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QUANTIZATION OF AFFINE COADJOINT ORBITS

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Abstract

Using twisted equivariant K-homology, E. Meinrenken defined the quantization of a q-Hamiltonian space as the pushforward of the fundamental class by a Morita morphism and obtained an element in the Verlinde algebra. This dissertation explains a different way to obtain the quantization of a Hamiltonian loop group space.

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Chapter 1 Introduction

1.1 Geometric quantization and K-homology

Roughly speaking, geometric quantization refers to the process of turning a classical system, which is modeled as a symplectic manifold together with several functions called observables, into a quantum system consisting of a collection of skew adjoint operators on an appropriate Hilbert space.

To be explicit, let (M, ω) be a compact symplectic manifold. The symplectic 2-form ω enables us to define the Poisson bracket

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M), \quad \{f, g\} = \omega(V_f, V_g),$$

where V_f, V_g are the Hamiltonian vector fields defined by the formula $\iota_{V_f}\omega = df$ and $\iota_{V_g}\omega = dg$. The process of quantization should ideally produce a correspondence

$$Q : A \to \{\text{Skew adjoint operators on } H\},$$

such that $Q_{\{f,g\}} = [Q_f, Q_g]$ for $f, g \in A$ and $Q_1 = \frac{2\pi}{\sqrt{-1}}\,\mathrm{Id}$, where $A \subset C^\infty(M)$ is a much smaller subspace, closed under Poisson bracket.

Neither existence nor uniqueness of quantization can be easily resolved. See [59] or [28] for further details. Furthermore, even the precise meaning of appropriate Hilbert space is problematic.

Suppose $\mathcal{L}$ is a Hermitian line bundle over M with a Hermitian connection ∇ such that

$$\mathrm{curv}(\nabla) = 2\pi\sqrt{-1}\omega.$$

It is well known that such a line bundle exists if and only if the cohomology class of ω is integral, i.e., $[\omega] \in \mathrm{Image}(H^2(M,\mathbb{Z}) \to H^2(M,\mathbb{R}))$. In this case, the symplectic manifold (M,ω) is said to be pre-quantizable, and $\mathcal{L}$ is called a pre-quantum line bundle. Now we can try to define the associated quantum system. Let $\langle s_1, s_2\rangle_{\mathcal{L}}$ denote the Hermitian metric on $\mathcal{L}$, then one can define an inner product on $\Gamma(\mathcal{L})$ by the following formula,

$$\langle s_1, s_2\rangle = \int_M \langle s_1, s_2\rangle_{\mathcal{L}} \frac{\omega^n}{n!}, \quad s_1, s_2 \in \Gamma(\mathcal{L}),$$

where $n = \dim(M)/2$. Let $H = L^2(M,\mathcal{L})$ be the Hilbert space of L^2 sections of $\mathcal{L}$. For any smooth function $f \in C^\infty(M)$, define the associated unbounded skew-adjoint operator $Q_f(s)$ on H by the formula

$$Q_f(s) = \nabla_{V_f} s - 2\pi\sqrt{-1} f \cdot s, \quad s \in H.$$

One can check that $Q_{\{f,g\}} = [Q_f, Q_g]$ and $Q_1 = \frac{2\pi}{\sqrt{-1}}\,\mathrm{Id}$.

However quantum mechanics tells us that this Hilbert space H is too large. Additional structures such as polarizations may be introduced to reduce the size of H. However, in this dissertation, we will not discuss polarizations directly, but instead investigate quantization from the related point of view of K-theory or K-homology, using index theory.

Since M is a symplectic manifold, there exists a canonical Spin^c structure on M, with which one can construct the associated Spin^c Dirac operator, twisted by the line bundle $\mathcal{L}$. To be explicit, let J be a compatible almost complex structure on TM, i.e.,

$$J : TM \to TM, \quad J^2 = -\operatorname{Id},$$

and the formula

$$g(v, w) := \omega(v, Jw)$$

defines a Riemannian metric on TM. Let $T_{\mathbb{C}}M = TM \otimes \mathbb{C} = TM \oplus iTM$ be the complexification of TM. Extending J to $T_{\mathbb{C}}M$ by complex linearity, we have the decomposition

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M,$$

where $T^{1,0}M$ and $T^{0,1}M$ are the $+i$ and $-i$ eigen-subbundles of J respectively. Let $T^{(1,0),*}M$ and $T^{(0,1),*}M$ be the dual bundles. Analogously, on the exterior product of contangent bundle we have the decomposition

$$\Lambda^* T^*_{\mathbb{C}}M = \bigoplus \Lambda^{i,j} T^* M,$$

where $\Lambda^{i,j} T^* M = \Lambda^i (T^{(1,0),*}M) \otimes \Lambda^j (T^{(0,1),*}M)$.

Let $V \in T_{\mathbb{C}}M$, then $V = V_+ + V_- \in T^{1,0}M \oplus T^{0,1}M$. Define the operator

$$c : T_{\mathbb{C}}M \to \operatorname{End}(\Lambda^{0,*}T^*M), \quad c(V) = \sqrt{2}(\epsilon(V_+^*) - \iota(V_-)).$$

We can check that c satisfies the condition $c(V)^2 = -g(V, V)$, so it defines a Clifford action,

$$c : \mathbb{C}\mathrm{liff}(TM) \to \operatorname{End}(\Lambda^{0,*}T^*M).$$

Let $S = \Lambda^{0,*}T^*M \otimes \mathcal{L}$ be the spinor module twisted by the line bundle $\mathcal{L}$, and $\not{D}_{\mathcal{L}}$ be the Spin^c Dirac operator on S. See [39] for details.

Definition 1.1.1. Let (M, ω) be a compact symplectic manifold with a prequantum line bundle $\mathcal{L}$, then the geometric quantization is defined as the index of the Spin^c Dirac operator

$$Q(M) = \text{Index}(\not{D}_{\mathcal{L}}),$$

which is a finite dimensional virtual vector space.

Remark 1.1.2. Let G be a compact Lie group. If M is a G-equivariant symplectic manifold and $\mathcal{L}$ is a G-equivariant pre-quantum line bundle, one can define the geometric quantization similarly by taking the index of the associated Spin^c Dirac operator,

$$Q(M) = \text{Index}(\not{D}_{\mathcal{L}}) \in R(G),$$

where $R(G)$ is the representation ring of G.

The index is independent of the choice of almost complex structure on M. It is worth a mention that the Spin^c Dirac operator $\not{D}$ defines a class in the K-homology group of M,

$$[\not{D}] \in K_0^G(M).$$

It is a fundamental class of M, in the sense that every class in the K-homology group $K_0^G(M)$ arises from a twisted Spin^c Dirac operator $\not{D}_E$ for some complex vector bundle E, see [30] for details.

The defintion of quantization can be simplified in terms of K-homology, using the concept of fundamental class and Morita morphism (see Section 2.3.1). We observe that the line bundle $\mathcal{L}$ defines a Morita morphism between the trivial module bundles

$$\mathcal{E}_{\mathcal{L}} : (M, \underline{\mathbb{C}}) \to (\text{pt}, \mathbb{C}),$$

which in turn defines a homomorphism between the K-homology groups

$$\mathcal{E}_{\mathcal{L}*} : K_0^G(M) \to K_0^G(\text{pt}) \simeq R(G).$$

Definition 1.1.3. Let (M, ω) be a compact symplectic manifold with a pre-quantum line bundle $\mathcal{L}$, then the geometric quantization is defined as the push-forward of the fundamental class

$$Q(M) = \mathcal{E}_{\mathcal{L}*}([\not{D}]) \in R(G),$$

where $R(G)$ is the representation ring of G.

If furthermore M is assumed to be a Kähler manifold with a holomorphic pre-quantum line bundle $\mathcal{L}$, then the geometric quantization is just the Kähler quantization, see [28].

Being a Kähler manifold, M is a symplectic manifold with symplectic 2-form ω and at the same time a complex manifold with integral almost complex structure J. The spinor module bundle is $S = \Lambda^{0,*}(T^*M)$ and the Spin^c Dirac operator turns out to be the Dolbeault operator,

$$\not{\partial} = \bar{\partial} + \bar{\partial}^*.$$

From Hodge theory, the index of the twisted Dolbeault $\not{\partial}_{\mathcal{L}}$ can be written as

$$\text{Index}(\not{\partial}_{\mathcal{L}}) = \bigoplus H^{0,i}(M, \mathcal{L}),$$

where $H^{0,i}(M, \mathcal{L})$ denotes the i-th Dolbeault cohomology group.

Since the line bundle $\mathcal{L}$ is positive, by Kodaira vanishing theorem, the cohomology groups $H^{0,i}(M, \mathcal{L})$ vanish at all positive degrees. Hence the Hilbert space in the geometric quantization is the completed space of holomorphic sections of the holomophic line bundle.

Example 1.1.4. An important example is the coadjoint orbit $\mathcal{O}_\mu \subset \mathfrak{g}^*$, which is a canonical Kähler manifold with a pre-quantum line bundle $\mathcal{L}$. The quantization of

$\mathcal{O}_\mu$ is a generator of $R(G)$,

$$Q(\mathcal{O}_\mu) = [V_\mu] \in R(G),$$

where V_μ is the irreducible representation of G with the highest weight μ.

1.2 q-Hamiltonian spaces and twisted K-homology

Let G be a compact connected Lie group. We recall that a Hamiltonian G-space (M, ω_M, Φ) is a G-symplectic manifold M together with a G-equivariant moment map $\Phi : M \to \mathfrak{g}^*$ such that

$$\iota(X_M)\omega_M = \mathrm{d}\,\Phi(X), \quad X \in \mathfrak{g},$$

where X_M denotes the vector field generated by $X \in \mathfrak{g}$.

Alekseev, Malkin and Meinrenken introduced the concept of a q-Hamiltonian G-space with moment map taking values in the Lie group G in [2]. The definition of a q-Hamiltonian G-space is similar but it possesses some features that will be clear in the discussion of Hamiltonian LG-spaces. See Section 3.3.

Roughly speaking, a q-Hamiltonian G-space (N, ω_N, Ψ) is a G-manifold N together with an invariant 2-form ω_N and a G-equivariant map $\Psi : N \to G$ with some conditions on ω_N and Ψ. It should be pointed out that ω_N is not a symplectic form. For definition, see Chapter 3.

Examples of q-Hamiltonian G-spaces are conjugacy classes of G, which may not have any Spin^c structure. Under some non-degeneracy condition, one can construct a q-Hamiltonian space from a usual Hamiltonian G-space by taking the exponential map, see Section 3 of [2].

To quantize a q-Hamiltonian space, we need to introduce pre-quantization conditions. Let $\mathcal{A}_G$ be a G-equivariant Dixmier-Douady bundle whose Dixmier-

Douady class is a generator of $H^3_G(G,\mathbb{Z}) \simeq Z$. See [44].

Definition 1.2.1. A q-Hamiltonian G-space (N, ω_N, Ψ) is pre-quantizable at level k if there exists a G-equivariant Morita morphism

$$(\Psi, \mathcal{E}) : (N, \mathbb{C}) \to (G, \mathcal{A}_G^{-k}).$$

A Morita morphism is the counterpart of a pre-quantum line bundle. It is the geometric realization of the integral class (ω_N, η_G) in the relative cohomology group $H^3_G(\Psi)$.

Another key feature in q-Hamiltonian G-space is the existence of twisted Spin^c structure. In [1], Alekseev and Meinrenken constructed a distinguished G-equivariant Morita morphism

$$(\Psi, \mathcal{S}) : (N, \mathbb{C}\mathrm{liff}(TN)) \to (G, \mathcal{A}_G^{-h^\vee}).$$

After combining those two Morita morphisms and applying the K-homology, one obtains a G-equivariant homomorphism

$$\Psi_* : KK_G(C(N, \mathbb{C}\mathrm{liff}(TN)), \mathbb{C}) \to KK_G(C(G, \mathcal{A}_G^{-h^\vee - k})).$$

The K-homology group $KK_G(C(M, \mathbb{C}\mathrm{liff}(TN)), \mathbb{C})$ is twisted by the Dixmier-Douady bundle $\mathbb{C}\mathrm{liff}(TN)$. It has a distinguished class, called the Kasparov's Dirac element, denoted by $[N]$, see [34]. If additionally M is a Spin^c manifold, then one has the isomorphism

$$KK_G(C(N, \mathbb{C}\mathrm{liff}(TN)), \mathbb{C}) \simeq KK_G(C(N), \mathbb{C}),$$

and the Kasparov's Dirac element maps to the class defined by a Spin^c Dirac operator, see [44]. However, a general q-Hamiltonian space may not have any Spin^c

structures. It is for this reason the Kasparov's Dirac element enters the picture.

Now we are ready to define the quantization of a q-Hamiltonian G-space.

Definition 1.2.2. The quantization of a pre-quantizable q-Hamiltonian space (N, ω_N, Ψ) is defined as the push-forward of the Kasparov's Dirac element $[N] \in KK_G(C(N, \mathbb{C}\mathrm{liff}(TN)), \mathbb{C})$,

$$Q(N) = \Psi_*([N]).$$

Donovan and Karoubi [21] defined the twistings for K-theory by torsion classes in H^3, represented by Azumaya bundles. J. Rosenberg [53] generalized to non-torsion classes.

In [23] Freed, Hopkins and Teleman proved that the equivariant twisted K-theory is isomorphic to the Verlinde ring. By Poincaré duality (see Section 2.4.3), it is isomorphic to K-homology, which is more natural since there is no degree shift in the formula.

Theorem 1.2.3. *The equivariant twisted K-homology group is isomorphic to the Verlinde ring,*

$$KK_G(C(G, \mathcal{A}_G^{-h^\vee - k}), \mathbb{C}) \simeq R_k(G).$$

The Verlinde ring $R_k(G)$ was introduced by E. Verlinde in [58]. It is the representation ring of projective positive energy representations of the loop group LG at level k. We will focus on this topic in Chapter 3.

Alternatively, $R_k(G)$ may be interpreted as the quotient of $R(G)$ by the ideal $I_k(G)$ of characters vanishing at all points (see [44])

$$t_\lambda = \exp\left(\frac{\lambda + \rho}{h^\vee + k}\right).$$

The quotient map $q : R(G) \to R_k(G)$ can be understood from K-homology. By taking the identity $\mathrm{pt} = e \in G$ and restricting the Dixmier-Douady bundle $\mathcal{A}_G$ on it, we obtain the quotient homomorphism

$$q : KK_G(C(\mathrm{pt}, \mathcal{A}_G^{-h^\vee-k}|_{\mathrm{pt}}), \mathbb{C}) \to KK_G(C(G, \mathcal{A}_G^{-h^\vee-k}), \mathbb{C}).$$

Indeed, the left hand side is

$$KK_G(C(\mathrm{pt}, \mathcal{A}_G^{-h^\vee-k}|_{\mathrm{pt}}), \mathbb{C}) \simeq KK_G(C(\mathrm{pt}, \mathbb{K}|_{\mathrm{pt}}), \mathbb{C}) \simeq K_0^G(\mathrm{pt}) = R(G),$$

and right hand side is just by FHT isomorphism.

We have reviewed the basics of Hamiltonian and q-Hamiltonian G-spaces, as well as their geometric quantizations in terms of K-homology. We shall introduce Hamiltonian LG-space and study its geometric quantization.

Definition 1.2.4. A Hamiltonian LG-space (M, ω, Φ) at level 1 is a Banach manifold M together with an LG-action, an invariant 2-form $\omega \in \Omega^2(M)^{LG}$, and an equivariant moment map, $\Phi : M \to L\mathfrak{g}^*$ such that

(1) $\mathrm{d}\,\omega = 0$.

(2) The moment map Φ satisfies the condition

$$\iota(\xi_M)\omega = \mathrm{d}\int_{S^1} \Phi(\xi), \quad \xi \in L\mathfrak{g}.$$

(3) ω is weakly non-degenerate, i.e., the induced map $TM \to T^*M$ is injective.

Here affine coadjoint action of LG on $L\mathfrak{g}^*$ is given by

$$\gamma \cdot \mu = \mathrm{Ad}_\gamma\, \mu - \gamma'\gamma^{-1}.$$

If we interpret $L\mathfrak{g}^*$ as the space of connections on the trivial bundle $S^1 \times G$ over S^1, then the affine action is just the gauge transformation. It was shown in [2] that every Hamiltonian LG-space with proper moment map determines a q-Hamiltonian G-space and vice versa. In particular, the conjugacy classes of G correspond exactly to the affine coadjoint orbits of LG on $L\mathfrak{g}^*$.

For any $\mu \in L\mathfrak{g}^*$, let $\mathrm{Hol}_s : L\mathfrak{g}^* \to G$ be the unique solution of the initial value problem

$$\mathrm{Hol}_s(\mu)^{-1}\frac{\mathrm{d}}{\mathrm{d}\,s}\mathrm{Hol}_s(\mu) = \mu, \quad \mathrm{Hol}_0(\mu) = e.$$

The function Hol_s satisfies the equivariance condition

$$\mathrm{Hol}_s(\gamma \cdot \mu) = \gamma(0)\,\mathrm{Hol}_s(\mu)\gamma(s)^{-1}.$$

Finally let $\mathrm{Hol} = \mathrm{Hol}_1$ be the holonomy map, that is, evaluation at the first complete cycle. Then $\mathrm{Hol} : L\mathfrak{g}^* \to G$ is equivariant with respect to the evaluation homomorphism $LG \to G, \gamma \mapsto \gamma(0)$. It can be shown that the action of the based loop group ΩG on $L\mathfrak{g}^*$ is free, and the quotient map is just the holonomy map. In other words, $\mathrm{Hol} : L\mathfrak{g}^* \to G$ is the universal principal ΩG-bundle.

Since the ΩG action on $L\mathfrak{g}^*$ is free, its action on M is also free by the equivariance of the moment map, and hence M is a principal ΩG-bundle over a finite dimensional manifold $\mathrm{Hol}(M) := M/\Omega G$. This quotient $\mathrm{Hol}(M)$ carries a canonical structure of a q-Hamiltonian G-space, and the following diagram commutes,

$$\begin{array}{ccc} M & \xrightarrow{\tilde{\Phi}} & L\mathfrak{g}^* \\ {\scriptstyle \mathrm{Hol}}\downarrow & & \downarrow{\scriptstyle \mathrm{Hol}} \\ N & \xrightarrow{\Psi} & G. \end{array}$$

A Hamiltonian LG-space (M, ω, Φ) is pre-quantizable at level k if the cohomology class ω is integral, that is, there exists an $\tilde{L}G$-equivariant prequantum line bundle. Here $\tilde{L}G$ denotes the central extension of LG at level k.

1.3 Main results of the dissertation

In this dissertation, we are going to construct a fundamental class on a principal ΩG manifold M, using the complex structure on the fiber LG/G. From Fourier theory, we have the decomposition

$$L\mathfrak{g}_{\mathbb{C}} = \bigoplus_n \mathfrak{g}_{\mathbb{C}} z^n.$$

Let $\mathfrak{g}_+ = \bigoplus_{n>0} \mathfrak{g}_{\mathbb{C}} z^n$ and $\mathfrak{g}_- = \bigoplus_{n<0} \mathfrak{g}_{\mathbb{C}} z^n$. An element $e_a z^n \in \mathfrak{g}_{\mathbb{C}} z^n$ is denoted by e_a^n.

Let

$$\mathcal{S}_{\Omega\mathfrak{g}} = \bigwedge \mathfrak{g}_-^* \otimes \bigwedge \mathfrak{g}_{\mathbb{C}}^*$$

denote the (extended) spinor module. We introduce the Dolbeault operator $\bar{\partial}$, given by the formula

$$\bar{\partial} = 1/2 \sum_{a,m\geq 0} \epsilon(e_a^m)\,\mathrm{ad}(e_a^{-m}),$$

where the operator $\mathrm{ad}(e_a^{-m})$ is defined by the formula

$$\mathrm{ad}(e_a^{-m})(e_b^n) = \begin{cases} [e_a, e_b]^{n-m} & \text{if } n-m \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

A slightly different Dolbeault operator was used in [56] to prove a vanishing theorem for Lie algebra cohomology. The formal adjoint under some Hermitian metric is given by

$$\bar{\partial}^* = -1/2 \sum_{a;m\geq 0} \mathrm{ad}(e_a^m)\iota(e_a^{-m}).$$

Note that for any finite energy vector, $\bar{\partial}$ and $\bar{\partial}^*$ consist of only finitely many nonzero terms.

Let $D_{\mathrm{vert}} = \bar{\partial} + \bar{\partial}^*$, and we call it the vertical operator.

Theorem 1.3.1.

$$D_{\text{vert}}^2 = h^\vee \cdot (\mathrm{E} + 2 \cdot \deg) + 1/2 \sum_a \mathrm{ad}^2(e_a),$$

where E *is the energy operator, i.e.,* $\mathrm{E}(e_b^n) = ne_b^n$*, and* deg *is the degree operator.*

Corollary 1.3.2. *The unbounded operator* $D_{\text{vert}} = \bar{\partial}^* + \bar{\partial}$ *is essentially self-adjoint, has compact resolvent, and is Fredholm with index given by* $\mathbb{C}$*.*

The Dirac operator can be extended to the space of semi-infinite forms $\mathcal{S}^\infty$. The following proposition shows that D_{vert} does not commute with the LG action.

Proposition 1.3.3. *The square of the Dirac operator* D_{vert}^2 *satisfies the relation*

$$[\rho(\xi), D_{\text{vert}}^2] = h^\vee \rho(\xi'), \quad \xi \in L\mathfrak{g}$$

with the action of $\tilde{L}G$ *on the twisted spinor module* $\mathcal{S}^\infty$*.*

A similar phenomenon occurs in the paper [25], where a family of Dirac operators was constructed, and each of which is not $\tilde{L}G$-equivariant.

Next, using the vertical operator, we can form the differential operator on the pre-Hilbert space $\mathcal{H}_{Kas}^0 = C(N, \mathbb{C}\mathrm{liff}(TN)) \hat{\otimes} \mathcal{S}_{\Omega\mathfrak{g}}^* \hat{\otimes} \mathcal{S}_{\Omega\mathfrak{g}}$ given by the formula:

$$\not{D}_{Kas} = \not{D}_N \hat{\otimes} \mathrm{Id} \hat{\otimes} \mathrm{Id} + \mathrm{Id} \hat{\otimes} \mathrm{Id} \hat{\otimes} D_{\text{vert}},$$

where D_{vert} acts on $\mathcal{S}_{\Omega\mathfrak{g}}$ and $\not{D}_N$ is the operator for the Kasparov's Dirac element, acting on $C(N, \mathbb{C}\mathrm{liff}(TN))$. See Section 2.2.4.

Lemma 1.3.4. *The operator* $\not{D}_{Kas}$ *is an unbounded essentially selfadjoint odd operator on the Hilbert space* $\mathcal{H}_{Kas}$*.*

Theorem 1.3.5. *The operator*

$$\rho_{Kas}(a)(1 + D_{Kas}^2)^{-1}$$

is a compact operator, and the commutator

$$[\rho_{Kas}(a), \not{D}_{Kas}]$$

is bounded, that is, $(\mathcal{H}_{Kas}, \rho_{Kas}, \not{D}_{Kas})$ *defines an unbounded Kasparov KK-cycle. Hence,* $F_{Kas} = \not{D}_{Kas}(1 + \not{D}_{Kas}^2)^{-1/2}$ *defines a class in the K-homology group* $KK_G(C(N, \mathbb{C}\mathrm{liff}(TN))\hat{\otimes}\mathbb{K}(\mathcal{S}^*_{\Omega\mathfrak{g}}), \mathbb{C})$.

The bundle $\mathcal{L} \otimes \mathcal{H}^{-k}_{LG}$ is an ΩG-vector bundle. Since the action of ΩG is free, the quotient

$$\mathcal{E}_{Line} = (\mathcal{L} \otimes \mathcal{H}^{-k}_{LG})/\Omega G$$

is a bundle over N. It gives rise to Morita morphism

$$(\Psi, \mathcal{E}_{Line}) : (N, \mathbb{C}) \to (G, \mathcal{A}^{-k}_G).$$

Theorem 1.3.6. *For any conjugacy class* $N = \mathcal{C}_{\mathrm{Hol}(\mu)}$*, there is a distinguished* G*-equivariant Morita morphism*

$$(\Psi, \mathcal{E}_{Spin}) : (N, \mathbb{C}\mathrm{liff}(TN)) \to (G, \mathcal{A}^{-h^\vee}_G).$$

Tensoring with $\mathbb{K}(\mathcal{S}^*_{\Omega\mathfrak{g}})$ *yields*

$$(\Psi, \mathcal{E}_{Spin}\hat{\otimes}\mathcal{S}^*_{\Omega\mathfrak{g}}) : (N, \mathbb{C}\mathrm{liff}(TN)\hat{\otimes}\mathbb{K}(\mathcal{S}^*_{\Omega\mathfrak{g}})) \to (G, \mathcal{A}^{-h^\vee}_G).$$

By combining the twisted Spin^c structure and the pre-quantum line bundle, we obtain a Morita morphism

$$(\Psi, \mathcal{E}_{Line}\hat{\otimes}\mathcal{E}_{Spin}\hat{\otimes}\mathcal{S}^*_{\Omega\mathfrak{g}}) : (N, \mathbb{C}\mathrm{liff}(TN)\hat{\otimes}\mathcal{S}^*_{\Omega\mathfrak{g}}) \to (G, \mathcal{A}^{-h^\vee-k}_G),$$

where $\mathcal{A}^{-h^\vee-k}_G = L\mathfrak{g}^* \times_{\Omega G} \mathbb{K}(\mathcal{H}^{-h^\vee-k}_{LG})$.

Definition 1.3.7. The quantization class is defined as the KK-class of the tensor product of the Morita morphisms

$$[\mathcal{E}_{Line}\hat{\otimes}\mathcal{E}_{Spin}\hat{\otimes}\mathcal{S}^*_{\Omega\mathfrak{g}}] \in KK_G(C(G, \mathcal{A}_G^{-h^\vee-k}), C(N, \mathbb{C}\mathrm{liff}(TN))\hat{\otimes}\mathbb{K}(\mathcal{S}^*_{\Omega\mathfrak{g}})).$$

The quantization of an affine coadjoint orbit M is the Kasparov product of the fundamental class with the quantization class,

$$\mathcal{Q}(M) = [\mathcal{E}_{Line}\hat{\otimes}\mathcal{E}_{Spin}\hat{\otimes}\mathcal{S}^*_{\Omega\mathfrak{g}}] \otimes [F_{Kas}] \in KK_G(C(G, \mathcal{A}_G^{-h^\vee-k}), \mathbb{C}).$$

Theorem 1.3.8. *Let $\mu \in \Lambda_k^*$, and let $q : R(G) \to R_k(G)$ be the quotient map, then the quantization $\mathcal{Q}$ maps the fundamental class to the corresponding generator in the Verlinde ring $R_k(G)$,*

$$\mathcal{Q}([\mathcal{O}_\mu]) = q(V_\mu).$$

Hence the quantization map $\mathcal{Q}$ coincides with the one defined by Meinrenken in [44].

Chapter 2

K-theory and K-homology

2.1 K-theory

2.1.1 Topological K-theory

Topological K-theory was introduced by Atiyah and Hirzebruch [4], motivated by work of Grothendieck in algebraic geometry. The main reference to this section is the classic book [5] by Atiyah.

Let X be a compact Hausdorff topological space, and let $\mathrm{Vect}_{\mathbb{C}}(X)$ be the isomorphism classes of complex vector bundles over X. $\mathrm{Vect}_{\mathbb{C}}(X)$ is an abelian semigroup under the operation of the Whitney sum.

Definition 2.1.1. The *Atiyah-Hirzebruch K-theory group* $K^0(X)$ is defined to be the free abelian group generated by $\mathrm{Vect}_{\mathbb{C}}(X)$ modulo the subgroup generated by the elements of the form $E + F - E \oplus F$.

In short, the K-theory group $K^0(X)$ over X is the Grothendieck group of vector bundles. Every class in $K^0(X)$ is of the form $[E] - [F]$ for some $E, F \in \mathrm{Vect}_{\mathbb{C}}(X)$. Since X is compact, for every vector bundle H, there exists a vector bundle H' such that $H \oplus H' = \underline{\mathbb{C}}^n$ for some trivial bundle $\underline{\mathbb{C}}^n$. Hence every class is of the form $[E] - \underline{\mathbb{C}}^n$. K^0 is a contravariant homotopy functor by the pullback operation on vector bundles.

Example 2.1.2. (1) $K^0(\text{pt}) \simeq \mathbb{Z}$.
(2) $K^0(S^2) \simeq \mathbb{Z} \oplus \mathbb{Z}$.

Let $\iota : \text{pt} \to X$ be the inclusion of a base point in X. The *reduced K-theory group* $\tilde{K}^0(X)$ is defined to be the kernel of the induced map

$$\iota^* : K^0(X) \to K^0(\text{pt}).$$

Suppose X is a locally compact Hausdorff topological space. Let X^+ denote the one-point compactification of X. The K-theory group $K^0(X)$ is defined to be $\tilde{K}^0(X^+)$. Note that if X is compact, then $K^0(X)$ coincides with the previous definition. Moreover, the odd K-theory group is defined to be $K^{-1}(X) = K^0(\mathbb{R} \times X)$. Every element in $K^0(X)$ can be represented by a compactly supported and bounded complex of vector bundles over X, [29]

$$E_0 \leftarrow E_1 \leftarrow \dots \leftarrow E_n.$$

Now let G be a topological group. A G-space X is a topological space with a continuous G-action, i.e., the map $G \times X \to X$ is continuous. A G-equivariant map between two G-spaces is a continuous map which intertwines with the G-actions.

Definition 2.1.3. A *G-equivariant vector bundle* over a G-space X is a vector bundle $\pi : E \to X$ such that
(1) E is a G-space,
(2) π is G-equivariant, and
(3) the map $g : E_x \to E_{gx}$ is a vector space homomorphism for any $g \in G$ and $x \in X$.

Example 2.1.4. If G is a Lie group, and M is a G-manifold, then the tangent bundle TM is a G-equivariant vector bundle.

Definition 2.1.5. The *equivariant K-theory group* $K^0_G(X)$ over a compact Hausdorff space X is the Grothendieck group of G-vector bundles.

Lemma 2.1.6. *If the action of G on X is free, then G-equivariant vector bundles on X correspond bijectively to vector bundles on X/G. Hence we have*

$$K^0_G(X) \simeq K^0(X/G).$$

Proof. See [29]. □

Note that the G-equivariant theory group over a point is just the representation ring of G,

$$K^0_G(\text{pt}) = R(G).$$

If T be a maximal torus of G, then $K^0_G(G/T) = K^0_T(\text{pt}) = R(T)$, that is, any G-equivariant vector bundle over G/T is of the form $G \times_T V$ for some T-vector space V. In later sections, we will study the relation between $K^0_G(G/T)$ and $K^0_G(\text{pt})$ using index theory.

2.1.2 K-theory for operator algebras

A *Banach algebra* is an associative algebra over $\mathbb{C}$ which at the same time is a Banach space such that $\|xy\| \leq \|x\|\|y\|$.

Definition 2.1.7. A *C*-algebra* A is a Banach algebra over $\mathbb{C}$ equipped with an involution $^* : A \to A$ such that:

(1) $x^{**} = x$,

(2) $(x+y)^* = x^* + y^*$,

(3) $(\alpha x)^* = \overline{\alpha} x^*$,

(4) $\|x^* x\| = \|x\|\|x^*\|$. (C* identity)

An *ideal* J of a C*-algebra is a closed, two-sided ideal such that $a \in J$ implies $a^* \in J$. The quotient A/J is a C^* algebra in the quotient norm.

Example 2.1.8. Let X be a locally compact Hausdorff topological space, and Y a open subset. Let $C_0(X)$ be the space of complex-valued functions on X which vanish at infinity. Then $C_0(X)$ is a C^*-algebra without identity, and $J = C_0(Y)$ is an ideal of A. We have an exact sequence of C^*-algebras,

$$0 \to J \to A \to A/J \to 0.$$

A function between two topological spaces is *proper* if the inverse images of compact subsets are compact. A proper continuous map $f : Y \to X$ between two locally compact Hausdorff spaces Y and X induces a homomorphism of C*-algebras,

$$f^* : C_0(X) \to C_0(Y).$$

Definition 2.1.9. Let A be a unital C*-algebra. The *K-theory group* $K_0(A)$ of A is the abelian group with one generator $[p]$ for each projection in $M_n(A)$ with the relations:

(1) if p, q are projections, and p is joined by continuous path of projections to q in $M_n(A)$, then $[p] = [q]$,

(2) $[0] = 0$, and

(3) $[p] + [q] = [p \oplus q]$.

Alternatively, $K_0(A)$ is the abelian group with one generator for each isomorphism classes of finitely generated projective modules over A with the relations

$$[p] + [q] = [p \oplus q].$$

If A is nonunital, then $K_0(A)$ is defined to be the kernel of the induced homomorphism $K_0(A^+) \to K_0(\mathbb{C}) = \mathbb{Z}$, where A^+ is the unitalization of A.

Example 2.1.10. Suppose H is an infinite-dimensional separable Hilbert space, and let $\mathbb{B}(H)$, $\mathbb{K}(H)$ be the algebras of bounded operators and compact operators on H respectively. Then one has $K_0(\mathbb{B}(H)) = 0$, and $K_0(\mathbb{K}(H)) = \mathbb{Z}$, see [30].

Proposition 2.1.11 (Stability)**.** *For any C^*-algebra A,*

$$K_0(A) \simeq K_0(A \otimes \mathbb{K}(H)).$$

Proof. See [30]. □

The supspension SA of A is defined as the algebra of continuous functions $SA = C_0(\mathbb{R}) \otimes A$. Note that for $A = C_0(X)$, $SA = C_0(SX)$. Then the K-theory group in odd degree is given by $K_1(A) = K_0(SA)$. More generally, we can define $K_n(A) = K_0(C_0(\mathbb{R}^n) \otimes A)$, see [30].

Proposition 2.1.12 (Bott Periodicity)**.**

$$K_0(A) \simeq K_2(A).$$

Proof. See [30]. □

Proposition 2.1.13. *Let $J \subset A$ be a closed ideal of A, then there is an exact sequence*

$$\begin{array}{ccccc} K_0(J) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A/J) \\ \uparrow & & & & \downarrow \\ K_1(A/J) & \longleftarrow & K_1(A) & \longleftarrow & K_1(J) \end{array}$$

Proof. See [30]. □

Proposition 2.1.14. *If I_1, I_2 are two closed ideals of A such that $A = I_1 + I_2$. Then there exists the Mayer-Vietoris sequence*

$$\begin{array}{ccccc} K_0(I_1 \cap I_2) & \longrightarrow & K_0(I_1) \oplus K_0(I_2) & \longrightarrow & K_0(A) \\ \uparrow & & & & \downarrow \\ K_1(A) & \longleftarrow & K_1(I_1) \oplus K_1(I_2) & \longleftarrow & K_1(I_1 \cap I_2) \end{array}$$

Proof. See [31]. □

Recall that if $f : Y \to X$ is a proper continuous map, then there is an induced $*$-homomorphism $f^* : C_0(X) \to C_0(Y)$, which again induces $f^* : K_0(C_0(X)) \to K_0(C_0(Y))$. Hence K^0 is contravariant functor from the category of locally compact Hausdorff spaces with proper maps to the category of abelian groups. For convenience, $K_0(C_0(X))$ is also denoted as $K^0(X)$.

Example 2.1.15. Let $f : Y \subset X$ be the inclusion of a closed subset Y into a locally compact space X. Then f is proper, hence $f^* : K^0(X) \to K^0(Y)$.

Remark 2.1.16. Let $f : U \subset X$ be the inclusion of an open subset U into a locally compact space X. Note f is not proper, but $f^* : C_0(U) \to C_0(X)$ induces $f^* : K^0(U) \to K^0(X)$ (regard f as $f^+ : X^+ \to U^+$, sending the complement of U to the point at infinity).

2.2 Analytic K-homology

2.2.1 K-homology

In studying K-homology, it is more natural and convenient to introduce grading on C* algebras.

Definition 2.2.1. A C*-algebra A is *graded* if there exists an automorphism $\epsilon : A \to A$ such that $\epsilon^2 = 1$.

Let A_0 and A_1 be the eigenspaces of ϵ with corresponding eigenvalues 1 and -1 respectively, i.e.,

$$A_j = \{x \in A : \epsilon(x) = (-1)^j x\}.$$

Then we have the decomposition,

$$A = A_0 \oplus A_1.$$

Note that an ordinary C*-algebra may be viewed as a graded one by setting $A_1 = 0$. If H is a graded Hilbert space, then there is a natural grading on the algebras $\mathbb{B}(H)$ and $\mathbb{K}(H)$. If A and B are two graded C*-algebras, then we can form the graded tensor product $A\hat{\otimes}B$. For more details about graded C*-algebras, see [11].

Definition 2.2.2. Let A, B be two graded C*-algebras. A $*$-homomorphism $\rho : A \to B$ is a *graded $*$-homomorphism* if it preserves the grading, that is,

$$\rho(A_j) \subset B_j, \quad j = 0, 1.$$

A *representation* of a C*-algebra A is a pair (H, ρ) where H is a graded Hilbert space and $\rho : A \to \mathbb{B}(H)$ is a graded $*$-homomorphism.

Definition 2.2.3. Let A be a graded C^*-algebra. A *Fredholm module* is a triple (ρ, H, F) where

(1) $H = H_0 \oplus H_1$ is a graded Hilbert space,

(2) $\rho : A \to \mathbb{B}(H)$ is a graded $*$-homomorphism of A, and

(3) $F \in \mathbb{B}(H)$ is an odd operator,

such that the following operators are compact,

$$(F^2 - \mathrm{Id})\rho(a), \quad (F^* - F)\rho(a), \quad [F, \rho(a)],$$

for any $a \in A$.

Remark 2.2.4. Note that in the definition of Fredholm module, $[F, \rho(a)]$ is taken as graded commutator, i.e.,

$$[F, \rho(a)] = F\rho(a) - (-1)^{\deg(F)\cdot\deg(a)}\rho(a)F.$$

If $U : H' \to H$ is a unitary operator preserving the gradings, then $(U^*\rho U, H', U^*FU)$

is *unitarily equivalent to* (ρ, H, F). If there exists a family of Fredholm modules (ρ, H, F_t) parametrized by $t \in [0,1]$ such that the map $t \mapsto F_t$ is norm continuous, then (ρ, H, F_0) and (ρ, H, F_1) are *operator homotopic*.

Definition 2.2.5. The *Kasparov K-homology group* $K^0(A)$ is the abelian group with one generator for each unitary equivalence class of Fredholm modules over A such that

(1) $[x] = [y]$ if x, y are operator homotopic, and

(2) $[x] + [y] = [x \oplus y]$.

Definition 2.2.6. Let J be an ideal of A. A *relative Fredholm module* for $(A, A/J)$ is given by the (ρ, H, F) such that:

(1) H is a graded Hilbert space,

(2) $\rho : A \to B(H)$ is a graded $*$-homomorphism,

(3) F is a bounded operator of degree one and the following operators are compact,

$$(F^2 - \mathrm{Id})\rho(j), \quad (F^* - F)\rho(j), \quad [F, \rho(a)], \quad j \in J, a \in A.$$

If X is a locally compact topological space, the K-homology group $K^0(C_0(X))$ is denoted by $K_0(X)$. In a similar way, one can define relative Kasparov group $K^0(A, A/J)$. The excision map sends a relative Fredholm module over $(A, A/J)$ to a Fredholm module over J by restricting the represention ρ to J.

Theorem 2.2.7 (Excision)**.** *The excision map* $K(A, A/J) \to K(J)$ *is an isomorphism.*

Proof. See [30]. □

Theorem 2.2.8. *There exists a six-term exact sequence*

$$\begin{array}{ccccc} K^0(A/J) & \longrightarrow & K^0(A) & \longrightarrow & K^0(J) \\ \uparrow & & & & \downarrow \\ K^1(A, A/J) & \longleftarrow & K^1(A) & \longleftarrow & K^1(A/J) \end{array}$$

Proof. See [30]. □

Theorem 2.2.9. *Let I_1, I_2 be two closed ideals of A such that $I_1 + I_2 = A$. Then there exists the Mayer-Vietoris sequence*

$$\begin{array}{ccccc} K^0(A) & \longrightarrow & K^0(I_1) \oplus K^0(I_2) & \longrightarrow & K^0(I_1 \cap I_2) \\ \uparrow & & & & \downarrow \\ K^1(I_1 \cap I_2) & \longleftarrow & K^1(I_1) \oplus K^1(I_2) & \longleftarrow & K^1(A) \end{array}$$

Proof. See [30]. □

Theorem 2.2.10 (Stability)**.**

$$K^0(\mathbb{K}) \simeq K^0(\mathbb{C})$$

and more generally,

$$K^0(A \hat{\otimes} \mathbb{K}) \simeq K^0(A)$$

Proof. See [30]. □

Remark 2.2.11. (1) If $f : Y \to X$ is a proper continuous map, then it induces a $*$-homomorphism $C_0(X) \to C_0(Y)$, which in turn induces

$$f_* : K_0(Y) \to K_0(X).$$

Hence K_0 is a covariant functor under proper maps.

(2) The inclusion $\iota : U \to X$ of a open set into a locally compact space X is not

proper. But we can form a map $\iota^+ : X^+ \to U^+$ so that there is an induced map

$$\iota^* : K_0(X) \to K_0(U).$$

K_0 is a contravariant functor under the inclusion of open sets.

2.2.2 Elliptic differential operators

Let E be a complex vector bundle over a smooth manifold M.

Definition 2.2.12. A *first order linear differential operator* on E is a complex-linear map

$$D : \Gamma(E) \to \Gamma(E)$$

with the following properties:

(1) if s_1, s_2 are smooth sections of E which agree on open set $U \subset M$, then Ds_1 and Ds_2 agree on U, and

(2) in local coordinates, D is represented by a formula

$$D = \sum_j A_j \frac{\partial}{\partial x_j} + B,$$

where A_j, B are smooth matrix-value functions on U.

Definition 2.2.13. The *symbol* of D is the vector bundle morphism

$$\sigma(D) : T^*M \to \mathrm{End}(E),$$

defined by the formula

$$\sigma(D, df) = [D, f].$$

Note that if $df = \sum \xi_j dx_j$, then $\sigma(D, df) = \sum A_j \xi_j$.

Now suppose on E there is a Hermitian metric $\langle , \rangle_E$ and on M there is a smooth measure μ. Then there is an inner product on the space of compactly supported smooth sections $\Gamma_c(E)$ given by the formula,

$$\langle \phi, \psi \rangle = \int_M \langle \phi, \psi \rangle_E d\mu.$$

Let $L^2(E)$ be the Hilbert space formed by taking the completion of $\Gamma_c(E)$ with respect to this inner product.

Lemma 2.2.14. *There exists a unique differential operator, called the formal adjoint $D^\dagger : \Gamma(E) \to \Gamma(E)$ satisfying the formula*

$$\langle D\phi, \psi \rangle = \langle \phi, D^\dagger \psi \rangle, \quad \phi, \psi \in \Gamma_c(E).$$

The symbol of the formal adjoint $D^\dagger$ is given by $\sigma(D^\dagger) = -\sigma(D)^\dagger$.

Proof. See [30]. □

A first order linear differential operator is an unbounded operator. Here is some review of unbounded operators.

Suppose $T : \text{Domain}(T) \to H$ is an unbounded operator with the domain densely defined in H. T is *closed* if the graph of T is closed, and *closable* if it has an extension $\overline{T}$ which is closed. T is closable if and only if $(0, y) \in \overline{\text{Graph}(T)}$ implies $y = 0$ (i.e., $x_n \to 0$ and $Tx_n \to y$ imply $y = 0$).

Lemma 2.2.15. *Every differential operator D is closable.*

Proof. Let $\{\psi_n\}$ be a sequence in $C_c^\infty(M, E)$ with $\psi_n \to 0$ and $D\psi_n \to \phi$, then for any $f \in C_c^\infty(M, E)$, we have

$$\langle \phi, f \rangle = \lim \langle D\psi_n, f \rangle = \lim \langle \psi_n, D^\dagger f \rangle = 0.$$

Thus $\phi = 0$. □

Definition 2.2.16. An unbounded operator T is *symmetric* if

$$\langle Tx, y\rangle = \langle x, Ty\rangle, \quad \text{for all } x, y \in \text{Domain}(T).$$

Let D be a symmetric differential operator, the *minimal domain* of D is the domain of the closure $\overline{D}$.

Definition 2.2.17. Let $T : \text{Domain}(T) \to H$ be a densely defined unbounded operator. The *domain of the adjoint* T^* is the set of all $x \in H$ for which there is a $y \in H$ satisying

$$\langle Tz, x\rangle = \langle z, y\rangle \text{ for all } z \in \text{Domain}(T).$$

Let D be a symmetric differential operator, the *maximal domain* of D is the domain of the adjoint D^*.

For any $x \in \text{Domain}(T^*)$, y exists and is unique. Hence $T^*(x) = y$ is well-defined.

From the proof in Lemma 2.2.15, if T is symmetric, then T is closable, and $\text{Domain}(T) \subset \text{Domain}(\overline{T}) \subset \text{Domain}(T^*)$. If T is symmetric and closed, then $\text{Domain}(T) = \text{Domain}(\overline{T}) \subset \text{Domain}(T^*)$.

Definition 2.2.18. T is *essentially self-adjoint* if

$$\text{Domain}(T) \subset \text{Domain}(\overline{T}) = \text{Domain}(T^*).$$

T is *self-adjoint* if

$$\text{Domain}(T) = \text{Domain}(\overline{T}) = \text{Domain}(T^*).$$

If T is self-adjoint, then the operators $(T \pm i)$ are bounded below, so $(T \pm i)^{-1}$ are bounded linear operators on H. There exists a unique homomorphism from the C^* algebra of continuous bounded functions on $\mathbb{R}$ to the algebra of bounded linear operators on H, which maps $(x \pm i)^{-1}$ to $(T \pm i)^{-1}$.

Theorem 2.2.19. *An unbounded, self-adjoint operator T is unitarily equivalent to a direct sum of multiplication operators, each of which is of the form $M\phi(\lambda) = \lambda\phi(\lambda)$ on $L^2(\mathbb{R}, \mu)$ for some measure on $\mathbb{R}$.*

Proof. See [30]. □

Definition 2.2.20. An essentially self-adjoint operator D has *compact resolvent* if $f(D)$ is a compact operator for every $f \in C_0(\mathbb{R})$.

Let D be a self-adjoint operator with compact resolvent, then D has real eigenvalues λ_j with $\lambda_j \to \infty$ as $j \to \infty$.

Proposition 2.2.21. *Let D be a self-adjoint operator with compact resolvent, then $\lambda - D$ is Fredholm for any $\lambda \in \mathbb{C}$. In particular, D is Fredholm, that is, the kernel of D is finite dimensional and range is closed.*

Lemma 2.2.22. (1) *Every compactly supported, symmetric differential operator on an open manifold is essentially self-adjoint.*
(2) *Let D be a symmetric differential operator. If there is a proper function $f \in C^\infty(M)$ such that $[D, f]$ is a bounded operator, then D is essentially self-adjoint.*

Proof. See [30]. □

Definition 2.2.23. A differential operator D is *elliptic* if the $\sigma_D(x, \xi)$ is invertable for all nonzero $\xi \in T^*M$.

Proof. See [30]. □

Lemma 2.2.24. (1) *If D is a symmetric elliptic differential operator on a compact manifold, then $\phi(D)$ is a compact operator for every $\phi \in C_0(\mathbb{R})$.*
(2) *If D is an essentially selfadjoint differential operator on M and elliptic over an*

open subset $U \subset M$, then $\phi(D)\rho(f)$ is a compact operator for every $\phi \in C_0(\mathbb{R})$ and $f \in C_0(U)$.

Proof. See [30]. □

Definition 2.2.25. A smooth function $\chi : \mathbb{R} \to [-1, 1]$ is a *normalizing function* if it satisfies the conditions: χ is odd, $\chi(x) > 0$ if $x > 0$ and $\chi(x) \to 1$ if $x \to \infty$.

Note that D is of degree one, so is $\chi(D)$. One simple example is the function below,

$$\chi(x) = \frac{x}{\sqrt{1 + x^2}}.$$

Example 2.2.26 (Dolbeault Element)**.** Let N be a compact Riemannian manifold. The cotangent bundle T^*N has a canonical almost complex structure. Let $D = \bar{\partial} + \bar{\partial}^*$ be the Dolbeault operator on smooth forms with compact support $\Omega_c^*(T^*N)$. Let H be the Hilbert space of L^2-forms, graded by even and odd degree. Then D is essentially self-adjoint operator, and in Section 2.4 we shall see that $(H, \rho, D(1 + D^2)^{-1/2})$ is a Kasparov $(C_0(T^*N), \mathbb{C})$-cycle.

Lemma 2.2.27. *If $T_1 : \mathrm{Domain}(T_1) \to H_1$ and $T_2 : \mathrm{Domain}(T_2) \to H_2$ are selfadjoint operators with compact resolvent, where H_1 and H_2 are graded Hilbert spaces. Then the graded tensor product*

$$T_1 \hat{\otimes} \mathrm{Id} + \mathrm{Id} \hat{\otimes} T_2$$

is an essentially selfadjoint with compact resolvent.

Proof. See section 7.5.1 of [55]. □

2.2.3 Equivariant K-homology

Let G be a compact Lie group, acting on a graded C^*-algebra A by automorphisms preserving grading, such that $G \to A, g \mapsto g \cdot a$ is norm continuous for each $a \in A$.

A *G-Fredholm module* is a triple (ρ, H, F) where H is a G-equivariant graded Hilbert space, $\rho : A \to \mathbb{B}(H)$ is a G-equivariant graded $*$-homomorphism, and $F \in \mathbb{B}(H)$ is a G-equivariant operator of degree one, such that

$$(F^2 - 1)\rho(a), \quad (F^* - F)\rho(a), \quad [F, \rho(a)]$$

are compact operators for any $a \in A$.

Similarly, we can define G-equivariant unitary equivalence and G-equivariant operator homotopy. The *G-equivariant K-homology group* $K_G(A)$ is the abelian group generated by the unitary equivalent classes of Fredholm modules, subject to operator homotopy relations: $[x] = [y]$ if x, y are G operator homotopic.

Example 2.2.28. Consider the simplest example. For $M = \mathrm{pt}$, the G-equivariant K-homology is just the representation ring $R(G)$, i.e., $K^G(\mathrm{pt}) \simeq R(G)$. If $V \in R(G)$ is a module, then

$$H = V \oplus 0, \quad \rho : \mathbb{C} \to \mathbb{B}(H), \quad F = 0$$

is the corresponding Fredholm module in $K^G(\mathrm{pt})$. Conversely, if

$$H = H_0 \oplus H_1, \quad \rho : \mathbb{C} \to \mathbb{B}(H), \quad F = \begin{bmatrix} 0 & U^* \\ U & 0 \end{bmatrix}$$

is a Fredholm module, then $\mathrm{Ker}\, F|_{H_0} - \mathrm{Ker}\, F|_{H_1} \in R(G)$.

2.2.4 The Kasparov's Dirac element

Suppose N is a G-Riemannian manifold, ∇^c is a G-Clifford connection on the bundle $\mathrm{Cliff}(TN)$ and $\{e_i\}$ is a local orthonormal frame, see [39]. The Kasparov's Dirac element on a Riemannian manifold N is given by the triple $(\rho_N, \not{D}_N, \mathcal{H}_N)$, where

(1) $\mathcal{H}_N = \overline{C(N, \mathbb{C}\mathrm{liff}(TN))}$ is a $\mathbb{Z}/2$ graded Hilbert space, and

(2) $\rho_N : C(N, \mathbb{C}\mathrm{liff}(TN)) \to \mathbb{B}(\mathcal{H}_N)$ is a graded $*$-homomorphism given by Clifford multiplication from the left, and

(3) $\not{D}_N$ is a Dirac operator, locally given by

$$\not{D}_N(s) = \sum (-1)^{\deg(s)} \nabla^c_{e_i} s \cdot e_i, \quad s \in C(N, \mathbb{C}\mathrm{liff}(TN)).$$

Note that the coefficient $(-1)^{\deg(s)}$ is necessary because in the definition of Fredholm module graded commutator is used. Alternatively, we may take $\mathcal{H}_N = \overline{\Omega^*(N)}$ and $\not{D}_N = d + d^*$ the Hodge de-Rham operator (see Section 4 in [34]).

Recall that N is spin^c if and only if there exists a spinor module $\mathcal{S}$ such that $\mathcal{S} : \mathbb{C}\mathrm{liff}(TN) \to \mathbb{C}$ defines a Morita isomorphism, see [44]. If N is a complex manifold, then there is a canonical spinor module

$$\mathcal{S} = \Lambda^{0,*}(T^*_{\mathbb{C}} N)$$

and the Dolbeault operator $D = \bar{\partial} + \bar{\partial}^*$ defines a K-homology class

$$(L^2(N, \mathcal{S}), \rho, D(1 + D^2)^{-1/2})$$

in the K-homology group $K_0(N)$, see [30].

In addition, the spinor module $\mathcal{S}$ induces a map $K_G(C(N, \mathbb{C}\mathrm{liff}(TN))) \to K_G(N)$. For the construction, see Section 2.4.

Proposition 2.2.29. *Let N be a complex manifold with the canonical spin^c structure. The induced map*

$$K_G(C(N, \mathbb{C}\mathrm{liff}(TN))) \to K_G(N),$$

is an isomorphism, taking the Kasparov's Dirac element to the class defined by the Dolbeault operator.

Proof. See [44]. □

2.3 Twisted K-homology

2.3.1 Dixmier-Douady bundles

Let H be a separable complex Hilbert space, and $P = \mathbb{P}(H)$ the projective space. The structural group is the projective unitary group $PU(H)$ with strong operator topology. The algebras of bounded linear operators on P is canonically defined,

$$\mathbb{B}(P) = \mathbb{B}(H),$$

which is independent of the choice of the Hilbert space H. The dual P^* is the projective space of H^*, which can be identified with the space of closed hyperplanes in P. The tensor product of two projective spaces is defined as

$$P_1 \otimes P_2 = \mathbb{P}(HS(H_1^*, H_2)),$$

where HS is the space of Hilbert-Schmidt operators.

The unitary group $U(H)$ with norm topology is a real Banach Lie group, and is contractible according to Kuiper's theorem [36]. The projective unitary group $PU(H)$ with norm topology is a classifying space for $U(1)$, hence it is the Eilenberg-MacLane space $K(\mathbb{Z}, 2)$,

$$BU(1) = PU(H) = K(\mathbb{Z}, 2), \quad BPU(H) = K(\mathbb{Z}, 3).$$

Lemma 2.3.1. *Let H be a separable complex Hilbert space, and $PU(H)$ be the projective unitary group in strong operator topology. Then the action of $PU(H)$ on $\mathbb{K}(H)$ by conjugation,*

$$PU(H) \times \mathbb{K}(H) \to \mathbb{K}(H),$$

is continuous.

Proof. See [52]. □

There is an exact sequence of Lie groups

$$1 \to U(1) \to U(H) \to \mathrm{Aut}(\mathbb{K}(H)) \to 1.$$

Hence we have the isomorphism,

$$PU(H) \simeq \mathrm{Aut}(\mathbb{K}(H)),$$

where $\mathrm{Aut}(\mathbb{K}(H))$ is the $*$-automorphism group, in point-norm topology.

Definition 2.3.2. A *Dixmier-Douady bundle* $\mathcal{A}$ over N is a locally trivial bundle of C^*-algebras over N, such that each fiber $\mathcal{A}_x \simeq \mathbb{K}(H_x)$ for some Hilbert space H_x.

A Morita trivialization of $\mathcal{A} \to N$ is a bundle $\mathcal{E} \to N$ of Hilbert spaces together with an isomorphism $\mathcal{A} \to \mathbb{K}(\mathcal{E})$. The obstruction to the existence of a Morita trivialization is the *Dixmier-Douady class* $\mathrm{DD}(\mathcal{A}) \in H^3(N, \mathbb{Z})$.

Suppose $\mathcal{A}_i \to N_i, i = 1, 2$ are two Dixmier-Douady bundles modeled on $\mathbb{K}(H_i)$. A *Morita morphism*

$$(\Phi, \mathcal{E}) : (N_1, \mathcal{A}_1) \to (N_2, \mathcal{A}_2)$$

is a proper map $\Phi : N_1 \to N_2$ together with a $(\Phi^*\mathcal{A}_2, \mathcal{A}_1)$-bimodule bundle $\mathcal{E} \to N_1$, locally modeled on $\mathbb{K}(H_1, H_2)$. The existence of Mortia morphism is equivalent to $\mathrm{DD}(\mathcal{A}_1) = \Phi^* \mathrm{DD}(\mathcal{A}_2)$.

Suppose $(\Phi, \mathcal{E}) : (N_1, \mathcal{A}_1) \to (N_2, \mathcal{A}_2)$ and $(\Psi, \mathcal{F}) : (N_2, \mathcal{A}_2) \to (N_3, \mathcal{A}_3)$, then the composition is given by

$$(\Psi \circ \Phi, \Phi^*\mathcal{F} \otimes_{\Phi^*\mathcal{A}_2} \mathcal{E}) : (N_1, \mathcal{A}_1) \to (N_3, \mathcal{A}_3).$$

The opposite algebra bundle $\mathcal{A}^{\mathrm{op}}$ for $\mathcal{A}$ has the same vector bundle structure, but with opposite multiplication. $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$ is Morita trivializable (see [8]).

More generally, a graded Dixmier-Douady bundle is modeled on the graded algebra $\mathbb{K}(H)$ for some graded Hilbert space H. The obstruction to the existence of graded Dixmier-Douady bundle $\mathcal{A} \to N$ is a class $\mathrm{DD}(\mathcal{A}) \in H^1(N, \mathbb{Z}_2) \oplus H^3(N, \mathbb{Z})$. We can generalize the definitions to the G-equivariant case.

Example 2.3.3. If $E \to N$ is a Euclidean vector bundle of even rank, then $\mathcal{A} = \mathbb{C}\mathrm{liff}(E)$ is a graded Dixmier-Douady bundle. The H^3-component of $\mathrm{DD}(\mathcal{A})$ is the third integral Stiefel-Whitney class of E, and H^1-component measures the orientablity of E.

2.3.2 Twisted K-homology

Donovan and Karoubi [21] defined twistings for K-theory by torsion classes in H^3, represented by Azumaya bundles. J. Rosenberg [53] generalized to non-torsion classes, by defining the twisted K-theory for $(N, \mathcal{A})$ to be $K_0(C(N, \mathcal{A}))$, the K-theory of the C*-algebra of sections of $\mathcal{A}$.

Let $\mathcal{A} \to N$ be a G-equivariant graded Dixmier-Douady bundle. The equivariant K-homology group twisted by $\mathcal{A}$ for $(N, \mathcal{A})$ is given by

$$K_0^G(N, \mathcal{A}) = K_G^0(C_0(N, \mathcal{A})).$$

If $(\Phi, \mathcal{E}) : (N_1, \mathcal{A}_1) \to (N_2, \mathcal{A}_2)$ is a Morita morphism, then one can construct a group homomorphism (see Section 2.4)

$$K_0^G(N_1, \mathcal{A}_1) \to K_0^G(N_2, \mathcal{A}_2).$$

Hence K-homology is a covariant functor for Morita morphism.

Example 2.3.4. Suppose N is an even dimensional Riemannian G-manifold. The

Kasparov's Dirac element

$$\not{D}_N \in K_0^G(N, \mathbb{C}\mathrm{liff}(TN))$$

is a distinguished class in the equivariant twisted K-homology group.

2.3.3 Example: SU(2)

The equivariant twisted K-homology group for SU(2) can be calculated using Mayer-Vietories sequence for K-theory or K-homology (see [23] or [20]). It is worth going through this simple example. The equivariant twistings on SU(2) are classified by $H_G^3(G, \mathbb{Z}) \simeq \mathbb{Z}$. Let τ denote a class in $H_G^3(X, \mathbb{Z})$, and $K_0^G(X, \tau)$ denote the twisted K-homology group.

Let $U = \mathrm{SU}(2)_+ = \mathrm{SU}(2) - \{-I\}$ and $V = \mathrm{SU}(2)_- = \mathrm{SU}(2) - \{I\}$ be two covers of SU(2). Let $W = U \cap V \simeq G/T \times \mathbb{R}$. We have exact sequence in twisted K-homology,

$$\begin{array}{ccccc}
K_0^G(\pm I, \tau'') & \longrightarrow & K_0^G(G, \tau) & \longrightarrow & K_0^G(U, \tau') \\
\uparrow & & & & \downarrow \\
K_1^G(U, \tau') & \longleftarrow & K_1^G(G, \tau) & \longleftarrow & K_1^G(\pm I, \tau'')
\end{array}$$

Note that $\tau' \in H_G^3(U, \mathbb{Z}) \simeq H_G^3(G/T \times \mathbb{R}, \mathbb{Z}) \simeq H_T^3(pt, \mathbb{Z}) = 0$. In addition, $K_0^G(\pm I, \tau'') = R(G) \oplus R(G)$, $K_1^G(\pm I, \tau'') = 0$, $K_0^G(U, \tau') = K_0^G(U) = K_1^G(G/T) = 0$, and $K_1^G(U, \tau') = K_1^G(U) = K_0^G(G/T) = R(T)$.

The exact sequence then becomes

$$0 \to K_1^G(G, \tau) \to R(T) \to R(G) \oplus R(G) \to K_0^G(G, \tau) \to 0.$$

Let $\Phi : R(T) \to R(G) \oplus R(G)$, then $K_0^G(G, \tau) = \mathrm{Coker}(\Phi)$. It should be noticed that the sequence is compatible with $R(G)$-module structure.

Let L be the generator of $R(T)$, and ρ_n the irreducible representation of SU(2) with the highest weight $n \cdot \rho$. The the restriction map $R(G) \to R(T)$ takes $\rho_n \mapsto L^n + ... + L^{-n}$ and one has $R(G) \simeq R(T)^W$, where W is the Weyl group. On the other hand, we have holomorphic induction $R(T) \to R(G)$ sending L^n to ρ_n and L^{-n} to ρ_{n+2}. (This can be obtained from Serre duality; note that the canonical bundle $K_{G/T} = G \times_T \bigwedge^{Top}(\sum_{\alpha>0} \mathfrak{g}^*_{-\alpha})$ has the weight 2ρ).

$R(T)$ is a free $R(G)$-module of rank 2. Let $\Psi : R(G) \times R(G) \to R(T)$ be $R(G)$-isomorphism. Since $L^n = L\rho_{n-1} - \rho_{n-2}$,

$$\Psi(\rho_0, 0) = L; \quad \Psi(0, \rho_0) = -L^0$$

Thus for a typical element (ρ_n, ρ_m) we have $\Psi(\rho_n, \rho_m) = L \cdot \rho_n - \rho_m$. Now $\Phi(L) = (\rho_1, \rho_{1+k})$, and $\Phi(L^0) = (\rho_0, \rho_k)$, we have $\Phi \circ \Psi : R(G)^2 \to R(G)^2$ as a matrix

$$\Phi \circ \Psi = \begin{pmatrix} \rho_1 & \rho_{1+k} \\ -\rho_0 & -\rho_k \end{pmatrix}$$

in terms of basis $(\rho_0, 0)$ and $(0, \rho_0)$.

We could change the basis to $(\rho_0, 0)$ and (ρ_0, ρ_1). Then $\Psi(\rho_0, \rho_1) = L - \rho_1 = -L^{-1}$, and $\Psi \circ \Phi(\rho_0, \rho_1) = (\rho_1, \rho_{k+1}) + \rho_1(-\rho_0, -\rho_k) = (0, -\rho_{k-1})$. In terms of this basis, the matrix is

$$\Phi \circ \Psi = \begin{pmatrix} \rho_1 & \rho_{1+k} \\ 0 & -\rho_{k-1} \end{pmatrix}$$

Thus the twisted K-homology of SU(2) at level k is $R(G)/(\rho_{k-1})$, which is isomorphic to the Verlinde ring $R_{k-2}(G)$, see Chapter 5.

One can also work with Mayer-Vietoris sequence in K-homology with compact support.

2.4 KK-theory

2.4.1 Basic definitions

Definition 2.4.1. Let B be a graded C*-algebra. A pre-Hilbert module over B is a graded right B-module $\mathfrak{E}$ equipped with a B-valued function $\langle \cdot, \cdot \rangle : \mathfrak{E} \times \mathfrak{E} \to B$ such that

(1) $\langle \cdot, \cdot \rangle$ is sesquilinear.

(2) $\langle x, yb \rangle = \langle x, y \rangle b$.

(3) $\langle x, y \rangle = \langle y, x \rangle^*$.

(4) $\langle x, x \rangle \geq 0;$ and if $\langle x, x \rangle = 0,$ then $x = 0$.

(5) $\mathfrak{E}^j B^i \subset \mathfrak{E}^{i+j}$ and $\langle \mathfrak{E}^i, \mathfrak{E}^j \rangle \subset B^{i+j}$.

For $x \in \mathfrak{E}$, we define the norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$. If $\mathfrak{E}$ is complete, then $\mathfrak{E}$ is called a *Hilbert module* over B or *Hilbert B-module*.

Example 2.4.2. A Hilbert $\mathbb{C}$-module is a just Hilbert space. B itself is a Hilbert B-module with $\langle b_1, b_2 \rangle = b_1^* b_2$.

Definition 2.4.3. Let B be a graded C*-algebra. $\mathbb{B}(\mathfrak{E})$ is the set of all graded module homomorphisms $T : \mathfrak{E} \to \mathfrak{E}$ for which there is an adjoint module homomorphism $T^* : \mathfrak{E} \to \mathfrak{E}$ with

$$\langle Tx, y \rangle = \langle x, T^* y \rangle, \quad x, y \in \mathfrak{E}.$$

By closed graph theorem, each operator in $\mathbb{B}(\mathfrak{E})$ is a bounded operator, and $\mathbb{B}(\mathfrak{E})$ is C*-algebra. But $\mathbb{B}(\mathfrak{E})$ does not include all the bounded operators in general. Let $\mathbb{K}(\mathfrak{E})$ denote the closure of finite rank operators on $\mathfrak{E}$.

Definition 2.4.4. Let A, B be graded C* algebras. A Kasparov (A, B)-cycle is a triple $(\mathfrak{E}, \phi, T)$ where $\mathfrak{E}$ is a graded Hilbert B-module, $\phi : A \to \mathbb{B}(\mathfrak{E})$ is a graded

$*$-homomorphism and $T \in \mathbb{B}(\mathfrak{E})$ is an operator of degree one such that

$$\phi(a)(T - T^*), \quad \phi(a)(T^2 - \mathrm{Id}), \quad [\phi(a), T],$$

are compact for all $a \in A$. The KK-theory group $KK(A, B)$ is the abelian group of homotopy classes of Kasparov (A, B)-cylces.

Let X be a locally compact space, then

$$K_0(X) = KK(C_0(X), \mathbb{C}), \quad K^0(X) = KK(\mathbb{C}, C_0(X)).$$

If $f : A \to B$ is a graded homomorphism of C*-algebras, then $(B, f, 0)$ defines a class in $KK(A, B)$.

Example 2.4.5. Let $(\Phi, \mathcal{E}) : (N_1, \mathcal{A}_1) \to (N_2, \mathcal{A}_2)$ be a Morita morphism for Dixmier-Douady bundles $\mathcal{A}_1 \to N_1$ and $\mathcal{A}_2 \to N_2$, then the triple

$$(\mathfrak{E} = \overline{C_0(N_1, \mathcal{E})}, \rho, 0)$$

is a Kasparov $(C_0(N_2, \mathcal{A}_2), C(N_1, \mathcal{A}_1))$-cycle, where
(1) $\rho : C_0(N_2, \mathcal{A}_2) \to \mathbb{B}(\mathfrak{E})$ is locally given by the composition operation $\mathbb{K}(H_2) \times \mathbb{K}(H_1, H_2) \to \mathbb{K}(H_1, H_2)$, and
(2) The $C_0(N_1, \mathcal{A}_1)$-Hilbert module $\mathfrak{E}$ is locally given by

$$\langle , \rangle : \mathbb{K}(H_1, H_2) \times \mathbb{K}(H_1, H_2) \to \mathbb{K}(H_1), \quad (S, T) \mapsto S^*T.$$

Hence the Morita morphism induces an element

$$[(\mathfrak{E}, \rho, 0)] \in KK(C_0(N_2, \mathcal{A}_2), C_0(N_1, \mathcal{A}_1)).$$

2.4.2 Kasparov product

Theorem 2.4.6.

$$KK(A_1, B_1 \hat{\otimes} D) \times KK(D \hat{\otimes} A_2, B_2) \to KK(A_1 \hat{\otimes} A_2, B_1 \hat{\otimes} B_2).$$

Proof. See [11]. □

Example 2.4.7. Let M be a Spin^c manifold, and $\not{D}$ be the associated Spin^c Dirac operator. The index map is the Kasparov product

$$KK(\mathbb{C}, C(M)) \otimes KK(C(M), \mathbb{C}) \to KK(\mathbb{C}, \mathbb{C}), \quad ([E], [\not{D}]) \mapsto \mathrm{Index}(\not{D}_E).$$

where a vector bundle E is regarded as an element in $KK(\mathbb{C}, C(M))$, and $\not{D}_E$ is the Spin^c Dirac operator twisted by E.

Definition 2.4.8. $x \in KK(A, B)$ is a *KK-equivalence* if there is a $y \in KK(B, A)$ such that

$$xy = 1_A, \quad yx = 1_B.$$

Two C*-algebra A, B are called KK-equivalent if there exists a KK-equivalence in $KK(A, B)$.

If $x \in KK(A, B)$ is a KK-equivalence, then for any D, $KK(B, D) \to KK(A, D)$ and $KK(D, A) \to KK(D, B)$ given by Kasprarov product with x are isomorphisms. In particular, if A, B are trivally graded, then right multiplication by x gives

$$KK_*(\mathbb{C}, A) \simeq KK_*(\mathbb{C}, B),$$

and left multiplication by y gives

$$KK_*(A, \mathbb{C}) \simeq KK_*(B, \mathbb{C}).$$

Example 2.4.9. Let N be a Riemannian manifold, then $C_0(TN)$ is KK-equivalent to $C(N, \mathbb{C}\mathrm{liff}(TN))$.

Proposition 2.4.10. *Let A be a graded C^* algebra and $\mathbb{K}$ be the algebra of compact operators on a graded Hilbert space. Then A is KK-equivalent to $A\hat{\otimes}\mathbb{K}$,*

$$KK(A, \mathbb{C}) \simeq KK(A\hat{\otimes}\mathbb{K}, \mathbb{C}).$$

Proof. See page 158 in [11]. □

2.4.3 Poincaré duality

Kasparov proved the Poincaré duality using KK-theory in [34]. J. Tu proved the following general form of the Poincaré duality in [57].

Theorem 2.4.11. *Let G be a compact Lie group and M a G-manifold. Let A, B be graded G-C^*-algebras, and let $\mathcal{A}$ be a G-graded Dixmier-Douady bundle over M, then there exists an isomorphism*

$$KK_G(B, C_0(M, \mathcal{A})\hat{\otimes}D) \to KK_G(C_0(M, \mathbb{C}\mathrm{liff}(TM)\hat{\otimes}\mathcal{A}^{op})\hat{\otimes}B, D).$$

Proof. See [57]. □

In particular when $B = D = \mathbb{C}$, one has

$$KK(\mathbb{C}, C_0(M, \mathcal{A})) \simeq KK(C_0(M, \mathbb{C}\mathrm{liff}(TM)\hat{\otimes}\mathcal{A}^{op}), \mathbb{C}).$$

Example 2.4.12. Let M be an even dimensional Spin^c manifold, then the Spin^c Dirac operator defines a fundamental class $[M] \in KK(C(M), \mathbb{C})$, which depends on the Spin^c structure. The cap product with $[M]$ induces the Poincaré duality

$$KK(\mathbb{C}, C(M)) \to KK(C(M), \mathbb{C}).$$

The trivial line bundle $\mathbb{C}$ corresponds to the fundamental class $[M]$.

Example 2.4.13. If M is only a closed Riemannian manifold, then one can use the Kasparov's Dirac element $[M] \in KK(C(M, \mathbb{C}\text{liff}(TM), \mathbb{C})$ to form the Poincaré duality

$$KK(C, C(M)) \to KK(C(M, \mathbb{C}\text{liff}(TM), \mathbb{C}).$$

2.4.4 Unbounded KK-cycles

Baaj and Julg [7] showed how to define $KK(A, B)$ using unbounded KK-cycles. See also [11].

Definition 2.4.14. Let A and B be graded C* algebras. An unbounded Kasparov (A, B)-cycle is a triple $(\mathfrak{E}, \phi, D)$ where $\mathfrak{E}$ is a Hilbert B-module, $\phi : A \to \mathbb{B}(\mathfrak{E})$ is a graded $*$-homomorphism, and D is an odd self-adjoint operator on $\mathfrak{E}$, such that
(1) $\phi(a)(1 + D^2)^{-1} \in \mathbb{K}(\mathfrak{E})$ for any $a \in A$, and
(2) $[D, \phi(a)] \in \mathbb{B}(\mathfrak{E})$.

Example 2.4.15. The Dolbeault operator and the Kasparov's Dirac operator form unbounded KK-cycles.

Proposition 2.4.16. *Let A and B be graded C^*-algebras. If $(\mathfrak{E}, \phi, D)$ is an unbounded Kasparov (A, B)-cycle, then $(\mathfrak{E}, \phi, D(1 + D^2)^{-1/2})$ defines a bounded Kasparov (A, B)-cycle.*

Proof. We will sketch a proof for $B = \mathbb{C}$. The general situation is similar, see [11]. Let $F = D(1 + D^2)^{-1/2}$, then F is self-adjoint and odd. Since $F^2 - \mathrm{Id} = -(1 + D^2)^{-1} \in \mathbb{K}(\mathfrak{E})$, one has $\phi(a)(F^2 - \mathrm{Id}) \in \mathbb{K}(\mathfrak{E})$. Next, F can be written as

$$F = \frac{2}{\pi} \int_0^\infty D(1 + \lambda^2 + D^2)^{-1} d\lambda.$$

Then one has

$$[\phi(a), F] = \frac{2}{\pi} \int_0^\infty (1+\lambda^2+D^2)^{-1}((1+\lambda^2)[\phi(a), D]+D[\phi(a), D]D)(1+\lambda^2+D^2)^{-1} d\lambda.$$

Since $[\phi(a), D](1 + \lambda^2 + D^2)$ and $[\phi(a), D]D(1 + \lambda^2 + D^2)$ are compact by Rellich lemma, $[\phi(a), F]$ is compact. □

Chapter 3

Loop groups and representations

3.1 Loop groups

From now on, we assume that G is a compact, connected, simply connected simple Lie group.

Definition 3.1.1. The *loop group* LG of G is the set of smooth maps from the circle $S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$ to G with products taken pointwise. The Lie algebra of LG is the vector space $L\mathfrak{g}$ of loops on $\mathfrak{g}$, with brackets taken pointwise.

LG is a Fréchet Lie group modeled on the Fréchet space $L\mathfrak{g}$. Note that the diffeomorphism group $\mathrm{Diff}(S^1)$ of S^1 acts on LG by reparameterizing the loops. In particular, the rotation group $\mathrm{Rot}(S^1)$ acts on LG and $L\mathfrak{g}$.

Let ΩG be the *based loop group*, given by the kernel of the evaluation map $LG \to G, \gamma \mapsto \gamma(0)$.

Lemma 3.1.2. *LG is a semi-direct product*

$$LG \simeq \Omega G \rtimes G, \quad \gamma \mapsto (\gamma\gamma(0)^{-1}, \gamma(0))$$

where G is viewed as the constant loops in LG and the action of G on ΩG is given by conjugation.

Remark 3.1.3. We may define LG to be the group of maps $S^1 \to G$ of some fixed Sobolev class $s > 1$, $L\mathfrak{g} = \Omega^0(S^1, \mathfrak{g})$ its Lie algebra, and $L\mathfrak{g}^* = \Omega^1(S^1, \mathfrak{g})$ the space of Lie algebra-valued 1-forms of Sobolev class $s - 1$, viewed as the space of connections on the trivial principal bundle $S^1 \times G \to S^1$. Integration over S^1 gives a non-degenerate pairing between $L\mathfrak{g}^*$ and $L\mathfrak{g}$.

The action of $\mathrm{Rot}(S^1)$ induces a $\mathbb{Z}$-grading on the complexified Lie algebra $L\mathfrak{g}_{\mathbb{C}}$,

$$L\mathfrak{g}_{\mathbb{C}} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_{\mathbb{C}} z^n,$$

where $\mathfrak{g}_{\mathbb{C}} z^n$ consists of loops of the form $z \mapsto X z^n$ for $X \in \mathfrak{g}_{\mathbb{C}}$.

3.1.1 Loop groups and central extensions

Let $\mathbb{T} = \mathrm{U}(1) = \{z \in \mathbb{C} : |z| = 1\}$.

Definition 3.1.4. A *central extension* of LG by $\mathbb{T}$ is an exact sequence

$$1 \to \mathbb{T} \to \widetilde{L}G \to LG \to 1.$$

As a topological space, $\widetilde{L}G$ is a circle bundle over LG. The corresponding Lie algebra of a central extension $\widetilde{L}G$ is $L\mathfrak{g} \oplus \mathbb{R}$, which is an affine Kac-Moody algebra, see [33].

Isomorphism classes of extensions $\widetilde{L}\mathfrak{g}$ of $L\mathfrak{g}$ by $\mathbb{R}$ correspond exactly to the invariant symmetric bilinear forms on the Lie algebra $\mathfrak{g}$. Indeed, as a vector space, $\widetilde{L}\mathfrak{g} = L\mathfrak{g} \oplus \mathbb{R}$. If $\langle \cdot, \cdot \rangle$ is an invariant symmetric bilinear form on $\mathfrak{g}$, then the Lie bracket is given by

$$[(\xi, a), (\eta, b)] = ([\xi, \eta], \omega(\xi, \eta)), \quad \text{for } \xi, \eta \in L\mathfrak{g}, \text{ and } a, b \in \mathbb{R}, \tag{3.1}$$

where ω is given by

$$\omega(\xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} \langle \xi(\theta), \eta'(\theta) \rangle \, \mathrm{d}\, \theta. \tag{3.2}$$

Lemma 3.1.5. *The bilinear map $\omega : L\mathfrak{g} \times L\mathfrak{g} \to \mathbb{R}$ possesses the following properties:*

(1) $\omega(\xi, \eta) = -\omega(\eta, \xi)$ *for any* $\xi, \eta \in L\mathfrak{g}$.

(2) *It satisfies the cocycle condition:*

$$\omega([\xi, \eta], \zeta) + \omega([\eta, \zeta], \xi) + \omega([\zeta, \xi], \eta) = 0,$$

for any ξ, η *and* $\zeta \in L\mathfrak{g}$.

Proof. The first formula follows from integration by parts. The second formula follows from the Jacobi identity and the invariance of the symmetric bilinear form:

$$\langle [\xi, \eta], \zeta \rangle = \langle \xi, [\eta, \zeta] \rangle,$$

for any ξ, η and $\zeta \in L\mathfrak{g}$. □

Proposition 3.1.6. *Let Γ be a Lie group acting smoothly on a connected, simply connected manifold X. Let ω be an invariant integral closed 2-form on X. Then there is an central extension $\tilde{\Gamma}$ of Γ canonically associated to (X, ω). For every point $x \in X$, the Lie algebra of $\tilde{\Gamma}$ can be represented by the cocycle*

$$(\xi, \eta) \to \omega(\xi_x, \eta_x), \tag{3.3}$$

where ξ_x, η_x are the tangent vectors at $x \in X$ corresponding to the actions of the infinitesimal elements ξ, η of Γ.

Proof. See [50]. □

Since G is assumed to be simply connected, and from Lemma 3.1.2, $LG =$

$G \ltimes \Omega G$, we have

$$\pi_0(LG) = \pi_0(G) \times \pi_0(\Omega G) = \pi_0(G) \times \pi_1(G) = 0,$$

and

$$\pi_1(LG) = \pi_1(G) \times \pi_1(\Omega G) = \pi_1(G) \times \pi_2(G) = 0,$$

by the fact that $\pi_2(G)$ is trivial for any compact Lie group, see [15]. Therefore LG is connected and simply connected.

Let LG act on itself by left translation, then Proposition 3.1.6 implies that there is a one-to-one correspondence between the isomorphism classes of central extensions of LG and the equivalence classes in the third cohomology group,

$$H^2(LG, \mathbb{Z}) \cong H^3(G, \mathbb{Z}).$$

The Lie algebra 2-cocycle ω is a 2-form on $L\mathfrak{g}$, which is the tangent space of LG at the identity. Therefore by left translation it defines a left-invariant 2-form ω on LG, and the cocycle condition means that this differential form is closed.

Theorem 3.1.7. *Let G be a connected, simply connected compact Lie group. Then*
(1) *A 2-cocycle ω on $L\mathfrak{g}$ defines a central extension of LG if and only if $[\omega/2\pi] \in H^3(G, \mathbb{Z})$.*
(2) *If ω defines a central extension $\widetilde{L}G$, then the central extension is unique.*
(3) *Let $\widetilde{L}G$ be a central extension, then there is a unique action of $\mathrm{Diff}^+(S^1)$ on $\widetilde{L}G$ that covers its action on LG.*
(4) *A 2-cocycle satisfies the integrality condition if and only if the inner product $\langle H_\alpha, H_\alpha \rangle$ is an even integer for each coroot H_α of G.*

Proof. See [50]. □

Definition 3.1.8. Since G is simple, all invariant inner products on $\mathfrak{g}$ are proportional, and there is a smallest one that satisfies the integrality condition, which is called the *basic inner product*, and denoted by

$$B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}.$$

The corresponding extension is the *basic central extension.* All other invariant inner products that satisfy the integrality condition is an integer multiple of the basic one.

Remark 3.1.9. If G is not simple but just semisimple, then the universal central extension is no longer a circle bundle, but rather an extension by a torus $\mathbb{T}^d$, where d is the number of simple components of G.

3.1.2 Affine roots and affine Weyl group

We have seen that $\tilde{L}\mathfrak{g}$ forms a Lie algebra given a symmetric bilinear form $\langle \cdot, \cdot \rangle$, which induces an inner product on $L\mathfrak{g}$,

$$\langle \xi, \eta \rangle = \frac{1}{2\pi} \int_0^{2\pi} \langle \xi(\theta), \eta(\theta) \rangle \, \mathrm{d}\, \theta.$$

Lemma 3.1.10. *The adjoint action of $L\mathfrak{g}$ on $\tilde{L}\mathfrak{g}$ is given by*

$$\tilde{\mathrm{ad}}_\xi(\eta, b) = (\mathrm{ad}_\xi(\eta), \omega(\xi, \eta)),$$

and the corresponding adjoint action of LG on $\tilde{L}\mathfrak{g}$ is given by

$$\tilde{\mathrm{Ad}}_\gamma(\eta, b) = (\mathrm{Ad}_\gamma(\eta), b - \langle \gamma^{-1}\gamma', \xi \rangle).$$

Proof. The proof is straightforward. □

Lemma 3.1.11. *The coadjoint action of $\gamma \in LG$ on the dual space $\tilde{L}\mathfrak{g}^* = L\mathfrak{g}^* \oplus \mathbb{R}$ is given by*

$$\tilde{\mathrm{Ad}}_\gamma(\mu, k) = (\mathrm{Ad}_\gamma(\mu) - k\gamma'\gamma^{-1}, k), \tag{3.4}$$

for $\gamma \in LG$ and $\mu \in L\mathfrak{g}^*$.

Proof. We need to check that $\tilde{\mathrm{Ad}}_\gamma(\mu, k)(\eta, b) = (\mu, k)(\tilde{\mathrm{Ad}}_{\gamma^{-1}}(\eta, b))$, which is clear using the pairing $(\mu, k)(\eta, b) = \int_{S^1} \mu(\eta) - kb$. $\square$

Note that unlike the usual coadjoint action, the action in the formula above is shifted by a term $k\gamma'\gamma^{-1}$. Hence we call it the *affine coadjoint action.* We shall fix a level $k \neq 0$ and identify $L\mathfrak{g}^*$ with the affine hyperplane $L\mathfrak{g}^* \times \{k\}$.

Definition 3.1.12. The *affine coadjoint orbit* $\mathcal{O}_\mu$ through μ is the set

$$LG \cdot \mu = \{\tilde{\mathrm{Ad}}_\gamma \mu : \gamma \in LG\},$$

where the action $\tilde{\mathrm{Ad}}$ is the affine coadjoint action given in Equation 3.4.

We shall see that the affine coadjoint orbits $\mathcal{O}_\mu$ are canonically Kähler manifolds (see Section 3.3.2). They are infinite dimensional, but the ΩG action is free and the quotients are conjugacy classes of G.

Now consider the semidirect product $\mathrm{Rot}(S^1) \ltimes \widetilde{L}G$. The rotation group $\mathrm{Rot}(S^1)$ acts on $\widetilde{L}G$ naturally and covers the rotation action on LG (see Theorem 3.1.7). Let R_τ be the rotation of an angle τ, i.e.,

$$(R_\tau \cdot \gamma)(\theta) = \gamma(\theta - \tau), \quad \gamma \in LG,$$

then the product on $\mathrm{Rot}(S^1) \ltimes LG$ is given by

$$(R_\theta, \gamma) \cdot (R_\tau, \sigma) = (R_{\theta+\tau}, \gamma \cdot (R_\theta \cdot \sigma)), \quad \gamma, \sigma \in LG.$$

Lemma 3.1.13. *The Lie algebra of* $\mathrm{Rot}(S^1) \ltimes \widetilde{L}G$ *is* $\mathbb{R} \oplus L\mathfrak{g} \oplus \mathbb{R}$ *with the brackets given by*

$$[(x, \xi, a), (y, \eta, b)] = (0, [\xi, \eta] + x\eta' - y\xi', \omega(\xi, \eta)),$$

where ξ' and η' are the derivatives of ξ and η respectively.

Proof. See [50]. □

Define a bilinear form on $\mathbb{R} \oplus L\mathfrak{g} \oplus \mathbb{R}$ by

$$\langle (x, \xi, a), (y, \eta, b) \rangle = \langle \xi, \eta \rangle - xb - ya, \tag{3.5}$$

which is invariant under the adjoint action of the Lie algebra on itself.

The adjoint action of $\gamma \in LG$ on the Lie algebra $\mathbb{R} \oplus L\mathfrak{g} \oplus \mathbb{R}$ is given by the formula,

$$\gamma \cdot (x, \xi, a) = (x, \gamma \cdot \xi - x\gamma'\gamma^{-1}, a - \langle \gamma^{-1}\gamma', \xi \rangle + \frac{1}{2}x\langle \gamma^{-1}\gamma', \gamma^{-1}\gamma' \rangle). \tag{3.6}$$

Definition 3.1.14. The *affine Weyl group* is defined as

$$\tilde{W} = N(\mathrm{Rot}(S^1) \times T)/(\mathrm{Rot}(S^1) \times T),$$

where $N(\mathrm{Rot}(S^1) \times T)$ is the normalizer of $\mathrm{Rot}(S^1) \times T$ in $\mathrm{Rot}(S^1) \ltimes LG$.

Proposition 3.1.15. *Let W be the Weyl group of G, and $\check{T} = \mathrm{Hom}(\mathbb{T}, T)$ be the group of homomorphisms from $\mathbb{T}$ to T. Then the affine Weyl group $\tilde{W}$ of LG is the semidirect product*

$$W \ltimes \check{T} \subseteq (G/T) \ltimes \Omega G.$$

Proof. See [50]. □

The affine Weyl group acts on the Lie algebra $\mathbb{R} \oplus \mathfrak{t} \oplus \mathbb{R}$ of $\mathrm{Rot}(S^1) \oplus (T \oplus \mathbb{T})$ by adjoint action and on the dual $\mathbb{R} \oplus \mathfrak{t}^* \oplus \mathbb{R}$ by affine coadjoint action. That is, if $\gamma \in \check{T}$, then $\gamma(\theta) = \exp(\theta H)$ for some $H \in \mathfrak{t}$, denoted γ_H. Using the inner product we can identify $\mathfrak{t}$ and $\mathfrak{t}^*$, say λ_H is the corresponding element in $\mathfrak{t}^*$, then from equation (3.6) and (3.4), the adjoint action on $\mathbb{R} \oplus \mathfrak{t} \oplus \mathbb{R}$ is given by

$$\gamma_H \cdot (x, X, a) = (x, X - xH, a - \langle H, X \rangle + \frac{1}{2}x\|\lambda_H\|^2), \tag{3.7}$$

and the coadjoint action on $\mathbb{R} \oplus \mathfrak{t}^* \oplus \mathbb{R}$ is given by

$$\gamma_H \cdot (n, \lambda, h) = (n + \langle \lambda, \lambda_H \rangle + \frac{1}{2} h \|\lambda_H\|^2, \lambda + h\lambda_H, h). \tag{3.8}$$

The group $\mathrm{Rot}(S^1) \ltimes LG$ has a maximal abelian subgroup $\mathrm{Rot}(S^1) \times T$, where T is a maximal torus of G and is regarded as the constant loops in LG. The abelian group $\mathrm{Rot}(S^1) \times T$ acts on the complexified Lie algebra $\mathbb{C} \oplus L\mathfrak{g}_{\mathbb{C}}$ of $\mathrm{Rot}(S^1) \ltimes LG$ by adjoint action. Hence the complexified Lie algebra decomposes as

$$\mathbb{C} \oplus L\mathfrak{g}_{\mathbb{C}} = (\mathbb{C} \oplus \mathfrak{t}_{\mathbb{C}}) \oplus \bigoplus_{n \neq 0} \mathfrak{t}_{\mathbb{C}} \cdot z^n \oplus \bigoplus_{(n,\alpha)} \mathfrak{g}_\alpha z^n,$$

according to the characters of $\mathrm{Rot}(S^1) \times T$.

The *affine roots* of $\mathrm{Rot}(S^1) \ltimes LG$ consist of the following weights:

$$\{(n, \alpha) : n \in \mathbb{Z}, \alpha \text{ is a root of } G\} \text{ and } \{(n, 0) : n \neq 0\}.$$

The *positive affine roots* of $\mathrm{Rot}(S^1) \ltimes LG$ consist of the following roots:

$$\{(n, \alpha) : n > 0, \alpha \text{ is a root of } G\} \text{ and } \{(0, \alpha) : \alpha \text{ is a positive root of } G\}.$$

If $\{\alpha_i\}$ is a set of simple roots of G, then the *simple affine roots* of $\mathrm{Rot}(S^1) \ltimes LG$ consist of $\boldsymbol{\alpha_i} = (0, \alpha_i)$ as well as $\boldsymbol{\alpha_0} = (1, -\alpha_{max})$, where α_{max} is the maximal root of G.

Suppose $\{X_\alpha, X_{-\alpha}, H_\alpha\}$ span a Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Given a root (n, α), there is a corresponding $\mathfrak{su}(2)$ subalgebra of $\widetilde{L}\mathfrak{g}_{\mathbb{C}}$ generated by the loops $X_\alpha z^k$, $X_{-\alpha} z^{-k}$ and the coroot

$$H_{k,\alpha} = [X_\alpha z^k, X_{-\alpha} z^{-k}] = H_\alpha - \frac{1}{2} k \|H_\alpha\|^2 \in \mathfrak{t}_{\mathbb{C}} \oplus \mathbb{R}.$$

The reflection of a weight $\boldsymbol{\lambda} = (n, \lambda, h)$ through the hyperplane orthogonal to the root $\boldsymbol{\alpha} = (k, \alpha, 0)$ is given by

$$\begin{aligned} s_{k,\alpha}(n, \lambda, h) &= \boldsymbol{\lambda} - \boldsymbol{\lambda}(H_{k,\alpha})\boldsymbol{\alpha} \\ &= (n - \lambda(H_\alpha)k + \frac{1}{2}h\|H_\alpha\|^2 k^2, \lambda - \lambda(H_\alpha)\alpha + \frac{1}{2}h\|H_\alpha\|^2 k\alpha, h). \end{aligned}$$

These $s_{k,\alpha}$ are generated by the reflections $s_{0,\alpha}$, which act solely on the t^* componets and generate the Weyl group W of G, as well as the transformations,

$$t_\alpha(n, \lambda, h) = s_{1,\alpha}s_{0,\alpha}(n, \lambda, h) = (n + \lambda(H_\alpha) + \frac{1}{2}h\|H_\alpha\|^2 k^2, \lambda + hH_\alpha, h),$$

which when restricting to t^*, generates the $\check{T}$.

One may think of affine roots of G as linear functions on the Lie algebra $\mathbb{R} \oplus \mathfrak{t}$, or as *affine-linear* functions on $\mathfrak{t}$. Then W preserves the hyperplance $1 \times \mathfrak{t} \times 1$, and $\gamma_H \in \check{T}$ acts on it by translation by the vector $H \in \mathfrak{t}$.

The element $s_{k,\alpha}$ corresponds to the reflection through the hyperplane determined by the equation $\langle \lambda, \alpha \rangle = hk$. These hyperplanes divide $\mathfrak{t}^*$ into connected components called *alcoves*, and the affine Weyl group acts simply transitively on these alcoves.

3.2 Positive energy representations

The representations of LG we will consider are projective unitary representations on a Hilbert space $\mathcal{H}$. If

$$U : LG \to PU(\mathcal{H}) = U(\mathcal{H})/\mathbb{T},$$

is a projective representation, then

$$U_{\gamma_1} \cdot U_{\gamma_2} = c(\gamma_1, \gamma_2) U_{\gamma_1 \cdot \gamma_2},$$

where $c(\gamma_1, \gamma_2) \in \mathbb{T}$ is the projective multiplier. Any projective unitary representation of LG comes from a genuine representation of some central extension $\widetilde{L}G$.

3.2.1 Classification

The theory of positive energy representations of LG is similar to the representation theory of compact Lie groups.

Let $\mathcal{H}$ be a topological vector space. Recall that if $\mathbb{T}$ acts on $\mathcal{H}$ continuously, then a dense subspace of $\mathcal{H}$ decomposes into an algebraic direct sum

$$\mathcal{H} = \bigoplus_{n \in \mathbb{Z}} \mathcal{H}(n),$$

where each $\mathcal{H}(n)$ is a closed subspace of $\mathcal{H}$.

Definition 3.2.1. *A positive energy representation* of LG is a representation $(\rho, \mathcal{H})$ of $\mathrm{Rot}(S^1) \ltimes \tilde{L}G$

$$\rho : \mathrm{Rot}(S^1) \ltimes \tilde{L}G \to U(\mathcal{H}),$$

that satisfies the positive energy condition: $\mathcal{H}(n) = 0$ for $n < 0$.

In other words, $(\rho, \mathcal{H})$ is a projective representation of LG, with an intertwining action of $\mathrm{Rot}(S^1)$,

$$R_\theta U_\gamma R_\theta^{-1} = U_{R_\theta \gamma}, \tag{3.9}$$

and $\mathcal{H}$ satisfies the positive energy condition.

Theorem 3.2.2. *Up to essential equivalence, every representation of LG*

(1) *is projective,*

(2) *is completely reducible,*

(3) *is unitary,*

(4) *extends to a holomorphic projective repesentation of $LG_{\mathbb{C}}$,*

(5) *admits a (projective) intertwining action of* $\mathrm{Diff}^+(S^1)$.

Proof. See [50]. □

Remark 3.2.3. Since every representation of LG is completely reducible, it is sufficient to study the irreducible ones.

Let $\mathrm{Rot}(S^1) \times T \times \mathbb{T}$ be a maximal torus of $\mathrm{Rot}(S^1) \ltimes \widetilde{L}G$, where $\mathbb{T}$ is the circle in the central extension. The representation space $\mathcal{H}$ then decomposes into

$$\mathcal{H} = \bigoplus \mathcal{H}(n, \lambda, h), \tag{3.10}$$

where $\mathcal{H}(n, \lambda, h)$ is the weight space on which $\mathrm{Rot}(S^1) \times T \times \mathbb{T}$ acts by the character $(n, \lambda, h) \in \mathbb{Z} \times T \times \mathbb{Z}$. The characters that occur in (3.10) are called *weights* of the representation $(\rho, \mathcal{H})$.

If $(\rho, \mathcal{H})$ is an irreducible representation, then the weight space for the action of extension circle $\mathbb{T}$ has multiplicity one, so only one value h occurs in (3.10). The number h is called *the level* of the representation.

Proposition 3.2.4. *Let $\mathcal{H} = \bigoplus_{k \in \mathbb{Z}} \mathcal{H}(k)$ be the energy decomposition, then each $\mathcal{H}(k)$ is of finite dimension. In particular, each weight space $\mathcal{H}(n, \lambda, h)$ is of finite dimension.*

Proof. See [50]. □

The affine Weyl group

$$\begin{aligned}
\tilde{W} &= N(\mathrm{Rot}(S^1) \times T)/(\mathrm{Rot}(S^1) \times T) \\
&\cong N(\mathrm{Rot}(S^1) \times T \times \mathbb{T})/(\mathrm{Rot}(S^1) \times T \times \mathbb{T})
\end{aligned}$$

permutes the weights (n, λ, h) of the representation, with the formula given in Equation (3.8).

Theorem 3.2.5. *Let $(\rho, \mathcal{H})$ be an irreducible representation of LG, then*

(1) *There exists a unique lowest weight (n, λ, h) of $\mathcal{H}$, such that $\boldsymbol{\lambda} - \boldsymbol{\alpha}$ is not a weight of $(\rho, \mathcal{H})$ for any positive root $\boldsymbol{\alpha} = (k, \alpha) > 0$,*

(2) *The lowest weight is antidominant, i.e.,* $\boldsymbol{\lambda}(H_{k,\alpha}) \leq 0$ *for any positive coroot* $H_{k,\alpha}$.

Proof. See [50]. □

The lowest weight space is one-dimensional and generates $\mathcal{H}$. An antidominant weight (n, λ, h) lies in the fundamental Weyl alcove, and can be described by the formula,

$$-\frac{1}{2}h\|H_\alpha\|^2 \leqslant \lambda(H_\alpha) \leqslant 0, \tag{3.11}$$

for any positive root α.

The condition (3.11) of being antidominant is equivalent to the condition, λ is antidominant and

$$\langle \lambda, -\alpha_{max} \rangle \leq h,$$

where α_{max} is the highest root of the adjoint representation $\mathfrak{g}$. It follows that for a fixed minimum energy, there are only finitely many irreducible representations at each level. If the level $h = 0$, the representation is trivial.

Theorem 3.2.6. *The isomorphism classes of irreducible representations are parametrized by the set of antidominant weights.*

Proof. See [50]. □

Remark 3.2.7. The antidominant condition (3.11) does not involve the energy n. Since the irreducible representations with antidominant weights (n, λ, h) and (m, λ, h) are equivalent as representations of $\widetilde{L}G$ (they differ only by the multiplication by the character $R_\theta \mapsto e^{i(m-n)\theta}$), it is enough to investigate the antidominant weights of the form $(0, \lambda, h)$.

Definition 3.2.8. Suppose $\{\boldsymbol{\alpha}\}$ is a system of positive roots of LG, the *fundamental alcove* is the unique alcove such that $\langle \boldsymbol{\lambda}, \boldsymbol{\alpha} \rangle \leq 0$ for any $\boldsymbol{\alpha} > 0$.

If $\omega_1, \cdots, \omega_l$ are the fundamental weights of G, *i.e.*, $\omega_i(H_j) = \delta_{ij}$, then one can define the *fundamental weights* of LG as follows,

$$\begin{aligned}\boldsymbol{\omega}_0 &= (0, 0, 1),\\ \boldsymbol{\omega}_i &= (0, -\omega_i, \langle \omega_i, \alpha_{max} \rangle), \quad i = 1, \cdots, l.\end{aligned}$$

The representation corresponding to $\boldsymbol{\omega_0}$ is called *the basic representation*. It turns out that a weight is antidominant if and only if it is a positive integer linear combination of fundamental weights.

Example 3.2.9. Let $G = \mathrm{SU}(2)$, and consider the universal central extension of $L\mathrm{SU}(2)$ with the basic inner product B on $\mathfrak{su}(2)$. Then $\hat{T} \cong \mathbb{Z}$ with integer λ corresponding to $\mathrm{diag}(u, u^{-1}) \mapsto u^{\lambda}$. The fundamental antidominant weights are

$$\boldsymbol{\omega}_0 = (0, 0, 1),$$

$$\boldsymbol{\omega}_1 = (0, -1, 1).$$

The weights of the basic representation consist of all $(m, \lambda, 1)$ such that λ is even and $\lambda^2 \leq 2m$; the multiplicity of $(m, \lambda, 1)$ is the number of paritITions of $m - \frac{1}{2}\lambda^2$.

In general, since the inner product (3.5) is preserved under affine Weyl group action, the orbit of the lowest weight $\boldsymbol{\lambda}$ under the affine Weyl group consists of lattice points $\boldsymbol{\mu} = (m, \mu, h)$ such that

$$\|\boldsymbol{\mu}\|^2 = \|\mu\|^2 - 2mh = \|\boldsymbol{\lambda}\|^2.$$

The remaining weights satisfy the condition

$$\|\boldsymbol{\mu}\|^2 = \|\mu\|^2 - 2mh \leq \|\boldsymbol{\lambda}\|^2.$$

3.2.2 Borel-Weil theory

According to Borel-Weil theory, one can construct all the irreducible representations of G from the holomorphic line bundles on the homogeneous space G/T. In this section, we shall follow this approach and construct all the positive energy representations of LG using the holomorphic line bundles on the homogeneous space LG/T.

Let B^+ be a Borel subgroup of $G_{\mathbb{C}}$. The choice of B^+ provides a system of positive roots. The Lie algebra of B^+ is $\mathfrak{b}^+ = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha>0} \mathfrak{g}_\alpha$. Let $N^+ \subset B^+$ be the unipotent subgroup of B^+. The Lie algebra of N^+ is just $\mathfrak{n}^+ = \bigoplus_{\alpha>0} \mathfrak{g}_\alpha$.

After choosing B^+, we let $B^+G_{\mathbb{C}} \subseteq LG_{\mathbb{C}}$ be the subgroup, consisting of the loops of form

$$\gamma_0 + \gamma_1 z + \gamma_2 z^2 + \gamma_3 z^3 + \cdots ,$$

with $\gamma_0 \in B^+ \subseteq G_{\mathbb{C}}$. Then the homogeneous space $LG/T \cong LG_{\mathbb{C}}/B^+G_{\mathbb{C}}$ is an infinite dimensional complex manifold. In addition, we let $N^+G_{\mathbb{C}} \subseteq B^+G_{\mathbb{C}}$ be the subgroup consisting of loops such that $\gamma_0 \in N^+$.

Let $\tilde{N}^+G_{\mathbb{C}}$ and $\tilde{B}^+G_{\mathbb{C}}$ be the central extension induced from $\tilde{L}G_{\mathbb{C}}$, then

$$\tilde{B}^+G_{\mathbb{C}}/\tilde{N}^+G_{\mathbb{C}} \cong \tilde{T}_{\mathbb{C}}. \tag{3.12}$$

Remark 3.2.10. For loop group $LG_{\mathbb{C}}$, the Lie algebra of $B^+G_{\mathbb{C}}$ is

$$\mathfrak{b}^+\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{n>0} \mathfrak{t}_{\mathbb{C}} z^n \oplus \bigoplus_{\alpha>0} \mathfrak{g}_\alpha \oplus \bigoplus_{n>0} \mathfrak{g}_\alpha z^n,$$

which consists of complex torus and positive root spaces. The Lie algebra of $N^+G_{\mathbb{C}}$ consists of positive root spaces only,

$$\mathfrak{n}^+\mathfrak{g}_{\mathbb{C}} = \bigoplus_{n>0} \mathfrak{t}_{\mathbb{C}} z^n \oplus \bigoplus_{\alpha>0} \mathfrak{g}_\alpha \oplus \bigoplus_{n>0} \mathfrak{g}_\alpha z^n.$$

Let $\widetilde{L}G$ be a central extension of LG, then

$$LG/T \cong \widetilde{L}G/\tilde{T} \cong \widetilde{L}G_{\mathbb{C}}/\tilde{B}^{+}G_{\mathbb{C}}. \tag{3.13}$$

Each character $\boldsymbol{\lambda}$ of $\tilde{T} = T \times \mathbb{T}$ extends canonically to a holomorphic homomorphism

$$\boldsymbol{\lambda} : \tilde{B}^{+}G_{\mathbb{C}} \to \mathbb{C}^{\times}.$$

Therefore, one can define a holomorphic line bundle

$$L_{\boldsymbol{\lambda}} = \widetilde{L}G_{\mathbb{C}} \times_{\tilde{B}^{+}G_{\mathbb{C}}} \mathbb{C}$$

Note that $\widetilde{L}G_{\mathbb{C}}$ and $\mathrm{Rot}(S^1)$ act on the line bundle $L_{\boldsymbol{\lambda}}$, hence on the holomorphic sections $\Gamma_{hol}(L_{\boldsymbol{\lambda}})$.

Proposition 3.2.11 ([50])**.** *The line bundle $L_{\boldsymbol{\lambda}}$ possesses non-vanishing holomorphic sections if and only if the weight $\boldsymbol{\lambda}$ is antidominant.*

Proof. See [50]. □

Theorem 3.2.12. *Let $\boldsymbol{\lambda}$ be an antidominant weight, and $\Gamma_{hol}(L_{\boldsymbol{\lambda}})$ the space of holomorphic sections, which is a representation of $\widetilde{L}G$, then the representation is*
(1) *of positive energy;*
(2) *of finite type, that is, each component in the energy decomposition is of finite dimension;*
(3) *essentitally unitary;*
(4) *ireducible with lowest weight $\boldsymbol{\lambda}$ of zero energy.*

Proof. See [50]. □

Theorem 3.2.13. *Every irreducible representation of $\widetilde{L}G$ is essentially equivalent to some $\Gamma_{hol}(L_{\boldsymbol{\lambda}})$.*

Proof. See [50]. □

3.3 Hamiltonian LG-spaces

3.3.1 Introduction

Recall that a Hamiltonian G-space (M, ω_M, Φ) is a manifold M together with a G-action, an invariant 2-form $\omega_M \in \Omega^2(M)^G$, and an equivariant moment map $\Phi : M \to \mathfrak{g}^*$ such that

(1) $\mathrm{d}\,\omega_M = 0$.

(2) The moment map Φ satisfies the condition:

$$\iota(X_M)\omega_M = \mathrm{d}\,\Phi(X), \quad X \in \mathfrak{g},$$

where X_M denotes the vector field generated by $X \in \mathfrak{g}$.

(3) ω_M is non-degenerate.

From the definition, (M, ω) is a G-symplectic manifold. Suppose $\mathcal{L}$ is a Hermitian line bundle over M with a Hermitian connection ∇ such that

$$\mathrm{curv}(\nabla) = 2\pi\sqrt{-1}\omega.$$

It is well known that such a line bundle exists if the cohomology class of ω is integral, i.e., $[\omega] \in \mathrm{Image}(H^2(M, \mathbb{R}) \to H^2(M, \mathbb{Z}))$. In this case, the symplectic manifold (M, ω) is said to be pre-quantizable, and the line bundle $\mathcal{L}$ is called a pre-quantum line bundle. A G-symplectic manifold (M, ω) with a G-pre-quantum line bundle naturally gives rise to a moment map by Kostant formula and hence it forms a Hamiltonian G-space.

The concepts related to Hamiltonian G-spaces can be generalized to Hamiltonian LG-spaces with minor modification. Since the underlying spaces and groups are infinite dimensional, we proceed with care. With a slight abuse of notation, we use the symbols (M, ω_M, Φ) for both Hamiltonian G-spaces and LG-spaces.

Definition 3.3.1. A Hamiltonian LG-space (M, ω_M, Φ) at level k is a Banach manifold M together with an LG-action, an invariant 2-form $\omega_M \in \Omega^2(M)^{LG}$, and an equivariant moment map, $\Phi : M \to L\mathfrak{g}^*$ in which the action of LG on $L\mathfrak{g}^*$ is by affine coadjoint action with level k such that

(1) $\mathrm{d}\,\omega_M = 0$.

(2) The moment map Φ satisfies the condition

$$\iota(\xi_M)\omega_M = \mathrm{d}\int_{S^1} \Phi(\xi), \quad \xi \in L\mathfrak{g}.$$

(3) ω_M is weakly non-degenerate, i.e., the induced map $\omega_M : TM \to T^*M$ is injective.

Equivalently, a Hamiltonian LG-space (at level k) is a Hamiltonian $\widetilde{L}G$-space with central circle acting trivially on M and constant moment map k. If (M, ω_M, Φ) is a Hamiltonian LG-space at level k, then $(M, \omega_M/k, \Phi/k)$ is a Hamiltonian LG-space at level 1.

Much of the theory of compact Hamiltonian G-spaces carrries to Hamiltonian LG spaces if one assumes that the moment map Φ is proper, i.e., the inverse image of every compact set is compact.

One can similarly define pre-quantum line bundle $\mathcal{L}$ over Hamiltonian LG space (M, ω, Φ) by requiring that

$$\mathrm{curv}(\nabla) = 2\pi\sqrt{-1}\omega_M.$$

However, the line bundle $\mathcal{L}$ admits an $\tilde{L}G$-action, see Proposition 3.1.6.

3.3.2 Affine coadjoint orbits

We will show that there is a canonical Kähler structure on every affine coadjoint orbit.

Let $\mathcal{O}_\mu = LG \cdot \mu$ be an affine coadjoint orbit through $\mu \in L\mathfrak{g}^*$, then there is a LG-equivariant diffeomorphism,

$$\tilde{\mathrm{Ad}}(\mu) : LG/K_\mu \to \mathcal{O}_\mu, \quad [\gamma] \mapsto \mathrm{Ad}^*_\gamma(\mu) - \gamma'\gamma^{-1},$$

where K_μ is the isotropy subgroup $K_\mu = \{\gamma \in LG : \tilde{\mathrm{Ad}}_\gamma(\mu) = \mu\}$. At the infinitesimal level,

$$\tilde{\mathrm{ad}}(\mu) : (L\mathfrak{g}/\mathfrak{k}_\mu) \to T_\mu\mathcal{O}_\mu, \quad [\eta] \mapsto \mathrm{ad}^*_\eta(\mu) - \eta'.$$

where $\mathfrak{k}_\mu = \{\eta \in L\mathfrak{g} | \tilde{\mathrm{ad}}^*_\eta(\mu) = 0\}$ is the Lie algebra of isotropy subgroup.

As in the finite dimensional case, the tangent space $T_\mu\mathcal{O}_\mu$ is

$$\{\tilde{\mathrm{ad}}^*_\eta(\mu) | [\eta] \in L\mathfrak{g}/\mathfrak{k}\}.$$

Hence the vector field generated by $[\eta] \in L\mathfrak{g}/\mathfrak{k}$ at μ is

$$\eta^\sharp = \tilde{\mathrm{ad}}^*_\eta(\mu).$$

Definition 3.3.2. The symplectic 2-form is given by an analog to the Kirillov-Kostant-Souriau formula

$$\omega_\mu(\xi^\sharp, \eta^\sharp) = (\mu, 1)([\xi, \eta], \left\langle \xi', \eta \right\rangle) = \mu([\xi, \eta]) + \left\langle \xi', \eta \right\rangle.$$

Lemma 3.3.3. *If* $\xi, \eta \in L\mathfrak{g}$, *then*

$$[\xi, \eta]^\sharp_\mu = [\xi^\sharp_\mu, \eta] + [\xi, \eta^\sharp_\mu].$$

Proof. By direct calculation, one has

$$\begin{aligned}
[\xi, \eta]^\sharp_\mu =& \operatorname{ad}_{[\xi,\eta]}(\mu) - [\xi, \eta]' = [[\xi, \eta], \mu] - [\xi', \eta] - [\xi, \eta'] \\
=& [[\xi, \mu], \eta] + [\xi, [\eta, \mu]] - [\xi', \eta] - [\xi, \eta'] \\
=& [\xi^\sharp_\mu, \eta] + [\xi, \eta^\sharp_\mu].
\end{aligned}$$

□

Lemma 3.3.4. *Let* $\gamma \in LG$, *then* $\gamma_* : T_\mu \mathcal{O} \to T_{\gamma\cdot\mu}\mathcal{O}$ *is given by*

$$\gamma_* = \operatorname{Ad}_\gamma .$$

Proof. Let $\Gamma(s)$ be a curve in $\mathcal{O}$ such that $\Gamma(0) = \mu$ and $\Gamma'(0) = V \in T_\mu\mathcal{O}$, then $\gamma_*(V) = \frac{d}{ds}|_{s=0}(\operatorname{Ad}_\gamma(\Gamma(s)) - \gamma'\gamma^{-1}) = \operatorname{Ad}_\gamma(V)$. □

Lemma 3.3.5. *Let* $\gamma \in LG$ *and* $\xi \in L\mathfrak{g}$, *then*

$$\gamma_*(\xi^\sharp_\mu) = (\operatorname{Ad}_\gamma \xi)^\sharp_{\gamma\cdot\mu}.$$

Proof. From previous lemma, the left side is $\gamma_*(\xi^\sharp_\mu) = \operatorname{Ad}_\gamma(\xi^\sharp_\mu)$. The right side is

$$\begin{aligned}
(\operatorname{Ad}_\gamma \xi)^\sharp_{\gamma\cdot\mu} =& [\operatorname{Ad}_\gamma \xi, \gamma \cdot \mu] - (\operatorname{Ad}_\gamma \xi)' \\
=& [\operatorname{Ad}_\gamma \xi, \operatorname{Ad}_\gamma \mu] - [\operatorname{Ad}_\gamma \xi, \gamma'\gamma^{-1}] - (\operatorname{Ad}_\gamma \xi)' \\
=& \operatorname{Ad}_\gamma(\operatorname{ad}_\xi(\mu)) - [\operatorname{Ad}_\gamma \xi, \gamma'\gamma^{-1}] - (\operatorname{Ad}_\gamma \xi)'.
\end{aligned} \tag{3.14}$$

Note that

$$
\begin{aligned}
&- [\operatorname{Ad}_\gamma \xi, \gamma'\gamma^{-1}] - (\operatorname{Ad}_\gamma \xi)' \\
=& - [\gamma\xi\gamma^{-1}, \gamma'\gamma^{-1}] - (\gamma\xi\gamma^{-1})' \\
=& - \gamma\xi\gamma^{-1}\gamma'\gamma^{-1} + \gamma'\gamma^{-1}\gamma\xi\gamma^{-1} - \gamma'\xi\gamma^{-1} - \gamma\xi'\gamma^{-1} + \gamma\xi\gamma^{-1}\gamma'\gamma^{-1} \\
=& - \operatorname{Ad}_\gamma \xi',
\end{aligned} \tag{3.15}
$$

where we use the formula $(\gamma^{-1})' = -\gamma^{-1}\gamma'\gamma^{-1}$. Hence the right side is $\operatorname{Ad}_\gamma(\operatorname{ad}_\xi(\mu)) - \operatorname{Ad}_\gamma \xi' = \operatorname{Ad}_\gamma(\xi^\sharp_\mu)$, which completes the proof. □

Proposition 3.3.6. *The symplectic 2-form ω on $\mathcal{O}_\mu$ is LG-invariant, nondegenerate and closed, hence it defines a symplectic structure on the affine coadjoint orbit $\mathcal{O}_\mu$.*

Proof. First We want show ω is LG-invariant, i.e.,

$$\omega_{\gamma\cdot\mu}(\gamma_*(\xi^\sharp_\mu), \gamma_*(\eta^\sharp_\mu)) = \omega_\mu(\xi^\sharp_\mu, \eta^\sharp_\mu).$$

From the previous lemma, the left hand side is

$$
\begin{aligned}
&\omega_{\gamma\cdot\mu}((\operatorname{Ad}_\gamma \xi)^\sharp_{\gamma\cdot\mu}, (\operatorname{Ad}_\gamma \eta)^\sharp_{\gamma\cdot\mu}) \\
=& \left\langle \gamma\cdot\mu, [\operatorname{Ad}_\gamma \xi, \operatorname{Ad}_\gamma \eta] \right\rangle + \left\langle (\operatorname{Ad}_\gamma \xi)', \operatorname{Ad}_\gamma \eta \right\rangle \\
=& \langle \mu, [\xi, \eta] \rangle - \left\langle \gamma'\gamma^{-1}, \operatorname{Ad}_\gamma [\xi, \eta] \right\rangle + \left\langle (\operatorname{Ad}_\gamma \xi)', \operatorname{Ad}_\gamma \eta \right\rangle
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\left\langle (\operatorname{Ad}_\gamma \xi)', \operatorname{Ad}_\gamma \eta \right\rangle \\
=& \left\langle \operatorname{Ad}_\gamma \xi', \operatorname{Ad}_\gamma \eta \right\rangle + \left\langle \gamma'\gamma^{-1}(\operatorname{Ad}_\gamma \xi), \operatorname{Ad}_\gamma \eta \right\rangle - \left\langle (\operatorname{Ad}_\gamma \xi)\gamma'\gamma^{-1}, \operatorname{Ad}_\gamma \eta \right\rangle \\
=& \left\langle \xi', \eta \right\rangle + \left\langle [\gamma'\gamma^{-1}, \operatorname{Ad}_\gamma \xi], \operatorname{Ad}_\gamma \eta \right\rangle \\
=& \left\langle \xi', \eta \right\rangle + \left\langle \gamma'\gamma^{-1}, [\operatorname{Ad}_\gamma \xi, \operatorname{Ad}_\gamma \eta] \right\rangle
\end{aligned}
$$

Hence the left side is $\langle \mu, [\xi, \eta] \rangle + \langle \xi', \eta \rangle$ which is equal to $\omega_\mu(\xi_\mu^\sharp, \eta_\mu^\sharp)$. $\square$

Since the basic inner product B is nondegenerate, there exists a uniqe $\zeta_\mu \in L\mathfrak{g}$ such that

$$\left\langle \zeta_\mu, \xi \right\rangle = \mu(\xi).$$

Let $\mathfrak{m} = \mathfrak{k}^\perp$ the orthogonal complement of $\mathfrak{k}$ with resprect to B in $L\mathfrak{g}$, then

$$L\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}.$$

The tangent space $T_\mu(\mathcal{O}_\mu)$ is just $\mathfrak{m}$, on which the symplectic 2-form is $\omega(\xi, \eta) = \mu([\xi, \eta]) + \langle \xi', \eta \rangle$ for $\xi, \eta \in \mathfrak{m}$. Moreover, there exists a unique operator A on $\mathfrak{m}$ given by

$$\omega(\xi, \eta) = \langle A\xi, \eta \rangle.$$

The canonical complex structure is then defined by

$$J = A(A^*A)^{-1/2}$$

in the polar decomposition. Alternatively, one can consider the operator

$$A_\mu : L\mathfrak{g} \to L\mathfrak{g}^* \to L\mathfrak{g}, \quad \xi \mapsto d\xi - \mathrm{ad}_\xi^*(\mu) \mapsto \xi' + \mathrm{ad}_{\zeta_\mu}(\xi),$$

which has the kernel $\mathrm{Ker}(A) = \mathfrak{k}$ and the range $\mathrm{Range}(A) = T_\mu(\mathcal{O}_\mu)$.

Lemma 3.3.7. *Let $\xi \in L\mathfrak{g}$, $\mu \in L\mathfrak{g}^*$ and $\gamma \in LG$.*

$$A_\mu(\xi_\mu^\sharp) = (A_\mu \xi)_\mu^\sharp.$$

$$\mathrm{Ad}_\gamma A_\mu \xi = A_{\gamma \cdot \mu} \mathrm{Ad}_\gamma \xi.$$

Proof. In the first identity, the left hand side is

$$\begin{aligned} A_\mu(\xi^\sharp_\mu) =& A_\mu(\mathrm{ad}_\xi(\mu) - \xi') \\ =& \mathrm{ad}_\mu(\mathrm{ad}_\xi(\mu) - \xi') + (\mathrm{ad}_\xi(\mu) - \xi')' \\ =& [\mu, [\xi, \mu]] - [\mu, \xi'] + [\xi, \mu]' - \xi''. \end{aligned}$$

The right side is

$$\begin{aligned} (A_\mu \xi)^\sharp_\mu = (\mathrm{ad}_\mu \xi + \xi')^\sharp_\mu = [[\mu, \xi] + \xi', \mu] - [\mu, \xi]' - \xi'' \\ = [[\mu, \xi], \mu] + [\xi', \mu] - [\mu, \xi]' - \xi''. \end{aligned}$$

In the second identity, the right hand side is

$$\begin{aligned} & [\gamma\mu\gamma^{-1} - \gamma'\gamma^{-1}, \gamma\xi\gamma^{-1}] + (\gamma\xi\gamma^{-1})' \\ =& [\gamma\mu\gamma^{-1}, \gamma\xi\gamma^{-1}] - [\gamma'\gamma^{-1}, \gamma\xi\gamma^{-1}] + (\gamma\xi\gamma^{-1})' \\ =& \mathrm{Ad}_\gamma([\mu, \xi]) - \gamma'\gamma^{-1}\gamma\xi\gamma^{-1} + \gamma\xi\gamma^{-1}\gamma'\gamma^{-1} + \gamma'\xi\gamma^{-1} + \gamma\xi'\gamma^{-1} - \gamma\xi\gamma^{-1}\gamma'\gamma^{-1} \\ =& \mathrm{Ad}_\gamma([\mu, \xi] + \xi') = A_\mu(\xi^\sharp_\mu). \end{aligned}$$

□

Lemma 3.3.8. *The almost complex structure J is LG-invariant.*

Proof. We want to show

$$A_{\gamma\cdot\mu}(\gamma_* \xi^\sharp_\mu) = \gamma_*(A_\mu \xi^\sharp_\mu).$$

Note that from previous lemma, $\gamma_*(A_\mu\xi^\sharp_\mu) = \mathrm{Ad}_\gamma(A_\mu\xi^\sharp_\mu) = \mathrm{Ad}_\gamma((A_\mu\xi)^\sharp_\mu) = (\mathrm{Ad}_\gamma A_\mu\xi)^\sharp_{\gamma\cdot\mu}$, while the left side is $A_{\gamma\cdot\mu}(\gamma_*\xi^\sharp_\mu) = A_{\gamma\cdot\mu}(\mathrm{Ad}_\gamma \xi)^\sharp_{\gamma\cdot\mu} = (A_{\gamma\cdot\mu} \mathrm{Ad}_\gamma \xi)^\sharp_{\gamma\cdot\mu} = (\mathrm{Ad}_\gamma A_\mu\xi)^\sharp_{\gamma\cdot\mu}$ from previous lemma. □

Proposition 3.3.9. *The symplectic structure given by the 2-form ω and the complex structure J constructed above provide a canonical Kähler structure on the coadjoint* *orbit* $\mathcal{O}_\mu$.

Example 3.3.10. Suppose $\mu = 0$, then the stabilizer is G considered as constant loops, and the coadjoint orbit $\mathcal{O}_0$ is

$$\Omega G \to LG/G \to \mathcal{O}_0, \quad \gamma \mapsto -\gamma'\gamma^{-1}.$$

The tangent space $T_0\mathcal{O}_0$ at 0 is

$$\Omega\mathfrak{g} \to L\mathfrak{g}/\mathfrak{g} \to T_0\mathcal{O}_0, \quad \xi \mapsto -\xi'.$$

The KKS 2-form is then

$$\omega(\xi, \eta) = \left\langle \xi', \eta \right\rangle.$$

and the complex structure J comes from the energy operator

$$A\xi = \xi'.$$

If we complexified the tangent space, $\Omega\mathfrak{g}_{\mathbb{C}} = \oplus_{n\neq 0}\mathfrak{g}_{\mathbb{C}}z^n$, then the holomorphic part and anti-holomorphic part are $\oplus_{n>0}\mathfrak{g}_{\mathbb{C}}z^n$ and $\oplus_{n<0}\mathfrak{g}_{\mathbb{C}}z^n$ respectively.

Example 3.3.11. If μ is sufficiently generic so that the stabilizer is a maximal torus T. Since the set of conjugacy classes are classified by the fundamental alcove $\mathfrak{A}$, we can assume $\zeta_\mu \in \mathfrak{A}$. The tangent space $T_\mu\mathcal{O}_\mu$ is

$$L\mathfrak{g}/\mathfrak{t} \to T_\mu\mathcal{O}_\mu, \quad \xi \mapsto \mathrm{ad}^*_\xi(\mu) - \xi'.$$

Let $L\mathfrak{g}/\mathfrak{t} = \mathfrak{m} \oplus \Omega\mathfrak{g}$, then the operator A is

$$A\xi = \xi' + \mathrm{ad}_{\zeta_\mu}(\xi).$$

And the complex structure J defined by A has holomorphic part and antiholomorphic part are given by

$$\sum_{\alpha>0} \mathfrak{g}^\alpha + \sum_{n>0} \mathfrak{g}^{\mathbb{C}} z^n, \quad \sum_{\alpha<0} \mathfrak{g}^\alpha + \sum_{n<0} \mathfrak{g}^{\mathbb{C}} z^n.$$

3.3.3 q-Hamiltonian spaces

Alekseev, Malkin and Meinrenken introduced the concept of a q-Hamiltonian G-space with moment map taking values in the Lie group G, see [2].

Let θ^L and θ^R denote the left- and right-invariant Maurer-Cartan forms respectively. In a faithful matrix representations of G, $\theta^L = g^{-1}\,\mathrm{d}\,g$ and $\theta^R = \mathrm{d}\,gg^{-1}$. Let $\chi_G \in \Omega^3(G)$ denote the canonical closed bi-invariant 3-form on G,

$$\chi_G = \frac{1}{12}B(\theta^L, [\theta^L, \theta^L]) = \frac{1}{12}B(\theta^R, [\theta^R, \theta^R]),$$

where $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is the basic inner product.

Definition 3.3.12. A q-Hamiltonian G-space (N, ω_N, Ψ) is a G-manifold N together with an invariant 2-form $\omega_N \in \Omega^2(N)^G$ and a G-equivariant map $\Psi : N \to G$ in which G acts on itself by conjugation, such that

(1) The differential of ω_N is given by

$$\mathrm{d}\,\omega_N = -\Psi^*\chi.$$

(2) The moment map satisfies the condition

$$\iota(X_N)\omega_N = \frac{1}{2}\Psi^*(\theta^L + \theta^R, X), \quad X \in \mathfrak{g}.$$

(3) The kernel of ω_x at any $x \in N$ is given by

$$\mathrm{Ker}(\omega_N)_x = \{X_N : X \in \mathrm{Ker}(\mathrm{Ad}_{\Psi(x)} + \mathrm{Id})\}.$$

Example 3.3.13. Examples of q-Hamiltonian G-spaces are conjugacy classes of G. In fact, for every conjugacy class $\mathcal{C} \subset G$, the invariant 2-form $\omega_{\mathcal{C}} \in \Omega^2(\mathcal{C})^G$ is given by the formula

$$(\omega_{\mathcal{C}})_g(X_{\mathcal{C}}, Y_{\mathcal{C}}) = \frac{1}{2}(B(Y, \mathrm{Ad}_g X) - B(X, \mathrm{Ad}_g Y)),$$

where $X_{\mathcal{C}}$ and $Y_{\mathcal{C}}$ are the generating vector fields of $X, Y \in \mathfrak{g}$ respectively.

One may construct a q-Hamiltonian space from a usual Hamiltonian G-space, but the non-degeneracy condition of ω_N requires an extra condition on the differential of exponential map, namely, $\mathrm{d}_X \exp$ is bijective for all $X \in \Phi(N)$, see section 3 in [2].

To quantize q-Hamiltonian spaces, we need to introduce pre-quantizable conditions. Let $\mathcal{A}_G$ be a G-equivariant Dixmier-Douaday bundle whose Dixmier-Douaday class is a generator of $H^3_G(G, \mathbb{Z}) \simeq Z$, see [44].

Definition 3.3.14. A q-Hamiltonian G-space (N, ω_N, Ψ) is pre-quantizable at level k if there exists a G-equivariant Morita morphism

$$(\Psi, \mathcal{E}) : (N, \mathbb{C}) \to (G, \mathcal{A}_G^{-k}).$$

The Morita morphism is the counterpart to a pre-quantum line bundle. It is the geometric realization of the integral class (ω_N, η_G) in relative cohomology $H^3_G(\Psi)$.

It is shown in [2] that every Hamiltonian LG-space with proper moment map determines a q-Hamiltonian G-space and vice versa. In particular, the conjugacy classes of G correspond exactly to the affine coadjoint orbits of LG on $L\mathfrak{g}^*$.

For any $\mu \in L\mathfrak{g}^*$, let $\mathrm{Hol}_s : L\mathfrak{g}^* \to G$ be the unique solution to the initial value problem

$$\mathrm{Hol}_s(\mu)^{-1} \frac{\mathrm{d}}{\mathrm{d}\, s} \mathrm{Hol}_s(\mu) = \mu/k, \quad \mathrm{Hol}_0(\mu) = e.$$

The function Hol_s satisfies the equivariance condition

$$\mathrm{Hol}_s(\gamma \cdot \mu) = \gamma(0)\,\mathrm{Hol}_s(\mu)\gamma(s)^{-1}. \tag{3.16}$$

Let $\mathrm{Hol} = \mathrm{Hol}_{2\pi}$ be the holonomy map, that is, evaluation at the first complete cycle. Then $\mathrm{Hol} : L\mathfrak{g}^* \to G$ is equivariant with respect to the evaluation homomorphism $LG \to G, \gamma \to \gamma(0)$. It can be shown that the action of based loop group ΩG on $L\mathfrak{g}^*$ is free, and the quotient map is just the holonomy map. In other words, $\mathrm{Hol} : L\mathfrak{g}^* \to G$ is the universal principal ΩG-bundle.

Since the ΩG action on $L\mathfrak{g}^*$ is free, its action on M is also free by equivariance of the moment map, and hence M is a principal ΩG-bundle over a finite dimensional manifold $\mathrm{Hol}(M) := M/\Omega G$. This quotient $\mathrm{Hol}(M)$ carries a canonical structure of a q-Hamiltonian G-space, and the following diagram commutes,

$$\begin{array}{ccc} M & \xrightarrow{\tilde{\Phi}} & L\mathfrak{g}^* \\ {\scriptstyle \mathrm{Hol}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Hol}} \\ N & \xrightarrow{\Psi} & G. \end{array}$$

Example 3.3.15. Let $M = \Omega G$, then $\mathrm{Hol}(M) = \mathrm{pt}$, and if $M = LG/T$, then $\mathrm{Hol}(M) = G/T$.

Chapter 4

Quantization of affine coadjoint orbits

4.1 Dirac operators

4.1.1 Preliminaries

Let $\mathfrak{g}$ be a simple Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ its complexification. Let B be the basic inner product on $\mathfrak{g}$ and $\{e_a\}$ a basis of $\mathfrak{g}_{\mathbb{C}}$. Let $\{e_a^*\}$ be the corresponding dual basis, so that $B(e_a, e_b^*) = \delta_{a,b}$.

The *dual Coxeter number* $h^\vee$ of $\mathfrak{g}$ is given by the formula

$$\frac{1}{2}\sum_a \operatorname{ad}(e_a^*)\operatorname{ad}(e_a)X = h^\vee X, \quad X \in \mathfrak{g}_{\mathbb{C}}.$$

This invariant quantity plays an important role in this dissertation.

Example 4.1.1. Let $G = \mathrm{SU}(2)$, and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(2,\mathbb{C})$. The basic inner product is given by $B(X,Y) = \operatorname{trace}(XY)$. Let

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then we have $[E, F] = H, [H, E] = 2E$ and $[H, F] = -2F$. Note that the set $\{H, E, F\}$ forms a basis for $\mathfrak{g}_{\mathbb{C}}$, and one can check that $B(E, F) = 1, B(H, H) = 2$, and $B(E, H) = B(F, H) = B(E, E) = B(F, F) = 0$. Hence the dual basis is given by $\{H^* = \frac{1}{2}H, E^* = F, F^* = E\}$. The dual Coxeter number then can be obtained by calculating the Casimir element (see [35])

$$\mathrm{Cas} = \frac{1}{2}(H^*H + E^*E + F^*F) = (\frac{1}{4}H^2 + \frac{1}{2}EF + \frac{1}{2}FE).$$

The Casimir element is in the center of the universal enveloping algebra $U\mathfrak{g}_{\mathbb{C}}$, hence it is a scalar. In particular, $\mathrm{Cas}(H) = \frac{1}{2}([E, [F, H]] + [F, [F, H]]) = 2H$. This example shows that the dual Coxeter number $h^\vee$ for SU(2) is 2.

In general, let $\pi : \mathfrak{g} \to End(V)$ be a complex representation of $\mathfrak{g}$ on a vector space V. The *Casimir operator* associated to π is defined by

$$\mathrm{Cas}_\pi = \frac{1}{2}\sum_a \pi(e_a^*)\pi(e_a).$$

By Schur's lemma, Cas_π is a scalar on the irreducible subspace V_λ with the highest weight λ. One can calculate the scalar by applying the Casimir operator to the highest weight vector.

Proposition 4.1.2. *On an irreducible subspace with the highest weight λ the Casimir is given by the scalar*

$$\mathrm{Cas}_\pi = \frac{1}{2}(\|\lambda + \rho\|^2 - \|\rho\|^2).$$

In particular, for adjoint representation, one has $\mathrm{Cas}_{\mathrm{ad}} = \frac{1}{2}(\|\alpha_{max} + \rho\|^2 - \|\rho\|^2) = B(\alpha_{max}, \rho) + 1$. *Hence we have*

$$B(\alpha_{max}, \rho) + 1 = h^\vee.$$

Proof. See G. Segal in [29]. □

Example 4.1.3. The dual Coxeter number for SU(n) is

$$h^\vee = n.$$

Indeed, the maximal root is $\alpha_{max} = [1, 0, 0, ..., -1]$, and the half sum of positve roots is given by

$$\rho_G = \frac{1}{2}[n-1, n-2, ..., -(n-2), -(n-1)].$$

So $h^\vee = B(\alpha_{max}, \rho_G) + 1 = \frac{1}{2}(2n-2) + 1 = n$.

For simplicity, we shall assume $\{e_a\}$ is an orthonormal basis of $\mathfrak{g}$ with respect to $-B$. Note that B is negative definite since G is compact with trivial center, see Section 6.2 in [35]. Then we have

$$-\frac{1}{2}\sum_a [e_a, [e_a, X]] = h^\vee X, \quad X \in \mathfrak{g}_{\mathbb{C}}.$$

Example 4.1.4. Let $G = \mathrm{SU}(2)$, then the matrices $e_1 = \frac{i}{\sqrt{2}}H$, $e_2 = \frac{i}{\sqrt{2}}(E-F)$, and $e_2 = \frac{i}{\sqrt{2}}(E+F)$ form an orthonormal basis of $\mathfrak{g} = \mathfrak{su}(2)$ with respect to $-B(X,Y) = -\operatorname{trace}(XY)$.

The following proposition will be frequently used.

Proposition 4.1.5. *Let $\{e_a\}$ be an orthonormal basis of $\mathfrak{g}$ with respect to some* ad*-invariant inner product, then the following identity holds*

$$\sum_a \mathfrak{B}([e_a, X], e_a) = \sum_a \mathfrak{B}(e_a, [X, e_a]),$$

where $\mathfrak{B}$ is any bilinear map $\mathfrak{g} \times \mathfrak{g} \to V$ taking values in a vector space V.

Proof. Let $\langle \cdot, \cdot \rangle$ denote the ad-invariant inner product on $\mathfrak{g}$. Then we have

$$\begin{aligned}
\sum_a \mathfrak{B}([e_a, X], e_a) &= \sum_{a,b} \mathfrak{B}(\langle [e_a, X], e_b \rangle e_b, e_a) \\
&= \sum_{a,b} \mathfrak{B}(e_b, \langle [e_a, X], e_b \rangle e_a) \\
&= \sum_{a,b} \mathfrak{B}(e_b, \langle e_a, [X, e_b] \rangle e_a) \\
&= \sum_b \mathfrak{B}(e_b, [X, e_b]) = \sum_a \mathfrak{B}(e_a, [X, e_a])
\end{aligned}$$

□

4.1.2 Vertical operators

In this section, we are going to construct a fundamental class on a principal ΩG manifold M, using the complex structure from the fiber LG/G. From Fourier theory, we have the decomposition

$$L\mathfrak{g}_{\mathbb{C}} = \bigoplus_n \mathfrak{g}_{\mathbb{C}} z^n.$$

Let $\mathfrak{g}_+ = \bigoplus_{n>0} \mathfrak{g}_{\mathbb{C}} z^n$ and $\mathfrak{g}_- = \bigoplus_{n<0} \mathfrak{g}_{\mathbb{C}} z^n$. An element $e_a z^n \in \mathfrak{g}_{\mathbb{C}} z^n$ is denoted by e_a^n.

As before, let B denote the basic inner product on $\mathfrak{g}$, and let $\{e_a\}$ be an orthonormal basis of $\mathfrak{g}$ with respect to $-B$. It induces a Hermitian metric on $\mathfrak{g}_{\mathbb{C}} z^n$, given by the formula

$$\langle e_a^n | e_b^m \rangle = \delta_{a,b} \delta_{n,m}.$$

We also have the identification $\mathfrak{g}_+ \to \mathfrak{g}_-^*$, given by $(e_a^n, e_b^{-m}) = \langle e_a^n | e_b^m \rangle$.

Definition 4.1.6. Let

$$\mathcal{S} = \bigwedge \mathfrak{g}_-^* \otimes \bigwedge \mathfrak{g}_{\mathbb{C}}^*$$

denote the (extended) spinor module.

With the metric defined above, we have $\mathcal{S} \simeq \bigwedge \mathfrak{g}_+ \otimes \bigwedge \mathfrak{g}_{\mathbb{C}} = \bigwedge \mathfrak{g}_{\mathbb{C}}[z]$.

Definition 4.1.7. The (algebraic) Dolbeault operator $\bar{\partial}$ is given by the formula

$$\bar{\partial} = 1/2 \sum_{a, m \geq 0} \epsilon(e_a^m) \operatorname{ad}(e_a^{-m}),$$

where $\operatorname{ad}(e_a^{-m})$ is defined as

$$\operatorname{ad}(e_a^{-m})(e_b^n) = \begin{cases} [e_a, e_b]^{n-m} & \text{if } n - m \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 4.1.8. (1) The term $\operatorname{ad}(e_a^{-m})$ in the formula above comes from the Levi-Civita connection on the homogeneous space. It is not exactly the adjoint action, but instead the adjoint action followed by projection. See [25].
(2) For any form of finite energy, the operator $\bar{\partial}$ consists of only finitely many terms, hence it is well-defined.

The algebraic Dolbeault operator we define here is slightly different from the one used in [56] by Teleman in studying the Lie algebra cohomology. The motivation of our definition is that we are trying to study the ΩG-invariant form on the principal ΩG-bundle M. In the simplest situation, when the manifold M under consideration is the affine coadjoint orbit $M = LG/G$ through $0 \in L\mathfrak{g}^*$, we may identify the invariant forms with the exterior algebra of $\Omega\mathfrak{g}^*$,

$$\Omega^*(M)^{\Omega G} = \bigwedge \Omega\mathfrak{g}^*.$$

Moreover, $M = LG/G$ is a complex manifold, which enables us to define the Dolbeault operator. For technical reasons (see Theorem 4.1.18), we enlarge the spinor module by tensoring a finite dimensional part $\bigwedge \mathfrak{g}_{\mathbb{C}}$.

Lemma 4.1.9. *The following identities hold:*

$$\operatorname{ad}(e_a^{-m})^* = -\operatorname{ad}(e_a^m),$$

$$\epsilon(e_a^m)^* = \iota(e_a^{-m}).$$

Proof. On $\mathfrak{g}_+$, the metric is given by $\left\langle e_b^n | e_c^l \right\rangle = \delta_{b,c}\delta_{n,l}$. Since

$$\left\langle \operatorname{ad}(e_a^{-m}) e_b^n | e_c^l \right\rangle = B([e_a, e_b], e_c)\delta_{n-m,l}$$

and

$$\left\langle e_b^n | -\operatorname{ad}(e_a^m) e_c^l \right\rangle = -B(e_b, [e_a, e_c])\delta_{n,m+l},$$

the identity follows from the ad-invariance of the basic inner product on $\mathfrak{g}$. The second identity can be obtained similarly. □

Lemma 4.1.10. *The formal adjoint under the Hermitian metric is given by*

$$\bar{\partial}^* = -1/2 \sum_{a;m\geq 0} \operatorname{ad}(e_a^m)\iota(e_a^{-m}).$$

Proof. The proof is obtained by applying the previous lemma. □

Note that for any form of finite energy, $\bar{\partial}^*$ consists of finitely many nonzero terms, hence the operators $\bar{\partial}^*$ is well-defined.

To simplify notation in the calculation, we will use graded commutator in this chapter,

$$[A, B] = AB - (-1)^{\deg(B)\cdot\deg(A)} BA.$$

Hence if A or B is even, the graded commutator is the same as the usual one. For example $\operatorname{ad}(e_a^m)$ is considered even while $\iota(e_b^{-n}), \epsilon(e_b^n)$ and $\bar{\partial}$ are odd.

Lemma 4.1.11. *The following identities hold:*

$$[\mathrm{ad}(e_a^m), \iota(e_b^{-n})] = \iota(\mathrm{ad}(e_a^m)e_b^{-n}),$$
$$[\mathrm{ad}(e_a^m), \epsilon(e_b^n)] = \epsilon(\mathrm{ad}(e_a^m)e_b^n).$$

Proof. On $\mathfrak{g}_+$, for the first formula we have

$$[\mathrm{ad}(e_a^m), \iota(e_b^{-n})](e_c^l) = -B(e_b, [e_a, e_c])\delta_{n,m+l}$$

and

$$\iota(\mathrm{ad}(e_a^m)e_b^{-n})(e_c^l) = B([e_a, e_b], e_c)\delta_{n-m,l}.$$

Note that in the above formulas, if $m \geq n$, then both sides vanish. The general case is obtained by induction. For the second formula,

$$\begin{aligned}[\mathrm{ad}(e_a^m), \epsilon(e_b^n)](\phi) &= \mathrm{ad}(e_a^m)(e_b^n \wedge \phi) - e_b^n \wedge \mathrm{ad}(e_a^m)(\phi) \\ &= \mathrm{ad}(e_a^m)(e_b^n) \wedge \phi + e_b^n \wedge \mathrm{ad}(e_a^m)(\phi) - e_b^n \wedge \mathrm{ad}(e_a^m)(\phi) \\ &= \epsilon(\mathrm{ad}(e_a^m)e_b^n)(\phi).\end{aligned}$$

The proof is complete. □

Lemma 4.1.12. *The following identities hold:*

$$[\bar{\partial}, \epsilon(e_b^n)] = \epsilon(\bar{\partial}(e_b^n)),$$
$$[\bar{\partial}, \iota(e_b^{-n})] = \mathrm{ad}(e_b^{-n}),$$
$$[\bar{\partial}^*, \epsilon(e_b^n)] = -\,\mathrm{ad}(e_b^n),$$
$$\bar{\partial}^2 = 0, \quad (\bar{\partial}^*)^2 = 0.$$

Proof. For the first formula we have

$$2[\bar{\partial}, \epsilon(e_b^n)] = \sum_{m\geq 0} \epsilon(e_a^m)\,\mathrm{ad}(e_a^{-m})\epsilon(e_b^n) + \sum_{m\geq 0} \epsilon(e_b^n)\epsilon(e_a^m)\,\mathrm{ad}(e_a^{-m})$$
$$= \sum_{m=0}^{n} \epsilon(e_a^m)\epsilon([e_a, e_b]^{n-m}).$$

For the second formula we have

$$2[\bar{\partial}, \iota(e_b^{-n})](e_c^l) = \iota(e_b^{-n}) \sum (e_a^m \wedge [e_a, e_c]^{l-m})$$
$$= [e_b, e_c]^{l-n} - \sum_a B(e_b, [e_a, e_c])e_a^{l-n}$$
$$= 2\,\mathrm{ad}(e_b^{-n})(e_c^l).$$

The third formula is obtained by taking adjoint. The last two formulas follow by direct calculation and taking adjoint. □

Proposition 4.1.13.

$$[\bar{\partial}^*, [\bar{\partial}, \epsilon(e_b^n)]] = h^\vee(n+1)\epsilon(e_b^n) + \sum_{m=0}^{n} \epsilon(e_a^{-n})\,\mathrm{ad}([e_a, e_b]^{n-m}).$$

Proof. We have

$$2[\bar{\partial}^*, [\bar{\partial}, \epsilon(e_b^n)]] = \sum_{m=0}^{n} [\bar{\partial}^*, \epsilon(e_a^m)\epsilon([e_a, e_b]^{n-m})]$$
$$= \sum_{m=0}^{n} \Big([\bar{\partial}^*, \epsilon(e_a^m)]\epsilon([e_a, e_b]^{n-m}) - \epsilon(e_a^m)[\bar{\partial}^*, \epsilon([e_a, e_b]^{n-m})]\Big)$$
$$= \sum_{m=0}^{n} \Big(-\,\mathrm{ad}(e_a^m)\epsilon([e_a, e_b]^{n-m}) + \epsilon(e_a^m)\,\mathrm{ad}([e_a, e_b]^{n-m})\Big)$$
$$= -\sum_{m=0}^{n} \epsilon([e_a, [e_a, e_b]]^n) + \epsilon([e_a, e_b]^{n-m})\,\mathrm{ad}(e_a^m)$$
$$+ \sum_{m=0}^{n} \epsilon(e_a^m)\,\mathrm{ad}([e_a, e_b]^{n-m})$$
$$= 2h^\vee(n+1)\epsilon(e_b^n) + 2\sum_{m=0}^{n} \epsilon(e_a^m)\,\mathrm{ad}([e_a, e_b]^{n-m}).$$

The proof is complete. ∎

Proposition 4.1.14.

$$[\bar{\partial}, [\bar{\partial}^*, \epsilon(e_b^n)]] = -\sum_{m=0}^{n-1} \epsilon(e_a^m)\,\mathrm{ad}([e_a, e_b]^{n-m}).$$

Proof. First note that $[\bar{\partial}^*, \epsilon(e_b^n)] = -\,\mathrm{ad}(e_b^n)$.

$$\begin{aligned}
2[\bar{\partial}, [\bar{\partial}^*, \epsilon(e_b^n)]] &= 2[\mathrm{ad}(e_b^n), \bar{\partial}] \\
&= \sum_{m\geq 0} [\mathrm{ad}(e_b^n), \epsilon(e_a^m)\,\mathrm{ad}(e_a^{-m})] \\
&= \sum_{m\geq 0} [\mathrm{ad}(e_b^n), \epsilon(e_a^m)]\,\mathrm{ad}(e_a^{-m}) + \epsilon(e_a^m)[\mathrm{ad}(e_b^n), \mathrm{ad}(e_a^{-m})] \\
&= \sum_{m\geq 0} \epsilon([e_b, e_a]^{m+n})\,\mathrm{ad}(e_a^{-m}) - \sum_{m\geq 0} \epsilon(e_a^m)\,\mathrm{ad}([e_a, e_b]^{n-m}) \\
&\qquad + \sum_{m\geq 0} \epsilon(e_a^m)([\mathrm{ad}(e_b^n), \mathrm{ad}(e_a^{-m})] - \mathrm{ad}([e_b, e_a]^{n-m})) \\
&= \sum_{m=0}^{n-1} \epsilon(e_a^m)\,\mathrm{ad}([e_a, e_b]^{n-m}) \\
&\qquad + \sum_{m\geq 0} \epsilon(e_a^m)([\mathrm{ad}(e_b^n), \mathrm{ad}(e_a^{-m})] - \mathrm{ad}([e_b, e_a]^{n-m})).
\end{aligned}$$

Here we are using the identity

$$\begin{aligned}
\sum_{m\geq 0} \epsilon([e_b, e_a]^{m+n})\,\mathrm{ad}(e_a^{-m}) &= \sum_{m\geq 0} \epsilon(e_a^{m+n})\,\mathrm{ad}([e_a, e_b]^{-m}) \\
&= \sum_{m\geq n} \epsilon(e_a^m)\,\mathrm{ad}([e_a, e_b]^{n-m}).
\end{aligned}$$

Next we shall compute the sum

$$\sum_{m\geq 0} \epsilon(e_a^m)([\mathrm{ad}(e_b^n), \mathrm{ad}(e_a^{-m})] - \mathrm{ad}([e_b, e_a]^{n-m})). \tag{4.1}$$

Note that by the Jacobi identity

$$([\mathrm{ad}(e_b^n), \mathrm{ad}(e_a^{-m})] - \mathrm{ad}([e_b, e_a]^{n-m}))(e_c^l) = \begin{cases} -[e_b, [e_a, e_c]]^{n-m+l} & \text{if } l+1 \leq m \leq l+n, \\ 0 & \text{else.} \end{cases}$$

So the sum will be equal to the following formula:

$$\begin{aligned}
&\sum_{m\geq 0} \epsilon(e_a^m)([\mathrm{ad}(e_b^n), \mathrm{ad}(e_a^{-m})] - \mathrm{ad}([e_b, e_a]^{n-m})) \\
=& -\sum_{m\geq 0} \sum_{l\geq m-n; p\geq 0}^{m-1} \epsilon(e_a^m)\epsilon([e_b, [e_a, e_c]]^{n-m+l})\iota(e_c^{-l})) \\
=& \sum_{l\geq 0} \sum_{m=l+1}^{l+n} \epsilon([e_a, e_c]^m)\epsilon([e_b, e_a]^{n-m+l})\iota(e_c^{-l})) \\
=& \sum_{l\geq 0} \sum_{m=l+1}^{l+n} \epsilon([e_a, e_b]^{n-m+l})\epsilon([e_a, e_c]^m)\iota(e_c^{-l})) \\
=& \sum_{l\geq 0} \sum_{m=1}^{n} \epsilon([e_a, e_b]^{n-m})\epsilon([e_a, e_c]^{m+l})\iota(e_c^{-l})) \\
=& \sum_{m=1}^{n} \epsilon([e_a, e_b]^{n-m}) \, \mathrm{ad}(e_a^m)) \\
=& \sum_{m=0}^{n-1} \epsilon([e_a, e_b]^m) \, \mathrm{ad}(e_a^{n-m})).
\end{aligned}$$

Therefore we have

$$2[\bar{\partial}, [\bar{\partial}^*, \epsilon(e_b^n)]] = 2 \sum_{m=0}^{n-1} \epsilon([e_a, e_b]^m) \, \mathrm{ad}(e_a^{n-m}).$$

□

Definition 4.1.15. The unbounded operator $D_{\text{vert}} = \bar{\partial} + \bar{\partial}^*$ is called the vertical operator.

Since $\bar{\partial}^2 = 0$ and $(\bar{\partial}^*)^2 = 0$, we have $D_{\text{vert}}^2 = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$.

Proposition 4.1.16.

$$[D^2_{\text{vert}}, \epsilon(e^n_b)] = h^\vee(n+1)\epsilon(e^n_b) + \sum_a \epsilon(e^n_a)\,\mathrm{ad}([e_b, e_a]).$$

Proof. Combining previous two propositions gives the result. □

Proposition 4.1.17.

$$\left[\frac{1}{2}\sum_a \mathrm{ad}(e_a)^2, \epsilon(e^n_b)\right] = -h^\vee\epsilon(e^n_b) + \sum_a \epsilon(e^n_a)\,\mathrm{ad}([e_b, e_a]).$$

Proof. Applying the formulas in Lemma 4.1.11, we have

$$\begin{aligned}
&\left[\frac{1}{2}\sum_a \mathrm{ad}(e_a)^2, \epsilon(e^n_b)\right] \\
=&\frac{1}{2}\sum_a (\epsilon([e_a, e_b]^n)\,\mathrm{ad}(e_a) + \mathrm{ad}(e_a)\epsilon([e_a, e_b]^n)) \\
=&\frac{1}{2}\sum_a \epsilon([e_a, e_b]^n)\,\mathrm{ad}(e_a) + \frac{1}{2}\sum_a [e_a, [e_a, e_b]]^n \\
&\qquad + \frac{1}{2}\sum_a \epsilon([e_a, e_b]^n)\,\mathrm{ad}(e_a) \\
=&\sum_a \epsilon(e^n_a)\,\mathrm{ad}([e_b, e_a]) - h^\vee\epsilon(e^n_b).
\end{aligned}$$

□

Theorem 4.1.18.

$$D^2_{\text{vert}} = h^\vee \cdot (\mathrm{E} + 2\cdot \deg) + 1/2\sum_a \mathrm{ad}^2(e_a),$$

where E *is the energy operator, i.e.,* $\mathrm{E}(e^n_b) = ne^n_b$*, and* deg *is the degree operator.*

Proof. We shall prove the theorem by induction. Note that $D^2 = [\bar\partial, \bar\partial^*]$ using graded commutator, and

$$[D^2, \epsilon(e^n_b)] = [\bar\partial, [\bar\partial^*, \epsilon(e^{-m}_a)]] + [\bar\partial^*, [\bar\partial, \epsilon(e^n_b)]].$$

From Propositioin 4.1.16, we have

$$[D^2, \epsilon(e_b^n)] = h^\vee(n+1)\epsilon(e_b^n) + \sum_a \epsilon(e_a^n)\,\mathrm{ad}([e_b, e_a]).$$

In addition, from Proposition 4.1.17

$$\left[\frac{1}{2}\sum_a \mathrm{ad}(e_a)^2, \epsilon(e_b^n)\right] = -h^\vee \epsilon(e_b^n) + \sum_a \epsilon(e_a^n)\,\mathrm{ad}([e_b, e_a]).$$

Hence

$$[D^2 - \frac{1}{2}\sum_a \mathrm{ad}(e_a)^2, \epsilon(e_b^n)] = h^\vee(n+2)\epsilon(e_b^n).$$

For any $\phi \in \bigwedge^* \mathfrak{g}_+$, we have

$$\begin{aligned}
&(D^2 - \frac{1}{2}\sum_a \mathrm{ad}(e_a)^2)(e_b^n \wedge \phi) \\
=&[D^2 - \frac{1}{2}\sum_a \mathrm{ad}(e_a)^2, \epsilon(e_b^n)]\phi + \epsilon(e_b^n)(D^2 - \frac{1}{2}\sum_a \mathrm{ad}(e_a)^2)\phi \\
=&h^\vee(n+2)(e_b^n \wedge \phi) + h^\vee(\mathrm{E}(\phi) + 2\deg(\phi))(e_b^n \wedge \phi) \\
=&h^\vee(\mathrm{E}(e_b^n \wedge \phi) + 2\deg(e_b^n \wedge \phi))(e_b^n \wedge \phi).
\end{aligned}$$

Therefore,

$$D^2 - 1/2\sum_a \mathrm{ad}^2(e_a) = h^\vee \cdot (\mathrm{E} + 2 \cdot \deg).$$

The proof is complete. □

Example 4.1.19. For a 1-form $\phi = e_b^n$, we have

$$\begin{aligned}
D^2_{\mathrm{vert}}(e_a^n) &= h^\vee(n+2)e_b^n + 1/2\sum_a [e_a, [e_a, e_b^n]] \\
&= h^\vee(n+2)e_b^n - h^\vee e_b^n, \\
&= h^\vee \cdot (\mathrm{E} + 1)(e_b^n).
\end{aligned}$$

Recall that by definition an essentially selfadjoint unbounded operator D has compact resolvent if $f(D)$ is compact for every $f \in C_0(\mathbb{R})$.

Theorem 4.1.20. *The unbounded operator $D_{\mathrm{vert}} = \bar{\partial}^* + \bar{\partial}$ is essentially self-adjoint, has compact resolvent and is Fredholm with index given by $\mathbb{C}$.*

Proof. From Theorem 4.1.18, $D^2 = h^\vee \cdot (\mathrm{E} + 2\deg) - \mathrm{Cas}_{\mathrm{ad}}$, hence on an irreducible subspace V_λ with the highest weight λ, we have

$$\begin{aligned} D^2 &= h^\vee \cdot (\mathrm{E} + 2\deg) + 1/2\|\rho\|^2 - 1/2\|\lambda + \rho\|^2 \\ &= h^\vee \cdot (\mathrm{E} + 2\deg) - 1/2\|\lambda\|^2 - B(\lambda, \rho). \end{aligned}$$

Suppose $V_\lambda \subset (\bigwedge^{q_1} \mathfrak{g}_{\mathbb{C}} z^{l_1}) \otimes (\bigwedge^{q_2} \mathfrak{g}_{\mathbb{C}} z^{l_2}) \otimes ... \otimes (\bigwedge^{q_k} \mathfrak{g}_{\mathbb{C}} z^{l_k})$ is an irreducible subspace, where $l_1, ..., l_k$ are distinct nonnegative intergers and $q_1, ..., q_k \leq \dim \mathfrak{g}_{\mathbb{C}}$ are positive integers. Let λ be the highest weight given by

$$\lambda = \alpha_{n_1^1} + ... + \alpha_{n_{q_1}^1} + ... + \alpha_{n_1^k} + ... + \alpha_{n_{q_k}^k},$$

where $\alpha_{n_1^i}, ..., \alpha_{n_{q_i}^i}, i = 1, 2, ..., k$ are the weights on $\bigwedge^{q_i} \mathfrak{g}_{\mathbb{C}} z^{l_i}$. Then we have $\sum_{i=1}^k q_i l_i \leq \mathrm{E}$ and $\sum_{i=1}^k q_i \leq \deg$.

Next, consider $-1/2\|\lambda\|^2 - B(\lambda, \rho)$. Using the inequality $\|v + w\|^2 \leq 2\|v\|^2 + 2\|w\|^2$ and the identity $\|\alpha_{max}\|^2 = 2$, we obtain

$$1/2\|\lambda\|^2 \leq \|\alpha_{n_1^1}\|^2 + ... + \|\alpha_{n_{q_1}^1}\|^2 + ... + \|\alpha_{n_1^k}\|^2 + ... + \|\alpha_{n_{q_k}^k}\|^2 \leq 2\deg.$$

From Proposition 4.1.2, we have $B(\alpha_{max}, \rho) = h^\vee - 1$, so

$$B(\lambda, \rho) \leq \deg \cdot B(\alpha_{max}, \rho) = \deg \cdot (h^\vee - 1).$$

Thus

$$D^2 \geq h^\vee \cdot (\mathrm{E} + 2\deg) - 2\deg - \deg \cdot (h^\vee - 1) = h^\vee \cdot \mathrm{E} + \deg \cdot (h^\vee - 1),$$

which implies D^2 vanishes only on $\mathbb{C}$, the zero energy space of zero degree.

$\square$

4.1.3 Semi-infinite forms

From last chapter, the square of the vertical operator D_{vert} is roughly the energy operator on the extended spinor module $\mathcal{S} = \bigwedge(\mathfrak{g}_{\mathbb{C}}[z])$.

Lemma 4.1.21. *The vertical operator D_{vert} commutes with the action G on the spinor module $\mathcal{S}$.*

Proof. Let $e_b \in \mathfrak{g}$. Put $\bar{\partial} = \frac{1}{2}\sum_{m\geq 0}\bar{\partial}_m$, with

$$\bar{\partial}_m = \sum_a \epsilon(e_a^m)\,\mathrm{ad}(e_a^{-m}).$$

Then for each $m \geq 0$, we can check that

$$\begin{aligned}
[\bar{\partial}_m, \rho(e_b)] =& \sum_a [\epsilon(e_a^m), \rho(e_b)]\,\mathrm{ad}(e_a^{-m}) + \epsilon(e_a^m)[\mathrm{ad}(e_a^{-m}), \rho(e_b)] \\
=& \sum_a \epsilon([e_a, e_b]^m)\,\mathrm{ad}(e_a^{-m}) + \epsilon(e_a^m)\,\mathrm{ad}([e_a, e_b]^{-m}) \\
=& \sum_a \epsilon(e_a^m)\,\mathrm{ad}([e_b, e_a]^{-m}) + \epsilon(e_a^m)\,\mathrm{ad}([e_a, e_b]^{-m}) \\
=& 0.
\end{aligned}$$

Hence $[\bar{\partial}, \rho(e_b)] = 0$. Taking the adjoint yields $[\bar{\partial}^*, \rho(e_b)] = 0$. This completes the proof. $\square$

Definition 4.1.22. The space of semi-infinite forms $\bigwedge^{\infty - p, q} = \bigwedge^q(\mathfrak{g}_{\mathbb{C}}[z]) \otimes \bigwedge^{\infty - p}\mathfrak{g}_-$ is spanned by the monomials of the form

$$\psi = e_{a_1}^{n_1} \wedge ... \wedge e_{a_q}^{n_q} \wedge e_{b_1}^{m_1} \wedge e_{b_2}^{m_2} \wedge ... \wedge e_{b_k}^{m_k} \wedge ...,$$

where $n_1, ..., n_q \geq 0$, $m_1, m_2, ... < 0$, and the set $\{e_{b_k}^{m_k}\}$ consists of all elements in the basis of $\mathfrak{g}_-$ except p elements, and when k is sufficiently large, m_k is nonincreasing.

In this chapter, we shall extend the vertical operator D_{vert} to the space of semi-infinite forms.

Consider the Clifford algebra $\mathbb{C}\text{liff}(L\mathfrak{g} \oplus L\mathfrak{g}^*)$ generated by the relations

$$[e_a^m, e_b^n] = 0, \quad [(e_a^m)^*, (e_b^n)^*] = 0,$$

$$[(e_a^m)^*, e_b^n] = \langle (e_a^m)^*, e_b^n \rangle = \delta_{a,b}\delta_{m,n}.$$

The Clifford algebra $\mathbb{C}\text{liff}(L\mathfrak{g} \oplus L\mathfrak{g}^*)$ acts on $\bigwedge^{\infty-p,q}$ by the formulas,

$$\epsilon(e_c^l)\psi = e_c^l \wedge e_{a_1}^{n_1} \wedge ... \wedge e_{a_q}^{n_q} \wedge e_{b_1}^{m_1} \wedge e_{b_2}^{m_2} \wedge ... \wedge e_{b_k}^{m_k} \wedge ..., \quad l \in \mathbb{Z},$$

and

$$\begin{aligned}\iota((e_c^l)^*)\psi = &\sum_{k=1}^{q} (-1)^{k-1} \langle (e_c^l)^*, e_{a_k}^{n_k} \rangle e_{a_1}^{n_1} \wedge ... \wedge \hat{e}_{a_k}^{n_k} \wedge ... \wedge e_{a_q}^{n_q} \wedge e_{b_1}^{m_1} \wedge e_{b_2}^{m_2} \wedge ... \wedge e_{b_k}^{m_k} \wedge ... \\ &+ \sum_{k=1}^{\infty} (-1)^{q+k-1} \langle (e_c^l)^*, e_{b_k}^{m_k} \rangle e_{a_1}^{n_1} \wedge ... \wedge e_{a_q}^{n_q} \wedge e_{b_1}^{m_1} \wedge e_{b_2}^{m_2} \wedge ... \wedge \hat{e}_{b_k}^{m_k} \wedge ... \wedge e_{b_k}^{m_k} \wedge ...\end{aligned}$$

Lemma 4.1.23. *The following identities holds,*

$$[\epsilon(Xz^m), \epsilon(Yz^n)] = 0, \quad [\iota((Xz^n)^*), \iota((Yz^n)^*)] = 0,$$

$$[\iota(Xz^m), \epsilon((Yz^n)^*)] = \langle Xz^m, (Yz^n)^* \rangle.$$

for $Xz^m, Yz^n \in L\mathfrak{g}, (Xz^n)^*, (Yz^n)^* \in L\mathfrak{g}^*$.

Proof. The proof is obtained by direct computation. □

For our convenience, we choose a vacuum vector

$$|0\rangle = e_{c_1}^{-1} \wedge e_{c_2}^{-1} \wedge ... \wedge e_{c_N}^{-1} \wedge e_{c_1}^{-2} \wedge e_{c_2}^{-2} \wedge ... \wedge e_{b_N}^{-2} \wedge ...,$$

where $\{e_{c_k}\}_{k=1}^N$ is an orthonormal basis of $\mathfrak{g}$ and the order is fixed. There is a projective action of LG on the space of semi-infinite forms. First, if $Xz^l \in \mathfrak{g}_{\mathbb{C}} z^l$ for $l \neq 0$, then

$$\rho(Xz^l)\psi = \sum_{k=1}^{q} e_{a_1}^{n_1} \wedge \ldots \wedge \operatorname{ad}_{Xz^l} e_{a_k}^{n_k} \wedge \ldots \wedge e_{a_q}^{n_q} \wedge e_{b_1}^{m_1} \wedge e_{b_2}^{m_2} \wedge \ldots \wedge e_{b_k}^{m_k} \wedge \ldots$$
$$+ \sum_{k=1}^{\infty} e_{a_1}^{n_1} \wedge \ldots \wedge e_{a_q}^{n_q} \wedge e_{b_1}^{m_1} \wedge e_{b_2}^{m_2} \wedge \ldots \wedge \operatorname{ad}_{Xz^l} e_{b_k}^{m_k} \wedge \ldots \wedge e_{b_k}^{m_k} \wedge \ldots$$

Note that $\rho(Xz^l)$ is well-defined if Xz^l for $l \neq 0$. Note also that $\rho(Xz^l)\,|0\rangle = 0$ for $l \leq 0, X \in \mathfrak{g}_{\mathbb{C}}$.

Lemma 4.1.24.

$$[\rho(Xz^l), \iota((Yz^k)^*)] = \iota(\operatorname{ad}_{Xz^l}(Yz^k)^*),$$

$$[\rho(Xz^l), \epsilon(Yz^k)] = \epsilon(\operatorname{ad}_{Xz^l} Yz^k),$$

for $Xz^l \in \mathfrak{g}z^l, l \neq 0$, *and* $(Yz^k)^* \in (\mathfrak{g}z^k)^*$.

Proof. The proof can be obtained by direct calculation. See [22]. □

Next, the action of $X \in \mathfrak{g}_{\mathbb{C}}$ on $|0\rangle$ is defined by

$$\rho(X)(|0\rangle) = 0,$$

and on a general element ψ it is obtained from the formulas in Lemma 4.1.24.

Example 4.1.25. Let $e_c^l \wedge |0\rangle = \epsilon(e_c^l)\,|0\rangle$, then $\rho(X)(e_c^l \wedge |0\rangle) = \rho(X)\epsilon(e_c^l)\,|0\rangle = \epsilon(e_c^l)\rho(X)\,|0\rangle + \epsilon(\operatorname{ad}_X(e_c^l))\,|0\rangle = \operatorname{ad}_X(e_c^l) \wedge |0\rangle$.

We shall focus on the space

$$\mathcal{S}^{\infty,q} = \bigwedge^{q}(\mathfrak{g}_{\mathbb{C}}[z]) \otimes \bigwedge^{\infty} \mathfrak{g}_{-}.$$

With this notation, $\mathcal{S}^{\infty,q}$ can be regarded as $\mathcal{S}^{\infty,q} = \bigwedge^q(\mathfrak{g}_{\mathbb{C}}[z]) \otimes \mathbb{C} \cdot |0\rangle$. Finally, let

$$\mathcal{S}^{\infty} = \oplus_{q\geq 0}\mathcal{S}^{\infty,q}.$$

It is viewed as the twisted spinor module obtained by twisting $\mathcal{S}$ by the line $\mathbb{C} \cdot |0\rangle$.

Proposition 4.1.26. *The action of LG on $\mathcal{S}^{\infty}$ is projective, with level $2h^{\vee}$.*

Proof. We only need to show that for any $H \in \mathfrak{t}_{\mathbb{C}}$,

$$[\rho(Hz^n), \rho(Hz^{-n})] = 2h^{\vee} \cdot n \cdot B(H, H).$$

One can check that by direct calculation,

$$[\rho(Hz^n), \rho(Hz^{-n})] |0\rangle = \rho(Hz^{-n})\rho(Hz^n) |0\rangle = n \sum_{\alpha\in R} \alpha^2(H) |0\rangle .$$

Further computation shows

$$\begin{aligned}
\sum_{\alpha\in R} \alpha^2(H) &= \sum_{\alpha\in R} B([H, [H, E_\alpha], E_\alpha^*) \\
&= - \sum_{\alpha\in R} B([H, E_\alpha], [H, E_\alpha^*]) \\
&= \sum_{\alpha\in R} B([E_\alpha^*, [E_\alpha, H], H) \\
&= 2h^{\vee} B(H, H).
\end{aligned}$$

Hence the proof is complete. ∎

Remark 4.1.27. In [37] Landweber constructed a spinor representation for LG with projective level given by $h^{\vee}$. Hence semi-infinite forms may be regarded as the square of the spinor representation.

In addition, the Dolbeault operator $\bar{\partial}$ extends to $\mathcal{S}^\infty$ naturally by the formula,

$$\bar{\partial}(\phi \otimes |0\rangle) = \bar{\partial}\phi \otimes |0\rangle \,.$$

The fomal adjoint is given by

$$\bar{\partial}^*(\phi \otimes |0\rangle) = \bar{\partial}^*\phi \otimes |0\rangle \,,$$

in which we identify the dual of $\mathbb{C} \cdot |0\rangle$ with itself.

Proposition 4.1.28. *The Dirac operator D^2_{vert} satisfies the identity*

$$[D^2_{\mathrm{vert}}, \rho(\xi)] = h^\vee \rho(\xi').$$

with the action of $\tilde{L}G$ on the twisted spinor module $\mathcal{S}^\infty$.

Proof. Since $\rho(\xi)$ preserves degree and it commutes with $\mathrm{Cas}_{\mathrm{ad}}$, we have for $\xi = e_a^n \in L\mathfrak{g}_{\mathbb{C}}$,

$$\begin{aligned}
[D^2_{\mathrm{vert}}, \rho(\xi)]\psi =& h^\vee([\mathrm{E}, \rho(\xi)] + 2[\deg, \rho(\xi)] - [\mathrm{Cas}_{\mathrm{ad}}, \rho(\xi)])\psi \\
=& h^\vee[\mathrm{E}, \rho(\xi)]\psi \\
=& h^\vee(\mathrm{E}(\rho(\xi)\psi) - \mathrm{E}(\psi))\rho(\xi)\psi \\
=& h^\vee \rho(\xi')\psi.
\end{aligned}$$

□

Remark 4.1.29. In [23], D. Freed, M. Hopkins and C. Teleman introduced a family of Dirac operators $\{\not{D}_\mu\}$ parametrized by $L\mathfrak{g}^*$. Interestingly, for fixed $\mu \in L\mathfrak{g}^*$, each operator $\not{D}_\mu$ is not $\tilde{\Omega}G$-equivariant, but the map $\mu \mapsto \not{D}_\mu$ is $\tilde{\Omega}G$-equivariant.

4.1.4 The Kasparov's Dirac element

In this section, we review the construction of Kasparov's Dirac element on a finite dimensional Riemannian manifold, and then introduce Kasparov's Dirac element on an affine coadjoint orbit.

Suppose N is a G-Riemannian manifold, and ∇^c is a G-Clifford connection on the bundle $\mathrm{Cliff}(TN)$ and $\{e_i\}$ a local orthonormal frame, see [39]. The Kasparov's Dirac element on a Riemannian manifold N is given by the triple $(\rho_N, \not{D}_N, \mathcal{H}_N)$, where

(1) $\mathcal{H}_N = \overline{C(N, \mathbb{C}\mathrm{liff}(TN))}$ is a $\mathbb{Z}/2$ graded Hilbert space,

(2) $\rho_N : C(N, \mathbb{C}\mathrm{liff}(TN)) \to \mathbb{B}(\mathcal{H}_N)$ is a graded $*$-homomorphism given by Clifford multiplication from the left, and

(3) $\not{D}_N$ is a Dirac operator, locally given by

$$\not{D}_N(s) = \sum (-1)^{\deg(s)} \nabla^c_{e_i} s \cdot e_i, \quad s \in C(N, \mathbb{C}\mathrm{liff}(TN)).$$

Alternatively, we may take $\mathcal{H}_N = \overline{\Omega^*(N)}$ and $\not{D}_N = d + d^*$ the Hodge de-Rham operator (see Section 4 in [34]).

For notational convenience, the vertical Dirac operator defined in previous section is denoted by D_{vert}, and the spinor module $\bigwedge \mathfrak{g}_{\mathbb{C}}[z]$ is denoted by $\mathcal{S}_{\Omega\mathfrak{g}}$. With D_{vert} and $\not{D}_N$, we will be able to define a fundamental class $(\rho_{Kas}, \not{D}_{Kas}, \mathcal{H}_{Kas})$ in the K-homology group

$$KK_G(C(N, \mathbb{C}\mathrm{liff}(TN) \hat{\otimes} \mathbb{K}(\mathcal{S}^*_{\Omega\mathfrak{g}})), \mathbb{C}).$$

First the graded Hilbert space $\mathcal{H}_{Kas}$ is the completed Hilbert space

$$\mathcal{H}_{Kas} = \overline{C(N, \mathbb{C}\mathrm{liff}(TN) \hat{\otimes} \mathcal{S}^*_{\Omega\mathfrak{g}} \hat{\otimes} \mathcal{S}_{\Omega\mathfrak{g}})},$$

in which $\mathcal{S}^*_{\Omega\mathfrak{g}} \hat{\otimes} \mathcal{S}_{\Omega\mathfrak{g}}$ is interpreted as the graded Hilbert space of Hilbert-Schmidt operators on $\mathcal{S}_{\Omega\mathfrak{g}}$.

Next the graded homomorphism

$$\rho_{Kas} : C(N, \mathbb{C}\mathrm{liff}(TN)\hat{\otimes}\mathbb{K}(\mathcal{S}^*_{\Omega\mathfrak{g}})) \to \mathbb{B}(\mathcal{H}_{Kas})$$

is given by the multiplication of operator-valued sections and $\mathbb{K}(\mathcal{S}^*_{\Omega\mathfrak{g}})$ acts on the factor $\mathcal{S}^*_{\Omega\mathfrak{g}}$. Note that $\mathbb{K}(\mathcal{S}^*_{\Omega\mathfrak{g}})$ is a graded algebra.

Finally, let $\not{D}_{Kas}$ be the differential operator on the space

$$C(N, \mathbb{C}\mathrm{liff}(TN)\hat{\otimes}\mathbb{K}(\mathcal{S}^*_{\Omega\mathfrak{g}}))\hat{\otimes}\mathcal{S}_{\Omega\mathfrak{g}}$$

given by

$$\not{D}_{Kas} = \not{D}_N \hat{\otimes}\,\mathrm{Id}\,\hat{\otimes}\,\mathrm{Id} + \mathrm{Id}\,\hat{\otimes}\,\mathrm{Id}\,\hat{\otimes} D_{\mathrm{vert}},$$

where D_{vert} acts on $\mathcal{S}_{\Omega\mathfrak{g}}$ and $\not{D}_N$ acts on $C(N, \mathbb{C}\mathrm{liff}(TN))$.

Lemma 4.1.30. *The operator $\not{D}_{Kas}$ is an essentially selfadjoint unbounded operator of degree one on the Hilbert space $\mathcal{H}_{Kas}$.*

Proof. by Lemma 2.2.27, $\not{D}_{Kas}$ is essentially selfadjoint. □

Proposition 4.1.31. *The operator*

$$\rho_{Kas}(a)(1 + \not{D}^2_{Kas})^{-1}$$

is a compact operator, and the commutator

$$[\rho_{Kas}(a), \not{D}_{Kas}]$$

is bounded. Hence $(\mathcal{H}_{Kas}, \rho_{Kas}, \not{D}_{Kas})$ defines an unbounded Kasparov KK-cycle.

Proof. For the first part. Note that for an essentially self-adjoint operator D, $(1+D^2)^{-1}$ is compact if and only if e^{-D^2} is compact. From last section, the operator

$e^{-D^2_{\mathrm{vert}}}$ is compact. From Chapter 1, Section 2.2.4, we know that $(1+D_N^2)^{-1}$ is compact, hence $e^{-D_N^2}$ is compact. It shows that $e^{-D_N^2}\hat{\otimes}e^{-D^2_{\mathrm{vert}}}$ is a compact operator, hence $\rho_{Kas}(a)(1+\not{D}^2_{Kas})^{-1}$ is compact.

For the second part. Let $a \in \Gamma(TN), T \in \mathbb{K}(\mathcal{S}^*_{\Omega\mathfrak{g}})$, the graded commutator

$$[\rho_{Kas}(a\hat{\otimes}T), \not{D}_{Kas}] = [\rho_N(a), \not{D}_N]\hat{\otimes}T\hat{\otimes}\,\mathrm{Id}$$

is a bounded operator from Section 2.2.4 of Chapter 1. □

Theorem 4.1.32. *Let* $F_{Kas} = D_{Kas}(1+D^2_{Kas})^{-1/2}$*, then*

$$[F_{Kas}] \in KK_G(C(N, \mathbb{C}\mathrm{liff}(TN)\hat{\otimes}\mathbb{K}(\mathcal{S}^*_{\Omega\mathfrak{g}})), \mathbb{C}).$$

Proof. Since $(\mathcal{H}_{Kas}, \rho_{Kas}, D_{Kas})$ is an unbounded KK-cycle, we obtain a KK-class in $KK_G(C(N, \mathbb{C}\mathrm{liff}(TN)), \mathbb{C})$ by the techniques from Chapter 1, □

Definition 4.1.33. The class $[F_{Kas}]$ is called the Kasparov's Dirac element of the principal ΩG-manifold M, and is denoted by $[M]$.

4.2 Pre-quantization

4.2.1 Pre-quantum line bundles

Let (M, ω, Φ) be a Hamiltonian LG space at level $k \geq 0$ with proper moment map $\Phi : M \to L\mathfrak{g}^*$.

Definition 4.2.1. A pre-quantum line bundle $\mathcal{L}$ on (M, ω, Φ) is an $\widetilde{L}G$-equivariant Hermitian line bundle with an invariant connection ∇ satisfying the condition

$$\omega = \frac{\sqrt{-1}}{2\pi}\nabla^2,$$

and the Kostant's formula,

$$L_{\xi_M} s = \nabla_{\xi_M} s - 2\pi\sqrt{-1}\Phi(\xi)s, \quad s \in \Gamma(\mathcal{L}).$$

The central circle of $\widetilde{L}G$ acts on the fibers of $\mathcal{L}$ with the weight $k \in \mathbb{Z}$.

A twisting may be constructed from loop group central extension. The G-equivariant DD-bundles over G are classified by $H^3_G(G,\mathbb{Z}) \simeq \mathbb{Z}$. Let $\mathcal{H}_{LG}$ be a projective positive energy representation of LG at level 1. The projective cocycle defines a class in $H^2(\Omega G,\mathbb{Z})$, which is isomorphic to $H^3(G,\mathbb{Z})$ by transgression map in cohomology. See page 48 in [50].

Lemma 4.2.2. *Let $\mathcal{H}_{LG}$ be a projective positive energy representation of LG at level* 1*. The associated Dixmier-Douady bundle*

$$\mathcal{A}_G = L\mathfrak{g}^* \times_{\Omega G} \mathbb{K}(\mathcal{H}_{LG})$$

is a generator in $H^3_G(G,\mathbb{Z})$.

Proof. With a slight abuse of notation, we use H for the Hilbert space $\mathcal{H}_{LG}$ temporarily in the proof. Since $L\mathfrak{g}^*$ is a principal ΩG-bundle and $L\mathfrak{g}^*$ is contractible, G is a classifying space of ΩG, i.e., $B\Omega G \simeq G$. Let $K(\mathbb{Z},n)$ the Eilenberg-MacLane space, and $PU(H)$ be the projective unitary group with strong operator topology. Then $\mathrm{Aut}(\mathbb{K}(H)) \simeq PU(H) \simeq K(\mathbb{Z},2)$. The continuous group homomorphism $\Omega G \to \mathrm{Aut}(\mathbb{K}(H))$ yields a continuous map

$$B\Omega G \to B\,\mathrm{Aut}(\mathbb{K}(H)),$$

i.e., $f : G \to BPU(H) \simeq K(\mathbb{Z},3)$.

Recall that $H^3(G;\mathbb{Z}) \simeq [G;K(Z,3)]$. The invariant of $\mathcal{A}_G$ in $H^3(G,\mathbb{Z})$ is given by the homotopy class of f. Hence the transgression $H^3(G,\mathbb{Z}) \to H^2(\Omega G,\mathbb{Z})$ takes the invariant of $\mathcal{A}_G$ to the class of basic central extension $\tilde{\Omega}G$. See also [6]. $\square$

Example 4.2.3. Suppose $G = SU(2)$ with two covers $U = SU(2) - \{-I\}$, and $V = SU(2) - \{I\}$. $L\mathfrak{g}^*$ is identified with $L\mathfrak{g}$ by basic inner product. Let $\log : U \to \mathfrak{g}$ the inverse of exponential map. Note that if $g \in V$ then $-g \in U$. For a regular element $g \in U \cap V$, let $\sigma_g(t) = \exp(tX_\rho)$ where $e^{X_\rho} = -I$ and $X_\rho = aX$ for some positive scalar a. The principal ΩG-bundle $L\mathfrak{g}$ on U can be trivialized as

$$L\mathfrak{g}|_U \to U \times \Omega G, \quad \xi \to (\mathrm{Hol}(\xi), \phi_U(\xi)),$$

where $\phi_U(\xi) = \gamma$ is defined by $\gamma \cdot \log(Hol(\xi)) = \xi$. Similarly, the principal ΩG-bundle $L\mathfrak{g}$ on V can be trivialized as

$$L\mathfrak{g}|_V \to V \times \Omega G, \quad \xi \to (\mathrm{Hol}(\xi), \phi_V(\xi)),$$

where $\phi_V(\xi) = \gamma$ is defined by $\gamma \cdot (\log(-\mathrm{Hol}(\xi))) = \sigma \cdot \xi$. Thus we obtain a transition map

$$\phi_{UV} : U \cap V \to \Omega G, \quad g \mapsto -\sigma_g(t).$$

After choosing the standard maximal torus $T \subset SU(2)$, we have a G-equivariant diffeomorphism $U \cap V \to G/T \times T_{(0,\rho)}$. The transition map is then

$$\phi : G/T \times T_{(0,\rho)} \to \Omega G, \quad (gT, \tau) \mapsto -\exp(t\,\mathrm{ad}_g\,\rho).$$

Hence we obtain a G-equivariant continuous map $\phi : G/T \to \Omega G$. Now consider the map

$$\begin{array}{ccccc} P & \longrightarrow & \tilde{\Omega}G & \longrightarrow & U(H) \\ \downarrow & & \downarrow & & \downarrow \\ G/T & \longrightarrow & \Omega G & \longrightarrow & PU(H) \end{array}$$

It is clear that the pull-back P of $\tilde{\Omega}G$ on G/T is a principal $U(1)$ bundle such that class $[P] \in H^1_G(G/T, U(1)) = H^2_G(G/T, \mathbb{Z}) = \mathbb{Z}$ corresponds to the projective cocycle in $\tilde{\Omega} G$.

Remark 4.2.4. A concrete construction of DD-bundles over G can be found in [44] by Meinrenken.

Let $\mathcal{H}^{-k}_{LG}$ be a negative energy representation of LG of level $-k$, so that the dual is positive energy representation of level k. Let

$$\mathcal{A}^{-k}_G = L\mathfrak{g}^* \times_{\Omega G} \mathbb{K}(\mathcal{H}^{-k}_{LG}) \tag{4.2}$$

be the G-equivariant Dixmier-Douaday bundle over G. Its cohomology class in $H^3(G)$ depends on the level only and is mapped to the number $-k$ under the identification $H^3(G) \simeq \mathbb{Z}$.

The bundle $\mathcal{L} \otimes \mathcal{H}^{-k}_{LG}$ is an ΩG-vector bundle. Since the action of ΩG is free, the quotient

$$\mathcal{E}_{Line} = (\mathcal{L} \otimes \mathcal{H}^{-k}_{LG})/\Omega G$$

is a bundle over N. It gives rise to Morita morphism

$$(\Psi, \mathcal{E}_{Line}) : (N, \underline{\mathbb{C}}) \to (G, \mathcal{A}^{-k}_G).$$

Example 4.2.5. Let Λ^*_k be the set of level k weights. The coadjoint orbit $\mathcal{O}_\mu = LG \cdot \mu$ is pre-quantizable if and only if $\mu \in \Lambda^*_k$. The pre-quantum line bunde is the associated line bundle

$$\mathcal{L} = \widetilde{L}G \times_{\widetilde{L}G_\mu} \mathbb{C}_{\mu,k}.$$

4.2.2 Twisted Spin^c structures

Recall that the Riemannian manifold N has a twisted Spin^c structure, i.e., there exists a Morita module $\mathcal{E}_{Spin}$ such that $\mathbb{C}\mathrm{liff}(TN) \simeq_{\mathcal{E}_{Spin}} \Phi^* \mathcal{A}^{h^\vee}_G$.

The construction of the Morita modules involves Dirac structure and Dirac morphism. Bursztyn and Crainic in [16] observed that the conditions $\iota_{\xi_M}\omega_M = d\int_{S^1}\Phi(\xi)$, and $\mathrm{Ker}(\omega_M) = 0$ give rise to a strong Dirac morphism

$$(d\Psi, \omega_N) : (\mathbb{T}N, TN) \to (\mathbb{T}G, E_G),$$

where $\mathbb{T}N = TN \oplus T^*N$ and E_G is the Cartan-Dirac structure over G.

A. Alekseev and E. Meinrenken in [1] constructed a Dirac-Dixmier-Douady functor, which associates to any Dirac structure a Dixmier-Douady bundle and to any strong Dirac morphism of Dirac structures a Morita morphism.

Theorem 4.2.6. *Let G be a compact, connected Lie group. For any q-Hamiltonian G-space (N, ω_N, Ψ), there exists a distinguished 2-isomorphism class of G-equivariant Morita morphisms*

$$(\Psi, \mathcal{E}_{Spin}) : (N, \mathbb{C}\mathrm{liff}(TN)) \to (G, \mathcal{A}_G^{-h^\vee}).$$

Proof. See Section 5 in [1]. □

Remark 4.2.7. When $G = 1$ so that (N, ω_N) is a symplectic manifold, then twised spin^c structure is the standard spin^c coming from the symplectic structure ω_N.

Let $\mathrm{Hol} : M \to N$ be the holonomy map, defined in Section 4.3 of Chapter 3. Now suppose M is just an affine coadjoint orbit $M = \mathcal{O}_\mu = LG \cdot \mu$, then the holonomy manifold $\mathrm{Hol}(M)$ is a conjugacy class $N = \mathcal{C}_{\mathrm{Hol}(\mu)} \subset G$.

Furthermore, Since coadjoint orbits are classified by the closed fundamental Weyl chamber, we can assume $\mu \in \Lambda$, i.e., a constant loop in $\mathfrak{g}^*$. In particular, $\mathrm{Hol}(\mu) = \exp(\mu)$, where we use the basic inner product B to identity $\mathfrak{g}$ and $\mathfrak{g}^*$.

Let $K \subset LG$ be the isotropy subgroup of μ and $H \subset G$ the isotropy subgroup of $\mathrm{Hol}(\mu)$. Then K and H are isomorphic [50],

$$K \to H, \quad \gamma \mapsto \gamma(0).$$

The inverse map is given by

$$H \to K, \quad g \mapsto \mathrm{Hol}_s^{-1} \cdot g \cdot \mathrm{Hol}_s,$$

where $\mathrm{Hol}_s : \mathbb{R} \to G$ is the function defined in Section 4.3 of Chapter 3. Since $\mathrm{Hol}_s = \exp(s\mu)$, we have the map

$$g \mapsto \exp(-s\mu) g \exp(s\mu),$$

which is in general not constant.

Lemma 4.2.8. *If μ is generic, i.e., it does not lies in wall of the fundamental Weyl alcove $\mathfrak{A}$, then the isotropy subgroup K consists of constant loops.*

Proof. If μ is generic, then $e^X g = g e^X$ implies $e^{tX} g = g e^{tX}$ for any nonzero $X \in \mathfrak{g}$ and any $g \in G$. □

Lemma 4.2.9. *The following diagram commutes*

$$\begin{array}{ccc} LG/K & \xrightarrow{\widetilde{\mathrm{Ad}}.(\mu)} & \mathcal{O}_\mu \\ {\scriptstyle \mathrm{Hol}}\downarrow & & \downarrow{\scriptstyle \mathrm{Hol}} \\ G/H & \xrightarrow{\mathrm{Ad}.(u)} & \mathcal{C}_{\mathrm{Hol}(\mu)} \end{array}$$

where the left vertical map is the evaluation $\mathrm{Hol} : [\gamma] \mapsto [\gamma(0)]$.

Proof. The proof follows from Equation (3.16) in Chapter 3. Then tangent bundle of N □

Since $N = \mathcal{C}_{\mathrm{Hol}(\mu)}$ is a homogeneous space, twisted Spin^c structure can be constructed explicitly.

Theorem 4.2.10. *For any conjugacy class $N = \mathcal{C}_{\mathrm{Hol}(\mu)}$, there is a distinguished G-equivariant Morita isomorphism*

$$(\Psi, \mathcal{E}_{Spin}) : (N, \mathbb{C}\mathrm{liff}(TN)) \to (G, \mathcal{A}_G^{-h^\vee}).$$

Proof. The Morita module $\mathcal{E}_{Spin}$ was constructed by E. Meinrenken in [43] . Since $N = G/H$, we have $TN = G \times_H \mathfrak{h}^\perp$ and $\mathbb{C}\mathrm{liff}(TN) = G \times_H \mathbb{C}\mathrm{liff}(\mathfrak{h}^\perp)$. On the other hand, we have $\Psi^*(\mathcal{A}_G^{-h^\vee}) = G \times_H \mathbb{K}(\mathcal{H}_{LG}^{-h^\vee})$ by the formula (4.2). Hence one only need to construct a H-equivariant Morita module

$$\mathbb{C}\mathrm{liff}(\mathfrak{h}^\perp) \simeq \mathbb{K}(\mathcal{H}_{LG}^{-h^\vee}).$$

One key observation is that the central extension $\tilde{H}$ of H defined by diagram

$$\begin{array}{ccc} \tilde{H} & \longrightarrow & \mathrm{Spin}^c(\mathfrak{h}^\perp) \\ \downarrow & & \downarrow \\ H & \longrightarrow & \mathrm{SO}(\mathfrak{h}^\perp) \end{array}$$

has the weight $-h^\vee$. Hence if we let $\mathcal{S}_\mathfrak{h}$ be a Spinor module over $\mathbb{C}\mathrm{liff}(\mathcal{S}_\mathfrak{h}^\perp)$, then

$$\mathcal{E}_{Spin} = \mathcal{S}_\mathfrak{h} \otimes \mathcal{H}_{LG}^{-h^\vee}.$$

□

Finally, we tensor the twisted Spin^c structure with the trivial Morita module

$$\mathcal{S}_{\Omega\mathfrak{g}}^* : (N, \mathbb{K}(\mathcal{S}_{\Omega\mathfrak{g}}^*)) \to (G, \underline{\mathbb{C}}),$$

and obtain

$$(\Psi, \mathcal{E}_{Spin} \hat{\otimes} \mathcal{S}_{\Omega\mathfrak{g}}^*) : (N, \mathbb{C}\mathrm{liff}(TN) \hat{\otimes} \mathbb{K}(\mathcal{S}_{\Omega\mathfrak{g}}^*)) \to (G, \mathcal{A}_G^{-h^\vee}).$$

4.3 Quantization

4.3.1 Verlinde ring

The set $R^k(LG)$ of positive energy projective representations of LG at a fixed level k has an obvious group structure by taking the direct sum. However, unlike the representation ring $R(G)$ of G, the usual tensor product will not provide a ring structure for $R^k(LG)$ at the fixed level. E. Verlinde in [58] discovered the fusion product under which $R^k(LG)$ becomes a ring. This is called the Verlinde ring. See also [10].

Alternatively, the Verlinde ring $R^k(LG)$ may be described algebraically(see Appendix D in [44]). Let $\Lambda = \mathrm{Hom}(\mathrm{U}(1), T) \subset \mathfrak{t}$ be the integral lattice and $\Lambda^* = \mathrm{Hom}(T, \mathrm{U}(1))$ be the (real) weight lattice. The level k fusion ideal $I_k(G) \subset R(G)$ is the ideal of characters vanishing at all points

$$t_\lambda = \exp\left(\frac{\lambda+\rho}{h^\vee + k}\right).$$

The level k fusion ring is the quotient ring

$$R_k(G) = R(G)/I_k(G).$$

Freed-Hopkin-Telemann Theorem [23] shows that there is a remarkable relation between twisted K-homology and Verlinde ring.

Theorem 4.3.1. *The equivariant twisted K-homology group is isomorphic to the Verlinde ring,*

$$KK_G(C(G, \mathcal{A}_G^{-h^\vee - k})) \simeq R_k(G).$$

Proof. See for example [23], or [43]. □

The quotient map $q : R(G) \to R_k(G)$ can be understood from K-homology. By taking the identity $\mathrm{pt} = e \in G$ and restricting the Dixmier-Douady bundle $\mathcal{A}_G$ on it, we obtain the quotient homomorphism

$$q : KK_G(C(\mathrm{pt}, \mathcal{A}_G^{h^\vee+k}|_{\mathrm{pt}}), \mathbb{C}) \to KK_G(C(G, \mathcal{A}_G^{h^\vee+k}), \mathbb{C}),$$

for the left hand side is $KK_G(C(\mathrm{pt}, \mathcal{A}_G^{h^\vee+k}|_{\mathrm{pt}}), \mathbb{C}) \simeq KK_G(C(\mathrm{pt}, \mathbb{K}|_{\mathrm{pt}}), \mathbb{C}) \simeq K_0^G(\mathrm{pt}) = R(G)$, and right hand side is just by FHT isomorphism.

4.3.2 Quantization of affine coadjoint orbits

By combining the twisted Spin^c structure and pre-quantum line bundle, we obtain a Morita morphism

$$(\Psi, \mathcal{E}_{Line}\hat{\otimes}\mathcal{E}_{Spin}\hat{\otimes}\mathcal{S}^*_{\Omega\mathfrak{g}}) : (N, \mathbb{C}\mathrm{liff}(TN)\hat{\otimes}\mathbb{K}(\mathcal{S}^*_{\Omega\mathfrak{g}})) \to (G, \mathcal{A}_G^{-h^\vee-k}),$$

where $\mathcal{A}_G^{-h^\vee-k} = L\mathfrak{g}^* \times_{\Omega G} \mathbb{K}(\mathcal{H}_{LG}^{-h^\vee-k})$.

Definition 4.3.2. The *quantization class* is defined as the KK-class of the tensor produdct of the Morita morphisms

$$[\mathcal{E}_{Line}\hat{\otimes}\mathcal{E}_{Spin}\hat{\otimes}\mathcal{S}^*_{\Omega\mathfrak{g}}] \in KK_G(C(G, \mathcal{A}_G^{-h^\vee-k}), C(N, \mathbb{C}\mathrm{liff}(TN))\hat{\otimes}\mathbb{K}(\mathcal{S}^*_{\Omega\mathfrak{g}})).$$

The *quantization of an affine coadjoint orbit M* is the Kasparov product of the fundamental class with quantization class,

$$\mathcal{Q}(M) = [\mathcal{E}_{Line}\hat{\otimes}\mathcal{E}_{Spin}\hat{\otimes}\mathcal{S}^*_{\Omega\mathfrak{g}}] \otimes [F_{Kas}] \in KK_G(C(G, \mathcal{A}_G^{-h^\vee-k}), \mathbb{C}).$$

Let Λ_k^* be the set of level k weights, i.e., weights $\mu \in \mathfrak{t}^*$ such that μ/k is in the fundamental Weyl alcove.

Theorem 4.3.3. *Let $\mu \in \Lambda_k^*$ and let $q : R(G) \to R_k(G)$ be the quotient map, then the quantization map $\mathcal{Q}$ sends the fundamental class to the corresponding generator* *in the Verlinde ring* $R_k(G)$,

$$\mathcal{Q}([\mathcal{O}_\mu]) = q(V_\mu).$$

Hence the quantization map $\mathcal{Q}$ *coincides with the one defined by Meinrenken in* [44].

Proof. We prove the theorem following the methods from [43]. First of all, by stability of KK-theory, we have the identification in K-homology

$$KK_G(A\hat{\otimes}\mathbb{K}, \mathbb{C}) \simeq KK_G(A, \mathbb{C}),$$

where A is any G-equivariant graded C* algebra, and $\mathbb{K}$ is the algebra of compact operators on a graded Hilbert space. See Section 5.2 in Chapter 2.

In particular, Kasparov's Dirac element

$$[F_{Kas}] \in KK_G(C(N, \mathbb{C}\mathrm{liff}(TN)\hat{\otimes}\mathbb{K}(\mathcal{S}^*_{\Omega\mathfrak{g}})), \mathbb{C})$$

corresponds to the element

$$[F^0_{Kas}] \in KK_G(C(N, \mathbb{C}\mathrm{liff}(TN)), \mathbb{C}),$$

where F^0_{Kas} is the bounded operator on the space $C(N, \mathbb{C}\mathrm{liff}(TN))\hat{\otimes}\mathcal{S}_{\Omega\mathfrak{g}}$ defined the formula,

$$\not{D}^0_{Kas} = \not{D}_N \hat{\otimes}\,\mathrm{Id} + \mathrm{Id}\,\hat{\otimes} D_{\mathrm{vert}}.$$

Furthermore, since the index of the vertical operator D_{vert} is just one, the K-homology class of $\not{D}^0_{Kas}$ is the same as Kasparov's Dirac element

$$[\not{D}_N] \in KK_G(C(N, \mathbb{C}\mathrm{liff}(TN)), \mathbb{C}).$$

From Lemma 4.2.9 the conjugacy N is isomorphic to G/H where H is the isotropy subgroup. Hence we identify N with G/H and consider the G-equivariant moment map,

$$\Psi : G/H \to G.$$

With this identification, the Morita module $\mathcal{E}_{Line} \otimes \mathcal{E}_{Spin}$ on G/H is then

$$\mathcal{E}_{Line} \otimes \mathcal{E}_{Spin} = G \times_H \mathbb{K}(\mathbb{C}_\mu \otimes \mathcal{S}_{\mathfrak{h}} \otimes \mathcal{H}_{LG}^{-h^\vee - k}).$$

Let $\psi : G/T \to G/H, gT \mapsto gH$ be the G-equivariant map. The pull-back of $\mathbb{C}\mathrm{liff}(TG/H)$ under ψ is Morita isomorphic to $\mathbb{C}$, with Morita module $G \times_T \mathcal{S}_{\mathfrak{h}}^*$. Let $\tilde{\Psi} = \Psi \circ \psi : G/T \to G$ be the composition, then $\tilde{\Psi}^*(\mathcal{A}_G^{-h^\vee - k})$ is Morita isomorphic to $\mathbb{C}$, with Morita module

$$\tilde{\mathcal{E}} = G \times_T (\mathbb{C}_\mu \otimes (\mathcal{S}_{\mathfrak{h}}^* \otimes_{\mathbb{C}\mathrm{liff}(\mathfrak{h}^\perp)} \mathcal{S}_{\mathfrak{h}}) \otimes \mathcal{H}_{LG}^{-h^\vee - k}) = G \times_T (\mathbb{C}_\mu \otimes \mathcal{H}_{LG}^{-h^\vee - k}).$$

On the other hand, $\tilde{\Psi}$ is G-equivariant homotopic to the constant map onto the identity element $e \in G$. Denote the constant map by $\hat{\Psi} : G/T \xrightarrow{\pi} e \xrightarrow{\iota} G$, viewed as the composition of the obvious maps π and ι. Then $\hat{\Psi}^*(\mathcal{A}_G^{-h^\vee - k})$ is Morita isomorphic to $\mathbb{C}$, with Morita module

$$\hat{\mathcal{E}} = G \times_T (\mathcal{H}_{LG}^{-h^\vee - k}).$$

Therefore, we have two Morita modules $\tilde{\mathcal{E}}$ and $\hat{\mathcal{E}}$ for $(G/T, \mathbb{C}) \to (G, \mathcal{A}_G^{-h^\vee - k}))$, and they differ by the line bundle $G \times_T \mathbb{C}_\mu$.

Now consider the following diagram

$$\begin{array}{ccccc}
KK_0(C(G/T), \mathbb{C}) & \xrightarrow{\psi_*} & KK_G(C(N, \mathbb{C}\mathrm{liff}(TN)), \mathbb{C}) & & \\
 & & & \searrow^{\Psi_*} & \\
\downarrow{\scriptstyle \otimes \mathbb{C}_\mu} & & & & KK_G(C(G, \mathcal{A}_G^{-h^\vee - k}), \mathbb{C}). \\
 & & & \nearrow_{\iota_*} & \\
K_0(C(G/T), \mathbb{C}) & \xrightarrow{\pi_*} & KK_G(\mathbb{C}, \mathbb{C}) & &
\end{array}$$

Note that ψ_* sends the fundamental class $[G/T] \in KK_0(C(G/T), \mathbb{C})$ to $[\not{D}]_N$, which is mapped to an element in twisted K-homology under the quantization map Ψ_*. On the other hand, $[G/T]$ is sent to $[G/T] \otimes \mathbb{C}_\mu$, an element twisted by the line bundle in the first vertical map. The push-forward π_* is just the index map, hence $[G/T] \otimes \mathbb{C}_\mu$ is mapped to $V_\mu \in R(G)$. As noted before, ι_* is the quotient map $R(G) \to R_k(G)$. Since diagram commutes, the element $[\not{D}]_N$ is sent to the generator $[V_\mu] \in R_k(G)$.

The proof is complete. $\square$

Chapter 5
Further directions

The geometric interpretation of the fundamental class $[M]$ is that it is similiar to the Kasparov's Dirac element over a finite dimensional Riemannian manifold. By choosing an ΩG-equivariant connection, the Clifford bundle is $\mathbb{C}\mathrm{liff}(TM) \simeq \mathbb{C}\mathrm{liff}(VM) \otimes \mathbb{C}\mathrm{liff}(HM)$. Taking the ΩG invariant sections and noting that $VM \simeq M \times \Omega\mathfrak{g}$, we have formally

$$C(M, \mathbb{C}\mathrm{liff}(TM))^{\Omega G} \simeq C(M, \mathbb{C}\mathrm{liff}(\Omega\mathfrak{g}))^{\Omega G} \otimes_{C(N)} \mathbb{C}\mathrm{liff}(TN),$$

in which we may identify $\mathbb{C}\mathrm{liff}(\Omega\mathfrak{g}) \simeq \mathcal{S}^{\infty *} \otimes \mathcal{S}^{\infty}$. We may also replace $\mathbb{C}\mathrm{liff}(\Omega\mathfrak{g})$ by $\mathbb{K}(\mathcal{S}^{\infty *})$, so that we obtain the DD bundle $\mathcal{A}_N^{-2h^\vee}$. Note that it is the pull-back of the DD bundle $\Phi^*(\mathcal{A}_G^{-2h^\vee})$. In addition, the twisted Spin^c structure implies there exists a Morita module $\mathcal{E}$ such that $\mathbb{C}\mathrm{liff}(TN) \simeq_{\mathcal{E}} \Phi^*(\mathcal{A}_G^{-h^\vee})$. Hence $\mathbb{C} \simeq_S \mathbb{C}\mathrm{liff}(TN)\hat{\otimes}\mathbb{C}\mathrm{liff}(TN) \simeq_{\mathcal{E}\otimes\mathcal{E}} \Phi^*(\mathcal{A}_G^{-2h^\vee})$, that is, $\mathcal{A}_N^{-2h^\vee}$ is Morita trivialized.

We introduce the vertical operator D_{vert} on the spinor module $\mathcal{S}_{\Omega\mathfrak{g}} = \bigwedge \mathfrak{g}_{\mathbb{C}}[z]$, and extend D_{vert} to the space of semi-infinite forms $\mathcal{S}^{\infty}$. Then one can define an alternative Hilbert space for the fundamental class $[M]$ as follows

$$\mathcal{H}'_{Kas} = \overline{C(M, \mathrm{Hol}^*(\mathbb{C}\mathrm{liff}(TN))\hat{\otimes}\mathcal{S}^{\infty}\hat{\otimes}\mathcal{H}_{LG}^{-2h^\vee})^{\Omega G}}$$

It is the space of sections of the bundle

$$(M \times_{\Omega G} (\mathcal{S}^\infty \otimes \mathcal{H}_{LG}^{-2h^\vee})) \otimes \mathrm{Cliff}(TN).$$

The alternative operator $\not{D}'_{Kas}$ would be

$$\not{D}'_{Kas} = \mathrm{Hol}^*(\not{D}_N) \hat{\otimes} \mathrm{Id} \hat{\otimes} \mathrm{Id} + \mathrm{Id} \otimes D_{\mathrm{vert}} \hat{\otimes} \mathrm{Id}\,.$$

However, the $\tilde{L}G$ action on $\mathcal{S}^\infty$ does not commute with the vertical operator $\not{D}_{\mathrm{vert}}$, which impies that $\not{D}'_{Kas}$ does not act on the $\mathcal{H}'_{Kas}$.

The noncommutativity of $\not{D}'_{Kas}$ and the $\tilde{L}G$ action cannot be resolved easily, for as long as $\not{D}'^2_{Kas}$ involves an energy operator, the noncommutativity occurs.

One potential solution is to follow Freed, Hopkins and Teleman's approach in [23]. Instead of focusing on one single operator, we may construct a family of differential operators parametrized by an appropriate space.

Bibliography

[1] A. Alekseev and E. Meinrenken. Dirac structures and Dixmier-Douady bundles. *Int. Math. Res. Not. IMRN*, (4):904-956, 2012.

[2] A. Alekseev, A. Malkin, and E. Meinrenken. Lie group valued moment maps. *J. Differential Geom.*, 48(3):445-495, 1998.

[3] A. Alekseev, E. Meinrenken, and C. Woodward. The Verlinde formulas as fixed point formulas. *J. Symplectic Geom.*, 1(1):1-46, 2001.

[4] M. F. Atiyah and F. Hirzebruch. Riemann-Roch theorems for differentiable manifolds. *Bulletin of the American Mathematical Society*, 65:276-281, 1959.

[5] M. F. Atiyah. *K-theory.* Benjamin, New York, 1967.

[6] M. F. Atiyah and G.B. Segal. Twisted K-theory. *Ukrainian Math. Bull.* 1, 2004.

[7] S. Baaj, and P. Julg. Théorie bivariante de Kasparov et opérateurs non bornés dans les C*-modules hilbertiens. *C.R. Acad. Sci. Paris Sér. I Math.* 296(21):875-878, 1983.

[8] P. Baum, A. Carey and B. Wang. K-cycles for twisted K-homology. *Journal of K-theory: K-theory and its Applications to Algebra, Geometry, and Topology*, 12, pp 69-98, 2013.

[9] N. Berline, E. Getzler and M. Vergne. *Heat kernels and Dirac operators*, Springer-Verlag, New York-Heidelberg-Berlin, 1992.

[10] A. Beauville. Conformal blocks, fusion rules and the Verlinde formula, 1994. Available as arXiv:alg-geom/9405001.

[11] B. Blackada. *K-theory for operator algebras*, volume 5 of *Mathematical Sciences Research Institute Publications.* Springer-Verlag, New York-Heidelberg-Berlin, 1986.

[12] M. Bordemann, M. Forger, and H. Romer, *Homogeneous Kähler manifolds: Paving the way towards new supersymmetry sigma models*, 1986.

[13] R. Bott. The index theorem for homogeneous differential operators. In S.S. Cairns, editor, *Differential and Combinatorial Topology*: a symposium in honor of Marston Morse, page 167-186. Princeton University Press, Princeton, N.J., 1965.

[14] R. Bott and L.W. Tu. *Differential forms in algebraic topology.* Graduate Texts in Mathematics No. 82, Springer Verlag, 1982.

[15] T. Bröcker and T. Dieck. *Representations of compact Lie groups.* Graduate Texts in Mathematics No. 98, Springer Verlag, 1985.

[16] H. Bursztyn and M. Crainic. Dirac structures, momentum maps, and quasi-Poisson manifolds. The breadth of symplectic and Poisson geometry, *Progr. Math.,* vol. 232, 1-40, Birkhauser Boston, Boston, MA 2005.

[17] A. Cannas da Silva. *Lectures on symplectic geometry.* Lecture notes in Math. 1764, Springer, Berlin, 2001. MR1853077 (200i:53105).

[18] J. B Conway. *A course in functional analysis.* Graduate Texts in Mathematics No. 96, Springer Verlag, 1985.

[19] J.J. Duistermaat, *The heat kernel Lefschetz fixed point formula for the spin-c Dirac operator,* Birkhauser Boston, Boston, MA, 1996.

[20] D.E. Evans and T. Gannon. Modular invariants and twisted equivariant K-theory. *Communications in number theory and physics*, vol 3, no. 2, pp. 209-296, 2009.

[21] P. Donovan and M. Karoubi. Graded brauer groups and K-theory with local coefficients. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, Volume 38, Issue 1, pp 5-25, January 1970.

[22] I.B. Frenkel, H. Garland and G.J. Zuckerman. Semi-infinite cohomology and string theory. *Proc. Natl. Acad. Sci. USA*, Vol. 83, pp. 8442-8446, 1986.

[23] D. S. Freed, M. J. Hopkins and C. Teleman. Loop groups and twisted K-theory II. *J. Amer. Math. Soc.*, 26(3):595-644, 2013.

[24] D. S. Freed, M. J. Hopkins and C. Teleman. Loop groups and twisted K-theory III. *Ann. of Math. (2)*, 174(2):947-1007, 2011.

[25] D.S. Freed. Flat manifolds and infinite dimensional Kähler geometry. *Infinite Dimensional Groups with Applications*, edited by Victor Kac, 1985.

[26] D.S. Freed. Geometry of loop groups. *J. Differential Geom.* Volume 28, Number 2, 223-276, 1988.

[27] V. Guillemin and S. Sternberg. Geometric quantization and multiplicities of group representations. *Invent. Math.*, 67, no. 3, 515-538, 1982.

[28] V. Guillemin, V. Ginzburg and Y. Karshon. *Moment Maps, Cobordisms, and Hamiltonian Group Actions.* Mathematical surveys and monographs, vol 98, 2002.

[29] G. B. Segal. Equivariant K-theory. *Inst. Hautes Etudes Sci. Publ. Math.*, 34:129-151, 1968.

[30] N. Higson and J. Roe. *Analytic K-homology.* Oxford University Press, 2000.

[31] N. Higson and E. Guentner. Group C*-algebras and K-theory. *Noncommutative geometry*, Lectures given at C.I.M.E. Summer School held in Martina Franca, Italy, September 3-9, 2000. Springer-Verlag Berlin Heidelberg, 2004.

[32] A. Huckleberry. Introduction to group actions in symplectic and complex geometry. *Infinite Dimensional Kähler Manifolds*, DMV Seminar Band Volume 31, pp 1-129, 2001.

[33] Victor G. Kac. *Infinite-Dimensional Lie Algebras.* Cambridge University Press, Cambridge, 1994.

[34] G.G. Kasparov. Equivariant KK-theory and the Novikov conjecture. *Inventiones Mathematicae*, 91:147-201, 1988.

[35] A. Kirillov Jr. *An Introduction to Lie Groups and Lie Algebras.* Cambridge University Press, Cambridge, 2008.

[36] N. Kuiper. Contractibility of the unitary group in Hilbert space. *Topology*, 3:19-30, 1964.

[37] G.D. Landweber. *Dirac operators on loop spaces.* Ph.D. Thesis, Harvard University, Cambridge, Massachusetts(1999). Electronically available at http://math.bard.edu/greg/LoopDirac.pdf

[38] G. D. Landweber. Multiplets of representations and Kostant's Dirac operator for equal rank loop groups. *Duke Math. J.*, 110(1):121-160, 2001.

[39] H. B. Lawson and M. Michelsohn. *Spin geometry.* Princeton University Press, 1989.

[40] E. Meinrenken, C. Woodward. *Fusion of Hamiltonian loop group manifolds and cobordism*, 1997. Also available as arXiv:dg-ga/9707019.

[41] E. Meinrenken and C. Woodward. Hamiltonian loop group actions and Verlinde factorization. *J. Differential Geom.*, 50(3):417-469, 1998.

[42] E. Meinrenken and C. Woodward. Canonical bundles for Hamiltonian loop group manifolds. *Pacific J. Math.*, 198(2):477-487, 2001.

[43] E. Meinrenken. On the quantization of conjugacy classes. *Enseign. Math. (2)*, 55(1-2):33-75, 2009. Also available as arXiv:0707.3963.

[44] E. Meinrenken. Twisted K-homology and group-valued moment maps. *Int. Math. Res. Not. IMRN*, (20):4563-4618, 2012.

[45] S. Mohrdieck and R. Wendt, *Twisted conjugacy classes, coadjoint orbits of loop groups and D-branes in the WZW-model*, 2004. Available as arXiv:math/0303118.

[46] K. Neeb. Infinite-dimensional groups and their representations. *Infinite Dimensional Kähler Manifolds*, DMV Seminar Band Volume 31, pp 131-178, 2001.

[47] H. Posthuma. Dirac induction for loop groups. *Lett. Math. Phys.*, 95(1):89-107, 2011.

[48] G. Pederson. *C^*-algebras and Their Automorphism Groups*. London Math. Soc. Monographs 14, Academic Press, London-New York, 1979.

[49] R. J. Plymen and P. L. Robinson. *Spinors in Hilbert space*. volume 114 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1994.

[50] A. Pressley and G. Segal. *Loop groups*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1986. Oxford Science Publications.

[51] M. A. Rieffel. Dirac operators for coadjoint orbits of compact Lie groups. *Münster J. of Math.*, 2, 265-298, 2009.

[52] I. Raeburn and D. P. Williams. *Morita Equivalence and Continuous-trace C^*-algebras.* Mathematical Surveys and Monographs, 1998.

[53] J. Rosenberg. Continuous-trace algebras from the bundle theoretic point of view. *J. Austral. Math. Soc. Ser.* A 47, no. 3, 368-381, 1989.

[54] M. Schottenloher. *A Mathematical introduction into conformal field theory.* Springer-Verlag, Lecture Notes of Physics 759, 2. Ed., 2008.

[55] K. Schmüdgen. *Unbounded Self-adjoint Operators on Hilbert Space.* Graduate Texts in Mathematics No. 265, Springer Verlag, 2012.

[56] C. Teleman. Lie algebra cohomology and the fusion rules. *Comm. Math. Phys.*, Volume 173, Number 2, 265-311, 1995.

[57] J. Tu. Twisted K-theory and Poincaré duality. *American Mathematical Society*, 362(3):1269-1278, 2008.

[58] E. Verlinde. Fusion rules and modular transformations in 2D conformal field theory. *Nuclear Physics* B300 [FS22], 360-376, 1988.

[59] N. M. J. Woodhouse. *Geometric quantization.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1992.

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