nonlocal entropy solution.

In our numerical experiments, we taking \( g \) as the numerical flux of Finite Difference scheme for local Burger’s equation, so that our nonlocal conservation law can be seen as a “nonlocal Burger’s equation”. We perform five groups of experiments for the nonlocal scheme (3.57) with this \( g \), using different initial data \( u_0 \) and kernels \( \omega^\delta \):

(i) Fixing \( \delta \) and refining \( \Delta x \), the numerical solution of nonlocal Burger’s equation converges;
(ii) Fixing \( \Delta x \) and refining \( \delta \), the numerical solution of nonlocal Burger’s equation converges to Finite Difference solution of local Burger’s equation;
(iii) Fixing the number of interaction cells, \( r = \lfloor \frac{\delta}{\Delta x} \rfloor \), and refining both \( \delta \) and \( \Delta x \) at the same time, the numerical solution of nonlocal Burger’s equation converges to the entropy solution of local Burger’s equation;
(iv) Fixing \( \delta \) and using piecewise constant initial condition \( u_0 \), we observe the propagation of the discontinuities;
(v) Fixing \( \delta \) and using smooth initial condition \( u_0 \), we look at the time evolution of nonlocal solutions for different kernel \( \omega^\delta \). The numerical solution appears to develop no shocks when \( \frac{\omega^\delta(h)}{h} \in L^1(\mathbb{R}) \), which is consistent with Theorem 3.4.8, and develop shocks when \( \frac{\omega^\delta(h)}{h} \notin L^1(\mathbb{R}) \), suggesting that the condition (3.47) is not only sufficient but also necessary for this theorem.

The paper is organized as follows: Section 2 reviews some relevant facts of local conservation laws. Section 3 introduces the nonlocal conservation laws and shows the uniqueness of its entropy solution. Section 4 investigates some basic model properties, and proves that the under certain condition of kernel \( \omega^\delta \), nonlocal solution develops no shock in finite time under smooth initial condition. In Section 5 we introduce our numerical scheme, with different choices of numerical flux \( g \). Section 6 proves that under the nonlocal CFL condition, the numerical solution of our scheme converges to the unique entropy solution of the nonlocal

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(local) conservation law when $\delta$ is fixed and $\Delta x \to 0$ (when $\delta$ and $\Delta x$ both go to zero). Numerical experiments are in Section 7. We summarize our work in section 8.

### 3.2 Local hyperbolic conservation laws

In this section we briefly review some preliminaries of local conservation laws that are relevant to our work of nonlocal models. It consists of two parts. The first part states some basic properties of local conservation law, and the second part is about its Finite Difference schemes.

We start with the definition of the “entropy solution” of local conservation law (3.4).

**Definition 3.2.1.** An entropy solution of the classical conservation law (3.4) on $\mathbb{R} \times (0, T)$ is a function $u \in L^\infty(\mathbb{R} \times (0, T))$ such that

$$
\int_0^T \int_\mathbb{R} |u - c| \phi_t + \text{sgn}(u - c)(f(u) - f(c))\partial_x \psi dx dt \geq 0,
$$

for every $\phi \in C^1_0(\mathbb{R} \times (0, T))$ with $\phi \geq 0$ and every $c \in \mathbb{R}$. \hfill (3.7)

We refer to (3.7) as the entropy inequality for the local conservation law, where $\eta(u, c)$ is the Kružkov’s entropy function, and the corresponding Kružkov’s entropy flux is

$$
q(u, c) = \text{sgn}(u - c)(f(u) - f(c)). \hfill (3.8)
$$

### 3.2.1 Basic facts

Some basic facts concerning local hyperbolic conservation laws:

- The evolution of spontaneous shock discontinuities with requires weak solution (in distributional sense) of (3.4).
- The existence of possibly infinitely many weak solutions.
- The entropy solution can be realized as a viscosity limit solution, $u = \lim_{\varepsilon \to 0} u^\varepsilon$, where $\varepsilon$ is the viscosity parameter, and $u^\varepsilon$ solves

$$
u^\varepsilon_t + [f(u^\varepsilon)]_x = (Qu^\varepsilon)_x, \quad \varepsilon Q > 0.$$

• Scalar conservation law (3.4) has Maximum Principle.

• $L^1$ contraction: If $u^1$ and $u^2$ are two entropy solutions of scalar conservation law (3.4), then

$$||u^2(\cdot, t) - u^1(\cdot, t)||_{L^1(x)} \leq ||u^2_0 - u^1_0||_{L^1(x)}.$$  \hspace{1cm} (3.9)

By the Crandall-Tartar lemma [45], the $L^1$ construction property is equivalent with monotonicity of the solution operator, assuming that this operator is conservative.

• TV bound. The solution operator of (3.4) is translational invariant. For scalar entropy solution $u(x, t)$, the $L^1$ contraction in (3.9) yields the Total Variance (TV) bound [49].

$$||u(\cdot, t)||_{BV} \leq ||u_0(\cdot)||_{BV}, \quad ||u(\cdot, t)||_{BV} = \sup_{\Delta x \geq 0} \frac{||u(\cdot + \Delta x, t) - u(\cdot, t)||}{\Delta x}.$$  \hspace{1cm} (3.10)

3.2.2 Numerical methods: Finite Difference schemes

There are mainly five categories of numerical methods for approximate solutions of the classical nonlinear conservation laws:

• Finite Difference methods

• Finite Element schemes

• Finite Volume schemes

• Spectral approximation

• Kinetic formulations

Among these methods, Finite Difference methods are the most widely used methods, and also most closely related to our work. We consider only the Finite Difference schemes in our discussion.

The fundamental building block for the construction of approximate solutions in one dimensional case is the solution of Riemann’s problem, where the initial data is piecewise constant:

$$u(x, 0) = \begin{cases} u_l, & x < 0, \\ u_r, & x > 0. \end{cases} \hspace{1cm} (3.11)$$
For Riemann problems of local conservation law (3.4), consider numerical scheme (3.5). It is a conservative scheme, in the sense that

\[ \sum_j u_j^{n+1} = \sum_j u_j^n, \quad \text{for all } n. \quad (3.12) \]

The numerical flux \( g \) in (3.5) is Lipschitz continuous, and is consistent with flux function \( f \) in conservation law (3.4), in the sense that \( g(u, u) = f(u) \). Different options of numerical flux \( g \) leads to different schemes. For instance,

- Godunov’s scheme. In 1959, Godunov [54] proposed a way to make use of the characteristic information within the framework of a conservative method. In Godunov’s method, we start with a numerical grid function \( u^n \) at time \( t_n \). We then define a piecewise-constant function \( \tilde{u} \). At time \( t = t^n \), \( \tilde{u}(x, t^n) \) equals to the value of numerical solution \( u_j^n \) in the grid cell \( (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}) \). In the time interval \( (t^n, t^{n+1}) \), \( \tilde{u} \) on \( (t^n, t^{n+1}) \) is defined as the true entropy solution of (3.4) with piecewise constant initial data \( \tilde{u}(x, t^n) \) at initial time \( t^n \). (3.4) with piecewise constant initial data \( \tilde{u}(x, t^n) \) can be seen as a family of Riemann’s problems, whose solutions will not interact as long the the time step \( \Delta t \) is small enough to satisfy the CFL condition

\[ \frac{\Delta t}{\Delta x} \max_j |f'(u_j^n)| \leq \frac{1}{2}. \quad (3.13) \]

After obtaining \( \tilde{u} \) over \( (t^n, t^{n+1}) \), we define the numeral solution \( u_j^{n+1} \) as the cell average of \( \tilde{u} \) at \( t^{n+1} \). After simplification of computation, the Godunov ‘s scheme is in form of (3.5) with flux \( g(u_j^n, u_j^{n+1}) \) defined as the value of \( \tilde{u} \) along the line \( x = x_{j+\frac{1}{2}} \).

Godunov scheme plays a pivotal role in the Finite Difference methods for solving scalar nonlinear conservation laws. It is the forerunner of a large class of upwind which are evolved in terms of exact or approximate Riemann solvers.

- Lax-Friedrich’s scheme:

\[ u_j^{n+1} = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) - \frac{\Delta t}{2\Delta x}(f(u_{j+1}^n) - f(u_{j-1}^n)). \quad (3.14) \]
The scheme can be rewritten into the conservative form (3.5) by taking
\[ g(u^n_j, u^n_{j+1}) = \frac{1}{2} (f(u^n_j) + f(u^n_{j+1})) - \frac{\Delta x}{2\Delta t} (u^n_{j+1} - u^n_j) \] (3.15)

Next we will introduce some key ingredients for convergence of numerical solutions:

- **Conservative scheme.** A scheme for (3.4) is conservative, if it can be written into the form (3.5).

- **Consistent scheme:** a scheme in the form of (3.5) is consistent, if the numerical flux \( g \) satisfies \( g(u, u) = f(u) \).

- **Monotone scheme.** A scheme is monotone if
  \[ u^0_j \leq u^0_{j+1} \quad \text{for all} \quad j \quad \Rightarrow \quad u^n_j \leq u^n_{j+1} \quad \text{for all} \quad n, j. \] (3.16)

It’s well known that if \( g \) is monotone, i.e., non-decreasing on the first argument, and non-increasing on the second argument, then under the local CFL condition
\[ \frac{\Delta t}{\Delta x} \left( \left| \frac{\partial g(a, b)}{\partial a} \right| + \left| \frac{\partial g(a, b)}{\partial b} \right| \right) \leq 1, \] (3.17)
scheme (3.5) is monotone.

- **Monotonicity preserving.**
  \[ u^0_j \leq u^0_{j+1} \quad \text{for all} \quad j \quad \Rightarrow \quad u^n_j \leq u^n_{j+1} \quad \text{for all} \quad n, j. \] (3.18)

- **TVD:**
  \[ \sum_j |u^n_{j+1} - u^n_{j+1}| \leq \sum_j |u^n_{j+1} - u^n_{j+1}|, \] (3.19)

- **Maximum Principle:**
  \[ \min_k u^n_k \leq u^n_j \leq \max_k u^n_k, \quad \text{for all} \quad n, j. \] (3.20)

A conservative monotone scheme is automatically monotonicity preserving, TVD and have Maximum Principle [55].
Early constructions of approximate solutions for scalar conservation laws, most notably – Finite Difference methods, utilize this monotonicity to construct convergent schemes, [46,47]. Monotone approximations are limited, however, to first order accuracy [77]. The limitation of first order accuracy for monotone approximations, can be avoided if $L^1$ contractive solutions are replaced with the weaker requirement of bounded variation solutions.

Constructions of scalar entropy solutions by TVD approximations were used in the pioneering works of Ol’einik [50], Vol’pert [49], Kružkov [51], and Crandall [52]. In one dimensional case, the TVD property (3.10) enables to construct convergent difference schemes of high-order resolution; Harten initiated the construction of high-resolution TVD schemes in [53]. A whole generation of TVD schemes was developed during the beginning of 80’s. Our current work focus on monotone schemes with first-order accuracy, and the high-resolution schemes could be the next interesting topic to look at in near future.

Now we are ready to state the theorem of numerical convergence for local conservation law (3.4) citelax1960systems.

**Theorem 3.2.2.** *(Numerical convergence of local conservation law)*

Consider a sequence of of grids indexed by $l = 1, 2, \ldots$, with mesh parameters $(\Delta x)_l, (\Delta t)_l \to 0$ as $l \to \infty$. Let $U_l(x,t)$ denote the numerical approximation computed with a consistent and conservative method (3.5) on the $l$th grid. Also assume that $g$ is Lipschitz continuous. Suppose the $U_l$ converge to a function $u$ as $l \to \infty$, in the sense that made precise below. Then $u(x,t)$ is the entropy solution of conservation law (3.4).

We will assume that we have convergence of $U_l$ to $u$ in the following sense:

1. Over every bounded set $\Omega = [a,b] \times [0,T]$ in $x$-$t$ space,

\[
\|U_l - u\|_{1,\Omega} = \int_0^T \int_a^b |U_l(x,t) - u(x,t)| dx dt \to 0, \quad \text{as } l \to +\infty. \tag{3.21}
\]

2. Uniformly bounded: for some constant $C$,

\[
\|u^\Delta\|_{L^\infty(\mathbb{R} \times \mathbb{R}^+)} + \|u\|_{L^\infty(\mathbb{R} \times \mathbb{R}^+)} \leq C. \tag{3.22}
\]
3.3 Nonlocal conservation law

Now we introduce our formulation of nonlocal conservation law (3.23) and its entropy solution.

We propose the following scalar nonlocal conservation laws in 1-dimension:

\[
\frac{\partial u}{\partial t} + \int_0^\delta \left( \frac{g(u, \tau_h u) - g(\tau_{-h} u, u)}{h} \right) \omega^\delta(h) dh = 0, \quad x \in \mathbb{R}, \; t \in [0, T],
\]

where \( \delta > 0 \) is the nonlocal horizon parameter. For any real \( h \), the translation operator \( \tau_h \) is defined by

\[
\tau_h v(x) = v(x + h)
\]

for any function \( v = v(x) \) on \( x \in \mathbb{R} \). So \( \tau_h u(x, t) = u(x + h, t) \). The kernel \( \omega^\delta : \mathbb{R} \to [0, +\infty) \) is a nonnegative density, supported in \((0, \delta)\):

\[
\omega^\delta \geq 0, \quad \omega^\delta \text{ is supported on } [0, \delta], \quad \int_0^\delta \omega^\delta(h) dh = 1.
\]

It can be seen as \( \omega^\delta(h) = \frac{1}{\delta} \rho \left( \frac{h}{\delta} \right) \), where \( \rho \) is a non-negative density function supported on \([0, 1]\). In this paper we assume \( \omega^\delta \in C^2(0, \delta) \).

The two-point flux \( g = g(u_1, u_2) \) is assumed to satisfy the following conditions:

(i) \( g \) is consistent with a local flux \( f \):

\[
g(u, u) = f(u).
\]

(ii) \( g : W^{1, \infty}(\mathbb{R}) \times W^{1, \infty}(\mathbb{R}) \to W^{1, \infty}(\mathbb{R}) \) and its partial derivatives, denoted as \( g_1 \) and \( g_2 \), are Lipschitz continuous, with Lipschitz constant \( C \):

\[
||g(a, b) - g(c, d)||_\infty + \sum_{i=1}^2 ||g_i(a, b) - g_i(c, d)||_\infty \leq C(||a - c||_\infty + ||b - d||_\infty).
\]

(iii) \( g \) is nondecreasing with respect to the first argument, and nonincreasing to the second argument:

\[
g_1(u_1, u_2) := \frac{\partial g}{\partial u_1}(u_1, u_2) \geq 0, \quad g_2(u_1, u_2) := \frac{\partial g}{\partial u_2}(u_1, u_2) \leq 0.
\]
(iv) Partial derivatives \( g_i \) are bounded in \( L^\infty(\mathbb{R}) \),

\[
\|g_i(a,b)\|_\infty \leq C(\|a\|_\infty + \|b\|_\infty). \quad i = 1, 2,
\]

where constant \( C \) only depends on \( g \).

Denote operator \( L \) as

\[
L^\delta(u)(x) := \int_\mathbb{R} g(u(x),u(x+h)) - g(u(x-h),u(x)) \frac{\omega^\delta(h)}{h} dh.
\]

(3.30)

For convenience we sometimes omit the dependence on \( \delta \), and write \( L^\delta \) as \( L \), and \( \omega^\delta \) as \( \omega \).

In the paper we focus on the following initial-value problem of nonlocal conservation laws:

\[
\begin{cases}
  u_t + L(u) = 0, & (x, t) \in \mathbb{R} \times [0, T], \\
  u(x, 0) = u_0(x), & x \in \mathbb{R}.
\end{cases}
\]

(3.31)

Later we will show that, nonlocal problem (3.31) is an appropriate generalization of the local problem (3.4).

In the remaining part of this work, we sometimes write \( u(x, t) \) as \( u(x) \) and \( \omega^\delta \) as \( \omega \), but readers should always keep in mind that \( u \) depends on time \( t \) and kernel \( \omega \) depend on horizon parameter \( \delta \). Moreover, we use \( g_i \) \((i = 1, 2)\) to denote the partial derivative of \( g \) with respect to the first or second argument.

### 3.3.1 Nonlocal entropy inequality and entropy solutions

**Definition 3.3.1.** We say \( u \) is an entropy solution of nonlocal conservation law (3.31), if \( u \in L^\infty(\mathbb{R} \times [0, T]) \cap C(0, T; L^1(\mathbb{R})) \) and it satisfies the following Kružkov-type entropy inequality:

\[
\int_0^T \int_\mathbb{R} |u - c| \phi_t dx dt + \int_0^T \int_\mathbb{R} \frac{\tau_h \phi - \phi}{h} q(u, \tau_h u) \omega(h) dh dx dt \geq 0
\]

(3.32)

for every \( \phi \in C^1_0(\mathbb{R} \times [0, T]) \) with \( \phi \geq 0 \) and any constant \( c \in \mathbb{R} \). Here \( \tau \) is the translation operator defined in (3.24), \( q \) is the nonlocal entropy flux corresponding to the entropy function.
\( \eta(u, c) = |u - c| \), defined as
\[
q(a, b; c) = g(a \lor c, b \lor c) - g(a \land c, b \land c).
\] (3.33)

Equivalently,
\[
q(a, b; c) = \text{sgn}(b - a) \left\{ \frac{\text{sgn}(a - c) + \text{sgn}(b - c)}{2}[g(a, b) - g(c, c)] + \frac{\text{sgn}(a - c) - \text{sgn}(b - c)}{2}[g(c, b) - g(a, c)] \right\},
\]

where we set \( \text{sgn}(0) = 1 \). In particular, by the consistency of \( g \) (3.26),
\[
q(u, u; c) = g(u \lor c, u \lor c) - g(u \land c, u \land c) = f(u \lor c) - f(u \land c) = \text{sgn}(u - c)[f(u) - f(c)],
\]
which is consistent with the local entropy flux \( q \) (3.8). The inequality (3.32) is referred to as the entropy inequality for nonlocal conservation law (3.31).

**Lemma 3.3.2.** Let \( q \) be defined in (3.33), with \( g \) being Lipschitz continuous (3.27). Then there holds:

(i) the Lipschitz continuity of \( q \):
\[
|q(a, b) - q(c, d)| \leq C(|a - c| + |b - d|).
\] (3.34)

(ii) the boundedness of \( q \):
\[
|q(a, b; c)| \leq C(|a| + |b| + |c|),
\] (3.35)

**Proof.** (i) Since \( g \) is Lipschitz continuous(3.27) , so it \( q \) by its definition:
\[
|q(a, b) - q(c, d)| \leq C(|a - c| + |b - d|).
\]

(ii)
\[
|q(a, b; c)| = |g(a \lor c, b \lor c) - g(a \land c, b \land c)| \leq C_g(|a \lor c - a \land c| + |b \lor c - b \land c|) \]
\[ C_g(|a - c| + |b - c|) \leq C(|a| + |b| + |c|) \]

\[ \text{Theorem 3.3.3. (Uniqueness of entropy solution for nonlocal conservation law)} \]

Denote \( \Pi_T := \mathbb{R} \times [0, T] \). Let \( g \) be Lipschitz continuous (3.27), and \( u, v \) be two entropy solutions of nonlocal conservation law (3.31) with initial data \( u_0, v_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \), respectively. Then: for almost all \( t \in (0, T) \),

\[ ||u(\cdot, t) - v(\cdot, t)||_{L^1(\mathbb{R})} \leq ||u_0 - v_0||_{L^1(\mathbb{R})}, \quad (3.36) \]

which implies \( v = u \) almost everywhere in \( \Pi_T \).

\[ \text{Proof.} \]

(1) Let \( \psi = \psi(x, t, y, s) \in C^\infty(\Pi_T \times \Pi_T) \) be a non-negative test function satisfying

\[ \psi(x, t, y, s) = \psi(y, t, x, s), \quad \psi(x, t, y, s) = \psi(x, s, y, t), \quad \psi(x + h, t, y, s) = \psi(x, t, y + h, s). \]

Let \( u = u(x, t) \) and \( v = v(y, s) \) be two entropy solutions of nonlocal conservation law (3.23). Consider entropy function \( \eta(u, c) = |u - c| \), and entropy flux \( q(a, b; c) \) defined in (3.33). First we take \( c = v(y, s) \) in the nonlocal entropy inequality for \( u \) (3.32), and integrate over \((y, s) \in \Pi_T\):

\[ \int_{\Pi_T \times \Pi_T} \eta(u(x, t), v(y, s)) \partial_x \psi(x, t, y, s) dw \]

\[ + \int_{\Pi_T \times \Pi_T} \int_0^\delta \frac{\psi(x + h, t, y, s) - \psi(x, t, y, s)}{h} q(u(x, t), u(x + h, t); v(y, s)) \omega(h) dh dw \geq 0, \]

where \( dw = dx dt dy ds \). Similarly, take \( c = u(x, t) \) in the nonlocal entropy inequality for \( v \), and integrate over \((x, t) \in \Pi_T\), and using the symmetry of \( \psi \), we have

\[ \int_{\Pi_T \times \Pi_T} \eta(v(y, s), u(x, t)) \partial_x \psi(x, t, y, s) dw \]

\[ + \int_{\Pi_T \times \Pi_T} \int_0^\delta \frac{\psi(x, t, y + h, s) - \psi(x, t, y, s)}{h} q(v(y, s), v(y + h, s); u(x, t)) \omega(h) dh dw \geq 0. \]

Adding these two inequalities, and using the properties of \( \psi \), we have

\[ \int_{\Pi_T \times \Pi_T} \eta(u(x, t), v(y, s)) (\partial_t + \partial_x) \psi(x, t, y, s) dw \]
+ \int_{\Pi_T \times \Pi_T} \int_0^h \frac{\psi(x + h, t, y, s) - \psi(x, t, y, s)}{h}
\left[q(u(x, t), u(x + h, t); v(y, s)) - q(v(y, s), v(y + h, s); u(x, t))\right] \omega(h) dhdw \geq 0.
(3.37)

Take
\psi(x, t, y, s) = \xi\left(\frac{x - y}{2}\right) \xi\left(\frac{t - s}{2}\right) \phi\left(\frac{x + y}{2}, \frac{t + s}{2}\right),
where \phi = \phi(x, t) \in C_0^\infty(\Pi_T \times P i_T) is a non-negative test function, \xi is a scaling of \xi:
\xi_\rho(x) = \frac{1}{\rho} \xi\left(\frac{x}{\rho}\right), \quad \rho > 0,
and \xi \in C_0^\infty(\mathbb{R}) is a non-negative function satisfying
\xi(x) = \xi(-x), \quad \xi(x) = 0 \text{ for } |x| \geq 1, \quad \text{and } \int_{\mathbb{R}} \xi(x) dx = 1.
As \rho \to 0, for the first term in (3.37),
\lim_{\rho \to 0} \int_{(\Pi_T)^2} \eta(u(x, t), v(y, s)) (\partial_t + \partial_s) \psi(x, t, y, s) d\omega = \int_{\Pi_T} \eta(u(x, t), v(x, t)) \partial_t \phi(x, t) dx dt,
the proof of which can be found in, e.g., proof of Theorem 1 in [78]. The second term in (3.37) goes to zero:
\lim_{\rho \to 0} \int_{\Pi_T \times \Pi_T} \int_0^\delta \frac{\psi(x + h, t, y, s) - \psi(x, t, y, s)}{h}
\left[q(u(x, t), u(x + h, t); v(y, s)) - q(v(y, s), v(y + h, s); u(x, t))\right] \omega(h) dhdw = 0,
(3.38)
which will be shown the second part of this proof. Hence sending \rho \to 0, (3.37) becomes
\int_{\Pi_T} \eta(u(x, t), v(x, t)) \partial_t \phi(x, t) dx dt \geq 0,
and this inequality implies \textit{L}^1\text{-contraction} (3.36) (see [78]).

(2) It now remains to show (3.38). To the second term in (3.37), employing change of
variable
\[ \tilde{x} = \frac{x + y}{2}, \quad z = \frac{x - y}{2}, \quad \tilde{t} = \frac{t + s}{2}, \quad \tau = \frac{t - s}{2}, \]

\[
\text{(thus } x = \tilde{x} + z, \quad y = \tilde{x} - z, \quad t = \tilde{t} + \tau, \quad s = \tilde{t} - \tau.\]

and use Lebesgue’s differentiation theorem, it becomes

\[
\lim_{\rho \to 0} \int_{(\Pi_T)} \int_0^\delta \xi_\rho \left( z + \frac{h}{2} \right) \frac{\xi_\rho(\tau) \phi \left( \tilde{x} + \frac{h}{2}, \tilde{t} \right) - \xi_\rho(z) \xi_\rho(\tau) \phi \left( \tilde{x}, \tilde{t} \right)}{h}
\]

\[
\left[ q \left( u(\tilde{x} + z, \tilde{t} + \tau), u(\tilde{x} + z + h, \tilde{t} + \tau); v(\tilde{x} - z, \tilde{t} - \tau) \right) - q \left( v(\tilde{x} - z, \tilde{t} - \tau), v(\tilde{x} - z + h, \tilde{t} - \tau); u(\tilde{x} + z, \tilde{t} + \tau) \right) \right] \omega(h) \, dh \, dw
\]

\[
= \int_{\Pi_T} \int_0^\delta \frac{\phi \left( \tilde{x} + \frac{h}{2}, \tilde{t} \right) - \phi \left( \tilde{x}, \tilde{t} \right)}{h}
\]

\[
\left[ q \left( u(\tilde{x}, \tilde{t}), u(\tilde{x} + h, \tilde{t}); v(\tilde{x}, \tilde{t}) \right) - q \left( v(\tilde{x}, \tilde{t}), v(\tilde{x}, \tilde{t}); u(\tilde{x}, \tilde{t}) \right) \right] \omega(h) \, dh \, d\tilde{x} \, d\tilde{t}. \tag{3.39}
\]

Taking \( \phi \) as follows

\[
\phi(x, t) = \chi(t) \varphi_n(x), \quad \chi \in C_0^\infty(0, T), \quad \varphi_n(x) = \int_{\mathbb{R}} \xi(x - y) 1_{|y| < n} \, dy, \quad n \in (1, \infty).
\]

then each \( \varphi_n \) is in \( C_c^\infty(\mathbb{R}) \), and vanishes on \( \{ x \in \mathbb{R} : ||x| - n| > 1 + \frac{\delta}{2} \} \). Moreover, noticing that

\[
\lim_{n \to \infty} \varphi_n(x) = \varphi_\infty(x) \equiv 1, \quad \sup_{x, h} \left| \frac{\varphi_n(x + h) - \varphi_n(x)}{h} \right| = C_n < \infty, \quad \text{and} \quad \sup_{x, h} \left| \frac{\varphi_\infty(x + h) - \varphi_\infty(x)}{h} \right| = 0,
\]

we have that \( \frac{\varphi_n(x + \frac{h}{2}) - \varphi_n(x)}{h} \) is uniformly bounded:

\[
\sup_n \sup_{x \in \mathbb{R}, h \in (0, \delta]} \frac{\varphi_n(x + h) - \varphi_n(x)}{h} < \infty. \tag{3.40}
\]

Letting \( n \to \infty \) in (3.39), by Lemma 3.3.2, (3.40) and (3.25) it becomes

\[
\lim_{n \to \infty} \left| \int_{\Pi_T} \int_0^\delta \chi(t) \frac{\varphi_n \left( x + \frac{h}{2} \right) - \varphi_n \left( x \right)}{h}
\]

\[
\left[ q \left( u(x, t), u(x + h, t); v(x, t) \right) - q \left( v(x, t), v(x, t); u(x, t) \right) \right] \omega(h) \, dh \, dx \, dt \right|.
\]

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\[
\leq C||x||_\infty \lim_{n \to 0} \int_{\Omega_T} \int_0^\delta \left(|u(x,t)| + |v(x,t)| + |u(x+h,t)| + |v(x+h,t)|\right) 1_{|x| - n < 1 + \frac{\delta}{2}} \omega(h) dh dx
\]
\[
= C||x||_\infty \lim_{n \to 0} \int_{\Omega_T} \left(|u(x,t)| + |v(x,t)|\right) 1_{|x| - n < 1 + \frac{\delta}{2}} dx dt
\]
\[
+ C||x||_\infty \lim_{n \to 0} \int_{\Omega_T} \int_0^\delta \left(|u(x+h,t)| + |v(x+h,t)|\right) 1_{|x| - n < 1 + \frac{\delta}{2}} \omega(h) dh dx dt
\]
\[
\leq C||x||_\infty \lim_{n \to 0} \int_{\Omega_T} \left(|u(x,t)| + |v(x,t)|\right) 1_{|x| - n < 1 + \frac{\delta}{2}} dx dt = 0,
\]
by the dominated convergence theorem since \(u\) and \(v\) belong to \(L^1(\Omega_T)\). Thus (3.38) is reached, and the proof finishes.

\[
\square
\]

### 3.4 Model Properties

In this part, we explore some basic properties for nonlocal conservation law (3.31), including the nonlocal flux, relation with local conservation law (3.4), Maximum Principle. We also proved that, under appropriate conditions, solutions of nonlocal conservation law do not develop shocks. (Theorem 3.4.8).

#### 3.4.1 Nonlocal Flux

To look at flux on interval \([a,b]\) at time \(t\), we integrate the model (3.23) with respect to \(x\) on \([a,b]\), change variable \(y = x - h\), to get

\[
\frac{d}{dt} \int_a^b u(x,t) dx + \int_a^b \int_0^\delta g(u(x), u(x+h)) \frac{\omega^\delta(h)}{h} dh dx - \int_a^b \int_0^\delta g(u(x-h), u(x)) \frac{\omega^\delta(h)}{h} dh dx = 0,
\]
\[
\frac{d}{dt} \int_a^b u(x,t) dx + \int_a^b \int_0^\delta g(u(x), u(x+h)) \frac{\omega^\delta(h)}{h} dh dx - \int_{a-h}^{b-h} \int_0^\delta g(u(y), u(y+h)) \frac{\omega^\delta(h)}{h} dh dy = 0,
\]
\[
\frac{d}{dt} \int_a^b u(x,t) dx + \int_0^\delta \int_{b-h}^{b} g(u(x), u(x+h)) \frac{\omega^\delta(h)}{h} dh dx - \int_0^\delta \int_{a-h}^{a} g(u(x), u(x+h)) \frac{\omega^\delta(h)}{h} dh dx = 0,
\]

which is analogous to the local case:

\[
\frac{d}{dt} \int_a^b u(x,t) dx + f(u(b)) - f(u(a)) = 0.
\]
3.4.2 Generalization of local conservation law

Now we show formally that the nonlocal conservation law (3.23) reduces to the local one (3.4). Denote $\delta$ as the Dirac delta function.

**Lemma 3.4.1.** Assume $g \in C^1(\mathbb{R} \times \mathbb{R})$, $u \in C^1(\mathbb{R})$, and that

$$\lim_{\delta \to 0} \int \omega_\delta(h) \to \delta(h)$$

in distribution sense. Then:

$$\lim_{\delta \to 0} \int \left[ g(u(x), u(x + h)) - g(u(x - h), u(x)) \right] \omega_\delta(h) \frac{dh}{h} = [f(u)]_x.$$

**Proof.** By assumption, for any $(x, t)$, $g(u(x), u(x + h)) - g(u(x - h), u(x))$, as a function of $h$, is in $C^1(\mathbb{R})$.

Noting that $C^1(\mathbb{R})$ is a test space for the Derac delta distribution, by (3.41), we have

$$\lim_{\delta \to 0} \int \frac{g(u(x), u(x + h)) - g(u(x - h), u(x))}{h} \omega_\delta(h) \frac{dh}{h} = [f(u)]_x.$$

The last equality comes from the consistency of $g$ with local flux $f$ (3.26).

The next lemma shows that if there is some good function $u$ satisfying a nonlocal entropy condition, then as $\delta \to 0$, $u$ also satisfies local entropy condition.

**Lemma 3.4.2.** Assume $g$ satisfies (3.28, 3.26, 3.27), and kernel $\omega_\delta$ satisfies condition (3.25). Let $u \in L^\infty(\mathbb{R} \times \mathbb{R}^+)$, $u(\cdot, t) \in BV(\mathbb{R})$ for any $t \in [0, T]$, and $u$ satisfies the nonlocal entropy inequality (3.32). Then: $u$ satisfies the local entropy inequality (3.7) as $\delta \to 0$. 

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Proof. Since $u$ satisfies the nonlocal entropy inequality (3.32),

$$\int_0^T \int_R \phi_t |u - c| \, dx \, dt + \int_0^T \int_R \int_0^\delta \frac{\phi(x + h, t) - \phi(x, t)}{h} q(u, \tau_h u) \omega^\delta(h) \, dh \, dx \, dt \geq 0$$

holds for every $\phi \in C^1_0(R \times [0, T])$ with $\phi \geq 0$ and any constant $c \in \mathbb{R}$. Recalling $q(u, u) = \text{sgn}(u - c)(f(u) - f(c))$, we only need to prove

$$\lim_{\delta \to 0} \int_0^T \int_R \int_0^\delta \frac{\phi(x + h, t) - \phi(x, t)}{h} q(u, \tau_h u) \omega^\delta(h) \, dh \, dx \, dt = \int_0^T \int_R \partial_x(\phi(x)) q(u, u) \, dx \, dt.$$

Suppose the compact support of $\phi$ is in interval $[a, b] \times [0, T]$. By boundedness of $q$ (3.35) and $\omega^\delta \in L^1(\mathbb{R})$ (3.25), we have

$$\left| \int_0^\delta \frac{\phi(x + h) - \phi(x)}{h} q(u, \tau_h u) \omega^\delta(h) \, dh \right| \leq \int_0^\delta C ||\phi_x||_{L^\infty(\mathbb{R})} ||u||_{L^\infty(\mathbb{R})} + 1) \omega^\delta(h) \, dh \leq C \int_0^\delta \omega^\delta(h) \, dh \leq C,$$

so we can apply dominated convergence theorem,

$$\lim_{\delta \to 0} \int_0^T \int_R \int_0^\delta \frac{\phi(x + h, t) - \phi(x, t)}{h} q(u, \tau_h u) \omega^\delta(h) \, dh \, dx \, dt = \int_0^T \int_R \lim_{\delta \to 0} \int_0^\delta \frac{\phi(x + h, t) - \phi(x, t)}{h} q(u, \tau_h u) \omega^\delta(h) \, dh \, dx \, dt = \int_0^T \int_R \partial_x(\phi(x)) q(u, u) \, dx \, dt,$$

where the last equality is based on the following fact

$$\lim_{\delta \to 0} \int_0^\delta [\phi(x + h, t) - \phi(x, t)] q(u, \tau_h u) \omega^\delta(h) \, dh = \partial_x(\phi(x)) q(u, u), \quad \text{a.e } (x, t) \in [a, b] \times [0, T],$$

which we are going to prove now.

Actually, for any $t \in [0, T]$, since $u(\cdot, t)$ is a BV function, $u(\cdot, t)$ is continuous with respect to $x$ almost everywhere on $\mathbb{R}$. Let $x \in [a, b]$ be a continuity point of $u(\cdot, t)$. Then for any
\( \varepsilon > 0 \), there exists a \( \delta^* > 0 \), such that for any \( \delta < \delta^* \),

\[
\left| \frac{\phi(x + h) - \phi(x)}{h} - \phi_x \right| \leq \varepsilon, \quad |u(x) - u(x + h)| \leq \varepsilon, \quad \text{a.e. } h \in (0, \delta],
\]

So by the boundedness and Lipschitz continuity of \( q \) (3.34, 3.35), we have

\[
\begin{align*}
&\left| \frac{\phi(x + h) - \phi(x)}{h} q(u, \tau_h u) - \phi_x q(u, u) \right| \\
\leq &\left| \frac{\phi(x + h) - \phi(x)}{h} - \phi_x \right| \cdot |q(u, \tau_h u)| + |\phi_x| \cdot |q(u, \tau_h u) - q(u, u)| \\
\leq &\left| \frac{\phi(x + h) - \phi(x)}{h} - \phi_x \right| C(||u||_{L^\infty(\mathbb{R})} + 1) + C|u - \tau_h u| \leq C\varepsilon, \quad \text{with } C = C(\phi, ||u||_{L^\infty}, g).
\end{align*}
\]

Therefore,

\[
\begin{align*}
&\left| \int_0^\delta \frac{\phi(x + h) - \phi(x)}{h} q(u, \tau_h u) \omega^\delta(h)dh - \phi_x q(u, u) \right| \\
\leq &\left| \int_0^\delta \frac{\phi(x + h) - \phi(x)}{h} q(u, \tau_h u) \omega^\delta(h)dh - \int_0^\delta \phi_x q(u, u) \omega^\delta(h)dh \right| \\
\leq &\left| \int_0^\delta \frac{\phi(x + h) - \phi(x)}{h} q(u, \tau_h u) - \phi_x q(u, u) \right| \omega^\delta(h)dh \\
\leq &\ C\varepsilon \int_0^\delta \omega^\delta(h)dh = C\varepsilon, \quad \text{where } C = C(\phi, ||u||_{L^\infty}, g),
\end{align*}
\]

which finishes the proof.

\[\square\]

### 3.4.3 Maximum Principle

We will show that, nonlocal conservation law (3.31) has Maximum Principle, just as in the local case.

Let \( \Omega \subset \mathbb{R} \) be a bounded domain and \( T \) be a finite terminal time, denote \( U_T = \overline{\Omega} \times [0, T] \), and \( \Gamma_T = (\partial \Omega \times (0, T]) \cup (\overline{\Omega} \times \{ t = 0 \}) \).
Lemma 3.4.3. Assume that \( \max_{U_T} u \) and \( \min_{U_T} u \) exist, and that for any \( x \in \Omega \), \( u(x,t) \in C^1[0,T] \). If
\[
 u_t \geq \mathcal{L} u \quad \text{on} \quad \Omega \setminus \partial \Omega \times (0,T) \tag{3.42}
\]
then
\[
 \min_{U_T} u = \min_{\Gamma_T} u. \tag{3.43}
\]
If
\[
 u_t \leq \mathcal{L} u \quad \text{on} \quad \Omega \setminus \partial \Omega \times (0,T) \tag{3.44}
\]
then
\[
 \max_{U_T} u = \max_{\Gamma_T} u. \tag{3.45}
\]

Proof. We prove it in three steps.

Step 1. We prove for the case when the inequality (3.42) is strictly “greater than”. Let \((x_0,t_0)\) be such that \( u(x_0,t_0) = \min_{U_T} u \). Suppose \((x_0,t_0) \in \Omega \setminus \partial \Omega \times (0,T) \), then \( u_t(x_0,t_0) \leq 0 \).

On the other hand, noting \( \omega \) is nonnegative, and that assumption (3.28) implies
\[
 g(u(x),u(x + h)) - g(u(x - h),u(x))
\]
\[
 = [g(u(x),u(x + h)) - g(u(x),u(x))] + [g(u(x),u(x)) - g(u(x - h),u(x))] \geq 0,
\]
one has
\[
 \mathcal{L}(u)(x) = \int_0^\delta [g(u(x),u(x + h)) - g(u(x - h),u(x))] \frac{\omega^\delta(h)}{h} dh \geq 0, \tag{3.46}
\]
which leads to a contradiction. So \((x_0,t_0) \in \Gamma_T \).

Step 2. When (3.42) holds, denote
\[
 u^\varepsilon(x,t) = u(x,t) + \varepsilon t
\]
with \( \varepsilon > 0 \), then \( u^\varepsilon_t(x,t) = u_t(x,t) + \varepsilon \), and
\[
 u^\varepsilon_t(x,t) > \int_0^\delta [g(u^\varepsilon(x),u^\varepsilon(x + h)) - g(u^\varepsilon(x - h),u^\varepsilon(x))] \frac{\omega^\delta(h)}{h} dh.
\]
By the argument made in step 1, \( \min_{U_T} u^\varepsilon = \min_{\Gamma_T} u^\varepsilon \). Letting \( \varepsilon \to 0 \), one gets \( \min_{U_T} u = \min_{\Gamma_T} u \).

Step 3. Replacing \( u \) by \(-u\) in the step 2, we immediately get (3.45). \( \square \)

Lemma 3.4.3 implies that the nonlocal Cauchy problem (3.31) have Maximum Principle,
i.e, the maximum or minimum value of $u$ appears on $\Gamma_T$, that is, either at the initial time or on $\partial \Omega$.

**Theorem 3.4.4. (Maximum principle)** Assume that $u$ solves the nonlocal Cauchy problem (3.31), $u(x,t) \in C^1([0,T])$ for any $x \in \Omega$, and $u$ obtains its maximum and minimum in $U_T$. Then

$$\max_{U_T} u = \max_{\Gamma_T} u, \quad \min_{U_T} u = \min_{\Gamma_T} u.$$ 

### 3.4.4 No shock formation for special kernels

In this part we show that, when $u_0 \in W^{1,\infty}(\mathbb{R})$, under appropriate assumptions, solutions of nonlocal conservation law (3.23) do not develop shocks in finite time.

For convenience of notation, denote

$$\tilde{\omega}(h) := \frac{\omega(h)}{h}. \quad (3.47)$$

**Lemma 3.4.5.** Assume $g$ satisfies condition (3.27) and (3.29), and that

$$\int_{\mathbb{R}} \tilde{\omega}(h) dh < +\infty. \quad (3.48)$$

where $\tilde{\omega}$ is defined in (3.47). Then: $\mathcal{L}$ defined in (3.30) is a bounded operator from $W^{1,\infty}(\mathbb{R})$ to $W^{1,\infty}(\mathbb{R})$, and for any $u \in W^{1,\infty}(\mathbb{R})$,

$$\left\| \mathcal{L}(u) \right\|_{L^\infty} \leq C\left\| \tilde{\omega} \right\|_{L^1} \left\| u \right\|_{L^\infty}. \quad (3.49)$$

$$\left\| \frac{d}{dx} \mathcal{L}(u) \right\|_{L^\infty} \leq C\left\| \tilde{\omega} \right\|_{L^1} \left\| u \right\|_{L^\infty} \left\| u' \right\|_{L^\infty}. \quad (3.50)$$

**Proof.** See proof in Lemma .1.1 in appendix. 

**Lemma 3.4.6.** Assume $g$ satisfies condition (3.27) and (3.29), and $\omega$ satisfies (3.48). Then: $\mathcal{L} : W^{1,\infty}(\mathbb{R}) \to W^{1,\infty}(\mathbb{R})$ is Lipschitz continuous, with

$$\left\| \mathcal{L}(u) - \mathcal{L}(v) \right\|_{W^{1,\infty}} \leq C\left\| \tilde{\omega} \right\|_{L^1} \left( 1 + \left\| u \right\|_{W^{1,\infty}} + \left\| v \right\|_{W^{1,\infty}} \right) \left\| u - v \right\|_{W^{1,\infty}}.$$
Proof. See proof in Lemma .1.2 in appendix.

Lemma 3.4.7. Fix a time interval \((0, T)\). Assume that initial condition \(u_0 \in C^\infty(\mathbb{R})\), \(g\) satisfies conditions (3.27, 3.29), and \(\omega\) satisfies (3.47). Let \(u\) be a solution of the nonlocal conservation law (3.31). Then for any \(t \in (0, T)\),

\[
||u(\cdot, t)||_{L^\infty} \leq ||u_0||_{L^\infty} e^{C||\tilde{\omega}||_{L^1} K_t},
\]

\[
\left\| \frac{\partial}{\partial x} u(\cdot, t) \right\|_{L^\infty} \leq \left\| \frac{du_0}{dx} \right\|_{L^\infty} e^{C||\tilde{\omega}||_{L^1} K_t},
\]

where \(K := ||u||_{L^\infty(\mathbb{R} \times (0, T))}\).

Proof. See proof in Lemma .1.3 in appendix.

Lemma 3.4.5- 3.4.7 lead to the well-posedness in \(C(0, T; W^{1,\infty}(\mathbb{R})) \cap W^{1,\infty}(\mathbb{R} \times (0, T))\) of our nonlocal conservation law (3.31), given by the following theorem.

Theorem 3.4.8. (No shock formation for special kernels) Let \(g\) satisfies condition (3.27) and (3.29), and \(u_0 \in W^{1,\infty}(\mathbb{R})\). If the kernel \(\omega^\delta\) satisfies conditions (3.25) and \(\frac{\omega(h)}{h}\) is integrable (3.47), then: for any finite time interval \((0, T)\), the nonlocal conservation law (3.31) has a unique solution in \(C(0, T; W^{1,\infty}(\mathbb{R})) \cap W^{1,\infty}(\mathbb{R} \times (0, T))\). It implies that the nonlocal solution \(u\) do not develop shocks in finite time.

Proof. The proof follows standard steps such as those for the existence of solutions for abstract ODE in Banach space. Local existence and uniqueness are obtained from local boundedness and local Lipschitz continuity of operator \(L\) shown in Lemma 3.4.5. Moreover, Maximum Principle in Theorem (3.4.4) and Lemma 3.4.7 gives the uniform boundedness of the solution in \(L^\infty(0, T; W^{1,\infty}(\mathbb{R}))\), thus the solution can be extended uniquely to a larger time interval. That is, the solution does not blows up.

\[\square\]

3.5 Numerical schemes for nonlocal model

In this section we present a monotone scheme (3.57) for nonlocal conservation law (3.31).
Denote $\Delta x$ and $\Delta t$ as the spacial and time grid-size, $I_j = [(j - \frac{1}{2}) \Delta x, (j + \frac{1}{2}) \Delta x)$ and $I^n = [n \Delta t, (n + 1) \Delta t)$ as the spacial and time cells, and grid points $x^n, t^n$ as the mid-point of $I_j$ and $I^n$. Denote $u^n_j$ as the numerical solution at grid point $(x_j, t^n)$.

### 3.5.1 Discretization of the integral

Fixing $(x, t) = x_j, t^n$, we look for an appropriate numerical integration for the integral in nonlocal equation (3.23):

$$
\int_0^\delta \frac{g(u(x), u(x+h)) - g(u(x-h), u(x))}{h} \omega_\delta(h) dh
$$

Equation (3.51)

In our work, we always take $h$-grid size $\Delta h = \Delta x$.

Since generally the integrand in the integral (3.51) could blow up at $h = 0$, normal quadrature like composite trapezoidal rule may not directly apply to the whole integral. To take care of this issue, we do the following.

**Case 1:** When $\delta \geq \Delta x$, we split the integration domain into two parts:

$$
\int_0^\delta \frac{g(u(x), u(x+h)) - g(u(x-h), u(x))}{h} \omega_\delta(h) dh = \int_0^{r \Delta x} + \int_{r \Delta x}^\delta
$$

where the integer $r$ is defined as the floor function

$$
r = \left\lfloor \frac{\delta}{\Delta x} \right\rfloor.
$$

Equation (3.52)

Since $\delta \geq \Delta x$, $r \geq 1$. Using the “Intermediate principle of Integration”, the first integral is approximated by

$$
\int_0^{r \Delta x} \frac{g(u(x), u(x+h)) - g(u(x-h), u(x))}{h} \omega(h) dh
$$

$$
= \sum_{k=1}^r \int_{(k-1)\Delta x}^{k\Delta x} \frac{g(u(x), u(x+h)) - g(u(x-h), u(x))}{h} \omega(h) dh
$$

50
\[ \sim \sum_{k=1}^{r} \frac{g(u_j, u_{j+k}) - g(u_{j-k}, u_j)}{k\Delta x} \int_{(k-1)\Delta x}^{k\Delta x} \omega(h) dh, \]

and similarly the second integral is approximated by
\[ \int_{\delta \Delta x}^\delta \frac{g(u(x), u(x+h)) - g(u(x-h), u(x))}{h} \omega(h) dh \sim \int_{\delta \Delta x}^0 \frac{1}{r \Delta x} \int_{r \Delta x}^\delta \omega(h) dh. \]

So the integral (3.51) is approximated by
\[ \sum_{k=1}^{r} [g_{j,j+k} - g_{j-k,j}] \left\{ \frac{1}{k\Delta x} \int_{(k-1)\Delta x}^{k\Delta x} \omega(h) dh + \frac{1_{r \geq 1, k=r}}{r \Delta x} \int_{r \Delta x}^\delta \omega(h) dh \right\}. \]

**Case 2:** When \( 0 < \delta \leq \Delta x, r = 0 \). We use the following approximation of the integral
\[ \int_{\delta \Delta x}^\delta \frac{g(u(x), u(x+h)) - g(u(x-h), u(x))}{h} \omega(h) dh \sim \frac{g_{j,j+1} - g_{j-1,j}}{\Delta x} \int_{0}^{\delta} \omega(h) dh = \frac{g_{j,j+1} - g_{j-1,j}}{\Delta x} \]

by noting that \( \omega \) satisfies (3.25).

Combining the above two cases, the integral (3.51) at \( (x, t) - (x_j, t^n) \) is approximated by
\[ \int_{0}^{\delta} \frac{g(u(x), u(x+h)) - g(u(x-h), u(x))}{h} \omega(h) dh \sim \sum_{k=1}^{r \vee 1} [g_{j,j+k} - g_{j-k,j}] W_k \] (3.53)

where we denote
\[ g_{i,j} := g(u_i, u_j). \] (3.54)

for convenience, and
\[ W_k = \frac{1}{k\Delta x} \int_{(k-1)\Delta x}^{k\Delta x} \omega(h) dh + \frac{1_{k=r}}{r \Delta x} \int_{r \Delta x}^{\delta} \omega(h) dh, \] (3.55)

or equivalently,
\[ \begin{cases} W_k = \frac{1}{k\Delta x} \int_{(k-1)\Delta x}^{k\Delta x} \omega(h) dh, & k = 1, \ldots, (r \vee 1 - 1), \\ W_r = \frac{1}{(r \vee 1)\Delta x} \int_{(r \vee 1 - 1)\Delta x}^{\delta} \omega(h) dh, & k = r \vee 1. \end{cases} \]

Noting \( r \) is defined in (3.52), each \( W_k \) depends on both \( \Delta x \) and \( \delta \).
Remark 3.5.1. Since $\omega$ satisfies (3.25), $W_k$ defined in (3.55) satisfies
\[
\Delta x \sum_{k=1}^{r+1} k W_k = \begin{cases} 
\sum_{k=1}^{r+1} f_k^{\Delta x} \omega(h) dh + \frac{1}{r \Delta x} \int_0^r g_{\delta} \omega(h) dh = f_\delta^\omega \omega(h) dh = 1, & \text{if } r \geq 1, \\
\int_0^r \omega(h) dh = \int_0^\delta \omega(h) dh = 1, & \text{if } r = 0.
\end{cases}
\]
So
\[
\Delta x \sum_{k=1}^{r+1} k W_k = 1, \quad \text{for any pair } (\Delta x, \delta).
\] (3.56)

3.5.2 Numerical scheme

We consider the following forward-in-time conservative scheme for nonlocal problem (3.31):
\[
\left\{ \begin{array}{l}
\frac{u_j^{n+1} - u_j^n}{\Delta t} + \sum_{k=1}^{r+1} [g_{j+k} - g_{j-k}] W_k = 0,

u_j^0 = \frac{1}{\Delta x} \int_{I_j} u_0(x) dx,
\end{array} \right.
\] (3.57)
where $I_j := [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$, and $W_k$ is defined in (3.55).

Equivalently, the first equation in (3.57) can be seen as
\[
u_j^{n+1} = H(u_j^{n-1}, \ldots, u_j^n, \ldots, u_{j+r}^n)
\] (3.58)
with $H$ defined as
\[
H(u_j^{n-1}, \ldots, u_j^n, \ldots, u_{j+r}^n) = u_j^n - \Delta t \sum_{k=1}^{r+1} [g_{j+k} - g_{j-k}] W_k.
\] (3.59)

In the following, we will refer scheme (3.57) or (3.58) as the “nonlocal scheme” for convenience.

Remark 3.5.2. Fix spacial mesh $\Delta x$, let $\delta < \Delta x$ and $\delta \to 0$, the first equation in scheme (3.57) reduces to
\[
u_j^{n+1} = u_j^n - \Delta t \frac{\Delta x}{\Delta x} [g(u_j, u_{j+1}) - g(u_{j-1}, u_j)],
\] (3.60)
which is exactly the standard Finite Difference scheme for local conservation law (3.4), where $g$ serves as the numerical flux function.
In the local case, different choices of numerical flux lead to different schemes, such as standard Godunov scheme, Linearized Riemann solvers such as Murmann-Roe scheme, central schemes such as the Lax-Friedrichs scheme, Rusanov scheme and Engquist-Osher scheme. It is expected that, by taking $g$ as such numerical fluxes, we get corresponding nonlocal versions of these local schemes.

**Definition 3.5.3.** A numerical method $u_j^{n+1} = H(u_{j-r}^n, \ldots, u_{j+r}^n)$ is called a **conservative** if

$$\sum_{j=-\infty}^{\infty} u_j^{n+1} = \sum_{j=-\infty}^{\infty} u_j^n, \quad \forall n.$$  \hfill (3.61)

**Definition 3.5.4.** We call a numerical scheme $u_j^{n+1} = H(u_{j-r}^n, \ldots, u_{j+r}^n)$ a **monotone scheme** if

$$v_j^n \geq u_j^n, \quad \forall j \quad \Rightarrow \quad v_j^{n+1} \geq u_j^{n+1}, \quad \forall j.$$  \hfill (3.62)

It is easy to see that: a conservative consistent scheme (3.58) is monotone, if $H$ is monotone. Namely, $H$ is nondecreasing with respect to each of its arguments:

$$\frac{\partial H(u_{j-r}^n, \ldots, u_{j+r}^n)}{\partial u_i^n} \geq 0, \quad \text{for all } i, j, \text{ and vector } u^n = (\ldots, u_{j-1}^n, u_j^n, u_{j+1}^n, \ldots).$$  \hfill (3.63)

In the next lemma we show that our scheme (3.58) is a monotone scheme under appropriate assumptions of $g$ and a nonlocal CFL condition.

**Lemma 3.5.5.** *(Monotone scheme)*

(i) Scheme (3.58) is conservative and consistent.

(ii) Suppose $g(a, b)$ satisfies (3.26, 3.28, 3.29) on $[B_1, B_2] \times [B_1, B_2]$, where $B_1 = \min_j \{u_j^0\}$, $B_2 = \max_j \{u_j^0\}$. Scheme (3.58) is monotone if the following CFL condition holds:

$$\left(\frac{\Delta t}{\Delta x}\right) \left(\sup_{B_1 \leq a, b \leq B_2} |g_1(a, b)| + \sup_{B_1 \leq a, b \leq B_2} |g_2(a, b)|\right) \leq 1.$$  \hfill (3.64)

(iii) Assume the CFL-condition (3.64) hold. Then scheme (3.58) has Maximum Principle:

$$B_1 \leq u_j^n \leq B_2, \quad \forall n, j.$$  \hfill (3.65)
Proof. (i) It is straightforward by definition.

(ii) It suffices to show that: $H$ defined in (3.59) is monotone, i.e, (3.63) holds. Since $g$ is monotone (3.28), it is straightforward that for any $k = 1, 2, \ldots, r$,

$$\frac{\partial H}{\partial u_{j-k}^n} \geq 0, \quad \frac{\partial H}{\partial u_{j+k}^n} \geq 0.$$

Moreover, under the assumption that $B_1 \leq u_j^n \leq B_2$ for any $n$ or $j$,

$$\frac{\partial H}{\partial u_j^n} = 1 - (\Delta t) \sum_{k=1}^{r} [g_1(u_{j+k}, u_{j-k}) - g_2(u_{j}, u_{j+k})] W_k$$

$$\geq 1 - \frac{\Delta t}{\Delta x} \left( \sup_{B_1 \leq a, b \leq B_2} |g_1(a, b)| + \sup_{B_1 \leq a, b \leq B_2} |g_2(a, b)| \right) \left( \Delta x \sum_{k=1}^{r} W_k \right)$$

$$\geq 1 - \left( \frac{\Delta t}{\Delta x} \right) \left( \sup_{B_1 \leq a, b \leq B_2} |g_1| + \sup_{B_1 \leq a, b \leq B_2} |g_2| \right) \geq 0,$$

The last two inequalities come from (3.56) and the CFL condition (3.64).

(ii) The Maximum Principle is a direct consequence of the monotonicity of $H$ (3.63) and consistency of $g$ (3.26). Actually,

$$u_j^{n+1} = H(u_{j-r}, \ldots, u_{j+r}) \leq H(\max_j \{u_j^n\}, \ldots, \max_j \{u_j^n\}) = \max_j \{u_j^n\},$$

$$u_j^{n+1} = H(u_{j-r}, \ldots, u_{j+r}) \geq H(\min_j \{u_j^n\}, \ldots, \min_j \{u_j^n\}) = \min_j \{u_j^n\}.$$

That is

$$\min_j \{u_j^n\} \leq u_j^{n+1} \leq \max_j \{u_j^n\}.$$

\[\square\]

3.5.3 “Nonlocal Lax-Friedrich’s scheme” and alternative flux $\tilde{g}$

In this part we propose a nonlocal version of Lax-Friedrich’s scheme, which, just as in the local case, can be seen as the forward-in-time scheme (3.57) with a new flux $\tilde{g}$.
Consider the following “nonlocal” Lax-Friedrich’s scheme
\[
\frac{u_j^{n+1} - \frac{1}{2} \sum_{k=1}^{r \vee 1} Q_k^r (u_{j+k}^n + u_{j-k}^n)}{\Delta t} + \sum_{k=1}^{r \vee 1} [g_{j,j+k} - g_{j-k,j}] W_k = 0. \tag{3.66}
\]

The coefficients \( \{Q_k^r\}_k \) may depend on \( r \), and are required to satisfy the following conditions:
\[
\sum_{k=1}^{r \vee 1} Q_k^r = 1, \tag{3.67}
\]
\[
Q_k^r \geq 0, \tag{3.68}
\]
and
\[
\sup_{r \geq 1} \sum_{k=1}^{r} Q_k^r k^2 < \infty. \tag{3.69}
\]

**Remark 3.5.6.** By lemma 3.5.9, the discretization
\[
\frac{u_j^{n+1} - \sum_{k=1}^{r \vee 1} Q_k^r (u_{j+k}^n + u_{j-k}^n)}{\Delta t} \sim u_t
\]
is an appropriate approximation of \( u_t \). When \( r = 1 \), \( Q_k^1 = 1 \), it recovers the classical Lax-Friedrich’s method:
\[
\frac{u_j^{n+1} - \frac{1}{2} (u_{j+1}^n + u_{j-1}^n)}{\Delta t} \sim u_t.
\]

**Remark 3.5.7.** Property (3.67) and (3.69) are needed in Lemma 3.5.9. The nonnegativity of \( Q_k^r \) in (3.68) is required in (3.73) since \( W_k \) need to be nonnegative.

**Remark 3.5.8.** An example of \( Q_k^r \) is:
\[
Q_k^r = \frac{C_r}{k^4}, \quad C_r = \left( \sum_{k=1}^{r} \frac{1}{k^4} \right)^{-1}. \tag{3.70}
\]
Note that the case \( Q_k^r \equiv C \) (with \( C \) being a constant independent of \( k \)) violates condition (3.69).

**Lemma 3.5.9.** *(Numerical consistency with \( u_t \))*
Let \( \delta \) be fixed, and \( Q_k^r \) satisfy condition (3.67) and (3.69). Suppose \( u = u(x,t) \in C^2(\mathbb{R} \times \mathbb{R}^+) \),
and the following partial derivatives of $u$ is uniformly bounded:

$$\sup_{(x,t) \in \mathbb{R} \times \mathbb{R}^+} u_{tt}(x,t) + \sup_{(x,t) \in \mathbb{R} \times \mathbb{R}^+} u_{xx}(x,t) < C,$$

for some constant $C$. Then: as $\Delta t$ and $\frac{(\Delta x)^2}{\Delta t} \to 0$,

$$\frac{u_{j+1}^n - \frac{1}{2} \sum_{k=1}^{r\vee} Q_k^n (u_{j+k}^n + u_{j-k}^n)}{\Delta t} \to u_t(x_j, t^n), \quad \text{for any } j, n, r. \quad (3.72)$$

**Proof.** Since $u(x,t) \in C^2(\mathbb{R} \times \mathbb{R}^+)$, by Taylor’s Theorem,

$$u_{j+1}^n = u_j^n + (u_t)_j^n \Delta t + \frac{u_{tt}(\xi_1)}{2}(\Delta t)^2, \quad \xi_1 \in (t^n, t^{n+1}),$$
$$u_{j+k}^n = u_j^n + (u_x)_j^n (k \Delta x) + \frac{u_{xx}(\xi_2)}{2}(k \Delta x)^2, \quad \xi_2 \in (x_j, x_{j+k}),$$
$$u_{j-k}^n = u_j^n - (u_x)_j^n (k \Delta x) + \frac{u_{xx}(\xi_3)}{2}(k \Delta x)^2, \quad \xi_3 \in (x_{j-k}, x_j).$$

Thus, by condition (3.67),

$$\frac{1}{2} \sum_{k=1}^{r\vee} Q_k^n (u_{j+k}^n + u_{j-k}^n) = \frac{1}{2} \sum_{k=1}^{r\vee} Q_k^n \left[ 2u_j^n + Q_k^n \frac{u_{xx}(\xi_2)}{2}(k \Delta x)^2 + \frac{u_{xx}(\xi_3)}{2}(k \Delta x)^2 \right]$$
$$= u_j^n + \frac{(\Delta x)^2}{4} \sum_{k=1}^{r\vee} Q_k^n k^2 \left( u_{xx}(\xi_2) + u_{xx}(\xi_3) \right).$$

So

$$\frac{u_{j+1}^n - \frac{1}{2} \sum_{k=1}^{r\vee} Q_k^n (u_{j+k}^n + u_{j-k}^n)}{\Delta t} = (u_t)_j^n + \frac{u_{tt}(\xi_1)}{2} \Delta t - \frac{(\Delta x)^2}{4 \Delta t} \sum_{k=1}^{r\vee} Q_k^n k^2 \left( u_{xx}(\xi_2) + u_{xx}(\xi_3) \right).$$

Since $u_{xx}(\xi_2) + u_{xx}(\xi_3)$ is uniformly bounded by assumption (3.71), and $\sum_{k=1}^{r\vee} Q_k^n k^2$ is bounded uniformly for $r$ by (3.69), equation (3.72) is reached as we send $\Delta t$ and $\frac{(\Delta x)^2}{\Delta t}$ to zero.

\[ \square \]

**Lemma 3.5.10.** Assume that $\{Q_k^n\}_k$ satisfies condition (3.68). Taking

$$Q_k^r = \Delta x (kW_k), \quad (3.73)$$

(3.67) is automatically satisfied due to (3.56), and the nonlocal Lax-Friedrich’s scheme (3.66)
can be written into the form of forward-in-time scheme \((3.57)\), with a new flux

\[
\tilde{g}(u_j, u_{j+k}) = g(u_j, u_{j+k}) - \frac{k \Delta x}{2 \Delta t} (u_{j+k} - u_j).
\]

\[(3.74)\]

Moreover, it is apparent that if \(g\) satisfies conditions \((3.26-3.29)\), so does \(\tilde{g}\).

**Proof.** The nonlocal Lax-Friedrich’s scheme \((3.66)\) can be rewritten as

\[
\frac{u^n_{j+1} - u^n_j}{\Delta t} = \frac{1}{\Delta t} \left[ \frac{1}{2} \sum_{k=1}^{r^1} Q_k^n (u^n_{j+k} + u^n_{j-k}) - \sum_{r=1}^r [g_{j,j+k} - g_{j-k,j}] W_k \right]
\]

Note that, by condition of \(Q_k^n\) \((3.67)\),

\[
\frac{1}{2} \sum_{k=1}^{r^1} Q_k^n (u^n_{j+k} + u^n_{j-k}) - u^n_j = \frac{1}{2} \sum_{k=1}^{r^1} Q_k^n (u^n_{j+k} + u^n_{j-k}) - \left( \sum_{k=1}^{r^1} Q_k^n \right) u^n_j
\]

\[= \frac{1}{2} \sum_{k=1}^{r^1} Q_k^n \left[ (u^n_{j+k} - u^n_j) - (u^n_j - u^n_{j-k}) \right].
\]

Plugging it in and using definition of \(Q_k^n\) \((3.73)\), one has

\[
\frac{u^n_{j+1} - u^n_j}{\Delta t} = \sum_{k=1}^{r^1} \left\{ \frac{Q_k^n}{2 \Delta t} \left[ (u^n_{j+k} - u^n_j) - (u^n_j - u^n_{j-k}) \right] - W_k [g(u^n_j, u^n_{j+k}) - g(u^n_{j-k} - u^n_j)] \right\}
\]

\[= \sum_{k=1}^{r^1} \frac{Q_k^n}{k \Delta x} \frac{k \Delta x}{2 \Delta t} \left[ (u^n_{j+k} - u^n_j) - (u^n_j - u^n_{j-k}) \right] - \sum_{k=1}^{r^1} W_k [g(u^n_j, u^n_{j+k}) - g(u^n_{j-k} - u^n_j)]
\]

\[= \sum_{k=1}^{r^1} W_k \left\{ \frac{k \Delta x}{2 \Delta t} (u^n_{j+k} - u^n_j) - \frac{k \Delta x}{2 \Delta t} (u^n_j - u^n_{j-k}) - [g(u^n_j, u^n_{j+k}) - g(u^n_{j-k} - u^n_j)] \right\}
\]

\[= - \sum_{k=1}^{r^1} W_k \left[ \tilde{g}(u^n_j, u^n_{j+k}) - \tilde{g}(u^n_{j-k} - u^n_j) \right]
\]

where in the the last equality, the new flux \(\tilde{g}\) is taken as

\[
\tilde{g}(u^n_j, u^n_{j+k}) = g(u^n_j, u^n_{j+k}) - \frac{k \Delta x}{2 \Delta t} (u^n_{j+k} - u^n_j).
\]

\[\square\]
3.6 Convergence of numerical solution and wellposedness of nonlocal entropy solution

3.6.1 Numerical convergence as \((\delta, \Delta x) \to (\delta, 0)\) or \((\delta, \Delta x) \to (0,0)\)

In this part, we consider the convergence of the numerical solutions to nonlocal scheme (3.57), under two kinds of limiting processes:

1. when horizon \(\delta\) is fixed and grid-size \(\Delta x \to 0\);
2. when \((\delta, \Delta x) \to (0,0)\).

In this chapter we always assume the ratio of grid-sizes \(\Delta t / \Delta x\) is a fixed number. Therefore, as \(\Delta x \to 0\), \(\Delta t \to 0\) at the same rate.

Theorem 3.6.10 shows that, if we fix \(\delta\), and send \(\Delta x\) to zero, then the numerical solution converges to the entropy solution of the “nonlocal” conservation law (3.31); however, if we send both \(\delta\) and \(\Delta x\) to zero, the numerical solution converges to the entropy solution of the “local” conservation law (3.4).

Let \(u^n_j\) be the numerical solution of scheme (3.57). Denote \(I_j = \left(\left(j - \frac{1}{2}\right) \Delta x, \left(j + \frac{1}{2}\right) \Delta x\right)\), and \(I^n = [n \Delta t, (n + 1) \Delta t)\). Define piecewise constant function \(u^{\Delta,\delta}\) using the grid function \(u^n_j\):

\[
u^{\Delta,\delta}(x, t) = \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} u^n_j 1_{I_j \times I^n}(x, t),
\]

where \(1_{I_j \times I^n}\) is the indicator function which takes value 1 when \((x,t) \in I_j \times I^n\), and 0 otherwise. Thus \(u^{\Delta,\delta}\) depends on the grid size \(\Delta x\), \(\Delta t\), and the horizon parameter \(\delta\). We sometimes write \(u^{\Delta,\delta}\) as \(u^{\Delta}\) to explicitly emphasize the dependence on \(\delta\).

Suppose \(\Delta x\) and \(\Delta t\) satisfy the CFL condition (3.64), then by the discrete Maximum Principle (3.65), we have

\[
||u^{\Delta,\delta}||_{L^\infty(\mathbb{R} \times \mathbb{R}^+)} \leq ||u_0||_{L^\infty(\mathbb{R})}.
\]

Theorem 3.6.1. (Convergence of numerical solution)

Fix a terminal time \(T\), and consider nonlocal scheme (3.57) on \([0, T]\). Assume \(u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap C([0, T]; BV(\mathbb{R}))\), \(g\) satisfies (3.28, 3.26, 3.27), and kernel \(\omega^\delta\) satisfies condition (3.25). Also assume \(\frac{\Delta t}{\Delta x}\) be fixed, and satisfies the CFL condition (3.64).
Then:
(i) Fixing $\delta > 0$, and sending $\Delta x$ to 0, $u^{\Delta, \delta}$ converges to the entropy solution of nonlocal conservation law (3.31), $u^{\text{nonl}}$, in $L^1_{\text{loc}}(\mathbb{R})$ uniformly for $t \in [0, T]$. More precisely,
\[
\lim_{\Delta x \to 0} \sup_{t \in [0, T]} \int_{\mathbb{R}} |u^{\Delta, \delta}(\cdot, t) - u^{\text{nonl}}(\cdot, t)| dx dt = 0. \tag{3.77}
\]

(ii) As $\delta$ and $\Delta x$ both go to 0, $u^{\Delta, \delta}$ converges to the entropy solution of local conservation law (3.4), $u^{\text{local}} : [0, T] \to L^1(\mathbb{R})$, in $L^1_{\text{loc}}(\mathbb{R})$ uniformly for $t \in [0, T]$. More precisely,
\[
\lim_{(\Delta x, \delta) \to (0, 0)} \sup_{t \in [0, T]} \int_{\mathbb{R}} |u^{\Delta, \delta}(\cdot, t) - u^{\text{local}}(\cdot, t)| dx dt = 0. \tag{3.78}
\]

Proof. Our proof mainly follows that the first part of proof for Theorem 1 in [46] by Crandall and Majda. We only need to make clear of a few facts as follows.

1) By Lemma 3.6.2, nonlocal scheme (3.57) can be rewritten into the conservative form (0.4) in [46]:
\[
u^{n+1}_j = u^n_j - \frac{\Delta t}{\Delta x} [g(u^{n}_{j-r+1}, \ldots, u^{n}_{j+r}) - g(u^n_{j-r}, \ldots, u^{n}_{j+r-1})], \tag{3.79}
\]
with $g$ being Lipschitz continuous and consistent, so nonlocal scheme (3.57) is conservative and consistent. (3.57) is also monotone by Lemma 3.5.5.

2) In Lemma 3.6.4, we prove the nonlocal version of Proposition 3.5 in [46]. Thus Corollary 3.6 in [46] also holds for our scheme (3.79).

3) Notice that equation (5.2) in [46] do not hold for our scheme (3.57), since as $\Delta x$ vanish, our scheme may have infinite propagation speed. Actually, at any time $t = K\delta t$, $K \in \mathbb{N}$, the value of $u^0_j$ has influenced the value of $u$ on $[x_j - D, x_j + D]$, where $D = \delta K$. Noticing that $\frac{\Delta t}{\Delta x}$ is a constant, say $c$, we have $D = \delta \frac{t}{c\Delta x} = \frac{\Delta t}{c\Delta x}$, so when $\Delta x$ go to zero (no matter $\delta$ is fixed or goes to zero), it is possible that $\frac{\delta}{\Delta x}$ go to $\infty$, thus $D$ could be unbounded.

4) Since equation (5.2) in [46] does not hold for our scheme (3.57), neither does equation (5.3) in [46]. That is, fixing a horizon $\delta$, for our scheme we only have precompactness in
$L_{loc}^1(\mathbb{R})$ instead of $L^1(\mathbb{R})$:

$$\{u^{\Delta,\delta}(\cdot,t) : 0 \leq t \leq T, 0 \leq \Delta x \leq 1\}$$

is precompact in $L_{loc}^1(\mathbb{R})$.

Therefore, following the proof therein, we have: there exists a subsequence $\{u^{\Delta_k,\delta_k}\}_{k}$ (where $(\Delta x)_k \rightarrow 0$ when $k \rightarrow \infty$) and a function $u^* \in L_{loc}^1(\mathbb{R})$, such that for any compact set $\Omega \subset \mathbb{R}$,

$$\lim_{k \rightarrow \infty} \max_{t \in [0,T]} \|u^{\Delta_k,\delta_k}(\cdot,t) - u^*(\cdot,t)\|_{L^1(\Omega)} = 0. \quad (3.80)$$

Since $\|u^{\Delta,\delta}(\cdot,t)\|_{L^1(\mathbb{R})} \leq \|u_0\|_{L^1(\mathbb{R})} < \infty$, $u^* \in L^1(\mathbb{R})$. By (i) of Proposition 3.6.6, $u^*$ is an entropy solution of the nonlocal conservation law, denoted as $u^{nonl}$. The uniqueness of nonlocal entropy solution (Theorem 3.3.3) guarantees the uniqueness of the limit function $u^*$, and thus we have the convergence of the whole sequence $u^{\Delta,\delta}$, reaching (3.77). Also, since

$$\|u^{\Delta,\delta}(\cdot,t)\|_{Z} \leq \|u_0\|_{Z} \text{ for all } t \in [0,T], \quad Z = L^\infty(\mathbb{R}), BV(\mathbb{R}),$$

$u^*$ satisfies

$$\|u^*(\cdot,t)\|_{Z} \leq \|u_0\|_{Z} \text{ for all } t \in [0,T]. \quad (3.81)$$

Similarly, by the precompactness of $\{u^{\Delta,\delta}(\cdot,t) : 0 \leq t \leq T, 0 \leq \Delta x, \delta \leq 1\}$ in $L_{loc}^1(\mathbb{R})$, there exists a subsequence $\{u^{\Delta_k,\delta_k}\}_{k}$ (where $((\Delta x)_k,\delta_k) \rightarrow (0,0)$ when $k \rightarrow \infty$) and a function $u^{**} \in L^1(\mathbb{R})$, such that for any compact set $\Omega \subset \mathbb{R}$,

$$\lim_{k \rightarrow \infty} \max_{t \in [0,T]} \|u^{\Delta_k,\delta_k}(\cdot,t) - u^{**}(\cdot,t)\|_{L^1(\Omega)} = 0. \quad (3.82)$$

By (ii) of Proposition 3.6.6, $u^{**}$ is an entropy solution of the local conservation law, denoted as $u^{local}$. The uniqueness of local entropy solution guarantees the uniqueness of the limit function $u^{**}$, and thus we have the convergence of the whole sequence $u^{\Delta,\delta}$, reaching (3.78). And

$$\|u^{**(\cdot,t)}\|_{Z} \leq \|u_0\|_{Z} \text{ for all } t \in [0,T]. \quad (3.83)$$

It finishes the first part of the proof.

\[\square\]

In the remaining part of this work, we sometimes write $u^*(x,t)$ and $u^{**}(x,t)$ as $u^*(x)$ and $u^{**}(x)$, especially in Proposition 3.6.7 and Lemma 3.6.8. but readers should always keep in
mind that both $u^*$ and $u^{**}$ depend not only on $x$ but also on time $t$.

**Lemma 3.6.2.** Let $g$ be consistent, monotone and Lipschitz continuous (3.26, 3.28, 3.27), also let the nonlocal CFL condition (3.64). Then: nonlocal scheme (3.57) can be rewritten into

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x} \left[ g(u_{j-r+1}, \ldots, u_{j+r}) - g(u_{j-r}, \ldots, u_{j+r-1}) \right], \quad (3.84)$$

where

$$g(u_{j-r}, \ldots, u_{j+r-1}) = \sum_{k=1}^{r} \sum_{l=1}^{k} g_{j-l,j-l+k} W_k \Delta x, \quad (3.85)$$

and $g$ is Lipschitz continuous, and consistent with the local flux $f$:

$$g(u, \ldots, u) = f(u). \quad (3.86)$$

**Proof.** In scheme (3.57),

$$g_{j+k} - g_{j-k,j} = \sum_{l=1}^{r} (g_{j+1-l,j+1-l+k} - g_{j-l,j-l+k}),$$

so

$$\sum_{k=1}^{r} \sum_{l=1}^{k} g_{j+k} W_k \Delta x = \sum_{k=1}^{r} \sum_{l=1}^{k} (g_{j+1-l,j+1-l+k} - g_{j-l,j-l+k}) W_k \Delta x$$

$$= \sum_{k=1}^{r} \sum_{l=1}^{k} g_{j+1-l,j+1-l+k} W_k \Delta x - \sum_{k=1}^{r} \sum_{l=1}^{k} g_{j-l,j-l+k} W_k \Delta x$$

$$= g(u_{j+1-r}, \ldots, u_{j+r}) - g(u_{j-r}, \ldots, u_{j+r-1}).$$

thus (3.84) is proved.

Since $g$ is Lipschitz continuous, so is $g$. The consistency of $g$ comes from the consistency of $g$ and normality of $W_k$ (3.56):

$$g(u, \ldots, u) = \sum_{k=1}^{r} \sum_{l=1}^{k} g(u, u) W_k \Delta x = f(u) \sum_{k=1}^{r} \sum_{l=1}^{k} W_k \Delta x = f(u). \tag*{\Box}$$

**Remark 3.6.3.** Note that, unlike the Finite Difference scheme for local conservation law,
this numerical flux \( g \) here depends on the grid size \( \Delta x \). This difference plays a crucial role when sending \( \Delta x \) to zero. Theorem 3.6.10, shows: when sending both \( \Delta x \) and \( \delta \) to zero, then the limit function \( u \) converges to the local entropy solution. However, if we fix \( \delta \) when \( \Delta x \) to zero, then the limit function \( u \) will converge to the nonlocal entropy solution.

**Lemma 3.6.4.** Assume \( u_0 \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R}) \). For nonlocal scheme (3.57), we have

\[
||H(u) - u||_{L^1(\Delta)} \leq C \Delta t ||u^0||_{BV(\Delta)}, \tag{3.87}
\]

with \( H \) is as in (3.59), and \( C \) is independent of \( \Delta t \) and \( \Delta x \). Here the discrete \( L^1 \) and \( BV \) norms are defined as

\[
||u||_{L^1(\Delta)} = \Delta x \sum_{j=-\infty}^{\infty} |u_j|, \quad ||u||_{BV(\Delta)} = \Delta x \sum_{j=-\infty}^{\infty} |u_{j+1} - u_j|.
\]

**Proof.** At any time level \( n \),

\[
||H(u) - u||_{L^1(\Delta)} = \sum_{j=-\infty}^{\infty} \left| \Delta t \sum_{k=1}^{r+1} (g_{j,j+k} - g_{j-k,j}) W_k \right| \Delta x
\]

\[
\leq \Delta x \Delta t \sum_{j=-\infty}^{\infty} \sum_{k=1}^{r+1} |g_{j,j+k} - g_{j-k,j}| W_k
\]

\[
\leq C \Delta x \Delta t \sum_{j=-\infty}^{\infty} \sum_{k=1}^{r+1} (|u_j - u_{j+k}| + |u_{j-k} - u_k|) W_k.
\]

Note that, initial data \( u^0 \in L^1(\mathbb{R}) \), by Lemma 3.5.5, scheme (3.58) admits Maximum Principle, so \( \{u^n\} \) is bounded, say by \( A \). Without loss of generality, assume \( r \geq 1 \). By Finibili Theorem, we can switch the order of summation, to get

\[
||H(u) - u||_{L^1(\Delta)} \leq C \Delta x \Delta t \left[ \sum_{j=-\infty}^{\infty} \sum_{k=1}^{r+1} |u_j - u_{j+k}| W_k + \sum_{j=-\infty}^{\infty} \sum_{k=1}^{r+1} |u_{j-k} - u_k| W_k \right]
\]

\[
\leq C \Delta x \Delta t \left[ \sum_{k=1}^{r+1} W_k \sum_{j=-\infty}^{\infty} |u_j - u_{j+k}| + \sum_{k=1}^{r+1} W_k \sum_{j=-\infty}^{\infty} |u_{j-k} - u_k| \right]
\]

\[
\leq 2C \Delta x \Delta t \sum_{j=-\infty}^{\infty} \sum_{k=1}^{r+1} |u_j - u_{j-k}| W_k
\]
\[
\leq C \Delta x \Delta t \sum_{k=1}^{r} W_k \sum_{l=1}^{\infty} \sum_{j=-\infty}^{k} |u_{j-l+1} - u_{j-l}|
\]
\[
= C \Delta x \Delta t \sum_{k=1}^{r} W_k \sum_{l=1}^{\infty} \sum_{j=-\infty}^{k} |u_{j-l+1} - u_{j-l}|
\]
\[
= C \Delta x \Delta t \sum_{k=1}^{r} W_k \sum_{l=1}^{\infty} |u_{j+1} - u_j|
\]
\[
\leq C \Delta x \Delta t \sum_{k=1}^{r} kW_k \sum_{j=-\infty}^{\infty} |u_{j+1} - u_j|
\]
\[
\leq C \Delta t \sum_{j=-\infty}^{\infty} |u_{j+1} - u_j| \left( \sum_{k=1}^{r} kW_k \Delta x \right) = C \Delta t \|u\|_{BV(\Delta)}, \quad \forall \Delta x.
\]

where the constant \(C\) only depends on \(g\), and the last inequality comes from (3.56).

\[\blacksquare\]

**Remark 3.6.5.** Lemma 3.6.4 mimics Proposition 3.5 in [46]. In Proposition 3.5 in [46], the constant \(C\) on the right hand side of (3.87) depends on \(r\), the number of cells involved in numerical flux \(g\). However, in Lemma 3.6.4, we are able to bound the left hand side of (3.87) in a way such that the coefficient \(C\) is independent of \(r\), and thus independent of \(\Delta x, \Delta t\) and \(\delta\). We are able to do it by virtue of the fact that for our nonlocal scheme, the numerical flux \(g\) depends on \(r\), while for local scheme, \(g\) is independent of \(r\).

**Proposition 3.6.6.** Let \(u_0 \in BV(\mathbb{R})\). Suppose all the assumptions of Theorem 3.6.10 hold, and \(u^{\Delta, \delta}\) is defined in (3.75).

(i) \(u^*\) in (3.80) is an entropy solution of the nonlocal conservation law.

(ii) \(u^{**}\) in (3.82) is an entropy solution of the local conservation law.

**Proof.** Since the CFL condition (3.64) holds, by Lemma 3.5.5, \(H\) in nonlocal scheme (3.57)

\[
u_{j+1}^n = H(u_{j-r}^n, \ldots, u_{j+r}^n)
\]

is nondecreasing with respect to each of its arguments. So for any constant \(c \in \mathbb{R}\),

\[
H(u_{j-r}^n \wedge c, \ldots, u_{j+r}^n \wedge c) \leq u_{j+1}^n \leq H(u_{j-r}^n \vee c, \ldots, u_{j+r}^n \vee c),
\]

\[
H(u_{j-r}^n \wedge c, \ldots, u_{j+r}^n \wedge c) \leq c \leq H(u_{j-r}^n \vee c, \ldots, u_{j+r}^n \vee c),
\]

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\[
\begin{align*}
u_j^{n+1} \lor c & \leq H(u_{j-r}^{n} \lor c, \ldots, u_{j+r}^{n} \lor c), \\
v_j^{n+1} \land c & \geq H(u_{j-r}^{n} \land c, \ldots, u_{j+r}^{n} \land c).
\end{align*}
\]

Subtracting the last two inequalities,

\[
\begin{align*}
u_j^{n+1} \lor c - v_j^{n+1} \land c & \leq H(u_{j-r}^{n} \lor c, \ldots, u_{j+r}^{n} \lor c) - H(u_{j-r}^{n} \land c, \ldots, u_{j+r}^{n} \land c), \\
v_j^{n+1} \lor c - v_j^{n+1} \land c & \leq u_j^n \lor c - \Delta t \sum_{k=1}^{\lfloor \frac{n}{1} \rfloor} \left[ g(u_j^n \lor c, u_{j+k}^n \lor c) - g(u_{j-k}^n \land c, u_j^n \land c) \right] W_k \\
& \quad - u_j^n \land c + \Delta t \sum_{k=1}^{\lfloor \frac{n}{1} \rfloor} \left[ g(u_j^n \land c, u_{j+k}^n \land c) - g(u_{j-k}^n \land c, u_j^n \land c) \right] W_k \\
|v_j^{n+1} - c| & \leq |u_j^n - c| - \Delta t \sum_{k=1}^{\lfloor \frac{n}{1} \rfloor} \left[ g(u_j^n \lor c, u_{j+k}^n \lor c) - g(u_{j-k}^n \land c, u_j^n \land c) \right] W_k \\
& \quad + \Delta t \sum_{k=1}^{\lfloor \frac{n}{1} \rfloor} \left[ g(u_{j-k}^n \lor c, u_j^n \lor c) - g(u_{j-k}^n \land c, u_j^n \land c) \right] W_k \\
\frac{|v_j^{n+1} - c| - |u_j^n - c|}{\Delta t} & \leq - \sum_{k=1}^{\lfloor \frac{n}{1} \rfloor} \left[ g(u_j^n, u_{j+k}^n; c) - g(u_{j-k}^n, u_j^n; c) \right] W_k,
\end{align*}
\]

where \( q \) is the nonlocal entropy flux defined in (3.33). Let

\[
\phi \in C^\infty_c(\mathbb{R}^+ \times \mathbb{R}), \quad \phi \geq 0, \quad \text{and denote } \phi^n_j := \frac{\phi(x_{j-\frac{1}{2}}) - \phi(x_{j+\frac{1}{2}})}{\Delta x}. \tag{3.88}
\]

Multiply the above inequality by \( \Delta t \Delta x \phi^n_j \geq 0 \), and sum with respect to \( j \) and \( n \), to get

\[
\begin{align*}
\Delta t \Delta x \sum_{n=0}^{N} \sum_{j=-\infty}^{\infty} \frac{|v_j^{n+1} - c| - |u_j^n - c|}{\Delta t} \phi^n_j & \leq 0, \tag{3.89}
\end{align*}
\]

Let \( \{(\Delta x)_k\}_k \) be the subsequence of \( \Delta x \) in (3.80). By (3.80), and with the same argument as in [46], the term (3.89) converges as follows

\[
\lim_{(\Delta x)_k \to 0} \Delta t \Delta x \sum_{n=0}^{N} \sum_{j=-\infty}^{\infty} \frac{|v_j^{n+1} - c| - |u_j^n - c|}{\Delta t} \phi^n_j = - \int_0^T \int_{\mathbb{R}} |u^* - c| \phi dxdt.
\]

Let \( \{\delta, (\Delta x)_k\}_k \) be the subsequence of \( \{\delta, \Delta x\} \) in (3.82). By (3.82), a similar argument
yields

\[
\lim_{\delta_k \to 0,(\Delta x)_k \to 0} \Delta t \Delta x \sum_{n=0}^{N} \sum_{j=-\infty}^{\infty} \frac{|u_{j+1}^n - c| - |u_j^n - c|}{\Delta t} \phi_j^n = \int_0^T \int_{\mathbb{R}} |u^{**} - c| \phi_t dx dt.
\]

The proof is finished by applying Proposition 3.6.7 to the term (3.90).

\[\blacksquare\]

**Proposition 3.6.7.** Assume \(u_0 \in BV(\mathbb{R})\) and \(\phi\) satisfies (3.88). Suppose all the assumptions of Proposition 3.6.6 hold. Let \(c \in \mathbb{R}\), and denote the nonlocal entropy flux \(q(a,b; c)\) as \(q(a,b)\) for convenience.

(i) Fixing \(\delta > 0\), and let \(\{(\Delta x)_t\}_t\) be the subsequence of \(\Delta x\) in (3.80). Then:

\[
\lim_{(\Delta x)_t \to 0} \Delta t \Delta x \sum_{n=0}^{N} \sum_{j=-\infty}^{\infty} \sum_{k=1}^{r<n} [q(u_j^n, u_{j+k}^n) - q(u_{j-k}^n, u_j^n)] W_k
= -\int_0^T \int_{\mathbb{R}} \phi(x) - \phi(x + \delta) q(u^*(x), u^*(x + \delta)) \frac{\omega(h)}{h} dh dx dt.
\]

(ii) Let \(\{\delta_t,(\Delta x)_t\}_t\) be the subsequence of \(\{\delta, \Delta x\}\) in (3.82). Then:

\[
\lim_{\delta_t \to 0,(\Delta x)_t \to 0} \Delta t \Delta x \sum_{n=0}^{N} \sum_{j=-\infty}^{\infty} \sum_{k=1}^{r<n} \phi_j^n [q(u_j^n, u_{j+k}^n) - q(u_{j-k}^n, u_j^n)] W_k
= -\int_0^T \int_{\mathbb{R}} \phi_x \text{sgn}(u^{**} - c) (f(u^{**}) - f(c)) dx dt.
\]

**Proof.** Using Lemma 3.6.9 and noting \(\Delta h = \Delta x\),

\[
\Delta t \Delta x \sum_{n=0}^{N} \sum_{j=-\infty}^{\infty} \phi_j^n \sum_{k=1}^{r<n} [q(u_j^n, u_{j+k}^n) - q(u_{j-k}^n, u_j^n)] W_k
= -\Delta t \Delta x \sum_{n=0}^{N} \sum_{j=-\infty}^{\infty} \sum_{k=1}^{r<n} (\phi_{j+k} - \phi_j) q(u_j^n, u_{j+k}^n) W_k
= -\Delta t \Delta x \sum_{n=0}^{N} \sum_{j=-\infty}^{\infty} \sum_{k=1}^{r<n} \left( \frac{\phi_{j+k} - \phi_j}{k \Delta x} \right) q(u_j^n, u_{j+k}^n) kW_k \Delta h
= -\Delta t \Delta h \Delta x \sum_{n=0}^{N} \sum_{j=-\infty}^{\infty} \sum_{k=1}^{r<n} \phi_j^n \phi_{j+k} W_k \int_{(k-1)\Delta x}^{k\Delta x} \omega(s) ds
= -\int_0^T \int_{\mathbb{R}} \int_0^\delta \frac{\phi_{\Delta}(x,t) - \phi_{\Delta}(x + h_{\Delta}(h),t)}{h_{\Delta}(h)} q(u^\Delta(x), u^\Delta(x + h_{\Delta}(h))) \omega_{\Delta}(h) dh dx dt.
\]
where $u^\Delta$ is defined in (3.75), and

$$
\phi^\Delta(x,t) = \sum_{j=-\infty}^{\infty} \sum_{n=0}^{\infty} \left( \frac{\phi(x_{j+\frac{1}{2}}) - \phi(x_{j-\frac{1}{2}})}{\Delta x} (x - x_{j-\frac{1}{2}}) + \phi(x_{j-\frac{1}{2}}) \right) 1_{[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]}(x,t),
$$

$$
\bar{\omega}_k := \frac{1}{\Delta x} \int_{(k-1)\Delta x}^{k\Delta x} \omega(s)ds,
$$

$$
\bar{\omega}^\Delta(h) = \sum_{k=1}^{r} \bar{\omega}_k 1_{[(k-1)\Delta x, k\Delta x]}(h) = \sum_{k=1}^{r} \left( \frac{1}{\Delta x} \int_{(k-1)\Delta x}^{k\Delta x} \omega(s)ds \right) 1_{[(k-1)\Delta x, k\Delta x]}(h),
$$

$$
h^\Delta(h) = \sum_{k=1}^{r} (k\Delta x) 1_{[(k-1)\Delta x, k\Delta x]}(h).
$$

Note that these functions are piecewise constant except $\phi^\Delta(x,t)$. $\phi^\Delta(x,t)$ is piecewise linear about $x$ on the interval $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$. Also notice the following facts:

a) \[ \lim_{\Delta x \to 0} h^\Delta(h) = h, \quad h \in (0, \delta] \] (3.94)

b) \[ \lim_{\Delta x \to 0} \bar{\omega}^\Delta(h) = \omega(h), \quad \forall h \in (0, \delta]; \] (3.95)

\[ ||\bar{\omega}^\Delta||_{L^1(0,\delta)} = \int_0^\delta \bar{\omega}^\Delta(h)dh = \sum_{k=1}^{r} \int_{(k-1)\Delta x}^{k\Delta x} \omega(s)ds \equiv 1, \quad \text{for all } \delta > 0. \] (3.96)

c) Since $\phi^\Delta(x,t) \to \phi(x,t)$ uniformly, $\phi$ is continuous, and $h^\Delta(h) \to h$

\[ \lim_{\Delta x \to 0} \phi^\Delta(x + h^\Delta(h),t) \to \phi(x + h,t) \quad \text{a.e } (x,t). \] (3.97)

d) \[ \frac{\phi^\Delta(x,t) - \phi^\Delta(x + h^\Delta(h),t)}{h^\Delta(h)} \leq C||\phi_x||_{L^\infty(\mathbb{R}x\mathbb{R}^+)} \] (3.98)

e) When (3.80) holds, apparently

\[ \lim_{k \to \infty} u^\Delta(x,t) = u^*(x,t), \quad \text{for any } t \in [0,T]. \] (3.99)

Moreover, by (3.81), $u^*(\cdot,t) \in BV(\mathbb{R})$, so $u^*$ has at most countably many discontinuities and is
continuous almost everywhere on \( \mathbb{R} \). Hence for almost everywhere \((x, t) \in [a - \delta, b + \delta] \times [0, T]\),

\[
\lim_{\Delta x \to 0} u^*(x + h^\Delta(h), t) = u^*(x + \lim_{\Delta x \to 0} h^\Delta(h), t) = u^*(x + h, t),
\]

by (3.94).

These facts will be useful in the proof of this proposition, as well as proof of Lemma 3.6.8.

(i) We start by proving (3.91). For convenience of notation, we just denote \((\Delta x)_l\) as \(\Delta x\), but we should always keep in mind: these \(\{\Delta x_l\}\) corresponds to the subsequence of \(u^\Delta\) that converges to \(u^*\) as in (3.80).

It suffices to show that, when \(\delta\) is fixed and \(\Delta x \to 0\), (3.93) converges as follows:

\[
\lim_{\Delta x \to 0} \int_0^T \int_0^\delta \frac{\phi^\Delta(x, t) - \phi^\Delta(x + h^\Delta(h), t)}{h^\Delta(h)} q(u^\Delta(x), u^\Delta(x + h^\Delta(h))) \omega^\Delta(h) dh dx dt = \int_0^\delta \int_0^T \frac{\phi(x) - \phi(x + h)}{h} q(u(x), u(x + h)) \omega(h) dh dx dt.
\]

Actually, applying dominated convergence theorem (conditions will be checked in a second) in (3.93), we have

\[
\lim_{\Delta x \to 0} \int_0^\delta \left( \int_{\mathbb{R}^+ \times \mathbb{R}} \frac{\phi^\Delta(x, t) - \phi^\Delta(x + h^\Delta(h), t)}{h^\Delta(h)} q(u^\Delta(x), u^\Delta(x + h^\Delta(h))) \omega^\Delta(h) dx dt \right) dh = \int_0^\delta \lim_{\Delta x \to 0} \left( \int_{\mathbb{R}^+ \times \mathbb{R}} \frac{\phi^\Delta(x, t) - \phi^\Delta(x + h^\Delta(h), t)}{h^\Delta(h)} q(u^\Delta(x), u^\Delta(x + h^\Delta(h))) \omega^\Delta(h) dx dt \right) dh
\]

\[
= \int_0^\delta \int_{\mathbb{R} \times \mathbb{R}} [\phi(x, t) - \phi(x + h, t)] q(u(x, t), u(x + h, t)) \omega^\delta(h) dx dt dh.
\]

It remains to check two conditions for dominated convergence theorem:

**Condition 1:** For a.e \( h \in (0, \delta]\),

\[
\lim_{\Delta x \to 0} \int_0^T \int_{\mathbb{R}} \frac{\phi^\Delta(x, t) - \phi^\Delta(x + h^\Delta(h), t)}{h^\Delta(h)} q(u^\Delta(x), u^\Delta(x + h^\Delta(h))) dx dt \omega^\Delta(h) = \int_0^T \int_{\mathbb{R}} \frac{\phi(x, t) - \phi(x + h, t)}{h} q(u^*(x, t), u^*(x + h, t)) dx dt \omega^\delta(h).
\]

This is based on Lemma 3.6.8 and (3.95).
Condition 2: there exists a function $Y \in L^1(0, \delta)$, such that
\[
\left| \int_0^T \int_\mathbb{R} \frac{\phi^\Delta(x, t) - \phi^\Delta(x + h^\Delta(h), t)}{h^\Delta(h)} q(u^\Delta(x), u^\Delta(x + h^\Delta(h))) \bar{\omega}^\Delta(h) dx dt \right| \leq Y(h).
\]
Actually,
\[
\left| \int_0^T \int_{a-\delta}^{b+\delta} \frac{\phi^\Delta(x, t) - \phi^\Delta(x + h^\Delta(h), t)}{h^\Delta(h)} q(u^\Delta(x), u^\Delta(x + h^\Delta(h))) \bar{\omega}^\Delta(h) dx dt \right|
\leq \bar{\omega}^\Delta(h) \cdot C \cdot \int_0^T \int_{a-\delta}^{b+\delta} \frac{\phi^\Delta(x, t) - \phi^\Delta(x + h^\Delta(h), t)}{h^\Delta(h)} dx dt
\leq \bar{\omega}^\Delta(h) \cdot C \cdot T(b - a + 2\delta) \| \phi_x \|_{L^\infty(\mathbb{R} \times \mathbb{R}^+)} = C_{\phi, u_0, g, \delta} \bar{\omega}^\Delta(h),
\]
by the boundedness about $\phi$ (3.97-3.98), boundeness of $q$ (3.35) and boundedness of $u^\Delta$ (3.76). Noting that $\bar{\omega}^\Delta$ is integrable (3.96), we can take
\[
Y(h) = C_{\phi, u_0, g, \delta} \bar{\omega}^\Delta(h).
\]
(ii) For convenience of notation, we just denote $\delta_1$ as $\delta$, and $(\Delta x)_1$ as $\Delta x$, but we should always keep in mind: these \{\delta_1, (\Delta x)_1\} corresponds to the subsequence of \{u^\Delta_1\} that converges to $u^{**}$ as in (3.82).

To prove (3.92), it suffices to show that, when $\delta$ and $\Delta x$ both go to zero, (3.93) converges as follows:
\[
\lim_{(\delta, \Delta x) \to (0, 0)} \int_0^T \int_\mathbb{R} \frac{\phi^\Delta(x, t) - \phi^\Delta(x + h^\Delta(h), t)}{h^\Delta(h)} q(u^\Delta(x), u^\Delta(x + h^\Delta(h))) \bar{\omega}^\Delta(h) dh dx dt
= \int_0^\infty \int_\mathbb{R} \phi_x(x, t) q(u^{**}(x), u^{**}(x)) dx dt = \int_0^T \int_\mathbb{R} \phi_x(x, t) sgn(u^{**} - c)(f(u^{**}) - f(c)) dx dt.
\]
Actually, for any $\varepsilon > 0$, when $\delta$ and $\Delta x$ are small enough,
\[
\left| \frac{\phi^\Delta(x, t) - \phi^\Delta(x + h^\Delta(h), t)}{h^\Delta(h)} q(u^\Delta(x), u^\Delta(x + h^\Delta(h))) - \phi_x(x) q(u^{**}(x), u^{**}(x)) \right|
\leq \left( \left| \frac{\phi^\Delta(x + h^\Delta(h)) - \phi^\Delta(x)}{h^\Delta(h)} - \frac{\phi(x + h) - \phi(x)}{h} \right| + \left| \frac{\phi(x + h) - \phi(x)}{h} - \phi_x(x) \right| \right) \left| q(u^\Delta(x), u^\Delta(x + h^\Delta)) \right|
+ |\phi_x(x)| \left| q(u^\Delta(x), u^\Delta(x + h^\Delta)) - q(u^{**}(x), u^{**}(x)) \right|
\leq 2\varepsilon C_{\phi, g} |u^\Delta(x)| + |u^\Delta(x + h^\Delta)| + 1 + C_{\phi, g} \left( |u^\Delta(x) - u^{**}(x)| + |u^\Delta(x + h^\Delta) - u^{**}(x + h^\Delta)| + |u^{**}(x + h^\Delta) - u^{**}(x)| \right)
\leq C_{\phi, g, u_0} \varepsilon + C_{\phi, g} \left( |u^\Delta(x) - u^{**}(x)| + |u^\Delta(x + h^\Delta) - u^{**}(x + h^\Delta)| + |u^{**}(x + h^\Delta) - u^{**}(x)| \right)
\]
\[ \leq C_{\phi,g,u_0} \varepsilon + C_{\phi,g} |u^\Delta(x + h^\Delta) - u^{**}(x + h^\Delta)|, \quad \text{a.e. (x, t) } \in \mathbb{R} \times \mathbb{R}^+. \]

The inequalities above are based on the boundedness of \( \phi \) (3.97-3.98), boundedness and Lipschitz continuity of \( q \) (3.34, 3.35), and convergence of \( u^\Delta \delta \) to \( u^{**} \) as \( \Delta x \to 0 \) (3.82).

Supposing the compact support of \( \phi \) is in interval \([a, b] \times [0, T] \). Using the above inequality, and noting that \( ||\bar{\omega}^{\delta}||_{L^1(\mathbb{R})} = 1 \) by (3.95),

\[
\begin{align*}
& \left| \int_0^T \int_{a-\delta}^{a+\delta} \int_0^\delta \frac{\phi^\Delta(x,t) - \phi^\Delta(x+h^\Delta(h),t)}{h^\Delta(h)} q(u^\Delta(x), u^\Delta(x+h^\Delta(h))) \bar{\omega}^\Delta(h) dh dx dt \\
& \quad - \int_0^T \int_{a-\delta}^{a+\delta} \int_0^\delta \phi_x(x) q(u^{**}(x), u^{**}(x)) dx dt \right|
\leq \int_0^T \int_{a-\delta}^{a+\delta} \int_0^\delta \left| \frac{\phi^\Delta(x,t) - \phi^\Delta(x+h^\Delta(h),t)}{h^\Delta(h)} q(u^\Delta(x), u^\Delta(x+h^\Delta(h))) \right| \bar{\omega}^\Delta(h) dh dx dt
\leq \int_0^T \int_{a-\delta}^{a+\delta} \int_0^\delta (C_{\phi,g,u_0} \varepsilon + C_{\phi,g} |u^\Delta(x + h^\Delta) - u^{**}(x + h^\Delta)|) \bar{\omega}^\Delta(h) dh dx dt
\leq C_{\phi,g,u_0} \varepsilon + \int_0^T \int_{a-\delta}^{a+\delta} \int_0^\delta |u^\Delta(y) - u^{**}(y)| \bar{\omega}^\Delta(h) dy dh dt
\leq C_{\phi,g,u_0} \varepsilon + \int_0^T \int_{a-\delta}^{a+\delta} |u^\Delta(y) - u^{**}(y)| dy dt \left( \int_0^\delta \bar{\omega}^\Delta(h) dh \right)
= C_{\phi,g,u_0} \varepsilon + \int_0^T \int_{a-\delta}^{a+\delta} |u^\Delta(x) - u^{**}(x)| dx dt.
\end{align*}
\]

By (3.82), when \( \Delta x \) is small enough,

\[
\int_0^T \int_{a-\delta}^{a+\delta} |u^\Delta(x) - u^{**}(x)| dx dt < \varepsilon,
\]

and thus

\[
\int_0^T \int_{a-\delta}^{a+\delta} \int_0^\delta \left| \frac{\phi^\Delta(x,t) - \phi^\Delta(x+h^\Delta(h),t)}{h^\Delta(h)} q(u^\Delta(x), u^\Delta(x+h^\Delta(h))) \right| \bar{\omega}^\Delta(h) dh dx dt \leq C_{\phi,g,u_0} \varepsilon,
\]

Now (3.92) is reached, and the proof is finished.

\[\square\]

**Lemma 3.6.8.** Suppose all the assumptions of Proposition 3.6.7 hold. Let \( u^{(\Delta x)i} \) be the subsequence that converges to \( u^* \) as in (3.80), and for convenience we denote \( \{(\Delta x)l\} \) as \( \Delta x \).

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Then for a.e $h \in (0, \delta)$, 

$$
\lim_{\Delta x \to 0} \int_0^T \int_{\mathbb{R}} \frac{\phi^\Delta(x,t) - \phi^\Delta(x + h^\Delta(h),t)}{h^\Delta(h)} q(u^\Delta(x),u^\Delta(x + h^\Delta(h))) \, dx \, dt
$$

$$
= \int_0^T \int_{\mathbb{R}} \frac{\phi(x,t) - \phi(x + h,t)}{h} q(u^*(x,t),u^*(x + h,t)) \, dx \, dt. 
$$

(3.101)

Proof. Denote the support of $\phi(x,t)$ is in $[a,b] \times [0,T]$, then (3.101) is equivalent to

$$
\lim_{\Delta x \to 0} \int_0^T \int_{a - \delta}^{b + \delta} \frac{\phi^\Delta(x,t) - \phi^\Delta(x + h^\Delta(h),t)}{h^\Delta(h)} q(u^\Delta(x),u^\Delta(x + h^\Delta(h))) \, dx \, dt
$$

$$
= \int_0^T \int_{a - \delta}^{b + \delta} \frac{\phi(x,t) - \phi(x + h,t)}{h} q(u^*(x,t),u^*(x + h,t)) \, dx \, dt. 
$$

By the fact

$$
f_n \to f \text{ a.e, } \sup_n |f_n| \leq C \mathbf{1}_E, \sup_x |f| < \infty, \quad g_n \to g \in L^1, \sup_{n,x} g < \infty \quad \implies \quad f_n g_n \to fg \text{ in } L^1, 
$$

we only need to show the following two conditions: for almost everywhere $h \in (0, \delta]$,

(i) 

$$
\left\{ \begin{array}{l}
\lim_{\Delta x \to 0} \frac{\phi^\Delta(x,t) - \phi^\Delta(x + h^\Delta(h),t)}{h^\Delta(h)} = \frac{\phi(x,t) - \phi(x + h,t)}{h} \quad \text{a.e } (x,t) \in \mathbb{R} \times \mathbb{R}^+ \\
\sup_{x,t} \left| \frac{\phi^\Delta(x,t) - \phi^\Delta(x + h^\Delta(h),t)}{h^\Delta(h)} \right| < C \mathbf{1}_E, \quad \sup_{x,t} \left| \frac{\phi(x,t) - \phi(x + h,t)}{h} \right| < \infty,
\end{array} \right. 
$$

where $E \subset \mathbb{R}$ is a bounded set.

(ii) 

$$
\lim_{\Delta x \to 0} \int_0^T \int_{a - \delta}^{b + \delta} |q(u^\Delta(x),u^\Delta(x + h^\Delta(h))) - q(u^*(x),u^*(x + h))| \, dx \, dt = 0
$$

$$
\sup_{\Delta x,x,t} |q(u^\Delta(x),u^\Delta(x + h^\Delta(h)))| < \infty.
$$

(i) is apparent by (3.97-3.98). Next we show (ii). The bounded of $q$ was obtained by the boundedness of $q$ (3.35) and the boundedness of $u^\Delta$ (3.76). By the Lipschitz continuity of $q$ (3.34), 

$$
\left| \int_0^T \int_{a - \delta}^{b + \delta} q(u^\Delta(x),u^\Delta(x + h^\Delta(h))) - q(u^*(x),u^*(x + h)) \, dx \, dt \right|
$$
\[
\leq \int_0^T \int_{a-\delta}^{a+\delta} |q(u^\Delta(x), u^\Delta(x+h^\Delta(h)) - q(u^*(x), u^*(x+h^\Delta(h)))| \, dx \, dt \\
+ \int_0^T \int_{a-\delta}^{a+\delta} |q(u^*(x), u^*(x+h^\Delta(h))) - q(u^*(x), u^*(x+h))| \, dx \, dt
\leq C \int_0^T \int_{a-\delta}^{a+\delta} |u^\Delta(x) - u^*(x)| + |u^\Delta(x+h^\Delta(h)) - u^*(x+h^\Delta(h))| \, dx \, dt \\
+ C \int_0^T \int_{a-\delta}^{a+\delta} |u^*(x+h^\Delta(h)) - u^*(x+h)| \, dx \, dt := I + II.
\]

For the first integral, assumption (3.80) yields
\[
\lim_{\Delta x \to 0} I = \lim_{\Delta x \to 0} \int_0^T \int_{a-\delta}^{a+\delta} |u^\Delta(x) - u^*(x)| + |u^\Delta(x+h^\Delta(h)) - u^*(x+h^\Delta(h))| \, dx \, dt
\leq \lim_{\Delta x \to 0} \int_0^T \int_{a-\delta}^{a+\delta} |u^\Delta(x) - u^*(x)| \, dx \, dt + \lim_{\Delta x \to 0} \int_0^T \int_{a-\delta + h^\Delta(h)}^{a+\delta + h^\Delta(h)} |u^\Delta(y) - u^*(y)| \, dy \, dt
\leq \int_0^T \int_{a-\delta}^{a+\delta} |u^\Delta(x) - u^*(x)| \, dx \, dt + \lim_{\Delta x \to 0} \int_0^T \int_{a-\delta}^{a+\delta + h} |u^\Delta(y) - u^*(y)| \, dy \, dt = 0,
\]
with the last equality coming from (3.80). For the second integral, by the boundedness of \( u^* \) (3.81), we can apply dominated convergence theorem:
\[
\lim_{\Delta x \to 0} II = \int_0^T \int_{a-\delta}^{a+\delta} \lim_{\Delta x \to 0} |u^*(x+h^\Delta(h)) - u^*(x+h)| \, dx \, dt = 0,
\]
the last equality is due to the (almost everywhere) continuity of \( u^* \) (3.100). So condition (ii) is reached. It completes the proof of this lemma.

\[
\text{Lemma 3.6.9. (Summation by parts for discretization of the integral)}
\]
\[
\sum_{j=-\infty}^{\infty} \sum_{k=1}^{r} (q_{j+k} - q_{j-k}) W_k \phi_j = - \sum_{j=-\infty}^{\infty} \sum_{k=1}^{r} (\phi_{j+k} - \phi_j) q_{j+k} W_k
\]  
\[
(3.103)
\]

\text{Proof.}
\[
\sum_{j=-\infty}^{\infty} \sum_{k=1}^{r} (q_{j+k} - q_{j-k}) W_k \phi_j
= \sum_{j=-\infty}^{\infty} \sum_{k=1}^{r} (q_{j+k} - q_{j-k}) W_k \phi_j
= \left[ \cdots + (q_{-1,0} - q_{-2,1}) W_1 \phi_{-1} + (q_{-1,1} - q_{-3,-1}) W_2 \phi_{-1} + \cdots + (q_{-1,-1+r} - q_{-1,-1}) W_r \phi_{-1} + \cdots \right]
\]
Theorem 3.6.10. (Main Convergence Theorem) Assume \( u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \), and all other assumptions are the same as Theorem 3.6.1 (That is, to extract the assumption \( u_0 \in C([0,T];BV(\mathbb{R})) \)). Then: the conclusion of Theorem 3.6.1 still holds.

Proof. The proof is the same as the second part of proof for Theorem 1 in [46], and our Theorem 3.6.1 is used to replace the first part of proof for Theorem 1 in [46].

\[ \sum_{j=-\infty}^{\infty} q_{j,j+1} W_1(\phi_j - \phi_{j+1}) + \sum_{j=-\infty}^{\infty} q_{j,j+2} W_2(\phi_j - \phi_{j+2}) + \cdots \]

3.6.2 Existence and uniqueness of the nonlocal entropy solution

A re-examination of the proof of Theorem 3.6.10 shows the existence of the entropy solution for nonlocal conservation law (3.31), \( u^{\text{nonl}} \). (The uniqueness of nonlocal entropy solution was given by Theorem 3.3.3.)

Theorem 3.6.11. (Existence and uniqueness of the nonlocal entropy solution) In nonlocal conservation law (3.31), assume \( g \) satisfies (3.26, 3.28, 3.27), and kernel \( \omega^\delta \) satisfies condition (3.25). Then, for every initial data \( u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \), there is a unique entropy solution \( u^{\text{nonl}} \) of (3.31) with \( u(0) = u_0 \), such that

(a) If \( u_0 \in BV(\mathbb{R}) \), \( t \to u^{\text{nonl}}(\cdot,t) \) is Lipschitz continuous into \( L^1(\mathbb{R}) \) and

\[ ||u^{\text{nonl}}(\cdot,t)||_{BV(\mathbb{R})} \leq ||u_0||_{BV(\mathbb{R})}. \]
3.7 Numerical experiments

In this section, we will perform numerical experiments on scheme (3.57) for our nonlocal conservation law (3.31). We will pick \( g \) as the Godunov flux function or Engquist-Osher flux function corresponding to the Finite Difference scheme of the classical Burger’s equation (3.4), such that our nonlocal conservation law (3.31) can be seen as a “nonlocal Burger’s equation”.

We will perform five groups of numerical experiments for the nonlocal scheme (3.57), each with different initial data \( u_0 \) and kernels \( \omega_\delta \):

(i) Fix \( \delta \), refine \( \Delta x \). The numerical solution of nonlocal Burger’s equation converges;

(ii) Fixing \( \Delta x \) and refining \( \delta \). The numerical solution of nonlocal Burger’s equation converges to the numerical solution of local Burger’s equation;

(iii) Fixing the number of interaction cells, \( r = \left\lfloor \frac{\delta}{\Delta x} \right\rfloor \), and refining both \( \delta \) and \( \Delta x \) at the same time. The numerical solution of nonlocal Burger’s equation converges to the entropy solution of local Burger’s equation;

(iv) Shock formation. Fixing \( \delta \), and using smooth initial condition \( u_0^3 \), we study, under different kernel \( \omega_\delta \), whether the nonlocal solutions develop shocks in finite time.

(v) Propagation of discontinuity. We consider piecewise constant initial data \( u_0^1 \) and \( u_0^2 \), and observe the time evolution of nonlocal solution under different kernels.

3.7.1 Set up \( g, u_0 \) and \( \omega_\delta \)

3.7.1.1 Setup \( g \).

In the first three experiments, we will take \( g \) as the Godunov flux \( g^{\text{God}} \) for the Finite Difference scheme of the classical Burger’s equation \( u_t + \left( \frac{u^2}{2} \right)_x = 0 \). Then the Godunov flux function
corresponding to the Burger’s flux $f(\theta) = \frac{\theta^2}{2}$, denoted as $g^{\text{God}}$, is

$$g^{\text{God}}(u^n_j, u^n_{j+1}) = \begin{cases} \min_{u^n_j \leq \theta \leq u^n_{j+1}} \frac{\theta^2}{2}, & \text{if } u^n_j \leq u^n_{j+1}, \\ \max_{u^n_{j+1} \leq \theta \leq u^n_j} \frac{\theta^2}{2}, & \text{if } u^n_j \geq u^n_{j+1}. \end{cases}$$

Since $f$ is strictly convex, the above formula reduces to

$$g^{\text{God}}(u^n_j, u^n_{j+1}) = \max \{ f(\max(u^n_j, \theta^*)) , f(\min(u^n_{j+1}, \theta^*)) \},$$

where $\theta^*$ is the unique local minimum of $f$, $\theta^* = 0$. That is,

$$g^{\text{God}}(a, b) = \frac{1}{2} \max \{ (a^+)^2 , (b^-)^2 \}. \quad (3.104)$$

In the last two experiments we will take $g$ as the Engquist-Osher flux $g^{\text{EO}}$ associated with the Burger’s flux $f$:

$$g^{\text{EO}}(a, b) = \int_0^b \min(f'(s), 0) ds + \int_0^a \max(f'(s), 0) ds + f(0)$$
$$= \frac{a^2}{2} \mathbf{1}_{a > 0}(a) + \frac{b^2}{2} \mathbf{1}_{b < 0}, \quad (3.105)$$

In both cases, our nonlocal model (3.31) can be seen as a nonlocal Burger’s equation.

3.7.1.2 Setup kernel $\omega^\delta$.

We focus on kernels $\omega^\delta$ that are power functions. Let $p > -1$, consider the following density function $\rho$, and the corresponding kernel $\omega^\delta(h) = \frac{1}{3} \rho \left( \frac{h}{3} \right)$:

$$\rho(h) = (1 + p) h^p \mathbf{1}_{(0,1)}(h), \quad \omega^\delta(h) = \left( \frac{1 + p}{3^{1+p}} \right) h^p \mathbf{1}_{(0,\delta)}(h). \quad (3.106)$$

3.7.1.3 Setup initial condition $u_0$.

In the experiments we will use three initial conditions, each is periodic with a cycle in $[-1,1]$. The first two are piecewise-constant functions for Riemann problem, the third is a smooth function:
\[ u_0^1(x) = \begin{cases} 1, & x \in [-1, 0) \\ 0, & x \in [0, 1] \end{cases} \quad (3.107) \]

\[ u_0^2(x) = \begin{cases} -1, & x \in [-1, 0) \\ 1, & x \in [0, 1] \end{cases} \quad (3.108) \]

\[ u_0^3(x) = \sin \pi x, \quad x \in [-1, 1]. \quad (3.109) \]

The initial data \( u_0^j \) in the discrete level is taken as the cell average as in (3.57).

### 3.7.1.4 More on the experiments

Notice:

(1) In our experiments, we will investigate how the following elements would affect the numerical solutions: time \( T \), horizon \( \delta \), grid-size \( \Delta x \), initial data \( u_0^i \), and \( p \), the power parameter that determines kernel \( \omega^\delta \) in (3.106).

(2) In all experiments we fix \( \Delta x \Delta t = 4 \), which makes nonlocal CFL condition (3.64) hold, for both \( g^{God} \) and \( g^{EO} \), and for all \( p > -1 \).

(3) We only consider the case when \( \delta \) is a multiple of cell size \( \Delta x \), such that \( r = \lfloor \frac{\delta}{\Delta x} \rfloor \) is a nonnegative integer.

(4) In all our plots, black curve represents the initial data.

### 3.7.2 Experiment 1: Fix \( \delta \), refine \( \Delta x \)

In Fig 3.1, we fix horizon \( \delta = 0.2 \), time \( T = 0.6 \), and refine \( \Delta x \). Columns of Figure 3.1 correspond to different choices of initial data \( u_0^i \), while the rows indicate different \( p \) values. In each plot, the color blue, cyan, green, yellow, purple and red correspond to \( \Delta x = \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128} \) and \( \frac{1}{256} \) respectively. The plots show that, as we refine \( \Delta x \), the curves get closer and appear to converge.

It is also interesting to observe how the \( p \) value impacts the nonlocal solutions under the same \( \delta \) at the same time \( T \). In general, the smaller \( p \) is, the more the nonlocal solutions look
like the solution of local Burger’s equation. More specifically, for $u_0^1$, when $p$ decreases from 1 to $-0.9$, the discontinuity introduced by $u_0^1$ flats out less, and the nonlocal solution appears steeper near $x = 0.3$. While the $u_0^2$ case is the opposite: when $p$ decrease, the nonlocal solution encroach on the discontinuity of $u_0^2$ in a faster way, and the nonlocal solution appears smoother near $x = 0$. For the smooth initial data $u_0^3$, the smaller $p$ is, the more nonlocal solution looks like a shock (We will examine whether a shock indeed forms in Experiment 4.)

Table 3.1 - 3.2 list the numerical errors in $L^1$, $L^\infty$ and BV norms, for $p = 1$, $p = 0$ and $p = -0.9$ respectively, with the smooth initial data $u_0^3$. The “true” nonlocal solution $u_{true}^{nonlocal}$ is actually a numerical solution computed on a fine mesh, with $\Delta x = 1/512$ and $\Delta t = \Delta x/4$. The table shows that, as we refine $\Delta x$, the error is decreasing, no matter which norm is used. In particular, for the $p = -0.9$ case, the $L^\infty$ and BV norms do not reduce as significantly as the $p = 1$ and $p = 0$ case. It may be due to that there is shock formation near $x = 0$ when $p = -0.9$ (see Experiment 4 for more details).

| Error $||u_{numerical}^\delta - u_{true}^\delta||$, $p = 1$ | $dx$ | $L^1$ | $L^\infty$ | BV |
|---|---|---|---|---|
| kernel $\omega_1^\delta$ | | | | |
| $dx = 1/8$ | 0.1929 | 0.5619 | 1.3261 |
| $dx = 1/16$ | 0.0944 | 0.3125 | 0.7579 |
| $dx = 1/32$ | 0.0458 | 0.1983 | 0.4721 |
| $dx = 1/64$ | 0.0214 | 0.1120 | 0.2634 |
| $dx = 1/128$ | 0.0093 | 0.0539 | 0.1261 |
| $dx = 1/256$ | 0.0031 | 0.0196 | 0.0455 |

Table 3.1: Errors of nonlocal numerical solutions from nonlocal “true” solution, with $p = 1$, $\delta = 0.2$ and $T = 0.6$ are fixed.
Table 3.2: Errors of nonlocal numerical solutions from nonlocal “true” solution, with $p = 0$ and $p = -0.9$ respectively. $\delta = 0.2$ and $T = 0.6$ are fixed.
Fig. 3.1: Fix $\delta = 0.2$ and $T = 0.6$, refine $\Delta x$. The color blue, cyan, green, yellow, purple and red correspond to $\Delta x = \frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, $\frac{1}{64}$, $\frac{1}{128}$ and $\frac{1}{256}$, respectively.
3.7.3 Experiment 2: Fix $\Delta x$, refine parameter $\delta$

In Fig 3.2, we fix $\Delta x = \frac{1}{200}$, $T = 0.5$, refine $\delta$. The blue, cyan, green and purple curves correspond to $\delta = 64\Delta x$, $16\Delta x$, $4\Delta x$ and $\Delta x$. From the plot we see that, as we refining $\delta$, the curves gets closer the numerical solution of classical Burger’s equation. Table 3.3 and 3.4 list the numerical errors in $L^1$, $L^\infty$ and BV norms, when using $p = 0.25$, $p = 0$ and $p = -0.5$, with $T = 0.2$, $\delta = 0.2$ fixed. It is shown that, as we refine $\delta$, the error in each norm decreases.

$u_{\text{local numerical}}$ is a Finite Difference solution of classical Burger’s equation, and it is computed on a fine mesh of $\Delta x = \frac{1}{1600}$ and $\Delta t = \frac{\Delta x}{4}$, using classical Godunov method.

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$L^1$</th>
<th>$L^\infty$</th>
<th>BV</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.32</td>
<td>0.1237</td>
<td>0.1436</td>
<td>0.7929</td>
</tr>
<tr>
<td>0.16</td>
<td>0.0682</td>
<td>0.0821</td>
<td>0.6251</td>
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<td>0.08</td>
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<td>0.0894</td>
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<tr>
<td>0.04</td>
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<td>0.0721</td>
<td>0.3246</td>
</tr>
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<tr>
<td>0.01</td>
<td>0.0096</td>
<td>0.0357</td>
<td>0.1039</td>
</tr>
</tbody>
</table>

Table 3.3: Errors of nonlocal numerical solutions from local true solution, with $p = 0$

<table>
<thead>
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<th>$\delta$</th>
<th>$L^1$</th>
<th>$L^\infty$</th>
<th>BV</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.32</td>
<td>0.1371</td>
<td>0.1563</td>
<td>0.8691</td>
</tr>
<tr>
<td>0.16</td>
<td>0.0750</td>
<td>0.0904</td>
<td>0.6578</td>
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<tr>
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<td>0.0439</td>
<td>0.0965</td>
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</tr>
<tr>
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<td>0.0808</td>
<td>0.3656</td>
</tr>
<tr>
<td>0.02</td>
<td>0.0138</td>
<td>0.0558</td>
<td>0.1999</td>
</tr>
<tr>
<td>0.01</td>
<td>0.0108</td>
<td>0.0404</td>
<td>0.1172</td>
</tr>
</tbody>
</table>

Table 3.4: Errors of nonlocal numerical solutions from local true solution, with $p = -0.5$ and $p = 0.25$
(a) $p = 0.5, u_0^1$
(b) $p = 0.5, u_0^2$
(c) $p = 0.5, u_0^3$
(d) $p = 0, u_0^1$
(e) $p = 0, u_0^2$
(f) $p = 0, u_0^3$
(g) $p = -0.25, u_0^1$
(h) $p = -0.25, u_0^2$
(i) $p = -0.25, u_0^3$

Fig. 3.2: Fix $\Delta x$, refine $\delta$. 
3.7.4 Experiment 3: Fix $r$, refine both $\Delta x$ and $\delta$ at the same time

Figure 3.3-3.5 use $p = 0.25$, $p = 0$ and $p = -0.5$, respectively. In each case, we fix $r = 2$ and $r = 5$, refine $\delta$ into half each time, thus $\Delta x$ is refined into half accordingly. The blue, cyan, green, purple and red curves correspond to $\Delta x = 0.2$, 0.1, 0.05, 0.025 and 0.0125 at $T = 0.5$.

As the plots shows, for all three initial data $u_0^i$ and $p$, no matter what value $r$ is, as $\delta$ and $\Delta x$ are refined at the same time, the nonlocal numerical solutions get closer to the corresponding Finite Difference solution of classical Burger’s equation.

For each initial data, comparing the plots with the same $p$ but different $r$ values, we find that: the larger $r$ is, the smoother the nonlocal solution is. It is because, since $\Delta x$ is fixed, larger $r$ corresponds to larger nonlocal horizon $\delta$, and thus this stronger nonlocality leads to stronger smoothing effect.

Also, if we compare the plots with the same $r$ but different $p$ values, we observe similar phenomenon as Experiment 1: when $p$ decreases, the more the nonlocal solutions look like the solution of local Burger’s equation. More specifically, the discontinuity introduced by $u_0^1$ flats out less; the discontinuity introduced by $u_0^2$ is encroached faster; and nonlocal solution associated with $u_0^3$ tends to be more like a shock.
Fig. 3.3: Use $p = 0.25$. Fix $r$, refine $\Delta x$.

Fig. 3.4: Use $p = 0$. Fix $r$, refine $\Delta x$. 
Fig. 3.5: Use $p = -0.5$. Fix $r$, refine $\Delta x$. 

(a) $r = 2$, $u_0^1$  
(b) $r = 2$, $u_0^2$  
(c) $r = 2$, $u_0^3$  
(d) $r = 5$, $u_0^1$  
(e) $r = 5$, $u_0^2$  
(f) $r = 5$, $u_0^3$
### 3.7.5 Experiment 4: Shock formation

In this experiment, we focus on smooth initial data $u_0^3$, and look at whether the nonlocal solution develop shocks under different $p$ values.

Figure 3.6 shows the time evolution of nonlocal solution under different $p$, with horizon $\delta = 0.2$ fixed. The first and second rows of plots correspond to $\delta = 0.2$ and $\delta = 0.1$ respectively, while plots in the same column share the same $p$ value: $p = 1$, $p = 0$ or $p = -0.9$. In each plot, the color blue, cyan, green, yellow, orange, light pink, purple and red curves correspond to nonlocal solution at time $T = 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4$ and $1.6$, respectively. In all these plots we use grid-size $\Delta x = \frac{1}{400}$, because a further refinement of $\Delta x$ shows no visible difference in any of these six plots.

We make the following observations from Figure 3.6:

1) All six plots in Figure 3.6 demonstrate the damping effect, but the damping effect of $p = -0.9$ cases are less significant compared to the $p = 0$ and $p = 1$ cases.
2) As time goes (from blue to red), the curves seem to be steeper and steeper at $x = 0$.
3) Comparing the plots in the same column of Figure 3.6 (i.e, the same $p$ with different $\delta$), we see that: for each color (i.e, at each time), the smaller $\delta$ corresponds to steeper curve around $x = 0$. It is because: smaller $\delta$ makes the kernel $\omega^{\delta}$ more like the Dirac delta function, thus the nonlocal solution behave more like the local solution, which is a N-wave.
4) Comparing the plots in the same row of Figure 3.6 (that is, the same $\delta$ with different $p$), we see that: the closer $p$ is to $-1$, the steeper the curve is around $x = 0$. It is because: when $p$ gets closer to $-1$, the kernel $\omega^{\delta}(h) \sim h^p$ is more like the Dirac delta function, thus the nonlocal solution behave more like the local solution.

In each of the six plots in Figure 3.6, the nonlocal solutions seem to be steeper as time goes, but would they really develop “shocks”? To look into this, under different $p$ and $\delta$ values, we record slopes of the curves near $x = 0$ associated with different $\Delta x$ and time $T$, as shown in Table 3.5-3.7. Since we always take the total number of cells $N$ on $x \in [-1, 1]$ to be an even number, we utilize the following two ways to compute slopes near $x = 0$, one is symmetric around $x = 0$, the other is a one-sided slope:

\[
\text{Slope 1} = \frac{u\left(\frac{\Delta x}{2}\right) - u\left(-\frac{\Delta x}{2}\right)}{h}, \quad \text{Slope 2} = \frac{u\left(\frac{3\Delta x}{2}\right) - u\left(\frac{\Delta x}{2}\right)}{h}.
\]
If the slopes almost doubles as we refine $\Delta x$ into half, then we have reason to believe there are shock formations. The value of $u(-\frac{3}{4})$ is also shown to verify the numerical convergence.

Above all, in all cases of Table 3.5 through Table 3.7, we see that $u(-\frac{3}{4})$ values do not change much as we refine $\Delta x$, which means the grid-size $\Delta x$ is small enough for these computations. We pick $x = -\frac{3}{4}$ as the spot to observe numerical convergence, because from the plots we believe $x = -\frac{3}{4}$ is always away from the shock area (if any).

First look at whether there is a shock formation for the cases of $p = 1$ and $p = 0$ (Table 3.5 and 3.6). Before time $T = 1.4$, the slopes increases with time; after time $T = 1.4$, there is a decrease of the slope values. Although such decrease may be partially due to the damping effect, we believe the slopes would not increase after the time $T = 1.4$. More importantly, at any time $T$, as we refine $\Delta x$, neither slope 1 nor slope 2 doubles. It suggests that when $p = 1$ or $p = 0$, nonlocal solution do not develop shocks. This is consistent with Theorem (3.4.8), which implies, when $p > 0$, there should be no shock developed for smooth initial data.

While for the case $p = -0.9$ in Table 3.7, before $T = 0.5$, the slopes increases with time; after $T = 0.5$, the slope values decrease, which may result from the damping effect. Starting from time $T = 0.4$, both slope 1 and slope 2 tend to double as we refine $\Delta x$. So we believe there is shocks, and we guess that the shocks forms between $T = 0.3$ and $T = 0.4$.

Now it’s natural to ask the following question: for the $p = -0.9$ case, at what time in interval $[0.3, 0.4]$ does the shock forms?

We thus take a closer look the nonlocal solutions during $T \in [0.3, 0.4]$ with $p = -0.9$, and $\delta = 0.2, 0.1$. Figure 3.7 shows the time evolution of nonlocal solution under different $\delta$, with $p = -0.9$ and $\Delta x = \frac{1}{4000}$. The color blue, light blue, cyan, green, yellow, orange, light pink, purple and red curves correspond to nonlocal solution at time $T = 0.31, 0.32, 0.33, \ldots, 0.38$ and 0.39, respectively. From the plots in Figure 3.7, we see that $T \in (0.3, 0.4)$ does look like the time interval when the shock (if any) forms. Table 3.8 shows the slope evolutions. We see that, as we refine $\Delta x$ from $\frac{1}{200}$ to $\frac{1}{400}$, the slope is multiplied by a factor which is close to but less than 2. Similarly as we refine $\Delta x$ from $\frac{1}{400}$ to $\frac{1}{800}$. Supposing there is no damping, then when the shock forms, we expect the slope to double when we refine $\Delta x$. That is, we expect a factor of 2 for the slope. So we want to check: at what time the slope factor gets closed

85
enough to 2—this time is the shock formation time. (Here we focus on the factor of Slope 1).

In the plots in Figure 3.8, the y-axis represents the time, from 0.31 to 0.39, and the x-axis represents the slope factor. The colors blue, cyan, green, ..., pink, red represent the time $T = 0.31, 0.32, 0.33, ..., 0.38, 0.39$, respectively. In each color, the square on the left hand side represents the factor of slope 1 when $\Delta x = \frac{1}{200}$ is refined to $\Delta x = \frac{1}{400}$, and the square on the right represents the factor of slope 1 when $\Delta x = \frac{1}{400}$ is refined to $\Delta x = \frac{1}{800}$.

It shows that, at each time point, the slope factor between $\Delta x = \frac{1}{400}$ and $\Delta x = \frac{1}{800}$ is always closer to 2, compared with the slope factor between $\Delta x = \frac{1}{200}$ and $\Delta x = \frac{1}{400}$. It makes sense since the finer grid gives more accurate solution, and thus is capable of better capturing the shock behavior.

At what time the shock forms is determined by our definition of “the slope factor is close enough to 2”. For instance, say we consider the slope factor to be close enough to 2 when it is above 1.8, then based on the plots, $T^* = 0.34$ is the shock formation time for $\delta = 0.2$, and $T^* = 0.33$ is the shock formation time for $\delta = 0.1$. By Figure 3.8 we see that, no matter what our definition of “close to 2” is, the shock formation time for $\delta = 0.2$ is always larger than that for $\delta = 0.1$. Both these two shock formation time should be larger than the classical Burger’s equation with initial data $u_0^3$, which is $T^* = 1/\pi \approx 3.18$. 
Fig. 3.6: Time Evolution of nonlocal solution, with $\Delta x = \frac{1}{400}$.

(a) $p = 1, \delta = 0.2$
(b) $p = 0, \delta = 0.2$
(c) $p = -0.9, \delta = 0.2$
(d) $p = 1, \delta = 0.1$
(e) $p = 0, \delta = 0.1$
(f) $p = -0.9, \delta = 0.1$

Fig. 3.7: Time evolutions of nonlocal solutions during $T \in [0.3, 0.4]$.

(a) $p = -0.9, \delta = 0.2$
(b) $p = -0.9, \delta = 0.1$
Fig. 3.8: Factors of slope 1 when $\Delta x$ is refined. The y-axis represents time, and the x-axis represents the slope factor. The colors blue, cyan, green, ..., pink, red represent the time $T = 0.31, 0.32, 0.33, ..., 0.38, 0.39$, respectively. In each color, the square on the left represents the factor of slope 1 when $\Delta x = \frac{1}{200}$ is refined to $\Delta x = \frac{1}{400}$, and the square on the right represents the factor of slope 1 when $\Delta x = \frac{1}{400}$ is refined to $\Delta x = \frac{1}{800}$.

![Graph](image)

Table 3.5: $p = 1$, time evolutions of the curve slopes near $x = 0$.  

<table>
<thead>
<tr>
<th>$p = -0.9$, delta=0.2</th>
<th>$T = 0.2$</th>
<th>$T = 0.4$</th>
<th>$T = 0.6$</th>
<th>$T = 0.8$</th>
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<th>$T = 1.2$</th>
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<td>slope 2</td>
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<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

| $u(-\frac{3}{2})$    |           |           |           |           |           |           |           |           |
| $dx = 1/200$         | 0.5088    | 0.4024    | 0.3344    | 0.2868    | 0.2516    | 0.2244    | 0.2027    | 0.1850    |
| $dx = 1/400$         | 0.5106    | 0.4037    | 0.3353    | 0.2875    | 0.2521    | 0.2248    | 0.2031    | 0.1853    |
| $dx = 1/800$         | 0.5115    | 0.4043    | 0.3357    | 0.2878    | 0.2524    | 0.2250    | 0.2032    | 0.1854    |

<table>
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<th>$p = -0.9$, delta=0.1</th>
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<td>-47.23</td>
<td>-41.81</td>
</tr>
</tbody>
</table>

| $u(-\frac{3}{2})$    |           |           |           |           |           |           |           |           |
| $dx = 1/200$         | 0.4874    | 0.3735    | 0.3035    | 0.2561    | 0.2219    | 0.1959    | 0.1756    | 0.1592    |
| $dx = 1/400$         | 0.4893    | 0.3747    | 0.3044    | 0.2567    | 0.2224    | 0.1963    | 0.1759    | 0.1594    |
| $dx = 1/800$         | 0.4903    | 0.3754    | 0.3048    | 0.2571    | 0.2226    | 0.1965    | 0.1760    | 0.1595    |
|\( p = 0, \delta = 0.2 \)| \( T = 0.2 \) | \( T = 0.4 \) | \( T = 0.6 \) | \( T = 0.8 \) | \( T = 1 \) | \( T = 1.2 \) | \( T = 1.4 \) | \( T = 1.6 \) |
|---|---|---|---|---|---|---|---|
| **slope 1** | \( dx = 1/200 \) | \(-9.91\) | \(-26.06\) | \(-47.43\) | \(-61.57\) | \(-65.47\) | \(-63.62\) | \(-59.67\) | \(-55.34\) |
| | \( dx = 1/400 \) | \(-10.10\) | \(-27.90\) | \(-55.65\) | \(-80.17\) | \(-92.38\) | \(-94.42\) | \(-91.22\) | \(-86.04\) |
| | \( dx = 1/800 \) | \(-10.21\) | \(-29.00\) | \(-61.49\) | \(-96.62\) | \(-121.17\) | \(-132.13\) | \(-133.42\) | \(-129.51\) |
| **slope 2** | \( dx = 1/200 \) | \(-9.74\) | \(-24.59\) | \(-40.34\) | \(-44.87\) | \(-40.52\) | \(-34.84\) | \(-30.11\) | \(-26.5994\) |
| | \( dx = 1/400 \) | \(-10.06\) | \(-26.96\) | \(-50.35\) | \(-64.51\) | \(-64.50\) | \(-57.79\) | \(-50.30\) | \(-43.9816\) |
| \( u(-\frac{x}{2}) \) | \( dx = 1/200 \) | \(0.4989\) | \(0.3887\) | \(0.3193\) | \(0.2716\) | \(0.2367\) | \(0.2099\) | \(0.1888\) | \(0.1716\) |
| | \( dx = 1/400 \) | \(0.5008\) | \(0.3899\) | \(0.3202\) | \(0.2722\) | \(0.2372\) | \(0.2103\) | \(0.1891\) | \(0.1719\) |
| | \( dx = 1/800 \) | \(0.5017\) | \(0.3905\) | \(0.3206\) | \(0.2726\) | \(0.2374\) | \(0.2105\) | \(0.1892\) | \(0.1720\) |

**Table 3.6:** \( p = 0 \), time evolutions of the curve slopes near \( x = 0 \).

<table>
<thead>
<tr>
<th>( p = 0, \delta = 0.1 )</th>
<th>( T = 0.2 )</th>
<th>( T = 0.4 )</th>
<th>( T = 0.6 )</th>
<th>( T = 0.8 )</th>
<th>( T = 1 )</th>
<th>( T = 1.2 )</th>
<th>( T = 1.4 )</th>
<th>( T = 1.6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>slope 1</strong></td>
<td>( dx = 1/200 )</td>
<td>(-13.27)</td>
<td>(-58.05)</td>
<td>(-99.99)</td>
<td>(-100.56)</td>
<td>(-91.04)</td>
<td>(-81.44)</td>
<td>(-71.21)</td>
</tr>
<tr>
<td></td>
<td>( dx = 1/400 )</td>
<td>(-13.69)</td>
<td>(-68.65)</td>
<td>(-143.36)</td>
<td>(-154.88)</td>
<td>(-142.58)</td>
<td>(-128.12)</td>
<td>(-115.34)</td>
</tr>
<tr>
<td></td>
<td>( dx = 1/800 )</td>
<td>(-13.94)</td>
<td>(-76.40)</td>
<td>(-192.47)</td>
<td>(-231.12)</td>
<td>(-219.65)</td>
<td>(-199.19)</td>
<td>(-179.86)</td>
</tr>
<tr>
<td><strong>slope 2</strong></td>
<td>( dx = 1/200 )</td>
<td>(-11.90)</td>
<td>(-36.05)</td>
<td>(-37.80)</td>
<td>(-31.85)</td>
<td>(-27.66)</td>
<td>(-24.44)</td>
<td>(-21.87)</td>
</tr>
<tr>
<td></td>
<td>( dx = 1/400 )</td>
<td>(-12.83)</td>
<td>(-49.54)</td>
<td>(-61.92)</td>
<td>(-51.75)</td>
<td>(-44.43)</td>
<td>(-39.14)</td>
<td>(-39.93)</td>
</tr>
<tr>
<td></td>
<td>( dx = 1/800 )</td>
<td>(-13.42)</td>
<td>(-62.10)</td>
<td>(-83.79)</td>
<td>(-70.57)</td>
<td>(-61.17)</td>
<td>(-54.97)</td>
<td>(-49.62)</td>
</tr>
<tr>
<td>( u(-\frac{x}{2}) )</td>
<td>( dx = 1/200 )</td>
<td>(0.4823)</td>
<td>(0.3660)</td>
<td>(0.2953)</td>
<td>(0.2479)</td>
<td>(0.2158)</td>
<td>(0.1882)</td>
<td>(0.1682)</td>
</tr>
<tr>
<td></td>
<td>( dx = 1/400 )</td>
<td>(0.4843)</td>
<td>(0.3673)</td>
<td>(0.2962)</td>
<td>(0.2485)</td>
<td>(0.2143)</td>
<td>(0.1885)</td>
<td>(0.1684)</td>
</tr>
<tr>
<td></td>
<td>( dx = 1/800 )</td>
<td>(0.4852)</td>
<td>(0.3679)</td>
<td>(0.2966)</td>
<td>(0.2488)</td>
<td>(0.2145)</td>
<td>(0.1887)</td>
<td>(0.1686)</td>
</tr>
</tbody>
</table>

**Table 3.7:** \( p = -0.9 \), time evolutions of the curve slopes near \( x = 0 \).
Table 3.8: \( p = -0.9 \), evolutions of the curve slopes during \( T \in [0.3, 0.4] \).
3.7.6 Experiment 5: Time evolution of discontinuities

In this experiment, we use piecewise-constant initial data $u_1^0$ and $u_2^0$, and observe how the discontinuity in the initial data deforms or propagates with time.

Figure 3.9 shows the time evolution of nonlocal solution, under different $p$ and piece-wise constant initial data, with horizon $\delta = 0.2$ fixed. The $i$-th row of plots ($i = 1, 2$) correspond to piecewise initial data $u_i^0$, while plots in the same column share the same $p$ value: $p = 1$, $p = 0$ or $p = -0.9$. In each plot, the color blue, cyan, green, yellow, orange, light pink, purple and red curves correspond to nonlocal solution at time $T = 0.2$, 0.4, 0.6, 0.8, 1, 1.2, 1.4 and 1.6, respectively. In all these plots we use grid-size $\Delta x = \frac{1}{400}$, because a further refinement of $\Delta x$ shows no visible difference in any of these six plots.

First look at the $u_1^0$ cases in plots (a-c). 1) From blue to red curves, the damping effect is apparent, no matter what $p$ is. 2) the nonlocal solution always convects towards the right, which is consistent with the local Burger’s equation case. 3) For both $p = 1$ and $p = 0$, the discontinuity in initial data near $x = 0$ are quickly smooth out, while for $p = -0.9$, the discontinuity seems to maintain as time goes. 4) if we compare curves of the same color (that is, nonlocal solutions associated with different $p$ at the same time $t$), we will see that: as $p$ gets closer to $-1$, the nonlocal solution appears to be more and more like a shock.

Now look at the $u_2^0$ cases in plots (d-f). 1) again the damping effect is obvious. 2) the smaller $p$ is, the quicker the discontinuity at $x = 0$ disappears. 3) the smaller $p$ is, the steeper the curve is at $x = \pm 1$. 

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Fig. 3.9: Time evolution of nonlocal solution $u$, with $\delta = 0.2$ and $\Delta x = \frac{1}{400}$. The color blue, cyan, green, yellow, orange, light pink, purple and red curves correspond to nonlocal solution at time $T = 0.2, 0.4, 0.6, 0.8, ..., 1.4$ and $1.6$, respectively.
3.8 Summary and future work

We proposed a class of nonlocal conservation laws in such a way that it represents a reasonable generalization of the local conservation law. The nonlocal conservation law shares some important properties of the local conservation law, such as the conservation property and Maximum Principle, and nonlocal conservation law reduces to its local counterpart when we take special kernel. We proved the uniqueness and existence of the nonlocal conservation law, and also proved that, under special kernels, the nonlocal solution develop no shocks with smooth initial condition. Numerically, we proposed a monotone scheme, the solution of which, as the horizon $\delta$ is fixed and $\Delta x \to 0$, converges to the entropy solution of the nonlocal conservation law, while as both $\delta$ and $\Delta x$ vanish, converges to the entropy solution of the local conservation law. Results of numerical experiments are consistent with these theories.

There are several directions that may deserve further exploration:

- Sufficient and necessary condition for no shocks. In Theorem 3.4.8, we only provided a condition under which the nonlocal solution does not develop shock for smooth initial condition. Experiment 4 suggest that this condition may be not only sufficient, but also necessary. It would be interesting to prove it theoretically.

- Higher order scheme. In the local case, the accuracy order of a monotone scheme is at most 1. For our nonlocal monotone scheme (3.57), Experiment 2 suggests that its accuracy order is also no larger than 1. So it could be interesting to explore higher-order numerical schemes.

- Nonlocal-in-time. The the integral in our nonlocal conservation law is on the spacial variable, that is, the nonlocality is spatial. We could also try to non-localize the time variable, that is, replace the partial derivative $u_t$ by an integral involving some kernel function.

- Higher dimension problems. Our work focused on scalar $u$ with $x \in \mathbb{R}$ (our results actually hold for $x \in \mathbb{R}^N$). But when $u$ is a vector, how to propose a system of nonlocal conservation laws such that it is consistent with the local case, and its entropy solution converges to the entropy solution of the classical system of conservation law?
.1 Wellposedness

In this part, $u'$ is used to denote the derivative of $y$ on $x$, and we sometimes write $|| \cdot ||_{L^\infty(\mathbb{R})}$ as $|| \cdot ||_\infty$, and $|| \cdot ||_{L^1(\mathbb{R})}$ as $|| \cdot ||_1$.

Lemma .1.1. Assume $g$ satisfies condition (3.27) and (3.29), and $\omega$ satisfies (3.48). Then: $\mathcal{L} : W^{1,\infty}(\mathbb{R}) \rightarrow W^{1,\infty}(\mathbb{R})$ is a bounded operator, with

$$s||\mathcal{L}(u)||_{L^\infty} \leq C ||\tilde{\omega}||_{L^1} ||u||_{L^\infty},$$

$$\left\| \frac{d}{dx} \mathcal{L}(u) \right\|_{L^\infty} \leq C ||\tilde{\omega}||_{L^1} ||u'||_{L^\infty} ||u||_{L^\infty}.$$

Proof.

$$\mathcal{L}(u)(x) = \int_{\mathbb{R}} [g(u(x), u(x + h)) - g(u(x - h), u(x))] \frac{\omega(h)}{h} dh$$

$$= \int_{\mathbb{R}} [g(u(x), u(x + h)) - g(u(x), u(x))] \frac{\omega(h)}{h} dh$$

$$+ \int_{\mathbb{R}} [g(u(x), u(x)) - g(u(x - h), u(x))] \frac{\omega(h)}{h} dh$$

$$= \int_{\mathbb{R}} \left[ \frac{g(u(x), u(x + h)) - g(u(x), u(x))}{h} \right] \frac{u(x + h) - u(x)}{h} \omega(h) dh$$

$$+ \int_{\mathbb{R}} \left[ \frac{g(u(x), u(x)) - g(u(x - h), u(x))}{h} \right] \frac{u(x) - u(x - h)}{h} \omega(h) dh$$

$$||\mathcal{L}(u)||_{L^\infty} \leq C \left\| \int_{\mathbb{R}} [u(x + h) - u(x)] \frac{\omega(h)}{h} dh \right\|_{L^\infty} + C \left\| \int_{\mathbb{R}} [u(x) - u(x - h)] \frac{\omega(h)}{h} dh \right\|_{L^\infty}$$

$$\leq C ||\tilde{\omega}||_1 ||u||_{L^\infty},$$

where $C$ only depends on $g$.

$$\left\| \frac{d}{dx} \mathcal{L}(u)(x) \right\|_{L^\infty} \leq \int_{\mathbb{R}} \left\| \frac{d}{dx} [g(u(x), u(x + h)) - g(u(x - h), u(x))] \right\|_{L^\infty} \frac{\omega(h)}{h} dh,$$
where, by (3.29),
\[
\begin{align*}
\left\| \frac{d}{dx} [g(u(x), u(x+h)) - g(u(x-h), u(x))] \right\|_\infty \\
= \left\| g_1(u(x), u(x+h))u'(x) + g_2(u(x), u(x+h))u'(x + h)
- g_1(u(x-h), u(x))u'(x - h) - g_2(u(x-h), u(x))u'(x) \right\|_\infty \\
\leq \|u'\|_\infty (\|g_1(u, \tau_h u)\|_\infty + \|g_1(\tau_{-h} u, u)\|_\infty + \|g_2(u, \tau_h u)\|_\infty + \|g_2(\tau_{-h} u, u)\|_\infty) \\
\leq C\|u'\|_\infty \|u\|_\infty.
\end{align*}
\]
So
\[
\left\| \frac{d}{dx} [g(u(x), u(x+h)) - g(u(x-h), u(x))] \right\|_\infty \leq C \|\tilde{\omega}\|_1 \|u'\|_\infty \|u\|_\infty.
\]

\[\square\]

**Lemma 1.2.** Assume \(g\) satisfies condition (3.27) and (3.29), and \(\omega\) satisfies (3.48). Then: \(\mathcal{L} : W^{1,\infty}(\mathbb{R}) \to W^{1,\infty}(\mathbb{R})\) is Lipschitz continuous, with
\[
\|\mathcal{L}(u) - \mathcal{L}(v)\|_{W^{1,\infty}} \leq C\|\tilde{\omega}\|_1 \left(1 + \|u\|_{W^{1,\infty}} + \|v\|_{W^{1,\infty}}\right) \|u - v\|_{W^{1,\infty}}.
\]

**Proof.**
\[
\begin{align*}
\|\mathcal{L}(u) - \mathcal{L}(v)\|_\infty &= \left\| \int_{\mathbb{R}} [g(u(x), u(x+h)) - g(u(x-h), u(x)) - g(v(x), v(x+h) + g(v(x-h), v(x)))] \frac{\omega(h)}{h} dh \right\|_\infty \\
&\leq \int_{\mathbb{R}} \|g(u(x), u(x+h)) - g(u(x-h), u(x)) - g(v(x), v(x+h) + g(v(x-h), v(x))}\|_\infty \frac{\omega(h)}{h} dh \\
&\leq C\|u - v\|_\infty \|\tilde{\omega}\|_1,
\end{align*}
\]
the last equation is based on the Lipschitz continuity of \(g\) (3.27).
\[
\left\| \frac{d}{dx} \mathcal{L}(u) - \frac{d}{dx} \mathcal{L}(v) \right\|_\infty \leq \int_{\mathbb{R}} \left[ \left\| g_1(u(x), u(x+h))u'(x) - g_1(v(x), v(x+h))v'(x) \right\|_\infty \\
+ \left\| g_2(u(x), u(x+h))u'(x + h) - g_2(v(x), v(x+h))v'(x + h) \right\|_\infty \\
+ \left\| -g_1(u(x-h), u(x))u'(x - h) + g_1(v(x-h), v(x))v'(x - h) \right\|_\infty \right]
\]

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Then:

\[ \| -g_2(u(x-h), u(x))u'(x) + g_2(v(x-h), v(x))v'(x) \|_\infty \frac{\omega(h)}{h} dh \]

\[ \leq \int_{\mathbb{R}} (A + B + C + D) \frac{\omega(h)}{h} dh, \]

where

\[ A = \| g_1(u(x), u(x+h))u'(x) - g_1(v(x), v(x+h))v'(x) \|_\infty \]

\[ \leq \| g_1(u(x), u(x+h))u'(x) - g_1(u(x), u(x+h))v'(x) \|_\infty \]

\[ + \| g_1(u(x), u(x+h))v'(x) - g_1(v(x), v(x+h))v'(x) \|_\infty \]

\[ \leq \| g_1(u(x), u(x+h)) \|_\infty \| u' - v' \|_\infty + \| g_1(u(x), u(x+h)) - g_1(v(x), v(x+h)) \|_\infty \| v' \|_\infty \]

\[ \leq C \| u \|_\infty \| u' - v' \|_\infty + C \| u - v \|_\infty \| v' \|_\infty. \]

Similar for B, C and D. So

\[ \left\| \frac{d}{dx} \mathcal{L}(u) - \frac{d}{dx} \mathcal{L}(v) \right\|_\infty \leq C \| \tilde{\omega} \|_1 \left( 1 + \| u \|_{W^{1,\infty}} + \| v \|_{W^{1,\infty}} \right) \| u - v \|_{W^{1,\infty}}. \]

**Lemma 1.3.** Assume \( g \) is Lipschitz continuous and satisfies condition (3.27, 3.29), and \( \omega \) satisfies (3.48). Let \( u \in C(0,T; W^{1,\infty}(\mathbb{R})) \) satisfies the nonlocal conservation law (3.31). Then:

\[ \| u(\cdot, t) \|_\infty \leq \| u_0 \|_\infty e^{C\| \tilde{\omega} \|_1 K_t}, \]

\[ \left\| \frac{\partial}{\partial x} u(\cdot, t) \right\|_\infty \leq \left\| \frac{du_0}{dx} \right\|_\infty e^{C\| \tilde{\omega} \|_1 K_t}, \]

where \( K := \| u \|_{L^\infty(\mathbb{R} \times (0,T))} \).

**Proof.** Since \( u_0 \in L^\infty(\mathbb{R}) \), by the Maximum Principle in Theorem 3.4.4, \( K < +\infty \). Take \( L^\infty \) norm on both sides of the nonlocal equation (3.23),

\[ \frac{d}{dt} \| u(\cdot, t) \|_\infty = \int_{\mathbb{R}} \| g(u(x), u(x+h)) - g(u(x+h), u(x)) \|_\infty \frac{\omega(h)}{h} dh \]

\[ \leq C \int_{\mathbb{R}} (\| u(x) - u(x-h) \|_\infty + \| u(x+h) - u(x) \|_\infty) \frac{\omega(h)}{h} dh \]

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\[ \leq C \|\tilde{\omega}\|_1 \|u(\cdot, t)\|_\infty. \]

Take partial derivative of \(x\) on both sides of the nonlocal equation, and then take \(L^\infty\) norm. By Lemma 3.4.5, we get

\[
\frac{d}{dt} \left\| \frac{\partial u}{\partial x} \right\|_\infty = \left\| \frac{\partial}{\partial x} \mathcal{L}(u) \right\|_\infty \leq C \|\tilde{\omega}\|_1 \|u(\cdot, t)\|_\infty \|u'(\cdot, t)\|_\infty.
\]
Bibliography


Vita
Zhan Huang

Zhan Huang was born in 1983 in Guangxi Province, the People’s Republic of China. In 1999, she enrolled in the No. 3 High School in the city of Nanning. There she was trained in science major. In 2002, she was admitted by the Department of mathematics in Beijing Normal University. She obtained the Bachelor’s and Master’s degree in Applied Mathematics from Beijing Normal University, in 2006 and 2009 respectively. Then she was accepted by the Ph.D. program of Applied Mathematics in The Pennsylvania State University, where her research was focused on nonlocal models for linear and nonlinear differential equations. She is expected to graduate on December 2015.