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C*-ALGEBRAS IN KIRILLOV THEORY

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Abstract

In this dissertation, I study connections between C^*-algebra theory and the representation theory of simply connected nilpotent Lie groups, specifically the Kirillov theory. If G is a connected, simply connected, nilpotent Lie group, then Kirillov's famous theorem gives an explicit bijection between the set of equivalence classes of unitary irreducible representations of G and the set of coadjoint orbits of G in $\mathfrak{g}^*$, the dual of the Lie algebra of G. Inspired by this, and by the Plancherel theorem, I introduce two new C^*-algebras. The first is an algebra of operators on $L^2(G)$ and the second is an algebra of operators on $L^2(\mathfrak{g}^*)$. I formulate the conjecture that they are isomorphic, prove the conjecture in the case of Heisenberg group (which is the crucial building block for general nilpotent Lie groups) and examine the prospects for the conjecture in other cases.

Contents

Acknowledgements **vi**

1 Introduction **1**

2 Preliminaries **17**

2.1 Harmonic analysis 17

2.2 C*-algebras 33

2.3 Kirillov theory 40

3 A conjectural Kirillov isomorphism in C*-algebra theory **53**

3.1 Definition of $A(G)$ 54

3.2 Definition of $A(\mathfrak{g}^*)$ 56

3.3 The conjecture 58

3.4 The case of the Heisenberg group 60

3.5 Other nilpotent groups 68

4 Structure of $A(G)$ for the Heisenberg group **76**

4.1 The unitary dual of a product 76

4.2 The diagonal in the unitary dual of a product 77

4.3 The spectrum of $A(G)$ for the Heisenberg group 80

5 Structure of $A(\mathfrak{g}^*)$ for the Heisenberg group **87**

5.1 Alternative definition of $A(\mathfrak{g}^*)$ 87

5.2 Representations of $G \ltimes \mathfrak{g}$ 90

5.3 Spectrum of $A(\mathfrak{g}^*)$ for the Heisenberg group 94

6 Summary and directions for future work **106**

Bibliography **108**

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Chapter 1

Introduction

This thesis is about representation theory, specifically about the "Kirillov theory" (also known as the "orbit method" or the "method of coadjoint orbits"). This method provides an explicit bijection between the **unitary dual space** $\widehat{G}$ of G, i.e., the set of equivalence classes of unitary irreducible representations of G, and the **coadjoint orbits** $\mathfrak{g}^*/\operatorname{Ad}^*(G)$ of the action of G on the dual vector space $\mathfrak{g}^*$ of its Lie algebra $\mathfrak{g}$.

The method was originally discovered in 1962 by Alexandre Aleksandrovich Kirillov for connected and simply connected nilpotent Lie groups in his paper [16]. Since then Kirillov theory has been extended with modifications to many other classes of groups. For example, it is true after modification for compact Lie groups by the Cartan-Weyl highest weight theory, and for simply connected type I solvable groups by the results of Auslander and Kostant in [1] and [2]. See [30] for more details on how Kirillov theory has been developed in other settings.

Nilpotent Lie Groups and Algebras

A finite-dimensional real Lie algebra $\mathfrak{g}$ is said to be **nilpotent** if its descending central series eventually vanishes. That is, we inductively define

$$\mathfrak{g}^{(1)} = \mathfrak{g}, \quad \mathfrak{g}^{(n+1)} = [\mathfrak{g}, \mathfrak{g}^{(n)}],$$

and $\mathfrak{g}$ is nilpotent if $\mathfrak{g}^{(n+1)} = 0$ for some n. Each $\mathfrak{g}^{(k)}$ is an ideal in $\mathfrak{g}$. For the least n such that $\mathfrak{g}^{(n+1)} = 0$ but $\mathfrak{g}^{(n)} \neq 0$ we call $\mathfrak{g}$ an **n-step nilpotent** Lie algebra. In other words, $\mathfrak{g}$ is n-step nilpotent if and only if all brackets of at least $n+1$ elements of $\mathfrak{g}$ are 0 but not all brackets of n elements are zero. Observe that an n-step nilpotent Lie algebra has a nonempty center since $\mathfrak{g}^{(n)}$ is central.

There is another way to think about nilpotent Lie algebras. Consider the ascending central series, which is defined inductively by

$$\mathfrak{g}_{(1)} = \mathfrak{z}(\mathfrak{g}), \quad \mathfrak{g}_{(i)} = \{X \in \mathfrak{g} : [X, \mathfrak{g}] \subseteq \mathfrak{g}_{(i-1)}\},$$

where $\mathfrak{z}(\mathfrak{g})$ is the center of $\mathfrak{g}$. Then $\mathfrak{g}$ is an n-step nilpotent Lie algebra if and only if $\mathfrak{g} = \mathfrak{g}_{(n)}$ and $\mathfrak{g} \neq \mathfrak{g}_{(n-1)}$.

A **nilpotent Lie group** G is one whose Lie algebra $\mathfrak{g}$ is nilpotent. There is also an equivalent definition of nilpotent Lie group. The descending central series for the group G is defined by

$$G^{(1)} = G, \quad G^{(j+1)} = [G, G^{(j)}]$$

where the bracket $[H, K]$ is a subgroup generated by all $hkh^{-1}k^{-1}, h \in H, k \in K$. G is said to be nilpotent if $G^{(j)} = \{e\}$ for some j. It is true that $G^{(j)}$ are Lie subgroups of G and the Lie algebra of $G^{(j)}$ is $\mathfrak{g}^{(j)}$. See §1.2 in [5] for more details.

The Lie algebra $\mathfrak{n}_n$ of strictly upper triangular $n \times n$ matices is an $(n-1)$-step nilpotent Lie algebra of dimension $n(n-1)/2$ and its center is one-dimensional. We write the corresponding Lie group with Lie algebra $\mathfrak{n}_n$ as N_n. Every connected, simply connected nilpotent Lie group has a faithful embedding as a closed subgroup of N_n for some n. See §1.2 in [5] for more details.

If $\mathfrak{g}$ is nilpotent, so are all subalgebras and quotient algebras of $\mathfrak{g}$. But it is false that if $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ such that $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are nilpotent, then $\mathfrak{g}$ is nilpotent.

Every connected nilpotent Lie group is **unimodular** (a left Haar measure is also right-invariant). If a nilpotent Lie group G is both connected and simply connected, then the exponential map $\exp : \mathfrak{g} \to G$

$$\exp X = \sum_{j=0}^{\infty} \frac{X^j}{j!}$$

is an analytic diffeomorphism taking Lebesgue measure on $\mathfrak{g}$ to a left-invariant Haar measure on G. See §1.2 in [5] for more details.

For any matrix Lie group G with Lie algebra $\mathfrak{g}$, the low order terms of the Campbell-Baker-Hausdorff formula are explicitly given by

$$\begin{aligned} \log(\exp X \cdot \exp Y) = X + Y &+ \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] \\ &- \frac{1}{48}[Y, [X, [X, Y]]] - \frac{1}{48}[X, [Y, [X, Y]]] \\ &+ (\text{commutators in five or more terms}), \end{aligned}$$

where $X, Y \in \mathfrak{g}$. Note that this series ends after a finite number of steps for a nilpotent Lie group. See Chapter 3 in [14] for more details.

Another important example in the theory of nilpotent Lie groups and algebras is the $(2n+1)$-dimensional **Heisenberg algebra**, which is denoted by

$\mathfrak{h}_n$. It is the Lie algebra with vector space basis $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\}$ whose pairwise brackets are all zero except for

$$[X_i, Y_i] = Z, \qquad 1 \leqslant i \leqslant n.$$

The matrix realization for Heisenberg algebra is to let $zZ + \sum_{i=1}^{n}(x_i X_i + y_i Y_i)$ correspond to $(n+2) \times (n+2)$ matrix

$$\begin{pmatrix} 0 & x_1 & \ldots & x_n & z \\ & \cdot & & & y_1 \\ & & \cdot & & \vdots \\ & & & \cdot & y_n \\ 0 & & & & 0 \end{pmatrix}.$$

Note that the Heisenberg algebra is a two-step nilpotent Lie algebra and $\mathfrak{n}_3 = \mathfrak{h}_1$. Heisenberg Lie algebra got its name because it has a structure reflecting Heisenberg's canonical commutation relations in quantum mechanics (see [9] for more details). In [15], Roger Howe explains the role of Heisenberg group in harmonic analysis.

In 1958, Dixmier [7] listed all nilpotent Lie algebras (up to isomorphism) of dimension $\leqslant 5$. We give this list below and write only the nonzero brackets $[x_i, x_j]$ for a basis $x_1, \ldots, x_n$ where $i < j$.

- Dimension 1: only the Abelian one;
- Dimension 2: only the Abelian one;
- Dimension 3: Only the Hesienberg Lie algebra $\mathfrak{h}_1$ with the bracket $[x_1, x_2] = x_3$;
- Dimension 4: Only $\mathfrak{k}_3$ with the brackets $[x_1, x_2] = x_3, [x_1, x_3] = x_4$;
- Dimension 5: there is six algebras
  - $\mathfrak{g}_{5,1}$ with the brackets $[x_1, x_2] = x_5, [x_3, x_4] = x_5$;

- $\mathfrak{g}_{5,2}$ with the brackets $[x_1, x_2] = x_4, [x_1, x_3] = x_5$;
- $\mathfrak{g}_{5,3}$ with the brackets $[x_1, x_2] = x_4, [x_1, x_4] = x_5, [x_2, x_3] = x_5$;
- $\mathfrak{g}_{5,4}$ with the brackets $[x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5$;
- $\mathfrak{g}_{5,5}$ with the brackets $[x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5$;
- $\mathfrak{g}_{5,6}$ with the brackets $[x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5, [x_2, x_3] = x_5$;

Nielsen [23] extended the list to dimension $\leqslant 6$ in 1983. There are uncountably many isomorphism classes for dimension $\geqslant 7$; see Skjelbred [28]. The complete classification of nilpotent Lie algebras of dimension 7 over algebraically closed fields and $\mathbb{R}$ can be found in Gong [13]. Seeley provided all 7-dimensional nilpotent Lie algebras over $\mathbb{C}$ in [27].

Representation Theory

A **unitary representation** of a topological group G is a continuous homomorphism π from G into the group $U(\mathcal{H}_\pi)$ of unitary operators on some nonzero Hilbert space $\mathcal{H}_\pi$ (the unitary group is equipped with the strong operator topology). So it is a map $\pi : G \to U(\mathcal{H}_\pi)$ satisfying

$$\pi(xy) = \pi(x)\pi(y), \quad \pi(x^{-1}) = \pi(x)^{-1} = \pi(x)^*,$$

and $x \mapsto \pi(x)u$ is continous from G to $\mathcal{H}_\pi$ for any $u \in \mathcal{H}_\pi$. The dimension of $\mathcal{H}_\pi$ is called the **dimension** or **degree** of π.

Basic examples of unitary representation are the left and right regular representations. Let G be a locally compact group endowed with a left Haar measure ds. For every $s \in G$, let $\lambda(s)$ be the operator in $L^2(G)$ defined by

$$(\lambda(s)f)(x) = f(s^{-1}x),$$

where $f \in L^2(G), x \in G$. λ is called the **left regular representation** of G in $L^2(G)$. Similarly the **right regular representation** ρ is defined by

$$(\rho(s)f)(x) = \Delta(s)^{\frac{1}{2}} f(xs),$$

for every $s \in G$ where $\Delta : G \to \mathbb{R}$ is the modular function of G.

Let π_1 and π_2 be unitary representations of G. An **intertwining operator** for π_1 and π_2 is a bounded linear map $T : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$ such that for all $x \in G$, we have

$$T\pi_1(x) = \pi_2(x)T.$$

The unitary representations π_1 and π_2 are **(unitarily) equivalent** if there is a unitary intertwining operator. We write $\pi_1 \simeq \pi_2$. It is true that $\lambda \simeq \rho$. We therefore speak sometimes of the regular representation of G, without specifying left or right.

If the only closed subspaces of $\mathcal{H}_\pi$ invariant under $\pi(G)$ are 0 and $\mathcal{H}_\pi$, then π is said to be **(topologically) irreducible**. We write the set of equivalence classes of unitary irreducible representations of G by $\widehat{G}$ and the equivalence class of a unitary irreducible representation π by $[\pi]$. Every locally compact group G has enough unitary irreducible representations to separate points of G:

1.1 Theorem (The Gelfand-Raikov Theorem)**.** *If x and y are two distinct points in a locally compact group G, there is a unitary irreducible representation π of G such that $\pi(x) \neq \pi(y)$.*

One of the basic questions in representation theory is to describe all the unitary irreducible reprentations of G, up to equivalence. Many mathematicans have been working on the determination of $\widehat{G}$ for various types of groups G. The answer depends strongly on the structure of G. For connected, simply connected nilpotent Lie groups, a beautiful description of the dual $\widehat{G}$ was found by Kirillov, and will be described next.

Kirillov Theory

Let G be a nilpotent Lie group with Lie algebra $\mathfrak{g}$, and denote the vector space dual of $\mathfrak{g}$ by $\mathfrak{g}^*$. There is an action of G on $\mathfrak{g}^*$ called the **contragradient** of the adjoint action, or the coadjoint action Ad^*. It is defined as follows:

$$((\mathrm{Ad}^* x)l)(Y) = l((\mathrm{Ad}\, x^{-1})Y), \quad Y \in \mathfrak{g}, l \in \mathfrak{g}^*, \text{ and } x \in G.$$

A **coadjoint orbit**, denoted by $\mathfrak{g}^*/\mathrm{Ad}^*(G)$, is an orbit of the Lie group G in the space $\mathfrak{g}^*$.

The **stabilizer** subgroup of G associated with $l \in \mathfrak{g}^*$ is defined by

$$R_l = \{x \in G \mid (\mathrm{Ad}^* x)l = l\}.$$

If G is a nilpotent Lie group and $l \in \mathfrak{g}^*$, then the stabilizer R_l is connected and

$$r_l = \{X \in \mathfrak{g} \mid (\mathrm{ad}^* X)l = 0\}$$

is its Lie algebra and $R_l = \exp r_l$. See Lemma 1.3.1 in [5] for the proof. There is also another way to describe this Lie subalgebra r_l of Lie algebra $\mathfrak{g}$. Each $l \in \mathfrak{g}^*$, defines a natural bilinear form $B_l : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$,

$$B_l(X, Y) = l([X, Y]), \quad X, Y \in \mathfrak{g}.$$

Then B_l is skew-symmetric, i.e., $B_l(X, Y) = -B_l(Y, X)$. The **radical** of B_l is, by definition, $\{Y \in \mathfrak{g} : B_l(X, Y) = 0 \text{ for all } X \in \mathfrak{g}\}$ which coincides with r_l. If $\mathfrak{g}$ is a Lie algebra and $l \in \mathfrak{g}^*$, its radical r_l has even codimension in $\mathfrak{g}$. Hence coadjoint orbis are of even dimension. See Lemma 1.3.2 in [5] for proof.

Now let V be a real vector space with a skew-symmetric symplectic bilinear form B. Its **isotropic subspaces** W are those such that $B(w, w') = 0$, for all $w, w' \in W$. It is known that maximal isotropic subspaces exist and have the same dimension:

$$\frac{1}{2}\dim(V/\operatorname{rad} B) + \dim(\operatorname{rad} B) = \frac{1}{2}(\dim V + \dim \operatorname{rad} B),$$

where $\operatorname{rad} B := \{x \in V : B(x, y) = 0 \text{ for all } y \in V\}$. In other words, they have codimension $k = \frac{1}{2}(\dim V/\operatorname{rad} B)$ and lie halfway between the radical $\operatorname{rad} B$ and V. Note that $\operatorname{rad} B$ is contained in them.

In particular, if $V = \mathfrak{g}, l \in \mathfrak{g}^*$, and $B = B_l$ then we call subalgebras $\mathfrak{m} \subseteq \mathfrak{g}$ that are isotropic for B_l and have codimension $k = \frac{1}{2}(\dim V/\operatorname{rad} B)$ as **polarizing subalgebras** or **maximal subordinate subalgebras** for l. Given $l \in \mathfrak{g}^*$, the radical r_l is uniquely determined, but there can be many polarizing subalgebras $\mathfrak{m}$ as we shall see in examples below. There is no systematic way to construct all of $\mathfrak{m}$ and this is one of the complications of the theory. See §1.3 in [5] for more details.

Let G be a connected, simply connected nilpotent Lie group. We choose a polarizing subalgebra $\mathfrak{m}$ for l and let $M = \exp \mathfrak{m}$. This is a Lie subgroup of G. Then $l|_{\mathfrak{m}}$ is an algebra homomorphism from $\mathfrak{m}$ to $\mathbb{R}$. Define a map from M to S^1 by

$$\chi_{l,M}(\exp Y) = e^{2\pi i l(Y)},$$

for $Y \in \mathfrak{m}$. The isotropy condition insures that $\chi_{l,M}$ defines a 1-dimensional representation of the subgroup $M = \exp \mathfrak{m}$. Maximal isotropy insures that $\chi_{l,M}$ induces to an irreducible unitary representation of G. Then we write $\pi_{l,M} = \operatorname{Ind}_M^G \chi_{l,M}$ as the induced representation from M to G. It can be shown that $\pi_{l,M}$ is independent of the choice of maximal subordinate subalgebra up to equivalence. So we may write π_l for $\pi_{l,M}$. See §2.2 and

§2.4 in [5] for more details.

1.2 Theorem. *Let G be a connected, simply connected, nilpotent Lie group. Given $l \in \mathfrak{g}^*$. The is equivalence class of π_l depends only on the orbit of l under the coadjoint action and the map*

$$\mathcal{O}_l \mapsto [\pi_l], \quad \mathfrak{g}^*/\operatorname{Ad}^*(G) \to \widehat{G}$$

is a bijection from the set of coadjoint orbits to the dual $\widehat{G}$ of G. It is a homeomorphism with respect to the natural quotient topology on the set of orbits and the Fell topology on $\widehat{G}$.

This theorem is proved by Kirillov in [16], except for the fact that the map $[\pi_l] \mapsto \mathcal{O}_l$ is a continuous map, which was proved by Ian Brown 11 years later in [4] in 1973. One of the main ingredients for the proof of the above theorem is Kirillov's Lemma. In many theorems concerning Kirillov theory, proofs are given by induction on the dimension of the group G (or equivalently on the dimension of $\mathfrak{g}$.) If the center $\mathfrak{z}(\mathfrak{g})$ has dimension greater than 1, we usually factor out a proper ideal $\mathfrak{h} \subseteq \mathfrak{z}(\mathfrak{g})$ and solve the problem on a quotient algebra of lower dimension. If $\dim \mathfrak{z}(\mathfrak{g}) = 1$, we use Kirillov's Lemma. This lemma shows that, for any noncommutative nilpotent Lie algebra with one-dimensional center, it contains a specially placed Heisenberg Lie subalgebra.

1.3 Lemma (Kirillov's Lemma)**.** *Let $\mathfrak{g}$ be a noncommutative nilpotent Lie algebra whose center $\mathfrak{z}(\mathfrak{g})$ is one-dimensional. Then $\mathfrak{g}$ can be decomposed as*

$$\mathfrak{g} = \mathbb{R}Z \oplus \mathbb{R}Y \oplus \mathbb{R}X \oplus \mathfrak{w} = \mathbb{R}X \oplus \mathfrak{g}_0$$

a vector space direct sum, where

1. $\mathbb{R}Z = \mathfrak{z}(\mathfrak{g})$;

2. $[X, Y] = Z$;

3. $\mathfrak{g}_0 = \mathbb{R}Y \oplus \mathbb{R}Z \oplus \mathfrak{w}$ *is the centralizer of* Y*, and an ideal in* $\mathfrak{g}$.

Let's consider the Kirillov Theory in the case of Heisenberg Lie group H_n. It can be shown that the coadjoint orbits in $\mathfrak{h}_n^*$ are

1. The singleton sets in $Z^\perp$ where $Z^\perp = \{l \mid l(Z) = 0\}$;

2. The hyperplanes $\lambda Z^* + Z^\perp$ for $\lambda \neq 0$, where

$$Z^*(Z) = 1, \quad Z^*(X) = 0, \quad Z^*(Y) = 0.$$

The complete classification of the irreducible unitary representations of Heisenberg Lie group H_n is the consequence of the famous Stone-von Neumann theorem (See Rosenberg in [26] or Prasad in [25] for more details on Stone-von Neumann theorem). The proof of the classification can be found in Corwin and Greenleaf [5] or in Folland [9], [10].

1.4 Proposition. *Every irreducible unitary representation of* H_n *is unitarily equivalent to one and only one of the following representations:*

1. *For* $l \in Z^\perp$*, the one-dimensional representation is*

$$\pi(\exp(W)) = e^{2\pi i l(W)}, \quad \textit{for } W \in \mathfrak{h}_n;$$

2. *For* $\lambda \in \mathbb{R}\backslash\{0\}$*, the corresponding representation on* $L^2(\mathbb{R}^n)$ *is defined by*

$$[\pi(\exp(xX))f](t) = f(t + \lambda x)$$
$$[\pi(\exp(yY))f](t) = e^{2\pi i y \cdot t} f(t)$$
$$[\pi(\exp(zZ))f](t) = e^{2\pi i \lambda z} f(t),$$

where $\{X, Y, Z\}$ *is a basis of Heisenberg Lie algebra* $\mathfrak{h}_n$.

Hilbert-Schmidt Operators

Let $\mathcal{H}$ be a Hilbert space and $T \in B(\mathcal{H})$ where $B(\mathcal{H})$ is the set of all bounded linear operators on $\mathcal{H}$. T is called **positive** if $\langle Tu, u\rangle \geqslant 0$ for all $u \in \mathcal{H}$. Note that T^*T is always a positive operator for any bounded operator T on $\mathcal{H}$, so we can define

$$|T| = \sqrt{T^*T}.$$

Now let $\mathcal{H}$ be a separable Hilbert space. Suppose T is a positive operator on $\mathcal{H}$. We say that T is **trace-class** if T has an orthonormal eigenbasis $\{e_n\}$ with eigenvalues $\{\lambda_n\}$ (where $\lambda_n \geqslant 0$), and $\sum \lambda_n < \infty$.

An operator $T \in B(\mathcal{H})$ is called **trace-class** if the positive operator $|T|$ is trace class. If T is trace-class, we set

$$\operatorname{tr}(T) = \sum \langle Tx_n, x_n\rangle,$$

where $\{x_n\}$ is any orthonormal basis for $\mathcal{H}$. The set of trace-class operators is a two-sided $*$-ideal in $B(\mathcal{H})$ (see p. 258 in [10] for proof).

An operator $T \in B(\mathcal{H})$ is called **Hilbert-Schmidt** if T^*T is trace-class. The inner product on the the space of all Hilbert-Schmidt operators can be defined as

$$\langle T, S\rangle = \operatorname{tr}(S^*T) = \sum \langle Tx_n, Sx_n\rangle,$$

for any orthonormal basis $\{x_n\}$. It can be shown that T is Hilbert-Schmidt if and only if $\|T\| = \sum \|Tx_n\|^2 < \infty$ for any orthonormal basis $\{x_n\}$.

The set of Hilbert-Schmidt operators form a two-sided $*$-ideal in $B(\mathcal{H})$. It is indeed a Hilbert space, which can be shown to be naturally isometrically isomorphic to the tensor product of Hilbert spaces $\mathcal{H} \otimes \mathcal{H}^*$, where $\mathcal{H}^*$ is the dual vector space of $\mathcal{H}$. See §7.3 and appendix 2 in [10] for more details.

Direct Integral Decompositions

Let $(A, \mathcal{M})$ be a measurable space equipped with a σ-algebra $\mathcal{M}$. A family $\{\mathcal{H}_\sigma\}_{\sigma \in A}$ of nonzero separable Hilbert spaces indexed by A is called a **field** of Hilbert spaces over A, and an element of $\prod_{\alpha \in A} \mathcal{H}_\alpha$ - that is, a map f on A such that $f(\alpha) \in \mathcal{H}_\alpha$ for each α - is called a **vector field** on A. We write the inner product and norm on $\mathcal{H}_\alpha$ by $\langle \cdot, \cdot \rangle_\alpha$ and $\| \cdot \|_\alpha$. A **measurable field of Hilbert spaces** over A is a field of Hilbert spaces $\{\mathcal{H}_\alpha\}$ together with a countable set $\{e_j\}_1^\infty$ of vector fields with the following properties:

1. the functions $\alpha \mapsto \langle e_j(\alpha), e_k(\alpha) \rangle_\alpha$ are measurable for all j, k,
2. the linear span of $\{e_j(\alpha)\}_1^\infty$ is dense in $\mathcal{H}_\alpha$ for each α.

Given a measurable field of Hilbert spaces $\{\mathcal{H}_\alpha\}, \{e_j\}$ on A, a vector field f on A will be called **measurable** if $\langle f(\alpha), e_j(\alpha) \rangle_\alpha$ is a measurable function on A for each j.

Now let $\{\mathcal{H}_\alpha\}, \{e_j\}$ be a measurable field of Hilbert spaces over A, and suppose μ is a measure on A. The **direct integral** of the spaces $\mathcal{H}_\alpha$ with respect to μ, denoted by

$$\int^{\oplus} \mathcal{H}_\alpha \, d\mu(\alpha),$$

is the space of measurable vector fields f on A such that

$$\|f\|^2 = \int \|f(\alpha)\|_\alpha^2 \, d\mu(\alpha) < \infty.$$

$\int^{\oplus} \mathcal{H}_\alpha \, d\mu(\alpha)$ is a Hilbert space with inner product

$$\langle f, g \rangle = \int \langle f(\alpha), g(\alpha) \rangle_\alpha \, d\mu(\alpha).$$

See §7.4 in [10] for more details. Let's consider the following two examples:

1.5 Example. Consider the quotient space $\mathfrak{g}^*/\operatorname{Ad}^*(G)$ with the quotient topology and measurable space structure. According to Kirillov (see §2.7 in [19]), under the identification

$$\widehat{G} \cong \mathfrak{g}^*/\operatorname{Ad}^*(G),$$

the Plancherel measure (see Theorem 2.5 below) gives a measure σ on $\mathfrak{g}^*/\operatorname{Ad}^*(G)$ such that

$$\int_{\mathfrak{g}^*} f(l)\, dl = \int_{\mathfrak{g}^*/\operatorname{Ad}^*(G)} \left(\int_{\mathcal{O}} f(l)\, d\mu(l) \right) d\sigma(\mathcal{O}),$$

for all nonnegative measurable functions f on $\mathfrak{g}^*$. We obtain a measurable field of Hilbert spaces on $\mathfrak{g}^*/\mathcal{O}$ by setting $\mathcal{H}_{\mathcal{O}} = L^2(\mathcal{O})$ and by constructing measurable sections from all measurable functions $f : \mathfrak{g}^* \to \mathbb{C}$ with $\int_{\mathfrak{g}^*} |f(l)|^2\, dl < \infty$ by defining $f_{\mathcal{O}} = f|_{\mathcal{O}}$.

1.6 Example. The unitary dual $\widehat{G}$ has the structure of a measurable space. The spaces of Hilbert-Schmidt operators on $\mathcal{H}_\pi (\pi \in \widehat{G})$ have a structure of a measurable field and there is a direct integral decomposition (as in Theorem 2.5 below).

The Abstract Plancherel Theorem

The Plancherel Theorem is, roughly speaking, the decomposition of the biregular representation of locally compact group G as a direct integral of irreducible representations. Now let G be a unimodular locally compact group. Recall that the right ρ and the left λ regular representations of G on $L^2(G)$ are defined by

$$\rho(x)f(y) = f(yx), \quad \lambda(x)f(y) = f(x^{-1}y).$$

We can combine both representations to obtain a new representation β of $G \times G$ on $L^2(G)$. It is defined by

$$\beta(x, y)f(z) = f(x^{-1}zy).$$

and we call β the **bi-regular representation** of G (although it is actually a representation of $G \times G$).

For a second countable, unimodular, postliminal group G, there is a measurable field of irreducible representations over $\widehat{G}$ such that the representation at the point $p \in \widehat{G}$ belongs to equivalence class p (see §7.5 in [10]). Hence we identify the points of $\widehat{G}$ with the representations in this field. Therefore, if $f \in L^1(G)$, we define the Fourier transform of f to be the measurable field of operators over $\widehat{G}$ given by

$$\widehat{f}(\pi) = \int f(x)\pi(x^{-1})\, dx.$$

We want to think $\widehat{f}(\pi)$ as an element of $H_\pi \otimes \mathcal{H}_{\overline{\pi}}$. However, $H_\pi \otimes \mathcal{H}_{\overline{\pi}}$ can be identified with the space of Hilbert-Schmidt operators (see Appendix 2 and §7.3 in [10]). It turns out that $\widehat{f}(\pi)$ is Hilbert-Schmidt for a suitably large class of f's and π's. The proof of the theorem below may be found in §18.8 in [8] or §4.3 in [5].

1.7 Theorem (The Abstract Plancherel Theorem)**.** *Suppose G is a second countable, unimodular, postliminal group. There is a measure μ, called* ***Plancherel measure****, on $\widehat{G}$, uniquely determined once the Haar measure on G is fixed, with the following properties. The Fourier transform $f \mapsto \widehat{f}$ maps $L^1(G) \cap L^2(G)$ into $\int^{\oplus} \mathcal{H}_\pi \otimes \mathcal{H}_{\overline{\pi}}\, d\mu(\pi)$, and it extends to a unitary map from $L^2(G)$ onto $\int^{\oplus} \mathcal{H}_\pi \otimes \mathcal{H}_{\overline{\pi}}\, d\mu(\pi)$ that intertwines the bi-regular representation β with $\int^{\oplus} \pi \otimes \overline{\pi}\, d\mu(\pi)$.*

The C*-algebras $A(G)$ and $A(\mathfrak{g}^*)$

The first C*-algebra, denoted $A(G)$, acts on the Hilbert space $L^2(G)$ and is related to the decomposition of $L^2(G)$ into irreducible representations. Recall that the group G acts on $L^2(G)$ by both the left and right regular representations, and we shall use both in the construction of $A(G)$. Namely we shall define $A(G)$ to be the image of the group C^*-algebra of $G \times G$ under the "bi-regular" representation of $G \times G$ on $L^2(G)$.

According to the abstract Plancherel theorem, the Hilbert space $L^2(G)$ decomposes as an integral of the Hilbert spaces $\mathcal{H}_\pi \otimes \mathcal{H}_{\overline{\pi}}$, as π ranges over the irreducible unitary representations of G. Here $\mathcal{H}_{\overline{\pi}}$ is the complex conjugate of the Hilbert space $\mathcal{H}_\pi$ associated to the representation π. We see in this way that the C^*-algebra $A(G)$ is very closely related to the unitary representation theory of G.

The second C^*-algebra is denoted $A(\mathfrak{g}^*)$ and it acts on the Hilbert space $L^2(\mathfrak{g}^*)$. It is the image under the natural coadjoint/multiplication action of the crossed product C^*-algebra $C^*(G, C_0(\mathfrak{g}^*))$. Another way of describing $A(\mathfrak{g}^*)$ is to use the Fourier transform isomorphism

$$L^2(\mathfrak{g}) \cong L^2(\mathfrak{g}^*),$$

under which $A(\mathfrak{g}^*)$ becomes conjugate to the natural coadjoint/translation representation of the group $G \ltimes \mathfrak{g}$ on $L^2(\mathfrak{g})$.

As we saw from the example above, the Hilbert space $L^2(\mathfrak{g}^*)$ decomposes as an integral of the Hilbert spaces $L^2(\mathcal{O})$ associated to the coadjoint orbits $\mathcal{O}$ of G. This decomposition is also a decomposition of $L^2(\mathfrak{g}^*)$ into irreducible representations of $A(\mathfrak{g}^*)$. We see in this way that the C^*-algebra $A(\mathfrak{g}^*)$ is very closely related to the space of coadjoint orbits in $\mathfrak{g}^*$.

The above remarks motivate the conjecture that $A(G)$ and $A(\mathfrak{g}^*)$ are isomorphic.

Outline and Main Results

In Chapter 2, we breifly review the the essential definitions, theorems and examples related to harmonic analysis, C*-algebras and Kirillov theory for nilpotent Lie group.

In Chapter 3, we explicitly define the two C*-algebras $A(G)$ and $A(\mathfrak{g}^*)$. Then we will prove the conjecture for the Heisenberg group H_1 using the exponential map, which gives a unitary isomorphism from $L^2(G)$ to $L^2(\mathfrak{g})$. We will show that the unitary isomorphism conjugates $A(G)$ onto $A(\mathfrak{g}^*)$. Unfortunately this is not true for all nilpotent groups, as we shall show at the end of this chapter.

In Chapter 4, we investigate the structure of the C^*-algebra $A(G)$ in the case of the Heisenberg group H_1. We shall determine the spectrum of the C^*-algebra, and use the information we find to decompose $A(G)$ as a C^*-algebra extension. We have already proved the main conjecture for the Heisenberg group in previous chapter, but since the method of proof does not carry over to more general nilpotent groups, we hope that a better understanding of the structure of $A(G)$ may eventually lead to proofs of the conjecture in more cases.

In Chapter 5, we shall do the same as in Chapter 4 for the C^*-algebra $A(\mathfrak{g}^*)$.

The conjecture that $A(G)$ is isomorphic to $A(\mathfrak{g}^*)$ remains open, in general. A goal of future work will be to determine decompositions of the two C*-algebras by ideas in cases beyond the Heisenberg group, with the aim of using this structure to prove the isomorphism.

Chapter 2

Preliminaries

2.1 Harmonic analysis

Locally Compact Groups

A topological space X is called **locally compact** if every point of X has a compact neighbourhood. So a compact space is automatically locally compact. A **topological group** is a group G together with a topology on G under which the group operations are continuous; that is, the maps

$$(x, y) \mapsto xy \quad \text{and} \quad x \mapsto x^{-1}$$

are both continous for all $x, y \in G$. Any group is a topological group under the discrete topology and in fact is locally compact. The theory of locally compact groups therefore embraces the theory of ordinary groups.

Let H be a subgroup of the topological group G. Let q be the natural quotient map of G onto the left coset space G/H. We impose the quotient topology on G/H; that is, $U \subset G/H$ is open if and only if $q^{-1}(U)$ is open in G. Then q maps open sets in G to open sets in G/H; that is $q(U)$ is open if U is open. It is known that if G is a T_1 space then G is Hausdorff. If G is not a T_1 space then $G/\overline{\{e\}}$ is a Hausdorff topological group.

It is because of this result, it is often unnecessary (for example in representation theory) to assume that a topological group is Hausdorff. With this in mind, therefore the term **locally compact group** will from now on mean a topological group whose topology is locally compact and Haudorff. Common examples are $\mathbb{R}^n, (\mathbb{R}/\mathbb{Z})^n$ and all closed subgroups of the group, $GL(n, \mathbb{R})$, of invertible linear transformation of $\mathbb{R}^n$.
References: §2.1 in [10], §28 in [21].

The Haar Measure

Let G be a second countable, locally compact group (every connected Lie group is second countable). A **left (invariant) Haar measure** on G is a nonzero Borel measure μ on G with $\mu(xE) = \mu(E)$ for every Borel subset E of G and every $x \in G$. A similar definition is made for right (invariant) Haar measure. If μ is a Borel measure on the locally compact group G, and we define $\tilde{\mu}(E) = \mu(E^{-1})$, then it is easy to show that μ is a left Haar measure if and only if $\tilde{\mu}$ is a right Haar measure. The question to ask is about the existence and uniqueness of Haar measure. There is a simple argument to prove that every Lie group has a Haar measure. It is also well known but not easy to prove that

1. Every locally compact group G posseses a left Haar measure μ.

2. Left Haar measure is unique up to multiplication by a positive constant; that is if λ and μ are left Haar measures on G, there exists $c \in (0, \infty)$ such that $\mu = c\lambda$.

3. $\mu(U) > 0$ for every nonempty Borel open subset U of G.

4. G is compact if and only if $\mu(G) < \infty$. Therefore Haar measure is customarily normalized so that $\mu(G) = 1$ ($\int 1 \, d\mu = 1$) for any compact

groups G.

Some examples of Haar measure for various groups are listed below.

1. A Haar measure on $(\mathbb{R}, +)$ which takes the value 1 on the unit interval $[0,1]$ is equal to the restriction of Lebesgue measure to the Borel subsets of $\mathbb{R}$.

2. $\dfrac{dx}{|x|}$ is a Haar measure on the multiplicative group $\mathbb{R}^{\times}$.

3. $\dfrac{dx\,dy}{x^2+y^2}$ is a Haar measure on the multiplicative group $\mathbb{C}^{\times}$ with coordinates $z = x + iy$.

4. $\dfrac{dT}{|\det T|^n}$ is a left and right Haar measure on $GL(n,\mathbb{R})$ where dT is Lebesgue measure on the vector space of all real $n \times n$ matrices.

5. $dx\,dy\,dz$ is a Haar measure on the 3-dimensional Heisenberg group

$$H_1 = \left\{ \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

In general, left invariant Haar measure need not be also right invariant. Now let G be a locally compact group with left Haar measure λ. Fix $x \in G$, we define $\lambda_x(E) = \lambda(Ex)$ for any Borel subset E of G. Then λ_x is again a left Haar measure. Therefore by the uniqueness of left Haar measure, there is a positive number $\Delta(x)$ such that $\lambda_x = \Delta(x)\lambda$. The function $\Delta : G \to (0, \infty)$ is called the **modular function** of G. It is the fact that a modular function is a continuous homomorphism from G to the multiplicative group $\mathbb{R}\backslash\{0\}$; that is

$$\Delta(e) = 1, \quad \Delta(xy) = \Delta(x)\Delta(y), \quad \Delta(x^{-1}) = \Delta(x)^{-1}, \text{ for all } x, y \in G.$$

If we write $d\lambda(x)$ by dx, we have

$$d(yx) = dx, \quad d(xy) = \Delta(y)\,dx, \quad dx^{-1} = \Delta(x)^{-1}\,dx, \text{ for all } y \in G.$$

In other words,

$$\begin{aligned}\int_G f(yx)\,dx &= \int_G f(x)\,dx \\ \int_G f(x^{-1})\,dx &= \int_G \frac{f(x)}{\Delta(x)}\,dx \\ \Delta(y)\int_G f(x)\,dx &= \int_G f(x)\,d(xy) = \int_G f(xy^{-1})\,dx,\end{aligned}$$

for all $y \in G$ and $f \in C_c(G)$, where $C_c(G)$ is the set of all continous functions on G with a compact support.

G is said to be **unimodular**, if $\Delta \equiv 1$, that is, if left Haar measure is also right Haar measure. Abelian groups, discrete groups, compact groups and connected nilpotent Lie groups are all unimodular.

References: §2.2 in [10], §28- §30 in [21].

Group Algebras

From now on we assume that each locally compact group G is equipped with a fixed left Haar measure λ and we write dx for $d\lambda(x)$ when there is no ambiguity. Let $L^1(G)$ be the space of absolutely integrable functions with respect to left Haar measure. If $f, g \in L^1(G)$, their convolution product is defined by

$$\begin{aligned}(f * g)(x) &= \int f(y)g(y^{-1}x)\,dy \\ &= \int f(xy)g(y^{-1})\,dy \\ &= \int f(y^{-1})g(yx)\Delta(y^{-1})\,dy \\ &= \int f(xy^{-1})g(y)\Delta(y^{-1})\,dy.\end{aligned}$$

Using Fubini's theorem, one can show that this integral is absolutely convergent for almost every x and that

$$\|f * g\|_1 \leqslant \|f\|_1 \|g\|_1.$$

Note that when G is unimodular, the term $\Delta(y^{-1})$ disappears. Convolution can be extended from L^1 to other L^p and we have the following result: Suppose $1 \leqslant p \leqslant \infty$, $f \in L^1$ and $g \in L^p$, we have $f * g \in L^p$ and

$$\|f * g\|_p \leqslant \|f\|_1 \|g\|_p.$$

If G is unimodular, the same conclusions hold with $f * g$ replaced by $g * f$. We shall use this when $p = 2$.

The involution is defined by the relation

$$f^*(x) = \Delta(x)^{-1}\overline{f(x^{-1})}.$$

It is the fact that $f^* \in L^1$ if and only if $f \in L^1$ and $\|f^*\|_1 = \|f\|_1$. With the convolution and the involution defined above, $L^1(G)$ becomes a Banach $*$-algebra, called the L^1 **group algebra** of G. Moreover, it is well-known that

1. G is Abelian if and only if $L^1(G)$ is Abelian.

2. If G is separable then so is $L^1(G)$.

3. $L^1(G)$ has an approximate identity (See the definition in "C*-algebras" section) Specifically, let U be any neighborhood of e in G, and let ψ_U be a function such that $\operatorname{supp}\psi_U$ is compact and contained in U, $\psi_U \geqslant 0, \psi_U(x^{-1}) = \psi_U(x)$, and $\int \psi_U = 1$. Then $\{\psi_U \mid U\}$ is an approximate identify for $L^1(G)$, i.e.,

$$\|\psi_U * f - f\|_1 \to 0, \quad \|f * \psi_U - f\|_1 \to 0$$

for all $f \in L^1(G)$.

4. $L^1(G)$ has an identity if and only if G is discrete. If e is an identity of G then the function $\delta_e(x)$ which is 1 at $x = e$ and zero else where is an identify of $L^1(G)$. In this case, we write $l^1(G)$. It is well-known that the group algebra $\mathbb{C}G$ consisting of all finite sum $\sum_{s \in G} \alpha_x \delta_s$ form a dense subalgebra of $l^1(G)$.

References: §13.2 in [8], §2.5 in [10], §31 in [21].

Representation Theory

Let G be a locally compact group. A **unitary representation** of G is a homomorphism π from G into the group $U(\mathcal{H}_\pi)$ of unitary operators on some nonzero Hilbert space $\mathcal{H}_\pi$ that is continous in the strong operator topology. In other words, a unitary representation is a map $\pi : G \to U(\mathcal{H}_\pi)$ satisfying

$$\pi(xy) = \pi(x)\pi(y), \quad \pi(x^{-1}) = \pi(x)^{-1} = \pi(x)^*,$$

and $x \mapsto \pi(x)u$ is continous from G to $\mathcal{H}_\pi$ for all $u \in \mathcal{H}_\pi$. The Hilbert space $\mathcal{H}_\pi$ is called the **representation space** of π and its dimension of $\mathcal{H}_\pi$ is called the **dimension** or **degree** of π. It is worth nothing that the strong and weak topologies coincide on $U(\mathcal{H}_\pi)$. In the above, we can therefore replace the strong topology by the weak topology throughout.

If f is a function on the topological group G and $y \in G$, then we define the **left** and **right translates** of f through y by

$$L_y(f(x)) = f(y^{-1}x), \quad R_y(f(x)) = f(xy).$$

Note that we are using y^{-1} in L_y and y in R_y to make both $y \mapsto L_y$ and $y \mapsto R_y$ group homomorhisms from G to the group of linear operators on the space of functions on G; that is $L_{xy} = L_x L_y$ and $R_{xy} = R_x R_y$.

Now fix a Haar measure on G, it is easy to check that convolutions have the following behavior under translations:

$$L_z(f * g) = (L_z f) * g, \quad R_z(f * g) = f * (R_z g).$$

Basic examples of unitary representation are the left and right regular representations arising from the action of G on itself by left or right translation. Left translations give the **left regular representation** λ of G on $L^2(G)$, which is defined by

$$(\lambda(x)f)(y) = L_x f(y) = f(x^{-1}y),$$

where $f \in L^2(G), x, y \in G$. Likewise, the right translation operators R_x define a unitary representation $\widetilde{\rho}$ on $L^2(G, \mu_r)$ where μ_r is right Haar measure on G. They can also be built into a unitary representation ρ on $L^2(G)$ with left Haar measure μ_L.

$$\begin{aligned}
(\widetilde{\rho}(x)f)(y) &= R_x f(y) = f(yx) \\
(\rho(x)f)(y) &= \Delta(x)^{\frac{1}{2}} R_x f(y) = \Delta(x)^{\frac{1}{2}} f(yx),
\end{aligned}$$

where $\Delta : G \to \mathbb{R}$ is the modular function of G. Both are called the **right regular representation** of G. If G is unimodular, both are the same.

If π_1 and π_2 are unitary representations of G, an **intertwining operator** from π_1 to π_2 is a bounded linear map $T : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$ such that for all $x \in G$, we have

$$T\pi_1(x) = \pi_2(x)T.$$

The set of all intertwining operators is denoted by $\mathcal{C}(\pi_1, \pi_2)$. If $\mathcal{C}(\pi_1, \pi_2)$ contains a unitary operator U then π_1 and π_2 are **(unitarily) equivalent** and we write $\pi_1 \simeq \pi_2$. We write $\mathcal{C}(\pi)$ for $\mathcal{C}(\pi, \pi)$. This space is called **communtant** or **centralizer** of π. It consists of all bounded operators on

$\mathcal{H}_\pi$ that commute with $\pi(x)$ for all $x \in G$. It is straightforward to check that

1. $\widetilde{\rho} \simeq \rho$ and $f \mapsto \Delta^{\frac{1}{2}} f$ is an intertwining operator.

2. $\widetilde{\rho} \simeq \lambda$ and $Uf(x) = f(x^{-1})$ is an intertwining operator.

3. $\lambda \simeq \rho$ and $f'(x) = \Delta(x)^{-\frac{1}{2}} f(x^{-1})$ is an intertwining operator.

We therefore use sometimes of the **regular representation** of G without specifying left or right.

A closed subspace $\mathcal{M}$ of $\mathcal{H}_\pi$ is called an **invariant subspace** for π if $\pi(x)\mathcal{M} \subset \mathcal{M}$ for all $x \in G$. If $\mathcal{M} \neq \{0\}$ is invariant, the restriction of π to $\mathcal{M}$,

$$\pi^{\mathcal{M}}(x) = \pi(x)|_{\mathcal{M}},$$

defines a representation of G on $\mathcal{M}$. We call it a **subrepresentation** of π.

Let π be a unitary representation of G and $u \in \mathcal{H}_\pi$, the closed linear span $\mathcal{M}_u$ of $\{\pi(x)u \mid x \in G\}$ in $\mathcal{H}_\pi$ is called the **cyclic subspace** generated by u. $\mathcal{M}_u$ is invariant under π. If $\mathcal{M}_u = \mathcal{H}_\pi$, u is called a **cyclic vector** for π. If π has a cyclic vector, π is called a **cyclic representation**. If $\{\pi_i\}_{i\in I}$ is a family of unitary representations, their **direct sum** $\oplus\pi_i$ is the repesentation π on $\mathcal{H} = \oplus\mathcal{H}_\pi$, defined by

$$\pi(x)(\sum v_i) = \sum \pi_i(x)v_i, \quad (v_i \in \mathcal{H}_{\pi_i}).$$

It is well-known that every unitary representation is a direct sum of cyclic representations. Also the following conditions are equivalent:

(i) the only closed subspaces of $\mathcal{H}_\pi$ invariant under $\pi(G)$ are 0 and $\mathcal{H}_\pi$;

(ii) every non-zero vector of $\mathcal{H}_\pi$ is a cyclic vector for π.

If these conditions are satisfied, and, in addition, $\mathcal{H}_\pi \neq 0$, then π is said to be **topologically irreducible** or simply **irreducible**. If G is Abelian, then every irreducible representation of G is one-dimensional. We denote the set of equivalence classes of unitary irreducible representations of G by $\widehat{G}$ and the equivalence class of a unitary irreducible representation π by $[\pi]$. We also have

2.1 Theorem (Schur's Lemma)**.** *Suppose π_1 and π_2 are unitary irreducible representations of G. If π_1 and π_2 are equivalent then $\mathcal{C}(\pi_1, \pi_2)$ is one-dimensional; otherwise, $\mathcal{C}(\pi_1, \pi_2) = \{0\}$.*

For a locally compact group G, it has enough unitary irreducible representations to separate points according to the Gelfand-Raikov theorem.

2.2 Theorem (The Gelfand-Raikov Theorem)**.** *If x and y are two distinct points in a locally compact group G, there is a unitary irreducible representation π of G such that $\pi(x) \neq \pi(y)$.*

The unitary irreducible representations of a locally compact group G are the building blocks of the harmonic analysis associated to G. One of the basic questions of harmonic analysis on G is to describe all the unitary irreducible representations of G, up to equivalence. Many mathematicans have been working on the determination of $\widehat{G}$ for various types of groups G. The answer depends strongly on the structure of G. For connected, simply connected nilpotent Lie groups, a beautiful description of the dual $\widehat{G}$ was found by Kirillov. We shall discuss it later in this chapter.

References: §13.1 in [8], §3.1 in [10].

Representations of a Group and its Group Algebra

When π is a unitary representation of G, it induces a representation $\widetilde{\pi}$ of $L^1(G)$ by integration: If $f \in L^1(G)$, we define the bounded operator $\widetilde{\pi}(f)$ on $\mathcal{H}_\pi$ by

$$\widetilde{\pi}(f) = \int f(x)\pi(x)\,dx.$$

This operator-valued integral is intepreted as follows: For any $u \in \mathcal{H}_\pi$, we define $\widetilde{\pi}(f)u$ by specifying its inner product with an arbitrary $v \in \mathcal{H}_\pi$, which the vector is given by

$$\langle\widetilde{\pi}(f)u, v\rangle = \int f(x)\langle\pi(x)u, v\rangle\,dx. \tag{2.1}$$

Since $\langle\pi(x)u, v\rangle$ is a bounded continuous function of $x \in G$, the integral on the right is the ordinary integral of a function in $L^1(G)$. We have $|\langle\widetilde{\pi}(f)u, v\rangle| \leqslant \|f\|_1\|u\|\|v\|$, so $\pi(f)$ is a bounded linear operator on $\mathcal{H}_\pi$ with norm

$$\|\widetilde{\pi}(f)\| \leqslant \int |f(x)|\,dx = \|f\|_1.$$

It can be shown that the map $f \mapsto \widetilde{\pi}(f)$ is a nondegenerate $*$-representation of $L^1(G)$ on $\mathcal{H}_\pi$ (means the closed linear span of $\pi(L^1(G))\mathcal{H}$ is $\mathcal{H}$). For example, if λ is the left regular representation of G, then $\widetilde{\lambda}(f)$ is simply a convolution with f on the left:

$$\widetilde{\lambda}(f)g = \int f(y)L_y(g)\,dy = f * g.$$

In this case nondegeneracy follows from the existence of approximate units. This representation is called the **left regular representation** of $L^1(G)$ in $L^2(G)$.

On the other hand, suppose $\widetilde{\pi}$ is a nondegenerate $*$-representation of $L^1(G)$ on the Hilbert space $\mathcal{H}$. Then π arises from a unique unitary represenation of G on $\mathcal{H}$ according to (2.1).

There is therefore a bijective correspondence between the collection of all unitary representations of G and the collection of all nondegenerate $*$-representations $(\pi, \mathcal{H})$ of $L^1(G)$. Indeed it can be shown further that there is a bijective correspondence between the unitary irreducible representations of G and the topologically irreducible representations of $L^1(G)$ because the topological irreducibility of the nondegenerate representation π' of $L^1(G)$ is equivalent to that of an associated unitary representation π of G.

References: Chapter VII in [6], §13.3 in [8], §3.2 in [10].

Induced Representation

Let G be a locally compact group, H a closed subgroup, $q : G \to G/H$ the canonical quotient map, σ a unitary representation of H on $\mathcal{H}_\sigma$, and $C(G, \mathcal{H}_\sigma)$ the space of continuous functions from G to $\mathcal{H}_\sigma$. Let's consider the following space of vector-valued functions:

$$\begin{aligned}\mathcal{H}_0 = \{f \in C(G, \mathcal{H}_\sigma) \mid q(\operatorname{supp} f) &\text{ is compact and} \\ &f(x\xi) = \sigma(\xi^{-1})f(x) \text{ for } x \in G, \xi \in H\}.\end{aligned}$$

Next proposition tells us that this set is nonempty and how its elements actually look like.

2.3 Proposition. *If $\alpha : G \to \mathcal{H}_\sigma$ is continuous with compact support, the the function*

$$f_\alpha(x) = \int_H \sigma(\eta)\alpha(x\eta)\, d\eta$$

belongs to $\mathcal{H}_0$ and is uniformly continuous on G. Moreover, every element of $\mathcal{H}_0$ if of the form f_α for some $\alpha \in C_c(G, \mathcal{H}_\sigma)$.

If G/H admits invariant measure μ (This is true for any unimodular group. In general, see Theorem 2.49 p. 57 in [10].), then for $f, g \in \mathcal{H}_0$,

$\langle f(x), g(x)\rangle_\sigma$ depends only on the coset $q(x)$ of x and it defines a function in $C_c(G/H)$ which can be integrated with respect to μ, and so we set

$$\langle f, g\rangle = \int_{G/H} \langle f(x), g(x)\rangle_\sigma \, d\mu(xH).$$

This is an inner product on $\mathcal{H}_0$ and it is preserved by left translations. Let $\mathcal{H}$ be the Hilbert completion of $\mathcal{H}_0$. Then the left translations of G on $\mathcal{H}_0$ ($f \mapsto L_x f$) extend to the unitary operators on $\mathcal{H}$. It is indeed a unitary representation of G. We call it the **representation induced** by σ and denote it by $\operatorname{Ind}_H^G \sigma$.

References: Chapter 6 in [10], Chapter 2 in [5], or Chapter 13 in [17].

Mackey Machine

Let G be a locally compact group and N a nontrivial closed Abelian normal subgroup of G. Let G act on N by conjugation. This induces an action of G on the dual group $\widehat{N}$, $(x, \nu) \to x\nu$, defined by

$$\langle n, x\nu\rangle = \langle x^{-1}nx, \nu\rangle \quad \text{where } x \in G, \nu \in \widehat{N}, n \in N.$$

For each $\nu \in \widehat{N}$, we denote by G_ν the **stabilizer** of ν,

$$G_\nu = \{x \in G \mid x\nu = \nu\},$$

which is a closed subgroup of G, and we denote by $\mathcal{O}_\nu$ the **orbit** of ν:

$$\mathcal{O}_\nu = \{x\nu \mid x \in G\}.$$

We shall say that G acts **regularly** on $\widehat{N}$ if the following two conditions are satisfied.

1. The orbit space is **countably separated**, that is, there is a countable family $\{E_j\}$ of G-invariant Borel sets in $\widehat{N}$ such that each orbit in $\widehat{N}$ is the intersection of all the E_j's that contain it.

2. For each $\nu \in \widehat{N}$, the natural map $xG_\nu \to x\nu$ from G/G_ν to $\mathcal{O}_\nu$ is a homeomorphism.

When G is second countable, these two conditions are actually equivalent.

Let G be a topological group. The **semidirect product** of two closed subgroups N and H if N is normal in G and the map $(n, h) \to nh$ from $N \times H$ to G is a homeomorphism; in this case we write $G = N \ltimes H$. Every element of G can be written uniquely as nh with $n \in N$ and $h \in H$, and the group law is

$$(n_1, h_1)(n_2, h_2) = (n_1[h_1 n_2 h_1^{-1}], h_1 h_2).$$

If N is Abelian, for $\nu \in \widehat{N}$ we define the **little group** H_ν associated to ν to be

$$H_\nu = G_\nu \cap H.$$

Since $G_\nu \supset N$, we then have $G_\nu = N \ltimes H_\nu$ and $H_\nu \cong G_\nu / N$. The character ν always extends to a homomorphism $\tilde{\nu} : G_\nu \to \mathbb{T}$ by the formula

$$\tilde{\nu}((n, h)) = \nu(n) = \langle n, \nu \rangle.$$

To see this, consider

$$\tilde{\nu}((n_1, h_1), (n_2, h_2)) = \tilde{\nu}((n_1[h_1 n_2 h_1^{-1}], h_1 h_2)) = \nu((n_1[h_1 n_2 h_1^{-1}]),$$

and since $h_1 \in H_\nu$,

$$\tilde{\nu}((n_1, h_1), (n_2, h_2)) = \nu(n_1 n_2) = \nu(n_1)\nu(n_2) = \tilde{\nu}((n_1, h_1))\tilde{\nu}((n_2, h_2)).$$

If $\nu \in \widehat{N}$ and ρ is an irreducible representation of H_ν, we obtain an irreducible representation of G_ν, which we denote by $\nu \otimes \rho$, by setting

$$(\nu \otimes \rho)((n, h)) = \nu(n)\rho(h),$$

and every irreducible representation σ of G_ν such that $\sigma(n) = \langle n, \nu \rangle I$ for $n \in N$ is of this form. Moreover $(\nu \otimes \rho)|_{H_\nu} = \rho$, and $\nu \otimes \rho$ is equivalent to $\nu \otimes \rho'$ if and only if ρ is equivalent to ρ'. The irreducible representation of $G = N \ltimes H$ can be completely classified in terms of irreducible representations of N (i.e., the characters $\nu \in \widehat{N}$) and the irreducible representations of their little groups H_ν by the following theorem:

2.4 Theorem. *Suppose $G = N \ltimes H$, where N is Abelian and G acts regularly on $\widehat{N}$. If $\nu \in \widehat{N}$ and ρ is an irreducible representation of H_ν, then $\mathrm{Ind}_{G_\nu}^{G}(\nu \otimes \rho)$ is an irreducible representation of G, and every irreducible representation of G is equivalent to one of this form. Moreover $\mathrm{Ind}_{G_\nu}^{G}(\nu \otimes \rho)$ and $\mathrm{Ind}_{G_{\nu'}}^{G}(\nu' \otimes \rho')$ are equivalent if and only if ν and ν' belong to the same orbit, say $\nu' = x\nu$, and $h \to \rho(h)$ and $h \to \rho(x^{-1}hx)$ are equivalent representations of H_ν.*

References: §6.6 in [10].

Hilbert-Schmidt Operators

Let $\mathcal{H}$ be a Hilbert space and $T \in B(\mathcal{H})$ where $B(\mathcal{H})$ is the set of all bounded linear operators on $\mathcal{H}$. T is called **positive** if $\langle Tu, u \rangle \geqslant 0$ for all $u \in \mathcal{H}$. Note that T^*T is always a positive operator for any bounded operator T on $\mathcal{H}$, so we can define

$$|T| = \sqrt{T^*T}.$$

Now let $\mathcal{H}$ be a separable Hilbert space. Suppose T is a positive operator on $\mathcal{H}$. We say that T is **trace-class** if T has an orthonormal eigenbasis $\{e_n\}$ with eigenvalues $\{\lambda_n\}$ (where $\lambda_n \geqslant 0$), and $\sum \lambda_n < \infty$.

An operator $T \in B(\mathcal{H})$ is called **trace-class** if the positive operator $|T|$ is trace class. If T is trace-class, we set

$$\mathrm{tr}(T) = \sum \langle Tx_n, x_n \rangle,$$

where $\{x_n\}$ is any orthonormal basis for $\mathcal{H}$. The set of trace-class operators is a two-sided $*$-ideal in $B(\mathcal{H})$

An operator $T \in B(\mathcal{H})$ is called **Hilbert-Schmidt** if T^*T is trace-class. The inner product on the the space of all Hilbert-Schmidt operators can be defined as

$$\langle T, S\rangle = \mathrm{tr}(S^*T) = \sum \langle Tx_n, Sx_n\rangle,$$

for any orthonormal basis $\{x_n\}$. It can be shown that T is Hilbert-Schmidt if and only if $\|T\| = \sum \|Tx_n\|^2 < \infty$ for any orthonormal basis $\{x_n\}$.

The set of Hilbert-Schmidt operators form a two-sided $*$-ideal in $B(\mathcal{H})$. It is indeed a Hilbert space, which can be shown to be naturally isometrically isomorphic to the tensor product of Hilbert spaces $\mathcal{H} \otimes \mathcal{H}^*$, where $\mathcal{H}^*$ is the dual vector space of $\mathcal{H}$.

References: §7.3 and Appendix 2 in [10].

Direct Integral Decompositions

Let $(A, \mathcal{M})$ be a measurable space equipped with a σ-algebra $\mathcal{M}$. A family $\{\mathcal{H}_\sigma\}_{\sigma \in A}$ of nonzero separable Hilbert spaces indexed by A is called a **field** of Hilbert spaces over A, and an element of $\prod_{\alpha \in A} \mathcal{H}_\alpha$ - that is, a map f on A such that $f(\alpha) \in \mathcal{H}_\alpha$ for each α - is called a **vector field** on A. We write the inner product and norm on $\mathcal{H}_\alpha$ by $\langle \cdot, \cdot \rangle_\alpha$ and $\|\cdot\|_\alpha$. A **measurable field of Hilbert spaces** over A is a field of Hilbert spaces $\{\mathcal{H}_\alpha\}$ together with a countable set $\{e_j\}_1^\infty$ of vector fields with the following properties:

1. the functions $\alpha \mapsto \langle e_j(\alpha), e_k(\alpha)\rangle_\alpha$ are measurable for all j, k,

2. the linear span of $\{e_j(\alpha)\}_1^\infty$ is dense in $\mathcal{H}_\alpha$ for each α.

Given a measurable field of Hilbert spaces $\{\mathcal{H}_\alpha\}, \{e_j\}$ on A, a vector field f on A will be called **measurable** if $\langle f(\alpha), e_j(\alpha)\rangle_\alpha$ is a measurable function on A for each j.

Now let $\{\mathcal{H}_\alpha\}, \{e_j\}$ be a measurable field of Hilbert spaces over A, and suppose μ is a measure on A. The **direct integral** of the spaces $\mathcal{H}_\alpha$ with respect to μ, denoted by

$$\int^{\oplus} \mathcal{H}_\alpha \, d\mu(\alpha),$$

is the space of measurable vector fields f on A such that

$$\|f\|^2 = \int \|f(\alpha)\|_\alpha^2 \, d\mu(\alpha) < \infty.$$

$\int^{\oplus} \mathcal{H}_\alpha \, d\mu(\alpha)$ is a Hilbert space with inner product

$$\langle f, g \rangle = \int \langle f(\alpha), g(\alpha) \rangle_\alpha \, d\mu(\alpha).$$

References: §7.4 in [10].

The Abstract Plancherel Theorem

The Plancherel Theorem is, roughly speaking, the decomposition of the bi-regular representation of locally compact group G as a direct integral of irreducible representations. Now let G be a unimodular locally compact group. Recall that the right ρ and the left λ regular representations of G on $L^2(G)$ are defined by

$$\rho(x)f(y) = f(yx), \quad \lambda(x)f(y) = f(x^{-1}y).$$

We can combine both representations to obtain a new representation β of $G \times G$ on $L^2(G)$. It is defined by

$$\beta(x, y)f(z) = f(x^{-1}zy).$$

and we call β the **bi-regular representation** of G (although it is actually a representation of $G \times G$).

For a second countable, unimodular, postliminal group G, there is a measurable field of irreducible representations over $\widehat{G}$ such that the representation at the point $p \in \widehat{G}$ belongs to equivalence class p. Hence we identify the points of $\widehat{G}$ with the representations in this field. Therefore, if $f \in L^1(G)$, we define the Fourier transform of f to be the measurable field of operators over $\widehat{G}$ given by

$$\widehat{f}(\pi) = \int f(x)\pi(x^{-1})\,dx.$$

We want to think $\widehat{f}(\pi)$ as an element of $H_\pi \otimes \mathcal{H}_{\overline{\pi}}$. However, $H_\pi \otimes \mathcal{H}_{\overline{\pi}}$ can be identified with the space of Hilbert-Schmidt operators. It turns out that $\widehat{f}(\pi)$ is Hilbert-Schmidt for a suitably large class of f's and π's.

2.5 Theorem (The Abstract Plancherel Theorem)**.** *Suppose G is a second countable, unimodular, postliminal group. There is a measure μ, called **Plancherel measure**, on $\widehat{G}$, uniquely determined once the Haar measure on G is fixed, with the following properties. The Fourier transform $f \mapsto \widehat{f}$ maps $L^1(G) \cap L^2(G)$ into $\int^\oplus \mathcal{H}_\pi \otimes \mathcal{H}_{\overline{\pi}}\,d\mu(\pi)$, and it extends to a unitary map from $L^2(G)$ onto $\int^\oplus \mathcal{H}_\pi \otimes \mathcal{H}_{\overline{\pi}}\,d\mu(\pi)$ that intertwines the bi-regular representation β with $\int^\oplus \pi \otimes \overline{\pi}\,d\mu(\pi)$.*

References: Appendix 2, §7.3, and §7.5 in [10], §18.8 in [8], §4.3 in [5].

2.2 C*-algebras

Group C*-algebras and Fell Topology

Let A be a Banach algebra over $\mathbb{C}$. If A admits a map $x \mapsto x^*$ with the following properties:

$$(\lambda x + \mu y)^* = \overline{\lambda}x^* + \overline{\mu}y^*, \quad (xy)^* = y^*x^*, \quad (x^*)^* = x,$$

for all $x, y \in A$, $\lambda, \mu \in \mathbb{C}$, then A is called a **Banach $*$-algebra**. If an involution $*$ of A satisfies the additional condition:

$$\|x^*x\| = \|x\|^2,$$

for all $x \in A$ then A is called a **C*-algebra**. An **approximate identity** or **approximate unit** for a C*-algebra A is an increasing net $(u_\lambda)_{\lambda \in \Lambda}$ of positive elements in the closed unit ball of A such that $a = \lim_\lambda au_\lambda$ for all $a \in A$. Equivalently, $a = \lim_\lambda u_\lambda a$ for all $a \in A$.

Let A be a Banach $*$-algebra with an approximate identity. For each $x \in A$, we define

$$\|x\|' = \sup\{\|\pi(x)\| \mid \pi \text{ is a } *\text{-representation of } A\}.$$

For each $x \in A$, it is known that

$$\begin{aligned}\|x\|' &= \sup\{\|\pi(x)\| \mid \pi \text{ is a topologically irreducible } *\text{-representation of } A\} \\ &= \sup\{\rho(x^*x)^{\frac{1}{2}} \mid \rho \text{ is a continuous positive form of norm } \leqslant 1 \text{ on } A\} \\ &= \sup\{\rho(x^*x)^{\frac{1}{2}} \mid \rho \text{ is a pure state on } A\}.\end{aligned}$$

The map $x \mapsto \|x\|'$ is a seminorm on A satisfying

$$\|x\|' \leqslant \|x\|, \quad \|xy\|' \leqslant \|x\|'\|y\|', \quad \|x^*\|' = \|x\|', \quad \|x^*x\|' = \|x\|'^2,$$

for any $x, y \in A$. Let N be the set of $x \in A$ such that $\|x\|' = 0$. This is a closed, self-adjoint, two-sided ideal of A , and $\|\cdot\|'$ becomes a C^*-norm on the quotient A/N. Endowed with this norm, A/N satisfies all the C*-algebra axiom except that A/N is not complete in general. The completion of $B = (A/N, \|\cdot\|')$ is called the **enveloping C^*-algebra** of A.

The canonical map of A into B is a norm-reducing $*$-algebra morphism whose image is dense in B. When A is a C*-algebra, we have $\|x\|' = \|x\|$

and A may be identified with its own enveloping C*-algebra. Also we have the following:

2.6 Proposition. *Let A be a Banach $*$-algebra with an approximate identity, B the enveloping C*-algebra of A and τ the canonical map of A into B.*

1. *If π is a $*$-representation of A, there is a unique $*$-representation ρ of B such that $\pi = \rho \circ \tau$, and $\rho(B)$ is the C*-algebra generated by $\pi(A)$.*

2. *The map $\pi \mapsto \rho$ is a bijection of the set of $*$-representations of A onto the set of $*$-representations of B.*

3. *ρ is nondegenerate if and only if π is nondegenerate.*

4. *ρ is topologically irreducible $*$-representation if and only if π is a topologically irreducible $*$-representation.*

5. *If f is a continuous positive form on A, there is a unique positive form on g on B such that $f = g \circ \tau$. Moreover $\|g\| = \|f\|$.*

6. *The map $f \mapsto g$ is a bijection of the set of continuous positive forms on A onto the set of positive forms on B.*

Now let G be a locally compact group. Note that the $L^1(G)$ norm is not a C*-algebra norm and so $L^1(G)$ is not a C*-algebra. Since $L^1(G)$ is a Banach $*$-algebra with an approximate identity, we can form its enveloping C*-algebra. We call this the **(full) C*-algebra of the group** G and it is denoted $C^*(G)$. If G is discrete, then $C^*(G)$ admits an identity element. If G is separable, $C^*(G)$ is also separable. For $f \in L^1(G)$,

$$\|f\|' = \sup \|\pi(f)\|$$

where π run through the set of nondegenerate $*$-representations of $L^1(G)$, or the set of unitary representations of G. Then $f \mapsto \|f\|'$ is a seminorm on $L^1(G)$, and indeed, a norm since $L^1(G)$ admints an injective representation (the left regular representation of $L^1(G)$ in $L^2(G)$ is injective). The C*-algebra of G is simply the completion of $L^1(G)$ for this supremum norm. By the proposition above, any $*$-representation of $L^1(G)$ extends uniquely to $*$-representation of $C^*(G)$. Therefore we establish a bijective correspondence between unitary representation of G and nondegenerate $*$-representations of $C^*(G)$. In other words, this is a bijection between $C^*(G)\hat{}$ and $\widehat{G}$ where $C^*(G)\hat{}$ is the set of all nondegenerate $*$-representations of $C^*(G)$. If π is such a representation, its kernel

$$\ker(\pi) = \{f \in C^*(G) \mid \pi(f) = 0\}$$

is a closed two-sided ideal of $C^*(G)$. If π is irreducible, then $\ker(\pi)$ is called a **primitive ideal** of $C^*(G)$. The space of all primitive ideals of $C^*(G)$ is denoted by $\mathrm{Prim}(G)$; that is

$$\mathrm{Prim}(G) = \{\ker(\pi) \mid \pi \in \widehat{G}\}.$$

For any nonempty subset S of $\mathrm{Prim}(G)$, we define $\overline{S} \subset \mathrm{Prim}(G)$ by

$$\overline{S} = \{\mathcal{I} \in \mathrm{Prim}(G) \mid \mathcal{I} \supset \bigcap_{\mathcal{J} \in S} \mathcal{J}\}.$$

Then for any $S, T \subset \mathrm{Prim}(G)$, it can be shown that

$$\overline{\varnothing} = \varnothing, \quad S \subset \overline{S}, \quad \overline{\overline{S}} = \overline{S}, \quad \overline{S \cup T} = \overline{S} \cup \overline{T}.$$

It follows from a theorem of Kuratowski (See p.119 in [10]) that there is a unique topology on $\mathrm{Prim}(G)$ such that $\overline{S}$ is the closure of S for all $S \subset \mathrm{Prim}(G)$. This topology is called **hull-kernel topology** or **Jacobson topology**. It can be shown that

1. This topology is T_0.

2. S is a closed subset of $\mathrm{Prim}(G)$ if and only if S is exactly the set of primitive ideals containing some (fixed) subset of $C^*(G)$.

3. Let $I \in \mathrm{Prim}(G)$. Then $\{I\}$ is closed in $\mathrm{Prim}(G)$ if and only if I is maximal among primitive ideals.

Let G be a locally compact group and π an irreducible representation of G. Then the kernel $\ker(\pi) \in \mathrm{Prim}(G)$ depends only on the equivalence class of π, and the map $[\pi] \mapsto \ker(\pi)$ is a surjection from $\widehat{G}$ onto $\mathrm{Prim}(G)$. We can therefore pull back the hull-kernel topology on $\mathrm{Prim}(G)$ to $\widehat{G}$; that is, open sets on $\widehat{G}$ is of the form $\{[\pi] \mid \ker(\pi) \in U\}$ where U is open in $\mathrm{Prim}(G)$. This topology is called the **Fell topology** on $\widehat{G}$.

References: §2.7, §3.1, §13.9 in [8], §7.1 in [10], §5.4 in [22].

Liminal, Postliminal

A unitary representation π of G is **primary** if the center of $\mathcal{C}(\pi)$ is trivial, i.e., consists of scalar multiples of I. By Schur's lemma, every irreducible representation is primary. The group G is said to be **type I** if every primary representation of G is a direct sum of copies of some irreducible representation. Connected nilpotent Lie groups are type I.

Let $K(\mathcal{H})$ be the set of all compact operators from $\mathcal{H}$ to itself. A C*-algebra A is said to be **liminal** or **CCR** if $\pi(A) \subset K(\mathcal{H}_\pi)$ for every irreducible representation π; that is, $\pi(f)$ is compact operator for every irreducible $*$-representation of A. Equivalently, $\pi(A) = K(\mathcal{H}_\pi)$. CCR is an acronym for "completely continuous representations" ("completely continuous operator" was an old terminology for compact operator) and liminal is a French synonym for CCR invented by Dixmier. A C*-algebra A is said to be **postliminal** if $K(\mathcal{H}) \subset \pi(A)$. The postliminal C*-algebras are also called **GCR** or **Type I C*-algebras**. Every liminal C*-algebra is postliminal. If A is a liminal C*-algebra, then its C*-subalgebra and its quoteint C*-algebras are liminal also. The converse is false in general. On the other hand, A is a postliminal C*-algebra if and only if I and A/I are postliminal where I is a closed ideal in A.

The group G is called **liminal** (resp. **postliminal**) if $C^*(G)$ is liminal (resp. postliminal). In this case, G is liminal if and only if $\pi(f)$ is compact whenever π is irreducible and $f \in L^1(G)$, since $L^1(G)$ is dense in $C^*(G)$. Abelian groups, compact groups and connected nilpotent Lie group are liminal. If G is a second countable locally compact group, the following are equivalent:

(i) G is type I.

(ii) The Fell topology on $\widehat{G}$ is T_0.

(iii) The map $[\pi] \mapsto \ker(\pi)$ from $\widehat{G}$ to $\mathrm{Prim}(G)$ is bijective.

(iv) G is postliminal.

It is also known that a locally compact group G is liminal if and only if the Fell topology on $\widehat{G}$ is T_1. Compare item 3 from previous section.
References: §7.2 in [10], [12], §5.6 in [22].

Weak Containment, Amenability and Reduced Group C*-algebras

If $\varphi : A \to B$ is a linear map between C*-algebras A and B, then φ is said to be **positive** if $\varphi(A^+) \subset B^+$, where A^+ is the set of positive elements of A. A **state** on a C*-algebra A is a positive linear functional on A of norm one. Let $(\pi, \mathcal{H})$ be a representation of A. A state (or positive linear functional)

ρ on A is said to be **associated with** π, if there exists $\xi \in \mathcal{H}$ such that

$$\rho(a) = \langle \pi(a)\xi, \xi \rangle$$

for all $a \in A$. Now let π_1, π_2 be two representations of A. We say that π_1 is **weakly contained** in π_2 (or π_2 **weakly contains** π_1), if $\ker \pi_2 \subset \ker \pi_1$. The **support** of π is the set of $\sigma \in \widehat{G}$ which are weakly contained in π. It can be shown that the following statements are equivalent:

(i) π_1 is weakly contained in π_2;

(ii) Each positive functional on A associated with π_1 is a weak*-limit of sums of positive functionals associated with π_2;

(iii) Each state on A associated with π_1 is a weak*-limit of states which are sums of positive functionals associated with π_2.

In terms of group, π_1 is weakly contained in π_2 if every positive definite matrix coefficient $\langle \pi_1(x)\xi, \xi \rangle$ can be approximated uniformly on compact subsets of G by finite sums of positive definite matrix coefficients $\langle \pi_2(x)f_i, f_i \rangle$ where $f_i \in L^2(G)$.

Let G be a locally compact group. A group G is called **amenable** if there exist a left translation invariant mean m for G, that is, a state on $L^\infty(G)$ (view $L^\infty(G)$ as a C*-algebra) such that

$$m(L_s f) = m(f),$$

for all $s \in G, f \in L^\infty(G)$ where L_s is left translation action of G on $L^\infty(G)$. Discrete groups and Abelian groups are amenable. So are compact groups. Indeed, if μ is an invariant Haar measure on G with $\mu(G) = 1$, then

$$m(f) = \int f(s)\, d\mu(s), \quad \forall f \in L^\infty(G),$$

is an invariant mean on $L^\infty(G)$.

Let λ be the left regular representation of $L^1(G)$ on $L^2(G)$. Then $\|f\|_r = \|\lambda(f)\|$ (where $f \in L^1(G)$) is a C*-norm on $L^1(G)$. The completion of $(L^1(G), \|\cdot\|_r)$ is called the **reduced C*-algebra of the group** G, and it is denoted by $C_r^*(G)$. In other words,

$$C_r^*(G) = \overline{\lambda(L^1(G))}.$$

If G is a discrete amenable group, then $C^*(G)$ and $C_r^*(G)$ are isomorphic. For any locally compact group G, the following statements are also equivalent:

(i) G is amenable;

(ii) Any $*$-representation of $C^*(G)$ is weakly contained in its left regular representation, where the left regular representation of $C^*(G)$ is the unique extension of the left regular representation of $L^1(G)$;

(iii) The left regular representation of $C^*(G)$ is faithful;

(iv) There is an isomorphism between $C^*(G)$ and $C_r^*(G)$.

Reference: Chapter VII in [6], Chapter 16 in [20].

2.3 Kirillov theory

Nilpotent Lie Groups and Nilpotent Lie Algebras

A finite-dimensional real Lie algebra $\mathfrak{g}$ is a finite dimensional vector space over $\mathbb{R}$ on which there is a bilinear form named the **bracket** and denoted by $[\cdot,\cdot]$ with the following properties

$$[x, y] = -[y, x], \quad [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0,$$

for all $x, y, z \in \mathfrak{g}$. The latter equality is the **Jacobi identity**. The algebra of all $n \times n$ square matrices is an example of Lie algebra with the bracket $[A, B] = AB - BA$. A subspace $\mathfrak{h}$ is said to be an **ideal** if $[x, h] \in \mathfrak{h}$ for all $x \in \mathfrak{g}$, and all $h \in \mathfrak{h}$. A linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism if $\phi([x, y]) = [\phi(x), \phi(y)]$ for all $x, y \in \mathfrak{g}$.

Let $\mathfrak{g}$ be a Lie algebra. The **descending central series** of $\mathfrak{g}$ is defined inductively by

$$\mathfrak{g}^{(1)} = \mathfrak{g}, \quad \mathfrak{g}^{(n+1)} = [\mathfrak{g}, \mathfrak{g}^{(n)}] = \mathbb{R}\text{- span}\{[X, Y] : X \in \mathfrak{g}, Y \in \mathfrak{g}^{(n)}\}$$

It follows that $\mathfrak{g} = \mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \supset \ldots$, as the name suggests, and $\phi(\mathfrak{g}^{(n)}) \subset \mathfrak{h}^{(n)}$ for all n if $\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism. Also $[\mathfrak{g}^{(p)}, \mathfrak{g}^{(q)}] \subset \mathfrak{g}^{(p+q)}$, for all intergers p and q. In particular, $\mathfrak{g}^{(k)}$ is an ideal in $\mathfrak{g}$ for all k.

We say that $\mathfrak{g}$ is a **nilpotent Lie algebra** if its descending central series eventually vanishes, i.e., there is an integer n such that $\mathfrak{g}^{(n+1)} = 0$. If n is the minimum positive integer such that $\mathfrak{g}^{(n+1)} = 0$ but $\mathfrak{g}^{(n)} \neq 0$. Then $\mathfrak{g}$ is said to be n**-step nilpotent**. Therefore $\mathfrak{g}$ is n-step nilpotent if and only if all brackets of at least $n + 1$ elements of $\mathfrak{g}$ are 0 but not all brackets of n elements are. If $\mathfrak{g}$ is nilpotent, so are all subalgebras and quotient algebras of $\mathfrak{g}$. However it is not true that if $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ such that $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are nilpotent, then $\mathfrak{g}$ is necessarily nilpotent. The counter example is $\mathfrak{g} = \mathbb{R}\text{- span}\{X, Y\}$ with $[X, Y] = X$, and $\mathfrak{h} = \mathbb{R}\text{- span}\{X\}$. If $\mathfrak{g}$ is n-step nilpotent, $\mathfrak{g}^{(n)}$ is central. Therefore $\mathfrak{g}$ always has a nonempty center.

A connected simply-connected Lie group G is said to be **nilpotent** if its Lie algebra $\mathfrak{g}$ is nilpotent. A nilpotent Lie group G is one whose Lie algebra $\mathfrak{g}$ is nilpotent. There is an equivalent definition of nilpotent Lie group. The descending central series for the group G is defined by

$$G^{(1)} = G, \quad G^{(j+1)} = [G, G^{(j)}]$$

where the bracket $[H, K]$ is a subgroup generated by all $hkh^{-1}k^{-1}, h \in H, k \in K$. G is said to be nilpotent if $G^{(j)} = \{e\}$ for some j. It can be shown that $G^{(j)}$ are Lie subgroups of G and the Lie algebra of $G^{(j)}$ is indeed $\mathfrak{g}^{(j)}$. So the Lie group and Lie algebra definitions coincide. There is also another way to think about nilpotent Lie algebra using ascending central series. The **ascending central series** of Lie algebra $\mathfrak{g}$ is defined inductively by

$$\mathfrak{g}_{(1)} = \mathfrak{z}(\mathfrak{g}), \quad \mathfrak{g}_{(i)} = \{X \in \mathfrak{g} : [X, \mathfrak{g}] \subseteq \mathfrak{g}_{(i-1)}\},$$

where $\mathfrak{z}(\mathfrak{g})$ is the center of $\mathfrak{g}$. Each $\mathfrak{g}_{(i)}$ is an ideal and $\mathfrak{z}(\mathfrak{g}) = \mathfrak{g}_{(1)} \subset \mathfrak{g}_{(2)} \subset \ldots$, as the name suggests. The Lie algebra $\mathfrak{g}$ is n-step nilpotent if and only if $\mathfrak{g} = \mathfrak{g}_{(n)} \neq \mathfrak{g}_{(n-1)}$. Now let's explore a few examples of nilpotent Lie algebra.

1. $\mathbb{R}^n$ with the trivial bracket ($[X, Y] = 0$ for all X, Y) is an Abelian nilpotent Lie algebra. This is (up to isomorphism) the only one-step nilpotent Lie algebra.

2. The $(2n + 1)$-dimensional **Heisenberg algebra**, denoted by $\mathfrak{h}_n$, is the Lie algebra with basis $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\}$ whose pairwise brackets are equal to zero except for

$$[X_i, Y_i] = Z, \quad 1 \leqslant i \leqslant n.$$

There is a matrix realization for Heisenberg algebra in which $zZ + \sum_{i=1}^{n}(x_i X_i + y_i Y_i)$ corresponds to the $(n + 2) \times (n + 2)$ matrix

$$\begin{pmatrix} 0 & x_1 & \ldots & x_n & z \\ & \cdot & & & y_1 \\ & & \cdot & & \vdots \\ & & & \cdot & y_n \\ 0 & & & & 0 \end{pmatrix}.$$

Note that the Heisenberg algebra is a two-step nilpotent Lie algebra.

3. The $(n+1)$-dimensional Lie algebra $\mathfrak{k}_n$ is the Lie algebra with basis $\{X, Y_1, \ldots, Y_n\}$ and brackets all equal to zero except for

$$[X, Y_i] = Y_{i+1}, \quad 1 \leqslant i \leqslant n-1.$$

The matrix realization is obtained by letting $xX + \sum_{i=1}^{n} y_i Y_i$ correspond to $(n+1) \times (n+1)$ matrix

$$\begin{pmatrix} 0 & x & 0 & \ldots & \ldots & 0 & y_n \\ & 0 & x & 0 & \ldots & \ldots & y_{n-1} \\ & & 0 & \ddots & \ddots & \ldots & \vdots \\ & & & 0 & x & 0 & y_3 \\ & & & & 0 & x & y_2 \\ & & & & & 0 & y_1 \\ & & & & & & 0 \end{pmatrix}.$$

Note that $\mathfrak{k}_n$ is an n-step nilpotent Lie algebra and that $\mathfrak{k}_2 = \mathfrak{h}_1$.

4. Let $\mathfrak{n}_n$ be the Lie algebra of strictly upper triangular $n \times n$ matices. This is an $(n-1)$-step nilpotent algebra of dimension $n(n-1)/2$ and its center is one-dimensional. Note that $\mathfrak{n}_3 = \mathfrak{h}_1$.

Nilpotent Lie groups are usually denoted by capital letters, and the corresponding Lie algebra is denoted by the corresponding lower case gothic letter. For example we use H_n, K_n and N_n for the nilpotent Lie groups corresponding to nilpotent Lie algebras $\mathfrak{h}_n$, $\mathfrak{k}_n$ and $\mathfrak{n}_n$. Next are the classical, basic theorems about nilpotent Lie algebras. The first one is a special case of Ado's Theorem (every finite-dimensional Lie algebra over $\mathbb{C}$ is isomorphic to a subalgebra of $\mathrm{gl}(n, \mathbb{C})$).

2.7 Theorem (Birkhoff Embedding Theorem)**.** *Let $\mathfrak{g}$ be a nilpotent Lie algebra over $\mathbb{R}$. Then there is a finite-dimensional vector space V and an injection $i : \mathfrak{g} \to \mathrm{gl}(V)$, such that, for all $X \in \mathfrak{g}$, $i(X)$ is nilpotent.*

2.8 Theorem (Engel's Theorem)**.** *Let* $\mathfrak{g}$ *be a Lie algebra and let* $\alpha : \mathfrak{g} \to \mathrm{gl}(V)$ *be a homomorphism such that* $\alpha(X)$ *is nilpotent for all* $X \in \mathfrak{g}$*. Then there exists a flag (Jordan-Holder series) of subspaces*

$$\{0\} = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V, \quad \text{with } \dim V_j = j,$$

such that $\alpha(X)V_j \subseteq V_{j-1}$ *for all* $j \geqslant 1$ *and all* $X \in \mathfrak{g}$*. In particular,* $\alpha(\mathfrak{g})$ *is a nilpotent Lie algebra.*

So every nilpotent Lie algebra has a faithful embedding in $\mathfrak{n}_n$ for some n. If G is a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$, then the exponential map from $\mathfrak{g}$ to G becomes the ordinary exponential map and the adjoint map $\mathrm{Ad}\, x$ becomes $A \mapsto xAx^{-1}$ for $A \in \mathfrak{g}, x \in G$.

Connected, simply connected nilpotent Lie groups have several nice properties as we list some of them below.

2.9 Theorem. *Let* G *be a connected, simply connected nilpotent Lie group with Lie algebra* $\mathfrak{g}$*.*

1. $\exp : \mathfrak{g} \to G$ *is an analytic diffeomorphism. It carries the Lebesgue measure on* $\mathfrak{g}$ *to a left-invariant Haar measure on* G*. This measure is also right-invariant. In other words,* G *is unimodular.*

2. *The Campbell-Baker-Hausdorff formula holds for all* $X, Y \in \mathfrak{g}$*. The low order terms in Campell-Baker-Hausdorff formula is*

$$\begin{aligned} X * Y = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] \\ - \frac{1}{48}[Y,[X,[X,Y]]] - \frac{1}{48}[X,[Y,[X,Y]]] \\ + \textit{(commutators in five or more terms)}. \end{aligned}$$

where $X * Y = \log(\exp X \cdot \exp Y)$*, where* $X, Y \in \mathfrak{g}$*. This series is a finite sum when* G *is a nilpotent Lie group.*

3. *Every connected Lie subgroup of G is closed, and also simply connected and nilpotent.*

4. *G has a faithful embedding as a closed subgroup of N_n for some n.*

5. *Let $\mathfrak{g}$ be a nilpotent Lie algebra and let $\mathfrak{z}$ be the center of $\mathfrak{g}$. Then* $\exp(\mathfrak{z})$ *is the center of G.*

From now on, when we talk about a nilpotent Lie group, we will always assume it is connected and simply-connected. Since exp is a diffeomorphism of $\mathfrak{g}$ onto G, we can use it to transfer coordinates from $\mathfrak{g}$ to G. If we use exp to identify $\mathfrak{g}$ to G then the group multiplication becomes the Campbell-Baker-Hausdorff product

$$\exp(X * Y) = \exp X \cdot \exp Y, \quad \text{for all } X, Y \in \mathfrak{g}.$$

If $\mathfrak{g}$ is equipped with coordinates associated with a linear basis, the corresponding coordinates in G will be called **exponential coordinates**.
Reference: §1.1, §1.2 in [5].

Elements of Kirillov Theory

The next result, Kirillov's lemma, is a key ingredient for the representation theory for nilpotent Lie groups. This lemma asserts that a noncommutative nilpotent Lie algebra with one-dimensional center is closely related in structure to the Heisenberg Lie algebra.

2.10 Lemma (Kirillov's Lemma)**.** *Let $\mathfrak{g}$ be a noncommutative nilpotent Lie algebra whose center $\mathfrak{z}(\mathfrak{g})$ is one-dimensional. Then $\mathfrak{g}$ can be decomposed as*

$$\mathfrak{g} = \mathbb{R}Z \oplus \mathbb{R}Y \oplus \mathbb{R}X \oplus \mathfrak{w} = \mathbb{R}X \oplus \mathfrak{g}_0$$

a vector space direct sum, where

$$\mathbb{R}Z = \mathfrak{z}(\mathfrak{g}), \ \textit{with } [X, Y] = Z;$$

and $\mathfrak{g}_0 = \mathbb{R}Y \oplus \mathbb{R}Z \oplus \mathfrak{w}$ *is the centralizer of* Y*, and an ideal.*

Let G be a nilpotent Lie group with Lie algebra $\mathfrak{g}$, and denote the dual vector space of $\mathfrak{g}$ by $\mathfrak{g}^*$. The group G acts on $\mathfrak{g}^*$ by the contragradient of the adjoint map, or the **coadjoint map** Ad^*; defined by,

$$((\mathrm{Ad}^*\, x)l)(Y) = l((\mathrm{Ad}\, x^{-1})Y), \quad Y \in \mathfrak{g}, l \in \mathfrak{g}^*, \text{ and } x \in G.$$

A **coadjoint orbit** is an orbit of the Lie group G in the space $\mathfrak{g}^*$. The differential $\mathrm{d}(\mathrm{Ad}^*)_e$, of the coadjoint map at the unit $e \in g$ is written $\mathrm{ad}^* : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}^*)$, and is given by

$$((\mathrm{ad}^*\, x)l)(Y) = l([Y, X]) = l(\mathrm{ad}(-X)Y), \ X, Y \in \mathfrak{g}, \ l \in \mathfrak{g}^*.$$

The **stabilizer** subgroup of G associated with $l \in \mathfrak{g}^*$ is defined by

$$R_l = \{x \in G \mid (\mathrm{Ad}^*\, x)l = l\}.$$

If G is a nilpotent Lie group and $l \in \mathfrak{g}^*$, then the stabilizer R_l is connected and

$$r_l = \{X \in \mathfrak{g} \mid (\mathrm{ad}^*\, X)l = 0\} \subset \mathfrak{g}$$

is its Lie algebra. Thus $R_l = \exp r_l$. See Lemma 1.3.1 in [5] for proof. There is also another way to describe this Lie subalgebra r_l of Lie algebra $\mathfrak{g}$. For each $l \in \mathfrak{g}^*$, defines a natural bilinear form $B_l : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$,

$$B_l(X, Y) = l([X, Y]), \quad X, Y \in \mathfrak{g}.$$

Then B_l is skew-symmetric, i.e., $B_l(X, Y) = -B_l(Y, X)$. The **radical** of B_l is, by definition, $\{Y \in \mathfrak{g} : B_l(X, Y) = 0 \text{ for all } X \in \mathfrak{g}\}$ which coincides with r_l. The next lemma is helpful for finding the radical.

2.11 Lemma. *If $\mathfrak{g}$ is a Lie algebra and $l \in \mathfrak{g}^*$, its radical r_l has even codimension in $\mathfrak{g}$. Hence coadjoint orbis are of even dimension.*

See Lemma 1.3.2 in [5] for proof. Now let V be a real vector space with a skew-symmetric symplectic bilinear form B, its **isotropic subspaces** W are those such that $B(w, w') = 0$, for all $w, w' \in W$. It can be shown that maximal isotropic subspaces exist and have the same dimension:

$$\frac{1}{2}\dim(V/\operatorname{rad}B) + \dim(\operatorname{rad}B) = \frac{1}{2}(\dim V + \dim\operatorname{rad}B),$$

where $\operatorname{rad}B := \{x \in V : B(x, y) = 0 \text{ for all } y \in V\}$. In other words, they have codimension $k = \frac{1}{2}(\dim V/\operatorname{rad}B)$ and lie halfway between the radical $\operatorname{rad}B$ and V. Note that $\operatorname{rad}B$ is contained in them.

In particular, if $V = \mathfrak{g}, l \in \mathfrak{g}^*$, and $B = B_l$ then we call subalgebras $\mathfrak{m} \subseteq \mathfrak{g}$ that are isotropic for B_l and have codimension $k = \frac{1}{2}(\dim V/\operatorname{rad}B)$ as **polarizing subalgebras** or **maximal subordinate subalgebras** for l. Given $l \in \mathfrak{g}^*$, the radical r_l is uniquely determined, but there can be many polarizing subalgebras $\mathfrak{m}$ as we shall see in examples below. There is no systematic way to construct all of $\mathfrak{m}$ and this is one of the complications of the theory.

2.12 Example. Consider $G = \mathbb{R}^n$ (the Abelian case). We have $\mathfrak{g} \cong \mathbb{R}^n$, with trivial bracket. Then for all $x \in G$ and $X \in \mathfrak{g}$, we have $(\operatorname{Ad}x)X = X$. Thus $\operatorname{Ad}x = I$ and so $\operatorname{Ad}^* x = I$ for all $x \in G$. Therefore coadjoint orbits in $\mathfrak{g}^*$ are points. For all $l \in \mathfrak{g}^*$, we have $R_l = G$. So $r_l = \mathfrak{g}$. Therefore the unique polarizing subalgebra for l is $\mathfrak{g}$, since $\dim r_l = \dim\mathfrak{g}$.

2.13 Example. Let $G = H_n$, $\mathfrak{g} = \mathfrak{h}_n$. By using matrix realization, we write $w \in G, W \in \mathfrak{g}$ by $(n+2) \times (n+2)$ matrices

$$w = \begin{pmatrix} 1 & x_1 & \dots & x_n & z \\ & 1 & & & y_1 \\ & & \ddots & & \vdots \\ & & & 1 & y_n \\ 0 & & & & 1 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & a_1 & \dots & a_n & c \\ & \cdot & & & b_1 \\ & & \cdot & & \vdots \\ & & & \cdot & b_n \\ 0 & & & & 0 \end{pmatrix}.$$

Then

$$\begin{aligned}(\operatorname{Ad} w^{-1})W &= w^{-1}Ww \\ &= \begin{pmatrix} 1 & -x_1 & \dots & -x_n & -z + x \cdot y \\ & 1 & & & -y_1 \\ & & \ddots & & \vdots \\ & & & 1 & -y_n \\ 0 & & & & 1 \end{pmatrix} Ww \\ &= \begin{pmatrix} 0 & a_1 & \dots & a_n & c + y \cdot a - x \cdot b \\ & & & & b_1 \\ & & & & \vdots \\ & & & & b_n \\ 0 & & & & 0 \end{pmatrix}.\end{aligned}$$

where $\cdot$ is the inner product in $\mathbb{R}^n$. Let $\{Z, Y_1, \dots, Y_n, X_1, \dots, X_n\}$ be a basis for $\mathfrak{g} = \mathfrak{h}_n$. Then $\{Z^*, Y_1^*, \dots, Y_n^*, X_1^*, \dots, X_n^*\}$ is a dual basis for $\mathfrak{g}^*$. Let $W \in \mathfrak{g}$. So

$$W = cZ + \sum_{i=1}^{n} (a_i X_i + b_i Y_i).$$

Let $l \in \mathfrak{g}^*$. Then $l = \gamma Z^* + \sum_{i=1}^{n} (\beta_j Y_j^* + \alpha_j X_j^*) = l_{\alpha,\beta,\gamma}$. So

$$l(W) = c\gamma + \sum_{j=1}^{n} (\alpha_j a_j + \beta_j b_j).$$

Thus if $w = \exp(zZ + \sum_{i=1}^{n}(y_iY_i + x_iX_i))$, we get

$$\begin{aligned}(\mathrm{Ad}^*(w)l_{\alpha,\beta,\gamma})(W) &= l(\mathrm{Ad}(w^{-1})W) \\ &= l\big[(c + \sum_{j=1}^{n}(y_ja_j - x_jb_j))Z + \sum_{j=1}^{n}(a_jX_j + b_jY_j)\big] \\ &= c\gamma + \sum_{j=1}^{n}(y_ja_j - x_jb_j)\gamma + \sum_{j=1}^{n}(a_j\alpha_j + b_j\beta_j) \\ &= c\gamma + \sum_{j=1}^{n}(a_j(\alpha_j + y_j\gamma) + b_j(\beta_j - x_j\gamma)) \\ &= l_{\alpha+y\gamma,\beta-\gamma x,\gamma}(W)\end{aligned}$$

Case I: For $\gamma \neq 0$, we get

$$(\mathrm{Ad}^*\, G)l_{\alpha,\beta,\gamma} = \{l_{\alpha',\beta',\gamma} \mid \alpha', \beta' \in \mathbb{R}^n\}$$

which are $2n$-dimensional orbits in $\mathfrak{g}^*$ of the form $\gamma Z^* + \mathfrak{z}^\perp$ where $\mathfrak{z}^\perp = \{l \in \mathfrak{g}^* : l(Z) = 0\}$. In this case $r_l = \mathbb{R}Z$. Since $\dim \mathfrak{m} = \frac{1}{2}(\dim \mathfrak{g} + \dim r_l) = \frac{1}{2}((2n+1)+1) = n+1$ and $r_l \subseteq \mathfrak{m}$, we need to add n more vectors. In this case m is not unique. Few examples of polarizing subalgebras are $\mathfrak{m} = \mathbb{R}Z + \mathbb{R}\text{ -span}\{X_1, \ldots, X_n\}$, $\mathfrak{m} = \mathbb{R}Z + \mathbb{R}\text{ -span}\{Y_1, \ldots, Y_n\}$ and $\mathfrak{m} = \mathbb{R}Z + \mathbb{R}\text{ -span}\{X_1, \ldots, X_k\} + \mathbb{R}\text{ -span}\{Y_{k+1}, \ldots, Y_n\}$ for $1 \leqslant k \leqslant n-1$.

Case II: For $\gamma = 0$, we get

$$(\mathrm{Ad}^*\, G)l_{\alpha,\beta,0} = \{l_{\alpha,\beta,0}\}$$

which are the points orbits in $\mathfrak{z}^\perp = \mathbb{R}X^* + \mathbb{R}Y^* \subseteq \mathfrak{g}^*$. In this case, $r_l = \mathfrak{g}$ is the unique polarizing subalgebra for l.

Now we are ready to see the Kirillov theory. Let's recall the notations here. Let $\mathfrak{g}^*$ be the dual vector space of $\mathfrak{g}$ and G acts on $\mathfrak{g}^*$ by the coadjoint action $\mathrm{Ad}^*(G)$. Give $l \in \mathfrak{g}^*$, let B_l be the linear form $B_l(X, Y) = l([X, Y])$

and let r_l be its radical. Choose a polarizing subalgebra $\mathfrak{m}$ for l and let $M = \exp \mathfrak{m}$. Define the map from M to S^1 by

$$\chi_{l,M}(\exp Y) = \exp 2\pi i l(Y),\ Y \in \mathfrak{m},$$

This is a one-dimensional representation of M, since $l([\mathfrak{m}, \mathfrak{m}]) = 0$. Then we denote the induced representation from M to G by $\pi_{l,M} = \operatorname{Ind}_M^G \chi_{l,M}$.

We know before that a polarizing subalgebra always exists and indeed we can find one that $\pi_{l,M}$ is irreducible.

2.14 Theorem. *Let $l \in \mathfrak{g}^*$. Then there exists a polarizing subalgebra $\mathfrak{m}$ for l such that $\pi_{l,M}$ is irreducible.*

The next theorem allows us to write π_l instead of $\pi_{l,M}$ since it is independent of the choices of polarizing subalgebras for l.

2.15 Theorem. *Let $l \in \mathfrak{g}^*$, and let $\mathfrak{m}, \mathfrak{m}'$ be two polarizing subalgebras for l. Then $\pi_{l,M} \cong \pi_{l,M'}$. In particular, $\pi_{l,M}$ is irreducible whenever $\mathfrak{m}$ is polarizing subalgebra for l.*

Next theorem tells us that all unitary irreducible representation of G must be in the form of π_l up to equivalence class. So we have

$$\widehat{G} = \{[\pi_l] \mid l \in \mathfrak{g}^*\},$$

where $[\pi_l]$ is the equivalence class of π_l.

2.16 Theorem. *Let π be any irreducible unitary representation of G. Then there is an $l \in \mathfrak{g}^*$ such that $\pi_l \cong \pi$.*

The following theorem gives a nice connection between unitary equivalent irreducible representations and the coadjoint orbits in $\mathfrak{g}^*$.

2.17 Theorem. *Let* $l, l' \in \mathfrak{g}^*$. *Then* $\pi_l \cong \pi_{l'} \Leftrightarrow l$ *and* l' *are in the same* $\mathrm{Ad}^*(G)$*-orbit in* $\mathfrak{g}^*$. *In other words,*

$$[\pi_l] = [\pi_{l'}] \Leftrightarrow \text{ there is an } x \in G \text{ such that, } l' = (\mathrm{Ad}^*\, x)l.$$

Combining these four theorems, we receive

2.18 Theorem (Kirillov Theory)**.** *There is a bijection between the coadjoint orbits* $\mathfrak{g}^*/\,\mathrm{Ad}^*(G)$ *and the collection of equivalence classes of irreducible unitary representations of* G *which is denoted by* $\widehat{G}$. *Indeed,*

$$\mathfrak{g}^*/\,\mathrm{Ad}^*(G) \cong \widehat{G}$$

via the map $l \mapsto [\pi_{l,M}]$ *which is independent of the choices of polarizing subalgebras* $\mathfrak{m}$.

The proofs of the Theorem 2.14-2.17 have some similarities. The proofs are all by induction on $\dim G$ and considered in 2 cases. In case 1, we suppose that there is 1-dimensional central subalgebra $\mathfrak{h}$ on which $l = 0$. The general idea is to apply induction hypothesis to $\mathfrak{g}/\mathfrak{h}$ and lift it back to $\mathfrak{g}$. For second case which we assume that $\mathfrak{g}$ has a 1-dimensional center on which l is nontrivial, we apply Kirillov's lemma and induction in stages in all proofs. Roughly speaking, the Heisenberg Lie group is the building block for general nilpotent Lie group. Now let's consider the exposition of Kirillov Theory in $\mathbb{R}^n$ and Heisenberg Lie group H_n.

2.19 Example. Consider $G = \mathbb{R}^n$. It is well known that $\widehat{G} \cong \mathbb{R}^n$, with $\lambda \in \mathbb{R}^n$ corresponding to the 1-dimensional representation $\chi_\lambda : G \to S^1$ defined by

$$\chi_\lambda(x) = e^{2\pi i \lambda \cdot x}.$$

Also $\mathfrak{g} = \mathbb{R}^n$, $\mathfrak{g}^* = \mathbb{R}^n$ and $\operatorname{Ad} x$, $\operatorname{Ad}^* x$ are the identity map for all $x \in G$ from example 2.12. Thus $\mathfrak{g}^*/\operatorname{Ad}^*(G) = \mathbb{R}^n$ and the map $l \mapsto \pi_l$ is simply the map $l \mapsto \chi_l$.

Let's consider the Kirillov Theory in the case of Heisenberg Lie group H_n. We showed in example 2.13 that the adjoint orbits in $\mathfrak{h}_n^*$ are

1. The singleton sets in $Z^\perp$ where $Z^\perp = \{l \mid l(Z) = 0\}$;

2. The hyperplanes $\lambda Z^* + Z^\perp$ for $\lambda \neq 0$, where

$$Z^*(Z) = 1, \quad Z^*(X) = 0, \quad Z^*(Y) = 0.$$

The complete classification of the irreducible unitary representations of Heisenberg Lie group H_n is the consequence of the famous Stone-von Neumann theorem (see Rosenberg in [26] or Prasad in [25] for more details on Stone-von Neumann theorem). The proof of the classification can be found in Corwin and Greenleaf [5] or in Folland [9], [10].

2.20 Proposition. *Every irreducible unitary representation of H_n is unitarily equivalent to one and only one of the folowing representations:*

1. *For $l \in Z^\perp$, the one-dimensional representation is*

$$\pi(\exp(W)) = e^{2\pi i l(W)}, \quad \text{for } W \in \mathfrak{h}_n;$$

2. *For $\lambda \in \mathbb{R}\backslash\{0\}$, the corresponding representation on $L^2(\mathbb{R}^n)$ is defined by*

$$[\pi(\exp(xX + yY + zZ))f](t) = e^{2\pi i(t\cdot y + \lambda z)} f(t + x)$$

where X, Y, Z are basis of Heisenberg Lie algebra $\mathfrak{h}_n$.

Reference: Chapter 2 in [5].

Chapter 3

A conjectural Kirillov isomorphism in C*-algebra theory

In this chapter we shall associate two C*-algebras to a simply connected nilpotent group G, and, inspired by Kirillov theory, we shall conjecture that they are isomorphic. We shall check the conjecture for the Heisenberg group.

The first C*-algebra, denoted $A(G)$, acts on the Hilbert space $L^2(G)$ and is related to the decomposition of $L^2(G)$ into irreducible representations. Recall that the group G acts on $L^2(G)$ by both the left and right regular representations, and we shall use both in the construction of $A(G)$. Namely we shall define $A(G)$ to be the image of the group C^*-algebra of $G \times G$ under the "bi-regular" representation of $G \times G$ on $L^2(G)$.

According to the abstract Plancherel theorem, the Hilbert space $L^2(G)$ decomposes as an integral of the Hilbert spaces $H_\pi \otimes H_{\overline{\pi}}$, as π ranges over the irreducible unitary representations of G. Here $H_{\overline{\pi}}$ is the complex conjugate of the Hilbert space H_π associated to the representation π. We see in this way that the C^*-algebra $A(G)$ is very closely related to the unitary representation theory of G.

The second C^*-algebra is denoted $A(\mathfrak{g}^*)$ and it acts on the Hilbert space $L^2(\mathfrak{g}^*)$. It is the image under the natural coadjoint/multiplication action of the crossed product C^*-algebra $C^*(G, C_0(\mathfrak{g}^*))$. Another way of describing $A(\mathfrak{g}^*)$ is to use the Fourier transform isomorphism

$$L^2(\mathfrak{g}) \cong L^2(\mathfrak{g}^*),$$

under which $A(\mathfrak{g}^*)$ becomes conjugate to the natural coadjoint/translation representation of the group $G \ltimes \mathfrak{g}$ on $L^2(\mathfrak{g})$.

As we noted in the introduction, the Hilbert space $L^2(\mathfrak{g}^*)$ decomposes as an integral of the Hilbert spaces $L^2(\mathcal{O})$ associated to the coadjoint orbits of G. This decomposition is also a decomposition of $L^2(\mathfrak{g}^*)$ into irreducible representations of $A(\mathfrak{g}^*)$. We see in this way that the C^*-algebra $A(\mathfrak{g}^*)$ is very closely related to the space of coadjoint orbits in $\mathfrak{g}^*$.

The above remarks motivate the conjecture. We will prove the conjecture for the Heisenberg group at the end of the chapter using the exponential map, which gives a unitary isomorphism from $L^2(G)$ to $L^2(\mathfrak{g})$. We will show that the unitary isomorphism conjugates $A(G)$ onto $A(\mathfrak{g}^*)$. Unfortunately this is almost certainly not true for all nilpotent groups in view of Remark 3.14.

3.1 Definition of $A(G)$

For each $g_1, g_2 \in G$, we define a map

$$T_{g_1,g_2} : L^2(G) \to L^2(G), \quad \text{by} \quad f \mapsto {}^{g_1}f^{g_2},$$

where ${}^{g_1}f^{g_2}(g) = f(g_1^{-1} g g_2)$ for every $g \in G$. Therefore

$$(T_{g_1,g_2} f)(g) = f(g_1^{-1} g g_2).$$

3.1 Remark. T is a unitary representation of $G \times G$ on $L^2(G)$ by sending $(g_1, g_2) \mapsto T_{g_1,g_2}$. To see that T_{g_1,g_2} is a homomorphism, consider

$$\begin{aligned}
[T_{g_1,g_3}(T_{g_2,g_4}f)](g) &= (T_{g_2,g_4}f)(g_1^{-1}gg_3) \\
&= f(g_2^{-1}g_1^{-1}gg_3g_4) \\
&= f\left((g_1g_2)^{-1}g(g_3g_4)\right) \\
&= (T_{(g_1g_2,g_3g_4)}f)(g) \\
&= (T_{(g_1,g_3)(g_2,g_4)}f)(g).
\end{aligned}$$

In other words, $T_{g_1,g_3} \circ T_{g_2,g_4} = T_{(g_1,g_3)(g_2,g_4)}$. Also T_{g_1,g_2} is a unitary operator, since

$$\begin{aligned}
\langle T^*_{g_1,g_2}f, h\rangle &= \langle f, T_{g_1,g_2}h\rangle \\
&= \int_G f(g)\overline{T_{g_1,g_2}h}(g)\,dg \\
&= \int_G f(g)\overline{h}(g_1^{-1}gg_2).
\end{aligned}$$

Changing variables by setting $k = g_1^{-1}gg_2$. Then $g = g_1kg_2^{-1}$ and $dk = dg$. Therefore

$$\begin{aligned}
\langle T^*_{g_1,g_2}f, h\rangle &= \int_G f(g_1kg_2^{-1})\overline{h}k\,dk \\
&= \langle T_{g_1^{-1},g_2^{-1}}f, h\rangle \\
&= \langle T_{(g_1,g_2)^{-1}}f, h\rangle = \langle T^{-1}_{(g_1,g_2)}f, h\rangle
\end{aligned}$$

In other words, $T^*_{g_1,g_2} = T^{-1}_{g_1,g_2}$.

By the remark above, we therefore obtain a representation

$$C^*(G \times G) \to B(L^2(G)).$$

We define $A(G)$ as the image under this map. This is automatically closed, so a C*-algebra. In other words, $A(G) \subset B(L^2(G))$ is the C*-algebra closure of all the operators

$$T = \int_G \int_G \varphi(g_1, g_2) T_{g_1, g_2} \, dg_1 \, dg_2,$$

where $\varphi \in C_c^\infty(G \times G)$. This means that

$$\langle T f_1, f_2 \rangle = \int_G \int_G \varphi(g_1, g_2) \langle T_{g_1 g_2} f_1, f_2 \rangle_{L^2(G)} \, dg_1 \, dg_2,$$

for all $f_1, f_2 \in L^2(G)$.

3.2 Definition of $A(\mathfrak{g}^*)$

For each $g \in G, X \in \mathfrak{g}$, we define a map

$$S_{g,X} : L^2(\mathfrak{g}) \to L^2(\mathfrak{g}), \quad f \mapsto {}^g f^X,$$

where ${}^g f^X(Y) = f(g^{-1} Y g - X)$ for all $Y \in \mathfrak{g}$. Hence

$$(S_{g,X}(f))(Y) = f(\mathrm{Ad}_{g^{-1}}(Y) - X) = f(g^{-1} Y g - X).$$

3.2 Remark. S is a unitary representation of $G \ltimes \mathfrak{g}$ on $L^2(\mathfrak{g})$ by $(g, X) \mapsto S_{g,X}$. To see this, let $(g_1, X_1), (g_2, X_2) \in G \ltimes \mathfrak{g}$. We consider

$$\begin{aligned}
\left(S_{g_1,X_1}(S_{g_2,X_2} f)\right)(Y) &= (S_{g_2,X_2} f)(g_1^{-1} Y g_1 - X_1) \\
&= f(g_2^{-1}(g_1^{-1} Y g_1 - X_1) g_2 - X_2) \\
&= f((g_1 g_2)^{-1} Y (g_1 g_2) - (g_2^{-1} X_1 g_2 + X_2)) \\
&= (S_{g_1 g_2, g_2^{-1} X_1 g_2 + X_2} f)(Y) \\
&= (S_{(g_1,X_1)(g_2,X_2)} f)(Y),
\end{aligned}$$

therefore $S_{(g_1,X_1)} S_{(g_2,X_2)} = S_{(g_1,X_1)(g_2,X_2)}$. Moreover for any $f, h \in L^2(\mathfrak{g})$,

$$\begin{aligned}
\langle S_{g,X}^* f, h \rangle &= \langle f, S_{g,X} h \rangle \\
&= \int_{\mathfrak{g}} f(Z) \overline{S_{g,X} h}(Z) \, dZ \\
&= \int_{\mathfrak{g}} f(Z) \overline{h}(g^{-1} Z g - X) \, dZ.
\end{aligned}$$

Let $Y = g^{-1}Zg - X$. Then $Z = g(Y + X)g^{-1}$, $dZ = dY$ and so

$$\begin{aligned}
\langle S^*_{g,X} f, h\rangle &= \int_{\mathfrak{g}} f(g(Y+X)g^{-1})\overline{h}(Z)\,dZ \\
&= \int_{\mathfrak{g}} f(gYg^{-1} + gXg^{-1})\overline{h}(Z)\,dZ \\
&= \int_{\mathfrak{g}} (S_{g^{-1},-gXg^{-1}} f)(Z)\overline{h}(Z)\,dz \\
&= \langle S_{(g,X)^{-1}} f, h\rangle = \langle S^{-1}_{(g,X)} f, h\rangle.
\end{aligned}$$

Therefore $S^*_{g,X} = S^{-1}_{g,X}$. In other words, $S_{g,X}$ is a unitary homomorphism from $L^2(\mathfrak{g})$ to $L^2(\mathfrak{g})$.

From the remark above, therefore we get a representation

$$C^*(G \ltimes \mathfrak{g}) \to B(L^2(\mathfrak{g})).$$

We define $A(\mathfrak{g}^*)$ as the image under this map. So $A(\mathfrak{g}^*) \subset B(L^2(\mathfrak{g}))$ is the C*-algebra closure of all the operators

$$S = \int_{\mathfrak{g}} \int_G \psi(g, X) S_{g,X}\, dg\, dX,$$

where $\psi \in C_c^\infty(G \ltimes \mathfrak{g})$. This means that

$$\langle Sf_1, f_2\rangle = \int_{\mathfrak{g}} \int_G \psi(g, X)\langle S_{g,X} f_1, f_2\rangle_{L^2(\mathfrak{g})}\, dg\, dX,$$

for all $f_1, f_2 \in L^2(\mathfrak{g})$.

3.3 Remark. Alternatively, we can define $A(\mathfrak{g}^*)$ by using $L^2(\mathfrak{g}^*)$ since Fourier transform is a unitary isomorphism from $L^2(\mathfrak{g})$ to $L^2(\mathfrak{g}^*)$ by $f \mapsto \widehat{f}$, where

$$\widehat{f}(l) = \int_{\mathfrak{g}} f(X) e^{il(X)}\, dX$$

for a suitable choice of translation invariant measure on $\mathfrak{g}^*$. Here we can choose dX so that $X \mapsto \exp(X)$ from $\mathfrak{g}$ to G is measure preserving. See Remark 5.1 in Chapter 5.

3.3 The conjecture

We conjecture that there is an isomorphism between $A(G)$ and $A(\mathfrak{g}^*)$ at least in some cases. We will prove the conjecture for the Heisenberg group H_1 in the next section using the exponential map, which gives a unitary isomorphism from $L^2(G)$ to $L^2(\mathfrak{g})$. Unfortunately this method almost certainly fails for more general nilpotent groups as we shall also see at the end of this chapter.

Recall that for any nilpotent Lie group G and its Lie algebra $\mathfrak{g}$,

$$\exp : \mathfrak{g} \to G$$

is a measure-preserving diffeomorphism. This gives a unitary isomorphism map

$$U : L^2(G) \to L^2(\mathfrak{g}), \quad f \mapsto f \circ \exp,$$

for every $f \in L^2(G)$. Its adjoint $U^* = U^{-1}$ is defined by $f \mapsto f \circ \log$. Consequently, we obtain a map

$$UT_{g_1,g_2}U^{-1} : L^2(\mathfrak{g}) \to L^2(\mathfrak{g}),$$

for each pair of elments $g_1, g_2 \in G$. For any $f \in L^2(\mathfrak{g})$, we have

$$(UT_{g_1,g_2}U^{-1})(f) = UT_{g_1,g_2}(f \circ \log) = U(\ {}^{g_1}(f \circ \log)^{g_2}) = \ {}^{g_1}(f \circ \log)^{g_2} \circ \exp .$$

Therefore

$$(UT_{g_1,g_2}U^{-1}(f))(Y) = f(\log(g_1(\exp Y)g_2)),$$

for any $Y \in \mathfrak{g}$.

3.4 Remark. Note that if $x \in G \times G$ and $x = (g_1^{-1}, g_2)$ then

$$(T_x f)(g) = f(g_1 g g_2).$$

Similarly, if $z \in G \ltimes \mathfrak{g}$ and $z = (g^{-1}, -X)$ then

$$(S_z f)(Y) = f(gYg^{-1} + X).$$

To simplify our calculation, we shall use the following notations from now on

$$(T_{g_1,g_2} f)(g) = f(g_1^{-1} g g_2), \quad (S_{g,X}(f))(Y) = f(gYg^{-1} + X).$$

Next lemma is straightforward to prove. And we shall use in the next section.

3.5 Lemma. *For $g_1, g_2, g_3, g_4 \in G$ and $X_1, X_2 \in \mathfrak{g}$, we have*

$$\begin{aligned} \left(UT_{g_3,g_4}U^{-1}\right)\left(UT_{g_1,g_2}U^{-1}\right) &= UT_{g_1g_3,g_4g_2}U^{-1} \\ S_{g_2,X_2}S_{g_1,X_1} &= S_{g_1g_2,g_1X_2g_1^{-1}+X_1}. \end{aligned}$$

Proof. Let $g_1, g_2, g_3, g_4 \in G$, $f \in L^2(\mathfrak{g})$ and $X \in \mathfrak{g}$. We have

$$\begin{aligned} \left(\left(UT_{g_3,g_4}U^{-1}\right)\left(UT_{g_1,g_2}U^{-1}\right)(f)\right)(X) &= \left(\left(UT_{g_1,g_2}U^{-1}\right)(f)\right)(\log(g_3(\exp X)g_4)) \\ &= f(\log(g_1 \exp(\log(g_3(\exp X)g_4))g_2)) \\ &= f(\log(g_1g_3(\exp X)g_4g_2)) \\ &= \left(\left(UT_{g_1g_3,g_4g_2}U^{-1}\right)(f)\right)(X). \end{aligned}$$

Also, for any $Y \in \mathfrak{g}$, we have

$$\begin{aligned} \left(S_{g_2,X_2}S_{g_1,X_1}(f)\right)(Y) &= S_{g_2,X_2}\left(S_{g_1,X_1}(f)\right)(Y) \\ &= \left(S_{g_1,X_1}(f)\right)(g_2Y{g_2}^{-1} + X_2) \\ &= f\left(g_1(g_2Y{g_2}^{-1} + X_2){g_1}^{-1} + X_1\right) \\ &= f(g_1g_2(Y){g_2}^{-1}{g_1}^{-1} + g_1X_2{g_1}^{-1} + X_1) \\ &= \left(S_{g_1g_2,g_1X_2{g_1}^{-1}+X_1}(f)\right)(Y). \end{aligned}$$

□

3.4 The case of the Heisenberg group

Now we focus only on the Heisenberg group H_1 and its Lie algebra $\mathfrak{h}_1$. For simplicity, we write

$$(x, y, z) \text{ for } \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \in H_1, \text{ and}$$

$$[a, b, c] \text{ for } \begin{pmatrix} 0 & a & c \\ & 0 & b \\ & & 0 \end{pmatrix} \in \mathfrak{h}_1.$$

3.6 Lemma. *For all $a, b, c, x, y, z \in \mathbb{R}$, we have*

1. $\exp[a, b, c] = (a, b, c + \frac{ab}{2})$
2. $\log(x, y, z) = [x, y, z - \frac{xy}{2}]$
3. $(x, y, z)(a, b, c) = (x + a, y + b, z + c + xb)$
4. $(x, y, z)^{-1} = (-x, -y, xy - z)$
5. $[a, b, c](x, y, z) = [a, b, ay + c]$
6. $(x, y, z)[a, b, c] = [a, b, xb + c]$
7. $[x, y, z] + [a, b, c] = [x + a, y + b, z + c]$

Proof. We shall show only first two identities since the rest is straightforward. Let a, b, c, x, y, z be real numbers.

$$\begin{aligned} \exp[a, b, c] &= \sum_{n=0}^{\infty} \frac{[a, b, c]^n}{n!} = I + [a, b, c] + \frac{[a, b, c]^2}{2} + 0 \\ &= I + [a, b, c] + \frac{1}{2}[0, 0, ab] = (a, b, c + \frac{ab}{2}). \end{aligned}$$

Also, we have

$$\begin{aligned}\log(x,y,z) &= \log[I-(I-(x,y,z))] = \sum_{n=1}^{\infty} -\frac{(I-(x,y,z))^n}{n} \\ &= -(I-(x,y,z)) - \frac{(I-(x,y,z))^2}{2} - 0 \\ &= [x,y,z] - \frac{1}{2}[-x,-y,-z]^2 \\ &= [x,y,z] - \frac{1}{2}[0,0,xy] = [x,y,z-\frac{1}{2}xy]. \qquad \square\end{aligned}$$

3.7 Lemma. *Let* $(u,v,w),(x,y,z),(x',y',z') \in H_1$, *and* $[\alpha,\beta,\gamma],[a,b,c] \in \mathfrak{h}_1$. *For any* $f \in L^2(\mathfrak{g})$, *we have*

$$\begin{aligned}&(UT_{(x,y,z)(x',y',z')}U^{-1}(f))([a,b,c]) \\ &\quad = f([x+a+x', y+b+y', z+c+z'+\frac{ab}{2}+xb+xy'+ay' \\ &\qquad - \frac{(x+a+x')(y+b+y')}{2}]) \\ &\quad = f([x+a+x', y+b+y', z+c+z'+\frac{1}{2}(xy'-xy-x'y-x'y') \\ &\qquad + \frac{x-x'}{2}b - \frac{y-y'}{2}a]) \\ &(S_{(u,v,w),[\alpha,\beta,\gamma]}(f))([a,b,c]) = f([a+\alpha, b+\beta, c+\gamma+ub-av]).\end{aligned}$$

Proof. By using identities in Lemma 3.6, we have

$$
\begin{aligned}
&(UT_{(x,y,z)(x',y',z')}U^{-1}(f))([a,b,c]) \\
&\quad = f(\log((x,y,z)(\exp[a,b,c])(x',y',z'))) \\
&\quad = f(\log((x,y,z)(a,b,c+\frac{ab}{2})(x',y',z'))) \\
&\quad = f(\log((x+a,y+b,z+c+\frac{ab}{2}+xb)(x',y',z'))) \\
&\quad = f(\log((x+a+x',y+b+y',z+c+z'+\frac{ab}{2}+xb+xy'+ay')) \\
&\quad = f([x+a+x',y+b+y',z+c+z'+\frac{ab}{2}+xb+xy'+ay' \\
&\qquad -\frac{(x+a+x')(y+b+y')}{2}]) \\
&\quad = f([x+a+x',y+b+y',z+c+z'+\frac{1}{2}(xy'-xy-x'y-x'y') \\
&\qquad +\frac{x-x'}{2}b-\frac{y-y'}{2}a]).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
(S_{(u,v,w),[\alpha,\beta,\gamma]}(f))([a,b,c]) &= f((u,v,w)[a,b,c](u,v,w)^{-1}+[\alpha,\beta,\gamma]) \\
&= f((u,v,w)[a,b,c](-u,-v,uv-w)+[\alpha,\beta,\gamma]) \\
&= f([a,b,ub+c](-u,-v,uv-w)+[\alpha,\beta,\gamma]) \\
&= f([a,b,-av+ub+c]+[\alpha,\beta,\gamma]) \\
&= f([a+\alpha,b+\beta,c+\gamma+ub-av]). \qquad \square
\end{aligned}
$$

3.8 Lemma. *For any* $(u,v,w),(\alpha,\beta,\gamma),(u',v',w'),(\alpha',\beta',\gamma') \in H_1$, *and* $[\alpha,\beta,\gamma],[\alpha',\beta',\gamma'] \in \mathfrak{h}_1$, *we have*

$$
\begin{aligned}
&\left(UT_{(u',v',w')(\alpha',\beta',\gamma')}U^{-1}\right)\left(UT_{(u,v,w)(\alpha,\beta,\gamma)}U^{-1}\right) \\
&\qquad\qquad = UT_{(u+u',v+v',w+w'+uv')(\alpha'+\alpha,\beta'+\beta,\gamma'+\gamma+\alpha'\beta)}U^{-1},
\end{aligned}
$$

$$
S_{(u',v',w')[\alpha',\beta',\gamma']}S_{(u,v,w)[\alpha,\beta,\gamma]}7 = S_{(u+u',v+v',w+w'+uv')[\alpha+\alpha',\beta+\beta',\gamma+\gamma'+u\beta'-v\alpha']}.
$$

Proof. We use the relationships we found in Lemma 3.5, then apply identities in Lemma 3.6 to obtain

$$\begin{aligned}\left(UT_{(u',v',w'),(\alpha',\beta',\gamma')}U^{-1}\right)&\left(UT_{(u,v,w)(\alpha,\beta,\gamma)}U^{-1}\right)\\ &= UT_{(u,v,w)(u',v',w'),(\alpha',\beta',\gamma')(\alpha,\beta,\gamma)}U^{-1}\\ &= UT_{(u+u',v+v',w+w'+uv'),(\alpha'+\alpha,\beta'+\beta,\gamma'+\gamma+\alpha'\beta)}U^{-1}.\end{aligned}$$

Also, we have

$$\begin{aligned}S_{(u',v',w')[\alpha',\beta',\gamma']}&S_{(u,v,w)[\alpha,\beta,\gamma]}\\ &= S_{(u,v,w)(u',v',w'),(u,v,w)[\alpha',\beta',\gamma'](u,v,w)^{-1}+[\alpha,\beta,\gamma]}\\ &= S_{(u+u',v+v',w+w'+uv'),[\alpha',\beta',u\beta'+\gamma'](-u,-v,uv-w)+[\alpha,\beta,\gamma]}\\ &= S_{(u+u',v+v',w+w'+uv'),[\alpha',\beta',-\alpha'v+u\beta'+\gamma']+[\alpha,\beta,\gamma]}\\ &= S_{(u+u',v+v',w+w'+uv')[\alpha+\alpha',\beta+\beta',\gamma+\gamma'+u\beta'-v\alpha']}.\end{aligned}$$ □

3.9 Proposition. *For the Heisenberg group H_1 and its Lie algebra $\mathfrak{h}_1$, every UTU^{-1} is equal to some S and every $U^{-1}SU$ is equal to some T. More explicitly, we have*

$$UT_{(x,y,z)(x',y',z')}U^{-1} = S_{(\frac{x-x'}{2},\frac{y-y'}{2},z')[x+x',y+y',z+z'+\frac{1}{2}(xy'-xy-x'y-x'y')]} \quad (3.1)$$

$$U^{-1}S_{(u,v,w),[\alpha,\beta,\gamma]}U = T_{(u+\frac{\alpha}{2},v+\frac{\beta}{2},\gamma+uv+\frac{\alpha\beta}{4}-\frac{u\beta}{2}+\frac{\alpha v}{2}-w),(-u+\frac{\alpha}{2},-v+\frac{\beta}{2},w)}. \quad (3.2)$$

In other words, there are polynomial maps $P : H_1 \times H_1 \to H_1 \times \mathfrak{h}_1$ and $Q : H_1 \times \mathfrak{h}_1 \to H_1 \times H_1$, which are defined by

$$\begin{aligned}&P(x,y,z,x',y',z')\\ &\quad= \left(\frac{x-x'}{2},\frac{y-y'}{2},z',x+x',y+y',z+z'+\frac{1}{2}(xy'-xy-x'y-x'y')\right)\\ &Q(u,v,w,\alpha,\beta,\gamma)\\ &\quad= \left(u+\frac{\alpha}{2},v+\frac{\beta}{2},\gamma+uv+\frac{\alpha\beta}{4}-\frac{u\beta}{2}+\frac{\alpha v}{2}-w,-u+\frac{\alpha}{2},-v+\frac{\beta}{2},w\right)\end{aligned}$$

satisfying

1. $P(Q(g,X)) = (g,X)$ *and* $Q(P(g_1,g_2)) = (g_1,g_2)$

2. $UT_{g_1,g_2}U^{-1} = S_{P(g_1,g_2)}$

3. $U^{-1}S_{g,X}U = T_{Q(g,X)}$,

for all $g, g_1, g_2 \in H_1$ *and* $X \in \mathfrak{h}_1$. *Moreover, the maps* P *and* Q *are diffeomorphisms.*

Proof. The identity (3.1) is easily followed by Lemma 3.7:

$$\begin{aligned}
&\left(S_{(\frac{x-x'}{2},\frac{y-y'}{2},z')[x+x',y+y',z+z'+\frac{1}{2}(xy'-xy-x'y-x'y')]}(f)\right)([a,b,c]) \\
&\qquad = f([x+a+x', y+b+y', z+c+z'+\frac{1}{2}(xy'-xy-x'y-x'y') \\
&\qquad\quad + \frac{x-x'}{2}b - \frac{y-y'}{2}a]) \\
&\qquad = (UT_{(x,y,z)(x',y',z')}U^{-1}(f))([a,b,c]),
\end{aligned}$$

for all $f \in L^2(\mathfrak{h}_1)$ and $[a,b,c] \in \mathfrak{h}_1$. Similarly, by applying Lemma 3.7, we have

$$\begin{aligned}
&(UT_{(u+\frac{\alpha}{2},v+\frac{\beta}{2},\gamma+uv+\frac{\alpha\beta}{4}-\frac{u\beta}{2}+\frac{\alpha v}{2}-w),(-u+\frac{\alpha}{2},-v+\frac{\beta}{2},w)}U^{-1}(f))([a,b,c]) \\
&\quad = f([a+\alpha, b+\beta, \gamma+uv+\frac{\alpha\beta}{4}-\frac{u\beta}{2}+\frac{\alpha v}{2}+c+\dagger+ub-va]),
\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
\dagger = \frac{1}{2}\Big[&\left(u+\frac{\alpha}{2}\right)\left(-v+\frac{\beta}{2}\right) - \left(u+\frac{\alpha}{2}\right)\left(v+\frac{\beta}{2}\right) \\
&- \left(-u+\frac{\alpha}{2}\right)\left(v+\frac{\beta}{2}\right) - \left(-u+\frac{\alpha}{2}\right)\left(-v+\frac{\beta}{2}\right)\Big],
\end{aligned}$$

which can be simplified to

$$\dagger = \frac{1}{2}\left(-2uv - \alpha v + u\beta - \frac{\alpha\beta}{2}\right). \tag{3.4}$$

Therefore

$$\begin{aligned}
&\gamma + uv + \frac{\alpha\beta}{4} - \frac{u\beta}{2} + \frac{\alpha v}{2}c + \dagger + ub - va \\
&\quad = \gamma + uv + \frac{\alpha\beta}{4} - \frac{u\beta}{2} + \frac{\alpha v}{2} + c + \frac{1}{2}\left(-2uv - \alpha v + u\beta - \frac{\alpha\beta}{2}\right) + ub - va \\
&\quad = c + \gamma + ub - av. \qquad (3.5)
\end{aligned}$$

Substitute (3.5) into (3.3),

$$\begin{aligned}
&(UT_{(u+\frac{\alpha}{2}, v+\frac{\beta}{2}, \gamma+uv+\frac{\alpha\beta}{4}-\frac{u\beta}{2}+\frac{\alpha v}{2}-w),(-u+\frac{\alpha}{2}, -v+\frac{\beta}{2}, w)}U^{-1}(f))([a,b,c]) \\
&\qquad = f([a+\alpha, b+\beta, c+\gamma+ub-av]) \\
&\qquad = (S_{(u,v,w),[\alpha,\beta,\gamma]}(f))([a,b,c]),
\end{aligned}$$

for all $f \in L^2(\mathfrak{h}_1)$ and $[a,b,c] \in \mathfrak{h}_1$. Therefore (3.2) is followed. Now let's check the conditions $P \circ Q = \text{Id}$ and $Q \circ P = \text{Id}$. We shall consider only at the 6th cooordinate of $P \circ Q$ and the third coordinate of $Q \circ P$ since the rest is easy to see. Using (3.4),

$$\begin{aligned}
P(Q(\gamma)) &= \left(\gamma + uv + \frac{\alpha\beta}{4} - \frac{u\beta}{2} + \frac{\alpha v}{2}\right) + \dagger \\
&= \left(\gamma + uv + \frac{\alpha\beta}{4} - \frac{u\beta}{2} + \frac{\alpha v}{2}\right) + \frac{1}{2}\left(-2uv - \alpha v + u\beta - \frac{\alpha\beta}{2}\right) \\
&= \gamma.
\end{aligned}$$

Also, we have

$$Q(P(z)) = z + \frac{1}{2}\left(xy' - xy - x'y - x'y'\right) + *, \qquad (3.6)$$

where

$$\begin{aligned}
* = \frac{1}{2}\Big[&\left(\frac{x-x'}{2}\right)\left(\frac{y-y'}{2}\right) + \frac{(x+x')(y+y')}{4} \\
&- \left(\frac{x-x'}{2}\right)\left(\frac{y+y'}{2}\right) - \left(\frac{x+x'}{2}\right)\left(\frac{y-y'}{2}\right)\Big]
\end{aligned}$$

which can be simplified to

$$* = \frac{1}{2}\left(-xy' + xy + x'y + x'y'\right). \tag{3.7}$$

Substitute (3.7) into (3.6),

$$Q(P(z)) = z + \frac{1}{2}\left(xy' - xy - x'y - x'y'\right) + \frac{1}{2}\left(-xy' + xy + x'y + x'y'\right) = z.$$

Now we shall consider the determinant of Jacobian matrices J_P and J_Q as follow:

$$\begin{aligned}
J_P &= \begin{vmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ \frac{y'-y}{2} & \frac{-x-x'}{2} & 1 & \frac{-y-y'}{2} & \frac{x-x'}{2} & 1 \end{vmatrix} \\
&\overset{\substack{\frac{1}{2}R_4+R_1\to R_1 \\ \frac{1}{2}R_5+R_2\to R_2}}{=} \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ \frac{y'-y}{2} & \frac{-x-x'}{2} & 1 & \frac{-y-y'}{2} & \frac{x-x'}{2} & 1 \end{vmatrix} \\
&\overset{C_3\leftrightarrow C_6}{=} (-1)\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ \frac{y'-y}{2} & \frac{-x-x'}{2} & 1 & \frac{-y-y'}{2} & \frac{x-x'}{2} & 1 \end{vmatrix} \\
&= -1.
\end{aligned}$$

Similarly, we have

$$J_Q = \begin{vmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ v-\frac{\beta}{2} & u+\frac{\alpha}{2} & -1 & \frac{\beta}{4}+\frac{v}{2} & \frac{\alpha}{4}-\frac{u}{2} & 1 \\ -1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{vmatrix}$$

$$\overset{\substack{-R_4+R_1\to R_1 \\ -R_5+R_2\to R_2}}{=} \begin{vmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ v-\frac{\beta}{2} & u+\frac{\alpha}{2} & -1 & \frac{\beta}{4}+\frac{v}{2} & \frac{\alpha}{4}-\frac{u}{2} & 1 \\ -1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{vmatrix}$$

$$\overset{R_3\leftrightarrow R_6}{=} (-1) \begin{vmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 & 0 & \frac{1}{2} & 0 \\ v-\frac{\beta}{2} & u+\frac{\alpha}{2} & -1 & \frac{\beta}{4}+\frac{v}{2} & \frac{\alpha}{4}-\frac{u}{2} & 1 \end{vmatrix}$$

$$= -1.$$ □

By proposition above, it follows easily that two C*-algebras $A(G)$ and $A(\mathfrak{g}^*)$ are isomorphic in the case of Heisenberg group H_1.

3.10 Corollary. *For Heisenberg group $G = H_1$ and its Lie algebra $\mathfrak{g} = \mathfrak{h}_1$,*

$$UA(G)U^{-1} = A(\mathfrak{g}).$$

Consequently, U is an isomorphism from $A(G)$ onto $A(\mathfrak{g}^*)$.

Proof. By using the properties from previous proposition, we have

$$\begin{aligned} U\left(\int_G\int_G \varphi(g_1,g_2)T_{g_1g_2}\,dg_1\,dg_2\right)U^{-1} &= \int_G\int_G \varphi(g_1,g_2)UT_{g_1g_2}U^{-1}\,dg_1\,dg_2 \\ &= \int_G\int_G \varphi(g_1,g_2)S_{P(g_1,g_2)}\,dg_1\,dg_2 \\ &= \int_G\int_{\mathfrak{g}} \varphi(Q(g,X))S_{g,X}|\det J_P|\,dg\,dX \\ &= \int_G\int_{\mathfrak{g}} \varphi(Q(g,X))S_{g,X}\,dg\,dX, \end{aligned}$$

for any $\varphi \in C_c^\infty(G\times G)$. □

3.5 Other nilpotent groups

In this section, we shall see that the same proof fails for Lie group K_3. For Lie group K_3 and its Lie algebra $\mathfrak{k}_3$, we write

$$(w,x,y,z) \quad \text{for} \quad \begin{pmatrix} 1 & w & \frac{w^2}{2} & z \\ & 1 & w & y \\ & & 1 & x \\ & & & 1 \end{pmatrix} \in K_3,$$

$$[a,b,c,d] \quad \text{for} \quad \begin{pmatrix} 0 & a & 0 & d \\ & 0 & a & c \\ & & 0 & b \\ & & & 0 \end{pmatrix} \in \mathfrak{k}_3, \text{ and}$$

$$[a,b,c,d]_e \quad \text{for} \quad \begin{pmatrix} 0 & a & e & d \\ & 0 & a & c \\ & & 0 & b \\ & & & 0 \end{pmatrix}.$$

Similar to what we did for Heisenberg group H_1, we start by couple lemmas involving calculation to obtain formulas for $UT_{g_1,g_2}U^{-1}$ and $S_{g,X}$.

3.11 Lemma. *Let* $(w,x,y,z),(w',x',y',z')\in K_3$, *and* $[a,b,c,d]\in\mathfrak{k}_3$. *Then*

$$\begin{aligned}
\exp[a,b,c,d] &= (a,b,c+\frac{ab}{2},d+\frac{ac}{2}+\frac{a^2b}{6}) \\
\log(w,x,y,z) &= [w,x,y-\frac{wx}{2},z-\frac{wy}{2}+\frac{w^2x}{12}] \\
(w,x,y,z)(w',x',y',z') &= (w+w',x+x',y+y'+wx',z+z'+wy'+\frac{w^2x'}{2}) \\
(w,x,y,z)^{-1} &= (-w,-x,wx-y,-\frac{w^2x}{2}+wy-z) \\
[a,b,c,d](w,x,y,z) &= [a,b,c+ax,d+ay]_{wa} \\
(w,x,y,z)[a,b,c,d] &= [a,b,c+wb,d+wc+\frac{w^2b}{2}]_{wa}.
\end{aligned}$$

Proof. Again, we shll show only the first two identities.

$$\begin{aligned}
\exp[a,b,c,d] &= \sum_{n=0}^{\infty}\frac{[a,b,c,d]^n}{n!} \\
&= I+[a,b,c,d]+\frac{[a,b,c,d]^2}{2!}+\frac{[a,b,c,d]^3}{3!}+0 \\
&= I+\begin{pmatrix}0 & a & 0 & d\\ & 0 & a & c\\ & & 0 & b\\ & & & 0\end{pmatrix}+\frac{1}{2}\begin{pmatrix}0 & 0 & a^2 & ac\\ & 0 & 0 & ab\\ & & 0 & 0\\ & & & 0\end{pmatrix}+\frac{1}{6}\begin{pmatrix}0 & 0 & 0 & a^2b\\ & 0 & 0 & 0\\ & & 0 & 0\\ & & & 0\end{pmatrix} \\
&= \begin{pmatrix}1 & a & \frac{a^2}{2} & d+\frac{ac}{2}+\frac{a^2b}{6}\\ & 1 & a & c+\frac{ab}{2}\\ & & 1 & b\\ & & & 1\end{pmatrix} \\
&= (a,b,c+\frac{ab}{2},d+\frac{ac}{2}+\frac{a^2b}{6}).
\end{aligned}$$

Also, we have

$$\begin{aligned}
\log(w,x,y,z) &= \log I - (I-(w,x,y,z)) \\
&= \sum_{n=1}^{\infty} -\frac{(I-(w,x,y,z))^n}{n} \\
&= -(I-(w,x,y,z)) - \frac{(I-(w,x,y,z))^2}{2} - \frac{(I-(w,x,y,z))^3}{3} - 0 \\
&= -\begin{pmatrix} 0 & -w & -\frac{w^2}{2} & -z \\ & 0 & -w & -y \\ & & 0 & -x \\ & & & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & 0 & w^2 & wy+\frac{w^2x}{2} \\ & 0 & 0 & wx \\ & & 0 & 0 \\ & & & 0 \end{pmatrix} \\
&\quad - \frac{1}{3}\begin{pmatrix} 0 & 0 & 0 & -w^2x \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & w & 0 & z-\frac{wy}{2}+\frac{w^2x}{12} \\ & 0 & w & y-\frac{wx}{2} \\ & & 0 & x \\ & & & 0 \end{pmatrix} \\
&= [w,x,y-\frac{wx}{2}, z-\frac{wy}{2}+\frac{w^2x}{12}].
\end{aligned}$$

□

3.12 Lemma. *For Lie group K_3 and its Lie algebra $\mathfrak{k}_3$, we have*

$$\begin{aligned}
&(UT_{(w',x',y',z)(w',x',y',z')}U^{-1}(f))[a,b,c,d] \\
&\quad = f([a+w+w', b+x+x', c+y+y'+\frac{1}{2}(wb+wx'+ax') \\
&\qquad - \frac{1}{2}(ax+wx+w'b+w'x+w'x'), *]),
\end{aligned}$$

where

$$\begin{aligned}
* = d+z+z'+\frac{1}{2}(wc+wy'+ay') - \frac{1}{2}(ay+wy+w'c+w'y'+w'y) \\
- \frac{1}{3}(w'wb+w'wx'+w'ax') + \frac{1}{6}(wax'+wax+w'ax+w'wx) \\
+\frac{1}{12}(w^2x'+a^2x'+w^2b+w'^2b+a^2x+w^2x+w'^2x+w'^2x') - \frac{1}{12}(wab+w'ab)
\end{aligned} \tag{3.8}$$

and

$$(S_{(w,x,y,z),[\alpha\beta\gamma\delta]}(f))([a,b,c,d]) \\ = f([a+\alpha, b+\beta, c+\gamma+wb-xa, d+\delta+wc-ya+\frac{w^2b}{2}]),$$

for all $(w,x,y,z),(w',x',y',z') \in K_3,$ *and* $[\alpha,\beta,\gamma,\delta] \in \mathfrak{k}_3.$

Proof. Using identities we obtained in Lemma 3.11,

$$\begin{aligned}(S_{(w,x,y,z),[\alpha\beta\gamma\delta]}(f))([a,b,c,d]) &= f((w,x,y,z)[a,b,c,d](w',x',y',z') + [\alpha,\beta,\gamma,\delta]) \\ &= f([a,b,c+wb,d+wc+\frac{w^2b}{2}]_{wa}(-w,-x,wx-y,-\frac{w^2x}{2}+wy-z) \\ &\quad + [\alpha,\beta,\gamma,\delta]) \\ &= f([a,b,c+wb-xa,d+wc-ya+\frac{w^2b}{2}] + [\alpha,\beta,\gamma,\delta]) \\ &= f([a+\alpha,b+\beta,c+\gamma+wb-xa,d+\delta+wc-ya+\frac{w^2b}{2}])\end{aligned}$$

And also

$$\begin{aligned}&(UT_{(w,x,y,z)(w',x',y',z')}U^{-1}(f))([a,b,c,d]) \\ &= f(\log((w,x,y,z)(\exp[a,b,c,d])(w',x',y',z'))) \\ &= f(\log((w,x,y,z)(a,b,c+\frac{ab}{2},d+\frac{ac}{2}+\frac{a^2b}{6})(w',x',y',z'))) \\ &= f(\log((w+a,x+b,y+c+\frac{ab}{2}+wb,z+d+\frac{ac}{2}+\frac{a^2b}{6}+wc+\frac{wab}{2}+\frac{w^2b}{2}) \\ &\quad (w',x',y',z'))) \\ &= f(\log((a+w+w',b+x+x',c+y+y'+\frac{ab}{2}+wb+(w+a)x',\dagger))),\end{aligned}$$

where

$$\dagger = d+z+z'+\frac{ac}{2}+\frac{a^2b}{6}+wc+\frac{wab}{2}+\frac{w^2b}{2}+(w+a)y'+\frac{(w+a)^2x'}{2}.$$

Therefore

$$
\begin{aligned}
&(UT_{(w,x,y,z)(w',x',y',z')}U^{-1}(f))([a,b,c,d]) \\
&\qquad = f([a+w+w', b+x+x', c+y+y'+\frac{ab}{2}+wb+(w+a)x' \\
&\qquad\quad -\frac{(a+w+w')(b+x+x')}{2}], \nabla) \\
&\qquad = f([a+w+w', b+x+x', c+y+y'+\frac{1}{2}(wb+wx'+ax') \\
&\qquad\quad -\frac{1}{2}(ax+wx+w'b+w'x+w'x'), \nabla]),
\end{aligned}
$$

where

$$
\begin{aligned}
\nabla &= \dagger - \frac{(a+w+w')(c+y'+y+\frac{ab}{2}+wb+(w+a)x')}{2} \\
&\quad + \frac{(a+w+w')^2(b+x+x')}{12} \\
&= *.
\end{aligned}
$$

□

The following proposition tells us that the group K_3 can not be handled in the same way as the Heisenberg group for the purposes of our conjecture.

3.13 Proposition. *For Lie group K_3 and its Lie algebra $\mathfrak{k}_3$, there is some T that $U^{-1}TU \neq S$ for any S. On the other hand, there is some S that $S \neq UTU^{-1}$ for any T.*

Proof. Suppose that we can write $UT_{(1,1,0,0)(0,0,0,0)}U^{-1}$ as a certain S, which means we can find $k, l, m, n, \alpha, \beta, \gamma, \delta$, such that

$$S_{(k,l,m,n)[\alpha,\beta,\gamma,\delta]} = U^{-1}T_{(1,1,0,0)(0,0,0,0)}U.$$

From previous lemma, we have

$$
\begin{aligned}
&(S_{(k,l,m,n),[\alpha\beta\gamma\delta]}(f))([a,b,c,d]) \\
&\qquad = f([a+\alpha, b+\beta, c+\gamma+kb-la, d+\delta+kc-ma+\frac{k^2b}{2}]), \\
&(UT_{(1,1,0,0)(0,0,0,0)}U^{-1}(f))[a,b,c,d] \\
&\qquad = f([a+1, b+1, c+\frac{b}{2}-\frac{a}{2}-\frac{1}{2}, d+\frac{c}{2}+\frac{a}{6}+\frac{b}{12}+\frac{a^2}{12}+\frac{1}{12}-\frac{ab}{12}]).
\end{aligned}
$$

for any $a, b, c, d \in \mathbb{R}$. Consequently,

$$
\begin{aligned}
a+\alpha &= a+1 \\
b+\beta &= b+1 \\
c+\gamma+kb-la &= c+\frac{b}{2}-\frac{a}{2}-\frac{1}{2} \\
d+\delta+kc-ma+\frac{k^2b}{2} &= d+\frac{c}{2}+\frac{a}{6}+\frac{b}{12}+\frac{a^2}{12}+\frac{1}{12}-\frac{ab}{12},
\end{aligned}
$$

for all $a, b, c, d \in \mathbb{R}$. First two equations give $\alpha = \beta = 1$. Also $\gamma = -\frac{1}{2}, k = \frac{1}{2}$ and $l = \frac{1}{2}$ by third equation. Substitute $k = \frac{1}{2}$ into the last equation, we then have

$$
\delta - ma = \frac{a}{6} - \frac{b}{24} + \frac{a^2}{12} + \frac{1}{12} - \frac{ab}{12}.
$$

This is a contradiction since δ and m are simply constants and they are independent of variable b.

Now suppose that we can write $S_{(1,1,1,0)[2,2,0,0]}$ as a certain UTU^{-1}, which means we can find $w, x, y, z, w', x', y', z'$, such that

$$
UT_{(w,x,y,z)(w',x',y',z')}U^{-1} = S_{(1,1,1,0)[2,2,0,0]}.
$$

From previous lemma, we have

$$
(S_{(1,1,1,0)[2,2,0,0]}(f))([a,b,c,d]) = f([a+2, b+2, c+b-a, d+c-a+\frac{b}{2}]).
$$

Therefore if $*$ is the term in

$$\begin{aligned} a + w + w' &= a + 2 \\ b + x + x' &= b + 2 \\ c + y + y' + \frac{w - w'}{2} b - \frac{x - x'}{2} a + \frac{wx' - wx - w'x - w'x'}{2} &= c + b - a \\ * &= d + c - a + \frac{b}{2}, \end{aligned} \tag{3.9}$$

for all $a, b, c, d \in \mathbb{R}$ where $*$ is the one in (3.8). First two equations give $w + w' = 2$ and $x + x' = 2$, while the third equation gives $w - w' = 2$, $x - x' = 2$ and

$$y + y' + \frac{wx' - wx - w'x - w'x'}{2} = 0. \tag{3.10}$$

Then we have $w = x = 2$ and $w' = x' = 0$ which transform (3.10) to $y + y' = 2$ and (3.8) to

$$\begin{aligned} * &= d + z + z' + \frac{1}{2}(2c + 2y' + ay') - \frac{1}{2}(ay + 2y) + \frac{1}{6}(4a) \\ &\quad + \frac{1}{12}(4b + 2a^2 + 8) - \frac{1}{12}(2ab) \\ &= d + z + z' + c + y' + \frac{ay'}{2} - \frac{ay}{2} - y + \frac{2a}{3} + \frac{b}{3} + \frac{a^2}{6} + \frac{2}{3} - \frac{ab}{6} \\ &= d + z + z' + c + (1 + \frac{a}{2})(y' - y) + \frac{2a}{3} + \frac{b}{3} + \frac{a^2}{6} + \frac{2}{3} - \frac{ab}{6}. \end{aligned} \tag{3.11}$$

Then we substitute (3.11) into (3.9), use $y' = 2 - y$ and then simplify it to obtain

$$0 = z + z' + (2 + a)(1 - y) + \frac{5a}{3} - \frac{b}{6} + \frac{a^2}{6} + \frac{2}{3} - \frac{ab}{6},$$

for $a, b, c, d \in \mathbb{R}$. This is again a contradiction since z, z' and y are simply constants and they are independent of variable b. □

3.14 Remark. This proposition does not conclusively show that U fails to conjugate $A(G)$ into $A(\mathfrak{g}^*)$, but it offers strong evidence that this is so. In any case, the method of proof that we used for the Heisenberg group H_1 certainly fails for K_3.

Chapter 4

Structure of $A(G)$ for the Heisenberg group

The purpose of this chapter is to investigate the structure of the C^*-algebra $A(G)$ in the case of the Heisenberg group (in the next chapter we shall do the same for the C^*-algebra $A(\mathfrak{g}^*)$). We shall determine the spectrum of the C^*-algebra, and use the information we find to decompose $A(G)$ as a C^*-algebra extension. We have already proved the main conjecture for the Heisenberg group, but since the method of proof does not carry over to more general nilpotent groups, we hope that a better understanding of the structure of $A(G)$ may eventually lead to proofs of the conjecture in more cases.

4.1 The unitary dual of a product

There is a natural map from $\widehat{G_1} \times \widehat{G_2}$ into $\widehat{G_1 \times G_2}$ defined by

$$([\pi_1], [\pi_2]) \mapsto [\pi_1 \otimes \pi_2].$$

It is indeed a well-defined injection map. In what follows we shall use the following more precise theorem of Wulfsohn (see [31] for proof and details).

4.1 Theorem (Wulfsohn). *If either G_1 or G_2 is postliminal, then the mapping*

$$([\pi_1], [\pi_2]) \mapsto [\pi_1 \otimes \pi_2]$$

is a homeomorphism from $\widehat{(G_1)_r} \times \widehat{(G_2)_r}$ onto $(\widehat{G_1 \times G_2})_r$.

4.2 The diagonal in the unitary dual of a product

Since $A(G)$ is a quotient of the C*-algebra of $G \times G$, its spectrum can be viewed as a closed subset of the unitary dual of $G \times G$. In this section we shall begin by considering the "diagonal" representations $\pi \otimes \pi^*$. Our first goal is to show that they lie in the spectrum of $A(G)$.

4.2 Theorem. *Let G be an amenable liminal Lie group. If π is an irreducible representation of G then $\pi \otimes \pi^*$ is weakly contained in the bi-regular representation β. So it is in the dual of $A(G)$.*

Proof. We'll prove by contradiction. We suppose that $\pi \otimes \pi^*$ is not weakly contained in β. Then

$$\ker(\beta) \not\subset \ker(\pi \otimes \pi^*).$$

Consequently, there is a basic open set in the product topology of $\widehat{G} \times \widehat{G}$ (thanks to previous theorem),

$$U_{I_1} \times U_{I_2} = \{(\pi_1, \pi_2) \mid \pi_1(I_1) \neq 0, \text{ and } \pi_2(I_2) \neq 0\}$$

containing $\pi \otimes \pi^*$ and not intersecting Supp(β). As indicated, $U_{I_1} \times U_{I_2}$ corresponds to an ideal $I_1 \otimes I_2 \subset C^*(G \times G)(= C^*(G) \otimes C^*(G))$. Since the open set $U_{I_1} \times U_{I_2}$ is disjoint from β, we have

$$\beta(I_1 \otimes I_2) = 0.$$

If $f \in C_c^\infty(G)$, then we define $\overline{f} \in C_c^\infty(G)$ by

$$\overline{f}(g) = \overline{f(g)},$$

for all $g \in G$. The map $f \mapsto \overline{f}$ extends to a map

$$C^*(G) \to C^*(G).$$

So we have that $\pi(f) = 0$ if and only if $\pi^*(\overline{f}) = 0$. Now we let

$$J_1 = I_1 \cap \overline{I_2} \quad \text{and} \quad J_2 = I_2 \cap \overline{I_1}.$$

Then we have $J_1 \subset I_1$, $J_2 \subset I_2$ and $J_1 = \overline{J_2}$. Therefore

$$\beta(J_1 \otimes J_2) = \beta(J_1 \otimes \overline{J_1}) = 0.$$

By liminality, $\pi[I_1] = K(\mathcal{H}_\pi)$ and $\pi[I_2] = K(\mathcal{H}_{\pi^*})$. Choose $i_1 \in I_1$ so that $\pi(i_1) \neq 0$ and similarly choose $i_2 \in I_2$, so that

$$\pi^*(i_2) = \pi(i_1).$$

Then consider the element

$$\overline{i_2}^* i_1 \in \overline{I_2} \cdot I_1 = \overline{I_2} \cap I_1 = J_1.$$

Here we use the fact that $I \cap J = I \cdot J$ for any ideals I, J in C*-algebra. Therefore

$$\pi(\overline{i_2}^* i_1) = \pi(\overline{i_2}^*)\pi(i_1) = \pi^*(i_2)^*\pi(i_1) = \pi(i_1)^*\pi(i_1) \neq 0.$$

Therefore $\pi(J_1) \neq 0$. Consequently, $\beta(J_1 \otimes \bar{J}_1) = 0$. So we have $J_1 \neq 0$, yet

$$\beta(J_1 \otimes \overline{J_1}) = 0.$$

Pick any nonzero $x \in J_1$. Then for any $y \in C_c^\infty(G)$, we have $yy^* \in C_c^\infty(G)$ and so $yy^* \in L^2(G)$. In addition, $\beta(x \otimes \overline{x})(yy^*) = xyy^*x^*$. So

$$xyy^*x^* = 0$$

for all $y \in C_c^\infty(G)$. The convolution xyy^*x^* is a smooth function in $L^2(G)$, and

$$(xyy^*x^*)(e) = \|xy\|^2_{L^2(G)}.$$

Here we use the fact that if $f \in L^2(G)$ is any function then

$$(f * f^*)(e) = \int f(g)f^*(g^{-1})\,dg = \int f(g)\overline{f}(g)\,dg = \int |f(g)|^2\,dg = \|f\|^2_{L^2(G)}.$$

So $xy = 0$ for all $y \in C_c^\infty(G)$ and hence $x = 0$ in $C_c^\infty(G)$. This is a contradiction. Note that when we wrote xyy^*x^*, we meant $\beta(x \otimes \overline{x})(yy^*) \in L^2(G)$. We used the fact that this is equal to the convolution of $\lambda(x)y \in L^2(G)$ with $(\lambda(x)y)^* \in L^2(G)$. That is

$$\beta(x \otimes \overline{x})(yy^*) = (\lambda(x)y)(\lambda(x)y)^*.$$ □

4.3 Remark. Given a Hilbert space $\mathcal{H}$, its dual space $\mathcal{H}^*$ is the space of all bounded linear functional from $\mathcal{H}$ to $\mathbb{C}$. Also its complex conjugate space $\overline{\mathcal{H}}$ is the same set as Hilbert space $\mathcal{H}$ with the following rules for addition, scalar multiplication and inner product:

$$\overline{v+w} := \overline{v} + \overline{w}, \quad \alpha\overline{v} := \overline{\overline{\alpha}v}, \quad \langle \overline{v}, \overline{w} \rangle_{\overline{\mathcal{H}}} := \langle w, v \rangle_{\mathcal{H}}.$$

Its complex conjugate space $\overline{\mathcal{H}}$ is the same as its dual space $\mathcal{H}^*$ via the map $v \mapsto (w \mapsto \langle w, v \rangle)$. Therefore, if π is a representation of G on $\mathcal{H}$, we can identify a **contragredient representation** π^* of G on $\mathcal{H}_{\pi^*} = \mathcal{H}^*$ with a **complex conjugate representation** $\overline{\pi}$ of G on $\mathcal{H}_{\overline{\pi}} = \overline{\mathcal{H}}$, where

$$[\pi^*(g)(\varphi)](v) = \varphi(\pi(g^{-1})v) \qquad (\varphi \in \mathcal{H}^*, v \in \mathcal{H})$$

$$\overline{\pi}(g)v = \overline{\pi(g)v} \qquad (v \in \overline{\mathcal{H}}).$$

We have now shown that certain representations of $G \times G$ belong to the spectrum of $A(G)$. Our next goal is to show that certain other representations *do not* belong to the spectrum. We shall do this by studying the center of the group G.

4.4 Theorem. *If $\pi_1 \otimes \pi_2^*$ is weakly contained in α then $\pi_1|_{\mathfrak{z}(G)} = \pi_2|_{\mathfrak{z}(G)}$.*

Proof. Let π_1 and π_2 be unitary representations of G on H_1 and H_2 respectively. Let $v_1 \in H_1$ and $v_2 \in H_2$. Assume that $\pi_1 \otimes \pi_2^*$ is weakly contained in bi-regular representation β. Therefore there are unit vectors $f_n \in L^2(G)$ such that

$$\langle \beta(g_1, g_2) f_n, f_n \rangle \to \langle \pi_1(g_1)v_1, v_1 \rangle \langle \overline{\pi_2(g_2)v_2}, \overline{v_2} \rangle$$

which is

$$\langle \beta(g_1, g_2) f_n, f_n \rangle \to \langle \pi_1(g_1)v_1, v_1 \rangle \langle v_2, \pi_2(g_2)v_2 \rangle \tag{4.1}$$

for all $g_1, g_2 \in G$ as $n \to \infty$. For $g_1 = g_2 \in \mathfrak{z}(G)$, we have

$$(\beta(g_1, g_2) f_n)(\gamma) = f_n(g_1^{-1}\gamma g_2) = f_n(g_1^{-1}\gamma g_1) = f_n(\gamma).$$

Also by Schur's lemma, $\pi_1|_{\mathfrak{z}(G)} = z_1 \operatorname{Id}_{\mathfrak{z}(G)}$ and $\pi_2|_{\mathfrak{z}(G)} = z_2 \operatorname{Id}_{\mathfrak{z}(G)}$, where z_1, z_2 are unit vectors in $\mathbb{C}$. Consider (4.1) particularly when $g_1 = g_2 \in \mathfrak{z}(G)$, it becomes

$$\langle f_n, f_n \rangle \to \langle z_1 v_1, v_1 \rangle \langle v_2, z_2 v_2 \rangle \text{ as } n \to \infty \quad \Rightarrow \quad z_1 \overline{z_2} = 1.$$

Multiplying z_2 to both sides of the equation, we obtain $z_1 = z_2$. In other words, $\pi_1|_{\mathfrak{z}(G)} = \pi_2|_{\mathfrak{z}(G)}$. □

4.3 The spectrum of $A(G)$ for the Heisenberg group

In the previuos section we obtained some information about the spectrum $A(G)$. In this section, we shall use the fact that the spectrum is a *closed* subset of the dual of $G \times G$ to complete the computation of the spectrum.

Let G be a Heisenberg group H_1 and its Lie algebra $\mathfrak{g} = \mathfrak{h}_1$. Recall that there are two inequivalent classes of unitary irreducible representations

$\pi_c : G \to U(L^2(\mathbb{R}))$ and $\pi_{ab} : G \to U(\mathbb{C})$ which are defined by

$$(\pi_c(g)f)(t) = e^{icz}e^{ixt}f(t - cy)$$
$$\pi_{ab}(g) = e^{i(ax+by)},$$

where $a, b, c \in \mathbb{R}$ and for $g = (x, y, z) \in H_1$. We shall show that $\{\pi_{\alpha\beta} \otimes \pi^*_{\gamma\delta} \mid \alpha, \beta, \gamma, \delta \in \mathbb{R}\}$ is weakly contained in $\{\pi_c \otimes \pi^*_c \mid c \in \mathbb{R}\backslash\{0\}\}$. To do that, we need the following lemma.

4.5 Lemma. *We have*

$$\int_{\mathbb{R}} e^{\alpha x^2+\beta x+\gamma}\, dx = \sqrt{\frac{\pi}{-\alpha}} \cdot e^{-\frac{\beta^2}{4\alpha}+\gamma},$$

where $\alpha < 0, \beta$ *and* γ *are constants.*

Proof. Note that

$$\alpha x^2 + \beta x + \gamma = \alpha\left(x + \frac{\beta}{2\alpha}\right)^2 + \left(-\frac{\beta^2}{4\alpha} + \gamma\right).$$

We write $\delta = \frac{\beta}{2\alpha}$ and $\epsilon = -\frac{\beta^2}{4\alpha} + \gamma$. Then

$$\int_{\mathbb{R}} e^{\alpha x^2+\beta x+\gamma}\, dx = \int e^{\alpha(x+\delta)^2+\epsilon}\, dx = e^{\epsilon}\int e^{\alpha(x+\delta)^2}\, dx.$$

Let $u = \sqrt{-\alpha}(x + \delta)$. Then $du = \sqrt{-\alpha}\, dx$ and $u^2 = -\alpha(x + \delta)^2$. Then

$$\begin{aligned}\int_{\mathbb{R}} e^{\alpha x^2+\beta x+\gamma}\, dx &= e^{\epsilon}\int e^{-u^2}\frac{du}{\sqrt{-\alpha}} = \frac{e^{\epsilon}}{\sqrt{-\alpha}}\int e^{-u^2}\, du \\ &= \frac{e^{\epsilon}}{\sqrt{-\alpha}}\sqrt{\pi} = \left(\sqrt{\frac{\pi}{-\alpha}}\right)e^{-\frac{\beta^2}{4\alpha}+\gamma}.\end{aligned}$$

The second to last equality uses the fact that $\int_{\mathbb{R}} e^{-x^2}\, dx = \sqrt{\pi}$. ∎

Before completing our calculation of the spectrum of $A(G)$, we prove a simpler result concerning the dual of G rather than the dual of $G \times G$.

4.6 Lemma. $\{\pi_{ab} \mid a, b \in \mathbb{R}\}$ *is weakly contained in* $\{\pi_c \mid c \in \mathbb{R}\backslash\{0\}\}$.

Proof. Given $(a, b) \in \mathbb{R}^2$, it suffices to find $f \in L^2(\mathbb{R})$ such that

$$\langle \pi_c(g) f, f \rangle \to \langle \pi_{ab}(g) 1, 1 \rangle, \text{ for all } g \in G \text{ as } c \to 0.$$

We claim that $f(t) = \sqrt[4]{\frac{2}{\pi c}} \cdot e^{-\frac{(a-t)^2}{c} - i\frac{b}{c}t}$ does the job. For $g = (x, y, z)$, we have

$$\begin{aligned}
\langle \pi_c(g) f, f \rangle &= \int (\pi_c(g) f)(t) \overline{f}(t)\, dt \\
&= \int e^{icz} e^{ixt} f(t - cy) \overline{f}(t)\, dt \\
&= \int e^{icz} e^{ixt} \left(\sqrt[4]{\frac{2}{\pi c}} \cdot e^{-\frac{(a-(t-cy))^2}{c} - i\frac{b}{c}(t-cy)} \right) \left(\sqrt[4]{\frac{2}{\pi c}} \cdot e^{-\frac{(a-t)^2}{c} + i\frac{b}{c}t} \right) dt \\
&= \sqrt{\frac{2}{\pi c}} \cdot e^{iby+icz} \int e^{ixt} e^{-\frac{(a-(t-cy))^2}{c} - \frac{(a-t)^2}{c}}\, dt. \qquad (4.2)
\end{aligned}$$

Note

$$\begin{aligned}
ixt - \frac{(a-(t-cy))^2}{c} - \frac{(a-t)^2}{c} &= ixt - \frac{1}{c}\left((a-t+cy)^2 + (a-t)^2\right) \\
&= ixt - \frac{1}{c}\left(2(a-t)^2 + 2(a-t)cy + c^2y^2\right) \\
&= ixt - \frac{1}{c}\left(2a^2 - 4at + 2t^2 + 2acy - 2tcy + c^2y^2\right) \\
&= \left(-\frac{2}{c}\right) t^2 + \left(\frac{4a}{c} + 2y + ix\right) t \\
&\quad + \left(-\frac{2a^2}{c} - 2ay - cy^2\right).
\end{aligned}$$

Now we apply Lemma 4.5,

$$\begin{aligned}
\int e^{ixt} e^{-\frac{(a-(t-cy))^2}{c} - \frac{(a-t)^2}{c}}\, dt &= \sqrt{\frac{\pi}{\frac{2}{c}}} \cdot e^{-\frac{(\frac{4a}{c}+2y+ix)^2}{4(-\frac{2}{c})} + (-\frac{2a^2}{c} - 2ay - cy^2)} \\
&= \sqrt{\frac{\pi c}{2}} \cdot e^{\frac{c}{8}(\frac{16a^2}{c^2} + 4y^2 - x^2 + \frac{16ay}{c} + 4ixy + \frac{8aix}{c}) - \frac{2a^2}{c} - 2ay - cy^2} \\
&= \sqrt{\frac{\pi c}{2}} \cdot e^{-\frac{cy^2}{2} - \frac{x^2 c}{8} + \frac{icxy}{2} + iax}. \qquad (4.3)
\end{aligned}$$

Then we substitute (4.3) into (4.2),

$$\begin{aligned}\langle \pi_c(g)f, f\rangle &= \sqrt{\frac{2}{\pi c}} \cdot e^{iby+icz}\left(\sqrt{\frac{\pi c}{2}} \cdot e^{-\frac{cy^2}{2}-\frac{x^2c}{8}+\frac{icxy}{2}+iax}\right)\\ &= e^{iby+iax}e^{icz-\frac{cy^2}{2}-\frac{x^2c}{8}+\frac{icxy}{2}},\end{aligned}$$

which converges to $e^{i(ax+by)} = \langle \pi_{ab}(g)1, 1\rangle$ as $c \to 0$. $\square$

4.7 Proposition. *$\{\pi_{\alpha\beta} \otimes \pi^*_{\gamma\delta} \mid \alpha, \beta, \gamma, \delta \in \mathbb{R}\}$ is weakly contained in $\{\pi_c \otimes \pi^*_c \mid c \in \mathbb{R}\backslash\{0\}\}$. In other words, we can find ${}^{\gamma\delta}h^{\alpha\beta}_c \in L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$ such that*

$$\langle(\pi_c(g_1) \otimes \pi^*_c(g_2))^{\gamma\delta}\, h^{\alpha\beta}_c, {}^{\gamma\delta}\, h^{\alpha\beta}_c\rangle \to \pi_{\alpha\beta}(g_1)\pi^*_{\gamma\delta}(g_2),$$

as $c \to 0$ for all $g_1, g_2 \in H_1$.

Proof. Let $g_1 = (x, y, z)$ and $g_2 = (u, v, w)$. Define $f^{\alpha\beta}_c$ as follows:

$$f^{\alpha\beta}_c(t) = \sqrt[4]{\frac{2}{\pi c}} \cdot e^{-\frac{(\alpha-t)^2}{c}}e^{-i\frac{\beta}{c}t}.$$

From the proof of Lemma 4.6, we have

$$\langle \pi_c(g_1)f^{\alpha\beta}_c, f^{\alpha\beta}_c\rangle = e^{i\alpha x+i\beta y}e^{icz-\frac{cy^2}{2}-\frac{x^2c}{8}+\frac{icxy}{2}}.$$

If we can show that

$$\langle\overline{\pi^*_c(g_2)f^{\gamma\delta}_c}, \overline{f^{\gamma\delta}_c}\rangle = e^{-i\gamma u-i\delta v}e^{-icw-\frac{cv^2}{2}-\frac{icuv}{2}-\frac{u^2c}{8}},$$

then we are done since

$$\begin{aligned}&\left\langle(\pi_c(g_1) \otimes \pi^*_c(g_2))(f^{\alpha\beta}_c \otimes f^{\gamma\delta}_c), f^{\alpha\beta}_c \otimes f^{\gamma\delta}_c\right\rangle\\ &\qquad= \left\langle \pi_c(g_1)f^{\alpha\beta}_c \otimes \pi^*_c(g_2)f^{\gamma\delta}_c, f^{\alpha\beta}_c \otimes f^{\gamma\delta}_c\right\rangle\\ &\qquad= \left\langle \pi_c(g_1)f^{\alpha\beta}_c, f^{\alpha\beta}_c\right\rangle\left\langle\overline{\pi^*_c(g_2)f^{\gamma\delta}_c}, \overline{f^{\gamma\delta}_c}\right\rangle\\ &\qquad= \left(e^{i\alpha x+i\beta y}e^{icz-\frac{cy^2}{2}-\frac{x^2c}{8}+\frac{icxy}{2}}\right)\\ &\qquad\quad\cdot\left(e^{-i\gamma u-i\delta v}e^{-icw-\frac{cv^2}{2}-\frac{icuv}{2}-\frac{u^2c}{8}}\right)\end{aligned}$$

which converges to $e^{i\alpha x+i\beta y}e^{-i\gamma u-i\delta v}=\pi_{\alpha\beta}(g_1)\pi^*_{\gamma\delta}(g_2)$, as $c\to 0$. Now, let's find the formula for $\pi^*_c(g_2)f^{\gamma\delta}_c$. Consider

$$\begin{aligned}
\langle\overline{\pi^*_c(g_2)f^{\gamma\delta}_c},\overline{f^{\gamma\delta}_c}\rangle &= \langle f^{\gamma\delta}_c,\pi_c(g_2)f^{\gamma\delta}_c\rangle \\
&= \int f^{\gamma\delta}_c(t)\overline{\left(\pi_c(g_2)f^{\gamma\delta}_c\right)}(t) \\
&= \int f^{\gamma\delta}_c(t)e^{-icw}e^{-iut}\overline{f^{\gamma\delta}_c}(t-cv)\,dt \\
&= \int\left(\sqrt[4]{\frac{2}{\pi c}}\cdot e^{-\frac{(\gamma-t)^2}{c}-i\frac{\delta}{c}t}\right)e^{-icw}e^{-iut} \\
&\quad\cdot\left(\sqrt[4]{\frac{2}{\pi c}}\cdot e^{-\frac{(\gamma-(t-cv))^2}{c}+i\frac{\delta}{c}(t-cv)}\right)\,dt \\
&= \sqrt{\frac{2}{\pi c}}\cdot e^{-icw}e^{-i\delta v}\int e^{-iut}e^{-\frac{(\gamma-t+cv)^2}{c}-\frac{(\gamma-t)^2}{c}}\,dt.
\end{aligned}$$

Note that

$$\begin{aligned}
-iut-&\frac{(\gamma-t+cv)^2}{c}-\frac{(\gamma-t)^2}{c} \\
&= -iut-\frac{1}{c}\left((\gamma-t+cv)^2+(\gamma-t)^2\right) \\
&= -iut-\frac{1}{c}\left(2(\gamma-t)^2+2(\gamma-t)cv+c^2v^2\right) \\
&= -iut-\frac{1}{c}\left(2\gamma^2-4\gamma t+2t^2+2\gamma cv-2tcv+c^2v^2\right) \\
&= \left(-\frac{2}{c}\right)t^2+\left(\frac{4\gamma}{c}+2v-iu\right)t+\left(-\frac{2\gamma^2}{c}-2\gamma v-cv^2\right).
\end{aligned}$$

Again we apply Lemma 4.5,

$$\int e^{-iut} e^{-\frac{(\gamma-t+cv)^2}{c}-\frac{(\gamma-t)^2}{c}}\, dt$$

$$= \sqrt{\frac{\pi}{\frac{2}{c}}} \cdot e^{-\frac{(\frac{4\gamma}{c}+2v-iu)^2}{4(-\frac{2}{c})}+(-\frac{2\gamma^2}{c}-2\gamma v-cv^2)}$$

$$= \sqrt{\frac{\pi c}{2}} \cdot e^{\frac{c}{8}(\frac{16\gamma^2}{c^2}+\frac{16\gamma v}{c}-\frac{8\gamma iu}{c}+4v^2-4viu-u^2)-\frac{2\gamma^2}{c}-2\gamma v-cv^2}$$

$$= \sqrt{\frac{\pi c}{2}} \cdot e^{\frac{2\gamma^2}{c}+2\gamma v-i\gamma u+\frac{cv^2}{2}-\frac{icuv}{2}-\frac{u^2c}{8}-\frac{2\gamma^2}{c}-2\gamma v-cv^2}$$

$$= \sqrt{\frac{\pi c}{2}} \cdot e^{-i\gamma u-\frac{cv^2}{2}-\frac{icuv}{2}-\frac{u^2c}{8}}.$$

Therefore

$$\overline{\langle \pi_c^*(g_2) f_c^{\gamma\delta}, \overline{f_c^{\gamma\delta}} \rangle} = \sqrt{\frac{2}{\pi c}} \cdot e^{-icw} e^{-i\delta v} \left(\sqrt{\frac{\pi c}{2}} \cdot e^{-i\gamma u-\frac{cv^2}{2}-\frac{icuv}{2}-\frac{u^2c}{8}} \right)$$
$$= e^{-i\gamma u-i\delta v} e^{-icw-\frac{cv^2}{2}-\frac{icuv}{2}-\frac{u^2c}{8}},$$

as we claimed above. So the proof is complete. □

4.8 Corollary. *For $a, b, d \in \mathbb{R}$, $c \in \mathbb{R}\backslash\{0\}$ and $c \neq d$, $\pi_c \otimes \pi_{ab}^*, \pi_{ab} \otimes \pi_c^*$ and $\pi_c \otimes \pi_d^*$ are not in $\widehat{A(G)}$*

Proof. The center $\mathfrak{z}(G)$ of Heisenberg group is $\{(0,0,z) \mid z \in \mathbb{R}\}$. By the definition, we have

$$\pi_c \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} = e^{icz}\, \mathrm{Id}_{\mathfrak{z}(G)}, \quad \pi_{ab} \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} = \mathrm{Id}_{\mathfrak{z}(G)},$$

for all $z \in \mathbb{R}$. Therefore $\pi_c \neq \pi_{ab}$ for all nonzero real number c and $\pi_c \neq \pi_d$ if $c \neq d$. Then this corollary immediately follows from Theorem 4.4. □

Let's summary what we have shown so far

1. All $\pi_c \otimes \pi_c^*$ and $\pi_{\alpha\beta} \otimes \pi_{\alpha\beta}^*$ are in the spectrum of $A(G)$.

2. $\pi_c \otimes \pi_{ab}^*, \pi_{ab} \otimes \pi_c^*$ and $\pi_c \otimes \pi_d^*$ are *not* in the spectrum of $A(G)$.

3. $\{\pi_{\alpha\beta} \otimes \pi_{\gamma\delta}^* \mid \alpha, \beta, \gamma, \delta \in \mathbb{R}\}$ is weakly contained in $\{\pi_c \otimes \pi_c^* \mid c \in \mathbb{R}\backslash\{0\}\}$.

Combine these results altogether, we have the following

4.9 Theorem. *The spectrum of $A(G)$ is the union of $\{\pi_c \otimes \pi_c^* \mid c \in \backslash\{0\}\}$ and $\{\pi_{\alpha\beta} \otimes \pi_{\gamma\delta}^* \mid \alpha, \beta, \gamma, \delta \in \mathbb{R}\}$.*

4.10 Theorem. *Let G be a Heisenberg group H_1, $\widehat{A(G)}$ is the closure of the set $\{\pi_c \otimes \pi_c^* \mid c \neq 0\}$ in $(\widehat{G \times G})$.*

Chapter 5

Structure of $A(\mathfrak{g}^*)$ for the Heisenberg group

In this chapter we shall continue the theme of the previous chapter by calculating the spectrum of $A(\mathfrak{g}^*)$ and using the information obtained to decompose the C^*-algebra $A(\mathfrak{g}^*)$ as an extension of more elementary C^*-algebras. Of course the results will parallel the work in the previous chapter, since the two C^*-algebras have been proved to be isomorphic in Chapter 3. But our methods here, while they follow those of the previous chapter, will be independent of them.

5.1 Alternative definition of $A(\mathfrak{g}^*)$

In this section, we shall explain how to view $A(\mathfrak{g}^*)$ as the C*-algebra generated by a unitary representation of the semidirect product group $G \ltimes g$ on $L^2(\mathfrak{g}^*)$.

We define the map $\alpha : G \ltimes \mathfrak{g} \to U(L^2(\mathfrak{g}^*))$ by

$$\alpha(g, X)f = \alpha|_G(g)\,(\alpha|_{\mathfrak{g}}(X)f)\,,$$

where $g \in G$ and $X \in \mathfrak{g}$. For $f \in L^2(\mathfrak{g}^*)$ and $l \in \mathfrak{g}^*$,

$$(\alpha|_G(g)f)(l) = f(\operatorname{Ad}^*_{g^{-1}} l), \quad (\alpha|_{\mathfrak{g}}(X)f)(l) = e^{il(X)} f(l),$$

where $\operatorname{Ad}^*_g l = l(\operatorname{Ad}_{g^{-1}})$. Therefore, by the definition, we have

$$\begin{aligned}(\alpha(g,X)f)(l) &= [\alpha|_G(g)(\alpha|_{\mathfrak{g}}(X)f)](l) \\ &= (\alpha|_{\mathfrak{g}}(X)f)(\operatorname{Ad}^*_{g^{-1}} l) \\ &= e^{i(\operatorname{Ad}^*_{g^{-1}} l)(X)} f(\operatorname{Ad}^*_{g^{-1}} l) \\ &= e^{il(gXg^{-1})} f(\operatorname{Ad}^*_{g^{-1}} l).\end{aligned}$$

5.1 Remark. As we pointed out in Chapter 3, this is an alternative definition of $A(\mathfrak{g}^*)$ via a unitary isomorphism Fourier transform from $L^2(\mathfrak{g})$ to $L^2(\mathfrak{g}^*)$ by $f \mapsto \widehat{f}$, where

$$\widehat{f}(l) = \int_{\mathfrak{g}} f(X) e^{il(X)}\, dX.$$

We also write $\mathcal{F}(f)$ for $\widehat{f}$. In other words, the following diagram commutes:

$$\begin{array}{ccc} f \in L^2(G) & \xrightarrow{\quad\mathcal{F}\quad} & \widehat{f} \in L^2(\mathfrak{g}^*) \\ {\scriptstyle S_{g,X}}\big\downarrow & & \big\downarrow{\scriptstyle \alpha} \\ S_{g,X}f \in L^2(G) & \xrightarrow{\mathcal{F}} & \alpha \circ \mathcal{F} = \mathcal{F} \circ S_{g,X} \in L^2(\mathfrak{g}^*) \end{array}$$

This is true by considering Fourier transform of $S_{g,X}f$. Recall that

$$S_{g,X}f : Y \mapsto f(g^{-1}Yg - X).$$

Then for any $l \in \mathfrak{g}^*$,

$$\mathcal{F}(S_{g,X}f)(l) = \int_{\mathfrak{g}} e^{il(Y)} (S_{g,X}f)(Y)\, dY = \int_{\mathfrak{g}} e^{il(Y)} f(g^{-1}Yg - X)\, dY.$$

Let $Z = g^{-1}Yg - X$. Then $dZ = dY$ and $Y = g(Z+X)g^{-1}$ and so

$$\begin{aligned}
\mathcal{F}(S_{g,X}f)(l) &\int_{\mathfrak{g}} e^{il(g(Z+X)g^{-1})} f(Z)\, dZ \\
&= e^{il(gXg^{-1})} \int_{\mathfrak{g}} e^{il(gZg^{-1})} f(Z)\, dZ \\
&= e^{il(gXg^{-1})} \int_{\mathfrak{g}} e^{i\,\mathrm{Ad}^*_{g^{-1}}\, l(Z)} f(Z)\, dZ \\
&= e^{il(gXg^{-1})} \widehat{f}(\mathrm{Ad}^*_{g^{-1}}\, l) \\
&= \alpha(\mathcal{F}f)(l).
\end{aligned}$$

5.2 Remark. For $g_1, g_2 \in G$, $\mathrm{Ad}^*_{g_2^{-1}}\,\mathrm{Ad}^*_{g_1^{-1}} = \mathrm{Ad}^*_{(g_1g_2)^{-1}}$ To see this, consider

$$(\mathrm{Ad}^*_{g_2^{-1}}\,\mathrm{Ad}^*_{g_1^{-1}}\, l)(X) = \mathrm{Ad}^*_{g_1^{-1}}\, l(g_2Xg_2^{-1}) = l(g_1g_2Xg_1^{-1}g_2^{-1}) = \mathrm{Ad}^*_{(g_1g_2)^{-1}}\, l(X),$$

for any $X \in \mathfrak{g}$.

5.3 Remark. α is a unitary representation. To see that $\alpha(g,X)$ is homomorphism, we consider

$$\begin{aligned}
\left[\alpha(g_1,X_1)(\alpha(g_2,X_2)f)\right](l) &= e^{il(g_1X_1g_1^{-1})}(\alpha(g_2,X_2)f)(\mathrm{Ad}^*_{g_1^{-1}}\, l) \\
&= e^{il(g_1X_1g_1^{-1})} e^{i\,\mathrm{Ad}^*_{g_1^{-1}}\, l(g_2X_2g_2^{-1})} f(\mathrm{Ad}^*_{g_2^{-1}}\,\mathrm{Ad}^*_{g_1^{-1}}\, l) \\
&= e^{il(g_1X_1g_1^{-1})} e^{il(g_1g_2X_2(g_1g_2)^{-1})} f(\mathrm{Ad}^*_{g_2^{-1}}\,\mathrm{Ad}^*_{g_1^{-1}}\, l) \\
&= e^{il(g_1X_1g_1^{-1}+g_1g_2X_2(g_1g_2)^{-1})} f(\mathrm{Ad}^*_{g_2^{-1}}\,\mathrm{Ad}^*_{g_1^{-1}}\, l) \\
&= e^{il(g_1g_2(g_2^{-1}X_1g_2+X_2)(g_1g_2)^{-1}} f(\mathrm{Ad}^*_{(g_1g_2)^{-1}}\, l) \\
&= (\alpha(g_1g_2, g_2^{-1}X_1g_2+X_2)f)(l) \\
&= (\alpha\left((g_1,X_1)(g_2,X_2)\right)f)(l).
\end{aligned}$$

Therefore, for any $(g_1,X_1),(g_2,X_2) \in G \ltimes \mathfrak{g}$,

$$\alpha(g_1,X_1)\alpha(g_2,X_2) = \alpha\left((g_1,X_1)(g_2,X_2)\right).$$

Additionally $\alpha(g,X)$ is a unitary map, which is $\alpha(g,X)^* = \alpha(g,X)^{-1}$. Indeed, for any $f, h \in L^2(\mathfrak{g}^*)$,

$$\begin{aligned}\langle \alpha(g,X)^* f, h\rangle &= \langle f, \alpha(g,X)h\rangle \\ &= \int_{\mathfrak{g}^*} f(l)\overline{\alpha(g,X)h}(l)\, dl \\ &= \int_{\mathfrak{g}^*} f(l) e^{-il(gXg^{-1})}\overline{h}(\mathrm{Ad}^*_{g^{-1}} l)\, dl.\end{aligned}$$

Changing variables by setting $k = \mathrm{Ad}^*_{g^{-1}}$. Then $l = \mathrm{Ad}^*_g k$, and $l(gXg^{-1}) = k(X)$. Therefore

$$\begin{aligned}\langle \alpha(g,X)^* f, h\rangle &= \int_{\mathfrak{g}^*} e^{-ik(X)} f(\mathrm{Ad}^*_g k)\overline{h}(k)\, dk \\ &= \int_{\mathfrak{g}^*} e^{ik(g^{-1}(-gXg^{-1})g)} f(\mathrm{Ad}^*_g k)\overline{h}(k)\, dk \\ &= \int_{\mathfrak{g}^*} \left[\alpha(g^{-1}, -gXg^{-1})f\right](k)\overline{h}(k)\, dk \\ &= \langle \alpha(g^{-1}, -gXg^{-1})f, h\rangle \\ &= \langle \alpha((g,X)^{-1})f, h\rangle = \langle \alpha(g,X)^{-1} f, h\rangle.\end{aligned}$$

From the remark above, we further obtain a representation

$$C^*(G \ltimes \mathfrak{g}) \to B(L^2(\mathfrak{g}^*)).$$

The C*-algebra $A(\mathfrak{g}^*)$ is the image under this map.

5.2 Representations of $G \ltimes \mathfrak{g}$

Let G be the Heisenberg group H_1 and $\mathfrak{g} = \mathfrak{h}_1$ its Lie algebra. We shall use Mackey Machine to obtain all irreducible unitary representations of $G \ltimes \mathfrak{g}$. Firstly, recall that all characters $\varphi : \mathfrak{g} \to S^1$ are in the following form

$$\varphi_{abc}([u,v,w]) = e^{i(au+bv+cw)},$$

for $[u,v,w] \in \mathfrak{g}$.

5.4 Remark. Let $l \in \mathfrak{g}^*$ and $\{X, Y, Z\}$ a basis of H_1. Then $l_{abc} = aX^* + bY^* + cZ^*$ where X^*, Y^*, Z^* are dual basis of H_1^*. Also the character φ_{abc} corresponds to l_{abc} by

$$\varphi_{abc} = e^{il_{abc}}.$$

Next two lemmas will be used to find the stabilizer subgroup of G with respect to φ_{abc} for all $a, b, c \in \mathbb{R}$.

5.5 Lemma. *For $g = (x, y, z) \in G$, we have*

$$\begin{aligned} \operatorname{Ad}_g^* \varphi_{abc} &= \varphi_{a+cy,b-cx,c} \\ \operatorname{Ad}_{g^{-1}}^* \varphi_{abc} &= \varphi_{a-cy,b+cx,c}. \end{aligned}$$

Proof. For $u, v, w \in \mathbb{R}$, we have

$$\begin{aligned} \operatorname{Ad}_g^* \varphi_{abc}([u, v, w]) &= \varphi_{abc}\left(g^{-1}[u, v, w]g\right) \\ &= \varphi_{abc}\left((-x, -y, xy - z)[u, v, w](x, y, z)\right) \\ &= \varphi_{abc}([u, v, uy + w - xv]) \\ &= e^{i(au+bv+c(uy+w-xv))} \\ &= e^{i((a+cy)u+(b-cx)v+cw)} \\ &= \varphi_{a+cy,b-cx,c}([u, v, w]). \end{aligned}$$

The second equation can be shown similarly. □

Now we introduce the notation $\simeq$ to denote orbit equivalence in $\mathfrak{g}^*$. That is, if l_1 and l_2 are in the same coadjoint orbit then we write $l_1 \simeq l_2$. In other words, $l_2 = \operatorname{Ad}_g^*(l_1)$ for some $g \in G$.

5.6 Lemma. *For any nonzero c, $\varphi_{abc} \simeq \varphi_{a'b'c}$. In particular, $\varphi_{abc} \simeq \varphi_{00c}$.*

Proof. If $\varphi_{abc} \simeq \varphi_{a'b'c'}$ then $\varphi_{a'b'c'} = \mathrm{Ad}_g^* \varphi_{abc}$. Therefore, for all $u, v, w \in \mathbb{R}$, we have

$$\varphi_{a'b'c'}([u, v, w]) = \mathrm{Ad}_g^* \varphi_{abc}([u, v, w]).$$

Applying previous lemma with $g = (x, y, z) \in G$, we have

$$a'u + b'v + c'w = (a + cy)u + (b - cx)y + cw.$$

This is possible if and only if $c = c'$. If both are zero, we then have $a = a'$ and $b = b'$ which is not an interesting case. If $c = c'$ are nonzero, then we have,

$$a + cy = a', \quad b - cx = b' \quad \Rightarrow \quad y = \frac{a' - a}{c}, \quad x = \frac{b - b'}{c}.$$

Therefore $\varphi_{abc} \simeq \varphi_{a'b'c}$, as desired. The second statement is immediate followed. □

Now we are ready to classify all of the stabilizer subgroup of G with respect to φ_{abc}.

5.7 Lemma. *Let $a, b, c \in \mathbb{R}$. The stabilizer subgroups of G with respect to φ_{abc} are*

$$G_{\varphi_{abc}} = \begin{cases} \mathfrak{z}(G) = \{(0, 0, z) \mid z \in \mathbb{R}\} & \text{if } c \neq 0 \\ G & \text{if } c = 0. \end{cases}$$

Proof. By the definition of stabilizer subgroup and previous lemma,

$$\begin{aligned} G_{\varphi_{abc}} &= \{g \in G \mid \varphi_{abc}(X) = \mathrm{Ad}_g^* \varphi_{abc}(X) \text{ for all } X \in \mathfrak{g}\} \\ &= \{g \in G \mid \varphi_{abc}(X) = \varphi_{abc}(g^{-1}Xg) \text{ for all } X \in \mathfrak{g}\} \\ &= \{g \in G \mid \varphi_{abc}([u, v, w]) = \varphi_{abc}(g^{-1}[u, v, w]g) \text{ for all } [u, v, w] \in \mathfrak{g}\} \\ &= \left\{(x, y, z) \in G \mid e^{i(au+bv+cw)} = e^{i((a+cy)u+(b-cx)v+cw)} \text{ for all } u, v, w \in \mathbb{R}\right\} \\ &= \{(x, y, z) \in G \mid a = a + cy, b = b - cx \text{ for all } u, v, w \in \mathbb{R}\} \\ &= \{(x, y, z) \in G \mid cy = 0, cx = 0\}. \end{aligned}$$

If $c = 0$, then there is no condition on x, y, z. So $G_{\varphi_{abc}} = G$. If $c \neq 0$, then $cy = cx = 0$ implies $y = x = 0$. Therefore $G_{\varphi_{abc}} = \mathfrak{z}(G)$. $\square$

Recall that the inequivalent unitary irreducible representations of G are of the form

$$\left(\pi_\gamma((x, y, z))f\right)(t) = e^{i(\gamma z + xt)} f(t - \gamma y)$$
$$\pi_{\alpha\beta}((x, y, z)) = e^{i(\alpha x + \beta y)}.$$

Now we apply Mackey machine process to obtain all representations on $G \ltimes \mathfrak{g}$. The results are shown below.

5.8 Theorem. *The unitary dual of $G \ltimes \mathfrak{g}$ consists of the following three families of irreducible unitary representations:*

- $\mathrm{Ind}_{G_\varphi \ltimes \mathfrak{g}}^{G \ltimes \mathfrak{g}} \pi_{\lambda,c}$, *where λ is any real number and c is any nonzero real number, and $\pi_{\lambda,c}$ is the one-demensional representation given by*

$$\pi_{\lambda,c}((0, 0, z), [u, v, w]) = e^{i(\lambda z + cw)},$$

  for any $(0, 0, z) \in \mathfrak{z}(G)$, $[u, v, w] \in \mathfrak{g}$. Here $\varphi = \varphi_{00c}$.

- $\pi_{\gamma,ab}$, *where γ, a, b are any real numbers, is the representation on $L^2(\mathbb{R})$ given by*

$$\left[\pi_{\gamma,ab}((x, y, z), [u, v, w])f\right](t) = e^{i(\gamma z + xt)} e^{i(au + bv)} f(t - \gamma y),$$

  for any $(x, y, z) \in G$, $[u, v, w] \in \mathfrak{g}$, $f \in L^2(\mathbb{R})$, and $t \in \mathbb{R}$.

- $\pi_{\alpha\beta,ab}$, *where α, β, a, b are any real numbers, is the one-dimensional representation given by*

$$\pi_{\alpha\beta,ab}((x, y, z), [u, v, w]) = e^{i(\alpha x + \beta y)} e^{i(au + bv)},$$

  for any $(x, y, z) \in G$, $[u, v, w] \in \mathfrak{g}$.

In other words, $\widehat{G \ltimes \mathfrak{g}} = \{\mathrm{Ind}_{G_\varphi \ltimes \mathfrak{g}}^{G \ltimes \mathfrak{g}} \pi_{\lambda,c}, \pi_{\gamma,ab}, \pi_{\alpha\beta,ab}\}$.

Proof. If $c \neq 0$, by previous lemma, $G_{\varphi_{abc}} = \mathfrak{z}(G) = \{(0,0,z) \mid z \in \mathbb{R}\}$. In this case, $\widehat{G_{\varphi_{abc}}} = \widehat{\mathfrak{z}(G)} = \{\pi^\lambda\}$, where

$$\pi^\lambda((0,0,z)) = e^{i\lambda z}, \text{ for all } z \in \mathbb{R}.$$

Since φ_{abc} and φ_{00c} are in the same orbit by Lemma 5.6, it follows that $\pi^\lambda \otimes \varphi_{abc}$ is equivalent to $\pi^\lambda \otimes \varphi_{00c}$. We shall use the latter for simplicity. For $g = (0,0,z) \in \mathfrak{z}(G)$ and $X = [u,v,w] \in \mathfrak{g}$, then

$$(\pi^\lambda \otimes \varphi_{00c})(g,X) = \pi^\lambda(g)\varphi_{abc}(X) = e^{i(\lambda z + cw)}.$$

We write $\pi_{\lambda,c}$ for $\pi^\lambda \otimes \varphi_{00c}$ from now on. Now let's consider when $c = 0$, we learn from previous lemma that $G_{\varphi_{abc}} = G$. Then for $g = (x,y,z) \in G$ and $X = [u,v,w] \in \mathfrak{g}$, we have

$$\begin{aligned}\left[(\pi_\gamma \otimes \varphi_{ab0})(g,X)f\right](t) &= [\pi_\gamma(g)\varphi_{ab0}(X)f](t) = e^{i(\gamma z + xt)}f(t-\gamma y)e^{i(au+bv)} \\ (\pi_{\alpha\beta} \otimes \varphi_{ab0})(g,X) &= \pi_{\alpha\beta}(g)\varphi_{ab0}(X) = e^{i(\alpha x + \beta y)}e^{i(au+bv)}.\end{aligned}$$

We write $\pi_{\gamma,ab}$ for $\pi_\gamma \otimes \varphi_{ab0}$ and $\pi_{\alpha\beta,ab}$ for $(\pi_{\alpha\beta} \otimes \varphi_{ab0})$. Then induced representations of $\pi_{\lambda,c}$, $\pi_{\gamma,ab}$, $\pi_{\alpha\beta,ab}$ from $G_\varphi \ltimes \mathfrak{g}$ to $G \ltimes \mathfrak{g}$ form the complete list of representations on $G \ltimes \mathfrak{g}$ up to equivelence class by Mackey machine process. □

5.3 Spectrum of $A(\mathfrak{g}^*)$ for the Heisenberg group

The next question we consider is "Which irreducible unitary representations of $G \ltimes \mathfrak{g}$ are weakly contained in α?" First we shall exclude some representations. The following calculation is analogous to Theorem 4.4.

5.9 Theorem. *If* $\pi \in \widehat{G \ltimes \mathfrak{g}}$ *is weakly contained in* α *then*

$$\pi\left((0,0,z),0\right) = \mathrm{Id}, \quad \textit{for all } z \in \mathbb{R}.$$

Proof. Let π be weakly contained in α. Then there are unit vectors f_n and v in $L^2(\mathfrak{g}^*)$ satisfying

$$\langle \alpha(g, X) f_n, f_n \rangle \to \langle \pi(g, X) v, v \rangle$$

as $n \to \infty$ for all $g \in G, X \in \mathfrak{g}$. Let $g = (0, 0, z) \in \mathfrak{z}(G)$ and $X = 0$. Then we have,

$$\langle \alpha(g, X) f_n, f_n \rangle = \int_{\mathfrak{g}^*} f_n(\mathrm{Ad}_g^* \xi) \overline{f_n(\xi)}\, d\xi = \int_{\mathfrak{g}^*} f_n(\xi) \overline{f_n(\xi)}\, d\xi = \|f_n\|^2_{L^2(\mathfrak{g}^*)} = 1,$$

for all n. On the other hand, $\pi|_{\mathfrak{z}(G)}$ is a character. Therefore, by Schur's lemma, $\pi\,(g, X) = c_z \,\mathrm{Id}$. Consequently,

$$\langle \pi(g, X) v, v \rangle = c_z \langle v, v \rangle = c_z.$$

By the definition of weak containment above, $c_z = 1$. Therefore

$$\pi\,((0, 0, z), 0) = \mathrm{Id},$$

as desired. □

5.10 Corollary. *$\pi_{\gamma,ab}$ is not weakly contained in α.*

Proof. Since $\pi_\gamma \otimes \pi_{ab0}\,((0, 0, z), 0) = e^{i\gamma z}$, which is not the identity. Therefore $\pi_{\gamma,ab}$ is not weakly contained in α from theorem above. □

Next we consider whether $\mathrm{Ind}_{G_\varphi \ltimes \mathfrak{g}}^{G \ltimes \mathfrak{g}} \pi_{\lambda,c}$ is weakly contained in α by using the same criteria. In order to to resolve this issue, we need to find a formula for the action of $G \ltimes \mathfrak{g}$ in the induced representation. Recall that the multiplication in $G \ltimes \mathfrak{g}$ is defined by

$$(g_1, X_1)(g_2, X_2) = (g_1 g_2, g_2^{-1} X_1 g_2 + X_2),$$

where $(g_1, X_1), (g_2, X_2) \in G \ltimes \mathfrak{g}$.

Now let G be the Heisenberg group H_1 and $\mathfrak{g} = \mathfrak{h}_1$ its Lie algebra. Let $g_1 = (x_1, y_1, z_1)$, $g_2 = (x_2, y_2, z_2) \in G$ and $X_1 = [u_1, v_1, w_1], X_2 = [u_2, v_2, w_2] \in \mathfrak{g}$. Then

$$
\begin{aligned}
&(g_1, X_1)(g_2, X_2) = (g_1 g_2, g_2^{-1} X_1 g_2 + X_2) \\
&= ((x_1, y_1, z_1)(x_2, y_2, z_2), (x_2, y_2, z_2)^{-1}[u_1, v_1, w_1](x_2, y_2, z_2) + [u_2, v_2, w_2]) \\
&= ((x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2), (-x_2, -y_2, x_2 y_2 - z_2)[u_1, v_1, w_1](x_2, y_2, z_2) \\
&\quad + [u_2, v_2, w_2]) \\
&= ((x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2), [u_1, v_1, -x_2 v_1 + w_1](x_2, y_2, z_2) + [u_2, v_2, w_2]) \\
&= ((x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2), [u_1, v_1, u_1 y_2 + w_1 - x_2 v_1] + [u_2, v_2, w_2]) \\
&= ((x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2), [u_1 + u_2, v_1 + v_2, w_1 + w_2 + u_1 y_2 - x_2 v_1]).
\end{aligned}
$$

With $x_1 = x, y_1 = y, z_2 = z, u_2 = u, v_2 = v, w_2 = w$ and setting the rest equal to zero, we obtain the following identity:

$$((x, y, z), [u, v, w]) = ((x, y, 0), [0, 0, 0])((0, 0, z), [u, v, w]), \tag{5.1}$$

where $((x, y, 0), [0, 0, 0]) \in G \ltimes \mathfrak{g}$ and $((0, 0, z), [u, v, w]) \in G_\varphi \ltimes \mathfrak{g}$. It is easy to check that $(g, X)^{-1} = (g^{-1}, -gXg^{-1})$. Therefore

$$
\begin{aligned}
((0, 0, z), [u, v, w])^{-1} &= ((0, 0, z)^{-1}, -(0, 0, z)[u, v, w](0, 0, z)^{-1}) \\
&= ((0, 0, -z), -(0, 0, z)[u, v, w](0, 0, -z)) \\
&= ((0, 0, -z), -[u, v, w](0, 0, -z)) \\
&= ((0, 0, -z), -[u, v, w]) \\
\therefore ((0, 0, z), [u, v, w])^{-1} &= ((0, 0, -z), [-u, -v, -w]). \qquad (5.2)
\end{aligned}
$$

By definition, the induced representation acts on a Hilbert completion of the space W of continuous functions from $G \ltimes \mathfrak{g}$ to $\mathbb{C}$ satisfying

$$f((g_1, X_1)(g_2, X_2)) = \pi_{\lambda, c}((g_2, X_2)^{-1}) f((g_1, X_1)),$$

for every $(g_1, X_1) \in G \ltimes \mathfrak{g}$ and $(g_2, X_2) \in G_\varphi \ltimes \mathfrak{g}$. Using (5.1) and (5.2), we change this condition to

$$\begin{aligned} f(((x,y,z),[u,v,w])) &= f(((x,y,0),[0,0,0])((0,0,z),[u,v,w])) \\ &= \pi_{\lambda,c}(((0,0,z),[u,v,w])^{-1})f((x,y,0),[0,0,0]) \\ &= \pi_{\lambda,c}((0,0,-z),[-u,-v,-w])f((x,y,0),[0,0,0]) \\ &= e^{i(\lambda(-z)+c(-w))}f((x,y,0),[0,0,0]) \\ &= e^{-i(\lambda z+cw)}f((x,y,0),[0,0,0]). \end{aligned} \tag{5.3}$$

Now we write $\sigma = \mathrm{Ind}_{G_\varphi \ltimes \mathfrak{g}}^{G \ltimes \mathfrak{g}} \pi_{\lambda,c}$. By the construction of induced representation, σ acts on W by

$$\begin{aligned} [\sigma((g,X))f]\,(h,Y) &= f\left((g,X)^{-1}(h,Y)\right) \\ &= f\left((g^{-1},-gXg^{-1})(h,Y)\right) \\ &= f\left((g^{-1}h, h^{-1}(-gXg^{-1})h+Y)\right) \\ \therefore [\sigma((g,X))f]\,(h,Y) &= f\left((g^{-1}h, -h^{-1}(gXg^{-1})h+Y)\right), \end{aligned} \tag{5.4}$$

for $(g,X),(h,Y) \in G \ltimes \mathfrak{g}$ and $f \in W$. In the Heisenberg group, we write now $g = (x,y,z), h = (x',y',z') \in G$ and $X = [u,v,w], Y = [u',v',w'] \in \mathfrak{g}$. Then

$$\begin{aligned} g^{-1}h &= (x,y,z)^{-1}(x',y',z') \\ &= (-x,-y,xy-z)(x',y',z') \\ &= (x'-x, y'-y, z'-z+xy-xy') \\ &= (x'-x, y'-y, z'-z-x(y'-y)). \end{aligned} \tag{5.5}$$

And also

$$
\begin{aligned}
& -h^{-1}(gXg^{-1})h+Y \\
&= -(x',y',z')^{-1}\left((x,y,z)[u,v,w](x,y,z)^{-1}\right)(x',y',z')+[u',v',w'] \\
&= -(-x',-y',x'y'-z')\left([u,v,xv+w](-x,-y,xy-z)\right)(x',y',z')+[u',v',w'] \\
&= -(-x',-y',x'y'-z')[u,v,-uy+xv+w](x',y',z')+[u',v',w'] \\
&= -[u,v,-uy+xv-x'v+w](x',y',z')+[u',v',w'] \\
&= -[u,v,uy'-uy+xv-x'v+w]+[u',v',w'] \\
&= [-u,-v,-uy'+uy-xv+x'v-w]+[u',v',w'] \\
&= [u'-u,v'-v,w'-uy'+uy-xv+x'v-w] \\
&= [u'-u,v'-v,w'-w-u(y'-y)+v(x'-x)]. \qquad (5.6)
\end{aligned}
$$

Combine (5.3) – (5.6) altogether, we obtain

$$
\begin{aligned}
& [\sigma((g,X))f]\,(h,Y) \\
&= f((x'-x,y'-y,z'-z-x(y'-y)),[u'-u,v'-v,w'-w-u(y'-y)+v(x'-x)]) \\
&= e^{-i\{\lambda(z'-z-x(y'-y))+c(w'-w-u(y'-y)+v(x'-x))\}}f((x'-x,y'-y,0),[0,0,0]), \qquad (5.7)
\end{aligned}
$$

where $g=(x,y,z)$, $h=(x',y',z')\in G$, and $X=[u,v,w],Y=[u',v',w']\in\mathfrak{g}$. Particularly, if $g_1=(0,0,z)$ and $X=0$ (zero matrix),

$$[\sigma((g_1,X))f]\,(h,Y)=e^{-i\{\lambda(z'-z)+cw'\}}f((x',y',0),[0,0,0]). \qquad (5.8)$$

Also with $g_2=(0,0,0)$ (identity matrix) and $X=0$, we have

$$[\sigma((g_2,X))f]\,(h,Y)=e^{-i\{\lambda z'+cw'\}}f((x',y',0),[0,0,0]). \qquad (5.9)$$

But (5.4) tells us that

$$[\sigma((g_2,X))f]\,(h,Y)=f\left((g_2^{-1}h,-h^{-1}(g_2Xg_2^{-1})h+Y)\right)=f((h,Y)),$$

which means $\sigma((g_2, X)) = \mathrm{Id}$. Therefore, by combining (5.8) and (5.9),

$$[\sigma((g_1, X))f](h, Y) = e^{i\lambda z}[\sigma((g_2, X))f](h, Y) = e^{i\lambda z} f((h, Y)).$$

Therefore, we obtain

5.11 Lemma. $\mathrm{Ind}_{G_\varphi \ltimes \mathfrak{g}}^{G \ltimes \mathfrak{g}} \pi_{\lambda,c}((0, 0, z), 0) = e^{i\lambda z}$, *where* $\varphi = \varphi_{00c}$.

By applying Theorem 5.9,

5.12 Corollary. $\mathrm{Ind}_{G_\varphi \ltimes \mathfrak{g}}^{G \ltimes \mathfrak{g}} \pi_{\lambda,c}$ *is not weakly contained in* α *when* $\lambda \neq 0$.

Having excluded certain irreducible unitary representations of $G \ltimes \mathfrak{g}$ from the spectrum of $A(\mathfrak{g}^*)$ we shall now prove that certain others do belong to the spectrum. First we shall show that $\mathrm{Ind}_{G_\varphi \ltimes \mathfrak{g}}^{G \ltimes \mathfrak{g}} \pi_{0,c}$ is weakly contained in α. In order to prove this, we introduce new notation $\mathcal{O}_c \subset \mathfrak{g}^*$. It is defined by

$$\mathcal{O}_c = \{l_{abc} \mid a, b \in \mathbb{R}\}.$$

It corresponds to the coadjoint orbit of φ_{00c} which is $\{\varphi_{abc} \mid a, b \in \mathbb{R}\}$ by Lemma 5.6 and Remark 5.4. Also the map $\pi_{\mathcal{O}_c}$ is defined by using the same formula as α. Therefore $\pi_{\mathcal{O}_c} : G \ltimes \mathfrak{g} \to U(L^2(\mathcal{O}_c))$ is defined by

$$\pi_{\mathcal{O}_c}(g, X)f = \pi_{\mathcal{O}_c}|_G(g)\left(\pi_{\mathcal{O}_c}|_{\mathfrak{g}}(X)f\right),$$

where $f \in L^2(\mathcal{O}_c)$. More explicitly, we have

$$[\pi_{\mathcal{O}_c}(g, X)f](l) = e^{il(gXg^{-1})} f(\mathrm{Ad}^*_{g^{-1}} l).$$

From now on we write (a, b) for l_{abc} since c is fixed. If $g = (x, y, z) \in G$ and $X = [u, v, w] \in \mathfrak{g}$, then by Lemma 5.5

$$[\pi_{\mathcal{O}_c}(g, X)f](a, b) = e^{i(au+bv+c(-uy+w+xv))} f(a - cy, b + cx). \tag{5.10}$$

We shall prove the following two results.

5.13 Theorem. *$\pi_{\mathcal{O}_c}$ is weakly contained in α.*

Proof. Fix $c_0 \in \mathbb{R}\backslash\{0\}$. Given $h \in L^2(\mathcal{O}_{c_0})$ with $\|h\| = 1$, we define its extension on $\mathfrak{g}^*$ by

$$\widetilde{h}_\epsilon(l) = h(l)\varphi_\epsilon\left(l([0,0,1]) - c_0\right),$$

where $\varphi_\epsilon : \mathbb{R} \to \mathbb{R}^+$ has support on $|c - c_0| < \epsilon$ and $\int_{\mathbb{R}} \varphi_\epsilon(x)^2\, dx = 1$. In particular, if $l = l_{abc}$, we have

$$\widetilde{h}_\epsilon(a,b,c) = h(a,b,c_0)\varphi_\epsilon(c - c_0).$$

Here we write (a,b,c) for l_{abc}. We shall show that

$$\langle \alpha(g,X)\widetilde{h}_\epsilon, \widetilde{h}_\epsilon\rangle \to \langle \pi_{\mathcal{O}_{c_0}}(g,X)h, h\rangle, \tag{5.11}$$

for all $g \in G$ and $X \in \mathfrak{g}$, when $\epsilon \to 0$. Let $g = (x,y,z) \in G$ and $X = [u,v,w] \in \mathfrak{g}$. Similar to what we have done in (5.10),

$$\left[\pi_{\mathcal{O}_c}(g,X)h\right](a,b,c) = e^{i(au+bv+c_0(-uy+w+xv))}h(a - c_0 y, b + c_0 x, c_0).$$

Therefore the RHS of (5.11) is

$$\iint e^{i(au+bv+c_0(-uy+w+xv))}h(a - c_0 y, b + c_0 x, c_0)\overline{h}(a,b,c_0)\, da\, db.$$

On the other hand,

$$\left[\alpha(g,X)\widetilde{h}_\epsilon\right](a,b,c) = e^{i(au+bv+c(-uy+w+xv))}h(a - cy, b + cx, c_0)\varphi_\epsilon(c - c_0).$$

by the definition of α and $\widetilde{h}_\epsilon$. As a result, the LHS of (5.11) is

$$\iiint e^{i(au+bv+c(-uy+w+xv))}h(a - cy, b + cx, c_0)\overline{h}(a,b,c_0)\varphi_\epsilon(c - c_0)^2\, da\, db\, dc.$$

Therefore $\langle \alpha(g,X)\widetilde{h}_\epsilon, \widetilde{h}_\epsilon\rangle - \langle \pi_{\mathcal{O}_{c_0}}(g,X)h, h\rangle$ is

$$\iiint e^{i(au+bv)}\overline{h}(a,b,c_0)\varphi_\epsilon(c - c_0)^2\Big[e^{ic(-uy+w+xv)}h(a - cy, b + cx, c_0) \\ - e^{ic_0(-uy+w+xv)}h(a - c_0 y, b + c_0 x, c_0)\Big]\, da\, db\, dc$$

By Cauchy-Schwarz inequality,

$$|\langle \alpha(g,X)\widetilde{h}_\epsilon, \widetilde{h}_\epsilon\rangle - \langle \pi_{\mathcal{O}_{c_0}}(g,X)h,h\rangle|^2 \leqslant A\cdot B, \tag{5.12}$$

where

$$\begin{aligned} A &= \iiint |e^{i(au+bv)}\overline{h}(a,b,c_0)|^2 \varphi_\epsilon(c-c_0)^2\,da\,db\,dc \\ B &= \iiint \varphi_\epsilon(c-c_0)^2 \big|e^{ic(-uy+w+xv)}h(a-cy,b+cx,c_0) \\ &\qquad - e^{ic_0(-uy+w+xv)}h(a-c_0y,b+c_0x,c_0)\big|^2\,da\,db\,dc. \end{aligned}$$

Note that $A = \left(\iint |\overline{h}(a,b,c_0)|^2\,da\,db\right)\left(\int \varphi_\epsilon(c-c_0)^2\,dc\right) = 1$. We can assume that $h \in C_c^\infty(\mathcal{O}_{c_0})$. Then the function

$$c \mapsto \Big[(a,b) \mapsto e^{ic(-uy+w+xv)}h(a-cy,b+cx,c_0)\Big]$$

is continuous. So for any $\eta > 0$, there is $\epsilon > 0$ so that if $|c-c_0| < \epsilon$,

$$\iint \big|e^{ic(-uy+w+xv)}h(a-cy,b+cx,c_0) - e^{ic_0(-uy+w+xv)}h(a-c_0y,b+c_0x,c_0)\big|^2\,da\,db < \eta.$$

Hence $B < \eta \int \varphi_\epsilon(c-c_0)^2\,dc = \eta$. Therefore $\pi_{\mathcal{O}_c}$ is weakly contained in α by (5.12). □

5.14 Theorem. $\mathrm{Ind}_{G_\varphi\ltimes\mathfrak{g}}^{G\ltimes\mathfrak{g}}\pi_{0,c}$ *is unitarily equivalent to* $\pi_{\mathcal{O}_c}$.

Proof. By the definition of induced representation, the Hilbert space of $\mathrm{Ind}_{G_\varphi\ltimes\mathfrak{g}}^{G\ltimes\mathfrak{g}}\pi_{0,c}$ can be identified with $L^2\left(\left\{\begin{pmatrix}1&a&0\\0&1&b\\0&0&1\end{pmatrix}, a,b\in\mathbb{R}\right\}\right)$ (equivalently $L^2(\mathbb{R}^2)$). This is true since there is an isomorphism from $G\ltimes\mathfrak{g}/G_\varphi\ltimes\mathfrak{g}$ onto $\left\{\begin{pmatrix}1&a&0\\0&1&b\\0&0&1\end{pmatrix}, a,b\in\mathbb{R}\right\}$ via

$$\left(\begin{pmatrix}1&a&0\\0&1&b\\0&0&1\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\right)G_\varphi\ltimes\mathfrak{g} \leftarrow\!\shortmid \begin{pmatrix}1&a&0\\0&1&b\\0&0&1\end{pmatrix}.$$

We write (a,b) for $\begin{pmatrix}1 & a & 0\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix}$ and σ for $\mathrm{Ind}_{G_\varphi\ltimes\mathfrak{g}}^{G\ltimes\mathfrak{g}}\,\pi_{0,c}$. Let $g=(x,y,z)\in G$ and $X=[u,v,w]\in\mathfrak{g}$. With new identification together with (5.4),

$$\begin{aligned}
&[\sigma((g,X))f]\,(a,b)\\
&\quad= f\left(g^{-1}(a,b,0), -(a,b,0)^{-1}(gXg^{-1})(a,b,0)\right)\\
&\quad= f\left((x,y,z)^{-1}(a,b,0), -(a,b,0)^{-1}[u,v,-uy+xv+w](a,b,0)\right)\\
&\quad= f\left((-x,-y,xy-z)(a,b,0), -(-a,-b,ab)[u,v,-uy+xv+w+ub]\right)\\
&\quad= f\left((a-x,b-y,xy-z-xb),[-u,-v,-w+uy-ub+av-xv]\right).
\end{aligned}$$

Then we apply (5.3) with $\lambda=0$ to obtain

$$[\sigma(g,X)f]\,(a,b)=e^{i(-cav+cbu-cuy+cw+cxv)}f(a-x,b-y).$$

after simplification. Now we define a map U from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$ by

$$(Uf)(a,b)=c^{-1}f(-c^{-1}b,c^{-1}a).$$

It is straightforward to check that its inverse U^{-1} is

$$(U^{-1}f)(a,b)=cf(cb,-ca).$$

Moreover U is a unitary map since $U^*=U^{-1}$. To see this, consider

$$\begin{aligned}
\langle U^*f,g\rangle &= \langle f,Ug\rangle\\
&= \int f(a,b)c^{-1}\overline{g}(-c^{-1}b,c^{-1}a)\,da\,db\\
&= \int f(cy,-cx)c^{-1}\overline{g}(x,y)c^2\,dx\,dy\\
&= \int cf(cy,-cx)\overline{g}(x,y)\,dx\,dy\\
&= \langle U^{-1}f,g\rangle.
\end{aligned}$$

Moreover U intertwines between σ and $\pi_{\mathcal{O}_c}$ since

$$\begin{aligned}\left[U\sigma(g,X)U^{-1}f\right](a,b) &= c^{-1}[\sigma(g,X)U^{-1}f](-c^{-1}b, c^{-1}a)\\ &= c^{-1}\left[e^{i(bv+au-cuy+cw+cxv)}U^{-1}f(-c^{-1}b-x, c^{-1}a-y)\right]\\ &= c^{-1}\left[e^{i(bv+au-cuy+cw+cxv)}cf(a-cy, b+cx)\right]\\ &= e^{i(bv+au-cuy+cw+cxv)}f(a-cy, b+cx)\\ &= [\pi_{\mathcal{O}_c}(g,X)f](a,b),\end{aligned}$$

for all $(a,b)\in\mathbb{R}^2$. Consequently $\mathrm{Ind}_{G_\varphi\ltimes\mathfrak{g}}^{G\ltimes\mathfrak{g}}\pi_{0,c}$ is unitarily equivalent to $\pi_{\mathcal{O}_c}$.

□

Combine two theorems above together, we get the result which is roughly analogous to Theorem 4.2.

5.15 Corollary. $\mathrm{Ind}_{G_\varphi\ltimes\mathfrak{g}}^{G\ltimes\mathfrak{g}}\pi_{0,c}$ *is weakly contained in* α.

Then we shall give the relationship between two representations in $A(\mathfrak{g}^*)$.

5.16 Theorem. $\{\pi_{\alpha\beta,\gamma\delta} \mid \alpha,\beta,\gamma,\delta\in\mathbb{R}\}$ *is weakly contained in* $\{\pi_{\mathcal{O}_c} \mid c\neq 0\}$ *and therefore it is weakly contained in* $\{\mathrm{Ind}_{G_\varphi\ltimes\mathfrak{g}}^{G\ltimes\mathfrak{g}}\pi_{0,c} \mid c\neq 0\}$.

Proof. Define a function $f_c\in L^2(\mathcal{O}_c)$ by

$$f_c(a,b)=\sqrt{\frac{2}{\pi c}}\cdot e^{\frac{i\alpha b}{c}-\frac{i\beta a}{c}}e^{-\frac{(a-\gamma)^2}{c}-\frac{(b-\delta)^2}{c}},$$

for $a,b\in\mathbb{R}$. Therefore, we have

$$f_c(a-cy,b+cx)=\sqrt{\frac{2}{\pi c}}\cdot e^{\frac{i\alpha(b+cx)}{c}-\frac{i\beta(a-cy)}{c}}e^{-\frac{(a-cy-\gamma)^2}{c}-\frac{(b+cx-\delta)^2}{c}}.$$

Let $g = (x, y, z) \in G$ and $X = [u, v, w] \in \mathfrak{g}$. Then

$$\begin{aligned}
&\langle \pi_{\mathcal{O}_c}((g, X)) f_c, f_c \rangle \qquad (5.13)\\
&= \iint [\pi_{\mathcal{O}_c}(g, X) f_c](a, b) \overline{f_c}(a, b)\, da\, db \\
&= \iint e^{i(au+bv+c(-uy+w+xv))} f_c(a - cy, b + cx) \overline{f_c}(a, b)\, da\, db \\
&= e^{i(\alpha x+\beta y)} e^{ic(-uy+w+xv)} \left(\sqrt{\frac{2}{\pi c}}\right)^2 \left(\int e^{iau} e^{-\frac{(a-\gamma)^2}{c} - \frac{(a-cy-\gamma)^2}{c}}\, da\right) \\
&\quad \left(\int e^{ibv} e^{-\frac{(b-\delta)^2}{c} - \frac{(b+cx-\delta)^2}{c}}\, db\right). \qquad (5.14)
\end{aligned}$$

By using similar arguments in the proof of Lemma 4.6, we obtain

$$\int e^{iau} e^{-\frac{(a-\gamma)^2}{c} - \frac{(a-cy-\gamma)^2}{c}}\, da = \sqrt{\frac{\pi c}{2}} \cdot e^{-\frac{cy^2}{2} - \frac{u^2 c}{8} + \frac{icuy}{2} + i\gamma u}, \qquad (5.15)$$

and

$$\int e^{ibv} e^{-\frac{(b-\delta)^2}{c} - \frac{(b+cx-\delta)^2}{c}}\, db = \sqrt{\frac{\pi c}{2}} \cdot e^{-\frac{cx^2}{2} - \frac{v^2 c}{8} - \frac{icvx}{2} + i\delta v}. \qquad (5.16)$$

Now we substitute (5.15) and (5.16) in (5.14),

$$\begin{aligned}
&\langle \pi_{\mathcal{O}_c}((g, X)) f_c, f_c \rangle \\
&= e^{i(\alpha x+\beta y)} e^{ic(-uy+w+xv)} e^{-\frac{cy^2}{2} - \frac{u^2 c}{8} + \frac{icuy}{2} + i\gamma u} e^{-\frac{cx^2}{2} - \frac{v^2 c}{8} - \frac{icvx}{2} + i\delta v} \\
&= e^{i(\alpha x+\beta y)} e^{i(\gamma u+\delta v)} e^{ic(-uy+w+xv) - \frac{cy^2}{2} - \frac{u^2 c}{8} + \frac{icuy}{2} - \frac{cx^2}{2} - \frac{v^2 c}{8} - \frac{icvx}{2}}.
\end{aligned}$$

It converges to $e^{i(\alpha x+\beta y)} e^{i(\gamma u+\delta v)} = \langle \pi_{\alpha\beta,\gamma\delta}(g, X) 1, 1 \rangle$, as $c \to 0$. □

Consequently, we obtain the following theorem for $A(\mathfrak{g}^*)$ which is analogous to Theorems 4.9 and 4.10 for $A(G)$.

5.17 Theorem. *The spectrum of* $A(\mathfrak{g}^*)$ *is the union of* $\{\mathrm{Ind}_{G_\varphi \ltimes \mathfrak{g}}^{G \ltimes \mathfrak{g}} \pi_{0,c} \mid c \neq 0\}$ *and* $\{\pi_{\alpha\beta,\gamma\delta} \mid \alpha, \beta, \gamma, \delta \in \mathbb{R}\}$.

5.18 Theorem. *The spectrum of* $A(\mathfrak{g}^*)$ *is the closure of the set*

$$\{\mathrm{Ind}_{G_\varphi \ltimes \mathfrak{g}}^{G \ltimes \mathfrak{g}} \pi_{0,c} \mid c \neq 0\}.$$

5.19 Remark. The underlying idea is that

$$\begin{aligned} \pi_c \otimes \pi_c^* \in \widehat{A(G)} \quad &\text{corresponds to} \quad \mathrm{Ind}_{G_\varphi \ltimes \mathfrak{g}}^{G \ltimes \mathfrak{g}} \pi_{0,c} \in \widehat{A(\mathfrak{g}^*)}; \\ \pi_{\alpha\beta} \otimes \pi_{\gamma\delta}^* \in \widehat{A(G)} \quad &\text{corresponds to} \quad \pi_{\alpha\beta,ab} \in \widehat{A(\mathfrak{g}^*)}. \end{aligned}$$

Chapter 6

Summary and directions for future work

In this thesis we have formulated and examined a conjecture in C*-algebras and representation theory that is inspired by Kirillov's orbit theorem for irreducible unitary representations of nilpotent groups, but different from it. We constructed two C*-algebras, one related to the decomposition of $L^2(G)$ into irreducible representations of $G \times G$, and one related to the decomposition of $\mathfrak{g}^*$ into coadjoint orbits. Our conjecture is that the two C*-algebras are isomorphic. We proved the conjecture for the Heisenberg group, and examined it further detail in that case.

When G is the Heisenberg group there is a natural, Lie-theoretic proof of the conjecture using the exponential map. This seems to be very special to the Heisenberg group, and so we began the program of examining the conjecture from a more detailed representation-theoretic perspective. We explained how (in the case of the Heisenberg group) both C*-algebras decompose into simpler parts through C*-algebra extensions. Naturally we expect to see the same structure in more general cases, and the first example to be considered in future work will be the four-dimensional group K_3

introduced in Chapter 2.

After these calculations are done, we can hope to use the theory of C*-algebra extensions (initiated by Brown Douglas and Fillmore and developed much more fully by Kasparov) to settle our isomorphism conjecture in at least some new cases. The idea here is that there are very few ways of re-assembling a C*-algebra from the component parts that appear in extensions of the sort that are seen in the analysis of $A(G)$ and $A(\mathfrak{g}^*)$. But we have scarcely begun to examine the issues that are involved here in any detail. Much work remains to be done.

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