ESSAYS IN DIRECTED SEARCH WITH PRIVATE INFORMATION

A Dissertation in Economics
by
Seyed Mohammadreza Davoodalhosseini

© 2015 Seyed Mohammadreza Davoodalhosseini

Submitted in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

August 2015
The dissertation of Seyed Mohammadreza Davoodalhosseini was reviewed and approved* by the following:

Neil Wallace  
Professor of Economics, The Pennsylvania State University  
Dissertation Co-Adviser, Co-Chair of Committee

Shouyong Shi  
Professor of Economics, The Pennsylvania State University  
Dissertation Co-Adviser, Co-Chair of Committee

Manolis Galenianos  
Professor of Economics, Royal Holloway, University of London  
Special Member

Edward Green  
Professor of Economics, The Pennsylvania State University

Anthony Kwasnica  
Professor of Business Economics, The Pennsylvania State University

Robert C. Marshall  
Professor of Economics, The Pennsylvania State University  
Head of the Department of Economics

*Signatures are on file in the Graduate School.
Abstract

This dissertation consists of two chapters, each of which is based on a theoretical research paper. The two chapters are related in that in both chapters I study models of directed (competitive) search with private information. However, some details of the environments and the questions addressed in each chapter are different. In Chapter 1, I mainly address a normative question about the (in)efficiency of equilibrium, while the main question in Chapter 2 is how to characterize the equilibrium and what properties the equilibrium has.

In Chapter 1, I characterize the constrained efficient or planner’s allocation in a directed search model with private information. In this economy, buyers post contracts and sellers with private information observe all postings and direct their search toward their preferred contract. Then buyers and sellers match bilaterally and trade. I define a planner whose objective is to maximize social welfare subject to the information and matching frictions of the environment. I show in my main result that if the market economy fails to achieve the first best, then the planner, using a direct mechanism, achieves strictly higher welfare than the market economy. I also derive conditions under which the planner achieves the first best. I show that the planner can implement the direct mechanism by imposing submarket-specific taxes and subsidies on buyers conditional on trade (sales tax).

In an asset market application, I show that in general the efficient sales tax schedule is non-monotone in the price of assets. This non-monotonicity makes the implementation of the direct mechanism difficult in practice. I show that if in addition to sales tax the planner can use entry tax, submarket-specific taxes and subsidies imposed on buyers conditional on entry to each submarket whether they find a match or not, then the planner can implement the direct mechanism by using monotone tax schedules, increasing sales tax and decreasing entry tax.

In Chapter 2, I study a model in which firms invest in capital and post wages, and heterogeneous workers, who have private information about their skills, choose where to apply. Workers and firms match bilaterally. Each matched agent gets an exogenous payoff from the match before wages are paid. Each of these payoffs
displays complementarity in capital and skill. I derive conditions under which the market exhibits PAM, positive assortative matching. Under these conditions, the firms over-invest in capital compared to the first best, because capital is used as a screening device.

I show that the fact that firms need to screen workers in the presence of private information changes the predictions of the model compared to that in the presence of complete information. In particular, I show that the complete information allocation exhibiting PAM is not necessary nor sufficient for the market allocation to exhibit PAM.
# Table of Contents

List of Figures viii

List of Tables ix

Acknowledgments x

Chapter 1
Constrained Efficiency with Search and Information Frictions 1

1.1 Introduction ................................................. 1
  1.1.1 Related Literature ................................... 5

1.2 Model ......................................................... 6
  1.2.1 Environment ........................................... 6
  1.2.2 Equilibrium Definition ............................... 8
  1.2.3 Planner’s Problem ..................................... 9
  1.2.4 Implementation ....................................... 11

1.3 Characterization ............................................. 15
  1.3.1 Complete Information Allocation or First Best ........ 15
  1.3.2 Results .................................................. 17

1.4 Example 1: Asset Market with Lemons ....................... 21
  1.4.1 Explanation of the Results ........................... 23
  1.4.2 What If There Are No Gains from Trade for Some Types? 25

1.5 Example 2: The Rat Race .................................... 26

1.6 Extension: Asset Market with a Continuous Type Space .... 27
  1.6.1 Environment ........................................... 28
  1.6.2 Complete Information Allocation or First Best .......... 29
  1.6.3 Planner’s Problem ..................................... 30
  1.6.4 Characterization ....................................... 31
    1.6.4.1 Characterization of the Constrained Efficient Allocation .......... 31
    1.6.4.2 Characterization of Equilibrium Allocation .................. 34
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>A.6.2 Even If $c'(.) \geq 0$ and $h'(.) \geq 0$, the Optimal Tax Schedule</td>
<td>129</td>
</tr>
<tr>
<td>May Not Be Monotone</td>
<td></td>
</tr>
<tr>
<td>A.6.3 What If the Complete Information Allocation Is Not Achievable</td>
<td>130</td>
</tr>
<tr>
<td>A.7 Proof of Proposition 5</td>
<td>133</td>
</tr>
</tbody>
</table>

## Appendix B

### Proofs of Chapter 2

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>B.1 Proof of Proposition 6</td>
<td>138</td>
</tr>
<tr>
<td>B.2 Rest of the Proofs for Discrete Type Space</td>
<td>148</td>
</tr>
<tr>
<td>B.3 Proofs of the Continuous Type Case</td>
<td>157</td>
</tr>
</tbody>
</table>

### Bibliography

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bibliography</td>
<td>162</td>
</tr>
</tbody>
</table>
## List of Figures

1.1 Proof steps of Theorem 2 in a schematic diagram. ........................................ 42  
1.2 Resource allocation in the asset market with lemons in a schematic diagram. ................................................................. 43  
1.3 Indifference curves in the asset market with lemons. ......................... 44  
1.4 Illustrating sufficient conditions for the planner to achieve the first best. ................................................................. 45  
1.5 Illustrating sufficient conditions for the equilibrium allocation to be separating. ................................................................. 46  
1.6 Payoff structure in Example 1. .......................................................... 47  
1.7 Equilibrium and efficient level of price, market tightness and tax for Example 1. ................................................................. 48  
1.8 Sellers’ expected payoff in Example 1. ............................................. 49  
1.9 Payoff structure in Example 2. .......................................................... 50  
1.10 Equilibrium and efficient level of price, market tightness and tax for Example 2. ................................................................. 51  
1.11 Sellers’ expected payoff in Example 2 ............................................. 52  
   
2.1 Comparison between equilibrium and efficient allocations in Example 3. ................................................................. 84  
2.2 Workers’ payoff in equilibrium and efficient allocations in Example 3. ................................................................. 85  
2.3 Checking for incentive compatibility constraint in Example 3. ............ 86  
2.4 Comparison between equilibrium and efficient allocations in Example 4. ................................................................. 87  
2.5 Workers’ payoff in equilibrium and efficient allocations in Example 4. ................................................................. 88  
2.6 Checking for incentive compatibility constraint in Example 4. ............ 89  
2.7 Comparison between equilibrium and efficient allocations in Example 5. ................................................................. 90  
2.8 Workers’ payoff in equilibrium and efficient allocations in Example 5. ................................................................. 91  
2.9 Checking for incentive compatibility constraint in Example 5. ............ 92
## List of Tables

1.1 Comparison between equilibrium and efficient allocations in the asset market with lemons. ........................................ 42
1.2 Comparison between equilibrium and efficient allocations in the rat race. ................................................................. 43
1.3 Characterization of equilibrium under different conditions ........................................ 43
I would like to extend my greatest appreciation to Dr. Neil Wallace for his tremendous support and encouragement he has shown during the last three years. Whenever I had an idea, he patiently listened to me and asked me guiding questions so I had to think harder and to make myself clear. I have learned from him how to do research. He has read my drafts, attended my presentations repeatedly and generously provided me with detailed feedback on my research papers and on my presentations. He has been a perfect intellectual role model for me.

I owe a large debt to Dr. Manolis Galenianos for his support. Although he was in London, UK, he kindly accepted to continue talking to me via Skype regularly. He was honest enough to point out my weaknesses to me and to push me to work harder, and caring and supportive enough to see my potential even when I could not.

I would like to express my gratitude to Dr. Shouyong Shi. He has been very helpful in the development of this dissertation, especially in its final stages and when I was on the market. He was listening carefully in order to give me smart and practical comments which brought positive changes I could hardly imagine.

I would like to thank Dr. Kala Krishna. I began my research under her supervision. She spent a lot of time with me to discuss my vague ideas and to let me find my fields of interests. Dr. Ed Green, Dr. Venky Venkateswaran, Dr. Kalyan Chatterjee, Dr. Vijay Krishna and Dr. Ruillin Zhu were all very helpful in shaping my thoughts about economic theory. Their time is greatly appreciated.

I have benefited from discussions with David Jinkins and Yu Awaya, especially in the first three years of the graduate school. David also generously helped me revise parts of this dissertation. Furthermore, I have enjoyed discussions with Rakesh Chaturvedi, Chun-Ting Chen, Joosung Lee, Wataru Nowaza, Hoonsik Yang and many others. I really appreciate their help.

I am extremely grateful to my mother, Zahra, and my father, Hossein, for being wonderful friends for me when I needed friends, for being supportive when I was trying to prepare for Iranian University Entrance Exam in high school,
being mentors when I wanted to switch my major from Electrical Engineering to Economics in college, for encouraging me when I decided to study abroad and for supporting me emotionally when I was studying toward my Ph.D. degree. They sacrificed their joys of life to provide me and my sister a calm and comfortable environment in home so we can pursue our goals. I also thank my sister, Azadeh. She has been always kind and inspiring to me. I have to also remember my loving grandmother, Tooran. She loved to see me graduate. She passed just months ago before we can celebrate my graduation together. I wish to pay my great respects to her.

Hamideh, my wife, my best friend, companion and supporter, has been an unending source of love and patience. She supported me in hard times and I want to express my deepest gratitude to her. It would have been impossible for me to complete this dissertation without her in my life.

And finally, “All the praises and thanks are to Allah, the Lord of all the worlds.” (Quran 1:2)
Dedication

To My Parents
And to Hamideh
Chapter 1  
Constrained Efficiency with Search and Information Frictions

1.1 Introduction

There are search frictions and private information in asset, labor, housing and other markets. For example consider markets for assets which are traded over the counter (OTC) like mortgage-backed securities, structured credit products and corporate bonds. It is natural to think that sellers in these markets have private information about the value of their assets. Also, they must incur search costs to find buyers for their assets.

Specially after 2008, there has been a lot of discussion about the role of private information in causing the financial crisis and consequently, many policy questions have arisen. One of these questions is whether subsidizing asset purchases is a good policy or not from a social point of view. No paper has studied socially efficient policies in this context, although some papers like [3], [4] and [5] have studied positive implications of those policies. In particular, [3] shows that if there are fire sales in the asset market, subsidizing the purchase of low price assets increases the liquidity of all assets in the market. In an application of my model, I contribute to this literature by studying the socially efficient policy in an environment similar to [3]. I characterize the optimal taxation policy in the asset market and show that in general the optimal tax schedule is non-monotone in the price of assets. In particular, I show if there are fire sales in the asset market, then taxing high price assets and subsidizing low price ones is not optimal.
From a theoretical point of view, this paper studies the constrained efficient allocation in economies with directed (competitive) search and adverse selection. My environment is the same as one in Guerrieri, Shimer and Wright (2010) (GSW henceforth). They define and characterize equilibrium and show its existence and uniqueness. I define and characterize the planner’s problem for this environment and show in my main result that if the equilibrium fails to achieve the first best,\(^1\) then the equilibrium is generally constrained inefficient. That is, the planner can achieve strictly higher welfare than the equilibrium.\(^2\) I also derive sufficient conditions under which the planner can achieve the first best.

In this economy, there is a large number of homogeneous buyers on one side of the market whose population is endogenously determined through free entry. There is a fixed population of sellers on the other side of the market who have private information about their types. Buyers and sellers match bilaterally and trade in different locations, called submarkets. In each submarket, there are search frictions in the sense that buyers and sellers on both sides get matched generally with probability less than one.

In order to define the planner’s problem for this environment, I take a mechanism design approach. The planner’s objective is to maximize social welfare and he is subject to the same information and search frictions present in the market economy. That is, the planner cannot observe types of sellers and also cannot force sellers or buyers to participate in the mechanism that the planner designs. In the language of mechanism design, the planner faces incentive compatibility of sellers, participation constraints of sellers and buyers and his own budget-balance condition. That is, the net amount of transfers that the planner makes to agents must be non-positive.

To implement this mechanism, all the planner needs to do is to impose submarket-specific sales taxes and subsidies on buyers in each submarket conditional on trade.\(^3\) The timing of actions are otherwise the same: Having observed the schedule of sales tax, buyers first choose a submarket and then sellers observe all open submarkets.

\(^1\)The first best allocation is the solution to the planner’s problem when the planner faces only search frictions, but he has complete information about the type of agents.

\(^2\)In three examples, GSW introduce some pooling or semi-pooling allocations that Pareto dominate the equilibrium allocations. They do not characterize the constrained efficient allocation nor do they define it.

\(^3\)It is discussed in the paper that if there are not positive gains from trade for some types, lump sum transfers to sellers may be also needed to implement the direct mechanism.
(the sumbarkets that some buyers have selected) and choose where to go. Then buyers and sellers trade if they find a match. Note that the set of open submarkets in the planner’s implementation may be different than that in the equilibrium. Also, the equilibrium allocation is a feasible allocation for the planner, because the revenue that the planner makes over each submarket is zero in the equilibrium allocation.

To understand how the planner can achieve strictly higher welfare than the market economy, I study some examples. In the first one in Section 1.4, I study an asset market with lemons. Sellers have one indivisible asset which is of two types: high and low. The high-type asset is more valuable to both buyers and sellers. GSW show that there exists a unique separating equilibrium in which different types trade in different submarkets. High-type sellers prefer the higher price submarket with lower probability of matching (submarket two), while low-type sellers are just indifferent between the two submarkets. Low-type sellers are willing to sacrifice price for probability of trade, because they do not want to get stuck with their “lemons”. On the other hand, high-type sellers do not want to sacrifice price, because their assets are more valuable to them in case of being unmatched. Probability of matching in this example, in fact, is used as a screening device.

The planner can do better than equilibrium in the following way. Beginning from the equilibrium allocation which is feasible for the planner, the planner subsidizes sellers in submarket one (low-type sellers) by a small amount so that their incentive compatibility constraint for choosing submarket two becomes slack. Now more buyers enter submarket two to get matched with previously unmatched high-type sellers. Therefore, welfare increases due to the formation of new matches. To finance subsidies to the sellers in submarket one (low-type sellers), the planner taxes sellers in submarket two (high-type sellers). The planner keeps subsidizing low-type and taxing high-type sellers until he achieves the first best, in which high-type sellers also get matched with probability one, or participation constraint of high-type sellers binds. The same idea goes through even if there are more than two types.

To understand the nature of inefficiency in the market economy, consider the externalities implied by having one more buyer in a submarket. First, it decreases the probability that other buyers in that submarket are matched. Second, it in-
creases the probability that other sellers are matched in that submarket. In the presence of complete information, it is well established in the literature that buyers entering the market in the directed search setting can internalize these externalities by choosing the “right” price (contract), if sellers’ types are observable and contractible and if buyers can commit to their postings. However, the change in the payoff of sellers following the entry of one more buyer in a submarket has another effect in this environment which is absent in the complete information case. This change will affect the incentive compatibility constraints that buyers who want to attract other types of sellers face, thus affecting the set of feasible submarkets that other buyers can enter to attract those sellers.

Buyers in the market economy do not take into account the effect of their entry on the contracts posted on other submarkets and consequently on the payoff of sellers in other submarkets. The planner internalizes these externalities by imposing appropriate taxes on agents and therefore can do better than the market economy. The extent to which the planner can improve efficiency depends on the details of the environment. In my second result, I derive sufficient conditions under which the planner can eliminate distortions completely to achieve the first best.

In the second example in Section 1.5, I characterize the constrained efficient allocation in a version of the rat race (originally studied by [13]) and compare my results with GSW who solve for the equilibrium allocation in this environment. There are two types of workers. High-type workers incur less cost for working longer hours and generate higher output compared to low-type workers. Also the marginal output with respect to hours of work that high-type workers generate is higher. In equilibrium, high-type workers work inefficiently for longer hours than they would work under complete information and get matched with inefficiently higher probability. The planner, in contrast to the market economy, achieves the first best. He pays low-type workers higher wages and high-type workers lower wages than what they would get under complete information. These subsidies (to low-type workers) and taxes (on high-type workers) are needed to ensure that low-type workers do not have any incentive to apply to the submarket that high-type

---

4The efficiency of competitive search equilibrium in the presence of complete information is probably the most important result in this literature. In the random search setting, in contrast, the equilibrium is generally inefficient because the entrants generally fail to internalize the aforementioned externalities. See the following papers for directed search models and their efficiency properties: [1, 2, 6–11]. See [12] for a random search model.

5My paper is also related to the classic adverse selection models like [14, 15].
workers apply to. Moreover, if the share of high-type workers in the population is sufficiently high, the planner’s allocation even Pareto dominates the equilibrium allocation.

In the asset market example explained above, the trades which involve high-type (or equivalently high price) assets are taxed and other trades are subsidized. I am interested to know whether this observation can be generalized to more realistic environments or not. To answer this question, I extend the two-type asset market to a continuous type one, which is a static version of [3], and derive sufficient conditions under which the planner can achieve the first best. The optimal submarket-specific sales tax that implements the optimal mechanism is not generally monotone in the price of assets. This feature makes it hard for the planner to implement this mechanism in the real world, partly because implementing a non-monotone tax schedule requires the planner to have precise information about the details of the the economy, but this requirement is unlikely to be met in the real world applications. For example, with a non-monotone tax schedule little misspecification of the model by the planner can lead to significant losses in efficiency. Ideally, the tax schedule should be independent of the details of the economy.

In the next step, I show that imposing two types of taxes, not only sales tax but also submarket-specific entry tax, which is imposed on buyers conditional on entry to each submarket regardless of whether they find a match or not, solves the non-monotonicity problem. That is, the planner can always design monotone tax schedules, decreasing entry tax and increasing sales tax, to implement the direct mechanism.

1.1.1 Related Literature

[16] and [17] study constrained efficient allocation in environments with directed search and private information. [16] shows that the competitive search equilibrium is constrained inefficient in a dynamic setting, if the economy is not on the steady state path. However in both papers, the agents who search (workers) do not have ex-ante private information. After they get matched with firms, they learn their types which become their private information.

[18] study a model with directed search and private information and show that the equilibrium is constrained inefficient. There are two important differences
between their paper and mine. First, the information friction in their paper is moral hazard, because the public cannot observe whether the workers have searched or not and if so, toward which type of firms. In contrast, the information friction in my paper is adverse selection. Second, workers are risk averse in their paper, in contrast to sellers in my paper who are risk neutral. The inefficiency result in their paper relies on the risk aversion assumption. Therefore, the channels through which inefficiency arises in the two papers are different.

[19] study a model in which sellers with private information post contracts, in contrast to GSW in which the uninformed side of the market posts contracts. They investigate the potentially conflicting roles of prices: the signaling role and the search directing role. Aside from some details,\footnote{For example in their model, sellers choose the quality of their products. The quality, then, becomes their private information.} the notion of constrained efficiency defined in this paper and the ideas behind that (that the planner can make transfers across agents or equivalently across submarkets) apply to their model as well, because the environments are similar, although they have a different trading mechanism.

The paper is organized as follows. In Section 1.2, I develop the environment of the model and define the planner’s problem. In Section 1.3, I characterize the planner’s allocation and state my main results. In Section 1.4, I study a two-type asset market example, characterize the planner’s allocation and compare it with the equilibrium allocation. I also explain the nature of inefficiency in the market economy and discuss why and how the planner can allocate resources more efficiently than the market economy. In Section 1.5, I study a version of the rat race. In Section 1.6, I study an asset market with a continuous type space to characterize the efficient tax schedule. Section 1.7 concludes. All proofs appear in the appendix.

1.2 Model

1.2.1 Environment

Consider an economy with two types of agents, buyers and sellers and $n + 1$ goods where $n \in \mathbb{N}$. Goods 1, 2, ..., $n$ are produced by sellers and consumed by buyers,
while good $n + 1$ is a numeraire good and is produced and consumed by everyone. Let $a \equiv (a^1, a^2, ..., a^n) \in \mathbb{A} \subset \mathbb{R}^n$ be a vector where $\mathbb{A}$ is compact, convex and non-empty. Component $k$ of this vector, $a^k$, denotes the quantity of good $k$. For example in a labor market, $a$ can be a positive real number denoting the hours of work. When I say an agent produces (or consumes) $a$, I mean that the agent produces (or consumes) $a^1$ units of good 1, $a^2$ units of good 2 and so on.

There is a measure 1 of sellers. A fraction $\pi_i > 0$ of sellers are of type $i \in \{1, 2, ..., I\}$. Type is seller’s private information. On the other side of the market, there is a large continuum of homogenous buyers who can enter the market by incurring cost $k > 0$. After buyers enter the market, buyers and sellers are allocated to different submarkets (described below). Matching is bilateral. After they match, they trade.

There are search frictions in this environment. By search frictions I mean that sellers generally get to match with the buyers they have chosen with probability less than one. Matching occurs in submarkets which are simply some locations for trades. Matching technology determines the probability that sellers and buyers in each submarket get matched. If the ratio of buyers to sellers in one submarket is $\theta \in [0, \infty]$, then the buyers are matched with probability $q(\theta)$. Symmetrically, matching probability for sellers is $m(\theta) \equiv \theta q(\theta)$. As is standard in the literature, I assume that $m$ is non-decreasing and $q$ is non-increasing. Both $m(\cdot)$ and $q(\cdot)$ are continuous.

Sellers’ and buyers’ payoff functions are quasi-linear in the numeraire good. The payoff of a buyer who enters the market from consuming $a$ and producing $t \in \mathbb{R}$ units of the numeraire good is $v_i(a) - t - k$ if matched with a type $i$ seller and is $-k$ if unmatched. The payoff of a type $i$ seller from producing $a$ and consuming $t \in \mathbb{R}$ units of the numeraire good is $u_i(a) + t$ if matched with a buyer and is 0 otherwise.

The difference between payoff functions in this paper and in GSW is that I assume quasi-linear preferences, while they do not make this assumption. The reason that I impose quasi-linearity assumption is that I want to do welfare analysis and I want to use taxes and subsidies. If the preferences are not quasi-linear, the weight that the planner assigns to buyers and sellers might become important.
1.2.2 Equilibrium Definition

First, let me briefly describe how the market economy works, the special case in which the planner does nothing. Then I describe the planner's problem. The definition of equilibrium is taken completely from GSW.

Submarkets in the market economy are characterized by \( y \equiv (a, p) \) where \( a \in A \) denotes the vector of goods 1 to \( n \) to be produced by sellers in this submarket and \( p \in \mathbb{R} \) is the amount of the numeraire good to be transferred from buyers to sellers. No submarket which would deliver buyers a strictly positive payoff is inactive in the equilibrium. If there was such a submarket, some buyers would have already entered that submarket to exploit that opportunity. On the other side of the market, sellers observe all \((a, p)\) pairs posted in the market, anticipate the market tightness at each submarket and then direct their search toward one which delivers them the highest expected payoff.

Let \( \gamma_i(y) \) denote the share of sellers that are type \( i \) in the submarket denoted by \( y \), with \( \Gamma(y) \equiv \{ \gamma_1(y), ..., \gamma_i(y), ..., \gamma_I(y) \} \in \Delta^I \) where \( \Delta^I \) is an \( I \)-dimensional simplex, that is, \( 0 \leq \gamma_i(y) \leq 1 \) for all \( y \) and \( \sum_{i=1}^I \gamma_i(y) = 1 \). To make the notation clear for the rest of the paper, the first component of \( y \) is denoted by \( a \), rather than \( y_1 \) and the second component is denoted by \( p \) rather than \( y_2 \). Similarly if the submarket is denoted by \( y' \), the first and second components of \( y' \) are denoted by \( a' \) and \( p' \).

**Definition 1.** An equilibrium, \( \{Y, \lambda, \theta, \Gamma\} \), is a measure \( \lambda \) on \( Y \equiv A \times \mathbb{R} \) with support \( Y^P \), a function \( \theta : Y \rightarrow [0, \infty] \), and a function \( \Gamma : Y \rightarrow \Delta^I \) which satisfies the following conditions:

(i) **Buyers’ profit maximization and free entry**

For any \( y \in Y \),

\[
q(\theta(y)) \sum_i \gamma_i(y)(v_i(a) - p) \leq k,
\]

with equality if \( y \in Y^P \).

(ii) **Sellers’ optimal search**

Let \( U_i = \max \left\{ 0, \max_{y' \in Y^P} \left\{ m(\theta(y'))(u_i(a') + p') \right\} \right\} \) and \( U_i = 0 \) if \( Y^P = \emptyset \). Then for any \( y \in Y \) and \( i \), \( U_i \geq m(\theta(y))(u_i(a) + p) \) with equality if \( \theta(y) < \infty \) and \( \gamma_i(y) > 0 \). Moreover, if \( u_i(a) + p < 0 \), either \( \theta(y) = \infty \) or \( \gamma_i(y) > 0 \).

(iii) **Market clearing**
For all $i$, $\int_Y f_Y \frac{\gamma_i(y)}{\theta'(y)} d\lambda(y) \leq \pi_i$, with equality if $U_i > 0$.

Let me make several brief comments about the equilibrium definition. For further details, see GSW. Equilibrium condition (i) states that buyers should not earn a strictly positive profit from entering any submarket (on- or off-the-equilibrium-path). That is, there are no opportunities for trade unexploited in the equilibrium. If buyers’ expected payoff in one submarket is strictly negative, no buyer enters that submarket. If that is strictly positive, more buyers will enter that submarket and the market tightness will be changed. Therefore, for all markets that the planner wants to be open, buyers must get exactly 0 expected payoff. A buyer has to incur entry cost $k$ if he wants to enter submarket $y$. Then, he gets matched with a type $i$ seller with probability $\gamma_i(y)$ from which he gets a payoff of $v_i(a)$ in terms of the numeraire good, and pays $p$ units of the numeraire good to the seller.

Equilibrium condition (ii) is composed of two parts. The first part states that among all submarkets in the equilibrium, $y \in Y^P$, sellers choose to go to a submarket which maximizes their payoff. The second part imposes some restrictions on beliefs regarding the market tightness and composition of types for off-the-equilibrium-path, $y \notin Y^P$. The market tightness for off-the-equilibrium-path is set such that sellers who choose to go to those posts do not gain by doing so relative to their equilibrium payoff. Also, this restriction with respect to the composition of types states that if buyers believe that some types would apply to an off-the-equilibrium-path post, then those types should be exactly indifferent between the payoff they get from that post relative to their equilibrium payoff. Equilibrium condition (iii) is straightforward.

### 1.2.3 Planner’s Problem

I define a planner whose objective is to maximize the weighted average of the payoff to sellers. The planner faces the same information and search frictions present in the market economy. The planner uses a direct mechanism to allocate resources. In the direct mechanism and thanks to the revelation principle, sellers report their types to the planner and then the planner allocates them to a 4-tuple $(\tilde{\alpha}_i, \tilde{\rho}_i, \tilde{s}_i, \tilde{\theta}_i)$. Here, $\tilde{\alpha}_i$ is the vector of production of goods 1 to $n$ to be produced by sellers who report type $i$, $\tilde{\rho}_i$ is the amount of the numeraire good transferred to them.

---

8Note that buyers get payoff 0 either in the market economy or under the planner’s allocation.
conditional on finding a match, $\tilde{s}_i$ is the amount of the numeraire good transferred to them unconditionally and $\tilde{\theta}_i$ is the average number of buyers assigned to them. The planner maximizes his objective function subject to incentive compatibility of sellers, participation constraint of sellers and his budget-balance condition.

**Definition 2.** A feasible mechanism is a set $\{(\tilde{a}_i, \tilde{p}_i, \tilde{s}_i, \tilde{\theta}_i)\}_{i \in \{1, 2, \ldots, I\}}$ such that the following conditions hold:

1. **Incentive Compatibility of Sellers**
   
   For all $i$ and $j$,
   
   $$U_i \equiv m(\tilde{\theta}_i)(u_i(\tilde{a}_i) + \tilde{p}_i) + \tilde{s}_i \geq m(\tilde{\theta}_j)(u_i(\tilde{a}_j) + \tilde{p}_j) + \tilde{s}_j.$$

2. **Participation Constraint of Sellers**
   
   For all $i$,
   
   $$U_i \geq 0.$$

3. **Planner’s Budget-Balance**

   $$\sum_{i=1}^{I} \pi_i \left[ m(\tilde{\theta}_i)(v_i(\tilde{a}_i) - \tilde{p}_i) - k\tilde{\theta}_i - \tilde{s}_i \right] \geq 0.$$

Two points are worth mentioning about this definition. The first one is that the participation constraint or individual rationality of buyers here is taken implicitly into account by condition (3). Consider the following scenario. The planner charges buyers some participation fee. Once a buyer agrees to participate, the planner assigns the buyer to get matched with one type of sellers according to a uniform distribution. There are $\pi_i \tilde{\theta}_i$ number of buyers who are assigned to type $i$ sellers and overall there are $\sum \pi_j \tilde{\theta}_j$ buyers who participate. Therefore, the probability that a buyer is assigned to type $i$ sellers is $\frac{\pi_i \tilde{\theta}_i}{\sum \pi_j \tilde{\theta}_j}$. Therefore, the expected benefit of the buyer from entering the market and getting matched with type $i$ is $\frac{\pi_i \tilde{\theta}_i}{\pi \tilde{\theta}_j} (q(\tilde{\theta}_i)v_i(\tilde{a}_i) - k)$. On the other hand, each type $j$ seller needs to get paid $\tilde{p}_j$ units of the numeraire good conditional on matching and $\tilde{s}_j$ units unconditionally. Therefore, overall $\sum \pi_j (m(\tilde{\theta}_j)\tilde{p}_j + \tilde{s}_j)$ amount of the numeraire good is needed to compensate sellers. Since the planner does not spend any resources from his own pocket, each participating buyer should pay $\frac{\sum \pi_j (m(\tilde{\theta}_j)\tilde{p}_j + \tilde{s}_j)}{\sum \pi \tilde{\theta}_j}$. In order for buyers to participate in the direct mechanism, the benefit that each buyer gets ex-ante,
\[ \frac{\pi_i \tilde{\theta}_i}{\sum \pi_j \tilde{\theta}_j} (q(\tilde{\theta}_i) v_i(a_i) - k), \]
should weakly exceed the amount of the numeraire good that the buyer should pay, \[ \frac{\sum \pi_j (m(\hat{\theta}_j) \hat{p}_j + \hat{s}_j)}{\sum \pi_j \hat{\theta}_j}. \]
Condition (3) in the definition summarizes this requirement.

The second point is that in this definition, I did not allow the planner to use lotteries. By lotteries I mean that after agents report their types, the planner allocates, say, type \( i \) sellers to different 4-tuples, \((\tilde{a}, \tilde{p}, \tilde{s}, \tilde{\theta})\) and \((\tilde{a}', \tilde{p}', \tilde{s}', \tilde{\theta}')\), with positive probability where these 4-tuples may deliver type \( i \) sellers different payoffs. If the planner can use lotteries, then the planner may be able to achieve even higher welfare than what he achieves in the constrained efficient allocation here,\(^9\) because he would have one more tool.\(^{10}\)

**Definition 3.** A constraint efficient mechanism is a feasible mechanism which maximizes the planner’s objective function. That is, the planner solves the following problem:

\[
\max \left\{ \pi_i U_i \right\}_{i \in \{1, 2, \ldots, I\}} \sum_i \pi_i U_i
\]

s. t. \( \{(\tilde{a}_i, \tilde{p}_i, \tilde{s}_i, \tilde{\theta}_i)\}_{i \in \{1, 2, \ldots, I\}} \) is a feasible mechanism.

As far as the notation is concerned, whenever a variable has \( \tilde{\} \) in the paper, it shows that the variable is an element of a direct mechanism. CE represents the constrained efficient allocation, FB represents the first best allocation and EQ represents the equilibrium allocation.

**1.2.4 Implementation**

To implement the direct mechanism, the planner is assumed to have the power to impose taxes and subsidies on agents. It turns out that imposing two types of taxes are sufficient for the planner to implement the direct mechanism discussed above. First, the planner chooses a tax amount for each submarket. This tax will be imposed on buyers conditional on trade, \( t(a, p) : A \times \mathbb{R} \to \mathbb{R} \). The results will

\(^9\)The lotteries may help the planner to achieve higher welfare if the objective function of the planner is not concave or if the constraint set is not convex.

\(^{10}\)To elaborate, in my first result, I show that if the equilibrium does not achieve the first best, then the planner achieves strictly higher welfare than the equilibrium without using any lotteries. Adding another tool can only make this result stronger. In my second result, I derive conditions under which the planner achieves the first best without using any lotteries. Since the planner achieves the first best, adding another tool does not change this result.
not change if, instead, taxes are imposed on sellers. Second, the planner can make lump sum transfers, \( T \in \mathbb{R}_+ \) units of the numeraire good, to sellers.\(^{11}\) Note that any post in the market economy is a special case of this description with \( t(y) = 0 \) for all submarkets and \( T = 0 \).

The planner may want agents not to trade in some submarkets, despite the fact that agents in the market economy want to trade in those submarkets. In such a case, the planner can impose sufficiently high amount of tax on trade in those submarkets. Aside from the ability to make these transfers, the market economy and the planner face the same restrictions: Amount of goods to be produced by sellers or payments to be made by buyers cannot be conditioned on the type of sellers. The ex-ante payoff of buyers in both cases should be 0 to ensure that buyers want to participate and also to ensure that there is no excess entry into any submarket. Also in both cases sellers choose submarkets which maximize their expected payoff or stay out. Some sellers’ payoffs from entering any open submarket, the submarkets that some buyers choose to go, is non-positive, so they will not apply to any submarket. I call these sellers non-participants. They just receive \( T \).

The planner faces a budget constraint (or a budget-balance condition as called in the mechanism design literature) similar to that in the direct mechanism. This condition states that the net amount of transfers that the planner makes to agents should not exceed 0. Notice that in the market economy, it is not possible to transfer funds (the numeraire good) from one submarket to another. That is, all the surplus generated in any submarket belongs to sellers in that submarket. Under the planner’s allocation, on the other hand, sellers might get a higher or lower payoff than the surplus they generate. In short, cross-subsidization across submarkets is possible.

As defined earlier, let \( y \equiv (a, p) \) denote a submarket. An allocation \( \{\lambda, Y^P, \theta, \Gamma, t, T\} \) is a distribution \( \lambda \) over \( Y \) with support \( Y^P \) (so \( Y^P \) is the set of open submarkets),

\(^{11}\)Without loss of generality, I assume that \( T \) must be positive. If \( T \) is negative and some types do not participate, i.e., do not apply to any submarket, then sellers’ participation constraint is violated. If all types participate, then one can easily incorporate \( T \) into prices, that is, one can change \( p_i \) to \( p_i + \frac{T}{m_i(\theta_i)} \), to replace negative \( T \) by 0. Therefore, it is without loss of generality to assume that \( T \) is positive.

\(^{12}\)It is possible that some types get a negative payoff from active sub-markets, so they prefer not to apply to any submarket. However they distort the allocation for other types via the incentive compatibility constraint. The planner is allowed to pay them \( T \) to reduce this distortion.
the ratio of buyers to sellers for each submarket \( \theta : Y \to [0, \infty] \), the distribution of types in each submarket \( \Gamma : Y \to \Delta^I \), the amount of tax (in terms of the numeraire good) to be imposed on buyers at each submarket conditional on trade, \( t : Y \to \mathbb{R} \), and finally the amount of the numeraire good to be transferred to sellers in a lump sum way, \( T \in \mathbb{R}_+ \). Because the planner faces some constraints, only some allocations are implementable for the planner. The definition of an implementable allocation is given below.

**Definition 4.** A planner’s allocation \( \{\lambda, Y^P, \theta, \Gamma, t, T\} \) is implementable if it satisfies the following conditions:

(i) **Buyers’ maximization and free entry**
For any \( y \in Y \),
\[
q(\theta(y)) \sum_i \gamma_i(y)(v_i(a) - p - t(y)) \leq k,
\]
with equality if \( y \in Y^P \).

(ii) **Sellers’ maximization**
Let \( U_i \equiv \max\{0, \max_{y' \in Y^P} \{m(\theta(y'))(u_i(a') + p')\}\} + T \) and \( U_i = T \) if \( Y^P = \emptyset \). For any \( y \in Y \) and \( i \),
\[
m(\theta(y))(u_i(a) + p) + T \leq U_i,
\]
with equality if \( \gamma_i(y) > 0 \) and \( \theta(y) < \infty \). If \( u_i(a) + p < 0 \), then \( \theta(y) = \infty \) or \( \gamma_i(y) = 0 \).

(iii) **Feasibility or market clearing**
For all \( i \), \( \int_{Y^P} \frac{\gamma_i(y)}{\theta(y)} d\lambda(\{y\}) \leq \pi_i \), with equality if \( U_i > T \).

(iv) **Planner’s budget constraint**
\[
\int_{Y^P} q(\theta(y)) t(y) d\lambda(\{y\}) \geq T.
\]

The definition of implementable allocation is similar to the definition of equilibrium. Regarding condition (i), when buyers want to choose a submarket, they form beliefs regarding market tightness and composition of types at each submarket. The restriction on these beliefs are also exactly the same as those in the definition of equilibrium. Note that here buyers not only need to make payment to sellers but also to the planner. Conditions (ii) and (iii) are exactly the same as their counterparts in the equilibrium definition. Condition (iv), the budget-balance condition, is self-explanatory.
Condition (ii) summarizes two constraints, sellers’ participation constraint and sellers’ incentive compatibility constraint. To make the exposition easier, for any given allocation, define $X_i$ as follows: $X_i \equiv \{(\theta(y), a) | y \equiv (a, p) \in Y^P, \gamma_i(y) > 0\}$. Denote elements of $X_i$ by $(\theta_i, a_i)$. In words, $\theta_i$ is the market tightness of a submarket to which type $i$ applies with strictly positive probability and $a_i$ is the production level of that submarket. Sellers’ maximization constraint implies that for any $i, j$, $(\theta_i, a_i) \in X_i$ and $(\theta_j, a_j) \in X_j$:

$$m(\theta_i)(u_i(a_i) + p_i) \geq m(\theta_j)(u_i(a_j) + p_j) \quad (IC).$$

I call this constraint IC or incentive compatibility constraint. This constraint is equivalent to condition (1) in the definition of feasible mechanism.

**Definition 5.** A constrained efficient allocation is an implementable allocation which maximizes welfare among all implementable allocations. That is, a constrained efficient allocation solves the following problem:

$$\max_{\{\lambda, Y^P, \theta, \Gamma, t, T\}} \sum_i \pi_i U_i$$

s.t. $\{\lambda, Y^P, \theta, \Gamma, t, T\}$ is implementable.

I show in Lemma 1 that the way I define the constrained efficient allocation here is without loss of generality. That is, a planner who uses a direct mechanism as defined earlier achieves the same welfare level as the planner in Definition 5.

**Lemma 1.** Given any feasible mechanism, there is an associated implementable allocation under which all types get exactly the same payoff as in the direct mechanism.

Since implementation of a direct mechanism requires a large amount of communication, which is unrealistic in many economic applications, a proper implementation should be close to the real world applications as much as possible. The way that I have formulated the implementation of the direct mechanism here has this feature, because the elements of the implementable allocation have natural interpretations. For example, $t$ can be interpreted as submarket-specific sales tax. Lemma 1 guarantees that all technical results that I derive by utilizing the direct mechanisms can be naturally implemented in the real world applications.
Given any equilibrium \( \{ \lambda, Y, \theta, \Gamma \} \), I construct an allocation called \textbf{equilibrium allocation} \( \{ \lambda^{EQ}, Y^{EQ}, \theta^{EQ}, \Gamma^{EQ}, t^{EQ}, T^{EQ} \} \) where \( \lambda^{EQ} = \lambda \), \( Y^{EQ} = Y \), \( \theta^{EQ}(y) = \theta(y) \), \( \Gamma^{EQ}(y) = \Gamma(y) \), \( t^{EQ}(y) = 0 \) for all \( y \) and \( T^{EQ} = 0 \). The only difference is that I added zero taxes to the definition of equilibrium. The equilibrium allocation is implementable, because sellers’ maximization condition and buyers’ profit maximization and zero profit condition are satisfied following their counterparts in the definition of equilibrium. The planner’s budget constraint is also trivially satisfied, because \( t^{EQ} = 0 \) for all \( y \in Y^{EQ} \) and \( T^{EQ} = 0 \). When I say equilibrium allocation, I mean the implementable allocation which is constructed from equilibrium objects as above.

Finally, when I refer to equilibrium in the paper, I mean the notion of equilibrium which was discussed above where the uninformed side of the market posts contracts. I do not mean a notion of equilibrium with signaling in which the informed side of the market posts contracts, unless otherwise noted.\(^{13}\)

\section{1.3 Characterization}

I first study the complete information case as a benchmark and then present my main results.

\subsection{1.3.1 Complete Information Allocation or First Best}

As a benchmark, consider an otherwise the same environment as introduced above except that the type of sellers is common knowledge. Since buyers have complete information about the type of sellers, the submarkets in the market economy are not only indexed by the level of production and price but also by the type of sellers that the buyers want to meet. The buyers, who contemplate what submarket to enter to attract type \( i \) sellers, enter a submarket which maximizes the payoff of type \( i \) subject to the free entry condition. (See [7] for further explanation.) If there is any submarket that would deliver type \( i \) sellers a higher payoff, some buyers would enter that submarket and then, sellers would strictly prefer that

\(^{13}\)I conjecture that my first result regarding the inefficiency of equilibrium will hold even if another notion of equilibrium is considered in which the informed side of the market posts the terms of trade. Of course, one needs to impose some reasonable restrictions on off-the-equilibrium-path beliefs similar to those proposed by [20].
submarket. Therefore, buyers who attract type $i$ solve the following problem in
the market economy with complete information:

$$\max_{\theta, a, p} \left\{ m(\theta)(p + u_i(a)) \right\}$$

s.t. $q(\theta)(v_i(a) - p) \geq k$.

Denote the solution to this problem by $(\theta^{FB}_i, a^{FB}_i, p^{FB}_i)^{14}$. It is easy to see that the
constraint of the problem must hold with equality, so after eliminating $p$ from the
maximization problem, one can write the payoff of type $i$ sellers from participating
in the market in the complete information case as $\max_{\theta, a} \left\{ m(\theta)(v_i(a) + u_i(a)) - k\theta \right\}$. Let $U^{FB}_i$ be the payoff of type $i$ in the complete information case. Then $U^{FB}_i$ is
calculated as follows:

$$U^{FB}_i = \max_{\theta, a} \left\{ m(\theta)(v_i(a) + u_i(a)) - k\theta \right\}.$$

Notice that the objective function, $m(\theta)(v_i(a) + u_i(a)) - k\theta$, is exactly equal
to the surplus created by a type $i$ seller. Thus, the planner who observes types
of sellers solves exactly the same problem as buyers in the market economy with
complete information. This is the core of the argument in the literature which
states that the market economy decentralizes the planner’s allocation under com-
plete information. As already stated, there are many papers in the literature with
different environments but with this common theme that when agents on one side
of the market compete with each other in posting contracts and commit to them,
then the market decentralizes the planner’s allocation, if the contract space is rich
enough. See [1, 2, 7, 8, 10, 11, 21].^{15} This highlights the importance of my results
that when private information is introduced into the model, the predictions of the
model regarding the efficiency of the equilibrium completely change. For more
explanation, see Section 1.4.1.

---

^{14} All that matters for the first best allocation is the level of production and probability of
matching. Since transfers is not part of the first best allocation, having superscript of $FB$ for
price in $p^{FB}_i$ is somewhat misleading. More precisely, $p^{FB}_i$ is the payment that buyers make
to sellers in the market with complete information. I do not want to introduce a new notation
for the market with complete information, so I keep $p_i$ with superscript of $FB$ throughout the
paper to refer to the payment that buyers make to type $i$ sellers in the market with complete
information.

^{15} If the contract space is not rich enough, the equilibrium might be constrained inefficient,
like [22].
If $U_i^{FB} \geq 0$, the planner wants type $i$ to get matched with probability $m(\theta_i^{FB})$ and to produce $a_i^{FB}$. If $U_i^{FB} < 0$, then type $i$ sellers do not participate in the market. The planner does not want them to participate, either. In this paper, when I say that the planner achieves the complete information allocation or achieves the first best, I mean that there exists an implementable allocation in which type $i$ sellers get matched with probability $m(\theta_i^{FB})$ and produce $a_i^{FB}$ for all $i$.

### 1.3.2 Results

As already seen, the equilibrium allocation is feasible for the planner. It is immediately followed that the planner can achieve the level of welfare which is at least as much as that in the market economy. Theorem 1 states that the planner can achieve strictly higher welfare.

Let $\bar{Y} \equiv \bigcup_i \bar{Y}_i$ where

$$\bar{Y}_i \equiv \{(a,p)\mid (a,p) \in A \times R, q(0)(v_i(a) - p) \geq k, \text{ and } u_i(a) + p \geq 0\}.$$ 

If $(a,p) \notin \bar{Y}$, then no type will be attracted to this submarket in the market economy. Also for the future reference, let $\bar{A}$ be defined as follows:

$$\bar{A} \equiv \{a\mid (a,p) \in \bar{Y} \text{ for some } p \in \mathbb{R}\}.$$ 

**Assumption 1.**

1. **Strict Monotonicity:** For all $a \in \bar{A}$, $v_1(a) < v_2(a) < ... < v_I(a)$.
2. **Sorting:** For all $i$, $a \in \bar{A}$ and $\epsilon > 0$, there exists $a' \in B_i(a) \equiv \{a' \in A \mid \|a - a'\|_2 < \epsilon\}$ such that

$$u_j(a') - u_j(a) < u_h(a') - u_h(a) \text{ for all } j \text{ and } h \text{ with } j < i \leq h.$$ 

3. **Technical assumption:** Function $m(\theta)(u_i(a) + v_i(a)) - k\theta$ has only one local maximum on its domain, $(\theta, a) \in \mathbb{R}_+ \times \mathbb{A}$.

**Theorem 1 (Result 1).** Suppose Assumption 1 holds. Also assume that all types with positive gains from trade (all $i$ with $U_i^{FB} > 0$) get a strictly positive payoff in the equilibrium. If the equilibrium fails to achieve the first best, then the planner achieves strictly higher welfare than the equilibrium.
Some remarks about the assumptions are in order. A standard single crossing condition states that the indifference curves of different types must cross only once. The sorting assumption here (which is the same as in GSW) is in a sense a local crossing condition, because it allows \( a' \) to be greater than \( a \) for some \( a \) and less than \( a \) for other \( a \). Moreover, it is in a sense stronger than single crossing condition, because it states that given any \( a \), there exists an \( a' \) with such a property. The requirement that all types with positive gains from trade must be active in the equilibrium is satisfied if there are positive gains from trade for all types. In an example in Section 4, I will make it clear why this assumption is necessary for this result.

The idea of the proof is as follows. I begin from the equilibrium allocation, propose a direct mechanism which is basically a perturbation of the equilibrium allocation in a particular way and then show that the proposed allocation is feasible and achieves strictly higher welfare than the equilibrium allocation.

I first explain how the equilibrium is constructed. Under similar conditions (weak monotonicity and sorting), GSW prove that the equilibrium for type \( i \) is characterized by maximizing the payoff of type \( i \), subject to the free entry condition and the incentive compatibility constraint of all lower types. That is, type \( j < i \) should not get a higher payoff if he chooses the submarket that type \( i \) chooses. They prove that this equilibrium is unique in terms of payoffs.

Let \( \{\lambda^{EQ}, Y^{EQ}, \theta^{EQ}, \Gamma^{EQ}, t^{EQ}, T^{EQ}\} \) denote the equilibrium allocation where \( Y^{EQ} \equiv \{y_1^{EQ}, y_2^{EQ}, \ldots, y_I^{EQ}\} \). Also let \( U_i^{EQ} \) denote the utility that type \( i \) gets in the equilibrium. In this explanation, assume that all types are active in the equilibrium, \( U_i^{EQ} > 0 \) (which is the case if there are positive gains from trade for all types). Since the equilibrium does not achieve the first best, there exists a type, say type \( i \), which creates the surplus that is less than the first best level. It implies that at least one IC constraint in the problem for type \( i \) is binding in the equilibrium. For example, suppose type \( j \) is indifferent between \( y_j^{EQ} \) and \( y_i^{EQ} \) with \( j < i \).

The planner begins from a direct mechanism in which each type is allocated the same \((a, p, \theta)\) as in the equilibrium. Since all types are active in the equilibrium, I assume without loss of generality that the unconditional transfer, \( s \), for all types is initially set equal to 0, that is, \( \tilde{s}_l = 0 \) for all \( l \). In order to improve welfare, the planner subsidizes all types lower than \( i \) identically by a small amount, \( \epsilon > 0 \). That is, \( \tilde{s}_h = \epsilon \) for all \( h < i \). Now constraints of the maximization problem for
type $i$ become slack, so the planner can find another triple $(a', p', \theta')$ such that the surplus generated by type $i$ increases. Therefore, the payoff of type $i$ strictly increases.

Now consider type $i + 1$. The planner solves the maximization problem for type $i + 1$ again. That is, he maximizes the payoff of type $i + 1$ subject to the free entry condition and the incentive compatibility constraint of all lower types. Since all lower types including type $i$ get a strictly higher payoff than the equilibrium allocation now, the maximization problem for type $i + 1$ is now less constrained, so the planner can achieve weakly higher welfare from type $i + 1$ as well. The planner keeps doing the same thing for all types above $i$ and assigns them new $(a, p, \theta)$ triples. The welfare of the population has increased so far, because type $i$ has generated strictly higher surplus and all types $i + 1$ to $I$ have generated weakly higher surplus. To satisfy the budget-balance condition, the planner imposes an identical tax on all types so that IC constraints are not affected. Making transfers across agents does not change the welfare of the population, therefore, the welfare level now is strictly higher than that in the equilibrium.

In the next proposition, I provide sufficient conditions for the planner to achieve the first best. Before that, let me introduce some definitions. I say that $a' \geq a$ if $a'^k \geq a^k$ for all $k \in \{1, 2, ..., n\}$, that is, if $a'$ is greater than $a$ component by component. Function $g : A \times \{1, 2, ..., I\} \to \mathbb{R}$ has increasing differences in $(a, i)$ if for $a' \geq a$, $g(a', i) - g(a, i)$ is weakly increasing in $i$. Function $g : A \times \{1, 2, ..., I\} \to \mathbb{R}$ is supermodular in $a$ if for all $a, b \in A$, $g(a, i) + g(b, i) \leq g(a \lor b, i) + g(a \land b, i)$.

**Assumption 2.** The following conditions hold:

1. Monotonicity of $u$ in $i$: $u_1(a) \leq u_2(a) \leq ... \leq u_I(a)$ for all $a \in \bar{A}$.

2. $u$ has increasing differences in $(a, i)$.

3. $u + v$ has increasing differences in $(a, i)$.

4. Supermodularity of $f$ in $a$ where $f_i(a) \equiv u_i(a) + v_i(a)$ for all $a \in A$ and $i$.

5. Either (a) holds or (b) and (c) hold:

   (a) Monotonicity of $v$ in $i$: $v_1(a) \leq v_2(a) \leq ... \leq v_I(a)$ for all $a \in \bar{A}$.

   (b) Monotonicity of $f$ in $i$: $f_1(a) \leq f_2(a) \leq ... \leq f_I(a)$ for all $a \in \bar{A}$. 

19
(c) Sufficient gains from trade for all types:

\[ U_i^{FB} \geq m(\theta_{i-1}^{FB})(u_i(a_i^{FB}) - u_{i-1}(a_{i-1}^{FB}))\frac{\sum_{j=i}^{I} \pi_j}{\pi_i}, \text{ for } i > 1 \text{ and } U_1^{FB} \geq 0. \]

**Theorem 2** (Result 2). Under Assumption 2, the planner achieves the first best.

Part 1 of Assumption 2 simply states that the payoff of higher types is higher than lower types for any given level of production. Part 2 is a standard increasing differences property.\(^{16}\) If \( u \) is differentiable, this assumption implies that given a level of production, the marginal payoff of higher types with respect to the level of production of good \( k \in \{1, 2, ..., n\} \) is higher than that of lower types.\(^{17}\) Similarly, part 3 states that the marginal surplus with respect to the level of production of good \( k \) that higher types create is higher than the associated marginal surplus that lower types create. Part 4 is a standard supermodularity condition which states that the marginal surplus created by type \( i \) with respect to the level of production of good \( k \) is increasing in the level of production of good \( l \) (\( k \neq l \)). In part 5, I require either of the two following conditions. For any given level of production, buyers weakly prefer higher types of sellers. If this assumption is not satisfied, I require that \( u_i + v_i \) is increasing in \( i \) for any production level (in part 5(b)) and also \( \frac{\sum_{j=i}^{I} \pi_j}{\pi_i} \) is less than some threshold for every \( i > 1 \).

The proof follows a guess-and-verify approach. I first guess that the planner can achieve the first best under the conditions in Assumption 2. Then I ensure that all conditions for feasibility are satisfied. See Figure 1.1 for the illustration of the proof.

The planner achieves the first best if there exists a feasible mechanism in which type \( i \) sellers get matched with probability \( m(\theta_{i-1}^{FB}) \) and produce \( a_i^{FB} \). I need to find a set of transfers which together with \((\theta_i^{FB}, a_i^{FB})\) satisfy IC constraints. To find such a set, I show that if part 1 of Assumption 2 holds and if transfers are such that local downward IC constraints are satisfied and are binding, then all IC constraints are satisfied. By local downward IC constraint I mean that type \( i \) should not gain by reporting type \( i - 1 \). Moreover, I show that by this construction

\(^{16}\)This property is equivalent to the single crossing condition (which is also called Spence-Mirrlees condition) for a broad class of functions. See [23] for a full discussion about these properties and the relationship between them.

\(^{17}\)I do not impose differentiability assumption, though.
method, the amount of transfers to the lowest type \((p_1)\) determines the amount
of transfers for all other types. A set of transfers that satisfies local downward
IC constraints exists if \((\theta^{FB}_i, a^{FB}_i)\) is increasing in \(i\) and also if \(u\) has increasing
differences property in \((a, i)\) (part 2 of Assumption 2). See Theorem 7.1 and 7.3

According to Theorem 5 in [23], if \(u_i + v_i\) satisfies parts 3 and 4 of Assumption 2,
then \(a^{FB}_i \equiv \max_{a \in A}\{u_i(a) + v_i(a)\}\) is increasing in \(i\), that is, \(a^{FB}_i \geq a^{FB}_{i-1}\). If \(u_i + v_i\) is increasing in \(i\) (part 1 and 5(a) or 5(b) of Assumption 2), then \(m(\theta)(u_i(a^{FB}_i) + v_i(a^{FB}_i)) - k\theta\) satisfies increasing differences property in \((\theta, i)\). Also, \(m(\theta)(u_i(a^{FB}_i) + v_i(a^{FB}_i)) - k\theta\) is trivially supermodular in \(\theta\) because \(\theta\) is one-dimensional. Therefore, again according to Theorem 5 in [23], \(\arg \max_{\theta} \{m(\theta)(u_i(a^{FB}_i) + v_i(a^{FB}_i)) - k\theta\}\) will be also increasing in \(i\). Hence, \((\theta^{FB}_i, a^{FB}_i)\) is increasing in \(i\). Then the planner
adjusts \(\tilde{p}_1\) such that all types get a positive payoff. This implies that one can
set \(\tilde{s}_i = 0\) for all \(i\). Given the transfer scheme (the prices to be paid to sellers),
the planner ensures that the budget constraint holds with equality by making
identical transfers to all types. In future sections, I will make it clear by a couple
of examples the mechanism through which the planner can improve welfare relative
to the market economy and how he might achieve the first best.

1.4 Example 1: Asset Market with Lemons

So far I have considered a general framework. In the following two sections, I
study two examples from [26] and characterize the constrained efficient allocation
for them and compare them with the associated equilibrium allocations. At the end
of this section, I provide some intuition on how and why the planner can increase
welfare by using appropriate transfers. Also, I explain the nature of inefficiency in
the models of directed search with adverse selection.

The first example is an asset market with lemons (in the spirit of [14]). There
are two types of assets, with value \(c_i\) to the seller and \(h_i\) to the buyer. Both \(c_i\) and
\(h_i\) are in terms of a numeraire good. The payoff of a buyer matched with a type \(i\)
seller is \(\alpha h_i - t - k\) where \(\alpha\) is the probability that the buyer gets the asset from
the seller and \(t\) is the amount of the numeraire good that he pays (either to the
planner or to sellers) in terms of the numeraire good. The payoff of a type \(i\) seller
matched with a buyer is \(-\alpha c_i + t\) where \(\alpha\) is the probability that the seller gives
the asset to the buyer and \( t \) is the amount of the numeraire good he consumes. The buyer’s payoff is \(-k\) if unmatched. As a special case of the original setting, here: \( I = 2, n = 1, a \equiv \alpha, u_i(\alpha) = -\alpha c_i \) and \( v_i(\alpha) = \alpha h_i \). The matching function is \( m(\theta) = \min\{1, \theta\} \), that is, the short side of the market gets matched for sure. Following GSW, I also make the following assumptions:

**Assumption 3.** In the asset market with lemons,

1. \( 0 < h_1 < h_2 \) and \( 0 < c_1 < c_2 \).
2. For \( i = 1, 2 \), \( c_i < b_i \equiv h_i - k \).

**Proposition 1.** In the asset market with lemons, the planner achieves strictly higher welfare than the equilibrium. If \( \pi_1 b_1 + \pi_2 b_2 \geq c_2 \), then the planner achieves the first best. See full characterization of the constraint efficient allocation in Table 1.1.

The first part of this proposition is a special case of Theorem 1 and states that the planner achieves strictly higher welfare than the market economy. Then, in order to fully characterize the constraint efficient allocation, I separate two cases: \( \pi_1 b_1 + \pi_2 b_2 \geq c_2 \) and \( \pi_1 b_1 + \pi_2 b_2 < c_2 \). Specially in the first case, I show that the planner achieves the first best. This claim is stronger than Theorem 2, because the requirements are weaker. It is easy to check that in order for Assumption 2 to be satisfied,\(^{18}\) one needs \( b_2 - c_2 < b_1 - c_1 \) and \( \frac{\pi_2 c_2}{\pi_1} \leq b_1 - c_1 \), which are stronger requirements than the requirement in Proposition 1 (\( \pi_1 b_1 + \pi_2 b_2 \geq c_2 \)).

In the second and third columns of Table 1.1 the equilibrium outcomes under complete information and under private information are described respectively. In the fourth and fifth columns, I describe the planner’s allocation under different conditions.

Since there are positive gains from trade for both types according to part 2 of Assumption 3, under complete information the planner wants both types to get matched with probability 1 (\( \theta_1^{FB} = \theta_2^{FB} = 1 \)) and also trade with probability 1 (\( \alpha_1^{FB} = \alpha_2^{FB} = 1 \)). As already discussed under complete information, the market decentralizes the first best allocation.

\(^{18}\)In order to apply Theorem 2 to this setting, first switch the label of type one and type two. Now see that parts 1 to 4 of Assumption 2 are easily satisfied. It is just left to check Assumption 2 parts 5(b) and 5(c). Part 5(b) is satisfied if \( h_2 - c_2 < h_1 - c_1 \). Part 5(c) is satisfied if \( U_2^{FB} \geq 0 \) and \( U_2^{FB} \geq \frac{m(1)(-c_1 + c_2)}{\pi_1} \).
In the equilibrium with private information, different types trade in different submarkets. In submarket one, price is lower, but probability of matching is higher compared to submarket two ($p_{1}^{EQ} < p_{2}^{EQ}$). The market tightness is used as a screening device here. The probability of matching for type two is distorted so that type one would not want to apply to submarket two, although the price is higher there. The equilibrium allocation is independent of the distribution of types.

If $\pi_{1}b_{1} + \pi_{2}b_{2} \geq c_{2}$, then the planner achieves the first best through a pooling allocation. See the fourth column of Table 1.1. In this allocation, the planner does not need to use any transfers. All he needs to do is to restrict the entry of buyers to other submarkets by imposing large taxes on those submarkets and have all sellers trade in a pooling submarket with $p = \pi_{1}b_{1} + \pi_{2}b_{2}$ and $t = 0$. This allocation cannot be sustained as an equilibrium, because buyers would have incentives to open a new submarket with a higher price to attract only high type sellers from the pool, i.e. cream skimming. But then the probability that high quality sellers get matched will be reduced compared to the first best and the planner does not want that. This is why the planner imposes large taxes on other submarkets.

Now assume that $\pi_{1}b_{1} + \pi_{2}b_{2} < c_{2}$. The planner’s allocation in this case is reported in the fifth column of Table 1.1. Type two would get less than 0 under the pooling allocation, so pooling both types is not feasible. Therefore, the first best is not achievable via a pooling allocation. The first best is not achievable through any separating allocation either, because if $\alpha_{1} = \alpha_{2} = \theta_{1} = \theta_{2} = 1$, then the payment to sellers in both submarkets should be the same to satisfy IC condition. If the payments in both submarkets are equal, then this allocation is pooling, but it is already shown that the pooling allocation is not feasible. The same explanation is illustrated via indifference curves in Figure 1.3.

### 1.4.1 Explanation of the Results

To understand how the planner can achieve strictly higher welfare than the equilibrium in the asset market with lemons, assume as a thought experiment that the planner begins from the equilibrium allocation, which is feasible for the planner, and wants to increase welfare. In the equilibrium type one is indifferent between choosing submarket one and submarket two. Although some type two sellers are unmatched in submarket two, buyers do not enter submarket two any more, be-
cause more entry will make submarket two strictly preferable for type one, thus leading to entry of type one to submarket two. Nevertheless, matching with type one sellers in submarket two with positive probability is not worthwhile for buyers given the high price that buyers need to pay in submarket two. In short, the IC constraints that buyers face do not allow more buyers to enter submarket two.

To improve efficiency, the planner increases the net payment to type one, $p_1$, so that IC constraint of type one for choosing submarket two becomes slack.\textsuperscript{19} That is, type one strictly prefers submarket one over submarket two following this subsidy. Now more buyers have incentives to enter submarket two to get matched with previously unmatched sellers of type two. To finance subsidies to sellers in submarket one (type one sellers), the planner must tax sellers in submarket two (type two sellers). The planner keeps increasing $p_1$ and decreasing $p_2$ until one of the following happens. Either he achieves the first best, which is the case in the pooling allocation where both types trade with probability 1, or participation constraints of type two sellers bind, that is, type two sellers get exactly payoff 0. The former happens if $\pi_1 b_1 + \pi_2 b_2 \geq c_2$ and the latter happens if $\pi_1 b_1 + \pi_2 b_2 < c_2$. Figure 1.2 illustrates this point. Although I explained the main idea through a two-type example, the intuition is the same in the general n-type setting, or even in a continuous type setting which will be discussed in Section 6.

The main difference between planner’s allocation and the equilibrium allocation is that in the equilibrium, the payment to sellers is exactly equal to the payment that buyers make. Also, because of the free entry condition, the buyers get 0, so the sellers get the whole surplus in every submarket. However, it is feasible for the planner to give sellers in one submarket more and sellers in other submarkets less than the surplus they generate. The only constraint that the planner faces is the budget constraint over all submarkets. That is, the amount of transfers that buyers pay must be equal to amount of transfers that sellers receive over all submarkets.

Entry of more buyers in a submarket creates two types of externalities on others. First, it decreases the probability of matching of other buyers in that submarket.

\textsuperscript{19}The way that the planner implements the mechanism is to subsidize buyers in submarket one, that is, $t_1 < 0$. Since there is zero profit condition for buyers in each submarket, buyers in submarket one pays the net amount of $b_1$, anyway, which is equal to $p_1 + t_1$. But $t_1 < 0$, so $p_1 > b_1$. In other words, when the planner imposes tax or subsidy on buyers in one submarket, it is as if the planner imposes that tax or subsidy on sellers.
Second, it changes the payoff of sellers in that submarket. In the complete information case, the change in the payoff of sellers in one submarket does not affect the payoff of sellers in other submarkets. In fact, under complete information, the negative externality that entrants impose on other buyers is exactly offset by the amount of positive externalities that they impose on sellers and therefore the equilibrium allocation is efficient. When there is private information, the change in the payoff of one type of sellers alters the IC constraints that other buyers face in other submarkets, thus affecting the set of feasible contracts that those buyers can offer to attract other types of sellers. This, in turn, will affect the payoff of other sellers in other submarkets. The buyers in the market economy does not take this effect into account. The planner, in contrast, internalizes these externalities and therefore is able to increase welfare.

The inability of buyers to internalize the externalities they create on others in equilibrium is similar to the situation in random search models with ex-post bargaining (like [12]) in which the share of the surplus that buyers get is exogenously fixed, so the outcome is generally inefficient. Here, although the division of the surplus to buyers and sellers is not exogenously fixed, it is endogenously pinned down by two constraints that IC and free entry impose on the allocation, so it is generally unlikely that the constrained efficiency is achieved by equilibrium. The planner can internalize these externalities, because he is not constrained by the free entry condition at each submarket, so he can make the buyers’ share of the surplus satisfy Hosios condition (See [27]).

1.4.2 What If There Are No Gains from Trade for Some Types?

GSW show that in the asset market with lemons, if there are no gains from trade only for type one, that is, $b_1 - c_1 < 0$ and $b_2 - c_2 > 0$, then the entire market will shut down. I show that in this case the planner cannot help. See Appendix, Page 117, for the proof.

The intuition is as follows. Type two is not active in the equilibrium, so given IC of type one, the highest payoff that type two can get in the market is negative, so type two chooses not to participate in the market. Therefore, both IC are binding (both types get zero payoff anyway.) The trick that worked in the proof of Theorem 1 is not effective here, because any direct subsidies intended for type
one equally attracts type two sellers, so type two would also prefer to report to be type one. The takeaway message is that if the distortion is so severe that inactivity of one type in equilibrium leads to inactivity of other types, then the planner may not be able to help.

1.5 Example 2: The Rat Race

In this section I study another example from GSW, the rat race, which was originally discussed in [13]. The main reason that I include this example is that the first best here is achievable only through a separating allocation, in contrast to the previous example (asset market with lemons) where the first best was achievable through a pooling allocation (if \( \pi_1 b_1 + \pi_2 b_2 \geq c_2 \)). The planner here achieves the first best by separating different types and using appropriate transfers.

There are two types of workers (as sellers) on one side and firms (as buyers) on the other side of the market. The payoff of a type \( i \) worker matched with a firm from \( a \) hours of work and consuming \( t \) units of the numeraire good is \( t - \phi_i(a) \). The worker’s payoff is 0 if unmatched. The payoff of a firm matched with a type \( i \) worker when the worker works for \( a \) hours and the firm pays \( t \) units of the numeraire good (either to the worker or to the planner) is \( v_i(a) - t - k \). The firm’s payoff is \(-k\) if unmatched. As a special case of the original setting, here \( I = 2, n = 1 \) and \( u_i(a) = -\phi_i(a) \). Matching function \( m(\theta) \) is strictly concave and twice differentiable. I make the following assumptions:

Assumption 4. In the rat race example,

1. \( \phi_i \) is differentiable, increasing, strictly convex and \( \phi_i(0) = \phi_i'(0) = 0 \).
2. For all \( a \), \( \phi_1(a) = \tau \phi_2(a) \) where \( \tau > 1 \).
3. \( v_i \) is differentiable, increasing and strictly concave.
4. For all \( a \), \( v_1(a) \leq v_2(a) \) and \( v_1'(a) \leq v_2'(a) \).

Remember from Theorem 2 that Assumption 2 is sufficient for the planner to achieve the first best. I argue here that if Assumption 4 holds, then Assumption 2 is automatically satisfied and therefore I can use that result. Part 1 of Assumption 2 is satisfied because \(-\phi_1(a) < -\phi_2(a) \) for all \( a \). Part 2 of Assumption 2 (increasing differences property of \( u(.) \)) is satisfied because \(-\phi_2(a) - (-\phi_1(a)) \) is increasing in
a due to the assumption that \( \tau > 1 \). Part 3 of Assumption 2 (increasing differences property of \( u(\cdot) + v(\cdot) \)) is satisfied because \( v_2(a) - \phi_2(a) - (v_1(a) - \phi_1(a)) \) is increasing in \( a \). Part 4 of Assumption 2 (supermodularity of \( u(\cdot) + v(\cdot) \) in \( a \)) is trivially satisfied because \( a \) is just one-dimensional. Part 5(a) of Assumption 2 (monotonicity of \( v(\cdot) \)) is satisfied because \( v_2(a) \geq v_1(a) \).

**Proposition 2.** If Assumption 4 holds and \( U_i^{FB} > 0 \) for all \( i \), then the planner achieves the first best. See the fourth column of Table 1.2 for the full description of the constrained efficient allocation.

This result is a special case of Theorem 2. The planner subsidizes type one (\( p_1^{CE} > p_1^{FB} \)) and taxes type two (\( p_2^{CE} < p_2^{FB} \)) to achieve efficiency. By offering this schedule of transfers, allocating the low type workers higher wage and the high type workers lower wage than their wages in the equilibrium with complete information, the planner discourages type one workers from applying to submarket two, thus reducing the cost of private information.

An interesting point about this result is that the planner achieves the first best regardless of the distribution of types. The intuition is that if the planner sets payments such that type one gets at least payoff 0, then the planner can make positive amount of money over each submarket.

GSW make the same assumptions except that they they do not impose \( v_1'(a) \leq v_2'(a) \). When \( U_2^{FB} - U_1^{FB} \geq (\tau - 1)m(\theta_2^{FB})\phi_2(a_2^{FB}) \), then the equilibrium does not achieve the first best. They propose a pooling allocation which Pareto dominates the equilibrium allocation if \( \pi_1 \) is sufficiently small, although the pooling allocation does not achieve the first best. As stated earlier, the planner achieves the first best regardless of \( \pi_1 \). Moreover, if \( \pi_1 \) is sufficiently small, then the planner’s allocation Pareto dominates the equilibrium allocation.

### 1.6 Extension: Asset Market with a Continuous Type Space

The model studied in this section is an extension of that in Section 4 to a continuous type space. In Section 4, the efficient tax schedule requires high price assets to be taxed and low price assets to be subsidized. An interesting question is whether
the tax schedule which implements the constraint efficient mechanism is generally monotone in the price of assets or not.

This extension is interesting not only because it makes it possible to consider cases in which the value of assets to sellers does not have the same order as the value of assets to buyers, but also because I can answer some relevant policy questions about the optimal taxation in the asset markets.\textsuperscript{20} Also, studying this case makes it possible for us to compare the planner’s allocation with the equilibrium allocation in [3]. The setting in this section is basically a static version of Chang’s environment, and fortunately the main ideas regarding the equilibrium and the planner’s allocation are captured in this static case.\textsuperscript{21} Since this environment is not a special case of the original setting in Section 2, I need to define the constrained efficient allocation again. The main ideas discussed so far will be used similarly for this case as well, but the mathematical tools used to characterize the planner’s allocation will be different.

1.6.1 Environment

There is a continuum of measure one of heterogeneous sellers indexed by \( z \in Z \equiv [z_L, z_H] \subset \mathbb{R} \), with \( F(z) \) denoting the measure of sellers with types below \( z \). \( F \) is continuously differentiable and strictly increasing in \( z \) and \( F' \) is its derivative. Type \( z \) is seller’s private information. Similar to the original setting, buyers’ and sellers’ payoffs are quasi-linear. A buyer’s payoff who enters the market and gets matched with a type \( z \) is \( h(z) - t - k \) where \( t \) denotes the amount of a numeraire good that he produces and \( h(z) \) is the value of the asset to the buyer in terms of the numeraire good. His payoff is \(-k\) if unmatched. The payoff of a type \( z \) seller matched with a buyer is \( t - c(z) \) where \( t \) denotes the amount of the numeraire good that he

\textsuperscript{20}I could do the same exercise with a discrete type space with more than two types, but the technical analysis with a continuous type space is simpler.

\textsuperscript{21}In a dynamic setting, the planner has some intertemporal considerations, because the distribution of types in the population does not necessarily remain the same over time, because some types get matched more quickly than others and exit the market. This observation raises a new and interesting tradeoff, whether the planner wants to have low types find a match early or he wants to have all types together all the way to the end. The analysis of the dynamic setting is beyond the scope of this paper. Since the equilibrium allocation is distribution free, the equilibrium analysis is much easier than the analysis of the planner’s problem in the dynamic case. However, if one assumes in the dynamic setting that when sellers sell their assets, they are endowed a new asset with the same quality, the same results can be obtained from the dynamic setting as in the static setting, because the distribution of types does not change over time.
consumes and \( c(z) \) is the value of the asset to the seller in terms of the numeraire good. His payoff is 0 if unmatched. Functions \( h: Z \to R \) and \( c: Z \to R \) are twice continuously differentiable. Matching function \( m(\cdot) \) is increasing, strictly concave and twice differentiable with strictly decreasing elasticity. I also assume throughout this section that there are positive gains from trade for all types. Similar to the discrete type space, it turns out that all types will be active both in equilibrium and in the constrained efficient allocation.

### 1.6.2 Complete Information Allocation or First Best

Here, I mostly follow the discussion of the complete information case for the discrete type space in Section 3. Consider the market economy with the complete information. The buyers who attempt to attract type \( z \) sellers solve the following problem:

\[
U^\text{FB}(z) = \max_{\theta,p} \{ m(\theta)(p - c(z)) \}
\]

subject to \( q(\theta)(h(z) - p) \geq k \).

Let \( \theta^\text{FB}(z) \) and \( p^\text{FB}(z) \) denote the market tightness and the price that solve this problem. I assume that \( U^\text{FB}(z) > 0 \) for all \( z \), that is, that there are positive gains from trade for all types. Similar to the discrete type case, \( U^\text{FB}(z) = \max_{\theta} \{ m(\theta)(h(z) - c(z)) - k\theta \} \). Also \( \theta^\text{FB}(z) \) solves

\[
m'(\theta)(h(z) - c(z)) = k, \quad \text{(1.1)}
\]

for both the planner and the market economy with complete information. The left hand side of Equation 1.1 is the marginal benefit of adding one more buyer to the submarket composed of \( z \) sellers. The right hand side is the marginal cost of doing that. The planner keeps adding buyers to each submarket until the marginal cost and marginal benefit become equal.\(^{22}\)

---

\(^{22}\)To verify that the Hosios condition \([27]\) is satisfied in the market with complete information with directed search, I calculate the share of the surplus that sellers get in equilibrium:

\[
\frac{p^\text{FB}(z) - c(z)}{h(z) - c(z)} = \frac{U^\text{FB}(z)}{m(\theta^\text{FB}(z))(h(z) - c(z))} = \frac{m(\theta^\text{FB}(z))(h(z) - c(z)) - k\theta^\text{FB}(z)}{m(\theta^\text{FB}(z))(h(z) - c(z))} = -\frac{\theta^\text{FB}(z)q'(\theta^\text{FB}(z))}{m(\theta^\text{FB}(z))(h(z) - c(z))} = \eta(\theta^\text{FB}(z)),
\]

29
1.6.3 Planner’s Problem

Because our focus in this section is the shape of the optimal tax schedule, whether the optimal tax schedule be monotone in the price of assets or not, the interesting concept to study is the concept of implementable allocation and not direct mechanism. The main definition of implementable allocation in this section is a straightforward modification of definition in Section 2 to allow for a continuous type space. The definition of constrained efficient mechanism and also Lemma 1 can be modified in a straightforward way too. However, I do not repeat them here for the sake of brevity. (See Appendix, Section A.6.) Specifically, it can be proved in exactly the same fashion as in Lemma 1 that for any feasible mechanism, there exists an associated implementable allocation under which all types get exactly the same payoff as in the direct mechanism.

Definition 6. An implementable allocation, \( \{P, G, \theta, \mu, t, T\} \), is a measure \( G \) on the set of all possible prices, \( P \equiv \mathbb{R}_+ \), with support \( P \), a tightness function, \( \theta : P \rightarrow [0, \infty] \), a conditional density function of buyers’ beliefs regarding the type of sellers who would apply to \( p \), \( \mu : P \times \mathbb{Z} \rightarrow [0, 1] \), a tax function denoting the amount of tax to be imposed on buyers at each submarket conditional on trade, \( t : P \rightarrow \mathbb{R} \), and finally the amount of the numeraire good to be transferred to sellers in a lump sum way, \( T \in \mathbb{R}_+ \), which satisfies the following conditions:\(^{23}\)

(i) **Buyers’ profit maximization and free entry**

For any \( p \in P \),

\[
q(\theta(p))\left[ \int h(z)\mu(z|p)dz - p - t(p) \right] \leq k,
\]

with equality if \( p \in P \).

(ii) **Sellers’ optimal search**

Let \( U(z) = \max \left\{ 0, \max_{p' \in P} \left\{ m(\theta(p'))(p' - c(z)) \right\} \right\} + T \) and \( U(z) = T \) if \( P = \emptyset \).

Then for any \( p \in P \) and \( z \), \( U(z) \geq m(\theta(p))(p - c(z)) + T \) with equality if \( \theta(p) < \infty \) and \( \mu(z|p) > 0 \). Moreover, if \( p - c(z) < 0 \), either \( \theta(p) = \infty \) or \( \mu(z|p) > 0 \).

where the third equality follows from Equation 1.1. Hosios condition states that a necessary condition for the efficiency of any allocation is that the share of the surplus that type \( z \) sellers get from the match, \( \frac{p - c(z)}{h(z) - c(z)} \), for any \( z \) must be equal to the elasticity of matching function with respect to the number of sellers. As shown above, the equilibrium allocation under complete information satisfies this property.

\(^{23}\) In this section, since I have assumed that there are positive gains from trade for all types, it is easy to check that \( T \) is redundant. That is, the welfare level will not be lower without \( T \).
(iii) **Feasibility or market clearing**
For all \( z \), \( \int_P \frac{\mu(z|p)}{q(p)} \, dG(p) \leq F'(z) \), with equality if \( U(z) > T \).

(iv) **Planner’s budget constraint**
\[
\int_P q(\theta(p)) t(p) \, dG(p) \geq T.
\]

**Definition 7.** A constrained efficient allocation is an implementable allocation which maximizes welfare among all implementable allocations. That is, a constrained efficient allocation solves the following problem:
\[
\max_{\{P,G,\theta,\mu,t,T\}} \int U(z) \, dF(z)
\]
\[
s. \ t. \ \{P,G,\theta,\mu,t,T\} \text{ is implementable},
\]
where \( U(z) \) is define in part (ii) of Definition 6.

1.6.4 **Characterization**

1.6.4.1 **Characterization of the Constrained Efficient Allocation**
To find a direct mechanism which solves the planner’s problem, I use somewhat a backward approach. I first guess that the planner can achieve the first best. That is, the planner can maximize his objective function for each type separately. Then I find a set of prices such that sellers’ IC conditions are satisfied. Given this set of prices, I derive sufficient conditions under which the planner’s budget constraint and participation constraint of all types hold simultaneously. To find an implementable allocation which implements this direct mechanism, I calculate taxes in such a way that buyers’ maximization and free entry (condition (i) in Definition 6) for on-the-equilibrium-path prices are satisfied. Finally I construct taxes and beliefs for off-the-equilibrium-path prices. Checking for other conditions in Definition 6, then, would be easy.

**Assumption 5.** For all \( z \), \( c'(z) > 0 \) and either

1. \( h'(z) \leq 0 \) for all \( z \), or
2. \( h'(z) \leq c'(z) \) and \( \psi\left(\frac{k}{h(z) - c(z)}\right) \left[\frac{h(z) - c(z)}{c'(z)}\right] \geq \frac{F(z)}{F'(z)} \) for all \( z \), where \( \psi(.) \equiv \eta(m^{-1}(.) \) and \( \eta(\theta) \equiv -\frac{\theta q'(\theta)}{q(\theta)} \).

31
Proposition 3. If Assumption 5 holds and $U^{FB(z)} > 0$ for all $z$, then the planner achieves the first best.

Given any implementable allocation, define correspondence $\Omega(z)$ as follows

$$\Omega(z) \equiv \{(p, \theta(p), t(p)) \text{ such that } \mu(z|p) > 0\}.$$ 

Denote the elements of $\Omega(z)$ by $(\tilde{p}(z), \tilde{\theta}(z), \tilde{t}(z))$ showing the price, market tightness and the tax amount (imposed on buyers) of the submarkets to which type $z$ applies with a strictly positive probability. Basically, $\tilde{p}(z)$, $\tilde{\theta}(z)$ and $\tilde{t}(z)$ are elements of a direct mechanism.\footnote{Here, having $\tilde{t}(z)$ as a tax amount in the direct mechanism is an abuse of notation. This is because I have defined the direct mechanism in such a way that transfers are made only to sellers and the planner just ensures that buyers get an ex-ante payoff 0. Therefore, $\tilde{t}(z)$ should be interpreted as the tax amount that buyers should pay in the implementable allocation if they are matched with type $z$ sellers. The reason that I define it as a function of $z$ is because it makes the analysis simpler and more intuitive.} Similar to the notation in previous sections, $\tilde{x}(z)$ denotes the variable $x$ allocated to type $z$ in a direct mechanism, whether the direct mechanism be used for a constraint efficient allocation or an equilibrium allocation.

It is shown in the proof of Proposition 3 that all types trade in submarkets with different market tightness, therefore the allocation is separating and $\tilde{p}^{CE}(z)$, $\tilde{\theta}^{CE}(z)$ and $\tilde{t}^{CE}(z)$ are just functions (as opposed to correspondences) of $z$. It is also shown in the proof that these variables are given as follows:

$$\tilde{\theta}^{CE}(z) = \theta^{FB}(z) \text{ for all } z,$$

$$\tilde{p}^{CE}(z) = c(z) + \frac{U(z_H) + \int_{z}^{z_H} \frac{m(\tilde{\theta}^{CE}(z_0))c'(z_0)dF(z_0)}{m(\tilde{\theta}^{CE}(z))}}{m(\tilde{\theta}^{CE}(z))} \text{ for all } z, \quad (1.2)$$

where

$$U(z_H) = \int \left[m(\tilde{\theta}^{CE}(z))(h(z) - c(z)) - k\tilde{\theta}^{CE}(z) - m(\tilde{\theta}^{CE}(z))c'(z)\frac{F(z)}{F'(z)}\right]dF(z),$$

and

$$\tilde{t}^{CE}(z) = h(z) - \tilde{p}^{CE}(z) - \frac{k}{q(\tilde{\theta}^{CE}(z))} \text{ for all } z. \quad (1.3)$$

To have a rough idea how the planner can undo the effects of private information and achieves the first best, I proceed by analyzing the incentive compatibility problem that the planner faces. I assume (without loss of generality) that sellers
are allocated to different submarkets through a direct mechanism. That is, if a type $z$ agent reports $\hat{z}$, his payoff is given by $m(\bar{\theta}(\hat{z}))(\bar{p}(\hat{z}) - c(z))$. In a direct mechanism, agents of type $z$ choose a $\hat{z}$ which maximizes their payoff:

$$\max_{\hat{z}} \{ m(\bar{\theta}(\hat{z}))(\bar{p}(\hat{z}) - c(z)) \}. \tag{1.4}$$

I keep the assumption that $c'(z) > 0$ throughout this section, so the seller’s payoff function, $m(\theta)(p(z) - c(z))$, satisfies single crossing condition. (See Theorem 7.3 in [24].) As already discussed in the sketch of the proof of Theorem 2, $\bar{\theta}(z)$ being decreasing in $z$ implies that there exists a set of transfers to sellers that satisfies IC. Now assume $h'(z) \leq 0$ for all $z$ or $h'(z) \leq c'(z)$ for all $z$. In either case, $\theta^{FB}(z)$ is decreasing in $z$ according to Equation 1.1. Therefore, if $\bar{\theta}(z)$ is set to be equal to $\theta^{FB}(z)$ for all $z$, one can find such transfers. Then according to Envelope theorem, one can calculate these transfers as given in Equation 1.2.

Since $\bar{\theta}(z)$ is strictly decreasing here (because $\theta^{FB}(z)$ is strictly decreasing), then the associated implementable allocation must be separating, so the amount of tax that should be imposed on buyers in each submarket, $\bar{\ell}(z)$, will be a function (not a correspondence) of $z$ and can be easily calculated by buyers’ profit maximization and free entry condition. I provide sufficient conditions in the proof such that the planner’s budget constraint also holds. If $h'(z) \leq 0$, then the planner has enough resources to distribute among agents regardless of the distribution. If $h'(z) \leq 0$ is not satisfied for some $z$ but $h'(z) - c'(z) \leq 0$ still holds for all $z$, then I need another condition (in part 2 of Assumption 5) which relates the distribution of types to the payoff and matching functions to ensure that the planner’s budget constraint is satisfied.

Proposition 3 analyzes just one possible case for the planner’s problem where monotonicity constraint (that $\bar{\theta}(z)$ should be decreasing in $z$) and participation constraint for almost all types are not binding. Analyzing other cases where monotonicity constraint or participation constraint is binding does not add much insight to the analysis, so I skip it. As an example, I solve the planner’s problem for the case where participation constraint is binding in Appendix, Section A.6.3. For the case where monotonicity constraint is binding, one can use existing techniques from mechanism design literature to bunch multiple types. The characterization in that case is available upon request.
1.6.4.2 Characterization of Equilibrium Allocation

I report the results of a static version of [3] here and compare the equilibrium allocation with the planner’s one. Definition of equilibrium is similar to the definition of implementable allocation here, but with the restriction that taxes and transfers must be all equal to 0. I do not repeat the definition of equilibrium here to save space. See [3] for more details on the equilibrium definition. I study a static model while she studies a dynamic model. To see why I consider a static model, see Footnote 21. Chang assumes that utility of holding the asset until finding a buyer is different across different types of assets. Similarly, I assume sellers with high \( z \) values their assets more (\( c' > 0 \)).

**Assumption 6.** \( c'(z) > 0 \) and \( h'(z) \geq 0 \) for all \( z \).

**Proposition 4** (Equivalent to Proposition 1 in [3]). If Assumption 6 holds and if \( U_{FB}(z) > 0 \) for all \( z \), then there exists a unique equilibrium. The equilibrium is separating. The market tightness solves the differential equation 1.6. The initial condition is given by \( \tilde{\theta}_{EQ}(z_L) = \theta_{FB}(z_L) \). Prices are given by \( \tilde{p}_{EQ}(z) = h(z) - \frac{k}{q(\tilde{\theta}_{EQ}(z))}. \)

I explain the logic behind her result. See her paper for the formal proof. First note that the IC constraints that agents face in the market economy are the same as those the planner faces, therefore I can use Equation 1.4 to describe IC constraints for analyzing equilibrium too. The only difference is that the prices are different in the market economy, because they are pinned down by the free entry condition. Chang shows that any equilibrium under Assumption 6 is separating, so free entry implies that \( \tilde{p}_{EQ}(z) = h(z) - \frac{k}{q(\tilde{\theta}_{EQ}(z))} \) for all \( z \). Therefore, the payoff of type \( z \) in the market economy, denoted by \( U_{EQ}(z) \), is calculated as follows:

\[
U_{EQ}(z) = \max_{\hat{z}} \{ m(\tilde{\theta}_{EQ}(\hat{z}))(h(\hat{z}) - c(z)) - k\tilde{\theta}_{EQ}(\hat{z}) \},
\]

where the objective function is the payoff of type \( z \) if he reports type \( \hat{z} \). FOC with respect to \( \hat{z} \) (together with the assumption of differentiability of \( \tilde{\theta}(z) \)) yields

\[
\left[ m'(\tilde{\theta}_{EQ}(z))(h(z) - c(z)) - k \right] \frac{d\tilde{\theta}_{EQ}(z)}{dz} + m(\tilde{\theta}_{EQ}(z))h'(z) = 0,
\]

where I used the fact that at the solution, \( \hat{z} = z \) due to IC.
With respect to the initial condition, roughly speaking, the market delivers the complete information payoff to the type which has the most incentive to deviate. For example, when \( h' \geq 0 \), the lowest type has the most incentive to deviate, so his allocation is set to the complete information level, i.e., \( \tilde{\theta}^{EQ}(z_L) = \theta^{FB}(z_L) \). The necessary condition for IC and the initial condition for the differential equation are depicted in Table 1.3 for different assumptions, where \( c' \) and \( h' \) are both positive or both negative, or only one of them is positive.

### 1.6.4.3 Disagreement in the Ranking of Assets between Buyers and Sellers or Two-Dimensional Private Information

[3] assumes in another part of her paper that sellers have another dimension of private information. Some sellers get liquidity shocks so they need to sell their assets quickly. What is relevant to our discussion is that following this extension, it is possible that function \( h \) has a strict local maximum, keeping the assumption \( c'(z) > 0 \) fixed. If \( h \) has a strict local maximum, she proves that full separation in the market is not possible. Also, she derives some conditions under which an equilibrium with fire sales exists, where many low type sellers and some high type sellers who need liquidity sell their assets with a lower price but very quickly. My characterization, in contrast, shows if \( h'(z) \leq c'(z) \) and if part 2 of Assumption 5 or Assumption 9 holds, even if \( h \) has a local maximum, then the constrained efficient allocation is separating, that is, the planner wants different types to trade in different submarkets. This case is depicted in Figure 1.4 where \( h'(z) \) is drawn in terms of \( c'(z) \) for all \( z \).

Now suppose \( h'(z) - c'(z) \leq 0 \) is violated for some \( z \). For example, \( h - c \) has one local minimum, but \( h'(z) \geq 0 \) and \( c'(z) > 0 \) both hold, as depicted in Figure 1.5. The equilibrium in this case is separating. The planner’s allocation, in contrast, involves some pooling, because monotonicity constraint (that \( \tilde{\theta}^{CE}(z) \) should be decreasing in \( z \)) cannot be satisfied through any separating allocation.\(^{25}\) The bottom line is that pooling of types occurs under different conditions in the planner’s allocation and the equilibrium allocation.\(^{26}\) The welfare level in the planner’s

---

\(^{25}\)Solving explicitly for the planner’s allocation in this case does not give us new insights, so I skip its analysis. For example, see the appendix of Chapter 7 in [24].

\(^{26}\)Roughly speaking, the planner is concerned with the surplus from the match not the value of the match to buyers only, so in the conditions regarding the planner’s allocation, usually \( h - c \) shows up. The buyers in the equilibrium are concerned with the value of the assets to themselves,
problem is strictly higher than that in the equilibrium by the same argument made in the proof of Theorem 1 even if the planner does not achieve the first best.

### 1.6.5 Examples of the Optimal Taxation

In this section, I present two examples in order to compare the first best (FB), equilibrium (EQ) and constrained efficient (CE) allocations and to figure out what types should be taxed and what types should be subsidized.

**Example 1.** Model parameters: \( m(\theta) = 1 - e^{-\theta} \), \( Z = [9, 10.5] \subset R \), \( c(z) = z \), \( h(z) = 0.04(z - 6.5)(z - 7)(z - 10) + 17 \), \( k = 1 \), and \( F(.) \) is uniform.

Here, \( c' > 0 \), \( h' > 0 \) and \( h' - c' < 0 \). It is easy to check that part 2 of Assumption 5 is satisfied, therefore Proposition 3 holds. Hence, the market tightness at the constrained efficient allocation is given by \( \tilde{\theta}_{CE}(z) = \theta_{FB}(z) = m'^{-1}\left(\frac{h(z) - c(z)}{k}\right) = \ln\left(\frac{h(z) - c(z)}{k}\right) \). Then, \( \tilde{p}_{CE}(z) \) and \( \tilde{t}_{CE}(z) \) are derived from Equation 1.2 and Equation 1.3. The net payment that buyers make in the constrained efficient allocation, \( \tilde{p}_{CE}(z) + \tilde{t}_{CE}(z) \), is equal to \( p_{FB}(z) \equiv h(z) - \frac{k}{q(\theta_{FB}(z))} \). Regarding equilibrium allocation, \( \tilde{\theta}_{EQ}(z) \) is derived from differential equation 1.6 with the initial condition \( \tilde{\theta}_{EQ}(9) = \theta_{FB}(9) \). The price that buyers pay in equilibrium is \( \tilde{p}_{EQ}(z) = h(z) - \frac{k}{q(\theta_{EQ}(z))} \).

In Figure 1.6, \( h(z) \) and \( c(z) \) in the left graph and \( h(z) - c(z) \) in the right graph are depicted. In the upper part of Figure 1.7, \( p(z) \) and \( \theta(z) \) for all three cases (FB, EQ and CE) are depicted. In this example, since Assumption 5 holds, \( \tilde{\theta}_{CE} \) is equal to \( \theta_{FB} \). On the other hand, \( \tilde{\theta}_{EQ} \) is less than \( \theta_{FB} \), because market tightness is basically the tool that buyers in the market economy use to screen high type sellers. Low type sellers prefer to sell their assets more quickly, because they do not want to get stuck with their “lemons.” Consequently, \( \tilde{p}_{EQ} \) is greater than \( p_{FB} \). Also, \( \tilde{p}_{CE} \) is higher for lower types and lower for higher types compared to \( p_{FB} \). Since the market tightness is the same in FB and CE, the price that buyers should pay in CE should be the same as that in FB in order for buyers’ profit maximization and free entry condition to be satisfied. On the other hand, \( \tilde{p}_{CE} \) is the payment that sellers should receive in CE. Therefore, the amount of tax that buyers should pay, \( \tilde{t}_{CE} \), is just equal to the difference, \( p_{FB} - \tilde{p}_{CE} \). In the lower left part of Figure...
1.7, \( \hat{t}^{CE}(z) \) is drawn in terms of \( z \). Also, \( \tilde{t}^{CE}(z) \) is drawn in terms of \( \hat{p}^{CE}(z) \) in the lower right part of the figure. In Figure 1.8, the payoff to sellers of different assets in EQ, FB and CE is depicted. As seen from the figure, the constrained efficient allocation Pareto dominates the equilibrium allocation.

The predictions of this model regarding monotonicity of \( \hat{t}^{CE}(z) \) in terms of \( z \) is the same as predictions of the simple two-type example studied in Section 4. As a result, one might think that if buyers and sellers agree on the ranking of assets, the monotonicity of the optimal tax schedule must be a general result. However, this is not true. Specifically, I show in an example that if \( h'(z) \geq 0 \) for all \( z \) and \( h'(z_L) = 0 \), then the optimal tax schedule is not monotone in the price of assets. Specifically, \( \frac{dt^{CE}(p)}{dp} \bigg|_{p=p_L} < 0 \) and \( \frac{dt^{CE}(p)}{dp} \bigg|_{p=p_0} > 0 \) for some other \( p_0 \). The proof is in Appendix (Section A.6.2).

Next, I study another example where \( h \) has a local maximum and therefore separation of types in equilibrium is not possible, as explained in the last subsection. Also, the tax schedule will be non-monotone in the price of assets.

**Example 2.** Model parameters: \( m(\theta) = 1 - e^{-\theta} \), \( Z = [6, 10.5] \subset R \), \( c(z) = z \), \( h(z) = 0.04(z - 6.5)(z - 7)(z - 10) + 17 \), \( k = 1 \), and \( F(.) \) is uniform.

Note that functions \( c \) and \( h \) are both the same as in Example 1. Only the domain (\( Z \)) is different. Now, \( h \) has a strict local maximum, so the equilibrium will not be separating. Similar to the previous example, Assumption 5 is satisfied, therefore Proposition 3 holds. Hence, the market tightness at the constrained efficient allocation is similarly given by \( \tilde{\theta}^{CE}(z) = \theta^{FB}(z) = m^{-1}\left( \frac{h(z) - c(z)}{k} \right) = \ln\left( \frac{h(z) - c(z)}{k} \right) \).

According to the algorithm proposed by [3], I calculate one semi-pooling equilibrium where types \( z \in [6, 9) \) trade in a pool with a low price but with high probability. Types \( z \in (9, 10.5] \) trade in separating submarkets. Type \( z = 9 \) is indifferent between the pool and one of the separating submarkets. Prices are calculated similarly as explained in Example 1.

In Figure 1.9, the value of assets to buyers (left), the value of assets to sellers (middle) and the surplus generated by each type (right) are depicted. In the upper part of Figure 1.10, price and market tightness for EQ, FB and CE are depicted. Similar to the previous example, market tightness in CE is the same as that in FB. Market tightness in EQ for types \( z \in [6, 9) \) is higher than that in FB and is less for other types. Taxes are calculated in the same way as in Example 1. An interesting
fact here is that the amount of tax imposed on buyers is neither monotone in the type of sellers that buyers meet (the lower left graph in Figure 1.10), nor in the price paid to sellers (the lower right graph in the same figure) or buyers (not shown in this figure). Finally in Figure 1.11, the payoff to sellers of different assets in EQ, FB and CE is depicted similarly to Figure 1.8.

1.6.6 Sales Tax and Entry Tax

As shown in the previous subsection, even if buyers and sellers agree on the ranking of assets, it is still possible that the tax schedule imposed on buyers conditional on trade is not monotone in the price of assets. One disadvantage of a non-monotone tax schedule is that it is extremely hard to implement it in the real world. Although it is usually assumed in the literature (including in this dissertation) that the planner has precise information about the distribution of types and the payoff structure of assets, but ideally one wants to reduce the dependence of what the planner should do on the details of the economy. If the tax schedule is non-monotone in the price of assets, this dependence is crucial. In contrast, if the tax schedule is monotone, possible errors in implementation may cause less inefficiencies. This is because there exists exactly one price with the property that trades with prices above that should be taxed and other trades should be subsidized.27

Given that a monotone tax schedule is desirable, in this subsection I suggest another tax schedule in addition to the tax schedule discussed so far. Therefore, buyers will be subject to two types of taxes, one is conditional on entry to each submarket (entry tax) and the other one conditional on trade (sales tax). The definition of implementable tax schedule should be slightly modified to include both types of taxes. See Appendix, Section A.7 for the details. I show in the following proposition that in general any feasible mechanism can be implemented by a decreasing entry tax and an increasing sales tax both in the price of assets.

Proposition 5 (Implementation of the direct mechanism with monotone entry and sales tax). Assume \( c'(z) > 0 \) for all \( z \). Take any feasible mechanism. Assume that all types get a strictly positive payoff and also that the market tightness allocated...
to different types is all different. Then there exists an associated implementable allocation with monotone tax schedules in the price of assets, decreasing entry tax and increasing sales tax, such that all types get the same payoff as what they get in the feasible mechanism.

To understand why \( \tilde{t}(z) \) may not be monotone in \( z \) in absence of entry tax and how entry tax may solve this problem, I write the free entry condition as follows, given the fact that the allocation is separating: \( q(\tilde{\theta}^{CE}(z))(h(z) - \tilde{p}^{CE}(z) - \tilde{t}^{CE}(z)) = k \). Therefore,

\[
\tilde{t}^{CE}(z) = h(z) - \frac{k}{q(\tilde{\theta}^{CE}(z))} - \tilde{p}^{CE}(z).
\]

The term \( \frac{k}{q(\tilde{\theta}^{CE}(z))} \) is decreasing in \( z \) because \( \tilde{\theta}^{CE}(z) \) is decreasing in \( z \). I show now that \( \tilde{p}^{CE}(z) \) is strictly increasing in \( z \).\textsuperscript{28} According to Equation 1.2, one can write:

\[
\frac{d[m(\tilde{\theta}^{CE}(z))\tilde{p}^{CE}(z)]}{dz} = m'(\tilde{\theta}^{CE}(z))\frac{d\tilde{\theta}^{CE}(z)}{dz} - c(z).
\]

Hence

\[
\frac{d\tilde{p}^{CE}(z)}{dz} = - \frac{m'(\tilde{\theta}^{CE}(z))}{m(\tilde{\theta}^{CE}(z))} \frac{d\tilde{\theta}^{CE}(z)}{dz} (\tilde{p}^{CE}(z) - c(z)). \tag{1.7}
\]

But \( \tilde{p}^{CE}(z) - c(z) \) is strictly positive. Otherwise, type \( z \) will be inactive, thus contradicting the assumption that all types are active. Since \( \tilde{\theta}^{CE}(z) \) is strictly decreasing, the right hand side of the above equation is strictly positive, that is, \( \tilde{p}(z) \) is strictly increasing.

Hence, in general it is not guaranteed that \( \tilde{t}^{CE}(z) \) is monotone in \( z \). The idea to make \( \tilde{t}^{CE}(z) \) monotone is to add an entry tax for each submarket, \( \tilde{t}_e(z) \), so the free entry condition can be written as follows:

\[
\tilde{t}(z) = h(z) - \frac{k + \tilde{t}_e(z)}{q(\tilde{\theta}(z))} - \tilde{p}(z).
\]

If \( \tilde{t}_e(z) \) is constructed to be decreasing sufficiently fast in \( z \), then the effect of \( \frac{k + \tilde{t}_e(z)}{q(\tilde{\theta}(z))} \) dominates the effect of \( \tilde{p}(z) \) and so \( \tilde{t}(z) \) becomes increasing in \( z \).

One important point here is that since entry tax will be collected even before

\textsuperscript{28}The derivation below holds not only for the constrained efficient allocation but also for any allocation that satisfies IC and sellers’ participation constraint.
buyers get to find a match, it cannot be less than \(-k\); for otherwise, buyers would not have incentive to participate in the allocation. In other words, if the entry tax for a submarket is less than \(-k\), then buyers pay the tax (basically get this subsidy) and make a positive profit net of the entry cost, \(\tilde{t}_e + k\), and then do not participate in the matching stage which gives them a strictly negative payoff. Therefore, \(k + \tilde{t}_e(z) \geq 0\) is another constraint for the construction of the optimal tax schedule that I take into account in the proof.

**Corollary 1.** Take any constrained efficient mechanism (which is a direct mechanism). Then there exists an associated constrained efficient allocation (which is an implementable allocation) such that all types get exactly the same payoff as in the direct mechanism and the entry tax is decreasing and the sales tax is increasing in the price of assets.

### 1.7 Conclusion

I have characterized the constrained efficient allocation in an environment with directed search and adverse selection. Under similar assumptions that GSW make to characterize the unique equilibrium, the planner can achieve strictly higher welfare than the equilibrium if the equilibrium fails to achieve the first best. Under a different assumption (Assumption 2), the planner can even achieve the first best. The main idea is that the planner tries to use transfers rather than market tightness or production level to have incentive constraints satisfied.

In the market economy, the buyers do not take into account the effect of their entry on the set of feasible submarkets available to buyers who want to attract other types of sellers. Entry of a buyer to a submarket changes the payoff of sellers in that submarket and this in turn, through incentive compatibility constraints, changes the set of feasible contracts that buyers can post in other submarkets and eventually changes the payoff of sellers in other submarkets. The planner takes this externality into account and therefore, he is able to increase welfare by imposing appropriate taxes and subsidies.

I illustrated my results in different examples. In an asset market example in Section 6, I showed that if the value of assets to sellers is increasing and the surplus created by assets is decreasing in the type of assets \((c' > 0\text{ and } h' - c' \leq 0)\), then the planner can achieve the first best by subsidizing low price assets and taxing high
prices ones in a large class of environments. The optimal tax schedule, however, is not generally monotone in the price of assets, e.g., when buyers and sellers do not agree on the ranking of assets (which happens if some sellers of high quality assets are financially distressed so they are in an urgent need to sell their assets).

If directed search with adverse selection is a good framework to capture what happened in OTC markets during the recent financial crisis, as [3] and [4] use the same framework to analyze these markets, then my results imply that it is not an optimal policy to subsidize the purchase of all low price assets when there are fire sales in asset markets. That is, asset subsidy programs may have not been the best policy from a social point of view (although it may have increased liquidity of assets). Then I showed that if buyers are subject to two types of taxes, not only sales tax but also entry tax, then there exist monotone tax schedules, increasing sales tax and decreasing entry tax, which implement the constrained efficient mechanism.

I have assumed in this paper that agents match bilaterally. An important question is that if one considers a more general framework and allows several sellers to meet with a buyer so that sellers face some competition after meeting a buyer, whether it induces sellers to reveal their types less costly? And importantly, does the equilibrium remain constrained inefficient? In a work in progress, I study a similar environment but with many-on-one meetings. Buyers post mechanisms (which possibly depend on the number of sellers who will show up and on their reports) and commit to them. For example buyers might post second price auctions with reserved prices. I want to characterize both equilibrium and the constrained efficient allocation in such an environment. My conjecture is that the equilibrium will remain constrained inefficient. This is yet to be verified.
Figure 1.1. Schematic diagram indicating the proof steps of Theorem 2. SM and IDP refer to supermodularity and increasing differences property respectively. M+S refers to [23], L+M refers to [25].

<table>
<thead>
<tr>
<th></th>
<th>Complete information (FB)</th>
<th>Equilibrium</th>
<th>Constrained efficient if $\pi_1 b_1 + \pi_2 b_2 \geq c_2$</th>
<th>Constrained efficient if $\pi_1 b_1 + \pi_2 b_2 &lt; c_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_1, \alpha_2$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\theta_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>1</td>
<td>$\frac{b_1-c_1}{b_2-c_1}$</td>
<td>$-$</td>
<td>$\frac{\pi_1 (b_1-c_1)}{c_2-\pi_1 c_1-\pi_2 b_2}$</td>
</tr>
<tr>
<td>$p_1$</td>
<td>$b_1$</td>
<td>$b_1$</td>
<td>$\pi_1 b_1 + \pi_2 b_2$</td>
<td>$\frac{\pi_1 (b_1-c_1) + \pi_2 (c_2-b_2)}{c_2-\pi_1 c_1-\pi_2 b_2}$</td>
</tr>
<tr>
<td>$p_2$</td>
<td>$b_2$</td>
<td>$b_2$</td>
<td>$-$</td>
<td>$\frac{\pi_2 (b_2-c_2)}{c_2-\pi_1 c_1-\pi_2 b_2}$</td>
</tr>
<tr>
<td>$t_1$</td>
<td>$-$</td>
<td>$-$</td>
<td>0</td>
<td>$\frac{-\pi_2 (b_2-c_2) (b_1-c_1)}{c_2-\pi_1 c_1-\pi_2 b_2}$</td>
</tr>
<tr>
<td>$t_2$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$b_2 - c_2$</td>
</tr>
<tr>
<td>$U_1$</td>
<td>$b_1 - c_1$</td>
<td>$b_1 - c_1$</td>
<td>$\pi_1 b_1 + \pi_2 b_2 - c_1$</td>
<td>$\frac{\pi_1 (b_1-c_1) (c_2-c_1)}{c_2-\pi_1 c_1-\pi_2 b_2}$</td>
</tr>
<tr>
<td>$U_2$</td>
<td>$b_2 - c_2$</td>
<td>$\frac{b_1-c_1}{b_2-c_1} (b_2 - c_2)$</td>
<td>$\pi_1 b_1 + \pi_2 b_2 - c_2$</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 1.1. Comparison between different allocations in the asset market with lemons. $t_1$ and $t_2$ denote the tax amount levied on buyers in submarket one and two in the implementation of the constrained efficient allocation. $U_1$ and $U_2$ denote the payoff of type one and two in different allocations. If $\pi_1 b_1 + \pi_2 b_2 \geq c_2$, the planner can achieve the first best through a pooling allocation where both types trade in one submarket with price equal to $\pi_1 b_1 + \pi_2 b_2$. If $\pi_1 b_1 + \pi_2 b_2 < c_2$, the first best is not achievable. The constrained efficient allocation is implemented in the market through a separating allocation.
Figure 1.2. This schematic diagram illustrates how the planner allocates resources. Dashed lines show the flow of funds. In equilibrium, type one is indifferent between two submarkets, while type two strictly prefers submarket two. To improve welfare, the planner taxes sellers in submarket two and subsidizes sellers in submarket one so that type one sellers are discouraged from applying to submarket two (the higher price submarket). Now, more buyers enter submarket two and the outcome approaches the first best.

<table>
<thead>
<tr>
<th>Condition</th>
<th>Necessary condition for I.C</th>
<th>Initial conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 &lt; c'(z)$ and $0 &lt; h'(z)$</td>
<td>$\frac{d\theta^{EQ}}{dz} &lt; 0$</td>
<td>$\hat{\theta}^{EQ}(z_L) = \theta^{FB}(z_L)$</td>
</tr>
<tr>
<td>$0 &lt; c'(z)$ and $0 &gt; h'(z)$</td>
<td>$\frac{d\theta^{EQ}}{dz} &lt; 0$</td>
<td>$\hat{\theta}^{EQ}(z_H) = \theta^{FB}(z_H)$</td>
</tr>
<tr>
<td>$0 &gt; c'(z)$ and $0 &lt; h'(z)$</td>
<td>$\frac{d\theta^{EQ}}{dz} &gt; 0$</td>
<td>$\hat{\theta}^{EQ}(z_L) = \theta^{FB}(z_L)$</td>
</tr>
<tr>
<td>$0 &gt; c'(z)$ and $0 &gt; h'(z)$</td>
<td>$\frac{d\theta^{EQ}}{dz} &gt; 0$</td>
<td>$\hat{\theta}^{EQ}(z_H) = \theta^{FB}(z_H)$</td>
</tr>
</tbody>
</table>

Table 1.2. The rat race results. In equilibrium, the probability of finding a match and hours of work of type two workers are distorted upward compared to the first best allocation. The planner subsidizes type one ($t_1 < 0$) and taxes type two ($t_2 < 0$) to correct the distortions and achieve the first best.

Table 1.3. Equilibrium allocation in different cases in the asset market with continuous type space
Figure 1.3. The indifference curves of buyers and sellers are illustrated here when $\pi_1 b_1 + \pi_2 b_2 < c_2$. CE, FB, EQ represent constrained efficient, first best and equilibrium allocations. In the equilibrium allocation, the market tightness for type two is less than 1. Intersection of indifference curve of type one and indifference curve of buyers in submarket two determines $\theta_{EQ}^2$. At this point, type one is indifferent between both submarkets. The planner makes subsidies to buyers at submarket one ($t_1 < 0$), thus pushing buyers’ indifference curves in that submarket upward. Because of zero profit condition for buyers, eventually type one sellers get a higher payoff than equilibrium. The planner taxes buyers in submarket two ($t_2 > 0$) to raise funds for subsidies made to type one. Now, the market tightness that the planner assigns to type two is increased compared to that in equilibrium.
Figure 1.4. Assume that $h(z)$ has a strict local maximum but $h'(z) - c'(z) \leq 0$ for all $z$. Also assume that the distribution is such that part 2 of Assumption 5 (or Assumption 17 in Appendix A) holds. Because the value of assets to sellers with higher $z$ is not monotone in $z$, the equilibrium will involve some pooling. [3] shows this point formally in her Proposition 5. However, the planner’s allocation is separating. The planner can actually achieve the first best according to Proposition 3. Symmetrically, if $c'(z) < 0$ and $h'(z) - c'(z) \geq 0$, and if a similar condition to part 2 of Assumption 5 or Assumption 17 holds, then the planner will get a separating allocation.
Equilibrium allocation is separating if \((c'(z), h'(z))\) lies in one quadrant for all \(z\).

\[ h'(z) - c'(z) = 0 \]

**Figure 1.5.** Here, \(h - c\) has an interior local minimum and \(h'(z) \geq 0\) for all \(z\). Since \(h'(z) \geq 0\) for all \(z\), then the equilibrium allocation is separating. However, the planner’s allocation is pooling, because monotonicity constraint is not satisfied. Indeed, the planner wants to pool all types higher than a threshold in one submarket.
Figure 1.6. The payoff structure of different assets for the model parameters in Example 1 is depicted here. In the left graph, the value of type $z$ asset to buyers, $h(z)$, (in blue) and the value of type $z$ to sellers, $c(z)$, (in red) are depicted. Gains from trade, $h(z) - c(z)$, is depicted in the right graph.
Figure 1.7. Model parameters are defined in Example 1. In the upper left graph, the price that sellers get in FB (in green), in CE (in red) and in EQ (in dashed blue) are depicted. In the upper right graph, market tightness for each type in FB and CE (in green) and in EQ (in blue) are depicted. In the lower left graph, the optimal level of submarket-specific taxes that buyers should pay, $\tilde{t}_{CE}(z)$, is depicted in terms of $z$. In the lower right graph $t(z)$ is depicted in terms of the price that sellers get, $\tilde{p}_{CE}(z)$. It is observed here that the efficient tax schedule is monotone in the type or price of assets.
Figure 1.8. Model parameters are defined in Example 1. The expected payoff to sellers in FB (in green), in CE (in dotted red) and in EQ (in dashed blue) are depicted. CE Pareto dominates EQ in this example.
Figure 1.9. The payoff structure of different assets for the model parameters in Example 2 is depicted here. In the left graph, the value of type $z$ asset to buyers, $h(z)$, in the middle graph the value of type $z$ asset to sellers, $c(z)$, and in the right graph the gains from trade, $h(z) - c(z)$, are depicted.
Figure 1.10. This figure is similar to Figure 1.7 but with model parameters defined in Example 2. In the upper left graph, the price that sellers get in FB (in green), in CE (in dashed red) and in EQ (in blue) are depicted. In the upper right graph, market tightness for each type in FB and CE (in green) and in EQ (in blue) are depicted. In the lower left graph $t(z)$ is depicted in terms of $z$ and in the lower right graph $\tilde{t}_{CE}(z)$ is depicted in terms of $\tilde{p}_{CE}(z)$. The efficient tax schedule is non-monotone in the type or price of assets.
Figure 1.11. This figure is similar to one in Figure 1.8 but with model parameters defined in Example 2. The expected payoff to sellers in FB (in green), in CE (in dashed red) and in EQ (in blue) are depicted. CE Pareto dominates EQ in this example, too.
Chapter 2  
Directed Search with Complementarity and Adverse Selection

2.1 Introduction

In the tradition of assignment literature following [28], this paper studies the assignment patterns between firms and workers in an environment with search and information frictions. There is a fixed population of heterogeneous workers who have private information about their skill levels (denoted by $z$). On the other side of the market, there are a lot of ex-ante homogeneous firms who can enter the market by investing in capital (and capital level is denoted by $x$). The level of capital that firms choose is observable to all market participants. After firms enter the market and invest, they post wages (denoted by $w$) and workers direct their search toward their preferred combination of wage and capital level. Wages are not conditional on the type of workers, because the workers’ skills are not observable to firms. Then workers and firms match bilaterally and trade. After they trade, each side gets a share of the surplus. The firm gets $b(x, z) - w$ and the worker gets $e(x, z) + w$ from the joint surplus of $b(x, z) + e(x, z)$. Both $b$ and $e$ functions display complementarity in the level of firm’s capital and the worker’s skill.

Shi (2001), [1], studies the complete information version of this model in which the type of workers is observable and contractible. In this case, if the firms can commit to what they post, as usually assumed in the directed search literature, then the initial ownership of the surplus is immaterial for the equilibrium allocation, because the competition between firms in posting contracts leads firms to set the
transfers (wages) such that the payoff that each side gets from the match equals the marginal contribution of that side to the match.\textsuperscript{1} That is, the market allocation with complete information achieves the first best, the allocation that the planner with complete information chooses. He also derives the necessary and sufficient condition under which the first best allocation exhibits PAM, positive assortative matching. This condition states that the level of complementarity between capital and skill in the joint surplus of the match must be sufficiently high.

In this paper, I take a closer look at the effects of private information on this economy and study the patterns of sorting and also how these patterns are changed compared to them under complete information. When workers have private information, the initial ownership of the surplus before transfers matters substantially and the allocation does not generally achieve the first best. In particular in the first result, I characterize the conditions under which the market economy exhibits PAM. That is, workers with high skill levels get matched with firms with high amount of capital. I also show that over-investment occurs in the market. That is, the level of capital allocated to a worker with a given skill is higher than that under complete information.

In the second result, I show that the presence of sufficient level of complementarity between factors of production is not necessary nor sufficient for the equilibrium allocation with private information to exhibit PAM. I provide several examples to illustrate this point. In Example 3, the first best allocation features NAM, negative assortative matching, while the market allocation with private information features PAM. In Example 5, the first best allocation features PAM while the market allocation with private information does not. Example 3 suggests that one reason that PAM may be prevalent in many real world applications\textsuperscript{2} is not necessarily because the level of complementarity between factors of production in the joint surplus is high enough. Rather, PAM might be the result of competition between firms trying to attract high skill workers by offering them jobs with a high capital level, because a high skill worker achieves higher marginal payoff (before transfers) from these jobs compared to a low skill worker. In other words, the capital level is

\textsuperscript{1}In [1], the firms not only post their own capital level and the wage that they would pay, but also they post the type of the worker that they want to hire and commit to that. That is, even if a worker with higher skill applies for the job, the firm does not accept that worker.

\textsuperscript{2}There is a vast literature on the identification of sorting from wage data and whether sorting is actually observed in the data or not. For example, see [29–35].
used as a screening device for uninformed firms to attract high skill workers with private information.

Furthermore, I characterize the first best allocation and the constrained efficient allocation. [1] characterizes the first best allocation with urn-ball matching function. I characterize that for a general matching function. Regarding the constrained efficient allocation, I take its definition and follow its characterization method from Chapter 1 and derive sufficient conditions under which the planner who faces the same information and search frictions can achieve the first best allocation.

To provide some intuition for my results, I first explain the economic forces at play when there is complete information. As stated earlier and as [1] formally proves it, the market allocation and the planner’s allocation are the same under complete information, so I just explain how the planner allocates resources under complete information. There are two economic forces, one acting for and one acting against PAM. The first one is that because there is complementarity in the joint surplus, the planner tends to match high skill workers with firms with a high amount of capital (complementarity effect). The second one is that because there are search frictions, the planner would like high skill workers to get matched with high probability, even if they get matched with firms with a low amount of capital (“trading security” effect as [2] call it). Similarly for firms with high amount of capital, the planner would like them to get matched with high probability, even if they get matched with low skill workers. Here, the planner basically wants high capital firms and high skill workers to be utilized as much as possible.

When there is private information, incentive compatibility of workers should be also satisfied, so high skill workers must be matched either with high capital firms (screening through capital allocation) or they must be matched with high probability of trade, regardless of the type of the firm they get matched with (screening through market tightness). Complementarity effect and screening through capital allocation are forces toward PAM and trading security effect and screening through market tightness are forces against PAM.

Screening through capital allocation is a force toward PAM, because the factors of production are complementary in the payoff of the worker from the match. Due to the presence of complementarity in the worker’s payoff, the marginal payoff of the worker from the match with respect to capital is higher for high skill workers. Therefore, one way to screen high skill workers is to offer them a high capital level.
Note that this effect has nothing to do with the complementarity of the factors of production in the joint surplus from the match. Screening through market tightness is a force against PAM, because the payoff of workers from the match is increasing in their skills. Therefore, another way to satisfy incentive compatibility of workers is to offer high skill workers higher probability of matching in the market.

My sufficient conditions in the first result characterize a parameter region in which the effects of complementarity and screening through capital allocation dominate the effects of trading security and screening through market tightness. Under complete information, only complementarity and trading security effects are present. Under private information, if the effect of screening through capital allocation is sufficiently stronger than the effect of screening through market tightness, then the market with private information exhibits PAM, but the first best allocation may not exhibit PAM (because the complementarity effect may not dominate the trading security effect). This observation explains why the first best allocation exhibiting PAM is not necessary for the market allocation to exhibit PAM.

On the other hand if the effect of screening through market tightness is sufficiently stronger than the effect of screening through capital allocation, then the market with private information does not exhibit PAM, although the first best allocation may exhibit PAM (because the complementarity effect may be sufficiently stronger than the trading security effect). This observation explains why the first best allocation exhibiting PAM is not sufficient for the market allocation to exhibit PAM.

In contrast to most (if not all) papers in the directed search literature that emphasize the tradeoff between prices (wages) and probability of matching (employment) for workers, my results suggest that workers in the market might be compensated only through non-monetary payoffs from the match, not through higher wages nor through higher probability of matching. The effect of non-monetary payoffs on shaping the patterns of assignment is new in this literature to the best of my knowledge.

To interpret the initial ownership of the surplus, consider the firm-worker relationship. The share of the firm from the match before making transfers is the amount of production. The share of the worker from the match before making transfers can be disutility of work, what the worker learns from the job or the gains that the worker gets from making a professional network while on the job.
Although I use the interpretation of the firm-worker relationship, my results can be applied to other examples such as the relationship between owners of laboratories and researchers, the relationship between universities and students, and the relationship between loan providers (with different conditions to provide loan) and owners of projects. Before this paper, [36] study the effects of the initial ownership of the surplus on efficiency of the pre-match investment in absence of search frictions and discuss it in several examples. Specifically, they call the payoffs that the two sides of the match get before any transfers are made “premuneration values.”

Last but not least, I have emphasized the role of adverse selection both in this introduction and also in my results, but it is extremely important to note that all results go through in the environment with complete information if firms cannot post contracts contingent on the type of workers. There are at least two reasons for inability of firms to post contingent contracts. One reason is that the type of workers may not be contractible even though it is observable. Another reason is that even if the type of workers is both observable and contractible, regulations and laws may prohibit firms from posting contracts contingent on those characteristics.\(^3\)

2.1.1 Related Literature

This is the first paper which incorporates several features together in one model: complementarity between factors of production in the payoff of both sides of the match, private information, pre-match investment and search frictions. Because this paper has these different ingredients, naturally it shares common features with different strands of literature. I cannot do justice to all contributions in the related literature, so I just mention some papers closer to mine.

As stated earlier, the closest paper is [1]. He studies the complete information version of my model. He is also the first who incorporates complementarity between factors of production in a directed search model and studies its implications for assignment patterns.

In the assignment literature, [28] studies patterns of matching in an environment without search frictions or private information. [2] study an environment with directed search with complementarity and complete information.\(^4\) The population

\(^3\)In the real world, if the type is observable, then firms and workers may be engaged in ex-post bargaining to exhaust gains from trade. In this paper, I do not allow for this possibility.

\(^4\)More precisely in [2], buyers (like workers in my paper) may have private information about
in both sides of the market is fixed in their model, but in my model the distribution of firms is endogenously determined by free entry. [37] study the patterns of assignment in an environment with random search and without private information. [10] and [11] study environments with directed search and complementarity in the output. The firms can observe the type of the workers after getting matched and before trade. [38] considers an economy with directed search in which workers’ types are observable but not contractible. The firms cannot offer contingent wages, but they pick the best worker among their applicants.

Other papers study pre-match investment. [8] study pre-match investment in an environment with directed search but without complementarity and without private information. [39] study a model with parents’ investment in wealth of their children. This wealth can improve the quality of the spouses that the children can marry.

[40] and [36] also study a model with private information and pre-match investment. They discuss premuneration values in several examples. One common theme between their papers and mine is that when there is private information, the initial ownership of the surplus is important and the initial ownership has non-trivial effects on investment and efficiency. The most important difference is that they do not have search frictions. As noted earlier, in the market with search frictions, trading security motive is a force against PAM and can change the implications of the model. This effect is not present in their model.

This paper is also related to another strand of literature which introduces private information into directed search. [19] study an environment in which sellers with private information post contracts to attract uninformed buyers (signaling). In a work closer to this paper, [26] study an environment with directed search and private information in which the uninformed side (like firms in my setup) posts contracts to attract workers with private information (screening). In my paper, firms can enter the market with heterogeneous amount of capital, so firms can be heterogeneous when workers apply. In their model, firms are homogeneous even after they enter the market, so it is not possible to explore the patterns of assignments in their model.

The paper is organized as follows. In Section 1.2, I describe the environment their types, but their private information does not affect the payoff of sellers, therefore private information does not have a bite. If this was not the case, then one could not handle private information with their characterization method.
and define the equilibrium. In Section 1.3, I derive sufficient conditions for PAM and explain important parts of my characterization method. In Section 1.4, I characterize the first best allocation. In Section 1.5, I provide some examples, state the second result and develop some intuition for my results. In Section 1.6, I define and characterize the constrained efficient allocation and derive sufficient conditions under which the planner achieves the first best. Section 1.7 concludes. Some proofs are in the main body of the text and the rest of them are in the appendix.

2.2 Model

2.2.1 Environment

Consider an economy with two types of agents, firms and workers and with three goods, capital, $Z$ (labor force) and a numeraire good (money). The capital good is produced by firms and the labor force is supplied by workers. The numeraire good is produced and consumed by both agents. There is a measure 1 of workers. Each worker is endowed with one unit of indivisible labor. A fraction $\pi_i$ of workers are of type $i$ with skill level $z_i$ where $z_i \in \mathbb{Z} \equiv \{z_1, z_2, ..., z_I\} \subseteq \mathbb{R}$ and $z_1 < z_2 < ... < z_I$. The type is worker's private information. On the other side of the market, there is a large continuum of ex-ante homogenous firms. The measure of firms who choose to enter the market is endogenous and is determined by free entry. If a firm wants to enter the market with $x \in \mathbb{R}_+$ units of capital, he has to incur $C(x)$ where $C : \mathbb{R}_+ \to \mathbb{R}_+$.

The preferences of both firms and workers are quasi-linear. If a type $i$ worker is matched with a firm with capital level $x$, and if the worker consumes $p \in \mathbb{R}$ units of the numeraire good, then the payoff of the worker from the match is $p + e(x, z_i)$ where $e : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$. His payoff is 0 if unmatched. Regarding the payoff of the firms, a firm's payoff is 0 if the firm does not enter the market. If the firm enters the market with capital level $x$ and gets matched with a type $i$ worker, and if the firm produces $p \in \mathbb{R}$ units of the numeraire good, then the firm's payoff is $b(x, z_i) - p - C(x)$ where $b : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$. If the firm does not get matched with any worker, the firm's payoff is $-C(x)$.

After firms enter the market, firms post a trading price and their capital level,
Having observed the joint distribution of prices and capital levels, workers direct their search toward their preferred \((p, x)\) which maximizes their expected payoff. If they get matched, they trade with the posted price.

There are search frictions in this environment in the sense that workers generally are matched with the firms they have chosen with probability less than one. Matching occurs in submarkets which are some locations for trade and are characterized by \((p, x)\). Matching technology determines the probability that workers and firms in each submarket get matched. If the ratio of firms offering \((p, x)\) to workers seeking \((p, x)\) is \(\theta \in [0, \infty]\), then the firms are matched with probability \(q(\theta)\). Symmetrically, matching probability for workers is \(m(\theta) \equiv \theta q(\theta)\).

Let \(\gamma_i(y)\) denote the share of workers that are type \(i\) in the submarket denoted by \(y \equiv (p, x)\), with \(\Gamma(y) \equiv \{\gamma_1(y), ..., \gamma_i(y), ..., \gamma_I(y)\} \in \Delta I\) where \(\Delta I\) is an I-dimensional simplex, that is, \(\sum \gamma_i(y) = 1\) and \(\gamma_i(y) \geq 0\) for all \(i\). To summarize, the expected payoff of a type \(i\) worker from entering a submarket \((p, x)\) with market tightness \(\theta\) is

\[
m(\theta)(p + e(x, z_i)),
\]

and the expected payoff of a firm from entering a submarket \((p, x)\) with market tightness \(\theta\) is

\[
q(\theta)(\sum_i \gamma_i(p, x)b(x, z_i) - p) - C(x).
\]

2.2.2 Equilibrium Definition

Let \(\bar{Y}\) be defined as \(\bar{Y} \equiv \bigcup_i \bar{Y}_i\), where

\[
\bar{Y}_i \equiv \{(p, x) \mid q(0)(b(x, z_i) - p) \geq C(x), \text{ and } p + e(x, z_i) \geq 0\}.
\]

Also for future references, let \(\bar{X}\) be defined as follows:

\[
\bar{X} \equiv \{x \mid (p, x) \in \bar{Y} \text{ for some } p \in \mathbb{R}\}.
\]

If \((p, x) \notin \bar{Y}\), then no type will be attracted to this submarket. This is because even if the firms in this submarket get matched with the highest possible probability, neither the firm nor the worker gets as much payoff as the (firm's) entry cost or the (worker's) outside option, respectively. Therefore, without loss of generality
we can restrict our attention to \((p, x)\) only in \(\bar{Y}\) throughout the paper. Note that 
\(\bar{Y}\) is compact, because of the following reason: \(y \equiv (p, x) \in \bar{Y}\) iff 
\(-e(x, z_i) \leq p \leq b(x, z_i) - \frac{C(x)}{q(0)}\) for some \(i\). Due to the assumption that 
\(b_{xx} + e_{xx} < C_{xx}\) (according to part 5 of Assumption 7 which will come later),
for each \(i\) there exists \(\bar{x}_i\) such that 
\(-e(x, z_i) > b(x, z_i) - \frac{C(x)}{q(0)}\) for \(x \geq \bar{x}_i\) and 
\(-e(x, z_i) \leq b(x, z_i) - \frac{C(x)}{q(0)}\) for \(x \leq \bar{x}_i\). Hence 
\(\bar{Y}_i = \{(p, x)| x \in [0, \bar{x}_i], \text{ and } p \in [-e(x, z_i), b(x, z_i) - \frac{C(x)}{q(0)}]\}\) is compact. Since 
\(\bar{Y}\) is the union of a finite number of compact sets, \(\bar{Y}\) is also compact.

Equilibrium definition here is taken from [26] and is developed to allow for heterogeneous
level of investment in capital on the side of firms. An equilibrium allocation, which is defined formally below, describes a set of open submarkets \(Y^P\),
the distribution of firms over open submarkets \(\lambda\), the ratio of firms to workers \(\theta\) at
all submarkets, and finally the distribution of types \(\Gamma\) at all submarkets.

**Definition 8 (Equilibrium Definition).** A competitive search equilibrium, \(\{Y^P, \lambda, \theta, \Gamma\}\),
is a measure \(\lambda\) on \(Y \equiv \mathbb{R} \times \mathbb{R}_+\) with support \(Y^P\), a function \(\theta : Y \to [0, \infty]\), and
a function \(\Gamma : Y \to \Delta^I\) which satisfies (i)-(iii):

(i) **Profit Maximization and Free Entry of Firms**

For any \((p, x) \in Y\),

\[ q(\theta(p, x)) \sum_i \gamma_i(p, x)(b(x, z_i) - p) \leq C(x), \]

with equality if \((p, x) \in Y^P\).

(ii) **Workers’ Maximization**

Let \(U_i \equiv \max\{0, \max_{(p', x') \in Y^P} \{m(\theta(p', x'))(p' + e(x', z_i))\}\}\) and \(U_i = 0\) if \(Y^P = \emptyset\). For any \((p, x) \in Y\) and \(i\),

\[ m(\theta(p, x))(p + e(x, z_i)) \leq U_i, \]

with equality if \(\gamma_i(p, x) > 0\) and \(\theta(p, x) < \infty\). If \(p + e(x, z_i) < 0\), then \(\theta(p, x) = \infty\) or \(\gamma_i(p, x) = 0\).

(iii) **Market clearing**

For all \(i\), \(\int_{Y^P} \frac{\gamma_i(p, x)}{q(p, x)} d\lambda\{(p, x)\} \leq \pi_i\), with equality if \(U_i > 0\).

I say a submarket \((p, x)\) is active or open in equilibrium if \((p, x) \in Y^P\). I also say
type \(i\) is active if type \(i\) applies to some submarket with strictly positive probability.
Condition (i) implies that all firms who enter the market get an expected payoff 0. That is, the expected benefit from posting \((p, x)\) is exactly equal to the cost of investment \(C(x)\). Also there exists no profitable opportunity left unexploited in the market. That is, if a submarket is not open, it means that buyers cannot get strictly positive payoff by choosing that submarket. Condition (ii) implies that workers choose one submarket which maximizes their expected payoff among all open submarkets. This definition allows for randomization between different submarkets if the worker is indifferent between them. The workers choose their outside option (which delivers them payoff 0) if their payoff in the market is less than 0. If type \(i\) gets a strictly positive payoff by applying to a submarket, then the third condition implies that measure of type \(i\) workers over all submarkets is exactly equal to \(\pi_i\), which is the measure of type \(i\) in the population.

When firms want to enter the market, they need to form beliefs regarding the market tightness and the composition of types over all submarkets, not only those posted in the equilibrium. [26] impose the following restrictions on the beliefs (similar to [41] and [42]). Take a submarket \((p, x)\) which is not open in the equilibrium. If this submarket cannot deliver at least the equilibrium payoff for any type even with the highest possible probability of matching, then I set \(\theta = \infty\) at this submarket and the beliefs about the composition of types at that submarket is unrestricted. Now suppose there exists a market tightness such that some types get payoffs higher than their equilibrium payoffs. \(\theta(p, x)\) should be set to the highest possible number such that no worker gets a strictly higher payoff than his equilibrium payoff. As a thought experiment, let the market tightness be \(\infty\) at the beginning. Some workers apply to this submarket, thus driving the market tightness down. The workers keeps applying to this submarket until no worker is strictly better off relative to his equilibrium payoff. Then, the firms can assign positive probability to type \(i\) applying to this submarket only if type \(i\) is indifferent between this submarket and the submarket he chooses in equilibrium.

Given these beliefs, there shall not exist any \((p, x)\) combination off-the-equilibrium-path which would deliver the firms strictly positive payoff (according to condition (i) of Equilibrium definition). If such a \((p, x)\) existed, some firms would have entered those submarkets to exploit that opportunity.

I have implicitly assumed in the definition of the equilibrium that entering the market and posting a price occur simultaneously. Since workers do not act between
these two actions of the firms, changing this assumption to one under which firms invest first and post a price next will not change my main results, although the equilibrium definition and especially restrictions on off-the-equilibrium-path beliefs should be modified.

2.2.2.1 More on the Equilibrium Definition

There are at least two ways to interpret this search environment. The first one is to consider an economy with finite number of workers ($N$), each of which is of type $i$ with probability $\pi_i$. On the other side of the market, the number of firms, $M$, is endogenously determined. Firms enter the market and post $(p, x)$ combinations simultaneously. Having observed all postings, workers choose the firms to which they want to apply, only based on the combination $(p, x)$ that the firms have posted and independent of the entity of the firms. That is, if two firms have posted the same $(p, x)$, then the probability that type $i$ applies to each firm is equal. Whether the results in this paper are the limit results of the aforementioned environment when $N$ goes to infinity is an interesting question and is left for future research. For example, [43] studies a model with pre-match investment by both sides of the market. Surprisingly in his paper, the results in the limit game is not the same as the results in the same economy but with continuum of players. There are important differences, however, between his paper and mine, which makes the analysis of the finite-economy version of my model still interesting: In his paper there are no search frictions and also PAM in his paper is an assumption not a result.

Another way to think about this problem, following [9], is to assume that there are some market makers who set up submarkets with 0 cost. Suppose there is a perfect competition among them. Market makers compete with each other so that no submarket which would deliver firms a positive payoff is inactive in the equilibrium. If there was such a submarket, some market makers would have already set up that submarket to exploit that opportunity.

2.3 Characterization

First I prove the existence and uniqueness of equilibrium (in terms of payoffs). To do so, I use the characterization method in [26]. Then, I address my main
results. Before I proceed, note that my model is not a special case of the model in [26]. In their model, cost of entering the market is fixed. In my model, cost of entry depends on the capital level that the firm chooses. Therefore, after firms enter the market, they are not homogeneous any more. Technically, the payoff function of firms cannot be written in the form of $q(\theta)v_1(p, x) - k$ where $\theta$ is the market tightness for submarket $(p, x)$ and $k$ is a constant. In my model, cost of entering the market, $C(x)$, is also endogenous. Therefore, I need to modify their characterization method to be able to use it in my model.

In what follows, I first introduce some regularity assumptions that I need for existence and uniqueness. These assumptions will be maintained throughout the discussion. I will introduce more assumptions for other results later.

Assumption 7.

1. (Standard assumptions on the matching function) Both $q(.)$ and $m(.)$ are twice continuously differentiable. Also $q'(.) < 0$, $m'(.) > 0$ and $m''(.) < 0$ and $m(.)$ has a strictly decreasing elasticity.

2. (Positive gains from trade for all types) For all $i$, $\max_{\theta,x} \left[ m(\theta)f(x, z_i) - \theta C(x) \right] > 0$, where $f(x, z) \equiv b(x, z) + e(x, z)$ and $\arg \max_{\theta,x} \left[ m(\theta)f(x, z_i) - \theta C(x) \right]$ is unique.

3. Functions $e_x$, $e_z$, $e_{xz}$, $b_x$, $b_z$ and $b_{xz}$ exist and are all positive for any $x, z$. Also $e(0, z) = 0$ and $b(0, z) = 0$ for all $z$.

4. Functions $C'(.)$ and $C''(.)$ exist and $C(0) = 0$, $C'(x) > 0$ and $C''(x) > 0$ for all $x \in \bar{X}$.

5. Functions $e_{xx}$ and $b_{xx}$ exist and $b_{xx}(x, z) + e_{xx}(x, z) < C''(x)$ for all $z \in [z_1, z_I]$ and for all $x \in \bar{X}$.

Part 1 of Assumption 7 is standard and most of matching functions commonly used in the literature satisfy this property. (See [2]). Part 2 states that there are positive gains from trade for all types. I make this assumption to ensure that all types will be active in the equilibrium (as will be shown below). If there are not positive gains from trade for some types, then all results go through but the exposition becomes more complicated without further insights. Part 3 states that the payoff of the worker from the match $e(.,.)$ and the payoff of the firm from the match $b(.,.)$ are also increasing in the firm’s capital level and the worker’s skill. Capital and worker’s skill are complementary in $e(.,.)$ and $b(.,.)$, that is, $e_{xz} > 0$
and $b_{xz} > 0$. Part 4 is standard. Part 5 is one way to ensure that there is an upper bound for the capital level that the firms choose in this economy. Even if the probability of matching for firms approaches its maximum ($\theta \to 0$), it is not worthwhile for them to acquire an unbounded level of capital, because $C$ grows faster than the amount of the surplus generated from any match even with the highest type.

I establish the existence and uniqueness of the equilibrium in the following proposition.

**Proposition 6.** Suppose parts 1, 3, 4 and 5 of Assumption 7 hold. Then competitive search equilibrium exists and the workers’ payoffs ($\{U_i\}_{1 \leq i \leq I}$) are unique. Moreover, if part 2 of Assumption 7 also holds, then all types get a strictly positive payoff in the market, that is, $U_i > 0$ for all $i$.

The proof of the proposition is constructive and is based on a set of maximization problems:

**Problem 1 ($P_i$).**

$$\max_{\theta \in [0, \infty], (p, x) \in Y} \{m(\theta)(p + e(x, z_i))\}$$

s. t. $q(\theta)(b(x, z_i) - p) \geq C(x)$ and $m(\theta)(p + e(x, z_j)) \leq U_j$ for all $j < i$.

Denote the value of the objective function of Problem $P_i$ by $U_i$ and the triple that solves Problem $P_i$ given $(U_1, \ldots, U_{i-1})$ by $(\theta_i, p_i, x_i)$. The constraint for type $j$ in Problem $P_i$ implies that type $j < i$ does not get more than $U_j$ if he chooses the submarket that type $i$ chooses. Hereafter, I call this constraint $IC_{ji}$, incentive compatibility constraint when type $j$ applies to submarket $i$. For type $z_i$ the triple $(\theta_i, p_i, x_i)$ should maximize his payoff, subject to the free entry condition and the incentive compatibility constraint of only lower types.

Let Problem $P$ denote the larger problem of solving Problem $P_i$ for all $i$. My main task in this paper is to characterize $(\theta_i, p_i, x_i)$. I show in the following theorem that in equilibrium each type applies to only one submarket and only one type applies to each submarket. Also, I show that $x_i$ is increasing in $i$, that is, the equilibrium exhibits PAM. In the third part of the theorem, I show that over-investment occurs, that is, $x_i$ is always bigger than the level of capital allocated to type $i$ in the first best allocation. The first best allocation is the allocation that
the planner chooses if the planner has complete information. For future references, let

\[
(\theta^F_B, x^F_B) \in \arg\max_{\theta,x} \{m(\theta)(b(x,z_i) + e(x,z_i)) - \theta C(x)\},
\]

\[
U^F_B = \max_{\theta,x} \{m(\theta)(b(x,z_i) + e(x,z_i)) - \theta C(x)\},
\]

where superscript \(FB\) represents the first best. [1] and [2] have shown that the market with complete information decentralizes the first best. Therefore, in the market with complete information, type \(i\) gets matched with probability \(m(\theta^F_B)\) with a firm with capital level \(x^F_B\). I make the following assumption and maintain it throughout the discussion in this section.

**Assumption 8** (Multiplicative separability of workers’ payoff). There exist strictly increasing functions \(A : \mathbb{R}_+ \to \mathbb{R}_+\) and \(D : \mathbb{R} \to \mathbb{R}_+\) such that

\[
e(x,z) = A(x)D(z),
\]

and \(A(0) = 0\). \(A(\cdot)\) is twice continuously differentiable.

This assumption imposes a restriction on the functional form of the payoff function of workers. Function \(e(x,z)\) must be multiplicatively separable in the capital level and the worker’s type, like Cobb-Douglas. I use this assumption mainly to characterize IC schemes and also to prove concavity of the problem. (See Lemma 2.) The following assumptions are also needed at some stages of the proof.

**Assumption 9.** \(\frac{1}{q(\theta)}\) is convex in \(\theta\), \(q'(0) < 0\) and \(|q''(0)| < \infty\).

**Assumption 10.**

1. For all \(x \in \bar{X}\), \(\max_{\theta} \left\{ \frac{m''(\theta)q'(\theta)}{q^2(\theta)m''(\theta)} \right\} \frac{C'(x)}{C(x)} < \frac{\alpha'(x)}{\alpha(x)}\).

2. For all \(x \in \bar{X}\) and for all \(z \in [z_1, z_I]\), \(\frac{b_{x,z}}{b_{x,z}} \geq \frac{\alpha'(x)}{\alpha(x)}\).

**Assumption 11** (Concavity assumptions).

\(^5\)I show here that part 1 of Assumption 7 and Assumption 9 imply that

\[
\max_{\theta} \left\{ \frac{m'(\theta)q'(\theta)}{q(\theta)m''(\theta)} \right\} \leq 0.5,
\]

and because \(m' = \theta q' + q\), so \(\max_{\theta} \left\{ \frac{m''(\theta)q'(\theta)}{q^2(\theta)m''(\theta)} \right\} \leq 0.5\). To see why, note that by Assumption 9,
1. For all \( x \in \bar{X} \) and for all \( z \in [z_1, z_I] \), \( \frac{b_x(x,z)}{b(x,z)} - \frac{A''(x)}{A(x)} < \frac{2A'(x)}{A(x)} \left( \frac{b_x(x,z)}{b(x,z)} - \frac{A'(x)}{A(x)} \right) \).

2. For all \( x \in \bar{X} \), either
   \[
   (a) \quad A(x)(A''(x)C(x) + 2A'(x)C'(x)) > 4A^2(x)C(x) \text{ and } C''(x) \geq \frac{(A''(x)C(x) + 2A'(x)C'(x))^2}{8C(x)A''(x)},
   \]
   \[
   (b) \quad \text{or } A(x)(A''(x)C(x) + 2A'(x)C'(x)) \leq 4A^2(x)C(x) \text{ and } C''(x) \geq \frac{A''(x)C(x) + 2A'(x)C'(x)}{A(x)} - \frac{2A^2(x)C(x)}{A^2(x)}.
   \]

**Assumption 12** (Other concavity assumptions).

1. For all \( x \in \bar{X} \) and all \( z \in [z_1, z_I] \), \( \frac{b_{xx}(x,z)}{b_x(x,z)} \leq \frac{e_{xx}(x,z)}{e_x(x,z)} \).
2. Either \( A''(x) < 0 \) for all \( x \in \bar{X} \), or \( A''(x) \geq 0 \) and \( \frac{C''(x)}{C'(x)} - \frac{A''(x)}{A'(x)} \geq 0 \) for all \( x \in \bar{X} \).

**Theorem 3** (Main result). Suppose Assumption 7, Assumption 8, Assumption 9, Assumption 10 and Assumption 11 hold. Then,

1. For any \( i \) there exists a unique \((p_i, x_i) \in Y^P\) such that \( \gamma_i(p_i, x_i) = 1 \).
2. \( \text{(PAM) The level of capital allocated to type } i \text{ in the equilibrium, } x_i, \text{ is increasing in } i. \)
3. \( \text{(Over-investment) } x_i \geq x_i^{FB}. \)

I need to explicitly characterize the solution to Problem \( P \). I show in the proof of Proposition 6 (in Lemma 9 in the appendix) that the first constraint in Problem \( P_1 \) must be binding. Therefore, one can eliminate \( p \) from Problem \( P_1 \) and write the problem only in terms of \( \theta \) and \( x \):

\[
\max_{\theta \in [0, \infty], x \in X} \{ m(\theta)(b(x, z_i) + e(x, z_i)) - \theta C(x) \}
\]

s. t. \( m(\theta)(b(x, z_i) + e(x, z_j)) - \theta C(x) \leq U_j \) for all \( j < i \).

The main technical challenge in this paper is to characterize the solution to this problem. It is hard to work with this problem because it has \( i - 1 \) constraints and specially because it is not convex. That is, the objective function is not generally
\[
\frac{1}{q} \quad \text{is convex, therefore, } \left[ -\frac{q'(\theta)}{q(\theta)} \right]^t \leq 0, \text{ so } q''q - 2q^2 \leq 0, \text{ or equivalently, } \theta q'' - 2\theta q^2 \leq 0. \text{ But}
\]
\[
0 \geq \theta q'' - 2\theta q^2 = (\theta q'' + 2q')q - 2m'q = m''q - 2m'q'.
\]
That is, \( \max_{\theta} \{ \frac{f'(\theta)}{q(\theta)m'(\theta)} \} \leq 0.5 \).
concave nor quasi-concave in \((\theta, x)\). I proceed by proving a couple of lemmas to characterize the solution to this problem to prove the theorem. The first step toward the characterization of the problem is to show that in any equilibrium, the “monotonicity constraint” is satisfied (Lemma 2). That is,

\[ m(\theta_i)A(x_i) \geq m(\theta_{i-1})A(x_{i-1}) \]

for any selection of \((\theta_i, x_i)\) and \((\theta_{i-1}, x_{i-1})\) and \(i \geq 2\). This lemma, then, helps us to simplify the constraint set of Problem \(P_i\) substantially to get a problem which is still not convex but has effectively only two constraints (Lemma 3). Then, Lemma 4 establishes the uniqueness of the solution to Problem \(P\) in terms of the contracts posted in the market. Note that the uniqueness of equilibrium in terms of payoffs, not in terms of the posted contracts, has been shown in Proposition 6. In Lemma 5, I show that the equilibrium allocation exhibits PAM. Finally in Lemma 6, I show that firms over-investment in capital compared to the first best allocation.

### 2.3.1 Proof Steps

Let

\[ S_i \equiv \{m(\theta(p, x))A(x) \mid \gamma_i(p, x) > 0, (p, x) \in Y^P\} \]

For all \(i\), denote elements of \(S_i\) by \(s_i\). This set is composed of the products of probability of matching and \(A(x)\) for submarkets to which type \(i\) applies in the equilibrium. Also let \(\bar{s}_i \equiv \sup\{s \mid s \in S_i\}\).

**Lemma 2** (monotonicity). Under assumption 8, \(s_i \geq s_j\) for all \(i, j\) with \(i > j\) and for any \(s_i \in S_i\) and \(s_j \in S_j\).

**Proof.** Consider the maximization problems of type \(i\) and type \(j\) workers in the equilibrium, respectively:

\[ m(\theta_j)(A(x_j)D_i + p_j) \leq m(\theta_i)(A(x_i)D_i + p_i), \]
\[ m(\theta_i)(A(x_i)D_j + p_i) \leq m(\theta_j)(A(x_j)D_j + p_j). \]

By combining two conditions together one yields

\[ (m(\theta_j)A(x_j) - m(\theta_i)A(x_i))D_i \leq m(\theta_i)p_i - m(\theta_j)p_j \leq (m(\theta_j)A(x_j) - m(\theta_i)A(x_i))D_j \]
Without loss of generality assume that $i > j$, so $D_i > D_j$. Therefore,

$$m(\theta_j)A(x_j) \leq m(\theta_i)A(x_i),$$
or equivalently, $s_i \leq s_j$. The proof is complete. \hfill \square

Consider the following problem.

**Problem 2** ($Q_i(\bar{s})$).

$$\max_{\theta \in [0, \infty], x \in \bar{X}} \{m(\theta)(b(x, z_i) + e(x, z_i)) - \theta C(x)\}$$

s. t. $m(\theta)(b(x, z_i) + e(x, z_{i-1})) - \theta C(x) \leq U_{i-1}$ and $m(\theta)A(x) \geq \bar{s}$.

Now consider Problem $P_i$. Given $\bar{s}_{i-1}$ and $\{U_j\}_{j \in \{1, \ldots, i-1\}}$, if one ignores $IC_{ji}$ constraints for all $j < i - 1$ and instead considers the monotonicity constraint for type $i$, $m(\theta)A(x) \geq \bar{s}_{i-1}$, then Problem $Q_i(\bar{s}_{i-1})$ is obtained. The objective function in this problem is the surplus that type $i$ creates. The first constraint requires that type $i - 1$ does not prefer submarket $i$ over his own submarket.

The next lemma states that solving Problem $P_i$ is equivalent to solving Problem $Q_i(\bar{s}_{i-1})$. This is a huge step toward characterization of Problem $P_i$.

**Lemma 3.** Set $\bar{s}_0 = 0$. Take $\bar{s}_{i-1}$ and $\{U_j\}_{j \in \{1, \ldots, i-1\}}$ as given.

1. Suppose $(\theta_i, x_i)$ solves Problem $Q_i(\bar{s}_{i-1})$. Then $(\theta_i, x_i, b(x_i, z_i) - \frac{C(x_i)}{q(\theta_i)})$ is a solution to Problem $P_i$.

2. Suppose $(\theta_i, x_i, p_i)$ solves Problem $P_i$, then $(\theta_i, x_i)$ is a solution to Problem $Q_i(\bar{s}_{i-1})$.

Here, I give a sketch of the proof why the upward local IC ($IC_{i-1,i}$) and the monotonicity constraint are sufficient to characterize Problem $P_i$. I argue by induction that if $(\theta_k, x_k)$ solves $Q_k(\bar{s}_{k-1})$ for all $k$, then for any $i$ and $j < i$, $(\theta_i, x_i, b(x_i, z_i) - \frac{C(x_i)}{q(\theta_i)})$ satisfies all $IC_{j,i}$ constraints in $P_i$. The complete proof comes in the appendix.

For $i = 1$, Problem $P_i$ and Problem $Q_i(0)$ are the same. Consider $i > 1$. By induction hypothesis, any solution to Problem $Q_j(\bar{s}_{j-1})$ is feasible for $P_j$ for all $1 \leq j < i$. Now, I want to show that any solution to Problem $Q_i(\bar{s}_{i-1})$ is also feasible
for Problem $P_i$. Define $t_k \equiv m(\theta_k)(b(x_k, z_k) - \frac{C(x_k)}{q(\theta_k)})$ for all $k$. Now I show that $(\theta_i, x_i, b(x_i, z_i) - \frac{C(x_i)}{q(\theta_i)})$ is in the constraint set of $P_i$, that is, $t_i + s_i D_j - t_j - s_j D_j \leq 0$:

$$t_i + s_i D_j - t_j - s_j D_j$$

$$= \sum_{k=j}^{i-1} \left[ \frac{(t_{k+1} + s_{k+1} D_k)}{\theta_k} - (t_k + s_k D_k) \right] + \frac{(D_k - D_j)(s_k - s_{k+1})}{\theta_k} \leq 0 \text{ from 1st constraint of } Q_{k+1}(\bar{s}_k)$$

$$\leq 0 \text{ from 2nd constraint of } Q_{k+1}(\bar{s}_k) \quad (2.1)$$

So far, our problem has been simplified a lot, because it has now just two constraints. However, the objective function $m(b + e) - \theta C$ in Problem $Q_i(.)$ by itself is generally not concave nor even quasi-concave, therefore, it is not possible to show the uniqueness of the solution by strict (quasi-) concavity of the objective and convexity of the constraint set without taking the constraints explicitly into account.

To address this challenge, the problem needs to be simplified further, so I derive $\theta$ from $m(\theta)A(x) = s$ in terms of $x$ and $s$, eliminate $\theta$ from Problem $Q_i(\bar{s}_{i-1})$ completely and then write the objective function as follows:

$$\Pi(s, x, z) = b(x, z) + s D(z) - m^{-1}(s \frac{s}{A(x)})C(x). \quad (2.2)$$

Let $\delta_i \equiv D_i - D_{i-1}$. Then Problem $Q_i(\bar{s})$ is the same as the following problem:

**Problem 3** $(R_i(\bar{s}))$.

$$\max_{s, x} \Pi(s, x, z)$$

s. t. $\Pi(s, x, z) \leq U_{i-1} + \delta_i s$ and $\bar{s} \leq s \leq A(x)$.

The requirement of $s \leq A(x)$ comes from $s = m(\theta)x \leq A(x)$. However, this constraint is never binding, for otherwise the value of the objective function goes to $-\infty$ because $\theta$ goes to $\infty$.

In the next lemma, I characterize the solution to Problem $R_i(\bar{s}_{i-1})$ which is equivalent to Problem $Q_i(\bar{s}_{i-1})$. In particular, I show that there is a unique $(\theta_i, x_i)$ that solves Problem $R_i(\bar{s}_{i-1})$. This problem is much simpler than the original Problem $P_i$, because it has just two constraints and also because it has a special form, where the objective is repeated in the LHS of the first constraint.
Lemma 4. Suppose Assumption 8, Assumption 9 and Assumption 11 hold. If \((x_i, s_i)\) solves Problem \(R_i(\bar{s}_{i-1})\), then \(s_i\) is unique. Also,

\[
x_i \in \arg \max_{A^{-1}(s) \leq x} \Pi(s_i, x, z_i),
\]

and \(x_i\) is unique too.

The main question is that how \(x_i\) in the equilibrium changes with \(i\) through a rather complex maximization Problem \(P\). So far, it has been shown that solving Problem \(R_i(s_{i-1})\) is equivalent to solving Problem \(Q_i(s_{i-1})\) for all \(i\). Lemma 4 is useful, because it allows us to characterize this problem in two separate steps. The first step is to solve the unconstrained maximization problem \(\max_x \Pi(s, x, z_i)\) for any \(s\). The second step is to find the value of \(s\) which solves Problem \(R_i(s_{i-1})\). Let

\[
\bar{x}(s, z) \equiv \arg \max_{A^{-1}(s) \leq x} \Pi(s, x, z).
\]

In what follows, I show that \(\bar{x}(s, z)\) is increasing in \(s\) and \(z\). I have already shown that \(s_i\) is also increasing in \(i\), therefore, \(\bar{x}(s_i, z_i)\) also increasing in \(i\). This is exactly the PAM property that we were after.

Lemma 5. Under Assumption 10, \(\bar{x}(s, z)\) is increasing in \(s\) and \(z\).

The idea of the proof is as follows. First note that function \(\Pi\) satisfies increasing differences property in \((z; x)\), that is, \(\frac{\partial^2 \Pi}{\partial x \partial z} > 0\) due to part 2 of Assumption 10. However, it is hard (or impossible) to prove that \(\Pi\) has increasing differences in \((s; x)\) for all \((s, x)\). Therefore, instead of showing this property for all \(s\), I just show it at optimal points. That is, I show in the proof that \(\frac{\partial^2 \Pi}{\partial x \partial s} |_{x=\bar{x}(s, z)} > 0\) for all \(s\). Then I use this fact to derive a contradiction if \(\bar{x}(s, z)\) is not increasing in \(s\) for some \(s\).

To complete the proof of Theorem 3, I show that \(s_i\) is greater than \(m(\theta_i^{FB}) A(x_i^{FB})\) (the counterpart of \(s_i\) under complete information). This fact and the fact that \(\bar{x}(s, z)\) is increasing in \(s\) are used to show that the capital level assigned to type \(i\) in the equilibrium is greater than that under complete information. That is, \(\bar{x}(s_i, z_i) \geq \bar{x}(s_i^{FB}, z_i) = x_i^{FB}\). This is exactly the over-investment (compared to the complete information case) property that we were after.

Lemma 6. For all \(i\), \(s_i \geq s_i^{FB}\) where \(s_i^{FB} \equiv m(\theta_i^{FB}) A(x_i^{FB})\).
2.4 Complete Information Case

In this section, I study the market allocation with complete information. As already mentioned, the market with complete information decentralizes the first best (or planner’s allocation). For this reason, I use complete information allocation and first best interchangeably. In the market with complete information, firms post contracts not only describing \( x \) and \( p \) but also \( z \). That is, firms commit to hire only a particular type of workers.

In this section, I also change the environment to one with a continuous type space, only because this makes the exposition simpler.\(^6\) This also helps me to use the techniques that are easier to handle with a continuous type space compared to a discrete type space. Instead of assuming that there are \( I \) types, I assume here that there is a continuum of measure one of heterogeneous workers indexed by \( z \in Z \equiv [z_L, z_H] \subset \mathbb{R}_+ \), with \( F(z) \) denoting the measure of workers with types below \( z \). \( F \) is differentiable and strictly increasing in \( z \) and \( F'(.,.) \) is its density function. Every thing else in the environment is the same as in the original setting.

[1] studies the complete information case of this environment with urn-ball matching function. Here, I derive Shi’s results with a general matching function. For the submarket in which type \( z \) is active, the market designer or the planner solves the following problem:

\[
\max_{\theta, x} \{ m(\theta)(b(x, z) + e(x, z)) - \theta C(x) \} \tag{2.3}
\]

Let \( f(x, z) \equiv b(x, z) + e(x, z) \). The first order conditions (FOC) require:

\[
m'(\theta) f(x, z) = C(x), \tag{2.4}
\]

\[
q(\theta) f(x, z) = C'(x). \tag{2.5}
\]

I derive \( \theta \) from the first FOC and plug it in the second one to yield:

\[
q(m'^{-1}(\frac{C(x)}{f(x, z)})) = \frac{C''(x)}{f_x(x, z)}. \]

\(^6\)Results in the previous section hold true for any distribution of types. Therefore, even if types become arbitrarily close to each other, like a continuous type space, results should generalize to that case too.
One can solve this equation for \( x \) and then plug it in Equation 2.5 or 2.4 to solve it for \( \theta \), so one can derive \( \theta^{FB}(z) \) and \( x^{FB}(z) \) from these two first order conditions. The superscript \( \text{FB} \) refers to complete information allocation or first best. I usually drop the superscript in this section if there is no danger of confusion.

The derivative of the latter equation with respect to \( z \) is given by:

\[
\frac{f' C' - C' f_x}{f^2} \frac{dx}{dz} - \frac{C f_x}{f^2} \frac{q'(m^{-1}(\frac{C_2}{f}))}{m''(m^{-1}(\frac{C_2}{f}))} = \frac{f_x C'' - C' f_{xx}}{f_x^2} \frac{dx}{dz} - \frac{C' f_{xx}}{f_x^2},
\]

or equivalently,

\[
\frac{f_x C'' - C' f_{xx}}{f_x^2} - \frac{f C' - C f_x}{f^2} \frac{q'}{m''} \frac{dx}{dz} = \frac{C' f_{xx}}{f_x^2} - \frac{C f_x q'}{f^2 \cdot m''},
\]

or

\[
\frac{f_x C'' - C' f_{xx}}{f_x^2} > \frac{f C' - C f_x}{f^2} \frac{m'(\cdot) q'(\cdot) C' f}{q(\cdot) m''(\cdot) C f_x},
\]

where I multiplied the RHS by \( \frac{m'(\cdot) C' f}{q(\cdot) C f_x} \). This operation does not change the inequality because this term is equal to one according to the first order conditions.

The following assumption guarantees that the inequality is satisfied:

**Assumption 13.** For all \( x \in \bar{X} \) and \( z \in [z_L, z_H] \),

\[
\frac{C''(x)}{C'(x)} - \frac{f_{xx}(x,z)}{f_{x}(x,z)} > \max_{\theta} \left\{ \frac{m'(\theta) q'(\theta)}{q(\theta) m''(\theta)} \right\}.
\]

Regarding the RHS of Equation 2.6, I want to show \( \frac{C' f_{xx}}{f_x^2} > \frac{C f_x q'}{f^2 \cdot m''} \). I use the same trick as before by multiplying the RHS of the latter equation by \( \frac{m'(\cdot) C' f}{q(\cdot) C f_x} \) which equals to 1. Then one gets \( \frac{ff_{xx}}{f_x f} \geq \frac{m'q'}{q m''}. \) The following assumption ensures that this inequality holds.7

\footnote{One can write the LHS of the equation in the assumption as follows: \( \frac{\frac{dx}{dz} (ln(\frac{C_2}{f}))}{dx (ln(\frac{C_2}{f}))} \).

Then the assumption is equivalent to \( \frac{\frac{dx}{dz} (ln(\frac{C_2}{f}))}{dx (ln(\frac{C_2}{f}))} > \gamma \) where \( \gamma \equiv \max_{\theta} \left\{ \frac{m'(\theta) q'(\theta)}{q(\theta) m''(\theta)} \right\} \). Rewriting the last...}
Assumption 14. For all $x \in \bar{X}$ and $z \in [z_L, z_H]$,  
\[
\frac{f(x, z)f_{xx}(x, z)}{f_x(x, z)f_z(x, z)} \geq \max_{\theta} \left\{ \frac{m'(\theta)q'(\theta)}{q(\theta)m''(\theta)} \right\}.
\] (2.8)

Proposition 7 ( [1] with general matching function). Under Assumption 7, 13 and 14, there exists a unique equilibrium. The equilibrium allocation decentralizes the first best allocation and $\theta^{FB}(z)$ and $x^{FB}(z)$ are obtained from Equation 2.4 and 2.5. Prices in the equilibrium are given by free entry:  
\[
p(z) = b(x^{FB}(z), z) - \frac{C(x^{FB}(z))}{q(\theta^{FB}(z))}.
\]
Furthermore, the allocation exhibits PAM.

Proof. The sketch of the proof was given above. For details, see [1].

2.4.1 Comparison to [1] and [2]

Proposition 7 extends the analysis of [2] to an environment in which the distribution of firms is determined endogenously by free entry. They have the same assumption as Assumption 14, but they do not have any assumption on $C(.)$ (like Assumption 13), because $C(.)$ is endogenous in their paper. In contrast, in my paper the distribution of firms is endogenous.

My analysis also extends analysis of [1] to an environment with a general matching function. [1] has the same problem but he adopts urn-ball matching function, i.e., $m(\theta) = 1 - e^{-\theta}$. Shi writes $\theta(.)$ on the equilibrium path in terms of $C(.)$ and $f(.)$ and simplifies the RHS of Equation (2.6). Then Assumption 1 part (v) and Equation (5.1) in his paper are used to show that the coefficient of $\frac{dx}{dz}$ and the RHS of Equation (2.6) are both positive. Since I work with a general matching function, I could not use this simplification.

inequality yields $\frac{d}{dx} \left( \ln \left( \frac{C'}{f_x} \right) - \gamma \ln \left( \frac{C'}{f} \right) \right) > 0$, or  
\[
\frac{d}{dx} \left\{ \frac{C'}{f_x} \left( \frac{C'}{f} \right)^{\gamma} \right\} > 0.
\]
In order to develop some intuition for the results in this section and especially for sufficiency of Assumption 14, suppose for now that Assumption 13 is satisfied, so the coefficient of $\frac{\partial^2}{\partial z^2}$ in Equation 2.6 is positive. (Assumption 13, roughly speaking, states that $C''$ must be sufficiently positive.) On the RHS of Equation 2.6, there are two terms. Suppose first that there is no complementarity in the surplus, that is, $f_{xz}(x, z) = 0$ for all $(x, z)$. Mathematically speaking, the RHS becomes negative, so the assignment exhibits NAM, negative assortative matching. To understand the economics of this observation better, note that since there is no complementarity in the surplus, the only important thing for the planner is to match high skill workers with higher probability, regardless of the capital level of the firm assigned to that type of workers. Of course, the best candidate to provide this “trading security” (as [2] put it) is low capital firms, because employing low level of capital is relatively cheaper to provide this trading security. The same argument can be applied for firms with high amount of capital. If the planner wants some firms to enter with high amount of capital, the planner wants to ensure that those firms will be utilized with high probability and so it is necessary to assign a great number of workers to match with them. Again and similar to the previous case, the cheapest way to do that is to assign low skill workers to match with high capital firms.

Now suppose $f_{xz} > 0$. Since there is complementarity between factors of production, skill and capital, the planner tends to match high skill workers with high capital firms, because the total surplus produced in matches between $(x', z')$ and $(x, z)$ with $x < x'$ and $z < z'$ is greater than that produced in matches between $(x', z)$ and $(x, z')$. If Assumption 14 is satisfied, the complementarity effect dominates the trading security effect and so the assignment will be PAM. I will show in the next section, however, that this assumption is not necessary nor sufficient for the market allocation with private information to exhibit PAM.

### 2.5 Examples and Explanation of the Results

So far, I have shown that under assumptions in Theorem 3, the equilibrium allocation with private information exhibits PAM. In this section, I introduce several examples to show that the first best allocation exhibiting PAM is not necessary nor sufficient for the market allocation with private information to exhibit PAM. After these examples, I explain the intuition behind this observation in Section
2.5.4. One important point is that equilibrium exhibits PAM because firms use capital as a screening device to separate different types of workers.

2.5.1 The First Best Allocation Exhibiting PAM Is Not Necessary for the Market Allocation to Exhibit PAM

I proceed with an example and characterize the equilibrium and first best allocation for that. An interesting fact is that here the first best allocation does not exhibit PAM, but the equilibrium allocation does. As discussed in the previous section, if the amount of complementarity between capital and labor is not sufficiently large, the first best allocation may not exhibit PAM. Although here the level of complementarity is not enough, the equilibrium with private information exhibits PAM.

Example 3. Parameter values:

\[ m(\theta) = \frac{\theta}{1 + \theta} \]

\[ C(x) = 0.1x^{1.8} \]

\[ e(x, z) = x^{0.5}(z + 1)^{0.2} \]

\[ b(x, z) = 5(0.75x^{0.8} + 0.25z^{0.8})^{\frac{1}{0.8}} \]

\[ z_i \in \{2, 2.5, ..., 6\}. \]

Under these parameter values, the equilibrium with private information exhibits PAM, but the first best allocation exhibits NAM. The associated graphs are depicted in Figure 2.1, Figure 2.2 and Figure 2.3.

2.5.2 Both Efficient and Market Allocations Exhibit PAM

Example 4. Parameter values:

\[ m(\theta) = \frac{\theta}{(1 + \theta^3)^{\frac{1}{3}}} \]

\[ C(x) = 0.1x^2 \]
\[ e(x, z) = x^{0.5}(z + 1)^{0.6} \]
\[ b(x, z) = 5x^{0.95}(z + 2)^{0.1} \]
\[ z_i \in \{2, 2.5, ..., 6\}. \]

Under these parameter values, the equilibrium with private information and also the first best allocation exhibit PAM. Note that here all conditions of Theorem 3 are satisfied. The associated graphs are depicted in Figure 2.4, Figure 2.5 and Figure 2.6.

### 2.5.3 The First Best Allocation Exhibiting PAM Is Not Sufficient for the Market Allocation to Exhibit PAM

The following example shows a case opposite to Example 3 where the first best allocation exhibits PAM but market with private information does not exhibit PAM. (Actually the allocation is not assortative.) As observed in Figure 2.7, high skill workers are not compensated by higher capital level nor higher wages. Rather, they get matched with higher probability (lower probability of unemployment).

One crucial difference between this case and Example 3 is that here, the marginal payoff of high skill workers with respect to capital is not much higher compared to that of low skill workers, so firms cannot screen high skill workers by allocating them higher amount of capital. Rather, the firms use market tightness to screen high skill workers, because that is cheaper.

**Example 5. Parameter values:**

\[ m(\theta) = \frac{\theta}{(1 + \theta^{1.5})^{\frac{3}{2}}} \]
\[ C(x) = 0.1x^2 \]
\[ e(x, z) = x^{0.1}(z + 1)^{0.6} \]
\[ b(x, z) = 5x^{0.95}(z + 2)^{0.1} \]
\[ z_i \in \{2, 2.5, ..., 6\}. \]

---

8I have solved exactly the same example but for \( z_i \in \{2, 4, 6\} \). The results, which are similar to the current example, are not reported here for the sake of brevity.
Under these parameter values, the equilibrium with private information does NOT exhibit PAM, but the first best allocation exhibits PAM. The associated graphs are depicted in Figure 2.7, Figure 2.8 and Figure 2.9.

2.5.4 Explanation of the Results and the Role of Assumptions

I investigated the economic forces at play when there is complete information in the previous section. Now, I explore the forces which shape the patterns of sorting in the presence of private information. There are two factors which are important for workers: $m(\theta)p$ which is the expected payment they get, and $s \equiv m(\theta)A(x)$ which is the coefficient of $D(z)$. Since $D(z)$ is increasing in $z$, incentive compatibility condition requires that $m(\theta)A(x)$ in equilibrium to be increasing in $z$ (Lemma 2).

Now suppose a firm wants to allocate a fixed amount of $m(\theta)p$ and a fixed $s$ to type $i$ to have IC satisfied. Given that, part 1 of Assumption 10 ensures that the surplus generated in the match with a type $i$ worker given $s$ (surplus function conditional on $s$, $\Pi(s, x, z_i)$) satisfies increasing differences property in $x$ and $z$. That is, the higher the type of a worker, the higher the amount of capital to be allocated to that worker (holding $s$ constant). Moreover, part 2 of Assumption 10 together with Assumption 9 ensures that the surplus generated in the match with a type $i$ worker given $s$ locally satisfies increasing differences property in $x$ and $s$.

On the other hand, in order for an allocation to satisfy IC, $s_i$ must be increasing in $i$. Increasing differences property of the surplus of firms conditional on delivering a particular $s$ to workers together with $s$ being increasing in the type of workers implies that the allocation must be PAM. I call this effect of private information on the assignment pattern “screening” effect. It is a classic result in many strands of literature that private information gives rise to existence of a separating equilibrium in the context of labor, financial and other markets. Nevertheless, to the best of my knowledge, it has not been documented in the previous literature that private information may lead to the assortative matching between factors of production.

If firms have complete information about workers, there will be no requirement for $s$ to be increasing. In fact, for the complete information allocation to exhibit PAM, it is not even necessary to have the increasing differences properties discussed above.\footnote{Actually, it is not possible to establish increasing differences property in $(x; z)$ or in $(x; \theta)$ for the unconditional surplus functions, $m(h + e) - \theta C$.} However, the assignment with complete information exhibits
PAM by making Assumption 13 and Assumption 14 as addressed in Proposition 7. It becomes clear from the explanation above that if those increasing differences properties hold, even if FB does not exhibit PAM, due to the fact that \( s \) must be increasing in workers’ types, then the equilibrium allocation with private information exhibits PAM. It also becomes clear that if Assumption 10 fails to hold, even if FB is PAM, the equilibrium with private information may not exhibit PAM.

Regarding the role of other assumptions, first note that almost all matching functions usually used in the literature satisfy Assumption 9, like so-called urn-ball \( (m(\theta) = 1 - e^{-\theta}) \) or telegraph \( (m(\theta) = \frac{\theta}{1+\theta}) \) matching functions. Second, I used Assumption 8 to ensure that considering IC of only type \( i-1 \) for the maximization problem of type \( i \) is sufficient. One can characterize the equilibrium and derive analytical results for another case in which \( e \) is additively separable, too. I have not included those results to save space.

### 2.6 Constrained Efficient Allocation

Similar to Section 2.4, I work with the continuous type space, only because it is easier to work with. In order to define the problem of a constrained planner, who faces the same information and search frictions, I use the language of direct mechanism. For a detailed discussion on the planner’s problem in environments with directed search and adverse selection, see Chapter 1, in which I showed that the constrained efficient allocation generally leads to strictly higher welfare than equilibrium. I derive sufficient conditions for this environment under which the planner can achieve the first best. Consistent with the main finding in Chapter 1, the welfare in the constrained efficient allocation is higher than that in the equilibrium allocation, because both market tightness and capital level in equilibrium are distorted relative to their first best level for almost all types.

Thanks to the revelation principle, it is without loss of generality to assume that workers report their types to the planner and then they are assigned a level of investment, \( x(z) \), a level of transfer conditional on finding a match, \( p(z) \), a level of unconditional transfer, \( T(z) \), and a market tightness (which determines the number of firms which will be matched with the worker in average), \( \theta(z) \). To be more precise, an allocation is a correspondence \( \Phi(z) : Z \rightarrow \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+^2 \) and its elements are denoted by \( (x(z), p(z), T(z), \theta(z)) \). Unconditional transfer \( T(\cdot) \)
is assumed to be non-negative without loss of generality. If $T(z)$ is negative for a type, say type $z$, and that type is not active, that is, $\theta(z) = 0$, then workers’ participation constraint is violated. If all types are active, then for any $z$ with negative $T(z)$, one can change $p(z)$ to $p(z) + \frac{T(z)}{m(\theta(z))}$ and set $T(z) = 0$. Therefore, assuming $T(z)$ to be non-negative is without loss of generality. Note that if all types are active, then $T(z)$ can be set even equal to 0 for all $z$ without loss of generality. Because this assumption is satisfied here, in what follows, I assume $T(z) = 0$ for all $z$ unless otherwise noted. I proceed by introducing the definition of feasible allocations.

**Definition 9.** An allocation is feasible if it satisfies the following conditions:

1. *(Incentive Compatibility of Workers)* For all $z$ and $\hat{z}$,
   
   $U(z) \equiv m(\theta(z))(p(z)+e(x(z),z))+T(z) \geq U(z,\hat{z}) \equiv m(\theta(\hat{z}))(p(\hat{z})+e(x(\hat{z}),z))+T(\hat{z})$.

2. *(Participation Constraint of Workers)* For all $i$,
   
   $U(z) \geq 0$.

3. *(Planner’s Budget Balance)*
   
   $\int [m(\theta(z))(b(x,z) - p(z)) - \theta(z)C(x(z)) - T(z)]dF(z) \geq 0$.

**Definition 10.** A constraint efficient allocation is a feasible allocation which maximizes the ex-ante weighted average payoff of workers. That is, a constrained efficient allocation solves the following problem:

$$\max_{\phi(\cdot)} \int U(z)dF(z)$$

$$s. t. \Phi(.) \text{ is feasible},$$

where $U(z)$ is define in part 1 of Definition 9.

**2.6.1 Results**

The main result in this section is that the constrained planner can achieve the first best, although he faces private information. The main economic intuition is
that the planner can make cross-subsidization between types which is not possible in the market economy. I will give a sketch of the formal proof below and the complete proof comes in the appendix. For more informal discussion, see Chapter 1.

**Assumption 15.** For all \( x \in \bar{X} \) and \( z \),

\[
\frac{C''(x)}{C(x)} - \frac{f_{x}(x,z)}{f(x,z)} = \frac{f(x,z)f_{xx}(x,z)}{f(x,z)f_{x}(x,z)} > \frac{f(x,z)f_{xx}(x,z)}{f(x,z)f_{x}(x,z)} \geq \max_{\theta} \left\{ \frac{m'(\theta)q'(\theta)}{q(\theta)m''(\theta)} \right\}.
\]

**Assumption 16.** For all \( x \in \bar{X} \) and \( z \),

\[
\psi\left( \frac{C(x)}{f(x,z)} \right) \geq 1 - F(z),
\]

where \( \psi(.) \equiv \eta(m'^{-1}(.)) \) and \( \eta(\theta) \equiv \frac{\theta q'(\theta)}{q(\theta)} \).

**Proposition 8.** Under Assumption 15 and Assumption 16, the planner achieves the first best. This allocation exhibits PAM.

According to this proposition, \( \theta(.) \) and \( x(.) \) are the same as those in the complete information allocation, so they are derived simultaneously from Equation 2.4 and Equation 2.5. Therefore, when I refer to \( \theta(z) \) and \( x(z) \), I mean the market tightness and capital level of the submarket that is allocated to type \( z \) workers in the first best allocation. The price allocated to type \( z \) is pinned down by the following equation

\[
p(z) = -e(x(z), z) + \frac{U(z_L) + \int_{z_0}^{z} m(\theta(z_0))e_z(x(z_0), z_0)dz_0}{m(\theta(z))}, \tag{2.9}
\]

where

\[
U(z_L) = \int [m(\theta(z_0))\left[ b(x(z_0), z_0) + e(x(z_0), z_0) - e_z(x(z_0), z_0)\frac{1 - F(z_0)}{F'(z_0)} \right] - \theta(z_0)C(x(z_0)) ]F'(z_0)dz_0. \tag{2.10}
\]

I give a sketch of the proof here. The proof is based on a guess-and-verify approach. The market tightness and the capital level allocated to different types in the first best are known. The goal is to find a set of transfers that satisfies all conditions.
of feasibility. I first show that under Assumption 15, both $x(z)$ and $\theta(z)$ are increasing in $z$. That is, higher types get matched with higher probability and are also matched with firms with higher amount of capital. $x(z)$ is increasing in $z$ according to Proposition 7, because Assumption 14 and Assumption 15 are implied by Assumption 15. To show that $\theta(z)$ is also increasing in $z$, take the derivative of Equation 2.5 with respect to $z$ and substitute $\frac{dx}{dz}$ from Equation 2.6 to calculate $\frac{d\theta}{dz}$. After some tedious algebra, one can check that under the left side inequality in Assumption 15, $\frac{d\theta}{dz}$ is also positive.

According to [24] (theorem 7.1 and 7.3), monotonicity of $\theta(z)$ and $x(z)$ in $z$ are necessary and sufficient conditions for any allocation to satisfy IC. Both $\theta(z)$ and $x(z)$ are increasing in $z$, so there exists a set of transfers such that IC is satisfied. To derive this set of transfers, remember that function $U(.)$ is an envelope function, because $U(z) = \max \hat{z} U(z, \hat{z})$. Therefore, according to [44], $U(.)$ can be written as follows: $U(z) = U(z_L) + \int_{z_L}^{z} \frac{\partial U(z, z_0)}{\partial z} dz_0 = U(z_L) + \int_{z_L}^{z} m(\theta(z_0)) e_z(x(z_0), z_0) dz_0$. Now the set of transfers, $p(.)$, can be recovered from $U(z) = m(\theta(z))(p(z) + e(x(z), z))$ and write $p(z)$ in terms of $\theta(.)$ and $x(.)$ to derive Equation 2.9. Finally, to ensure that the planner’s budget constraint holds, I substitute $p(z)$ in the budget constraint, do a little algebra and show that the constraint is satisfied by using Assumption 16.

### 2.7 Conclusion

I explore the determinants of assignment patterns between workers and firms in economies with search frictions, private information and two sided heterogeneity. This paper has two main contributions. The technical contribution is that I solve a rather complicated maximization problem with a non-concave objective function and a non-convex constraint set. The techniques used in the characterization of the equilibrium can be used in similar mechanism design and search theory problems. The main difficulty in the proof is to identify the type whose incentive compatibility constraint may become binding. I impose a multiplicative separability structure on the payoff function of workers form the match and show that the firms who want to attract a particular type of workers, say type $i$, need to only worry about the incentive compatibility constraint of type $i - 1$. This observation makes our characterization simpler, because the constraint set of firms’ maximization problem
becomes smaller. As a result, a modified problem can be obtained which has a concave objective function under some conditions.

The second contribution is that I identify an economic force for assortative matching which was not discussed in the literature before. This force, which I call “screening” effect, might lead to PAM or NAM depending on the payoff structure and matching function of the environment. In my main result, I derive sufficient conditions under which the equilibrium allocation exhibits PAM. This force is only present when there is private information or when the type of workers is not contractible. The main message is that even though the level of complementarity between factors of production may not be sufficiently high so that FB may not exhibit PAM, but the firms who want to attract high skill workers need to offer them jobs with higher amount of capital so as to screen them from other types of workers.

My results has some implications for identification of assignment patterns from wage data. In Example 3 and Example 4, I indicated that even if the matching between firms and workers is positive assortative, wage may be decreasing in the type of workers and the type of firms involved in the match. Therefore, it is not possible in general to identify sorting from wage data. In Example 3, the correlation between wage and capital level is negative, but the assignment is neither PAM nor NAM. These observations reiterates the main message of [35] that sign of sorting cannot be identified from wage data. [35], however, do not have directed search in their setting, nor do they have private information.

In order to make the model testable and closer to data, one possible extension is to study a full-fledged labor market with dynamics and on-the-job search in which type of workers is their private information at the beginning of the employment. As time goes by, the employer receives signals regarding the type of the employee and may want to fire the employee at some point in time. Still hiring a low type worker is costly but not irreversible. The worker, on the other hand, keeps searching on the job to expand its outside opportunities in the case of being fired or to find a better job. Of course, the intensity of on-the-job search depends on the type of workers. The main questions that I would like to explore are as follows: (i) What is the assignment pattern between workers and firms and how predictions of the model will be changed compared to the static case? (ii) Can this model explain the dynamics of unemployment in the data? These questions are left for future
Figure 2.1. The solid lines represent equilibrium allocation and the dashed lines represent the first best allocation. With parameter values in Example 3, the market exhibits PAM, but the first best allocation exhibits NAM. In equilibrium, high skill workers are compensated not through high wages nor through high probability of matching. Rather, they get matched with high capital firms. The marginal payoff of high skill workers in capital is higher than that of low skill workers. Therefore, it is possible for firms to screen high skill workers by offering them higher amount of capital.

research.
Figure 2.2. With parameter values in Example 3, unsurprisingly, the utility of workers in the equilibrium with private information (solid line) is less than that in the first best allocation (dashed line).
Figure 2.3. With parameter values in Example 3, we check that the incentive compatibility constraints of workers are satisfied. This figure depicts $U(z_0, z)$ for 3 arbitrary values of $z_0$. Function $U(z_0, z)$ is the payoff of a type $z_0$ worker if he applies to the submarket to which type $z$ workers apply to.
Figure 2.4. With parameter values in Example 4, both equilibrium (solid lines) and the first best (dashed line) allocations exhibit PAM. Firms with high amount of capital are matched with high skill workers. The wage of high skill workers is lower than that of low skill ones.
Figure 2.5. With parameter values in Example 4, the utility of workers in the equilibrium with private information (solid line) and complete information (dashed line).
Figure 2.6. With parameter values in Example 4, we check similarly as previous example that the incentive compatibility constraints of workers are satisfied.
Figure 2.7. The solid lines represent equilibrium allocation and the dashed lines represent complete information. With parameter values in Example 5, the market allocation is not assortative, but the complete information allocation exhibits PAM. High skill workers are compensated through high probability of matching. The payoff of high skill workers is higher than that of low skill workers given a level of capital. Therefore, it is possible for firms to screen high skill workers by offering them higher probability of matching.
Figure 2.8. With parameter values in Example 5, the utility of workers in the equilibrium with private information (solid line) is, again unsurprisingly, less than that under complete information (dashed line).
The payoff of type $z_0$ if he reports $z$

Figure 2.9. With parameter values in example 5, we check similarly as previous examples that the incentive compatibility constraints of workers are satisfied. This figure depicts $U(z_0, z)$ for 3 arbitrary values of $z_0$. Function $U(z_0, z)$ is defined as the payoff of a type $z_0$ worker if he applies to the submarket to which type $z$ workers apply to.
Appendix A
Proofs of Chapter 1

A.1 Direct Mechanism: Definitions and Proofs

I take a constrained efficient mechanism. I show by construction that there exists a constrained efficient allocation associated with the direct mechanism which delivers the same welfare. Before I get to the results, note that the budget constraint in the constrained efficient mechanism is always binding. Otherwise, one can increase all \( \tilde{s}_i \) by an identical small amount. Then all other conditions continue to be met, but welfare strictly increases.

Proof. Proof of Lemma 1

Given a constrained efficient mechanism \( \{(\tilde{a}_i, \tilde{p}_i, \tilde{s}_i, \tilde{\theta}_i)\}_{i \in \{1,2,\ldots,I\}} \), I construct an implementable allocation which delivers the same welfare. First, define \( N \) to be the set of all types who get matched with strictly positive probability in the direct mechanism, that is, \( N \equiv \{i | \tilde{\theta}_i > 0\} \). Second, set

\[
T = \begin{cases} 
\tilde{s}_i & \text{if } \exists i \text{ such that } \tilde{\theta}_i = 0 \\
0 & \text{otherwise}
\end{cases}
\]

Third, for all \( i \in N \) construct \( y_{k_i} \) as follows:

\[
y_{k_i} = (\tilde{a}_i, \tilde{p}_i + \frac{\tilde{s}_i - T}{m(\tilde{\theta}_i)}).
\]
and also set \( t(y_{k_i}) = v_i(\bar{a}_i) - \bar{p}_i - \frac{\bar{s}_i - T}{m(\bar{\theta}_i)} - \frac{k}{q(\bar{\theta}_i)} \) and

\[
Y^P = \{ y_{k_i} \mid i \in N \}, \quad \theta(y_{k_i}) = \bar{\theta}_i, \quad \gamma_i(y_{k_i}) = 1, \quad \lambda(\{ y_{k_i} \}) = \pi_i \bar{\theta}_i, \quad (A.1)
\]

For any other submarket \( y \notin Y^P \), define \( K(y) = \{ j \mid u_j(a) + p > 0 \} \) to denote the types which get a strictly positive payoff by applying to to \( y \). If \( K(y) \neq \emptyset \) and \( \min_{j \in K(y)} \left\{ \frac{U_j}{u_j(a) + p} \right\} < \bar{m} \equiv \lim_{\theta \to \infty} m(\theta) \), then set \( \theta(y) \) such that

\[
m(\theta(y)) = \min_{j \in K(y)} \left\{ \frac{U_j}{u_j(a) + p} \right\}.
\]

If the latter equation holds for several \( \theta(y) \), then pick the greatest one. If \( K(y) = \emptyset \), or \( \min_{j \in K(y)} \left\{ \frac{U_j}{u_j(a) + p} \right\} \geq \bar{m} \), then set \( \theta(y) = \infty \). To define the composition function for \( y \notin Y^P \), define \( n = \min \{ \arg \min_{j \in K(y)} \left\{ \frac{U_j}{u_j(a) + p} \right\} \} \) and set \( \gamma_n(y) = 1 \). If \( K(y) = \emptyset \), then \( \Gamma(y) \) can be chosen arbitrarily, so for example set \( \gamma_1(y) = 1 \). Also for \( y \notin Y^P \), set \( t(y) = \max_{i, a} v_i(a) - p \).

\( T \) is well-defined, because if there are more than one \( i \) with \( \bar{\theta}_i = 0 \), then \( \bar{s}_i \) must be the same for all of them, for otherwise, sellers’ incentive compatibility constraint in the definition of feasible mechanism is violated. All \( y_{k_i} \) are also well-defined, because \( \bar{\theta}_i \) cannot be equal to 0. Moreover, \( t(y) \) is well-defined too, since no \( \bar{\theta}_i \) can be equal to \( \infty \). If \( \bar{\theta}_i \) goes to \( \infty \) for some \( i \), then the planner’s budget-balance condition will be violated, because the planner needs to spend infinite amount of resources to finance entry of buyers (the left hand side of the planner’s budget constraint goes to \( -\infty \)).

If there exist \( i \) and \( j \) (\( i \neq j \)) such that \( y_{k_i} = y_{k_j} \), I show below that \( m(\bar{\theta}_i) = m(\bar{\theta}_j) \). Assume without loss of generality that \( \bar{\theta}_i \leq \bar{\theta}_j \). Then I keep \( y_{k_i} \) and remove \( y_{k_j} \) and let \( \gamma_i(y_{k_i}) = \frac{\pi_i}{\pi_i + \pi_j} \), \( \gamma_j(y_{k_i}) = \frac{\pi_j}{\pi_i + \pi_j} \), \( \lambda(\{ y_{k_i} \}) = (\pi_i + \pi_j) \bar{\theta}_i \) and \( t(y_{k_i}) = \frac{\pi_i v_i(\bar{a}_i) + \pi_j v_j(\bar{a}_i)}{\pi_i + \pi_j} - \bar{p}_i - \frac{\bar{s}_i - T}{m(\bar{\theta}_i)} - \frac{k}{q(\bar{\theta}_i)} \). Now I show that \( m(\bar{\theta}_i) = m(\bar{\theta}_j) \).

According to sellers’ incentive compatibility constraint (for type \( i \) to report \( j \),
one can write:¹

\[ m(\tilde{\theta}_i)(u_i(\tilde{a}_i) + \tilde{p}_i) + \tilde{s}_i \geq m(\theta_j)(u_i(\tilde{a}_j) + \tilde{p}_j) + \tilde{s}_j = m(\tilde{\theta}_j)(u_i(\tilde{a}_i) + \tilde{p}_i + \frac{\tilde{s}_j - T}{m(\theta_j)}) + T \]

\[ = m(\tilde{\theta}_j)(u_i(\tilde{a}_i) + \tilde{p}_i + \frac{\tilde{s}_j - T}{m(\theta_j)}) + T, \]

where the second equality follows from the assumption that \( y_{ki} = y_{kj} \). This implies that \( (m(\tilde{\theta}_i) - m(\theta_j))(u_i(\tilde{a}_i) + \tilde{p}_i + \frac{\tilde{s}_i - T}{m(\theta_j)}) \geq 0 \). If \( \tilde{\theta}_k = 0 \) for all \( k \), so \( T = 0 \) by construction, and due to the participation constraint for type \( i \), \( m(\tilde{\theta}_i)(u_i(\tilde{a}_i) + \tilde{p}_i + \tilde{s}_i) \geq 0 \). Therefore, \( m(\tilde{\theta}_i) \geq m(\tilde{\theta}_j) \). Now assume that there exists \( k \) such that \( \tilde{\theta}_k = 0 \).

IC constraint for type \( i \) to report \( k \) implies that \( m(\tilde{\theta}_i)(u_i(\tilde{a}_i) + \tilde{p}_i + \tilde{s}_i) = \tilde{s}_k = T \) for all \( i \). Therefore, whether there exists \( k \) with \( \tilde{\theta}_k = 0 \) or not, \( u_i(\tilde{a}_i) + \tilde{p}_i + \tilde{s}_i - \frac{T}{m(\theta_j)} \geq 0 \). Thus, \( m(\tilde{\theta}_i) \geq m(\tilde{\theta}_j) \). Similarly by considering the sellers’ incentive compatibility constraint for \( j \) to report \( i \), we can get \( m(\tilde{\theta}_j) \geq m(\tilde{\theta}_i) \). Therefore, \( m(\tilde{\theta}_j) = m(\tilde{\theta}_i) \).

The proposed allocation is implementable because of the following reasons. Regarding sellers’ maximization condition, I first show that \( U_i \geq m(\theta(y))(u_i(a) + p) + T \) for all \( y \in Y^P \) and \( i \in N \). We know that if \( y \in Y^P \), then there exists \( i \in N \) such that \( y = y_{ki} \). Therefore I need to show that

\[ m(\theta_{ki})(u_i(a_{ki}) + p_{ki}) \geq m(\theta_{kj})(u_i(a_{kj}) + p_{kj}) \text{ for } j \in N, \quad (A.2) \]

and

\[ m(\theta_{ki})(u_i(a_{ki}) + p_{ki}) \geq T. \quad (A.3) \]

To show the above inequalities, note that

\[ m(\theta_{ki})(u_i(a_{ki}) + p_{ki}) = m(\tilde{\theta}_i)(u_i(\tilde{a}_i) + \tilde{p}_i + \tilde{s}_i - T \geq m(\tilde{\theta}_i)(u_i(\tilde{a}_j) + \tilde{p}_j + \tilde{s}_j - T \]

for all \( j \). The equality follows from the construction of \( y_{ki} \). The inequality follows

¹I stated in the main body of the paper that if one allows the planner to use direct mechanism with lotteries and randomization, then there might be some loss of generality in formulating the problem as formulated in the definition of constrained efficient allocation. This is exactly where we can see why. If the planner uses randomization, then incentive compatibility holds not necessarily for each \((\tilde{a}_i, \tilde{p}_i, \tilde{s}_i, \tilde{\theta}_i)\), but holds in expectation. Therefore, if there are two \( y_k \) and \( y_{kj} \) which are equal and types \( i \) and \( j \) are allocated to them with positive probability, it might be the case that \( \theta_{ki} \) and \( \theta_{kj} \) are not equal, thus we cannot construct an implementable allocation from that direct mechanism. As stated earlier, the planner might want to use lotteries if his objective function is not concave or his constraint set is not convex.
from the incentive compatibility condition in the direct mechanism. The right hand side equals to 0 if \( j \notin N \) and equals to \( m(\theta_{kj})(u_i(a_{kj}) + p_{kj}) \) if \( j \in N \) due to the construction of \( y_{kj} \). Thus, Equation A.2 is established. For Equation A.3, if there exists \( j \) such that \( j \notin N \), then \( \tilde{\theta}_j = 0 \), \( \tilde{s}_j = T \) and the equation is established. If \( N = \{1, 2, ..., I\} \), then \( T = 0 \) and \( m(\theta_{ki})(u_i(a_{ki}) + p_{ki}) \geq 0 \) due to the participation constraint in the direct mechanism.

Now I show that \( U_i \geq m(\theta(y))(u_i(a) + p) + T \) for all \( y \in Y^P \) and \( i \notin N \), so I need to show that

\[
m(\theta_{kj})(u_i(a_{kj}) + p_{kj}) + T \leq T \quad \text{for all } j \in N.
\]

But

\[
m(\theta_{kj})(u_i(a_{kj}) + p_{kj}) + T = m(\tilde{\theta}_j)(u_i(\tilde{a}_j) + \tilde{p}_j) + \tilde{s}_j \leq m(\tilde{\theta}_i)(u_i(\tilde{a}_i) + \tilde{p}_i) + \tilde{s}_i = \tilde{s}_i = T
\]

for all \( j \in N \). The first equality follows from the construction of \( y_{kj} \) and \( j \in N \). The inequality follows from the incentive compatibility condition in the direct mechanism. The next equality holds because \( \tilde{\theta}_i = 0 \) since \( i \notin N \). The last equality holds due to the construction of \( T \) and \( i \notin N \).

Now I show that condition (ii) is satisfied. By construction of \( \theta(.) \) and \( \Gamma(.) \) and as shown above, \( U_i \geq m(\theta(y))(u_i(a) + p) + T \) for all \( y \in Y^P \) with equality if \( \theta(y) < \infty \) and \( \gamma_i(y) > 0 \). The inequality also holds for \( y \notin Y^P \) due to the construction of \( \theta(.) \) and \( \Gamma(.) \). Also by construction of \( N \), for any \( i \in N \), \( m(\theta(y))(u_i(a) + p) > 0 \). Given \( y \), if \( u_i(a) + p < 0 \) for some \( i \), then \( i \notin K(y) \). Thus, if \( K(y) = \emptyset \), then \( \theta(y) = \infty \). If \( K(y) \neq \emptyset \), then \( \gamma_{\alpha_i}(y) = 1 \) for some \( n \in K(y) \), therefore \( \gamma_{\alpha_i}(y) = 1 \).

**Buyers’ profit maximization and free entry condition** holds due to the following reasons. Consider first \( y \in Y^P \). Remember that for \( y \in Y^P \), there exists \( i \in N \) such that \( y = y_{ki} \). But for all \( i \in N \), \( q(\theta_{ki})(v_i(a_{ki}) - p_k - t_k) - T = k \). Therefore, the buyers’ profit maximization and free entry condition holds with equality for \( y \in Y^P \). Now consider \( y \notin Y^P \). Then \( q(\theta(y))\Sigma\gamma_{\alpha_i}(y)(v_i(a) - p) - t(y) < k \) due to the choice of \( t(y) \). Therefore condition (i) is satisfied.\(^2\)

**Feasibility condition** is obviously satisfied following the construction of \( \lambda \).

\(^2\)It is immediately clear from this step of the proof that the restrictions on off-the-equilibrium-path beliefs do not play any role in our analysis. That is, any other off-the-equilibrium-path beliefs would work with the taxes that we chose. This is because the planner does not face any restriction on the tax amount that he can impose.
Planner’s budget constraint is satisfied because:

\[ \int q(\theta)td\lambda(y) - T \geq \sum_{i \in N} q(\tilde{\theta}_i)(v_i(\tilde{a}_i) - p_i - \frac{\tilde{s}_i - T}{m(\tilde{\theta}_i)} - \frac{k}{q(\tilde{\theta}_i)})\pi_i - T \]

\[ = \sum_{i \in N} q(\tilde{\theta}_i)(v_i(\tilde{a}_i) - p_i - \frac{\tilde{s}_i}{m(\tilde{\theta}_i)} - \frac{k}{q(\tilde{\theta}_i)})\pi_i - \sum_{i \notin N} T \]

\[ = \sum_{i \in N} [m(\tilde{\theta}_i)(v_i(\tilde{a}_i) - p_i) - \tilde{s}_i - k\tilde{\theta}_i]\pi_i + \sum_{i \notin N} [m(\tilde{\theta}_i)(v_i(\tilde{a}_i) - p_i) - \tilde{s}_i - k\tilde{\theta}_i]\pi_i \geq 0. \]

The first inequality holds due to the construction of implementable allocation and due to the following reason. As mentioned earlier, it might be the case that several types have the same \( y_{k_i} \). I showed that \( m(\theta_{k_i}) = m(\theta_{k_j}) \) and if \( \theta_{k_i} \neq \theta_{k_j} \), then I chose the lowest one. The first inequality follows. The second equality holds due to the fact that for \( i \notin N \), \( \tilde{\theta}_i = 0 \) and \( \tilde{s}_i = T \). The last inequality holds due to the budget-balance condition in the definition of feasible mechanism.

A.2 Proof of Theorem 1

Proof of Theorem 1. I begin from equilibrium allocation and modify it to improve welfare. Consider a type \( i \) seller who gets a strictly positive payoff in the equilibrium and does not produce \( a_i^{FB} \) or does not get matched with probability \( m(\theta_i^{FB}) \). Such a type exists because the equilibrium fails to achieve the first best. Let \( \tilde{i} \) denote this type. I define a set of problems, similar but not the same as one in GSW, and characterize its solution. From that solution, I construct a feasible mechanism and show that it yields higher welfare for the planner than the equilibrium allocation.

According to Proposition 3 in their paper, GSW show that the following set of problems characterizes the equilibrium.

Problem 4 \((P_i(0))\).

\[ \max_{\theta \in [0,\infty],[a,p] \in \mathcal{Y}} \{m(\theta)(u_i(a) + p)\} \]

subject to

\[ q(\theta)(v_i(a) - p) \geq k, \]

\[ m(\theta)(u_j(a) + p) \leq \bar{U}_j(0) \text{ for all } j < i. \]
More precisely, define problem $P(0)$ to be the set of problems $P_i(0)$ for all $i$. Let $\tilde{U}_i(0)$ be the value of the objective function in problem $P_i(0)$ given $(\tilde{U}_1(0), \tilde{U}_2(0), ..., \tilde{U}_{i-1}(0))$ if $\tilde{U}_i(0)$ is strictly greater than 0 and $\tilde{U}_i(0) = 0$ otherwise. Denote by $I^*(0) \subseteq \{1, 2, ..., I\}$ the set of types such that the constraint set in $P_i(0)$ is non-empty and $\tilde{U}_i(0)$ is strictly greater than 0. For any $i \in I^*(0)$, let $(\tilde{\theta}_i(0), \tilde{a}_i(0), \tilde{p}_i(0))$ denote the solution to problem $P_i(0)$ given $(\tilde{U}_1(0), \tilde{U}_2(0), ..., \tilde{U}_{i-1}(0))$. Now, consider another set of problems which is basically a perturbation of the above problem in a specific way. We will see why all variables in the above and below problems are written as functions of 0 and $\epsilon$, respectively.

**Problem 5** ($P_i(\epsilon), \epsilon > 0$).

$$\max_{\theta \in [0, \infty], (p,a) \in V} \{ m(\theta)(u_i(a) + p) + \delta_i \}$$

subject to

$$q(\theta)(v_i(a) - p) \geq k,$$

and

$$m(\theta)(u_j(a) + p) + \delta_i \leq \tilde{U}_j(\epsilon) \text{ for all } j < i,$$

where $\delta_i = \begin{cases} \epsilon & \text{if } i < \tilde{i} \text{ or } i \notin I^*(0) \\ 0 & \text{otherwise} \end{cases}$.

Similarly as above, define problem $P(\epsilon)$ to be the set of problems $P_i(\epsilon)$ for all $i$. Let $\tilde{U}_i(\epsilon)$ be the value of the objective function in problem $P_i(\epsilon)$ given $(\tilde{U}_1(\epsilon), \tilde{U}_2(\epsilon), ..., \tilde{U}_{i-1}(\epsilon))$ if $\tilde{U}_i(\epsilon)$ is strictly greater than $\epsilon$ and $\tilde{U}_i(\epsilon) = \epsilon$ otherwise. Denote by $I^*(\epsilon) \subseteq \{1, 2, ..., I\}$ the set of types such that the constraint in $P_i(\epsilon)$ is non-empty and $\tilde{U}_i(\epsilon)$ is strictly greater than $\epsilon$. For any $i \in I^*(\epsilon)$, let $(\tilde{\theta}_i(\epsilon), \tilde{a}_i(\epsilon), \tilde{p}_i(\epsilon))$ denote the solution to problem $P_i(\epsilon)$ given $(\tilde{U}_1(\epsilon), \tilde{U}_2(\epsilon), ..., \tilde{U}_{i-1}(\epsilon))$.

Since the constraint of $P_i(\epsilon)$ is exactly the same as $P_i(0)$ for types below $\tilde{i}$, they get exactly $\tilde{U}_i(\epsilon) = \tilde{U}_i(0) + \epsilon$. (It is easy to see it by induction.) Also for types above $\tilde{i}$ who are non-participant, that is, $i > \tilde{i}$ and $i \notin I^*(0)$, then $\tilde{U}_i(\epsilon) = \tilde{U}_i(0) + \epsilon = \epsilon$.

For type $\tilde{i}$, problem $P_i(\epsilon)$ maximizes the objective function given $(\tilde{U}_1(\epsilon), \tilde{U}_2(\epsilon), ..., \tilde{U}_{\tilde{i}-1}(\epsilon))$. Since type $\tilde{i}$ does not achieve the first best in equilibrium, some constraints must be binding at $P_i(0)$, thus Problem $P_i(\epsilon)$ yields strictly higher value of the objective function than problem $P_i(0)$, because those constraints are now

98
relaxed. To elaborate, according to Lemma 7, the first constraint is always binding so we can eliminate $p$ from the problem and rewrite the problem as follows:

$$\max_{\theta \in [0, \infty), (\cdot, a) \in Y} \{m(\theta)(u_i(a) + v_i(a)) - k\theta\}$$

subject to $m(\theta)(u_j(a) + v_i(a)) - k\theta \leq \bar{U}_j(0)$ for all $j < i$.

Due to the assumption that $m(\theta)(u_i(a) + v_i(a)) - k\theta$ has a single peak, locally relaxing the constraint improves welfare if the solution is not at the peak.

For types above $\bar{i}$ who are active in the equilibrium, the objective function is weakly higher than the equilibrium allocation. I show the latter claim by induction. Fix $j > \bar{i}$ and assume that for all $k$ such that $\bar{i} \leq k < j$, then $\bar{U}_k(\epsilon) \geq \bar{U}_k(0)$, that is, the value of the objective functions is higher than that in the equilibrium. This implies that the constraint set in problem $P_j(\epsilon)$ is bigger for all $k$ with $\bar{i} \leq k < j$. On the other hand for $k < \bar{i}$, we already know that the constraint set is bigger by definition of $P_k(\epsilon)$. Hence, the value of the objective function in $P_j(\epsilon)$ should be weakly higher than that in equilibrium.

To summarize, so far I have proved the following:

$$\bar{U}_i(\epsilon) = \bar{U}_i(0) + \epsilon$$

for all $i < \bar{i}$ or $i \notin I^*(0)$,

$$\bar{U}_\bar{i}(\epsilon) > \bar{U}_\bar{i}(0),$$

and $\bar{U}_i(\epsilon) \geq \bar{U}_i(0)$ for all $i > \bar{i}$ and $i \in I^*(0)$. \hfill (A.4)

Set $\epsilon_1 = \min\{U_i(0)|i \geq \bar{i}, i \in I^*(0)\}$. In Lemma 7, I will derive another upper bound for $\epsilon$ called $\epsilon_2$. Let $\epsilon$ be in $(0, \min(\epsilon_1, \epsilon_2))$. For example fix

$$\epsilon = \frac{1}{2} \min(\epsilon_1, \epsilon_2).$$ \hfill (A.5)

For this $\epsilon$, all types who participate in the allocation ($i \in I^*(0)$) get a strictly positive payoff in the solution to Problem $P_i(\epsilon)$.
Now I propose the following direct mechanism:

\[
(\tilde{a}_i, \tilde{p}_i, \tilde{s}_i, \theta_i) = \begin{cases} 
(\bar{a}_i(0), \bar{p}_i(0), \epsilon - \tilde{\epsilon}, \theta_i(0)) & \text{if } 1 \leq i < \bar{i} \text{ and } i \in I^*(0) \\
(\bar{a}_i(\epsilon), \bar{p}_i(\epsilon) - \frac{\epsilon}{m(\theta_i(\epsilon))}, \epsilon - \tilde{\epsilon}, \theta_i(\epsilon)) & \text{if } \bar{i} \leq i \leq I \text{ and } i \in I^*(0) ,
(\bar{a}_1(0), \bar{p}_1(0), \epsilon - \tilde{\epsilon}, 0) & \text{if } i \notin I^*(0)
\end{cases}
\]

where \( \tilde{\epsilon} \equiv \epsilon(\sum_{i=1}^{\bar{i}-1} \pi_i + \sum_{i=\bar{i}+1, i \notin I^*(0)} \pi_i) \). Note that for \( i \notin I^*(0), \tilde{a}_i \) and \( \tilde{p}_i \) are arbitrary, because \( \theta_i \) is set to be 0.

It is important to note that \( I^*(0) = I^*(\epsilon) \) due to the following reasons. Regarding types below \( \bar{i} \), since they just get lump sum transfers which is common across all types, their incentives to participate or not does not change. Therefore, if their payoff in the equilibrium is less than 0 so they do not apply to any submarket, they remain inactive also under the proposed allocation.

For types \( \bar{i} \) and above, if they participate in equilibrium, they want also to participate in the new allocation due to the choice of \( \epsilon \). Now, suppose they do not participate in equilibrium. According to the assumption that we made that all types with positive gains from trade will be active in equilibrium, their non-participation in the equilibrium means that they could not generate a strictly positive payoff. Therefore, relaxing constraints do not help them to generate a strictly positive payoff.

I have made this assumption that all types who get a strictly positive payoff in the complete information case will get a strictly positive payoff in the equilibrium. If there are positive gains from trade for all types, then all types will get a strictly positive payoff in the equilibrium (according to Proposition 4 in GSW). Therefore, the above assumption will be automatically satisfied.

I show below that allocation \( \{(\tilde{a}_i, \tilde{p}_i, \tilde{s}_i, \theta_i)\}_{i \in \{1, 2, \ldots, I\}} \) is feasible and yields strictly higher welfare than the equilibrium allocation.

**Incentive Compatibility of Sellers**

Type \( i \) gets payoff \( m(\theta_i) (u_i(\tilde{a}_i) + \tilde{p}_i) + \tilde{s}_i \) which equals to

\[
\begin{cases} 
 m(\tilde{a}_i(0))(u_i(\tilde{a}_i(0)) + \tilde{p}_i(0)) + \epsilon - \tilde{\epsilon} & \text{if } 1 \leq i < \bar{i} \text{ and } i \in I^*(0) \\
 m(\tilde{a}_i(\epsilon))(u_i(\tilde{a}_i(\epsilon)) + \tilde{p}_i(\epsilon)) - \tilde{\epsilon} & \text{if } \bar{i} \leq i \leq I \text{ and } i \in I^*(0) .
\epsilon - \tilde{\epsilon} & \text{if } i \notin I^*(0)
\end{cases}
\]
Note that the market tightness and production level that are allocated to agents are the same as those in Problem $P_i(\epsilon)$. The only difference is that all types now get $\tilde{\epsilon}$ less compared to their payoff in Problem $P_i(\epsilon)$. Since all types are taxed by the identical amount of $\tilde{\epsilon}$, incentives are not affected, therefore, in order to check for incentive compatibility in the proposed allocation, we can check whether incentive compatibility holds in Problem $P_i(\epsilon)$ or not.

By construction, all upward IC hold, because they are explicitly taken into account as constraints of Problem $P_i(\epsilon)$. Moreover, in Lemma 7 below, I show that if $\epsilon < \tilde{\epsilon}_2$, then all downward IC are also satisfied. Therefore, incentive compatibility of sellers is satisfied.

**Participation Constraint of Sellers**

The payoff to sellers is summarized as follows:

\[
\begin{cases}
U_i(0) + \epsilon - \tilde{\epsilon} & \text{if } 1 \leq i < \tilde{i} \text{ and } i \in I^*(0) \\
U_i(\epsilon) - \tilde{\epsilon} & \text{if } \tilde{i} \leq i \leq I \text{ and } i \in I^*(0) \\
\epsilon - \tilde{\epsilon} & \text{if } i \notin I^*(0)
\end{cases}
\]

For $1 \leq i < \tilde{i}$ or $i \in I^*(0)$, their payoff will be weakly higher than $\epsilon - \tilde{\epsilon}$ which is obviously positive. For $\tilde{i} \leq i \leq I$ and $i \in I^*(0)$, $U_i(\epsilon) \geq \epsilon$ by the choice of $\epsilon$ in Equation A.5, and because $\epsilon > \bar{\epsilon}$, therefore, $U_i(\epsilon) \geq \bar{\epsilon}$.

**Planner’s Budget Constraint**

\[
\sum \pi_i [m(\bar{\theta}_i)(v_i(\bar{a}_i) - \bar{p}_i) - k\bar{\theta}_i - \bar{s}_i] \sum \pi_i [m(\tilde{\theta}_i)(v_i(\tilde{a}_i) - \tilde{p}_i) - k\tilde{\theta}_i - \tilde{s}_i] 
\]

\[
= \sum_{i \in \{1, 2, \ldots, \bar{i} - 1\} \cap I^*(0)} \pi_i \tilde{\theta}_i(0) [q(\tilde{\theta}_i(0))(v_i(\tilde{a}_i(0)) - \tilde{p}_i(0))] \\
+ \sum_{i \in \{\bar{i}, \bar{i} + 1, \ldots, I\} \cap I^*(0)} \pi_i \left[ \tilde{\theta}_i(\epsilon) [q(\tilde{\theta}_i(\epsilon))(v_i(\tilde{a}_i(\epsilon)) - \tilde{p}_i(\epsilon))] + \epsilon \right] - (\epsilon - \tilde{\epsilon}) = 0
\]

The two terms with accolades under them are equal to 0 due to Lemma 7. The whole expression equals 0 due to the definition of $\tilde{\epsilon}$. 
Last Step: Calculating Welfare

I calculate welfare from the direct mechanism:

$$\sum \pi_i U_i = \sum_{i \in \{1,2,\ldots,i-1\} \cap I^*(0)} \pi_i (U_i(0)+\epsilon-\tilde{\epsilon}) + \sum_{i \in \{i,i+1,\ldots,I\} \cap I^*(0)} \pi_i (U_i(0)-\epsilon) + \sum_{i \notin I^*(0)} \pi_i (\epsilon-\tilde{\epsilon})$$

$$> \sum_{i \in \{1,2,\ldots,i-1\} \cap I^*(0)} \pi_i (U_i(0)+\epsilon-\tilde{\epsilon}) + \sum_{i \in \{i,i+1,\ldots,I\} \cap I^*(0)} \pi_i (U_i(0)-\epsilon) + \sum_{i \notin I^*(0)} \pi_i (\epsilon-\tilde{\epsilon})$$

$$= \sum \pi_i U_i^{EQ}.$$

The inequality follows from Equation A.6. The proof is now complete, because we have found a feasible mechanism that yields higher welfare than the equilibrium. Of course, this allocation is implementable due to Lemma 1.

I prove in the following lemma that at the solution to Problem $P(\epsilon)$, the first constraint in $P_i(\epsilon)$ should be binding. Also, I show that sellers are not attracted to submarkets designed for higher types, if $\epsilon$ is chosen sufficiently small. This lemma is similar to Lemma 1 in GSW, but I prove a stronger claim. I prove that higher types are strictly worse off if they apply to submarkets designed for lower types. That is, downward IC cannot be binding. The reason that I get a stronger result is that I assume strict monotonicity for $v_i(a)$ in $i$ for every $a$ with $a \in \bar{A}$, while they just assume weak monotonicity.

Lemma 7. There exist $I^*(\epsilon) \subseteq \{1,2,\ldots,I\}$, $\{\bar{U}_i(\epsilon)\}_{i \in \{1,2,\ldots,I\}}$ and $\{(\bar{\theta}_i(\epsilon),\bar{a}_i(\epsilon),\bar{p}_i(\epsilon))\}_{i \in I^*(\epsilon)}$ that solve problem $P(\epsilon)$. Also, there exists $\bar{\epsilon}_2 > 0$ such that for every $\epsilon \in [0,\bar{\epsilon}_2)$, the following holds at any solution for Problem $P_i(\epsilon)$ for $i \in I^*(\epsilon)$:

$$q(\bar{\theta}_i(\epsilon))(v_i(\bar{a}_i(\epsilon)) - \bar{p}_i(\epsilon)) = k,$$

$$m(\bar{\theta}_i(\epsilon))(u_j(\bar{a}_i(\epsilon)) + \bar{p}_i(\epsilon)) + \delta_i < \bar{U}_j(\epsilon) \text{ for all } j > i,$$

where $\delta_i = \begin{cases} \epsilon & \text{if } i < \bar{i} \text{ or } i \notin I^*(0) \\ 0 & \text{otherwise} \end{cases}$.

Proof.

Part 1: Existence of a solution to Problem $P(\epsilon)$

First fix $\epsilon$ and set $I^*(\epsilon) = \emptyset$. For $i = 1$, the objective function is continuous and the constraint set is compact. If the constraint is empty, set $\bar{U}_1(\epsilon) = \epsilon$. Otherwise,
since the objective function is continuous and the constraint set is compact, $P_1(\epsilon)$ has a solution and a unique maximum. If the value of the maximum is less than $\epsilon$, again set $\bar{U}_1(\epsilon) = \epsilon$. Otherwise, denote by $(\bar{\theta}_1(\epsilon), \bar{a}_1(\epsilon), \bar{p}_1(\epsilon))$ one of the maximizers and add 1 to the set $I^*(\epsilon)$.

I proceed by induction. By induction hypothesis, I have found $(\bar{U}_1(\epsilon), \bar{U}_2(\epsilon), ..., \bar{U}_{i-1}(\epsilon))$ and also $(\bar{\theta}_i(\epsilon), \bar{a}_i(\epsilon), \bar{p}_i(\epsilon))$ for all $i \in I^*(\epsilon)$. Again, if the constraint is empty, set $\bar{U}_i(\epsilon) = \epsilon$. Otherwise, $\bar{U}_i(\epsilon)$ is well-defined and unique. If the value of the maximum is less than $\epsilon$, again set $\bar{U}_i(\epsilon) = \epsilon$. Otherwise, denote by $(\bar{\theta}_i(\epsilon), \bar{a}_i(\epsilon), \bar{p}_i(\epsilon))$ one of the maximizers and add $i$ to the set $I^*(\epsilon)$.

**Part 2: The first constraint in $P_i(\epsilon)$ (the free entry condition) is binding**

Assume by way of contradiction that the constraint is not binding for some $i \in I^*(\epsilon)$. First note that $\bar{\theta}_i(\epsilon) > 0$ because $\bar{U}_j(\epsilon) > \delta_j$ for all $j \in I^*(\epsilon)$. According to part 2 of Assumption 1 (sorting), for every $\tau > 0$, there exists an $a' \in B_\tau(\bar{a}_i(\epsilon))$ such that

$$u_i(a') > u_i(\bar{a}_i(\epsilon)) \quad (A.7)$$

and $u_j(a') < u_j(\bar{a}_i(\epsilon))$ for all $j < i$. \hspace{1cm} (A.8)

Set $\tau > 0$ sufficiently small such that $q(\bar{\theta}_i(\epsilon))(v_i(a') - \bar{p}_i(\epsilon)) \geq k$ for all $B_\tau(\bar{a}_i(\epsilon))$. Now consider $(\bar{\theta}_i(\epsilon), a', \bar{p}_i(\epsilon))$. The first constraint in Problem $P_i(\epsilon)$ is satisfied following the choice of $\tau$ and other constraints are satisfied because $m(\bar{\theta}_i(\epsilon))(u_j(a') + \bar{p}_i(\epsilon)) + \delta_i < m(\bar{\theta}_i(\epsilon))(u_j(\bar{a}_i(\epsilon)) + \bar{p}_i(\epsilon)) + \delta_i \leq \bar{U}_j(\epsilon)$ for all $j < i$. But the value of the objective function is now higher: $m(\bar{\theta}_i(\epsilon))(u_i(a') + \bar{p}_i(\epsilon)) + \delta_i > m(\bar{\theta}_i(\epsilon))(u_i(\bar{a}_i(\epsilon)) + \bar{p}_i(\epsilon)) + \delta_i$, which is a contradiction with $(\bar{\theta}_i(\epsilon), \bar{a}_i(\epsilon), \bar{p}_i(\epsilon))$ being a solution to problem $P_i(\epsilon)$.

**Part 3: Incentive compatibility for all types when $\epsilon = 0$**

Fix $i$ such that $i \in I^*(\epsilon)$. In this part, I show that incentive compatibility holds at $\epsilon = 0$, that is,

$$m(\bar{\theta}_i(0))(u_j(\bar{a}_i(0)) + \bar{p}_i(0)) < \bar{U}_j(0) \text{ for all } i \in I^*(0), j > i.$$ 

Assume by way of contradiction that there exists $n$ such that $n > i$ and $m(\bar{\theta}_i(0))(u_n(\bar{a}_i(0)) + \bar{p}_i(0)) \geq \bar{U}_n(0)$. Denote the smallest such $n$ by $h$. That
is,
\[
m(\bar{\theta}_i(0))(u_j(\bar{a}_i(0)) + \bar{p}_i(0)) < \bar{U}_j(0) \quad \text{for all } i \leq j < h,
\]
(A.9)
\[
m(\bar{\theta}_i(0))(u_h(\bar{a}_i(0)) + \bar{p}_i(0)) \geq \bar{U}_h(0).
\]
(A.10)

Now I show that \((\bar{\theta}_i(0), \bar{a}_i(0), \bar{p}_i(0))\) is feasible for problem \(P_h(0)\). The first constraint in problem \(P_h(0)\) is satisfied because
\[
q(\bar{\theta}_i(0))(v_h(\bar{a}_i(0)) - \bar{p}_i(0)) > q(\bar{\theta}_i(0))(v_i(\bar{a}_i(0)) - \bar{p}_i(0)) \geq k,
\]
where the first inequality follows from part 1 of Assumption 1 (strict monotonicity).\(^3\) Also, \(m(\bar{\theta}_i(0))(u_j(\bar{a}_i(0)) + \bar{p}_i(0)) \leq \bar{U}_j(0)\) holds true for any \(j \) with \(i < j < h\) according to Equation A.9, and holds true for any \(j\) with \(j \leq i\), because \((\bar{\theta}_i(0), \bar{a}_i(0), \bar{p}_i(0))\) is feasible for problem \(P_i(0)\).

According to part 2 of Assumption 1 (sorting), there exists \(b \in \bar{A}_i(0)\) sufficiently close to \(a_i\), such that \(q(\bar{\theta}_i(0))(v_h(b) - \bar{p}_i(0)) \geq k\), and
\[
u_h(b) > u_h(\bar{a}_i(0)),
\]
(A.11)
and \(u_j(b) < u_j(\bar{a}_i(0))\) for all \(j < h\).
(A.12)

Now, the claim is that \((\bar{\theta}_i(0), b, \bar{p}_i(0))\) is feasible for problem \(P_h(0)\) but delivers strictly higher utility for type \(h\). First, the first constraint is satisfied by choice of \(b\). Second, all incentive compatibility constraints are satisfied because \(m(\bar{\theta}_i(0))(u_j(b) + \bar{p}_i(0)) < m(\bar{\theta}_i(0))(u_j(\bar{a}_i(0)) + \bar{p}_i(0)) \leq \bar{U}_j(0)\) for all \(j < i\), where the weak inequality follows from the fact that \((\bar{\theta}_i(0), \bar{a}_i(0), \bar{p}_i(0))\) is feasible for problem \(P_i(0)\). The value of the objective function is greater than \(\bar{U}_h(0)\) because \(m(\bar{\theta}_i(0))(u_h(b) + \bar{p}_i(0)) > m(\bar{\theta}_i(0))(u_h(\bar{a}_i(0)) + \bar{p}_i(0)) \geq \bar{U}_h(0)\), which contradicts with \(\bar{U}_h(0)\) being a maximizer of \(P_h(0)\).

**Part 4: Existence of a neighborhood \([0, \epsilon_2]\) such that incentive compatibility for all types is satisfied if \(\epsilon \in [0, \epsilon_2]\)**

First of all, it is easy to see that similar to the argument in previous part, the

\(^3\)I show here that \(a_i(0) \in A\) so we can use strict monotonicity of \(v_i\). Since \((\bar{\theta}_i(0), \bar{a}_i(0), \bar{p}_i(0))\) is feasible for Problem \(P_i(0)\), so \(q(\bar{\theta}_i(0))(v_i(\bar{a}_i(0)) - \bar{p}_i(0)) \geq k\). But \(\bar{\theta}_i(0) \geq 0\) so \(q(0)(v_i(\bar{a}_i(0)) - \bar{p}_i(0)) \geq k\). Also \(m(\bar{\theta}_i(0))(u_i(\bar{a}_i(0)) + \bar{p}_i(0)) + \delta_i = U_i(0)\), but \(U_i(0) > \delta_i\) by construction of \(I^*(0)\), therefore \(u_i(\bar{a}_i(0)) + \bar{p}_i(0) \geq 0\). Hence \(\bar{a}_i(0) \in A\).
first constraint in Problem $P_i(\epsilon)$ must be binding. Therefore, we can eliminate $\bar{p}_i(\epsilon)$ to write the problem in the following form:

$$\max_{\theta \in [0, \infty), (\cdot, a) \in \bar{Y}} \{ m(\theta)(u_i(a) + v_i(a)) - k\theta + \delta_i \}$$

s. t. $m(\theta)(u_j(a) + v_i(a)) - k\theta + \delta_i \leq \bar{U}_j(\epsilon)$ for all $j < i$.

Our goal here is to apply theorem of the maximum to this problem and show that $\bar{U}_h(\epsilon)$ is continuous in $\epsilon$ and $\bar{a}_h(\epsilon), \bar{\theta}_h(\epsilon)$ and $\bar{p}_h(\epsilon)$ are all upper hemi-continuous in $\epsilon$. I proceed by induction on $h$.

For $h = 1$, it is trivial because the constraint set is independent of $\epsilon$. Also the constraint set is compact, due to the following reasons. With respect to $a$, note that $a \in \bar{A}$ and $\bar{A}$ is compact. Regarding $\theta$, I show that we can assume without loss of generality that $\theta$ lies in a closed interval which is a subset of $R_+$. Suppose by way of contradiction that $\theta$ is unbounded. Because $m(\theta) \leq 1$ and $u_i(a) + v_i(a)$ is bounded and $k > 0$, the objective function goes to $-\infty$. Therefore, we can restrict our attention to an interval $[0, M]$ for some $M \in R_+$. As a result, we can assume without loss of generality that $(\theta, a) \in [0, M] \times \bar{A}$.

The objective function is continuous in $(\theta, a)$ and $\epsilon$. Therefore, $\bar{U}_1(\epsilon)$ is continuous in $\epsilon$, and $\bar{a}_1(\epsilon), \bar{\theta}_1(\epsilon)$ and $\bar{p}_1(\epsilon)$ are all upper hemi-continuous in $\epsilon$.

Now consider $h > 1$. By induction hypothesis, $\bar{U}_j(\epsilon)$ are continuous in $\epsilon$ for all $j < h$, therefore, the constraint set is continuous in $\epsilon$ too. With a similar argument as above, we can conclude that the constraint set is a compact valued and continuous correspondence in $\epsilon$. Therefore, $\bar{U}_h(\epsilon)$ is continuous in $\epsilon$ and $\bar{a}_h(\epsilon), \bar{\theta}_h(\epsilon)$ and $\bar{p}_h(\epsilon)$ are all upper hemi-continuous in $\epsilon$ for all $h$. Both $\bar{\theta}_h(\epsilon)$ and $\bar{a}_h(\epsilon)$ are UHC in $\epsilon$, and $m(.)$ and $u_i(.)$ are continuous functions, therefore $m(\bar{\theta}_h(\epsilon))$ and $u_i(\bar{a}_h(\epsilon))$ are UHC in $\epsilon$. (See [45], Theorem 17.23.)

Define $e_{k,i}(\epsilon)$ as follows: $e_{k,i}(\epsilon) = \bar{U}_k(\epsilon) - m(\bar{\theta}_i(\epsilon))(u_k(\bar{a}_i(\epsilon)) + p_i(\epsilon)) - \delta_i$. Since $e_{k,i}(\epsilon)$ is just sum of some UHC correspondences, $e_{k,i}(\epsilon)$ itself is also UHC. But $e_{k,i}(\epsilon)|_{\epsilon=0} > 0$ according to part 3. I show below that because $e_{k,i}(\epsilon)$ is UHC in an interval close to 0 and its value at 0 is strictly positive, there must exist a neighborhood $[0, \epsilon_{k,i}]$ for some $\epsilon_{k,i} > 0$ such that $e_{k,i}(\epsilon)$ is strictly positive, too. Now, set

$$\bar{\epsilon}_2 = \min_{i, k > i} \epsilon_{k,i}.$$
That is, there exists a neighborhood \([0, \bar{\epsilon}_2]\) such that if \(\epsilon \in [0, \bar{\epsilon}_2]\), then higher types are strictly worse off by reporting a lower type.

To show that for any \(i\) and \(k > i\) there must exist a neighborhood \([0, \epsilon_{k,i}]\) for some \(\epsilon_{k,i} > 0\) such that \(e_{k,i}(\epsilon)\) is strictly positive, suppose by way of contradiction that there does not exist such a neighborhood. That is, there exists \(i\) and \(k > i\) such that for any \(\epsilon > 0\), there exists function \(\tilde{e}(\epsilon) \in e_{k,i}(\epsilon)\) with \(\tilde{e}(\epsilon) \leq 0\). Consider \(\{\epsilon_n\}_{n \in \mathbb{N}}\) where \(\epsilon_n = \frac{1}{n}\). Since \(e_{k,i}(\cdot)\) is UHC and because \(\epsilon_n \to 0\), there exists a convergent sub-sequence \(\{\tilde{e}_n\}_{n \in \mathbb{N}}\) of \(\{\tilde{e}(\epsilon_n)\}_{n \in \mathbb{N}}\) such that its limit point is in \(e_{k,i}(0)\). This is a contradiction, because \(e_{k,i}(0) > 0\) but \(\tilde{e}_n \leq 0\) for all \(n\), so its limit point cannot be a strictly positive number.

The proof is complete because I have shown that there exists a \(\bar{\epsilon}_2\) such that \(e(\epsilon) > 0\) for any \(\epsilon \in [0, \bar{\epsilon}_2]\). Therefore, all incentive compatibility constraints are satisfied.

\[\blacksquare\]

### A.3 Proof of Theorem 2

**Proof.** I will construct a feasible mechanism in which type \(i\) sellers get matched with probability \(m(\theta_i^{FB})\) and produce \(a_i^{FB}\). Under part 5 (a) or 5 (b) of Assumption 2, it can be easily shown that \(U_i^{FB}\) is increasing in \(i\). Let \(\hat{i}\) denote the highest type of sellers without gains from trade. Then all types 1, 2, ..., \(\hat{i}\) are inactive, that is, they are matched with probability 0. Given this observation, I assume that there are positive gains from trade for all types and then I construct a feasible mechanism that achieves the first best for all these types. If there are not positive gains from trade for some types, the same construction method with little adjustments can be used to establish the proof.

Consider the following direct mechanism:

\[
\{(a_i^{FB}, \tilde{p}_i, \tilde{s}_i, \theta_i^{FB})\}_{i \in \{1, 2, ..., I\}},
\]

where \(\tilde{p}_1 = -u_1(a_1^{FB})\) and \(\tilde{p}_i\) is defined for \(i \geq 2\) recursively as follows:

\[
m(\theta_i^{FB})(\tilde{p}_i + u_i(a_i^{FB})) = m(\theta_i^{FB})(\tilde{p}_{i-1} + u_i(a_{i-1}^{FB})). \tag{A.13}
\]

Also, \(\tilde{s}_i = \tilde{s} \equiv \sum_{j=1}^I \pi_j \left[ m(\theta_j^{FB})(v_j(a_j^{FB}) - \tilde{p}_j) - k\theta_j^{FB} \right]\) for all \(i\). I just need to
show that conditions for feasibility are satisfied.

Incentive Compatibility of Sellers

I prove that this condition is satisfied in four steps:

**Step 1:** \((\theta_i^{FB}, a_i^{FB})\) is increasing in \(i\).

I use Assumption 2. First, \(a_i^{FB}\) is increasing in \(i\) because \(u_i(a) + v_i(a)\) satisfies increasing differences property in \((a, i)\) and also because \(u_i(a) + v_i(a)\) is supermodular in \(a\). (See Theorem 5 in [23]). Furthermore, \(m(\theta)(u_i(a_i^{FB}) + v_i(a_i^{FB})) - k\theta\) satisfies increasing differences property in \((\theta, i)\), because \(m\) is increasing, because \(u_i(a) + v_i(a)\) is increasing in \(i\) and because \(a_i^{FB}\) is increasing in \(i\). Hence, \(\theta_i^{FB}\) is increasing in \(i\).

**Step 2:** Local IC constraints are satisfied.

In equation A.13, \(\tilde{p}_i\) is set such that all local downward incentive compatibility constraints are satisfied and binding. That is, for all \(i \geq 2\) type \(i\) is indifferent between reporting \(i\) and \(i - 1\). Now, I show that sellers’ maximization constraint is satisfied.

First, I show that type \(i - 1\) weakly prefers to report \(i - 1\) over \(i\) (local upward incentive compatibility). That is,

\[
m(\theta_i^{FB})(\tilde{p}_i + u_{i-1}(a_i^{FB})) \leq m(\theta_{i-1}^{FB})(\tilde{p}_{i-1} + u_{i-1}(a_{i-1}^{FB})). \tag{A.14}
\]

I begin from the left hand side:

\[
m(\theta_i^{FB})(\tilde{p}_i + u_{i-1}(a_i^{FB})) = m(\theta_i^{FB})(\tilde{p}_i + u_i(a_i^{FB})) - m(\theta_i^{FB})(u_i(a_i^{FB}) - u_{i-1}(a_i^{FB}))
\]

\[
= m(\theta_{i-1}^{FB})(\tilde{p}_{i-1} + u_{i-1}(a_{i-1}^{FB})) + m(\theta_i^{FB})(u_i(a_i^{FB}) - u_{i-1}(a_{i-1}^{FB})) - m(\theta_i^{FB})(u_i(a_i^{FB}) - u_{i-1}(a_i^{FB}))
\]

\[
\leq m(\theta_{i-1}^{FB})(\tilde{p}_{i-1} + u_{i-1}(a_{i-1}^{FB})) + m(\theta_{i-1}^{FB})(u_i(a_i^{FB}) - u_{i-1}(a_{i-1}^{FB}) - u_i(a_i^{FB}) + u_{i-1}(a_i^{FB}))
\]

The first equality follows from the construction of \(\tilde{p}_i\) (Equation A.13). The first inequality follows from the fact that \(\theta_i\) and \(u_i(\cdot)\) are both increasing in \(i\). The
second inequality follows from increasing differences property of \(u\) in \((a, i)\) and also from the fact that \(a_{i-1}^{FB} \leq a_i^{FB}\) (component by component).

Second, I calculate \(\tilde{p}_i\) in terms of \(\bar{p}_1\):

\[
m(\theta_i^{FB})(\tilde{p}_i + u_i(a_i^{FB})) = m(\theta_{i-1}^{FB})(\bar{p}_{i-1} + u_i(a_{i-1}^{FB})) = m(\theta_{i-1}^{FB})(\tilde{p}_{i-1} + u_{i-1}(a_{i-1}^{FB})) + m(\theta_{i-1}^{FB})(u_i(a_i^{FB}) - u_{i-1}(a_{i-1}^{FB})) = m(\theta_{i-1}^{FB})(\tilde{p}_{i-1} + u_{i-1}(a_{i-1}^{FB})) + K_i(\theta_{i-1}^{FB}, a_{i-1}^{FB}) - K_{i-1}(\theta_{i-1}^{FB}, a_{i-1}^{FB}), \quad (A.15)
\]

where \(K_i(\theta, a)\) is defined as follows:

\[
K_i(\theta, a) \equiv m(\theta)u_i(a).
\]

Using telescoping technique yields the following equation for all \(i \geq 2\):

\[
m(\theta_i^{FB})(\tilde{p}_i + u_i(a_i^{FB})) = m(\theta_1^{FB})(\tilde{p}_1 + u_1(a_1^{FB}))+\sum_{j=2}^{i}[K_j(\theta_j^{FB}, a_j^{FB})-K_{j-1}(\theta_{j-1}^{FB}, a_{j-1}^{FB})]. \quad (A.16)
\]

Step 3: Other upward IC constraints are satisfied.

Now, I show that for all \(i\) and \(k\) with \(k \leq i - 1\), type \(k\) does not gain by reporting \(i\), that is,

\[
m(\theta_k^{FB})(\tilde{p}_k + u_k(a_k^{FB})) \geq m(\theta_i^{FB})(\tilde{p}_i + u_i(a_i^{FB})).
\]

Note that

\[
m(\theta_k^{FB})(\tilde{p}_k + u_k(a_k^{FB})) - m(\theta_i^{FB})(\tilde{p}_i + u_i(a_i^{FB})) = \sum_{j=k}^{i-1} \left[ m(\theta_j^{FB})(\tilde{p}_j + u_j(a_j^{FB})) - m(\theta_{j+1}^{FB})(\tilde{p}_{j+1} + u_j(a_{j+1}^{FB})) + m(\theta_{j+1}^{FB})u_j(a_{j+1}^{FB}) - m(\theta_j^{FB})u_j(a_j^{FB}) - m(\theta_{j+1}^{FB})u_k(a_{j+1}^{FB}) + m(\theta_j^{FB})u_k(a_j^{FB}) \right] \\
\geq \sum_{j=k}^{i-1} \left[ m(\theta_{j+1}^{FB})u_j(a_{j+1}^{FB}) - m(\theta_j^{FB})u_j(a_j^{FB}) - m(\theta_{j+1}^{FB})u_k(a_{j+1}^{FB}) + m(\theta_j^{FB})u_k(a_j^{FB}) \right]
\]

108
\[ \sum_{j=k}^{i-1} \left[ m(\theta^F_j)(u_j(a^F_{j+1}) - u_k(a^F_{j+1}) - u_j(a^F_j) + u_k(a^F_j)) \right] \geq 0. \]

The first equality is derived by doing some algebra and using telescoping technique. The first inequality uses the fact that type \( i - 1 \) weakly prefers to report \( i - 1 \) over \( i \) (See equation A.14). The second inequality uses \( \theta^F_{j+1} \geq \theta^F_j \) and also the fact that \( u_i \) is increasing in \( i \) for \( a \in \bar{A} \). The last inequality is the implication of the increasing differences property of \( u \) (part 1 of Assumption 1) and the fact that \( a^F_{j+1} \geq a^F_j \).

**Step 4: Other downward IC constraints are satisfied.**

Again, I show that type \( k \) does not gain by reporting \( i \). If \( i + 1 \leq k \), I use the same technique as above:

\[ m(\theta^F_k)(\bar{p}_k + u_k(a^F_k)) - m(\theta^F_i)(\bar{p}_i + u_k(a^F_i)) \]
\[ = \sum_{j=i+1}^{k} \left[ m(\theta^F_j)(\bar{p}_j + u_j(a^F_j)) - m(\theta^F_{j-1})(\bar{p}_{j-1} + u_j(a^F_j)) \right] \]
\[ + m(\theta^F_{j-1})u_j(a^F_{j-1}) - m(\theta^F_j)u_j(a^F_j) - m(\theta^F_{j-1})u_k(a^F_{j-1}) + m(\theta^F_j)u_k(a^F_j) \]
\[ \geq \sum_{j=i+1}^{k} \left[ m(\theta^F_{j-1})u_j(a^F_{j-1}) - m(\theta^F_j)u_j(a^F_j) \right] \]
\[ - m(\theta^F_{j-1})u_k(a^F_{j-1}) + m(\theta^F_j)u_k(a^F_j) \]
\[ \geq \sum_{j=i+1}^{k} \left[ m(\theta^F_{j-1})(u_j(a^F_{j-1}) - u_k(a^F_{j-1}) - u_j(a^F_j) + u_k(a^F_j)) \right] \geq 0. \]

The first inequality is again derived by using telescoping technique. The first inequality follows from construction of \( \bar{p}_i \). The second inequality uses \( \theta^F_{j+1} \geq \theta^F_j \) and also the fact that \( u_i \) is increasing in \( i \) for every \( a \). The last inequality is the implication of increasing differences property of \( u \) in \((a,i)\) and the fact that \( a^F_{j+1} \geq a^F_j \).

**Participation constraint**

To show \( U_i = m(\theta^F_i)(\bar{p}_i + u_i(a^F_i)) + \bar{s}_i \geq 0 \) for all \( i \), consider Equation A.16.
The first term in the right hand side is zero following the construction of \( \tilde{p}_1 \). The summation is positive following the assumption that \( u_i \) is increasing in \( i \) for every \( a \). Also, I show below that \( \tilde{s} \) is always positive.

\[
\tilde{s} = \sum \pi_i \left[ m(\theta_i^{FB})(v_i(a_i^{FB}) - \tilde{p}_i) - k\theta_i^{FB} \right] \\
= \sum_{i=1}^{I} \pi_i \left[ m(\theta_i^{FB})(v_i(a_i^{FB}) + u_i(a_i^{FB})) - k\theta_i^{FB} \right] - \sum_{i=1}^{I} \pi_i \left[ m(\theta_i^{FB})(\tilde{p}_i + u_i(a_i^{FB})) \right] \\
= \sum_{i=1}^{I} \pi_i \left[ m(\theta_i^{FB})(v_i(a_i^{FB}) + u_i(a_i^{FB})) - k\theta_i^{FB} \right] \\
- \sum_{i=1}^{I} \pi_i \left[ \sum_{j=2}^{i} \left[ K_j(\theta_j^{FB} - \theta_{j-1}^{FB}, a_j - a_{j-1}) \right] \right] - m(\theta_1)(\tilde{p}_1 + u_1(a_1)). \quad (A.17)
\]

For the fourth equality, I used the definition of \( \tilde{p}_i \) from Equation A.16.

**First, suppose part 5(a) of Assumption 2 holds.**

To prove that the participation constraint is satisfied, it is sufficient to show that the right hand side of Equation A.17 is positive for all \( i \). I proceed with induction on \( i \). If \( i = 1 \), the right hand side of the equation is equal to 0 by the choice of \( \tilde{p}_1 \). For \( i = 2 \):\(^4\)

\[
m(\theta_2)(v_2(a_2) + u_2(a_2)) - k\theta_2 \geq m(\theta_1)(v_2(a_1) + u_2(a_1)) - k\theta_1 \\
\geq m(\theta_1)(u_2(a_1) - u_1(a_1)) + m(\theta_1)(v_2(a_1) + u_1(a_1)) - k\theta_1 \\
\geq m(\theta_1)(u_2(a_1) - u_1(a_1)) + m(\theta_1)(v_1(a_1) + u_1(a_1)) - k\theta_1 + \\
\geq m(\theta_1)(u_2(a_1) - u_1(a_1)) + m(\theta_1)(u_2(a_1) + \tilde{p}_1)
\]

The first inequality holds true due to the fact that \( \theta_1 \) and \( a_1 \) are feasible for the second type maximization problem \( \max_{\theta,a} \{ m(\theta)(v_2(a) + u_2(a)) - k\theta \} \). The second inequality holds because \( v_i(.) \) is increasing in \( i \). The last inequality holds due to the construction of \( \tilde{p}_1 \).

Now assume that the induction hypothesis for type \( i - 1 \) is correct. Then, I

\(^4\)Establishing the claim for \( i = 2 \) is redundant, but I just do it here to make clear the main idea used in the general case (for \( i > 1 \).
show that the hypothesis will be correct for type $i$ as well. Let me remind you that the induction hypothesis states that $m(\theta_i^{FB})(v_i(a_i^{FB}) + u_i(a_i^{FB})) - k\theta_i^{FB} \geq m(\theta_1)(\bar{p}_1 + u_1(a_1)) + \sum_{j=2}^{i} [K_j(\theta_{j-1}^{FB}, a_{j-1}) - K_{j-1}(\theta_{j-1}^{FB}, a_{j-1})]$. 

$m(\theta_i^{FB})(v_i(a_i^{FB}) + u_i(a_i^{FB})) - k\theta_i^{FB} \geq m(\theta_i^{FB})(v_i(a_i^{FB}) + u_i(a_i^{FB})) - k\theta_i^{FB}$

$= m(\theta_{i-1}^{FB})(v_i(a_{i-1}^{FB}) + u_{i-1}(a_{i-1}^{FB})) - k\theta_{i-1}^{FB} + m(\theta_i^{FB})(u_i(a_i^{FB}) - u_{i-1}(a_{i-1}^{FB}))$

$\geq m(\theta_{i-1}^{FB})(v_{i-1}(a_{i-1}^{FB}) + u_{i-1}(a_{i-1}^{FB})) - k\theta_{i-1}^{FB} + m(\theta_i^{FB})(u_i(a_i^{FB}) - u_{i-1}(a_{i-1}^{FB}))$

$\geq m(\theta_1)(\bar{p}_1 + u_1(a_1)) + \sum_{j=2}^{i-1} [K_j(\theta_{j-1}^{FB}, a_{j-1}) - K_{j-1}(\theta_{j-1}^{FB}, a_{j-1})]$

$+ K_i(\theta_i^{FB}, a_i^{FB}) - K_{i-1}(\theta_{i-1}^{FB}, a_{i-1}^{FB})$

$= m(\theta_1)(\bar{p}_1 + u_1(a_1)) + \sum_{j=2}^{i} [K_j(\theta_{j-1}^{FB}, a_{j-1}) - K_{j-1}(\theta_{j-1}^{FB}, a_{j-1})]$

Similar to the case for $i = 2$, the first inequality holds because $\theta_i^{FB}$ and $a_{i-1}^{FB}$ are feasible for type $i$ maximization problem (max$_{\theta,a}\{m(\theta)(v_i(a) + u_i(a)) - k\theta\}$). The second inequality holds because $v_i$ is increasing in $i$. The last inequality holds due to the induction hypothesis.

**Second, suppose parts 5(b) and 5(c) of Assumption 2 hold, instead.**

Here, I cannot show that the terms inside the sigma in Equation A.17 are positive for each $i$. Rather, I need to algebraically simplify the right hand side of Equation A.17 as follows. To simplify the notation, I use $\Delta = \sum_{k=1}^{I} \pi_k$. 

$$\tilde{s} = \sum \pi_i \left[ m(\theta_i^{FB})(v_i(a_i^{FB}) - \bar{p}_i) - k\theta_i^{FB} \right]$$

$$= \sum_{i=1}^{I} \pi_i \left[ m(\theta_i^{FB})(v_i(a_i^{FB}) + u_i(a_i^{FB})) - k\theta_i^{FB} \right] - \sum_{i=1}^{I} \pi_i \left[ m(\theta_i^{FB})(\bar{p}_i + u_i(a_i^{FB})) \right]$$

$$= \sum_{i=1}^{I} \pi_i \left[ m(\theta_i^{FB})(v_i(a_i^{FB}) + u_i(a_i^{FB})) - k\theta_i^{FB} \right]$$

$$- \sum_{i=1}^{I} \pi_i \left[ \sum_{j=2}^{i} [K_j(\theta_{j-1}^{FB}, a_{j-1}) - K_{j-1}(\theta_{j-1}^{FB}, a_{j-1})] \right] - m(\theta_1)(\bar{p}_1 + u_1(a_1))$$

111
\[
= \sum_{i=1}^{I} \pi_i \left[ m(\theta_i^{FB})(v_i(a_i^{FB}) + u_i(a_i^{FB})) - k\theta_i^{FB} \right] \\
- \sum_{j=2}^{I} \left[ \pi_j \left( K_j(\theta_{j-1}, a_{j-1}) - K_{j-1}(\theta_{j-1}, a_{j-1}) \right) \frac{\Delta_j}{\pi_j} \right] - m(\theta_1)(\tilde{p}_1 + u_1(a_1)) \\
= \sum_{i=2}^{I} \left[ m(\theta_i^{FB})(v_i(a_i^{FB}) + u_i(a_i^{FB})) - k\theta_i^{FB} - \left( K_j(\theta_{j-1}, a_{j-1}) - K_{j-1}(\theta_{j-1}, a_{j-1}) \right) \frac{\Delta_j}{\pi_i} \right] \\
+ \pi_1 U_1^{FB} - m(\theta_1)(\tilde{p}_1 + u_1(a_1)) \geq 0.
\]

For the second equality, I changed the order of summations for the double sigma term and then used the definition of \( \Delta_i \). I used Assumption 2 part 3 and choice of \( \tilde{p}_1 \) to establish the last inequality.

**Budget constraint**

This condition is trivially satisfied due to the construction of \( \tilde{s} \).

The proposed allocation achieves the maximum among all feasible allocations, because the level of \( \theta \) and \( a \) assigned to every type is exactly equal to that under the first best, so it is not possible to increase the value of the objective function any more. This concludes the proof.

### A.4 Proofs of the Asset Market with Lemons

**Proof of Asset market with lemons if \( \pi_1b_1 + \pi_2b_2 \geq c_2 \).**

Consider the following direct mechanism: \( \{(\alpha_i, p_i, s_i, \theta_i)\}_{i \in \{1,2\}} \) with \( \alpha_i = 1, p_i = \pi_1 b_1 + \pi_2 b_2, \tilde{s}_i = 0, \theta_i = 1 \) for all \( i \). I suppress \( \tilde{\cdot} \) for the proofs in the asset market with lemons to make the notation simpler.

Incentive compatibility of sellers is clearly satisfied, because both types get the same \( (\alpha_i, p_i, s_i, \theta_i) \). Also, both types get a positive payoff, so participation constraint of sellers is also obviously satisfied. Planner’s budget-balance is also trivially satisfied. The objective function is maximized because the \( \theta \) and \( \alpha \) allocated to both types is the same as what they get under complete information. The proof is complete.

**Proof of Asset market with lemons when \( \pi_1b_1 + \pi_2b_2 \leq c_2 \).**

Here the first best is not achievable through a pooling allocation, because type
two gets a strictly negative payoff in the pooling allocation, therefore, pooling allocation is not feasible. If $b_2 - c_2$ is greater than $b_1 - c_1$, part 5(b) of Assumption 2 is violated. If $b_2 - c_2$ is less than or equal to $b_1 - c_1$, then it is easy to check that although part 5(b) is satisfied, part 5(c) is violated. Therefore, it is not possible to use Theorem 2. Hence, I need to solve the planner’s problem completely by taking all constraints into account. I proceed in 6 steps. In the first step, I use a direct mechanism to write down the planner’s problem. In the second step, I show that the market tightness for both types must be strictly positive. In the third step, I show that the market tightness for both types must be less than or equal to 1. In the fourth step, I show that $\alpha$ (probability that the seller gives the asset to the buyer) for both types must be equal to 1. In the fifth step, I show that market tightness for type one must be equal to 1. In the last step, I calculate the market tightness for type two. This will conclude the characterization of the constrained efficient mechanism.

**Step 1: Formulating the problem and simplifying it**

Let $\{(\alpha_i, p_i, s_i, \theta_i)\}_{i \in \{1, 2\}}$ denote the allocation with the direct mechanism. For now, I assume $s_i = 0$ for all $i$. In the proof, since there are positive gains from trade for both types, I can show that both types must be active. Therefore, the assumption that $s_i = 0$ is without loss of generality. That is, if $s_i \neq 0$ for some $i$, we can change $p_i$ to $p_i + \frac{s_i}{m(\theta_i)}$ and set $s_i = 0$. Therefore, the planner’s problem can be written as follows:

**Problem 6** (Asset market with lemons, 1).

$$\max \sum_{i \in \{1, 2\}} \pi_i \min\{\theta_i, 1\}(p_i - \alpha_i c_i),$$

subject to

$$\min\{\theta_1, 1\}(p_1 - \alpha_1 c_1) \geq \min\{\theta_2, 1\}(p_2 - \alpha_2 c_1) \quad (IC-12),$$

$$\min\{\theta_2, 1\}(p_2 - \alpha_2 c_2) \geq \min\{\theta_1, 1\}(p_1 - \alpha_1 c_2) \quad (IC-21),$$

$$\min\{\theta_1, 1\}(p_1 - \alpha_1 c_1) \geq 0 \quad (IR-1),$$

$$\min\{\theta_2, 1\}(p_2 - \alpha_2 c_2) \geq 0 \quad (IR-2),$$
\[ \sum_{i=1}^{2} \pi_i \min\{\theta_i, 1\}(\alpha_i h_i - p_i) - k\theta_i \geq 0 \quad (BB). \]

In the first (second) line, I ensure that type one (two) does not want to report type two (one). I call this constraint IC-12 (IC-21). In the third (fourth) line, I ensure that type one (two) gets a strictly positive payoff. I call this constraint IR-1 (IR-2). The last line is planner’s budget constraint.

Notice that the planner’s budget constraint must be binding. If not binding, we can distribute the extra resources in a lump sum way and identically among both types to increase the value of the objective function, while keeping all other constraints satisfied. Therefore, we can write from BB that

\[ \sum_{i=1}^{2} \pi_i \min\{\theta_i, 1\}(\alpha_i h_i - p_i) = \sum_{i=1}^{2} \pi_i \min\{\theta_i, 1\} \alpha_i h_i - k\theta_i. \]

Hence we can write the objective function as

\[ \sum_{i=1}^{2} \pi_i \min\{\theta_i, 1\} \alpha_i (h_i - c_i) - k\theta_i. \]

**Step 2: \( \theta_1 > 0 \) and \( \theta_2 > 0 \)**

If both \( \theta_1 \) and \( \theta_2 \) are 0 then the welfare level equals 0. But this is not possible because we know at least that equilibrium allocation is feasible and delivers strictly positive utility. To rule out the case that one of them is 0, note that IC-12 and IC-21 together imply that:

\[ (m(\theta_1)\alpha_1 - m(\theta_2)\alpha_2)c_1 \leq m(\theta_1)p_1 - m(\theta_2)p_2 \leq (m(\theta_1)\alpha_1 - m(\theta_2)\alpha_2)c_2. \quad (A.18) \]

But \( c_1 < c_2 \), therefore,

\[ m(\theta_1)\alpha_1 \geq m(\theta_2)\alpha_2. \quad (A.19) \]

If \( \theta_1 = 0 \), then \( \theta_2 \) must be 0 as well, and this leads to 0 level of welfare. Nevertheless, this cannot be part of a planner’s allocation, given the fact that the equilibrium allocation is feasible and delivers strictly positive welfare. Thus \( \theta_1 > 0 \). If \( \theta_2 = 0 \), then it is easy to check that the maximum possible welfare in this case (even if \( \theta_1 = 1 \)) is less than the level of welfare under the proposed solution. Therefore, \( \theta_2 > 0 \).

Let \( r_i \equiv \min\{\theta_i, 1\}p_i \) for all \( i \). For any \( \theta_i \) and \( r_i \in \mathbb{R} \), we can find a unique \( p_i \in \mathbb{R} \) which solves the maximization problem. From now on, we work with \( r_i \) instead of \( p_i \) because it simplifies the analysis. Therefore, we can rewrite the
Problem as follows:

**Problem 7** (Asset market with lemons, 2).

\[
\max_{\{\theta_i, \alpha_i, r_i\}\}_{i=1,2} \sum_{i=1}^{2} \pi_i (\min\{\theta_i, 1\} \alpha_i (h_i - c_i) - k\theta_i),
\]

subject to

\[
\begin{align*}
 r_1 - \min\{\theta_1, 1\} \alpha_1 c_1 &\geq r_2 - \min\{\theta_2, 1\} \alpha_2 c_1 \quad (IC-12), \\
r_2 - \min\{\theta_2, 1\} \alpha_2 c_2 &\geq r_1 - \min\{\theta_1, 1\} \alpha_1 c_2 \quad (IC-21), \\
r_1 - \min\{\theta_1, 1\} \alpha_1 c_1 &\geq 0 \quad (IR-1) \\
r_2 - \min\{\theta_2, 1\} \alpha_2 c_2 &\geq 0 \quad (IR-2) \text{ and } \\
\sum_{i=1}^{2} \pi_i (\min\{\theta_i, 1\} \alpha_i h_i - k\theta_i - r_i) &= 0 \quad (BB).
\end{align*}
\]

**Step 3:** \(\theta_1 \leq 1\) and \(\theta_2 \leq 1\)

Suppose \(\theta_i > 1\) for some \(i\). Then consider the following: \(\theta_i' = 1, r_i' = r_i + k(\theta_i - 1)\pi_i\) and \(r_j' = r_j + k(\theta_i - 1)\pi_i\) where \(j \neq i\). Therefore, if I replace \(\theta_i, r_1\) and \(r_2\) by \(\theta_i', r_1'\) and \(r_2'\) respectively, I can increase the value of the objective function by \(k(\theta_i - 1)\). Also, the new solution satisfies all the constraints because of the following: Obviously, IC-12 and IC-21 are still satisfied, because the change in \(r_1\) is the same as the change in \(r_2\) and also \(\min\{\theta_1, 1\}\) and \(\min\{\theta_2, 1\}\) have not changed. IR-1 and IR-2 are satisfied because \(r_1' > r_1\) and \(r_2' > r_2\). BB is also satisfied by construction of \(r_1'\) and \(r_2'\). A contradiction. Therefore, for all \(i \in \{1, 2\}\), \(\theta_i \leq 1\).

**Step 4:** \(\alpha_1 = \alpha_2 = 1\)

Suppose \(\alpha_i < 1\) for some \(i\). Let \(\alpha_i'\) be defined such that \(\alpha_i'(\theta_i - \epsilon)\) equals \(\alpha_i\theta_i\), where \(0 < \epsilon < \theta_i(1-\alpha_i)\). Fix \(\epsilon\) and consider the following: \(\theta_i' = \theta_i - \epsilon, r_i' = r_i + k\epsilon\pi_i\) and \(r_j' = r_j + k\epsilon\pi_i\) where \(j \neq i\).

Now, if I replace \(\alpha_i, \theta_i, r_1\) and \(r_2\) by \(\alpha_i', \theta_i', r_1'\) and \(r_2'\) respectively, I can increase the value of the objective function by \(k\epsilon\). I show that the new solution satisfies all the constraints because of the following: Obviously, IC-12 and IC-21 are still satisfied, because \(\min\{\theta_i, 1\}\alpha_i = \min\{\theta_i', 1\}\alpha_i'\). IR-1 and IR-2 are satisfied
because \( r'_1 > r_1 \) and \( r'_2 > r_2 \). BB is also satisfied by construction of \( r'_1 \) and \( r'_2 \). A contradiction. Therefore, for all \( i \in \{1, 2\} \), \( \alpha_i = 1 \).

For simplicity, I write the planner’s problem again incorporating the results so far:

**Problem 8** (Asset market with lemons, 3).

\[
\max \{ \pi_i (\theta_i (h_i - c_i) - k \theta_i) \mid \theta_i, r_i \}_{i=1,2} \sum_{i=1}^2 \pi_i (\theta_i (h_i - c_i) - k \theta_i),
\]

subject to

\[
\begin{align*}
    r_1 - \theta_1 c_1 & \geq r_2 - \theta_2 c_1 \quad (IC-12), \\
    r_2 - \theta_2 c_2 & \geq r_1 - \theta_1 c_2 \quad (IC-21), \\
    r_1 - \theta_1 c_1 & \geq 0 \quad (IR-1), \\
    r_2 - \theta_2 c_2 & \geq 0 \quad (IR-2), \\
    \sum_{i=1}^2 \pi_i (\theta_i h_i - k \theta_i - r_i) &= 0 \quad (BB).
\end{align*}
\]

**Step 5: \( \theta_1 = 1 \)**

First note that \( \theta_1 \geq \theta_2 \) following Equation A.19 and because \( \alpha_1 = \alpha_2 = 1 \) according to step 4. By way of contradiction, assume that \( \theta_1 < 1 \) at a solution. I consider two cases. First, assume that IR-2 is not binding. I propose the following: \( \theta'_i = \theta_i + \epsilon \) for all \( i \) where \( \epsilon \in (0, \min \{1 - \theta_1, \frac{r_2 - \theta_2 c_2}{\pi_2 - \pi_1 b_1 - \pi_2 b_2}\}) \). Fix \( \epsilon \) and let \( r'_i = r_i + (\pi_1 b_1 + \pi_2 b_2) \epsilon \) for all \( i \). It is easy to check that all constraints are satisfied, but the value of the objective function now has increased by \((\pi_1 (b_1 - c_1) + \pi_2 (b_2 - c_2)) \epsilon\), a contradiction. Note that I used Equation A.19 to ensure that \( \theta_2 + \epsilon < 1 \).

Second, assume that IR-2 is binding. I propose the following: \( \theta'_i = \theta_i + \epsilon \) where \( \epsilon < 1 - \theta_1 \), \( r'_1 = r_1 + b_1 \epsilon \). It is again easy to check that all constraints are satisfied. The only tricky thing here is to check that IC-21 is satisfied. But the LHS in IC-21 is fixed. The RHS increases by \( \epsilon (b_1 - c_2) \) which is a negative number, so IC-21 is not violated. (Note that \( b_1 - c_2 < 0 \), otherwise \( \pi_1 b_1 + \pi_2 b_2 > \pi_1 c_2 + \pi_2 c_2 = c_2 \) which contradicts the initial assumption that \( \pi_1 b_1 + \pi_2 b_2 < c_2 \)). But the value of the objective function now has increased by \( \pi_1 b_1 \epsilon \), a contradiction.
Step 6: Calculating $\theta_2$ and the rest of unknowns

I write $r_1$ from the budget constraint in terms of other variables, specially $r_2$:

$$r_1 = b_1 + \frac{\pi_2}{\pi_1} \theta_2 b_2 - \frac{\pi_2}{\pi_1} r_2$$

(A.20)

Now, one can write Equation A.18 as follows after replacing $r_1$ from the above equation:

$$(1 - \theta_2) c_1 \leq b_1 + \frac{\pi_2}{\pi_1} \theta_2 b_2 - \frac{r_2}{\pi_1} \leq (1 - \theta_2) c_2.$$  

(A.21)

First, note that IR-1 is implied by IC-12 and IR-2. Second, I argue that IR-2 must be binding at the solution. By way of contradiction, suppose not. Then only Equation A.21 is sufficient to determine $\theta_2$. But in order to maximize the objective function, I need to choose the highest possible $\theta_2$ consistent with Equation A.21, which is $\theta_2 = 1$. But according to equation A.21, $r_2 = \pi_1 b_1 + \pi_2 b_2$ and $r_2 > c_2$ from IR-2, which is a contradiction with $\pi_1 b_1 + \pi_2 b_2 < c_2$. Therefore, IR-2 is binding.

Third, since IR-2 is binding, I replace $r_2$ by $\theta_2 c_2$ and rewrite equation A.21 again:

$$(1 - \theta_2) c_1 \leq b_1 + \theta_2 c_2 (\pi_2 b_2 - c_2) \leq (1 - \theta_2) c_2.$$  

(A.22)

Now, it is easy to see that the right inequality in A.22 is satisfied for any $\theta_2 \in [0, 1]$, because $b_1 < c_2$. In order to maximize the objective function, I need to find the maximum value for $\theta_2$ under which the left inequality in Equation A.22 is satisfied ($((1 - \theta_2) c_1 \leq b_1 + \frac{\theta_2}{\pi_1} (\pi_2 b_2 - c_2))$. This implies that

$$\theta_2 = \frac{\pi_1 (b_1 - c_1)}{c_2 - \pi_2 b_2 - \pi_1 c_1}.$$  

The proof is complete, because I have found the values for $\alpha_i$, $\theta_i$ and $r_i$. I can calculate values of $p_i$ from $\theta_i$ and $r_i$ and check that they are the same as them in Table 1.1. Note that $t_i$ in Table 1.1 is calculated such that buyers’ free entry and zero profit condition is satisfied for each submarket in the decentralized economy.

\[\square\]

What if there are no gains from trade for some types?

Because the proof is similar to the the previous proof up to step 5, I do not repeat
those steps here, so I begin from Problem 8. I want to show that \( \theta_1 = \theta_2 = 0 \) at the solution. First note that Equation A.19, implies that \( \theta_1 \geq \theta_2 \). Also note that IR-1 is implied by IC-12 and IR-2, so we ignore IR-1. If \( \theta_1 = 0 \), then \( 0 \leq \theta_2 \leq \theta_1 = 0 \) and the proof is complete. Therefore, by way of contradiction assume that \( \theta_1 > 0 \). In the first step below, I show that \( \theta_2 < \theta_1 \) (with strict inequality). In the second step, I show that IC-12 is not binding. Then I propose a new set of \( \{(\theta_1', \theta_2'), (\theta_2, r_2)\} \) such that all constraints are satisfied, but the value of the objective function is increased.

**Step 1: \( \theta_2 < \theta_1 \)**

Suppose to the contrary that \( \theta_2 \geq \theta_1 \), but \( \theta_2 \) cannot exceed \( \theta_1 \) as mentioned above, so \( \theta_2 = \theta_1 \). IC-12 and IC-21 together imply that \( r_1 = r_2 \). Then, BB implies that \( r_2 = (\pi b_1 + \pi b_2) \theta_2 \). The latter together with IR-2 implies that \( (\pi b_1 + \pi b_2 - c_2) \theta_2 \geq 0 \). But \( \pi b_1 + \pi b_2 - c_2 < 0 \), therefore \( \theta_2 = 0 \) and so \( \theta_1 = 0 \). This is a contradiction with \( \theta_1 > 0 \).

**Step 2: IC-21 is binding**

By way of contradiction, suppose IC-21 is not binding. Consider \( \theta_1' = \theta_1 - \epsilon \) and \( r_1' = r_1 - b_1 \epsilon \) with \( \epsilon > 0 \). Since \( \theta_1 > 0 \) and IC-21 is not binding, we can find a sufficiently small \( \epsilon \) such that \( \theta_1' > 0 \) and IC-21 still holds. Now, it is easy to check that \( \{(\theta_1', r_1'), (\theta_2, r_2)\} \) is feasible for Problem 8, but it leads to higher value for the objective function than \( \{(\theta_1, r_1), (\theta_2, r_2)\} \). Notice that we used \( b_1 - c_1 < 0 \) to check that IC-12 is satisfied.

**Step 3: IC-12 is not binding**

Suppose by way of contradiction that IC-12 is binding, then following step 2 (stating that IC-21 is binding), it is easy to check that \( r_1 = r_2 \) and \( \theta_1 = \theta_2 \). Then BB implies that \( r_2 = (\pi b_1 + \pi b_2) \theta_2 \). According to IR-2, \( (\pi b_1 + \pi b_2 - c_2) \theta_2 \geq 0 \). But \( \pi b_1 + \pi b_2 - c_2 < 0 \), so \( \theta_1 = \theta_2 = 0 \). This is a contradiction, so IC-12 is not binding.

**Step 4: \( \theta_1 = 0 \)**

We have assumed \( \theta_1 > 0 \). Now, we want to get a contradiction. Now, consider \( \{(\theta_1', r_1'), (\theta_2, r_2')\} \) where \( \theta_1' = \theta_1 - \epsilon \) with \( \epsilon > 0 \), \( r_1' = r_1 - (\pi b_1 + \pi b_2) \epsilon \) and
\[ r'_2 = r_2 + \pi_1(c_2 - b_1)\epsilon. \] Since \( \theta_1 > 0 \) and IC-12 is not binding, we can find a sufficiently small \( \epsilon \) such that \( \theta'_1 > 0 \) and IC-12 still holds. Now it is easy to check that \( \{(\theta'_1, r'_1), (\theta_2, r'_2)\} \) is feasible for Problem 8, but it leads to higher value for the objective function than \( \{(\theta_1, r_1), (\theta_2, r_2)\} \). A contradiction, so the proof is complete. \( \square \)

### A.5 Proof of the Rat Race

*Proof.* This proposition is basically a special case of Theorem 2. It is straightforward to check that all conditions are satisfied. Specially note that, part 5(a) of Assumption 2 is satisfied, therefore, we do not need any assumption on the distribution of types. \( \square \)

### A.6 Asset Market with a Continuous Type Space

Here I define feasible mechanism which is exactly similar to its counterpart with discrete type space (Definition 2). The planner allocates each (reported) type a market tightness, \( \tilde{\theta} : Z \rightarrow \mathbb{R}_+ \), a transfer conditional on finding a match, \( \tilde{p} : Z \rightarrow \mathbb{R} \), and an unconditional transfer, \( \tilde{s} : Z \rightarrow \mathbb{R} \).

**Definition 11.** A feasible mechanism is a set \( \{(\tilde{p}(.), \tilde{s}(.), \tilde{\theta}(.))\} \) such that the following conditions hold:

1. **(Incentive Compatibility of Sellers)** For all \( z \) and \( \hat{z} \),
   \[ U(z) \equiv m(\tilde{\theta}(z))(\tilde{p}(z) - c(z)) + \tilde{s}(z) \geq U(z, \hat{z}) \equiv m(\tilde{\theta}(\hat{z}))(\tilde{p}(\hat{z}) - c(z)) + \tilde{s}(\hat{z}). \]

2. **(Participation Constraint of Sellers)** For all \( i \),
   \[ U(z) \geq 0. \]

3. **(Planner’s Budget-Balance)**
   \[ \int [m(\tilde{\theta}(z))(h(z) - \tilde{p}(z)) - k\tilde{\theta}(z) - \tilde{s}(z)]dF(z) \geq 0. \]
Definition 12. A constraint efficient mechanism is a feasible mechanism which maximizes the planner’s objective function.

The ideas used here are the same as those used in the discrete type space and specially similar to the asset market with lemons. However, mathematical tools that I use here are different, because the state space is continuous.

One way of proving Proposition 3 is to take a Guess-And-Verify approach. I guess that the first best is achievable. Then I check whether conditions for feasibility are satisfied. One problem is that if the first best is not achievable (like conditions in Proposition 9), this approach does not work, because checking for feasibility is not sufficient, since there might be other implementable allocations which deliver a higher value of the objective function for the planner. Therefore, in order to be able to use a general solution method, I first characterize the incentive compatible schemes, as is common in the mechanism design literature. Then, I work with a modified problem in which sellers’ maximization condition has been replaced by some other constraints (monotonicity and Envelope condition).

In the first step, note that similar to the discrete type space, the budget-balance constraint must be satisfied with equality at the constrained efficient mechanism. Otherwise, the planner can distribute extra resources identically among all types. No constraint will be changed, but all types get a strictly higher payoff and therefore the planner can improve welfare. Now I write the planner’s problem into the following form.

Problem 9.

\[
\max_{\theta(z), p(z)} \int \left[ m(\theta(z))(h(z) - c(z)) - k\theta(z) \right] dF \\
\text{s.t. } z \in \arg \max \hat{z} U(z, \hat{z}), \quad (IC) \\
U(z, \hat{z}) \geq 0 \quad (IR),
\]

\[
\text{and } \int \left[ m(\theta(z))(h(z) - p(z)) - k\theta(z) - ss(\hat{z}) \right] dF = 0 \quad (BB),
\]

in which \( U(z, \hat{z}) \equiv m(\theta(\hat{z}))(p(\hat{z}) - c(z)) + ss(\hat{z}) \).

Note that no transfer appears in the objective function, because we have assumed that all types participate in the mechanism and also we have replaced \( \int [m(\theta(z))p(z)] dF \) by \( \int [m(\theta(z))h(z) - k\theta(z)] dF \) from the budget-balance condition.
Characterizing the Incentive Compatible Schemes

By assumption $c(z)$ is strictly monotone in $z$. The first two parts of the following lemma state that $c'(z) \frac{d \tilde{\theta}(z)}{dz} \leq 0$ is necessary and sufficient for any allocation which satisfies IC. Necessity is clear. Sufficiency means that there exists transfer schedules $\tilde{p}(.)$ and $\tilde{s}(.)$ such that the direct mechanism, $\{(\tilde{p}(.), \tilde{s}(.), \tilde{\theta}(.))\}$, satisfies IC. The third part characterizes $U(z)$ for any allocation which satisfies IC.

**Lemma 8** (Necessary and sufficient condition for $\tilde{\theta}(z)$ to be implementable). Assume that $c(z)$ is strictly monotone in $z$.

1. Take any mechanism $\{(\tilde{p}(.), \tilde{s}(.), \tilde{\theta}(.))\}$ that satisfies IC. If $\tilde{\theta}(z)$ is a piecewise $C^1$ function, then $c'(z) \frac{d \tilde{\theta}(z)}{dz} \leq 0$ wherever $\tilde{\theta}(z)$ is differentiable at $z$.

2. Consider any piecewise $C^1$ function $\tilde{\theta}(z)$ satisfying $c'(z) \frac{d \tilde{\theta}(z)}{dz} \leq 0$. Then there exists transfer schedules $\tilde{p}(.)$ and $\tilde{s}(.)$ such that the mechanism $\{(\tilde{p}(.), \tilde{s}(.), \tilde{\theta}(.))\}$ satisfies IC.

3. If mechanism $\{(\tilde{p}(.), \tilde{s}(.), \tilde{\theta}(.))\}$ satisfies IC, then $U(z)$ must satisfy

$$U(z) = U(z_H) + \int_{z_0}^{z_H} m(\tilde{\theta}(z_0))c'(z_0)dz_0.$$

Proof. Define $V(X, R, z) \equiv Xc(z) + R$, $x(z) \equiv -m(\theta(z))$ and $r(z) \equiv m(\tilde{\theta}(z))\tilde{p}(z) + \tilde{s}(z)$. Obviously, $U(z, \hat{z}) = V(x(\hat{z}), r(\hat{z}), z)$.

Following [24], Theorem 7.1, a necessary condition for $x(.)$ to satisfy IC is

$$\frac{\partial}{\partial z} \left[ \frac{\partial V}{\partial X} \frac{\partial V}{\partial R} \right] \frac{dx}{dz} \geq 0,$$

whenever $x(.)$ is differentiable at $z$. But

$$\frac{\partial}{\partial z} \left[ \frac{\partial V}{\partial X} \right] \frac{dx}{dz} = \frac{\partial}{\partial z} \left( -m'(\tilde{\theta}(z)) \right) \frac{d \tilde{\theta}(z)}{dz}.$$

Also $c'(.) > 0$ and $m'(.) \geq 0$, therefore, the necessary condition is equivalent to

$$c'(z) \frac{d \tilde{\theta}(z)}{dz} \leq 0.$$  \hspace{1cm} (A.23)

According to [24] Theorem 7.3, a sufficient condition for $x(.)$ to satisfy IC is that $\frac{dx(z)}{dz} \geq 0$, or equivalently, $c'(z) \frac{d \tilde{\theta}(z)}{dz} \leq 0$.

---

5Briefly, the idea of the proof for necessity is that the second order condition for IC maximiza-
For the third part of the lemma, I use corollary 1 from [44]. This result states that if $\tilde{\theta}(z)$ satisfies IC, then $U(.)$ can be written as follows:

$$U(z) = U(z_H) - \int_z^{z_H} \frac{\partial U(z_0, z_0)}{\partial z} dz_0 = U(z_H) + \int_z^{z_H} m(\tilde{\theta}(z_0))c'(z_0)dz_0. \quad (A.24)$$

This equation is derived from the envelope theorem and is standard in mechanism design literature. The requirements of the result of [44] that we need to check are as follows:

1. $U(z, \hat{z})$ is differentiable and absolutely continuous in $z$.
   This is satisfied because $c$ is assumed to be twice differentiable.

2. $\sup_{\hat{z}} \left| \frac{\partial U(z, \hat{z})}{\partial z} \right|$ is integrable.
   This is satisfied because $\sup_{\hat{z}} \left| \frac{\partial U(z, \hat{z})}{\partial z} \right| \leq \left| c'(z) \right| < M$ for some $M \in \mathbb{R}$, because $c'(.)$ is continuous and is defined over a compact set $[z_L, z_H]$.

3. $\tilde{\theta}(z)$ is obviously non-empty.

We know from IC that $U(z) = m(\tilde{\theta}(z))(\tilde{p}(z) - c(z)) + \tilde{s}(z)$ for all $z$. All types are active, because as we will verify it later, the first best is achievable and there are positive gains from trade for all types. Therefore, from now on we can assume without loss of generality that $\tilde{s}(z) = 0$ for all types. (Otherwise, we can change $\tilde{p}(z)$ to $\tilde{p}(z) + \frac{\tilde{s}(z)}{m(\tilde{\theta}(z))}$.) I substitute $U(.)$ from Equation A.24 into $U(z) = m(\tilde{\theta}(z))(\tilde{p}(z) - c(z))$ to derive transfers:

$$\tilde{p}(z) = c(z) + \frac{U(z_H) + \int_z^{z_H} m(\tilde{\theta}(z_0))c'(z_0)dz_0}{m(\tilde{\theta}(z))} \quad (A.25)$$

Now, I use budget-balance condition to derive $U(z_H)$:

$$0 = \int \left[ m(\tilde{\theta}(z))[h(z) - p(z)] - k\tilde{\theta}(z) \right] F'(z)dz$$

$$= \int \left[ m(\tilde{\theta}(z))[h(z) - c(z)] - k\tilde{\theta}(z) - m(\tilde{\theta}(z))(\tilde{p}(z) - c(z)) \right] F'(z)dz$$

$$= \int \left[ m(\tilde{\theta}(z))[h(z) - c(z)] - k\tilde{\theta}(z) - \int_z^{z_H} m(\tilde{\theta}(z_0))c'(z_0)dz_0 - U(z_H) \right] F'(z)dz$$

The problem (max $\hat{z} U(z, \hat{z})$) should hold. For sufficiency, the proof goes by contradiction. The proof of this lemma is standard in mechanism design literature thus omitted from here.
\[ = \int \left[ m(\tilde{\theta}(z))(h(z) - c(z)) - k\tilde{\theta}(z) - m(\tilde{\theta}(z))c'(z) \frac{F(z)}{F'(z)} \right] F'(z)dz - U(z_H) \]

The third equality follows from Equation A.25. The fourth equality uses the relationship between \(U(z)\) and \(\tilde{p}(z)\) and also Equation A.24. The fifth equality is established using integration by parts.\(^6\) Therefore,

\[ U(z_H) = \int \left[ m(\tilde{\theta}(z))(h(z) - c(z) - c'(z) \frac{F(z)}{F'(z)}) - k\tilde{\theta}(z) \right] F'(z)dz \quad (A.26) \]

According to Equation A.24 and because \(c'(.) > 0\), if \(U(z_H) \geq 0\), then \(U(z) \geq 0\) for all \(z\). Hence, the following inequality,

\[ \int \left[ m(\tilde{\theta}(z))(h(z) - c(z) - c'(z) \frac{F(z)}{F'(z)}) - k\tilde{\theta}(z) \right] F'(z)dz \geq 0, \quad (A.27) \]

implies that planner’s budget constraint and participation constraint of all types are satisfied.

So far I have reduced IC constraint in the planner’s problem to two conditions A.23 and A.25. Planner’s budget constraint and participation constraint of all types are also summarized in Equation A.27. Therefore, thanks to Lemma 8, I can rewrite the planner’s problem as follows to derive \(\tilde{\theta}(z)\) and \(\tilde{p}(z)\).

**Problem 10. Planner’s problem**

\[
\begin{align*}
\max_{\theta(z), p(z)} & \int \left[ m(\theta(z))[h(z) - c(z)] - k\theta(z) \right] F'(z)dz \\
\text{s. t.} & \quad c'(z) \frac{d\theta(z)}{dz} \leq 0, \\
U(z_H) & \equiv \int \left[ m(\tilde{\theta}(z))(h(z) - c(z) - c'(z) \frac{F(z)}{F'(z)}) - k\tilde{\theta}(z) \right] F'(z)dz \geq 0
\end{align*}
\]

From now one, I work with this problem and characterize the solution to this problem.

---

\(^6\)For any differentiable functions \(F\) and \(G\), if \(G(z_H) = 0\), and \(F(z_L) = 0\) we will have: \(\int_{z_H}^{z_L} F'(z)G(z)dz = -\int_{z_L}^{z_H} F(z)G'(z)dz\) using integration by parts. In the above equality, set \(\int_{z_H}^{z_L} m(\tilde{\theta}(z_0))c'(z_0)dz_0\).
A.6.1 Proof of Proposition 3

I propose the following direct mechanism as a solution to this problem (for Proposition 3):

\[ \tilde{\theta}^{CE}(z) = \theta^{FB}(z), \quad (A.28) \]

\[ \tilde{p}^{CE}(z) = c(z) + \frac{U(z_H) + \int_{z_0}^{z_H} m(\theta^{FB}(z_0))c'(z_0)dz_0}{m(\theta^{FB}(z))}, \quad (A.29) \]

where \( U(z_H) = \int \left[ m(\theta^{FB}(z))(h(z) - c(z)) - m(\theta^{FB}(z))c'(z) \frac{F(z)}{F'(z)} \right] F'(z)dz \) and \( \tilde{s}^{CE}(z) = 0 \) for all \( z \).

Later on, I construct an associated implementable allocation with the above market tightness and transfers. Specifically, I construct off-the-equilibrium-path beliefs for that.

**Proof of Proposition 3 under Part 1 of Assumption 5**

*Proof.* I prove Proposition 3 under the first assumption, \( h'(.) \leq 0 \), using somewhat a backward approach. I first guess that the planner can achieve the first best. That is, the planner can maximize his objective function point-wise (for each \( z \), separately). What I need to do then is to check that the two constraints of Problem 10, monotonicity constraint and \( U(z_H) \geq 0 \), are satisfied.

The first best level of market tightness, \( \theta^{FB}(z) \), is given by

\[ m'(\theta^{FB}(z))(h(z) - c(z)) - k = 0. \quad (A.30) \]

By differentiating it with respect to \( z \), one yields

\[ \frac{d\theta^{FB}(z)}{dz} = -\frac{k(h'(z) - c'(z))}{m''(\theta^{FB}(z))(h(z) - c(z))^2}. \quad (A.31) \]

By assumption, \( h'(.) - c'(.) \leq 0 \) and \( m''(.) \leq 0 \), so \( \frac{d\theta^{FB}(z)}{dz} \) is negative. Hence, \( c'(z)\frac{d\theta(z)}{dz} \leq 0 \) constraint in problem 10 is satisfied. Now I calculate \( U(z_H) \) and show that it is positive. From equation A.26, one can write

\[ U(z_H) = \int \left[ m(\tilde{\theta}(z))(h(z) - c(z) - c'(z) \frac{F(z)}{F'(z)} - k\tilde{\theta}(z)) \right] F'(z)dz \]

124
\[
= \int \left[ - \int_{\tilde{z}}^{z_H} m(\theta^{FB}(z_0))(h'(z_0) - c'(z_0))dz_0 + U^{FB}(z_H) \right] \\
- m(\theta^{FB}(z))c'(z) \frac{F(z)}{F'(z)} F'(z)dz
\]

\[
= - \int \left[ m(\theta^{FB}(z))(h'(z) - c'(z)) + m(\theta^{FB}(z))c'(z) \right] F(z)dz + U^{FB}(z_H)
\]

\[
= - \int m(\theta^{FB}(z))h'(z)F(z)dz + U^{FB}(z_H) \geq 0
\]

The second equality uses the fact that \( \tilde{\theta}(z) = \theta^{FB}(z) \) and also the fact that

\[
\frac{d}{dz} \left[ \max_{\theta}(m(\theta)(h(z) - c(z)) - k\theta) \right] = m(\theta^{FB})(h'(z) - c'(z)).
\]

The third equality is derived by using integration by parts. The inequality holds because \( h'(z) < 0 \) by assumption and \( U^{FB}(z_H) \geq 0 \) because there are positive gains from trade for all types. Both constraints in Problem 10 are satisfied. Also, because the proposed allocation for the solution is the first best allocation, we do not need to check that any other allocation achieves higher welfare, because this is the highest possible welfare. This completes the proof that the first best is achievable by a feasible mechanism. \( \square \)

Note that in order to show that \( \theta^{FB}(z) \) is decreasing, it was sufficient to have \( h'(.) - c'(.) \leq 0 \) (according to Equation B.10). In the next part of the proposition, I replace the assumption \( h'(.) \leq 0 \) with a weaker assumption, \( h'(.) - c'(.) \leq 0 \). To satisfy \( U(z_H) \geq 0 \), I need another assumption summarized in part 2 of Assumption 5.

**Proof of Proposition 3 under Part 2 of Assumption 5**

**Proof.** Now suppose part 2 of Assumption 5 holds. The proof is similar to the previous part. Because \( h'(z) - c'(z) \) is negative, according to Equation B.10, the first constraint in Problem 10 is satisfied. We just need to show that \( U(z_H) \) is positive. Again from equation A.26, one can write

\[
U(z_H) = \int \left[ m(\theta^{FB}(z))[h(z) - c(z)] - k\theta^{FB}(z) - m(\theta^{FB}(z))c'(z) \frac{F(z)}{F'(z)} \right] F'(z)dz.
\]

\( \text{The proposition and its proof can be written in the exactly same fashion if instead } c(.) \text{ is strictly decreasing and } h(.) - c(.) \text{ is increasing. The result are not reported to save space.} \)
A sufficient condition for the integral to be positive is that the sum of the terms in the brackets is always positive. That is, for all \( z \):

\[
\frac{m(\theta^{FB}(z))[h(z) - c(z)] - k\theta^{FB}(z) - m(\theta^{FB}(z))c'(z)\frac{F(z)}{F'(z)}}{m(\theta^{FB}(z))} \geq 0.
\]

But at the solution \( m'(\theta^{FB}(z))[h(z) - c(z)] = k \), therefore

\[
\frac{m(\theta^{FB}(z))[h(z) - c(z)] - k\theta^{FB}(z)}{m(\theta^{FB}(z))} = \frac{m(\theta^{FB}(z))[h(z) - c(z)] - m'(\theta^{FB}(z))[h(z) - c(z)]\theta^{FB}(z)}{m(\theta^{FB}(z))} = -\frac{\theta^{FB}(z)q'(\theta^{FB}(z))}{q(\theta^{FB}(z))}(h(z) - c(z)).
\]

Hence, for \( U(z_H) \) to be positive, it is sufficient to have:

\[
\eta(\theta^{FB}(z)) \frac{h(z) - c(z)}{c'(z)} \geq \frac{F(z)}{F'(z)} \quad \text{for all } z.
\]

From \( m'(\theta^{FB}(z))[h(z) - c(z)] = k \), I can write \( \theta^{FB}(z) = m'^{-1}\left(\frac{k}{h(z) - c(z)}\right) \). Replacing \( \theta^{FB}(\cdot) \) in the sufficient condition yields

\[
\psi\left(\frac{k}{h(z) - c(z)} \frac{h(z) - c(z)}{c'(z)} \right) \geq \frac{F(z)}{F'(z)}
\]

which is the same as the left hand side of part 2 of Assumption 5. This concludes the proof. \( \square \)

\( \ddot{p}(z) \) is increasing in \( z \)

I take a derivative of Equation A.25 with respect to \( z \) to get the following:

\[
\frac{d\ddot{p}(z)}{dz} = -\frac{m'(\ddot{\theta}(z))}{m(\theta(z))} \frac{d\ddot{\theta}(z)}{dz} (\ddot{p}(z) - c(z)) \geq 0 \quad \text{(A.32)}
\]

The inequality holds because \( \ddot{\theta}(z) \) is decreasing in \( z \) following the fact that the allocation satisfies IC. Moreover, \( \ddot{p}(z) - c(z) \) is positive following the fact that the allocation satisfies the participation constraint.

**Constructing an implementable allocation from the direct mechanism**

So far, I have constructed the direct mechanism for Proposition 3. I construct the associated implementable allocation \( \{P, G, \theta, \mu, t, T\} \) as follows.

\[
P \equiv [p_L, p_H] \subseteq \mathbb{P} = \mathbb{R}_+ \quad \text{where } p_L \equiv \ddot{p}^{CE}(z_L) \quad \text{and } p_H \equiv \ddot{p}^{CE}(z_H)
\]

\( ^8 \)Note that when \( c'(\cdot) < 0 \) and \( h'(\cdot) - c'(\cdot) \geq 0 \), a similar result can be obtained.
and $\tilde{p}^{CE}$ is given by Equation A.29.9 The market tightness for this allocation is given by

$$
\begin{align*}
\theta(p) &= \begin{cases} 
\infty & \text{for } p \leq c(z_L) \\
\min\{1, \frac{U(z_L)}{p-c(z_L)}\} & \text{for } p \in (c(z_L), p_L) \\
\theta^{FB}(\tilde{p}^{CE-1}(p)) & \text{for } p \in [p_L, p_H] \\
\min\{1, \frac{U(z_H)}{p-c(z_H)}\} & \text{for } p \in (p_H, \infty) 
\end{cases}
\end{align*}
$$

m(\theta(p)) = \begin{cases} 
\int_{p_L}^{p} \theta(p)F'(\tilde{p}^{CE-1}(p))dp & \text{for } p \in [p_L, p_H] \\
1 & \text{for } p > p_H 
\end{cases}

The rest of elements are given as follows:

$$
G(p) = \begin{cases} 
0 & \text{for } p < p_L \\
\int_{p_L}^{p} \theta(p)F'(\tilde{p}^{CE-1}(p))dp & \text{for } p \in [p_L, p_H] \\
1 & \text{for } p > p_H 
\end{cases}
$$

$$
t(p) = \begin{cases} 
h(z_L) - p & \text{for all } p < p_L \\
h(\tilde{p}^{CE-1}(p)) - p - \frac{k}{q(\theta(p))} & \text{and } p \in [p_L, p_H] \\
h(z_H) - p & \text{for } p > p_H 
\end{cases}
$$

$$
\int \mu(z)pdz = 1 \text{ for all } p, \text{ and } \mu(z)p = \begin{cases} 
0 & \text{for } p < p_L \text{ and } z \neq z_L \\
0 & \text{for } p \neq \tilde{p}(z) \text{ and } p \in [p_L, p_H] \\
0 & \text{for } p > p_H \text{ and } z \neq z_H 
\end{cases}
$$

$$
T = 0.
$$

The construction is straightforward. We allocate all types the same market tightness and transfer that they were given in the direct mechanism. For construction of off-the-equilibrium-path beliefs, if $p < p_L$, then the only type attracted to this post is $z_L$. Therefore, $\mu(z)p = 0$ for all $z \neq z_L$, and $\mu(z)p$ has a mass point at $z = z_L$. Similarly if $p > p_H$, then the only type attracted to this post is $z_H$. Therefore, $\mu(z)p = 0$ for all $z \neq z_H$. Given the above beliefs, we construct the tax amount for all $p$ such that buyers get a net profit of exactly 0 for $p \in P$ and $-k$ for $p \notin P$. Note that choice of $t$ is not unique for $p \notin P$. We could construct $t$ differently such that buyers get any non-positive amount of profit for $p \notin P$. $G(p)$ is easily constructed given the construction of $\theta(.)$.

9I showed above that $\tilde{p}(z)$ is strictly increasing in $z$. Also $\tilde{p}(z)$ is continuous, therefore the set of prices in the constructed implementable mechanism is $P \equiv [p_L, p_H]$. 

127
Now I check the conditions of implementability. The buyers’ maximization and free entry condition is satisfied due to the construction of $t$ (easy to check). Feasibility or market clearing is also trivially satisfied due to the construction of $G$. The budget-balance condition is satisfied due to the choice of $U(z_H)$.

Regarding the sellers’ optimal search condition, first note that the restriction on off-the-equilibrium-path beliefs is equivalent to:

$$m(\theta(p)) = \min\{1, \inf_{z \in \{z|c(z)<p\}} \frac{U(z)}{p-c(z)}\},$$

if $\{z|c(z)<p\}$ is non-empty. Otherwise, set $\theta(p) = \infty$. Now it is easy to see that sellers’ optimal search is also satisfied due to the construction of $\theta(p)$. The only thing worth explaining here is why only $z_L$ is attracted to any price less than $p_L$ (and similarly why only $z_H$ is attracted to any price greater than $p_H$). To see why, I begin by writing the incentive compatibility condition for any feasible mechanism:

$$m(\tilde{\theta}(z_L))(\tilde{p}(z_L) - c(z)) \leq U(z) \text{ for all } z.$$

After using the fact that $U(z_L) = m(\tilde{\theta}(z_L))(\tilde{p}(z_L) - c(z_L))$, we can write:

$$U(z_L) - U(z) \leq m(\tilde{\theta}(z_L))(c(z) - c(z_L)) \text{ for all } z.$$

Therefore,

$$U(z_L) - U(z) \leq m(\tilde{\theta}(z_L))(c(z) - c(z_L)) = \frac{U(z_L)}{\tilde{p}(z_L) - c(z_L)}(c(z) - c(z_L))$$

$$\leq \frac{U(z_L)}{p - c(z_L)}(c(z) - c(z_L)) \text{ for all } z \text{ and for } p \in (c(z_L), p(z_L)),$$

or equivalently,

$$\frac{U(z_L)}{p - c(z_L)} \leq \frac{U(z)}{p - c(z)} \text{ for all } z \text{ and for } p \in (c(z_L), p(z_L)).$$

Therefore by setting $m(\theta(p))$ to be equal to $\frac{U(z_L)}{p - c(z_L)}$, the restriction on off-the-equilibrium-path beliefs is satisfied.

\footnote{See [3] for a more detailed discussion.}
Understanding Part 2 of Assumption 5 better

It is easy to show that if \(1/q(\theta)\) is convex, \(\eta(\theta)\) is increasing in \(\theta\). Also, \(m'\) is decreasing in \(\theta\) by assumption. Hence, \(\psi(\cdot)\) is a decreasing function. The second part of Assumption 5 states that for a given distribution, a given \(z\) and a given value for \(c'(z)\), \(h(z) - c(z)\) should be sufficiently high or \(k\) should be sufficiently low. The intuition is that the surplus generated by type \(z\) should be sufficiently high (or the entry cost sufficiently low) to provide enough resources for the planner to implement the first best allocation. This assumption is exactly the counterpart of part 3 of Assumption 2 in the discrete type case.

A.6.2 Even If \(c'(\cdot) \geq 0\) and \(h'(\cdot) \geq 0\), the Optimal Tax Schedule May Not Be Monotone

Assume that part 2 of Assumption 5 holds, so FB is achievable and \(\tilde{\theta}(z) = \theta^{FB}(z)\). I suppress the superscript \(CE\) in this section to reduce the notation. I calculate \(m(\tilde{\theta}(z))t(z)\), take its derivative with respect to \(z\) and then show that if \(c'(\cdot) \geq 0\), \(h'(\cdot) \geq 0\) with strict inequality for a positive measure of \(z\) and \(h'(z_L) = 0\), then \(\frac{dt(p)}{dp} \big|_{p=p_L} < 0\).

\[
m(\tilde{\theta}(z))t(z) = m(\theta^{FB}(z))(h(z) - c(z)) - k\theta^{FB}(z) - U(z_H) - \int_{z}^{z_H} m(\theta^{FB}(z_0))c'(z_0)dz_0
\]

\[
= U^{FB}(z) - U(z_H) - \int_{z}^{z_H} m(\theta^{FB}(z_0))c'(z_0)dz_0
\]

The derivative of \(m(\theta^{FB}(z))t(z)\) with respect to \(z\) is given by:

\[
\frac{\partial}{\partial z}[m(\tilde{\theta}(z))t(z)] = \frac{\partial}{\partial z} U^{FB}(z) + m(\theta^{FB}(z))c'(z)
\]

\[
= m(\theta^{FB}(z))(h'(z) - c'(z)) + m(\theta(z))c'(z) = m(\theta^{FB}(z))h'(z) \geq 0 \quad (A.33)
\]

The second equality is derived by applying Envelope theorem to the following maximization problem: \(U^{FB}(z) = \max_{\theta} \{m(\theta)(h(z) - c(z)) - k\theta\}\). The inequality holds by assumption.

Similar to the previous proof, it is easier to work with the direct mechanism. Because \(\tilde{\theta}(\cdot)\) is strictly decreasing, then the associated implementable allocation
must be separating. I show below that \( \tilde{t}(z) \) is decreasing in \( z \) at \( z = z_L \). Then it is readily concluded that tax function must be also decreasing in the price at \( p = p_L \), because the allocation is separating and price in each submarket is strictly increasing in the type applying to that submarket.

According to Equation A.33,

\[
\frac{d[m(\tilde{\theta}(z))\tilde{t}(z)]}{dz} = m(\tilde{\theta}(z))h'(z),
\]

therefore

\[
\tilde{t}'(z) = h'(z) - \frac{m'(\tilde{\theta}(z))d\tilde{\theta}(z)}{m(\theta(z))d(z)}\tilde{t}(z)
\]

Consider this equality for \( z = z_L \). Given the assumption that \( h'(z_L) = 0 \) and given the fact that \( \tilde{\theta}'(z) < 0 \), it is sufficient to show that \( \tilde{t}(z_L) < 0 \). Then it follows that \( \tilde{t}(z_L) < 0 \). To calculate \( \tilde{t}(z) \), I use the planner’s budget-balance condition to write:

\[
\int m(\tilde{\theta}(z))\tilde{t}(z)dF(z) = 0.
\]

Let \( \chi(.) \equiv m(\tilde{\theta}(z))\tilde{t}(z) \). Then,

\[
0 = \int \chi(z)dF(z) = -\chi(z)(1 - F(z))\bigg|_{z_L}^{z_H} + \int \chi'(z)(1 - F(z))dz
\]

by using integration by parts. Therefore

\[
\chi(z_L) = -\int \chi'(z)(1 - F(z))dz < 0.
\]

The inequality holds because \( \chi'(z) = m(\tilde{\theta}(z))h'(z) \) from Equation A.33 and the fact that \( h'(z) \geq 0 \). Also, the inequality is strict because \( h'(z) > 0 \) for some \( z \). But \( \tilde{t}(z_L) = \frac{\chi(z_L)}{m(\theta(z_L))} < 0 \) by definition of \( \chi(.) \). The proof is complete.

A.6.3 What If the Complete Information Allocation Is Not Achievable

I keep the assumption that \( c'(z) > 0 \) and \( h'(z) - c'(z) \leq 0 \), but now assume that the distribution of types is such that the planner cannot achieve the first best. I show in the next proposition that the probability of matching for almost all types must be distorted (relative to the first best) so that IC and budget constraint are both satisfied.
Proposition 3 requires $m(\theta^FB(z))[h(z) - c(z) - c'(z)\frac{F'(z)}{F(z)}] - k\theta^FB(z)$ to be positive (if $h'(z) \leq 0$ is not satisfied for some $z$). However, if this expression is negative for some types, then the following result (Proposition 9) requires at least $h(z) - c(z) - c'(z)\frac{F'(z)}{F(z)}$ to be positive for all $z$. Also this proposition requires $h(z) - c(z) - c'(z)\frac{F'(z)}{F(z)}$ to be decreasing in $z$ to ensure that the monotonicity constraint is satisfied ($\frac{d\tilde{t}}{dz} \leq 0$).

Note that generally $\tilde{t} CE(z) \neq 0$ for almost all types. This implies that although the first best is not achievable under the premises of this proposition, the planner can use transfers effectively to achieve higher welfare than the equilibrium. The intuition is the same as in the simple two-type example.

**Assumption 17.** For all $z$, $h(z) - c(z) - c'(z)\frac{F'(z)}{F(z)} > 0$ and $\frac{d}{dz}[h(z) - c(z) - c'(z)\frac{F'(z)}{F(z)}] \leq 0$.

**Proposition 9.** Assume $c'(z) > 0, h'(z) - c'(z) \leq 0$ and $U^FB(z) > 0$ for all $z$. Also, suppose Assumption 17 holds. If the first best is not achievable, then there exists a $\nu > 0$ such that the market tightness $\tilde{t} CE(z)$ solves the following equations:

$$m'(\theta(z)) \left[ h(z) - c(z) - \frac{\nu}{1+\nu} c'(z)\frac{F(z)}{F'(z)} \right] = k,$$ \quad (A.34)

$$\int \left[ m(\theta(z)) \left[ h(z) - c(z) - c'(z)\frac{F(z)}{F'(z)} \right] - k\theta(z) \right] F'(z)dz = 0.$$ \quad (A.35)

Moreover, $\tilde{s} CE(z) = 0$ without loss of generality and $\tilde{p} CE(z)$ is obtained similarly as before:

$$\tilde{p} CE(z) = c(z) + \frac{U(z_H) + \int_z^{z_H} m(\tilde{t} CE(z_0))c(z_0)dz_0}{m(\tilde{t} CE(z))},$$

where $U(z_H) = 0$.

**Proof.** In this case by assumption, the complete information allocation is not achievable. Therefore, the guess-and-verify approach does not work. To solve for planner’s problem, consider Problem 10. I first ignore monotonicity constraint. I form the Lagrangian and derive first order condition (FOC). Then I verify that the monotonicity constraint (and consequently IC) is also satisfied. Denote the
Lagrangian by $\mathcal{L}$ and the Lagrangian multiplier by $\nu$:

$$
\mathcal{L} = \int [m(\theta(z))(h(z) - c(z)) - k\theta(z) + \nu[m(\theta(z))(h(z) - c(z)) - k\theta(z) - m(\theta(z))c'(z)\frac{F(z)}{F'(z)} - U(z_H)]]F'(z)dz. \quad (A.36)
$$

The FOC with respect to $\theta(z)$ for all $z$ is given by:

$$
m'(\theta(z))(h(z) - c(z)) - k + \nu(m'(\theta(z))(h(z) - c(z)) - k) - \nu m'(\theta(z))c'(z)\frac{F(z)}{F'(z)} = 0.
$$

It can be simplified to conform to Equation A.34 exactly.

According to the assumptions of the proposition, $h - c$ and $h - c - c'\frac{F(z)}{F'(z)}$ are decreasing in $z$. Also, $\nu$ is non-negative, so $\frac{1}{1+\nu} (h(z) - c(z)) + \frac{\nu}{1+\nu} (h(z) - c(z) - c'(z)\frac{F(z)}{F'(z)}) = h(z) - c(z) - \nu c'(z)\frac{F(z)}{F'(z)}$ is also decreasing in $z$. Therefore, FOC implies that $\theta(z)$ is decreasing in $z$ as well. As a result, the monotonicity constraint $(c'(z)\frac{d\theta(z)}{dz} \leq 0)$ is satisfied. The first best is not achievable, so if the planner allocates all types the market tightness $\theta^{FB}(z)$ (and corresponding transfers from equation A.25), then the derived value for $U(z_H)$ becomes negative. (Otherwise the first best would be achievable).\(^{11}\)

Assume there exists $\nu > 0$ such that the FOC and BB both hold. Since the objective function is strictly concave in $\theta(\cdot)$, and the objective function is just the sum of concave functions, the objective function is also concave in $\theta(\cdot)$.\(^{12}\) Because

\(^{11}\)Among all allocations in which all types are active, the proposed allocation is globally optimum, because the objective function is concave and the constraint set is convex. However, since FB is not achievable, we need to compare the level of welfare from this case (where all types are active) with another case where some types are inactive. Notice that since $U(z)$ is decreasing in $z$, if some types are not active, then all types above them will not be active either. Therefore, there is a threshold under which all types are active and above which all types are not active. Then, we can apply the same method to the set of types below the threshold in order to calculate welfare. To show that exclusion of some types does not increase welfare, note that we can write the value of the planner’s objective function as

$$
\int [m(\theta(z))(h(z) - c(z)) - k\theta(z)]F'(z)dz = \int m(\theta(z))c'(z)F(z)dz.
$$

The equality follows from Equation A.35 (which is implied by the fact that $U(z_H) = 0$). Since $c'(z) > 0$ for all $z$, if some types are excluded, then the value of the objective function will be strictly lower. Therefore, it is never optimal to exclude some types.

\(^{12}\)To show this point, consider a simpler version where the objective is a function of two variables, that is, $g(x_1, x_2) = f(x_1) + h(x_2)$. Also assume $f(\cdot)$ and $h(\cdot)$ are concave in $x_1$ and $x_2$.
of concavity of the objective function, the FOC is sufficient for the solution. Hence, it only remains to show that such \( \nu > 0 \) exists.

Note that \( \theta(\cdot) \) obtained from Equation A.34 is continuous in \( \nu \). Accordingly, the LHS of Equation A.35 is continuous in \( \nu \) as well. I need to show that the LHS of Equation A.35 is negative at \( \nu = 0 \) and is positive when \( \nu \to \infty \).

If \( \nu = 0 \), then \( \theta(z) = \theta^{FB}(z) \) is the solution to Equation A.34. The first best is not achievable, so the other constraint in Problem 10 must be violated:

\[
\int \left[ m(\theta^{FB}(z)) \left[ h(z) - c(z) - c'(z) \frac{F(z)}{F'(z)} \right] - k\theta(z) \right] F'(z) dz < 0.
\]

If \( \nu \to \infty \),

\[
\int \left[ m(\theta(z)) \left[ h(z) - c(z) - c'(z) \frac{F(z)}{F'(z)} \right] - k\theta(z) \right] F'(z) dz
\]

\[
= \lim_{\nu \to \infty} \int \left[ m(\theta(z)) \left[ h(z) - c(z) - \frac{\nu}{1 + \nu} c'(z) \frac{F(z)}{F'(z)} \right] - k\theta(z) \right] F'(z) dz
\]

\[
= \int \left[ \frac{km(\theta(z))}{m'(\theta(z))} - k\theta(z) \right] F'(z) dz = k \int \frac{m(\theta(z)) - \theta(z)m'(\theta(z))}{m'(\theta(z))} F'(z) dz
\]

\[
= -k \int \frac{\theta(z)^2 q'(\theta(z))}{m'(\theta(z))} F'(z) dz > 0,
\]

where the second equality is derived from the FOC for any \( \nu \), that is, \( m'(\theta(z))[h(z) - c(z) - c'(z) \frac{F(z)}{F'(z)}] = k \), and the last inequality follows from \( q' < 0 \) and \( m' > 0 \). According to intermediate value theorem, there exists a strictly positive \( \nu \) which satisfies A.35.

\[\square\]

A.7 Proof of Proposition 5

First I define the implementable allocation with two types of taxes similar to the definition of implementable allocation in Definition 6. One difference is that here respectively. I want to show that \( g \) is concave in \((x_1, x_2)\). To show that, I form the Hessian as follows:

\[
\begin{bmatrix}
    f'' & 0 \\
    0 & h''
\end{bmatrix}
\]

Since \( f'' \) and \( h'' \) are both negative, the determinant of Hessian is negative. Therefore \( g \) is concave.
there is also another tax that buyers should pay when they want to enter each submarket, \( t_e(p) \). This tax is collected before agents find a match. Due to this difference, we have to take into account the fact that the amount of entry tax for the open submarkets, \( p \in P \), cannot exceed \(-k\), otherwise buyers don’t have incentive to stay in the matching stage, i.e., they will leave after the entry tax (subsidy, in fact) is paid. This is reflected in condition (i) of the following definition.

Another difference is that here the set of admissible prices is assumed to be \((c(z_L), \infty)\). This is because for any \( p \leq c(z_L)\), no seller would have incentive to apply to that submarket, because the seller would get a negative payoff. Therefore, we just assume that such a price cannot be posted. This assumption is made to avoid some technical difficulties.

**Definition 13.** An implementable allocation, \( \{P, G, \theta, \mu, t, t_e, T\} \), is a measure \( G \) on the set of all possible prices, \( \mathbb{P} \equiv (c(z_L), \infty) \), with support \( P \), a function \( \theta : \mathbb{P} \to [0, \infty] \), a conditional density function of buyers’ beliefs regarding sellers’ types who would apply to \( p \), \( \mu : \mathbb{P} \times \mathbb{Z} \to [0, 1] \), a tax function denoting the amount of tax to be imposed on buyers at each submarket conditional on trade, \( t : \mathbb{P} \to \mathbb{R} \), another tax function denoting the amount of tax to be imposed on buyers at each submarket conditional on entry, \( t_e : \mathbb{P} \to \mathbb{R} \), and finally the amount of the numeraire good to be transferred to sellers in a lump sum way, \( T \in \mathbb{R}_+ \), which satisfies the following conditions:

(i) **Buyers’ profit maximization, free entry and no commitment**
For any \( p \in \mathbb{P} \),
\[
q(\theta(p))[\int h(z)\mu(z|p)dz - t(p)] \leq k + t_e(p),
\]
with equality if \( p \in P \). Also,
\[
0 \leq k + t_e(p)
\]
for any \( p \in P \).

(ii) **Sellers’ optimal search**
Let \( U(z) = \max \left\{ 0, \max_{p' \in \mathbb{P}} \left\{ m(\theta(p'))(p' - c(z)) \right\} \right\} + T \) and \( U(z) = T \) if \( P = \emptyset \).
Then for any \( p \in \mathbb{P} \) and \( z \), \( U(z) \geq m(\theta(p))(p - c(z)) + T \) with equality if \( \theta(p) < \infty \) and \( \mu(z|p) > 0 \). Moreover, if \( p - c(z) < 0 \), either \( \theta(p) = \infty \) or \( \mu(z|p) > 0 \).

(iii) **Feasibility or market clearing**
For all \( z \), \( \int_P \frac{\mu(z|p)}{\theta(p)}dG(p) \leq F'(z) \), with equality if \( U(z) > T \).
Planner’s budget constraint

\[ \int_P [q(\theta(p)) t(p) + t_e(p)] dG(p) \geq T. \]

Let \(\{ (\tilde{p}(.), \tilde{s}(.), \tilde{\theta}(.)) \} \) denote the direct mechanism with the properties given in Proposition 5. Since all types get a strictly positive payoff and also that the market tightness allocated to different types is all different, if \(\tilde{s}(z) \neq 0\) for some \(z\), we can substitute \(\tilde{p}(z)\) by \(\tilde{p}(z) + \frac{\tilde{s}(z)}{m(\tilde{\theta}(z))}\) for almost all types (because \(\tilde{\theta}(z) \neq 0\) for almost all types). Therefore, we can assume without loss of generality that \(\tilde{s}(z) = 0\) for almost all \(z\). Furthermore, to avoid technical difficulties, assume that the \(\tilde{p}(z)\) and \(\tilde{\theta}(z)\) are both differentiable in \(z\).

As shown in part 1 of Lemma 8, \(c'(z) \frac{d\tilde{\theta}(z)}{dz} \leq 0\) for all \(z\) is a necessary condition for any mechanism which satisfies IC. Since \(\tilde{\theta}(z)\) is different for different types by assumption, \(\tilde{\theta}(z)\) must be a strictly decreasing function in \(z\). Also in the proof of Proposition 3, it was shown that \(\tilde{p}(z)\) is given by the following equation in any mechanism that satisfies IC:

\[ \tilde{p}(z) = c(z) + \frac{U(z_H) + \int_z^{z_H} m(\tilde{\theta}(z_0)) c'(z_0) dz_0}{m(\tilde{\theta}(z))}, \]  

(A.38)

where \(U(z_H) = \int [m(\tilde{\theta}(z))(h(z) - c(z)) - k\tilde{\theta}(z) - m(\tilde{\theta}(z)) c'(z) F(z)] dF(z)\). According to Equation A.32, we have \(\tilde{p}'(z) = \frac{m'(\tilde{\theta}(z))}{m(\tilde{\theta}(z))} \frac{d\tilde{\theta}(z)}{dz} (\tilde{p}(z) - c(z))\) which implies that \(\tilde{p}(z)\) is strictly increasing in \(z\) with the assumption of differentiability of \(\tilde{\theta}(z)\). This is because \(\tilde{p}(z) - c(z)\) is strictly positive, (otherwise that type will get a negative payoff which contradicts the assumption that all types get a strictly positive payoff), and also because \(\tilde{\theta}(z)\) is strictly decreasing. Moreover, \(\tilde{p}(z)\) is continuous, therefore the set of prices in the constructed implementable mechanism is \(P \equiv [p_L, p_H]\) where \(p_L \equiv p(z_L)\) and \(p_H \equiv p(z_H)\).

I construct the allocation \(\{ P, G, \theta, \mu, t, t_e, T \}\) as follows and show that if \(M\) and \(M'\) are chosen sufficiently large, this allocation is implementable and \(t_e(p)\) is strictly decreasing and \(t(p)\) is strictly increasing in \(p\). The market tightness for
this allocation is given by: \[13\]

\[
\begin{cases}
    m(\theta(p)) = \min\{1, \frac{U(z_L)}{p-c(z_L)}\} & \text{for } p \in (c(z_L), p_L) \\
    \theta(p) = \tilde{\theta}(\tilde{p}^{-1}(p)) & \text{for } p \in [p_L, p_H] \\
    m(\theta(p)) = \min\{1, \frac{U(z_H)}{p-c(z_H)}\} & \text{for } p \in (p_H, \infty)
\end{cases}
\]

The rest of elements are given as follows:

\[
G(p) = \begin{cases}
    0 & \text{for } p \in (c(z_L), p_L) \\
    \int_{p_L}^{p} \theta(p)F'(\tilde{p}^{-1}(p))dp & \text{for } p \in [p_L, p_H] \\
    1 & \text{for } p > p_H
\end{cases}
\]

\[
t_e(p) = \begin{cases}
    -k + M(p_H - p) & \text{for } p \in (c(z_L), p_H) \\
    -k & \text{for } p \in (p_H, \infty)
\end{cases}
\]

\[
t(p) = \begin{cases}
    h(z_L) - p - \frac{k+t_e(p)}{\theta(p)} & \text{for all } p \in (c(z_L), p_L) \\
    h(\tilde{p}^{-1}(p)) - p - \frac{k+t_e(p)}{\theta(p)} & \text{and } p \in [p_L, p_H] \\
    t(p_H) + M'(p - p_H) & \text{for } p > p_H
\end{cases}
\]

\[
\int \mu(z|p)dz = 1 \text{ for all } p \text{ and } \mu(z|p) = \begin{cases}
    0 & \text{for } p < p_L \text{ and } z \neq z_L \\
    0 & \text{for } p \neq \tilde{p}(z) \text{ and } p \in [p_L, p_H] \\
    0 & \text{for } p > p_H \text{ and } z \neq z_H
\end{cases}
\]

\[T = 0.\]

Now I check the conditions of implementability. The buyers’ profit maximization and free entry condition is satisfied due to the construction of \(t\) and \(t_e\) (easy to check). Feasibility (or market clearing) is also trivially satisfied due to the construction of \(G\). The budget-balance is satisfied due to the choice of \(U(z_H)\). Sellers’ optimal search condition is satisfied and the argument is exactly similar to one in page 128, so I skip it.

---

13 If there are more than one \(\theta(p)\) consistent with the above equation, then choose the largest one. This does not happen here though, because we have assumed in this section that \(m\) is strictly increasing.
Regarding monotonicity of taxes, it is obvious that \( t_*(p) \) is decreasing in \( p \) for any \( p \in [p_L, p_H] \) for any \( M > 0 \). It is just left to show that \( t(p) \) is increasing in \( p \).

I take a derivative of \( t(p) \) with respect to \( p \):

\[
 t'(p) = h'(\tilde{p}^{-1}(p)) \frac{d(\tilde{p}^{-1}(p))}{dp} - 1 + M \frac{q(\theta(p)) + q'(\theta(p))\theta'(p)(p_H - p)}{q(\theta(p))^2}.
\]

Now, define

\[
 M_1 \equiv \inf \left\{ 4, \sup_{p \in [p_L, p_H]} \frac{1 - h'(\tilde{p}^{-1}(p))d(\tilde{p}^{-1}(p))}{q(\theta(p)) + q'(\theta(p))\theta'(p)(p_H - p)} \right\}.
\]

\( M_1 \) is a lower bound for \( M \). Note that 4 is just an arbitrary positive number. Also, the third expression in the min has been derived similarly to the second expression but for the case with \( p \in (c_L, p_L] \). I want to show that \( M_1 < \infty \), so I need to show that the second and third expressions in the min are less than \( \infty \). Note that if \( q(\theta(p)) \to 0 \), then the expression goes to 0, therefore I just need to show that \( \frac{d(\tilde{p}^{-1}(p))}{dp} > -\infty \). I have already calculated \( \frac{dp}{dz} \) in Equation A.32 and have shown that \( \frac{dp}{dz} > 0 \) for all \( z \). Therefore, \( \frac{d(\tilde{p}^{-1}(p))}{dp} \) which is just the inverse of \( \frac{dp}{dz} \) is always positive too. Since \( z \) lies in a compact interval, \( 1 - h'(.)\frac{dp}{dz} \) is not greater than 1 and the proof in this part is complete.

For \( p \in (c(z_L), p_L) \), we can similarly find \( M_2 > 0 \) such that if \( M > M_2 \), then \( t(p) \) is strictly increasing. For \( p > p_H \), again we can similarly find \( M_3 > 0 \) such that if \( M' > M_3 \), then \( t(p) \) is strictly increasing. Since \( t(p) \) is continuous by construction, therefore the fact that it is increasing in different intervals implies that it is increasing in the entire domain.
Appendix B
Proofs of Chapter 2

B.1 Proof of Proposition 6

As stated in the text, I follow [26] closely in this proof. Lemma 9 to 13 here correspond to Lemma 1 and Prop. 1 to 4 in [26], respectively.

I repeat Problem $P_i$ here from the text just for the ease of reference. Let Problem $P$ denote the larger problem of solving Problem $P_i$ for all $i$. I say that $I^*, \{U_i\}_{i \in \mathbb{Z}^*}, \{((\theta_i, p_i, x_i))\}_{i \in I^*}$ solve $P_i$ where these objects are defined as follows. Let $I^* \subset \mathbb{Z}$ denote the set of types such that the constraint in $P_i$ is non-empty and the maximized value of the objective function is strictly positive given $(U_1, ..., U_{i-1})$. For any $i \in I^*$, denote by $(\theta_i, p_i, x_i)$ the solution to $P_i$ given $(U_1, ..., U_{i-1})$ and denote by $U_i$ the maximized value to Problem $P_i$. Set $U_i = 0$ for $i \notin I^*$.

In Lemma 9 below, I show that at the solution to Problem $P_i$, the free entry condition binds and also all incentive compatibility constraints are satisfied (not only for $j < i$). Specifically, higher types are not attracted to submarkets that

---

1Major differences are in step 2 and step 3 of Lemma 9, verification of equilibrium condition (ii) in Lemma 10, step 1 and step 5 of Lemma 11 and the proof of Lemma 10.

2It is important to note that sometimes I define a notation that is already defined in the text only for the ease of reference. The repetition of a definition does not mean that they are different, unless stated otherwise.
lower types are attracted to. Lemma 10 states that any equilibrium is constructed by solving Problem $P$. Lemma 11 states that we can construct an equilibrium from any solution to Problem $P$. Since Problem $P$ has a recursive structure and each Problem $P_i$ has a unique solution, the existence and uniqueness of equilibrium follows directly from Lemma 10 and 11. Finally in Lemma 12, I introduce sufficient conditions under which all types are active in equilibrium, that is, $U_i > 0$ for all $i$.

**Lemma 9.** A solution to Problem $P (I^*, \{U_i\}_{i \in \mathbb{Z}}$ and $\{(\theta_i, p_i, x_i)\}_{i \in I^*})$ exists. At any solution, 

$$q(\theta_i)(b(x_i, z_i) - p_i) = C(x_i)$$

$$m(\theta_i)(p_i + e(x_i, z_j)) \leq U_j \text{ for all } j \text{ and } i \in I^*.$$

**Proof.**

**Step 1: Existence of a solution**

I use induction to prove the existence. For $i = 1$, if the constraint set is empty, then set $U_1 = 0$. Otherwise, note that the constraint set is compact and the objective function is continuous, therefore, there exists a solution and a unique maximum. If the maximum value is not strictly positive, then set $U_1 = 0$. Otherwise, set $U_1$ equal to the maximum value.

Now fix $i > 1$ and consider Problem $P_i$. By induction hypothesis, so far we have found $(\theta_j, p_j, x_j)$ for all $j \in I^*$ with $j < i$ and also $(U_1, U_2, ..., U_{i-1})$. If the constraint set is empty, then again set $U_i = 0$. Otherwise, with a similar argument as above, there exists a solution and a unique maximum. If the maximum value is not strictly positive, then set $U_i = 0$. Otherwise, set $U_i$ equal to the maximum value.

**Step 2: The first constraint in Problem $P_i$ is binding.**

I prove this part by way of contradiction. Suppose that for some $i \in I^*$, the first constraint is not binding, that is, $q(\theta_i)(b(x_i, z_i) - p_i) > C(x_i)$. Since $U_i = m(\theta_i)(p_i + e(x_i, z_i))$ is strictly positive, $m(\theta_i)$ is also strictly positive and $(p_i, x_i) \in \bar{Y}$. Therefore, we can find $\epsilon > 0$ sufficiently small such that for all $(p', x') \in B_\epsilon(p_i, x_i) \equiv \{(p', x') \mid (p', x') - (p_i, x_i) < \epsilon\}$:

$$q(\theta_i)(b(x', z_i) - p') > C(x').$$
Since $e_{x^2} > 0$, we have $e(x'', z_i) - e(x, z_i) > e(x'', z_j) - e(x, z_j)$ for all $j < i$ and $x'' > x$. Hence, for any $x'' > x_i$ there exists $p''$ such that $e(x'', z_i) - e(x_i, z_i) > p_i - p'' > e(x'', z_j) - e(x_i, z_j)$. Since $e$ is continuous in $x$, there exists $(p'', x'') \in B_i(p_i, x_i)$ such that

$$p'' + e(x'', z_i) > p_i + e(x_i, z_i),$$

$$p'' + e(x'', z_j) < p_i + e(x_i, z_j) \text{ for all } j < i. \quad (B.1)$$

The triple $(\theta_i, p'', x'')$ satisfies all the constraints of Problem $P_i$ because of the following reasons: The first constraint is satisfied because $(p'', x'') \in B_i(p_i, x_i)$. The second constraint is satisfied, because $m(\theta_i)(p'' + e(x'', z_j)) < m(\theta_i)(p_i + e(x_i, z_j)) \leq U_j$ for all $j < i$, where the last inequality holds because $(\theta_i, p_i, x_i)$ is feasible for Problem $P_i$.

However, the the value of objective function is now strictly higher: $m(\theta_i)(p'' + e(x'', z_i)) > m(\theta_i)(p_i + e(x_i, z_i)) = U_i$. This is a contradiction with $(\theta_i, p_i, x_i)$ being a solution to Problem $P_i$.

**Step 3: The second constraint in Problem $P_i$ holds for all $j$.**

Again, I prove the claim by way of contradiction. Suppose that for some $i \in I^*$, there exists $j > i$ such that $m(\theta_i)(p_i + e(x_i, z_j)) > U_j$. Denote by $h$ the smallest $j$ that has this property. Since $U_i = m(\theta_i)(p_i + e(x_i, z_i))$ is strictly positive, $m(\theta_i)$ is also strictly positive and $p_i + e(x_i, z_i) > 0$. Also, it was shown in the previous step that the first constraint is binding, $q(\theta_i)(b(x_i, z_i) - p_i) = C(x_i)$. Therefore, $q(0)(b(x_i, z_i) - p_i) \geq C(x_i)$. It follows that $(p_i, x_i) \in \bar{Y}$.

I argue below that $(\theta_i, p_i, x_i)$ is feasible for Problem $P_h$:

(a) Free entry condition is satisfied because $q(\theta_i)(b(x_i, z_i) - p_i) \geq q(\theta_i)(b(x_i, z_i) - p_i) = C(x_i)$ where the inequality holds following the assumption that $b_z > 0$ and $h > i$ and also the fact that $(p_i, x_i) \in \bar{Y} \subset \bar{Y}$.

(b) $m(\theta_i)(p_i + e(x_i, z_k)) \leq U_k$ holds for all $k < h$ due to the following reasons: It holds for $k < i$ because $(\theta_i, p_i, x_i)$ is feasible for Problem $P_i$. It holds for $k = i$ due to the definition of $U_i$ and the fact that $i \in I^*$. It also holds for $i < j < h$ due to the construction of $h$ as $h$ was selected to be the lowest index such that the second constraint is violated.

Finally, $(\theta_i, p_i, x_i)$ is feasible and yields strictly higher value than $U_h$, because $m(\theta_i)(p_i + e(x_i, z_h)) > U_h$, which is a contradiction with $U_h$ being the maximum
value of Problem $P_h$.

\[ \Box \]

**Lemma 10.** Suppose $I^*$, $\{U_i\}_{i \in \mathbb{I}}$ and $\{(\theta_i, p_i, x_i)\}_{i \in I^*}$ are a solution to Problem $P$. Then there exists an associated competitive search equilibrium, $\{Y^{eq}, \lambda^{eq}, \theta^{eq}, \Gamma^{eq}\}$, in which type $i$ gets exactly $U_i$, $Y^{eq} = \{y_i\}_{i \in I^*}$ where $y_i = (p_i, x_i)$, $\theta^{eq}(y_i) = \theta_i$ for any $i \in I^*$ and $\gamma_i(y_i) = 1$.

**Proof.** Given $I^*$, $\{U_i\}_{i \in \mathbb{I}}$ and $\{(\theta_i, p_i, x_i)\}_{i \in I^*}$, I construct the equilibrium as follows:

(a) $Y^{eq} = \{y_i\}_{i \in I^*}$ where $y_i = (p_i, x_i)$.
(b) $\theta^{eq}(y_i) = \theta_i$ for any $i \in I^*$.
(c) $\lambda^{eq}(\{y_i\}) = \pi_i \theta_i$ for any $i \in I^*$.
(d) For any other $(p, x)$, denote by $J(p, x) = \{j | p + e(x, z_j) > 0\}$ the set of types that achieve positive utility from $(p, x)$. If $J(p, x) = \emptyset$ or $\min_{j \in J(p, x)} \frac{U_j}{p + e(x, z_j)} < m(\infty)$, then set $\theta^{eq}(p, x) = \infty$. Otherwise, set

\[ m(\theta^{eq}(p, x)) = \min_{j \in J(p, x)} \frac{U_j}{p + e(x, z_j)}. \]

If this equation has multiple solution for $\theta^{eq}(p, x)$, choose the largest one.
(e) $\gamma^{eq}_i(y_i) = 1$ for any $i \in I^*$. For any other $(p, x)$, if $J(p, x) = \emptyset$, then $\Gamma^{eq}(p, x)$ is chosen arbitrarily, for example set $\gamma^{eq}_1(p, x) = 1$. Otherwise, define $h$ to be the smallest element of $\arg \min_{j \in J(p, x)} \frac{U_j}{p + e(x, z_j)}$. Then, set $\gamma_h(p, x) = 1$. In what follows, I show that all equilibrium conditions are satisfied.

**Equilibrium condition (i)** is satisfied for any $y_i$ and $i \in I^*$, because $(\theta_i, p_i, x_i)$ solves Problem $P_i$ and so $q(\theta_i)(b(x_i, z_i) - p_i) = C(x_i)$ according to Lemma 9. Now I want to show that Equilibrium condition (i) is satisfied for any arbitrary post $(p, x)$. I proceed by way of contradiction. Suppose there exists a $(p, x)$ with $q(\theta^{eq}(p, x)) \Sigma_i \gamma^{eq}_i(p, x)(b(x, z_i) - p) > C(x)$. This implies that $q(\theta^{eq}(p, x)) > 0$ and consequently $\theta^{eq}(p, x) < \infty$. This also implies that there exists $j$ such that $\gamma^{eq}_j(p, x) > 0$ and $q(\theta^{eq}(p, x))(b(x, z_j) - p) > C(x)$. Note that $\gamma^{eq}_j(p, x) > 0$ and $\theta^{eq}(p, x) < \infty$. It follows from the construction of equilibrium objects that $j$ must be the smallest element of $\arg \min_{k \in J(p, x)} \frac{U_k}{p + e(x, z_k)}$. Therefore, for $l < j$, if $p + e(x, z_l) > 0$, then $m(\theta^{eq}(p, x))(p + e(x, z_l)) < U_j$. If $p + e(x, z_l) < 0$, then $m(\theta^{eq}(p, x))(p + e(x, z_l)) \leq U_j$ because $U_j \geq 0$. Therefore, $(\theta^{eq}(p, x), p, x)$ is feasi-
ble for Problem $P_j$, but the first constraint is not binding. According to Lemma 9, there must exist $(\theta', p', x')$ which is also feasible for Problem $P_j$, but the value of the objective functions is greater, that is, $m(\theta')(p' + e(x', z_j)) > U_j \geq 0$. This contradicts the fact that $U_j$ is the maximum value of Problem $P_j$.

**Equilibrium condition (ii).** First note that $m(\theta^q(p, x))(p + e(x, z_i)) \leq U_i$ for all $(p, x) \in Y$ with equality if $\theta^q(p, x) < \infty$ and $\gamma^q_i(y) > 0$, given the way that we construct $\theta^q$ and $\gamma^q$. Also, for $i \in I^*$, $U_i = m(\theta_i)(p_i + e(x_i, z_i)) > 0$. For any $(p, x) \in Y$ with $p_i + e(x_i, z_i) < 0$ for some $i$, then $i \notin J(p, x)$. Hence, if $J(p, x) = \emptyset$ then $\theta^q(p, x) = \infty$, and if $J(p, x) \neq \emptyset$ then $\gamma^q_k(p, x) = 1$ for some $k \in J(p, x)$, which implies that $\gamma^q_i(p, x) = 0$.

**Equilibrium condition (iii)** is obviously satisfied following the construction of $\lambda^q$.

\[\square\]

**Lemma 11.** Take a competitive search equilibrium, $\{Y^q, \lambda^q, \theta^q, \Gamma^q\}$. Denote by $U^q_i$ the payoff of type $i$ in the equilibrium. Denote by $I^eq$ the set of types that get a strictly positive payoff in the equilibrium. For each $i \in I^eq$, a sub-market $(p, x) \in Y^q$ with $\theta^q(p, x) < \infty$ and $\gamma^q_i(p, x) > 0$ exists. Also, take any $\{(\theta_i, p_i, x_i)\}_{i \in I^eq}$ with $\theta_i = \theta^q(p_i, x_i) < \infty$ and $\gamma^q_i(p_i, x_i) > 0$. Then $I^eq$, $\{U^q_i\}_{i \in \mathbb{Z}}$ and $\{(\theta_i, p_i, x_i)\}_{i \in I^eq}$ solve Problem $P$.

**Proof.** The existence part is easy to show. It follows from equilibrium condition (i) that for any $(p, x) \in Y^q$, $q(\theta^q(y)) > 0$ and consequently, $\theta^q(y) < \infty$. Also, it follows from $U^q_i > 0$ that $\gamma^q_i(p, x) > 0$ for some $(p, x) \in Y^q$. Therefore, there exists a sub-market $(p, x) \in Y^q$ with $\theta^q(p, x) < \infty$ and $\gamma^q_i(p, x) > 0$ for each $i \in I^eq$.

The main task in this proof is to show that for any $i \in I^eq$ and $(\theta_i, p_i, x_i) \in Y^q$ with $\theta_i = \theta^q(p_i, x_i) < \infty$ and $\gamma^q_i(p_i, x_i) > 0$, $(\theta_i, p_i, x_i)$ solves Problem $P_i$. I proceed in the following 5 steps:

1. **The first constraint in Problem $P_i$ is satisfied**, $q(\theta_i)(b(x_i, z_i) - p_i) \geq C(x_i)$.

   Fix $i \in I^eq$ and $(\theta_i, p_i, x_i) \in Y^q$ with $\theta_i = \theta^q(p_i, x_i) < \infty$ and $\gamma^q_i(p_i, x_i) > 0$. First, $i \in I^eq$ implies that $U^q_i > 0$. Second, $U^q_i = m(\theta_i)(e(x_i, z_j) + p_i)$ by equilibrium condition (ii). $U^q_i > 0$ implies that $m(\theta_i) > 0$. Now assume by of
contradiction that \( q(\theta)(b(x_i, z_i) - p_i) < C(x_i) \). According to equilibrium condition (i), \( q(\theta_i)\gamma_j^eq(p_i, x_i)(b(x_i, z_j) - p_i) = C(x_i) \). Therefore, there must exist another type, \( l \), such that \( q(\theta_l)(b(x_i, z_l) - p_i) > C(x_i) \) and \( \gamma_j^eq(p_i, x_i) > 0 \). This implies that \( q(0)(b(x_i, z_l) - p_i) > C(x_i) \). Also, \( \theta_i < \infty \) and \( \gamma_j^eq(p_i, x_i) > 0 \) together with equilibrium condition (ii) imply that \( m(\theta_i)(e(x_i, z_l) + p_i) \geq 0 \). Therefore, \( (p_i, x_i) \in \tilde{Y}_l \).

Now, define \( \epsilon > 0 \) to be sufficiently small such that \( q(\theta_i)(b(x, z_l) - p) > C(x) \) for all \( (p, x) \in B_\epsilon(p_i, x_i) \). Similar to part 2 of Lemma 9, we use the fact that \( e_{xz} > 0 \). Since \( (p_i, x_i) \in \tilde{Y}_l \), there exists \( (p'', x'') \in B_\epsilon(p_i, x_i) \) such that

\[
p'' + e(x'', z_j) > p_i + e(x_i, z_j) \quad \text{for all} \quad j \geq l,
\]

\[
p'' + e(x'', z_j) < p_i + e(x_i, z_j) \quad \text{for all} \quad j < l.
\]

Also note that \( (p'', x'') \in \tilde{Y}_l \), because \( p'' + e(x'', z_l) > p_i + e(x_i, z_l) \geq 0 \) and \( q(0)(b(x'', z_l) - p'') \geq q(\theta_i)(b(x'', z_l) - p'') > C(x'') \).

Define \( \theta'' = \theta^{eq}(p'', x'') \). Then,

\[
m(\theta'')(p'' + e(x'', z_l)) \leq U_i^{eq} = m(\theta_i)(p_i + e(x_i, z_l)) < m(\theta_i)(p'' + e(x'', z_l)).
\]

The weak inequality follows from the definition of \( \theta'' \) and \( U_i^{eq} \). The equality holds because \( \theta_i < \infty \) and especially because type \( l \) goes to \( (p_i, x_i) \) with positive probability \( (\gamma_j^eq(p_i, x_i) > 0) \). The strict inequality follows from construction of \( (p'', x'') \). It immediately follows that \( \theta'' < \theta_i \) (because \( p'' + e(x'', z_l) > 0 \) and \( m(.) \) is weakly increasing).

For all \( j < l \), if \( p'' + e(x'', z_j) < 0 \), then the second equilibrium condition implies that \( \gamma_j^eq(p'', x'') = 0 \). Otherwise, \( m(\theta'')(p'' + e(x'', z_j)) < m(\theta_i)(p_i + e(x_i, z_j)) \leq U_j^{eq} \). The strict inequality follows from \( m(\theta'') < m(\theta_i) \) and also the construction of \( (p'', x'') \). The weak inequality follows from equilibrium condition (i). Hence, \( m(\theta'')(p'' + e(x'', z_j)) < U_j^{eq} \) implies \( \gamma_j^eq(p'', x'') = 0 \) for all \( j < l \).

Now,

\[
q(\theta'') \sum_j \gamma_j^eq(p'', x'')(b(x'', z_j) - p'') \geq q(\theta'')(b(x'', z_l) - p'') \geq q(\theta_i)(b(x'', z_l) - p''') > C(x'').
\]

The LHS is the firm’s profit from posting \( (p'', x'') \). The first inequality uses
\( \gamma_j^{eq}(p'', x'') = 0 \) for all \( j < l \), the fact that \( b \) is increasing in \( z \) and also that \((p'', x'') \in \tilde{Y}_i \subset \hat{Y}\). The second inequality uses \( \theta'' < \theta_i \). The last inequality uses the construction of \((p'', x'') \). This shows that the firm gets more profit from posting \((p'', x'') \) than what the firm gets in the equilibrium. This contradiction completes the proof in this step.

2. The second constraint in Problem \( P_i \) is satisfied, \( m(\theta_i)(e(x_i, z_j) + p_i) \leq U_j^{eq} \) for all \( j \).

This step is straightforward. Again, fix \( i \in I^{eq} \) and \((\theta_i, p_i, x_i) \in Y^{eq} \) with \( \theta_i = \theta^{eq}(p_i, x_i) < \infty \) and \( \gamma_i^{eq}(p_i, x_i) > 0 \). It follows from equilibrium condition (ii) that \( m(\theta_i)(e(x_i, z_j) + p_i) \leq U_j^{eq} \) for all \( j \).

3. Type \( i \) gets exactly \( U_i^{eq} \) from the triple \((\theta_i, x_i, p_i)\).

Again, similar to previous step, fix \( i \in I^{eq} \) and \((\theta_i, p_i, x_i) \in Y^{eq} \) with \( \theta_i = \theta^{eq}(p_i, x_i) < \infty \) and \( \gamma_i^{eq}(p_i, x_i) > 0 \). By way of contradiction assume that there exists another triple \((\theta, p, x)\) which is in the constraint set of Problem \( P_i \) but delivers strictly higher payoff than \((\theta_i, x_i, p_i)\). That is, \( q(\theta)(e(x, z_i) + p) \geq C(x) \), \( m(\theta)(e(x, z_j) + p) \leq U_j^{eq} \) for all \( j < i \) and \( m(\theta)(e(x, z_i) + p) > U_i^{eq} \).

Before we proceed, note that \((p, x) \in \tilde{Y} \), because of the following: First, \( m(\theta)(e(x, z_i) + p) > U_i^{eq} \) implies that \( m(\theta) > 0 \) and \( e(x, z_i) + p > 0 \). Second, \( q(\theta)(b(x, z_i) - p) \geq C(x) > 0 \) implies that \( q(\theta) > 0 \) and \( b(x, z_i) - p \geq 0 \) and

\[3\text{Note that } q(\theta) > 0 \text{ holds with inequality. Assume by way of contradiction that } q(\theta) = 0. \text{ This implies that } \theta = \infty. \text{ This also implies that } C(x) = 0, \text{ and consequently, } x = 0. \text{ Since } x = 0, \text{ from the constraints in Problem } P_i, \text{ we have } m(\theta)p \leq U_j \text{ for all } j < i. \text{ But } U_i = m(\theta)p, \text{ therefore, }\]

\[ U_i \leq U_j \text{ for all } j < i. \]

Now consider the IC for type \( i \) to go to a submarket for type \( j < i \). According to Lemma 9, we must have \( U_i \geq m(\theta_j)(p_j + e(x_j, z_i)) = m(\theta_j)(p_j + e(x_j, z_i)) + m(\theta_j)(e(x_j, z_i) - e(x_j, z_j)) = U_j + m(\theta_j)e(x_j, z_i) - e(x_j, z_j) \). But \( U_i \leq U_j \), so we must have \( \theta_j = 0 \) and consequently \( x_j = 0 \). However, \( \theta_j \) cannot be equal to 0, because all types get strictly positive payoff in the equilibrium according to Proposition 1. Therefore, \( x_j = 0 \). Now from \( q(\theta_j)(h(x_j) - p_j) \geq C(x_j) \) and from the fact that \( p_j > 0 \), it follows that \( q(\theta_j) = 0 \) or \( \theta_j = \infty \). Notice that \( \theta_j = \infty \) holds true for any
particularly \(q(0)(b(x, z_i) - p) \geq C(x)\).

We can choose \(p'\) smaller but sufficiently close to \(p\) and \(x' = x\) such that the following holds true:
\[
\frac{U_i^{eq}}{m(\theta)} - e(x', z_i) < p' < p.
\]

To summarize, so far we have found \((p', x')\) such that \(q(\theta)(b(x', z_i) - p') - C(x') > q(\theta)(b(x, z_i) - p) - C(x) \geq 0, m(\theta)(e(x', z_j) + p') \leq U_j^{eq}\) for all \(j < i\) and \(m(\theta)(e(x', z_i) + p') > U_i^{eq}\). Now, we can find a sufficiently small \(\epsilon > 0\) such that for all \((p, x) \in B_\epsilon(p', x')\), \(q(\theta)(b(x, z_i) - p) > C(x)\) and \(m(\theta)(e(x, z_i) + p) > U_i^{eq}\). Similar to step 2 of the proof of Lemma 9, according to Equation B.1, there exists \((p'', x'') \in B_\epsilon(p', x')\) such that
\[
p'' + e(x'', z_i) > p' + e(x', z_i),
\]
\[
p'' + e(x'', z_j) < p' + e(x', z_j) \text{ for all } j < i.
\]

Therefore, \(q(\theta)(b(x'', z_i) - p'') > C(x'')\), \(m(\theta)(e(x'', z_j) + p'') < U_j^{eq}\) for all \(j < i\) and \(m(\theta)(e(x'', z_i) + p'') > U_i^{eq}\).

Finally, I show that posting \((p'', x'')\) in the market yields strictly positive profit for firms while other conditions for the equilibrium are satisfied.

First, note that according to equilibrium condition (ii),
\[
U_i^{eq} \geq m(\theta^{eq}(p'', x''))(e(x'', z_i) + p'').
\]

This together with \(m(\theta)(e(x'', z_i) + p'') > U_i^{eq}\) implies that \(m(\theta) > m(\theta^{eq}(p'', x''))\).
This in turn implies that \(q(\theta^{eq}(p'', x''))(b(x'', z_i) - p'') > C(x'')\) and also that \(\theta^{eq}(p'', x'') < \infty\).

Second, I show that types below \(i\) will not choose \((p'', x'')\), that is, \(\gamma_j^{eq}(p'', x'') = 0\) for all \(j < i\). By way of contradiction suppose that there exists a type \(j < i\) with \(\gamma_j^{eq}(p'', x'') > 0\). It follows from equilibrium condition (ii) and also from \(\theta^{eq}(p'', x'') < \infty\) that \(e(x'', z_j) + p'' \geq 0\). But
\[
m(\theta)(e(x', z_j) + p') > m(\theta^{eq}(p'', x''))(e(x'', z_j) + p''),
\]
selection of submarkets for type \(j\), thus condition (iii) in equilibrium definition is violated. This is a contradiction.
because it was already shown that \( m(\theta) > m(\theta^{eq}(p'', x'')) \) and \( e(x', z_j) + p' > e(x'', z_j) + p'' \) by construction of \((p'', x'')\). It is required by equilibrium condition (ii) that \( U_j^{eq} = m(\theta)(e(x', z_j) + p') \) and consequently \( U_j^{eq} > m(\theta^{eq}(p'', x''))(e(x'', z_j) + p'') \). Since \( \theta^{eq}(p'', x'') < \infty \), equilibrium condition (ii) implies that \( \gamma_j^{eq}(p'', x'') = 0 \). Now, we calculate the profit from posting \((p'', x'')\):

\[
q(\theta^{eq}(p'', x'')) \sum_j \gamma_j^{eq}(p'', x'')(b(x'', z_j) - p'') \geq q(\theta^{eq}(p'', x''))(b(x'', z_i) - p'') > C(x''),
\]

where the weak inequality comes from the monotonicity of \( b \) in \( z \) and also from the fact that \( \gamma_j^{eq}(p'', x'') = 0 \) for all \( j < i \). This is the desired contradiction and completes this step.

5. For any \( i \notin I^{*eq} \), the constraint set in Problem \( P_i \) is empty or the maximum value of the problem is weakly negative.

Suppose by way of contradiction that the constraint set in Problem \( P_i \) is non-empty and also that the maximum value of the problem is strictly positive. That is, there exists a triple \((\theta, p, x)\) such that \( q(\theta)(b(x, z_i) - p) \geq C(x), m(\theta)(e(x, z_i) + p) \leq U_j^{eq} \) for all \( j < i \) and \( m(\theta)(e(x, z_i) + p) > U_i^{eq} = 0 \). The proof is exactly similar to the previous step: First, we find \((p', x')\) such that \( q(\theta)(b(x', z_i) - p') > C(x') \), \( m(\theta)(e(x', z_j) + p') \leq U_j^{eq} \) for all \( j < i \) and \( m(\theta)(e(x', z_i) + p') > 0 \). Then, we find \((p'', x'')\) such that \( q(\theta)(b(x'', z_i) - p'') > C(x'') \), \( m(\theta)(e(x'', z_j) + p'') < U_j^{eq} \) for all \( j < i \) and \( m(\theta)(e(x'', z_i) + p'') > 0 \), so we make sure that no type below \( i \) applies to \((p'', x'')\). Finally, we show that posting \((p'', x'')\) is a profitable deviation. A contradiction.

\[\square\]

**Lemma 12.** Competitive search equilibrium exists and the equilibrium payoffs are unique.

**Proof.** This lemma is straightforward given previous lemmas. In Lemma 9, it was shown that Problem \( P \) has a solution. Then, we showed in Lemma 10 that if \( I^* \), \( \{U_i\}_{i \in Z} \) and \( \{(\theta_i, p_i, x_i)\}_{i \in I^*} \) are a solution to Problem \( P \), then there is an associated equilibrium. Therefore, existence of an equilibrium is established.

Regarding uniqueness, it was shown in Lemma 11 that given any equilibrium, \( \{Y^{eq}, \lambda^{eq}, \theta^{eq}, \Gamma^{eq}\} \), the payoff that type \( i \) gets in the equilibrium, \( U_i^{eq} \), is the maximum value of Problem \( P_i \) for all \( i \in I^* \) and equals to 0 for \( i \notin I^* \). In Lemma 9, we
showed that the maximum value of Problem $P_i$ is unique. Therefore, the payoff that each type gets in equilibrium must be unique.

\[ \]

**Lemma 13.** Assume that for any $i$, there exists some $x$ such that $q(0)(b(x, z_i) + e(x, z_i)) > C(x)$. Then $U_i^{eq} > 0$ for all $i$ in any competitive search equilibrium. Specially, for any $i$ there exists a pair $(p, x)$ with $\theta^{eq}(p, x) < \infty$ and $\gamma_i^{eq}(p, x) > 0$.

**Proof.** Fix $i = 1$. There exists $x_1$ such that $q(0)(b(x_1, z_1) + e(x_1, z_1)) > C(x_1)$. Therefore, we can find a $p_1 \in \mathbb{R}$ such that $q(0)(b(x_1, z_1) + p_1) > C(x_1)$ and $p_1 + e(x_1, z_1) > 0$. Now, fix $\theta_1$ such that $q(\theta_1)(b(x_1, z_1) + p_1) = C(x_1)$. Obviously, $(\theta_1, p_1, x_1)$ satisfies the constraint of Problem $P_1$ and delivers a strictly positive payoff, $m(\theta_1)(e(x_1, z_1) + p_1) > 0$, which means that $U_1^{eq}$ must be strictly positive.

I proceed by induction. Fix $i > 1$ and assume that $U_j^{eq} > 0$ for all $j < i$. Again, there exists $x_i$ such that $q(0)(b(x_i, z_i) + e(x_i, z_i)) > C(x_i)$. Therefore, $p_i \in \mathbb{R}$ can be found such that $q(0)(b(x_i, z_i) + p_i) > C(x_i)$ and $p_i + e(x_i, z_i) > 0$. Now, fix $\theta_i > 0$ such that $q(\theta_i)(b(x_i, z_i) + p_i) \geq C(x_i)$ and $m(\theta_i)(p_i + e(x_i, z_j) \leq U_j^{eq}$ for all $j < i$. This is possible, because $q(.)$ is weakly decreasing, $m(.)$ is weakly increasing and $m(0) = 0$. Therefore, $(\theta_i, p_i, x_i)$ satisfies the constraints of Problem $P_i$ and delivers $m(\theta_i)(e(x_i, z_i) + p_i) > 0$ which means that $U_i^{eq}$ must be strictly positive. This completes the proof.

**Remark 1.** The following conditions are sufficient for Lemma 13 to hold: There are positive gains from trade for all types and $m$ is concave and differentiable.

**Proof.** Let $(\theta_i^{FB}, x_i^{FB}) = \arg \max_{\theta, x} [m(\theta)(b(x_i, z_i) + e(x, z_i)) - \theta C(x)]$. The value of the objective function is strictly positive because there are positive gains from trade for all types. Consider the following maximization problem: $\max_{\theta} [m(\theta)(b(x_i^{FB}, z_i) + e(x_i^{FB}, z_i)) - \theta C(x_i^{FB})]$. Concavity of $m$ and the fact that the maximum value is strictly positive imply that $m'(\theta)(b(x_i^{FB}, z_i) + e(x_i^{FB}, z_i)) = C(x_i^{FB})$ at $\theta_i^{FB}$. But $m'(\theta) = \theta q'(\theta) + q(\theta) < q(\theta)$ for all $\theta > 0$, where the inequality holds because $q'(\theta) < 0$. Therefore, $q(0)(b(x_i^{FB}, z_i) + e(x_i^{FB}, z_i)) \geq q(\theta_i^{FB})(b(x_i^{FB}, z_i) + e(x_i^{FB}, z_i)) > C(x_i^{FB})$. The proof is complete because for every $i$ we have found some $x$ such that $q(0)(b(x, z_i) + e(x, z_i)) > C(x)$.
B.2 Rest of the Proofs for Discrete Type Space

Proof of Lemma 3.

Part 1

I prove it by induction. For $i = 1$, the two problems, $P_1$ and $Q_1(0)$, are the same, so the statement is trivial. Fix $i > 1$. By induction hypothesis, if $(\theta_j, x_j)$ solves Problem $Q_j(\bar{s}_{j-1})$, then $(\theta_j, x_j, b(x_j, z_j) - C(x_j)_{q(\theta_j)})$ is a solution to Problem $P_j$ for all $j < i$. We want to show that the statement is true for $i$, so we need to show that (i) $(\theta_i, x_i, b(x_i, z_i) - C(x_i)_{q(\theta_i)})$ is feasible for Problem $P_i$, (ii) $(\theta_i, x_i, b(x_i, z_i) - C(x_i)_{q(\theta_i)})$ achieves the maximum in Problem $P_i$.

Regarding (i), I have already shown the feasibility of $(\theta_i, x_i)$ for Problem $P_i$ in the text following Lemma 3 (esp. in Equation 2.1).

To show (ii), suppose Problem $P_i$ has a solution, $(\theta', x', p')$, that achieves strictly higher payoff than $(\theta_i, x_i, b(x_i, z_i) - C(x_i)_{q(\theta_i)})$. To get a contradiction, I show that $(\theta', x')$ satisfies all constraints of Problem $Q_i(\bar{s}_{i-1})$. The first constraint is trivially satisfied, because it’s common in Problem $Q_i(\bar{s}_{i-1})$ and Problem $P_i$. Moreover, the solution to Problem $P$ characterizes the equilibrium and so according to Lemma 2, the monotonicity constraint must hold, that is, $m(\theta')A(x') \geq m(\theta_{i-1})A(x_{i-1})$ for all $\theta_{i-1}$ and $x_{i-1}$, thus $m(\theta')A(x') \geq \bar{s}_{i-1}$, so the second constraint in Problem $Q_i(\bar{s}_{i-1})$ is also satisfied. The last constraint, $m(\theta')A(x') \leq A(x')$ is also trivially satisfied because $m(\theta') \leq 1$. This is a contradiction, because $(\theta', x')$ is feasible for Problem $Q_i(\bar{s}_{i-1})$ but delivers strictly higher payoff than $(\theta_i, x_i)$.

Part 2

Suppose $(\theta_i, x_i, p_i)$ solves Problem $P_i$, we want to show that $(\theta_i, x_i)$ is a solution to Problem $Q_i(\bar{s}_{i-1})$. Again, we use induction. For $i = 1$, the two problems are the same, so the statement is trivial. By induction hypothesis, any solution to $P_j$ is a solution to $Q_j(\bar{s}_{j-1})$ for $j < i$. In order to show that the statement is true for $i$, we need to show that (i) $(\theta_i, x_i)$ is feasible for Problem $Q_i(\bar{s}_{i-1})$, (ii) $(\theta_i, x_i)$ achieves the maximum in Problem $Q_i(\bar{s}_{i-1})$.

Regarding (i), first note that by induction hypothesis, any solution to Problem $P_{i-1}$ is a solution to Problem $Q_{i-1}(\bar{s}_{i-1})$, and by part 1, any solution to Problem $Q_{i-1}(\bar{s}_{i-1})$ is a solution to Problem $P_{i-1}$. Therefore, $\bar{s}_{i-1}$ calculated based on the solution to Problem $P_{i-1}$ is the same as that calculated based on the so-
olution to Problem $Q_{i-1}(\bar{s}_{i-1})$. Second, it has been shown that Problem $P$ characterizes equilibrium allocation, so monotonicity constraint will hold. Specifically, $m(\theta_i)A(x_i) \geq m(\theta_{i-1})A(x_{i-1})$ for all $\theta_{i-1}$ and $x_{i-1}$, thus $m(\theta_i)A(x_i) \geq \bar{s}_{i-1}$.

To show (ii), suppose Problem $Q_i(\bar{s}_{i-1})$ has a solution, $(\theta', x')$, that achieves strictly higher payoff than $(\theta_i, x_i)$. To derive a contradiction, I show that $(\theta', x', p')$, where $p' \equiv b(x', z_i) - \frac{C(x')}{q(\theta')}$, satisfies all constraints in Problem $P_i$ but delivers higher payoff than $(\theta_i, x_i, b(x_i, z_i) - \frac{C(x_i)}{q(\theta_i)})$.

The first constraint is satisfied due to the construction of $p' = b(x', z_i) - \frac{C(x')}{q(\theta')}$.

Now I show that the other constraint, $m(\theta')(b(x', z_i) + e(x', z_j)) - \theta'C(x') \leq U_j$, holds for all $j < i$. The constraint is trivially satisfied for $j = i - 1$, because it is the same for Problem $P_i$ and Problem $Q_i(\bar{s}_{i-1})$. Now assume by way of contradiction that the constraint is violated for some $j < i - 1$. Fix $j$ such that $m(\theta')(b(x', z_i) + e(x', z_j)) - \theta'C(x') > U_j$. Then

$$U_{i-1} \geq m(\theta')(b(x', z_i) + e(x', z_{i-1})) - \theta'C(x')$$

$$= m(\theta')(b(x', z_i) + e(x', z_j)) - \theta'C(x') + m(\theta')(e(x', z_{i-1}) - e(x', z_j))$$

$$> U_j + m(\theta')(e(x', z_{i-1}) - e(x', z_j)) = U_j + m(\theta')A(x')(D_{i-1} - D_j)$$

$$\geq U_j + m(\theta_{i-1})A(x_{i-1})(D_{i-1} - D_j) = U_j + m(\theta_{i-1})(e(x_{i-1}, z_{i-1}) - e(x_{i-1}, z_j)) \quad (B.2)$$

The first and last inequalities follow from the fact that the first and second constraints in Problem $Q_i(\bar{s}_{i-1})$ hold, respectively. The strict inequality is based on our hypothesis that the constraint in Problem $P_i$ is violated for type $j$.

Remember the induction hypothesis that for all $k \leq i - 1$, Problem $Q_k(\bar{s}_{k-1})$ and Problem $P_i$ have the same solution. Specifically one of the constraints in Problem $P_{i-1}$ is as follows

$$U_{i-1} + m(\theta_{i-1})(e(x_{i-1}, z_{i-1}) - e(x_{i-1}, z_j)) \leq U_j \text{ for all } j < i - 1,$$

which obviously contradicts the inequality in Equation B.2. The proof is complete, because we have found a feasible $(\theta', x', p')$ for Problem $P_i$ which achieves higher utility than the problem’s maximized value, a contradiction.

$\square$

*Proof of Lemma 4*. First, I write Problem $P_i$ in another form which is extremely
convenient for the purpose of this proof:

**Problem 12.**

\[
\max_{\theta \in [0, \infty], x \in X, U_i \in \mathbb{R}_+} U_i \\
\text{s.t. } U_i \leq m(\theta)(b(x, z_i) + e(x, z_i)) - \theta C(x) \tag{B.3}
\]

and \( U_i \leq U_j + m(\theta)(e(x, i) - e(x, j)) \) for all \( j < i \). \tag{B.4}

If no constraint in Equation B.4 is binding, then \( U_i = m(\theta)(b(x_{FB}, z_i) + e(x_{FB}, z_i)) - \theta^{FB} C(x_{FB}) \) and there is a unique maximizer according to part 2 of Assumption 7. If some constraints in Equation B.4 are binding, then the first best level is not achievable. If at least one constraint in Equation B.4 is binding for \( j \leq i - 2 \), then I show that there is a unique solution to Problem \( P \) (and consequently Problem \( Q_i(s_{i-1}) \) for all \( i \)). If the constraint in Equation B.4 is binding for \( j = i - 1 \), then I show separately that the solution to Problem \( P \) for \( s \) is unique.

**Step 1: If at least one constraint in Equation B.4 is binding for \( j \leq i - 2 \), then the solution for \( s \) is unique.**

The first claim here is that if at least one constraint in Equation B.4 is binding for \( j \leq i - 2 \), that is, \( U_i = U_j + s_i(D_i - D_j) \), then the constraint must be binding for \( i - 1 \) as well and also \( s_i = s_{i-1} \).

But

\[
U_{i-1} - U_j \leq s_{i-1}(D_{i-1} - D_j) \leq s_i(D_{i-1} - D_j)
\]

\[
= s_i(D_i - D_j) - s_i(D_i - D_{i-1}) = U_i - U_j - s_i(D_i - D_{i-1}) \leq U_{i-1} - U_j
\]

The first inequality comes from the constraint with index \( j \) in the problem for type \( i - 1 \). It was already shown in Lemma 2 that \( s_i \) is increasing in \( i \) for all \( i \), so the second inequality holds. The second equality holds because the constraint with index \( j \) is binding in the problem for type \( i \) by assumption. The last inequality holds, because in the problem for type \( i \), the constraint must hold for \( i - 1 \).

Since the right hand side and the left hand side are equal, so all weak inequalities hold with equality. Therefore, \( s_i = s_{i-1} \) which implies that \( s_i \) is unique. It also implies that

\[
U_i - s_i(D_i - D_{i-1}) = U_{i-1}
\]
from the last inequality. Therefore, both constraints in Problem $Q_i(\bar{s}_{i-1})$ must be binding in this case.

**Step 2:** If the constraint in Equation B.4 is binding for $j = i - 1$, then the solution for $s$ is unique.

In order to prove this part, I use another representation of the problem in Problem $R_i(\bar{s}_{i-1})$ as defined in the text. For the ease of reference, I repeat it here:

**Problem 13 ($R_i(\bar{s})$).**

\[
\max_{s,x} \Pi(s, x, z_i)
\]

s. t. $\Pi(s, x, z_i) \leq U_{i-1} + \delta_i s$ and $\bar{s} \leq s \leq A(x)$.

Suppose by way of contradiction that there are multiple values for $s$ which solve Problem $R_i(\bar{s}_{i-1})$, say $s_A$ and $s_B$. Without loss of generality, assume $s_B > s_A$. Both $s_A$ and $s_B$ are feasible, so both of them are greater than $s_{i-1}$. Therefore, $s_B > s_A \geq s_{i-1}$. Hence, for $s_B$, the constraint $s_B \geq s_{i-1}$ is not binding, so the other constraint should be binding (otherwise there will be a unique solution). Since $s_B$ and $s_A$ are solutions to Problem $R_i(\bar{s}_{i-1})$, hence $\Pi(s_B, x_B, z_i) = U_{i-1} + \delta_i s_B$ and $\Pi(s_A, x_A, z_i) = U_{i-1} + \delta_i s_A$ for some $x_B$ and $x_A$, respectively. But

\[
\Pi(s_B, x_B, z_i) = U_{i-1} + \delta_i s_B > U_{i-1} + \delta_i s_A \geq \Pi(s_A, x_A, z_i)
\]

The first equality holds because we have shown above that the this constraint must be binding if the first best is not achievable. The second inequality follows from the fact that $(s_A, x_A)$ is feasible. But this is a contradiction with $(s_A, x_A)$ being a solution to Problem $R_i(\bar{s}_{i-1})$. Therefore, the solution to Problem $Q_i(\bar{s}_{i-1})$ for $s$ is unique.

**Step 3:** $\Pi(\cdot, x, \cdot)$ is strictly concave in $x$.

We need to show that $\Pi(s, x, \theta)$ is strictly concave in $x$ so it will have at most one maximizer given $s$.\(^4\) The first derivative of $\Pi$ with respect to $x$ is calculated

---

\(^4\)I could alternatively eliminate $x$ from the objective function and write the objective function only in terms of $\theta$ (instead of $x$). Then I would use Assumption 12 to establish concavity of the objective function.
as follows:

\[
\frac{\partial \Pi(s, x, z)}{\partial x} = \frac{b_x(x, z)A(x) - b(x, z)A'(x)}{A^2(x)} - \rho\left(\frac{s}{A(x)}\right)C'(x) + \rho'\left(\frac{s}{A(x)}\right)\frac{sA'(x)}{A^2(x)}C(x),
\]

where I used \(\rho(.) \equiv m^{-1}(.)\). The second derivative is also calculated as follows:

\[
\frac{\partial^2 \Pi(s, x, z)}{\partial x^2} = \frac{s b_{xx}(x, z)A(x) - b(x, z)A''(x)}{A^2(x)} - 2sA'(x)\frac{b_x(x, z)A(x) - b(x, z)A'(x)}{A^3(x)}
\]

\[
+ \rho'\left(\frac{s}{A(x)}\right)\frac{sA'(x)}{A^2(x)}C'(x) - \rho\left(\frac{s}{A(x)}\right)C''(x)
\]

\[
+ s\rho''\left(\frac{s}{A(x)}\right)\left(\frac{C''(x)A'(x) + C(x)A''(x)}{A^2(x)} - 2\frac{C(x)A'^2(x)}{A^3(x)}\right) - s^2 \rho''\left(\frac{s}{A(x)}\right)\frac{C(x)A'^2(x)}{A^4(x)}
\]

\[
= m(.) \frac{b_{xx}(x, z)A(x) - b(x, z)A''(x)}{A(x)} - 2m(.)A'(x)\frac{b_x(x, z)A(x) - b(x, z)A'(x)}{A^2(x)}
\]

\[
+ m(.) \frac{A'(x)}{m'(x)} A(x) C'(x) \theta C''(x)
\]

\[
+ m(.) \frac{C''(x)A'(x) + C(x)A''(x)}{A(x)} - 2C(x)A'^2(x)
\]

\[
+ \frac{m^2(.)m''(.)C(x)A'^2(x)}{m^3(.)A^2(x)}
\]

To derive the last equality, I just substituted \(\rho\) and its derivatives according to the following equations: \(\rho'\left(\frac{s}{A(x)}\right) = \theta\), \(\rho'\left(\frac{s}{A(x)}\right) = \frac{1}{m'(\theta)}\), \(\rho''\left(\frac{s}{A(x)}\right) = -\frac{m''(\theta)}{m^3(.)}\). The sum of the first two terms in the RHS of the above equation is positive due to part 1 of Assumption 11. I show below that the rest of the expression is positive according to part 2 of Assumption 11.
\[
-\theta \left\{ C''(x) - \frac{q(.)}{m'(.)} \left( 2C'(x)A'(x) + C(x)A''(x) \right) + \left( \frac{q(.)}{m'(.)} \right)^2 \frac{2C(x)A^2(x)}{A'(x)} \right\} 
\]

The second equality is derived after doing some algebra to rearrange terms. We need to show that the whole expression in accolades is positive. The lower line is positive because \( m''(.) \geq 0 \) and \( \max_{\theta} \left\{ \frac{q'(\theta)m'(\theta)}{q(\theta)m''(\theta)} \right\} \leq 0.5 \) under part 1 of Assumption 7 and Assumption 9. The upper line in the last accolades is a polynomial of degree 2 in \( q(\cdot)m'(\cdot) \). But \( q(\cdot)m'(\cdot) \) is greater than or equal to 1 for any \( \theta \), that is, \( q(\cdot)m'(\cdot) \in [1, \infty) \).

Therefore, we can separate two cases to derive sufficient conditions which ensure that the polynomial is always positive. If

\[
\frac{2C'(x)A'(x) + C(x)A''(x)}{A(x)} < 1,
\]

then the polynomial is positive for any \( \theta \) if it is positive for \( \frac{q(.)}{m'(.)} = 1 \). Therefore, we need the following condition:

\[
C''(x) - \frac{2C'(x)A'(x) + C(x)A''(x)}{A(x)} + \frac{2C(x)A^2(x)}{A'(x)} \geq 0.
\]

In contrast, if

\[
\frac{2C'(x)A'(x) + C(x)A''(x)}{A(x)} \geq 1,
\]

then then the polynomial is positive if it is positive at its trough:

\[
C''(x) - \frac{\left( \frac{2C'(x)A'(x) + C(x)A''(x)}{A(x)} \right)^2}{\frac{8C(x)A^2(x)}{A'(x)}} \geq 0.
\]

After some simplifications, one can check that these conditions are the same as those in part 2 of Assumption 11. Intuitively speaking, in both cases we require \( C'(x) \) to be sufficiently convex.

**Step 4:**

If \((s_B, x_B)\) solves Problem \(Q_i(s_{i-1})\) for type \(i\), then \(x_B \in \arg \max_{A^{-1}(s_B) \leq x} \Pi(s_B, x, z_i)\).
Suppose \((s_B, x_B)\) solves the problem but \(x_B \notin \arg \max_{A^{-1}(s_B) \leq x} \Pi(s_B, x, z_i)\). First note that the constraint is never binding, for otherwise, the objective function goes to \(-\infty\) (because \(\lim_{a \to -1} m^{-1}(a) = \infty\)). Since \(\Pi\) is differentiable and concave in \(x\), it follows that \(\Pi_x(s_B, x_B, z_i) \neq 0\). Given that, I construct the following function:

\[
L(s) \equiv \Pi(s, \bar{x}(s), z_i)
\]

where

\[
\bar{x}(s) = \frac{-\Pi_x(s_B, x_B, z_i) + \frac{\delta_i}{2} (s - s_B) + x_B}{\Pi_x(s_B, x_B, z_i)}.
\]

I show that there exists \(s_C\) sufficiently close to \(s_B\) such that \((s_C, \bar{x}(s_C))\) is feasible for Problem \(Q_i(\bar{s}_{i-1})\) but delivers higher payoff. To show that, I calculate \(L'(s)\) at \(s_B\):

\[
L'(s)\big|_{s=s_B} = \Pi_x(.) + \Pi_x(.) \frac{d\bar{x}(s)}{ds}
= \Pi_x(s_B, x_B, z_i) \left( -\frac{\Pi_x(s_B, x_B, z_i) + \frac{\delta_i}{2} (s - s_B) + x_B}{\Pi_x(s_B, x_B, z_i)} \right) = \frac{\delta_i}{2} > 0
\]

Fix \(s_C > s_B\) sufficiently close to \(s_B\) such that \(L(s_B) < L(s_C) < U_{i-1} + \delta_i s_C\). This is possible because \(L(s_B) = \Pi(s_B, x_B, z_i) = U_{i-1} + \delta_i s_B\) and because \(0 < L'(s) < \delta_i\). Now I show that \((s_C, \bar{x}(s_C))\) satisfies constraints of Problem \(Q_i(\bar{s}_{i-1})\) and yields higher utility than \((s_B, \bar{x}(s_B))\).

First, since \(L(s_C) < U_{i-1} + \delta_i s_C\), the first constraint is satisfied. Second, \(s_C > s_B > s_{i-1}\) is satisfied, because \(s_C > s_B \geq s_{i-1}\) due to the construction of \(s_C\) and feasibility of \(s_B\). \(s_C < A(x_C)\) is satisfied because \(\arg \max_{x} \{\Pi(s_B, x, z_i)\}\) is continuous in \(s_B\), because \(A(x)\) is continuous in \(x\), and because \(s_C\) is sufficiently close to \(s_B\). The value of the objective function is higher, \(\Pi(s_C, \bar{x}(s_C), z_i) > \Pi(s_B, \bar{x}(s_B), z_i)\), because \(L(s_B) < L(s_C)\), which is a contradiction with \((s_B, x_B)\) being a maximizer of Problem \(Q_i(\bar{s}_{i-1})\).

**Step 5: There exists a unique \((x, \theta)\) that solves Problem \(Q_i(s)\).**

I have already shown that there exists a unique solution \(s\) which solves Problem \(Q_i(\bar{s}_{i-1})\). Also, I have shown that \(\Pi(., x, \cdot)\) is strictly concave in \(x\) and if \((s_B, x_B)\) solves Problem \(R_i(\bar{s}_{i-1})\) for type \(i\), then \(x_B \in \arg \max_{A^{-1}(s_B) \leq x} \Pi(s_B, x, z_i)\). Therefore, there is a unique \(x_B\) which solves Problem \(Q_i(s)\). Also, \(s = m(\theta) A(x)\). Since \(s\) and \(x\) are unique at the solution and \(m\) is strictly increasing, then \(\theta\) must be
Proof of lemma 5. We have already shown that any solution to Problem $P_i$ in terms of $(x_i, s_i)$ satisfies $x_i \in \arg \max_{A^{-1}(s_i) \leq x} \Pi(s_i, x, z_i)$. To show that $\bar{x}(s, z)$ is increasing in $s$ and $z$, we need to show that the objective function, $\Pi(s, x, z)$, satisfies the increasing differences property in $(z; x)$ and $(s; x)$, i.e., $\partial^2 \Pi / \partial z \partial s > 0$ and $\partial^2 \Pi / \partial x \partial s > 0$. But $\partial^2 \Pi / \partial x \partial z = s A(x) - b_z A(x)$ which is positive because $s > 0$ and also following part 2 of Assumption 10.

Now I calculate $\partial^2 \Pi / \partial x \partial s$. Remember that $\rho \equiv m - 1$.

$$\partial \Pi(s, x, z) / \partial x = b_z(x, z)A(x) - b(x, z)A'(x) + \rho A(x)C'(x) + \rho' A(x) s A'(x) C(x)$$

Since $\Pi(.)$ is differentiable in $x$, and since $\bar{x}(s, z)$ is a maximizer of $\Pi$ according to Lemma 4, the first order condition is necessary, therefore,

$$\frac{\partial \Pi}{\partial x} \bigg|_{x=\bar{x}(s,z)} = 0. \quad (B.5)$$

Then, $\partial^2 \Pi / \partial x \partial s$ is calculated as follows:

$$\frac{\partial^2 \Pi}{\partial x \partial s} = \frac{b_z(x, z)A(x) - b(x, z)A'(x)}{A^2(x)} - \rho A(x) C'(x) A(x)$$

$$+ C(x) A'(x) \left( \rho A(x) + \rho' A(x) \right) \left( \rho'' A(x) \right).$$

I show that $\partial^2 \Pi / \partial x \partial s > 0$ at the optimal point:

$$\frac{\partial^2 \Pi}{\partial x \partial s} = \frac{\rho C'}{s} - \frac{\rho' C'}{A} + \frac{\rho'' C A'(s)}{A^3} = \frac{\theta C'}{s} - \frac{C'}{Am'} - \frac{CA'm''}{A^2m'^3} = \frac{C'}{Aq} - \frac{C'}{Am'} - \frac{CA'm''}{A^2m'^3}$$

$$= \frac{C'}{AqM'} - \frac{CA'm''}{A^2m'^3} > 0. \quad (B.6)$$

For the first equality, I used the first order condition in Equation B.5 for the optimal point. The inequality holds due to Assumption 10.

Following theorem of the maximum, we know that $\bar{x}(s, z)$ is a UHC correspondence. We have already proved that $\bar{x}(s, z)$ is unique given $s$ and $z$, so $\bar{x}(s, z)$ is a continuous function of $s$. Assume by way of contradiction that $\bar{x}(s, z)$ is not unique too.
weakly increasing in $s$ for some $s_0$. It follows that there exists an $\epsilon_1 > 0$ such that for any $s \in (s_0, s_0 + \epsilon_1)$: $\bar{x}(s, z) < x_0 \equiv \bar{x}(s_0, z)$.

We know from Equation B.6 that $\frac{\partial \Pi}{\partial x}(x_0, s_0, z) > 0$. Since $\frac{\partial \Pi}{\partial x}(x_0, s_0, z) = 0$, there exists $\epsilon_2 > 0$ such that for all $s_2 \in (s_0, s_0 + \epsilon_2)$,

$$\frac{\partial \Pi}{\partial x}(x_0, s_2, z) > 0. \quad \text{(B.7)}$$

Therefore, we can find $s_2 \in (s_0, s_0 + \min\{\epsilon_1, \epsilon_2\})$ such that

$$\frac{\partial \Pi}{\partial x}(\bar{x}(s_2, z), s_2, z) \geq \frac{\partial \Pi}{\partial x}(\bar{x}(s_0, z), s_2, z). \quad \text{(B.8)}$$

This is possible because $\frac{\partial \Pi}{\partial x}(x, s_2, z)\bigg|_{x=\bar{x}(s_2, z)} < 0$ and $\bar{x}(s_2, z) < \bar{x}(s_0, z)$. But the LHS of Equation B.8 is 0 and the RHS is strictly positive according to Equation B.7. A contradiction. This completes the proof.

\[\square\]

**Proof of Lemma 6.** Define

$$H_i(s) \equiv \max\{0, \max_{A^{-1}(s) \leq x} \Pi(s, x, z_i)\}. \quad \text{(B.9)}$$

The constraint captures the fact that $s = m(\theta)A(x)$ and $m(\theta) \leq 1$. Consider the following set $T_i = \{s | H_i(s) > 0\}$. Note that $s_i^{FB} \equiv m(\theta_i^{FB})A(x_i^{FB}) \in T_i$, because there are positive gains from trade for all types given part 2 of Assumption 7. Also note that $H_i(s)$ is an envelop function. According to Milgrom and Segal [44], $H_i(s)$ is continuous in $s$.

By way of contradiction, assume that $s_A$ solves Problem $R_i(s_{i-1})$ but $s_A < s_i^{FB}$. There must exist $s_D \in T_i$ such that $s_i^{FB} < s_D$ and $H_i(s_D) - U_{i-1} - \delta_i s_D < 0$. Otherwise, it implies that $H_i(s)$ grows infinitely large when $s$ goes to infinity. But this is a contradiction with $s_i^{FB}$ being a global maximizer of $R$.

Because the FB is not achievable, (otherwise $s_A = s_i^{FB}$,) $H_i(s_i^{FB}) - U_{i-1} - \delta_i s_i^{FB} > 0$ or $s_i^{FB} < s_{i-1}$. But $s_i^{FB} > s_A > s_{i-1}$, so $H_i(s_i^{FB}) - U_{i-1} - \delta_i s_i^{FB} > 0$. Because $s_A$ is feasible for Problem $Q_i(s_{i-1})$, so $H_i(s_A) - U_{i-1} - \delta_i s_A \leq 0$.

Notice that $H_i(s)$ is continuous in $s$ according to Envelop theorem. Therefore, $H_i(s) - U_{i-1} - \delta_i s$ is continuous in $s$. Since this function is positive at $s_i^{FB}$ and negative at $s_D$, according to intermediate value theorem, there exists $s_E \in (s_i^{FB}, s_D)$.
such that \( H_i(s_E) - U_{i-1} - \delta_i s_E = 0 \). This contradicts with \( s_A \) being a maximizer of the problem, because \( s_E \) is feasible but delivers higher utility than \( s_A \):

\[
H_i(s_E) = U_{i-1} + \delta_i s_E > U_{i-1} + \delta_i s_A \geq H_i(s_A)
\]

where the strict inequality follows from the construction of \( s_E \) and the fact that \( s_E > s_i^{FB} > s_A \). \qed

### B.3 Proofs of the Continuous Type Case

**Proof of Proposition 8.** A typical way to solve mechanism design problems is to characterize IC schemes first and then solve a somewhat relaxed problem and then check that IC constraints are satisfied. However in this proof, in order to save space, I use a guess-and-verify approach to shorten the proof. Since the candidate for the constrained efficient allocation is the complete information allocation (or first best), we just need to check that this candidate is a feasible allocation, because then by definition, it is impossible that any other allocation to achieve strictly higher welfare.

The candidate for constrained efficient allocation is as follows:

\[
\theta(z) = \theta^{FB}(z),
\]

\[
x(z) = x^{FB}(z),
\]

\[
T(z) = 0,
\]

\[
p(z) = -e(x(z), z) + \frac{U(z_L) + \int_{z_L}^{z} m(\theta^{FB}(z_0))e_z(x^{FB}(z_0), z_0) dz_0}{m(\theta^{FB}(z))},
\]

where

\[
U(z_L) = \int \left\{ m(\theta(z_0)) \left[ b(x(z_0), z_0) + e(x(z_0), z_0) - e_z(x(z_0), z_0) \frac{1 - F(z_0)}{F'(z_0)} \right] - \theta(z_0)C(x(z_0)) \right\} dz_0.
\]

Now we check that all three conditions for feasibility are satisfied:
1. Incentive Compatibility of Workers

As a reminder, I denote by \( U(z, \hat{z}) \) the payoff of type \( z \) if he reports type \( \hat{z} \). That is \( U(z, \hat{z}) = m(\theta(\hat{z}))c(x(\hat{z}), z) + r(\hat{z}) \) where \( r(\hat{z}) \equiv m(\theta(z))p(z) \). In the first step below, I show that \( \frac{d\theta(z)}{dz} \geq 0 \). Then I use it to show that \( U(z, \hat{z}) - U(z, z) \leq 0 \) for all \( z \) and \( \hat{z} \).

**Step 1:** \( \frac{d\theta(z)}{dz} \geq 0 \)

In order to derive \( \frac{d\theta(z)}{dz} \), I take the derivative of Equation 2.5 with respect to \( z \) to get:

\[
\theta'q'f_x + qf_{xz} = x'(C'' - qf_x).
\]

Note that the algebra is tedious, so I shorten the notation by using \( \theta' \) instead of \( \theta'(z) \equiv \frac{d\theta(z)}{dz} \), \( x' \equiv \frac{dx}{dz} \) instead of \( x'(z) \), \( f_x \) instead of \( f_x(x(z), z) \), so on and so forth. I substitute \( x' \) from Equation 2.6 to calculate \( \theta' \):

\[
\theta'q'f_x + qf_{xz} = \frac{C'f_{zz}}{f_x^2} - \frac{Cf_z q'}{f_x^2 m''} (C'' - qf_x). \tag{B.10}
\]

I also substituted \( q(\theta(z)) \) in the RHS of the above equation by \( \frac{C'}{f_x} \) from FOC in Equation 2.5. Let \( H \) denote the big denominator in the above equation to simplify the algebra. That is, \( H \equiv \frac{f_x C'' - C' f_{xx}}{f_x^2} - \frac{f_x C' f_{xx}}{f_x^2 m''} q' \). Now I rewrite Equation B.10 as follows:

\[
\theta'q'f_x H = \left[ \frac{C' f_{zz}}{f_x^2} - \frac{C f_z q'}{f_x^2 m''} \right] (C'' - f_{xx} \frac{C'}{f_x}) - q f_{xz} \frac{f_{xx} C'}{f_x} - f_x (C' - C' \frac{f_x}{f}) \frac{q'}{m''}.
\]

\[
= \frac{f_{xz} (C'' - f_{xx} \frac{C'}{f_x})}{f_x^2} \left( C' - qf_x \right) - C f_x \frac{q'}{f} \frac{q'}{f} \frac{f_{xx} C'}{f_x} \leq 0.
\]

The second equality is derived by just rearranging the terms. The inequality holds following from the left side inequality in Assumption 15.

Note that \( H \) is the coefficient of \( \frac{dx}{dz} \) in Equation 2.6. I showed in the text that this coefficient is positive. Also, \( q' \) is negative and \( f_x \) is positive, so \( \theta' \) is positive.

**Step 2:** \( U(z, \hat{z}) - U(z, z) \leq 0 \) for all \( z \)

I assume that \( \hat{z} > z \) without loss of generality. The analysis of the opposite case
According to the first order condition in Equation 2.4, we have

\[ U(z, \hat{z}) - U(z, z) = \int_{z}^{\hat{z}} \frac{\partial U(z, z_0)}{\partial \hat{z}} \, dz_0 \]

\[ = \int_{z}^{\hat{z}} \{ m(\theta(z_0)) e_x(x(z_0), z) \, \frac{dx(z_0)}{dz} + m'(\theta(z_0)) e(x(x(z_0), z) \, \frac{d\theta(z_0)}{dz} + r'(z_0) \} \, dz_0 \]

\[ = \int_{z}^{\hat{z}} \{ m(\theta(z_0)) (e_x(x(x(z_0), z) - e_x(x(z_0), z_0)) \, \frac{dx(z_0)}{dz} \]

\[ + m'(\theta(z_0)) (e(x(x(z_0), z) - e(x(z_0), z_0)) \, \frac{d\theta(z_0)}{dz} \} \, dz_0. \]

The third equality follows from the first order condition for type \( z_0 \) (\( \frac{\partial U(z_0, z_0)}{\partial z} = 0 \)). I have shown in Proposition 6 that \( \frac{dx(z)}{dz} \geq 0 \). Also I showed in step 1 that \( \frac{d\theta(z)}{dz} \geq 0 \). Since both \( e_z \) and \( e_{xz} \) are positive, so \( e(x(x(z_0), z) - e(x(z_0), z_0) \) and \( e_x(x(z_0), z) - e_x(x(z_0), z_0) \) are negative for all \( z \leq z_0 \). Therefore, the integrand in the above equation is negative and so the whole expression is negative.

2. Participation Constraint of Workers

To check that the payoff of all workers are positive, first I show that for the lowest type \( U(z_L) \geq 0 \). Therefore, I just need to show that the terms inside the integral in Equation 2.10 are positive, that is,

\[ m(\theta^{FB}(z)) [b(x^{FB}(z), z) + e(x^{FB}(z), z)] - \theta^{FB}(z)C(x^{FB}(z)) \]

\[ - m(\theta^{FB}(z)) e_z(x^{FB}(z), z) \frac{1 - F(z)}{F'(z)} \geq 0 \]

for all \( z \), or equivalently

\[ \frac{m(\theta^{FB}(z)) [b(x^{FB}(z), z) + e(x^{FB}(z), z)] - \theta^{FB}(z)C(x^{FB}(z))}{m(\theta^{FB}(z))} \geq e_z(x^{FB}(z), z) \frac{1 - F(z)}{F'(z)}. \]

According to the first order condition in Equation 2.4, we have

\[ m'(\theta^{FB}(z)) [b(x^{FB}(z), z) + e(x^{FB}(z), z)] = C(x^{FB}(z)), \]

therefore,

\[ \frac{m(\theta^{FB}(z)) [b(x^{FB}(z), z) + e(x^{FB}(z), z)] - \theta^{FB}(z)C(x^{FB}(z))}{m(\theta^{FB}(z))} \]

159
\[
\frac{m(\theta^{FB}(z))[b(x^{FB}(z), z) + e(x^{FB}(z), z)] - m'(\theta^{FB}(z))[b(x^{FB}(z), z) + e(x^{FB}(z), z)]\theta(z)}{m(\theta^{FB}(z))}
\]

\[
= -\frac{\theta^{FB}(z)q'(\theta^{FB}(z))}{q(\theta^{FB}(z))}[b(x^{FB}(z), z) + e(x^{FB}(z), z)].
\]

Let \(\eta(\theta) \equiv -\frac{\theta q'(\theta)}{q(\theta)}\). For \(U(z_L)\) to be positive, it is sufficient to have:

\[
\eta(\theta^{FB}(z))\frac{b(x^{FB}(z), z) + e(x^{FB}(z), z)}{e_z(x^{FB}(z), z)} \geq 1 - F(z)
\]

One can write

\[
\theta^{FB}(z) = m^{-1}\left(\frac{C(x^{FB}(z))}{b(x^{FB}(z), z) + e(x^{FB}(z), z)}\right)
\]

from the first order condition (Equation 2.4). Replacing \(\theta^{FB}(\cdot)\) in the sufficient condition yields

\[
\psi\left(\frac{C(x^{FB}(z))}{b(x^{FB}(z), z) + e(x^{FB}(z), z)}\right)\frac{b(x^{FB}(z), z) + e(x^{FB}(z), z)}{e_z(x^{FB}(z), z)} \geq 1 - F(z)
\]

which is true according to Assumption 16.

To show \(U(z) \geq 0\) for \(z > z_L\), I use corollary 1 from [44]. It states that if an allocation satisfies IC, then the payoff of type \(z\), \(U(z)\), can be written as follows:

\[
U(z) = U(z_L) + \int_{z_L}^{z} \frac{\partial U(z_0, z_0)}{\partial z} dz_0 \quad (B.11)
\]

This equation, which is standard in the mechanism design literature, is derived from envelope theorem. There are three requirements that we need to check for this result:

1. \(U(z, \hat{z})\) is differentiable and absolutely continuous in \(z\).
   This is satisfied because \(e\) is assumed to be twice differentiable in \(z\).

2. \(\sup_{\hat{z}} \left| \frac{\partial U(z, \hat{z})}{\partial z} \right|\) is integrable.
   This is satisfied because \(\sup_{\hat{z}} \left| \frac{\partial U(z, \hat{z})}{\partial z} \right| \leq |c'(z)| < M\) for some \(M \in \mathbb{R}\), because \(c'(\cdot)\) is continuous and is defined over a compact set \([z_L, z_H]\).

3. \(\theta(z)\) is non-empty.
   This is trivially satisfied because \(\theta(z) = \theta^{FB}(z)\) and \(\theta^{FB}(z)\) exists.

Hence,

\[
U(z) = U(z_L) + \int_{z_L}^{z} \frac{\partial U(z_0, z_0)}{\partial z} dz_0
\]
\[ = U(z_L) + \int_{z_L}^{z} m(\theta^{FB}(z_0))e_z(x^{FB}(z_0), z_0)dz_0 \geq U(z_L) \geq 0. \]

The first inequality holds because \( e_z(.,.) \geq 0 \) and the second one holds as we showed above.

3. Planner’s budget balance

\[
\int \left[ m(\theta^{FB}(z))[b(x^{FB}(z), z) - p(z)] - \theta^{FB}(z)C(x^{FB}(z)) \right] F'(z)dz
\]
\[
= \int \left[ m(\theta^{FB}(z))[b(x^{FB}(z), z) + e(x^{FB}(z), z)] - \theta^{FB}(z)C(x^{FB}(z)) \right.
\]
\[
- m(\theta^{FB}(z))(p(z) + e(x^{FB}(z), z)) \left] F'(z)dz \right.
\]
\[
= \int \left[ m(\theta^{FB}(z))[b(x^{FB}(z), z) + e(x^{FB}(z), z)] - \theta^{FB}(z)C(x^{FB}(z)) \right.
\]
\[
- \int_{z_L}^{z} m(\theta(z_0))e_z(x^{FB}(z_0), z_0)dz_0 \left] F'(z)dz \right.
\]
\[
= \int \left[ m(\theta^{FB}(z))[b(x^{FB}(z), z) + e(x^{FB}(z), z) - e_z(x^{FB}(z), z)\frac{1 - F(z)}{F'(z)}] \right.
\]
\[
- \theta^{FB}(z)C(x^{FB}(z)) \left] F'(z)dz \right.
\]
\[
- U(z_L) = 0.
\]

The second equality follows from construction of \( p(z) \). The third equality derives by using integration by parts. The last equality is due to the definition of \( U(z_L) \) in Equation 2.10.
Bibliography


Vita
Seyed Mohammadreza Davoodalhosseini

• PERSONAL
  – Date of Birth: 20/09/1985
  – Place of Birth: Esfahan, Iran
• EDUCATION
  – Ph.D., Department of Economics, The Pennsylvania State University, 2009-2015
  – M.Sc., Economics, Sharif University of Technology, 2007-2009
• FEILDS OF INTEREST
  – Primary: Macroeconomics, Search Theory
  – Secondary: Labor Economics, Mechanism Design
• RESEARCH
  – Constrained Efficiency with Search and Information Frictions
  – Directed Search with Complementarity and Adverse Selection
  – Directed Search toward Heterogeneously Informed Buyers
• CONFERENCE AND SEMINAR PRESENTATIONS
  – Southwest Search and Matching Group, UCLA, Nov. 2014
  – Cornell-Penn State Macroeconomics workshop, Sep. 2014
  – EconCon Conference, Princeton University, Aug. 2012
• HONORS AND AWARDS
  – Sharif Award for the highest GPA among Graduate students in Economics and Management sciences (Sharif University of Technology), 2009
  – Ranked third among 450,000 participants in the nationwide university entrance exam (Iran), 2003
• PROFESSIONAL ACTIVITIES
  – Referee for International Economic Review