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QUANTIZATION OF COADJOINT ORBITS VIA POSITIVITY OF KIRILLOV'S CHARACTER FORMULA

A Dissertation in
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Abstract

Kirillov proved his character formula for simply connected nilpotent Lie groups in 1962 and conjectured its universality. The validity of this conjecture has been verified for some other classes of Lie groups, most notably for the case of tempered representations of reductive Lie groups by Rossmann.

In this dissertation we explain how Kirillov's character formula can be used in the quantization of coadjoint orbits. First we prove a positivity property of Kirillov's character formula for some classes of Lie groups, including nilpotent Lie groups, which possess real polarizing subgroups. Then we use this positivity property to construct group representations following the ideas of Gelfand, Naimark, and Segal. Finally we discuss several approaches to proving positivity in the absence of real polarizations.

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Dedication

To my family for their constant support

Chapter 1

Introduction

Background

The idea of classifying irreducible representations of a group G, and studying characters of representations of G is a classical theme in representation theory and harmonic analysis dating back to the early days of the subject.

Since Kirillov's fundamental paper [15] in 1962, the *orbit method* has played an important role in the theory of Lie groups in both directions. In [15] Kirillov proved that coadjoint orbits of a connected simply connected nilpotent Lie group correspond, under quantization, to the equivalence classes of its irreducible unitary representations. The theory of geometric quantization due to Kirillov, Kostant [21], and Souriau [30] has shown that this close connection extends to many other groups. For instance, if G is a compact semisimple Lie group, the orbit method establishes a correspondence between the integral coadjoint orbits of G and $\widehat{G}$, the set of equivalence classes of irreducible unitary representations of G, given by the Borel-Weil theorem [16].

Two important parts of the orbit method philosophy regarding the relation between irreducible unitary representations and coadjoint orbits are given in the table below.

Table 1.1: The Orbit Method User's Guide

Representation Theory	Symplectic Geometry
$\widehat{G}$	$\mathfrak{g}^*/G$
$\operatorname{Tr}\pi_{\mathcal{O}}(\exp X)$	$j(X)^{-1/2}\int_{\mathcal{O}} e^{i\langle \ell, X\rangle+\sigma}$

Kirillov's character formula says that the characters of irreducible unitary representations of a Lie group G "should" be given by an equation of the form

$$\operatorname{Tr}\pi_{\mathcal{O}}(\exp X) = j(X)^{-1/2}\int_{\mathcal{O}} e^{i\langle \ell, X\rangle+\sigma} \tag{1.1}$$

where $\mathcal{O}$ is the coadjoint orbit in $\mathfrak{g}^*$ corresponding to $\pi_{\mathcal{O}} \in \widehat{G}$, σ is a canonical symplectic measure on $\mathcal{O}$, and j is the analytic function on $\mathfrak{g}$ defined by the formula

$$j(X) = \det\left(\frac{\sinh(\operatorname{ad} X/2)}{\operatorname{ad} X/2}\right).$$

The equation (1.1) is a character formula in the sense of Harish-Chandra and should be interpreted as an equation of distributions on a certain space of test functions on $\mathfrak{g}$.

Kirillov proved his character formula for simply connected nilpotent and simply connected compact Lie groups [15, 16] and conjectured its universality. The validity of this conjecture has been verified for some other classes of Lie groups, most notably for the case of tempered representations of reductive Lie groups by Rossmann [26]. Moreover, Atiyah-Bott [2] and Berline-Vergne [4] following the work of Duistermaat-Heckman [7], have shown that for compact Lie groups, Kirillov's character formula is equivalent to the Weyl character formula.

Let π be a traceable irreducible unitary representation of a Lie group G.

Then the Harish-Chandra character of π is a positive distribution in the sense that for any suitable test function f,

$$\operatorname{Tr} \pi(f * f^*) \geq 0.$$

The Gelfand-Naimark-Segal (GNS) construction makes the positivity of the Kirillov character formula remarkable, since GNS implies, roughly speaking, that any positive linear functional on a Lie group is the character of a group representation. Note that even when the character formula is positive, it does not obviously determine a representation, since the exponential map is neither injective nor surjective in general. This leads to the following question by Higson [14] "Is there a useful concept of partial representation corresponding to the partially-defined Kirillov character?"

In Chapters 3, 4, and 5 we explore the positivity of Kirillov's character formula for the classes of nilpotent, compact, and reductive Lie groups, respectively. In Chapter 6, we use the GNS construction to quantize the coadjoint orbits of nilpotent Lie groups where the character formula is positive.

Main Results

When G is a connected simply connected nilpotent Lie group, the validity of (1.1) implies that Kirillov's character formula has to be positive as well. In Chapter 3, we prove this result directly and without first constructing any underlying representation.

Theorem. *Kirillov's character* (1.1) *is a positive distribution for nilpotent Lie groups. More precisely, if G is a connected nilpotent Lie group with a coadjoint* *orbit* $\mathcal{O} \subset \mathfrak{g}^*$*, then for all* $f \in \mathcal{S}(G)$*,*

$$\chi(f * f^*) = \int_{\mathcal{O}} \widehat{f * f^*} \, d\mu_{\mathcal{O}} \geq 0,$$

where the Fourier transform is computed on the Lie algebra as follows: for $F \in \mathcal{S}(G)$ *and* $\psi \in \mathfrak{g}^*$*,*

$$\widehat{F}(\psi) = \int_{\mathfrak{g}} F(\exp X) e^{i\psi(X)} \, dX.$$

We first prove this result for the important case of the Heisenberg group where computations are short and insightful, and then we provide a separate proof for the general case.

In Chapters 4 and 7, we discuss several approaches to proving the positivity of (1.1) for compact Lie groups.

Theorem. *Let* G *be a connected compact Lie group. Then if* $\mathcal{O}$ *is an integral coadjoint orbit, Kirillov's character formula* (1.1) *defines a positive distribution for any* $f \in C^{\infty}(G)$ *supported in a sufficiently small neighborhood of the identity.*

Many coadjoint orbits from non-nilpotent Lie groups fall into the scope of the method used in Chapter 3 for nilpotent Lie groups. In Chapter 5, we make some adjustments to our previous methods and explain how they can be applied to prove the positivity (1.1) when G is a reductive Lie group by focusing on the special linear group $\mathrm{SL}(2, \mathbb{R})$. Let us write

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

for a basis of the three dimensional Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ of 2×2 traceless matrices.

Theorem. *Kirillov's character formula* (1.1) *is a positive distribution for* $\mathcal{O} =$

$\mathrm{Ad}^*(\mathrm{SL}(2,\mathbb{R}))H^*$ *and for any* $f \in C_c^\infty(\mathrm{SL}(2,\mathbb{R}))$ *supported in a sufficiently small neighborhood of the identity.*

In Chapter 6 we briefly discuss how the positivity property of Kirillov's character can be used to construct representations of Lie groups.

Chapter 2

Background

In this chapter we fix some notation and collect several standard facts that we will use freely in the sequel.

2.1 The KKS and Liouville Forms on Coadjoint Orbits

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and dual Lie algebra $\mathfrak{g}^*$. The group G acts on $\mathfrak{g}$ via the adjoint action

$$\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$$

whose differential is

$$\mathrm{ad}\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$$

where $\operatorname{ad}(X)(Y) = [X, Y]$ for $X, Y \in \mathfrak{g}$. The ***coadjoint action*** of G on $\mathfrak{g}^*$, $\operatorname{Ad}^* \colon G \to \operatorname{GL}(\mathfrak{g}^*)$, is the transpose or dual of the adjoint action:

$$(\operatorname{Ad}^* g) f(X) = f((\operatorname{Ad} g^{-1})X)$$

for $g \in G$, $X \in \mathfrak{g}$, and $f \in \mathfrak{g}^*$. The differential of the coadjoint action Ad^* is $\operatorname{ad}^* \colon \mathfrak{g}^* \to \operatorname{End}(\mathfrak{g}^*)$ given by

$$(\operatorname{ad}^* X) f(Y) = f((-\operatorname{ad} X)Y) = f([Y, X])$$

for $X, Y \in \mathfrak{g}$ and $f \in \mathfrak{g}^*$.

Any coadjoint orbit is a symplectic submanifold of $\mathfrak{g}^*$ with a canonical symplectic form sometimes called the ***Kirillov-Kostant-Souriau (KKS) symplectic structure*** described as follows.

Let G act on $\mathfrak{g}^*$ by the coadjoint action, and let $\mathcal{O} \subset \mathfrak{g}^*$ be an orbit of this action. Choose $\ell \in \mathcal{O}$ and denote by K the restriction of the coadjoint action to $\mathcal{O}$:

$$\begin{aligned} K \colon G &\to \mathcal{O} \\ g &\mapsto g \cdot \ell = \operatorname{Ad}^*(g)\ell. \end{aligned}$$

The differential of this map at the identity, namely $K_{*,e} \colon \mathfrak{g} \to T_\ell \mathcal{O}$, induces a linear isomomorphism $\tilde{K}_{*,e}$ between $\mathfrak{g}/\mathfrak{r}_\ell$ and $T_\ell \mathcal{O}$, where $\mathfrak{r}_\ell$ is the ***radical (or kernel)*** of the bilinear map

$$\mathfrak{g} \times \mathfrak{g} \to \mathbb{R}, \quad (X, Y) \mapsto \ell([X, Y]).$$

Hence we can transfer the nondegenerate alternating bilinear map B_ℓ on $\mathfrak{g}/\mathfrak{r}_\ell$ defined by $B_\ell(X, Y) = \ell([X, Y])$ to a nondegenerate alternating bilinear form

ω_ℓ on $T_\ell\mathcal{O}$, such that

$$\omega_\ell \colon T_\ell\mathcal{O} \times T_\ell\mathcal{O} \to \mathbb{R}, \quad \omega_\ell\left(\tilde{K}_{*,e}X, \tilde{K}_{*,e}Y\right) = B_\ell(X,Y).$$

The assignment $\ell \mapsto \omega_\ell$ clearly gives a 2-form ω on $\mathcal{O}$. We summarize the basic properties of ω in the next proposition whose complete proof can be found in [18].

Proposition 2.1.1. *$(\mathcal{O}, \omega)$ is a symplectic manifold. Furthermore, ω is G-invariant and the inclusion map $\Phi_G \colon \mathcal{O} \hookrightarrow \mathfrak{g}^*$ is a moment map, that is, Φ_G is G-equivariant and satisfies*

$$d\Phi_G^\xi = \iota(\xi^{\mathcal{O}})\omega, \quad \xi \in \mathfrak{g}$$

Here $\xi^{\mathcal{O}}$ is the vector field generated by ξ and $\Phi_G^\xi = \langle \Phi_G, \xi \rangle \colon \mathcal{O} \to \mathbb{R}$ is the ξ-coordinate of Φ_G.

Suppose $\dim \mathcal{O} = 2n$. Then

$$\mu = \frac{1}{n!(2\pi)^n} \underbrace{\omega \wedge \cdots \wedge \omega}_{n \text{ times}} \in \Lambda^{2n}(\mathcal{O})$$

gives a top form on the coadjoint orbit. We shall refer to μ as the canonical ***Liouville measure*** on $\mathcal{O}$.

Remark 2.1.2. It is convenient to work with the differential form (of mixed degree)

$$\exp\omega = 1 + \omega + \frac{1}{2!}\omega \wedge \omega + \frac{1}{3!}\omega \wedge \omega \wedge \omega + \cdots .$$

With the convention that $\int_M \alpha = 0$ if the degree of α is different than the dimension of M, the Liouville measure, the term of maximal degree in $\exp(\omega/2\pi)$ is given by the integration of $\exp(\omega/2\pi)$.

We warn the reader that the definition of the Liouville measure in some references might differ from ours by a factor of $(2\pi)^n$. This is merely due to the existence of slightly different versions of the Fourier transform of functions.

2.2 Trace-class Operators

We define 'traceable' operators in analogy with the traditional way of defining Lebesgue integrable functions where we first make sense of $\int f\,d\mu$ for $f \geq 0$ and then say that an arbitrary function f is integrable if $\int |f|\,d\mu < \infty$.

Similarly, we define the trace of a positive linear operator[1] $T \in B(\mathcal{H})$ on a separable Hilbert space $\mathcal{H}$ by $\operatorname{Tr} T = \sum_i \langle Te_i, e_i\rangle$ where $\{e_i\}$ is an orthonormal basis of $\mathcal{H}$. It is easy to check that this definition is basis independent. Generally, $T \in B(\mathcal{H})$ is said to be a ***trace-class*** operator if $\operatorname{Tr}|T| < \infty$. Here as usual $|T|$ is the positive operator $\sqrt{T^*T}$ associated with the bounded operator T. In this case, the trace of T is defined by

$$\operatorname{Tr} T = \sum_i \langle Te_i, e_i\rangle$$

which turns out to be finite and independent of the chosen basis for $\mathcal{H}$. An equivalent, but coordinate-free, way of defining trace-class operators is to say that they are precisely the operators that are the composition of two Hilbert-Schmidt operators.

2.3 Locally Compact Groups

It is well known that every locally compact group G possesses a left Haar measure $d\lambda$ which is unique up to a multiplicative constant. The ***modular func-***

[1] Recall that a self-adjoint linear operator $T \in \mathcal{B}(\mathcal{H})$ is said to be ***positive*** if $\langle Tv, v\rangle \geq 0$ for all $v \in \mathcal{H}$ or equivalently if the spectrum of T is a subset of nonnegative real numbers.

tion $\Delta\colon G \to (0,\infty)$ is the Radon-Nikodym derivative of left Haar measure with respect to right Haar measure, normalized to be 1 at the identity element, measures the extent to which λ fails to be right-invariant. More precisely,

$$d\lambda(xy) = \Delta(y)\, d\lambda(x)$$

for x and y in G. Another way of writing this equation is

$$\Delta(y) \int_G f(xy)\, d\lambda(x) = \int_G f(x)\, d\lambda(x)$$

for all $f \in L^1(G)$. We also have

$$\int_G f(x^{-1})\, d\lambda(x) = \int_G f(x)\Delta(x^{-1})\, d\lambda(x),$$

or in other words,

$$d\lambda(x^{-1}) = \Delta(x^{-1})\, d\lambda(x).$$

Remark 2.3.1. A word of caution is in order regarding the modular function. The notation that we have used here is standard but not universal. Some authors denote by $\Delta(g)$ (or sometimes $\delta(g)$) what we would call $\Delta(g^{-1})$.

If G is a Lie group, one can show that $\Delta(g) = |\det \mathrm{Ad}(g^{-1})|$. This implies, for instance, that any connected nilpotent Lie group is ***unimodular***, that is, the modular function is constant and everywhere equal to one.

Definition 2.3.2. Let G be a locally compact group with Haar measure $d\mu$. Given functions f and g on G, their ***convolution*** $f * g$ is the function on G

defined by

$$\begin{aligned} f * g(x) = L(f)(g)(x) &= \int_G f(y)g(y^{-1}x)\, d\mu(y) \\ &= \int_G f(xy)g(y^{-1})\, d\mu(y) \end{aligned}$$

whenever one (and hence both) of these integrals makes sense.

For instance if $f \in L^1(G)$ and $g \in L^p(G)$ for $p \in [0, \infty]$, then $f * g$ is defined almost everywhere and $f * g \in L^p(G)$.

2.4 The Harish-Chandra Character

It is well known that an irreducible representation of a compact Lie group is completely determined (up to equivalence) by its character. It was Harish-Chandra who realized through his study of infinite-dimensional representations that the correct way to think of characters of Lie groups is as distributions.

Example 2.4.1. Let G be a finite group acting by translation on $V = L^2(G)$. Then the character of this group representation is

$$\chi_{\mathrm{reg}}(g) = \begin{cases} 0 & \text{if } g \neq e \\ \dim V = |G| & \text{if } g = e. \end{cases}$$

Thus for infinite G we might expect χ_{reg} to be zero away from e, and infinite at e, that is, to be the Dirac delta function.

It is easy to make sense of this in the case of the regular representation of $G = \mathbb{T}$ and get the most basic example of a ***Harish-Chandra (or global) character***.

Example 2.4.2. The group $\mathbb{T}$ acts on S^1 by $e^{i\theta} \cdot z = e^{i\theta} z$, and thus on $L^2(S^1)$ by $L(e^{i\theta})(f)(z) = f(e^{-i\theta} z)$. Now we know that $L^2(S^1)$ has an orthonormal basis $\{ z^n \}_{n\in\mathbb{Z}}$ and $\mathbb{T}$ acts diagonally on this basis: $L(e^{i\theta}) z^n = e^{-in\theta} z^n$, so that we obtain

$$\operatorname{Tr} L(e^{i\theta}) = \sum_{n\in\mathbb{Z}} e^{-in\theta}.$$

Of course, this sum does not converge in the usual sense. However it converges weakly to $2\pi\delta_1$ in the sense of distributions, where δ_1 is the Dirac delta function at the identity.

Let (π, V_π) be a unitary representation of a locally compact group G. Then π induces a continuous homomorphism of Banach $*$-algebras via Bochner integration from $L^1(G)$ to the space of bounded operators $\mathcal{B}(V_\pi)$. Here the key observation is that $L^1(G)$ is a Banach algebra under convolution, which is just the usual group ring $\mathbb{C}G$ if G is finite, and that $L^1(G)$ has a natural isometric (conjugate-linear) involution $f \mapsto f^*$, where

$$f^*(x) = \overline{f(x^{-1})}\Delta(x^{-1}).$$

By abuse of notation, we denote this homomorphism by π,

$$\pi \colon L^1(G) \to \mathcal{B}(V_\pi), \quad \pi(f) = \int_G f(g)\pi(g)\, dg.$$

For instance, in Example 2.4.2 the matrix of $L(f)$ with respect to the basis $\{ z^n \}_{n\in\mathbb{Z}}$ is the diagonal matrix with the Fourier coefficients of f on its diagonal. Therefore, $\operatorname{Tr} L(f)$ is finite for $f \in C^\infty(G)$ and $f \mapsto \operatorname{Tr} L(f) = \delta_1(f)$ defines a distribution.

Inspired by the above examples, we can define the character of (π, V_π) to be the distribution $f \mapsto \pi(f)$ provided that the operator $\pi(f)$ is trace class.

Thus it is important to know when the operators $\pi(f)$ are traceable. The main results are as follows.

Theorem 2.4.3 (Harish-Chandra [11, 12])**.** *Let G be a connected semi-simple Lie group with finite center. Then for every irreducible unitary representation π of G and every $f \in C_c^\infty(G)$, the operator $\pi(f)$ is trace class and the map $f \mapsto \operatorname{Tr}\pi(f)$ is continuous, that is, it is a distribution on G.*

Theorem 2.4.4 (Kirillov)**.** *Let G be a nilpotent Lie group. Then for every irreducible representation π of G and every Schwartz function $f \in \mathcal{S}(G)$, the operator $f \mapsto \operatorname{Tr}\pi(f)$ is trace class and the map $f \mapsto \operatorname{Tr}\pi(f)$ is a tempered distribution on G.*

It follows that since $\pi(f * f^*) = \pi(f)\pi(f)^*$, the Harish-Chandra character of (π, V_π), whenever defined, is a ***convolution-positive*** (or ***positive*** for short) distribution in the sense that

$$\operatorname{Tr}\pi(f * f^*) = \operatorname{Tr}\pi(f)\pi(f)^* = \|\pi(f)\|_{\mathrm{HS}}^2 \geq 0. \tag{2.1}$$

2.5 Kirillov's Character Formula

2.5.1 The Abelian Case via Pontryagin Duality

Let G be a Locally compact abelian group and consider its convolution algebra $L^1(G)$. Then the Pontryagin dual of G, denoted by $\widehat{G}$, is canonically homeomorphic to the maximal ideal space $\mathcal{M}_{L^1(G)}$ of the commutative Banach algebra $L^1(G)$. In fact, the map $\chi \mapsto m_\chi$ from the dual group $\widehat{G}$ to the maximal ideal space $\mathcal{M}_{L^1(G)}$ with

$$m_\chi(f) = \widehat{f}(\chi)$$

is a homeomorphism. Here the Fourier transform $\hat{f}\colon \widehat{G} \to \mathbb{C}$ of a function $f \in L^1(G)$ is defined as

$$\hat{f}(\chi) = \int_G f(x)\overline{\chi(x)}\, dx.$$

Suppose π is a representation of the abelian group $\mathbb{R}^n$. Since the trace map $f \mapsto \operatorname{Tr}\pi(f)$ is a nonzero multiplicative functional on $L^1(\mathbb{R}^n)$ and $\widehat{\mathbb{R}^n} \simeq \mathbb{R}^n$, after a suitable identification, we conclude that

$$\operatorname{Tr}\pi(f) = \widehat{f}(x) \tag{2.2}$$

for some point $x \in \mathbb{R}^n$ depending only on the representation π. If we think of the right-hand side of the equation (2.2) as an integral of the Fourier transform over the singleton set $\{x\}$, then we get the Kirillov character formula in this setting.

For the case of the circle group $G = \mathbb{T}$, since the dual group $\widehat{\mathbb{T}} \simeq \mathbb{Z}$ is discrete, the above argument shows that we need to impose some integrality condition for the orbits over which we integrate (or evaluate) the Fourier transform to obtain the trace.

2.5.2 The General Case

Kirillov's character formula says that the characters χ of irreducible unitary representations of a Lie group G "should" be given by an equation of the form

$$\chi(\exp X) = j(X)^{-1/2} \int_{\mathcal{O}} e^{i\ell(X)}\, d\mu_{\mathcal{O}}(\ell) \tag{2.3}$$

where $\mathcal{O}$ is a coadjoint orbit in $\mathfrak{g}^*$, $\mu_{\mathcal{O}}$ is the canonical Liouville measure on $\mathcal{O}$, and j is the analytic function on $\mathfrak{g}$ defined by the formula

$$j(X) = \det\left(\frac{\sinh(\operatorname{ad} X/2)}{\operatorname{ad} X/2}\right);$$

this is the Jacobian of the exponential map $\exp\colon \mathfrak{g} \to G$ for a unimodular group G. This character formula should be interpreted as an equation of distributions on a certain space of test functions on $\mathfrak{g}$ as follows. For all smooth functions f compactly supported in a sufficiently small neighborhood of the origin in $\mathfrak{g}$,

$$\operatorname{Tr}\int_{\mathfrak{g}} f(X)\pi(\exp X)\,dX = \int_{\mathcal{O}}\int_{\mathfrak{g}} e^{i\ell(X)} f(X) j(X)^{-1/2}\,dX\,d\mu_{\mathcal{O}}(\ell) \qquad (2.4)$$

where π is the representation with character χ.

Definition 2.5.1. Suppose G is a Lie group and $f \in \mathfrak{g}^*$. An ***integral orbit datum at*** f is an irreducible unitary representation $(\pi, \mathcal{H}_\pi)$ of the isotropy group G_f subject to the condition

$$\pi(\exp X) = e^{if(X)} \cdot \mathrm{id}_{\mathcal{H}_\pi}, \quad X \in \mathfrak{g}_f.$$

The orbit $G \cdot f$ is called ***integral*** if it admits an integral orbit datum.

Theorem 2.5.2 (Kirillov)**.** *Suppose G is a connected (but not necessarily simply connected) nilpotent Lie group. Then the irreducible unitary representations of G are in natural one-to-one correspondence with the integral orbits of G.*

Theorem 2.5.3 (Kirillov)**.** *Let G be a connected, simply connected nilpotent Lie group and π an irreducible unitary representation of G. Then there is a coadjoint orbit $\mathcal{O}$ in $\mathfrak{g}^*$ such that* (2.4) *holds.*

Kirillov proved his character formula for simply connected nilpotent and simply connected compact Lie groups [15, 16] and conjectured its universality.

The validity of this conjecture has been verified for some other classes of Lie groups, most notably for the case of tempered representations of reductive Lie groups by Rossmann [26].

2.6 The Stationary Phase Method

In this section we review the ***stationary phase method*** that is used in studying the asymptotic behavior of oscillatory integrals of the form

$$I(t) = \int_{\mathbb{R}} g(x) e^{itf(x)}\, dx. \tag{2.5}$$

Here we assume that the ***phase*** and ***amplitude*** functions f and g are real-valued and sufficiently smooth and that g is compactly supported over the real line.

For large $t > 0$ the main contribution into the integral (2.5) is given by the neighborhood of the critical points of f where the derivative vanishes. Let x_0 be such a critical point. Then one can approximate $f(x)$ near x_0 by the first two terms of the Taylor series

$$f(x) = f(x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \cdots .$$

The main contribution of the critical point x_0 into the integral $I(t)$ is given by the Gaussian integral

$$I_0(t) = g(x_0) e^{itf(x_0)} \int_{\mathbb{R}} e^{\frac{1}{2} f''(x_0)(x-x_0)^2}\, dx$$

that can be computed explicitly as

$$I_0(t) = g(x_0) e^{i(tf(x_0) + \sigma \frac{\pi}{4})} \left(\frac{2\pi}{t|f''(x_0)|} \right)^{\frac{1}{2}},$$

where σ denotes sign of the second derivative $f''(x_0)$.

A similar formula holds for multivariable functions with nodegenerate critical points, that it, critical points at which the Hessian matrix is invertible.

Adding up the contributions of all (finitely many) critical points of the phase function, we get the following approximation of $I(t)$ for $t \gg 0$:

$$I(t) = \left(\frac{2\pi}{t}\right)^{\frac{n}{2}} \sum_i g(x_i) e^{itf(x_i)} \frac{e^{i\sigma_i \frac{\pi}{4}}}{|\det d^2_{x_i} f|^{\frac{1}{2}}} + O(t^{-\frac{n}{2}-1}) \tag{2.6}$$

as $t \to \infty$. Here n is the number of the variables of f and g, and $\sigma = \sigma_+ - \sigma_-$ is the ***signature*** of the Hessian matrix, defined as the difference of the numbers of positive and negative eigenvalues of the Hessian matrix.

Example 2.6.1. The inner product function

$$\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$$

$$(x, y) \mapsto \langle x, y \rangle$$

has a critical point at $(\mathbf{0}, \mathbf{0})$ with signature 0. Therefore, by the stationary phase formula (2.6) applied to a compactly supported function ψ we get the equality

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \psi(x, y) e^{it\langle x, y\rangle}\, dx\, dy = \left(\frac{2\pi}{t}\right)^2 \psi(\mathbf{0}, \mathbf{0}) + O(t^{-3}) \text{ as } t \to \infty.$$

Note that this particular example illustrates an instance of the "*exact* stationary phase formula," meaning that the error term $O(t^{-3})$ is indeed zero, as we know by the Fourier inversion formula. For another example of the exact stationary phase see the Subsection A.2 in the Appendix.

In 1982 Duistermaat and Heckman [7] found a symmetry principle giving a geometric explanation for such exactness in the stationary phase formula. The Duistermaat–Heckman Theorem was later generalized by Berline–Vergne [4] and

Atiyah–Bott [2] and also was used to establish the equivalence of the Kirillov and Weyl character formulae for compact Lie groups. We will not go into the details of such results here, but instead we shall explain how the Fourier inversion formula which is an 'exact formula' can be derived from the principle of stationary phase. For the sake of illustration, let us go back to our above example and make a shift in the inner product function to obtain

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \psi(x,y)e^{it\langle x,y-y_0\rangle}\,dx\,dy = \left(\frac{2\pi}{t}\right)^2 \psi(\mathbf{0},y_0) + O(t^{-3}) \text{ as } t\to\infty.$$

To extract an 'exact formula' out of this, first replace $\psi(x,y)$ with the product $f(x)g(y)$ of two smooth compactly supported functions f and g:

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x)g(y)e^{it\langle x,y-y_0\rangle}\,dx\,dy = \left(\frac{2\pi}{t}\right)^2 f(\mathbf{0})g(y_0) + O(t^{-3}).$$

Next make the change of variable $x \mapsto x/t$ and multiply both sides by t^2 to simplify:

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f\left(\frac{x}{t}\right) g(y)e^{i\langle x,y-y_0\rangle}\,dx\,dy = (2\pi)^2 f(\mathbf{0})g(y_0) + O(t^{-1}). \tag{2.7}$$

Clearly, the left-hand side of (2.7) equals $\int_{\mathbb{R}^2} f\left(\frac{x}{t}\right)\hat{g}(x)e^{-i\langle x,y_0\rangle}\,dx$. Thus if we choose f such that $f(\mathbf{0}) = 1$ and let $t\to\infty$ we recover the Fourier inversion formula

$$g(y_0) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \hat{g}(x)e^{-i\langle x,y_0\rangle}\,dx.$$

Chapter 3

Nilpotent Lie Groups

Nilpotent Lie groups and their representations have been studied extensively in the literature. From the many papers by Corwin, Greanleaf, Lipsman, Pukanzsky and others we only cite [19], [24], and [29] and refer the reader to [6] for a more comprehensive list of bibliographies.

3.1 The Heisenberg Group

The Heisenberg group plays a fundamental role in many areas of harmonic analysis, differential equations, number theory, and quantum physics. It also gives a mathematical formulation of the Heisenberg uncertainty principle of quantum mechanics, and reveals close connections to the harmonic oscillator.

In this section we shall develop the representation theory of the Heisenberg group following the works of Stone, von Neumann, and Kirillov, and at the end we give a direct proof of the positivity of Kirillov's character formula without invoking any representation theoretic results.

Consider the time-frequency translations on $L^2(\mathbb{R})$:

$$T_x f(t) = f(t+x), \quad M_y f(t) = e^{ity} f(t).$$

We have $T_x M_y = e^{ixy} M_y T_x$, so the collection of operators

$$\{\, e^{iz} M_y T_x \mid x, y, z \in \mathbb{R} \,\}$$

forms a group, essentially the Heisenberg group. More precisely, the ***real Heisenberg group*** H is $\mathbb{R}^3$ equipped with the group law

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + xy').$$

We remark that the Heisenberg group H can be realized by the 3×3 upper triangular matrices

$$H = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\}$$

with the usual matrix multiplication. In fact, thanks to a theorem of Engel [6], we can view any nilpotent Lie group as an abstract generalization of such matrix groups of unipotent upper triangular matrices. We can also think of nilpotent Lie groups as somehow arising from nilpotent Lie algebras that we now define.

Definition 3.1.1. The ***descending central series*** of a Lie algebra $\mathfrak{g}$ is defined inductively by

$$\mathfrak{g}^{(1)} = \mathfrak{g},$$
$$\mathfrak{g}^{(n+1)} = [\mathfrak{g}, \mathfrak{g}^{(n)}].$$

We say that $\mathfrak{g}$ is an n-step ***nilpotent Lie algebra*** if $\mathfrak{g}^{(n+1)} = 0$ but $\mathfrak{g}^{(n)} \neq 0$.

The Lie algebra $\mathfrak{h}$ of the Heisenberg group has three generators X, Y, and Z satisfying the *canonical commutation relation* (CCR)

$$[X, Y] = Z.$$

An important example of the Heisenberg pair is given by the multiplication and differentiation operators

$$Pf(x) = \frac{d}{dx} f(x), \quad Qf(x) = i\lambda x f(x) \tag{3.1}$$

which satisfy the relation $[P, Q] = i\lambda$, $\lambda \neq 0$. In fact, by a celebrated theorem of Stone and von Neumann, the equation (3.1) serves as a model example for the Heisenberg relations.

Theorem 3.1.2 (Stone-von Neumann)**.** *A pair of antisymmetric (unbounded) operators P and Q in the Hilbert space $\mathcal{H}$ satisfying the Heisenberg commutation relations $[P, Q] = i\lambda I$, $\lambda \neq 0$ can be realized as a differentiation and a multiplication:*

$$Pf(x) = \frac{d}{dx} f(x), \quad Qf(x) = i\lambda x f(x),$$

on scalar or vector-valued functions $f \in L^2(\mathbb{R})$. In other words, there exists a unitary intertwining map $W \colon \mathcal{H} \to L^2(\mathbb{R})$ that sends Q to $i\lambda x$ and P to d/dx.

It follows from the Stone-von Neumann Theorem that the irreducible unitary representations of the Heisenberg group H fall into two classes:

- The one-dimensional representations (characters) of the commutative quotient group $H/Z \simeq \mathbb{R}^2$,

$$\chi_{r,s}(a, b, c) = e^{i(ra+st)} \text{ for } (a, b, c) \in H,$$

and,

- A one-parameter family of infinite-dimensional representations π_t with $t \in \mathbb{R}$ realized in the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$ by operators

$$\pi_t(a, b, c)\phi(x) = e^{i(bx+c)t}\phi(x + a).$$

To apply Theorem 3.1.2 to irreducible unitary representations of the Heisenberg group H, we observe that any such representation π restricted to the center $Z = Z(H)$ must be scalar by Schur's Lemma; $T_{|Z} = e^{it}$. If $t = 0$, then π factors through the representation of the commutative quotient group H/Z, so it becomes a character. If $t \neq 0$, generators of the Lie algebra of H obey the Heisenberg CCR. So by Theorem 3.1.2, we get the subgroups $\{(a, 0, 0)\}$ and $\{(0, b, c)\}$, acting by translations and modulations on $L^2(\mathbb{R})$ as we claimed above.

Remark 3.1.3 (Historical Remarks)**.** The Heisenberg commutation relations first appeared in the context of quantum mechanics. The *states* of a quantum system are usually described by vectors in a Hilbert space $\mathcal{H}$, while the *observables* are given by certain (typically symmetric, but often unbounded) operators in $\mathcal{H}$. Examples include the so-called *position* and *momentum* operators

$$Pf(x) = \frac{d}{dx}f(x), \quad Qf(x) = i\lambda x f(x) \text{ on } \mathcal{H} = L^2(\mathbb{R}).$$

The precise knowledge of an observable A at state ψ is attainable only for special states: the *eigenvectors* of A. Thus, measurability (or observability) of A becomes paramount to its diagonalization.

Obviously, any pair of commutating observables can be simultaneously diagonalized, that is, observed to any degree of precision. However, noncommuting observables like position $\{Q_i\}$ and momenta $\{P_i\}$ cannot be diagonalized.

It was observed experimentally that the position and momentum of an electron cannot be accurately measured at once; the product of errors always remained greater than the Plank constant $\hbar$. This led Heisenberg to state his famous *Uncertainty Principle* of the quantum theory in the form of the commutation relation

$$[P, Q] = i\hbar.$$

3.1.1 The Symplectic Structure of the Coadjoint Orbits

Let G be a group acting on $\mathfrak{g}^*$ by the coadjoint action. We shall always regard the coadjoint orbits as injectively immersed submanifolds of $\mathfrak{g}^*$ diffeomorphic to G/G_f, where $G_f = \{\, g \in G \mid \mathrm{Ad}^*(g)f = f \,\}$ is the isotropy group of the coadjoint action at a point f in the orbit.

We now describe the tangent vectors to coadjoint orbits. For a coadjoint orbit $\mathcal{O}$ and $f \in \mathcal{O}$,

$$F(t) = \mathrm{Ad}^*(g(t))f$$

is a curve in $\mathcal{O}$ with $F(0) = f$ where $g(t)$ is a curve in G. Differentiating $F(t)$ at $t = 0$ we get

$$F'(0) = \mathrm{ad}^*(\xi)f \qquad \text{where } \xi = g'(0) \in \mathfrak{g}.$$

Thus we have the vector space isomorphism

$$T_f\mathcal{O} \cong \{\, \mathrm{ad}^*(\xi)f \mid \xi \in \mathfrak{g} \,\} \subset \mathfrak{g}^*.$$

Let the Heisenberg group H act on $\mathfrak{h}^*$ by the coadjoint action. Then for a coadjoint orbit $\mathcal{O} \subset \mathfrak{h}^*$ and $\xi \in \mathfrak{h}$ we can define the ***vector field*** $\xi^{\mathcal{O}}$ ***generated***

by ξ as follows:

$$\xi_f^{\mathcal{O}} = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}^*(\exp t\xi)f, \quad \text{for } f \in \mathcal{O}$$
$$= \mathrm{ad}^*(\xi)f = -f([\xi, \cdot]) \in T_f\mathcal{O}.$$

Now let X, Y, and Z be the standard basis vectors for $\mathfrak{h}$ satisfying the CCR and assume that $f(Z) = \gamma \neq 0$ so that the coadjoint orbit $\mathcal{O}$ is the plane $Z^* = \gamma$ in the $X^*Y^*Z^*$-coordinate system in $\mathfrak{h}^*$.

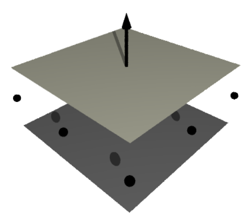

Figure 3.1: Coadjoint orbits for the Heisenberg group

Then for the Lie algebra elements

$$\xi = \xi_1 X + \xi_2 Y + \xi_3 Z$$

and

$$\eta = \eta_1 X + \eta_2 Y + \eta_3 Z$$

one computes $f([\xi, \eta]) = \gamma(\xi_1\eta_2 - \xi_2\eta_1)$, and therefore

$$\omega_f(\xi_f^{\mathcal{O}}, \eta_f^{\mathcal{O}}) = f([\xi, \eta])$$
$$= \gamma(\xi_1\eta_2 - \xi_2\eta_1). \tag{3.2}$$

On the other hand, writing out $\xi_f^{\mathcal{O}}$ and $\eta_f^{\mathcal{O}}$ in terms of basis vectors in $\mathfrak{h}^*$ we find

$$\xi_f^{\mathcal{O}} = \gamma(\xi_2 X^* - \xi_1 Y^*) \tag{3.3}$$

$$\eta_f^{\mathcal{O}} = \gamma(\eta_2 X^* - \eta_1 Y^*). \tag{3.4}$$

Putting equations (3.2), (3.3), and (3.4) together and identifying $\mathfrak{h}^{**}$ with $\mathfrak{h}$ we obtain the KKS symplectic form of the coadjoint orbit $\mathcal{O}$, namely $\omega = \frac{1}{\gamma} dX \wedge dY$. We shall use the Liouville form

$$\mu = \frac{\omega}{2\pi} = \frac{1}{2\pi\gamma} dX \wedge dY \tag{3.5}$$

as the volume form on $\mathcal{O}$ when integrating functions on $\mathcal{O}$.

One might wonder why the multiplicative constant γ appears in the KKS symplectic form ω. In the next section we will give a justification for this using the Plancherel Formula for the Heisenberg group.

3.1.2 Positivity of Kirillov's Character

Let $f \in \mathcal{S}(H)$ be a function in the Schwartz space of the Heisenberg group and identify the maximal coadjoint orbit $\mathcal{O}$ and the Lie algebra $\mathfrak{h}$ with the plane $\mathbb{R}^2 \times \{\gamma\} = \{ (\alpha, \beta, \gamma) \mid \alpha, \beta \in \mathbb{R} \}$ and $\mathbb{R}^3 = \{ (a, b, c) \mid a, b, c \in \mathbb{R} \}$, respectively. Then

$$\begin{aligned}
\chi_{\mathcal{O}}(f) &= \int_{\mathcal{O}} \int_{\mathfrak{h}} f(\exp X) e^{i(a\alpha + b\beta + c\gamma)}\, dX\, d\mu_{\mathcal{O}} \\
&= \frac{1}{2\pi|\gamma|} \int_c \int_{(\alpha,\beta)} \int_{(a,b)} f(\exp X) e^{i(a\alpha + b\beta + c\gamma)}\, da\, db\, d\alpha\, d\beta\, dc \quad \text{by (3.5)} \\
&= \frac{2\pi}{|\gamma|} \int_c f(\exp(0,0,c)) e^{ic\gamma}\, dc
\end{aligned}$$

where in the last step we have used the exact stationary phase formula, or simply the Fourier Inversion Theorem.

Thus we have found the following simplified form of Kirillov's formula for the maximal coadjoint orbit $\mathcal{O}$ of the Heisenberg group

$$\chi_{\mathcal{O}}(f) = \frac{2\pi}{|\gamma|} \int_{\mathbb{R}} f(\exp tZ) e^{it\gamma}\, dt, \quad f \in \mathcal{S}(H). \tag{3.6}$$

Now checking the positivity of character formula or equivalently the right-hand side of the above equality is easy. First note that

$$\chi_{\mathcal{O}}(f * f^*) = \frac{2\pi}{|\gamma|} \int_{\mathbb{R}} \int_{H} f(\exp tZ \cdot h)\overline{f(h)} e^{it\gamma}\, d\lambda\, dt$$

where $d\lambda = dx \wedge dy \wedge dz$ is the Haar measure on H after the usual identification of H with $\mathbb{R}^3$. Setting $h = \exp(xX + yY + zZ)$, that is possible due to the surjectivity of $\exp\colon \mathfrak{h} \to H$, and using the fact that $Z \in Z(H)$ the last expression equals

$$\begin{aligned}
&\frac{2\pi}{|\gamma|} \int_{\mathbb{R}} \int_{\mathbb{R}^3} f(\exp(xX + yY + (z+t)Z))\overline{f(\exp(xX + yY + zZ))} e^{it\gamma}\, d\lambda\, dt \\
&= \frac{2\pi}{|\gamma|} \int_{\mathbb{R}} \int_{\mathbb{R}^3} f(\exp(xX + yY + (z+t)Z)) e^{i(z+t)\gamma}\, \overline{f(\exp(xX + yY + zZ)) e^{iz\gamma}}\, d\lambda\, dt \\
&= \frac{2\pi}{|\gamma|} \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}} f(\exp(xX + yY + zZ)) e^{iz\gamma}\, dz \right|^2 dy\, dx \geq 0.
\end{aligned}$$

Kirillov's character is positive for point-orbits essentially by the discussion in

Subsection 2.5.1 about the abelian groups. To be more precise,

$$\begin{aligned}\chi_{\mathcal{O}}(f * f^*) = \int_{\mathfrak{h}} f * f^*(\exp X) e^{i\ell(X)}\, dX &= \int_H f * f^*(x) e^{i\ell(\log x)}\, d\lambda(x) \\ &= \int_H \int_H f(xy)\overline{f(y)}\, d\lambda(y) e^{i\ell(\log x)}\, d\lambda(x) \\ &= \int_H \int_H f(x)\overline{f(y)} e^{i\ell(\log xy^{-1})}\, dx\, dy \\ &= \left| \int_H f(x) e^{i\ell(\log x)}\, dx \right|^2 \geq 0.\end{aligned}$$

The last equality is due to the fact that if $\xi, \eta \in \mathfrak{h}$, then

$$\exp \xi \exp \eta = \exp(\xi + \eta + \frac{1}{2}[\xi, \eta])$$

and moreover $\ell([\xi, \eta]) = 0$, since $[\xi, \eta] \in Z(\mathfrak{h})$. Therefore, the map from G to $\mathbb{C}$ defined by $\exp(X) \mapsto \ell(X)$ is a group homomorphism.

3.1.3 The Plancherel Theorem

In this section we state a version of the Plancherel Formula for the Heisenberg group. This provides an answer to the question that we asked earlier about the existence of the multiplicative constant in the KKS symplectic form (3.5).

Theorem 3.1.4 (Plancherel Theorem)**.** *Let G be a second countable, unimodular, locally compact group of type I. There is a unique measure μ on $\widehat{G}$ such that for $f \in L^1(G) \cap L^2(G)$ one has*

$$\|f\|_2^2 = \int_{\widehat{G}} \|\pi(f)\|_{\mathrm{HS}}^2\, d\mu(\pi).$$

The measure μ is called the **Plancherel measure** *on $\widehat{G}$ associated with the Haar measure of G.*

Remark 3.1.5. If G is a locally compact abelian group, then the classical

Plancherel theorem shows that the Plancherel measure is just the (suitably normalized) Haar measure of the dual group $\widehat{G}$.

The Plancherel formula for a nilpotent Lie group G follows from the abelian Plancherel for the vector space $\mathfrak{g}$:

$$
\begin{aligned}
f(e) = f \circ \exp(\mathbf{0}) &= \int_{\mathfrak{g}^*} \widehat{f \circ \exp}(\ell)\, d\ell \\
&= \int_{\mathcal{O} \in \mathfrak{g}^*/G} \left(\int_{\ell \in \mathcal{O}} \widehat{f \circ \exp}(\ell)\, d\mu_{\mathcal{O}}(\ell) \right) dm(\mathcal{O})
\end{aligned}
$$

where dm is the quotient measure of the Lebesgue measure $d\ell$ by the Liouville measure $d\mu_{\mathcal{O}}$. Thus we obtain

$$
f(e) = \int_{\mathcal{O} \in \mathfrak{g}^*/G} \operatorname{Tr} \pi_{\mathcal{O}}(f)\, dm(\mathcal{O}).
$$

Proposition 3.1.6 (A Plancherel Type Formula). *Let $f \in \mathcal{S}(H)$ be a Schwartz function. Then*

$$
f(I) = \int_{\mathbb{R}^\times} \chi_{\mathcal{O}_\gamma}(f) |\gamma|\, d\gamma.
$$

This implies that the Plancherel measure of H is $|\gamma|\, d\gamma$ and that the set of one-dimensional representations in $\widehat{H}$ has Plancherel measure zero.

Proof. The proof is an immediate consequence of Equation (3.6) and the Euclidean Fourier Inversion Theorem.

$$
\begin{aligned}
\int_{\mathbb{R}^\times} \chi_{\mathcal{O}_\gamma}(f) |\gamma|\, d\gamma &= \frac{1}{2\pi} \int_{\mathbb{R}^\times} \int_{\mathbb{R}} f(\exp tZ) e^{it\gamma}\, dt\, d\gamma \qquad \text{by Equation (3.6)} \\
&= f(\exp tZ)\Big|_{t=0} = f(I).
\end{aligned}
$$

□

3.2 Positivity of Kirillov's Character for Nilpotent Lie Groups

Let $\mathfrak{g}$ be a real finite dimensional Lie algebra and let $f \in \mathfrak{g}^*$. Denote by B_f the alternating bilinear form on $\mathfrak{g}$ defined by

$$B_f(X, Y) = f([X, Y]), \quad X, Y \in \mathfrak{g}.$$

Definition 3.2.1. A subalgebra $\mathfrak{m} \subset \mathfrak{g}$ is said to be a ***real polarization*** at $f \in \mathfrak{g}^*$ if $\mathfrak{m}$ is a maximal totally isotropic subspace for B_f, that is, $f([X, \mathfrak{m}]) = 0$ if and only if $X \in \mathfrak{m}$.

A subalgebra $\mathfrak{h}$ with $f([\mathfrak{h}, \mathfrak{h}]) = 0$ is said to be ***subordinate*** to f. If $\mathfrak{g}$ is nilpotent, maximal subordinate subalgebras are real polarizations, but that need not be so in general.

Lemma 3.2.2.

(a) Let V be a $2n$-dimensional vector space with a nondegenerate alternating bilinear form. Then any isotropic subspace of V which is maximal under inclusion has dimension n.

(b) Let $\mathfrak{g}$ be a nilpotent Lie algebra and $\ell \in \mathfrak{g}^*$. Then there exists a maximal isotropic subalgebra associated to the alternating bilinear form given by $\langle X, Y \rangle = \ell([X, Y])$. Moreover, by the previous part, any such subalgebra lies halfway between the radical $\mathfrak{r}_\ell$ and the Lie algebra $\mathfrak{g}$.

Remark 3.2.3. Maximal isotropic subalgebras, whose existence is guaranteed for nilpotent Lie algebras, are sometimes called polarizing subalgberas in the literature.

Let $\mathfrak{m}$ be a polarizing subalgebra subordinate to ℓ. In the light of the above lemma,

$$\dim M/R_\ell = \dim \mathfrak{g}/\mathfrak{m}.$$

Motivated by the equality of dimensions, we formulate the following about coadjoint actions.

Lemma 3.2.4. *Let G be a nilpotent Lie group with Lie algebra $\mathfrak{g}$ and a polarizing subalgebra $\mathfrak{m}$ subordinate to an element $\ell \in \mathfrak{g}^*$. Then the mapping*

$$\begin{aligned} \theta \colon M/R_\ell &\to (\mathfrak{g}/\mathfrak{m})^* \\ m \bmod R_\ell &\mapsto m \cdot \ell - \ell. \end{aligned}$$

is a diffeomorphism.

The proof of this lemma hinges upon a technical result due to Chevalley and Rosenlicht.

Theorem 3.2.5 (Chevalley-Rosenlicht)**.** *Let G be a connected Lie group acting unipotently on a real vector space V. For each $v \in V$ there are $X_1, \dots, X_n \in \mathfrak{g}$ such that*

$$G \cdot v = \{(\exp x_1 X_1 \dots \exp x_n X_n) \cdot v \mid x_1, \dots, x_n \in \mathbb{R}\}.$$

The map

$$(x_1, \dots, x_n) \mapsto (\exp x_1 X_1 \dots \exp x_n X_n) \cdot v$$

is a diffeomorphism between $\mathbb{R}^n$ and the orbit $G \cdot v$ which is a closed submanifold of V.

Proof. See [6, Theorem 3.1.4]. □

We now turn to the proof of Lemma 3.2.4.

Proof of Lemma 3.2.4. By the identity $\operatorname{ad}^*(e^X) = e^{\operatorname{ad}^* X}$, or the Baker-Campbell-Hausdorff formula,

$$m \cdot \ell(P) = \ell(P) \quad \text{ for } P \in \mathfrak{m} \text{ and } m \in M. \tag{3.7}$$

Using this it is a straightforward matter to check that θ is well defined. Injectivity is clear since $m_1 \cdot \ell - \ell = m_2 \cdot \ell - \ell$ implies $m_1 m_2^{-1} \in R_\ell$ and hence $m_1 \bmod R_\ell = m_2 \bmod R_\ell$. Surjectivity of θ, on the other hand, is equivalent to surjectivity of its lift $\tilde{\theta}$ to the group M defined by $m \mapsto m \cdot \ell - \ell$. Computing the differential of the equivariant map $\tilde{\theta}$ at a point $g \in M$ one infers that

$$\tilde{\theta}_{*,g}(X) = \frac{d}{dt}\Big|_{t=0} \tilde{\theta}(g \exp tX) = g \cdot ((\operatorname{ad}^* X)\ell) \quad \text{ where } X \in T_g M,$$

and

$$\begin{aligned} \operatorname{rank} \tilde{\theta}_{*,g} &= \dim T_g M - \dim\{X \in T_g M \mid (\operatorname{ad}^* X)\ell = 0\} \\ &= \dim \mathfrak{m} - \dim \mathfrak{r}_\ell. \end{aligned}$$

Consequently, $\tilde{\theta}$ is a submersion, indeed a local diffeomorphism, and hence an open map. On the other hand, we know from Theorem 3.2.5 that the image of $\tilde{\theta}$ is a closed submanifold of $(\mathfrak{g}/\mathfrak{m})^*$ and the surjectivity of θ follows. We emphasize that it is exactly in the last step that we are using the fact that G is nilpotent to conclude that the image of $\tilde{\theta}$ is closed. $\square$

Theorem 3.2.6 (Weil's Formula)**.** *Let G be a locally compact group, and let H be a closed subgroup. There exists a G-invariant Radon measure $\nu \neq 0$ on the quotient G/H if and only if the modular functions Δ_G and Δ_H agree on H. In this case, the measure ν is unique up to a positive scalar. Given Haar measures on G and H, there is a unique choice for ν such that for every $f \in C_c(G)$ one* *has the quotient integral formula*

$$\int_G f(g)\,dg = \int_{G/H}\int_H f(xh)\,dh\,d\nu(xH).$$

We will have an occasion for using a slightly more general form of this theorem involving a tower of three groups which we now state.

Corollary 3.2.7. *Let G be a locally compact group with closed subgroups H and K such that $K \subset H \subset G$. Then*

$$\Delta_G|_H = \Delta_H, \quad \Delta_H|_K = \Delta_K,$$

if and only if there exist nonzero suitably normalized invariant Radon measures on the quotient spaces such that the equality

$$\int_{G/K} f(g)\,dgK = \int_{G/H}\int_{H/K} f(xh)\,dhK\,dxH$$

holds for any $f \in C_c(G/K)$.

Proof. First assume the equality of restrictions of modular functions. Let $F \in C_c(G)$ and apply Theorem 3.2.6 twice to obtain

$$\begin{aligned}\int_G F(t)\,dt &= \int_{G/K}\int_K F(gk)\,dk\,dgK \\ &= \int_{G/H}\int_H F(xs)\,ds\,dxH = \int_{G/H}\left(\int_{H/K}\int_K F(xhk)\,dk\,dhK\right)dxH.\end{aligned}$$

Fix $f \in C_c(G/K)$ and choose a function $\alpha \in C_c(G)$ with the property that for every $gK \in \text{supp}\,(f)$ we have $\int_K \alpha(gk)\,dk = 1$, then substitute $F = f\alpha$. For the standard proof of existence of such α we refer to [8, Lemma 2.47].
The converse is immediate from the first part of Theorem 3.2.6. $\square$

For a geometric proof of this ‘chain rule for integration’ formula see [13, Proposition 1.13].

Remark 3.2.8. Our next goal is proving that, with suitable choices of measure, the map θ of Lemma 3.2.4 is measure preserving. Here we collect some of standard conventions and results about installing measures on various spaces that we shall need shortly.

1. (Lebesgue measure on vector spaces) Let V be an n-dimensional vector space with a basis $\mathcal{B} = \{b_1, \ldots, b_n\}$. Then one can put a Lebesgue measure on V via the isomorphism $T\colon \mathbb{R}^n \to V$ sending the standard basis vector e_i of $\mathbb{R}^n$ to b_i. For $f \in C_c(V)$ define

$$\int_V f \, d\mu = \int_{\mathbb{R}^n} f \circ T \, dx.$$

   The usual properties of Lebesgue measure such as translation invariance are easy to check.

2. (The Lebesgue measure on the Lie algebra of a nilpotent Lie group is invariant under the adjoint action.) It is well known and easy to see that for locally compact Hausdorff spaces X and Y and a homeomorphism[1] Φ,

$$\int_X f^\Phi \, d\mu = \int_Y f \, d\mu_\Phi \qquad \text{(Change of variables formula)}$$

   holds for any Borel function f where f^Φ is the pull back of f given by $f^\Phi(x) = f(\Phi(x))$ and μ_Φ is the push forward of the Borel measure μ given by $\mu_\Phi(S) = \mu(\Phi^{-1}(S))$.

   Fix a Haar measure μ on a nilpotent Lie group G, then according to the

[1] The change of variables formula is true with the weaker assumption of measurability of Φ; however we shall assume that Φ is a homeomorphism so that μ and μ_Φ enjoy the same measure theoretic properties such as regularity.

change of variables formula,

$$\int_{\mathfrak{g}} f\, d\mu_{\log} = \int_G f \circ \log\ d\mu.$$

It is not hard to see, for instance, by direct computations in strong Malcev bases as in [6, Theorem 1.2.10], that $\mu_{\log}$ is a Lebesgue measure. For any $g \in G$,

$$\int_G f(\mathrm{Ad}(g)\log x)\, d\mu(x) = \int_G f(\log x)\, d\mu(x)$$

using the power series expansion of logarithm and unimodularity of G. This proves our claim. An immediate consequence that we mention for later use is the invariance of the Lebesgue measure on $\mathfrak{g}^*$ under the coadjoint action.

3. Let G be a nilpotent Lie group and H a closed subgroup of G. Then the Lebesgue measure on $\mathfrak{g}/\mathfrak{h}$ is invariant under the adjoint action of group H.

   For $f \in C_c(\mathfrak{g}/\mathfrak{h})$ we proceed as in the proof of Corollary 3.2.7 and choose $\alpha \in C_c(\mathfrak{g})$ such that for any $Y_{\mathfrak{g}}+\mathfrak{h} \in \mathrm{supp}(f)$ we have $\int_{\mathfrak{h}} \alpha(Y_{\mathfrak{g}}+Y_{\mathfrak{h}})\, d\gamma(Y_{\mathfrak{h}}) = 1$. Let $\pi\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ denote the natural projection map. Then it is clear that

$$\int_{\mathfrak{g}/\mathfrak{h}} f\, d\mu = \int_{\mathfrak{g}} (f \circ \pi)\alpha\, d\lambda.$$

   Thus, by the invariance of Lebesgue measures under the adjoint action that we discussed above, for any $h \in H$,

$$\begin{aligned}\int_{\mathfrak{g}/\mathfrak{h}} f(\mathrm{Ad}(h)X)\, d\mu(X) &= \int_{\mathfrak{g}} f(\mathrm{Ad}(h)\pi(Y))\alpha(Y)\, d\lambda(Y)\\ &= \int_{\mathfrak{g}} f(\pi(Y))\alpha(\mathrm{Ad}(h^{-1})Y)\, d\lambda(Y)\\ &= \int_{\mathfrak{g}/\mathfrak{h}} f\, d\mu(Y_{\mathfrak{g}}) \underbrace{\int_{\mathfrak{h}} \alpha(\mathrm{Ad}(h^{-1})(Y_{\mathfrak{g}} + Y_{\mathfrak{h}}))\, d\gamma(Y_{\mathfrak{h}})}_{=1}\\ &= \int_{\mathfrak{g}} f(\pi(Y))\alpha(Y)\, d\lambda(Y) = \int_{\mathfrak{g}/\mathfrak{h}} f(X)\, d\mu(X)\end{aligned}$$

as we wanted to show.

4. It follows from the definition of $\mathfrak{m}$ that $\mathfrak{m}/\mathfrak{r}_\ell \subset \mathfrak{g}/\mathfrak{r}_\ell$ is a Lagrangian subspace for B_ℓ and the assignment ϕ defined by $X \mapsto B_\ell(\cdot, X)$ is a canonical linear isomorphism between $\mathfrak{m}/\mathfrak{r}_\ell$ and $(\mathfrak{g}/\mathfrak{m})^*$.

5. Any nonzero Lebesgue measure λ on $\mathfrak{m}/\mathfrak{r}_\ell$ can be pushed forward to an M-invariant measure $\lambda_{\widetilde{\exp}}$ on M/R_ℓ via the exponential map. Also one can use the linear isomorphism ϕ described above to transfer the measure λ to $(\mathfrak{g}/\mathfrak{m})^*$.

6. (Dual Measure) Let V be a vector space equipped with a Lebesgue measure μ. Then we can uniquely determine a dual measure λ on V^* via the Fourier transform. More precisely, if

$$\widehat{\phi}(f) = \int_V \phi(X) e^{if(X)}\, d\mu(X),$$

then we choose λ so that

$$\phi(\mathbf{0}) = \int_{V^*} \widehat{\phi}(f)\, d\lambda(f),$$

and therefore $V \oplus V^*$ carries a canonical measure in the sense that if we scale the Lebesgue measure on V by a nonzero constant c, then the dual measure on V^* is scaled by c^{-1}. We shall be interested in the following decomposition of a Lie algebra

$$\mathfrak{g} = \mathfrak{g}/\mathfrak{m} \oplus (\mathfrak{g}/\mathfrak{m})^* \oplus \mathfrak{r}_\ell$$

which helps us assign a measure to R and obtain canonical measures on G and hence on G/R. (cf. Weil's Formula, Theorem 3.2.6)

Theorem 3.2.9 ([6, Theorem 1.1.13])**.** *Let* $\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_k$ *be subalgebras of a nilpotent Lie algebra* $\mathfrak{g}$ *with* $\dim \mathfrak{g}_j = n_j$*. Then*

(a) there exists a basis $\{ X_1, \ldots, X_n \}$ *of* $\mathfrak{g}$ *such that*

  (i) for each m*,* $\mathrm{Span}\{ X_1, \ldots, X_m \}$ *is a subalgebra of* $\mathfrak{g}$*, and*

  (ii) $\mathrm{Span}\{ X_1, \ldots, X_{n_j} \} = \mathfrak{g}_j$ *for* $1 \leq j \leq k$*.*

(b) If $\mathfrak{g}_j$ *are ideals of* $\mathfrak{g}$*, the above basis can be chosen so that each* $\mathrm{Span}\{ X_1, \ldots, X_m \}$ *is an ideal of* $\mathfrak{g}$*.*

A basis satisfying these properties is called a ***Malcev basis*** for $\mathfrak{g}$ through $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$.

Lemma 3.2.10. *With the above choices of measure, the map* θ *of Lemma 3.2.4 is measure preserving.*

Proof. Let $\{D_1, \ldots, D_r, X_1, \ldots, X_k\}$ be a Malcev basis for $\mathfrak{m}$ through $\mathfrak{r}_\ell = \mathrm{Span}\{D_1, \ldots, D_r\}$. Identify $\mathfrak{m}/\mathfrak{r}_\ell$ with the subspace $\mathrm{Span}\{X_1, \ldots, X_k\}$ of $\mathfrak{m}$. Now we can define

$$\begin{aligned} \widetilde{\exp} \colon \mathfrak{m}/\mathfrak{r}_\ell &\to M/R_\ell \\ x_1X_1 + \cdots + x_kX_k &\mapsto \exp(x_1X_1) \cdots \exp(x_kX_k)R_\ell \end{aligned}$$

which is automatically well defined and gives a polynomial relation between the domain and the codomain.

Consider the following diagram summarizing our above measure assignment to the quotient spaces.

$$\begin{tikzcd}
(\mathfrak{m}/\mathfrak{r}_\ell, \lambda) \arrow[r, "\widetilde{\exp}"] \arrow[d, "\phi"', "\cong"] & (M/R_\ell, \lambda_{\widetilde{\exp}}) \arrow[d, "\theta"] \\
((\mathfrak{g}/\mathfrak{m})^*, \lambda_\phi) & ((\mathfrak{g}/\mathfrak{m})^*, \lambda_\phi)
\end{tikzcd}$$

Define the change of basis transformation $T\colon \mathbb{R}^k \to \mathfrak{m}/\mathfrak{r}_\ell$ as before. By virtue of the change of variables formula, for any function $f \in C_c((\mathfrak{g}/\mathfrak{m})^*)$,

$$\int_{M/R_\ell} f \circ \theta \, d\lambda_{\widetilde{\exp}} = \int_{\mathfrak{m}/\mathfrak{r}_\ell} f \circ \theta \circ \widetilde{\exp} \, d\lambda = \int_{\mathbb{R}^k} f \circ \theta \circ \widetilde{\exp} \circ T \, dx.$$

To show that θ is measure preserving, we have to check that these integrals are equal to

$$\int_{(\mathfrak{g}/\mathfrak{m})^*} f \, d\lambda_\phi = \int_{\mathfrak{m}/\mathfrak{r}_\ell} f \circ \phi \, d\lambda = \int_{\mathbb{R}^k} f \circ \phi \circ T \, dx.$$

It suffices to prove that the Jacobian determinant of the diffeomorphism

$$T^{-1} \circ \phi^{-1} \circ \theta \circ \widetilde{\exp} \circ T$$

is constant, everywhere equal to one. Observe that

$$\begin{aligned}
\mathbb{R}^k \ni x = \sum x_i e_i &\overset{T}{\mapsto} X = \sum x_i X_i \overset{\widetilde{\exp}}{\mapsto} \widetilde{\exp}(X) \overset{\theta}{\mapsto} \widetilde{\exp}(X) \cdot \ell - \ell \\
&= \sum y_i(x) \phi(X_i) \overset{\phi^{-1}}{\mapsto} \sum y_i(x) X_i \overset{T^{-1}}{\mapsto} y = \sum y_i(x) e_i \in \mathbb{R}^k
\end{aligned}$$

where we have used the fact that $\phi(X_i)$ are basis vectors for $(\mathfrak{g}/\mathfrak{m})^*$. Choose a basis $\{X_{k+1}, \dots, X_{2k}\}$ for $\mathfrak{g}/\mathfrak{m}$ such that its dual basis in $(\mathfrak{g}/\mathfrak{m})^*$ is precisely $\{\phi(X_1), \dots, \phi(X_k)\}$, that is, with the usual notation from linear algebra of dual spaces, $X_{k+i}^* = \phi(X_i)$. Now we are ready to compute the Jacobian at the origin.

$$\begin{aligned} y_i(x) = \theta \circ \widetilde{\exp} \circ T(X_{k+i}) &= \langle \mathrm{Ad}^* \, \widetilde{\exp}(x_1 X_1 + \cdots + x_k X_k)\ell - \ell, X_{k+i} \rangle \\ &= \langle \ell, (\mathrm{Ad}\, \widetilde{\exp}(-x_1 X_1 - \cdots - x_k X_k) - i_d) X_{k+i} \rangle \end{aligned}$$

$$\begin{aligned} \frac{\partial y_i}{\partial x_j}(\mathbf{0}) &= \frac{d}{dx_j} \langle \ell, (e^{-\operatorname{ad}(x_j X_j)} - i_d) X_{k+i} \rangle \\ &= -\ell[X_j, X_{k+i}] = \phi(X_j)(X_{k+i}) = X_{k+j}^*(X_{k+i}) = \delta_{ij} \end{aligned}$$

Thus,

$$\frac{\partial(y_1, \dots, y_k)}{\partial(x_1, \dots, x_k)}(\mathbf{0}) = I_k.$$

Our next objective is to prove that the latter equality implies that the Jacobian determinant is constant, everywhere equal to one .

We can push forward λ_ϕ to an M-invariant measure on M/R_ℓ via θ^{-1}. To check M-invariance of $(\lambda_\phi)_{\theta^{-1}}$, observe that for $p \in M$,

$$p\theta^{-1}(\psi) = \theta^{-1}(p \cdot \psi + p \cdot \ell - \ell) \tag{3.8}$$

and therefore

$$\begin{aligned} \int_{M/R_\ell} f(px)\, d\lambda_{\theta^{-1}\circ\phi}(x) &= \int_{(\mathfrak{g}/\mathfrak{m})^*} f(p\theta^{-1}(\psi))\, d\lambda_\phi(\psi) \\ &= \int_{(\mathfrak{g}/\mathfrak{m})^*} f(\theta^{-1}(p \cdot \psi))\, d\lambda_\phi(\psi) \\ &= \int_{(\mathfrak{g}/\mathfrak{m})^*} f(\theta^{-1}(\psi))\, d\lambda_\phi(\psi) = \int_{M/R_\ell} f(x)\, d\lambda_{\theta^{-1}\circ\phi}(x) \end{aligned}$$

where the second and third equalities hold by Equation (3.8) and translation invariance of the Lebesgue measure, and part (2) of Remark 3.2.8, respectively.

But the M-invariant measure $\lambda_{\widetilde{\exp}}$ on M/R_ℓ is unique up to a positive scalar by Theorem 3.2.6. Therefore,

$$\lambda_{\theta^{-1}\circ\phi} = (\lambda_\phi)_{\theta^{-1}} = c\lambda_{\widetilde{\exp}}, \qquad \text{for some } c \in \mathbb{R}^+$$

and our calculation of the Jacobian at the origin shows that $c = 1$, as claimed. $\square$

Remark 3.2.11. Each y_i in the above proof is a polynomial in x, thanks to the Baker-Campbell-Hausdorff formula. However, since the Jacobian determinant $\det \frac{\partial(y_1,\ldots,y_k)}{\partial(x_1,\ldots,x_k)}$ is a nonzero constant, the Jacobian Conjecture implies that the map

$$x \mapsto (y_1(x), \ldots, y_k(x))$$

has a regular inverse. This is to say that the coadjoint map θ has a polynomial inverse.

For a nilpotent Lie group G and $\ell \in \mathfrak{g}^*$, let χ_ℓ denote Kirillov's character corresponding to the coadjoint orbit of ℓ. That is,

$$\chi_\ell(f) = \int_{\mathcal{O}_\ell} \widehat{f \circ \exp}(\phi)\, d\mu(\phi), \quad f \in \mathcal{S}(G).$$

Proposition 3.2.12. *Let G be a connected nilpotent Lie group with a polarizing subalgebra $\mathfrak{m}$. Then for any $f \in \mathcal{S}(G)$ and $\ell \in \mathfrak{g}^*$,*

$$\chi_\ell(f) = \int_{G/M} \int_{\mathfrak{m}} f(e^{\mathrm{Ad}(x)P}) e^{i\ell(P)}\, dP\, dxM. \tag{3.9}$$

Proof. First we use the diffeomorphism $G/R_\ell \cong \mathcal{O}_\ell$ to make a change of variables:

$$\chi_\ell(f) = \int_{\mathcal{O}_\ell} \int_{\mathfrak{g}} f(\exp X) e^{i\phi(X)}\, dX\, d\mu(\phi)$$
$$= \int_{G/R_\ell} \int_{\mathfrak{g}} f(\exp X) e^{ig\cdot\ell(X)}\, dX\, dgR_\ell,$$

where we write $g \cdot \ell = \mathrm{Ad}^*(g)\ell$ for the left action of G on the dual space $\mathfrak{g}^*$. Let $\mathfrak{m}$ be a maximal isotropic subalgebra of $\mathfrak{g}$ with $M = \exp \mathfrak{m}$ containing the isotropy group R_ℓ. Corollary 3.2.7 allows us to write the latter expression as

$$\int_{G/M} \int_{M/R_\ell} \int_{\mathfrak{g}} f(\exp X) e^{ixm\cdot\ell(X)}\, dX\, dmR_\ell\, dxM.$$

Since the measure on $\mathfrak{g}$ is G-invariant we can use the change of variable $X \mapsto x \cdot X$ where dot denotes the adjoint action. This together with a measure decomposition on the Lie algebra gives

$$\int_{G/M} \int_{M/R_\ell} \int_{\mathfrak{g}/\mathfrak{m}} \int_{\mathfrak{m}} f(e^{x\cdot(P+Y)}) e^{im\cdot\ell(P+Y)}\, dP\, dY\, dmR_\ell\, dxM.$$

Define

$$F_f^x(Y) = \int_{\mathfrak{m}} f(e^{x\cdot(P+Y)}) e^{i\ell(P+Y)}\, dP.$$

In view of (3.7) the above integration can be stated in the more compact form

$$\int_{G/M} \int_{M/R_\ell} \int_{\mathfrak{g}/\mathfrak{m}} F_f^x(Y) e^{im\cdot\ell(Y) - i\ell(Y)}\, dY\, dmR_\ell\, dxM.$$

Now we apply the measure-preserving diffeomorphism introduced in Lemma 3.2.4 to make a change of variables and to transform the second domain of integration from M/R_ℓ to $(\mathfrak{g}/\mathfrak{m})^*$:

$$\int_{G/M}\int_{M/R_\ell}\int_{\mathfrak{g}/\mathfrak{m}} F_f^x(Y)e^{im\cdot\ell(Y)-i\ell(Y)}\,dY\,dmR_\ell\,dxM$$
$$= \int_{G/M}\int_{(\mathfrak{g}/\mathfrak{m})^*}\int_{\mathfrak{g}/\mathfrak{m}} F_f^x(Y)e^{i\gamma(Y)}\,dY\,d\gamma\,dxM.$$

by the Fourier Inversion Theorem this equals

$$\int_{G/M} F_f^x(\mathbf{0})dxM = \int_{G/M}\int_{\mathfrak{m}} f(e^{x\cdot P})e^{i\ell(P)}\,dP\,dxM.$$

$\square$

When G is a connected nilpotent group, the notions of Harish-Chandra character and Kirillov character coincide as distributions over $\mathcal{S}(G)$, and therefore if follows from the Equation (2.1) that Kirillov's character has to be a positive distribution. We are now ready to prove this result directly and without giving any reference to the underlying representation.

Theorem 3.2.13. *Let G be a connected (but not necessarily simply connected) nilpotent Lie group and and let $f \in \mathcal{S}(G)$ be any Schwartz function. Then*

$$\chi_\ell(f * f^*) \geq 0$$

for any ℓ in a coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$.

Proof. For convenience we switch to the Lie group M in the equation (3.9) by choosing $p \in M$ such that $P = \log p$. Then

$$\chi_\ell(f) = \int_{G/M}\int_{M} f(xpx^{-1})e^{i\ell(\log p)}\,dp\,dxM$$

since $\exp_{*,X} = e^X\left(\frac{\mathrm{id} - e^{-\operatorname{ad} X}}{\operatorname{ad} X}\right)$ and $\det\left(\frac{\mathrm{id} - e^{-\operatorname{ad} X}}{\operatorname{ad} X}\right) = 1$ by nilpotency of $\operatorname{ad} X$.

Thus,

$$
\begin{aligned}
\chi_\ell(f * f^*) &= \int_{G/M} \int_M f * f^*(xpx^{-1}) e^{i\ell(\log p)}\, dp\, dxM \\
&= \int_{G/M} \left(\int_M \int_G f(xpx^{-1}g) f^*(g^{-1}) e^{i\ell(\log p)}\, dg\, dp \right) dxM \\
&= \int_{G/M} \left(\int_M \int_G f(xpx^{-1}g) \overline{f(g)} e^{i\ell(\log p)}\, dg\, dp \right) dxM.
\end{aligned}
$$

Applying the change of variable $g \mapsto xg^{-1}$ to the innermost integral and using the unimodularity of G the last integral simplifies to

$$
\int_{G/M} \left(\int_M \int_G f(xpg^{-1}) \overline{f(xg^{-1})} e^{i\ell(\log p)}\, dg\, dp \right) dxM.
$$

Next we apply the quotient integral formula to G to decompose the measure over M and G/M

$$
\int_{G/M} \left(\int_M \int_{G/M} \int_M f(xpq^{-1}y^{-1}) \overline{f(xq^{-1}y^{-1})} e^{i\ell(\log p)}\, dq\, dyM\, dp \right) dxM.
$$

Finally we change the order of the two integrations in the middle and use the change of variable $p \mapsto p^{-1}q$, and by unimodularity of M we find

$$
\begin{aligned}
&\chi_\ell(f * f^*) \\
&\quad = \int_{G/M} \int_{G/M} \int_M \int_M f(xp^{-1}y^{-1}) \overline{f(xq^{-1}y^{-1})} e^{i\ell\left(\log p^{-1}q\right)}\, dq\, dp\, dyM\, dxM \\
&\quad = \int_{G/M} \int_{G/M} \left| \int_M f(xp^{-1}y^{-1}) e^{-i\ell(\log p)}\, dp \right|^2 dyM\, dxM \geq 0.
\end{aligned}
$$

$\square$

Chapter 4

Compact Lie Groups

Localization theorems of Atiyah–Bott [2] and Berline–Vergne [4] that can be regarded as generalizations of the *exact stationary phase method* due to Duistermaat and Heckman [7] express integrals of equivariant forms as sums over fixed points and provide a method for proving Kirillov's character formula for compact Lie groups by showing its equivalence to the Weyl character formula.

In this chapter we study the positivity of Kirillov's character formula for compact Lie groups, and as before we start with the simplest non-abelian example, in this case the special unitary group SU(2). Of course, validity of Kirillov's character formula for compact Lie groups implies its positivity over the integral coadjoint orbits.

4.1 SU(2)

4.1.1 Coadjoint Orbits

Let us temporarily assume that G is a semisimple Lie group. Since semisimplicity is equivalent to non-degeneracy of the *Cartan-Killing form*

$$\begin{aligned} \kappa\colon \mathfrak{g} \times \mathfrak{g} &\to \mathbb{R} \\ (X, Y) &\mapsto \operatorname{Tr}(\operatorname{ad} X \circ \operatorname{ad} Y) \end{aligned}$$

we can identify $\mathfrak{g}$ and $\mathfrak{g}^*$ via the pairing

$$\begin{aligned} \iota\colon \mathfrak{g} &\to \mathfrak{g}^* \\ X &\mapsto \kappa_X \text{ where } \kappa_X(Y) = \kappa(X, Y). \end{aligned}$$

Then for any $g \in G$ we have

$$\begin{aligned} \kappa_{\mathrm{Ad}(g)X}(Y) &= \kappa(\mathrm{Ad}(g)X, Y) \\ &= \kappa(X, \mathrm{Ad}(g^{-1})Y) && \text{since } \kappa \text{ is Ad-invariant} \\ &= \kappa_X(\mathrm{Ad}(g^{-1})Y) \\ &= \mathrm{Ad}^*(g)(\kappa_X)(Y) \end{aligned}$$

and thus we see that for a semisimple Lie group G the coadjoint orbits in $\mathfrak{g}^*$ correspond to the Adjoint orbits of G in $\mathfrak{g}$. This is reflected in the commutativity of the following diagram:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota"] \arrow[d, "\mathrm{Ad}(g)"'] & \mathfrak{g}^* \arrow[d, "\mathrm{Ad}^*(g)"] \\ \mathfrak{g} \arrow[r, "\iota"'] & \mathfrak{g}^* \end{tikzcd}$$

Example 4.1.1. For $G = \mathrm{SU}(2)$ the covering map $\Phi\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$ and the Cartan-Killing form give the following equivariant identifications

$$\mathfrak{su}(2)^* \cong \mathfrak{su}(2) \cong \mathfrak{so}(3)$$

and therefore the coadjoint orbits of SU(2) are in correspondence with the Adjoint orbits of SO(3) which are concentric spheres in $\mathbb{R}^3$.

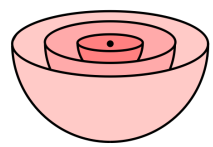

Figure 4.1: Integral coadjoint orbits for the group SU(2)

To define the norm of a Lie algebra element $X \in \mathfrak{su}(2)$ we think of X as a two-by-two Hermitian matrix of trace zero

$$X = \begin{bmatrix} it & z \\ -\overline{z} & -it \end{bmatrix} \quad t \in \mathbb{R}, z \in \mathbb{C}$$

and let $\|X\| = (t^2 + \|z\|)^{1/2}$. We shall use this normalization in the future without further comment.

4.1.2 The Symplectic Structure of the Coadjoint Orbits

Let the group SU(2) act on $\mathfrak{su}(2)^*$ by the coadjoint action. For a coadjoint orbit $\mathcal{O} \subset \mathfrak{su}(2)^*$ and $\xi \in \mathfrak{su}(2)$ we define the *vector field* $\xi^{\mathcal{O}}$ generated by ξ as usual:

$$\begin{aligned} \xi^{\mathcal{O}}_f &= \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}^*(\exp t\xi) f, \quad \text{for } f \in \mathcal{O} \\ &= \mathrm{ad}^*(\xi) f = -f([\xi, \cdot]) \in T_f\mathcal{O}. \end{aligned}$$

Now let i, j, and k be the standard basis vectors for $(\mathbb{R}^3, \times) \cong (\mathfrak{su}(2), [\cdot,\cdot])$ satisfying the cyclic relations

$$[i, j] = k, [j, k] = i, [k, i] = j$$

and assume momentarily that $\mathcal{O} \subset (\mathbb{R}^3)^*$ and $f = ai^* + bj^* + ck^* \in \mathcal{O}$. Then for the Lie algebra elements

$$\xi = \xi_1 i + \xi_2 j + \xi_3 k$$

and

$$\eta = \eta_1 i + \eta_2 j + \eta_3 k$$

one computes

$$f([\xi, \eta]) = \xi_1\eta_2 c - \xi_1\eta_3 b + \xi_2\eta_3 a - \xi_2\eta_1 c + \xi_3\eta_1 b - \xi_3\eta_2 a$$

and therefore

$$\begin{aligned} \sigma_f(\xi_f^{\mathcal{O}}, \eta_f^{\mathcal{O}}) &= f([\xi, \eta]) \\ &= \xi_1\eta_2 c - \xi_1\eta_3 b + \xi_2\eta_3 a - \xi_2\eta_1 c + \xi_3\eta_1 b - \xi_3\eta_2 a. \end{aligned} \tag{4.1}$$

On the other hand, writing out $\xi_f^{\mathcal{O}}$ and $\eta_f^{\mathcal{O}}$ in terms of basis vectors in $\mathfrak{h}^*$ we find

$$\xi_f^{\mathcal{O}} = (\xi_2 c - \xi_3 b)i^* + (\xi_3 a - \xi_1 c)j^* + (\xi_1 b - \xi_2 a)k^* \tag{4.2}$$

$$\eta_f^{\mathcal{O}} = (\eta_2 c - \eta_3 b)i^* + (\eta_3 a - \eta_1 c)j^* + (\eta_1 b - \eta_2 a)k^*. \tag{4.3}$$

For $f \in \mathcal{O}$ as before, assume $f(i) = a \neq 0$. Putting equations (4.1), (4.2), and (4.3) together and identifying $(\mathbb{R}^3)^{**}$ with $\mathbb{R}^3$ we obtain the KKS symplectic form of the coadjoint orbit $\mathcal{O} \subset (\mathbb{R}^3)^*$:

$$\sigma_f = \frac{dj \wedge dk}{a}.$$

A similar calculation at other points in $\mathcal{O}$ yields

$$\sigma = \begin{cases} \frac{dy\wedge dz}{x} & \text{for } x \neq 0, \\ \frac{dz\wedge dx}{z} & \text{for } y \neq 0, \\ \frac{dx\wedge dy}{z} & \text{for } z \neq 0 \end{cases}$$

which in spherical coordinates is equivalent to $\omega = r \sin\phi d\phi \wedge d\theta$ where r is the radius of the coadjoint orbit $\mathcal{O}$.

Remark 4.1.2. To transfer the above calculation of the KKS form from $(\mathbb{R}^3)^*$ to $\mathfrak{su}(2)^*$ we need to take into account the fact that the usual isomorphism of Lie algebras between $\mathfrak{su}(2)$ and $\mathbb{R}^3$ is not an isometry and in fact it changes the norm by a factor of 2; see, for instance, the Appendix. Therefore, the canonical KKS form on $\mathcal{O} \subset \mathfrak{su}(2)^*$ is given by

$$\omega = \begin{cases} \frac{dy\wedge dz}{2x} & \text{for } x \neq 0, \\ \frac{dz\wedge dx}{2z} & \text{for } y \neq 0, \\ \frac{dx\wedge dy}{2z} & \text{for } z \neq 0. \end{cases}$$

We shall use the Liouville form

$$\mu = \omega/2\pi \tag{4.4}$$

as the volume form on $\mathcal{O}$ when integrating functions on $\mathcal{O}$.

4.1.3 The Plancherel Theorem

According to the Plancherel Theorem, or the Fourier Inversion Theorem, if G is a compact Lie group and f is a continuously differentiable function on G, then

$$f(e_G) = \sum_{[\pi]\in\widehat{G}} (\dim\pi)\operatorname{Tr}\pi(f). \tag{4.5}$$

For $G = \mathrm{SU}(2)$, we know that the unitary dual contains representations of any dimension $n \in \mathbb{N}$ and that there are one-to-one correspondences

$$\mathbb{N} \longleftrightarrow \widehat{\mathrm{SU}(2)} \longleftrightarrow \text{ integral coadjoint orbits.}$$

If f is a smooth function on SU(2) supported sufficiently close to the identity element, then for any unitary irreducible representation $[\pi] \in \widehat{\mathrm{SU}(2)}$, the trace can be computed by Kirillov's character formula over some integral coadjoint orbit $\mathcal{O}_n$. Thus the Plancherel Formula (4.5) can be rewritten as

$$\begin{aligned} f(I) &= \sum_{[\pi]\in\widehat{\mathrm{SU}(2)}} (\dim\pi)\operatorname{Tr}\pi(f) \\ &= \sum_{n=1}^{\infty} n\chi_{\mathcal{O}_n}(f). \end{aligned} \tag{4.6}$$

We shall give a direct proof of (4.6) in Proposition 4.1.7.

The following lemma explains what happens in the above equation if the sum is taken over all of the coadjoint orbits of SU(2), rather than only the integral ones.

Proposition 4.1.3 (A Continuous Plancherel Type Formula)**.** *Let U be a sufficiently small neighborhood of $\mathbf{0} \in \mathfrak{su}(2)$ such that the restriction of the exponential function* $\exp\colon \mathfrak{su}(2) \to \mathrm{SU}(2)$ *to U is a diffeomorphism onto* $\exp(U)$*. Let*

$f \in C^\infty(\mathrm{SU}(2))$ *be a function with* $\operatorname{supp}(f) \subset \exp(U)$*. Then*

$$f(I) = 2\int_0^\infty r\chi_{\mathcal{O}_r}(f)\,dr. \tag{4.7}$$

Proof. The Liouville form of $\mathcal{O}_r$ computed in (4.4) is $1/4\pi r$ times the area form of a sphere of radius r with respect to the Euclidean metric. Thus,

$$\begin{aligned}
2\int_0^\infty r\chi_{\mathcal{O}_r}(f)\,dr &= 2\int_0^\infty \int_{\mathcal{O}_r}\int_U \sqrt{j(X)}f(\exp X)e^{i\ell(X)}\,dX\,r\,dr\,d\mu(\ell) \\
&= \frac{1}{2\pi}\int_{\mathfrak{su}(2)^*}\int_U \sqrt{j(X)}f(\exp X)e^{i\langle Y,X\rangle}\,dX\,dY \\
&= \frac{1}{2\pi}\int_{\mathfrak{su}(2)^*}\int_{\mathfrak{su}(2)} \sqrt{j(X)}f(\exp X)\mathbf{1}_U e^{i\langle Y,X\rangle}\,dX\,dY
\end{aligned}$$

by the Euclidean Fourier Inversion Theorem this equals

$$= \sqrt{j(X)}f(\exp X)\mathbf{1}_U\Big|_{X=\mathbf{0}} = f(I).$$

$\square$

We mention in passing that in the above computation $\sqrt{j(X)}$ could be any smooth function p with $p(\mathbf{0}) = 1$.

4.1.4 Positivity of Kirillov's Character

We begin with reviewing a classical theorem of H. Weyl and applying it to the group SU(2). Weyl proved this result together with some other formulae in a series of three papers in 1925–1926.

Theorem 4.1.4 (Weyl Integration Formula)**.** *Let T be a maximal torus of the compact connected Lie group G, and let invariant measures on G, T, and G/T*

be normalized as in Theorem 3.2.6. Then every $f \in C(G)$ satisfies

$$\int_G f(g)\, dg = \frac{1}{|W_G(T)|} \int_T D(t) \int_{G/T} f(gtg^{-1})\, dgT\, dt$$

where $D(t) = \det\left(\left[\mathrm{id} - \mathrm{Ad}(t^{-1})\right]\big|_{\mathfrak{g}/\mathfrak{t}}\right)$ and $W_G(T)$ is the Weyl group of G.

Example 4.1.5. In this example we compute the density function D and the Weyl group for $G = \mathrm{SU}(2)$. In this case, since the Cartan subalgebra $\mathfrak{t}$ equals $i\mathbb{R}$, with respect to the basis $\mathcal{B} = \{\, j + \mathfrak{t}, k + \mathfrak{t}\,\}$ for $\mathfrak{su}(2)/\mathfrak{t}$, we have

$$\left[\mathrm{id} - \mathrm{Ad}(t^{-1})\right]_{\mathcal{B}} = \begin{bmatrix} 1 - \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & 1 - \cos 2\theta \end{bmatrix}$$

and so

$$\begin{aligned} D(t) &= \det \begin{bmatrix} 1 - \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & 1 - \cos 2\theta \end{bmatrix} \\ &= (1 - \cos 2\theta)^2 + (\sin 2\theta)^2 = 2(1 - \cos 2\theta) = 4 \sin^2 \theta. \end{aligned}$$

The normalizer of T in $\mathrm{SU}(2)$ is the disjoint union

$$N_{\mathrm{SU}(2)}(T) = T \bigsqcup \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} T$$

thus the Weyl group $W_{\mathrm{SU}(2)}(T) = N_{\mathrm{SU}(2)}(T)/T$ is isomorphic to the two-element group $\mathbb{Z}/2\mathbb{Z}$.

Recall from Proposition 3.2.12 that when G is a nilpotent Lie group with a polarizing subgroup M, Kirillov's character for the coadjoint orbit $\mathcal{O}$ containing a functional $\ell \in \mathfrak{g}^*$ equals the distribution

$$f \mapsto \int_{G/M} \int_M f(xpx^{-1}) e^{i\ell(\log p)}\, dp\, dxM, \qquad f \in \mathcal{S}(G).$$

Now we state and prove an analogue of this proposition for SU(2) in the absence of a polarizing subgroup.

Proposition 4.1.6. *Let U be a sufficiently small neighborhood of $\mathbf{0} \in \mathfrak{su}(2)$ such that the restriction of the exponential function $\exp\colon \mathfrak{su}(2) \to \mathrm{SU}(2)$ to U is a diffeomorphism onto $\exp(U)$. Let $f \in C(\mathrm{SU}(2))$ be a function with $\mathrm{supp}(f) \subset \exp(U)$. Then*

$$\chi_{\mathcal{O}}(f) = 2 \int_{\mathrm{SU}(2)/T} \int_T f(xtx^{-1}) \sin(r\theta) \sin\theta\, dt\, dxT$$

where $t = \mathrm{diag}(e^{i\theta}, e^{-i\theta})$, $T = \{\, \mathrm{diag}(e^{i\theta}, e^{-i\theta}) \mid \theta \in \mathbb{R} \,\}$ is a maximal torus in $\mathrm{SU}(2)$, and $\mathcal{O}$ is a coadjoint orbit of radius r.

Proof. We interpret Kirillov's character formula (2.4) as the distribution

$$f \mapsto \chi_{\mathcal{O}}(f) = \int_{\mathcal{O}} \int_U \sqrt{j(X)} f(\exp X) e^{i\ell(X)}\, dX\, d\mu(\ell)$$

where the coadjoint orbit $\mathcal{O}$ passes through $\ell \in \mathfrak{g}^*$ with $\|\ell\| = r$ and μ is the Liouville form of $\mathcal{O}$ as in Equation (4.4). Explicit computation, carried out in the Appendix, shows that $\sqrt{j(X)} = \frac{\sin\|X\|}{\|X\|}$ and that $\int_{\mathcal{O}} e^{i\ell(X)}\, d\mu(\ell) = \frac{\sin r\|X\|}{\|X\|}$. Thus

$$\chi_{\mathcal{O}}(f) = \int_U f(\exp X) \frac{\sin\|X\|}{\|X\|} \frac{\sin(r\|X\|)}{\|X\|}\, dX.$$

Since $\det \exp_{*,0} = j(X) = \frac{\sin^2 \|X\|}{\|X\|^2}$ this equals

$$= \int_{\mathrm{SU}(2)} f(x) \frac{\sin(r\|\log x\|)}{\sin\|\log x\|}\, dx$$

and by the Weyl integration formula calculations in Example 4.1.5 we obtain

$$\begin{aligned}
&= \int_T \frac{1}{|W_{\mathrm{SU}(2)}(T)|} \int_{\mathrm{SU}(2)} f(xtx^{-1}) \frac{\sin(r\|\log xtx^{-1}\|)}{\sin\|\log xtx^{-1}\|} D(t)\, dxT\, dt \\
&= 2 \int_{\mathrm{SU}(2)/T} \int_T f(xtx^{-1}) \sin(r\theta) \sin\theta\, dt\, dxT.
\end{aligned}$$

The last step is justified by the equalities

(i) $\|\log xtx^{-1}\| = \|\log t\|$, and

(ii) $\sin(r|\theta|) \sin|\theta| = \sin(r\theta)\sin\theta$.

The first equality holds because $x \in \mathrm{SU}(2)$ and the second one is valid since the right-hand side is an even function of θ. □

With this proposition at hand we can give a direct proof of the Plancherel Formula (4.6).

Proposition 4.1.7 (A Plancherel Type Formula)**.** *Let U be a sufficiently small neighborhood of $\mathbf{0} \in \mathfrak{su}(2)$ such that the restriction of the exponential function* $\exp\colon \mathfrak{su}(2) \to \mathrm{SU}(2)$ *to U is a diffeomorphism onto* $\exp(U)$*. Let $f \in C^\infty(\mathrm{SU}(2))$ be a function with* $\mathrm{supp}(f) \subset \exp(U)$*. Then*

$$f(I) = \sum_{n=1}^{\infty} n\chi_{\mathcal{O}_n}(f).$$

For $f = g * g^*$ the proposition immediately implies the following positivity result:

$$0 \le \|g\|_2^2 = g * g^*(I) = \sum_{n=1}^{\infty} n\chi_{\mathcal{O}_n}(g * g^*).$$

Proof. Define $F_f(\theta) = \sin\theta \int_{\mathrm{SU}(2)/T} f(xtx^{-1})\, dxT$ where t denotes $\mathrm{diag}(e^{i\theta}, e^{-i\theta})$. Now use Proposition 4.1.6 to compute:

$$\begin{aligned}\sum_{n=1}^{\infty} n\chi_{\mathcal{O}_n}(f) &= 2\sum_{n=1}^{\infty} n \int_T \int_{\mathrm{SU}(2)/T} f(xtx^{-1}) \sin\theta \sin(n\theta)\, dxT\, dt \\ &= 2\sum_{n=1}^{\infty} \int_T F_f(\theta) n \sin(n\theta)\, dt\end{aligned}$$

integrating by parts we get

$$= 2\sum_{n=1}^{\infty} \int_T \frac{dF_f(\theta)}{d\theta} \cos(n\theta)\, dt$$

assuming that dt is normalized so that T has unit volume, this simplifies to

$$= \frac{dF_f(\theta)}{d\theta}\bigg|_{\theta=0} = f(I)$$

Note that F_f is an odd function of θ and hence its derivative is an even function. Therefore, the sum of the Fourier coefficients in the above Fourier (cosine) series is given by evaluation at 0. $\square$

In the special case of $r = 1$, $\chi_{\mathcal{O}_1}(f) = \int_{\mathrm{SU}(2)} f(x)\, dx$ and thus $\chi_{\mathcal{O}_1}$ is easily seen to be positive:

$$\chi_{\mathcal{O}_1}(f * f^*) = \int_{\mathrm{SU}(2)} \int_{\mathrm{SU}(2)} f(xy)\overline{f(y)}\, dy\, dx = \left| \int_{\mathrm{SU}(2)} f(x)\, dx \right|^2 \geq 0$$

as expected since the sphere of radius one corresponds to the trivial representation of SU(2). To discuss the positivity of $\chi_{\mathcal{O}_r}$ for other values of r we need the following lemma.

Lemma 4.1.8. *Let $c \in \mathbb{R}$ be any real number. Then the assignment*

$$f \mapsto \int_{\mathrm{SU}(2)/T} \int_T f(xtx^{-1}) e^{ic\theta} \, dt \, dxT$$

defines a positive distribution, where $t = \operatorname{diag}(e^{i\theta}, e^{-i\theta})$, and $T = \{ \operatorname{diag}(e^{i\theta}, e^{-i\theta}) \mid \theta \in \mathbb{R} \}$ is a maximal torus in $\mathrm{SU}(2)$. In fact,

$$\begin{aligned} &\int_{\mathrm{SU}(2)/T} \int_T f * f^*(xtx^{-1}) e^{ic\theta} \, dt \, dxT = \\ &\int_{\mathrm{SU}(2)/T} \int_{\mathrm{SU}(2)/T} \left| \int_T f(xt^{-1}y^{-1}) e^{-ic\theta} \, dt \right|^2 dyT \, dxT \end{aligned} \tag{4.8}$$

Proof. The proof of Theorem 3.2.13, namely the positivity of Kirillov's character for nilpotent Lie groups, gives, *mutatis mutandis*, the following. We repeat the argument and include the details for completeness and easy reference.

$$\begin{aligned} &\int_{\mathrm{SU}(2)/T} \int_T f * f^*(xtx^{-1}) e^{ic\theta} \, dt \, dxT \\ &\qquad = \int_{\mathrm{SU}(2)/T} \left(\int_T \int_{\mathrm{SU}(2)} f(xtx^{-1}g) f^*(g^{-1}) e^{ic\theta} \, dg \, dt \right) dxT \\ &\qquad = \int_{\mathrm{SU}(2)/T} \left(\int_T \int_{\mathrm{SU}(2)} f(xtx^{-1}g) \overline{f(g)} e^{ic\theta} \, dg \, dt \right) dxT. \end{aligned}$$

Applying the change of variable $g \mapsto xg^{-1}$ to the innermost integral and using the unimodularity of $\mathrm{SU}(2)$ the last integral simplifies to

$$\int_{\mathrm{SU}(2)/T} \left(\int_T \int_{\mathrm{SU}(2)} f(xtg^{-1}) \overline{f(xg^{-1})} e^{ic\theta} \, dg \, dt \right) dxT.$$

Next we apply the quotient integral formula to $\mathrm{SU}(2)$ to decompose the measure over T and $\mathrm{SU}(2)/T$

$$\int_{\mathrm{SU}(2)/T}\left(\int_T\int_{\mathrm{SU}(2)/T}\int_T f(xts^{-1}y^{-1})\overline{f(xs^{-1}y^{-1})}e^{ic\theta}\,ds\,dyT\,dt\right)\,dxT$$

where $s=\operatorname{diag}(e^{i\phi},e^{-i\phi})\in T$. Finally we change the order of the two integrations in the middle and use the change of variable $t\mapsto t^{-1}s$, and by unimodularity of T we find

$$\begin{aligned}&\int_{\mathrm{SU}(2)/T}\int_{\mathrm{SU}(2)/T}\int_T\int_T f(xt^{-1}y^{-1})\overline{f(xs^{-1}y^{-1})}e^{ic(\phi-\theta)}\,ds\,dt\,dyT\,dxT\\&=\int_{\mathrm{SU}(2)/T}\int_{\mathrm{SU}(2)/T}\left|\int_T f(xt^{-1}y^{-1})e^{-ic\theta}\,dt\right|^2\,dyT\,dxT\geq 0.\end{aligned}\tag{4.9}$$

□

It is tempting to assume that Lemma 4.1.8 combined with Proposition 4.1.6 should immediately imply the positivity of $f\mapsto\chi_{\mathcal{O}}(f)$ for any coadjoint orbit $\mathcal{O}\subset\mathfrak{g}^*$. However, since

$$\sin(r\theta)\sin\theta=\frac{e^{i(r-1)\theta}+e^{-i(r-1)\theta}}{2}-\frac{e^{i(r+1)\theta}+e^{-i(r+1)\theta}}{2}$$

if we apply Lemma 4.1.8 to this trigonometric identity directly, we will end up with the *difference* (rather than the sum) of two non-negative quantities which a priori might be negative.

Remark 4.1.9. Recall that integral coadjoint orbits in $\mathfrak{su}(2)^*$ are in one-to-one correspondence with the irreducible unitary representation of SU(2). Hence one can argue, on the basis of Atiyah-Bott and Berline-Vergne localization theorems, that since Kirillov's character formula is equivalent to the Weyl character formula when dealing with integral coadjoint orbits, the assignment $f\mapsto\chi_{\mathcal{O}}(f)$ is indeed a positive distribution. We conjecture, however, that the Kirillov's character is positive even for non-integral coadjoint orbits.

Chapter 5

Reductive Lie Groups

Rossmann has shown that for a semisimple Lie group Kirillov's formula is valid for characters of irreducible tempered representations.[1] Characters of non-tempered irreducible representations, on the other hand, usually do not arise as Fourier transforms of invariant measures on coadjoint orbits. Rossmann proposes a different type of integral formula: irreducible characters-even of non-unitary representations- can be expressed as Fourier transforms of certain cycles in coadjoint orbits of the complexified group. Schmid and Vilonen have proved some Rossmann type integral formulas in [28].

5.1 $\mathrm{SL}(2, \mathbb{R})$

Many coadjoint orbits of general Lie groups admit real polarizations and they fall into the scope of the method outlined in Chapter 3 for nilpotent Lie groups. Because of the importance of this result for the special linear group $\mathrm{SL}(2, \mathbb{R})$, we state a separate theorem for this case and we will prove it in this section.

[1] A representation is called ***tempered*** if its Harish-Chandra character is tempered as a distribution in the sense of Schwartz.

To streamline the notation, write

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{5.1}$$

for a basis of the three dimensional Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of 2×2 traceless matrices.

5.1.1 Quasi-invariant Measures on Quotient Spaces

To carry out similar computations as in the case of nilpotent Lie groups, we need to be able to decompose the Haar measure on the locally compact group G over a closed subgroup H and G/H even when G/H does not admit a G-invariant measure.

Definition 5.1.1. Let G be a locally compact group and H a closed subgroup of G. A Radon measure μ on G/H is called ***quasi-invariant*** under G if there exist functions λ_g defined on G/H such that

$$\int_{G/H} f(\Lambda_g x)\, d\mu(xH) = \int_{G/H} f(x)\lambda_g(x)\, d\mu(xH)$$

for all $g \in G$ and $f \in C_c(G/H)$ where $\Lambda_g(xH) = g^{-1}xH$.

Note that if $\lambda_g = 1$ for all $g \in G$, then μ is invariant under G, and hence quasi-invariance extends the notion of invariance.

The next theorem is standard and it generalizes Weil's Formula, Theorem 3.2.6, that we used for nilpotent Lie groups.

Theorem 5.1.2 (Mackey-Bruhat)**.** *Let G be a locally compact group. Given a closed subgroup H of G, there is always a continuous, strictly positive solution ρ of the functional equation*

$$\rho(xh) = \rho(x)\frac{\Delta_H(h)}{\Delta_G(h)}, \quad x \in G, h \in H. \tag{5.2}$$

Moreover, there is a quasi-invariant measure $d_\rho xH$ on G/H such that

$$\int_G f(g)\rho(g)\,dg = \int_{G/H}\int_H f(xh)\,dh\,d_\rho xH.$$

Remark 5.1.3. Henceforth we will always assume that our quotient spaces are equipped with measures as in Theorem 5.1.2 and we will drop the index ρ in the measure. A short calculation shows that in this situation

$$\lambda_g(xH) = \frac{\rho(gx)}{\rho(x)}, \quad x, g \in G.$$

See Definition 5.1.1 and, for instance, [25, Proposition 8.1.4]. If G/H carries a G-invariant measure, then we shall assume that $\rho \equiv 1$; this happens, for instance, when we study quotients diffeomorphic to coadjoint orbits $G/R \cong \mathcal{O}$ which are naturally equipped with a Liouville form.

Example 5.1.4. In this example we compute the ρ function for two pairs of groups consisting of the Lie group $G = \mathrm{SL}(2, \mathbb{R})$ and its closed subgroups M and R that we introduce below.

Consider the basis $\{H, X, Y\}$ for the Lie algebra $\mathfrak{g}$ as in (5.1) and let $\ell = H^*$ be the functional dual to H with respect to this basis. Then a maximal isotropic subalgebra of $\mathfrak{g}$ subordinate to ℓ is given by the upper triangular matrices

$$\mathfrak{m} = \mathrm{Span}\{H, X + Y\}.$$

Let M denote the subgroup generated by $\mathfrak{m}$. A short calculation shows that

$$M = \exp \mathfrak{m} = \left\{ \begin{bmatrix} a & * \\ 0 & a^{-1} \end{bmatrix} \middle| \ a > 0 \right\}$$

$$R = \{g \in G \mid \mathrm{Ad}^*(g)\ell = \ell\} = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \middle| \ a \in \mathbb{R}^\times \right\}.$$

To solve the functional equation (5.2) for ρ for the pairs (G, M) and (M, R) we need to know the modular functions of M and R. That's what we compute next.

$$\Delta_M(t) = \det \mathrm{Ad}(t^{-1}), \quad t \in M$$

$$\Delta_M \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} = a^{-2}, \quad a > 0.$$

Also,

$$\Delta_R(t) = \det \mathrm{Ad}(t^{-1}), \quad t \in R$$

$$\Delta_R \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} = 1, \quad a \in \mathbb{R}^\times.$$

Since G is unimodular, $\Delta_G \equiv 1$, and therefore,

$$\begin{aligned} \rho_{(G,M)}(gp) &= \rho_{(G,M)}(g)\frac{\Delta_M(p)}{\Delta_G(p)} \qquad \text{by (5.2)} \\ &= \rho_{(G,M)}(g)a^{-2}, \qquad \text{for } g \in G, \text{ and } p = \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \in M. \end{aligned} \tag{5.3}$$

The function $\rho_{(G,M)}\colon G \to \mathbb{R}$ defined by

$$\rho_{(G,M)} \begin{bmatrix} g_1 & g_2 \\ g_3 & g_4 \end{bmatrix} = \frac{1}{g_1^2 + g_3^2}$$

clearly solves the functional equation (5.3). Likewise, since $\Delta_R \equiv 1$ as we saw,

$$\begin{aligned} \rho_{(M,R)}(pr) &= \rho_{(M,R)}(p)\frac{\Delta_R(r)}{\Delta_G(r)} && \text{by (5.2)} \\ &= \rho_{(M,R)}(p)a^2, && \text{for } p \in M, \text{ and } r = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \in R. \end{aligned} \tag{5.4}$$

The function $\rho_{(M,R)}\colon M \to \mathbb{R}$ defined by

$$\rho_{(M,R)} \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} = a^2$$

is easily seen to solve the functional equation (5.4).

Now we record a corollary to Theorem 5.1.2 that will be used in proving our main result in this section.

Corollary 5.1.5. *Let G be a locally compact group with closed subgroups H and K such that $K \subset H \subset G$. Then there exist suitably normalized quasi-invariant measures on the quotient spaces such that the equality*

$$\int_{G/K} f(g)\, dgK = \int_{G/H} \int_{H/K} f(xh) \frac{\rho_{(G,K)}(xh)}{\rho_{(G,H)}(xh)\rho_{(H,K)}(h)}\, dhK\, dxH$$

holds for any $f \in C_c(G/K)$.

Proof. Let $F \in C_c(G)$ and apply Theorem 5.1.2 twice to obtain

$$\begin{aligned}
\int_G F(t)\,dt &= \int_{G/K}\int_K \frac{F(gk)}{\rho_{(G,K)}(gk)}\,dk\,dgK \\
&= \int_{G/H}\int_H \frac{F(xs)}{\rho_{(G,H)}(xs)}\,ds\,dxH \\
&= \int_{G/H}\left(\int_{H/K}\int_K \frac{F(xhk)}{\rho_{(G,H)}(xhk)\rho_{(G,K)}(hk)}\,dk\,dhK\right)dxH.
\end{aligned}$$

Fix $f \in C_c(G/K)$ and choose a function $\alpha \in C_c(G)$ with the property that for every $gK \in \operatorname{supp}(f)$ we have $\int_K \alpha(gk)\,dk = 1$, then substitute $F = f\alpha\rho_{(G,K)}$. For the standard proof of existence of such α we refer to [8, Lemma 2.47]. The above computation implies

$$\begin{aligned}
\int_{G/K} f(g)\,dgK &= \int_{G/H}\int_{H/K}\int_K \frac{f(xhk)\alpha(xhk)\rho_{(G,K)}(xhk)}{\rho_{(G,H)}(xhk)\rho_{(G,K)}(hk)}\,dk\,dhK\,dxH \\
&= \int_{G/H}\int_{H/K} f(xh)\int_K \frac{\alpha(xhk)\rho_{(G,K)}(xh)\frac{\Delta_K(k)}{\Delta_G(k)}}{\rho_{(G,H)}(xh)\frac{\Delta_H(k)}{\Delta_G(k)}\rho_{(G,K)}(h)\frac{\Delta_K(k)}{\Delta_H(k)}}\,dk\,dhK\,dxH \\
&= \int_{G/H}\int_{H/K} f(xh)\frac{\rho_{(G,K)}(xh)}{\rho_{(G,H)}(xh)\rho_{(H,K)}(h)}\,dhK\,dxH.
\end{aligned}$$

□

For a Lie group G with Lie algebra $\mathfrak{g}$ write $j_{\mathfrak{g}}$ for the Jacobian of the exponential map $\exp\colon \mathfrak{g} \to G$. According to a fundamental result in the theory of Lie groups by F. Schur (not to be confused with I. Schur)

$$j_{\mathfrak{g}}(X) = \det \exp_{*,X} = \det\left(\frac{\mathrm{id} - e^{-\operatorname{ad} X}}{\operatorname{ad} X}\right), \quad \text{where } X \in \mathfrak{g} \text{ and } \operatorname{ad} X\colon \mathfrak{g} \to \mathfrak{g}.$$

In the next example we compute this function for $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ and one of its subalgebras and indicate a a relation between the two.

Example 5.1.6. Consider the maximal isotropic subalgebra

$$\mathfrak{m} = \operatorname{Span}\{\, X + Y, H \,\}$$

subordinate to $\ell = H^*$ consisting of upper triangular matrices in $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$. For $W = aH + b(X + Y)$, the matrix of $\operatorname{ad}_{\mathfrak{m}} W \colon \mathfrak{m} \to \mathfrak{m}$ with respect to the basis $\mathcal{B}' = \{\, X + Y, H \,\}$ of $\mathfrak{m}$ is

$$[\operatorname{ad}_{\mathfrak{m}} W]_{\mathcal{B}'} = \begin{bmatrix} 2a & -2b \\ 0 & 0 \end{bmatrix}.$$

Therefore,

$$j_{\mathfrak{m}}(W) = \det\left(\frac{\operatorname{id} - e^{-\operatorname{ad}_{\mathfrak{m}} W}}{\operatorname{ad}_{\mathfrak{m}} W}\right) = \frac{1 - e^{-2a}}{2a} \tag{5.5}$$

by the Spectral Mapping Theorem applied to the eigenvalues of $\operatorname{ad}_{\mathfrak{m}} W$, namely $2a$ and 0.

Extend the basis $\mathcal{B}'$ of $\mathfrak{m}$ to the basis $\mathcal{B} = \{\, X + Y, H, X \,\}$ of $\mathfrak{g}$. Then the matrix of $\operatorname{ad}_{\mathfrak{g}} W \colon \mathfrak{g} \to \mathfrak{g}$ with respect to the basis $\mathcal{B}$ is

$$[\operatorname{ad}_{\mathfrak{g}} W]_{\mathcal{B}'} = \begin{bmatrix} 2a & -2b & 2a \\ 0 & 0 & 2b \\ 0 & 0 & -2a \end{bmatrix}.$$

Therefore,

$$j_{\mathfrak{g}}(W) = \det\left(\frac{\operatorname{id} - e^{-\operatorname{ad}_{\mathfrak{g}} W}}{\operatorname{ad}_{\mathfrak{g}} W}\right) = \left(\frac{1 - e^{-2a}}{2a}\right)\left(\frac{1 - e^{2a}}{-2a}\right) = \frac{(e^{a} - e^{-a})^2}{4a^2} \tag{5.6}$$

by the Spectral Mapping Theorem applied to the eigenvalues of $\operatorname{ad}_{\mathfrak{g}} W$, namely $2a$, 0, and $-2a$.

Equations (5.5), and (5.6) reveal an interesting relation between $j_{\mathfrak{m}}(W)$ and

$j_{\mathfrak{g}}(W)$:

$$j_{\mathfrak{g}}(W) = j_{\mathfrak{m}}^2(W)/\Delta_M(\exp W). \tag{5.7}$$

This turns out to play a key role in the proof of Theorem 5.1.7. Using the fact that $\Delta_M(\exp W) = \det \mathrm{Ad}_M(\exp -W) = \det e^{-\operatorname{ad}_{\mathfrak{m}} W}$ we obtain

$$\begin{aligned} j_{\mathfrak{m}}^2(W)/\Delta_M(\exp W) &= \det\left(\frac{\mathrm{id} - e^{-\operatorname{ad}_{\mathfrak{m}} W}}{\operatorname{ad}_{\mathfrak{m}} W}\right)^2 \det e^{\operatorname{ad}_{\mathfrak{m}} W} \\ &= \det\left(\frac{\sinh(\operatorname{ad}_{\mathfrak{m}} W/2)}{\operatorname{ad}_{\mathfrak{m}} W/2}\right)^2 \end{aligned}$$

Since $\mathrm{SL}(2,\mathbb{R})$ is unimodular, $\Delta_G \equiv 1$, and hence the left-hand side of (5.7) can be written in a similar fashion in terms of hyperbolic functions. Therefore we get the following neat reformulation of (5.7):

$$\det\left(\frac{\sinh(\operatorname{ad}_{\mathfrak{g}} W/2)}{\operatorname{ad}_{\mathfrak{g}} W/2}\right) = \det\left(\frac{\sinh(\operatorname{ad}_{\mathfrak{m}} W/2)}{\operatorname{ad}_{\mathfrak{m}} W/2}\right)^2, \qquad \text{for } W \in \mathfrak{m}.$$

5.1.2 Positivity of Kirillov's Character

Let $H \in \mathfrak{sl}(2,\mathbb{R})$ be as in (5.1). Then in a suitable coordinate system the coadjoint orbit $\mathcal{O} = \mathrm{Ad}^*(\mathrm{SL}(2,\mathbb{R}))H^*$ is a hyperboloid of one sheet.

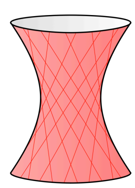

Figure 5.1: The coadjoint orbit $\mathcal{O} = \mathrm{Ad}^*(\mathrm{SL}(2,\mathbb{R}))H^*$

Before we embark on proving positivity of Kirillov's character for the orbit $\mathcal{O}$, let us mention that thanks to the existence of a real polarization, as in the case of nilpotent Lie groups, the coadjoint orbit $\mathcal{O}$ exhibits some affine structure, so we can expect Fourier analysis techniques to be very useful. In this way, the symplectic geometry can be seen as contributing in a significant way to our proof of the positivity of Kirillov's character.

Theorem 5.1.7. *Let U be a sufficiently small neighborhood of $\mathbf{0} \in \mathfrak{sl}(2,\mathbb{R})$ such that the restriction of the exponential function $\exp\colon \mathfrak{sl}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})$ to U is a diffeomorphism onto $\exp(U)$. Then Kirillov's character $f \mapsto \chi_{\mathcal{O}}(f)$ is a positive distribution for $\mathcal{O} = \mathrm{Ad}^*(\mathrm{SL}(2,\mathbb{R}))H^*$ and any $f \in C_c^\infty(\mathrm{SL}(2,\mathbb{R}))$ with $\mathrm{supp}(f) \subset \exp(U)$.*

Proof. To simplify the notation we shall write G and $\mathfrak{g}$ for $\mathrm{SL}(2,\mathbb{R})$ and $\mathfrak{sl}(2,\mathbb{R})$, respectively.

$$\chi_\ell(f) = \int_{\mathcal{O}_\ell} \sqrt{j_{\mathfrak{g}}(X)} f(\exp X)(\phi)\, dX\, d\mu(\phi)$$

using the diffeomorphism $\mathcal{O}_\ell \cong G/R_\ell$ we make a change of variables

$$= \int_{G/R_\ell} \int_{\mathfrak{g}} \sqrt{j_{\mathfrak{g}}(X)} f(\exp X) e^{ig\cdot\ell(X)}\, dX\, dgR_\ell$$

now we apply Corollary 5.1.5 with the computations carried out in Example 5.1.4

$$= \int_{G/M} \int_{M/R_\ell} \int_{\mathfrak{g}} \sqrt{j_{\mathfrak{g}}(X)} f(\exp X) e^{ixm\cdot\ell(X)} (x_{11}^2 + x_{21}^2)\, dX\, dmR_\ell\, dxM$$
$$= \int_{G/M} \int_{M/R_\ell} \int_{\mathfrak{g}/\mathfrak{m}} \int_{\mathfrak{m}} \sqrt{j_{\mathfrak{g}}(P+Y)} f(e^{x\cdot(P+Y)}) e^{im\cdot\ell(P+Y)} (x_{11}^2 + x_{21}^2)\, dP\, dP\, dmR_\ell\, dxM$$

Define $F_f^x(Y) = \int_{\mathfrak{m}} \sqrt{j_{\mathfrak{g}}(P+Y)} f(e^{x\cdot(P+Y)}) e^{im\cdot\ell(P+Y)} (x_{11}^2 + x_{21}^2)\, dP$. Then the last integral in the displayed formula equals

$$\begin{aligned}
\chi_\ell(f) &= \int_{G/M}\int_{M/R_\ell}\int_{\mathfrak{g}/\mathfrak{m}} F_f^x(Y)e^{im\cdot\ell(Y)-i\ell(Y)}\,dY\,dmR_\ell\,dxM\\
&= \int_{G/M}\int_{(\mathfrak{g}/\mathfrak{m})^*}\int_{\mathfrak{g}/\mathfrak{m}} F_f^x(Y)e^{i\gamma(Y)}\,dY\,d\gamma\,dxM\\
&= \int_{G/M} F_f^x(\mathbf{0})\,dxM\\
&= \int_{G/M}\int_{\mathfrak{m}} \sqrt{j_{\mathfrak{g}}(P)}f(e^{x\cdot P})e^{i\ell(P)}(x_{11}^2+x_{21}^2)\,dP\,dxM
\end{aligned}$$

where in the second to last equality we have used the Fourier Inversion Theorem. The change of variables formula applied to the restriction of exp to U implies that

$$\int_G \frac{f(g)}{j_{\mathfrak{g}}(\log g)}\,dg = \int_U f(\exp X)\,dX.$$

Therefore we have in fact simplified Kirillov's character formula to

$$\begin{aligned}
\chi_\ell(f) = \int_{G/M}\int_{\mathfrak{m}} \sqrt{j_{\mathfrak{g}}(P)}f(e^{x\cdot P})e^{i\ell(P)}(x_{11}^2+x_{21}^2)\,dP\,dxM =\\
\int_{G/M}\int_M \sqrt{j_{\mathfrak{g}}(\log p)}\frac{f(xpx^{-1})}{j_{\mathfrak{m}}(\log p)}(x_{11}^2+x_{21}^2)\,dp\,dxM
\end{aligned}$$

and this concludes the first part of the proof. Next we replace f with the convolution $f * f^*$ to prove the positivity of Kirillov's character:

$$\begin{aligned}
&\chi_\ell(f * f^*)\\
&= \int_{G/M}\int_M \frac{\sqrt{j_{\mathfrak{g}}(\log p)}}{j_{\mathfrak{m}}(\log p)} f * f^*(xpx^{-1})e^{i\ell(\log p)}(x_{11}^2+x_{21}^2)\,dp\,dxM
\end{aligned}$$

the next two equalities are simply obtained by writing out $f * f^*$

$$= \int_{G/M} \left(\int_M \int_G \frac{\sqrt{j_{\mathfrak{g}}(\log p)}}{j_{\mathfrak{m}}(\log p)} f(xpx^{-1}g) f^*(g^{-1}) e^{i\ell(\log p)} (x_{11}^2 + x_{21}^2)\, dg\, dp \right) dxM$$
$$= \int_{G/M} \left(\int_M \int_G \frac{\sqrt{j_{\mathfrak{g}}(\log p)}}{j_{\mathfrak{m}}(\log p)} f(xpx^{-1}g) \overline{f(g)} e^{i\ell(\log p)} (x_{11}^2 + x_{21}^2)\, dg\, dp \right) dxM.$$

Applying the change of variable $g \mapsto xg^{-1}$ to the innermost integral and using the unimodularity of G the last integral simplifies to

$$\int_{G/M} \left(\int_M \int_G \frac{\sqrt{j_{\mathfrak{g}}(\log p)}}{j_{\mathfrak{m}}(\log p)} f(xpg^{-1}) \overline{f(xg^{-1})} e^{i\ell(\log p)} (x_{11}^2 + x_{21}^2)\, dg\, dp \right) dxM.$$

Next we apply the quotient integral formula in Theorem 5.1.2 combined with the computations in Example 5.1.4 to G to decompose the measure over M and G/M

$$\int_{G/M} \left(\int_M \int_{G/M} \int_M \frac{\sqrt{j_{\mathfrak{g}}(\log p)}}{j_{\mathfrak{m}}(\log p)} f(xpq^{-1}y^{-1}) \overline{f(xq^{-1}y^{-1})} e^{i\ell(\log p)} \right.$$
$$\left. (x_{11}^2 + x_{21}^2)(y_{11}^2 + y_{21}^2) q_{11}^2\, dq\, dyM\, dp \right) dxM.$$

Finally we change the order of the two integrations in the middle and use the change of variable $p \mapsto p^{-1}q$, to find

$$\chi_\ell(f * f^*)$$
$$= \left(\int_{G/M} \int_{G/M} \int_M \int_M \frac{\sqrt{j_{\mathfrak{g}}(\log p^{-1}q)}}{j_{\mathfrak{m}}(\log p^{-1}q)} f(xp^{-1}y^{-1}) \overline{f(xq^{-1}y^{-1})} e^{i\ell\left(\log p^{-1}q\right)} \right.$$
$$\left. (x_{11}^2 + x_{21}^2)(y_{11}^2 + y_{21}^2) p_{11}^2\, dq\, dp\, dyM\, dxM \right)$$

and in the final step we make a crucial use of the Equation (5.7) about the relation between the functions $j_{\mathfrak{g}}$ and $j_{\mathfrak{m}}$ to finish the proof

$$= \Bigg(\int_{G/M} \int_{G/M} \left| \int_M f(xp^{-1}y^{-1}) e^{-i\ell(\log p)} p_{11}^2 \, dp \right|^2$$
$$(x_{11}^2 + x_{21}^2)(y_{11}^2 + y_{21}^2) dyM \, dxM \Bigg) \geq 0.$$

□

Example 5.1.8. Let E_{ij} be the 3×3 matrix with a one in the ij entry and zeros elsewhere. To simplify the notation, let us write F and G for $E_{11} - E_{22}$ and $E_{11} - E_{33}$, respectively. Then a basis for the eight-dimensional Lie algebra $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})$ is given by

$$\mathcal{B} = \{ F, G, E_{12}, E_{23}, E_{13}, E_{21}, E_{32}, E_{31} \}.$$

Consider the maximal isotropic subalgebra

$$\mathfrak{m} = \operatorname{Span}\{ F, G, E_{12}, E_{23}, E_{13} \}$$

subordinate to $\ell = F^*$ consisting of upper triangular matrices in $\mathfrak{sl}(3, \mathbb{R})$. For $W = fF + gG + e_3E_{12} + e_2E_{13} + e_1E_{23}$ in $\mathfrak{m}$, the matrix of $\operatorname{ad}_{\mathfrak{m}} W \colon \mathfrak{m} \to \mathfrak{m}$ with respect to the basis $\mathcal{B}' = \{ F, G, E_{12}, E_{23}, E_{13} \}$ of $\mathfrak{m}$ is

$$[\operatorname{ad}_{\mathfrak{m}} W]_{\mathcal{B}'} = \left[\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline -2e_3 & -e_3 & 2f + g & 0 & 0 \\ e_1 & -e_1 & 0 & g - f & 0 \\ -e_2 & -2e_2 & -e_1 & e_3 & f + 2g \end{array} \right].$$

Extend the basis $\mathcal{B}'$ of $\mathfrak{m}$ to the basis $\mathcal{B}$ of $\mathfrak{g}$. Then the matrix of $\operatorname{ad}_{\mathfrak{g}} W \colon \mathfrak{g} \to \mathfrak{g}$ with respect to the basis $\mathcal{B}$ is

$$[\operatorname{ad}_{\mathfrak{g}} W]_{\mathcal{B}} = \left[\begin{array}{c|ccc} & e_3 & 0 & -e_1 \\ & 0 & e_2 & e_1 \\ [\operatorname{ad}_{\mathfrak{m}} W]_{\mathcal{B}'} & 0 & 0 & e_2 \\ & -e_2 & 0 & 0 \\ & 0 & 0 & 0 \\ \hline & -2f-g & 0 & e_1 \\ \text{\Huge 0} & 0 & f-g & -e_3 \\ & 0 & 0 & -f-2g \end{array}\right].$$

Chapter 6

Quantization of Coadjoint Orbits

In this chapter we use a GNS-type method, based on the positivity of Kirillov's character, to construct some unitary representations of a given nilpotent Lie group.

As an evidence for the effectiveness of the orbit method in quantization we mention the following theorem due to Auslandar and Kostant.

Theorem. *Let G be a simply-connected solvable Lie group and $\mathcal{O}$ an integral coadjoint orbit. Then this orbit can be quantized to obtain an irreducible representation of G. Moreover, if G is of Type I, then any irreducible representation can be obtained in this way.*

6.1 Nilpotent Lie Groups

Let G be a connected, simply connected nilpotent Lie group. We know from Chapter 3 that Kirillov's character (1.1) defines a complex-valued distribution

$$\chi\colon C_c^\infty(G) \to \mathbb{C}$$

on $C_c^\infty(G)$. The positivity of Kirillov's character for connected nilpotent Lie groups, Theorem 3.2.13, implies that $\chi(f * f^*) \geq 0$ for each $f \in C_c^\infty(G)$ and hence the sesquilinear form

$$\langle f_1, f_2 \rangle_\chi = \chi(f_1 * f_2^*) \tag{6.1}$$

on $C_c^\infty(G)$ satisfies all the axioms for an inner product except for definiteness, that is, $\langle f, f \rangle_\chi = 0$ need not imply $f = 0$. The remaining axioms are enough to prove the Cauchy-Schwarz inequality

$$|\langle f_1, f_2 \rangle_\chi|^2 \leq \langle f_1, f_1 \rangle_\chi \langle f_2, f_2 \rangle_\chi, \tag{6.2}$$

and it follows that the set

$$N = \{\, f \in C_c^\infty(G) \mid \langle f, f \rangle_\chi = 0 \,\}$$

is a vector subspace of $C_c^\infty(G)$. The formula

$$\langle f_1 + N, f_2 + N \rangle_\chi = \langle f_1, f_2 \rangle_\chi \tag{6.3}$$

defines an inner product on the quotient space $C_c^\infty(G)/N$ and we let $\mathcal{H}_\chi$ denote the Hilbert space completion of $C_c^\infty(G)/N$.

Lemma 6.1.1. *The left- and right-translations preserve the sesquilinear form* (6.1)*:*

$$\langle \lambda_g f_1, \lambda_g f_2 \rangle_\chi = \langle f_1, f_2 \rangle_\chi$$
$$\langle \rho_g f_1, \rho_g f_2 \rangle_\chi = \langle f_1, f_2 \rangle_\chi$$

Proof. Since G is unimodular and $f_1 * f_2^*(x) = \int_G f_1(xy)\overline{f_2(y)}\, dy$, the right-translation invariance of the sesquilinear form is clear. On the other hand,

$$\begin{aligned} \lambda_g f_1 * (\lambda_g f_2)^*(x) &= \int_G f_1(g^{-1}xy)\overline{f_2(g^{-1}y)}\, dy \\ &= \int_G f_1(g^{-1}xgy)\overline{f_2(y)}\, dy \end{aligned}$$

and the left-translation invariance of the sesquilinear form, namely,

$$\langle \lambda_g f_1, \lambda_g f_2 \rangle_\chi = \langle f_1, f_2 \rangle_\chi$$

follows from the conjugation invariance of the character formula. □

Corollary 6.1.2. *The operators*

$$\begin{aligned} &L_g \colon \mathcal{H}_\chi \to \mathcal{H}_\chi, & &L_g[f] = [\lambda_g f] \\ &R_g \colon \mathcal{H}_\chi \to \mathcal{H}_\chi, & &R_g[f] = [\rho_g f] \end{aligned}$$

give rise to the unitary representations $(L, \mathcal{H}_\chi)$ *and* $(R, \mathcal{H}_\chi)$ *of the group* G*.*

Proof. The well definedness and unitarity of the operators L_g and R_g are immediate from the Cauchy-Schwarz inequality (6.2) and Lemma 6.1.1, respectively. Moreover, the strong continuity of the left and right regular representations $(\lambda, C_c^\infty(G))$ and $(\rho, C_c^\infty(G))$ at $g = e_G$ is transferred to $(L, \mathcal{H}_\chi)$ and $(R, \mathcal{H}_\chi)$ via the inner product formula (6.3) and this completes the proof. □

Chapter 7

Future Directions

7.1 Compact Lie Groups

7.1.1 Positivity of Kirillov's Character

Lemma 7.1.1. *Let χ be a distribution of the compact group G satisfying the relations*

(a) $\chi^* = \chi$, and

(b) $\chi = c\,\chi * \chi$ for some $c > 0$.

Then χ is a positive distribution.

Proof. Let f be a test function and compute:

$$\langle \chi, f * f^* \rangle = \langle \chi * f^*, f^* \rangle$$

using the assumptions (a) and (b) this equals

$$
\begin{aligned}
&= c\langle \chi * \chi * f^*, f^* \rangle \\
&= c\langle \chi * f^*, \chi * f^* \rangle = c \int_G |(\chi * f^*)(g)|^2 \, dg \geq 0.
\end{aligned}
$$

□

Examples of distributions satisfying the conditions of the above lemma can be found in abundance by considering the characters of irreducible representations of G. To be more precise, let π and σ be irreducible unitary representations of a compact group G with characters χ_π and χ_σ, respectively. Then the Schur orthogonality relations [20] immediately imply that

$$\chi_\pi^* = \chi_\pi, \tag{7.1a}$$

$$\chi_\pi * \chi_\sigma = \begin{cases} 0 & \text{if } \pi \not\sim \sigma, \\ \frac{1}{\dim \pi} \chi_\pi & \text{if } \pi \sim \sigma \end{cases} \tag{7.1b}$$

and hence $(\dim \pi)\chi_\pi$ is an idempotent in $C(G)$. Now if we regard $\chi = \chi_\pi$ as a distribution over $C^\infty(G)$ we get a positive distribution by Lemma 7.1.1. This suggests using Lemma 7.1.1 for proving positivity of Kirillov's character.

Lemma 7.1.2. *Kirillov's character $\chi_\mathcal{O}$ satisfies the condition* (a) *of Lemma 7.1.1, namely $\chi_\mathcal{O}^* = \chi_\mathcal{O}$.*

Proof. We have to show that $\chi_\mathcal{O}^* = \chi_\mathcal{O}$, where $\langle \chi_\mathcal{O}^*, f \rangle = \overline{\langle \chi_\mathcal{O}, f^* \rangle}$ and as usual

$$\langle \chi_\mathcal{O}, f \rangle = \int_\mathcal{O} \int_U \sqrt{j(X)} f(\exp X) e^{i\ell(X)} \, dX \, d\mu(\ell).$$

This is clear since $f^*(x) = \overline{f(x^{-1})}$, and moreover $j(X) = \det\left(\frac{\sinh(\operatorname{ad} X/2)}{\operatorname{ad} X/2}\right)$ is an even function of X. □

However the problem of computing $\chi_{\mathcal{O}} * \chi_{\mathcal{O}}$ for integral coadjoint orbits of a compact Lie group seems to be computationally involved.

Here we sketch another approach to proving positivity of Kirillov's character. By the Plancherel Theorem for compact Lie groups,

$$\sum_{\text{integral } \mathcal{O}} (\dim \mathcal{O})\chi_{\mathcal{O}} = \delta_e.$$

We proved this in Proposition 4.1.7 for SU(2) by a direct computation. However since the right-hand side of this equality, namely the Dirac delta function, is a positive distribution, one might try to show the positivity of each individual $\chi_{\mathcal{O}}$ by choosing test functions for which only one of the summands on the left-hand side is nonzero.

7.1.2 Quantization of Coadjoint Orbits

An obvious obstacle in applying the GNS-type method that we used for nilpotent Lie groups in Chapter 6 for other classes of Lie groups is that, generally speaking, the exponential map is no longer a diffeomorphism and hence Kirillov's character formula is only a local formula. Therefore, we need to be able to extend Kirillov's character χ from

$$S = \{\, f \in C^\infty(G) \mid \operatorname{supp}(f) \subset V \,\}$$

to a positive distribution over $C^\infty(G)$.

The next lemma shows that χ is a bounded linear functional on S and so one can hope to be able to extend χ to a $C^\infty(G)$ in a norm-preserving way using the tools from Operator Theory.[1]

[1]Note that such an extension exists and is given by the Weyl character formula, but with this approach we are not allowed to take the positivity of Weyl's character for granted.

Lemma 7.1.3. *let G be a compact Lie group. Then*

$$|\chi(f)| \leq C \cdot \|f\|_{L^1(G)}$$

for all functions $f \in C^\infty(G)$ with $\operatorname{supp}(f) \subset V$ where $V = \exp U$ and U is an open neighborhood of $\mathbf{0}$ in $\mathfrak{g}$ over which the exponential map is a diffeomorphism.

Proof. The coadjoint orbits of G are compact being the continuous images of the compact space G under Ad^*. Therefore,

$$\begin{aligned}
|\chi(f)| &= \left| \int_{\mathcal{O}} \int_U \sqrt{j(X)} f(\exp X) e^{i\ell(X)}\, dX\, d\omega_{\mathcal{O}} \right| \\
&\leq \mathrm{Vol}(\mathcal{O}) \int_{\mathfrak{g}} \left| \sqrt{j(X)} f(\exp X) \right| dX \\
&\leq \mathrm{Vol}(\mathcal{O}) \int_V \left| \frac{f(x)}{\sqrt{j(\log x)}} \right| dx \\
&\leq C \cdot \|f\|_{L^1(G)}
\end{aligned}$$

where $C = \mathrm{Vol}(\mathcal{O}) \cdot \sup_{X \in U} |j(X)^{-1/2}|$. ∎

7.2 Possible Extensions of Theorem 5.1.7

A basic concept in symplectic geometry is polarization and in particular Lagrangian fibration. Recall that for a symplectic manifold W, a ***Lagrangian fibration*** $p\colon W \to B$ is a fibration such that every fiber is a Lagrangian submanifold of W. In this Lie theoretic context the map $p\colon G/R \to G/M$ gives a Lagrangian fibration for the isotropy and polarizing subgroups R and M of G.

The following "Theorem" is a plausible extension of Theorem 5.1.7 that we proved for $\mathrm{SL}(2, \mathbb{R})$.

Target Theorem 1. *Let G be a unimodular Lie group with Lie algebra $\mathfrak{g}$ and $\psi \in \mathfrak{g}^*$. Then Kirillov's character formula is positive over the coadjoint orbit $\mathcal{O}$*

of ψ for any function $f \in C_c^\infty(G)$ supported in a sufficiently small neighborhood of G if

(a) $\mathfrak{g}^*$ *admits a polarizing subalgebra* $\mathfrak{m}$ *subordinate to* ψ*,*

(b) The fibers of the Lagrangian map $p\colon G/R \to G/M$ are globally affine spaces, and

(c) $[\mathfrak{m}, \mathfrak{m}] \cap \mathfrak{r}_\psi = \{\mathbf{0}\}$.

The last condition above is about the "smallness" of $\mathfrak{r}_\psi$. This kind of restriction is definitely necessary since Kirillov's formula is valid only for "generic" orbits. For instance, in the case of point orbits, that is, when $\mathfrak{g} = \mathfrak{r}_\psi$ and the radical is as big as possible, the formula (1.1) is not correct.

It can certainly be the case that the group has no real polarization. This happens, for instance, for any compact group; even for SU(2) as $\mathfrak{su}(2)$ has no 2-dimensional subalgebra.

One might try to apply a similar method to complex polarizations; this is justified by the following result of Dixmier (see for instance [18]): for a Lie algebra $\mathfrak{g}$ over an algebraically closed field K the set of functionals in $\mathfrak{g}^*$ which admit a K-polarization contains a Zariski open subset and hence is dense in $\mathfrak{g}^*$.

The above fact about complex polarizations of complex Lie algebras suggests that some variant of the following Target Theorem should be true.

Target Theorem 2. *Let G be a complex unimodular Lie group with Lie algebra $\mathfrak{g}$. Then there exists a dense subset D of $\mathfrak{g}^*$ such that Kirillov's character formula* (1.1) *is positive for the coadjoint orbit $\mathcal{O}$ of any $\psi \in D$ and for any function $f \in C_c^\infty(G)$ supported in a sufficiently small neighborhood of G provided that*

(a) The fibers of the Lagrangian map $p\colon G/R \to G/M$ are globally affine spaces, and

(b) The coadjoint orbit $\mathcal{O}$ satisfies a maximality condition.

Appendix

A Some Computations

A.1 $j_{\mathfrak{su}(2)}$

For $X = ai + bj + ck$, the matrix of $\operatorname{ad} X \colon \mathbb{R}^3 \to \mathbb{R}^3$ with respect to the basis $\mathcal{B} = \{ i, j, k \}$ of $(\mathbb{R}^3, \times) \cong (\mathfrak{su}(2), [\cdot, \cdot])$ is

$$[\operatorname{ad} X]_{\mathcal{B}} = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}.$$

Therefore, letting $\|X\| = 2\sqrt{a^2 + b^2 + c^2}$,

$$\begin{aligned} j_{\mathfrak{su}(2)}(X) &= \det\left(\frac{\operatorname{id} - e^{-\operatorname{ad} X}}{\operatorname{ad} X}\right) = \left(\frac{1 - e^{-i\sqrt{a^2+b^2+c^2}}}{i\sqrt{a^2+b^2+c^2}}\right)\left(\frac{1 - e^{i\sqrt{a^2+b^2+c^2}}}{-i\sqrt{a^2+b^2+c^2}}\right) \\ &= \frac{\sin^2 \|X\|}{\|X\|^2} \end{aligned}$$

by the Spectral Mapping Theorem applied to the eigenvalues of $\operatorname{ad} X$, namely 0 and $\pm i\sqrt{a^2 + b^2 + c^2}$, over the field of complex numbers.

A.2 Fourier Transform of the Surface Measure of the Sphere

To prove Proposition 4.1.6 one needs to compute the integral $I(X) = \int_{\mathcal{O}} e^{i\ell(X)}\, d\mu(\ell)$, or equivalently

$$\frac{1}{4\pi r}\int_{S_r^2} e^{i\langle X,Y\rangle}\, d\sigma(Y)$$

where the surface measure σ is normalized so that S^2 has total mass one. We can assume without loss of generality that $X = \|X\|e_3$ using the rotational symmetry of σ. Thus by considering the spherical coordinates, we find

$$\begin{aligned} I(X) = \frac{1}{4\pi r}\int_{S_r^2} e^{i\|X\|Y_3}\, d\sigma(Y) &= \frac{1}{4\pi r}\int_0^{2\pi}\int_0^{\pi} e^{i\|X\|r\cos\phi} r^2 \sin\phi\, d\phi\, d\theta \\ &= \frac{\sin r\|X\|}{\|X\|}. \end{aligned}$$

This is an instance of the *exact* stationary phase method and the decay and asymptotics of $\hat{\mu}(X) = I(X)$ as $\|X\| \to \infty$ are studied in the literature.

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