ESSAYS ON INTERNATIONAL TRADE

A Dissertation in
Economics
by
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Abstract

This dissertation consists of three chapters.

**Chapter 1** is devoted to a theoretical analysis of the interaction between comparative advantage and international risk sharing in the presence of uninsured total factor productivity shocks (TFP). The overwhelming consensus in the theoretical literature is that access to international risk sharing in the presence of uninsured TFP shocks induces a country to specialize more in its comparative advantage industries. In Chapter 1 of my dissertation I demonstrate that the effect of financial integration on production patterns depends on preferences and on the structure of the variance-covariance matrix of TFP shocks present in the economy. Using a variant of the standard $2 \times 2$ Ricardian model with TFP shocks by Helpman and Razin (1978), I show that if TFP shocks affect each industry in all countries the same way, the standard assumption, then financial integration indeed leads to a more specialized production structure. However, if shocks are not correlated across countries and affect all industries in a country the same way, then the effect of financial integration is ambiguous. I also show that in the absence of international risk sharing the Helpman-Razin model generally has (discrete) multiple equilibria — an overlooked phenomenon. I build a framework with a continuum of goods in the spirit of Eaton-Kortum and show how the multiple equilibria can be numerically bounded. Using this framework I explore the welfare effects of financial integration through its impact on the production structure. This work can be seen as a first attempt to bring together the discussion of international risk sharing and trade in the context of quantitative trade models of comparative advantage.

**Chapter 2** is a joint work with Gary Lyn and Andrés Rodríguez-Clare. This Chapter studies the implications for trading economies of industry-level external economies of scale, also known as Marshallian externalities. This fundamental question has received little attention in the recent trade literature because of the large number of equilibria that typically occur in trade models with Marshallian externalities. In this work we build a version of such a model that yields a unique equilibrium and a standard gravity equation. The underlying structure of our model is isomorphic to that of a multi-industry Krugman model of firm-level economies of scale and so our uniqueness result extends to this setting as well. The welfare analysis reveals that if the conditions for uniqueness are satisfied then all countries gain from trade even when the strength of scale economies varies across industries. Moreover, the
presence of scale economies tends to decrease the gains from trade but increase the gains from trade liberalization.

Chapter 3 is devoted to a theoretical analysis of a model with a nested constant elasticity of substitution utility function and heterogeneous firms involved in price competition.\textsuperscript{1} Models of this type have become popular in the international trade literature in recent years. I show the continuity of the model as the elasticity of substitution between goods goes to infinity. This result contrasts with the conjecture of prior literature. Continuity of the model ensures consistency of its outcomes when the elasticity of substitution approaches infinity. Therefore, researchers who were reluctant to use this model because of the lack of proof of continuity can now rely on the result of my work to employ the model in their research.

\footnote{A part of this chapter is published in Kucheryavyy (2012).}
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Chapter 1

Comparative Advantage and International Risk Sharing: Together at Last

1.1 Introduction

This paper analyses the interaction between comparative advantage and international risk sharing in the presence of total factor productivity (TFP) shocks. I consider a world economy with multiple countries and multiple industries with country-industry-specific aggregate productivity shocks. International risk sharing allows agents residing in different countries to insure each other’s risks through financial markets. I use the term “financial integration” to describe the situation of unrestricted international risk sharing between countries. Comparative advantage in this world is defined in the expected terms — by comparing the ratios of expected aggregate productivities for industries across countries.

There are two central questions in this paper. The first question is whether financial integration leads to more or less specialization in production. The second question is what are the welfare benefits of financial integration through its impact on the production structure. This paper is divided into two logical parts corresponding to its central questions. In the first part I use a simple model to analyze the link between financial integration and specialization in production. This simple model is the standard 2-goods-2-countries Ricardian model of comparative advantage with TFP shocks, a variant of the model by Helpman and Razin (1978a). In the second part I present a quantitative model of international risk sharing and
comparative advantage in the spirit of Eaton and Kortum (2002), which preserves the effects of the simple model. I use the quantitative model to explore the potential welfare benefits of financial integration.

The tight relationship between the central questions of this paper can be seen by highlighting the main features of the existing literature on the topics of international risk sharing and comparative advantage. The international finance literature on the effects of international risk sharing usually considers models with exogenous production structure: either endowment economies or economies where each country produces a unique good. In these models the key role of international risk sharing is to smooth consumption.\(^1\) Next, the international trade literature on comparative advantage features endogenous production structure, but usually ignores the international finance aspects.\(^2\) Finally, the literature on trade under uncertainty of productivity gives only qualitative results on the impact of financial integration on specialization and it lacks empirical assessment of welfare gains from financial integration.\(^3\) The current paper speaks to the literature on gains from international risk sharing by combining the elements from the literatures on trade under uncertainty and comparative advantage and considering a model of international risk sharing with endogenous production structure.

The overwhelming consensus in both international finance and trade literature is that financial integration unambiguously leads to more specialization in production.\(^4\) The logic behind this view is simple: in the absence of access to international risk sharing countries hedge their risks through diversification of production. If perfect international risk sharing can be achieved, countries will insure their risks through financial markets and specialize more in their comparative advantage industries. This “intuitive” result led to some speculation in the literature that financial integration could bring countries substantial welfare benefits, because countries will be focusing on their comparative advantage industries.\(^5\)

The analysis of the first part of this paper reveals that, contrary to the conventional view, financial integration does not necessarily lead to more specialization in production. The

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\(^1\) See, for example, Lucas (1987) and Cole and Obstfeld (1991). Papers by Obstfeld (1994) and Acemoglu and Zilibotti (1997) are among a few studies that consider models of international risk sharing with endogenous production structure. But they ignore comparative advantage and international trade in goods.

\(^2\) See, for example, Eaton and Kortum (2002) and Costinot, Donaldson and Komunjer (2012).

\(^3\) See, for example, Helpman and Razin (1978a), Koren (2003) and Islamaj (2014).

\(^4\) For example, in Imbs (2004) “...financial integration may decrease (or increase) synchronization [of business cycles], but will also unambiguously induce specialization.” (Section I, p. 723). And in Kalemli-Ozcan et al. (2003) “The theoretical foundations for the effect of risk sharing on industrial specialization are well established...” (Section I, p. 904)

\(^5\) For example, in Islamaj (2014) “The heterogeneity in production introduced in this model suggests substantial gains in welfare from financial integration” (Section 1, p. 479).
outcome of financial integration crucially depends on the structure of TFP shocks prevailing in the economy and on preferences. A common assumption about the structure of TFP shocks made in the previous literature is that shocks are industry-specific, i.e., shocks affect one industry in all countries the same way.\textsuperscript{6} An example of this kind of shocks can be a banana virus hitting production of bananas around the world. I show that financial integration indeed leads to more specialized production under this assumption. However, if shocks are country-specific, i.e., if shocks affect all industries in one country the same way, then financial integration can lead to a more diversified production structure. An example of a country-specific shock is a labor strike, an earthquake, or an economic reform in a particular country.

The assumption that shocks are industry-specific might not reflect well the observed shocks in the data. Recent evidence by Koren and Tenreyro (2007) shows that country-specific shocks are as important as industry-specific shocks in explaining overall output volatility across countries. The existing literature on international risk sharing and comparative advantage has not carefully analyzed country-specific shocks because of technical difficulties associated with solving the corresponding models. In this paper I overcome these technical difficulties and provide an exhaustive analysis of the impact of financial integration on the production structure under different types of shocks.

The main mechanism that drives the outcomes with country-specific shocks is insurance through prices of goods.\textsuperscript{7} To understand this mechanism, consider an example with two countries, Europe and China, having technologies to produce two goods, cars and t-shirts. Suppose that Europe has expected comparative advantage in the production of cars, and China has expected comparative advantage in the production of t-shirts. Imagine that Europe has no shocks to productivity, while China can have either a low or a high country-specific productivity shock affecting the production of cars and t-shirts the same way. Both countries have one representative consumer. Consumer in each country needs to allocate her labor (the only factor of production) to the domestic production before the shock to China is realized. After the shock is realized, both countries produce with the technology they have and with the amount of labor they preallocated to the production. Markets for goods are competitive. Suppose that there are two variants of this world corresponding to two extreme

\textsuperscript{6}Helpman and Razin (1978a) and Koren (2003) are examples of papers that study the effects of financial integration and that assume industry-specific shocks.

\textsuperscript{7}Newbery and Stiglitz (1981) studied this mechanism in the context of partial equilibrium models of trade. Cole and Obstfeld (1991) and Heathcote and Perri (2013) are examples of papers from the international finance literature that studied insurance through prices in the context of models with exogenous production structure.
cases of international financial markets structure: financial autarky and financial integration. If countries are financially integrated, then the representative consumers can trade the full set of Arrow-Debreu securities before the shock to China is realized. And after the shock is realized, consumers deliver goods to each other according to the Arrow-Debreu contracts they hold. In the case of financial autarky, consumers trade goods on the spot after the shock is realized. Assume that shipping of goods around the world is costless.

The question we are interested in is what happens to labor allocations in the countries as we go from financial autarky to financially integrated economies. Let us consider the case of financial autarky. There are two states of the world: the bad shock to China and the good shock to China. Assume that consumers in both countries are risk-averse. Then consumers in China “care” relatively more about what happens in the bad state of the world. Since shock to China does not change its ex post comparative advantage, China is an exporter of t-shirts in both states of the world. Then, when output of t-shirts is low in China, the relative price of t-shirts is high. In that sense the production of t-shirts in China is insured through the relative price of t-shirts. Therefore, China has incentives to allocate more resources to the production of t-shirts. However, that might be inefficient for the world. Because, in the good state of the world, China might be producing too many t-shirts. The world might value cars more in that state of the world. When China and Europe financially integrate, this inefficiency is eliminated. Insurance through prices becomes irrelevant for China, because it gets insurance through financial markets. And the production structure in China reflects what is valued by the world in expectation. So, after financial integration China might become more diversified in production.

Financial integration might also make a less volatile country — which is Europe in our example — more diversified. Under financial autarky Europe earns income by producing cars domestically and selling them to China. After financial integration Europe invests into the production of t-shirts in China, because this investment offers a higher mean return. But now Europe’s income comes from the domestic production of cars and from the investment in t-shirts in China. Production of t-shirts in China is more volatile, and so Europe’s income might become more volatile. Europe can reduce this volatility by producing some t-shirts at home, because Europe has a safe technology for the production of t-shirts. Thus, after financial integration Europe can become more diversified as well.

This paper also shows that under financial autarky and country-specific shocks the Helpman-Razin model generally has multiple equilibria — an earlier overlooked phenomenon. Again, insurance through prices plays an important role in generating multiple equilibria.
To address the second central question of this paper about welfare, I built an Eaton and Kortum (2002) style model of comparative advantage and international risk sharing. I bring the Helpman-Razin model into the framework of Eaton and Kortum with two countries, several (finite number of) industries with a continuum of goods in each industry, and free trade. In this model any country-industry-specific shock affects the whole continuum of goods in the corresponding industry. The continuum of goods serves two purposes. First, it introduces smoothness of the model outcomes with respect to the change in parameters. This makes it possible to bring the resulting framework to data. Second, I show how any solution of the financial autarky version of the model can be characterized using only aggregate quantities (aggregate labor allocations, trade shares, etc). In this version of the model each good is either produced by only one country in any equilibrium or is a disputed good. There is a continuum of equilibria each of which corresponds to some split of the set of disputed goods between the countries. The set of disputed goods can be bounded quantitatively. In this sense the multiple equilibria from the original Helpman-Razin model become manageable. I show that the model with a continuum of goods preserves the effects of financial integration on the production structure from the model with two goods. Simulations of the model reveal that welfare gains from financial integration occurring through the adjustment of the production structure are unlikely to be quantitatively large relative to the gains from consumption smoothing.

The rest of the paper is organized as follows. Section 1.2 reviews the literature. Section 1.3 presents the model in the general case with a continuum of goods and describes the solution approach. Section 1.4 provides analysis of the Helpman-Razin model and Section 1.5 provides analysis of the model with a continuum of goods. Section 1.6 concludes.

1.2 Literature Review

The role of productivity uncertainty in the trade models is a classic topic in the international trade literature. Pomery (1984) provides a good overview of challenges and theoretical results in that literature. The current paper is most closely related to Turnovsky (1974), Helpman and Razin (1978a), Eaton (1979), and Grossman and Razin (1985). These papers address whether countries specialize according to their comparative advantage in the presence of uncertainty of productivity. It was recognized in that literature that Ricardian comparative advantage ceases to be the sole determinant of specialization patterns: risk-aversion of agents and the structure of productivity shocks play an important role as well. Naturally, the literature devoted a
lot of attention to seeking a set of assumptions that guarantee specialization according to comparative advantage. For example, Helpman and Razin (1978b) show that, if shocks to industries affect all countries in the same way, and if international trade in firm shares is allowed, then countries will specialize according to comparative advantage. So international trade in securities could make trade flow in the “right” direction. However, as Grossman and Razin (1985) demonstrated, if country-specific shocks are allowed, then expected trade flows might go in the direction opposite to that predicted by expected comparative advantage. In the current paper, I start off by recognizing that both production and trade patterns can be changed in a non-trivial way, if we introduce uncertainty. The assumption that each industry has a continuum of goods allows me to get away from the problems associated with the knife-edge predictions of directions of trade flows and patterns of specialization.

A classical example in the international trade literature studying the link between financial integration and specialization patterns is the book by Helpman and Razin (1978a) mentioned in the previous paragraph. In their work, Helpman and Razin (1978a) not only show what set of assumptions is needed to make countries specialize according to their comparative advantage, but they also compare the outcomes of the models with and without financial integration. The authors show theoretically that, no matter what is the production structure in the absence of international risk sharing, if production risk can be insured through financial markets, a country would specialize in its comparative advantage industry. The critical assumption which drives this result is that shocks are industry-specific. Three other classical examples coming from the macroeconomics literature are papers by Saint-Paul (1992), Obstfeld (1994), and Acemoglu and Zilibotti (1997), in which the authors explicitly built in a trade off between diversification of risks through production versus diversification through financial markets. As a logical outcome, all of these papers find that better financial markets are associated with more specialization in production. The most recent and closely related to the current paper are the papers by Koren (2003) and Islamaj (2014) which present Ricardian models with uncertainty of production and varying degrees of financial markets incompleteness. Both of these papers study the relationship between financial openness and industrial specialization, and both of these papers come to the same conclusion that financial integration leads to a more specialized production structure. Koren (2003) assumes only industry-specific shocks, which drives his results. At the same time, even though Islamaj (2014) assumes that industry shocks within countries can be correlated, he never analyses the cases with strong correlation between the shocks within a country and, as a consequence, he only gets that financial integration induces specialization.
The current paper is also distantly related to another strand of the “old” literature on trade and uncertainty represented by Newbery and Stiglitz (1984), Eaton and Grossman (1986), Dixit (1987, 1989a,b). In these papers the authors were mostly concerned with the question of optimality of trade in the presence of uncertainty. Newbery and Stiglitz (1984) and Eaton and Grossman (1986) are classical papers showing the trade in the presence of uncertainty might not be optimal and, thus, some form of government intervention (e.g., tariffs) might be justified. The main assumption driving such an outcome was that insurance markets were not functioning properly in the analyzed economies. The above cited papers by Avinash Dixit came as a response to Newbery and Stiglitz (1984) and Eaton and Grossman (1986). Dixit showed that for many conceivable reasons of failures of insurance markets, if these failures are modeled explicitly, then free trade is constrained-Pareto optimal. Hence, there is no role for the government intervention. In the model presented here, multiple equilibria in the case of financial autarky can be ordered in terms of welfare from the point of view of a particular country. Then, the government in this country might be willing to pursue some policies which move this country in a better equilibrium. On top of that, some of the equilibria can be Pareto-dominated for both countries. So, both countries can potentially pursue a joint policy which will make them both better off.

Besides the papers by Koren (2003) and Islamaj (2014) mentioned above, the most recent papers directly descending from the “old” literature on trade and uncertainty also include the papers by Di Giovanni and Levchenko (2010) and Caselli, Koren, Lisicky and Tenreyro (2014). Di Giovanni and Levchenko (2010) analyze what determines how risky is the structure of country exports. The paper by Caselli et al. (2014) has a motivation similar to the current work. They show that — contrary to the common view — openness to trade can lead to less volatility in output.

The role of prices (terms of trade) as an insurance mechanism is a recognized phenomenon in both international trade and international finance literature. Among the first works that analyzed the insurance role of term of trade in the context of international trade was the book by Newbery and Stiglitz (1981). A classical example in the international finance literature is the paper by Cole and Obstfeld (1991), which shows how insurance through financial markets can be substituted with insurance through terms of trade. As was discussed in the introduction, the international finance literature traditionally abstracts away from comparative advantage in production and considers either endowment economies, as in Cole and Obstfeld (1991), or economies where each country produces a unique tradeable good, as in Heathcote and Perri (2013), who also talk about the effect of insurance through prices. As
the analysis of the current paper reveals, by bringing the price insurance mechanism into an economy with an endogenous production structure, we can get a variety of interesting effects, which might help us in analyzing the consequences of financial integration in the real world.

More generally, the current paper can be viewed as a part of the literature on the effects of terms of trade on different aspects of economies. For example, Bhagwati (1958) shows how technological growth can lead to welfare losses through the terms-of-trade effects. Epifani and Gancia (2009) find that openness to trade can (inefficiently) increase the size of governments through a terms-of-trade externality.

Concerning the welfare effects, the international finance literature traditionally finds small welfare gains from international risk sharing. For example, Cole and Obstfeld (1991) find that the gains from financial integration in the model calibrated to the moments of the U.S. and Japanese data are about 0.2 percent of output per year. Another example is Lucas (1987), who also finds small gains from international risk sharing. In these studies gains come from consumption smoothing only, while a potential feedback of production structure on welfare is ignored. Also, in Cole and Obstfeld (1991) prices act as an insurance mechanism and reduce the role of financial markets, which results in small gains from international risk-sharing.

There are several studies which take into account the effect of international risk sharing on production structure. Among them are Obstfeld (1994) and Acemoglu and Zilibotti (1997), who find that gains from international risk-sharing can be substantial. These papers explicitly model the trade off between investing in high return but risky activities versus investing in low return but safer activities. In such environment countries use financial markets to insure their risks and, thus, are able to specialize in risky production technologies. One important limitation of these studies is that they ignore international trade in physical goods. In contrast to these studies, the current paper shows that a neoclassical model of international trade predicts relatively small gains from international risk sharing.

The empirical assessment of the relationship between financial integration and degree of specialization was performed by Koren (2003), Kalemli-Ozcan et al. (2003), Imbs (2004), Basile and Girardi (2010), and Bos et al. (2011). In all of these papers authors run regressions of various measures of risk sharing on indices of industrial specialization and find a positive relationship between these quantities. These results suggest that out of all effects of financial integration the ones which lead to more specialization might be the strongest in data. However, the evidence from these papers does not tell us what is the contribution of different effects of financial integration in the data. So, there is still a lot of room for an empirical work in this area.
1.3 The Framework

The economy consists of $N = 2$ countries. Country 1 is labeled as “Home” and country 2 is labeled as “Foreign”. Labor is the only factor of production. Country $n$ is endowed with $L_n$ units of labor, which are inelastically supplied. Labor is immobile across counties, but perfectly mobile across industries of one country at the stage when employment decisions are made. Each country can potentially produce a continuum of goods in each of the $G$ industries, where $G$ is a finite number. Goods are indexed by $\omega \in \Omega \equiv [0, 1]$ and industries are indexed by $g = 1, \ldots, G$.

**Uncertainty** The only sources of uncertainty in the economy are aggregate supply shocks: shocks which proportionally affect all efficiencies of production on the country-industry level. The state space of the international economy is denoted by $S$, which is assumed to be discrete. The probability of state of the world $s \in S$ is given by $h(s) > 0$, such that $\sum_{s \in S} h(s) = 1$.

Each state $s \in S$ of the world is described by the set of country-industry specific productivity shocks $A_{n,g}(s)$ for $g = 1, \ldots, G$ and $n = H, F$. When uncertainty is resolved, the realized state of the world is common knowledge.

The economy has two stages: before and after uncertainty is resolved. In the first stage financial markets are active, while in the second stage physical goods markets are active. Firms hire labor in the first stage by paying wages, which are not state contingent and cannot be carried over to the second stage. To pay the wages, firms sell claims on their output in the financial market. Consumers in the first stage sell their labor endowment to firms and use wage income to buy claims on endowments of physical goods in different states of the world in the second stage. When the second stage comes, firms use exactly the amount of labor they hired at the first stage. Actual production, international trade, and consumption of goods happen at the second stage.

I consider two cases of financial markets: complete markets and financial autarky. In the complete markets case countries are financially integrated, and consumers buy state-contingent goods for all states of the world. In the financial autarky case consumers can only buy claims on their domestic firms’ output (in other words, there is no international risk sharing in this case). In the rest of the paper, when I talk about the effects of financial integration, I formally mean the effects of switching the financial markets structure from financial autarky to complete markets.

**Production technology and expected comparative advantage.** Technology of production is linear and stochastic: one unit of labor in industry $g$ of country $n$ can produce
\( A_{n,g}(s)z_{n,g}^\omega > 0 \) units of good \( \omega \) in state \( s \). Here \( z_{n,g}^\omega \) is the deterministic component of technology. It is convenient to assume that \( E_s[A_{n,g}(s)] = 1 \), so that \( z_{n,g}^\omega \) is the expected efficiency of country \( n \) in producing good \( \omega \) from industry \( g \). The deterministic parts of technology, \( z_{n,g}^\omega \), are independently (across goods, industries, and countries) drawn from a Fréchet distribution \( F_{n,g}(x) \) with parameters \( T_{n,g} > 0 \) and \( \theta > 1 \):

\[
F_{n,g}(x) = e^{-T_{n,g}x^{-\theta}}.
\]

In the certainty case (e.g., in the Eaton-Kortum model), \( T_{n',g}/T_{n'',g} \) defines comparative advantage of country \( n' \) relative to country \( n'' \) in producing goods from industry \( g \). There are at least four different ways to extend this definition to the uncertainty case. We can say that country \( n' \) has expected comparative advantage in production of goods from industry \( g' \) relative to industry \( g'' \), if

(i) \( T_{n',g'} > T_{n'',g''} \); or

(ii) \( E_s\left[\frac{A_{n',g'}(s)T_{n',g'}}{A_{n'',g'}(s)T_{n'',g'}}\right] > E_s\left[\frac{A_{n',g''}(s)T_{n',g''}}{A_{n'',g''}(s)T_{n'',g''}}\right] \); or

(iv) \( \frac{A_{n',g'}(s)T_{n',g'}}{A_{n'',g'}(s)T_{n'',g'}} > \frac{A_{n',g''}(s)T_{n',g''}}{A_{n'',g''}(s)T_{n'',g''}} \) for any state \( s \); or

(iii) \( \frac{A_{n',g'}(s)T_{n',g'}}{A_{n'',g'}(s)T_{n'',g'}} \) is a mean-preserving spread of \( \frac{A_{n',g''}(s)T_{n',g''}}{A_{n'',g''}(s)T_{n'',g''}} \).

I follow the previous literature on trade and uncertainty and choose the first definition — in terms of the ratios of the means of productivities. Apart from staying as close as possible to the previous literature, the advantage of using this definition is that with this definition expected comparative advantage does not depend on the structure of shocks. So, we can talk about riskiness (i.e., volatility of shocks) and expected comparative advantage separately.

**International trade.** International trade is free.

**Preferences.** Denote by \( c_{n,g}^\omega(s) \) country \( n \) representative consumer’s consumption of good \( \omega \) from industry \( g \) in state \( s \). For any two countries \( n \) and \( \ell \), good \( \omega \) from industry \( g \) produced by country \( n \) in state \( s \) is a perfect substitute of good \( \omega \) from industry \( g \) produced by country \( \ell \) in state \( s \). Country \( n \)’s consumer combines consumption of goods \( \omega \) into an industry-level consumption aggregate, \( C_{n,g}(s) \), by the CES function with elasticity of substitution \( \sigma_g > 0 \):

\[
C_{n,g}(s) \equiv \left[ \int_{\omega \in \Omega} c_{n,g}^\omega(s) \frac{\sigma_{g+1}}{\sigma_g} d\omega \right]^{\frac{\sigma_g}{\sigma_{g+1}}}.
\]
Industry-level consumption aggregates are combined into a final consumption aggregate by the CES function with elasticity of substitution $\eta > 0$:

$$C_n(s) \equiv \left[ \sum_{g=1}^{G} \alpha_g \frac{1}{\eta} C_{n,g}(s)^{\frac{\eta+1}{\eta}} \right]^\frac{\eta}{\eta-1},$$

where $\alpha_g > 0$. In terms of this consumption aggregate, country $n$’s consumer utility is given by the CRRA utility function:

$$U[C_n(s)] \equiv \frac{1}{1-\rho} C_n(s)^{1-\rho},$$

where $\rho > 0$ is the coefficient of relative risk aversion.

**Market structure.** It is assumed that firms are perfectly competitive, which implies that their expected profits are zero. Since goods $\omega$ from industry $g$ produced by any two countries in state $s$ are perfect substitutes, and since trade is free, price in state $s$ of good $\omega$ from industry $g$ is the same in any country. Denote this price by $p_{\omega,g}(s)$. Next, denote by $\psi_n(s)$ the pricing kernel in country $n$. This pricing kernel is an unknown which is derived later.\(^8\) Country-$n$-producer hires the amount of labor, $l_{n,g}^\omega$, which maximizes its expected profit:

$$\max_{l_{n,g}^\omega} \sum_{s=1}^{S} \psi_n(s) p_{\omega,g}(s) A_{n,g}(s) z_{n,g}^\omega l_{n,g}^\omega - w_n l_{n,g}^\omega.$$

Labor is hired ex ante at the cost of unit wage $w_n$. Wages are paid before uncertainty is realized. The firm’s problem results in the following complementary slackness condition:\(^9\)

$$l_{n,g}^\omega \geq 0 \text{ complementary to } w_n - \sum_{s=1}^{S} \psi_n(s) p_{\omega,g}(s) A_{n,g}(s) z_{n,g}^\omega \geq 0.$$

Denote by $I_n(s)$ country-$n$-consumer’s second stage income, which comes from collecting all the dividends from the firm shares held by the country-$n$-consumer. Then the second-stage

\(^8\)Since I focus on only two extreme cases — complete markets and financial autarky — the pricing kernel is uniquely defined on a country level.

\(^9\)Variable $a \geq 0$ is complementary to variable $b \geq 0$, if $ab = 0$. 
expenditure on individual goods is given by:

\[ x_{n,g}^{\omega}(s) = \left( \frac{p_g^{\omega}(s)}{P_g(s)} \right)^{1-\sigma_g} X_{n,g}(s), \]
\[ X_{n,g}(s) = \left( \frac{P_g(s)}{P(s)} \right)^{1-\eta} \alpha_g I_n(s), \]
\[ P_g(s) = \left[ \int_{\Omega} p_g^{\omega}(s)^{1-\sigma_g} d\omega \right]^{\frac{1}{1-\sigma_g}}, \]
\[ P(s) = \left[ \sum_{g=1}^{G} \alpha_g P_g(s)^{1-\eta} \right]^{\frac{1}{1-\eta}}. \]

The indirect utility function is given by:

\[ V(I_n(s), P(s)) \equiv \frac{1}{1-\rho} \left( \frac{I_n(s)}{P(s)} \right)^{1-\rho}. \]

I continue this section with the formal description of the complete financial markets case. After that I formally describe the financial autarky case.

1.3.1 Complete Markets

The consumer in country \( n \) maximizes her expected utility by choosing state-contingent income:

\[
\max_{I_n(s)} E_s [V(I_n(s), P(s))]
\]
\[
\text{s.t.}
\sum_{s=1}^{S} \psi_n(s) I_n(s) = w_n L_n.
\]

In the case of complete financial markets the pricing kernel, \( \psi_n(s) \), is the same in all countries and can be normalized to 1. Then the consumer’s problem implies

\[ I_n(s) = h(s)^\frac{1}{\rho} \left( \frac{P(s)}{P} \right)^{\frac{\rho-1}{\rho}} w_n L_n, \]
\[ P \equiv \left( E_s \left\{ \left[ \frac{P(s)}{h(s)} \right]^{\frac{\rho-1}{\rho}} \right\} \right)^{\frac{\rho}{\rho-1}}. \]
It is also useful to have an expression of the expected utility:

$$E_s [V (I_n(s), P(s))] = \frac{1}{1-\rho} \left( \frac{w_n L_n}{P} \right)^{1-\rho}.$$

In the case of logarithmic utility ($\rho = 1$) we have:

$$I_n(s) = h(s) w_n L_n,$$

$$P = \exp \left\{ E_s \left[ \ln \frac{P(s)}{h(s)} \right] \right\},$$

$$E_s [V (I_n(s), P(s))] = \ln \left( \frac{w_n L_n}{P} \right).$$

### 1.3.2 Financial Autarky

Denote by $\zeta_{n,g}^\omega(s)$ country-$n$-consumer’s income in state $s$ from the domestic firm producing good $\omega$ from industry $g$. In equilibrium, $\zeta_{n,g}^\omega(s) = p_{g}^\omega(s) A_{n,g}(s) z_{n,g}^\omega P_{n,g}$. The total state-$s$ income of country-$n$ consumer is given by

$$I_n(s) \equiv \sum_{g=1}^{G} \int_{\Omega} \zeta_{n,g}^\omega(s) d\omega.$$

Consumer in country $n$ maximizes her expected utility by choosing her incomes from different goods produced by domestic firms:

$$\max_{I_n(s), \zeta_{n,g}^\omega(s)} E_s [V (I_n(s), P(s))]$$

s.t.

$$\sum_{s=1}^{S} \psi_n(s) I_n(s) = w_n L_n,$$

$$I_n(s) = \sum_{g=1}^{G} \int_{\Omega} \zeta_{n,g}^\omega(s) d\omega.$$

By normalizing the Lagrange multiplier of the first budget constraint to 1, the consumer problem gives the expression for the pricing kernel:

$$\psi_n(s) = \frac{h(s)}{I_n(s)} \left( \frac{I_n(s)}{P(s)} \right)^{1-\rho}. $$
1.3.3 Equilibrium System of Equations

Since the financial autarky and complete market cases have many similar model components, the systems of equations that we need to solve to find equilibria in these cases — the equilibrium systems of equations — share many equations. Therefore, it is instructive to write the equilibrium systems of equations for these cases simultaneously.

To solve the model, we need to find $p_\omega^g(s)$, $P_g(s)$, $P(s)$, $s_{ni,g}^\omega(s)$, $c_{n,g}^\omega(s)$, $X_{n,g}(s)$, $I_n(s)$, $l_{n,g}^\omega$, $w_n$, which satisfy the following system of equations:

\[
\sum_{n=1}^{N} c_{n,g}^\omega(s) = \sum_{n=1}^{N} A_{n,g}(s) z_{n,g}^\omega l_{n,g}^\omega, \quad g = 1, \ldots, G, \quad \omega \in \Omega, \quad s \in S;
\]

\[
l_{i,g}^\omega \geq 0 \quad \text{complementary to} \quad w_i - E_s \left[ \psi_i(s) p_\omega^g(s) A_{i,g}(s) z_{i,g}^\omega \right] \geq 0, \quad (1.1)
\]

\[
\sum_{g=1}^{G} \int_{\Omega} l_{i,g}^\omega d\omega = L_i, \quad i = 1, \ldots, N.
\]

Demand for individual goods, $c_{n,g}^\omega(s)$, is given by

\[
c_{n,g}^\omega(s) = \frac{p_\omega^g(s)^{-\sigma_g}}{P_g(s)^{1-\sigma_g}} X_{n,g}(s),
\]

\[
X_{n,g}(s) = \left( \frac{P_g(s)}{P(s)} \right)^{1-\eta} \alpha_g I_n(s),
\]

\[
P_g(s) = \left[ \int_{\Omega} p_\omega^g(s)^{1-\sigma_g} d\omega \right]^{\frac{1}{1-\eta}},
\]

\[
P(s) = \left[ \sum_{g=1}^{G} \alpha_g P_g(s)^{1-\eta} \right]^{\frac{1}{1-\eta}}.
\]

In the case of complete markets:

\[
\psi_n(s) I_n(s) = h(s)^{\frac{1}{\rho}} \left( \frac{\psi_n(s) P(s)}{P} \right)^{\frac{\rho-1}{\rho}} w_n I_n,
\]

\[
P = \left( \sum_{s=1}^{S} [\psi_n(s) P(s)]^{\frac{\rho-1}{\rho}} h(s)^{\frac{1}{\rho}} \right)^{\frac{\rho}{\rho-1}}.
\]
In the case of financial autarky:

\[
\psi_n(s) = \frac{h(s)}{I_n(s)} \left( \frac{I_n(s)}{P(s)} \right)^{1-\rho},
\]

\[
I_n(s) = \sum_{g=1}^{G} \int_{\Omega} p_{g}^{\nu}(s) A_{n,g}(s) \omega_{n,g,l} d\omega.
\]

### 1.3.4 Assumptions on the Structure of Shocks and Overview of Theoretical Implications

The main theoretical result of this paper is that financial integration can lead to a more diversified production structure. This result is in contrast with the conventional view that financial integration necessarily induces specialization. In this paper I show that the effect of financial integration actually depends on the structure of the variance-covariance matrix of TFP shocks affecting the economy. All interesting effects of financial integration can be described by focusing on two types of shocks: industry-specific and country-specific shocks. We can formally define these shocks in terms of the variance-covariance matrix. Figure 1.1 shows the general variance-covariance matrix of TFP shocks for the economy with 2 industries. The elements of this matrix are \( \nu_{n',g',n''g''} \equiv Cov[A_{n',g'}(s), A_{n'',g''}(s)] \). Figure 1.2 shows the variance-covariance matrices for industry-specific and country-specific shocks. In the case of industry-specific shocks, each shock affects one industry in each country the same way, and shocks to different industries are not correlated with each other. An example of this type of shock is a new banana virus hitting production of bananas in Africa and Latin America. Alternatively, industry-specific shocks can be thought of and modeled as world-wide demand shocks in industries. In the case of country-specific shocks, each shock affects all industries in one country the same way, and shocks to different countries are not correlated with each other. An example of this type of shock is a labor strike in Argentina.

![Figure 1.1: General variance-covariance matrix of TFP shocks, \( \nu_{n',g',n''g''} \equiv Cov[A_{n',g'}(s), A_{n'',g''}(s)] \)](image-url)
(a) Industry-specific shocks, $\nu_g$ is the variance of shocks in industry $g = 1, 2$

(b) Country-specific shocks, $\nu_n$ is the variance of shocks in country $n = H, F$

Figure 1.2: Special cases of the variance-covariance matrix of TFP shocks

In the rest of this paper I assume that there are only either industry-specific or country-specific shocks. I show by way of numerical examples, that in the case of industry-specific shocks financial integration induces specialization, while in the case of country-specific shocks financial integration can lead to a more diversified production structure. All the effects of financial integration can be demonstrated using the standard 2-good Ricardian model of Helpman and Razin (1978a) — these effects remain the same in the full model with a continuum of goods. Introduction of a continuum of goods into the Helpman-Razin model allows us to get smoothness of the model outcomes in response to changes in parameters, which, in turn, makes it possible to bring the Helpman-Razin model to data. One can think about the Helpman-Razin model as a special case of the framework presented in this paper, where each industry has just one good instead of a continuum of goods. Helpman and Razin (1978a) originally assumed only industry-specific shocks, and, as a consequence, came to a conclusion that financial integration necessarily induces specialization. I devote Section 1.4 to the analysis of the model by Helpman and Razin (1978a) with both industry-specific and country-specific shocks.

In the case of financial autarky with country-specific shocks the model presented here has multiple equilibria. The multiplicity of equilibria does not arise because we introduced a continuum of goods. Even the original framework by Helpman and Razin (1978a) has multiple equilibria, which was overlooked in the previous literature. In Section 1.4 I give an
example with multiple equilibria in the Helpman-Razin model and provide intuition behind the multiplicity. Since the Helpman-Razin model has just 2 goods, it has a finite number of equilibria, and it is generally hard to find all the equilibria in this model. This is one reason why the multiplicity of equilibria in the Helpman-Razin model was not discovered up to this point. Once we introduce a continuum of goods into each industry, we get a continuum of equilibria. In Section 1.5 I show that these equilibria can be elegantly characterized and become “manageable” in the framework of this paper, which is another advantage — apart from “smoothness” — of introducing the continuum assumption.

## 1.3.5 Solution Approach

As I mentioned above, in Section 1.4 I analyze the Helpman-Razin model for both financial autarky and complete market cases with both industry- and country-specific shocks. In Section 1.5 I look at the model with a continuum of goods. I use different approaches to solve different variants of the model, because none of the approaches is suitable for all situations. Apart from the fact the model generally has multiple equilibria, the main complication with solving the model is that it is very difficult to find explicit expressions for prices of individual goods in different states of the world. The reason for that is that, contrary to the certainty case, state-specific prices are not equal to state-specific marginal costs:

\[
\psi_n(s)p^\omega_g(s) \neq \frac{w_n}{A_n(s)z_{n,g}^\omega}. 
\]

We only have equivalence in expectation:

\[
E_s[\psi_n(s)p^\omega_g(s)A_{n,g}(s)] = \frac{w_n}{z_{n,g}^\omega}, 
\]

if good \( \omega \) from industry \( g \) is produced in country \( n \). That means that we have to explicitly include the complimentary slackness conditions (1.1):

\[
l^\omega_{i,g} \geq 0 \text{ complementary to } w_i - E_s[\psi_i(s)p^\omega_g(s)A_{i,g}(s)z_{i,g}^\omega] \geq 0,
\]

into the equilibrium system of equations. Solving such systems of equations is a challenging task.

To solve the Helpman-Razin model for the cases which have a unique equilibrium, I
use the complementary slackness solver PATH.\textsuperscript{10} To solve the Helpman-Razin model in the cases where it can potentially have multiple equilibria — the case of financial autarky with country-specific shocks, — I use a grid search on labor allocations to find all possible equilibria.

Next, in some cases with a continuum of goods it is possible to collapse the equilibrium system into a nonlinear system, which involves only aggregate quantities (aggregate labor allocations) and no complimentary slackness conditions. These are the cases of financial autarky with either industry- or country-specific shocks and the case of complete markets with industry-specific shocks. In these cases we can solve the model by the fixed point iteration on the aggregate quantities.

Unfortunately, a similar aggregation is very difficult — although possible in theory — in the case of complete financial markets with country-specific shocks. The difficulty with aggregating the model in this case is that we get a different set of expressions depending on particular values of model parameters. I explain this in detail in Section 1.5. To solve the model in this case, I use a completely different approach. I modify the model by artificially making it an Armington economy.\textsuperscript{11} Namely, I introduce an artificial requirement that each country \( n \) buys each good \( \omega \) from all countries. Varieties of good \( \omega \) produced by different countries are combined by the CES utility function with elasticity \( \sigma' \). The equilibrium system of the modified model does not involve complementary slackness conditions. As elasticity \( \sigma' \) goes to infinity, the artificial requirement disappears, and the modified model converges to the original model. Presumably, the solution of the modified model also converges to the solution of the original model. I assume that it is true. This solution approach can be applied in a broader context of trade models of comparative advantage and it is described in detail in Appendix A.

\textsuperscript{10}See Ferris and Munson (1999) for a description of the PATH solver. The corresponding programs — written in AMPL — are available on my web page, \url{http://www.personal.psu.edu/kzk145/}. One reason of why the Helpman-Razin model has not been carefully analyzed until now is that there were no tools to solve this model in the general case. PATH is one of the first nonlinear complementary slackness solvers which is able to solve the equilibrium systems of equations arising in the Helpman-Razin model.

\textsuperscript{11}I thank Michael Fabinger for giving me this idea.
1.4 Effects of Financial Integration and Increased Riskiness in the Helpman-Razin Model

In this section I analyze numerical examples with the Helpman-Razin model for the cases of industry- and country-specific shocks. The purpose of these examples is to demonstrate the effects of financial integration and increased riskiness (i.e., volatility of shocks) as well as to show multiplicity of equilibria in the simple environment of the Helpman-Razin model. Understanding the intuition behind these effects in the standard 2-good Ricardian model will lay the foundation for understanding the effects in the model with a continuum of goods.

The Helpman-Razin model can thought of as a special case of the framework presented in this paper with 2 industries and one good in each industry instead of a continuum, i.e., with \( G = 2 \) and \( \Omega = 1 \). Since both industries have just one good, for the purposes of the current section we can use the words “industry” and “good” interchangeably to refer to an industry. Since there is only one good in each industry, the elasticity of substitution between goods from an industry, \( \sigma_g \), does not play any role. So, we can forget about this parameter of the purposes of the current section. Also, let us now interpret \( T_{n,g} \) — the parameter of the Fréchet distribution from the model with a continuum of goods — as expected productivity, i.e., as the expected amount of good \( g \) which can be produced in country \( n \) with one unit of labor. For all examples below, each country is assumed to be endowed with one unit of labor, i.e., \( L_n = 1 \) for \( n = H, F \).

<table>
<thead>
<tr>
<th></th>
<th>Home</th>
<th>Foreign</th>
</tr>
</thead>
<tbody>
<tr>
<td>Good 1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Good 2</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 1.1: Expected productivities for Sections 1.4.1 and 1.4.2, \( T_{n,g} \)

In Section 1.4.1 below I analyze the case of industry-specific shocks, and Section 1.4.2 I analyze the case of country-specific shocks. In both sections I use the same parametrization for expected productivities, \( T_{n,g} \), which is given in Table 1.1. With this parametrization, Home has comparative advantage in good 1 and completely specializes in this good in the certainty case, while Foreign has absolute advantage in both goods and incompletely specializes in good 2 in the certainty case. This parametrization is chosen such that there is always a unique
equilibrium for different values of preference parameters and different volatilities of shocks at which I look in Sections 1.4.1 and 1.4.2. At the same time, with this parametrization we can see all interesting effects of financial integration and increased riskiness. I look at the case with multiple equilibria in Section 1.4.3.

1.4.1 Industry-Specific Shocks

Let good 1 be a “safe” good, i.e., not subject to any shocks. Let good 2 be subject to a shock that can take a high or a low value with equal probabilities:

\[ A_2(s) = \begin{cases} 
1 - a, & \text{if } s = 1, \\
1 + a, & \text{if } s = 2,
\end{cases} \]

where \( a \in [0, 1) \). The volatility of shock to good 2 is equal to \( a^2 \). So, the higher is \( a \), the larger is the volatility of shock to good 2, and the riskier is production of this good.

Figure 1.3 depicts labor allocations to good 1 as functions of parameter \( a \) for different values of parameters \( \eta \) and \( \rho \). Each of the graphs in Figure 1.3 captures two important aspects of the framework. The first aspect concerns the effects of changing the financial market structure: the graphs show what happens to labor allocations as we go from financial autarky to complete financial markets. The second aspect concerns the effects of increasing volatility of shock to good 2 (i.e., riskiness of production of good 2). Particular values of \( \eta \) and \( \rho \) are not of as much interest as their relation to 1: there are qualitative changes in effects of increased riskiness depending on whether \( \eta \) and/or \( \rho \) are smaller or greater than 1.

As it is clear from Figure 1.3, the effect of financial integration is qualitatively the same for all combinations of \( \eta \) and \( \rho \). As we go from financial autarky to complete financial markets, countries become more specialized: the solid curves are either outside of the dashed curves or coincide with the dashed curves on all graphs. This is exactly the effect predicted by the conventional view about the role of financial integration. The intuition behind the effect of financial integration in this case is standard. If countries are shut out from international risk-sharing, they hedge their risks by diversifying production. Once international risk-sharing is allowed, countries hedge their risks through financial markets and specialize more in their comparative advantage industries.

Let us now discuss the effects of increased riskiness in production of good 2. Let us start with Figure 1.3a, which shows labor allocation to good 1 when \( \eta > 1 \) and \( \rho > 1 \). In this case goods 1 and 2 are substitutes and risk-aversion is “high”. As we increase volatility of
Figure 1.3: Labor allocations to good 1 under industry-specific shocks

production of good 2, Foreign diverts labor from production of this good to production of good 1. Intuitively, as good 2 becomes riskier, it makes sense to substitute it with the safer good 1. Next, as Figure 1.3c shows, when risk-aversion is “low” and goods are substitutes ($\rho < 1$ and $\eta > 1$), the effect of increased riskiness is the same as in the case of “high”
risk-aversion, although it is less pronounced.

When goods are complements, i.e., when \( \eta < 1 \), increased riskiness in good 2 can have the opposite effect on labor allocations compared to the case with \( \eta > 1 \). With \( \eta < 1 \), we have two competing effects. On the one hand, since good 2 is important for consuming good 1, as good 2 becomes riskier, it is important to invest even more into this good, to make sure there is enough output in bad times. On the other hand, since production of good 2 is high in good times, there no need to devote much labor to this good. Then, depending on the level of risk-aversion, one effect can dominate the other. When risk-aversion is “high”, consumers care about bad outcomes much more than about good outcomes. So, as Figure 1.3b shows, as we increase volatility of shock to good 2, Foreign devotes even more labor to the production of this good. However, when risk-aversion is “low”, consumers do not care as much about bad outcomes. Then, as Figure 1.3d shows, the second effect can start dominating the first effect at some point. We see from Figure 1.3d that, up to some level of volatility, labor allocation to good 1 in Foreign falls only slightly, and then it starts increasing with increase in riskiness in good 2.

It is also important to mention that when preferences over goods are Cobb-Douglas, i.e., when \( \eta = 1 \), increased riskiness has no impact on labor allocations. In this case labor allocations are the same as in the certainty case under both complete financial markets and financial autarky for any distribution of shocks to goods 1 and 2. With Cobb-Douglas preferences and industry-specific shocks, the price ratio of goods 1 and 2 is inversely proportional to the ratio shocks to goods 1 and 2. Also, the expenditure share on each good does not depend on the state of the world. Therefore, prices just adjust in each state of the world, leaving the spending shares unchanged and, so, labor allocations are the same as in the certainty case.

The outcomes of labor allocations described in the current subsection are not entirely new and can mostly be found in other works. This subsection can be seen as the first comprehensive overview of the effects of financial integration and increased riskiness in the Helpman-Razin model. The rest of the paper is devoted to the results which, to the best of my knowledge, cannot be found elsewhere. Therefore, before we proceed, it is instructive to pause and briefly talk about how the above results relate to the existing literature.

Rothschild and Stiglitz (1971) were the first who coherently analyzed effects of increased riskiness in a set of standard economic examples: investing into risky versus safe assets; saving more for tomorrow versus consuming more today; choosing the amount of capital today to complement or substitute labor tomorrow; and several other examples. All of these examples have a common theme: the effect of increased riskiness can switch from one
direction to the other depending on preference parameters. In particular, Rothschild and Stiglitz (1971) showed that the effect of increased riskiness on portfolio allocation between safe and risky assets depends on whether the coefficient of relative risk aversion is smaller or greater than 1. In the same manner, Rothschild and Stiglitz (1971) showed that in a situation, where a firm faces uncertainty in productivity and needs to employ capital \textit{ex ante} and labor \textit{ex post}, the effect of increase in riskiness depends on whether capital and labor are complements or substitutes. Even though none of the examples from Rothschild and Stiglitz (1971) can directly be related to the Helpman-Razin model, the effects we observed in the current subsection are similar to the effects described in Rothschild and Stiglitz (1971).

Next, the effects of financial liberalization on labor allocations in a Ricardian model with uncertainty can be found in Helpman and Razin (1978a), Koren (2003), and Islamaj (2014). All of these papers have a conclusion that financial liberalization results in more specialized production structure. This is not surprising, because the assumptions made in these papers do not deviate enough from the assumption that countries are subject to only industry-specific shocks. For example, Helpman and Razin (1978a) and Koren (2003) literally work with only industry-specific shocks. Similarly, despite the fact that Islamaj (2014) introduces correlations between industry shocks, he only looks at examples where all industries in all countries are affected by shocks with the same or very similar volatilities. These assumptions are not enough to get more diversified production structure in response to financial liberalization. As we shall see in Section 1.4.2, such an outcome is possible under the assumption that shocks are country-specific and the volatility of shocks in one country is higher than in the other.

### 1.4.2 Country-Specific Shocks

Let us now look at examples of labor allocations similar to the ones we saw in the case of industry-specific shocks with the difference that now only Foreign is subject to a country-specific shock, while Home is a safe country. The shock to Foreign country can take one of two values — low or high — with equal probability:

\[
A_F(s) = \begin{cases} 
1 - a, & \text{if } s = 1, \\
1 + a, & \text{if } s = 2,
\end{cases}
\]

where \(a \in [0, 1)\). The larger is \(a\), the larger is the volatility of shock to Foreign, the riskier Foreign is.

Figure 1.4 depicts labor allocations to good 1 as functions of parameter \(a\) for different
values of parameters $\eta$ and $\rho$. A quick look at Figure 1.4 should convince us that the effect of financial integration on production patterns is ambiguous. On graphs (a), (c), and (d) of Figure 1.4, the solid curves are always either inside of the dashed curves or coincide with them. This means that in these particular cases financial integration leads to a weakly
more diversified production structure. At the same time, as it can be seen from graph (b), corresponding to \( \eta < 1 \) and \( \rho > 1 \), financial integration can also lead to more specialization even under country-specific shocks: there is a region on graph (b), where the solid curves are outside of the dashed curves. However, as I discuss it below, the reason for more specialization in this case is totally different from the case of industry-specific shocks, and the logic of the conventional view about the role of financial markets does not apply in this case.

Overall, there are two separate effects which can lead to more diversification under financial integration. One effect concerns production in Home, and the other effect concerns production in Foreign. In essence, these effects can be described as follows. First, after financial integration, Home becomes exposed to the Foreign risk by investing into production in Foreign. As a result, Home might be willing to reduce this risk by diversifying domestic production. Second, in the good state of the world Foreign can have so strong an absolute advantage in its comparative disadvantage good, that it might make sense for Foreign to allocate labor to production of this good. However, under financial autarky, Foreign’s terms of trade insure the risk in production of its comparative advantage good and act as a counter-insurance for its comparative disadvantage good. This counter-acts Foreign’s willingness to produce its comparative disadvantage good. Under financial integration, production of the comparative disadvantage good in Foreign is insured through financial markets, and so Foreign might be willing to produce more of this good.

It is best first to talk about the effects of increased riskiness, before we turn to the detailed discussion of the effects of financial integration.

**Effects of increased riskiness.** Let us start with the extreme case in which the volatility of output in Foreign is very high. To have a particular image in mind, look at the parts of the graphs in Figure 1.4 close to \( a = 1 \). In the good state of the world Foreign is very productive in both goods — it turns into a “big” economy which can potentially supply the world with both goods. But if the bad state of the world realizes, Foreign turns into a “small” economy. In this state Home dominates the world. What is the best labor allocation in this uncertain world under financial integration? The best labor allocation should somewhat resemble the trade autarky outcome! Indeed, with such production structure, Foreign will supply the world with both goods if it happens to be very productive, and Home will be just a “small” part of this world. Otherwise, Home will supply the world with both goods, and Foreign will be a “small” part of this world. The role of financial integration here is very straightforward: Foreign gets insurance in the bad state of the world and pays for that insurance in the good state of the world. Of course, Home and Foreign can still trade and neither of them is truly
a small open economy. So, their labor allocations would not go the full way to the trade autarky outcome. Nevertheless, both Home and Foreign production structure under financial integration would be diversified. This is what we observe in all graphs (a)-(d) in Figure 1.4: the solid curves for Home and Foreign are close to each other and to the 0.5 level on the vertical axis when \( a \) is close to 1.

Can a similar labor allocation pattern happen under financial autarky? Surely it can if, for example, agents are risk-neutral. In this extreme case there will be no difference at all between labor allocations under financial autarky and complete financial markets. Obviously then, if agents are not very risk averse, the financial autarky outcomes will still resemble the complete financial markets outcomes. This is what we observe on graphs (c) and (d) in Figure 1.4: on these graphs the dashed curves have a similar pattern to the solid curves when \( a \) is close to 1 (and, actually, for other values of \( a \) as well).

However, the picture is different, if agents are very risk-averse. Looking at the dashed curves on graphs (a) and (b) in Figure 1.4, we see that Home and Foreign differ in the way they response to increased risk in financial autarky for high values of \( a \). Home diversifies its production structure, while Foreign specializes in its comparative advantage good 2. To understand the difference in the outcomes, we need to understand three things, which follow one from another. First, productivity shocks to Foreign do not change its comparative advantage, and under financial autarky each country exports its comparative advantage good and imports the other good in any state of the world. At the same time, this two-way trade does not necessarily happen in each state of the world under financial integration. As we discussed it above, when volatility in Foreign is high, Home can be exporting both goods in the state with low output in Foreign and importing both goods in the other state under financial integration. Second, in our example — with no shocks to Home — the low output in Foreign hurts both countries. In other words, the bad state for Foreign is also bad for Home. Indeed, when Foreign’s output is low, Foreign is obviously worse off relative to the state with high output. But Home is worse off as well, because it faces a higher price of the importing good relative to the state with high output in Foreign. Finally, in the bad state of the world Home and Foreign are in an unequal situation. Home faces adverse terms of trade, while Foreign faces favorable terms of trade.\(^{12}\) Indeed, when output of good 2 in Foreign is low, the price of good 2 relative to the price of good 1 is high, because Foreign is the exporter of good 2. So, the price of good 2 relative to price of good 1 insures production of good 2

\(^{12}\)Terms of trade of country \( n \) are defined as the price of its export good relative to the price of its import good.
and acts as a counter-insurance for production of good 1 in both countries.

Then, if agents from the two countries are very risk-averse, they care more about the bad state of the world. To mitigate the adverse effect of the terms of trade in the bad state, Home diversifies production. At the same time, the terms of trade insure Foreign’s production of good 2, and so Foreign puts even more resources into this good.

Up to this point we were discussing the effects of increased riskiness in extreme cases with very high volatility in Foreign. Let us now look at the cases with low volatility in Foreign, i.e., at the cases with parameter $a$ being close to 0. For low values of output volatility Foreign remains the exporter of good 2 in any state of the world under both financial autarky and complete markets. Therefore, similar to the situation with high volatility of shocks, Foreign is insulated by the terms of trade from low productivity. This force pushes Foreign to put more resources into production of good 2. However, there is another force, pushing Foreign to put more resources into production of good 1. In the good state of the world Foreign has absolute advantage in good 1. And the higher is the volatility of output in Foreign, the stronger the absolute advantage in the good state of the world is. Therefore, Foreign can potentially benefit more from putting more resources into the production of good 1. Then, if Foreign is very risk-averse, the first force dominates and labor allocation to good 1 decreases in response to increased riskiness. This effect can be seen in Figures 1.4a and 1.4b for dashed curves before vertical line $A$ under financial autarky and for solid curves before vertical line $B$ under complete markets. If, on the other hand, Foreign is not so risk-averse, the second force dominates and labor allocation to good 1 increases in response to increased riskiness. This effect can be seen in Figures 1.4c and 1.4d for dashed curves before vertical line $B$ under financial autarky and for solid curves before vertical line $A$ under complete markets.

For intermediate values of shocks to Foreign, as $a$ increases, we first see the same effect as with low shocks, and after some point we see the same effect as with extremely high shocks. For example, on Figure 1.4a the effect with low shocks explains the parts of dashed curves (corresponding to financial autarky) before the vertical line $E$, and the effect with high shocks starts kicking in after that. Similarly, on the same figure the effect with low shocks applies to the parts of solid curves (corresponding to complete markets) before the vertical line $C$, and the effect with high shocks starts being relevant after that.

**Effects of financial integration.** Now we are ready to proceed with a detailed discussion of the effects of financial integration using our examples in Figure 1.4. In order to understand the effects of financial integration in the case of country-specific shocks, it helps to start with a trivial observation that the aggregate risk in our economy is always present under any
structure of financial markets. Indeed, financial markets do not magically eliminate shocks. The role of financial markets is to allow countries to share risks. This means that one country can insure the other country from bad outcomes. In our example Home is not affected by any shocks, while Foreign can have a low or a high productivity. Nevertheless, as we discussed it above, low productivity in Foreign hurts both countries. And it is actually not obvious, which country is in a worse situation in the bad state of the world: Home or Foreign. As a matter of fact, Foreign can potentially be in a relatively (to Home) better position, when Foreign’s output is low, because Foreign faces favorable terms of trade, which insure Foreign to at least some extent. This argument brings us to an important conclusion that Foreign might be providing insurance to Home in the bad state under financial integration, even though Foreign experiences low productivity in this state. Of course, the opposite might happen as well: Home might be insuring Foreign from the bad state of the world.

Knowing which country provides insurance is the key to understanding in which direction the production structure of both countries changes after financial integration. When we observe the behavior of countries under financial integration, we can easily check which country provides insurance by looking at trade balance in the bad state of the world. The country with trade surplus in the bad state of the world (i.e., the country for which the value of exports exceeds the value of imports) provides insurance. There is also a straightforward way to check which country is going to provide insurance under financial integration by looking at the volatility of mean-normalized consumption under financial autarky, where the volatility of mean-normalized consumption in country \( n \) is defined as

\[
v_n \equiv \text{Var} \left[ \frac{C_n(s)}{E[C_n(s)]} \right].
\]

The volatility of mean-normalized consumption represents the amount of risk a country is exposed to. The country with a lower \( v_n \) under financial autarky is in a relatively better position, and it is going to provide insurance to the other country under financial integration. Indeed, under complete markets, \( v_H = v_F \). Since the aggregate risk is present under any structure of financial markets, as we go from financial autarky to complete markets, the aggregate risk does not disappear but is “reallocated” between the two countries. One country is going to “take” risk — by providing insurance — to the other country. Clearly, the country with higher \( v_n \) is in more need for insurance, and it is going to get it from the other country. As a result, \( v_n \) of the insured country is going to fall and \( v_n \) of the other country is going to rise after financial integration.
Figure 1.5: Volatility of mean-normalized consumption under country-specific shocks

Figure 1.5 depicts the volatilities of mean-normalized consumption for the four combinations of parameters we consider in this section. As we can see from Figure 1.5, in cases (a), (c), and (d) Foreign is riskier (in the sense of having higher $v_n$) than Home for all values
of parameter $a$. And after financial integration Home “takes” some of the Foreign’s risk in these cases. In case (b) there is a region around $a = 0.6$ where Home is actually riskier than Foreign. In that region Foreign provides insurance to Home after financial integration. The reason for Home being riskier in case (b) is that Home faces unfavorable terms of trade in the bad state of the world, and Home cannot substitute away consumption of good 2 with consumption of good 1, because $\eta < 1$ and so goods 1 and 2 are complements.

Having figured out the exact role of financial markets in our environment, we should now easily be able to understand the effects of financial integration on production structure. In cases (a), (c), and (d) from Figure 1.5 Home “takes” risk from Foreign after financial integration. As a result, Home becomes weakly more diversified: to establish this, we apply exactly the same logic which we used to understand the effects of increased riskiness on Home’s production structure. At the same time, Foreign “off-loads” some of its risk to Home and becomes a less riskier country. Therefore, Foreign can exploit more the absolute advantage in good 1 in the good state of the world by devoting more resources to production of this good. As a consequence, Foreign also becomes more diversified in these cases. Again, this is exactly the same logic as we used earlier to understand the effects of increased riskiness on Foreign’s production structure — with the only difference that now we apply this logic in the direction of reducing riskiness.

In case (b) from Figure 1.5 Foreign “takes” Home’s risk in the region of values of $a$ around 0.6. In that region both countries become more specialized after financial integration (see the corresponding region in Figure 1.4b). The logic behind this outcome is totally different from the standard logic behind the conventional view of financial integration. The conventional view might mistakenly attribute increased specialization in Foreign to Home insuring Foreign so that Foreign can focus on production of its comparative advantage good. As we now understand, the effect is actually quite the opposite. Foreign specializes more in its comparative advantage good, because Foreign insures Home and needs to absorb Home’s risk by relying on the insurance through the terms of trade, which pushes Foreign to produce more of its insured good 2.

Let us close this subsection with some thoughts on the generality of the above results. Although I do not provide any mathematical proofs, some of the results above seem to be general. For example, when volatility of shocks to Foreign is high enough, financial integration shall always lead to a more diversified production structure, as long as there is a unique equilibrium under financial autarky. On the other hand, for low shocks the effect is certainly ambiguous for $\eta < 1$ and any value of $\rho$ (it is just happened so that in case (d)
from Figure 1.4 financial integration induces diversification for all values of \( a \); it is possible to construct examples where financial integration induces specialization for low enough values of \( a \) and \( \rho < 1 \) and \( \eta < 1 \). But even if the effect of financial integration is ambiguous and the conventional view can sometimes make correct predictions, there is a certain value in the analysis we have done: as we saw, the conventional view might be making its correct predictions for the wrong reasons.

### 1.4.3 Multiple Equilibria in the Case of Financial Autarky with Country-Specific Shocks

To the best of my knowledge, this paper is the first one to point out that the Helpman-Razin model can have multiple equilibria in the case of financial autarky. The analysis of Section 1.4.2 should convince us that multiple equilibria do not appear for every set of parameters. As we shall see in the following examples, the multiplicity occurs when the comparative advantage of countries is not “strong”, where the notion of “strong” depends on parameters (preferences, productivities, shocks) of the model.

**Example 1.** Let us consider an example with preference parameters \( \eta = 2 \) and \( \rho = 2 \). As in the previous subsection imagine that Foreign is subject to a low or a high shock with equal probabilities. Expected and state-specific productivities are provided in Table 1.2. The low shock is equal to 0.1, while the high shock is equal to 1.9 (\( a = 0.9 \) in terms of the parametrization from Section 1.4.2). With this parametrization Home has comparative advantage in good 1, but this comparative advantage is “weak”.

![Table 1.2: Expected and state-specific productivities in Example 1 from Section 1.4.3, \( T_{n,g} \) and \( A_{n,g}(s)T_{n,g} \)](image)
There are three equilibria under financial autarky in this economy and a unique equilibrium under complete markets. Labor allocations in all equilibria are provided in Table 1.3. Let us first look at equilibria under financial autarky. Equilibrium 1 is the “standard” equilibrium — similar to the ones we saw in Section 1.4.2, in which Foreign takes advantage of the terms of trade insurance by specializing in its comparative advantage good 2. Equilibrium 3 is similar to equilibrium 1 in the sense that in equilibrium 3 Foreign also exploits the terms of trade insurance. The difference is that in equilibrium 3 Foreign captures production of its comparative disadvantage good 1 and leaves little place for Home to produce good 1. In equilibrium 3 Foreign will not move more resources into production of good 2 because its terms of trade act as a counter-insurance for this good, and Home will not move more resources into production of good 1, because the relative price of this good is very low when Foreign’s productivity is high. Equilibrium 2 is an “intermediate” equilibrium, where both countries rely on diversification of production instead of terms of trade to insure themselves from low productivity in Foreign. In this equilibrium countries almost do not trade with each other in the bad state of the world and rely on their domestic production.

Let us now briefly discuss the complete markets equilibrium. In this equilibrium both countries diversify their production more than in all financial autarky equilibria. Nevertheless, we can directly compare only financial autarky equilibrium 1 with the complete markets equilibrium. In both of these equilibria countries incompletely specialize in their comparative advantage goods. At the same time, in financial autarky equilibria 2 and 3 countries incompletely specialize in their comparative disadvantage goods, and it is actually possible to find examples where in these kinds of equilibria under financial autarky countries diversify their production more than under financial integration.

<table>
<thead>
<tr>
<th></th>
<th>Labor allocations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Financial autarky</td>
</tr>
<tr>
<td></td>
<td>Equilibrium 1</td>
</tr>
<tr>
<td>Home Good 1</td>
<td>0.85</td>
</tr>
<tr>
<td>Good 2</td>
<td>0.15</td>
</tr>
<tr>
<td>Foreign Good 1</td>
<td>0.0</td>
</tr>
<tr>
<td>Good 2</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Table 1.3: Labor allocations in different equilibria in Example 1 from Section 1.4.3
Example 2. The multiplicity of equilibria under financial autarky arises because firms’ expected revenues from production of different goods are non-convex functions of labor allocations, if consumers are risk-averse. The simplest way to see the non-convexity is to look at an example with symmetric countries which are subject to i.i.d. shocks with the same variance.

![Figure 1.6: Expected revenues from production of goods 1 and 2](image)

Suppose that preferences over goods are Cobb-Douglas with equal expenditure shares (i.e., $\eta = 1$ and $\alpha_1 = \alpha_2 = 1/2$). Expected productivities are given by $T_{H,1} = T_{F,2} = T$ and $T_{H,2} = T_{F,1} = 1$. There are two independent shocks: one affects production of both goods in Home only, and the other affects Foreign only. Each of the shocks can take a low or a high value with equal probability. The low value is equal to $1 - a$ and the high value is equal to $1 + a$ for both shocks, where $a \in [0, 1)$. Since shocks are independent, there are four equally probable states of the world corresponding to the four combinations of values of Home and
Foreign shocks:

\[
(A_H(s), A_F(s)) = \begin{cases}
(1 - a, 1 - a), & \text{if } s = 1, \\
(1 - a, 1 + a), & \text{if } s = 2, \\
(1 + a, 1 - a), & \text{if } s = 3, \\
(1 + a, 1 + a), & \text{if } s = 4.
\end{cases}
\]

Let us search for symmetric equilibria only, i.e., for equilibria in which Home labor allocation to good 1 is equal to Foreign labor allocation to good 2, \(l_{H,1} = l_{F,2}\). Consider any symmetric labor allocation in Home and Foreign such that \(l_{H,1} = l_{F,2} = l\) and \(l_{H,2} = l_{F,1} = 1 - l\), where \(l \in [0, 1]\) (assuming that each country is endowed with one unit of labor). The goods market clearing conditions imply that prices as functions of \(l\) are given by:

\[
\frac{p_1(s; l)}{p_2(s; l)} = \frac{A_H(s)(1 - l) + A_F(s)Tl}{A_H(s)Tl + A_F(s)(1 - l)}.
\]

Income in state \(s\) in Home as a function of \(l\) is given by

\[
I_H(s; l) = A_H(s)p_1(s; l)Tl + A_H(s)p_2(s; l)(1 - l),
\]

and the expected revenues from production of goods 1 and 2 in Home as functions of \(l\) are given by

\[
r_{H,1}(l) = \sum_s h(s) A_H(s) I_H(s; l)^{-\rho} P(s; l)^{\rho - 1} T p_1(s; l),
\]

\[
r_{H,2}(l) = \sum_s h(s) A_H(s) I_H(s; l)^{-\rho} P(s; l)^{\rho - 1} p_2(s; l),
\]

where \(P(s; l)\) is the Cobb-Douglas price index.

Suppose that \(\rho = 5\) and \(T = 1.2\). Figure 1.6 depicts expected revenues \(r_{H,1}(l)\) and \(r_{H,2}(l)\) as functions of \(l\) for two parametrizations of shocks: \(a = 0\) and \(a = 0.9\). The case with \(a = 0\), depicted in Figure 1.6a, corresponds to the certainty case. In this case the revenue functions are convex, and the function for good 1 is always larger than the function for good 2. There is a unique equilibrium in this case with \(l = 1\): since revenue from good 1 is always larger than the revenue from good 2, Home allocates all labor to production of good 1. In the case with \(a = 0.9\), depicted in Figure 1.6b, the revenue functions are non-convex and intersect two times, resulting in three equilibria denoted by points \(A\), \(B\), and \(C\). The first two equilibria...
occur at the points of intersection of curves $r_{H,g1}(l)$ and $r_{H,g1}(l)$. The third equilibrium is the same as in the certainty case with Home completely specializing in good 1.

1.5 Theoretical Predictions for the Model with a Continuum of Goods

In this section I analyze the model with a continuum of goods. As I discussed it earlier, in the case of financial autarky with country-specific shocks the model presented here has a continuum of equilibria. I first use the variant of the model with only one industry to show how the continuum of equilibria arises and how it can be characterized. I also analyze of the variant of the model with one industry for the case of complete financial markets and provide welfare comparisons of gains from financial integration on top of gains from trade. After that I move to the analysis of the model with two industries.

Before we move on, it is useful to have the following Claim which applies to any structure of shocks, either case of financial markets, and which is true regardless of how many equilibria exist in the model:

Claim 1. Consider any country $i = H, F$ and any industry $g$. Suppose that goods $\omega', \omega'' \in \Omega$ from industry $g$ are produced by country $i$ only. Then, in both cases of financial markets and for an arbitrary structure of shocks,

$$\frac{p_{g}^{\omega'}}{p_{g}^{\omega''}}(s) = \left(\frac{\omega'^{\sigma'}_{i,g}}{\omega''^{\sigma'}_{i,g}}\right)^{-1} \quad \text{and} \quad \frac{l_{i,g}^{\omega'}}{l_{i,g}^{\omega''}} = \left(\frac{\omega'^{\sigma'}_{i,g}}{\omega''^{\sigma'}_{i,g}}\right)^{\sigma_{g}^{-1}},$$

where $s$ is any state of the world.

For proof see the Algebra Appendix. This claim says that for any goods, in which countries completely specialize, all prices and labor allocations are the same up to a good-specific normalization. This property is the same as in the certainty case, and it is going to be the key property which allows us to find expressions for aggregate quantities.

1.5.1 One Industry and Country-Specific Shocks

For the purposes of this subsection we can drop industry index $g$ to keep notation simple. Then shocks to countries are denoted by $A_{H}(s)$ and $A_{F}(s)$. Also, as usual, it is convenient to
assume that goods within each industry are ordered so that comparative advantage of Home falls with good index $\omega$, i.e., \( z_F^\omega / z_H^\omega \) is an increasing function of index $\omega$.

### 1.5.1.1 Financial Autarky

We can guess and verify that there exists a class of equilibria where goods are split into those produced by Home only and those produced by Foreign only (see the Algebra Appendix for all derivations). We focus only on these equilibria. Generally this framework results in equilibria where the chain of comparative advantage in the goods produced by one country breaks. I do not consider these equilibria. Let $B$ be the threshold dividing goods produced by Home and Foreign, so that goods $\omega$, such that \( z_F^\omega / z_H^\omega < B \) are produced by Home. We need to find the conditions which determine this threshold.

It follows from Claim 1 that prices are given by

\[
p^\omega(s) = \begin{cases} 
\Lambda_H(s) [z_H^\omega]^{-1}, & \text{if } \omega \text{ is produced by Home;} \\
\Lambda_F(s) [z_F^\omega]^{-1}, & \text{if } \omega \text{ is produced by Foreign;}
\end{cases}
\]

where $\Lambda_H(s)$ and $\Lambda_F(s)$ are some unknowns, which are part of the equilibrium solution. The analogs of $\Lambda_H(s)$ and $\Lambda_F(s)$ in the certainty case are Home and Foreign wages. It is important to note here that the above expressions for prices do not give us “out of equilibrium” prices: if a country does not produce some good $\omega$ in equilibrium, these expressions do not tell us what is the price of this good that would make this country willing to produce good $\omega$.

The ratio $\Lambda_F(s)/\Lambda_H(s)$ defines the state-specific competitiveness of Foreign relative to Home in the same manner as how in the certainty case the ratio of wages $w_F/w_H$ defines Foreign’s relative competitiveness. We can show that for any two states $s'$ and $s''$:

\[
\frac{\Lambda_F(s')/\Lambda_H(s')}{\Lambda_F(s'')/\Lambda_H(s'')} = \left( \frac{A_F(s')/A_H(s'')}{A_F(s')/A_H(s'')} \right)^{-\frac{1}{B}}.
\]

Hence, the Foreign’s relative competitiveness changes from state to state. In the certainty case Foreign’s competitiveness level is the supply side of the condition which determines the threshold between goods produces by Home and Foreign. Since in the uncertainty case Foreign’s competitiveness changes across states, while the threshold $B$ is determined \textit{ex ante}, the link between Foreign’s competitiveness and the threshold $B$ is indirect. It can be
expressed through the following quantity:

$$\lambda(s) \equiv \frac{\Lambda_H(s)B}{\Lambda_F(s)}.$$ 

In the certainty case the corresponding quantity is just 1.

Next, in the certainty case, the equilibrium on the goods market is the demand side of the condition which determines the threshold between goods produced by Home and Foreign. In the uncertainty case, the equilibrium on the goods market pins down the link $\lambda(s)$ between competitiveness level and the threshold:

$$\lambda(s) = \left( \frac{A_H(s)L_HT_H^{-1}}{A_F(s)L_FT_F^{-1}} B^{-\theta-1} \right)^{-\frac{1}{\sigma}},$$

while the threshold itself remains undetermined. This gives rise to the indeterminacy of equilibria. The link $\lambda(s)$ is directly related to Home’s terms of trade:

$$tot_H(s) \equiv \frac{P_{HH}(s)}{P_{FF}(s)} = \lambda(s) \left( \frac{T_H}{B^{-\theta}T_F} \right)^{1-\sigma},$$

where $P_{HH}(s)$ and $P_{FF}(s)$ are the price indices of goods produced by Home and Foreign. By the definition of threshold $B$, the higher is $B$, the more goods are produced by Home. Hence, as we can see from equations (1.2) and (1.3), the measure of goods produced by Home influences Home’s terms of trade. The multiplicity of equilibria arises because, by capturing some measure of goods on the margin for production, Home gets insurance for these goods through the terms of trade, leaving no place for Foreign to produce these goods. This is exactly the same channel that gives rise to multiplicity of equilibria in the case with two goods.

Using the Home and Foreign firms’ problems, we can find the conditions which put bounds on the threshold $B$. Recall that the Home and Foreign firms’ problems result in the following complementary slackness condition (1.1):

$$l_{n}^\omega \geq 0 \text{ complementary to } w_n - \sum_{s=1}^{S} \psi_n(s)p^\omega(s)A_n(s)z_{n}\geq 0.$$ 

Let us look at any good $\omega$ which Home does not produce in equilibrium, i.e., $l_{H}^\omega = 0$. The
Home’s firm complementary slackness condition implies that
\[
\sum_{s=1}^{S} \psi_H(s)p^\omega(s)A_H(s)z_H^\omega \leq w_H.
\]

Since good \(\omega\) is produced by Foreign, its price is \(p^\omega(s) = \Lambda_F(s) [z_F^\omega]^{-1}\). Substituting this price into the above inequality, we get:
\[
\sum_{s=1}^{S} \psi_H(s)A_H(s)\Lambda_F(s) \leq w_H.
\]

Similarly, by looking at any good \(\omega\) which Foreign does not produce in equilibrium, we get
\[
\sum_{s=1}^{S} \psi_F(s)A_F(s)\Lambda_H(s) \leq w_F.
\]

Since \(z_F^\omega / z_H^\omega\) is a continuous function of index \(\omega\), the above two inequalities for Home and Foreign hold for the threshold good \(\omega_B\) as well. Since \(B = z_F^\omega / z_H^\omega\), we can combine the inequalities for Home and Foreign to get the conditions for the threshold:
\[
\sum_{s=1}^{S} \psi_H(s)A_H(s)\Lambda_F(s) w_H \leq B \leq \sum_{s=1}^{S} \psi_F(s)A_F(s)\Lambda_H(s) w_F.
\]

The conditions for the threshold \(B\) are complex expressions, which involve endogenous quantities. We can derive that
\[
\sum_{s=1}^{S} \psi_H(s)A_H(s)\Lambda_F(s) w_H = L_H L_F \cdot \frac{1}{1 + \frac{I_F(s')}{I_H(s')}} \left[ \frac{\Xi_H(s')}{1 - \rho} \right],
\]
\[
\sum_{s=1}^{S} \psi_F(s)A_F(s)\Lambda_H(s) w_F = L_H L_F \cdot \frac{1}{1 + \frac{I_H(s')}{I_F(s')}} \left[ \frac{\Xi_H(s')}{1 - \rho} \right],
\]
where
\[
\Xi_H(s) = L_H T_H^{\frac{1}{\mu}} \pi_{HH} H (s) \left[ m_H^{\frac{1}{\mu}} + \frac{1}{\mu} \right].
\]
is a short-cut introduced for convenience, $\pi_{HH}(s)$ is the expenditure share of Home on Home’s goods, and $m_H$ is the measure of goods produced by Home. Contrary to the certainty case, $\pi_{HH}(s) \neq m_H$, as can be seen from the corresponding expressions:

$$m_H = \frac{T_H}{T_H + T_F B^{-\theta}},$$

$$m_F = \frac{T_F B^{-\theta}}{T_H + T_F B^{-\theta}},$$

and

$$\pi_{HH}(s) = \pi_{FH}(s) = \lambda(s)^{1-\sigma} T_H,$$

$$\pi_{HF}(s) = \pi_{FF}(s) = \frac{T_F B^{-\theta}}{\lambda(s)^{1-\sigma} T_H + T_F B^{-\theta}}.$$ 

Intuitively, since relative competitiveness of countries varies across states, the trade shares are different, while the share of goods produced by each country is determined ex ante.

It is convenient to introduce notation for bounds on $B$:

$$b_H \equiv \sum_{s=1}^{S} \psi_H(s) A_H(s) \Lambda_F(s),$$

$$b_F \equiv \sum_{s=1}^{S} \psi_F(s) A_F(s) \Lambda_H(s).$$

Expressions for $b_H$ and $b_F$ are relatively simple when $\eta = 1$, $\sigma = 1$, and $\rho = 1$:

$$b_H = \frac{L_F}{L_H} \cdot \frac{T_H}{T_F} \cdot B^\theta \cdot \left( \sum_{s'} h(s') \frac{A_H(s')}{A_F(s')} \right)^{-1},$$

$$b_F = \frac{L_F}{L_H} \cdot \frac{T_H}{T_F} \cdot B^\theta \cdot \sum_{s'} h(s') \frac{A_F(s')}{A_H(s')}.$$ 

Note that, when there is no uncertainty, we get the familiar expression $b_H = b_F = \frac{L_F}{L_H} \cdot \frac{T_H}{T_F} \cdot B^\theta$.

As it should be clear from the above discussion, $b_H$ and $b_F$ are endogenous quantities and they are different across different equilibria. The smallest value for $b_H$ is achieved in an equilibrium with $B = b_H$, while the largest value for $b_F$ is achieved in an equilibrium with $B = b_F$. Let us denote the corresponding values by $\bar{b}_H$ and $\bar{b}_F$. Then, in any equilibrium, goods $\omega$ such that $z^\omega_F / z^\omega_H < \bar{b}_H$ are produced by Home, and goods $\omega$ such that $z^\omega_F / z^\omega_H > \bar{b}_F$ are produced by Foreign. The set of goods between $\bar{b}_H$ and $\bar{b}_F$ can be labeled as a “set of
disputed goods”. Each particular value of $B$ determines the split in the set of disputed goods between Home and Foreign: goods $\omega$ such that $\bar{b}_H \leq z_F/\bar{z}_H \leq B$ are produced by Home, and the rest is produced by Foreign. Each equilibrium from the continuum can be associated with the percentage of disputed goods produced by Home given by $\kappa \equiv (B - \bar{b}_H)/(\bar{b}_F - \bar{b}_H)$. Having this notation for $\kappa$ gives us a way to compare equilibria for different values of volatility of shocks corresponding to the same value of $\kappa$. In what follows, I will call $\kappa$-equilibrium an equilibrium corresponding to a particular value of $\kappa$.

Let us now look at gains from trade. In any equilibrium they are given by:

$$GT_n(s) = \pi_{nn}(s)^{-\frac{1}{\sigma-1}}m_n^{-\frac{1}{\sigma-1}}.$$

There is an interesting economic intuition behind this expression. Let us focus on some particular equilibrium of our economy. Let us label the economy with this equilibrium by $O$ (“original”). Economy $O$ is described by the threshold $B$ dividing goods produced by Home and Foreign, by trade shares $\pi_{nn}(s)$, and by the measure of goods produced by country $n$, $m_n$. Now imagine an Armington economy, where each country is forced to produce their measures $m_H$ and $m_F$ of goods. In other words, in this economy each country is exogenously assigned to produce exactly the goods they produce in the economy $O$. Let us label this Armington economy by $A$. By the definition of economy $A$, in the trade autarky of economy $A$ countries produce their exogenously assigned measures of goods $m_H$ and $m_F$. Consequently, when in the economy $A$ countries open up to trade, the measures of goods they produce do not change, but the measures of goods they consume expand to the full set of goods of the economy $O$. Hence, the gains from trade in economy $A$ come from the love for variety and are given by $\pi_{nn}(s)^{-\frac{1}{\sigma-1}}$.

Next, let us look at the welfare gains from going from the trade autarky of economy $O$ to the trade autarky of economy $A$. These welfare gains come from two components. On the one hand, as we go from the trade autarky in $O$ to the trade autarky in $A$, we shrink the set of the consumed goods, and, therefore, suffer a welfare loss of $m_n^{\frac{1}{\sigma-1}}$, which, again, comes from the love for variety. On the other hand, there is now more labor available to produce each good. This gives as a welfare gain of $m_n^{-\frac{1}{\sigma}}$. Therefore, the overall welfare gain from going from the trade autarky of economy $O$ to the trade autarky of economy $A$, is given by $m_n^{-\frac{1}{\sigma}}/\sigma$. Finally, the welfare gains from going from the trade autarky of economy $O$ to the free

\footnotesize{13}I thank Andrés Rodríguez-Clare for sharing with me this intuition.
trade in economy $O$, can be decomposed into (i) welfare gains from going from the trade autarky of economy $O$ to the trade autarky of economy $A$; (ii) welfare gains from going from the trade autarky of economy $A$ to the free trade in economy $A$ (which is the same as the free trade in economy $O$). Hence, the overall gains from trade in economy $O$ are given by

$$\pi_{nn}(s)^{-\frac{1}{\sigma-1}} m_n^{-\frac{1}{\gamma} + \frac{1}{\sigma-1}}.$$  

1.5.1.2 Complete Financial Markets

![Graph showing labor allocations across individual goods](image)

Figure 1.7: Labor allocations across individual goods

In the case of complete financial markets, there are regions of incomplete specialization even if trade is free. Figure 1.7a shows how labor is allocated to individual goods when there is no uncertainty or in the case of financial autarky. In these cases production of goods is split between Home and Foreign. Figure 1.7b shows labor allocations to individual goods in the case of complete financial markets. There is a region of incomplete specialization in the middle. Countries export (or import) different sets of goods within that region in different states of the world. As the volatility of country-specific shocks increases, that region becomes wider. The intuition behind why this region of incomplete specialization occurs comes directly from the case of two goods. First, as countries financially integrate, they take some part of each others risk, and need to reduce that risk by diversifying their production. Second, after financial integration countries do not have to rely on the terms of trade insurance. Hence, they can exploit their absolute advantage in the marginal goods in the states when their productivity is high and rely on financial markets to insure from the states with low productivity.
Because of the regions of incomplete specialization, it is hard to get analytic expressions (price indices, the share of goods produced by each country, welfare, etc.) in the case of complete financial markets. These expressions have to be derived for each value of $\theta$ separately, and they have very different forms for different values of $\theta$. In the regions of incomplete specialization price of any good $\omega$ is a function of both Home and Foreign efficiencies, $z_H^\omega$ and $z_F^\omega$. For example, if there are no shocks to Home and the shock to Foreign can take only two values, prices in states 1 and 2 are given by:

$$p^\omega(1) = \left( A_F(2) \frac{w_H}{z_H^\omega} - \frac{w_F}{z_F^\omega} \right) (A_F(2) - A_F(1))^{-1},$$

$$p^\omega(2) = \left( \frac{w_F}{z_F^\omega} - A_F(1) \frac{w_H}{z_H^\omega} \right) (A_F(2) - A_F(1))^{-1}.$$

Expressions for price indices of such prices involve taking integrals of rational polynomial functions having $\theta$ in the powers of polynomials. While it is possible to find closed-form expressions for such integrals, these expressions have very different forms depending on the value of $\theta$. Therefore, to solve the complete financial markets case in the most general form, I use the algorithm described in Appendix A.

1.5.1.3 A Numerical Example

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma$</td>
<td>4</td>
</tr>
<tr>
<td>$\rho$</td>
<td>2</td>
</tr>
<tr>
<td>$\theta$</td>
<td>5</td>
</tr>
<tr>
<td>$T_n$</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 1.4: Parameter values for the example with one industry

Let us now look at a numerical example which illustrates the effect of financial integration and increased riskiness on the measure of goods produced in Home and Foreign as well as on welfare. The elasticity of substitution between goods from different industries, $\eta$, is irrelevant for this section, because we have only one industry. The values of other parameters are provided in Table 1.4. The uncertainty in the world is described by one shock, which applies to Foreign only, while Home has no shocks. The shock to Foreign can take either a high or a low value with equal probabilities and is parametrized by the number $a \in [0, 1)$ in the
following way:

\[
A_F(s) = \begin{cases} 
1 - a, & \text{if } s = 1 \\
1 + a, & \text{if } s = 2.
\end{cases}
\]

Below I show the measure of goods produced by each country and welfare as functions of the volatility of shock to Foreign. It is more revealing to look at the measures of goods and welfare as functions of log-volatility. I calculate this volatility as the variance of log deviation from the mean:

\[
V[\hat{A}_F(s)] = E\left[(\hat{A}_F(s) - E[\hat{A}_F(s)])^2\right],
\]

where

\[
\hat{A}_F(s) \equiv \log A_F(s) - \log E[A_F(s)],
\]

Such way of calculating volatilities in the static framework presented here corresponds to the volatility of growth rates (i.e., volatility of log deviations from a trend) in a truly dynamic framework.

![Graphs showing the share of goods produced by Home and Foreign](image)

**Figure 1.8:** Shares of goods produced by Home and Foreign

Given the parametrization described above, there is a continuum of equilibria in the case
of financial autarky. Each of these equilibria is characterized by the threshold $B$ and the corresponding value of $\kappa$. Each value of the threshold $B$ defines the share of goods produced by Home and Foreign, $m_H$ and $m_F$. These shares are depicted in Figure 1.8 in the gray area between the dashed lines, which correspond to the bounds $b_H$ and $b_F$ within which the threshold $B$ varies.

In the case of complete financial markets there is a unique equilibrium for each value of volatility in Foreign. The solid red lines in Figure 1.8 depict the shares of goods produced by Home and Foreign. These shares are calculated as part of the solution of the problem, and they include all goods produced by Home and Foreign — not only the goods in which Home and Foreign completely specialize. For example, in Figure 1.7b, the share of goods produced by Home is equal to the percent of goods from $[0, 1]$ for which the solid red line is above level zero. Among those goods, there is a large share of goods in which Home incompletely specializes.

Not surprisingly, Figure 1.8 reveals that the measures of goods produced by Home and Foreign in the model with a continuum of goods follow the same pattern as the amount of labor allocated to goods in the framework with two goods (compare Figure 1.8 to Figure 1.4a). In the case of financial autarky, increasing volatility in Foreign results in a smaller share of goods produced in Foreign and a bigger share of goods produced in Home in any $\kappa$-equilibrium. Intuitively, Foreign moves resources into production of its best goods because these goods are insured by the terms of trade. In the case of complete financial markets, increasing volatility in Foreign results in a bigger share of goods produced in both countries. In this case both countries can benefit from the state with high aggregate productivity in Foreign, if Foreign produces a wider range of goods. And both countries rely on production in Home when aggregate productivity in Foreign is low.

Also, as Figure 1.8 demonstrates, for any $\kappa$-equilibrium, financial integration leads to more diversified production in Home and Foreign in terms of the share of goods produced by the countries: the solid red lines are above the gray regions in Figures 1.8a and 1.8b. In this case the terms of trade insurance becomes not so important for Foreign, and Foreign exploits the comparative advantage in a wide range of goods when its productivity is high. At the same time, Home insures Foreign from the state with low productivity and, so, Home needs to absorb that additional risk by producing a wider range of goods.

Figure 1.9 shows the gains from trade under (i) financial autarky; (ii) complete financial markets with the production structure exogenously fixed to what it is under financial autarky; (iii) complete financial markets with endogenous production structure. These gains are
Figure 1.9: Expected gains from trade for Home and Foreign

calculated in the following way:

\[
GT_n \equiv \begin{cases} 
  
  \frac{E \left[ U \left( C_n^A(s) \right) \right]}{E \left[ U \left( C_n(s) \right) \right]} - 1, & \text{if } \rho > 1, \\
  \frac{E \left[ U \left( C_n(s) \right) \right]}{E \left[ U \left( C_n^A(s) \right) \right]} - 1, & \text{if } \rho \leq 1,
\end{cases}
\]  

(1.4)

where \( C_n^A(s) \) is country \( n \)’s aggregate consumption in the trade autarky and \( C_n(s) \) is country \( n \)’s aggregate consumption in the case of free trade with either complete markets or financial autarky.\(^{14}\) So, the expected gains from trade measure the expected welfare increase resulting from going from trade autarky to free trade.

Figure 1.9 reveals that gains from moving from financial autarky to complete financial markets with endogenous production structure can be substantial. However, these gains are for the big part driven by consumption smoothing (represented by the dashed curves corresponding to the gains from trade in the case of complete financial markets with exogenous production structure). Moreover, Foreign can lose from allowing its production structure

\(^{14}\)Recall that if \( \rho > 1 \), then \( U \left( C_n(s) \right) = \frac{1}{1-\rho}C_n(s)^{1-\rho} < 0 \), and so the larger absolute value of \( U \left( C_n(s) \right) \) means the lower utility in this case.
to adjust in the case of complete financial markets. This happens because adjustment of production structure in Foreign leads to more diversified production structure in Foreign.

Also, Figure 1.9 demonstrates that, for each level of volatility in Foreign, the multiple equilibria existing under financial autarky can be ordered in terms of welfare from the point of view of a particular country. For example, the welfare in Foreign is the highest when it produces all disputed goods, i.e., when $B = b_H$, which correspond to the green dashed line in Figure 1.9b. Intuitively, on the one hand, when Foreign’s aggregate productivity is low, the disputed goods produced by Foreign are insured through the terms of trade. On the other hand, when Foreign’s aggregate productivity is high, Foreign has comparative advantage in the disputed goods, and, hence, Foreign benefits from the fact that it produces these goods with productivities which are higher relative to Home’s productivities in these goods. Note that for any level of volatility in Foreign, Foreign is always better off from capturing production of all disputed goods. For Home the situation is different. When volatility in Foreign is low, Home benefits from producing all disputed goods: the dashed blue line is above the dashed green line in Figure 1.9a. In this case Home produces the disputed goods with a safe technology and does not face high prices on these goods in the bad state of the world (which happens if Foreign produces the disputed goods). However, if volatility in Foreign is high, Home benefits more if it does not produce the disputed goods: the dashed blue line is below the dashed green when volatility in Foreign is higher than 5. This happens because, by allowing Foreign to produce the disputed goods, Home benefits from high aggregate productivity in Foreign in the good state of the world, and this benefit offsets the losses from high prices on the disputed goods in the bad state of the world.

The welfare analysis we have just done reveals that countries might be willing to pursue industrial policies to move to a better equilibrium. Moreover, sometimes both countries can benefit from such industrial policies. Nevertheless, all these gains happen on the margin, and the biggest benefit is reaped when the countries open financially.

1.5.2 Multiple Industries and Industry-Specific Shocks

Let us now consider the case with multiple industries and industry-specific shocks only.\textsuperscript{15} This case is very different from what we have just seen. One important distinction is that under industry-specific shocks the equilibrium is unique for any structure of financial markets. The purpose of this section is to demonstrate that the main effects from the model with 2

\textsuperscript{15}See the Algebra Appendix for all derivations.
goods are in action on the aggregate level in the model with a continuum of goods. So, we are
interested to look at the effects of increased riskiness and financial integration on aggregate
labor allocations. Also, we will look at welfare to compare the outcomes with the previously
considered case of a single industry with country-specific shocks.

Since there are only industry-specific shocks, we can drop country index from the notation
for shocks: \( A_{n,g}(s) = A_g(s) \). One can show that in this case there is a unique equilibrium in
which goods in each industry are split into those produced only by Home and those produced
only by Foreign. Suppose that goods within each industry are ordered so that comparative
advantage of Home falls with good index \( \omega \). Let \( B_g \) be a threshold dividing goods produced
by Home and Foreign in industry \( g \), so that goods \( \omega \), such that \( \frac{z_{F,g}}{z_{H,g}} < B_g \), are produced
by Home. Then for both cases of international risk sharing

\[
B_g = \left( \frac{T_{H,g}}{T_{F,g}} \right)^{-\frac{1}{1+\theta}} \frac{L_{F,g}}{L_{H,g}}.
\]

The same expression is true for the certainty case as well. The difference between the certainty
case, complete financial markets, and financial autarky will be in the amount of labor, \( L_{n,g} \),
devoted to each industry.

Let us now look at trade shares and labor allocations. Trade shares are the same for all
states of the world:

\[
\pi_{HH,g} = \pi_{FH,g} = \frac{T_{H,g}}{T_{H,g} + T_{F,g}B_g^{-\theta}},
\]

\[
\pi_{HF,g} = \pi_{FF,g} = \frac{T_{F,g}B_g^{-\theta}}{T_{H,g} + T_{F,g}B_g^{-\theta}}.
\]

This is the same expression as in the certainty case. The difference is in the value of threshold,
\( B_g \). Allocation of Home labor for the case of complete financial markets is given by the
expression:

\[
\frac{L_{H,g'}}{L_{H,g''}} = \frac{\alpha_g'}{\alpha_{g''}} \left( \frac{T_{H,g'}}{T_{H,g''}} \right)^{\frac{1}{1+\rho}} \left( \frac{\pi_{HH,g'}}{\pi_{HH,g''}} \right)^{1-\frac{2}{1+\rho}} \left( \frac{\Gamma_{g'}}{\Gamma_{g''}} \right)^{\eta-1}
\times \left( \frac{\sum_{s'} h(s') A_{g'}(s')^{\frac{1}{n}} \Xi(s')}{\sum_{s'} h(s') A_{g''}(s')^{\frac{1}{n}} \Xi(s')} \right)^{\eta},
\]
where
\[ \Xi(s) \equiv \left[ \sum_{g} \alpha_{g'} \left( \alpha_{g'}^{-1} A_{g'}(s) T_{H,g}^{\frac{1}{\eta}} L_{H,g}^{\frac{-1}{\eta+1}} \right)^{\frac{n+1}{\eta}} \right]^{\frac{\eta}{\eta-1}}. \]

Similarly, for the case of financial autarky:
\[
\frac{L_{H,g'}}{L_{H,g''}} = \frac{\alpha_{g'}}{\alpha_{g''}} \left( \frac{T_{H,g'}}{T_{H,g''}} \right)^{\frac{n-1}{\sigma}} \left( \frac{\pi_{HH,g'}}{\pi_{HH,g''}} \right)^{1-\frac{n-1}{\sigma}} \left( \frac{\Gamma_{g'}}{\Gamma_{g''}} \right)^{\eta-1} \times \frac{\left( \sum_{s'} h(s') \left( 1 + \frac{I_{F}(s')}{I_{H}(s')} \right)^{\rho} A_{g'}(s')^{\frac{n-1}{\eta}} \Xi(s')^{\frac{1}{\eta-\rho}} \right)^{\eta}}{\left( \sum_{s'} h(s') \left( 1 + \frac{I_{F}(s')}{I_{H}(s')} \right)^{\rho} A_{g''}(s')^{\frac{n-1}{\eta}} \Xi(s')^{\frac{1}{\eta-\rho}} \right)^{\eta}},
\]

where \( \Xi(s) \) is given by the same expression as in the case of complete financial markets. Note that the expressions for labor allocation in the case of complete financial markets and financial autarky differ only in the term \((1 + I_{F}(s') / I_{H}(s'))^{\rho}\), which enters the sums in the case of financial autarky. If consumers are risk-neutral, \( \rho = 0 \), then the financial markets structure has no impact on labor allocations across industries (but the allocations are different from the certainty case). If preferences over industries are Cobb-Douglas, \( \eta = 1 \), then in both cases of financial markets:
\[
\frac{L_{H,g'}}{L_{H,g''}} = \frac{\alpha_{g'}}{\alpha_{g''}} \frac{\pi_{HH,g'}}{\pi_{HH,g''}},
\]

which does not depend on the shocks and is the same as in the certainty case.

Let us now look at the examples of labor allocations for the case of two industries, which are mirror images of each other in terms of parameters \( T \) of Fréchet distribution: \( T_{H,1} = T_{F,2} = 1 \) and \( T_{H,2} = T_{F,1} = 2 \). With this parametrization Home has comparative advantage in industry 2, while Foreign has comparative advantage in industry 1. All other parameters are identical for the two countries: \( \rho = 2, \theta = 5, \sigma_{1} = \sigma_{2} = 4, \eta = 0.8 \) or 2. Let industry 1 be a safe industry, while industry 2 be subject to a shock which can take a high or a low value with equal probabilities:
\[
A_{2}(s) = \begin{cases} 
1 - a, & \text{if } s = 0 \\
1 + a, & \text{if } s = 1,
\end{cases}
\]

where \( a \in [0,1) \).
Figure 1.10 demonstrates that the patterns of aggregate labor allocations in the model with a continuum of goods are similar to the patterns of labor allocation in the model with two goods (the only difference is that having continuum of goods allows us to achieve smoothness of the outcomes). First, financial integration makes both countries more specialized in their comparative advantage industries: the solid lines are outside of the dashed lines in both Figures 1.10a and 1.10b. Second, as volatility of shocks to industry 2 increases, both countries pull labor out of this industry if $\eta > 1$, and put even more labor into this industry if $\eta < 1$. Intuitively, similar to the case with 2 goods, if $\eta > 1$ and industry 2 becomes riskier, then it makes sense to substitute its goods with goods from the safer industry 1. On the other hand, if $\eta < 1$, industry 2 goods are important for consuming goods from industry 1. And so, as industry 2 becomes riskier, it is important to invest even more into this industry, to make sure there is enough output in bad times.

Figure 1.11 shows the expected gains from trade calculated using formula (1.4). As we can see from Figure 1.11, the expected gains from trade do not change much as volatility of shocks to industry 2 increases. The gains from financial integration on top of the gains from trade (measured as increase in welfare resulting from allowing international risk sharing) are small in this case as well. These small gains are related to the fact that the differences in comparative advantage across countries in this example are small. In order to obtain larger gains from financial integration, we need to have relatively big differences in comparative advantages.
advantage. However, empirically, we do not observe substantial differences in comparative advantage between countries. This is also the reason why the trade literature traditionally estimates small gains from trade openness.

### 1.5.3 Multiple Industries and Country-Specific Shocks

In this section we focus on the country-specific shocks and shut down the industry-specific shocks: $A_{n,g}(s) = A_n(s)$.\(^{16}\) Formal analysis of the economy with multiple industries is similar to analysis of the economy with a single industry, which we have already done. So, in this section we start right away with a numerical example.

Let us look at aggregate labor allocations and gains from trade in an example similar to the one we analyzed in the case of industry-specific shocks. All parameter values are the same as in the example with industry-specific shocks. The only difference is that now only Foreign is subject to a country-specific shock, while Home is a safe country. The shock to Foreign country can take one of two values — low or high — with equal probabilities:

$$A_F(s) = \begin{cases} 1 - a, & \text{if } s = 0 \\ 1 + a, & \text{if } s = 1, \end{cases}$$

where $a \in [0, 1)$.

Figure 1.12 depicts aggregate labor allocations in industry 2 for $\eta = 2$ and $\eta = 0.8$. Let us first look at Figure 1.12a corresponding to $\eta = 2$. The two green dashed curves in Figure 1.12a bound labor allocations in Foreign for all possible $\kappa$-equilibria. As we can

\(^{16}\)See the Algebra Appendix for all derivations.
Figure 1.12: Aggregate employment in the risky industry

see, these curves are very close to each other, meaning that there is not much variation in aggregate labor allocations in Foreign across different $\kappa$-equilibria. The variation in aggregate labor allocations across $\kappa$-equilibria is even smaller (almost non-existent) for Home: the blue dashed curves in Figure 1.12a basically coincide with each other. The situation is similar for the case with $\eta = 0.8$ depicted in Figure 1.12b.

Similar to the case with two goods, we see from Figure 1.12 that financial integration leads to more diversified production structure in terms of aggregate labor allocations: the solid curves are outside of the dashed curves on both Figures 1.12a and 1.12b. An increase in riskiness makes Foreign devote more labor to its comparative advantage industry 1 under financial autarky, because this industry is insured by the terms of trade: the green dashed curves are decreasing on both Figures 1.12a and 1.12b. In the case of $\eta = 2$, Home also devotes more resources to industry 1, to diversify its risks of possible adverse terms of trade in the state with low aggregate productivity in Foreign. In the case of $\eta = 0.8$, the response of aggregate labor allocations in Home to increased riskiness is non-monotonic. For small levels of volatility in Foreign, Home puts more labor into industry 2 in response to increased riskiness. This happens because, as riskiness increases, Foreign produces less of goods from industry 2, but both countries need these goods because they complement consumption of goods from industry 1. So, Home starts producing more of goods from industry 2. At the same time, Home faces adverse terms of trade in industry 2. So, Home has incentives to
move labor out of this industry. As riskiness in Foreign becomes too large, the terms of trade effect dominates, and the aggregate labor allocation in industry 2 in Home falls.

Under complete financial markets both countries diversify their aggregate labor allocations in response to increased riskiness: Home allocates more labor to industry 1 and Foreign allocates more labor to industry 2. Again, this is driven by the same effect, which we saw in the case of two goods: by diversifying production both countries can benefit from the state with high aggregate productivity in Foreign, and both countries rely on production in Home to hedge the risk of low aggregate productivity in Foreign.

Figure 1.13: Expected gains from trade for Home and Foreign for $\eta = 2$ and $\eta = 0.8$
1.6 Conclusions

This paper provides careful analysis of the standard $2 \times 2$ Ricardian model with uncertainty — a variant of the Helpman-Razin model. This analysis reveals that — contrary to the prevailing view — the effect of financial integration on production structure of countries is ambiguous: this effect depends on the structure of TFP shocks and preferences. Under industry-specific shocks countries can become more specialized after financial integration, while under country-specific shocks countries can become more diversified after financial integration. The analysis also shows that the Helpman-Razin model generally has multiple equilibria, which was overlooked in the earlier literature.

Introduction of the continuum of goods into the Helpman-Razin model preserves all interesting effects and makes the multiple equilibria “manageable”. Building the model with a continuum of goods is a critical step towards quantifying with data the effects of financial integration and increased riskiness as well as measuring the welfare gains from financial integration on top of the gains from trade. The international finance literature traditionally finds small gains from financial integration, but there is some speculation in this literature that an endogenous production structure can be a source of larger welfare gains. Preliminary welfare analysis of the framework with a continuum of goods presented in this paper shows that the gains from financial integration through its impact on the production structure are unlikely to be significant relative to consumption smoothing. The next natural step in research on the topic of the current paper is to bring the presented framework to data.
Chapter 2

External Economies and International Trade: A Quantitative Framework

2.1 Introduction

The field of international trade has made great strides in recent years by “mapping theory to data” in the new quantitative trade models (or so called “gravity models”). This has led to important insights into the consequences of globalization, but a fundamental issue has been missing: the role of localized and industry-specific external economies of scale. These externalities played a large role in the world economy at the time of Marshall (1890, 1930), and recent anecdotal and empirical evidence suggest that they play, if anything, an even larger role in the global economy today.¹

There is a good reason for this oversight. Early models yielded some discomforting results, including “a bewildering variety of [multiple] equilibria” (Krugman, 1995) so that trade patterns need not conform to comparative advantage, along with the “paradoxical implication that trade motivated by the gains from concentrating production need not benefit the participating countries” (Grossman and Rossi-Hansberg, 2010). At the heart of these “pathologies” lay the compatibility assumption of increasing returns and perfect competition (as thoroughly discussed in part 2 of the classic Econometrica surveys by Chipman, 1965), namely that firms take productivity as given even though it depends on total industry output. This leads to a circularity whereby the scale of an industry affects its productivity, while an

industry’s productivity affects its scale through the impact on the pattern of comparative advantage and specialization. In the standard analysis, this leads to multiple equilibria.\(^2\)

**Grossman and Rossi-Hansberg (2010, henceforth GRH)** recently proposed a two-country Ricardian model with national industry-level external economies of scale (or *Marshallian externalities*) which attacks this compatibility assumption head-on. Instead of perfect competition, GRH assume Bertrand competition so that now firms in each industry understand the implications of their decisions on industry output and productivity, ensuring that in equilibrium we have the “right” allocation of industries across countries in a similar fashion to that of the constant returns to scale framework of Dornbusch, Fischer and Samuelson (1977a).\(^3\) While the framework successfully eliminates the “pathologies” in a world free of trade costs, Lyn and Rodríguez-Clare (2013a,b) illustrate circumstances under which multiple equilibria arise in the presence of trade costs. Coupled with the fact that the equilibrium has mixed strategies for some levels of trade costs, the framework quickly becomes intractable, with little hope of extending it to a multi-country setting with trade frictions.

This paper presents a Ricardian model with Marshallian externalities that admits a unique equilibrium under reasonable parameter restrictions. Unlike GRH, we leave the compatibility assumption intact and approach the problem from a different angle by relaxing the implicit assumption in the standard framework (and in GRH) that firms within each industry are producing a homogeneous variety. In particular, we allow for intra-industry heterogeneity a la Eaton and Kortum (2002, henceforth EK) and find that this adds some “curvature” that helps in establishing uniqueness of equilibrium as long as the strength of Marshallian externalities is “not too high”. The framework yields the standard gravity-type equation and so provides a platform to assess quantitatively the importance of these externalities for the welfare effects of trade.\(^4\)

The system of equations that characterizes the equilibrium of the Ricardian multi-industry model with Marshallian externalities turns out to be isomorphic to the equilibrium system of the multi-industry Krugman (1980) model of product differentiation with internal economies of scale. The existence and uniqueness result that we prove for our Ricardian setting can then be seamlessly applied to the multi-industry Krugman model. As far as we know, we are

\(^2\)See early work exploring this by Graham (1923), Ohlin (1933), Matthews (1949), Kemp (1964), Melvin (1969), Markusen and Melvin (1981), and Ethier (1982a,b).

\(^3\)This is most clearly the case with frictionless trade, which the authors use to make their main points.

\(^4\)Our analysis restricts to the case of Marshallian externalities, which operate inside each industry. An alternative case is the one in which some of the externalities operate across industries. Yatsynovich (2014) has recently shown conditions under which a model with such cross-industry externalities exhibits a unique equilibrium.
the first to establish uniqueness of equilibrium in this case. Not surprisingly, the isomorphism extends also to the multi-industry Melitz (2003) model if the productivity distribution is Pareto and the fixed exporting costs are paid in the units of labor of the destination country.

The common mathematical structure that characterizes the equilibrium in all these multi-industry gravity models is governed by two elasticities that can vary across sectors: the trade elasticity and the elasticity of productivity with respect to industry size, which we will refer to as the scale elasticity. The condition for uniqueness is that (in all sectors) the product of these two elasticities is not higher than one. In the Ricardian model the scale elasticity is given directly by the strength of Marshallian externalities, so the condition for uniqueness is that the strength of these externalities is not higher than the inverse of the trade elasticity. In the Krugman or Melitz-Pareto models the scale elasticity is given by the inverse of the trade elasticity, hence we are always at the edge of the region of uniqueness. One can easily add flexibility to the Krugman and Melitz-Pareto models to break the tight link between the two elasticities. For example, if we allow the elasticity of substitution across varieties from different countries to differ from the elasticity of substitution across varieties from the same countries (with nested CES preferences) then the scale elasticity is given by the ratio of these two elasticities of substitution divided by the trade elasticity. Alternatively, one can allow for heterogeneity in worker ability across industries as in Galle, Rodríguez-Clare and Yi (2014). This introduces “inter-industry curvature” into the model and reduces the scale elasticity in all industries below the inverse of the trade elasticity, thereby helping ensure uniqueness.\(^5\)

The final section of the paper uses the unified framework to explore the implications of scale economies for the welfare effects of trade. We first establish that even if the scale elasticity differs across industries, for example because of cross-industry variation in the strength of Marshallian externalities in the Ricardian model, if we are in the region of uniqueness then all countries gain from trade. This is noteworthy in light of previous results with this type of model where countries could lose from trade. We next show that, perhaps surprisingly, if the scale elasticity is the same across industries, the gains from trade as defined in Arkolakis, Costinot and Rodríguez-Clare (2012, henceforth ACR) are lower with scale.

\(^{5}\)As we were finishing this draft, we became aware of a new paper by Mariano Somale (Somale, 2014) which makes some related points to the ones we make in this paper regarding the welfare gains from trade. The purpose of the two papers, however, is quite different: whereas Somale is interested in introducing sector specific innovation into a multi-sector Eaton and Kortum (2002) model (via mechanisms from Eaton and Kortum, 2001) to quantify its implications for welfare, we are interested in uniqueness of equilibrium, isomorphisms across models, and the implications of scale economies for welfare. Interestingly, although the model in Somale (2014) is dynamic, the balanced growth path is also characterized by the same system of equations as all the models that we consider in this paper. A important contribution of his work is the estimation of the scale elasticity using the implications of the model for home market effects.
In contrast, for a simple case that we can solve analytically, the gains from trade liberalization are higher with scale economies than without.\(^6\) Formally, the opposite consequences of scale effects on the gains from trade and on the gains from trade liberalization come from the fact that when we compute gains from trade we take trade shares (from the data) as given, while when we compute gains from trade liberalization we allow trade shares to endogenously respond to the decline in trade costs. The logic here is similar to the one in the standard one-industry gravity model, where ACR gains from trade decrease as the trade elasticity increases since this makes it easy to substitute foreign for domestic goods when we move to autarky, while a higher trade elasticity leads to higher gains from trade liberalization since it allows for a stronger response of trade shares to the decline in trade costs (see Melitz and Redding, 2014). Similar forces operate in our model at the level of specialization across industries, where stronger scale economies imply that moving back to autarky is easier while trade liberalization leads to larger specialization and higher gains.

### 2.2 A Ricardian Model with Marshallian Externalities

The basis of our model is a multi-industry EK model such as that developed by Costinot, Donaldson and Komunjer (2012, henceforth CDK), but extended to allow for Marshallian externalities. Formally, there are \(N\) countries indexed by \(n, i\) and \(l\), and \(K\) industries or sectors indexed by \(k\). Each industry is composed of a continuum of goods or varieties \(\omega \in \Omega \equiv [0, 1]\). The only factor of production is labor, which is immobile across countries and perfectly mobile within a country. We denote \(\bar{L}_i\) and \(w_i\) as the labor endowment and wage level in country \(i\).

Each country has a representative consumer with upper-tier Cobb-Douglas preferences and lower-tier CES preferences. In particular, utility in country \(i\) is given by

\[
U_i = \prod_{k=1}^{K} \left( \int_{\Omega} Q_{i,k}(\omega)^{(\sigma-1)/\sigma} d\omega \right)^{\sigma/(\sigma-1)} \beta_{i,k}
\]

(2.1)

where \(\beta_{i,k} \in (0, 1)\) with \(\sum_{k=1}^{K} \beta_{i,k} = 1\), and \(Q_{i,k}(\omega)\) is the quantity of good \(\omega\) in industry \(k\).

---

\(^6\) Our expression for the gains from trade for the multi-industry model with scale economies is a generalization of the formula for the gains from trade derived in Arkolakis, Costinot and Rodríguez-Clare (2012) and explored more fully in Costinot and Rodríguez-Clare (2015). Our generalization extends this formula to the case in which economies of scale are weaker than those in the standard multi-industry Krugman or Melitz-Pareto models.
that is consumed in country $i$. Demand in industry $k$ of country $i$ — henceforth simply industry $(i, k)$ — for good $\omega$ is then given by

$$Q_{i,k}(\omega) = p_{i,k}(\omega)^{-\sigma}[P_{i,k}]^{\sigma-1}X_{i,k},$$  \hspace{1cm} (2.2)

where $p_{i,k}(\omega)$ is the price of good $\omega$ in industry $(i, k)$, $P_{i,k} = (\prod_{i} p_{i,k}(\omega)^{1-\sigma} d\omega)^{1/(1-\sigma)}$ is the corresponding industry-level price index, and $X_{i,k} = \beta_{i,k} X_{i}$ with $X_{i}$ denoting total spending in country $i$. The price index in country $i$ is $P_{i} = \tilde{\beta}_{i} \prod_{k=1}^{K} [P_{i,k}]^{\alpha_{k}}$, where $\tilde{\beta}_{i}$ is a constant.

The production technology exhibits constant or increasing returns to scale due to national external economies of scale at the industry level (i.e., Marshallian externalities). In particular, labor productivity for good $\omega$ in industry $(i, k)$ is $z_{i,k}(\omega)L_{i,k}$, where $z_{i,k}(\omega)$ is an exogenous productivity parameter, $L_{i,k}$ is the total labor allocated to industry $(i, k)$, and $\phi_{k}$ is the industry-specific parameter that governs the strength of Marshallian externalities. We model $z_{i,k}(\omega)$ as in EK: $z_{i,k}(\omega)$ is independently drawn from a Fréchet distribution with shape parameter $\theta_{k}$ and scale parameter $T_{i,k}$.

Trade costs are of the standard iceberg type, so that delivering a unit of the industry-$k$-good from country $i$ to $n$ requires shipping $\tau_{ni,k} \geq 1$. Moreover, $\tau_{ni,k} = 1$ for all $i$ and $\tau_{nl,k} \leq \tau_{ni,k} \tau_{ld,k}$ for all $n$, $l$, and $i$.

There is perfect competition, and the positive effect of industry size on productivity, $L_{i,k}^{\phi_{k}}$, is external to the firm. Thus, firms take as given both prices and unit costs, which are given by $c_{ni,k}(\omega) = \tau_{ni,k} w_{i} z_{i,k}(\omega)L_{i,k}^{\phi_{k}}$. This implies that

$$p_{ni,k}(\omega) = c_{ni,k}(\omega).$$  \hspace{1cm} (2.3)

Since consumers can shop for the best deal around the world, prices must satisfy

$$p_{n,k}(\omega) = \min_{1 \leq i \leq I} \{ p_{ni,k}(\omega) \}.$$  

Let $X_{ni,k}$ denote the total expenditure of country $n$ on goods in industry $(i, k)$, and let $\lambda_{ni,k} = X_{ni,k}/\sum_{l=1}^{N} X_{nl,k}$ denote industry-level bilateral trade shares. Following the procedure in EK yields

$$\lambda_{ni,k}^{PC}(\omega, L) = \frac{S_{i,k}^{PC} L_{i,k}^{\alpha_{k}} (w_{i} \tau_{ni,k})^{\varepsilon_{k}^{PC}}}{\sum_{l} S_{i,k}^{PC} L_{i,k}^{\alpha_{k}} (w_{l} \tau_{nl,k})^{\varepsilon_{k}^{PC}}},$$  \hspace{1cm} (2.4)

where $S_{i,k}^{PC} \equiv T_{i,k}$, $\alpha_{k}^{PC} \equiv \theta_{k} \phi_{k}$, $\varepsilon_{k}^{PC} \equiv \theta_{k}$, $w \equiv (w_{1}, ..., w_{N})$ is the vector of wages,
\[ L \equiv \{L_{i,k}\}_{i=1,...,N;k=1,...,K} \] is the matrix of labor allocations. We use the \( PC \) superscript to emphasize that these are variables corresponding to the case of perfect competition (PC). This will be useful later on when we draw comparisons between the case of perfect and monopolistic competition.

At this point we are ready to introduce industry and labor market clearing conditions. Due to Marshallian externalities, it is possible that some industries have zero aggregate labor allocations, which is different from the standard CDK model. Therefore, we need to be careful when formulating the market clearing conditions in our model. By analogy with the standard Ricardian model, we require that in equilibrium the revenue per worker in any sector \((i,k)\) does not exceed the wage:

\[
G_{i,k}^{PC}(w, L) \equiv w_i - \frac{1}{L_{i,k}} \sum_n \lambda_{ni,k}^{PC}(w, L) \beta_{n,k} w_n \bar{L}_n \geq 0.
\] (2.5)

Then the equilibrium labor allocation in sector \((i,k)\) should satisfy the following complementary slackness condition:

\[
L_{i,k} \geq 0, \quad G_{i,k}^{PC}(w, L) \geq 0, \quad L_{i,k} G_{i,k}^{PC}(w, L) = 0.
\] (2.6)

Note that the last equality is the standard industry clearing condition from CDK. And since in that model the equilibrium labor allocations are strictly positive for all \((i,k)\), the first and second inequalities above are automatically satisfied in any equilibrium.

The labor-market clearing condition for any country \(i\) is just the same as in CDK:

\[
\sum_k L_{i,k} = \bar{L}_i.
\] (2.7)

The equilibrium of the economy is a \((w, L) \in \mathbb{R}_+^N \times \mathbb{R}_+^{NK}\) such that (2.6) holds for all \((i,k)\) and (2.7) holds for all \(i\). We denote the set of equilibria by \(\Upsilon\).

### 2.2.1 Characterizing Equilibrium

To characterize the equilibrium we proceed in two steps: we first characterize the equilibrium labor allocations given wages, and then we characterize wages that satisfy labor market clearing given the corresponding equilibrium allocations.

\footnote{In the next section we show that identical complementary slackness conditions arise from free entry in the context of a Krugman monopolistic-competition model.}
Two-Step Equilibrium Definition. The equilibrium labor allocations for some wage vector \( w \in \mathbb{R}^N_{++} \) are given by \( L \in \mathbb{R}^N_+ \) that satisfy (2.6) for all \((i,k)\). Let \( \mathcal{L}(w) \) be the set of such equilibrium allocations. An equilibrium of the economy is a wage vector \( w \in \mathbb{R}^N_{++} \) such that there exists an element \( L(w) \in \mathcal{L}(w) \) such that \( (w,L(w)) \in \Upsilon \).

Note that given wages, for each industry \( k \) we have a system of \( N \) nonlinear complementary slackness conditions in \( L_{i,k} \) for \( i = 1, ..., N \). For the first step we exploit the fact that this system is independent across \( k \). Thus, it is convenient to introduce some additional notation.

Interior, Corner and Complete Specialization Allocations. Let \( L_k \) be the labor allocation vector and \( \mathcal{L}_k \) be the set of such equilibrium allocations for each \( k \). An allocation \( L_k \) is an interior allocation if \( L_{i,k} > 0 \) for all \( i \); an allocation \( L_k \) is a corner allocation if \( L_{i,k} = 0 \) for at least one \( i \); and an allocation \( L_k \) is a complete specialization allocation if there is a unique \( i^*(k) \) such that \( L_{i,k} = 0 \) for all \( i \neq i^*(k) \).

2.2.1.1 Step 1: Equilibrium Labor Allocations

Before proceeding, it is important to note that our results for the case \( \alpha_k = 1 \) for some \( k \) will require an additional assumption on the matrix of trade costs:

Assumption 1. Matrix

\[
\begin{pmatrix}
\tau_{-\varepsilon_k}^{PC} & \cdots & \tau_{-\varepsilon_k}^{PC} \\
\tau_{11,k} & \cdots & \tau_{1N,k} \\
\vdots & & \vdots \\
\tau_{N1,k} & \cdots & \tau_{NN,k}
\end{pmatrix}
\]

is non-singular.

We will be explicit about where this assumption is used. Note also that this matrix is not invertible if trade is free (i.e., \( \tau_{ni,k} = 1 \) for all \( n \) and \( i \)). Given the previous definitions, we are now ready to state our first Proposition.

Proposition 1. If either (a) \( \alpha_k < 1 \), or (b) \( \alpha_k = 1 \) and Assumption 1 holds, then the set \( \mathcal{L}_k(w) \) is a singleton; if \( \alpha_k > 1 \), then the set \( \mathcal{L}_k(w) \) contains multiple allocations, including (but not necessarily limited to) one for each complete specialization allocation. Furthermore, the unique allocation in \( \mathcal{L}_k(w) \) is an interior allocation if \( \alpha_k < 1 \), while it may be an interior or a corner allocation if \( \alpha_k = 1 \).

\[^8\text{Note that there are many complete specialization allocations. For instance, it could be the case that production in industry 1 is concentrated solely in country 1 (i(1) = 1) or in country 2 (i(1) = 2), and so on.}\]
In the rest of this subsection, we provide a sketch of the proof — the full proof is in the Appendix. From the complementary slackness condition in (2.6), for each industry \( k \) and given wages we have a system of \( N \) non-linear equations in \( L_{i,k} \) for \( i = 1, \ldots, N \). To proceed, we suppress the sub-index \( k \) and introduce the following notation: 

\[
\begin{align*}
  x_i &\equiv w_i L_{i,k}, \\
  a_{ni} &\equiv S_{i,k}^{PC} (w_i \tau_{ni,k})^{-\alpha_k^{PC}} w_i^{-\alpha_k^{PC}}, \\
  \alpha &\equiv \alpha_k^{PC} \text{ and } b_n \equiv \beta_{n,k} w_n \bar{L}_n. 
\end{align*}
\]

Using these definitions and letting \( x \equiv (x_1, \ldots, x_N) \), we can write the key function \( G_{i,k}^{PC}(w, L) \) as

\[
G_i(x) \equiv 1 - \sum_n a_{ni} x_i^{\alpha-1} b_n. 
\]

(Note that we can ignore the argument \( w \) here since it is for now given.) The complementary slackness conditions that determine the equilibrium labor allocations for some wage vector are

\[
x_i \geq 0, \quad G_i(x) \geq 0, \quad x_i G_i(x) = 0, \quad i = 1, \ldots, N. \tag{2.8}
\]

We separately consider three cases: \( \alpha < 1, \alpha = 1, \text{ and } \alpha > 1 \).

2.2.1.1 Case \( \alpha < 1 \). Here we provide a sketch of the proof that, if \( \alpha < 1 \), then the set \( L(w) \) is a singleton and corresponds to an interior allocation.

First, we argue that a corner allocation cannot satisfy (2.8). This follows from the fact that \( \lim_{x_i \to 0} G_i(x) = -\infty \) implying that, if \( x_i = 0 \) for some \( i \), then the inequality \( G_i(x) \geq 0 \) cannot be satisfied.

Next, we prove existence of a solution to \( G(x) = 0 \) with \( x > 0 \), implying an interior solution to (2.8). The system \( G(x) = 0 \) for \( x > 0 \) can equivalently be written as the fixed point problem \( F(x) = x \), with

\[
F_i(x) \equiv \sum_n a_{ni} x_i^{\alpha} b_n. 
\]

We show there is a compact set \( \Omega \equiv [\varepsilon, \varepsilon]^N \) with \( 0 < \varepsilon < \bar{\varepsilon} \) such that, if \( x \in \Omega \), then \( F(x) \in \Omega \).

Given continuity of \( F(x) \), we then apply Brouwer’s fixed point theorem to establish existence of a solution to \( F(x) = x \). The restriction \( \alpha < 1 \) is critical in this step.

Finally, we establish that there is a unique solution to the system \( G(x) = 0 \) for \( x > 0 \) by showing that the Jacobian of the equivalent system:

\[
x_i^{1-\alpha} - \sum_n a_{ni} x_i^{\alpha} b_n = 0, \quad i = 1, \ldots, N
\]
is a $P$-matrix for all $x > 0$, and then by applying Theorem 4 from Gale and Nikaido (1965).

2.2.1.1.2 Case $\alpha = 1$. Here we provide a sketch of the proof that, if $\alpha = 1$, then the set $\mathcal{L}(w)$ contains a unique element, which may be an interior or a corner allocation.

The case with $\alpha = 1$ is qualitatively different from the case with $\alpha < 1$, because for $\alpha = 1$ we cannot generally find a compact set bounded away from zero axes, which is mapped by $F(x)$ into itself. In other words, for $\alpha = 1$, $F(x)$ operates on a set which generally includes points $x_i = 0$. As a result, the fixed point of $F(x)$ can generally be a corner allocation. Then, to prove that such corner allocation is a solution of (2.8), we need to prove explicitly that the inequality $G(x) \geq 0$ is satisfied. This is different from the case with $\alpha < 1$, where the fixed point of $F(x)$ is an interior allocation and, so, it automatically satisfies the inequality $G(x) \geq 0$ by turning it into equality. So, in the case of $\alpha = 1$ we need to explicitly work with the nonlinear complementary problem (henceforth NCP) (2.8).

In proving existence and uniqueness of solution of (2.8) we proceed in three steps. We consider a modified problem:

$$x_i \geq \delta, \quad G_i(x) \geq 0, \quad (x_i - \delta)G_i(x) = 0, \quad i = 1, \ldots, N, \quad (2.9)$$

and we first prove that there exists a unique solution of this problem for any $\delta > 0$. In the second step we establish existence of solution to (2.8) by showing that there exists a limit of solutions of (2.9) as $\delta \to 0$ and that this limit solves (2.8). Finally, in the third step we prove that the solution of (2.8) is unique.

In proving existence and uniqueness of a solution of (2.9) we rely on results from the literature on nonlinear complementary slackness problems. In particular, we use results from the book by Facchinei and Pang (2003). The key property for existence in case of $\delta > 0$ is continuity of $G$ on some open set containing the set in which we look for a solution. There is no such set in case of $\delta = 0$, because any such open set would necessarily contain point $x = 0$ inside of it, and function $G$ is not continuous in this point. For uniqueness in case of $\delta > 0$ we use the fact that the Jacobian of $G$ is a positive definite matrix for all points in a relevant open set. Invertibility of the trade costs matrix — Assumption 1 — is key here.

To show that the limit of solutions of (2.9) as $\delta \to 0$ is a solution of (2.8), we use the Arzelà-Ascoli theorem to establish that the family of functions $G$ and $F$ with $\delta > 0$ has a sequence which uniformly converges (as $\delta \to 0$) to the corresponding functions with $\delta = 0$.

A is a $P$-matrix if all its principal minors are positive.
And, hence, the limit of solutions of (2.9) is a solution of (2.8).

Finally, to show uniqueness of solution of (2.8), we employ the facts that \( G_i \) is a concave function for any \( i \) and that any solution \( x \) of (2.8) satisfies \( \sum_i x_i = \sum_i b_i \). These facts imply that the set of solutions of (2.8), given by

\[
\left\{ x \in \mathbb{R}^N_+ : \sum_i x_i = \sum_i b_i \text{ and } G(x) \geq 0 \right\},
\]

is a convex set. Then, if there are two distinct solutions of (2.8), there is a continuum of solutions of (2.8), which, as we show, is impossible. Here we again use Assumption 1, which implies that the Jacobian of any non-empty subset of functions \( G \) is a positive definite matrix.

#### 2.2.1.3 Case \( \alpha > 1 \).

Here we illustrate that multiple equilibria arise in the context of \( \alpha > 1 \). Notice, for instance, that any complete specialization allocation is an equilibrium allocation. To illustrate this, set choose some \( i^* \) and set \( x_i = 0 \) for \( i \neq i^* \). Then, for \( i^* \)

\[
G_{i^*}(x) = 1 - \sum_n \frac{a_{ni^*} x_{i^*}^{\alpha-1}}{a_{ni^*} x_{i^*}^{\alpha}} b_n = 1 - x_{i^*}^{-1} \sum_n b_n = 0,
\]

so

\[
x_{i^*} = \sum_n b_n.
\]

For \( i \neq i^* \), \( G_i(x) = 1 \geq 0 \), so that the conditions for the NCP (2.8) are satisfied.

Of course, these are only a few heuristic examples and there are many other possible equilibrium allocations within this setting. However, characterizing the complete set of equilibria for this case is not the focus of our analysis. For a more rigorous treatment grappling with the implications of multiple equilibria under different market structures within the standard framework with Marshallian externalities we refer the reader to Lyn and Rodríguez-Clare (2013a,b).

#### 2.2.1.2 Step 2: Equilibrium Wages

In what follows, we restrict the analysis to the case \( \alpha \leq 1 \). For this case, Proposition 1 establishes that the solution of the system of nonlinear complementary slackness conditions
(2.6) determines a function from wages to labor allocations, \( L(w) \), for \( w \in \mathbb{R}^N_{++} \). Letting

\[
Z_i(w) \equiv \sum_k L_{i,k}(w) - \bar{L}_i,
\]

be the excess labor demand in country \( i \) defined for all \( w \in \mathbb{R}^N_{++} \) and letting \( Z(w) \equiv (Z_1(w), ..., Z_N(w)) \), the labor-market clearing conditions for all countries can be written simply as

\[
Z(w) = 0.
\]

Proposition 2. If either (a) \( \alpha_k < 1 \), or (b) \( \alpha_k = 1 \) and Assumption 1 holds, then there exists a vector of wages \( w \in \mathbb{R}^N_{++} \) that satisfies (2.11).

Proof. To establish existence of solution of (2.11), it suffices to show that the following properties as outlined in Proposition 17.B.2 in Mas-Colell, Whinston and Green (1995, MWG) are satisfied: (i) \( Z(w) \) is continuous; (ii) \( Z(w) \) is homogeneous of degree zero; (iii) \( w \cdot Z(w) = 0 \) for all \( w \) (Walras’ law); (iv) There is an \( A > 0 \) such that \( Z_i(w) > -A \) for all \( i \) and \( w \); (v) If \( w^s \to w \) as \( s \to \infty \), where \( w \neq 0 \) and \( w_i = 0 \) for some \( i \), then \( \max \{ Z_1(w^s), ..., Z_N(w^s) \} \to \infty \) as \( s \to \infty \). Properties (ii)-(iv) are immediate. The proof of properties (i) and (v) are in the Appendix.

In what follows, we first provide sufficient conditions for a unique equilibrium in the case of two countries \( (N = 2) \). Later we show how our framework can be readily extended to any finite number of countries, and discuss additional complexities which arise in such a setting.

Proposition 3. Assume that \( N = 2 \) and either (a) \( \alpha_k < 1 \), or (b) \( \alpha = 1 \) and Assumption 1 holds. Then there exists a unique (normalized) vector of wages \( w \in \mathbb{R}^N_{++} \) that satisfies (2.11).

We prove this proposition by showing that the labor excess demand function \( Z(w) \) has the gross substitutes property for \( N = 2 \). Uniqueness of solution then follows from Proposition 17.F.3 from MWG.

Interestingly, except for the two-country case, our excess labor demand system does not, in general, satisfy the gross-substitutes property, and so this property can no longer be invoked for establishing a unique equilibrium for \( N > 2 \). With industry-level externalities one has to contend with additional complications that arise when there are more than two countries. In particular, while these externalities act to reinforce the gross substitutes property when there
are two countries, the same is not necessarily true for three or more countries. For instance, a rise in the wage in one country, say country 1, may reduce the demand for labor there, while at the same time raising the demand for labor in another country, say country 2, which is so far consistent with the gross substitutes property. The complexities arise from the fact that the increased labor demand in country 2 can generate productivity effects that can lead to increased exports to a third country, say country 3, which can, in turn, result in a fall in the demand for labor there. In other words, a rise in wages in country 1 can result in a fall in the demand for labor in country 3, thereby, violating the gross substitutes property.

While we have not yet been able to prove our uniqueness result in Proposition 3 for the case $N > 2$, extensive simulations indicate that the Jacobian of the corresponding system of equations is a $P1$-matrix. This implies that the equilibrium is, in fact, unique. Moreover, if we assume the production of an "outside" good industry, as is typically done, then uniqueness obtains when extended to allow for any finite number of countries. That is, once wages are pinned down by the "outside" good industry, the proof from Proposition 1 — which is valid for any finite $N$ — implies a unique allocation of labor across industries.

In the section immediately following, we first outline a version of a multi-industry Krugman model with possibly different elasticities of substitution across varieties from the same country than across varieties from different countries, and then use this as a means of establishing a link between the underlying equilibrium structure of a model with monopolistic competition and firm-level economies of scale with that of our Ricardian model with industry-level economies of scale.

## 2.3 Linking Krugman and Marshall

In this subsection we demonstrate that the underlying equilibrium system of our Ricardian model with perfect competition and industry-level external economies is identical to that of the Krugman model with monopolistic competition and firm-level economies of scale. We finish the section by briefly noting other isomorphisms.

We start by outlining a version of a multi-industry Krugman model which allows for an added layer of product differentiation — differences in elasticities of substitution across domestic and internationally produced varieties within a particular industry. Consistent with our previous framework, we maintain the same notation for countries, industries, goods, labor

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10 Matrix $A$ is a $P1$-matrix if all its principal minors are non-negative and exactly one principal minor is zero.
endowments and wages, and we assume labor is the only factor of production. Preferences are multi-tiered: upper-tier Cobb-Douglas across industries with weights $\beta_{i,k}$, CES across country bundles within an industry with elasticity $\eta_k$, and CES across goods within a country bundle with elasticity $\sigma_k$.

Let $A_{i,k}$ be the exogenous productivity in $(i,k)$ which is common across firms in that industry, $F_{i,k}$ denote the fixed costs (in terms of labor) associated with the production of any variety in $(i,k)$, and $M_{i,k}$ the measure of goods produced in $(i,k)$. Total expenditures by country $n$ on country $i$ varieties of industry $k$ are $X_{ni,k} = (P_{ni,k}/P_{n,k})^{1-\eta_k} X_{n,k}$, where $P_{ni,k} = M_{i,k}^{1/(1-\sigma_k)} (\bar{m}_k w_{i\tau_{ni}}/A_{i,k})$ is the price index in country $n$ of country $i$ varieties of industry $k$, $\bar{m}_k = \sigma_k / (\sigma_k - 1)$ is the CES mark-up, and $P_{n,k}^{1-\eta_k} = \sum_i P_{ni,k}^{1-\eta_k}$. As before, we have $X_{n,k} = \beta_{n,k} w_{nL_n}$.

Variable profits in $(i,k)$ are simply total industry revenues divided by $\sigma_k$. Letting $\Pi_{i,k}$ be total profits (net of fixed costs) in industry $(i,k)$, we then have $\Pi_{i,k} = \sum_n X_{ni,k}/\sigma_k - w_i M_{i,k} F_{i,k}$. Free entry entails $\Pi_{i,k} = 0$. Combined with the previous expression this implies that total revenues in industry $(i,k)$ equal total wage payments in that industry, hence $\sum_n X_{ni,k} = w_i L_{i,k}$. We must then have $M_{i,k} = L_{i,k}/\sigma_k F_{i,k}$. Combining the previous expressions we get

$$\lambda_{ni,k}^{MC}(w, L) = \frac{S_{i,k}^{MC} L_{i,k}^{\alpha_{MC}^{MC}} (w_i \tau_{ni,k})^{-\varepsilon_{MC}^{MC}}}{\sum_i S_{i,k}^{MC} L_{i,k}^{\alpha_{MC}^{MC}} (w_i \tau_{ni,k})^{-\varepsilon_{MC}^{MC}}},$$

where $S_{i,k}^{MC} \equiv A_{i,k}^{\eta_k-1} F_{i,k}^{\frac{1}{1-\sigma_k}}$, $\alpha_{MC}^{MC} \equiv \frac{\eta_k - 1}{\sigma_k - 1}$, $\varepsilon_{MC}^{MC} \equiv \eta_k - 1$, and $MC$ superscript is used to emphasize that these variables correspond to the case of monopolistic competition (MC). Note that, if $\sigma_k = \eta_k$, then the model collapses to the standard multi-industry Krugman model. If varieties of all countries are perfect substitutes (i.e., $\sigma_k \to \infty$ for all $k$), then the model collapses to the multi-sector Armington model with no Marshallian externalities since then $\alpha_{MC}^{MC} \to 0$, which itself is isomorphic to the multi-sector EK model.

To deal with the possibility of corner labor allocations under MC, i.e., $L_{i,k} = 0$, we require that profits per firm in industry $(i,k)$ be non-positive. This then leads to exactly the same expressions as in (2.5) and (2.6) (with superscript $PC$ replaced by $MC$). Formally, the equilibrium system in both the PC and MC models is given by the following common mathematical system that we use to solve for $w$ and $L$:

$$L_{i,k}(w, L) = L_{i,k} \text{ for all } i, k,$$
and
\[ \sum_k L_{i,k}(w, L) = \bar{L}_i, \]
where
\[ L_{i,k}(w, L) \equiv \frac{1}{w_i} \sum_n \lambda_{ni,k}(w, L) \beta_{n,k} w_n \bar{L}_n, \]
and
\[ \lambda_{ni,k}(w, L) \equiv \frac{S_{ni,k} L_{i,k}^{\epsilon_k \psi_k} (w_i \tau_{ni,k})^{-\epsilon_k}}{\sum_l S_{nl,k} L_{l,k}^{\epsilon_k \psi_k} (w_l \tau_{nl,k})^{-\epsilon_k}}. \]

Here \( \epsilon_k \) and \( \psi_k \equiv \alpha_k / \epsilon_k \) are the trade elasticity and scale elasticity in sector \( k \), respectively. In case of MC, scale elasticity is given by \((\sigma_k - 1)^{-1}\), while trade elasticity is \((\eta_k - 1)\).

It is then immediately evident that the definition of equilibrium under monopolistic competition is identical to the one under perfect competition. Hence, one can simply replace the superscript \( PC \) with \( MC \), and all the results under monopolistic competition then follow in a straightforward way. In Appendix C.1 we show that the above system in \( w \) and \( L \) determines the equilibrium in the Melitz model with a Pareto distribution, if fixed export costs are paid in labor of the importing country, and \( \eta_k \) can differ from \( \sigma_k \) (as in Feenstra et al., 2014). Also, in an Appendix available upon request, we show the isomorphis to the Krugman model with \( \eta_k = \sigma_k \) in all \( k \) but allowing for heterogeneity (modeled FrÁElch as in Hsieh, Hurst, Jones and Klenow, 2013) in worker ability across sectors as in Galle, Rodríguez-Clare and Yi (2014).

### 2.4 Gains from Trade: Ricardo, Krugman and Marshall

In this section we conduct a preliminary exploration of the implications of scale economies for the welfare effects of trade. We restrict the analysis to the case in which \( \alpha_k \leq 1 \) for all \( k \), so that there is a unique equilibrium. We take advantage of the unified model developed in the previous sections, noting that the multi-industry EK or Armington models corresponds to the case with \( \alpha_k = 0 \) for all \( k \), while the standard multi-industry Krugman model corresponds to the case with \( \alpha_k = 1 \) for all \( k \).

To obtain an ACR-type formula for the gains from trade, we assume that the equilibrium is interior so that all trade shares and labor allocations are strictly positive. Given Proposition 1, this assumption is not restrictive for the case with \( \alpha_k < 1 \) for all \( k \). If \( \alpha_k = 1 \), then the
possibility of corner allocations arises, in which case some of the formulas below are no longer valid.

Dropping the PC and MC superscript notation, real wages in both frameworks can be written as

$$\frac{w_n}{P_n} = \bar{p} \prod_k S_{n,k}^{\beta_n,k} / \varepsilon_k \cdot \prod_k \lambda_{nn,k}^{-\beta_n,k} / \varepsilon_k \cdot \prod_k (L_{n,k})^{\beta_n,k} \psi_k,$$

where $\bar{p}$ is some constant, and $\psi_k \equiv \alpha_k / \varepsilon_k$ is the scale elasticity in industry $k$, which takes value $\phi_k$ under PC and $(\sigma_k - 1)^{-1}$ under MC. Using the standard hat notation ($\hat{x} = x'/x$), given some change in trade costs $\{\hat{\tau}_{ni,k}\}$, the changes in real wages are then

$$\frac{\hat{w}_n}{\hat{P}_n} = \prod_k \hat{\lambda}_{nn,k}^{-\beta_n,k} / \varepsilon_k \cdot \prod_k (\hat{L}_{n,k})^{\beta_n,k} \psi_k.$$

The first term on the RHS of this expression is the standard multi-industry formula for gains from trade (with upper-tier Cobb-Douglas preferences), while the second term is an adjustment for Marshallian externalities under PC or entry effects along with love of variety under MC.

Following ACR, we define the gains from trade as the absolute value of percentage change in real income as we move from the observed equilibrium to autarky,

$$GT_n \equiv 1 - \frac{w_n^A/P_n^A}{w_n/P_n}.$$

We compute $GT_n$ by applying (2.12) and noting that for the move back to autarky we have $\hat{\lambda}_{nn,k} = 1/\lambda_{nn,k}$, and $\hat{L}_{n,k} = \beta_n,k/r_{n,k}$, where $r_{n,k} \equiv L_{n,k}/L_n$ denotes the industry revenue (or employment) shares in the observed equilibrium. Using $e_{n,k} \equiv X_{n,k}/X_n$ for observed industry expenditure shares, this leads to a formula for the gains from trade that depends only on the country’s observables $\lambda_{nn,k}$, $e_{n,k}$ and $r_{n,k}$ as well as the trade elasticity, $\varepsilon_k$, and the scale elasticity $\psi_k$,

$$GT_n = 1 - \prod_k \lambda_{nn,k}^{e_{n,k}/\varepsilon_k} \left( \frac{e_{n,k}}{r_{n,k}} \right)^{e_{n,k} \psi_k}.$$

The expression for the gains from trade in the standard PC model with no scale economies obtains from (2.13) by setting $\psi_k = 0$ for all $k$,

$$GT_n = 1 - \prod_k (\lambda_{nn,k})^{e_{n,k}/\varepsilon_k}.$$
Given trade shares and trade elasticities, the implications of scale economies vis-à-vis the standard multi-industry PC model depends on the term \( \prod_k \left( \frac{e_{nk}}{r_{nk}} \right)^\psi_k \). In the case in which the scale elasticity is the same across industries \( (\psi_k = \psi \text{ for all } k) \), we show that \( \prod_k \left( \frac{e_{nk}}{r_{nk}} \right)^\psi_k \) is always higher than one, which implies that economies of scale actually reduce the gains from trade. Moreover, we show that the gains from trade are also decreasing in \( \psi \). More generally, if \( \psi_k \) varies across \( k \), we need to think about the correlation between \( \frac{e_{nk}}{r_{nk}} \) and \( \psi_k e_{nk} \), which, in turn, is dependent on whether or not countries have a comparative advantage or a comparative disadvantage in industries with strong economies of scale. For instance, if \( \frac{e_{nk}}{r_{nk}} \) is negatively correlated with \( \psi_k e_{nk} \), then country \( n \) tends to specialize in industries with strong Marshallian externalities, thereby, potentially leading to larger gains from trade than in the absence of these externalities. The reverse, however, holds for a country in which \( \frac{e_{nk}}{r_{nk}} \) is positively correlated with \( \psi_k e_{nk} \), that is, a country with a comparative advantage in industries with weak economies of scale. In principle, such a country could possibly lose from opening up to international trade. We prove that this cannot be the case if \( \alpha_k \leq 1 \) for all \( k \) by showing that trade always implies lower industry price indexes, and, in turn, a lower overall price index than in autarky. We summarize these results in the following Proposition.

**Proposition 4.** Assume \( \alpha_k \leq 1 \). (i) All countries gain from trade. (ii) Assume that the parameters are such that the equilibrium is interior. Then the gains from trade are given by Equation (2.13). (iii) Assume again that the parameters are such that the equilibrium is interior and also that \( \psi_k = \psi \). Then the gains from trade are lower with economies of scale \( (\psi > 0) \), and these gains are decreasing in the extent of scale economies \( \psi \).

**Proof.** See the Appendix.

As a special application of these results, consider the standard multi-industry Krugman model, where the gains from trade are obtained from (2.13) by setting \( \psi_k = 1/\varepsilon_k \) for all \( k \),

\[
GT_n = 1 - \prod_k \left( \lambda_{nn,k} \left( \frac{e_{nk}}{r_{nk}} \right) \right)^{e_{nk}/\varepsilon_k}.
\]

Analogous to our analysis above, if \( \varepsilon_k = \varepsilon \) and \( \psi_k = \psi \) for all \( k \) then the gains from trade are lower for the standard multi-industry Krugman model (for which \( \psi = 1/\varepsilon \)) relative to

\[\text{We show that in the region of uniqueness there are always gains from trade. However, one can readily verify that within our setting the usual “pathologies” of multiple equilibria and the possibility of losses from trade (as illustrated by Ethier (1982a)) arise for } \alpha > 1. \text{ See Lyn and Rodriguez-Clare (2013a,b) for a rigorous treatment grappling with these issues.} \]
cases with $\psi < 1/\varepsilon$. More generally, if $\varepsilon_k$ varies across industries, then things again depend

the correlation between $\varepsilon_{n,k}$ and $\varepsilon_k$. Thus, a similar argument to the one above regarding

the correlation between $\varepsilon_{n,k}$ and the strength of Marshallian externalities applies. The only

distinction is that under monopolistic competition the scale effects arise from the firm entry.

2.4.1 A First Look at Gains: A Simple Case with Some Numbers

To illuminate the links across frameworks with different scale elasticities, it is useful to

consider the case of two industries and two mirror-image countries, and to consider a trade

liberalization ($\hat{\tau} < 1$). Of course, for mirror-image countries we know that wages will be the

same, so we can just normalize wages to one ($w = 1$) in both countries. For ease of exposition

we index countries $i = H, F$, where $H$ and $F$ represent Home and Foreign, respectively. Let

$\bar{L} = 2$, $\beta_{i,k} = 1/2$ for all $(i, k)$, and let $S_{H,1} = S_{F,2} = S$ and $S_{H,2} = S_{F,1} = 1$, for $S > 1$.

Hence, Home has the comparative advantage in industry 1, and Foreign in industry 2. We

assume that $\varepsilon_k = \varepsilon$ and $\psi_k = \psi$ for $k = 1, 2$.

![Figure 2.1: Industry specialization and economies of scale](image)

We are interested in understanding the implications of the scale elasticity, $\psi$, for industry

specialization and the gains from trade. In the spirit of the ACR approach, we first illustrate

that given trade shares, the gains from trade are decreasing in $\psi$. We then show that the
conclusion is reversed once we consider a trade liberalization exercise in which trade shares respond endogenously to these scale effects as we lower trade costs. There we find that the gains from trade liberalization are increasing in $\psi$.

For the ACR-type approach, the gains for Home are simply

$$GT_H = 1 - \left( \frac{\lambda_{HH,1}^{1/2} \cdot \lambda_{HH,2}^{1/2}}{r_{H,1} \cdot (1 - r_{H,1})} \right)^{\psi/2}.$$  

The term $\frac{1/2}{r_{H,1}} \cdot \frac{1/2}{1-r_{H,1}}$ is minimized at $r_{H,1} = 1/2$, and specialization according to comparative advantage implies $r_{H,1} > 1/2$, so it follows that this term is higher than one. Thus, given trade shares, gains are lower with scale effects ($\psi > 0$) than without ($\psi = 0$). It is also easy to see that these gains are decreasing in $\psi$.\textsuperscript{12}

\begin{figure}[h]
\centering
\begin{subfigure}{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{figure1a.png}
\caption{Scale economies and comparative advantage}
\end{subfigure}\hspace{0.5cm}
\begin{subfigure}{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{figure1b.png}
\caption{Scale economies and gravity}
\end{subfigure}
\caption{Gains from trade}
\end{figure}

Next we study the gains from trade liberalization, allowing for endogenous responses of trade shares to trade costs. In the terminology of ACR, this corresponds to an “ex-ante analysis” whereas the results for the gains from trade above correspond to an “ex-post analysis.” We set $\theta = 5$ and $\psi = \{0, 0.1, 0.2\}$. The case $\psi = 0.2$ implies $\alpha = 1$, as in the standard multi-industry Krugman model, whereas the case $\psi = 0$ corresponds to the standard multi-industry gravity model without scale economies. The case $\psi = 0.1$ implies $0 < \alpha < 1$

\textsuperscript{12}Given symmetry, the same conclusions hold for Foreign.
and corresponds to an intermediate case. Note that for all these cases we have that in autarky (i.e., if $\tau = \infty$) $L_{i,k} = 1$ for all $(i,k)$. As $\tau$ falls from $\infty$, country $H$ specializes in sector 1 and country $F$ specializes in sector 2, but the degree of specialization will be stronger with $\psi = 0.2$ than $\psi = 0.1$, and with $\psi = 0.1$ than $\psi = 0$, as illustrated in Figure 1. Figure 2 shows the implications for the gains from trade liberalization for each of these three cases. We see that the gains from trade increase with $\psi$. The intuition is simple — countries gain by specializing according to comparative advantage, but the concentration of production also allows for a greater exploitation of scale economies, which, in turn, generates additional efficiency gains.

### 2.5 Concluding Remarks

For almost two centuries since Alfred Marshall’s initial exposition, economists have been intrigued with understanding the implications of external economies of scale for trading economies. The discomfort, however, with the plethora of multiple equilibria and counterintuitive implications in early work relegated external economies to the “back-burner” of the recent trade literature. In this paper we bring this key issue to the forefront by building a quantitative multi-industry trade model with Marshallian externalities that yields a unique equilibrium and a standard gravity equation. Next we show that the underlying structure of our model is isomorphic to that of a multi-industry Krugman model of firm-level economies of scale, and so our results also establish uniqueness within this setting. As such, we unify the literature on increasing returns by catapulting this missing piece to its place alongside the workhorse model with firm-level economies of scale. We then investigate the welfare properties and discuss how the welfare effects of trade differ from the model without economies of scale. Finally, our framework provides a natural platform to assess more fully the quantitative importance of these externalities, a primary agenda of our on-going research.
Chapter 3

Continuity of a Model with a Nested CES Utility Function and Bertrand Competition

3.1 Introduction

Models with a nested constant elasticity of substitution (CES) utility function, heterogeneous firms, and Bertrand competition are becoming popular in the international trade literature. This framework generates variation in markups and serves as a natural way to model the so-called “pricing-to-market” and “incomplete pass-through”. The work by Bernard, Eaton, Jensen and Kortum (2003) is one of the examples of employing this structure. Variable markups in their work lead to the explanation of important facts about exporting firms. Another example is the paper by Atkeson and Burstein (2007) which uses this structure to reproduce many of the important aspects of international price movements. De Blas and Russ (2011) is a more recent example which relies on the nested CES preferences, heterogeneous firms, and Bertrand competition to match a rich set of stylized facts regarding firms’ pricing behavior.

In the papers mentioned above, goods within sectors are assumed to be perfect substitutes. One possible generalization is to consider imperfect substitutes instead. This brings more flexibility into the model and might be crucial for matching real-world data. If one wants to consider imperfect substitutes, she needs to be sure that the resulting model has a unique

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1 A part of this chapter is published in Kucheryavyy (2012).
equilibrium and that this model is continuous as the elasticity of substitution between goods goes to infinity. More precisely, the latter property means that an equilibrium price vector of the model with imperfect substitutes converges to that of the model with perfect substitutes as the elasticity of substitution goes to infinity. Continuity of the model with imperfect substitutes is important to ensure consistency of its outcomes as the elasticity of substitution goes to infinity.

As it turns out, continuity of the model with imperfect substitutes is not a trivial property. For example, Atkeson and Burstein (2008) mention\textsuperscript{2} that they have chosen Cournot competition instead of Bertrand partially because they were not sure if the model with Bertrand competition is continuous. This is a somewhat disturbing fact that one has to avoid models with Bertrand competition for that reason. Indeed, the choice of the form of competition (Cournot versus Bertrand) matters for predictions of a model. This argument goes back to the original paper by Joseph Bertrand (Bertrand, 1883). Among more recent examples is the paper by Eaton and Grossman (1986), who demonstrate that optimal trade and industrial policy under oligopoly crucially depend on market structure.

The purpose of the current paper is to prove formally that the model with imperfect substitutes is continuous at infinity. This proof should provide a foundation for the future use of the model.

Before diving into details, it is worth mentioning about possible reasons for discontinuity. The first reason is that an equilibrium price vector can diverge as the elasticity of substitution goes to infinity. This case cannot be ruled out right away, because, on the one hand, there is no closed-form solution for an equilibrium price vector in the case of imperfect substitutes. On the other hand, in some cases the system of equations which yields an equilibrium price vector does not have a fixed point when the elasticity of substitution is infinite.

The second possible reason for discontinuity is that an equilibrium price vector can converge to a limit which is different from an equilibrium of the model with perfect substitutes. The model with perfect substitutes is a modification of the classical model of Bertrand competition.\textsuperscript{3} In this model firms face discontinuous demand,\textsuperscript{4} while in the model with imperfect substitutes demand is continuous. Therefore, there is no immediate reason to believe that solution of one model converges to solution of the other.

The paper is organized as follows. Section 2 contains setup of the model and a proof that

\textsuperscript{2}See page 2013 in Atkeson and Burstein (2008).
\textsuperscript{3}See, for example, Mas-Colell \textit{et al.} (1995), Chapter 12.C, pages 388-389.
\textsuperscript{4}If a firm sets a price which is higher than that of its competitors, then it sells nothing. Otherwise it either gets the whole market or equally shares the market with other firms.
there exists a unique equilibrium of the model. Section 3 contains a more detailed discussion of possible issues with continuity as well as formal statement of the result about continuity. This result is then proved in Section 4.

3.2 The Model

3.2.1 Environment

Since the continuity issue is the same for both closed and open versions of the model, it is without loss of generality that the current paper focuses on the closed economy version of the model.

The economy is divided into a continuum of sectors indexed by $j \in [0, 1]$. Each sector has a finite number of firms, each of which produces one good. Firms are indexed by $k = 1, 2, \ldots, K_j$, where $K_j$ is the number of firms in industry $j$.

Consumers

There is one representative consumer in the economy whose utility is given by the nested CES function:

$$U = \left[ \int_0^1 Q_j^{(\eta-1)/\eta} \, dj \right]^{\eta/(\eta-1)}, \quad (3.1)$$

where

$$Q_j = \left[ \sum_{k=1}^{K_j} q_{jk}^{(\rho-1)/\rho} \right]^{\rho/(\rho-1)}. \quad (3.2)$$

Here $Q_j$ is the aggregated demand in sector $j$, $q_{jk}$ is the demand for good $k$ from sector $j$, $\rho$ is the elasticity of substitution between goods from the same sector, and $\eta$ is the elasticity of substitution between goods from different sectors. The following assumption is made in line with Atkeson and Burstein (2008):

**Assumption 2.** $1 < \eta < \rho$.

The consumer faces prices $p_{jk}$ on goods $k = 1, 2, \ldots, K_j$ from sectors $j \in [0, 1]$ and spends $X$ on the overall consumption. Her problem is to maximize (3.1) subject to budget constraint $\int_0^1 X_j \, dj = X$, where $X_j = \sum_{k=1}^{K_j} p_{jk} q_{jk}$ is spending on consumption of goods from sector $j \in [0, 1]$. 

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Firms

Producer of good \( k \) from sector \( j \) has constant unit cost of production \( c_{jk} \). Costs of production satisfy the following assumption:

Assumption 3. \( c_{jk} > 0 \) for all \( j \) and \( k \), and producers are ordered by their unit costs so that \( c_{jk} < c_{jl} \) for any \( j \) and \( k < l \).

Producers of goods from each sector \( j \) are involved in Bertrand competition, i.e., they first simultaneously announce prices and then they get profits depending on the sector-level vector of prices. The profit function of the producer of good \( k \) from sector \( j \) is

\[
\pi_{jk}(p_{jk}; p_{j,[-k]}) = (p_{jk} - c_{jk})q_{jk}(p_{jk}; p_{j,[-k]}),
\]

where \( p_{j,[-k]} \) is a vector of prices of firms from sector \( j \) different from \( k \), and \( q_{jk}(p_{jk}; p_{j,[-k]}) \) is firm \( k \)’s demand.

3.2.2 Equilibrium for the case of perfect substitutes

For the case of perfect substitutes (\( \rho \) is infinite) firm \( k \) from industry \( j \) faces the following demand:

\[
q_{jk}(p_{jk}; p_{j,[-k]}) = \begin{cases} 
\frac{X}{p_{jk}} \left( \frac{p_{jk}}{P_{perf}} \right)^{1-\eta}, & \text{if } p_{jk} = \min_l \{p_{jl}\}; \\
0, & \text{otherwise};
\end{cases}
\]

where

\[
P_{perf} = \left[ \int_0^1 \left( \min_l \{p_{jl}\} \right)^{1-\eta} \, dj \right]^{1/(1-\eta)}
\]

is the price index in the economy. With this demand, firm \( k \) chooses price \( p_{jk} \) so as to maximize its profit function (3.3) taking prices of the other firms in sector \( j \) as well as the economy-wide price index \( P_{perf} \) as given. It is straightforward to verify that in equilibrium the lowest cost firm (i.e., firm 1 in each sector \( j \)) gets the total market, while the other firms
sell nothing. One possible set of firms’ equilibrium prices yielding this outcome is

\[ p_{jk} = \begin{cases} \min \left\{ \frac{\eta}{\eta - 1} c_j, c_{j2} \right\}, & \text{if } k = 1; \\ c_{jk}, & \text{if } k \neq 1. \end{cases} \quad (3.5) \]

There are infinitely many other sets of equilibrium prices. For example, when \( \frac{\eta}{\eta - 1} c_{j1} > c_{j2} \), firms with indexes \( k > 2 \) can set any price \( p_{jk} > c_{j2} \) in equilibrium because they get zero profits anyway. This paper focuses on the equilibrium given by (3.5) because the equilibrium of the model with imperfect substitutes converges to it as the elasticity of substitution goes to infinity.

### 3.2.3 Equilibrium for the case of imperfect substitutes

A closed-form solution for the equilibrium pricing strategies cannot be obtained for the case when goods within sectors are imperfect substitutes (i.e., \( \rho \) is finite). Instead one can only formulate an equation in prices and claim that it has a unique solution which constitutes the equilibrium of the model.

For the case of finite \( \rho \), demand for good \( k \) from industry \( j \) is given by:

\[
q_{jk}(p_{jk}, p_{j,-k}) = X \frac{p_{jk}}{P_j} \left( \frac{p_{jk}}{P_j} \right)^{1-\rho} \left( \frac{P_j}{P_{imperf}} \right)^{1-\eta}, \quad (3.6)
\]

where

\[
P_j = \left[ \sum_{k=1}^{K_j} P_{jk}^{1-\rho} \right]^{1/(1-\rho)}
\]

is the price index in sector \( j \), and

\[
P_{imperf} = \left[ \int_0^1 P_j^{1-\eta} dj \right]^{1/(1-\eta)}
\]

is the price index in the economy. Note that demand function (3.6) is continuous in contrast to demand function (3.4) for the case of perfect substitutes, which is discontinuous.
Substituting (3.6) into the profit function (3.3) yields
\[ \pi_{jk}(p_{jk}; p_{j[-k]}) = (p_{jk} - c_{jk})^{\rho} P_j^{(\rho - \eta)} P_{imperf}^{(\eta - 1)} X. \]  
(3.7)

In equilibrium firm \( k \) chooses \( p_{jk} \) so as to maximize (3.7) taking prices of the other firms in sector \( j \) as well as the economy-wide price index \( P_{imperf} \) as given. Differentiating (3.7) with respect to \( p_{jk} \) and collecting terms gives
\[ \frac{\partial \pi_{jk}(p_{jk}; p_{j[-k]})}{\partial p_{jk}} = -\left( p_{jk} - \frac{\varepsilon(s_{jk})}{\varepsilon(s_{jk}) - 1} c_{jk} \right) (\varepsilon(s_{jk}) - 1) (p_{jk})^{\rho - 1} P_j^{(\rho - \eta)} P_{imperf}^{(\eta - 1)} X, \]  
(3.8)

where
\[ \varepsilon(s_{jk}) = \eta s_{jk} + \rho (1 - s_{jk}) \]  
(3.9)

is interpreted as the elasticity of demand with respect to prices, and
\[ s_{jk} = \frac{\sum_l p_{jl}^{1 - \rho}}{\sum_l p_{jl}^{-\rho}} \]  
(3.10)

is the market share of firm \( k \) in sector \( j \). Profit function (3.7) implies that \( p_{jk} \geq c_{jk} \) for all \( j \) and \( k \). Combined with the assumption that \( c_{jk} > 0 \), this gives that \( p_{jk} > 0 \) for all \( j \) and \( k \). This, in turn, implies that \( 0 \leq s_{jk} \leq 1 \) for all \( j \) and \( k \). Hence, \( \varepsilon(s_{jk}) > 1 \), because \( 1 < \eta < \rho \). Therefore, the right-hand side of (3.8) can be equal to zero if and only if
\[ p_{jk} = m(s_{jk}) c_{jk}, \]  
(3.11)

where
\[ m(s_{jk}) \equiv \frac{\varepsilon(s_{jk})}{\varepsilon(s_{jk}) - 1} \]  
(3.12)

is the mark-up. This is the first-order condition of the firm \( k \)'s maximization problem. Note that formulas (3.9), (3.10), and (3.11) correspond to formulas (19), (17), and (15) reported in Atkeson and Burstein (2008).

Clearly, \( m(s_{jk}) > 1 \) for all \( j \) and \( k \). So, for \( p_{jk} = c_{jk} \) the left-hand side of (3.11) is less...
than the right-hand side of (3.11). Also, one can find that

$$\frac{\partial m(s_{jk})}{\partial p_{jk}} = \frac{(1 - \rho)(\rho - \eta)s_{jk}(1 - s_{jk})}{(\varepsilon(s_{jk}) - 1)^2}.$$  

Since $1 < \eta < \rho$, the right-hand side of this expression is negative. Hence, as $p_{jk}$ increases to $\infty$, the right-hand side of (3.11) monotonically decreases to $\frac{\rho}{\rho - 1}c_{jk}$. Therefore, there is exactly one point of intersection, $p_{jk}^*$, of the left-hand side and the right-hand side of (3.11). Then (3.8) implies that the profit function (3.7) increases for $p_{jk} < p_{jk}^*$ and decreases for $p_{jk} > p_{jk}^*$. Hence, $p_{jk}^*$ is the unique maximum of the profit function (3.7).

Let bars over variables denote vectors, so that $\overline{c}_j \equiv (c_{j1}, \ldots, c_{jK_j})$, $\overline{p}_j \equiv (p_{j1}, \ldots, p_{jK_j})$, and $\overline{m}(\overline{s}_j)^T \equiv (m(s_{j1}), \ldots, m(s_{jK_j}))$, where $\overline{A}^T$ denotes the transpose of vector $\overline{A}$. Then for each sector $j$ a vector of equilibrium prices $\overline{p}_j$ can be found as a fixed point of the equation

$$\overline{p}_j = \overline{m}(\overline{s}_j)^T \times \overline{c}_j.$$  \hspace{1cm} (3.13)

Claim 2. There is exists at least one vector of equilibrium prices for the case of imperfect substitutes.

Proof. Substituting formula (3.9) for elasticity into formula (3.12) for markup gives

$$m(s_{jk}) = \frac{\eta s_{jk} + \rho(1 - s_{jk})}{\eta s_{jk} + \rho(1 - s_{jk}) - 1}.$$  

The right-hand side of this expression is increasing in $s_{jk}$ under the assumption that $\eta < \rho$. It was noted earlier that $0 \leq s_{jk} \leq 1$. Hence, $\frac{\rho}{\rho - 1} \leq m(s_{jk}) \leq \frac{\eta}{\eta - 1}$. Therefore, since the boundaries of $m(s_{jk})$ do not depend on prices, the right-hand side of equation (3.13) is a function which maps the compact set $\left[\frac{\rho}{\rho - 1}c_{j1}, \frac{\eta}{\eta - 1}c_{j1}\right] \times \cdots \times \left[\frac{\rho}{\rho - 1}c_{jK_j}, \frac{\eta}{\eta - 1}c_{jK_j}\right]$ into itself. Clearly this function is continuous in prices on this compact set. Therefore, one can apply the Brouwer fixed point theorem and conclude that there exists at least one solution of equation (3.13).

Claim 3. The vector of equilibrium prices for the case of imperfect substitutes is unique.

Proof. Suppose that there are two solutions of equation (3.13). Then they are characterized by different vectors of market shares $(s_{j1}', \ldots, s_{jK_j}')$ and $(s_{j1}'', \ldots, s_{jK_j}'')$ such that $s_{jk}' \neq s_{jk}''$.  

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for some $\kappa$. Without loss of generality, assume that $s_{jk}^\prime > s_{jk}^\prime\prime$ for some $\kappa$. This implies that $m(s_{jk}^\prime) > m(s_{jk}^\prime\prime)$, because $m(s_{jk})$ is increasing in $s_{jk}$. Hence, it follows from equation (3.11) that $p_{jk}^\prime > p_{jk}^\prime\prime$ or, equivalently, $\left(\frac{p_{jk}^\prime}{p_{jk}^\prime\prime}\right)^{-\rho} < 1$.

For any $k = 1, \ldots, K$ denote $r_{jk} \equiv \left(\frac{p_{jk}^\prime}{p_{jk}^\prime\prime}\right)^{-\rho}$. Renumber indexes $1, \ldots, K$ such that $r_{jk}$ is weakly increasing in index $k$ (i.e., $r_{jk} \leq r_{j,k+1}$).\(^5\) It follows from the previous paragraph that $r_{j1} < 1$ (because $r_{jk} < 1$ at least for some $k$). Denote $d_{jk}^\prime \equiv (p_{jk}^\prime)^{-\rho} + \cdots + (p_{jk,k+1}^\prime)^{-\rho}$ and $d_{jk}^\prime\prime \equiv (p_{jk}^\prime\prime)^{-\rho} + \cdots + (p_{jk,K}^\prime\prime)^{-\rho}$. Let us prove by induction that $d_{jk}^\prime/d_{jk}^\prime\prime < r_{j1}$ for all $k \geq 2$.

Consider $k = 2$. Since $r_{j1} < 1$, equation (3.11) implies that $s_{j1}^\prime > s_{j1}^\prime\prime$. This is equivalent to

$$\frac{(p_{j1}^\prime)^{-\rho}}{(p_{j1}^\prime)^{-\rho} + d_{j2}^\prime} > \frac{(p_{j1}^\prime\prime)^{-\rho}}{(p_{j1}^\prime\prime)^{-\rho} + d_{j2}^\prime\prime},$$

which implies that $d_{j2}^\prime/d_{j2}^\prime\prime < r_{j1}$.

Next, suppose that $d_{jk}^\prime/d_{jk}^\prime\prime < r_{j1}$ for $k = 1, \ldots, l$. Let us prove that $d_{j,l+1}^\prime/d_{j,l+1}^\prime\prime < r_{j1}$. Inequality $d_{jl}^\prime/d_{jl}^\prime\prime < r_{j1}$ is equivalent to

$$\frac{(p_{jl}^\prime)^{-\rho} + d_{j,l+1}^\prime}{(p_{jl}^\prime\prime)^{-\rho} + d_{j,l+1}^\prime\prime} < r_{j1}.$$

This inequality can, in turn, be rewritten as

$$r_{j1} \frac{d_{j,l+1}^\prime}{d_{j,l+1}^\prime\prime} > 1 + \frac{(p_{jl}^\prime)^{-\rho}}{d_{j,l+1}^\prime\prime} \left(1 - \frac{r_{j1}}{r_{jl}}\right).$$

Combining this inequality with $r_{j1} \leq r_{jl}$ gives that $d_{j,l+1}^\prime/d_{j,l+1}^\prime\prime < r_{j1}$.

So, by induction, $d_{jk}^\prime/d_{jk}^\prime\prime < r_{j1}$ for all $k \geq 2$. In particular, $d_{j,K}^\prime/d_{j,K}^\prime\prime < r_{j1}$, which is equivalent to $r_{j,K} < r_{j1}$. But this contradicts to the choice of indexes: they were chosen such that $r_{jk}$ is weakly increasing in $k$. So, there is only one solution to equation (3.13).

\[\square\]

Claims 2 and 3 establish that for each finite $\rho$ there is a unique solution to equation (3.13) which gives a vector of equilibrium prices for each sector $j$.

\(^5\)Note that the order of production costs is irrelevant for the proof of this claim.
3.3 Main Result

There are two closely related issues concerning continuity of the model with imperfect substitutes. First, it is unclear whether a vector of equilibrium prices converges to some limit as $\rho$ goes to $\infty$. Second, even if a vector of equilibrium prices converges, it is unclear whether its limit is the equilibrium of the model with perfect substitutes. To understand the sources of these issues, note first that the demand function (3.4) in the model with perfect substitutes is discontinuous in prices, while the demand function (3.6) in the model with imperfect substitutes is continuous in prices. This means that the two models have different underlying structures. Therefore, it is generally conceivable that equilibrium prices of the model with imperfect substitutes may not converge to the equilibrium prices of the model with perfect substitutes.

To get further intuition about the issues with continuity, note that equation (3.13) can be written as $p_j = F_j(p_j; \rho)$. For each finite $\rho$, function $F_j$ is continuous on the set $\Omega_j(\rho) = \left[\frac{\rho}{\rho - 1} c_{j1}, \frac{\eta}{\eta - 1} c_{j1}\right] \times \cdots \times \left[\frac{\rho}{\rho - 1} c_{jK}, \frac{\eta}{\eta - 1} c_{jK}\right]$ and, as was proved in Section 3, has a unique fixed point in this set.

Now consider $\rho \to \infty$. One can immediately see that $\lim_{\rho \to \infty} \Omega_j(\rho) = \left[c_{j1}, \frac{\eta}{\eta - 1} c_{j1}\right] \times \cdots \times \left[c_{jK}, \frac{\eta}{\eta - 1} c_{jK}\right]$. Let $p'_j \in \lim_{\rho \to \infty} \Omega_j(\rho)$ and suppose that $p'_{j1} < p'_{jk}$ for all $k \neq 1$. Then, since $p'_j$ is fixed, $\lim_{\rho \to \infty} (p'_{j1})^\rho = 0$ for all $k \neq 1$. Moreover, $\lim_{\rho \to \infty} \rho \left(p'_{j1} / p'_{jk}\right)^\rho = 0$ for all $k \neq 1$.

Then

$$\lim_{\rho \to \infty} s'_{j1} = \lim_{\rho \to \infty} \frac{1}{1 + \sum_{l \neq 1} (p'_{jl})^{\rho-1}} = 1,$$

and

$$\lim_{\rho \to \infty} \rho \left(1 - s'_{j1}\right) = \lim_{\rho \to \infty} \rho \left(\frac{\sum_{l \neq 1} (p'_{jl})^{1-\rho}}{\sum_l (p'_{jl})^{1-\rho}}\right) = \lim_{\rho \to \infty} \frac{\rho \sum_{l \neq 1} (p'_{jl})^{\rho-1}}{1 + \sum_{l \neq 1} (p'_{jl})^{\rho-1}} = 0.$$

Similarly, $\lim_{\rho \to \infty} s'_{jk} = 0$ and $\lim_{\rho \to \infty} \rho(1 - s'_{jk}) = \infty$ for all $k \neq 1$. Combining formulas (3.9), (3.11),

\footnote{This is a standard result from real analysis: $\lim_{x \to \infty} x a^x = 0$ for any real number $0 < a < 1$.}
and (3.12) with the limits above, one can obtain $\lim_{\rho \to \infty} \mathbf{F}_j(p_j'; \rho) = \left( \frac{\eta}{\eta - 1} c_{j_1}, c_{j_2}, \ldots, c_{j_K} \right)^T$.

Next, suppose that $\frac{\eta}{\eta - 1} c_{j_1} > c_{j_2}$ and consider another price vector $p_j'' \in \lim_{\rho \to \infty} \Omega_j(\rho)$ such that $p_j'' = p_j''_1$ and $p_j''_1 < p_j''_{jk}$ for all $k \neq 1, 2$. Since $\frac{\eta}{\eta - 1} c_{j_1} > c_{j_2}$, such price vector exists in $\lim_{\rho \to \infty} \Omega_j(\rho)$. Repeating calculations from the previous paragraph, one can get that

$$\lim_{\rho \to \infty} s''_{j_1} = \lim_{\rho \to \infty} s''_{j_2} = \frac{1}{2} \text{ and } \lim_{\rho \to \infty} s''_{jk} = 0 \text{ for all } k \neq 1, 2.$$  

Also, $\lim_{\rho \to \infty} \rho (1 - s''_{jk}) = \infty$ for all $k$. Therefore, $\lim_{\rho \to \infty} \mathbf{F}_j(p_j''; \rho) = \left( c_{j_1}, c_{j_2}, \ldots, c_{j_K} \right)^T$.

Since $\lim_{\rho \to \infty} \mathbf{F}_j(p_j'; \rho) \neq \lim_{\rho \to \infty} \mathbf{F}_j(p_j''; \rho)$ and $\lim_{\rho \to \infty} \mathbf{F}_j(p_j'; \rho)$, $\lim_{\rho \to \infty} \mathbf{F}_j(p_j''; \rho)$ remain constant as $p_j'$ and $p_j''$ are marginally varied, $\lim_{\rho \to \infty} \mathbf{F}_j(\cdot; \rho)$ is discontinuous. So, we cannot use the Brouwer fixed point theorem — as we did in the proof of Claim 2 — to infer that $\lim_{\rho \to \infty} \mathbf{F}_j(\cdot; \rho)$ has a fixed point. Moreover, it is actually possible to see that $\lim_{\rho \to \infty} \mathbf{F}_j(\cdot; \rho)$ does not have a fixed point when $\frac{\eta}{\eta - 1} c_{j_1} > c_{j_2}$. Indeed, one the one hand, a price vector like $p_j'$ cannot be a fixed point of $\lim_{\rho \to \infty} \mathbf{F}_j(\cdot; \rho)$ because $\lim_{\rho \to \infty} \mathbf{F}_{j_1}(p_j'; \rho) = \frac{\eta}{\eta - 1} c_{j_1} > c_{j_2} = \lim_{\rho \to \infty} \mathbf{F}_{j_2}(p_j'; \rho)$ while $p_{j_1}' < p_{j_2}'. \text{ One the other hand, a price vector like } p_j'' \text{ also cannot be a fixed point of } \lim_{\rho \to \infty} \mathbf{F}_j(\cdot; \rho) \text{ because } \lim_{\rho \to \infty} \mathbf{F}_{j_1}(p_j''; \rho) = c_{j_1} < c_{j_2} = \lim_{\rho \to \infty} \mathbf{F}_{j_2}(p_j''; \rho) \text{ while } p_{j_1}'' = p_{j_2}''$. Therefore, it is conceivable that an equilibrium price vector of the model with imperfect substitutes might diverge.

Fortunately, it is possible to prove convergence. The following theorem states this result formally.

**Theorem 1.** The equilibrium price vector for the case of imperfect substitutes obtained from equation (3.13) converges to the equilibrium price vector for the case of perfect substitutes given by (3.5) as $\rho$ goes to $\infty$.

To understand why there is convergence of one solution to the other, imagine that one is able to find a fixed point of function $F_j(\cdot; \rho)$ for each $\rho$. Let $p_j(\rho)$ be this fixed point. Substitute it into formula (3.10) for the first firm’s market share and consider expression $\rho (1 - s_{jk}(\rho))$ as $\rho$ goes to $\infty$. Suppose that one is also able to prove that $\rho (1 - s_{jk}(\rho))$ converges to some limit as $\rho$ goes to $\infty$, and she now wants to figure out what this limit is. In the examples above we had that $\lim_{\rho \to \infty} \rho (1 - s_{j_1}') = 0$ and, when $\frac{\eta}{\eta - 1} c_{j_1} > c_{j_2}$, $\lim_{\rho \to \infty} \rho (1 - s_{j_1}'') = \infty$. In contrast to that, as it turns out, $\lim_{\rho \to \infty} \rho (1 - s_{j_1}(\rho))$ is a positive constant when $\frac{\eta}{\eta - 1} c_{j_1} > c_{j_2}$. Furthermore, this positive constant is such that $\lim_{\rho \to \infty} m(s_{j_1}(\rho)) = \frac{c_{j_2}}{c_{j_1}}$, so that $\lim_{\rho \to \infty} p_{j_1}(\rho) = c_{j_2}$.  

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This, in turn, means that \( p_{j1}(\rho) = \min \left\{ \frac{\eta}{\eta - 1} c_{j1}, c_{j2} \right\} \) when \( \frac{\eta}{\eta - 1} c_{j1} > c_{j2} \). Establishing this fact is the final goal in the proof of convergence.

The key steps of the proof of convergence are the following. First, boundness of \( \rho(1 - s_{j1}) \) as \( \rho \) goes to \( \infty \) is established. This immediately implies that there exists a sequence \( \{\rho_n\}_{n \in \mathbb{N}} \) such that \( \rho_n \to \infty \) and \( \rho_n(1 - s_{j1,n}) \) converges to some limit, where \( s_{j1,n} \) is an equilibrium market share corresponding to \( \rho_n \). Second, it is proved that any convergent sequence \( \rho_n(1 - s_{j1,n}) \) converges to the same limit. This implies that \( \rho(1 - s_{j1}) \) itself converges. The expression for the limit of \( \rho(1 - s_{j1}) \) then will follow from the previous steps — it will give a solution of the model with perfect substitutes.

3.4 Proof of Continuity

Consider a particular sector \( j \). To economize on notation, skip index \( j \) for all variables for that sector.

**Lemma 1.** \( \rho(1 - s_1) \) is bounded, i.e., there exists a real number \( B \) such that \( \rho(1 - s_1) < B \) for all \( \rho \).

**Proof.** Suppose \( \rho(1 - s_1) \) is not bounded as \( \rho \) goes to \( \infty \). This means that one can find a sequence \( \{\rho_n\}_{n \in \mathbb{N}} \) such that \( \rho_n \to \infty \) and \( \rho_n(1 - s_{1,n}) \to \infty \) as \( n \to \infty \), where

\[
s_{1,n} = \frac{p_{1,n}^{1 - \rho_n}}{\sum_l p_{1,n}^{1 - \rho_n}}
\]

is firm 1’s market share corresponding to \( \rho_n \), and \( p_{k,n} \) for \( k = 1, 2, \ldots, K \) are firms’ equilibrium prices corresponding to \( \rho_n \). Then

\[
p_{1,n} = \left[ \frac{\eta s_{1,n} + \rho_n(1 - s_{1,n})}{\eta s_{1,n} + \rho_n(1 - s_{1,n}) - 1} \right] c_1 \to c_1 \text{ as } n \to \infty.
\]

Since \( c_1 < c_2 \), this implies that there exists a number \( N \) for which \( p_{1,n} < c_2 \) for all \( n \geq N \). Therefore, since \( p_k \geq c_k \) for any \( \rho \), there exists a real number \( 0 < A < 1 \) such that \( \frac{p_{1,n}}{p_{l,n}} < A \) for all \( n \geq N \) and \( l \neq 1 \). Then for all \( n \geq N \):

\[
\sum_{l \neq 1} \left( \frac{p_{1,n}}{p_{l,n}} \right)^{\rho_n - 1} < \sum_{l \neq 1} A^{\rho_n - 1} \to 0 \text{ as } n \to \infty,
\]

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and
\[ \rho_n \sum_{l \neq 1} \left( \frac{p_{1,n}}{p_{l,n}} \right)^{\rho_n - 1} < \rho_n \sum_{l \neq 1} A^{\rho_n - 1} \to 0 \text{ as } n \to \infty. \]

But then:
\[
\lim_{n \to \infty} \rho_n (1 - s_{1,n}) = \lim_{n \to \infty} \rho_n \left( \frac{\sum_{l \neq 1} p_{1,n}^{1-\rho_n}}{\sum_l p_{l,n}} \right) = \lim_{n \to \infty} \frac{\rho_n \sum_{l \neq 1} \left( \frac{p_{1,n}}{p_{l,n}} \right)^{\rho_n - 1}}{1 + \sum_{l \neq 1} \left( \frac{p_{1,n}}{p_{l,n}} \right)^{\rho_n - 1}} = 0.
\]

This contradicts \( \lim_{n \to \infty} \rho_n (1 - s_{1,n}) = \infty \). Thus, \( \rho(1 - s_1) \) is bounded.

\[ \square \]

**Lemma 2.** The following limits hold for market shares and prices:

(i) \( \lim_{\rho \to \infty} s_1 = 1, \lim_{\rho \to \infty} s_k = 0 \) for \( k \neq 1 \); and

(ii) \( \lim_{\rho \to \infty} p_k = c_k \) for \( k \neq 1 \).

*Proof.* Lemma 1 immediately implies that
\[
1 - s_1 = \frac{\rho(1 - s_1)}{\rho} < \frac{B}{\rho} \to 0 \text{ as } \rho \to \infty.
\]

So, \( \lim_{\rho \to \infty} s_1 = 1 \). This, in turn, implies that \( \lim_{\rho \to \infty} s_k = 0 \) for all \( k \neq 1 \). Hence, \( \lim_{\rho \to \infty} \rho(1 - s_k) = \infty \).

And, finally, for all \( k \neq 1 \):
\[
\lim_{\rho \to \infty} p_k = \lim_{\rho \to \infty} \left[ \frac{\eta s_k + \rho(1 - s_k)}{\eta s_k + \rho(1 - s_k) - 1} \right] c_k = c_k.
\]

\[ \square \]

**Lemma 3.** Consider any sequence \( \{\rho_n\}_{n \in \mathbb{N}} \) such that \( \rho_n \to \infty \) and \( \rho_n (1 - s_{1,n}) \to b \) as \( n \to \infty \), where \( b > 0 \). Then \( \lim_{n \to \infty} \frac{p_{1,n}}{p_{2,n}} = 1 \).

*Proof.* Note first that the result of Lemma 1 guarantees the existence of a sequence \( \{\rho_n\}_{n \in \mathbb{N}} \) such that \( \rho_n \to \infty \) and \( \rho_n (1 - s_{1,n}) \to b \) as \( n \to \infty \), where \( b \) is a real number.\(^7\) Consider any

\[7\text{This is also a standard result from real analysis: every bounded sequence has a subsequence converging to a finite limit.}\]
such sequence and suppose that $b > 0$. Then

$$\lim_{n \to \infty} p_{1,n} = \lim_{n \to \infty} \frac{\eta s_{1,n} + \rho_n (1 - s_{1,n})}{\eta s_{1,n} + \rho_n (1 - s_{1,n}) - 1} \frac{c_1}{c_1} = \frac{\eta + b}{\eta + b - 1} c_1,$$

where $\lim_{n \to \infty} s_{1,n} = 1$ by Lemma 2. Also, by Lemma 2 $\lim_{n \to \infty} p_{2,n} = c_2$. Therefore

$$\lim_{n \to \infty} \frac{p_{1,n}}{p_{2,n}} = \frac{(\eta + b)c_1}{(\eta + b - 1)c_2}.$$

It is clear that $\lim_{n \to \infty} \frac{p_{1,n}}{p_{2,n}} \leq 1$, otherwise $\lim_{n \to \infty} s_{1,n} = 0$, which is not possible by Lemma 2. Suppose that $\lim_{n \to \infty} \frac{p_{1,n}}{p_{2,n}} < 1$. Then there exist such real number $0 < A < 1$ and integer $N$ that $\frac{p_{1,n}}{p_{2,n}} < A$ for all $n > N$. Then $\lim_{n \to \infty} \rho_n (1 - s_{1,n}) = 0$ (the proof is analogous to that of Lemma 1). This contradicts with what was supposed in the beginning: $\lim_{n \to \infty} \rho_n (1 - s_{1,n}) = b > 0$. Hence, $\lim_{n \to \infty} \frac{p_{1,n}}{p_{2,n}} = 1$.

**Proof of Theorem 1.** Consider a sequence $\{\rho_n\}_{n \in \mathbb{N}}$ such that $\rho_n \to \infty$ and $\rho_n (1 - s_{1,n}) \to b$ as $n \to \infty$, where $b \geq 0$.

Consider the case when $\frac{\eta}{\eta - 1} c_1 > c_2$. If $b = 0$, then $\lim_{n \to \infty} p_{1,n} = \frac{\eta}{\eta - 1} c_1 > \lim_{n \to \infty} p_{2,n} = c_2$. This implies that $\lim_{n \to \infty} s_{1,n} = 0$, a contradiction to Lemma 2. Hence, $b > 0$. Lemma 3 then implies that $\lim_{n \to \infty} \frac{p_{1,n}}{p_{2,n}} = 1$ and, thus, $b = \frac{\eta c_1 - (\eta - 1)c_2}{c_2 - c_1}$.

Now consider the case when $\frac{\eta}{\eta - 1} c_1 \leq c_2$. Suppose that $b > 0$. Then Lemma 3 implies that $\lim_{n \to \infty} p_{1,n} = \lim_{n \to \infty} p_{2,n}$. By Lemma 2 $\lim_{n \to \infty} p_{2,n} = c_2$. Therefore, $\lim_{n \to \infty} p_{1,n} = c_2$. On the other hand, $\lim_{n \to \infty} p_{1,n} = \frac{\eta + b}{\eta + b - 1} c_1$. Combining these two equalities, one can get that $b = \frac{\eta c_1 - (\eta - 1)c_2}{c_2 - c_1} \leq 0$, which contradicts $b > 0$. Hence, in this case $b = 0$ and $\lim_{n \to \infty} p_{1,n} = \frac{\eta}{\eta - 1} c_1$.

The conclusions of the two previous paragraphs can be summarized as follows. For the case when $\frac{\eta}{\eta - 1} c_1 > c_2$ every converging sequence $\rho_n (1 - s_{1,n})$ converges to the same limit equal to $\frac{\eta c_1 - (\eta - 1)c_2}{c_2 - c_1}$ (with $\lim_{n \to \infty} p_{1,n} = c_2$); for the case when $\frac{\eta}{\eta - 1} c_1 \leq c_2$ every converging
sequence \( \rho_n(1 - s_{1,n}) \) converges to the same limit equal to 0. This means that

\[
\lim_{\rho \to \infty} \rho(1 - s_1) = \begin{cases} \eta c_1 - (\eta - 1)c_2, & \text{if } \frac{\eta}{\eta - 1} c_1 > c_2; \\ 0, & \text{if } \frac{\eta}{\eta - 1} c_1 \leq c_2; \end{cases}
\]

which yields

\[
\lim_{\rho \to \infty} p_k = \begin{cases} \min\left\{ \frac{\eta}{\eta - 1} c_1, c_2 \right\}, & \text{if } k = 1; \\ c_k, & \text{if } k \neq 1. \end{cases}
\]

This is a solution of the model with perfect substitutes. So, the continuity of the model with imperfect substitutes is established. 

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\^{8}\text{This is one more standard result from real analysis: a sequence converges to } a \text{ if and only if any convergent subsequence of this sequence converges to } a.
Appendix A

Solution Algorithm

A.1 System of Equations for the Modified Model

Let \( p_{n,g}^\omega(s) \) be the price of the good \( \omega \) produced by country \( n \) in industry \( g \) in state \( s \). We have to find \( p_{n,g}^\omega(s), p_g^\omega(s), P_g(s), x_{n,g}^\omega(s), X_{n,g}(s), I_n(s), \Pi_{n,g}^\omega(s), \Pi_{n,g}, l_{n,g}, w_n \), which solve the following system of equations:

\[
p_g^\omega(s) = \left[ \sum_{i=1}^{N} p_{i,g}^\omega(s)^{1-\sigma'} \right]^{\frac{1}{1-\sigma'}},
\]
\[
P_g(s) = \left[ \int_{\omega \in \Omega} p_g^\omega(s)^{1-\sigma} d\omega \right]^{\frac{1}{1-\sigma}},
\]
\[
P(s) = \left[ \sum_{g=1}^{G} \alpha_g P_g(s)^{1-\eta} \right]^{\frac{1}{1-\eta}},
\]
\[
x_{n,g}^\omega(s) = \left( \frac{p_g^\omega(s)}{P_g(s)} \right)^{1-\sigma} X_{n,g}(s),
\]
\[
X_{n,g}(s) = \left( \frac{P_g(s)}{P(s)} \right)^{1-\eta} \alpha_g I_n(s),
\]
\[
\Pi_{n,g}^\omega(s) = \left[ \frac{1}{A_{i,g}(s)} \sum_{n=1}^{N} p_{n,g}^\omega(s)^{\sigma'-1} x_{n,g}^\omega(s) \right]^{\frac{1}{\sigma}},
\]
\[
w_i = \left( \frac{1}{L_i} \sum_{g=1}^{G} \int_{\omega \in \Omega} \left[ z_{i,g}^\omega \right]^{\sigma'-1} \left[ \Pi_{i,g}^\omega \right]^{\sigma'} d\omega \right)^{\frac{1}{\sigma}},
\]
\[
l_{i,g}^\omega = \left[ z_{i,g}^\omega \right]^{\sigma'-1} \left( \frac{\Pi_{i,g}^\omega}{w_i} \right)^{\sigma'}.
\]
\[ p_{n,g}^\omega(s) = \frac{\Pi_{i,g}^\omega(s)}{\sum_{i=1}^{\omega} w_i z_{i,g}^\omega}. \]

For complete financial markets:

\[ \Pi_{n,g}^\omega = \sum_{s=1}^{S} A_{n,g}(s) \Pi_{n,g}^\omega(s), \]
\[ I_n(s) = \frac{h(s)^{\frac{1}{p}} P(s)^{\frac{p-1}{p}}}{\sum_{s=1}^{S} h(s)^{\frac{1}{p}} P(s)^{\frac{p-1}{p}}} w_n L_n. \]

For financial autarky:

\[ \Pi_{n,g}^\omega = E_s \left[ A_{n,g}(s) \Pi_{n,g}^\omega(s) I_n(s)^{-\rho} P(s)^{\rho-1} \right], \]
\[ I_n(s) = \sum_{g=1}^{G} A_{n,g}(s) \int_{\omega \in \Omega} p_{n,g}^\omega(s) z_{n,g}^\omega L_n d\omega. \]

### A.2 Discretization Scheme

The most straightforward way to solve the model in the general case is to discretize the continuum of goods, i.e., randomly draw a finite number of efficiencies \( z_{n,g}^\omega \) from corresponding Fréchet distributions.\(^1\) This is the strategy I follow in the current paper. However, the discretization brings two related problems, which we should be aware of.

First, the Fréchet distribution has infinite support and a fat tail. Any discretization scheme will necessarily truncate the support, throwing away some part of the tail. This introduces a truncation error into the solution, which can be substantial. To get a sense of this problem, imagine that we have a Dornbush-Fisher-Samuelson economy (Dornbusch et al. (1977b)) with the underlying Fréchet distribution of efficiencies of production and totally identical countries. Suppose that we take 1000 random draws of efficiencies for each country. There is a big chance that one of the countries (say, Home) will get a draw much further down the tail than the other country (Foreign). If that happens, Home becomes a much more technologically advanced economy. As a result, the trade shares and wages will be significantly altered.

Second, as it is well known, the standard error of an integral computed by Monte Carlo

\(^1\) An alternative approach can be to approximate price distributions with some basis functions.
integration is inversely proportional to $\sqrt{M}$, where $M$ is the number of draws.\footnote{See, for example, Section 8.2 in Judd (1998).} In other words, to get one more digit of accuracy in aggregate values, we need to take 100 times more draws. This exponential growth of the number of points needed to get accurate solutions can be a limiting factor in solving the model.

Clearly, the two problems mentioned above are not specific to the model presented here. Any model with a fat-tail distribution, which needs to be solved by discretization, suffers from these problems. A natural question to ask here is why would we want choose the Fréchet distribution in the current paper, if we solve the model numerically anyway? One reason for choosing the Fréchet distribution is to stay as close as possible to the existing trade models of comparative advantage (Eaton and Kortum (2002) and Costinot et al. (2012)). Another reason is that with the Fréchet distribution we can actually go a long way in characterizing the solution in the special cases (free trade and only country- or industry-specific shocks).

To mitigate the discretization errors, I propose the following approach. Suppose we choose $M$ discretization points. Then for every country-industry we first construct a non-random sample of efficiencies $\tilde{z}_{n,g}^m$ by taking $M$ equidistant points on the interval $[0, 1]$ and translating them into the Fréchet distribution:

$$
\tilde{z}_{n,g}^m = \left( -\frac{1}{T_{n,g}} \ln u^m \right)^{-\frac{1}{\theta}},
$$

$$
u^m = \frac{m - 0.5}{M}, \quad m = 1, \ldots, M.
$$

After that we make the sample pseudo-random by shuffling indices of goods for each country-industry, i.e., for each country-industry, we take a random permutation $(\omega_1, \ldots, \omega_M)$ of indices $(1, \ldots, M)$ and reassign efficiencies:

$$
z_{n,g}^{\omega_m} = \tilde{z}_{n,g}^m.
$$

The sample obtained this way has two advantages over the regular Monte Carlo sampling. First, it gives stable solutions: it preserves the order of countries in terms of their levels of technology. Second, it reduces the variance of numerical errors. Thus, we need fewer points to achieve the same level accuracy comparing to the regular Monte Carlo sampling. In the current work, I was able to achieve much better results by using this discretization scheme over the regular Monte Carlo sampling.
A.3 Coping with Complementary Slackness Conditions

Solving the above equilibrium system of equations is a non-trivial task, because it involves complementary slackness conditions. A direct way to cope with them is to use a complementary slackness solver. As of today, the best complementary slackness solver is PATH described in Ferris and Munson (1999). When applied to the above problem with a small number of discrete points (e.g., \( M = 500 \)), it gives great results. However, to get accurate aggregate solutions, we need at least 10,000 points. As we increase the number points, we increase the number of complementary slackness conditions. With 10,000 points the memory requirements for PATH and the time of solution become too restrictive to use PATH on, say, desktop computers, or laptops, or even on bigger machines.

In the current paper I use an indirect approach to cope with complementary slackness conditions. I modify the model by introducing an artificial requirement that each country \( n \) buys each good \( \omega \) from all countries. Varieties of good \( \omega \) produced by different countries are combined by the CES utility function with elasticity \( \sigma' \). The equilibrium system of the modified model does not involve complementary slackness conditions. As elasticity \( \sigma' \) goes to infinity, the artificial requirement disappears, and the modified model converges to the original model. Presumably, the solution of the modified model also converges to the solution of the original model. I assume that it is true.

We can find an approximate solution of the original model by solving a sequence of modified models corresponding to an increasing sequence of elasticities \( \{\sigma'_d\}_{d=1}^{D} \) and using the solution of the model with \( \sigma'_d \) as a starting point for the system of equations corresponding to \( \sigma'_{d+1} \). The value of \( \sigma'_D \) should be large enough so that the change in the aggregate outcomes from \( \sigma'_{D-1} \) to \( \sigma'_D \) is small. In my calculations I always use \( \sigma'_D = 2^{17} = 131,072 \). In practice, most of the time of the algorithm is spend in computing powers of \( \sigma' \) and \( 1/\sigma' \). Since we are free to choose any increasing sequence of \( \{\sigma'_d\}_{d=1}^{D} \), it is best to use powers of 2: \( \sigma'_d = 2^d \), \( d = 1, \ldots, D \). Taking any number to power \( \sigma' \) generally requires at least \( \log_2 \sigma' \) multiplications, with exactly \( \log_2 \sigma' \) multiplications when \( \sigma' \) is a power of 2. So, if we choose \( \sigma'_d = 2^d \), we need to make the least theoretically possible number of multiplications each time when we take powers of \( \sigma'_d \). The reduction in solving time can be significant: several times faster comparing to a random sequence of \( \sigma' \).

The nonlinear system for the modified model can and must be solved by the fixed point iteration. The reason for this is that for large values of \( \sigma' \) the gradients of the equations are numerically unstable. So, the regular Newton’s method cannot be applied here. When using
the fixed point iteration, we also need to use damping.

There is one more crucial detail which is important to know when using the suggested approach with increasing elasticities. Solving the equilibrium system requires evaluating expressions of the following kind:

\[
p = \left[ \sum_{i=1}^{N} p_i^{1-\sigma'} \right]^{\frac{1}{1-\sigma'}}.
\]

As \(\sigma'\) becomes large, it is likely that the direct evaluation of such expression will result in arithmetic overflow. Instead, one has to evaluate the following (theoretically equivalent) expression:

\[
p = p_{\text{min}} \left[ \sum_{i=1}^{N} \left( \frac{p_i}{p_{\text{min}}} \right)^{1-\sigma'} \right]^{\frac{1}{1-\sigma'}},
\]

with

\[p_{\text{min}} = \min_i \{p_i\} \,.
\]

Only with transformations of this kind we are able to use elasticities \(\sigma'\) as large as \(2^{17}\).
Appendix B

A Ricardian Model with External Economies

B.1 Existence and Uniqueness

B.1.1 Proof of Proposition 1 ($\alpha_k < 1$).

In this part of the proof we show that there is a unique solution to the nonlinear complementarity problem (2.8).

First, note that a corner allocation cannot be a solution of (2.8). This follows from the fact that $\lim_{x_i \to 0} G_i(x) = -\infty$ implying that, if $x_i = 0$ for some $i$, then the inequality $G_i(x) \geq 0$ cannot be satisfied.

Let us now prove that there exists a unique $x > 0$ satisfying (2.8), i.e., that there exists a unique interior solution to (2.8). For that it is enough to focus on proving that there exists a unique $x > 0$ such that $G_i(x) = 0$ for $i = 1, \ldots, N$.

Existence: The system $G_i(x) = 0$, $i = 1, \ldots, N$, for $x > 0$ can equivalently be written as the fixed point problem $F_i(x) = x$, $i = 1, \ldots, N$, with

$$F_i(x) \equiv \sum_n \frac{a_{ni}x_n^\alpha}{\sum_l a_{nl}x_l^\alpha} b_n.$$ 

We now proceed to show that there exist $0 < \varepsilon < \bar{\varepsilon}$ such that, for any $x \in [\varepsilon, \bar{\varepsilon}]^N$, $F_i(x) \in$
We start by noting that, using \( \lambda_{ni} \equiv \frac{a_{ni}x_i^{\alpha}}{\sum_l a_{nl}x_l^{\alpha}} \), for \( i \neq j \) we have

\[
\frac{\partial F_i}{\partial x_j} = -\frac{\alpha}{x_j} \sum_n \lambda_{nj} b_n < 0, \quad \text{while} \quad \frac{\partial F_i}{\partial x_i} = \frac{\alpha}{x_i} \sum_n \lambda_{ni} (1 - \lambda_{ni}) b_n > 0.
\]

Next, let \( \Omega \equiv [\varepsilon, \bar{\varepsilon}]^N \), with \( 0 < \varepsilon < \bar{\varepsilon} \), and note that if \( x \in \Omega \) then

\[
\sum_n a_{ni} b_n \leq \frac{\varepsilon}{\varepsilon + \varepsilon^{1-\alpha} s_{ni}} \leq \frac{\sum_l a_{nl} b_n}{\alpha (\varepsilon) \sum_{l \neq i} a_{nl}} \leq \varepsilon \]

The middle inequality is automatically satisfied if \( 0 < \varepsilon < \bar{\varepsilon} \). Letting \( s_{ni} \equiv \frac{1}{a_{ni}} \sum_{l \neq i} a_{nl} \), the first inequality is equivalent to

\[
1 \leq \frac{b_n}{\varepsilon + \varepsilon^{1-\alpha} s_{ni}};
\]

while for the second we need

\[
\frac{b_n}{\varepsilon + \varepsilon^{1-\alpha} s_{ni}} \leq 1.
\]

Let \( s_{\min} \equiv \min_{n,i} s_{ni} \) and \( s_{\max} \equiv \max_{n,i} s_{ni} \) and \( b \equiv \sum_n b_n \). Then we know that

\[
\frac{b_n}{\varepsilon + \varepsilon^{1-\alpha} s_{ni}} \geq \frac{b}{\varepsilon + \varepsilon^{1-\alpha} s_{\max}} \text{ for all } n, i.
\]

Hence, if \( \varepsilon \) and \( \bar{\varepsilon} \) are such that

\[
1 \leq \frac{b}{\varepsilon + \varepsilon^{1-\alpha} s_{\max}},
\]

then

\[
1 \leq \frac{b}{\varepsilon + \varepsilon^{1-\alpha} s_{\max}} \leq \sum_n \frac{b_n}{\varepsilon + \varepsilon^{1-\alpha} s_{ni}} \text{ for all } n, i.
\]

Similarly, we know that

\[
\sum_n \frac{b_n}{\varepsilon + \varepsilon^{1-\alpha} s_{ni}} \leq \frac{b}{\varepsilon + \varepsilon^{1-\alpha} s_{\min}}.
\]
Thus, if $\varepsilon$ and $\bar{\varepsilon}$ are such that

$$b + \frac{b}{\bar{\varepsilon}^{1-\alpha}s_{\min}} \leq 1,$$

then

$$\sum_n \frac{b_n}{\varepsilon + \varepsilon^{1-\alpha}s_{n_i}} \leq \frac{b}{\varepsilon + \varepsilon^{1-\alpha}s_{\min}} \leq 1$$

for all $n, i$.

So we need to do choose $\varepsilon$ and $\bar{\varepsilon}$ with $\varepsilon < \bar{\varepsilon}$ such that

$$\frac{b}{\varepsilon (1 + \eta^\alpha s_{\min})} \leq 1 \leq \frac{b}{\bar{\varepsilon}\eta (1 + \eta^{-\alpha}s_{\max})},$$

where $\eta \equiv \varepsilon / \bar{\varepsilon}$. Suppose we choose $\varepsilon$ and $\bar{\varepsilon}$ such that

$$\frac{b}{\varepsilon (1 + \eta^\alpha s_{\min})} = 1.$$

Then we have

$$\bar{\varepsilon} = \frac{b}{1 + \eta^\alpha s_{\min}}.$$

Plugging this into $1 \leq \frac{b}{\varepsilon\eta (1 + \eta^{-\alpha}s_{\max})}$ yields

$$1 \leq \frac{b}{[1 + \eta^\alpha s_{\min}] \eta (1 + \eta^{-\alpha}s_{\max})} = \frac{1 + \eta^\alpha s_{\min}}{\eta + \eta^{1-\alpha}s_{\max}}.$$

It is then enough to show that there is a $\eta < 1$ such that

$$\eta + \eta^{1-\alpha}s_{\max} - \eta^\alpha s_{\min} \leq 1.$$

If $\alpha < 1$ then $\lim_{\eta \to 0} \eta = 0$, $\lim_{\eta \to 0} \eta^{1-\alpha} = 0$, and $\lim_{\eta \to 0} \eta^\alpha = 0$. Hence the LHS converges to zero as $\eta \to 0$. This implies that we can always choose $\eta < 1$ such that this inequality is satisfied. Hence, we have a well defined fixed point mapping. Since $F(x)$ is continuous on $[\varepsilon, \bar{\varepsilon}]^N$ and $[\varepsilon, \bar{\varepsilon}]^N$ is compact, the Brower’s fixed point theorem implies that there exists a solution $x \in [\varepsilon, \bar{\varepsilon}]^N$ of $x = F(x)$.

**Uniqueness:** Suppose that there are two distinct allocations $x' \neq x''$ such that $G_i(x') = G_i(x'') = 0$, $i = 1, \ldots, N$. Let $\xi_i = \min \{x'_i, x''_i\}$ and $\zeta_i = \max \{x'_i, x''_i\}$ for $i = 1, \ldots, N$. Clearly, $x' \in [\xi, \bar{\xi}]$ and $x'' \in [\xi, \bar{\xi}]$. 

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Consider the mapping \( \widetilde{G} : [\ell, r]^N \to \mathbb{R}^N \) defined by
\[
\widetilde{G}_i(x) \equiv \frac{G_i(x)}{x_i^{\alpha-1}} = x_i^{1-\alpha} - \sum_n \frac{a_{ni}}{\sum_l a_{nl}x_i^\alpha} b_n.
\]

Since the domain of \( \widetilde{G} \) does not include zeros, for any point \( x \in [\ell, r]^N \), \( G(x) = 0 \) if and only if \( \widetilde{G}(x) = 0 \). Then, by our supposition \( \widetilde{G}(x') = \widetilde{G}(x'') = 0 \). To argue that this leads to a contradiction, we are going to prove that \( \widetilde{G} \) is a univalent mapping on its domain. For that, in turn, we will apply Theorem 4 from Gale and Nikaido (1965), which requires that the Jacobian of the mapping \( \widetilde{G} \) is a P-matrix for all \( x \in [\ell, r]^N \).

Denote by \( J \) the Jacobian of mapping \( \widetilde{G} \). Its elements are:
\[
J_{ii}(x) = (1 - \alpha)x_i^{-\alpha} + \alpha x_i^{-\alpha} x_i^{-1} \sum_n \lambda_{ni}^2 b_n,
\]
\[
J_{ij}(x) = \alpha x_i^{-\alpha} x_j^{-1} \sum_n \lambda_{ni}\lambda_{nj} b_n \text{ for } j \neq i.
\]

In matrix notation the Jacobian is
\[
J(x) = (1 - \alpha)X^{-\alpha} + \alpha X^{-\alpha} \Lambda^T B \Lambda X^{-1},
\]
where \( X \) is a diagonal matrix with \( x_i \) on the diagonal, \( I \) is an identity matrix, \( \Lambda \) is the matrix of trade shares with \( \lambda_{ni} \) in intersection of the \( n \)th row and \( i \)th column, and \( B \) is a diagonal matrix with \( b_n \) on the diagonal.

To prove that \( J(x) \) is a P-matrix, we first need several definitions and preliminary results. Let \( A \) be an \( n \times n \) matrix. Matrix \( A \) is a P-matrix, if all its principal minors are positive, and matrix \( A \) is a \( P_0 \)-matrix, if all its principal minors are non-negative. Suppose that \( A \) is a \( P_0 \)-matrix and \( D \) is a diagonal matrix with positive diagonal entries, then \( A + D \) is a P-matrix (see Theorem 7.8.7 from Bapat and Raghavan, 1997). With these preliminaries in hand we now show that \( J(x) \) is a P-matrix for all \( \alpha \in (0, 1) \) and any \( x \in [\ell, r]^N \). Note first that \( 0 \leq \alpha < 1 \) implies the matrix \( (1 - \alpha)X^{-\alpha} \) is a diagonal matrix with all positive entries. Therefore, it is sufficient to show that \( A \equiv X^{-\alpha} \Lambda^T B \Lambda X^{-1} \) is a \( P_0 \)-matrix. Let \( C \) be any set of indexes from 1 to \( N \) of size less than \( N - 1 \). Consider any principal submatrix \( A_{C,C} \) of \( A \) given by \( C \): matrix \( A_{C,C} \) is a submatrix of \( A \) which is obtained from \( A \) by removing rows and columns with indexes from \( C \). Matrix \( A_{C,C} \) can be written as
\[
A_{C,C} = X_{C,C}^{-\alpha} \Lambda_C^T B \Lambda_C X_{C,C}^{-1}.
\]
where $X_{C,C}$ is obtained from $X$ by removing rows and columns with indexes from $C$, and matrix $\Lambda_C$ is obtained from $\Lambda$ by removing columns with indexes from $C$. Determinant of $A_{C,C}$ is

$$\det (A_{C,C}) = \det \left( X_{C,C}^{-\alpha} \right) \det \left( \Lambda_C^T B \Lambda_C \right) \det \left( X_{C,C}^{-1} \right).$$

Since matrix $X$ is a diagonal matrix with all positive entries, $\det (X_{C,C}^{-\alpha}) > 0$ and $\det (X_{C,C}^{-1}) > 0$. At the same time it is easy to see that matrix $\Lambda_C^T B \Lambda_C$ is a positive semi-definite matrix.

Indeed, for any vector $z \neq 0$, $z^T \Lambda_C^T B \Lambda_C z = \| B^{\frac{1}{2}} \Lambda_C z \|_2^2 \geq 0$. Therefore, $\det (\Lambda_C^T B \Lambda_C) \geq 0$, which, in turn, implies that $\det (A_{C,C}) \geq 0$. In other words, we have showed that any principal minor of $A$ is non-negative. Therefore, $A$ is a $P_0$-matrix and $J(x)$ is a $P$-matrix. That $\tilde{G}$ is, indeed, univalent on $[\epsilon, \bar{\epsilon}]^N$ then follows from Theorem 4 in Gale and Nikaido (1965). Thus, there can be only one solution $x^* \in [\epsilon, \bar{\epsilon}]^N$ such that $\tilde{G}(x^*) = 0$. ■

### B.1.2 Proof of Proposition 1 ($\alpha_k = 1$).

We first show that there exists a unique solution to the problem (2.9) for any $\delta > 0$. Then, by taking $\delta$ to 0, we establish existence of solution of (2.8). Finally, we prove that the solution of (2.8) is unique.

**Existence and uniqueness for $\delta > 0$.** In the proof of uniqueness and existence we use results from Facchinei and Pang (2003, henceforth FP). The propositions from this book have their own notation, and it might not be obvious right away how these propositions relate to our problem. Therefore, it is worth making an investment into bringing notation of our problem in correspondence with the notation from FP.

Let us first make change of variables: $y_i \equiv x_i - \delta$. In this notation, the NCP problem in (2.9) is to find vector $y \in \mathbb{R}^N$ satisfying the following conditions:

$$y \geq 0, \quad \tilde{G}(y) \geq 0, \quad \text{and} \quad y^T \tilde{G}(y) = 0,$$

where

$$\tilde{G}_i(y) \equiv 1 - \sum_{n=1}^N \frac{a_{ni}}{\sum_l a_{nl} y_l + \delta \sum_l a_{nl} b_n} b_n.$$

This statement of our problem corresponds to Definition 1.1.5 from FP. Following FP, we denote this problem by $NCP(\tilde{G})$. There is an associated problem, called the variational inequality and which is to find vector $y \in \mathbb{R}^N_{+}$ such that

$$(z - y)^T \tilde{G}(y) \geq 0, \quad \forall z \in \mathbb{R}^N_{+}.\,$$
In notation of Definition 1.1.1 from FP, this problem is denoted as $VI(\mathbb{R}^N_+, \tilde{G})$. Since the set $\mathbb{R}^N_+$ is a particular example of a cone, we can apply Proposition 1.1.3 from FP to get that a vector $y$ solves $NCP(\tilde{G})$ if and only if $y$ solves $VI(\mathbb{R}^N_+, \tilde{G})$. So, for us $NCP(\tilde{G})$ and $VI(\mathbb{R}^N_+, \tilde{G})$ are equivalent problems. It is important, because most of propositions in FP are given in terms of the variational inequality problem.

In the propositions from FP which we use, we need $\tilde{G}$ to be continuous. This property is understood as continuity of $\tilde{G}$ on some open set containing $\mathbb{R}^N_+$ (see Section 1.1, page 2, in FP). Similarly for differentiability. It is easy to construct such set in our environment:

$$D \equiv \{ z \in \mathbb{R}^N : z_i > -\epsilon \text{ for } i = 1, \ldots, I \},$$

where $\epsilon > 0$ is some small number such that $\sum_{i=1}^N a_{nl}z_l > -\delta$ for $n = 1, \ldots, N$ and all $z \in D$. Clearly, $D$ is an open convex set and $\mathbb{R}^N_+ \subseteq D$. It is obvious that $\tilde{G}$ is continuous in $D$. Moreover, $\tilde{G}$ is continuously differentiable in $D$.

Let us move to the existence of solution of $NCP(\tilde{G})$. We are going to use Proposition 2.2.3 from FP. This proposition requires that $\tilde{G}$ is continuous on some open set containing $\mathbb{R}^N_+$. As it is discussed above, we have this property. Also Proposition 2.2.3 requires that a certain feasibility condition holds. Namely, there exists a vector $y^{ref} \in \mathbb{R}^N_+$ such that the set

$$L_\prec \equiv \{ y \in \mathbb{R}^N_+ : \tilde{G}(y)^T(y - y^{ref}) < 0 \}$$

is bounded. Take $y^{ref} = 0$, then

$$L_\prec = \left\{ y \in \mathbb{R}^N_+ : \sum_{i} y_i < \sum_{i} \frac{a_{ni}y_i}{\sum_{l} a_{nl}y_l + \delta \sum_{l} a_{nl}} b_n \right\}.$$

Since for $y \in \mathbb{R}^N_+$ we have that

$$\sum_{i} \frac{a_{ni}y_i}{\sum_{l} a_{nl}y_l + \delta \sum_{l} a_{nl}} b_n < \sum_{i} \sum_{n} b_n = N \sum_{n} b_n,$$

the set $L_\prec$ is bounded. So, conditions for Proposition 2.2.3 are satisfied, and hence $VI(\mathbb{R}^N_+, \tilde{G})$ has a solution. This, in turn, implies that $NCP(\tilde{G})$ has a solution.

For uniqueness we are going to employ the fact that the Jacobian of $\tilde{G}$ is a positive
definite matrix for all points in the open convex set \( D \) defined earlier. Denote

\[
\tilde{\lambda}_{ni} \equiv \frac{a_{ni}b_{n}^{\frac{1}{2}}}{\sum_{l}a_{nl}y_{l} + \delta \sum_{l}a_{nl}},
\]

and let \( \tilde{\Lambda} \) be an \( N \times N \) matrix with elements \( \tilde{\lambda}_{ni} \). Then the Jacobian of mapping \( \tilde{G} \) can be written as \( J = \tilde{\Lambda}^{T}\tilde{\Lambda} \). Recall that \( a_{ni} = S_{i}^{PC}(w_{i}^{\tau_{mi}})^{-\varepsilon_{PC}}w_{i}^{-1} \). So, we can write \( \tilde{\Lambda} = MR \), where \( M \) is a diagonal matrix with elements \( b_{n}^{\frac{1}{2}}/\left(\sum_{l}a_{nl}y_{l} + \delta \sum_{l}a_{nl}\right) \) on the diagonal; \( R \) is a diagonal matrix with elements \( S_{i}^{PC}w_{i}^{-\varepsilon_{PC}-1} \) on the diagonal, and

\[
\tilde{T} \equiv \begin{pmatrix}
\tau_{11}^{-\varepsilon_{PC}} & \cdots & \tau_{N1}^{-\varepsilon_{PC}} \\
\vdots & \ddots & \vdots \\
\tau_{N1}^{-\varepsilon_{PC}} & \cdots & \tau_{NN}^{-\varepsilon_{PC}}
\end{pmatrix}.
\]

Diagonal entries of matrices \( M \) and \( R \) are positive for all points \( y \in D \). Therefore, these matrices are invertible. Also, matrix \( \tilde{T} \) is invertible by Assumption 1. Hence, \( \tilde{\Lambda} \) is an invertible matrix. This, in turn, implies that \( J \) is a positive definite matrix for all \( y \in D \).

Then Proposition 2.3.2 from FP implies that function \( \tilde{G} \) is strictly monotone on \( D \). Finally, Theorem 2.3.3 from FP states that for a continuous and strictly monotone mapping operating on a closed convex subset of \( \mathbb{R}^{N} \) the corresponding variational inequality problem has at most one solution. Since \( \tilde{G} \) has all the properties needed for Theorem 2.3.3 from FP, we conclude that \( VI(\mathbb{R}^{N}, \tilde{G}) \) has at most one solution. This, again, implies that \( NCP(\tilde{G}) \) has at most one solution. Since earlier we argued that there exists a solution of \( NCP(\tilde{G}) \), this completes our proof.

**Existence for \( \delta = 0 \).** We prove existence of the solution to (2.8) by taking the limit of solutions of (2.9) for \( \delta > 0 \).

Define functions \( F_{\delta} : \mathbb{R}^{N}_{+} \to \mathbb{R}^{N} \) and \( F : \mathbb{R}^{N}_{+} \to \mathbb{R}^{N} \) by

\[
F_{\delta}(x) \equiv (x_{i} - \delta) - \sum_{n=1}^{N} \frac{a_{ni}(x_{i} - \delta)}{\sum_{l=1}^{N}a_{nl}x_{l}}b_{n}, \quad \text{and} \quad F(x) \equiv x_{i} - \sum_{n=1}^{N} \frac{a_{ni}x_{i}}{\sum_{l=1}^{N}a_{nl}x_{l}}b_{n}.
\]

As we showed above, for each \( \delta > 0 \) there is a unique solution of the NCP problem (2.9). Let us denote this solution by \( x_{\delta} \). Let us choose some number \( \Delta > 0 \) and consider the set
of solutions of (2.9) for all $\delta \in (0, \Delta]$. Suppose that $\inf \{\sum x^\delta_i : x^\delta \in \mathcal{X}\} = 0$. That means that for any $\epsilon > 0$ there exists $\delta \in (0, \Delta]$ with the corresponding $x^\delta \in \mathcal{X}$ such that $\sum x^\delta_i < \epsilon$. But then, by choosing $\epsilon$ sufficiently small, we can violate the constraint $G_i(x^\delta) \geq 0$, which contradicts to the choice of $x^\delta$ as solving the NCP problem (2.9). Hence, $\inf \{\sum x^\delta_i : x^\delta \in \mathcal{X}\} = \tilde{A} > 0$. Let $A \equiv \min \{\tilde{A}, \sum b_i\}$.

Next, consider the following compact set

$$\Gamma = \left\{ x \in \mathbb{R}_+^N : A \leq \sum_i x_i \leq N\Delta + \sum_i b_i \right\}.$$  

For any $\delta \in (0, \Delta]$ we have that $F^\delta(x^\delta) = 0$. Therefore

$$\sum_i x^\delta_i = N\delta + \sum_i \sum_{n=1}^N \frac{a_{ni}(x^\delta_i - \delta)}{\sum_{l=1}^N a_{nl}x^\delta_l} b_n \leq N\Delta + \sum_i b_i,$$

which implies that $x^\delta \in \Gamma$ for any $0 < \delta \leq \Delta$. Also, if there exists a solution of (2.8), this solution is necessarily in the set $\Gamma$, because for this solution $\sum_i x_i = \sum_i b_i$. So, let us restrict the domains of $F^\delta$ and $F$ to $\Gamma$.

Clearly, for any $\delta \in (0, \Delta]$, $F^\delta$ is uniformly bounded on $\Gamma$. Moreover, all partial derivatives of $F^\delta_i$ for each $i$ are also uniformly bounded on $\Gamma$. Hence, the family of functions $\{F^\delta\}_{\delta > 0}$ is equicontinuous and uniformly bounded on $\Gamma$. Therefore, by the Arzelà-Ascoli theorem, there is a sequence $\{\delta_h\}_{h=1}^\infty$ such that $\delta_h \to 0$ and $F^{\delta_h}$ uniformly converges to some function on $\Gamma$ as $h \to \infty$. Obviously, any converging sequence of $F^{\delta_h}$ should converge to $F$. Each element of this sequence has an associated point $x^{\delta_h} \in \Gamma$ which solves (2.9), and so $F^{\delta_h}(x^{\delta_h}) = 0$. Since $\Gamma$ is compact, by Bolzano-Weierstrass theorem there is a convergent subsequence of $x^{\delta_h}$. For simplicity of notation, let us index elements of this subsequence by $\delta_h$. Let $x \equiv \lim_{h \to \infty} x^{\delta_h}$.

Then

$$\|F(x)\| = \lim_{h \to \infty} \|F(x^{\delta_h})\| = \lim_{h \to \infty} \|F(x^{\delta_h}) - F^{\delta_h}(x^{\delta_h})\| \leq \lim_{h \to \infty} \sup_{z \in \Gamma} \|F(z) - F^{\delta_h}(z)\| = 0,$$

where the last equality is due to uniform convergence on $\Gamma$. Hence, $x$ is a solution of $F(x) = 0$.

Next, since $x^{\delta_h} \geq 0$ for all $\delta_h$, we have that $x \geq 0$. And, since $G(\cdot)$ is continuous on $\Gamma$ and $G(x^{\delta_h}) \geq 0$ for all $\delta_h$, we have that $G(x) \geq 0$. Hence, $x$ is a solution of (2.8).
Uniqueness for $\delta = 0$. Consider the following sets:

$$
S_1 \equiv \left\{ x \in \mathbb{R}_+^N : \sum_i x_i = \sum_i b_i \right\}, \text{ and }
S_2 \equiv \left\{ x \in \mathbb{R}_+^N , x \neq 0 : G(x) \geq 0 \right\}.
$$

Take any $x \in S \equiv S_1 \cap S_2$. We have

$$
0 = \sum_i (x_i - b_i) = \sum_i \left( x_i - \sum_n a_{ni} x_i \frac{b_n}{\sum_l a_{nl} x_l} \right) = \sum_i x_i G_i(x).
$$

Since $x_i \geq 0$ and $G_i(x) \geq 0$, the above sum can be zero only if every term of the sum is zero, i.e., $x_i G_i(x) = 0$ for all $i$. Hence, if $x \in S$, then $x$ solves (2.8).

Set $S_1$ is obviously convex. Let us show that set $S_2$ is also convex. For that we need to show that functions $G_i$ are concave for $x \in \mathbb{R}_+^N, x \neq 0$. Since $G_i$ is continuously differentiable for all $x \in \mathbb{R}_+^N, x \neq 0$, we just need to check that the corresponding Hessians are negative semi-definite matrices. The element $(r,s)$ of the Hessian of $G_i(x)$ is

$$
H_{rs}(x) = -2 \sum_{n=1}^N \frac{a_{ni} a_{nr} a_{ns}}{(\sum_l a_{nl} x_l)^3} b_n.
$$

For any vector $z \in \mathbb{R}^N$:

$$
z^T H(x) z = -2z^T \begin{pmatrix}
\sum_r z_r \sum_{n=1}^N \frac{a_{ni} a_{nl} a_{ns}}{(\sum_l a_{nl} x_l)^3} b_n \\
\vdots \\
\sum_r z_r \sum_{n=1}^N \frac{a_{ni} a_{nN} a_{ns}}{(\sum_l a_{nl} x_l)^3} b_n
\end{pmatrix}
= -2 \sum_r z_r \sum_{s} z_s \sum_{n=1}^N \frac{a_{ni} a_{nr} a_{ns}}{(\sum_l a_{nl} x_l)^3} b_n
= -2 \sum_{n} \frac{a_{ni}}{(\sum_l a_{nl} x_l)^3} b_n \left( \sum_r z_r a_{nr} \right)^2 \leq 0.
$$

Hence, $H(x)$ is a negative semi-definite matrix, and so $G_i(x)$ is concave function for all $x \in \mathbb{R}_+^N$. Hence, the set of solutions of (2.8), $S$, is a convex set.

Suppose that there are two distinct points $x', x'' \in S$. It cannot be the case that $x'$ and $x''$ are both interior allocations, because otherwise both of them solve the same problem (B.1) for $\delta = \min \{ \min_i \{ x'_i \} , \min_i \{ x''_i \} \}$. Since $S$ is a convex set, $\gamma x' + (1 - \gamma) x'' \in S$ for all $\gamma \in [0,1]$. Moreover, since $x' \neq x''$, by choosing any two distinct numbers $\gamma', \gamma'' \in (0,1)$,
we can get two distinct solutions of (2.8) with the same patterns of zeros. For simplicity of notation, denote these two solutions by $x'$ and $x''$. Suppose without loss of generality that $x'_i \neq 0$ for $i = 1, \ldots, I$ and $x'_i = 0$ for $i = I + 1, \ldots, N$. Similarly for $x''$.

Consider a subproblem of (2.9) in which we set $x_i = 0$ for $i = I + 1, \ldots, N$ and want to find $x_i \geq \delta$ such that $G_i(x) \geq 0$ and $x_i G_i(x) = 0$ for $i = 1, \ldots, I$ for which $\delta = \min \{ \min_{i=1,\ldots,I} \{ x'_i \}, \min_{i=1,\ldots,I} \{ x''_i \} \}$. With the same change of variables as we did in (B.1), the corresponding subproblem of (B.1) is now

$$y_i \geq 0, \quad \tilde{G}_i(y) \geq 0, \quad \text{and} \quad y_i \tilde{G}_i(y) = 0, \quad i = 1, \ldots, I,$$  

where $y \equiv (y_1, \ldots, y_I)$ and

$$\tilde{G}_i(y) \equiv 1 - \sum_{n=1}^N \frac{a_{ni}}{\sum_{l=1}^I a_{nl}y_l} - \delta \sum_{n=1}^N \frac{b_n}{\sum_{l=1}^I a_{nl}}.$$

By our supposition, there are two distinct solutions of this problem: $(x'_1 - \delta, \ldots, x'_I - \delta)$ and $(x''_1 - \delta, \ldots, x''_I - \delta)$. Let us show that it cannot be the case.

Clearly, $\tilde{G}$ is continuously differentiable in an open convex set

$$\mathcal{D} \equiv \{ z \in \mathbb{R}^N : z_i > -\epsilon \quad \text{for} \quad i = 1, \ldots, I \}$$

for some small enough $\epsilon > 0$ such that $\sum_{l} a_{nl}z_l > -\delta$ for $n = 1, \ldots, N$ and all $z \in \mathcal{D}$. Denote

$$\tilde{\lambda}_{ni} \equiv \frac{a_{ni}b_n^{\frac{1}{2}}}{\sum_{l=1}^I a_{nl}y_l + \delta \sum_{l=1}^N a_{nl}}, \quad n = 1, \ldots, N \quad \text{and} \quad i = 1, \ldots, I,$$

and let $\tilde{\Lambda}$ be an $N \times I$ matrix with elements $\tilde{\lambda}_{ni}$. Then the Jacobian of mapping $\tilde{G}$ is $J = \tilde{\Lambda}^T \tilde{\Lambda}$.

To prove that $J$ is a positive definite matrix, we need to show that all its leading principal minors are positive. Let us look at the determinant of the whole matrix $J$. Clearly $\det(J) \geq 0$. So, we need to show that $\det(J) \neq 0$, i.e., $\text{rank}(J) = I$. We know that $\text{rank}(J) = \text{rank} \left( \tilde{\Lambda} \right)$.

Matrix $\tilde{\Lambda}$ can be decomposed into a product of three matrices: $\tilde{\Lambda} = M\tilde{T}R$, where $M$ is a diagonal $N \times N$ matrix with diagonal elements

$$m_{nn} = \frac{b_n^{\frac{1}{2}}}{\sum_{l=1}^I a_{nl}y_l + \delta \sum_{l=1}^N a_{nl}};$$

These solutions will have zeros only where both $x'$ and $x''$ simultaneously have zeros.
$R$ is a diagonal $I \times I$ matrix with elements $S^\text{PC}_i w_i^{\varepsilon \text{PC}} - \varepsilon \text{PC}^{-1}$ on the diagonal, and

$$
\tilde{T} \equiv \begin{pmatrix}
\tau_{11}^{-\varepsilon \text{PC}} & \cdots & \tau_{1I}^{-\varepsilon \text{PC}} \\
\vdots & \ddots & \vdots \\
\tau_{NI}^{-\varepsilon \text{PC}} & \cdots & \tau_{NI}^{-\varepsilon \text{PC}}
\end{pmatrix}.
$$

Since, $M$ and $R$ are diagonal matrices with positive diagonals, $\text{rank}(M) = N$ and $\text{rank}(R) = I$. Therefore, $\text{rank}(M\tilde{T}R) = \text{rank}(\tilde{T})$. Assumption 1 implies that the columns of matrix $\tilde{T}$ are linearly independent. Hence, $\text{rank}(\tilde{T}) = I$, which, in turn, implies that $\text{rank}(\tilde{\Lambda}) = I$ and $\text{rank}(J) = I$, and so $\det(J) > 0$.

Repeating a similar argument for all other leading principal minors of $J$, we can establish that all of them are positive. Hence, $J$ is a positive definite matrix. Therefore, problem (B.2) has a unique solution. We get a contradiction, and so set $S$ is a singleton.

### B.1.3 Proof of Proposition 2, property (i).

Proposition 1 implies that for each industry $k$ there exists a unique function $L_k : \mathbb{R}^N_{++} \to \mathbb{R}^N_{++}$ which maps wages into equilibrium labor allocations in sector $k$. We need to prove that this function is continuous in its domain, i.e., we need to prove that $L_k(w)$ is continuous in each point $w \in \mathbb{R}^N_{++}$. Consider any such point $w$. Let us separately analyze the cases of (a) $\alpha_k < 1$ and (b) $\alpha_k = 1$.

(a) In the case of $\alpha_k < 1$, we know that $L_k(\cdot)$ is a positive vector in any open subset of $\mathbb{R}^N_{++}$ containing $w$. So, in the case of $\alpha_k < 1$, $L_k(\cdot)$ satisfies the system of equations $G_{i,k}(w, L_k) = 0$ with $i = 1, \ldots, N$, where $G_{i,k}$ is given by (2.5). If the Jacobian $\partial G_{i,k} / \partial L_{j,k}$ for $i = 1, \ldots, N$ and $j = 1, \ldots, N$ calculated in point $(w, L_k)$ is an invertible matrix, then the implicit function theorem would imply that $L_k(\cdot)$ is continuous in $w$.

Denote the Jacobian $\partial G_{i,k} / \partial L_{j,k}$ by $J_{i,k}(w, L_k)$. The elements of this Jacobian, calculated in point $(w, L_k(w))$, are

$$
J_{i,k}(w, L_k(w)) = (1 - \alpha_k)w_i L_{i,k}(w)^{-1} + \alpha_k L_{i,k}(w)^{-1} \sum_n \lambda_{ni,k}(w, L_k(w)) \beta_{n,k} w_n L_n,
$$

$$
J_{j,k}(w, L_k(w)) = \alpha_k L_{i,k}(w)^{-1} L_{j,k}(w)^{-1} \sum_n \lambda_{ni,k}(w, L_k(w)) \lambda_{nj,k}(w, L_k(w)) \beta_{n,k} w_n L_n,
$$

where $\lambda_{ni,k}(w, L_k(w))$ are the trade shares given by (2.4). Denote by $W$ the diagonal matrix with elements $w_i$ on the diagonal, by $L_k$ the diagonal matrix with elements $L_{i,k}(w)$ on the
diagonal, by $B_k$ the diagonal matrix with elements $\beta_{i,k}w_i\bar{L}_i$ on the diagonal, and by $\Lambda_k$ the matrix with elements $\lambda_{ni,k}(w, L_k(w))$. In this notation, the Jacobian is

$$J_k(w, L_k) = (1 - \alpha_k)WL_k^{-1} + \alpha_kL_k^{-1}\Lambda_k^TB_k\Lambda_kL_k^{-1}.$$ 

The matrix $L_k^{-1}\Lambda_k^TB_k\Lambda_kL_k$ is positive semi-definite and $WL_k^{-1}$ is a diagonal matrix with positive elements on the diagonal. Therefore, $J_k(w, L_k)$ is a positive-definite matrix. In particular, it is invertible. Therefore, by the implicit function theorem, $L_k(\cdot)$ is continuous in $w$.

(b) In the case of $\alpha_k = 1$, the implicit function theorem cannot be applied, because labor allocations $L_k(w)$ can be zero for some $i$ and they solve the nonlinear complementary slackness problem (2.6) in its general form. In this case, we need to rely on the theory of nonlinear complementary slackness problems to show continuity of $L_k(\cdot)$. A relevant result from this theory is Theorem 3.1 from Kyparisis (1986). According to this theorem, for continuity of $L_k(\cdot)$ it is sufficient to show that the Jacobian $\partial G_{i,k}/\partial L_{j,k}$ calculated in point $(w, L_k)$ is a positive definite matrix. In the case of $\alpha_k = 1$ the elements of this Jacobian are given by

$$J_{ij,k}(w, L_k(w)) = \sum_n \tilde{\lambda}_{ni,k}(w, L_k(w))\tilde{\lambda}_{nj,k}(w, L_k(w))\beta_{n,k}w_n\bar{L}_n,$$

where

$$\tilde{\lambda}_{ni,k}(w, L_k(w)) \equiv \frac{T_{i,k}(w, \tau_{ni,k})^{-\theta}}{\sum_l T_{l,k}(w, \tau_{nl,k})^{-\theta}L_{l,k}}.$$ 

From here it is clear that under Assumption 1 $J(w, L_k(w))$ is a positive definite matrix (for the formal proof of that fact see the proof of Proposition 1 for $\alpha = 1$). Hence, $L_k(\cdot)$ is continuous in $w$. ■

B.1.4 Proof of Proposition 2, property (v).

Here we focus on the case of $0 < \alpha_k \leq 1$. The case of $\alpha_k = 0$ is a simple repetition of the corresponding existence proof by Alvarez and Lucas (2007) for each individual industry.

Consider any wages sequence $\{w^s\}_{s=1}^{\infty}$ such that $w^s \to w$ as $s \to \infty$, where $w \neq 0$ is a finite vector of wages such that $w_i = 0$ for some $i$. If there exists some industry $(i,k)$
such that \( L_{i,k}(w^s) \to \infty \), then \( Z_i(w^s) \to \infty \) and so \( \max_i \{ Z_1(w^s), ..., Z_N(w^s) \} \to \infty \). Now suppose that there is no such industry and that for any industry \((i,k)\) the limit \( \lim_{s \to \infty} L_{i,k}(w^s) \) is a finite (non-negative) number.

Consider index \(j\) such that wage \( w^s_j\) converges to 0 weakly “faster” than other wages. Formally, index \(j\) is such that for any \(j' \neq j\) the limit \( \lim_{s \to \infty} w^s_j / w^s_{j'} \) is finite. Such index always exists because there is a finite number of indices. We have that, for any \(s\), \( L_{j,k}(w^s) \) satisfies the inequality

\[
 w^s_j \geq \sum_{n=1}^{N} \frac{T_{j,k} L_{j,k}(w^s)_{\alpha k - 1} \left[ T_{j,k} T_{n,j,k} \right]^{-\theta_k}}{T_{l,k} L_{l,k}(w^s)_{\alpha k} \left[ T_{l,k} T_{n,l,k} \right]^{-\theta_k}} \beta_{n,k} w_n \bar{L}_n. \tag{B.3}
\]

Clearly, the denominator of any term in the above summation (B.3) converges to a finite number (which can be either positive or zero). The numerator of any term in the summation (B.3) converges to either a finite positive number or to infinity. Also, there exists at least one index \(n\) such that \( \lim_{s \to \infty} w^s_n > 0 \). Then, for this index \(n\) the corresponding term in the summation (B.3) converges to either a finite positive number or to \( \infty \). This, in turn, implies that the whole sum in (B.3) converges to either a finite positive number or to \( \infty \). At the same time, the left-hand side of inequality (B.3) converges to 0. A contradiction. Hence, there is at least one industry \((i,k)\) such that \( L_{i,k}(w^s) \to \infty \), and so \( \max_i \{ Z_1(w^s), ..., Z_N(w^s) \} \to \infty \).

\[\blacksquare\]

**B.1.5 Proposition 3 for \(N = 2\).**

This proof proceeds by showing that \( Z(w) \) satisfies the gross substitutes property (GSP). Uniqueness of solution then follows from Proposition 17.F.3 from MWG.

Let us separately consider the cases (a) \(\alpha < 1\) and (b) \(\alpha = 1\).

(a) If \(\alpha < 1\), then for any \((i,k)\) we have that \( L_{i,k}(w) > 0 \) for any wage vector \(w \in \mathbb{R}^N_+\).
and $L_{i,k}(w)$ solves:

$$w_i L_{i,k}(w) = \sum_n \lambda_{n,i,k}(w, L(w)) \beta_{n,k} w_n \bar{L}_n.$$ 

By differentiating both sides of this expression by $w_j$, we can get a linear system of equations which determines partial derivatives of labor allocations w.r.t. wages:

$$
\begin{pmatrix}
\sum_k \frac{\partial L_{1,k}(w)}{\partial w_1} \\
\sum_k \frac{\partial L_{2,k}(w)}{\partial w_1}
\end{pmatrix}
\begin{pmatrix}
\sum_k \frac{\partial L_{1,k}(w)}{\partial w_2} \\
\sum_k \frac{\partial L_{2,k}(w)}{\partial w_2}
\end{pmatrix}. 
$$

We can then directly check that this Jacobian satisfies the GSP. Namely, that for all $k$ we have $\frac{\partial L_{i,k}(w)}{\partial w_i} < 0$ for $i = 1, 2$, and $\frac{\partial L_{i,k}(w)}{\partial w_j} > 0$ for $i \neq j$. That, in turns, obviously implies that $Z(w)$ satisfies the GSP.

As in other proofs, let us introduce additional notation to write in matrix form the linear system of equations determining the wages effects on labor allocations. Denote $x_{i,j,k} \equiv \frac{d \ln L_{i,k}(w)}{d \ln w_j}$, $q_{i,k} \equiv w_i L_{i,k}(w)$ and $e_{i,k} \equiv w_i \beta_{i,k} \bar{L}_i$. Let $E_k$ denote the diagonal matrix with elements $e_{i,k}$ along the diagonal, $Q_k$ the diagonal matrix with elements $q_{i,k}$ along the diagonal, $\Lambda_k$ the matrix of sector level expenditure shares $\lambda_{i,j,k}$, and $X_k$ the matrix of partials $x_{i,j,k}$. Finally, let $A_k \equiv \left((1 - \alpha_k) Q_k + \alpha_k \Lambda_k^T E_k \Lambda_k\right)$ and $B_k \equiv \left(\Lambda_k E_k + \theta_k \Lambda_k^T E_k \Lambda_k - (1 + \theta) Q_k\right)$. In this notation the system of partials is simply

$$A_k X_k = B_k.$$ 

It straightforward to check that $A_k$ is positive definite matrix with all positive elements, and $B_k$ is a matrix with negative diagonal and positive off-diagonal elements. Since $A_k$ is positive definite, the inverse exists and its determinant is positive. Moreover, $A_k^{-1} = \frac{1}{\det(A_k)} C_k^T$, where $C_k^T$ is the transpose of the matrix of cofactors $C_k$. Since all the elements of $A_k$ are positive, then $C_k$ is a matrix consisting of positive diagonal elements and negative off-diagonal elements. Moreover, $C_k^T$ and $A_k^{-1}$ have this property as well. One can then readily verify that for the $2 \times 2$ case, $A_k^{-1} B_k$ is a matrix with the same properties as $B_k$ — negative diagonal and positive off diagonal elements. Thus the Jacobian of wages effects on labor allocations, indeed, satisfies the GSP.

(b) If $\alpha_k = 1$, then $L_{i,k}(w)$ can be equal to 0 for some $(i, k)$ and we cannot establish
differentiability of labor allocations in that region. We will use a more direct approach to prove the GSP.

Assume that \( \mathbf{w}^\prime \) and \( \mathbf{w}'' \) are such that w.l.o.g. \( w''_i > w'_1 \) and \( w''_2 = w'_2 \). To show that \( Z(\mathbf{w}) \) satisfies the GSP, we need to show that \( Z_2(\mathbf{w}'') > Z_2(\mathbf{w}') \).

In general, given wage \( \mathbf{w} \) there are three possibilities for each sector \( k \): (1) \( L_{i,k} > 0 \) for \( i = 1,2 \); (2) \( L_{1,k} = 0 \) and \( L_{2,k} = \frac{1}{w_2} (\beta_{1,k}w_1L_1 + \beta_{2,k}w_2L_2) \); (3) \( L_{2,k} = 0 \) and \( L_{1,k} = \frac{1}{w_1} (\beta_{1,k}w_1L_1 + \beta_{2,k}w_2L_2) \). Let us consider these different cases.

(1) If for \( \mathbf{w}'' \) we are also in case (1), then we can apply the proof for the case of \( \alpha_k < 1 \) and conclude that \( L_{2,k}(\mathbf{w}'') > L_{2,k}(\mathbf{w}') \).

Next, suppose that for \( \mathbf{w}'' \) we are in case (2). We know that, as long as we are in case (1), \( L_{1,k}(\mathbf{w}) \) is a decreasing function of \( w_1 \), while \( L_{2,k}(\mathbf{w}) \) is an increasing function of \( w_1 \). And, as long as we are in case (2), \( L_{2,k}(\mathbf{w}) \) is an increasing function of \( w_1 \). Then, since \( L_k(\mathbf{w}) \) is continuous, there exists some \( \tilde{w}_i \) such that we are in case (1) for all wages \( (w_1, w'_2) \) with \( w'_1 \leq w_1 < \tilde{w}_1 \) and we are in case (2) for all wages \( (w_1, w''_2) \) with \( \tilde{w}_1 \leq w_1 \leq w''_1 \). This immediately implies that \( L_{2,k}(w''_1, w''_2) > L_{2,k}(\tilde{w}_1, w''_2) > L_{2,k}(w'_1, w'_2) = L_{2,k}(w'_1, w'_2) \).

Finally, since \( L_k(\mathbf{w}) \) is continuous, we cannot be in case (3) for \( \mathbf{w}'' \).

(2) Now suppose that \( \mathbf{w}' \) is such that we are in case (2). Then, because of continuity of \( L_k(\mathbf{w}) \), we can only be in case (2) for \( \mathbf{w}'' \). Therefore, again, \( L_{2,k}(w''_1, w''_2) > L_{2,k}(w'_1, w'_2) \).

(3) Finally, imagine that \( \mathbf{w}' \) is such that we are in case (3). Because of continuity of \( L_k(\mathbf{w}) \), we can only be in case (1) or (3) for \( \mathbf{w}'' \). If we are in case (1) for \( \mathbf{w}'' \), then \( L_{2,k}(w''_1, w''_2) > L_{2,k}(w'_1, w'_2) \). If we are in case (3) for \( \mathbf{w}'' \), then it must be the case that there exists some industry \( \tilde{k} \), for which we are in case (1) or (2) for \( \mathbf{w}' \). For any such industry we have \( L_{2,\tilde{k}}(w''_1, w''_2) > L_{2,\tilde{k}}(w'_1, w'_2) \).

From (1)-(3) we conclude that \( Z_2(\mathbf{w}'') > Z_2(\mathbf{w}') \). ■

We now proceed to sketch of our strategy for establishing uniqueness of wages for more than two countries.

### B.1.6 Extending Proposition 3 to \( N > 2 \)

Denote

\[
Z(\mathbf{w}) \equiv (Z_1(\mathbf{w}), \ldots, Z_N(\mathbf{w}))
\]

We need to solve the system of equations \( Z(\mathbf{w}) = 0 \). Of course, the Walras' law implies that we only need \( N - 1 \) equations of this system with some wage chosen as numéraire, but for
convenience of notation we will keep all $N$ equations. The Jacobian of this system is

$$DZ(w) = \begin{pmatrix}
\frac{\partial Z_1(w)}{\partial w_1} & \cdots & \frac{\partial Z_1(w)}{\partial w_N} \\
\vdots & \ddots & \vdots \\
\frac{\partial Z_N(w)}{\partial w_1} & \cdots & \frac{\partial Z_N(w)}{\partial w_N}
\end{pmatrix}.$$

Since all $N$ equations of the system $Z(w) = 0$ are linearly dependent, the determinant of the whole Jacobian is zero. At the same time, if we can show that any principal submatrix of this Jacobian of size $N - 1$ is a $P$-matrix, then Theorem 4 from Gale and Nikaido (1965) will imply that the system $Z(w) = 0$ has a unique solution up to some normalization.

We have

$$\frac{\partial Z_i(w)}{\partial w_j} = \sum_{k=1}^{K} \frac{\partial L_{i,k}(w)}{\partial w_j}.$$

Then

$$DZ(w) = \sum_{k=1}^{K} \begin{pmatrix}
\frac{\partial L_{1,k}(w)}{\partial w_1} & \cdots & \frac{\partial L_{1,k}(w)}{\partial w_N} \\
\vdots & \ddots & \vdots \\
\frac{\partial L_{N,k}(w)}{\partial w_1} & \cdots & \frac{\partial L_{N,k}(w)}{\partial w_N}
\end{pmatrix}$$

$$= \sum_{k=1}^{K} L_k X_k W^{-1} = \sum_{k=1}^{K} L_k A_k^{-1} B_k W^{-1}.$$

Matrix $B_k$ has a negative diagonal and positive off-diagonal elements. Its columns sum up to zero. Hence, $-B_k$ is a general $M$-matrix (see Section 2.3 in Gale and Nikaido, 1965).\(^3\) Our extensive numerical simulations show that $DZ(w)$ has the properties needed for uniqueness of $w$: any principal submatrix of $DZ(w)$ of size $N - 1$ is a $P$-matrix. The formal proof of this property is coming soon. ■

\(^3\)Matrix $A$ is a general $M$-matrix if $A + \epsilon I$ is an $M$-matrix for any $\epsilon > 0$.  

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B.2 Gains from Trade

B.2.1 Proof of Proposition 4

For part (i) it is sufficient to show that for any industry \( k \) the price index is lower with trade. Suppressing subindex \( k \), first note that we have

\[
\sum_n \lambda_n(L)E_n = w_iL_i
\]

where \( E_n \) is exogenous spending by country \( n \) and

\[
\lambda_n(L) \equiv \frac{T_iL_i^\alpha(w_{i\tau_{ni}})^{-\varepsilon}}{\sum_l T_l^\alpha(w_{l\tau_{nl}})^{-\varepsilon}}
\]

Note that

\[
P_n^{-\varepsilon} = \sum_l T_l^\alpha(w_{l\tau_{nl}})^{-\varepsilon}
\]

Thus, if we show that \( Z_n = \sum_l T_l^\alpha(w_{l\tau_{nl}})^{-\varepsilon} \) increases as we move from autarky to the trade equilibrium, we would then establish that \( P_n \) falls and hence welfare increases. Now, note that in autarky \( Z_n = T_nL_n^\alpha w_n^{-\varepsilon} \). Thus, we need to show that

\[
T_nL_n^\alpha w_n^{-\varepsilon} < \sum_l T_l^\alpha(w_{l\tau_{nl}})^{-\varepsilon}
\]

where \( L \) satisfies

\[
\sum_n \frac{T_iL_i^\alpha(w_{i\tau_{ni}})^{-\varepsilon}}{\sum_l T_l^\alpha(w_{l\tau_{nl}})^{-\varepsilon}}E_n = w_iL_i
\]

If \( \alpha = 0 \) then this is trivial. If \( \alpha = 1 \) then in an interior equilibrium the system for \( L \) is

\[
\sum_n \frac{T_i(w_{i\tau_{ni}})^{-\varepsilon}}{\sum_l T_l(w_{l\tau_{nl}})^{-\varepsilon}}E_n = w_i
\]

Using \( P_n^{-\varepsilon} = \sum_l T_l(w_{l\tau_{nl}})^{-\varepsilon} \) we then have (using \( w_i \) as numÃ­raire)

\[
\sum_n T_iP_i^\varepsilon E_i + \sum_{n \neq i} T_i\tau_{ni}^{-\varepsilon} P_n^\varepsilon E_n = 1
\]

Since in autarky we must have

\[
\sum_n T_iP_i^\varepsilon E_i = 1
\]
then this immediately implies that \( P^e_i \) must be lower with trade than in autarky. Now let’s extend this methodology to a case with \( \alpha \in (0, 1) \). Now we have

\[
T_i P^e_i E_i + \sum_{n \neq i} T_i r_{ni}^e P^e_n E_n = L_i^{1-\alpha}
\]

If \( L_i \) falls relative to autarky then \( P_i \) has to fall as well. And if \( L_i \) increases then note that

\[
\lambda_{ii} = \frac{T_i L_i^\alpha (w_i)^{-\varepsilon}}{\sum_l T_l L_l^\alpha (u_l T_{il})^{-\varepsilon}} = \frac{T_i L_i^\alpha}{P_i^{-\varepsilon}}
\]

This implies that

\[
P^e_i = \frac{\lambda_{ii}}{T_i L_i^\alpha}
\]

If \( L_i \) increases relative to autarky, then since \( \lambda_{ii} \) is obviously lower relative to autarky, we establish that \( P^e_i \) is lower relative to autarky.

Part (ii) follows in a straightforward way as outlined in the text of Subsection 4.

For Part (iii) the problem is to first (suppressing \( n \) subindex):

\[
\min_{r_1, \ldots, r_K} \left[ \prod_k \left( \frac{e_k}{r_k} \right)^{e_{n,k}} \right]^\psi,
\]

where \( r_k = 1 - \sum_{h \neq k} r_h \) and \( \sum_k e_k = 1 \).

One can then readily verify that the first order conditions for this minimization problem reduces to:

\[
\frac{e_k}{r_k} = \sum_{h \neq k} \frac{e_k}{r_h} \quad \text{for} \quad k = 1, \ldots, K.
\]

Manipulating this system yields \( \frac{e_k}{r_k} = \frac{e_k}{r_h} \) for all \( k \neq h \), which together with \( \sum_k e_k = 1 \) gives

\[
r_k = e_k \quad \text{for} \quad k = 1, \ldots, K.
\]

To show this is a global minimum, one can then readily verify that the Hessian is everywhere positive semi-definite. Thus, we have . Since for any \( r_{n,k} \) and \( e_{n,k} \) the function is increasing in \( \psi \), the latter part of (iii) follows. ■
Appendix C

Linking Melitz and Marshall

C.1 Linking Melitz and Marshall

We consider a generalized Melitz-Pareto framework, in which the elasticity of substitution between varieties from different countries, $\eta_k$, is different from the elasticity of substitution between varieties from the same country, $\sigma_k$.

Let us ignore the industry subscript for a moment. Let $\Omega_{ni}$ denote the set of goods that $i$ sells to $n$. The price index of these goods is $P_{ni} \equiv \left( \int_{\omega \in \Omega_{ni}} p_{ni}(\omega)^{1-\sigma} d\omega \right)^{\frac{1}{1-\sigma}}$. Let $M_i$ denote the entry in country $i$ and $\phi_{ni}^*$ denote the cutoff productivity such that $i$ exports to $n$ all goods with productivity higher than $\psi_{ni}^*$. We have

$$P_{ni}^{1-\sigma} = M_i \int_{\psi_{ni}^*}^{\infty} [p_{ni}(\psi)]^{1-\sigma} dG_i(\psi)$$

$$= \theta b_i^\theta M_i \left[ \frac{\sigma}{\sigma - 1} w_i \tau_{ni} \right]^{1-\sigma} \int_{\psi_{ni}^*}^{\infty} \psi^{\sigma-\theta-2} d\psi$$

$$= M_i \left[ \frac{\sigma}{\sigma - 1} w_i \tau_{ni} \right]^{1-\sigma} \frac{b_i^\theta (\psi_{ni}^*)^{\sigma-\theta-1}}{\theta - (\sigma - 1)}.$$  

The condition that determines the cutoff $\psi_{ni}^*$ is

$$\frac{1}{\sigma} \left( \frac{\sigma}{\sigma - 1} \cdot \frac{w_i \tau_{ni}}{\psi_{ni}^*} \right)^{1-\sigma} P_{ni}^{\sigma-1} \left( \frac{P_{ni}}{P_n} \right)^{1-\eta} E_n = c_{ni}.$$
where \( c_{ni}^m \) is the marketing cost for \( i \) in \( n \) and \( E_n \) is the market size in \( n \). This implies that

\[
\psi_{ni}^* = \frac{w_i \tau_{ni}}{P_{ni}} \left( \frac{\sigma}{\bar{\sigma}} \cdot \frac{c_{ni}^m}{E_n} \right)^{\frac{1}{\sigma-1}} \left( \frac{P_{ni}}{P_n} \right)^{\frac{1-\eta}{\sigma-1}},
\]

where \( \bar{\sigma} \equiv \left( \frac{\sigma}{\sigma-1} \right)^{1-\sigma} \). Plugging this expression into the expression for the price index yields

\[
P_{ni} = w_i \tau_{ni} \left( \frac{w_i \tau_{ni}}{P_{ni}} \left( \frac{E_n}{c_{ni}^m} \right)^{\frac{1}{\sigma-1}} \left( \frac{P_{ni}}{P_n} \right)^{\frac{1-\eta}{\sigma-1}} \right)^{\frac{\sigma}{\sigma-1}-1} \left( \frac{R_i}{w_i} \right)^{\frac{1}{\sigma-1}} \xi_i,
\]

where \( \xi_i \equiv \mu_i \left( \frac{\sigma-1}{\sigma} \right)^{1/(1-\sigma)} \).

Now let’s bring back the industry subscript \( k \) and use \( R_{i,k} = w_i L_{i,k} \) to get

\[
P_{ni,k} = w_i \tau_{ni,k} \left( \frac{w_i \tau_{ni,k}}{P_{ni,k}} \left( \frac{E_{n,k}}{c_{ni,k}^m} \right)^{\frac{1}{\sigma_k-1}} \left( \frac{P_{ni,k}}{P_{n,k}} \right)^{\frac{1-\eta_k}{\sigma_k-1}} \right)^{\frac{\sigma_k}{\sigma_k-1}-1} \frac{1}{L_{i,k}^{\frac{1}{\sigma_k}}} \xi_{i,k},
\]

Assuming that \( c_{ni,k}^m = w_n f_{ni,k}^m \) and using \( E_{n,k} = \beta_{n,k} w_n \bar{L}_n \), we get

\[
P_{ni,k} = w_i \tau_{ni,k} \left( \frac{w_i \tau_{ni,k}}{P_{ni,k}} \left( \frac{P_{ni,k}}{P_{n,k}} \right)^{\frac{1-\eta_k}{\sigma_k-1}} \right)^{\frac{\sigma_k}{\sigma_k-1}-1} \frac{1}{L_{i,k}^{\frac{1}{\sigma_k}}} \xi_{i,k},
\]
where

\[ \kappa_{n_i,k} \equiv \left[ \beta_{n_i,k} \bar{L}_{n_i} \right] \frac{\eta_k}{\sigma_k} \left( \frac{\sigma_k}{\sigma_k - 1} \right)^{\frac{1}{\sigma_k}} \xi_{i,k} . \]

Solving for \( P_{n_i,k} \), we get

\[ P_{n_i,k}^{\frac{\eta_k}{\sigma_k - 1}} \left( \frac{\eta_k}{\sigma_k - 1} \right) = \left( \omega \tau_{n_i,k} \right) \frac{\eta_k}{\sigma_k} \left( \frac{\sigma_k}{\sigma_k - 1} \right)^{\frac{1}{\sigma_k}} \kappa_{n_i,k} , \]

and hence

\[ P_{n_i,k}^{\frac{1-\eta_k}{\sigma_k - 1}} \left( \frac{1-\eta_k}{\sigma_k - 1} \right) = \left( \omega \tau_{n_i,k} \right)^{\frac{1-\eta_k}{\sigma_k - 1}} \frac{\sigma_k - 1}{\sigma_k - 1 + (\eta_k - \sigma_k + 1) \left( \frac{\eta_k - \sigma_k}{\eta_k - \sigma_k} \right)} . \]

Therefore, using \( P_{n_k}^{1-\eta_k} = \sum_i P_{n_i,k}^{1-\eta_k} \), we can write expressions for trade shares:

\[ \lambda_{n_i,k} = \left( \frac{P_{n_i,k}}{P_{n_k}} \right)^{\frac{1-\eta_k}{\sigma_k - 1}} \frac{\sigma_k - 1}{\sigma_k - 1 + (\eta_k - \sigma_k + 1) \left( \frac{\eta_k - \sigma_k}{\eta_k - \sigma_k} \right)} . \]

where

\[ S_{n_i,k} \equiv b_{i,k}^{\alpha_k} \left[ f_{n_i,k} \right]^{\alpha_k} \left( \frac{1-\eta_k}{\sigma_k - 1} \right) , \]

and

\[ \alpha_k \equiv \frac{\eta_k - 1}{\sigma_k - 1} \cdot \frac{\sigma_k - 1}{\sigma_k - 1 + (\eta_k - \sigma_k + 1) \left( \frac{\eta_k - \sigma_k}{\eta_k - \sigma_k} \right)} . \]

Note that if \( \eta_k = \sigma_k \), then \( \alpha_k = 1 \) and

\[ \lambda_{n_i,k} = \frac{b_{i,k}^{\theta_k} \left[ f_{n_i,k} \right]^{\alpha_k} \left( \frac{1-\eta_k}{\sigma_k - 1} \right) \left( \omega \tau_{n_i,k} \right)^{\theta_k} L_{i,k} \}}{\sum_i b_{i,k}^{\theta_k} \left[ f_{n_i,k} \right]^{\alpha_k} \left( \frac{1-\eta_k}{\sigma_k - 1} \right) \left( \omega \tau_{n_i,k} \right)^{\theta_k} L_{i,k} } , \]

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which is a multi-industry analog of expression (6) in Arkolakis et al. (2008).

Next, if $\eta_k < \sigma_k$, then $\alpha_k < 1$. Also, here the trade elasticity is $\varepsilon_k \equiv \theta_k \alpha_k$ and the scale elasticity is $\psi_k \equiv \alpha_k / \varepsilon_k = 1 / \theta_k$. 
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