TOWARDS IMPROVED NDE AND SHM METHODOLOGIES
INCORPORATING NONLINEAR STRUCTURAL FEATURES

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by
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Abstract

Ultrasound is widely employed in Nondestructive Evaluation (NDE) and Structural Health Monitoring (SHM) applications to detect and characterize damage/defects in materials. In particular, ultrasonic guided waves are considered a foremost candidate for in-situ monitoring applications. Conventional ultrasonic techniques rely on changes/discontinuities in linear elastic material properties, namely the Young’s modulus and shear modulus to detect damage. On the other hand, nonlinear ultrasonic techniques that rely on micro-scale nonlinear material/structural behavior are proven to be sensitive to damage induced microstructural changes that precede macro-scale damage and are hence capable of early damage detection. The goal of this thesis is to investigate the capabilities of nonlinear guided waves — a fusion of nonlinear ultrasonic techniques with the guided wave methodologies for early damage detection. To that end, the thesis focuses on two important aspects of the problem:

1. Wavemechanics - deals with ultrasonic guided wave propagation in nonlinear waveguides;

2. Micromechanics - deals with correlating ultrasonic response with micro-scale nonlinear material behavior.

For the development of efficient NDE and SHM methodologies that incorporate nonlinear structural features, a detailed understanding of the above aspects is indispensable.

In this thesis, the wavemechanics aspect of the problem is dealt with from both theoretical and numerical standpoints. A generalized theoretical framework is developed to study higher harmonic guided waves in plates. This was employed to study second harmonic guided waves in pipes using a large-radius asymptotic approximation. Second harmonic guided waves in plates are studied from a numerical standpoint. Theoretical predictions are validated and some key aspects of higher harmonic generation in waveguides are outlined. Finally, second harmonic guided waves in plates with inhomogeneous and localized nonlinearities are studied and some important aspects of guided wave mode selection are addressed.
The other part of the work focused on developing a micromechanics based understanding of ultrasonic higher harmonic generation. Three important aspects of micro-scale material behavior, namely tension-compression asymmetry, shear-normal coupling and deformation induced asymmetry are identified and their role in ultrasonic higher harmonic generation is discussed. Tension-compression asymmetry is identified to cause second (even) harmonic generation in materials. Then, shear-normal coupling is identified to cause generation of secondary waves of different polarity than the primary waves. In addition, deformation induced anisotropy due to the presence of residual stress/strain and its contribution to ultrasonic higher harmonic generation is qualitatively discussed. Also, the tension-compression asymmetry in the material is quantified using an energy based measure. The above measure is employed to develop a homogenization based approach amenable to multi-scale analysis to correlate microstructure with ultrasonic higher harmonic generation.

Finally, experimental investigations concerning third harmonic SH wave generation in plates are carried out and the effect of load and temperature changes on nonlinear ultrasonic measurements are discussed in the context of SHM. It was found that while nonlinear ultrasound is sensitive to micro-scale damage, the relative nonlinearity parameter may not always be the best measure to quantify the nonlinearity as it is subject to spurious effects from changes in environmental factors such as loads and temperature.
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Chapter 1  
Introduction

The need to assess the current state of structure, predict remaining useful life and take necessary steps concerning the maintenance and or replacement of structure or its components is a challenge common to civil, aerospace and nuclear industries. While it is always safe to replace the structures after their design-life to ensure structural safety and avoid any catastrophes, it is not necessarily a cost-effective approach especially when the structure has significant amount of remaining useful life left in it. This raises two important questions:

1. How do we assess the current state of a structure?

2. How do we predict the remaining useful life?

Both the questions need to be answered from a quantitative standpoint. To answer the first question we need to inspect the structures for any hot-spots or defects that may result in structural failure. The second question invariably relies on the first one in that the severity of these hot-spots dictate the remaining useful life of the material. Quantitative approaches that address the above questions have been developed; Nondestructive evaluation (NDE) and Structural Health Monitoring (SHM) (Farrar and Worden, 2007) (Chang et al., 2003) (Sohn et al., 2004) methodologies enable us to inspect and assess the current state of structure and damage prognosis (Farrar and Lieven, 2007) (Inman et al., 2005) approaches enable us to predict the remaining useful life. Since the focus of this thesis is on NDE and SHM methodologies we present a brief overview of these approaches for assessing the current state of structure.

Conventionally, NDE encompasses off-line inspection methodologies that are employed for schedule-based maintenance throughout the service life of the structures. On the other hand, SHM methodologies are capable of continuous monitoring of
structures and hence capable of condition-based maintenance thereby significantly reducing the life-cycle costs. Both the approaches rely on tools that enable one to non-invasively interrogate the structures for hot-spots. More often than not the hot-spots are invisible to the naked eye and hence the above methodologies employ waves like ultrasound, infrared and X-rays that can travel into the structures to be inspected. Of these, ultrasound is most common and widely used method for inspection of structural materials, namely metals and composites.

Ultrasonic waves are elastic waves (Graff, 1975) that travel in materials. Two types of ultrasonic waves (Rose, 1999) are generally used for inspection purposes; bulk waves that travel in unbounded media and guided waves that travel in confined structures, namely waveguides. While bulk-wave inspection is capable of point-by-point inspection, it usually is time-consuming for applications requiring rapid interrogation of structures. On the other hand, guided waves (Raghavan and Cesnik, 2007) (Rose, 2002) (Rose, 2004) are capable of rapid interrogation of structures due to their high penetration power. This makes them more amenable to SHM methodologies employing ultrasound. However, due to the multi-mode nature of guided waves, a proper guided wave mode and frequency choice is indispensable for efficient inspection.

While conventional ultrasonic methods that mainly rely on linear elastic material properties are capable of macro-scale damage detection, nonlinear ultrasonic methods (Jhang, 2009) (Zheng et al., 2000) (Matlack et al., 2014) are capable of detecting micro-scale damage induced microstructural changes that precede macro-scale damage. By nonlinear ultrasonics we refer to higher harmonic frequency components generated in the material due to the nonlinearity associated with the micro-scale material behavior. Figure 1 shows the schematic of harmonic generation in materials where a primary wave of frequency $\omega$ and amplitude $A_1$ generates a second harmonic with amplitude $A_2$ and a third harmonic with amplitude $A_3$. Hence, these harmonics are sensitive to micro-scale damage and can be used for monitoring damage progression in materials.
While much of the earlier work focused on using bulk-waves for nonlinear ultrasonics, the main focus of this thesis is on exploring the capabilities of nonlinear guided waves that combine the early damage detection capabilities of nonlinear ultrasound with superior inspection capabilities of guided waves for NDE and SHM. This would result in NDE and SHM methodologies that incorporate nonlinear structural features for early damage detection in structures.

Next, we discuss some preliminaries that enable better understanding of this work.

1.1 Preliminaries

In this section, we present some preliminaries on guided waves in plates, continuum mechanics and nonlinear ultrasonics.

1.1.1 Guided waves in plates

Consider a traction-free plate with the coordinate system as shown in Figure 1.2.

The governing equations of motion for the plate are given by
\[ \nabla \cdot \mathbf{T} = \rho \ddot{\mathbf{u}} \] (Balance of linear momentum)

\[ \mathbf{T} = \mathbf{C} \epsilon \] (Constitutive relation)

\[ \epsilon = \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^T) \] (Strain-displacement relation)

\[ \mathbf{T} \mathbf{n} = 0. \text{ on } x_2 = \pm h \] (traction-free boundary condition) \hspace{1cm} (1.1)

where \( \mathbf{T} \) denotes the Cauchy stress tensor, \( \rho \) denotes the density, \( \mathbf{C} \) denotes the tensor of elastic moduli, \( \epsilon \) denotes the linearized strain tensor and \( \mathbf{u} \) denotes the displacement field. For the case of isotropic linear elastic material, the above equations become (using index notation)

\[ T_{ij,j} = \rho \ddot{u}_i \] (Balance of linear momentum)

\[ T_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2 \mu \epsilon_{ij} \] (Constitutive relation)

\[ \epsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \] (Strain-displacement relation)

\[ T_{ij} n_j = 0. \] (traction-free boundary condition) \hspace{1cm} (1.2)

The above set of equations while selecting the displacement as the primary unknown yield the well-known Navier’s equation of motion given by

\[(\lambda + \mu)u_{j,ji} + \mu u_{i,jj} = \rho \ddot{u}_i \] (1.3)

with the traction-free boundary conditions.

Under the assumption of plane-strain in \( x_1-x_2 \) plane, we can use Helmholtz decomposition (Rose, 1999) for the displacement as follows

\[ \mathbf{u} = \nabla \phi + \nabla \times \psi \] \hspace{1cm} (1.4)

Here \( \phi = \phi(x_1, x_2) \) is a scalar field and \( \psi = (0, 0, \psi(x_1, x_2)) \) is a vector field. Substituting Eqn(1.4) in Eqn(1.3) we get

\[
\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} = \frac{1}{c_l^2} \frac{\partial^2 \phi}{\partial t^2} \\
\frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2} = \frac{1}{c_t^2} \frac{\partial^2 \psi}{\partial t^2}
\] \hspace{1cm} (1.5)
Since we are interested in guided waves propagating in $x_1$ direction, we assume time-harmonic dependence of $\phi(x_1, x_2) = \phi(x_2)e^{i(kx_1-\omega t)}$ and $\psi(x_1, x_2) = \psi(x_2)e^{i(kx_1-\omega t)}$. Here $\omega$ and $k$ denote the angular frequency and the wavenumber of the guided wave mode. Substituting these in Eqn(1.5), we get

$$\frac{d^2\phi(x_2)}{dx_2^2} + \left(\frac{\omega^2}{c^2} - k^2\right)\phi(x_2) = 0$$

$$\frac{d^2\psi(x_2)}{dx_2^2} + \left(\frac{\omega^2}{c^2} - k^2\right)\psi(x_2) = 0$$

(1.6)

Letting $p = \sqrt{\left(\frac{\omega^2}{c^2} - k^2\right)}$ and $q = \sqrt{\left(\frac{\omega^2}{c^2} - k^2\right)}$. The solutions for the above set of equations are given by

$$\phi(x_2) = A\cos(px_2) + B\sin(px_2)$$

$$\psi(x_2) = C\cos(px_2) + D\sin(qx_2).$$

(1.7)

Enforcing the appropriate traction-free boundary conditions given by

$$T_{12}(x_2 = h) = 0$$

$$T_{12}(x_2 = -h) = 0$$

$$T_{22}(x_2 = h) = 0$$

$$T_{22}(x_2 = -h) = 0.$$}

(1.8)

at the top and bottom surface of the plate, we get the following set of equations in $A$, $B$, $C$ and $D$ as follows

$$\mu(-2ikpA\sin(ph) + (k^2 - q^2)D\sin(qh)) = 0$$

$$-\lambda(k^2 + p^2)A\cos(ph) - 2\mu\left(p^2D\cos(ph) + ikD\cos(qh)\right) = 0$$

$$\mu(2ikpB\cos(ph) + (k^2 - q^2)C\cos(qh)) = 0$$

$$-\lambda(k^2 + p^2)B\sin(ph) - 2\mu\left(p^2C\sin(ph) + ikC\sin(qh)\right) = 0$$

(1.9)

The first two relations in Eqn(1.9) yield symmetric modes for which $B = 0$ and $C = 0$ and the latter two yield antisymmetric modes for which $A = 0$ and $D = 0$. 

5
The corresponding dispersion relations are given by

\[
\frac{\tan(qh)}{\tan(ph)} = \frac{-4k^2pq}{(q^2 - k^2)^2} \quad \text{Symmetric modes}
\]

\[
\frac{\tan(qh)}{\tan(ph)} = \frac{-(q^2 - k^2)^2}{4k^2pq} \quad \text{Antisymmetric modes}
\]

The above dispersion relations yield \((\omega, k)\) combinations that correspond to the guided wave modes propagating in the plate. These are termed as Rayleigh-Lamb (RL) modes and are polarized in the \(x_1 - x_2\) plane. The case of plane strain considered above is not the general one and there are other modes termed as Shear horizontal (SH) modes that are polarized in the \(x_3\) direction and propagate in the \(x_1\) direction. Using a similar procedure to that outlined above one can obtain the following dispersion relation for SH modes in an isotropic plate.

\[
qh = \frac{n\pi}{2}
\]

While the RL and SH modes are decoupled in an isotropic plate, it should be noted that they are coupled in anisotropic plates. The analysis in that case is based on a partial wave technique as outlined in Solie and Auld (1973). For more complex waveguides that involve multi-layered plates or complex geometries, numerical methods such as Global Matrix Method (GMM) (Rose, 1999) and Semi Analytical Finite Element Method (SAFE) (Hayashi et al., 2003).

1.1.1.1 Dispersion curves and wavestructures

In this section, we discuss two main aspects of guided waves in structures. These are

1. Dispersion curves

2. Wavestructures

Dispersion curves depict the frequency-wavenumber combinations at which propagating guided wave modes exist in the structures. Figure 1.3 shows the phase velocity dispersion curves for an aluminum plate \((E = 70 \text{ GPa}, \nu = 0.33)\). Symmetric modes, antisymmetric modes and the shear-horizontal modes are marked on the figures. Except for the fundamental modes \(S_0, A_0\) and \(SH_0\), all other modes
have a cut-off frequency below which they do not exist in the plate. Also, while the $S_0$ and $A_0$ modes converge to the Rayleigh wave speed, all the other modes converge to the transverse wave speed on the phase velocity dispersion curve. From experimental standpoint, the wave velocity measurements yield the group velocity for dispersive guided wave modes. The relation between phase velocity and group velocity is given by

$$c_p = \frac{c_p^2}{c_p - \frac{\partial c_p}{\partial (fd)}}$$  \hspace{1cm} (1.12)

where $c_p$ denotes the phase velocity, $c_g$ denotes the phase velocity and $fd$ denotes the frequency-thickness product.

![Figure 1.3: Phase velocity dispersion curves for an aluminum plate.](image-url)

Another important aspect of the guided waves is the wavestructure that depicts the through-thickness displacement field of the guided wave modes. The wavestructure of a mode dictates the sensitivity of it to particular kinds of defects. Figure 1.4 shows the wavestructure of $S_0$ mode at 0.5 MHz. The $x_1$ component of the displacement is denoted by $u$ and the $x_2$ component of the displacement is denoted by $v$. As can be seen, the displacement $u$ is symmetric through the thickness for the $S_0$ mode and is antisymmetric for the $A_0$ mode shown in Figure 1.5. Likewise, Figures 1.6 and 1.7 show the wavestructures of the $S_1$ mode (3.58 MHz) and $S_2$ mode (7.16 MHz).
Figure 1.4: Wavestructure of $S_0$ mode (0.5 MHz).

Figure 1.5: Wavestructure of $A_0$ mode (0.5 MHz).
Figures 1.6 and 1.7 show the wavestructures of the SH$_1$ and SH$_2$ modes respectively. The $x_3$ component of the displacement is denoted by $w$. Unlike the Rayleigh-Lamb modes whose wavestructure is dependent on the frequency for a particular mode, SH modes have the same wavestructure at all the frequencies.
1.1.2 Continuum mechanics

In this section we discuss some preliminaries on continuum mechanics and associated notation that is used in the rest of the thesis. For more details we refer the reader to
1.1.2.1 Kinematics

We consider an abstract body $B$ that occupies a reference configuration $B_\kappa$ in space as shown in Figure 1.10. The body is deformed to its current configuration denoted by $B_1$. We denote the position of the material particles in the reference configuration by $X$ and that in the current configuration by $x$. The deformation $\chi$ is the map $\chi : B_\kappa \rightarrow B_1$ so that we have $x = \chi(X)$. If we denote a sequence of deformations ordered in time by $\chi_t$, we can write the position of material particles as $x(t) = \chi_t(X) := \chi(X,t)$. The velocity of material particles is given by $v(X,t) = \frac{\partial \chi(X,t)}{\partial t}$.

We denote the deformation gradient by $F$ and is given by

$$F = \frac{\partial \chi(X,t)}{\partial X} = \frac{\partial x}{\partial X}. \quad (1.13)$$

Likewise, the displacement gradient is denoted by $H$ and is given by

$$H = \frac{\partial(x - X)}{\partial X} = \frac{\partial(u(X))}{\partial X} = F - I. \quad (1.14)$$

where $u(X,t) = x - X$ denotes the displacement and $I$ denotes the identity tensor.

Throughout the thesis, we use Lagrangian measure of strain denoted by $E$ and given by

$$E = \frac{1}{2}(F^TF - I) = \frac{1}{2}(H + H^T + H^TH). \quad (1.15)$$

Any new notation introduced later will be explained accordingly.

Figure 1.10: Schematic depicting current and reference configurations.
1.1.2.2 Balance laws

Under the assumption that the deformation is the only cause for the change in the mass density, one can write the balance of mass as

\[ \dot{\rho} + \rho \nabla_x \cdot \mathbf{v} = 0 \quad \text{(Balance of Mass)} \quad (1.16) \]

where \( \rho \) denotes the density of the material in the current configuration and the divergence \( \nabla_x \) is taken with respect to the coordinates in the current configuration. Likewise, we can write balance of linear and angular momentum in the current configuration under the assumption of the absence of body-forces as

\[ \nabla_x \mathbf{T} = \rho \frac{\partial \mathbf{v}(\mathbf{X}, t)}{\partial t} \quad \text{(Balance of Linear momentum)} \]
\[ \mathbf{T} = \mathbf{T}^T. \quad \text{(Balance of Angular momentum)} \quad (1.17) \]

where \( \mathbf{T} \) denotes the Cauchy stress tensor. Likewise, one can write the reference version of the balance of momenta as follows

\[ \nabla_{\mathbf{X}} \mathbf{S} = \rho_n \frac{\partial \mathbf{v}(\mathbf{X}, t)}{\partial t} \quad \text{(Balance of Linear momentum)} \]
\[ \mathbf{S}^T = \mathbf{F}^T \mathbf{S} \quad \text{(Balance of Angular momentum)} \quad (1.18) \]

Here \( \mathbf{S} \) denotes the first Piola-Kirchoff stress tensor and is related to the Cauchy stress by

\[ \mathbf{S} = \det(\mathbf{F}) \mathbf{T} \mathbf{F}^{-T}. \quad (1.19) \]

where \( \mathbf{F}^{-T} \) denotes the inverse-transpose of \( \mathbf{F} \).

1.1.2.3 Constitutive theory

Using the Coleman-Noll procedure and Second law of Thermodynamics, one can show that the first Piola-Kirchoff stress tensor for a Green elastic material (Ogden, 1997) can be obtained as

\[ \mathbf{S} = \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}} \quad (1.20) \]

where \( W(\mathbf{F}) \) denotes the strain energy density function of the material. Likewise one can obtain the following relation for the second Piola-Kirchoff stress tensor.
(\text{T}_{\text{RR}}).

\[
\text{T}_{\text{RR}} = \frac{\partial W}{\partial \text{E}} \quad (1.21)
\]

The first and Second Piola Kirchoff stresses are related by

\[
\text{S} = \text{F} \text{T}_{\text{RR}} \quad (1.22)
\]

We restrict ourselves to one specific form of \( W(\text{E}) \) used in Landau and Lifshitz (1986) and given by

\[
W(\text{E}) = \frac{1}{2} \lambda (tr(\text{E}))^2 + \mu tr(\text{E}^2) + \frac{1}{3} C(tr(\text{E}))^3 + Btr(\text{E})tr(\text{E}^2) + \frac{1}{3} Atr(\text{E}^3) \quad (1.23)
\]

The corresponding second Piola-Kirchoff stress tensor is given by (Eqn(1.21))

\[
\text{T}_{\text{RR}} = \lambda tr(\text{E}) + 2\mu \text{E} + C(tr(\text{E}))^2 \text{I} + Btr(\text{E}^2)\text{I} + 2Btr(\text{E})\text{E} + A\text{E}^2. \quad (1.24)
\]

Next, we discuss preliminaries on nonlinear ultrasonics that provide the context for this thesis.

1.1.3 Nonlinear ultrasonics

The broad discipline of nonlinear ultrasonics (Jhang, 2009) (Zheng et al., 2000) encompasses several techniques that rely on nonlinear material response to detect and characterize damage. Some of these include Nonlinear Resonant Ultrasound Spectroscopy (NRUS) (Muller et al., 2005), Nonlinear Elastic Wave Spectroscopy (NEWS) (Van Den Abeele et al., 2000b) (Van Den Abeele et al., 2000a) (Van Den Abeele et al., 2001) and Higher harmonic generation (Cantrell and Yost, 2001). While the first two methods rely on non-classical nonlinear effects (Delsanto, 2006), Higher harmonic generation relies on classical nonlinearity. Nevertheless, most of the above nonlinear techniques exploit the generation of other (higher or sub) harmonics of a fundamental wave traveling in a material to discern the damage state. The source of the nonlinear material behavior can stem from several factors, namely inherent elastic nonlinearity, dislocations (Hikata et al., 1965), contacting surfaces of micro-cracks (Contact Acoustic Nonlinearity (CAN))(Solodov et al., 2002), etc. Of interest to this work are classical nonlinear effects that are generated from nonlinear mechanisms operational due to the presence of dislocations leading
to localized plasticity, micro-scale damage and finally to macro-scale damage. These damage mechanisms are more relevant to fatigue, creep, radiation damage, etc. Next, we provide a review of earlier work concerning the use of ultrasonic higher harmonic generation for characterizing damage using classical nonlinear effects.

Early investigations (Breazeale and Thompson, 1963) (Breazeale and Ford, 1965) concerning the higher harmonic generation in solids focused on evaluating the contribution of elastic nonlinearity arising out of lattice anharmonicity. Experiments were carried out either on single crystals or polycrystalline samples with very low dislocation densities to isolate the elastic nonlinearity contribution. Likewise, other investigations (Hikata et al., 1965) (Hikata and Elbaum, 1966a) (Hikata et al., 1966b) (Hiki and Granato, 1966) focused on evaluating the contribution of dislocations to the second and third harmonic generation. These findings spurred significant interest in using higher harmonic generation for characterizing damage progression in materials. Nagy (1998) compared the sensitivities of linear and nonlinear material properties to damage progression in fatigue of metals, composites, adhesives, etc and concluded that the nonlinear elastic properties are much more sensitive to the damage than their linear counterparts. The work by Cantrell and his colleagues (Cantrell and Yost, 2001) (Cantrell, 2004) (Cantrell, 2006) provided a systematic framework for evaluating the contribution of dislocation substructures, namely monopoles and dipoles to second harmonic generation in metals. Experimental investigations also validated the theory in that the so called nonlinearity parameter ($\beta$) was found to increase with the number of cycles to failure during fatigue. Several other investigations– (Baby et al., 2008) concerning creep damage, (Matlack et al., 2012) concerning the radiation damage and (Kim and Lissenden, 2009) concerning the precipitates, demonstrated the effectiveness of higher harmonic generation in general and second harmonic generation in particular to monitor micro-scale damage. All the above investigations focused on using bulk-waves for generating second harmonics and hence to investigate micro-scale damage in materials. In this context, a simple 1D model is often employed to model the nonlinear material behavior. This is given by

$$\sigma = E\epsilon(1 + \frac{\beta}{2}\epsilon)$$

(1.25)

where ‘$E$’ is the Young’s modulus, $\sigma$ is the uniaxial stress, $\epsilon$ is the strain and $\beta$ is called the (acoustic) nonlinearity parameter that quantifies the nonlinear
behavior of the material. Using the balance of linear momentum and linearized strain-displacement relations, one arrives at the following equation of motion.

$$\rho \frac{\partial^2 u}{\partial t^2} = E \frac{\partial^2 u}{\partial x^2} (1 + \beta \frac{\partial u}{\partial x})$$  \hspace{1cm} (1.26)

Here, $u$ is the displacement and we used the strain-displacement relation $\epsilon = \frac{\partial u}{\partial x}$ in arriving at Eqn (1.26). Eqn (1.26) can be solved using the perturbation approach (Cantrell, 2004) with an initial condition $u(0, t) = A_1 \cos(\omega t)$ to obtain (up to second harmonic)

$$u(x, t) = \frac{\beta k^2 A_1^2 x}{8} + A_1 \cos(kx - \omega t) - \frac{\beta k^2 A_1^2 x}{8} \cos(2(kx - \omega t)).$$  \hspace{1cm} (1.27)

Here, $\omega$ is the frequency and $k$ is the wavenumber of the fundamental wave propagating in the material. Key observations that need to be made in this regard are:

1. The amplitude of the secondary mode ($\cos(2(kx - \omega t))$) is proportional to the square of the amplitude of the fundamental mode. This holds good only for the classical nonlinear effects.

2. It increases linearly (until a finite-distance after which perturbation approach is no longer valid) with the propagation distance $x$.

3. It is directly proportional to the nonlinearity parameter $\beta$ and hence indirectly quantifies the nonlinearity in the material behavior.

4. There exists a quasi-static (zero-frequency) pulse that also is proportional to $x$ (Narasimha et al., 2009).

Often, from an experimental standpoint, it is the relative nonlinearity parameter that is used to quantify nonlinearity and hence micro-scale damage in the material. This is given by $\frac{A_2}{A_1}$ where $A_2$ and $A_1$ are the amplitudes of the secondary and the fundamental modes respectively. It should be emphasized that the above derivation Eqn(1.27) and hence the definition of the relative nonlinearity parameter relied on the following assumptions:

1. Both the fundamental and secondary modes propagate as plane waves. No diffraction affects are included.
2. The material response is (nonlinear) elastic i.e., no dissipation or attenuations of the fundamental wave is taken into account.

3. The micro-scale damage only affects $\beta$ parameter but not the Young’s modulus $E$.

Hence the relative nonlinearity parameter is not necessarily an accurate measure and needs to be compensated for the above factors whenever necessary. More discussion on this can be found in Chapter 10. While the bulk waves have significantly been employed for second harmonic generation, use of Rayleigh waves and guided waves for micro-scale damage characterization recently received significant attention in the recent past. The theoretical problem of nonlinear Rayleigh wave propagation on the surface of an isotropic solid was carried out by Zabolotskaya (1992), where the evolution equations for the wave were obtained by using a Hamiltonian formalism. The waveform distortion and shock formation have been (numerically) demonstrated for steel. Nonlinear Rayleigh waves have been employed to characterize material damage in a nickel-based super alloy in Herrmann et al. (2006). A laser-based interferometer technique was used to receive Rayleigh waves. Plastic deformation and low-cycle fatigue tests were conducted on the specimens of the nickel-based alloy and the nonlinearity parameter was tracked for increasing levels of damage inflicted on the specimen. It showed a monotonic trend of increasing nonlinearity with damage. Recently (Zeitvogel et al., 2014), second harmonic generation from Rayleigh waves was used to characterize stress corrosion cracking in carbon steel. More details concerning the use of nonlinear bulk and Rayleigh waves to characterize micro-scale damage can be found in Matlack et al. (2014). Next, we discuss the previous research efforts focusing on the use of nonlinear guided waves for micro-scale damage characterization.

Theoretical investigations concerning the second harmonic generation of guided waves in plates were first carried out by Mingxi (1998) and Deng (1999). De Lima and Hamilton (2003) developed an approach based on the perturbation approach and normal-mode expansion to analyze guided wave second harmonic generation in plates. They put-forth two conditions, namely the phase-matching condition and the non-zero power-flux criterion for cumulative second harmonic generation. These results were further extended by Srivastava and Lanza di Scalea (2009) and Müller et al. (2010). However, the above studies did not identify the appropriate
guided wave modes suitable for cumulative second harmonic generation. Matsuda and Biwa (2011) and Chillara (2012) independently analyzed the phase-matching and non-zero power-flux conditions to arrive at a complete list of guided wave modes capable of cumulative second harmonic generation. In addition, a generalized approach was presented in Chillara and Lissenden (2012) to analyze interaction of guided wave modes in the plate that enables the prediction of the nature of all the harmonics generated in the plate including the sum and difference frequency components from a given fundamental excitation. More details concerning the mathematical formulation and a summary of the results can be found in chapter 4.

Experimental investigations concerning the use of nonlinear guided waves for micro-scale damage characterization were limited. The cumulative nature of second harmonic guided waves was demonstrated in Deng et al. (2005). The feasibility of using nonlinear guided waves was demonstrated in Bermes et al. (2007) by evaluating the nonlinearity parameters associated with guided waves in two different kinds of aluminum plates. Fatigue damage was characterized using second harmonic guided waves in Deng and Pei (2007). Use of second harmonic guided waves for characterizing plasticity driven material damage was investigated in Pruell et al. (2007). Second harmonic guided waves were employed for characterizing creep damage for metallic alloys in Xiang et al. (2012). Recently (Lissenden et al., 2014a), third harmonic generation from fundamental Shear-Horizontal modes were employed for characterizing localized plastic deformation in plates.

While both the theoretical and experimental capabilities are now developed to the extent where the use of nonlinear ultrasonic techniques in general and nonlinear guided waves in particular is feasible, some questions still need to be addressed for developing efficient NDE and SHM methodologies. These include understanding the

1. effect of localized micro-scale damage on higher harmonic generation of guided waves.

2. relation between microstructure and the nonlinear parameter evaluated from the experiments.

3. effect of operational and environmental parameters on nonlinear ultrasonic measurements.
This work aims to address the above questions from theoretical, numerical and experimental standpoints.

Next, we provide the scope and outline of the thesis.

1.2 Scope and outline of the thesis

As mentioned earlier, the main goal of this thesis is to explore the use of nonlinear guided waves for early damage detection in structures. Since the material state is inferred indirectly from the higher harmonics generated, it is essential that we understand the relation between micro-scale material behavior and its influence on wave propagation. To that end this thesis discusses two important aspects of the problem:

1. Wave mechanics
2. Micromechanics

The wave mechanics aspect of the problem focuses on understanding nonlinear guided wave propagation in waveguides. In particular, it addresses key aspects of guided wave mode selection for efficient generation of higher harmonics. On the other hand, the micromechanics aspect of the problem deals with correlating the microstructure with higher harmonics.

From an experimental standpoint, this thesis discusses the effect of operational and environmental parameters on nonlinear ultrasonic measurements. Below is a brief outline of the thesis.

Chapter 1 provides a brief introduction and discusses preliminaries on guided waves in plates, Continuum mechanics and nonlinear ultrasonics.

Chapter 2 discusses the Frequency Domain Finite Element (FDFE) approach for studying guided waves especially in inhomogeneous waveguides. Through a series of examples, the efficacy of FDFE for guided wave mode selection is demonstrated. This approach is later employed to study the acoustoelastic effect in wave guides.

Chapter 3 focuses on the guided wave acoustoelasticity - a nonlinear effect of stress on wave speeds. Theory, problem formulation and numerical simulations to quantify the effect are discussed.
Chapter 4 presents a generalized theoretical framework for assessing higher harmonic guided waves in plates. Aspects of guided wave mode selection for cumulative harmonic generation are discussed.

Chapter 5 develops a large radius asymptotic approximation for axis-symmetric guided wave modes in pipes. This is employed to analyze second harmonic guided wave generation in pipes.

Chapter 6 discusses nonlinear guided wave propagation in plates from a numerical standpoint. Some important issues concerning harmonic generation in plates are addressed. Then, role of geometric and material nonlinearities in harmonic generation is discussed. Finally, nonlinear guided waves in plates undergoing localized microstructural changes is discussed. Effect of damage distribution, spatial extent and intensity on higher harmonic generation is investigated.

Chapter 7 develops a phenomenological approach towards understanding relation between micro-scale material behavior and higher harmonic generation. Role of tension-compression asymmetry, shear-normal coupling and deformation induced anisotropy on higher harmonic generation is presented.

Chapter 8 develops a homogenization based approach to assess the contribution of micro-scale defects to higher harmonic generation. The approach is applied to micro-voids in the material and the results are discussed.

Chapter 9 discusses the effect of load and temperature changes on nonlinear ultrasonic measurements. First, guided wave acoustoelasticity is studied. Then, the results from third harmonic guided wave measurements are used to demonstrate the effect of load/temperature on the nonlinearity parameter. Implications on SHM methodologies incorporating nonlinear structural features is discussed.

Chapter 10 presents the summary and future scope of the work.
Chapter 2
Frequency domain finite element method for studying guided waves in complex waveguides

Introduction

In this chapter we present the use of frequency domain finite element method (FDFE) for studying guided waves in complex waveguides. First, a brief background along with the formulation of the elastic boundary value problem is presented in section 2.1. Then, some specific examples demonstrating the application of the method to both homogeneous and inhomogeneous waveguides is discussed in section 2.2. Factors affecting the numerical accuracy of the solution are discussed in section 2.3. Then, guided wave mode selection for inhomogeneous waveguides is discussed in section 2.4. Finally, conclusions are drawn in 2.5. The content of this chapter is predominantly from Chillara et al. (2015b) and Chillara and Lissenden (2014a).

2.1 Background and Formulation

As mentioned in the previous chapter, there exists infinitely many guided wave modes in the structure owing to the eigenvalue problem associated with it. Hence, ultrasonic guided wave mode selection is the key to the success of guided wave NDE and SHM methodologies as each mode exhibits varying degree of sensitivity to different kinds of defects in the structure. Two important aspects that characterize guided waves in a waveguide are dispersion curves and wavestructures. These dictate the choice of mode and frequency for a particular application. While simple geometries like pipe and plate admit analytical solutions for dispersion curves and wavestructures (Rose, 1999) (Graff, 1975), other complex waveguides require...
us to use numerical methods of which the finite-difference method and the finite element method (Moser et al., 1999) are by far the most common methods employed. However, for specific cases, spectral methods have proven to be more efficient in this regard as was highlighted in Adamou and Craster (2004). Other approaches such as SAFE have been employed to study guided wave modes in waveguides which are homogeneous in the wave propagation direction. Even though the time-domain finite element approach can be employed to study wave propagation, in many cases it requires a lot of computational effort especially in scenarios where the waveguides are complex in nature. In addition, some of the above methods like SAFE are not directly applicable for the waveguides that are inhomogeneous in the wave propagation direction. In particular, this is the case for functionally graded materials, inhomogeneous periodic waveguides and waveguide transitions (Puthillath et al., 2013). To this end, we explore the use of frequency domain finite element approach (FDFE) for studying guided waves in complex/inhomogeneous waveguides. Next, we present the problem formulation for the FDFE approach.

**Formulation**

The governing equations for a linear elastic waveguide are given by

\[ \nabla \cdot \sigma = \rho \ddot{u} \quad \text{(Balance of linear momentum)} \]
\[ \sigma = C \epsilon \quad \text{(Constitutive relation)} \]
\[ \epsilon = \frac{1}{2} (\nabla u + \nabla u^T) \quad \text{(Strain-displacement relation).} \quad (2.1) \]

along with appropriate boundary conditions, where \( \sigma \) denotes the Cauchy stress tensor, \( \rho \) denotes the density, \( C \) denotes the tensor of elastic moduli, \( \epsilon \) denotes the linearized strain tensor and \( u \) denotes the displacement field. The frequency domain finite element method solves the above system (Eqn 2.1) in the frequency domain. Solution in the frequency domain gives the characteristics of the system in terms of the frequency or the modal response function depicting the structural behavior of the waveguide.

To better elucidate this, we consider the simple example of a spring-mass system undergoing a forced oscillation as shown in Figure 2.1. The frequency response function of the mass can be written as

\[ H(\omega) = \frac{1}{\kappa - m \omega^2} \]

where \( \kappa \), \( m \) and \( \omega \) are the stiffness of the spring, mass and frequency respectively. The natural frequency of
the system can be obtained by solving \( \kappa - m\omega^2 = 0 \). A similar concept can be extended to a waveguide except that it has infinite material particles and hence infinite natural frequencies. For the guided wave propagation in a homogeneous structure, one can define such function in the form \( H(\omega, k) \), where \( \omega \) denotes the frequency of the mode and \( k \) denotes the wavenumber in the propagation direction. For example, guided wave propagation in isotropic homogeneous plate (Rose, 1999) can be characterized by \( H(\omega, k) = H_s(\omega, k)H_a(\omega, k)H_{SH}(\omega, k) \), where

\[
H_s(\omega, k) = \frac{\tan(qh)}{\tan(ph)} + \frac{4k^2pq}{(q^2 - k^2)^2} = 0 \quad \text{Symmetric modes}
\]

\[
H_a(\omega, k) = \frac{\tan(qh)}{\tan(ph)} + \left(\frac{q^2 - k^2}{4k^2pq}\right) = 0 \quad \text{Antisymmetric modes}
\]

\[
H_{SH}(\omega, k) = qh - \frac{n\pi}{2} = 0 \quad \text{Shear-Horizontal modes.} \quad (2.2)
\]

Here, \( p = \sqrt{\left(\frac{\omega}{c_l}\right)^2 - k^2} \) and \( q = \sqrt{\left(\frac{\omega}{c_t}\right)^2 - k^2} \) where \( c_l \) and \( c_t \) are the longitudinal and transverse wave speeds in the material.

![Spring-mass system](image)

**Figure 2.1**: Spring-mass system

The above function \( H(\omega, k) \) can be determined explicitly only in a very few cases. For the case of inhomogeneous waveguides or waveguides with complex geometries, it is not possible for a single wavenumber to capture the displacement field throughout the structure. Hence, it is an onerous task to identify appropriate guided wave mode suitable for NDE/SHM application.

FDFE attempts to address the above issue by solving Eqn(2.1) for time-harmonic excitations i.e., in the frequency domain. To be specific, it assumes \( \sigma = \sigma(x)e^{i\omega t} \) and \( u = u(x)e^{i\omega t} \) in which case the governing equation transforms as

\[
\nabla.(\sigma(x)) + \rho\omega^2 u(x) = 0.
\]

(2.3)

Note that the time-derivatives are eliminated due to the time-harmonic assumption.
and hence it can be thought of as a pseudo-static problem and can be solved with a lesser computational resource using FEM. More importantly, any inhomogeneities in the geometry or the material properties can be incorporated in the discretization. Eqn(2.3) needs to be solved at each frequency $\omega$ and hence the source influence (Rose, 1999) due to multi-mode guided wave excitation in the study is minimized.

2.2 FDFE-Examples

In this section we present several examples that demonstrate the efficacy of the approach for studying guided waves in complex waveguides. First, we consider guided waves in isotropic homogeneous elastic plates where the standard solutions of Rayleigh-Lamb and Shear-Horizontal modes are recovered. Then, simple examples where FDFE is employed for inhomogeneous waveguides are presented.

2.2.1 FDFE-Isotropic homogeneous plates

All the results presented in this section are obtained using a code written in MATLAB. We consider the excitation of SH and RL modes in a traction-free plate the schematic of which is shown in Figure 2.2. Unless otherwise specified, the length of the waveguide is set to 60 mm and its thickness is 1 mm. 120 elements are used to discretize along the $x_1$ direction and 10 elements are used to discretize along the $x_2$ direction. The displacement field is approximated by using linear shape functions over the quadrilateral (rectangular) elements obtained from the above discretization. The excitation is specified as a displacement boundary condition at the left end of the plate where top and bottom nodes are assumed fixed to eliminate rigid-body solutions. Depending on the nature of the excitation, different modes are excited in the structure. To excite a symmetric RL mode, we specify symmetric $U_1$ profile through the thickness and similarly to excite SH modes we specify appropriate $U_3$ displacement on the boundary. In all the simulations presented in this section, the excitation amplitude $U_2$ is set to zero.
Before presenting the results, we show the dispersion curves (Figure 2.3) for the plate with material properties ($E = 80 GPa$, $\nu = 0.35$ and $\rho = 2700 Kg/m^3$).

![Dispersion curves](image)

**Figure 2.3: Dispersion curve for the traction-free plate**

### 2.2.1.1 SH-mode excitation

Here, we present results obtained for three different excitations: uniform $U_3$ displacement at 0.5 MHz and linear, antisymmetric $U_3$ displacement through the thickness at 2 and 0.5 MHz. As RL and SH modes in isotropic plates are decoupled, we expect to have only SH modes in the solution obtained using FDFE.
BC-Uniform U₃ displacement at 0.5 MHz

A displacement BC of \( U₃ = 1 \) is applied at the left end of the plate and the nodal displacements are computed for a frequency of 0.5 MHz. Results obtained are shown in Figure 2.4. Here

**Figure 2.4a** shows a snapshot of the \( U₃ \) displacement throughout the domain as an intensity map.

**Figure 2.4b** shows \( U₃ \) displacement field through the thickness of the plate (commonly known as wavestructure); this is plotted for all cross-sections (vertical-slices) along the \( x₁ \) direction.

**Figure 2.4c** shows the plot of \( U₃ \) displacement with \( x₁ \) at each \( x₂ \) (horizontal slices); For this mode all are identical as the wavestructure is uniform through the thickness.

**Figure 2.4d** shows the wavenumber spectrum obtained by taking the spatial FFT of \( U₃ \) displacement along \( x₁ \) for each \( x₂ \) (horizontal slice).

Similar plots are shown for all the subsequent examples.

Except for the end-effects, the wavestructure shown in Figure 2.4b corresponds to the \( SH₀ \) mode. The wavenumber obtained from Figure 2.4b is \( k = 942.5 \text{ m}^{-1} \), which enables the computation of the phase velocity as \( c_p = \frac{\omega}{k} = 3.33 \text{ mm/µs} \). This is in very good agreement with the actual phase velocity (3.31 \text{ mm/µs}) of the \( SH₀ \) mode obtained from the dispersion curves in Figure 2.3. The wave field features obtained from FDFE and the good agreement of the phase velocities suggest that it is the \( SH₀ \) mode present in the plate—as it should be.
Figure 2.4: Symmetric $U_3$ excitation (0.5 MHz) (a) displacement field (b) wavestructure (c) displacement vs $x_1$ (d) spatial Fourier transform of $U_3$

**BC-Linear variation of $U_3$ from -1 to 1 through the thickness**

Linear antisymmetric displacement excitation at the left end of the plate gives the results shown in Figure 2.5 for a frequency of 2 MHz. The mode excited is the $SH_1$ mode. The wavenumber obtained from the spatial Fourier transform (Figure 2.5d) is $k = 1990\ m^{-1}$. The computed phase velocity is $c_p = \frac{\omega}{k} = 6.31\ mm/\mu s$, while the
actual phase velocity of the mode is 5.9 $mm/\mu s$ from Figure 2.3. This discrepancy in the phase velocity is due to insufficient discretization as will be discussed later.

Figure 2.5: Antisymmetric linear $U_3$ excitation (2MHz) (a) displacement field (b) wavestructure (c) displacement vs $x_1$ (d) spatial Fourier transform of $U_3$

The same displacement boundary conditions at 0.5 MHz give the results shown in Figure 2.6. Figure 2.6a shows that the displacement dies out rapidly along the $x_1$ direction. This is because we are trying to excite an antisymmetric SH mode at
0.5 MHz where, in fact, there is none. This example demonstrate the capability of FDFE in identifying the presence and absence of guided wave modes for a particular boundary condition.

![Figure 2.6: Antisymmetric linear $U_3$ excitation (0.5 MHz) (a) displacement field (b) displacement vs $x_1$](image)

### 2.2.1.2 Rayleigh-Lamb mode excitation

In this section, we present results obtained for RL mode excitation
BC-Uniform $U_1$ displacement through the thickness

A uniform $U_1 = 1$ displacement is prescribed as the displacement boundary condition at the left end of the plate and FDFE is employed to compute the displacement field at a frequency of 0.5 MHz. Figure 2.7a shows the displacements $U_1$ and $U_2$ as an intensity map. Figures (2.7b-2.7d) show the wave structures and displacements as a function of $x_1$ while Figure 2.7e shows the spatial FFT. The phase velocity of the mode is computed from the wavenumber and is $c_p = 6 \ mm/\mu s$. This is in good agreement with the actual phase velocity of the $S_0$ mode which is $c_p = 5.79 \ mm/\mu s$ from the dispersion curve in Figure 2.3.
Figure 2.7: Symmetric $U_1$ excitation at 0.5 MHz; (a) displacement field (b) $U_1$ wavestructure (c) $U_2$ wavestructure (d) displacement vs $x_1$ (e) spatial Fourier transform of $U_1$

BC-Sinusoidal $U_1$ displacement through the thickness

A displacement $U_1(0, x_2) = -\sin(2\pi(2x_2 + 1))$ is applied as boundary condition at the left end of the plate and FDFE is employed to compute the displacement field at 2 MHz. Due to the antisymmetric nature of the displacement field, it is easy to
expect that antisymmetric modes are excited in the plate. However, there are two antisymmetric modes namely, the $A_0$ and the $A_1$ modes at 2 MHz. This is clearly evident from Figure 2.8e where the FFT shows two distinct peaks corresponding to the $A_0$ and $A_1$ modes. The computed phase velocities for $A_0$ and $A_1$ modes are $3.4 \text{ mm/µs}$ and $10.9 \text{ mm/µs}$ respectively. The actual phase velocities from Figure 2.3 for the modes are $2.82 \text{ mm/µs}$ and $10.82 \text{ mm/µs}$. The larger error in the phase velocity of the $A_0$ mode is due to insufficient discretization as will be discussed later. This example demonstrates the ability of FDFE to capture the existence of multiple modes for a given boundary condition. This is important in studying guided wave mode conversions which are common near waveguide transitions.
Figure 2.8: Sinusoidal $U_1$ excitation at 2 MHz; (a) displacement field (b) $U_1$ wavestructure (c) $U_2$ wavestructure (d) displacement vs $x_1$ (e) spatial Fourier transform of $U_1$

2.2.2 FDFE-Inhomogeneous waveguides

In this section, we demonstrate the approach for two simple cases of inhomogeneous waveguides:
1. Bi-material plate - a plate composed of two materials perfectly bonded together through the thickness.

2. A functionally graded plate with linearly varying material properties along the length of the waveguide.

The material properties used in the simulation are shown in Table 2.1.

<table>
<thead>
<tr>
<th>Material</th>
<th>E (GPa)</th>
<th>$\nu$</th>
<th>$\rho$ (kg/m$^3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material 1</td>
<td>80</td>
<td>0.35</td>
<td>2700</td>
</tr>
<tr>
<td>Material 2</td>
<td>210</td>
<td>0.35</td>
<td>7800</td>
</tr>
</tbody>
</table>

**Bi-material plate**

The bi-material plate has material 1 on the left and material 2 on the right with a sharp interface in the middle.

**BC-Uniform $U_3 = 1$ through the thickness of the plate**

Displacement boundary conditions are applied to excite $SH_0$ mode (0.4 MHz) in material 1 at the left end of the plate. The $U_3$ displacement field plotted in Figure 2.9c shows that the acoustic impedance mismatch at the interface results in a reflection and a decrease in energy transmitted across the interface. Figure 2.9d shows the spatial FFT of the displacement field in both the materials. The wavenumbers are clearly different as the phase velocities in material 1 and material 2 are 3.31 $mm/\mu s$ and 3.16 $mm/\mu s$ respectively. As the phase velocities were close, a more refined discretization with 180 elements along the length while increasing the waveguide length to 160 mm was employed. The computed phase velocities are 3.6 $mm/\mu s$ and 3.2 $mm/\mu s$ in material 1 and material 2 respectively.
Figure 2.9: Uniform $U_3 = 1$ excitation (0.4 MHz) (a) displacement field (b) wavestructure (c) displacement vs $x_1$ (d) spatial Fourier transform of $U_3$

**BC-Linearly varying $U_3$ from -1 to 1 through the thickness of the plate**

Displacement boundary conditions are employed to excite SH$_1$ mode (1.9 MHz) at the left end of the plate. A waveguide length of 60 mm is chosen, 200 elements are used to discretize along the length and 10 elements are used along the thickness direction. The results obtained are shown in Figure 2.10. The actual phase velocities of this mode in material 1 and material 2 are 6.78 mm/µs and 5.68 mm/µs while the computed phase velocities are 7.1 mm/µs and 5.7 mm/µs respectively.
Functionally graded plate with linearly varying material properties

The material properties are linearly varied from the left end to the right end of the plate with the left end corresponding to the material 1 and the right end corresponding to the material 2. This gradation is implemented in such a way that each element is assumed to be homogeneous.
**BC-Uniform $U_1 = 1$ through the thickness of the plate**

Boundary conditions are imposed to excite the $S_0$ mode at 0.5 MHz at the left end of the plate. The length of the waveguide is chosen to be 60 mm and 120 elements are used to discretize along the length of the waveguide and 10 elements are used along the thickness direction. The results are shown in Figure 2.11. As can be seen in Figure 2.11d, the amplitude of the wave decreases with the propagation distance $x_1$ due to the continuous variation of the material properties.
Figure 2.11: Symmetric $U_1$ excitation at 0.5 MHz; (a) displacement field (b) $U_1$ wavestructure (c) $U_2$ wavestructure (d) displacement vs $x_1$ (e) spatial Fourier transform of $U_1$

2.3 Factors affecting the numerical accuracy of the solutions

In this section, we investigate the effect of two factors, namely the spatial discretization and the length of the waveguide on the solution obtained using FDFE. For simplicity, we study this using SH$_0$ mode excitation in a homogeneous plate.
2.3.1 Effect of spatial discretization

Here, we investigate the effect of spatial discretization while keeping the length of the waveguide constant at 60 mm. Figure 2.12 shows the result obtained for the SH\(_0\) mode at 0.2 MHz for varying number of elements i.e., 20, 40, 80, 120 and 160 which correspond to \(\lambda/5.52\), \(\lambda/11\), \(\lambda/22.1\), \(\lambda/33.1\) and \(\lambda/44.1\) respectively. Here \(\lambda\) denotes the wavelength of the SH\(_0\) mode (0.2 MHz). As can be seen, the displacement field has essentially converged for 80 elements (\(\lambda/22.1\)).

![Figure 2.12: Effect of spatial discretization on the solution for the SH\(_0\) mode (0.2 MHz)](image)

Similar study was done for a higher frequency as well i.e., SH\(_0\) mode at 0.5 MHz. The wavelength of the mode in this case is \(\lambda = 6.62\ mm\). Figure 2.13 shows the result obtained for 40, 80, 120, 160 and 200 elements which correspond to \(\lambda/4.4\), \(\lambda/8.8\), \(\lambda/13.2\), \(\lambda/17.6\) and \(\lambda/22.1\) respectively. The solutions appears to have reasonably converged for 120 elements (\(\lambda/13.2\)). Based on the above study, we recommend a maximum element size of \(\lambda/20\) to ensure good accuracy in solutions obtained using FDFE.
Before we end this section, we comment on the discrepancy in the phase velocity computed for the $A_0$ mode in Figure 2.8. The actual phase velocity of the $A_0$ mode is $2.82 \text{ mm/µs}$ at a frequency of 2 MHz which corresponds to a wavelength of 1.41 mm. The element-size used for that particular simulation was 0.5 mm which is not in agreement with our recommendation for maximum element size of $\lambda/20 - \lambda/10$. Simulation was rerun with a spatial discretization of $\lambda/10$ and the phase velocity was obtained to be $2.9 \text{ mm/µs}$ which is much closer to the actual value of $2.82 \text{ mm/µs}$.

### 2.3.2 Effect of waveguide length

Other factor that affects the solution from FDFE is the length of the waveguide (or structural domain) used for the study. This in fact has a greater effect on the wavenumber ($k$) which affects the computed phase velocity. Suppose that the length of the waveguide is ‘L’ and ‘N’ elements each with a length $\Delta x$ are used to discretize the domain. The resolution in wavenumber is given by $\frac{2\pi}{(N+1)\Delta x} \approx \frac{2\pi}{L}$. Hence, the longer the length of the structural domain, the higher is the resolution in the wavenumber. To demonstrate this, FDFE is run for waveguides of different length keeping the spatial discretization ($\Delta x$) constant.

Figure 2.14 shows the normalized FFT’s of the displacement field for the $SH_0$ mode (0.5 MHz).
mode at 0.2 MHz for different waveguide lengths of 30, 60 and 120 mm with $\Delta x = 1 \, mm$. Similar results are shown for the $SH_0$ mode at 0.5 MHz for different waveguide lengths of 30, 45 and 60 mm with $\Delta x = 0.5 \, mm$. Both the figures indicate that the peak of the FFT becomes narrower as the length of the waveguide increases. This is in line with our signal processing knowledge that the resolution in the frequency domain increases with increase in the length of the time-domain window used for taking FFT.

Figure 2.14: Effect of waveguide length on the solution for the $SH_0$ mode (0.2 MHz)

Figure 2.15: Effect of waveguide length on the solution for the $SH_0$ mode (0.5 MHz)

In the next section we outline how FDFE can be used for guided wave mode selection for inhomogeneous/complex waveguides.
2.4 Guided wave mode selection using FDFE

FDFE has previously been employed for a variety of problems involving scattering from defects, reflection and transmission across interfaces, etc (Drozdz, 2008) (Rajagopal et al., 2012) (Moreau et al., 2012). However, one other important application of FDFE that has not received sufficient attention is guided wave mode selection for NDE/SHM applications. Here, we briefly describe how FDFE can be employed for efficient guided wave mode selection especially for inhomogeneous waveguides.

Assuming that guided wave modes are reasonably well known at the location of the transducer, the excited waves travel along the waveguide and may get transformed into different modes depending on the nature of waveguide and/or due to the presence of defects or different interface/boundary conditions. For identifying a specific targeted type of defect, it is essential that we select the appropriate mode that is most affected by the presence of the defect we are interested in. Often, this requirement can be stated in terms of stresses/displacements or some combination of those at the location of defect. For the case of an inhomogeneous waveguide, the guided wave field features at the location of the defect can be very different than those at the transducer location. Hence it is essential that we study the interaction of different modes with the defect and then choose the appropriate mode that has the desired characteristics of the wave field at the location of the defect. This can be easily accomplished using FDFE as

1. We can study the response of the waveguide for all possible excitations.

2. Change boundary conditions or add damage at appropriate locations to assess their influence on wave propagation.

3. Obtain wavenumbers and phase velocities of the guided wave modes away from the location of the transducer.

Next, we present two examples where the FDFE method is employed to study guided wave mode conversions in an important class of inhomogeneous wave guides, namely waveguide transitions. The results obtained are qualitatively compared with those from time-domain finite element simulations. We present results obtained for two cases of waveguide transitions:
1. Stepped aluminum plate

2. Aluminum-Epoxy-Aluminum waveguide

The results presented here are from Chillara and Lissenden (2014a). All simulations were carried out in ABAQUS, a commercial finite element software.

2.4.1 Stepped aluminum plate

The schematic of the stepped aluminum plate is shown in Figure 2.16. Displacement boundary conditions were applied at the left end of the plate to excite the $S_0$ mode at 0.16 MHz.

![Figure 2.16: Schematic of the stepped aluminum plate](image)

Figure 2.16: Schematic of the stepped aluminum plate

Figure 2.17 shows the result obtained from FDFE for the $U_2$ displacement field in the plate. As can be seen, even though we excite the $S_0$ mode at the left end of the plate, it gets mode converted to the antisymmetric mode at the transition (step). This is clearly evident from the results obtained from the time-domain finite element simulation shown in Figure 2.18 where the antisymmetric $U_2$ displacement field at $t = 18 \mu s$ converts to symmetric $U_2$ displacement field on both sides of the transition.

![Figure 2.17: $U_2$ displacement field obtained from FDFE for the stepped aluminum plate](image)
2.4.2 Aluminum-Epoxy-Aluminum waveguide

Here we present results obtained for the Aluminum-Epoxy-Aluminum waveguide, the schematic of which is shown in Figure 2.19. The material properties used for simulation are shown in Table 2.2. The results obtained from FDFE are compared with those from the time-domain simulations for two modes, namely the $S_0$ mode 0.44 MHz and 1.11 MHz.

Table 2.2: Material properties of aluminum and epoxy used for simulation.

<table>
<thead>
<tr>
<th>Material</th>
<th>E (GPa) (Youngs modulus)</th>
<th>$\nu$ (Poisson’s ratio)</th>
<th>$\rho$ (g/cc)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminum</td>
<td>70</td>
<td>0.33</td>
<td>2.7</td>
</tr>
<tr>
<td>Epoxy</td>
<td>2.56</td>
<td>0.4</td>
<td>1.1</td>
</tr>
</tbody>
</table>

$S_0$ mode (0.44 MHz)

Figure 2.20 shows the ($\sigma_{11}$) stress field in the plate obtained from FDFE. Similar results are obtained from the time domain finite element simulation as shown in Figure 2.21.
Figure 2.20: $\sigma_{11}$ stress field for the $S_0$ mode (0.44 MHz) obtained from FDFE for the Aluminum-Epoxy-Aluminum waveguide

Figure 2.21: $\sigma_{11}$ stress field for the $S_0$ mode (0.44 MHz) obtained from time-domain finite element simulation for the Aluminum-Epoxy-Aluminum waveguide

$S_0$ mode (1.11 MHz)

Figure 2.22 shows the stress field ($\sigma_{11}$) obtained from the FDFE. Note the alternating pattern in the stress field which agrees very well with that obtained from the time-domain simulation as shown in Figure 2.23.

Figure 2.23: $\sigma_{11}$ stress field for the $S_0$ mode (1.11 MHz) obtained from time-domain finite element simulation for the Aluminum-Epoxy-Aluminum waveguide

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2.5 Conclusions

In this chapter we investigated the capability of FDFE to study guided waves in complex waveguides. Elastic boundary value problem was formulated and solved using FDFE. The approach was applied to isotropic homogeneous plates and well-known Rayleigh-Lamb and Shear-Horizontal modes were recovered depending on the nature of the boundary condition. A procedure to extract the wavenumber from the spatial FFT and hence evaluate phase velocity of guided wave modes was presented. The approach was also employed to study two simple cases of inhomogeneous waveguides, namely a bi-material plate and a functionally graded plate with linearly varying material properties. The effect of spatial discretization and the length of the structural domain on the solution obtained using FDFE was studied. It was found that for a given length of the waveguide, the finer the discretization the greater was the accuracy of the solution. On the other hand, for a given discretization, longer waveguide length provided higher resolution in the wavenumber. Finally, the results obtained from FDFE are qualitatively compared to those obtained from the time-domain finite element simulation for waveguide transitions. Both were found to be in good agreement.

FDFE has the following advantages when compared to the time-domain finite element simulation. They are:

1. It is computationally inexpensive.
2. Structural response of the waveguide to a single mode can be studied.
3. Phase velocities of the modes can be obtained as opposed to the group velocities in the time-domain finite element simulations.
4. Guided wave mode selection for inhomogeneous waveguides can be easily carried out.

Before we end this chapter we would like to note that for quantitative comparison of the solutions from FDFE with those from time-domain simulations, one needs to use some kind of absorbing boundary conditions as was illustrated in Rajagopal et al. (2012) and Drozd (2008). Also, one can use spectral elements (Gopalakrishnan et al., 2007) instead of standard finite elements to solve the PDE’s in the frequency domain. This appears an attractive option to be explored.
Chapter 3
Guided wave acoustoelasticity

Introduction

In this chapter we study the nonlinear effect of stress on the wave speeds which is more commonly referred to as the “Acoustoelastic” effect. First we present a brief review of earlier investigations concerning the study of acoustoelasticity. Then, we present the theory behind understanding the guided wave acoustoelastic effect in section 3.1. Some numerical results from the theory of presented in section 3.1 are presented in section 3.2. Here, we also discuss the applicability of FDFE introduced in chapter 2 to study the acoustoelastic effect. Finally, conclusions are drawn in section 3.4.

The early investigation concerning the acoustoelastic effect can be traced back to the work of Hughes and Kelly (1953) and Toupin and Bernstein (1961) where they investigated the effect of external stress on the elastic wave speeds in the material. Explicit expressions relating the wave speeds to the stresses in terms of Lame’s and Murnaghan constants were obtained using a second order deformation theory. Experiments were conducted for samples under hydrostatic pressure and it was concluded that the pressure resulted in increase of the wave speeds for most of the materials. Several investigations (Thurston and Brugger, 1964) (Egle and Bray, 1976) (Hiki and Granato, 1966) relied on using the acoustoelastic effect for determining the higher order elastic constants (Murnaghan constants) in materials especially in anisotropic single crystals. It should be noted that the earlier investigations employed bulk waves for studying the acoustoelastic effect in materials. However, Rayleigh waves have also been extensively employed (Jassby and Kishoni, 1983) (Duquennoy et al., 1999) (Gokhale, 2007) to study acoustoelastic effect. Theoretical treatments concerning the use of Rayleigh waves can be found
Use of ultrasonic guided waves for studying acoustoelastic effect has not been explored to a great extent except for a few representative works by Lanza di Scalea et al. (2003) and Gandhi et al. (2011). Gandhi (2010) studied the guided wave acoustoelastic effect in plates both from a numerical and an experimental standpoint. Dispersion curves for the plate in the presence of external load were obtained using the partial wave technique (Rose, 1999). Experimental investigations were carried out to verify the theory and both were found to be in good agreement. The goal of this chapter is to present a formulation for studying guided wave acoustoelastic effect. In addition, we illustrate the application of the FDFE to study the acoustoelastic effect in waveguides.

Next, we introduce the mathematical theory behind the guided wave acoustoelastic effect.

### 3.1 Guided wave acoustoelastic effect

The Acoustoelastic effect comes under the class of problems where a small deformation is superposed on a large deformation in the material. Here, the small deformation corresponds to a stress wave and a large deformation corresponds to the pre-stress in the material. The general theoretical approach for this class of problems can be found in Ogden (1997) and Norris (1998).

Consider an initially isotropic plate loaded as shown in Figure 3.1. Let the initial displacement field in the material (due to pre-stress) be \( u_1(X) \) and that due to the ultrasonic wave be \( u_2(X, t) \) so that the total displacement is

\[
u(X, t) = u_1(X, t) + u_2(X, t)
\]

This gives the total displacement gradient as

\[
H = H_1 + H_2
\]

where \( H_1 = \frac{\partial u_1}{\partial X} \) and \( H_2 = \frac{\partial u_2}{\partial X} \). The balance of linear momentum for the initial pre-stress and that with the ultrasonic wave are given by

\[
\text{Div}(S(H_1)) = 0
\]

and
\[
\text{Div}(S(H_1 + H_2)) = \rho_\kappa \ddot{u}_2
\] (3.4)

Figure 3.1: Schematic of the loaded plate

Subtracting Eqn(3.3) from Eqn(3.4), we get

\[
\text{Div}(S(H_1 + H_2)) - \text{Div}(S(H_1)) = \rho_\kappa \ddot{u}_2
\] (3.5)

The above equation when written for the present case of “small deformation superposed on a large one” where we enforce the assumption \( ||H_2|| < < ||H_1|| \) by considering \( H_2 \) to be a differential displacement gradient superposed on \( H_1 \), we get

\[
\text{Div} \left( \frac{\partial S}{\partial H} \bigg|_{H_1} H_2 \right) = \rho_\kappa \ddot{u}_2
\] (3.6)

Here, \( \frac{\partial S}{\partial H} \bigg|_{H_1} \) can be thought of as a fourth-order tensor of instantaneous/incremental elastic moduli (Ogden, 1997). For the guided wave acoustoelastic problem we have \( u(X) = u(X_2)e^{i(kX_1 - \omega t)} \) and the Eqn(3.6) along with the traction-free boundary condition reads as

\[
\text{Div} \left( \frac{\partial S}{\partial H} \bigg|_{H_1} H_2 \right) = \rho_\kappa \omega^2 u(X)
\]

\[
\left( \frac{\partial S}{\partial H} \bigg|_{H_2} \right) n_\kappa = 0
\] (3.7)

Here, \( n_\kappa \) denotes the unit outward normal to the surface of the plate. Using the constitutive model introduced in chapter 1 (Eqn(1.23)), we have
\[
\frac{\partial S_{ij}}{\partial H_{kl}} = \lambda \delta_{ij} \delta_{kl} + (\mu + BH_{pp})(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + \lambda(H_{ij} \delta_{kl} + H_{kl} \delta_{ij})
\]
\[
+ \lambda H_{pp} \delta_{ik} \delta_{jl} + \mu(H_{ij} + H_{jl}) \delta_{ik} + \mu(H_{ik} + H_{ki}) \delta_{jl} + \mu(H_{il} \delta_{jk} + H_{kj} \delta_{il})
\]
\[
+ 2CH_{pp} \delta_{ij} \delta_{kl} + B(H_{ik} + H_{ki}) \delta_{ij} + B(H_{ij} + H_{ji}) \delta_{kl}
\]
\[
+ A \left((H_{ik} + H_{ki}) \delta_{jl} + (H_{jl} + H_{lj}) \delta_{ik} + (H_{jk} + H_{kj}) \delta_{li} + (H_{il} + H_{li}) \delta_{jk}\right)
\]

in index notation where '\(\delta\)' denotes the Kronecker delta.

### 3.2 Numerical results

In this section we present numerical results demonstrating the acoustoelastic effect. First, results are presented using the expressions obtained using Eqn(3.8). We consider tri-axial stretch deformation in material given by

\[
x_1 = p_1 X_1, x_2 = p_2 X_2, x_3 = p_3 X_3;
\]

where \(p_1\), \(p_2\) and \(p_3\) are the stretches in the coordinate directions. The corresponding displacement gradient is given by

\[
H = \begin{bmatrix}
  p_1 - 1 & 0 & 0 \\
  0 & p_2 - 1 & 0 \\
  0 & 0 & p_3 - 1
\end{bmatrix}
\]

(3.10)

The instantaneous moduli are evaluated using Eqn(3.8) for different cases of pre-deformation \((H)\) and the dispersion curves for the plate are computed using the SAFE approach. The results are presented in the following section.

#### 3.2.1 Phase and group velocity dispersion curves

The phase velocity dispersion curves are computed for the plate with material properties shown in Table 3.1.
Table 3.1: Material properties in GPa used for obtaining the numerical results

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\mu$</th>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>25</td>
<td>-320</td>
<td>-200</td>
<td>-190</td>
</tr>
</tbody>
</table>

We present results for three cases of uniaxial stretch.

$p_1=1.001; \ p_2=1; \ p_3=1$

Figure 3.2 shows the dispersion curves for the plate under a uniaxial stretch $p_1 = 1.001$ where the percent change in the phase velocity is indicated as a colormap. Similarly, Figures (3.3 & 3.4) demonstrate that for $p_1 = 1.002$ and $p_1 = 1.003$. As can be observed the percent change in phase speed increases with increasing level of deformation.

Figure 3.2: Phase velocity dispersion curves with percent change in speed shown as a colormap for uniaxial stretch $p_1 = 1.001$ (a) fundamental modes (b) higher modes
\( p_1 = 1.002; \quad p_2 = 1; \quad p_3 = 1 \)

![Figure 3.3: Phase velocity dispersion curves with percent change in speed shown as a colormap for uniaxial stretch \( p_1 = 1.002 \) (a) fundamental modes (b) higher modes](image)

\( p_1 = 1.003; \quad p_2 = 1; \quad p_3 = 1 \)

![Figure 3.4: Phase velocity dispersion curves with percent change in speed shown as a colormap for uniaxial stretch \( p_1 = 1.003 \) (a) fundamental modes (b) higher modes](image)

### 3.2.2 Deformation induced anisotropy

Another important aspect of the acoustoelastic effect is the anisotropy induced due to the presence of the pre-deformation/stress i.e., the material which is initially
isotropic behaves as an anisotropic material under load. To illustrate this we again consider the case of a triaxial stretch. For the undeformed material, the elastic constant matrix in Voigt notation is given by

$$C_{\text{linear}} = \begin{bmatrix}
110 & 60 & 60 & 0 & 0 & 0 \\
60 & 110 & 60 & 0 & 0 & 0 \\
60 & 60 & 110 & 0 & 0 & 0 \\
0 & 0 & 0 & 25 & 0 & 0 \\
0 & 0 & 0 & 0 & 25 & 0 \\
0 & 0 & 0 & 0 & 0 & 25 \\
\end{bmatrix} \text{ GPa}$$

For uniaxial stretch ($p_1 = 1.002; p_2 = 1; p_3 = 1$), the incremental/instantaneous modulii from Eqn(3.8) are

$$C_{\text{incremental}} = \begin{bmatrix}
105.42 & 58.56 & 58.56 & 0 & 0 & 0 \\
58.56 & 107.76 & 59.24 & 0 & 0 & 0 \\
58.56 & 59.24 & 107.76 & 0 & 0 & 0 \\
0 & 0 & 0 & 24.32 & 0 & 0 \\
0 & 0 & 0 & 0 & 24.1 & 0 \\
0 & 0 & 0 & 0 & 0 & 24.1 \\
\end{bmatrix} \text{ GPa}$$

which corresponds to a transversely isotropic material.

For biaxial stretch ($p_1 = 1.002; p_2 = 1.001; p_3 = 1$), the incremental modulii are

$$C_{\text{incremental}} = \begin{bmatrix}
104.3 & 57.84 & 58.18 & 0 & 0 & 0 \\
57.84 & 105.47 & 58.52 & 0 & 0 & 0 \\
58.18 & 58.52 & 106.64 & 0 & 0 & 0 \\
0 & 0 & 0 & 23.87 & 0 & 0 \\
0 & 0 & 0 & 0 & 23.76 & 0 \\
0 & 0 & 0 & 0 & 0 & 23.65 \\
\end{bmatrix} \text{ GPa}$$

which corresponds to an orthotropic material. As can be seen, the degree of anisotropy is dependent on the state of stretch/stress in the material. However, the degree of anisotropy is small but can be quantified from wave speed dependence on the direction of wave propagation. Consider the plate with coordinate-axes as shown in Figure 3.5. We investigate the anisotropy using the phase speed dependence on the angle ‘θ’ of wave propagation direction in the $X_1$-$X_3$ plane.
Figure 3.5: Schematic of the plate with coordinate-axes

Figure 3.6 shows the phase speed of the $S_0$ mode ($k=1000 \ m^{-1}$) as a function of $\theta$ for both the cases of uniaxial and biaxial stretch. As can be seen it is $\pi$ periodic as expected and the amplitude of the phase velocity change increases with increasing deformation.
Another example we choose to demonstrate deformation induced anisotropy is the case of biaxial-stretch in the $X_1 - X_3$ plane. Figure 3.7 shows the phase velocity ($c_p$) versus $\theta$ for the following cases of biaxial stretch; $p_1 = 1.001$ and $p_2 = 1.001, 1.002, 1.003$. It should be noted that the case of equibiaxial stretch shows no anisotropy as expected from the elementary stress transformation laws (Mohr’s circle).
It should be noted that all the results presented above correspond to homogeneous pre-deformation/stress in the material. However, it is important to investigate the effect of inhomogeneous stress state on the acoustoelastic effect. This case of inhomogeneous pre-stress is not tractable analytically and numerical methods are the only resort for this problem. To that end we investigated guided wave acoustoelastic effect using FDFE.

### 3.2.3 FDFE-Guided wave acoustoelasticity

All the results shown in this section are from simulations carried out in COMSOL. The schematic of the plate used for the simulations is shown in Figure 3.8. To eliminate the effect of boundary conditions, we use perfectly matched layer (PML) on both sides of the plate. To illustrate this, we carried out FDFE simulations for $S_0$ mode (0.5 MHz) with two different boundary conditions, namely fixed-fixed and fixed-free boundary condition. Figure 3.9 shows the comparison of the solution and as can be seen they match very well illustrating the efficacy of PML in eliminating
the effect of boundary conditions.

Figure 3.8: Schematic of the plate with the PML

Figure 3.9: Comparison of the solution with fixed-fixed and fixed-free boundary condition.

Simulations are run for different levels of uniaxial stress (20 MPa, 60 MPa, 100 MPa and 140 MPa) in the \( x \) direction as shown in Figure 3.8. The excitation for the \( S_0 \) mode is given as a pressure boundary condition on the top and bottom surface of the plate between \( x = 0 \) and \( x = 0.02 \) m. Figure 3.10 shows the \( x \) component of displacement on the top-surface of the plate along the length of the waveguide. As can be seen, increasing load tends to decrease the wavelength of the mode as is evident in the increasing phase difference between the waveforms. Also, it should be noted that the load affects the amplitude of the wave which in this case tends to decrease it. This is because of the change in the mode excitability at different levels of stress. Similar results are shown for the \( S_0 \) mode (1 MHz) in Figure 3.11.

This study illustrates that FDFE can be employed to numerically study the acoustoelastic effect in structures. In particular the study can be carried out for any inhomogeneous pre-stress/strain in the material.
Figure 3.10: Effect of load on the $S_0$ mode (0.5 MHz).

Figure 3.11: Effect of load on the $S_0$ mode (1 MHz).
3.3 Experiments

Some preliminary experimental investigations were carried out to study guided wave acoustoelastic effect for Lamb waves in plates. The schematic of the plate used of the experiments are shown in Figure 3.12. Ten PZT disc transducers were bonded to a Aluminum 7075-T6 plate on a circle of diameter 406 ± 6 mm (16 ± 0.25 inch). The signals for each pair of transducers were acquired by sending through one and receiving through all the other pairs. The plate has dimensions 914.4 mm × 609.6 mm × 1.58 mm (36 in × 24 in × 1/16 in) is uniaxially loaded on a MTS tensile test machine as indicated in the schematic. The acoustoelastic effect is studied for $S_0$ mode at the frequency-thickness product 490 kHz-mm which corresponds to the the transducer resonance frequency of 310 kHz. The transducer data are acquired for various loads on the plate; baseline (no load), 5000 lbs, 7500 lbs, 10000 lbs, 12500 lbs, 15000 lbs. These correspond to the uniaxial stresses of 23 MPa, 34.47 MPa, 46 MPa, 57.5 MPa and 69 MPa on the plate.

![Figure 3.12: Schematic of the experimental setup for studying guided wave acoustoelasticity.](image)

Figure 3.13 shows the time-domain signals obtained from the 1-6 transducer pair for different loads. Clearly, the signals arrive later in time for increasing load which shows that the wave travels with a lower speed. Also shown in Figure 3.14 is the time of arrival of the signals for various loads. It can be observed that the time of flight varies almost linearly with the applied load. One thing of interest is to
compare the change in time due to the change in the propagation path under load with the change due to the acoustoelastic effect. For example, consider the case when 15,000 lbs i.e., 69 MPa of stress is applied on the plate. The change in the length of the wave propagation path due to the stress is $0.001 \times 406\text{mm} = 0.406\text{mm}$. The wave speed of $S_0$ mode is $5.35\text{ mm/µs}$ which corresponds to a change in time of flight of $\frac{0.406\text{mm}}{5.35} = 0.076\mu s$. However, the observed change in time of flight in the experiment (Figure 3.14) is $90.92 - 90.48 = 0.44\mu s$. Hence it can be concluded that the change in the time of flight is due to the nonlinearity in the material that is responsible for the acoustoelastic effect.

![Figure 3.13: Time domain signal for 1-6 transducer pair (a) full window $S_0$ mode (b) zoomed-in plot](image)

![Figure 3.14: Time of flight versus load for 1-6 pair.](image)
Figure 3.15 shows the time-domain signals for the 3-8 pair. As can be seen from the zoomed-in version of the figure, the change in time of flight for this case is much less compared to that for 1-6 pair.

![Figure 3.15: Time domain signal for 3-8 transducer pair (a) full window S₀ mode (b) zoomed-in plot](image)

Figure 3.16 shows the time-domain signals for the 10-5 pair and as can be seen from the zoomed-in version in Figure 3.16b, the signals for higher loads shift to the left due to the Poisson’s effect i.e., due to the compressive strain perpendicular to the loading direction. It should also be noted that at some angle between the 3-8 pair and 10-5 pair, the time of flight does not change with the load.

![Figure 3.16: Time domain signal for 10-5 transducer pair (a) full window S₀ mode (b) zoomed-in plot](image)

More detailed study on guided wave acoustoelasticity can be found in Gandhi 60
et al. (2011).

3.4 Conclusions

In this chapter we investigated the guided wave acoustoelastic effect from a theoretical standpoint. It was found that the pre-stress/strain influences the wave speeds in the material and hence the change in the wave speeds can be used to measure the residual stress in the materials. Instantaneous/incremental elastic moduli for a Murnaghan hyperelastic material were obtained and SAFE was used to determine the guided wave speeds as a function of the external load. Also, deformation induced anisotropy due to the pre-stress/strain was discussed in the context of uniaxial and bi-axial stretch deformation. In addition, use of FDFE for numerically studying the guided wave acoustoelastic effect was outlined. Some preliminary experimental results concerning the acoustoelastic effect for $S_0$ mode are discussed.
Chapter 4
Higher harmonic guided waves in isotropic weakly nonlinear elastic plates

Introduction

In this chapter we present the problem formulation for higher harmonic guided waves in plates. Second harmonic generation of guided wave modes along with some important aspects of mode selection for generating cumulative second harmonics is discussed in section 4.1. Then, a generalized theory of mode interaction is presented in section 4.2 to predict the nature of higher harmonic guided wave modes generated due to interaction of two or more primary guided waves. Finally, conclusions are drawn in section 4.3. The theoretical treatment and results presented in this chapter are mainly from Chillara and Lissenden (2012), Chillara (2012) and Chillara and Lissenden (2016).

4.1 Second harmonic guided waves: Theory and Problem formulation

4.1.1 Notation

The problem formulation for second harmonic guided wave generation heavily relies on the principles of continuum mechanics presented in Chapter 1. Here, we introduce some notation that we use throughout this chapter and elsewhere in the thesis. We consider a weakly nonlinear elastic material with a strain energy function (Landau and Lifshitz, 1986) defined by
\[ W(E) = \frac{1}{2} \lambda (tr(E))^2 + \mu tr(E^2) + \frac{1}{3} C (tr(E))^3 + Btr(E)tr(E^2) + \frac{1}{3} Atr(E^3) \quad (4.1) \]

where \( \lambda, \mu \) are Lames constants and \( A, B \) and \( C \) are higher order elastic constants (Norris, 1998) and \( E \) is the Lagrangian strain. Another equivalent version of the above model is the Murnaghan model given by

\[ W(E) = \frac{1}{2} \lambda (tr(E))^2 + \mu tr(E^2) + \frac{1}{3} (l+2m)(tr(E))^3 - mtr(E)((tr(E))^2 - tr(E^2)) + n \det(E) \quad (4.2) \]

where \( l, m \) and \( n \) are Murnaghan constants and \( tr() \) and \( \det() \) denote the trace and determinant respectively. Landau-Lifshitz constants \((A, B, C)\) and Murnaghan constants are related by \( l = B + C \), \( m = \frac{1}{2} A + B \) and \( n = A \) (Destrade and Ogden, 2010). We use these two equivalent forms of the strain energy functions throughout the thesis. However, all the analysis presented in this chapter is based on Eqn(4.1).

Before we proceed further, we obtain the expressions for first and second Piola-Kirchoff stress tensors and also some other expressions that are useful during discussion. From Eqn (4.1) and Eqn(1.24) we obtain the second Piola-Kirchoff stress tensor as

\[ T_{RR}(E) = \lambda tr(E) + 2 \mu E + C(tr(E))^2 I + Btr(E^2) I + 2Btr(E)E + AE^2. \quad (4.3) \]

The first Piola-Kirchoff stress tensor can be obtained from the relation \( S = FT_{RR} \).

To improve the clarity of the presentations that follow, we consider first and second Piola-Kirchoff stress tensors as explicit functions of displacement gradient \((H)\) and are denoted as \( S(H) \) and \( T_{RR}(H) \). \( T_{RR}(H) \) can be obtained by using Eqn(4.3) and Eqn(1.15) and is given (up to second order in \( H \)) by

\[
T_{RR}(H) = \frac{\lambda}{2} tr(H + H^T) + \mu(H + H^T) + \frac{\lambda}{2} tr(H^T H) I + C tr(H)^2 I + \mu H^T H \\
+ B tr(H)(H + H^T) + \frac{B}{2} tr(H^2 + H^T H) I + \\
\frac{A}{4} (H^2 + H^T H + H H^T + H^T H^2). \quad (4.4)
\]

Further, we decompose \( T_{RR}(H) \) into two parts, namely \( T_{RR}^L(H) \) and \( T_{RR}^{NL}(H) \) such that \( T_{RR}(H) = T_{RR}^L(H) + T_{RR}^{NL}(H) \). As indicated in the notation, \( T_{RR}^L(H) \)
is linear in \( H \) and \( T_{RR}^{NL}(H) \) is nonlinear in \( H \) and are given by

\[
\begin{align*}
T_{RR}^L(H) & = \frac{\lambda}{2} \text{tr}(H + H^T) + \mu(H + H^T) \\
T_{RR}^{NL}(H) & = \frac{\lambda}{2} \text{tr}(H^T \text{I} + Ctr(H)H^T + \mu H^T H + Btr(H)(H + H^T) + \\
& \quad \frac{B}{2} \text{tr}(H^T H + H^T H + HH^T + H^T^2) \tag{4.5}
\end{align*}
\]

Likewise, using \( S = FT_{RR} \), we can write \( S(H) = S^L(H) + S^{NL}(H) \) where

\[
\begin{align*}
S^L(H) & = T_{RR}^L(H) \quad \text{and} \\
S^{NL}(H) & = HT_{RR}^L(H) + T_{RR}^{NL}(H). \tag{4.6}
\end{align*}
\]

### 4.1.2 Problem Formulation

We start the balance of linear momentum in the referential form for a traction-free plate (Figure 4.1) in the absence of body forces. Using the notation introduced in the section 4.1.1, we get

\[
\text{Div}(S(H)) = \rho_\kappa \ddot{u} \\
Sn_\kappa = 0 \tag{4.7}
\]

where \( u \) denotes the displacement, \( n_\kappa \) denotes the unit normal to the surface of the plate in the reference configuration and \( \rho_\kappa \) denotes the density of the material in the reference configuration.

![Figure 4.1: Schematic of the traction free plate.](image)

We need to solve the Eqn(4.7) assuming that a primary wave is propagating in the material. We achieve this using a perturbation approach (De Lima and Hamilton, 2003). To that end, we decompose the displacement field as

\[
u = u_1 + u_2 \quad \text{with} \quad \|u_2\| \ll \|u_1\| \tag{4.8}
\]
where \( u_1 \) and \( u_2 \) correspond to the displacement field of primary and secondary wave fields respectively. So, the total displacement gradient is given by

\[
H = H_1 + H_2 \text{ with } ||H_2|| << ||H_1||
\]

(4.9)

where \( H_1 = \text{Grad}(u_1) \) and \( H_2 = \text{Grad}(u_2) \). Using Eqn (4.6), we get

\[
S(H) = S^L(H) + S^{NL}(H) \Rightarrow \\
S(H_1 + H_2) = S^L(H_1 + H_2) + S^{NL}(H_1 + H_2) \Rightarrow \\
S(H_1 + H_2) = S^L(H_1) + S^L(H_2) + S^{NL}(H_1 + H_2).
\]

(4.10)

Note that we used the linearity of \( S^L(H) \) in arriving at Eqn(4.10). Since we are interested in the solution for second harmonic, we retain only terms of second order in \( H_1 \) in the complex expression for \( S^{NL}(H_1 + H_2) \), and denote those terms by \( S^{NL}(H_1, H_1, 2) \) which correspond to self-interaction of the primary mode (Chillara and Lissenden, 2012). So from Eqn(4.10) we have

\[
S(H) = S^L(H_1) + S^L(H_2) + S^{NL}(H_1, H_1, 2).
\]

(4.11)

Substituting Eqns(4.11 & 4.8) in Eqn(4.7), we can obtain two separate problems for \( u_1 \) and \( u_2 \) as follows:

\[
\text{Div}(S^L(H_1)) - \rho_\kappa \ddot{u}_1 = 0 \\
S^L(H_1) n_\kappa = 0 \text{ and } \\
\text{Div}(S^L(H_2)) - \rho_\kappa \ddot{u}_2 = -\text{Div}(S^{NL}(H_1, H_1, 2)) \\
S^L(H_2) n_\kappa = -S^{NL}(H_1, H_1, 2) n_\kappa
\]

(4.12)

Now, consider \( u_1 = \text{Re}\{u_1(X_2)e^{i(kX_1-\omega t)}\} \), a propagating guided wave mode in the plate (can either be RL or SH mode), where \( \omega \) denotes the angular frequency and \( k \) denotes the wavenumber of the mode. The first problem in Eqn(4.12) is identically satisfied due to our assumption that \( u_1 \) is a propagating mode in the plate. The solution for \( u_2 \) is obtained using the normal mode expansion technique (Auld, 1990). As in De Lima and Hamilton (2003), we seek asymptotic expansions
of $S^L(H_1)$ and $u_2$ as follows

$$S^L(H_1) = \sum_{m=1}^{m=\infty} A_m(X_1) S_m$$

$$\dot{u}_2 = \sum_{m=1}^{m=\infty} A_m(X_1) v_m$$

where $S_m$ and $v_m$ denote the stress and velocity fields corresponding to the guided wave modes at $2\omega$. As shown by Auld (1990), $A_m(X_1)$ satisfies the following ordinary differential equation (Eqn(4.14)) for each $n$ such that $P_{mn} \neq 0$.

$$4P_{mn} \left( \frac{dA_m}{dX_1} - i k_n^* X_1 \right) = \left( f_n^{surf} + f_n^{vol} \right)$$

Here,

$$P_{mn} = -\frac{1}{4} \int_{-h}^{h} \left( \frac{S_m v_n^* + S_n v_m^*}{4} \cdot n_1 \right) dX_2$$

$$f_n^{surf} = -\frac{1}{2} S^{NL}(H_1, H_1, 2) v_n^* n_2 \bigg|_{-h}^{h}$$

$$f_n^{vol} = \frac{1}{2} \int_{-h}^{h} \text{Div}(S^{NL}(H_1, H_1, 2)) v_n^* dX_2.$$ (4.15)

We note that for every propagating mode $m$ used in the asymptotic expansion, there is only one mode $n = m$ such that $P_{mn} \neq 0$ and $k_m = k_n$. If $m$ corresponds to a non-propagating mode, then $k_m = k_n^*$. This ensures that the solution to Eqn(4.14) is well defined and given by

$$A_m(X_1) = \frac{-i \left( f_n^{surf} + f_n^{vol} \right)}{4P_{mn}} \left( \frac{e^{i k_n^* X_1} - e^{i 2k X_1}}{k_n^* - 2k} \right) \text{ if } k_n^* \neq 2k$$

$$A_m(X_1) = \frac{\left( f_n^{surf} + f_n^{vol} \right)}{4P_{mn}} X_1 \text{ if } k_n^* = 2k$$ (4.16)

Note that if the primary mode is a propagating mode and if there exists another propagating mode $n = m$ such that $k_n^* = k_n = 2k$, then the amplitude $A_m$ increases linearly with the propagation distance and is termed as cumulative harmonic. While this condition is satisfied at every frequency for bulk waves, only specific primary guided wave modes generate cumulative second harmonics. The two conditions a primary mode needs to satisfy for it to generate cumulative second harmonic are
(De Lima and Hamilton, 2003)

1. Phase-matching condition: Existence of a propagating guided wave mode at 
   \((2\omega, 2k)\) where \((\omega, k)\) is the primary mode and 

2. Non-zero power-flux criterion: \((f_n^{\text{surf}} + f_n^{\text{vol}}) \neq 0\) for that mode \(n\) such that 
   \(k_n^* = k_n = 2k\).

Note that the above analysis does not assume the nature of fundamental mode and 
and hence is applicable to both Rayleigh-Lamb (RL) and Shear-Horizontal (SH) modes. 
A complete list of guided wave modes that satisfy the phase matching condition are 
obtained by Chillara (2012) and Matsuda and Biwa (2011). A parity analysis needs 
to be carried out (Chillara, 2012) (Chillara and Lissenden, 2012) (Müller et al., 
2010) for identifying the set of primary modes that satisfy non-zero power-flux 
criterion. The formulation presented above in terms of displacement gradient(\(H\)) 
streamlines and simplifies the parity analysis. It can be shown (Srivastava and 
Lanza di Scalea, 2009) (De Lima and Hamilton, 2003) that only symmetric modes 
exist as cumulative second harmonics. Also, Shear-Horizontal waves are not capable 
of generating cumulative second harmonics. Hence, combining the results obtained 
for modes satisfying phase-matching and non-zero power-flux criterion, we can 
obtain a complete list of guided wave modes (Chillara, 2012) that are capable of 
generating cumulative second harmonics. These are summarized in Table 4.1, where 
\(c_p\) denotes the phase velocity of the primary mode, \(c_l\) denotes the longitudinal wave 
speed in the material, \(c_t\) denotes the transverse wave speed in the material and the 
numbers in the columns are the angular frequencies \(\omega\) of the primary mode capable 
of generating cumulative second harmonic.

Table 4.1: List of guided wave modes that generate cumulative second harmonics

<table>
<thead>
<tr>
<th>Primary mode</th>
<th>cut-off modes</th>
<th>(c_p = c_l)</th>
<th>(c_p = \sqrt{2}c_l)</th>
<th>mode-intersections</th>
</tr>
</thead>
<tbody>
<tr>
<td>symmetric</td>
<td>(\frac{n\pi c_t}{h}) (\frac{n\pi c_t}{h\sqrt{c_l^2 - c_t^2}}) (\frac{\sqrt{2n\pi c_t c_l}}{h\sqrt{2c_l^2 - c_t^2}}) All</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>antisymmetric</td>
<td>(\frac{(2n+1)\pi c_t}{2h}) (\ldots) (\frac{\sqrt{2n\pi c_t c_l}}{h\sqrt{2c_l^2 - c_t^2}}) All</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Until now we focused on the second harmonic guided wave generation in a plate. 
In the next section, we formulate the problem for studying guided wave mode 
interactions in a plate and outline a procedure to predict the nature of higher 
harmonics.
4.2 Generalized theory of guided wave mode interaction

The problem formulation for the generalized theory of mode interaction follows along the same lines as that for second harmonic guided waves presented in section 4.1. However, in this case, we start with two primary modes \( a \) and \( b \) with displacements \( u_a \) and \( u_b \). We consider the case when both \( a \) and \( b \) are RL modes. Hence, similar to Eqn(4.8) we get

\[
\begin{align*}
  u &= u_a + u_b + u_{aa} + u_{ab} + u_{bb} \\
\end{align*}
\] (4.17)

for the total displacement field, where \( u_{aa} \) and \( u_{bb} \) correspond to the displacement fields due to self-interaction of modes \( a \) and \( b \) respectively and \( u_{ab} \) corresponds to the displacement field due to mutual-interaction of modes \( a \) and \( b \). Likewise, we have

\[
\begin{align*}
  H &= H_a + H_b + H_{aa} + H_{ab} + H_{bb} \\
\end{align*}
\] (4.18)

where \( H_{aa}, H_{bb} \) and \( H_{ab} \) are the displacement gradients corresponding to \( u_{aa}, u_{bb} \) and \( u_{ab} \) respectively. One can formulate three different boundary value problems corresponding to \( u_{aa}, u_{bb} \) and \( u_{ab} \) as follows:

\[
\begin{align*}
  \text{Div}(S^L(H_{aa})) - \rho_{\kappa} \dot{u}_{aa} &= -\text{Div}(S^{NL}(H_a, H_a, 2)) \\
  S^L(H_{aa}) n_\kappa &= -S^{NL}(H_a, H_a, 2) n_\kappa \\

  \text{Div}(S^L(H_{bb})) - \rho_{\kappa} \dot{u}_{bb} &= -\text{Div}(S^{NL}(H_b, H_b, 2)) \\
  S^L(H_{bb}) n_\kappa &= -S^{NL}(H_b, H_b, 2) n_\kappa \\

  \text{Div}(S^L(H_{ab})) - \rho_{\kappa} \dot{u}_{ab} &= -\text{Div}(S^{NL}(H_a, H_b, 2)) \\
  S^L(H_{ab}) n_\kappa &= -S^{NL}(H_a, H_b, 2) n_\kappa \\
\end{align*}
\] (4.19)

The first two problems in Eqn(4.19) correspond to the self-interaction of modes \( a \) and \( b \) respectively and are analogous to the one discussed for second harmonic guided wave generation in section 4.1. The third one corresponds to the mutual-interaction of modes \( a \) and \( b \) and can be solved using the perturbation approach. First, we would like to note that \( S^{NL}(H_a, H_b, 2) \) has terms like \( e^{i((k_a + k_b)X_1 - (\omega_a + \omega_b)t)} \) and \( e^{i((k_a - k_b)X_1 - (\omega_a - \omega_b)t)} \). Hence, the phase-matching condition and non-zero power-flux criterion in this case become
1. Phase-matching: Existence of a propagating guided wave mode at \((\omega_a + \omega_b, k_a + k_b)\) (sum) or \((\omega_a - \omega_b, k_a - k_b)\) (difference) where \((\omega_a, k_a)\) and \((\omega_b, k_b)\) are the propagating primary modes.

2. Non-zero Power flux criterion: \(\left(f_n^{surf} + f_n^{vol}\right) \neq 0\) where

\[
\begin{align*}
    f_n^{surf} &= \frac{-1}{2} S^{NL}(H_a, H_b, 2)v_n^* n_2 |_{-h}^{h} \\
    f_n^{vol} &= \frac{1}{2} \int_{-h}^{h} \text{Div}(S^{NL}(H_a, H_b, 2)\cdot v_n^*) dX_2. 
\end{align*}
\] (4.20)

While determining all the guided wave primary modes that are capable of generating cumulative sum and difference frequency modes is an onerous task and need to be carried out numerically, a list of few primary modes is identified in Chillara and Lissenden (2012). But some interesting results can be obtained by examining non-zero power flux criterion using the parity analysis and is presented in detail in Chillara and Lissenden (2012). It was shown that

- Interaction of modes of identical nature results in cumulative symmetric modes and that between opposite nature results in cumulative antisymmetric modes provided the phase-matching condition is satisfied.

Note that the above results were obtained for interaction of RL modes but can be appropriately extended for SH modes as well. In fact, the above framework can be extended to study the nature of higher harmonic generation up to any order as was presented in Chillara and Lissenden (2012). Liu et al. (2013b) carried this out for third harmonic SH waves in plates.

4.3 Summary

In this chapter, theoretical problem formulation for second harmonic guided waves in weakly nonlinear elastic plates was presented. The formulation was carried out in terms of displacement gradient which enhances clarity and simplifies the analysis. Two conditions, namely phase-matching and non-zero power-flux criterion that guarantee the cumulative second harmonic generation from primary guided wave modes were presented. A complete list of guided wave modes that generate cumulative second harmonics were identified. A generalized framework to study
interaction of guided wave modes was presented. It was shown that the interaction of modes of identical nature result in symmetric modes and that between modes of opposite nature result in antisymmetric modes. A procedure to assess the nature of higher harmonics was briefly outlined.
Chapter 5  
Second harmonic guided waves in pipes using a large-radius asymptotic approximation for axisymmetric longitudinal modes

Introduction

In this chapter, we present theoretical problem formulation for second harmonic guided waves in pipes in section 5.1. Then, a large radius asymptotic approximation for axis-symmetric longitudinal modes is presented in section 5.2. Guided wave modes in pipes are classified as asymptotic symmetric and antisymmetric modes are compared with those for the plate. In section 5.3, we investigate second harmonic generation from axis-symmetric longitudinal modes in pipe using a large radius asymptotic approximation introduced in section 5.2. Finally, conclusions are drawn in section 5.4. The content presented in this chapter is predominantly from Chillara and Lissenden (2013).

5.1 Second harmonic guided wave modes in pipes: Problem formulation

In this section, we formulate the problem for studying second harmonic guided waves in pipes. As the problem needs to be formulated in the context of large deformation, it is essential that one distinguishes the current and reference configurations explicitly. While second harmonic guided waves in non-plate-like waveguides has received attention from de Lima and Hamilton (2005), Srivastava and Lanza di Scalea (2010) and Srivastava et al. (2010), the above aspect has not been addressed and we would like to take up this issue in this section. To that end, we start
with a pipe whose cylindrical polar coordinates in the reference configuration are denoted by \((R, \Theta, Z)\) and those in current configuration are denoted by \((r, \theta, z)\). Corresponding unit vectors in reference and current configurations are denoted by \((e_R, e_\Theta, e_Z)\) and \((e_r, e_\theta, e_z)\). The deformation gradient \(F\) is a two-point tensor and has the following form

\[
F = F_{rR}(e_r \otimes e_R) + F_{r\Theta}(e_r \otimes e_\Theta) + F_{rZ}(e_r \otimes e_Z) + \\
F_{\theta R}(e_\theta \otimes e_R) + F_{\theta \Theta}(e_\theta \otimes e_\Theta) + F_{\theta Z}(e_\theta \otimes e_Z) + \\
F_{z R}(e_z \otimes e_R) + F_{z \Theta}(e_z \otimes e_\Theta) + F_{z Z}(e_z \otimes e_Z)
\]  

(5.1)

where

\[
F_{rR} = \frac{\partial r}{\partial R}, \quad F_{r\Theta} = \frac{1}{R} \frac{\partial r}{\partial \Theta}, \quad F_{rZ} = \frac{\partial r}{\partial Z}, \\
F_{\theta R} = r \frac{\partial \theta}{\partial R}, \quad F_{\theta \Theta} = r \frac{\partial \theta}{\partial \Theta}, \quad F_{\theta Z} = r \frac{\partial \theta}{\partial Z}, \\
F_{z R} = \frac{\partial z}{\partial R}, \quad F_{z \Theta} = \frac{1}{R} \frac{\partial z}{\partial \Theta}, \quad F_{z Z} = \frac{\partial z}{\partial Z}.
\]  

(5.2)

Figure 5.1: Schematic of the pipe showing the coordinate system.

We consider hyperelastic material with strain energy function \((\text{Eqn}(4.1))\), as in chapter 4. For this choice of strain energy function, the second Piola-Kirchoff stress tensor \((T_{RR})\) is given by \(\text{Eqn}(4.2)\). The first Piola-Kirchoff stress is given by \(S = FT_{RR}\). Note that the first Piola-Kirchoff stress is a two-point tensor like that of the deformation gradient and has the following representation.

\[
S = S_{rR}(e_r \otimes e_R) + S_{r\Theta}(e_r \otimes e_\Theta) + S_{rZ}(e_r \otimes e_Z) + \\
S_{\theta R}(e_\theta \otimes e_R) + S_{\theta \Theta}(e_\theta \otimes e_\Theta) + S_{\theta Z}(e_\theta \otimes e_Z) + \\
S_{z R}(e_z \otimes e_R) + S_{z \Theta}(e_z \otimes e_\Theta) + S_{z Z}(e_z \otimes e_Z)
\]
\[ S_{\theta R}(e_{\theta} \otimes e_R) + S_{\theta \Theta}(e_{\theta} \otimes e_{\Theta}) + S_{\theta Z}(e_{\theta} \otimes e_Z) + \\
S_{z R}(e_z \otimes e_R) + S_{z \Theta}(e_z \otimes e_{\Theta}) + S_{z Z}(e_z \otimes e_Z). \] (5.3)

As in the problem formulation for the second harmonic guided waves in plates, we start with the balance of linear momentum in the reference configuration as follows

\[ \text{Div}(\mathbf{S}) = \rho_u \ddot{\mathbf{u}} \]

\[ \mathbf{S}_{\kappa} = \mathbf{0}. \] (5.4)

Three main issues arise in proceeding further with the problem:

1. The displacement \( \mathbf{u} \) needs to be referred using only one of the bases (preferably reference as the problem is formulated in the reference configuration); \((e_R, e_{\Theta}, e_Z)\) or \((e_r, e_{\theta}, e_z)\). But this is cumbersome as \( \mathbf{u} = (re_r + ze_z) - (Re_R + Ze_Z) \), which requires explicit relation for \( e_r \) in terms of \((e_R, e_{\Theta}, e_Z)\).

2. The bases \((e_R, e_{\Theta}, e_Z)\) or \((e_r, e_{\theta}, e_z)\) are not orthonormal to each other in that \( e_r e_R \neq 1, e_r e_{\Theta} \neq 0 \), etc. Hence standard matrix algebra used for representing operations on tensors cannot be employed in their native form.

3. The ‘Div’ operator in reference configuration has additional terms when compared to its counterpart in the current configuration leading to its complex expression given below (Eqn(5.5)). For a detailed derivation see the Appendix A.

\[
\text{Div}(\mathbf{S}) = \left\{ \frac{\partial S_{\theta R}}{\partial R} + \frac{S_{\theta R}}{R} + \frac{1}{R} \frac{\partial S_{\theta \Theta}}{\partial \Theta} + \frac{\partial S_{\theta Z}}{\partial Z} - \frac{S_{\theta R}}{R} \frac{\partial \Theta}{\partial R} - \frac{S_{\theta \Theta}}{R} \frac{\partial \Theta}{\partial \Theta} - \frac{S_{\theta Z}}{R} \frac{\partial \Theta}{\partial Z} \right\} e_r + \\
\left\{ \frac{\partial S_{\theta \Theta}}{\partial R} + \frac{S_{\theta \Theta}}{R} + \frac{1}{R} \frac{\partial S_{\theta \Theta}}{\partial \Theta} + \frac{\partial S_{\theta Z}}{\partial Z} + \frac{S_{\theta R}}{R} \frac{\partial \Theta}{\partial R} + \frac{S_{\theta \Theta}}{R} \frac{\partial \Theta}{\partial \Theta} + \frac{S_{\theta Z}}{R} \frac{\partial \Theta}{\partial Z} \right\} e_{\theta} + \\
\left\{ \frac{\partial S_{z R}}{\partial R} + \frac{S_{z R}}{R} + \frac{1}{R} \frac{\partial S_{z \Theta}}{\partial \Theta} + \frac{\partial S_{z Z}}{\partial Z} \right\} e_z \] (5.5)

However, for the special case of axis-symmetric longitudinal primary modes in the pipe, we have \( \theta = \Theta, \, e_r = e_R, \, e_{\theta} = e_{\Theta} \) and \( e_z = e_Z \). Hence \( \mathbf{u} = u_R(R, Z)e_R + u_z(R, Z)e_Z \) where \( u_R(R, Z) \) and \( u_z(R, Z) \) are the radial and axial components of the displacement. Also, the ‘Div’ operator in this case corresponds to the standard
divergence operator (in current configuration) usually employed except for that the
derivatives with respect to the current coordinates are replaced with those in the
reference coordinates. Owing to this simplification, we investigate second harmonic
generation from axis-symmetric longitudinal modes in pipes.

First, we begin with the displacement field for primary axis-symmetric longitudinal
mode given by \( u_1 \). Let \( u_2 \) denote the perturbation over \( u_1 \) which corresponds
to the secondary mode generated from the primary mode. The corresponding
displacement gradient is given by

\[
H = H_1 + H_2
\]  

(5.6)

where \( H_1 \) and \( H_2 \) are displacement gradients corresponding to primary and sec-
ondary modes respectively. Following the notation used in chapter 4, we can
formulate two different boundary value problems as follows

\[
\text{Div}(S^L(H_1)) - \rho_\kappa \ddot{u}_1 = 0 \\
S^L(H_1)n_\kappa = 0 \quad \text{and} \\
\text{Div}(S^L(H_2)) - \rho_\kappa \ddot{u}_2 = -\text{Div}(S^{NL}(H_1, H_1, 2)) \\
S^L(H_2)n_\kappa = -S^{NL}(H_1, H_1, 2)n_\kappa
\]  

(5.7)

The first problem is identically satisfied due to our assumption that \( u_1 \) is a propa-
gating primary mode and the second problem can be solved using a normal mode
expansion technique. However, since the nature of solutions of the problem is
already investigated for plates, it would be worthwhile to see if we can extend
those results for pipes, especially in the case where the radius of the pipe is large
compared to its thickness. To that end, we investigate a large-radius asymptotic
approximation for axis-symmetric longitudinal modes. In addition, it is of con-
siderable importance even from a practical standpoint as the radius of the pipes
encountered in general are much larger compared to the thickness as illustrated in
Table 5.1 for common schedule 40 pipes. In this context asymptotic approximation
refers to

- approximate dispersion curves for pipes and
- approximate wavestructures for the corresponding guided wave modes in
Table 5.1: Dimensions of some schedule 40 pipes

<table>
<thead>
<tr>
<th>Pipe(Sch 40)</th>
<th>OD(mm)</th>
<th>ID(mm)</th>
<th>Thickness</th>
<th>Thickness mean diameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>4” (102 mm)</td>
<td>114</td>
<td>102</td>
<td>6</td>
<td>0.056</td>
</tr>
<tr>
<td>8” (219 mm)</td>
<td>219</td>
<td>203</td>
<td>8</td>
<td>0.039</td>
</tr>
<tr>
<td>12” (324 mm)</td>
<td>324</td>
<td>303</td>
<td>10.3</td>
<td>0.033</td>
</tr>
<tr>
<td>18” (457 mm)</td>
<td>457</td>
<td>429</td>
<td>14.3</td>
<td>0.032</td>
</tr>
</tbody>
</table>

Some important aspects concerning dispersion curves for pipes with smaller thickness to radius ratio were addressed in Gridin et al. (2003) and Li and Rose (2006). In the next section we compare wave structures for plates and pipes using a large radius asymptotic approximation and obtain an error estimate for the approximation.

5.2 Large-radius asymptotic approximation for axis-symmetric longitudinal modes

In this section, we obtain a large-radius approximation for wave structures in pipes. Note that the analysis presented in this section is for a linear case and applies to the primary mode propagating in the plate. Hence we drop the subscript in $u_1$ and denote it as $u$ as the results obtained are valid for any generic propagating guided wave mode in the pipe. We start with the Navier’s equation of motion in a linear elastic solid given by

$$(\lambda + 2\mu)\text{Grad}(\text{Div}(u)) - \mu\text{Curl}(\text{Curl}(u)) = \rho_\kappa \ddot{u} \quad (5.8)$$

where ‘Grad’ and ‘Curl’ are taken with respect to coordinates in reference configuration. Next, we consider the Helmholtz decomposition of the displacement field in the pipe given by

$$u = \text{Grad}(\phi) + \text{Curl}(\psi) \quad (5.9)$$

where $\phi$ is the scalar potential and $\psi = (0, 0, \psi)$ is the vector potential. Substituting Eqn(5.9) into Eqn(5.8), we get (for axis-symmetric modes) (Rose, 1999)

$$\frac{\partial^2 \phi}{\partial R^2} + \frac{1}{R} \frac{\partial \phi}{\partial R} + \frac{\partial^2 \phi}{\partial Z^2} = \frac{1}{c_t^2} \frac{\partial^2 \phi}{\partial t^2}$$

75
If we consider \( \phi = \phi(R) e^{ikZ-\omega t} \) and \( \psi = \psi(R) e^{ikZ-\omega t} \), we get

\[
\frac{d^2 \phi}{dR^2} + \frac{1}{R} \frac{d \phi}{dR} + \left( \frac{\omega}{c_l} \right)^2 - k^2 \phi = 0
\]

\[
\frac{d^2 \psi}{dR^2} + \frac{1}{R} \frac{d \psi}{dR} + \left( \frac{\omega}{c_t} \right)^2 - k^2 \psi = 0
\]

(5.11)

where \( c_l \) and \( c_t \) are longitudinal and transverse wave speeds in the material respectively.

Now, consider a pipe with coordinate system as shown in Figure 5.1. If \( R_1 \) and \( R_2 \) denote the inner and outer radii, then the mean radius is given by \( R_m = \frac{R_2 + R_1}{2} \) and the half-thickness is given by \( h = \frac{R_2 - R_1}{2} \). We make a change of variable \( R = R_m + X_2 \), where \( X_2 \in [-h,h] \). If \( h \ll R_m \) then Eqn(5.11) can be approximated by

\[
\frac{d^2 \phi}{dX_2^2} + \frac{1}{R_m} \frac{d \phi}{dX_2} + p^2 \phi = 0
\]

\[
\frac{d^2 \psi}{dX_2^2} + \frac{1}{R_m} \frac{d \psi}{dX_2} + q^2 \psi = 0
\]

(5.12)

where \( p = \sqrt{\left( \frac{\omega}{c_l} \right)^2 - k^2} \) and \( q = \sqrt{\left( \frac{\omega}{c_t} \right)^2 - k^2} \). The above equations in the case of a plate \( (R_m \rightarrow \infty) \) are

\[
\frac{d^2 \phi}{dX_2^2} + p^2 \phi = 0
\]

\[
\frac{d^2 \psi}{dX_2^2} + q^2 \psi = 0.
\]

(5.13)

While the solutions of Eqn(5.13) along with the traction free boundary conditions represent guided wave modes in the plate, the same for Eqn(5.12) represent guided wave modes in pipe in an approximate sense. Also, the solutions of Eqn(5.13) correspond to a harmonic oscillator and the solutions for Eqn(5.12) correspond to a damped harmonic oscillator which decay along the \( R \sim X_2 \) direction. We next look at the nature of solutions for Eqn(5.12). As both the equations in Eqn(5.12) are similar, we just comment on the solutions for the equation involving
φ. Suppose \( e^{sX_2} \) is a solution and substituting it in Eqn(5.12), we get 

\[
 s^2 + \frac{s}{R_m} + p^2 = 0 \Rightarrow s = -\frac{1}{2R_m} + \sqrt{\left(\frac{1}{4R_m^2} - 1\right)}.
\]

If \( 2pR_m >> 1 \), then the two independent solutions are given by 

\[
 e^{\frac{-X_2}{2R_m}} \cos \left( p\sqrt{\left(1 - \frac{1}{4p^2R_m^2}\right)} X_2 \right) \quad \text{and} \quad e^{\frac{-X_2}{2R_m}} \sin \left( p\sqrt{\left(1 - \frac{1}{4p^2R_m^2}\right)} X_2 \right).
\]

For large \( R_m \), we have 

\[
 pR_m >> 1 \quad \text{the solutions approach} \quad \cos \left( pX_2 \right) \quad \text{and} \quad \sin \left( pX_2 \right).
\]

Hence we term 

\[
 e^{\frac{-X_2}{2R_m}} \cos \left( p\sqrt{\left(1 - \frac{1}{4p^2R_m^2}\right)} X_2 \right) \quad \text{as “asymptotically symmetric mode” and} \quad e^{\frac{-X_2}{2R_m}} \sin \left( p\sqrt{\left(1 - \frac{1}{4p^2R_m^2}\right)} X_2 \right) \quad \text{as “asymptotically antisymmetric mode”}.
\]

Next, we compare the solutions for plate (Eqn(5.13)) and pipe (Eqn(5.12)) and obtain an error estimate for the same.

### 5.2.1 Error estimate

Suppose \( \phi_a \) is the asymptotic approximation and \( \phi \) is the actual solution for Eqn(5.10). Then substituting \( \phi_a \) in left hand side of Eqn(5.10), we have

\[
 \left| \frac{d^2 \phi_a}{dR^2} + \frac{1}{R} \frac{d\phi_a}{dR} + p^2 \phi_a \right| = \left| \frac{d^2 \phi_a}{dR^2} + \frac{1}{R_m} \frac{d\phi_a}{dR} + p^2 \phi_a + \left( \frac{1}{R} - \frac{1}{R_m} \right) \frac{d\phi_a}{dR} \right|
\]

\[
 = \left| \left( \frac{1}{R} - \frac{1}{R_m} \right) \frac{d\phi_a}{dX_2} \right| \leq \frac{ph}{R_m - hR_m} e^{\frac{h}{2R_m}}.
\]

Also, if we define

\[
 \left| \frac{d^2 \phi_a}{dR^2} + \frac{1}{R} \frac{d\phi_a}{dR} + p^2 \phi_a \right| = \int_{-h}^{h} \left| \frac{d^2 \phi_a}{dR^2} + \frac{1}{R} \frac{d\phi_a}{dR} + p^2 \phi_a \right| dX_2,
\]

then using the previous estimate we obtain,

\[
 \left| \frac{d^2 \phi_a}{dR^2} + \frac{1}{R} \frac{d\phi_a}{dR} + p^2 \phi_a \right| \leq \frac{2ph^2}{R_m^2 - hR_m} e^{\frac{h}{2R_m}} \approx O \left( \left( \frac{h}{R_m} \right)^2 \right).
\]

Hence it appears that for \( \frac{h}{R_m} < \frac{1}{10} \) the asymptotic approximation provides a very good approximation for guided wave modes in the pipe. Figure 5.2 shows the comparison between the asymptotic solution and the plate solutions for three values of \( p = 0.5, 2, 4 \) mm\(^{-1}\) and two different values of \( R_m = 10, 20 \) mm.
Figure 5.2: Comparison between asymptotic antisymmetric and plate solutions (a), (c) and (e); asymptotic symmetric and plate solutions (b), (d) and (f)

Similar plots are shown for $p = 2, 4 \text{ mm}^{-1}$ and various values of $R_m$ in Figure 5.3. Clearly, as $R_m$ increases the asymptotic solution approaches that of the plate.
We would like to end this section by noting that the asymptotic solution presented in this section is not necessarily the one with least error but it suffices for our purpose i.e., to illustrate the asymptotically symmetric/antisymmetric nature of the guided wave modes in pipe. For example, one can consider asymptotic approximation for actual solutions of Eqn(5.11) which are the Bessel functions...
$J_0(pR)$ and $Y_0(pR)$. The asymptotic approximations are given by $\sqrt{\frac{2}{\pi R}} \cos(pR - \frac{\pi}{4})$ and $\sqrt{\frac{2}{\pi R}} \sin(pR - \frac{\pi}{4})$ respectively and the asymptotically symmetric and antisymmetric modes are given by $\sqrt{\frac{R_m}{R_m+X_2}} \cos(pX_2)$ and $\sqrt{\frac{R_m}{R_m+X_2}} \sin(pX_2)$ respectively. In the next section, we analyze the second harmonic generation in pipes using the asymptotic approximation obtained here.

### 5.3 Analysis of second harmonic guided waves in pipes using large -radius asymptotic approximation

Based on Eqn(5.7), it suffices to examine the nature of $S^{NL}(H_1, H_1, 2)$ to investigate second harmonic guided waves in pipes. To that end, we first compare the displacement gradient obtained from the plate solution $(\phi_p, \psi_p)$ and the asymptotic solution $(\phi_a, \psi_a)$. Let $u_a$ and $u_p$ denote the displacement fields obtained using the asymptotic and the plate solutions respectively. Correspondingly, let $H_a$ and $H_p$ denote the displacement gradient. We have

\[
H_a = \begin{bmatrix}
\frac{\partial u_{aR}}{\partial R} & 0 & iku_{aR} \\
0 & \frac{u_{aR}}{R} & 0 \\
\frac{\partial u_{aZ}}{\partial R} & 0 & iku_{aZ}
\end{bmatrix}
\]

\[
H_p = \begin{bmatrix}
\frac{\partial u_{p2}}{\partial X_2} & 0 & iku_{p2} \\
0 & 0 & 0 \\
\frac{\partial u_{p1}}{\partial X_2} & 0 & iku_{p1}
\end{bmatrix}
\] (5.14)

Here, $u_{aR}$ and $u_{aZ}$ denote the ‘R’ component and the ‘Z’ component of the displacement $u_a$ respectively. Likewise, $u_{p1}$, $u_{p2}$ denote the $X_1$, $X_2$ components of $u_p$. For brevity, we denote $u_{aR}$, $u_{aZ}$, $u_{p1}$ and $u_{p2}$ by $U_R$, $U_Z$, $U_1$ and $U_2$ respectively. In terms of $(\phi_a, \psi_a)$ and $(\phi_p, \psi_p)$ these are given by

\[
U_R = \frac{d\phi_a}{dR} + ik\psi_a; \quad U_Z = -\frac{d\psi_a}{dR} + ik\phi_a; \quad U_1 = -\frac{d\psi_p}{dX_2} + ik\phi_p; \quad U_2 = \frac{d\phi_p}{dX_2} + ik\psi_p.
\]
Using the above relations and Eqn(5.14), we get

\[
H_a = \begin{bmatrix}
\frac{d^2 \phi_a}{dR^2} + ik \frac{d \psi_a}{dR} & 0 & ik \left( \frac{d \phi_a}{dR} + ik \psi_a \right) \\
0 & \frac{1}{R} \left( \frac{d \phi_a}{dR} + ik \psi_a \right) & 0 \\
-\frac{d^2 \psi_a}{dR^2} + ik \frac{d \phi_a}{dR} - \frac{1}{R} \frac{d \psi_a}{dR} & 0 & ik \left( -\frac{d \phi_a}{dR} + ik \phi_a - \psi_a \right)
\end{bmatrix}
\]

\[
H_p = \begin{bmatrix}
\frac{d^2 \phi_p}{dX_2^2} + ik \frac{d \psi_p}{dX_2} & 0 & ik \left( \frac{d \phi_p}{dX_2} + ik \psi_p \right) \\
0 & 0 & 0 \\
-\frac{d^2 \psi_p}{dX_2^2} + ik \frac{d \phi_p}{dX_2} & 0 & ik \left( -\frac{d \psi_p}{dX_2} + ik \phi_p \right)
\end{bmatrix}
\]

When \( R_m \) is large, the terms containing \( \frac{1}{R} \) in \( H_a \) are much smaller compared to others and hence to compare \( H_a \) and \( H_p \) it suffices to compare the following:

1. \( \phi_a \) and \( \phi_p \)
2. \( \frac{d \phi_a}{dR} \) and \( \frac{d \phi_p}{dX_2} \)
3. \( \frac{d^2 \phi_a}{dR^2} \) and \( \frac{d^2 \phi_p}{dX_2^2} \)

We have

\[
\phi_a = e^{-\frac{X_2^2}{2R_m}} \cos \left( p \sqrt{\left( 1 - \frac{1}{4p^2 R_m^2} \right)} X_2 \right)
\]

for asymptotically symmetric modes and \( \phi_p = \cos(pX_2) \) for symmetric modes. Also,

\[
\frac{d \phi_a}{dR} = \frac{d \phi_a}{dX_2} = -e^{-\frac{X_2}{2R_m}} \sin(cX_2)c - \frac{1}{2R_m} e^{-\frac{X_2}{2R_m}} \cos(cX_2); \\
\frac{d \phi_p}{dX_2} = -p \sin(pX_2)
\]

\[
\frac{d^2 \phi_a}{dR^2} = \frac{d^2 \phi_a}{dX_2^2} = -\left\{ e^{-\frac{X_2}{2R_m}} c^2 \cos(cX_2) - \frac{c}{R_m} e^{-\frac{X_2}{2R_m}} \sin(cX_2) + \frac{1}{4R_m^2} e^{-\frac{X_2}{2R_m}} \cos(cX_2) \right\}
\]

\[
\frac{d^2 \phi_p}{dX_2^2} = -p^2 \cos(pX_2)
\]

where \( c = p \sqrt{\left( 1 - \frac{1}{4p^2 R_m^2} \right)} \). As we would like to know if the results for second harmonic generation in plates can be extended to pipes, we make comparison between \( (\phi_a, \phi'_a, \phi''_a) \) and \( (\phi_p, \phi'_p, \phi''_p) \) for a small value of \( p = 0.2 \text{ mm}^{-1} \) in Figures 5.4, 5.5. A small value of \( p \) is selected as it corresponds to symmetric modes at
longitudinal wave speed \( c_p = c_l \) and were found to generate cumulative second harmonics in plates (Chilara, 2012) (Liu et al., 2013a) (Matsuda and Biwa, 2011).

![Figure 5.4: Comparison for the first derivative of \( \phi \)](Image)

![Figure 5.5: Comparison for the second derivative of \( \phi \)](Image)

To further substantiate the claim regarding the asymptotic nature of guided wave modes in pipe, a comparison is made between the wave structures obtained using semi analytical finite element method (SAFE) for plate and pipe. We specifically choose \( S_1 \) and \( S_2 \) mode at longitudinal wave speed for the comparison. The thickness and the inner radius of the pipe were chosen to be 1 mm and 10 mm respectively. Figure 5.6 shows the comparison for the \( S_1 \) mode and Figure 5.7 shows that for the \( S_2 \) mode.
Clearly, there is not much of difference between the wavestructures as expected from the foregoing analysis. To investigate the second harmonic generation in pipes, we need to carry out a parity analysis for the term $S_{NL}(H_1, H_1, 2)$ as was carried out in Chillara and Lissenden (2012) and Müller et al. (2010). Without going into the details, it can be shown that $S_{NL}(H_1, H_1, 2)$ is of ‘Sym’ nature. Hence, the non-zero power-flux criterion for plates can be extended to pipes and it can be concluded that only asymptotically symmetric modes can be generated as cumulative second harmonics in pipes. In addition, due to the asymptotic nature of the guided wave modes in pipes, all the modes in the plate that generate cumulative second harmonics can be employed for generating cumulative second harmonic guided waves in pipes.
5.4 Conclusions

An accurate and generalized formulation for studying second harmonic guided waves in pipes was presented. Emphasis was laid on the correct use of ‘Div’ operator in the reference configuration. Second harmonic guided wave generation from axis-symmetric longitudinal waves in pipes was studied using a large-radius asymptotic approximation. Axis-symmetric guided wave modes in pipes were classified and approximated as asymptotically symmetric and antisymmetric modes. An error estimate for the approximation was presented. Finally, nature of second harmonic generation from pipes was investigated by examining the parity of the nonlinear forcing term in Eqn(5.7). It was concluded that only asymptotically symmetric modes can be efficiently generated as second harmonics from primary axis-symmetric longitudinal modes.
Chapter 6  
Nonlinear guided waves in plates: A numerical perspective

Introduction

In this chapter, we study nonlinear guided waves in plates from a numerical standpoint. A detailed study of second harmonic guided waves in homogeneous plates is presented in section 6.1. Then, second harmonic guided waves in plates with localized micro-scale damage is investigated in section 6.2. A comparison of the results obtained for the localized case with those from the homogeneous case is made whenever pertinent. Finally, conclusions are drawn in section 6.3. This chapter is mainly based on the work presented in Chillara and Lissenden (2014b), Chillara and Lissenden (2015b) and Chillara and Lissenden (2016).

While the understanding of nonlinear guided waves in plates has progressed, several issues related to the wave-mechanics aspects of the problem are yet to be addressed. These stem from the analytical approach adopted to study nonlinear guided waves in plates. Some of these issues are listed below:

1. The theoretical analysis is carried out for single frequency excitations but the experiments employ finite bandwidth excitations. Hence, the effect of bandwidth on the nonlinear guided waves needs to be understood.

2. Theoretical analysis is carried out using the perturbation approach as no closed-form solutions are available. The validity and the applicability of the solution obtained using this approach needs to be ascertained.

3. Group velocity matching in addition to phase velocity matching and non-zero power flux criteria was advocated in Müller et al. (2010) for cumulative

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harmonic generation. However, Deng (Deng et al., 2011) presented an experimental investigation where it was shown that group velocity matching is not a necessary condition. In fact, it would be worthwhile to investigate if higher harmonic modes from finite wave packets can propagate without group velocity matching as the transfer of the energy from the primary to higher modes is no longer possible once they separate from the primary mode.

We address some the above issues in section 6.1 using numerical simulations. In addition, the role of some important aspects of material behavior like that of material and geometric nonlinearity in the higher harmonic generation will be addressed as they were not a subject of earlier investigations. Moreover, all the previous theoretical studies assume that the nonlinearity/micro-scale damage is uniformly distributed in the material. However, the interaction of ultrasound with localized damage is quite different from that with distributed damage (Chillara and Lissenden, 2015b) and depends on several factors like damage distribution, its spatial extent and intensity. In section 6.2, we would like to examine and try to understand the role of each of the above aspects of damage and their interplay in the second harmonic generation of guided waves.

All the simulations presented in this chapter are carried out using the solid mechanics module in COMSOL, a commercial finite element package. We use the Murnaghan hyperelastic constitutive model (Eqn(4.2)) to depict the nonlinear elastic material. It should be noted that the constitutive model is prone to convergence issues at large deformations. However the stress amplitudes corresponding to ultrasonic waves are much smaller and do not pose such numerical convergence issues. The elastic constants used correspond to aluminum and are listed in Table 6.1. The schematic of the model used for the simulations is shown in Figure 6.1. The thickness of the plate is set to 1 mm in all the simulations. Appropriate displacement boundary conditions are applied at the left end of the plate to excite the intended modes. Throughout this chapter, we denote the x-component of the displacement with ‘u’ and the y-component of the displacement with ‘v’. The dispersion curves for the plate along with the primary modes (highlighted in red) used in the simulations are shown in Figure 6.2.
Table 6.1: Elastic constants in GPa used for simulation

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\mu$</th>
<th>$l$</th>
<th>$m$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>51</td>
<td>26</td>
<td>-250</td>
<td>-333</td>
<td>-350</td>
</tr>
</tbody>
</table>

Figure 6.1: Schematic of the model used for simulations

Figure 6.2: Dispersion curves for the aluminum plate

6.1 Nonlinear guided waves in homogeneous weakly nonlinear elastic plates

In this section we study second harmonic guided waves in plates from a numerical standpoint. To enhance clarity, this section is divided into several smaller subsections each devoted to one aspect of second harmonic guided wave generation in plates.
Cumulative second harmonic generation from the \( S_0 \) mode (0.5 MHz) and the \( S_1 \) mode (3.6 MHz)

Here, we compare the cumulative second harmonic generation from two modes, namely the \( S_0 \) mode (0.5 MHz) and the \( S_1 \) mode (3.6 MHz). Triangular elements with a maximum size of 0.1 mm are employed to discretize the domain along the wave propagation direction and a minimum of 15 elements are used along the thickness direction. A maximum time-step of 0.01 \( \mu s \) is used for the \( S_0 \) mode and 0.005 \( \mu s \) is used for the \( S_1 \) mode. A displacement amplitude of 1E-7 m is used for the boundary conditions for the \( S_0 \) mode and that of 2E-8 m is used for the \( S_1 \) mode. This choice ensures a stress wave of a few MPa - typical of an ultrasonic wave propagating in the material.

While the \( S_1 \) mode (3.6 MHz) satisfies both the phase-matching and non-zero power-flux conditions, the \( S_0 \) mode (0.5 MHz) is not phase-matched to the secondary mode. The phase velocity of the primary \( S_0 \) mode (0.5 MHz) is 5.34 mm/\( \mu s \) and that of the second harmonic \( S_0 \) mode (1 MHz) is 5.27 mm/\( \mu s \). On the other hand, the phase velocity of both the primary \( S_1 \) mode (3.6 MHz) and the secondary \( S_2 \) mode (7.2 MHz) is 6.17 mm/\( \mu s \). Figure 6.3 shows the amplitude of the second harmonic from the \( S_0 \) mode (0.5 MHz) and the Figure 6.4 shows the same for the \( S_1 \) mode (3.6 MHz) as a function of the normalized propagating distance. The normalization is carried out using the corresponding wavelength of the primary mode (\( \lambda_{S_0} = 10.68 \ mm, \lambda_{S_1} = 1.713 \ mm \)). Clearly, the second harmonic from the \( S_0 \) mode (0.5 MHz) is not cumulative as it starts to decrease after about (\( \frac{\lambda_{S_0}}{\lambda_{S_0}} = 10 \)). On the other hand, the second harmonic from the \( S_1 \) mode (3.6 MHz) is cumulative and increases linearly as shown in Figure 6.4. This is in agreement with the prediction from the perturbation approach that the \( S_1 \) mode (3.6 MHz) generates cumulative second harmonic.
Here, we investigate the effect of the bandwidth of the primary wave on the second harmonic generation. This is important as the theory presented in chapter 4 is for single frequency but in reality experimental investigations employ excitations that have a finite frequency bandwidth. While this is not a major issue for bulk-waves, guided waves due to their dispersive nature can lead to results that are quite different from their single-frequency counterparts. To that end, second harmonic generation from $S_1$ mode is investigated for three different bandwidths as shown in Figure 6.5. The bandwidth of the excitation is varied by changing the parameter $\alpha$ for the gaussian excitation describing the displacement boundary condition i.e., $u(x = 0, t) = 2 \times 10^{-8} e^{-(\alpha(t-4.5)^2)} \cos(2 \pi \times 3.6 \times (t - 4.5)) \text{m}$ and $v(x = 0, t) = 2 \times 10^{-8} e^{-(\alpha(t-4.5)^2)} \sin(2 \pi \times 3.6 \times (t - 4.5)) \text{m}$ where $t$ is in $\mu$s.
Simulations are run for three different values of \( \alpha = 0.4, 0.6 \) and 0.8. Figure 6.6a shows the amplitude of the second harmonic and Figure 6.6b shows the relative nonlinearity parameter \( \frac{A_2}{A_1} \) as a function of normalized propagation distance. As can be seen, the rate at which the amplitude of the second harmonic increases is not identical for all the three cases with the maximum rate of increase being for the case where the bandwidth is the least. On the other hand, the relative nonlinearity parameter shows an opposite trend with the maximum rate of increase for the larger bandwidth. This apparent discrepancy is due to the difference in the amplitude of the fundamental mode for the three cases as shown in Figure 6.5 where the amplitude decreases with increasing bandwidth.

![Graph showing the amplitude of the second harmonic and relative nonlinearity parameter](image)

**Figure 6.5:** \( S_1 \) mode (3.6 MHz) - FFT’s for \( u \) at \( x=40 \text{ mm} \)

![Graphs showing the second harmonic amplitude and relative nonlinearity parameter](image)

**Figure 6.6:** \( S_1 \) mode (3.6 MHz) (a) Second harmonic amplitude \( (A_2) \) (b) Relative nonlinearity parameter \( \frac{A_2}{A_1} \) versus normalized propagation distance.
Role of material and geometric nonlinearities

In this section, we compare the contribution of material and geometric nonlinearity to the second harmonic generation. To that end, simulations are carried out for three variants of the constitutive model in Eqn(4.2). These are:

1. Linear elastic material (LE) - No material or geometric nonlinearities are included i.e., \( l = m = n = 0 \) (Eqn(4.2)) and linearized strain \( E^l = \frac{1}{2} (H + H^T) \) is used as a strain measure.

2. Nonlinear (NL) - Both material and geometric nonlinearities are included i.e., \( l \neq 0, m \neq 0, n \neq 0 \) and Lagrangian strain \( (E) \) is used as a strain measure.

3. Geometrically nonlinear (NG) - Only geometric nonlinearity is included i.e., \( l = m = n = 0 \) and Lagrangian strain \( (E) \) is used as a strain measure.

First, we compare the results obtained for second harmonic generation from the \( S_0 \) mode (0.5 MHz) for the three variants, namely, LE, NL and NG of the constitutive model. Figure 6.7a shows the time domain signals for each of the three cases at \( x=50 \text{ mm} \). The time domain signals do not differ much from each other except for the fact that pulse in NL case arrives a bit later in time. On the other hand, the frequency domain plots shown in Figure 6.7b clearly show the second harmonic content for NL and NG cases. Clearly, the amplitude of the second harmonic is much higher for the NL case compared to the NG case and in fact it is about 3-4 times higher when compared to the NG case. Similar results are observed for the second harmonic generation from the \( S_1 \) mode (3.6 MHz) as shown in Figure 6.8 which shows FFT’s for LE, NL and NG cases at \( x = 50 \text{ mm} \). In this case, the NL case has a second harmonic amplitude of about 10 times that for NG case. The results presented here indicate that the second harmonic generation is dominated by the material nonlinearity as opposed to the geometric nonlinearity. Hence, geometrically linear theories incorporating material nonlinearity for material behavior seem a reasonable approximation for studying nonlinear guided waves in plates.
Effect of scaling Murnaghan elastic constants

In this section we investigate the effect of changing Murnaghan third order elastic constants on the second harmonic generation of guided waves. As the higher order constants represent nonlinearity arising out of micro-scale damage, this study reveals the effect of micro-scale damage on the second harmonic generation. We present the results obtained for second harmonic generation from the $S_0$ mode (0.5 MHz). Simulations are run varying Murnaghan elastic constants $(l, m, n)$ in Table 6.1 where each of them is scaled by a constant 1,2 and 4. Figure 6.9 shows the FFT’s for each of the cases at $x = 50$ mm. As can be seen, the amplitude of second
harmonic increases with the scaling factor used and in fact it is almost linearly proportional to the scaling constant used.

Figure 6.9: S₀ mode (0.5 MHz)-FFT’s obtained from time domain signals at x=50 mm for the three different scaling factors; 1, 2 and 4.

Second harmonic generation with group velocity mismatch

The results presented till now employed guided wave modes which are (almost) phase matched and hence the second harmonic guided wave mode propagates in phase with the primary mode. Hence to investigate the effect of group velocity on the second harmonic generation, simulations are run for two primary modes, namely the S₀ mode (1 MHz) and the A₀ mode (0.5 MHz).

S₀ mode (1 MHz)

The phase velocity of the S₀ mode (1 MHz) is 5.27 mm/µs and that of the secondary mode (S₀ mode (2 MHz)) is 4.71 mm/µs. As the mode is not phase matched, the harmonic generation is not cumulative. The amplitude of the displacement boundary condition is set to 10⁻⁶ m to discuss the results that are not easily decipherable at low amplitudes. Figure 6.10 shows the time domain signals at x=50 mm for both the LE and NL cases described earlier. The second harmonic seems to be separated from the primary mode as indicated in the inset. Also, it appears that the time domain pulses for LE and NL cases seem to coincide to a larger extent during the times u is increasing (particle velocity is positive) than during the times when u is decreasing (particle velocity is negative). This is due to the Tension-Compression asymmetry exhibited by the model (Eqn(4.2)) and will be discussed in chapter 7. Figure 6.11 shows the FFT’s for the time domain signals
in the Figure 6.10. The static (zero-frequency) component is much larger when compared to the second harmonic as the static component is always cumulative whereas second harmonic is not. The fluctuations in the amplitude at 2 MHz are due to the second harmonic tail in the time domain window (27 µs) used for taking the FFT.

Figure 6.10: $S_0$ mode (1 MHz)-Time domain signals at $x = 50$ mm for LE and NL cases.

Figure 6.11: $S_0$ mode (1 MHz)- FFT’s for the signals at $x = 50$ mm for LE and NL cases.
A₀ mode (0.5 MHz)

To further investigate the separation between primary and the second harmonic mode as observed for the S₀ mode (1 MHz) in the previous section, simulations are run using the A₀ mode (0.5 MHz) as the primary mode. The amplitude of the displacement boundary condition is increased to $10^{-5} \text{ m}$ and the length of the waveguide is increased to 200 mm to allow for longer propagation distance. Figure 6.12 shows a zoomed-in view of the time domain signal obtained at $x = 50 \text{ mm}$. Clearly, there are two distinct pulses; the larger one corresponds to the primary mode and arrives later (smaller group velocity), and the smaller one corresponds to the secondary mode (larger group velocity) arrives earlier in time. Also, they are clearly separated with the time-difference between them increasing with increasing propagation distance. Several conclusions can be drawn in this regard:

1. The second harmonic separates from the primary mode (Figure 6.12) and hence group velocity matching is not required for the higher harmonic generation.

![Figure 6.12: A₀ mode (0.5 MHz)- Time domain signals at x=40,80 and 120 mm.](image)

2. The second harmonic mode generated is the S₀ mode (1 MHz) as opposed to the A₀ mode (1 MHz) as evident from the through-thickness displacement profiles in Figure 6.13 at $x = 120 \text{ mm}$ during the times $t = 28-35 \mu\text{s}$ which is antisymmetric about the midplane.
3. It can be concluded that the second harmonic is continuously generated from the primary mode and once generated they can propagate independent of the primary mode. This can be explained with the following rationale. As observed here, the second harmonic separates from the primary mode and hence it propagates as a distinct pulse independent of the primary mode. Now, the residual primary mode can again generate second harmonic and this process repeats with the second harmonic pulses being separated from the primary mode when sufficient time has elapsed - as dictated by the group velocity mismatch between the primary and the secondary modes. Also, the separated harmonic can itself generate other higher harmonics. This continuous generation can be interpreted as coming from distributed higher harmonic point sources with intensities proportional to $f_n^{surf}$ and $f_n^{vol}$ on the surface and inside the volume of the waveguide. The theory of mode interaction presented in chapter 4 can be used to assess the nature of such higher harmonic generation. It should be noted that these are much smaller in magnitude when compared to the primary mode and pose an enhanced difficulty in detecting them in an experiment.

**Mode interaction**

Here, we demonstrate sum and difference frequency generation in plates via mode-interaction. This is achieved by mixing two modes, namely the $S_0$ mode (0.4 MHz) and the $S_0$ mode (1.1 MHz) to generate modes at $1.1+0.4=1.5$ MHz and $1.1-0.4=0.7$ MHz. Figure 6.14a shows the time domain signals at $x = 40$ and 60 mm and the
Figure 6.14b shows the FFT (log-scale) depicting the sum and different frequencies generated via interaction.

![FFT](image)

(a) Time domain signal at $x = 40$ and 60 mm. (b) FFT’s for signal at $x = 40$ and 60 mm.

Figure 6.14: Mode interaction $S_0$ mode (1.1 MHz) X $S_0$ mode (0.4 MHz) (a) Time domain signal at $x = 40$ and 60 mm. (b) FFT’s for signal at $x = 40$ and 60 mm.

### 6.2 Nonlinear guided waves in plates undergoing localized microstructural changes

Until now, we dealt with nonlinear guided waves in homogeneous isotropic plates. However, as described in chapter 1 the important application of higher harmonic generation is to be able to detect and characterize damage induced microstructural changes. In other words, the goal is to characterize micro-scale damage features that precede macro-scale damage initiation. To deal with this problem in its entirety, two interrelated aspects of the problem need to be understood. They are

1. Modeling micro-scale damage in the context of nonlinear ultrasound.
2. Interaction of ultrasound with micro-scale damage.

The remaining part of chapter 6 and the chapters 7 and 8 deal with the above aspects. First, we begin with a brief discussion regarding the notion of micro-scale damage in the context of nonlinear ultrasound.

Most of the earlier studies pertaining to nonlinear ultrasound focused on using the ‘$\beta$’ parameter introduced in chapter 1 as a measure of micro-scale damage in the material. In other words, the extent of the nonlinearity in the material behavior is assumed to be a measure of the micro-scale damage in the material. In particular,
this approach has been adopted for studying damage due to dislocations (Cantrell and Yost, 2001), micro-cracks (Nazarov and Sutin, 1997), etc. However, in general one can describe damage using three aspects. They are

1. Damage intensity
2. Damage distribution
3. Damage spatial extent

In the context of nonlinear ultrasound, damage intensity can be identified with the ‘$\beta$’ parameter or equivalently the higher order elastic constants ($A, B, C$ or $l, m, n$); the damage distribution can be identified with the homogeneous/isotropic nature of damage and the spatial extent of the damage can be identified with the volume-fraction of the damage.

While description of micro-scale damage in terms of the ‘$\beta$’ parameter is well studied, other ways of describing micro-scale damage may seem more relevant depending on the damage mechanism at work. For example, if one is interested in monitoring damage evolution from a hot spot, a more pertinent measure of damage is the spatial extent to which the damage has progressed. Hence from a modeling perspective, the measure of damage adopted can have a large impact on the understanding of the physical mechanism underlying the damage progression. Also, as mentioned earlier, most of the previous studies relied on the assumption that micro-scale damage is homogeneous. However, the results pertaining to higher harmonic generation may be quite different if the micro-scale damage is localized and not uniformly distributed in the wave propagation direction. To address some of the above issues, we study and compare two ways of modeling micro-scale damage, namely varying the Murnaghan constants ($l, m, n$) (damage intensity) and varying the spatial extent and distribution of damage. In particular, we consider the following cases:

1. Localized and non-uniform damage distribution along the wave propagation.
2. Localized through-thickness damage distribution in the plate

Since the damage in the context of nonlinear ultrasound is always studied as some kind of nonlinear behavior of the material arising out of different microstructural features, we use the terms ‘micro-scale damage’ and ‘nonlinearity’ synonymously.

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from now on. In the next section, we study the role and interplay of the three aspects of damage, namely distribution, intensity and spatial extent in second harmonic generation.

**Continuous versus periodic distribution of damage**

We present the results obtained for $S_0$ mode (0.5 MHz). The displacement boundary condition applied at the left end of the plate to excite the mode is given by $u(t) = 10^{-7} e^{-0.2(t-6)^2} \sin(2\pi 0.5(t - 6)) \text{m}$ where ‘$t$’ is in $\mu s$. Amplitude of the excitation is chosen such that it corresponds to a stress wave of a few MPa in the material. Triangular elements with a maximum size of 0.0002 m are used to discretize the domain and a maximum time step of 0.05 $\mu s$ is employed for time-stepping. Even though the $S_0$ mode (0.5 MHz) is not phase-matched and the second harmonic generation is not strictly cumulative (Figure 6.3), the reason behind choosing this mode is that we are interested in second harmonic generation when the size of the localized nonlinear zones is about the order of the wavelength, which in this case is $\lambda_{S0} = 10.68$ mm; this allows for easy representation with sufficient discretization in a finite element model.

![Figure 6.15: Schematic of the model used for simulating periodic damage distribution.](image)

Figure 6.15: Schematic of the model used for simulating periodic damage distribution.

Second harmonic generation from two cases of damage distribution, namely continuous and periodic distributions are compared. The schematic of the model used for simulations is shown in Figure 6.15. The effective length of the damaged zone is defined as the length of the zone comprising the ‘NL’ material. The cell-size of the ‘NL’ zone and the ‘LE’ zone are equal and hence the volume-fraction of the damage is 0.5. Simulations are run for four cases with different cell-sizes keeping the effective length of the damaged zone to 80 mm. This choice is based on Figure 6.3 where the amplitude of the second harmonic increases almost linearly
(cumulative) during the 80 mm of the propagation distance. The four cases stated above correspond to 1-80 mm cell, 4-20 mm cells, 8-10 mm cells and 16-5 mm cells. Figure 6.16 shows the FFT’s of the time domain signal for ‘u’ obtained at the receiver location i.e., at \( x = 200 \text{ mm} \) from the left end of the plate as indicated in Figure 6.15. As can be seen, the continuous distribution of damage results in the highest amount of second harmonic. Also, the second harmonic amplitude does not seem to depend on the cell-size of the ‘NL’ zone for a constant effective nonlinear zone. The results obtained here are for the cases where the cell-size of the ‘NL’ zone is about \((0.5-2)\ \lambda\) and may be quite different when they are much larger than the wavelength.

**Figure 6.16**: Comparison of second harmonic generation for periodic versus continuous distribution of damage

**Interplay between damage intensity and spatial extent of micro-scale damage**

Here, we study the interplay between damage intensity and spatial extent of the damage. As damage intensity corresponds to the higher order elastic constants or the Murnaghan constants \((l, m, n)\), we change them by scaling each by factors 1, 2 or 4. Correspondingly, we scale the length of the ‘NL’ zone so that their product is a constant equal to 80 mm. Figure 6.17 shows the FFT’s of time domain signals for ‘u’ at the end of ‘NL’ zone i.e., 80 mm, 40 mm and 20 mm for the cases where the scaling is 1, 2 and 4 respectively. As can be seen, the second harmonic is not necessarily the same for all the cases. Hence from a second harmonic generation point of view, it appears that the scaling the higher order constants and reducing the length of the damaged zone are not equivalent in describing the damage.
Figure 6.17: Second harmonic generation for different scaling constants keeping the effective nonlinear zone the same

To further investigate the above observation, cumulative second harmonic generation from the $S_1$ mode (3.6 MHz) is investigated for three different cases where $l$, $m$ and $n$ are scaled by factors 1, 2 and 4 as earlier. Figure 6.18a shows the amplitude of the second harmonic plotted versus normalized propagation distance. Clearly, the rate of increase is much higher for a scaling constant of 4 as expected. Furthermore, Figure 6.18b shows the normalized amplitude of the second harmonic where the normalization is carried out using the scaling constant used. As can be seen, all the curves do not coincide and hence it can be concluded that the scaling and reducing the length of the ‘NL’ zone are not equivalent for higher harmonic generation.

Figure 6.18: Cumulative second harmonic from the $S_1$ mode (3.6 MHz) for different scaling constants 1, 2 and 4 (a) $A_2$ (b) Normalized $A_2$
In the next section, we investigate the effect of localized through-thickness damage on the second harmonic generation.

**Effect of through-thickness damage distribution on the second harmonic generation of guided waves**

In this section we investigate the effect of through-thickness damage distribution on the second harmonic generation of guided waves. Simulations are run for two modes namely, the $S_0$ mode (0.5 MHz) and the $S_1$ mode (3.6 MHz). The through-thickness damage is changed by varying the percentage through-thickness nonlinearity as indicated in the Figure 6.19.

![Figure 6.19: Schematic of the model with through-thickness nonlinearity used for the simulation.](image)

**$S_0$ mode (0.5 MHz)**

We first present the results obtained for second harmonic generation from the $S_0$ mode at 0.5 MHz. Figure 6.20 shows the relative nonlinearity parameter at $x = 100$ mm versus percentage through-thickness nonlinearity. Clearly, it increases linearly with amount of damage through the thickness. Also, Figure 6.21 shows the relative nonlinearity parameter as a function of the propagating distance for varying levels of through-thickness damage. As can be seen, the nonlinearity parameter increases with the propagation distance for each of the cases. Moreover, the rate of increase of the nonlinearity parameter increases with increasing through thickness nonlinearity. Also, it should be noted that the 20 %-$20\%$ case which corresponds to 20% nonlinearity on the top and 20% nonlinearity on the bottom of the plate almost coincides with the case with 40% nonlinearity on the top. Hence, it appears that the second harmonic generation from the $S_0$ mode (0.5 MHz) is independent of the through-thickness damage location but only depends on the volume-fraction of the damage. This is clearly evident from the plots of the normalized relative nonlinearity parameter shown in Figure 6.22 where each curve is normalized with its value at $x = 20$ mm and all of them except one (20%) coincide. This is because,
for the 20% case, the energy from the primary mode is transferred to antisymmetric mode at the second harmonic in addition to the symmetric mode. This occurs due to the asymmetry in the top and bottom surface of the plate. However, this needs to be further investigated as the other cases do not seem to show this trend.

Figure 6.20: Relative nonlinearity parameter versus the percent through-thickness nonlinearity.

Figure 6.21: Relative nonlinearity parameter versus the percent through-thickness nonlinearity.
Figure 6.22: Normalized relative nonlinearity parameter versus the percent through-thickness nonlinearity.

It should be noted that the second harmonic generation from the $S_0$ mode (0.5 MHz) is independent of the location of the damage due to its uniform wavestructure through the thickness as indicated in Figure 6.23.

Next, we present the results obtained for second harmonic generation from the $S_1$ mode at 3.6 MHz.

**$S_1$ mode (3.6 MHz)**

Simulations similar to the one for the $S_0$ mode (0.5 MHz) are run for the $S_1$ mode (3.6 MHz) by varying through-thickness damage in the plate. Triangular elements are used to discretize the model in Figure 6.19 with a maximum element size of
0.1 mm along the wave propagation direction and at least 15 elements are used to discretize through the thickness. Displacement boundary conditions corresponding to the wavestructure for the $S_1$ mode (3.6 MHz) are applied at the left end of the plate with a gaussian modulated sinusoid excitation in time given by

\[ u(x = 0, t) = (2E - 8) e^{-(0.4(t-4.5)^2)} \cos(2\pi(3.6)(t - 4.5)) \, m \]

and

\[ v(x = 0, t) = (2E - 8) e^{-(0.4(t-4.5)^2)} \sin(2\pi(3.6)(t - 4.5)) \, m \]

where $t$ is in $\mu$s. Simulations are run for different cases of damage distribution through the thickness namely, 20%, 40%, 60%, 80%, 90%, 100% and 20%-20% which corresponds to 20% through-thickness damage on the top and 20% on the bottom. Unlike the $S_0$ mode (0.5 MHz), the wave structure for the $S_1$ mode is not uniform through the thickness as indicated in the Figure 6.24 which depicts the wavestructures of both the $S_1$ mode (3.6 MHz) and the $S_2$ mode (7.2 MHz). Hence, we expect the results to be different from those obtained for the $S_0$ mode (0.5 MHz).

Figure 6.24: Wavestructures (a) $S_1$ mode (3.6 MHz) (b) $S_2$ mode (7.2 MHz)

Figure 6.25 shows the relative nonlinearity parameter as a function of the propagation distance. Several observations are to be made in this regard.

1. Relative nonlinearity parameter is much higher when compared to that from the $S_0$ mode (0.5 MHz) due to the higher frequency and cumulative nature of the $S_1$ mode (3.6 MHz).
2. Relative nonlinearity parameter is not monotonic with the increasing volume-fraction of the through-thickness damage. Hence, it can be concluded that the second harmonic generation from the $S_1$ mode (3.6 MHz) is not exclusively dependent on the volume fraction of through-thickness damage.

3. The case of 20%-20% coincides with that for 100% and hence it can be concluded that the second harmonic generation is mainly due to the contribution from the surface than from the bulk of the material. In fact, it appears that the bulk contribution reduces the second harmonic generation a little as is evident from Figure 6.25 where the 20%-20% case surpasses the 100 % for $x > 30$ mm.

Figure 6.25: Cumulative second harmonic from the $S_1$ mode (0.5 MHz) for varying levels of through-thickness damage.

From the above study, it appears that the second harmonic generation from the $S_1 - S_2$ mode pair at longitudinal wave speed is more sensitive to the surface
damage and can be used to efficiently detect and characterize it.

6.3 Conclusions

In this chapter we investigated nonlinear guided wave propagation in plates from a numerical standpoint. To enhance clarity, the study was presented in two separate sections.

Section 6.1 dealt with the nonlinear guided waves in homogeneous isotropic plates. The predictions of the perturbations theory presented in the Chapter 4 were verified by comparing the second harmonic generation from the S₀ mode (0.5 MHz) and the S₁ mode (3.6 MHz). Effect of frequency bandwidth of the primary mode on the second harmonic generation was investigated. Role of material and geometric nonlinearities in the second harmonic generation of guided waves was investigated. It was found that the contribution of the material nonlinearity was much higher when compared to the geometric nonlinearity. Effect of scaling Murnaghan constants on the second harmonic generation from the S₀ mode (0.5 MHz) was investigated and it was found that the amplitude of the second harmonic is almost proportional to the scaling constant used. Second harmonic generation with group velocity mismatch between the primary and secondary modes revealed that the second harmonic is continuously generated from the primary mode and once generated it propagates independent of the primary mode. The phenomenon of mode interaction to generate sum and difference frequencies was demonstrated by mixing two modes; S₀ mode (1.1 MHz) and the S₁ mode (0.4 MHz).

Section 6.2 focused on the second harmonic generation in plates with localized micro-scale damage. A brief discussion on the interpretation of the micro-scale damage in the context of nonlinear ultrasound was presented. The effect and interplay between three aspects of damage namely, damage intensity, spatial extent and distribution on the second harmonic generation from the S₀ mode (0.5 MHz) was investigated. For a given length of the damaged zone, it was found that continuous distribution of damage resulted in highest second harmonic when compared to the periodic distribution of the damage. Second harmonic generation by scaling the Murnaghan constants and reducing the length of the ‘NL’ zone revealed that scaling the damage intensity and reducing the spatial extent of damage are not equivalent for second harmonic generation. Effect of through-thickness damage distribution on
the second harmonic generation from the $S_0$ mode (0.5 MHz) and the $S_1$ mode (3.6 MHz) was investigated. Second harmonic generation from the $S_0$ mode (0.5 MHz) was found to depend only on the volume-fraction of the through-thickness damage but not on the location of it. On the other hand, second harmonic generation from the $S_1$ mode (3.6 MHz) revealed some interesting findings listed below

1. Second harmonic generation is not exclusively dependent on the volume-fraction of the through-thickness damage.

2. Surface contribution to the second harmonic generation is much higher than the volumetric contribution and hence $S_1 - S_2$ mode pair at longitudinal wave speed can be used to efficiently detect and characterize surface damage.

In the next chapter we adopt a phenomenological approach to understanding the second harmonic generation and try to develop an approach to assess the contribution of micro-scale defects like micro-cracks and voids to the acoustic nonlinearity in chapter 8.
Chapter 7
A phenomenological approach towards understanding the relationship between micro-scale material behavior and ultrasonic higher harmonic generation

Introduction

Until now, we studied the wave-mechanics aspects of the problem of ultrasonic higher harmonic generation. In chapter 6, we briefly studied the role of distribution, intensity and spatial extent of micro-scale damage on second harmonic generation. It should be noted that the constitutive model (Eqns 4.1,4.2) employed till now is at a continuum level and does not specifically capture the behavior of the material at the micro-scale. As the goal of the ongoing research in nonlinear ultrasonics is to be able to characterize micro-scale damage, it is imperative that we understand the effect of different micro-structural features on ultrasonic higher harmonic generation. To that end, we adopt a phenomenological approach towards the problem and identify some important aspects of micro-scale material behavior that contribute to ultrasonic higher harmonic generation.

The content of this chapter is organized as follows. In section 7.1 we present a detailed background of the earlier work relevant to the present discussion. Then, we discuss some important aspects of material behavior responsible for ultrasonic higher harmonic generation. Finally, conclusions are drawn in section 7.3. This chapter is mainly based on the work presented in Chillara and Lissenden (2015a).
7.1 Background

Investigations concerning nonlinear ultrasound in solids can be traced back to the early works of Breazeale and Thompson (1963) and Breazeale and Ford (1965). The main cause for higher harmonic generation was attributed to lattice anharmonicity (Landau and Lifshitz, 1986) arising out of non-quadratic interatomic potential governing the motion of atoms in the solid. Further investigations by (Hikata et al., 1965) led to the conclusion that dislocations are another major cause for the nonlinear behavior of metals. Hikata and Elbaum (1966a) and Hikata et al. (1966b) investigated second and third harmonic generation from dislocations in metals. Both single and poly-crystals were part of their study. In addition, the effect of bias-stress (applied load) on harmonic generation and attenuation was studied. While the harmonic generation from single crystals showed significant dependence on the bias-stress, the effect of bias-stress on harmonic generation from polycrystals was not significant. It should be emphasized that the early work presented here considered two main sources of nonlinearity, namely the elastic nonlinearity of the crystals and the dislocations. The above finding that second/third harmonic generation was sensitive to the presence of dislocations provided impetus to use higher harmonic generation as a tool for monitoring micro-scale damage evolution during fatigue, creep, thermal aging etc.

Cantrell and Yost (2001) used second harmonic generation to characterize fatigue microstructures in Al 2024-T4. The nonlinearity parameter ‘β’ was found to monotonically increase with the number of fatigue cycles and was about 300 % higher for samples cyclically loaded up to 100k cycles when compared to those in virgin state. Cantrell (2004) studied the harmonic generation in cyclically stressed wavy slip metals from a theoretical standpoint. The contribution of substructural features namely the lattice elasticity, dislocation monopoles and dislocation dipoles to the nonlinearity parameter were evaluated separately. Cantrell (2006) compared the predictions of the model with the results obtained from experiments for both polycrystalline nickel and Al 2024-T4; they were found to be in good agreement in both the cases. In addition, it was concluded that the increase of nonlinearity parameter during the first 80 % of fatigue life is dominated by fatigue substructures and that in the last 20% is dominated by the crack growth. Kim et al. (2006) presented a model to quantify the acoustic nonlinearity due to the elastic-plastic
deformations in the materials. The model is based on the cumulative micro-plastic shear strains induced due to monotonic or cyclic loading of the material and was applied to calculate the acoustic nonlinearity of fatigued single-crystal copper. A monotonic increase in \( \beta \) with fatigue cycles was predicted. Cantrell and Yost (2013) compared their model (Cantrell, 2004) with that of Kim et al. (2006) and concluded that their model better captures the contribution of fatigue to acoustic nonlinearity. Recently, Cash and Cai (2012) used dislocation dynamics to study the effect of bias-stress on the acoustic nonlinearity parameter. They found that the acoustic nonlinearity parameter is dependent on the orientation of the dislocation line and these contradict some of the previous findings.

While the effect of fatigue on the acoustic nonlinearity parameter is studied rigorously, several other damage mechanisms such as thermal aging, precipitate formation (Kim and Lissenden, 2009), creep (Baby et al., 2008), radiation damage (Matlack et al., 2012) and embrittlement which induce microstructural changes have received less attention. Each of the above mechanisms is expected to have a different effect on the acoustic nonlinearity parameter. Also, the analytical treatments to determine the contributions of the sources of nonlinearity, typically dislocations, have all been focused on using \( \beta \) as a measure of nonlinearity. While this is an acceptable measure of nonlinearity, it should be noted that a thorough understanding of the effect of each damage mechanism on acoustic nonlinearity is necessary to quantitatively evaluate the material state, especially in the scenarios where two competing mechanisms are influencing the change in the nonlinearity. Hence, it is essential to have efficient modeling approaches to study and understand the effect of different sources of nonlinearity.

We believe that there is a need for multi-scale modeling to address the above issue. This is because the damage progression spans multiple length scales and theoretically the contribution of each microstructural feature to the acoustic nonlinearity needs to be evaluated separately. However, it should be noted that ultrasonic waves used to probe the damage are sensitive only to an averaged material response over the wavelength. For example, if the wavelength is of the order of millimeters, the wave will not be sensitive to microstructural features like a single micro-crack of size 1 \( \mu m \). On the other hand, the wave will be sensitive to the homogenized response of several micro-cracks distributed over a length of 100 \( \mu m \). In this regard, homogenized meso-scale models which describe averaged material response seem to
be more relevant to study ultrasonic higher harmonic generation.

To that end, we discuss some important aspects of meso-scale behavior of materials that contribute to ultrasonic higher harmonic generation. These aspects of material behavior are general in that they do not pertain to any specific micro-scale damage. However, contribution of any microstructural feature to harmonic generation can be studied via the aspects to be discussed in section 7.2.

7.2 Aspects of material behavior responsible for ultrasonic higher harmonic generation

7.2.1 Features of higher harmonics

Since, ultrasonic higher harmonic generation is investigated in the frequency domain, we first begin by making some simple observations concerning the features of time domain signals that have distinctive features in the frequency domain. First, consider the case of a distorted sinusoid shown in Figure 7.1 where the distortion is obtained by multiplying the positive-half-cycle of a sine-wave by factors 1, 0.8 and 0.6. Clearly, the FFT shows only even harmonics. On the other hand, the triangular and trapezoidal pulses shown in Figure 7.2 and Figure 7.3 respectively show only odd harmonics in their FFT’s. While the distorted sinusoid is asymmetric about the zero-position, the triangular and the trapezoidal waveforms are symmetric about the zero-position. Hence, it appears that the even harmonic generation is associated with the asymmetry and odd harmonic generation is associated with the symmetry of the time domain waveforms. Next, we investigate some aspects of material behavior that produce such distortions in displacement/stress/strain fields during wave propagation in the material.
Figure 7.1: Distorted sine wave showing (only) even higher harmonics

Figure 7.2: Triangular wave showing (only) odd higher harmonics

Figure 7.3: Trapezoidal wave showing (only) odd higher harmonics
7.2.2 Relationship between micro-scale material behavior and higher harmonic generation

As mentioned in the section 7.1, micro-scale damage has always been quantified using the $\beta$ parameter i.e., the coefficient of the $\epsilon^2$ term in Eqn (1.25). In other words, the contribution of any microstructural feature to second harmonic generation is evaluated via $\beta$. However, no study has been presented to correlate microstructural features to particular trends in higher harmonic generation. In this section we discuss some important aspects of micro-scale material behavior that provide a deeper insight into understanding the relation between microstructure and higher harmonic generation.

To enhance the clarity in presentation, we first slightly modify the notion of higher order elastic constants (Eqn 4.1) as they correspond to elastic nonlinearity and are more pertinent to single crystals. To that end, we denote the microstructure of the material by $\Gamma$ and assume that the material response is governed by a strain energy function of the form $W(E, \Gamma)$ given by

$$W(E, \Gamma) = \frac{1}{2}\lambda(tr(E))^2 + \mu tr(E^2) + \frac{1}{3}C(\Gamma)(tr(E))^3 + B(\Gamma)tr(E)tr(E^2) + \frac{1}{3}A(\Gamma)tr(E^3)$$

(7.1)

where the following assumptions are made:

1. Strain energy function is dependent on the microstructure $\Gamma$.
2. Third order elastic constants are dependent on the microstructure $\Gamma$.
3. Change in $\Gamma$ does not affect the second order elastic constants ($\lambda, \mu$)

While the above assumptions are reasonable in general, there may be cases where they may be violated; e.g., when there is a macro-scale yielding or when there is a temperature-induced phase transformation. In those cases, one may have to account for the dependence of $\lambda, \mu$ on $\Gamma$ or may have to deal with a constitutive model different from Eqn(7.1). In addition, the description of $\Gamma$ is dependent on the length-scale of the microstructural feature we are interested in. For example, if we are interested in evaluating the contribution of dislocations the acoustic nonlinearity, $\Gamma$ describes the distribution, loop-lengths and other pertinent parameters relevant to dislocation distribution. On the other hand, if one is interested in evaluating the contribution of elastic nonlinearity to the acoustic nonlinearity, $\Gamma$ pertains to
orientation, size and distribution of grains in the materials. In this regard, it should be noted that \( A(\Gamma) \), \( B(\Gamma) \) and \( C(\Gamma) \) correspond to higher order elastic constants of a material only in the case of single crystals in their virgin state i.e., without damage. On the other hand, for polycrystals or polycrystals with defects, \( A(\Gamma) \), \( B(\Gamma) \) and \( C(\Gamma) \) need to be obtained on a case-by-case basis. More details concerning the above discussion will be presented in chapter 8.

Next, we discuss three important aspect of material behavior using the constitutive model in Eqn(7.1). They are:

1. Tension-Compression asymmetry
2. Shear-Normal coupling
3. Deformation-induced anisotropy

From now on, the dependence of \( A(\Gamma) \), \( B(\Gamma) \) and \( C(\Gamma) \) on \( \Gamma \) is not indicated for brevity. Before we move further, we introduce some notation that is useful in future discussion. The second Piola-Kirchoff stress for the constitutive response defined by Eqn(7.1) is given by

\[
T_{RR}(\mathbf{E}) = \lambda \text{tr} (\mathbf{E}) + 2\mu \mathbf{E} + C(\text{tr}(\mathbf{E}))^2 \mathbf{I} + B \text{tr}(\mathbf{E}^2) \mathbf{I} + 2B \text{tr}(\mathbf{E}) \mathbf{E} + A \mathbf{E}^2. \quad (7.2)
\]

In addition, we introduce the stress associated with the linearized strain \( \mathbf{E}^l = \frac{1}{2}(\mathbf{H} + \mathbf{H}^T) \) as

\[
T_{RR}(\mathbf{E}^l) = \lambda \text{tr}(\mathbf{E}^l) \mathbf{I} + 2\mu \mathbf{E}^l + A \mathbf{E}^{l2} + 2B \text{tr}(\mathbf{E}^l) \mathbf{E}^l + B \text{tr}(\mathbf{E}^{l2}) \mathbf{I} + C(\text{tr}(\mathbf{E}^l))^2 \mathbf{I} \quad (7.3)
\]

and the stress associated with the linear material behavior as

\[
T'_{RR}(\mathbf{E}) = \lambda \text{tr}(\mathbf{E}) \mathbf{I} + 2\mu \mathbf{E}. \quad (7.4)
\]

So, \( T_{RR}(\mathbf{E}^l) \) captures the effect of material nonlinearity and \( T'_{RR}(\mathbf{E}) \) captures the effect of geometric nonlinearity.

Next, we discuss the role of tension-compression asymmetry in higher harmonic generation.
Tension-Compression asymmetry

The role of tension-compression asymmetry in higher harmonic generation is investigated using the constitutive model in Eqn 7.1. We first start with uniaxial deformations where \( \mathbf{u} = \begin{pmatrix} u(X_1) \\ 0 \\ 0 \end{pmatrix} \). The corresponding Lagrangian strain is given by

\[
\mathbf{E} = \begin{bmatrix}
\frac{\partial u_1}{\partial X_1} + \frac{1}{2} \left( \frac{\partial u_1}{\partial X_1} \right)^2 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\] (7.5)

From Eqns(7.2, 7.3 & 7.4), we get

\[
T_{RR}(\mathbf{E})_{11} = (\lambda + 2\mu)(\epsilon + \frac{1}{2}\epsilon^2) + (A + 3B + C)\epsilon^2
\]

\[
T_{RR}(\mathbf{E}^l)_{11} = (\lambda + 2\mu)(\epsilon) + (A + 3B + C)\epsilon^2
\]

\[
(T^l_{RR}(\mathbf{E}))_{11} = (\lambda + 2\mu)(\epsilon + \frac{1}{2}\epsilon^2).
\] (7.6)

where \( \epsilon = \frac{\partial u_1}{\partial X_1} \). Figure 7.4 shows the plot of \( T_{RR}(\mathbf{E})_{11} \) with \( \epsilon \) for various scaling constants (1, 2, 4, 8 and 16) of higher order constants shown in Table 7.1. Also shown is the curve for the ‘LE’ case where no material and geometric nonlinearities are included. Clearly, the response shows asymmetry between tension and compression phases for scaling constants of 4 and higher. In fact it shows asymmetry for a scaling constant of 1 as well; as it is evident from the values of stresses \( T_{RR}(\mathbf{E})_{11} \) at \( \epsilon = \pm 0.001 \) in Table 7.2. Also shown in the Table 7.2 are the values of \( T_{RR}(\mathbf{E}^l)_{11} \) and \( (T^l_{RR}(\mathbf{E}))_{11} \) that capture the contribution of material and geometric nonlinearity as described earlier. From this, it can be concluded that the asymmetry due to the geometric nonlinearity is much smaller when compared to that due to material nonlinearity and this in fact is the reason for smaller contribution of geometric nonlinearity to second harmonic generation as investigated in chapter 6. It should be noted that even such a small asymmetry is sufficient for second order effects like the second harmonic generation. Thus, it is shown that the constitutive model in Eqn(7.1) shows tension-compression asymmetry. Next, we show that such an asymmetry is responsible for even harmonic generation in materials.
Table 7.1: Elastic constants (in GPa) used for the study

<table>
<thead>
<tr>
<th>λ</th>
<th>μ</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>25</td>
<td>-320</td>
<td>-200</td>
<td>-190</td>
</tr>
</tbody>
</table>

Figure 7.4: $T_{RR}(E)_{11}$ vs $\epsilon$

Table 7.2: Contribution of material and geometric nonlinearities to the asymmetry in stress (MPa)

<table>
<thead>
<tr>
<th>Strain ($\epsilon$)</th>
<th>$T_{RR}(E)_{11}$</th>
<th>$T_{RR}(E')_{11}$</th>
<th>$(T'<em>{RR}(E))</em>{11}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>108.95</td>
<td>108.89</td>
<td>110.06</td>
</tr>
<tr>
<td>-0.001</td>
<td>-111.05</td>
<td>-111.11</td>
<td>-109.94</td>
</tr>
</tbody>
</table>

To do so, we first consider the simple case of periodic motion of a particle in a conservative force field and examine the conditions under which the displacement field exhibits higher harmonics. Suppose that the mass of the particle is $m$ and let $u(t)$ denote the displacement of the particle as a function of time. Suppose that the conservative force field is governed by a potential $U$. Suppose that the initial conditions of position and velocity specify a total energy $\xi$ that is constant throughout the motion i.e.,

$$T(u(t)) + U(u(t)) = \xi$$  \hspace{1cm} (7.7)
where $T$ denotes the kinetic energy of the particle i.e., $T(t) = \frac{1}{2}m\dot{u}(t)^2$ and $\dot{u}(t)$ denotes the velocity of the particle.

We now prove that if $U(u)$ is asymmetric then so is $u(t).$ The asymmetry of $U$ is defined by $U(a) \neq U(-a), \forall a \in [-b,b]$ for some $b \in \mathbb{R}$. Also observe that since $u(t)$ is periodic so is $\dot{u}(t)$.

Proof: Suppose $U(u(t))$ is asymmetric but $u(t)$ is symmetric. Then, for every $t_1 > 0$, $\exists t_2$ such that

$$u(t_1) = -u(t_2); |\dot{u}(t_1)| = |\dot{u}(t_2)|.$$ \hspace{1cm} (7.8)

Since $U(u(t_1)) + T(\dot{u}(t_1)) = U(u(t_2)) + T(\dot{u}(t_2))$ as the energy is conserved, we have $U(u(t_1)) = U(u(t_2))$, which is a contradiction to the assumption that $U$ is asymmetric. Thus, $u(t)$ is asymmetric.

A similar proof can be given to show that if $u(t)$ is asymmetric then so is $U(u)$.

Here we have shown that if the potential $U$ is asymmetric then $u(t)$ is asymmetric and vice-versa. Since the motion of a material particle during wave propagation is analogous to that under a conservative force field where the potential energy is replaced by the strain energy, it can be concluded that if the strain energy is asymmetric then so is the particle displacement and hence has even harmonics in accordance with the observation made in section 7.2.1.

To substantiate the above claim, we investigate the motion of a particle attached to a bilinear spring whose stiffness dictates the conservative force field. The schematic of the spring-mass system is shown in Figure 7.5. The stiffness of the spring in tension is held fixed ($k = 10$ N/m) and that in compression is varied ($k = 10, 8, 6$ and $4$ N/m) as shown in Figure 7.5. The corresponding particle displacement as a function of time and its FFT are shown in Figure 7.6. Clearly, the FFT’s show the zero-frequency component along with a small second harmonic component for cases where the stiffness in compression is different than that in tension. Also, it can be observed that the amplitude of the second harmonic (and zero-frequency) component increases with increasing difference in stiffness between tension and compression. Although not decipherable, there are odd harmonic frequency components in the FFT’s; these are clearly visible in Figure 7.7 which shows the result for the case where the stiffness in compression is $k = 1$ N/m. These odd harmonics arise due to the discontinuity in the stiffness at the mean
position. To better elucidate this, an elasto-plastic spring mass system whose force-displacement response is given by

\[
F = \begin{cases} 
-ku & \text{if } |u| < u_e \\
-k u_e \frac{|u|}{u_e} & \text{if } |u| \geq u_e 
\end{cases}
\] (7.9)

is chosen where \( k = 10 \text{ N/m} \) and \( u_e = 0.1 \text{ m} \). An initial displacement of 0.2 m is given to the particle. Figure 7.8 shows the results obtained where the FFT shows only the presence of the odd harmonic frequency components but no even harmonics. This is because the force-displacement response of the elasto-plastic spring-mass system in Eqn(7.9) is symmetric but has a discontinuity at \( u = u_e \).

![Schematic of the spring mass system](image)

**Figure 7.5:** Schematic of the spring mass system

![Displacement vs time and their corresponding FFT’s (normalized)](image)

**Figure 7.6:** Displacement vs time and their corresponding FFT’s (normalized) for the bilinear spring mass system with stiffness in tension (\( k = 10 \text{ N/m} \))
From the above study, it can be concluded that tension-compression asymmetry in the material is responsible for even higher harmonic generation in materials. Also, the degree of the asymmetry is proportional to the content of even harmonics and also the zero-frequency component. From the above observation, we can quantify the degree of asymmetry in the constitutive response of the material by a parameter
‘η’ defined as follows

\[
\eta = \frac{(W(-\mathbf{E}) - W(\mathbf{E}))}{(W(\mathbf{E}) + W(-\mathbf{E}))}
\] (7.10)

where ‘\(\mathbf{E}\)’ denotes the Lagrangian strain tensor. The physical significance of the above measure can be explained by considering the 1D nonlinear stress-strain relation in Eqn(1.25). The corresponding strain energy function can be written as

\[
W(\epsilon) = \frac{1}{2}E\epsilon^2 + \frac{\beta}{6}E\epsilon^3.
\] (7.11)

From Eqns (7.10 & 7.11) we have \(\eta = -\frac{1}{3}\beta\epsilon\). Thus, \(\eta\) can be thought of as an alternative measure of nonlinearity proportional to \(\beta\). Unlike \(\beta\) whose definition is applicable only for a uniaxial stress state, \(\eta\) can be evaluated for any multiaxial stress state during the wave propagation. In fact, \(\eta\) corresponds to the fraction of energy transferred to the even harmonics from the primary mode. Note that the above definition does not rely on a specific form of strain energy function and hence can be applied to harmonic generation from any microstructural feature as will be illustrated in chapter 8. However, it should be noted that the above measure is defined under the assumption that both the strain states \(\mathbf{E}\) and \(-\mathbf{E}\) correspond to admissible deformations. While it is not always the case in the context of large deformations, it holds under the “small displacement gradient” assumption appropriate for the wave propagation studies.

Until now tension-compression asymmetry was identified and proven to be a necessary and sufficient feature for even harmonic generation. Next, we identify some plausible microstructural features that contribute to such an asymmetry in the material behavior. In addition to dislocations and elastic nonlinearity, residual stress is an important factor that contributes to the asymmetry in the material response. When a material is residually stressed/strained, the mean position of oscillation shifts and hence the response is asymmetric and is dependent on the nature of the residual stress/strain (tensile/compressive). In addition, residual stress changes the instantaneous stiffness of the material as will be illustrated in the discussion on ‘deformation induced anisotropy’ to follow. Also, the contact nonlinearity (Nazarov and Sutin, 1997) arising out of crack opening and closure during wave propagation shows asymmetry where the tension and compression phases correspond to crack opening and closure respectively.
Shear-Normal coupling

Here, we discuss shear-normal coupling exhibited by the constitutive model in Eqn(7.1). This is an aspect of material behavior that received much less attention as most studies used 1D models for studying higher harmonic generation. By shear-normal coupling we mean that a simple shear (Ogden, 1997) deformation can occur only by applying a normal stress in addition to the shear stress. This phenomenon results in additional coupling terms as will be illustrated in this section. This phenomenon does not occur in isotropic linear elasticity. In fact, only linear elastic crystals having symmetry above orthotropic exhibit this behavior. To demonstrate the shear-normal coupling exhibited by Eqn(7.1) we start with an antiplane shear deformation as follows

\[
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}, \quad H =
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\frac{\partial u_3}{\partial X_1}, \frac{\partial u_3}{\partial X_2} & 0
\end{bmatrix}
\] (7.12)

The Lagrangian strain is given by

\[
E = \frac{1}{2} \begin{bmatrix}
\frac{(\partial u_3}{\partial X_1})^2 & \frac{\partial u_3}{\partial X_1} \frac{\partial u_3}{\partial X_2} & \frac{\partial u_3}{\partial X_2} \\
\frac{\partial u_3}{\partial X_1} \frac{\partial u_3}{\partial X_2} & \frac{(\partial u_3}{\partial X_2})^2 & \frac{\partial u_3}{\partial X_1} \\
\frac{\partial u_3}{\partial X_2} & \frac{\partial u_3}{\partial X_1} & 0
\end{bmatrix}.
\] (7.13)

For mathematical simplicity, we illustrate the shear-normal coupling using \(T_{RR}(E)\), but \(T_{RR}(E')\) and \(T_{RR}(E)\) are similar, just having more coupling terms. Eqns(7.4 & 7.13) yield

\[
T_{RR}(E) = \begin{bmatrix}
(\lambda + \mu) \left(\frac{\partial u_3}{\partial X_1}\right)^2 + \lambda \left(\frac{\partial u_3}{\partial X_2}\right)^2 & \mu \left(\frac{\partial u_3}{\partial X_1}\right) \left(\frac{\partial u_3}{\partial X_2}\right) & \mu \frac{\partial u_3}{\partial X_1} \\
\mu \left(\frac{\partial u_3}{\partial X_1}\right) \left(\frac{\partial u_3}{\partial X_2}\right) & (\lambda + \mu) \left(\frac{\partial u_3}{\partial X_2}\right)^2 + \lambda \left(\frac{\partial u_3}{\partial X_1}\right)^2 & \mu \frac{\partial u_3}{\partial X_2} \\
\mu \frac{\partial u_3}{\partial X_1} & \mu \frac{\partial u_3}{\partial X_2} & \lambda \left(\frac{\partial u_3}{\partial X_1}\right)^2 + \left(\frac{\partial u_3}{\partial X_2}\right)^2
\end{bmatrix}.
\] (7.14)

The non-zero terms of second order that appear in 11, 22, 33, 12 are called the coupling terms which are a characteristic of nonlinear elastic behavior and hence they do not appear in linear elasticity of isotropic materials. Also, these terms are responsible for the generation of secondary longitudinal waves from the
transverse waves (Goldberg, 1961) in the bulk wave propagation, and secondary Rayleigh-Lamb modes from primary Shear-Horizontal modes (Liu et al., 2013a) in the guided wave propagation in plates. The harmonic generation observed due to the shear-normal coupling is quite different in that it does not arise out of a pulse distortion but due to the generation of a secondary wave with orthogonal polarization that propagates independent of the primary wave. Next, we identify some plausible microstructural features responsible for shear-normal coupling.

Crystalline anisotropy is one factor that can contribute to shear-normal coupling especially in the case where grains have a crystal symmetry above orthotropic. Also, the presence of residual stress may lead to shear-normal coupling due to the ‘deformation induced anisotropy’ as will be discussed later. In addition, the motion of dislocation under the influence of an external stress may lead to shear normal coupling as expressed by Peach-Koehler equations (Weertman and Weertman, 1966).

**Deformation induced anisotropy**

Deformation induced anisotropy refers to the apparent anisotropy exhibited in the material response due to an external load/deformation acting on it. The external load here may not necessarily correspond to an applied load but may be a residual stress/strain depending on the context. In fact in the context of the current discussion, the latter aspect is more relevant where the meso-scale behavior is influenced by the presence of residual stress/strain. In this section we investigate how a pre-deformation/strain affects the response of the material described by Eqn(7.1).

Consider a tri-axial stretch given by $x_1 = p_1 X_1$, $x_2 = p_2 X_2$ and $x_3 = p_3 X_3$ where the $(x_1, x_2, x_3)$ and $(X_1, X_2, X_3)$ are the coordinates of the position of the material particles in the current and the reference configurations respectively. So, the displacement gradient is given by

$$
H = \begin{bmatrix}
p_1 - 1 & 0 & 0 \\
0 & p_2 - 1 & 0 \\
0 & 0 & p_3 - 1
\end{bmatrix}.
$$

The effective/incremental modulii (Ogden, 1997) for the material can be obtained
for the above deformed state and are given by (in index notation)
\[
\frac{\partial S_{ij}}{\partial H_{kl}} = \lambda \delta_{ij} \delta_{kl} + (\mu + BH_{pp})(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda (H_{ij}\delta_{kl} + H_{kl}\delta_{ij}) \\
+ \lambda H_{pp}\delta_{ij}\delta_{kl} + \mu (H_{ij} + H_{jl})\delta_{ik} + \mu (H_{ik} + H_{ki})\delta_{jl} + \mu (H_{il}\delta_{jk} + H_{kj}\delta_{il}) \\
+ 2CH_{pp}\delta_{ij}\delta_{kl} + B(H_{ik} + H_{ki})\delta_{ij} + B(H_{ij} + H_{ji})\delta_{kl} \\
+ \frac{A}{4} ((H_{ik} + H_{ki})\delta_{jl} + (H_{jl} + H_{lj})\delta_{ik} + (H_{jk} + H_{kj})\delta_{li} + (H_{il} + H_{li})\delta_{jk})
\] (7.16)

where ‘\( \delta \)’ denotes the Kronecker-delta and \( S \) denotes the first Piola-Kirchoff stress tensor. The incremental modulii evaluated using Eqn (7.16) are given for two examples of stretch. In Voigt notation, the elastic constants for linear elastic material (obtained from the quadratic part of \( W \) in Eqn 7.1) are

\[
C_{\text{linear}} = \begin{bmatrix}
110 & 60 & 60 & 0 & 0 & 0 \\
60 & 110 & 60 & 0 & 0 & 0 \\
60 & 60 & 110 & 0 & 0 & 0 \\
0 & 0 & 0 & 25 & 0 & 0 \\
0 & 0 & 0 & 0 & 25 & 0 \\
0 & 0 & 0 & 0 & 0 & 25
\end{bmatrix} \text{ GPa}
\]

given \( \lambda \) and \( \mu \) from Table 7.1.

For uniaxial stretch \((p_1 = 1.002; p_2 = 1; p_3 = 1)\), the incremental modulii (Eqn(7.16)) are

\[
C_{\text{incremental}} = \begin{bmatrix}
105.42 & 58.56 & 58.56 & 0 & 0 & 0 \\
58.56 & 107.76 & 59.24 & 0 & 0 & 0 \\
58.56 & 59.24 & 107.76 & 0 & 0 & 0 \\
0 & 0 & 0 & 24.32 & 0 & 0 \\
0 & 0 & 0 & 0 & 24.1 & 0 \\
0 & 0 & 0 & 0 & 0 & 24.1
\end{bmatrix} \text{ GPa}
\]

which corresponds to a transversely isotropic material. For biaxial stretch \((p_1 = 1.002; p_2 = 1.001; p_3 = 1)\), the incremental modulii are
\[
C_{\text{incremental}} = \begin{bmatrix}
104.3 & 57.84 & 58.18 & 0 & 0 & 0 \\
57.84 & 105.47 & 58.52 & 0 & 0 & 0 \\
58.18 & 58.52 & 106.64 & 0 & 0 & 0 \\
0 & 0 & 0 & 23.87 & 0 & 0 \\
0 & 0 & 0 & 0 & 23.76 & 0 \\
0 & 0 & 0 & 0 & 0 & 23.65
\end{bmatrix} \text{ GPa}
\]

which corresponds to an orthotropic material.

The above examples suggest that the material which is initially isotropic exhibits an anisotropic response as evident from the instantaneous/incremental modulii. It should be observed that the anisotropy due to the load is small but may be sufficient for higher order effects such as higher harmonic generation. Also, we assumed that the pre-deformation is homogeneous and the results may be different if the deformation is inhomogeneous. Next, we discuss the effect of pre-deformation on higher harmonic generation.

As in Cantrell (2004), we start with the relation (Eqn(7.17)) between the first Piola-Kirchhoff stress and components of the displacement gradient for a general anisotropic material

\[
S_{ij} = A_{ijkl}H_{kl} + A_{ijklmn}H_{kl}H_{mn} + A_{ijklmnqp}H_{kl}H_{mn}H_{pq} \tag{7.17}
\]

where, \(A_{ijkl}, A_{ijklmn}\) and \(A_{ijklmnqp}\) are the first, second and third order Huang coefficients. Suppose the pre-deformation generates a displacement gradient \(H^0\) in the material. Equation (7.17) when written as a Taylor series about \(H^0\) assumes the form

\[
S = S(H^0) + \frac{\partial S}{\partial H_{H^0}}(H - H^0) + \left( \frac{\partial^2 S}{\partial^2 H_{H^0}}(H - H^0) \right)(H - H^0) \tag{7.18}
\]

where \(\frac{\partial S}{\partial H_{H^0}}\) and \(\frac{\partial^2 S}{\partial^2 H_{H^0}}\) are the first and second-order Huang coefficients referred from the deformed state corresponding to \(H^0\) and are given below.

\[
\frac{\partial S_{ij}}{\partial H_{rs}} = \begin{align*}
A_{ijrs} & + A_{ijrsmn}H^0_{mn} + A_{ijklrs}H^0_{kl} \\
& + A_{ijrsmnqp}H^0_{mn}H^0_{pq} + A_{ijklrsqp}H^0_{kl}H^0_{pq} + A_{ijklmns}H^0_{kl}H^0_{mn}
\end{align*}
\]
\[
\frac{\partial^2 S_{ij}}{\partial H_{rs} H_{tw}} = (A_{stw} + A_{ijtwrs}) + (A_{ijtwspq} + A_{stwpq}) H_{pq}^0 \\
+ (A_{stw} + A_{ijtwms}) H_{mn}^0 + (A_{ijklsw} + A_{ijkltw}) H_{kl}^0 \tag{7.19}
\]

As can be seen, the first and second order Huang coefficients depend on $H^0$. While the first order coefficients describe the anisotropy in the material response (discussed earlier) the second order coefficients describe the effect of $H^0$ on second harmonic generation.

### 7.3 Conclusions

In this chapter we discussed harmonic generation in materials from a phenomenological viewpoint. A detailed discussion of earlier work relevant to the current discussion was presented and the motivation for the study was outlined. Three main aspects relevant to the meso-scale behavior of materials were discussed using the constitutive model in Eqn(7.1).

Tension-Compression asymmetry exhibited by the constitutive model was discussed and identified to be the cause for even higher harmonic generation in materials. The contribution of material and geometric nonlinearities to the asymmetry were computed and it was found that material nonlinearity has a much higher contribution to the asymmetry when compared to the geometric nonlinearity. A simple bilinear spring was considered as a mechanical analogue to illustrate the findings. The asymmetry in the stiffness of the spring was found to be proportional to amplitude of even harmonics including the zero-frequency component. The discontinuity of the stiffness at the mean position was found to be responsible for generation of odd harmonics. This was further reinforced by considering the example of an elasto-plastic spring-mass system which has symmetric response in tension and compression. The degree of asymmetry in the constitutive response of the material was quantified using a factor ‘$\eta$’. Relevant microstructural features responsible for tension-compression asymmetry were identified.

Shear normal coupling exhibited by the constitutive model in Eqn(7.1) was discussed. It was identified to be the cause for generation of secondary waves with polarization different from that of the primary wave. Some plausible microstructural features responsible for shear-normal coupling were identified.

Deformation induced anisotropy due to presence of residual stress/strain was
discussed. Also, a qualitative discussion of the role of pre-stress/strain on higher harmonic generation was presented.

The aspects of material behavior presented in this chapter are general in that the contribution of any microstructural features to higher harmonic generation can be evaluated via these. In addition, we outlined important features like tension-compression asymmetry that a meso-scale constitutive model should incorporate for it to capture even higher harmonic generation. Furthermore, the alternative measure ‘\(\eta\)’ introduced during the discussion on tension-compression asymmetry plays an important role in quantifying the contribution of micro-scale defects like cracks and voids to second harmonic generation. This would be the focus of our discussion in the next chapter.
Chapter 8
A homogenization based approach to assess the contribution of micro-scale defects to acoustic nonlinearity

Introduction

In chapter 7 we studied some important aspects of micro-scale material behavior that are responsible for ultrasonic higher harmonic generation. These can be regarded as manifestations of nonlinear micro-scale material behavior and hence they can be used to quantify the nonlinearity associated with microstructural features. This chapter attempts address this issue by developing a homogenization based approach to quantify the tension-compression asymmetry associated with micro-scale material behavior. The approach relies on the definition of the ‘η’ parameter introduced in chapter 7 and evaluates the contribution of microstructure to ultrasonic second harmonic generation.

The content of this chapter is organized as follows. First we present a brief discussion on some earlier work relevant to the present discussion in section 8.1. Then, we present the homogenization based approach from a mathematical standpoint and also discuss its physical relevance in section 8.2. Then, in section 8.3 we employ the approach to quantify the second harmonic generation from a material with micro-voids. Finally conclusions are drawn in section 8.4.

Parts of this chapter are from Chillara and Lissenden (2015c).

8.1 Background

As mentioned earlier in chapter 7, early investigations (Breazeale and Thompson, 1963) (Breazeale and Ford, 1965) (Hikata et al., 1965) (Hikata and Elbaum, 1966a)
(Hikata et al., 1966b) of ultrasonic higher harmonic generation put forth two main causes of nonlinear material behavior:

1. Elastic nonlinearity associated with non-quadratic atomic potential governing the atoms in the solid.

2. Dislocation motions under oscillating stress/strain fields associated with the propagating wave.

Hence the modeling approaches always focused on quantifying the contribution of the above features to acoustic nonlinearity. From a continuum standpoint, elastic nonlinearity is often modeled as higher order elastic constants in the strain energy associated with the elastic material. It should be noted that the higher order elastic constants are more relevant for single crystals or polycrystals with no damage or in their virgin state. Experimental measurements of these higher order elastic constants were carried out using acoustoelasticity (Thurston and Brugger, 1964) (Hiki and Granato, 1966) (Hirao et al., 1981). For polycrystals that contain grains oriented in specified/random orientation depending on the manufacturing process, the higher order elastic constants are to be regarded as some kind of an average of the grains in the polycrystal. Hence, modeling approaches for computing higher order elastic constants for polycrystals rely on some micro-mechanics (Nemat-Nasser and Hori, 1999) inspired effective medium or homogenization approaches. One such study was carried out in Lubarda (1997) for isotropic aggregates of cubic crystals. Voigt, Reuss and semi-consistent approaches for third order elastic constants were obtained and compared with those obtained from experiments. Unlike the estimates of second-order elastic constants which always lie between Voigt and Reuss estimates, the third order elastic constants do not necessarily obey this and hence the bounds on these effective third elastic constants are not well ascertained. The models for dislocations induced nonlinearity in material are based on the string model for dislocation employed originally in Eshelby (1949) and Granato and Lücke (1956). These were employed by Hikata et al. (1965), Hikata and Elbaum (1966a) and Hikata et al. (1966b) to model second and third harmonic generation of ultrasonic waves in metals. However, these were mainly 1D elastic models that quantify the nonlinearity in the material by the ‘β’ parameter as outlined in chapter 1. Relation between the ‘β’ parameter and the dislocation density, loop lengths, etc were obtained in Hikata and Elbaum (1966a) and Hikata et al. (1966b) from
a quasi-static consideration. Cantrell (2004) and Cantrell (2006) extended the above approach put-forth by Hikata and co-workers to evaluate the contribution of dislocation monopoles and dipoles to acoustic nonlinearity. These were employed to predict the evolution of ‘\( \beta \)’ parameter during the fatigue damage. However, it should be noted that this approach is based on adding the contributions of the dislocation substructures without accounting for their distribution or orientation. This shortcoming was recently pointed out in Cash and Cai (2012) where the effect of orientation-dependent line energy on the acoustic nonlinearity parameter was investigated and it was concluded that the orientation of the dislocation line has a significant effect on the \( \beta \) parameter. In this regard it should be concluded that the dislocation based modeling approaches without recourse to the distribution of them in material may yield inaccurate results. Also, as emphasized in chapter 7, the wavelength of the ultrasonic waves used to investigate nonlinearity in the material are only sensitive to the distributed damage on length scales in the range of 50-500 \( \mu m \). While the dislocation-based models necessarily provide insight into interaction of wave with dislocations, it is more pertinent to discuss the wave-material interaction at a meso-scale where the distribution of damage is taken in to account. To that end, this work develops a micromechanics inspired homogenization based approach to assess the contribution of micro-scale defects to acoustic nonlinearity. This is based on the tension-compression asymmetry discussed in chapter 7.

8.2 Approach

The approach is based on conventional micro-mechanics based homogenization approaches that yield effective material properties. We begin with a statistically homogeneous material whose Representative Volume Element(RVE) is as shown in Figure 8.1 where nonlinearities(NL) (here elastic) are embedded in a matrix of a linear elastic material. The volume of RVE is denoted by \( \Omega \) and its surface is denoted by \( \partial \Omega \). Let \( \Omega_{LE} \) and \( \Omega_{NL} \) denote the collection of the material particles in linear and nonlinear zones respectively. For simplicity, we assume that matrix material and the nonlinearities are isotropic. Also the distribution of nonlinearities are assumed to be such that the overall behavior of the material is isotropic. As we are interested in the effect of nonlinearities on an ultrasonic wave (that corresponds
to a stress field of few MPa) propagating in the material, we consider the problem under a linearized strain assumption. Even the nonlinearities are dealt with under this assumption in which case we are considering only the contribution of the material nonlinearity; it suffices as the contribution of material nonlinearity is much higher when compared to geometric nonlinearity as was shown in Chillara and Lissenden (2014b).

Next we describe the geometric nature of nonlinearities depicted in Figure 8.1. By nonlinearities (NL) we mean a weakly nonlinear elastic inclusion or a micro-scale defect (like void/crack) surrounded by a region where the material is nonlinear elastic in nature. Of particular interest to the current study is the latter case of micro-scale defects even though it can be considered as a special case of the former. We note that the nonlinearities are weakly nonlinear-elastic in nature—an assumption that is generally used for studying the nonlinear wave propagation in structural materials. Put mathematically, we assume that the strain energy functions defining the material response of the linear elastic material and that of the nonlinear elastic material are given by

$$W_{LE}(E) = \frac{1}{2}\lambda_{LE}(tr(E))^2 + \mu_{LE}tr(E^2)$$  \hspace{1cm} (8.1)
and

\[ W_{NL}(\mathbf{E}) = \frac{1}{2} \lambda_{NL}(tr(\mathbf{E}))^2 + \mu_{NL} tr(\mathbf{E}^2) + \frac{1}{3} C(tr(\mathbf{E}))^3 + B tr(\mathbf{E}) tr(\mathbf{E}^2) + \frac{1}{3} A(tr(\mathbf{E}^3)) \]  

(8.2)

respectively. Here, \( \lambda_{LE} \) and \( \mu_{LE} \) are the Lame’s constants of the linear elastic material and \( \lambda_{NL} \) and \( \mu_{NL} \) are the Lame’s constants of the nonlinear elastic material and \( A, B \) and \( C \) are the third order elastic constants of the nonlinear material. We note that the constitutive model used for describing the response of the nonlinear elastic material is the one widely used to study nonlinear wave propagation in structural metals. However, it should be noted that the approach presented here is general and can be applied with any constitutive model depicting the nonlinearity (NL) in the material so long as it is a weak nonlinearity.

As mentioned earlier, our approach is based on quantifying the tension-compression asymmetry in the material response of a homogenized material and hence its contribution to acoustic nonlinearity. So, we briefly present the formulation of the elastic boundary value problems (BVP) that enable us to determine the homogenized response of the material.

Two types of boundary conditions on RVE can be considered as follows:

1. BVP with traction prescribed on \( \partial \Omega \)
2. BVP with displacement prescribed on \( \partial \Omega \)

**BVP with traction prescribed on \( \partial \Omega \)**

\[ \nabla \cdot \sigma^\Sigma = 0 \text{ in } \Omega \]
\[ \sigma^\Sigma \mathbf{n} = \Sigma \mathbf{n} \text{ on } \partial \Omega \]  

(8.3)

Here \( \Sigma \) is the macro-stress (tensor) (Nemat-Nasser and Hori, 1999) and is uniform on \( \partial \Omega \) and \( \sigma^\Sigma \) is the stress tensor field that satisfies Eqn(8.3) at each point inside the RVE. Similarly, the variables \( \mathbf{u}^\Sigma \) and \( \epsilon^\Sigma \) denote the displacement field and strain field within RVE for a prescribed macro-stress ‘\( \Sigma \)’ on the boundary. In addition to the above mentioned boundary conditions, there are displacement and traction continuity or traction free boundary conditions accordingly as the nonlinearity is an inclusion or a micro-scale defect.
BVP with displacement prescribed on $\partial \Omega$

\[ \nabla \cdot \sigma^H = 0 \text{ in } \Omega \]
\[ u^H = Hx \text{ on } \partial \Omega \]  \hspace{1cm} (8.4)

Here ‘$H$’ denotes the displacement gradient (under the assumption that $||H|| << 1$) and ‘$x$’ denotes the position of a material particle on the boundary. And, $\Sigma^H$, $u^H$ and $e^H$ denote the stress field, displacement field and strain field within the RVE for a prescribed macro-displacement gradient on the boundary.

Each of the above problems is expected to yield an equivalent homogenized strain-energy function $W_{\text{hom}}(E)$, namely $W_{\text{hom}}(E; \Sigma)$ or $W_{\text{hom}}(E; H)$ accordingly as the macro-stress or macro-displacement gradient is prescribed. From hereon, the explicit dependence of $W_{\text{hom}}(E; \Sigma)$ on ‘$\Sigma$’ and $W_{\text{hom}}(E; H)$ on ‘$H$’ is dropped. As the overall behavior of the material is assumed to be isotropic and the nonlinearities are weakly nonlinear elastic, the functional form of $W_{\text{hom}}(E)$ can be expected to be

\[ W_{\text{hom}}(E) = \frac{1}{2} \lambda_{\text{hom}} (\text{tr}(E))^2 + \mu_{\text{hom}} \text{tr}(E^2) + \tilde{W}_{\text{hom}}(E) \]  \hspace{1cm} (8.5)

where $\lambda_{\text{hom}}$ and $\mu_{\text{hom}}$ are homogenized linear elastic constants and $\tilde{W}_{\text{hom}}(E)$ is a correction to account for the overall weakly nonlinear response of the material. The explicit functional form of $\tilde{W}_{\text{hom}}(E)$ depends on the constitutive model used to describe the nonlinearities. However, we note that $\tilde{W}_{\text{hom}}(E)$ contains higher-order terms in $E$ i.e., of order higher than 2 when viewed as a Taylor expansion.

Now we present the approach that can be used for assessing the contribution of the micro-scale defects to the acoustic nonlinearity. The goal of this homogenization based approach is to determine the contribution of nonlinearities to second (even) harmonic generation in materials. It was shown in chapter 7 that the amount of second harmonic generation in the material can be quantified by the asymmetry in the strain energy function. Explicitly, if ‘$E$’ is the strain-state corresponding to the maximum strain energy of a material particle during wave propagation, the quantity ‘$\eta$’ defined below is proportional to the second (even) harmonic content in the wave packet. It can be interpreted as the fraction of energy of the fundamental mode that is transferred to the higher harmonic modes during one cycle of wave propagation.

\[ \eta = (W(-E) - W(E))/(2W_{\text{lin, hom}}(E)) \]  \hspace{1cm} (8.6)
Here $W_{lin, hom}(E)$ corresponds to the quadratic part of the energy $W_{hom}(E)$ which depends only on the constants $\lambda_{hom}$ and $\mu_{hom}$.

We employ the above definition of $\eta$ to each point in RVE and arrive at an estimate of the fraction of energy transferred to the second harmonic as a function of size and volume fraction of the micro-scale defects. We first outline the approach in a general context and then implement the same to arrive at the above mentioned estimates for specific cases of micro-scale defects. The approach is outlined below:

**Step 1 : Linear elastic homogenization problem**

As the nonlinearities are assumed to be weakly nonlinear elastic and the influence of these are localized to the vicinity of the micro-scale defects, we first consider the linear homogenization problem wherein the matrix material is linear elastic and also the nonlinearities are approximated as linear elastic material. The interaction between the defects can be accounted by using the self-consistent approach. This procedure yields the constants $\lambda_{hom}$ and $\mu_{hom}$ in Eqn (8.5).

**Step 2 : Single defect/inclusion problem (Linear elastic)**

Now consider the problem of single defect/inclusion of the same geometric nature as the nonlinearity and compute the strain field ($\epsilon^{\Sigma}$ or $\epsilon^{H}$) in the vicinity. But, the material properties of the linear elastic matrix ($\lambda_{hom}$ and $\mu_{hom}$) are as determined in Step 1 and the linear elastic properties of the inclusion/defect are the same as employed in Step 1. The boundary conditions for this problem are going to be the same as those for the RVE, namely Eqns (8.3) or (8.4). This would give an estimate of the strain field in the vicinity of nonlinearity. Here, we would like to make a small observation i.e., $\epsilon^{-\Sigma} = -\epsilon^{\Sigma}$ and $\epsilon^{-H} = -\epsilon^{H}$ due to the linearity of the problem being considered in this step.

**Step 3 : Computing $\eta$**

The parameter $\eta$ is evaluated at each point in the vicinity of the nonlinearity using the strain field ($\epsilon^{\Sigma}$ or $\epsilon^{H}$) obtained from Step 2 i.e., we have a function $\eta(x)$ for every $x \in \Omega_{NL}$.

**Step 4 : Contribution to even harmonic generation**

A scalar parameter for evaluating the overall contribution to second harmonic
generation can be defined by taking the volume average of ‘$\eta$’ over $\Omega_{NL}$ normalized by the volume of RVE.

\[
\bar{\eta} = \frac{1}{\text{vol}(\Omega)} \int_{\Omega_{NL}} \eta(x) = \frac{N}{\text{vol}(\Omega)} \int_{\text{single nonlinear zone}} \eta(x) d\Omega
\]

\[
= \frac{1}{\text{vol}(\Omega)} \int_{\Omega_{NL}} (W_{NL}(\epsilon^{-\Sigma}) - W_{NL}(\epsilon^\Sigma))/ (2W_{lin hom}(E; \Sigma))d\Omega
\]

\[
= \frac{1}{\text{vol}(\Omega)(2W_{lin hom}(E; \Sigma))} \int_{\Omega_{NL}} (W_{NL}(\epsilon^{-\Sigma}) - W_{NL}(\epsilon^\Sigma))d\Omega \quad (8.7)
\]

where $N$ denotes the number of nonlinearities in RVE of volume $\Omega$. Also, $W(\epsilon^\Sigma)$ corresponds to energy density in the nonlinear zone evaluated when ‘$\Sigma$’ is prescribed as macro-stress. Similar definition can be employed for a prescribed macro displacement gradient as well. Also, it is clearly evident that we have a dependence of the parameter on the volume fraction (through $N$) of nonlinearities and hence it is sensitive to the microstructure of the RVE.

Before we proceed further, we would like to make an interesting observation which would serve as a proof to validate the above procedure. Till now, we have not explicitly relied on the functional form of $\tilde{W}_{hom}(E)$ or that of $W_{NL}(E)$ i.e., the above procedure is valid for nonlinearities governed by any general constitutive model for $W_{NL}(E)$ as long as it corresponds to a weak nonlinearity. Now, suppose that overall behavior of the RVE is such that

\[
\tilde{W}_{hom}(E) = \frac{1}{3} \tilde{C}(tr(E))^3 + \tilde{B}tr(E)tr(E^2) + \frac{1}{3} \tilde{A}(tr(E^3)).
\]

Also, assume that $W_{NL}(E)$ can be reasonably approximated to the second order in strain as follows:

\[
W_{NL}(E) = \frac{1}{2} \frac{\partial^2 W_{NL}(E)}{\partial E_{ij} \partial E_{kl}} \bigg|_{E=0} E_{ij} E_{kl} + \frac{1}{6} \frac{\partial^3 W_{NL}(E)}{\partial E_{ij} \partial E_{kl} \partial E_{mn}} \bigg|_{E=0} E_{ij} E_{kl} E_{mn}
\]

\[
= \frac{1}{2} C_{ijkl} E_{ij} E_{kl} + \frac{1}{6} C_{ijklmn} E_{ij} E_{kl} E_{mn} \quad (8.8)
\]

where $C_{ijkl} = \frac{\partial^2 W_{NL}(E)}{\partial E_{ij} \partial E_{kl}} \bigg|_{E=0}$ and $C_{ijklmn} = \frac{\partial^3 W_{NL}(E)}{\partial E_{ij} \partial E_{kl} \partial E_{mn}} \bigg|_{E=0}$. 

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Using the above notation and noting that $\epsilon^{-\Sigma} = -\epsilon^\Sigma$, we have

$$W_{NL}(\epsilon^\Sigma) = \frac{1}{2} C_{ijkl} \epsilon_{ij}^\Sigma \epsilon_{kl}^\Sigma + \frac{1}{6} C_{ijklmn} \epsilon_{ij}^\Sigma \epsilon_{kl}^\Sigma \epsilon_{mn}^\Sigma$$

$$W_{NL}(\epsilon^{-\Sigma}) = \frac{1}{2} C_{ijkl} \epsilon_{ij}^{-\Sigma} \epsilon_{kl}^{-\Sigma} - \frac{1}{6} C_{ijklmn} \epsilon_{ij}^{-\Sigma} \epsilon_{kl}^{-\Sigma} \epsilon_{mn}^{-\Sigma}$$  \hspace{1cm} (8.9)

If we stipulate the condition that the homogenized material and the material with nonlinearities have the same ‘$\alpha_1$’ (as evaluated from Eqn(8.7)) then we must have

$$\frac{1}{3} \tilde{C}(tr(E^\Sigma))^3 + \tilde{B} tr((E^\Sigma))^2 + \frac{1}{3} \tilde{A} tr((E^\Sigma))^3 = \frac{1}{\Omega} \int_{\Omega_{NL}} \frac{1}{6} C_{ijklmn} \epsilon_{ij}^\Sigma \epsilon_{kl}^\Sigma \epsilon_{mn}^\Sigma$$  \hspace{1cm} (8.10)

where $E^\Sigma$ is the homogeneous strain in the RVE made of homogenized material subject to the uniform traction boundary condition. Equation 8.10 implies that we are indeed equating the energy stored in the nonlinear zones to the energy corresponding to the higher order strain terms in the homogenized material. This is physically reasonable and in fact the basis for several homogenization approaches (linear/nonlinear).

We finally conclude this section with the following comments. The procedure described above does not yield effective or homogenized third-order elastic constants of the material. Instead it quantifies the tension-compression asymmetry which is directly related to the second-harmonic generation in materials. However, there have been numerous studies that obtain overall homogenized behavior of nonlinear elastic composites including bounds for these estimates. For example, Castañeda (1991) outlines the procedure to obtain effective properties for nonlinear isotropic composites with voids, rigid inclusions, etc. We note that this study is geared towards understanding the influence of microstructure on second harmonic generation and the procedure outlined above accomplishes it without recourse to evaluating the higher order constants i.e., $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$. In the next section we present some results obtained by applying the above approach to material with micro-voids.
8.3 Results

In this section, we present the results obtained by employing the above approach to evaluate the contribution of micro-voids to acoustic nonlinearity. Figure 8.2 shows the RVE for the material with micro-voids. The micro-voids are assumed to be uniformly dispersed in a matrix of linear elastic (LE) material. Also, each void is assumed to be surrounded by a region of nonlinear elastic (NL) material which corresponds to localized plastic zones that represent driving forces for the voids to grow or coalesce. The radius of the ‘NL’ zone is assumed to be an integer multiple of the radius of the void. The elastic constants for LE and NL zones are shown in Table 8.1.

Table 8.1: Elastic constants (in GPa) used for the study

<table>
<thead>
<tr>
<th>λ</th>
<th>μ</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>51</td>
<td>26</td>
<td>-320</td>
<td>-200</td>
<td>-190</td>
</tr>
</tbody>
</table>

We consider the 2D version of the problem under a plane-strain assumption. Figure 8.3 shows the linear elastic properties, namely the Young’s modulus (E) and the shear modulus (μ) as a function of void-fraction. They change by about 22% over a void-fraction range of 10%. On the other hand, the asymmetry parameter (α) defined in the previous section is shown in Figure 8.4 for different ‘NL’ zones. Clearly, the tension-compression asymmetry increases with increasing size of the ‘NL’ zone. Other important thing to be noted is the scale of the vertical axis in Figure 8.4a; it is of the order of $10^{-4}$ suggesting that the fraction of the energy transferred to the secondary mode from the primary mode is very small. Also, shown in Figure 8.4b is the normalized version of the Figure 8.4 where each curve is normalized by its corresponding value at 1% void-fraction. All of them coincide indicating that the normalized change in the acoustic nonlinearity is independent of the ‘NL’ zone used for the computation. This in fact suggests the robustness of the approach outlined in this chapter. Also, the normalized asymmetry increases by a factor of about 20 over the void fraction range of 10%. This indicates that the nonlinear elastic properties are much more sensitive to the micro-scale damage than their linear elastic counter parts and thus substantiating the use of nonlinear methodologies for monitoring micro-scale damage progression.
Figure 8.2: RVE of the material with micro-voids

Figure 8.3: Linear elastic modulii (in Pa) versus volume fraction; E - Young’s modulus and $\mu$ - shear modulus
Figure 8.4: (a) Asymmetry (b) Normalized asymmetry versus void fraction

8.4 Conclusions

In this chapter we presented a brief discussion on modeling approaches previously employed to quantify acoustic nonlinearity in materials. Most of the approaches focused on evaluating the contribution of dislocations which are defects at atomic scale. It has been pointed out that these approaches do not take into account the distribution and/or orientation of dislocations with reference to the stress wave propagation and this can lead to incorrect results as outlined in Cash and Cai (2012). Also, it was emphasized that the ultrasonic waves are sensitive only
to the distributed damage over a length-scale of the order of the wavelength and hence meso-scale models should more accurately predict the effect of microstructure on acoustic nonlinearity. To that end, a micromechanics inspired homogenization based approach was developed to quantify tension-compression asymmetry and hence second harmonic generation in the material. The important features of this approach are

1. It does not depend on the specific form of the strain energy density used to describe the nonlinearities.

2. Can be applied with multiple micro-scale damage features and hence evaluate the overall contribution of microstructure especially in the case where there are competing micro-mechanisms that contribute to acoustic nonlinearity.

3. Unlike most of the dislocation-based models that are 1D in nature, the approach presented here is based on a 3D version of the constitutive model and hence is valid in a general context of guided or Rayleigh wave propagation.

4. The approach is amenable for multi-scale numerical modeling in that the dislocation-based models can serve as input to characterize the nonlinearity associated with the nonlinear zones at a micro-scale which are then used in the approach outlined in this chapter to evaluate the overall response of the material.

The approach was employed to evaluate the contribution of micro-voids to acoustic nonlinearity in a 2D plane strain setting. It was found that the change in the nonlinear elastic properties quantified via asymmetry was much higher when compared to the change in the linear elastic material properties. Also, the normalized asymmetry was found to be independent of the size of the ‘NL’ zone used in the calculations suggesting that the approach is robust to choice of that parameter. These further substantiated the previous conclusions that the nonlinear elastic properties are more sensitive to microscale damage than their linear counterparts.

Future work should focus on employing the above approach for other micro-scale defects like micro-cracks, inclusions, etc.
Chapter 9  
Effect of load and temperature changes on nonlinear ultrasonic measurements: Implications for SHM

Introduction

Nonlinear ultrasonic methodologies have significant potential in ensuring a paradigm shift from schedule-based maintenance to condition-based maintenance of structures. They can also significantly reduce the life-cycle costs when efficiently employed in conjunction with existing SHM methodologies. Having recognized this need, considerable research efforts are being made to transfer the nonlinear ultrasonic techniques from laboratory investigations to on-site inspection or monitoring scenarios. In this regard, ultrasonic guided waves (Rose, 2002) (and surface waves) are being considered as a foremost candidate for SHM incorporating nonlinear ultrasonic methodologies. These approaches are envisaged to combine the inspection capabilities of guided waves with early damage detection capabilities of nonlinear ultrasound and use what are called nonlinear ultrasonic guided waves for SHM. In this context, nonlinear ultrasonic guided waves refer to higher harmonic frequency components (second and third) generated from the primary guided wave mode. Several experimental investigations demonstrated the capabilities of second (Bermes et al., 2007; Pruell et al., 2007, 2009; Xiang et al., 2014) and third harmonic (Lissenden et al., 2014a) ultrasonic guided waves for detecting damage induced microstructural changes. However, most of these investigations were carried out in a laboratory setting where the effect of environmental factors are kept to a minimum. So, caution must be exercised in extending the results from the above investigations to actual scenarios in the context of structural health monitoring.
To that end, the study presented in this chapter aims to delineate the effect of load and temperature changes on nonlinear ultrasonic measurements. In particular, the effect of the above parameters on the relative nonlinearity parameter that is widely used for quantifying nonlinearity in the material behavior is studied.

The content of this chapter is organized as follows. First, we briefly introduce the relative nonlinearity parameter that is widely used to quantify the material nonlinearity. Then, we discuss the experimental results that describe the effect of external static-load on the nonlinearity parameter. Following that we discuss the effect of (localized) temperature changes on the nonlinearity parameter and finally we present conclusions from this study. The content of this chapter is based on Chillara et al. (2015a).

9.1 Relative acoustic nonlinearity parameter revisited

In this section we discuss the preliminaries on acoustic nonlinearity in materials and focus on the definition of relative nonlinearity parameter widely used to quantify acoustic nonlinearity. We first begin with the schematic of harmonic generation in material as shown in Figure 9.1 where the primary wave of frequency $\omega$ generates higher harmonic frequencies $2\omega$ and $3\omega$ due to the nonlinearity in the material. If we assume that the primary wave has an amplitude $A_1$ and the second and third harmonics have an amplitude $A_2$ and $A_3$ then the relative nonlinearity parameters for second and third harmonics are defined by $\frac{A_2}{A_1}$ and $\frac{A_3}{A_1}$ respectively. For the sake of discussion let us consider the relative nonlinearity parameter for second harmonic i.e., $\frac{A_2}{A_1}$. It should be noted that most of the studies employ the relative nonlinearity parameter rather than the absolute nonlinearity parameter to quantify progressive micro-scale damage. As can be seen, the change in the relative nonlinearity parameter can be brought about by change in either $A_2$ or $A_1$ or both. Also, the theoretical derivation (Cantrell and Yost, 2001) that leads to the above definition of the relative nonlinearity parameter is based on the following assumptions:
1. the incident primary wave is a plane-wave,

2. there is no attenuation in the material and

3. the amplitude $A_1$ used in the relative nonlinearity parameter is the amplitude at the source of excitation but not at the receiver.

However, the same definition is employed in the context of experiments where in fact none of the above assumptions hold good. In particular, the incident wave is not a plane wave due to the finite size of the source, the material is attenuative and the amplitude at the source of excitation is generally unknown. To that end several studies (Blackburn and Breazeale, 1984; Hurley and Fortunko, 1997; Torello et al., 2015; Best et al., 2014) developed and employed compensation techniques to account for the above factors, namely diffraction and attenuation. In general, in addition to the above factors, various other factors such as ultrasonic couplant effects (Sun et al., 2006) and nonlinearity from the electronics affect the material nonlinearity measurements. In the context of structural health monitoring, various other environmental factors such as temperature and external loads affect the nonlinear ultrasonic measurements. The goal of this work is to study the effect of these on nonlinear ultrasonic measurements. Such a study is very important from a practical standpoint especially in developing efficient compensation methodologies to account for changes in the above factors. Next, we describe the results obtained from experiments concerning the effect of loads and temperature changes on third harmonic guided wave generation in plate-like specimens.

### 9.2 Experiments

As mentioned earlier, two independent experiments were conducted to determine the effect of load and temperature changes on nonlinear ultrasonic measurements. The schematic of the experimental setups are shown in Figure 9.2. For both the experiments, magnetostrictive transducers were used to excite and receive the waves. The transmitter has a coil-spacing of 3.6 mm and is actuated at 0.83 MHz to excite $SH_0$ mode in the plate. The receiver on the other hand has a coil-spacing of 1.2 mm and is tuned to receive a third harmonic at 2.49 MHz using an impedance matching network from RITEC. The distance between the transmitter and receiver was set to 228.5 mm (9 inches) or all the experiments. The setup was calibrated for
electronic/instrumentation nonlinearities as discussed in Lissenden et al. (2014b). The thickness and width of the specimens were 1 mm and 63.5 mm (2.5 inches) respectively. We first discuss the experiments and results concerning the effect of load.

9.2.1 Effect of load on nonlinear ultrasonic measurements

Experiments were conducted on two different kinds of aluminum plate-like specimens; Al 2024 ($\sigma_y = 320$ MPa) and Al 7075 (solution annealed) ($\sigma_y = 60-80$ MPa) where $\sigma_y$ is the yield strength as estimated from hardness tests. The specimens were loaded under increasing static uniaxial load in a MTS test machine and the ultrasonic data from the receiver is acquired using a Tektronix oscilloscope. The load range (as nominal stress) was between 7 MPa-105 MPa for both Al 2024 and Al 7075. It should be noted that the above load range is within the elastic limit for Al 2024 but induces slight plastic deformation on Al 7075. First, we discuss the results obtained for Al 2024 specimen. The specimen was loaded, unloaded and again reloaded for 15 times with ultrasonic data being acquired for each loading cycle. This ensured statistically significant trends in the data obtained. Figure 9.3a shows the normalized amplitude of the fundamental frequency as a function of the load and Figure 9.3b shows the normalized amplitude of the third harmonic with load for all the 15 sets. Clearly, the amplitude of the fundamental drops with load except for the low-stress region where the specimen is slightly flexed. The drop in
the amplitude is due to load-dependent attenuation in polycrystals. On the other hand, the amplitude of the third harmonic remains (statistically) constant over the load range. Also, Figure 9.3c shows the relative nonlinearity parameter $\frac{A_3}{A_1}$ with load and can be seen to increase with it. It should be emphasized that the change in the relative nonlinearity parameter is a spurious effect of the drop in the fundamental amplitude ($A_1$) but not due to any significant microstructural changes as the loading is well within the elastic limit. This is something we would like to highlight in this study that the use of relative nonlinearity parameter is prone to such spurious effects and caution must be exercised in using it especially in the context of SHM. In addition, Figure 9.3d shows the time of flight of the ultrasonic wave from transmitter to receiver as a function of load and can be seen to increase with load due to the acoustoelastic effect (Toupin and Bernstein, 1961).

![Graphs](image)

Figure 9.3: Figure 3 Normalized (a) $A_1$ (b) $A_3$ (c) $\frac{A_3}{A_1}$ and (d) time of flight with load for Al 2024 specimen
Similar experiments were carried out for the Al 7075 specimen but for 5 loading cycles. However, it should be noted that the load range (7 MPa-105 MPa) exceeds the yield strength of the material and hence induces a small plastic strain in the specimen. This is evident from the results of third harmonic amplitude measurements. While the fundamental amplitude drops with load (Figure 9.4a) as observed for the Al 2024 specimen, the third harmonic amplitude increases with load for all the 5 sets as shown in Figure 9.4b. Likewise, the relative nonlinearity parameter shown in Figure 9.4c increases with the load.

Figure 9.4: Normalized (a) A1 (b) A3 and (c) $\frac{A_3}{A_1}$ with load for Al 7075 (solution annealed) specimen
9.2.2 Effect of localized temperature changes on nonlinear ultrasonic measurements

To study the effect of localized temperature changes on nonlinear ultrasonic measurements, a hot plate was used to heat the Al 2024 specimen in the central region between the transmitter and receiver as depicted in Figure 9.2b. The hot plate was heated in steps of 10°C from 30°C to 160°C and the ultrasonic signal is acquired at the receiver. The plate is cooled to room temperature (27°C) and the process was repeated to acquire 8 sets of data. It should be noted that the localized heating from hot-plate results in temperature gradients along the wave propagation direction and also through the thickness of the plate. Figure 9.5a shows the top and bottom surface temperatures (measured using thermocouples) of the plate for different hot-plate temperatures. Figure 9.5b shows the temperature distribution with distance measured from the centre of the hot-plate when its temperature is set to 160°C. It should be noted that each data point in Figure 9.5b is an average of three measurements across the width of the plate. Also, the transmitter/receiver is located at a distance (≈115 mm) from the hot-plate and the effect of temperature on their performance is hence assumed to be minimal.

Figure 9.5: (a) Surface temperature versus hot-plate temperature (b) Temperature distribution along the specimen where x=0 is the middle of the hot-plate.

Figure 9.6a shows the amplitude of the fundamental wave with hot-plate temperature and clearly it decreases with increasing temperature. On the other hand, Figure 9.6b shows the amplitude of the third harmonic with temperature and does not show any observable trend. But, the relative nonlinearity parameter shown in
Figure 9.6c shows an increasing trend with temperature. It should again be noted that this increase is again a spurious effect of change in the fundamental amplitude with temperature but not due to any temperature induced microstructural changes or residual stress (Nucera and di Scalea, 2014) as the specimen is unconstrained. Also, shown in Figure 9.6d is the time of flight of the ultrasonic wave for different hot-plate temperatures and clearly there is a significant change in the wave speed with temperature.

![Figure 9.6: Normalized (a) A1 (b) A3 and (c) $A_3/A_1$ with hot-plate temperature for Al 2024 specimen](image)

### 9.3 Conclusions

In this chapter, we highlighted some of the shortcomings of the relative nonlinearity parameter especially in the context of structural health monitoring. To demonstrate this, we investigated the effect of load and temperature changes on nonlinear
ultrasonic measurements using experiments. Ultrasonic third harmonic generation from SH₀ mode was carried out for increasing static loads on two different aluminum specimens; Al 2024 and Al 7075 (solution annealed). While the amplitude of the fundamental mode was affected due to the load in both the cases, the third harmonic for Al 2024 almost remained constant over the load range. On the other hand, the third harmonic from Al 7075 specimen increased with load due to small amounts of accumulated plastic strain/residual stress. The change in the relative nonlinearity parameter for Al 2024 was found to be due to the spurious effect of the drop in the fundamental amplitude. Experiments were also carried out on Al 2024 specimen to investigate the effect of localized temperature changes on nonlinear ultrasonic measurements. The amplitude of the fundamental was found to decrease with increasing temperature. However, the third harmonic amplitude did not show any definite trend with increasing temperature. But, the relative nonlinearity parameter showed spurious increase with temperature due to the drop in the fundamental amplitude. The above study demonstrates that while the higher harmonics are sensitive to microstructural changes, the nonlinearity parameter may not necessarily be a reasonable measure to quantify it especially in the context of SHM. There is a definite need to develop efficient compensation methodologies to account for the changes in fundamental and third harmonics with external factors such as load and temperature and should be considered as a part of future work.
Chapter 10
Conclusions

10.1 Summary

The thesis focused on exploring the capabilities of nonlinear guided waves for early damage detection. Second harmonic guided waves in plates were studied from a theoretical standpoint. A complete list of guided wave modes capable of generating cumulative second harmonics were identified. The above framework was extended to study second harmonic guided waves from axis-symmetric longitudinal modes in pipes using a large-radius asymptotic approximation. It was found that the results obtained for plates can be appropriately extended to pipes using the notion of asymptotic modes developed in chapter 5. Nonlinear guided wave propagation in plates was analyzed from a numerical standpoint in chapter 6. The contribution of material and geometric nonlinearities to second harmonic generation, effect of scaling higher order constants, effect of group-velocity mismatch, etc. on second harmonic generation was investigated. Also, second harmonic generation in plates with localized nonlinearities was investigated and some important aspects of guided wave mode selection were discussed. Chapter 7 and chapter 8 dealt with the micromechanics aspects necessary to correlate microstructure with the acoustic nonlinearity in polycrystals. Tension-compression asymmetry, shear-normal coupling and deformation induced anisotropy were identified as important aspects of (micro-scale) nonlinear material behavior relevant to ultrasonic higher harmonic generation. An energy based measure was introduced to quantify the tension-compression asymmetry in material behavior. This was employed in chapter 8 to develop a homogenization based approach to assess the contribution of micro-scale defects to acoustic nonlinearity. Chapter 9 dealt with the experimental aspects of the thesis where the effect of load and temperature changes on nonlinear ultrasonic
measurements were studied in the context of SHM. It was found that the definition of relative nonlinearity parameter currently being employed is prone to spurious effects and efficient compensation methodologies need to be developed to account for these effects on nonlinear ultrasonic measurements.

10.1.1 Key contributions

1. Use of Frequency Domain Finite Element Approach was proposed for efficient guided wave mode selection for inhomogeneous waveguides.

2. A generalized theoretical framework for analyzing ultrasonic higher harmonic generation of guided waves was developed.

3. Extensive numerical studies were carried out to study important aspects of second harmonic guided wave generation in plates with homogeneous, nonhomogeneous and localized nonlinearities.

4. A micromechanics based understanding of ultrasonic higher harmonic generation was presented with an emphasis on meso-scale models.

5. A micromechanics inspired homogenization approach tailorable to multi-scale analysis was developed to correlate ultrasonic higher harmonic generation with microstructure.

6. Experimental investigations were carried out to study the effect of environmental factors, namely load and temperature changes on nonlinear ultrasonic measurements in the context of SHM.

10.2 Future work

Some important extensions for the current work are outlined below:

1. Employ the generalized theoretical framework developed for analyzing higher harmonic generation in composites.

2. Numerically study the effect of localized plasticity on second harmonic generation.
3. Develop a new constitutive model for studying third harmonic generation in elastic solids.

4. Employ the micromechanics based homogenization approach to assess the contribution of micro-scale defects like micro-cracks, inclusions, etc. to acoustic nonlinearity.

5. Develop efficient compensation methodologies to account for the effect of environmental factors on SHM methodologies incorporating nonlinear ultrasound.
Appendix A

‘Div’ in reference cylindrical coordinates

Let $X$ and $x$ denote the position of a material point in the reference and current configuration respectively. Let $(R, \Theta, Z)$ and $(r, \theta, z)$ denote the coordinates in reference and current configuration respectively. Correspondingly, the unit vectors in reference and current configurations are denoted by $\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_Z$ and $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$. Then we have,

\[ X = R\mathbf{e}_R + Z\mathbf{e}_Z \]
\[ x = r\mathbf{e}_r + z\mathbf{e}_z. \]  \hspace{1cm} (A.1)

From the definition of the deformation gradient($\mathbf{F}$), we have

\[ d\mathbf{x} = \mathbf{F}d\mathbf{X} \]  \hspace{1cm} (A.2)

$\mathbf{F}$ in the tensor notation can be written as

\[ \mathbf{F} = F_{rR}(\mathbf{e}_r \otimes \mathbf{e}_R) + F_{r\Theta}(\mathbf{e}_r \otimes \mathbf{e}_\Theta) + F_{rZ}(\mathbf{e}_r \otimes \mathbf{e}_Z) + \\
F_{\theta R}(\mathbf{e}_\theta \otimes \mathbf{e}_R) + F_{\theta \Theta}(\mathbf{e}_\theta \otimes \mathbf{e}_\Theta) + F_{\theta Z}(\mathbf{e}_\theta \otimes \mathbf{e}_Z) + \\
F_{zR}(\mathbf{e}_z \otimes \mathbf{e}_R) + F_{z\Theta}(\mathbf{e}_z \otimes \mathbf{e}_\Theta) + F_{zZ}(\mathbf{e}_z \otimes \mathbf{e}_Z) \]  \hspace{1cm} (A.3)

We also have

\[ d\mathbf{X} = dR\mathbf{e}_R + Rd\Theta\mathbf{e}_\Theta + dZ\mathbf{e}_Z \quad \text{and} \]
\[ d\mathbf{x} = dr\mathbf{e}_r + rd\theta\mathbf{e}_\theta + dz\mathbf{e}_z \]  \hspace{1cm} (A.4)
Using Eqns(A.2, A.3 & A.4) we get the following system of equations:

\[
\frac{\partial r}{\partial R}dR + \frac{\partial r}{\partial \Theta}d\Theta + \frac{\partial r}{\partial Z}dZ = F_{rR}dR + F_{r\Theta}Rd\Theta + F_{rZ}dZ \\
\frac{\partial r}{\partial R}dR + \frac{\partial r}{\partial \Theta}d\Theta + \frac{\partial r}{\partial Z}dZ = F_{rR}dR + F_{r\Theta}Rd\Theta + F_{rZ}dZ \\
\frac{\partial z}{\partial R}dR + \frac{\partial z}{\partial \Theta}d\Theta + \frac{\partial z}{\partial Z}dZ = F_{zR}dR + F_{z\Theta}Rd\Theta + F_{zZ}dZ.
\]

(A.5)

From Eqn(A.5), we obtain the components of \( F \) as

\[
F_{rR} = \frac{\partial r}{\partial R}; F_{r\Theta} = \frac{1}{R} \frac{\partial r}{\partial \Theta}; F_{rZ} = \frac{\partial r}{\partial Z} \\
F_{\theta R} = r \frac{\partial \theta}{\partial R}; F_{\theta \Theta} = \frac{r}{R} \frac{\partial \theta}{\partial \Theta}; F_{\theta Z} = r \frac{\partial \theta}{\partial Z} \\
F_{zR} = \frac{\partial z}{\partial R}; F_{z\Theta} = \frac{1}{R} \frac{\partial z}{\partial \Theta}; F_{zZ} = \frac{\partial z}{\partial Z}.
\]

(A.6)

### A.1 Divergence in cylindrical coordinates

In this section we obtain a general expression for Divergence of \( S \) (as indicated in the section Notation). By definition, \( \text{Div}(S) \) is a vector in the current configuration and satisfies

\[
\text{Div}(S^T a) = \text{Div}(S).a \quad \forall \text{ constant } a \text{ in the current configuration.} \quad (A.7)
\]

Let \( a = a_re_r + a_\theta e_\theta + a_z e_z \) be a constant vector in current configuration. Then, we have

\[
da = 0 \Rightarrow da_re_r + a_\theta d\theta e_\theta + da_\theta e_\theta - a_\theta d\theta e_r + da_z e_z = 0.
\]

This gives us

\[
da_r = a_\theta d\theta; \quad da_\theta = a_r d\theta; \quad da_z = 0.
\]

From the above set of equations we can arrive at the following relations:

\[
\frac{\partial a_r}{\partial r} = 0; \quad \frac{\partial a_r}{\partial \Theta} = 0; \quad \frac{\partial a_r}{\partial z} = 0 \\
\frac{\partial a_\theta}{\partial r} = 0; \quad \frac{\partial a_\theta}{\partial \Theta} = -a_r; \quad \frac{\partial a_\theta}{\partial z} = 0 \\
\frac{\partial a_z}{\partial r} = 0; \quad \frac{\partial a_z}{\partial \Theta} = 0; \quad \frac{\partial a_z}{\partial z} = 0.
\]

(A.8)
\( S \) can be represented as
\[
S = S_r(e_r \otimes e_r) + S_{r\theta}(e_r \otimes e_\theta) + S_{rZ}(e_r \otimes e_Z) + S_{\theta R}(e_\theta \otimes e_R) + S_{\theta \Theta}(e_\theta \otimes e_\Theta) + S_{\theta Z}(e_\theta \otimes e_Z) + S_{zR}(e_z \otimes e_R) + S_{z\Theta}(e_z \otimes e_\Theta) + S_{zZ}(e_z \otimes e_Z).
\]

So,
\[
S^T a = (S_r R a_r + S_{\theta R} a_\theta + S_{zR} a_z) e_R + (S_r e_r + S_{\theta \Theta} a_\theta + S_{z\Theta} a_z) e_\theta + (S_r e_r + S_{\theta z} a_\theta + S_{zZ} a_z) e_Z
\]
\[(A.10)\]

\[
\text{Div}(S^T a) = \frac{1}{R} \left( \frac{\partial}{\partial R} (R(S_r a_r + S_{\theta R} a_\theta + S_{zR} a_z)) + \frac{\partial}{\partial \Theta} (S_r e_r + S_{\theta \Theta} a_\theta + S_{z\Theta} a_z) + \frac{\partial}{\partial Z} (S_r e_r + S_{\theta z} a_\theta + S_{zZ} a_z) \right).
\]
\[(A.11)\]

We simplify each of the above terms in Eqn(A.11) by using Eqn(A.8) to get the final expression for \( \text{Div}(S) \). We show this procedure for the first term involving \( e_R \) in Eqn(A.11) and the other terms can be simplified in an analogous manner.

\[
\frac{1}{R} \frac{\partial}{\partial R} (R(S_r a_r + S_{\theta R} a_\theta + S_{zR} a_z)) = \frac{\partial S_r}{\partial R} a_r + \frac{\partial S_{\theta R}}{\partial R} a_\theta + \frac{\partial S_{zR}}{\partial R} a_z +
\]
\[
S_r \left( \frac{\partial a_r}{\partial r} + \frac{\partial a_r}{\partial \theta} + \frac{\partial a_r}{\partial Z} \right) +
S_{\theta R} \left( \frac{\partial a_\theta}{\partial r} + \frac{\partial a_\theta}{\partial \theta} + \frac{\partial a_\theta}{\partial Z} \right) +
S_{zR} \left( \frac{\partial a_z}{\partial r} + \frac{\partial a_z}{\partial \theta} + \frac{\partial a_z}{\partial Z} \right).
\]

Using the relations (Eqn(A.8)), one can rewrite the above equation as follows
\[
\frac{1}{R} \frac{\partial}{\partial R} (R(S_r a_r + S_{\theta R} a_\theta + S_{zR} a_z)) = \left( \frac{\partial S_r}{\partial R} - S_{\theta R} \frac{\partial \theta}{\partial R} \right) a_r + \left( \frac{\partial S_{\theta R}}{\partial R} + S_r \frac{\partial \theta}{\partial R} \right) a_\theta + \frac{\partial S_{zR}}{\partial R} a_z.
\]
\[(A.12)\]

Finally, we have
\[
\text{Div}(S^T a) = \left\{ \begin{array}{l}
\frac{S_r}{R} + \frac{\partial S_r}{\partial R} - S_{\theta R} \frac{\partial \theta}{\partial R} + \frac{1}{R} \frac{\partial S_{\theta R}}{\partial \Theta} - S_{r \Theta} \frac{\partial \Theta}{\partial R} + \frac{\partial S_{r Z}}{\partial \Theta} - S_{\theta Z} \frac{\partial \Theta}{\partial Z} \end{array} \right\} a_r +
\left\{ \begin{array}{l}
\frac{S_{\theta R}}{R} + \frac{\partial S_{\theta R}}{\partial R} + S_r \frac{\partial \theta}{\partial R} + \frac{1}{R} \frac{\partial S_r}{\partial \Theta} + S_{r \Theta} \frac{\partial \Theta}{\partial R} + \frac{\partial S_{r Z}}{\partial \Theta} + S_{\theta Z} \frac{\partial \Theta}{\partial Z} \end{array} \right\} a_\theta +
\]

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\[
\left\{ \frac{S_z R}{R} + \frac{\partial S_z R}{\partial R} + \frac{1}{R} \frac{\partial S_z \Theta}{\partial \Theta} + \frac{\partial S_z Z}{\partial Z} \right\} a_z = \text{Div}(S) \cdot a
\]  \hspace{1cm} (A.13)

Comparing the coefficients of \(a_r\), \(a_\theta\) and \(a_z\) on both sides of Eqn (A.13), we get

\[
\text{Div}(S) = \left\{ \frac{\partial S_{rR}}{\partial R} + \frac{S_{rR}}{R} + \frac{1}{R} \frac{\partial S_{r\Theta}}{\partial \Theta} + \frac{\partial S_{rZ}}{\partial Z} - S_{\theta R} \frac{\partial \theta}{\partial R} - \frac{S_{\theta \Theta}}{R} \frac{\partial \theta}{\partial \Theta} - S_{\theta Z} \frac{\partial \theta}{\partial Z} \right\} e_r + \\
\left\{ \frac{\partial S_{\theta R}}{\partial R} + \frac{S_{\theta R}}{R} + \frac{1}{R} \frac{\partial S_{\theta \Theta}}{\partial \Theta} + \frac{\partial S_{\theta Z}}{\partial Z} + S_{r R} \frac{\partial \theta}{\partial R} + \frac{S_{r \Theta}}{R} \frac{\partial \theta}{\partial \Theta} + S_{r Z} \frac{\partial \theta}{\partial Z} \right\} e_\theta + \\
\left\{ \frac{\partial S_z R}{\partial R} + \frac{S_z R}{R} + \frac{1}{R} \frac{\partial S_z \Theta}{\partial \Theta} + \frac{\partial S_z Z}{\partial Z} \right\} e_z \hspace{1cm} (A.14)
\]
Appendix B

Nontechnical abstract

The need to ensure structural safety has become more important than ever with many structures nearing the end of their design-life i.e., the duration for which they were intended to be functional. While it is safe to replace them with new ones, this is not a cost-effective approach. This is especially the case for structures in civil, aircraft and nuclear industries. Hence, it is of paramount importance to assess the state of these structures and determine if we can extend the life of these structures by taking appropriate measures. Non-Destructive Evaluation (NDE) and Structural Health Monitoring (SHM) are tools that enable us to make the above decisions. In particular, they enable us to detect and characterize the defects which are the “hot-spots” that result in the failure of structures. More often than not, these defects are invisible to the naked-eye and hence NDE and SHM methodologies employ waves that propagate within the structures to identify them. Ultrasonic waves are widely used for defect detection in structural materials. Conventional ultrasonic inspection can only detect defects with sizes greater than a certain threshold value, which depends on the frequency of the wave being used. However, in many applications such as those in the nuclear industry, it is very crucial that we identify these defects in their incipient stage to avoid any catastrophic failure that results in an irrevocable damage. To that end, this thesis investigates and explores the capabilities of what are called “nonlinear ultrasonic guided waves” for early damage detection in structures. Nonlinear ultrasonic guided waves are waves that propagate in plate-like structures due to the nonlinearity in the material behavior caused by the presence of micro-scale damage. The presence of damage distorts the propagating waves generating frequencies that are multiples of excitation frequency. This phenomenon can be used to characterize the intensity and extent of the damage in the material. The goal of this work is to investigate the potential of nonlinear ultrasonic guided waves from theoretical, numerical and experimental standpoints.
Bibliography


Vita

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Education

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<th>Year</th>
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