ENERGY HARVESTING WIRELESS NETWORKS:
TRANSMISSION POLICIES AND CODING SCHEMES

A Dissertation in
Electrical Engineering

by

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Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Doctor of Philosophy

August 2015
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Abstract

With the wide deployment of battery powered wireless devices, prolonging the lifetime of wireless networks is becoming ever more critical. Systems powered by batteries suffer from a limited lifetime, whereas networks consisting of energy harvesting nodes can survive indefinitely. Along with its unique benefits, energy harvesting introduces a new challenge in terms of managing the intermittently available energy. Energy harvesting wireless networks call for transmission policies and coding schemes tailored to their variable nature of available energy.

A fundamental concern in energy harvesting wireless nodes is to maximize system performance under the inherent energy constraints. This entails utilizing the limited capacity battery to adapt to variations in harvested energy. The first part of this thesis investigates power policies to maximize the throughput of energy harvesting nodes for various system models. First, a single-user communication scenario with an energy harvesting transmitter is considered. Subject to energy causality and battery capacity constraints derived from offline knowledge of energy harvests, the throughput maximizing power policy is found. In particular, properties of the optimal power policy are identified, and the benefit of power control for energy harvesting communications is demonstrated.

Next, extensions to inefficient energy storage, interfering transmitters, and energy cooperation are considered. In particular, for the inefficient energy storage case, the trade-off between storing and consuming energy as harvested is outlined, and the
optimal power policy is attained with thresholds that regulate battery utilization. Similar thresholds are observed to arise for the fading channel, and for the online problem where harvested energy is revealed causally to the transmitter. As a multi-transmitter setting, the interference channel with two energy harvesting transmitters and two receivers is studied. This configuration requires nodes to not only adapt to their own energy harvests, but to adapt to each others’ transmission policies as well. To solve the problem, an iterative algorithm based on directional water-filling is proposed, and is shown to admit simpler interpretations in specific interference regions. A new dimension to energy management in energy harvesting networks is introduced via wireless energy transfer, which has become increasingly efficient in the past decade. The impact of energy cooperation is studied for various channel models by finding the jointly optimal power allocation and energy cooperation policies. Through the use of a subset of optimal policies, this joint optimization problem is simplified and solved with methods similar to the non-cooperating case.

Intermittency of available energy leads to challenges in coding as well. In the second part of this thesis, a simple energy harvesting channel is studied from an information theoretic perspective, with the purpose of identifying the theoretical limits of communication under energy harvesting constraints at the codeword symbol level. The equivalence of this channel to a timing channel is utilized to propose encoding schemes and upper bounds that characterize its capacity within a small gap. Moreover, the amount of information revealed to the decoder about the energy harvesting process is compared for different encoding schemes, addressing the state amplification and state masking problems for energy harvesting channels.
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Acknowledgments

First and foremost, I would like to thank my academic advisor Professor Aylin Yener for her invaluable guidance and instruction towards my dissertation. It is truly a rare opportunity to work with someone so passionate, dedicated, and selfless about research and discovery. I was deeply inspired by her intellect, research discipline, and determination in the face of challenges, which are exceptionally complemented by her integrity and commitment to excellence. I am thankful both for her substantial contribution to this thesis, and for the professional philosophy that she has instilled in me. It has certainly been a transforming experience to work with Professor Yener throughout my graduate studies, for which I am sincerely grateful.

I would also like to extend my gratitude to Professor Şennur Ulukuş for our very productive collaboration, which has significantly influenced this thesis. I am thankful for the many times she graciously hosted me at the University of Maryland, and for the numerous hours she has dedicated to our research, both in person and over the phone. Her motivating attitude and guidance did not only yield well-received publications, but also broadened my perspective in many ways.

I sincerely thank Professor Tom La Porta, Professor Vishal Monga, and Professor Adam Smith for serving in my Ph.D. dissertation committee and for their valuable suggestions which helped guide and improve this thesis. I would also like to thank our department head Professor Kültekin Aydın for his continuing support, and all Penn State faculty members whose classes I had the privilege of attending.
I would like to express my gratitude to Dr. Ömür Özel, with whom we carried out a fruitful collaboration in the past five years. I thank him for hosting me several times at the University of Maryland, for the countless hours we have spent sharing and expanding our knowledge, and for his enthusiasm and friendship. Special thanks are due also to Burak Varan and Başak Güler, with whom I thoroughly enjoyed conducting research, and from whom I have learned a lot throughout our joint research efforts.

I am grateful to all friends with whom I shared an office with, shared a house with, and shared time with throughout my studies; including Dr. Taha Özdemir, Dr. Xiang He, Dr. Ertuğrul Çiftçioglu, Dr. Min Li, Dr. Ye Tian, Dr. Igor Stanojev, Dr. Ebrahim MolavianJazi, Mohamed Nafea, Ahmed Zewail, Mahmoud Ashour, Abdelrahman Ibrahim, Berk Gürakan, Adi Hajj-Ahmad, Dr. Ersen Ekrem, and many others.

Last but certainly not least, I would like to thank my parents Mücahit and Ayşen Tütüncüoğlu, my brother Atakan Tütüncüoğlu, and my extended family for their enduring confidence in me. More than anyone else, I would like to thank my beloved fiancée, Beril Kumcuoğlu, for her everlasting love, kindness, and patience. She is without a doubt the person who made all this possible by balancing my life throughout this taxing journey, and has been a constant source of joy and hope even at the hardest of times.
Chapter 1

Introduction

The ever-increasing demand for mobility, ubiquity, and longer lifetimes for wireless communication networks calls for a new design paradigm. A critical impediment to achieving these goals is energy [23], which is conventionally supplied via a battery, often replaced or recharged from the grid, resulting in nodes that are either short-lived, bulky, or tethered. Energy harvesting offers to mitigate these issues by enabling perpetual operation of wireless communication nodes using the energy gathered from the surroundings [44].

Recent advances in energy harvesting and the advocacy for green technologies are leading to significant interest in systems powered by harvested ambient energy. Such systems can utilize various energy sources such as solar, wind, or mechanical, and extend their lifetime while reducing the size of their batteries. As such, energy harvesting wireless networks promise virtually indefinite lifetime and true mobility for wireless nodes, meanwhile being environment friendly. Such ease of deployment and maintenance, combined with the growing demand for high rate communications and the trend towards an internet of things, foresee a rapid increase in the number of energy harvesting communication devices in the near future. Given these advantages, one can forecast that there shall be growing interest in wireless networks comprised of energy harvesting nodes.
With the benefits of energy harvesting comes inevitable challenges in power management. In particular, harvested energy is intermittent by nature, resulting in a varying instantaneous energy budget for communication. Efficient utilization of the available energy therefore requires tailored power management strategies that balance storing energy for future use and leaving room in the battery for future harvests. These strategies depend on multiple factors such as the capacity of the battery, variations in the communication channel, efficiency of energy storage, and the possibility of cooperation with other nodes sharing the medium. Therefore, performance of an energy harvesting wireless system is tied closely to its energy management policies. In addition to its challenges in power management, energy harvesting also brings forth unique challenges in coding. Although information theory provides extensive results for average power constrained [85] and amplitude constrained [89] channels, when available energy is varying at the symbol level, these results are no longer sufficient.

Towards understanding their fundamental performance limits, this thesis studies transmission policies and coding schemes in energy harvesting communication systems. In particular, this thesis identifies optimal transmission policies that maximize the throughput of wireless transmitter nodes powered solely by energy harvesting. Algorithms to find the power policies which maximize the throughput of a wireless system under various conditions such as finite and lossy energy storage, interference, and energy cooperation are developed. Moreover, a binary energy harvesting channel is studied, where individual channel input symbols are constrained by the available energy, and upper and lower bounds for its capacity are derived. This line of study offers new insights about the design and fundamental limits of energy harvesting communication nodes.
1.1 Power Allocation in Energy Harvesting Nodes

We first consider the problem of power allocation in energy harvesting communication systems. Energy harvesting nodes differ from their battery or grid powered counterparts in the variability of the energy available for operation. Namely, the energy available to the node is driven by the exogenous process from which the node perpetually obtains its energy. This calls for carefully designed power policies which are not only energy-efficient, but also adaptive to the variations in harvested energy.

There is a significant amount of literature in energy-efficient communications and networking of nodes with non-rechargeable batteries, from the perspective of various protocol layers, or cross-layer approaches, e.g., [27, 23, 106, 4, 39, 124, 9, 1, 52, 128, 19], including finite horizon scenarios with delay constraints [126, 125, 63, 64]. References most relevant to this thesis include [107, 126]. Energy-efficient communication under a deadline constraint is considered in [107], resulting in a packet scheduling algorithm that achieves minimum energy via lazy scheduling. Namely, it is shown that the choosing the minimum constant transmit power that departs the data packets at hand by the deadline or by the next packet arrival is the most energy-efficient policy. This is a consequence of low-power transmission being more energy-efficient. In [126], this idea is extended to energy-efficient rate control with individual packet deadlines and data buffer size limitations. The common objective of these work is to minimize the total amount of energy consumed by the communication system, without any restrictions on the total available energy or its variability in time.
Optimal power policies for energy harvesting nodes have attracted recent interest in the research community. One approach is to sustain a performance while preserving a balance between harvested and consumed energy, i.e., energy neutrality [46]. References [88, 24] provide energy neutrality by stabilizing an energy queue. This approach often calls for the energy queue to grow indefinitely large for optimal operation, thus is not applicable to nodes with limited battery capacity. A discrete-time battery model is introduced in [53] for the problem of energy aware routing in wireless networks powered by renewable energy. Reference [51] assumes a Markov model for battery state, and proposes a threshold algorithm to best allocate available energy to packets with different rewards. In [83], policies based on the energy-error probability trade-off are developed to maximize successful transmission probability while minimizing probability of running out of energy in an energy harvesting body sensor network. For a solar powered wireless network, [66] lays out sleep/wake-up strategies for various factors and determines optimal parameters of the solar energy harvest based strategy using a bargaining game model.

In this thesis, we will investigate the offline power allocation problem for an energy harvesting communication node, where the energy harvesting scenario is known to the node beforehand. This approach serves both as a solution for systems where harvested energy is predictable, and as a benchmark and a source of insight for the design of systems where it is not. We will also formulate and solve the online problem for comparison purposes. The offline power allocation problem is first formulated in [120], which considers the problem of minimizing the transmission completion time for a given number of data packets. This reference introduces the energy causality constraint, which serves to ensure that allocated energy does not exceed the harvested energy, and shows
that the optimal transmit power policy is non-decreasing and piecewise constant. This result is consistent with lazy scheduling [107], since the primary motivation for power management in this setting is the efficient use of harvested energy, which is achieved through constant transmit powers between energy harvests.

In Chapter 2, we build upon the model of [120] by introducing a finite-sized battery, i.e., restricting the amount of energy that can be stored at the transmitter. The implication of this constraint is a trade-off between low-power transmission and utilization of the battery. Namely, energy-efficient low-power communication is no longer truly efficient since it results in battery overflows, and thus wastes energy. For this case, in order to avoid battery overflows, the optimal power policy is no longer non-decreasing. Also shown in Chapter 2 is the relation between the problems of minimizing transmission completion time and maximizing throughput by a deadline, which are solved with identical power policies. These problems are subsequently extended to fading channels in [71], where the transmitter needs to adapt both to the variations in harvested energy and to the variations in the communication channel. This is achieved by a directional water-filling algorithm, which combines the insights of Chapter 2 with the water-filling approach for fading channels [26]; see also [41] for a treatment of the energy harvesting and fading case.

In addition to their limited storage capacity, another practical consideration for the batteries on energy harvesting transmitters is storage efficiency. Although intelligent management of harvested energy yields a better performance, it relies on storing and retrieving energy to and from the battery without any losses. In reality, batteries suffer
from many imperfections such as inefficient storing and retrieving, leakage, and degradation [90, 91, 43, 38]. Hence, it is necessary to consider such imperfections when designing the optimal transmit power policies. The impact of degradation and leakage in energy storage on the throughput maximizing policies is studied in [20, 54]. In Chapter 3, we consider storage inefficiency, i.e., the case where a fraction of the stored energy is lost. A consequence of this imperfection is the trade-off between saving the energy for more efficient future use, or consuming it as harvested without any storage losses. Indeed, the optimal policy found in Chapter 3 employs thresholds which identify when this trade-off is to the favor of consuming or storing harvested energy. This chapter also presents online policies, formulated as Markov decision processes, which exhibit similar threshold characteristics.

While the single-link model provides valuable insights about optimal power policies for energy harvesting nodes, an essential extension of this line of work is to more involved channel models. Some examples for such channels include the broadcast channel [116, 6, 75], multiple access channel [119, 14], the two-hop relay channel [34, 42, 67, 68, 69, 3, 2, 55, 108], the two-way relay channel [97], and channels with energy harvesting receivers [98, 56, 123]. In Chapter 4, we consider the interference channel [82, 84, 61] with two energy harvesting transmitters, which serves as a building block for larger setups with interfering transmitter-receiver pairs. This channel requires nodes to not only adapt to their own energy, but also to that of interfering energy harvesting nodes. The jointly optimal power policy for the two interfering nodes is found using an iterative and generalized version of the directional water-filling algorithm [71], demonstrating the interaction between energy harvesting nodes sharing a wireless medium. This algorithm
is subsequently extended to a general communication model which includes arbitrary energy consumption models that may arise in more involved channel models as well as energy harvesting receivers.

Multiple energy harvesting nodes introduce the opportunity of cooperation, and in particular, energy cooperation. Recent advances in wireless energy transfer [50, 47] enable mid-range energy cooperation between energy harvesting nodes up to a few meters apart. The ability to share energy alleviates the concerns related to energy outage, which is particularly beneficial when the performance of the system depends on the simultaneous operation of all of its nodes. The offline energy management problem with uni-directional energy cooperation, i.e., transfer from one node to the other only, is introduced in [37] for various channel models. This problem entails solving for the jointly optimal transmit power and energy transfer policies, which significantly increases the dimension of the problem. This leads to two-dimensional directional water-filling, where water flow between transmitters is enabled to represent energy transfer. Chapter 6 extends this problem to the case of bi-directional energy transfer, and proposes a transformation that allows the separation of the optimal energy cooperation problem and the optimal power allocation problem, thereby obtaining the optimal policy using a one-dimensional iterative directional water-filling algorithm.

1.2 Coding Schemes for Energy Harvesting Nodes

We next consider an energy harvesting communication channel, where the channel input in each channel use is constrained by the energy state of the transmitter. Since the battery resides at the transmitter, the energy state is available at the encoder causally,
but is not available at the decoder. This results in a channel with causally known state
information at the encoder. In such channels, if the state is independent and identically
distributed (i.i.d.) over time, and is independent of the channel inputs, then the capacity
is achieved using a set of coding functions known as Shannon strategies [86]. However,
in the energy harvesting channel, the battery state has memory due to the storage of
energy through channel uses. Furthermore, the energy state depends on past channel
inputs since each symbol consumes a different amount of energy. Therefore, Shannon
strategies of [86] do not guarantee optimality for this channel. This channel model
resembles that of reference [112], which has action dependent states that are controlled
by the encoder through a set of actions. However, different from [112], in the case of
an energy harvesting channel, actions and channel inputs are one and the same, i.e., the
two cannot be chosen independently. This yields a conflict between choosing inputs with
the purpose of communicating, and with the purpose of controlling the state.

The capacity of an energy harvesting channel is first considered in [73], which
shows that when the battery is infinitely large, variations in harvested energy does not
affect the capacity of the channel. It is shown that saving energy for a negligible fraction
of the codeword duration enables the use of an average power constrained codebook
for the rest of the transmission, thereby achieving the same capacity as the blocklength
goes to infinity. On the other extreme, the no battery case is considered in [72], where
the harvested energy acts as the instantaneous state of the encoder, and any unused
energy is lost. Reference [72] merges the results on amplitude constrained channels
[89] with those on channels with state causally available at the encoder [86] to address
this problem. However, the capacity of an energy harvesting channel with a finite-sized battery remains an open problem [57, 21, 45].

To better understand the energy harvesting channel with a finite-sized battery, Chapter 7 introduces the binary energy harvesting channel, which consists of a unit-sized battery harvesting binary amounts of energy and communicating through a binary noiseless channel. Despite its simplicity, this model retains the challenges originating from energy harvesting and finite energy storage, and thus its capacity is still unknown. In Chapter 7, upper and lower bounds for the capacity of this channel are found by leveraging its equivalence to a timing channel. Namely, lower bounds are obtained using a genie-aided method, and by quantifying the leakage of the state information to the receiver; and achievable rates are obtained via encoding schemes on the timing channel. As a result, the capacity of this channel is obtained within 0.03 bits per channel use, and the proposed achievable schemes are observed to outperform basic Shannon strategies that only consider the instantaneous battery state.

The output of an energy harvesting channel also reveals information about the energy state of the encoder, since channel input is limited by the available energy. This results in the decoder obtaining information about the energy harvesting process. Such a revelation may be desirable, e.g., if the energy harvests can be useful to the decoder for management or monitoring purposes, or may not be desirable, e.g., if the transmitter intends to keep its energy source or state private. These cases result in two problems, namely state amplification [49, 17] and state masking [59], in which the amount of information revealed to the decoder about the energy harvests is maximized or minimized, respectively. Chapter 8 considers these two problems for the binary energy harvesting
channel in Chapter 7. In particular, feasible pairs of message rate and harvested energy information rate are obtained for the achievable rates of Chapter 7, revealing their respective benefits in the state amplification and state masking problems.

1.3 Thesis Road Map

Chapters 2-6 consider transmission policies for energy harvesting networks in a larger time scale, whereas Chapters 7-8 focus on channel capacity and coding schemes under energy constraints resulting from energy harvesting. Chapter 2 considers the power allocation problem for a single-link energy harvesting setup with a finite-sized battery at the transmitter. It introduces the constraints of an energy harvesting transmitter, solves the throughput maximization problem, and shows its relation to the transmission completion time minimization problem. Chapter 3 extends the results of Chapter 2 to the case where storing the harvested energy incurs a loss due to battery efficiency. It also studies optimal online power policies, which can be employed without knowledge of future energy harvests. Chapter 4 considers the energy harvesting interference channel, which consists of two energy harvesting transmitters, and proposes an iterative algorithm to find the optimal power policy. Chapter 5 extends the iterative algorithm of Chapter 4 to a generalized communication model, and demonstrates this extension through an energy harvesting transmitter and energy harvesting receiver link. Chapter 6 introduces the energy cooperation dimension to the power allocation problem, and finds the jointly optimal transmit power and energy transfer policies for three multi-transmitter models with wireless energy transfer. Chapter 7 presents the binary energy harvesting channel, in which each symbol put to the communication channel is individually constrained
by the available energy. It derives upper and lower bounds on the capacity of this channel. Chapter 8 considers the problem of revealing as much (or as little) information as possible to the decoder about the energy harvesting process. It evaluates the state amplification and masking rates for the encoding schemes of Chapter 7 in the binary energy harvesting channel. Chapter 9 concludes the thesis and discusses possible future work in this subject.
Chapter 2

The Energy Harvesting Transmitter with a Finite-Sized Battery

2.1 Introduction

In this chapter, we study the power allocation problem for a basic wireless communication channel, which consists of a single energy harvesting transmitter communicating with a single receiver. In particular, we find the power policy for the energy harvesting transmitter that maximizes the throughput or minimizes the transmission completion time by adapting to the harvested energy.

References [107, 126] demonstrate that intelligent allocation of transmit power when the node is deadline or data constrained has notable benefits. Our aim in this chapter is to make a similar claim when the nodes are energy constrained due to energy harvesting. This problem is first studied in [120], where the authors consider an energy harvesting node, and find transmission policies that minimize the transmission completion time of a given amount of data. This work assumes an infinite-sized battery for the energy harvesting transmitter.

We consider the problem of maximizing the data transmitted by the energy harvesting node under a deadline constraint. Unlike previous work [120], we employ the realistic constraint that the battery of a mobile energy harvesting node has a finite energy storage capacity. We seek to find the optimum transmission policy under this energy
storage constraint as well as the energy causality constraint. That is to say, due to the finite-sized battery at the transmitter, any received energy that overflows its capacity is lost, and that energy cannot be expended prior to being harvested. We also show that the problem solved in this chapter is closely related to the problem of transmission completion time minimization considered in [120]. Specifically, we show that the solution to the former is identical to that of the latter for the same parameters, providing a solution to the latter under battery storage constraints as well. We develop the algorithm that finds the optimum power policy to solve the former. Then, using the relation between the two problems, we present a similar algorithm that finds the optimum power policy to minimize the transmission completion time. This work was published in [99].

2.2 System Model

We consider a single-link (point-to-point) continuous-time system, which consists of an energy harvesting transmitter T1 and a receiver R1. The transmitter node T1 transmits continuously and its rate can be varied at will via power control. Specifically, T1 can choose to transmit with finite power \( p(t) \) at any time instant \( t \), which we will refer to as the power policy of the node. This achieves a corresponding instantaneous rate \( r(p(t)) \) where \( r(.) \) is a non-negative, increasing, strictly concave function that we will refer to as the power-rate function. Using a power-rate function of this form is fairly common [120, 126], and is clearly valid for the additive white Gaussian noise (AWGN) channel. Other factors contributing to energy consumption are not explicitly taken into account in this model, however, some may be integrated in the power-rate function without violating its above stated properties. For example, a processing power linear or
concave in transmission rate could be added to $r(.)$ to yield a new power-rate function satisfying the same properties.

![Diagram of energy harvesting setup](image)

**Fig. 2.1:** The single-link communication setup with an energy harvesting transmitter.

The transmitter replenishes its energy throughout the transmission session via energy harvesting. The energy harvesting process is modeled as a discrete process with energy packets of size $E_i \geq 0$ arriving at time instances $d_i \geq 0$, $d_{i+1} > d_i$, as shown in Figure 2.2. The first energy harvest $E_1$ is conventionally at time $d_1 = 0$, representing the initial energy in the battery. We refer to the time interval between $d_i$ and $d_{i+1}$ as the $i$th *epoch*. We consider the case where the transmitter is equipped with a finite-sized battery. That is to say that the battery at node T1 can store up to $E_{\text{max}}$ amount of energy, and any energy exceeding this capacity is immediately lost. The storage capacity is considered to remain constant throughout the transmission horizon, i.e., battery wear or fatigue is
assumed to be much slower than the time scale of the problem. Due to the finite-sized battery, if the harvested energy $E_i$ is larger than the available space in the battery at time $d_i$, the battery is charged to maximum capacity and the remainder of the energy packet is discarded. Note that an instantaneous energy consumption requires infinite instantaneous power, which is not allowed by system definition. Thus, all harvested energy must first be stored before consumption. Based on this observation, it is safe to assume that $E_i \leq E_{\text{max}}$ for all $i$, as no more than an $E_{\text{max}}$ amount of a harvest can be utilized by the transmitter. Therefore, a truncation at $E_{\text{max}}$ for all $E_i$ values will be assumed.

\[ \begin{array}{cccccc}
E_1 & E_2 & E_3 & E_4 & E_5 & \ldots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
d_1 = 0 & d_2 & d_3 & d_4 & d_5 & t \\
\end{array} \]

Fig. 2.2: Continuous-time energy harvesting model with energy harvests $E_i$ at times $d_i$.

In this model, the amount of energy available to the transmitter at any time instant is constrained, either because a sufficient amount may not yet be harvested or cannot be stored in the battery. As such, we have an energy-feasibility constraint on power policy $p(t)$. We define an energy-feasible power policy as a bounded non-negative function over positive numbers $\mathbb{R}^+$ that ensures the battery state stays within
[0, E_{max}] at all times. This is represented by the energy causality and no-energy-overflow constraints, which are defined as follows:

**Definition 2.1.** A power policy \( p(t) \) satisfies the energy causality constraint if it does not consume more energy in \([0,t]\) than is harvested in \([0,t]\) for all \( t > 0 \), i.e.,

\[
\int_0^t p(\tau) d\tau \leq \sum_{n=1}^i E_n, \quad \forall i > 0, \ t \geq 0, \ d_i \leq t \leq d_{i+1}.
\]  

(2.1)

**Definition 2.2.** A power policy \( p(t) \) satisfies the no-energy-overflow constraint if the difference between the harvested and consumed energy in \([0,t]\) does not exceed the battery size \( E_{max} \) for all \( t > 0 \), i.e.,

\[
\sum_{n=1}^i E_n - \int_0^t p(\tau) d\tau \leq E_{max}, \quad \forall i > 0, \ t \geq 0, \ d_i \leq t \leq d_{i+1}.
\]  

(2.2)

We remark that power policies which lead to battery overflows are physically possible, but are discarded from the energy-feasible policies via Definition 2.2 due to being strictly suboptimal as shown in subsequent sections of this chapter. By combining the constraints in Definitions 2.1 and 2.2, we express the set of energy-feasible power policies as

\[
\mathcal{P} = \left\{ p(t) \geq 0 \left| 0 \leq \sum_{n=1}^i E_n - \int_0^t p(\tau) d\tau \leq E_{max}, \ \forall t \geq 0, \ d_i \leq t \leq d_{i+1} \right. \right\} 
\]

\[
= \left\{ p(t) \geq 0 \left| \sum_{n=1}^{i+1} E_n - E_{max} \leq \int_0^{d_{i+1}} p(\tau) d\tau \leq \sum_{n=1}^i E_n, \ i \geq 1 \right. \right\}. 
\]  

(2.3)
The second equality in (2.3) arises from the non-negativity of $p(t)$ and thus the non-decreasing nature of $\int_0^t p(\tau) d\tau$ in $t$, rendering only a small portion of the constraints in the former definition sufficient. This is a convex set of functions, i.e., any convex combination of two energy-feasible power policies is also energy-feasible.

We consider the offline optimization problem, i.e., we consider that $\{E_i\}$ and $\{d_i\}$ are available at the transmitter at the beginning of transmission. As a consequence, the transmitter can choose a power policy $p(t)$ at the beginning of the transmission, ensuring that energy feasibility is maintained for all $t$. This approach provides a benchmark for the performance of an energy harvesting communication settings, and is also applicable when harvested energy is predictable. A practical example of such a setting is a solar powered surveillance network. A continuous operation would be necessary for surveillance purposes, and the nodes would realistically be operating on finite batteries. Once the nodes are deployed, the solar energy harvesting process would be predictable enough to have good estimates of $E_i$ and $d_i$, yet significantly varying throughout the day.

2.3 Throughput Maximization Problem

Our goal is to find the optimum power policy for maximizing the total data transferred from T1 to R1 within a finite deadline $D$, which we refer to as the throughput maximization problem. We consider the case where the transmitter T1 has an infinite backlog of data. Hence, the only constraint imposed by the model on the power policy is energy-feasibility. The objective is to maximize the total number of bits departed in
the time interval \([0, D]\) over \(p(t)\). This problem can be expressed as:

\[
\max_{p(t)} \int_0^D r(p(t))dt, \quad \text{s.t. } p(t) \in \mathfrak{P}. \tag{2.4}
\]

In this section, we first establish the necessary properties of the optimal power policy. We then present an algorithm to obtain the policy that satisfies all the necessary conditions and show the optimality of this policy.

### 2.3.1 Necessary Conditions for Optimality

The following lemmas provide the necessary conditions for a power policy to be optimal. These conditions provide valuable insight into the development of the algorithm proposed in Section 2.3.2 and the proof of its optimality.

**Lemma 2.1.** Given the total amount of energy consumed in any time interval \([t_1, t_2]\], a constant power transmission policy is throughput optimal, i.e., among all transmission policies consuming energy \(E\) within \([t_1, t_2]\), the constant power transmission \(p(t) = \frac{E}{t_2 - t_1}\) departs the largest number of bits.

**Proof:** The proof of the optimality of a constant power transmission in an epoch can be found in [117, Lemma 2] or [6, Corollary 1]. Although defined for energy harvesting epochs, these proofs can be directly extended to any time interval \([t_1, t_2]\) within the transmission session when the energy constraints from harvests are reduced to a total energy constraint as in this Lemma. The key property for this lemma to hold is the concavity of the power-rate function \(r(p)\). ■
Remark 2.1. Lemma 2.1 also dictates that in the optimal policy, the transmit power, and thus the rate, remains constant between energy harvests for an energy harvesting scenario. Considering the $i$th epoch $[d_{i-1}, d_i]$ and fixing the total consumed energy within this epoch, the feasibility conditions in (2.3) are satisfied by any non-negative transmission policy, including a constant power policy. Hence, without loss of optimality, we can consider power policies that remain constant within each epoch.

Lemma 2.2. Any power policy yielding a battery overflow is strictly suboptimal.

Proof: The proof is by contradiction. It is clear that the battery cannot overflow without an energy harvest. Therefore, we begin by assuming that the battery overflows at some energy harvest instant $d_i$. Let the power policy resulting in this overflow be $p(t)$. Since $E_i$ is less than or equal to $E_{\text{max}}$ by system model and causes an overflow, the battery state just before $d_i$, i.e., at $d_i^-$, is strictly positive. This implies that $p(t)$ can be increased by a small amount $\epsilon > 0$ in $[d_{i-1}, d_i]$ without violating energy-feasibility. This strictly increases the throughput since the power-rate function $r(p)$ is increasing in $p$. The excess energy required, $\epsilon(d_{i-1}, d_i)$, is recovered by the overflow at $d_i$ for $\epsilon$ sufficiently small, and therefore the power policy after $d_i$ may remain unchanged. Since the throughput of this power policy is strictly larger than that of the original policy, the original policy cannot be optimal. ■

Lemma 2.3. In the optimal power policy, the transmit power does not change unless the battery is either full or completely depleted.
**Proof:** The proof is by contradiction. Due to Remark 2.1, the transmit power may only change at an energy harvest. Assume that the transmit power changes at $d_i$, i.e., $p(d_i^-) \neq p(d_i^+)$, and that the battery is neither full nor depleted at time $d_i$. Consider the interval $[d_i - \tau, d_i + \tau]$ for an arbitrarily small $\tau > 0$, where the policy consumes a total energy of $\tilde{E}$. Let $p^*(t) = \tilde{E}/2\tau$ be the constant power policy in $[d_i - \tau, d_i + \tau]$, which expends the same total energy. For $\tau$ sufficiently small, $p^*(t)$ is feasible since the battery is neither depleted nor full at $t$, and can allow consuming less or more energy in $[d_i - \tau, d_i]$. In particular, this change is feasible since both the energy causality constraint and the no-energy-overflow constraint are inactive at $d_i$. Due to Lemma 2.1, $p^*(t)$ departs strictly more bits than $p(t)$ in $[d_i - \tau, d_i + \tau]$ while consuming the same amount of energy. Thus, $p(t)$ cannot be optimal, and the optimal power policy must remain constant unless an energy feasibility constraint is active, i.e., the battery is either full or depleted. ■

Note that in the proof above, $p^*(t)$ might still be feasible when battery is full or depleted, specifically if employing $p^*(t)$ does not violate the active constraint. This observation forms the grounds of the next lemma:

**Lemma 2.4.** In the optimal power policy, the change in transmit power at an energy harvest instant $d_i$ cannot be positive (negative) unless the battery is depleted (full) at that time instant.

**Proof:** The proof is by contradiction. Consider the notation in the proof of Lemma 2.3. Battery not being depleted at time $d_i$ implies that the second inequality in (2.3) is not active. Thus, if $p(d_i^-) < p(d_i^+)$ holds, i.e., if transmit power is increasing at $d_i$, then
$p^*(t) > p(t)$. Hence replacing $p(t)$ with $p^*(t)$ on $[d_i - \tau, d_i + \tau]$ strictly increases the LHS of the inequality, which is feasible. This shows that if the battery is not depleted at $d_i$, a power policy increasing at $d_i$ cannot be optimal. Similarly, battery not being full at time $d_i$ implies that the first inequality in (2.3) is not active. Hence, a decreasing power policy, $p(d_i^-) > p(d_i^+)$, implies $p^*(t) < p(t)$; which renders replacing $p(t)$ with $p^*(t)$ on $[d_i - \tau, d_i + \tau]$ feasible. This shows that if the battery is not full at $d_i$, a power policy decreasing at $d_i$ cannot be optimal. 

**Corollary 2.1.** Lemmas 2.3 and 2.4 together imply that in the optimal power policy, transmit power decreases only at energy harvest instants when the battery is full and increases only at energy harvest instants when the battery is depleted.

**Lemma 2.5.** The optimal power policy expends all harvested energy by the end of the transmission, i.e., by time $D$.

**Proof:** Assume that with an energy-feasible power policy $p(t)$, an $\bar{E}$ amount of energy remains in the battery at time $D$. Without violating feasibility, the transmit power in the interval $[D - \tau, D]$ can be increased by $\bar{E}/\tau$, which is feasible for a sufficiently small $\tau > 0$. Since the power-rate function is increasing, this strictly improves the throughput, and therefore the original policy $p(t)$ cannot be optimal. 

We next utilize the properties obtained here to find the optimal power policy.

### 2.3.2 Optimal Transmission Policy

Let $N$ be the number of energy harvests satisfying $d_i < D$, and define an energy harvest at time $d_{N+1} = D$ as a dummy harvest indicating the end of transmission.
Note that the energy harvested at the dummy harvest \( E_{N+1} \) is arbitrary as it cannot be used by the transmitter. Hence, we let \( E_{N+1} = E^{\text{max}} \), which conveniently results in all feasible policies in (2.3) depleting the battery at \( D \) to avoid an overflow. This is in accordance with the optimality condition in Lemma 2.5, and hence does not affect the solution.

From Lemmas 2.1-2.4 in Section 2.3.1, we know that the power policy consists of intervals of constant power, changing only at a subset of the energy harvest instances \( d_i \). We denote the subsequence of \( \{d_i\} \) at which the transmit power changes as \( \{\bar{d}_i\} \), with \( \bar{d}_1 = 0 \) by definition. Moreover, let \( \tilde{N} \) be the number of elements in the \( \{\bar{d}_i\} \) subsequence satisfying \( \bar{d}_i < D \), and add \( \bar{d}_{\tilde{N}+1} = d_{N+1} = D \) to the subsequence as the final element. The optimal power policy then has the form

\[
p(t) = \begin{cases} 
p_i, & \bar{d}_i \leq t < \bar{d}_{i+1}, \ i = 1, \ldots, \tilde{N}, \\
0, & t > \bar{d}_{\tilde{N}} = D.
\end{cases}
\] (2.5)

Note that with the structure in (2.5), we represent the optimal continuous-time power policy \( p(t) \) as a finite sequence of constant power levels \( \{p_i\} \) and a finite sequence of time instances \( \{\bar{d}_i\} \) at which the power levels change.

We remark that once the first power level \( p_1 \) and its interval \([\bar{d}_1, \bar{d}_2]\) is determined, the remainder of the problem can be considered as a separate throughput maximization problem starting at \( t = \bar{d}_2 \). That is, for a fixed \( p_1 \) and \( \bar{d}_2 \), the optimal power policy is the one that maximizes the throughput for a modified problem with energy harvest times shifted by \( \bar{d}_2 \), an updated initial battery state \( \sum_{k|d_k \leq \bar{d}_2} E_k - \bar{d}_2 p_1 \), and an updated
deadline $D - \bar{d}_2$. This means that once the first block of the optimal policy is identified, the remaining blocks can be found recursively with the same algorithm, using updated parameters and shifted harvest times. Therefore, we focus on determining the optimal transmit power in the first block of epochs. The modified optimization problem described above will be referred to as the *shifted* optimization problem.

We define two sets of powers \( \{p_e[2], \ldots, p_e[N+1]\} \) and \( \{p_f[2], \ldots, p_f[N+1]\} \), as

\[
p_e[i] = \frac{\sum_{n=1}^{i-1} E_n}{d_i}, \quad p_f[i] = \frac{\sum_{n=1}^{i} E_n - E_{\text{max}}}{d_i}, \quad (2.6)
\]

for \( i = 2, \ldots, N + 1 \). Here, \( p_e[i] \) and \( p_f[i] \) are the constant transmit powers that result in an empty battery at \( d_i^- \) or a full battery at \( d_i^+ \), respectively, if employed throughout \([0, d_i]\). Note that these values need not be feasible, or even positive, but only serve comparison purposes. We then define the set \( \mathbf{P} = \{\mathbf{P}[2], \mathbf{P}[3], \ldots, \mathbf{P}[N+1]\} \) with each element representing the closed interval between corresponding elements of the two sets \( p_e \) and \( p_f \), i.e.,

\[
\mathbf{P}[i] = [p_f[i], p_e[i]] = \left\{ p \mid p_f[i] \leq p \leq p_e[i] \right\}, \quad (2.7)
\]

for \( i = 2, \ldots, N+1 \). This translates to a range of constant transmit powers for the interval \([0, d_i]\) that would be energy-feasible at \( d_i \) when the feasibility conditions at \( d_n < d_i \) are ignored. Note that the choice of \( E_{N+1} = E_{\text{max}} \) results in \( p_e[N+1] = p_f[N+1] \), and thus gives \( \mathbf{P}[N+1] = \{p_e[N+1]\} \).

For a constant transmit power in \([0, d_i]\) to be feasible, the power should lie in the range \( \mathbf{P}[n] \) for \( n = 2, \ldots, i \), i.e., the constant transmit power should be feasible through
all energy harvests it extends over. This yields an upper bound $i_{ub}$ on the length of the first constant power transmission, that can be calculated as

$$i_{ub} = \max \left\{ i \left| \bigcap_{n=2}^{i} P[n] \neq \emptyset, i = 2, \ldots, N + 1 \right. \right\}. \quad (2.8)$$

This gives us a range for feasible constant transmit powers and the maximum duration of the first constant power transmission.

Given an energy harvesting scenario and the corresponding $\{p_e[i]\}$ and $\{p_f[i]\}$ sets, assume that a constant transmit power $p_1$ and duration $d_m, m \leq i_{ub}$, is feasible. Then, this transmission satisfies $p_1 \in \bigcap_{n=2}^{m} P[n]$, ends with a full or depleted battery at $d_2 = d_m$ due to Corollary 2.1, and cannot extend beyond $d_{i_{ub}}$, as it is rendered infeasible at $d_{i_{ub}+1}$ by one of the constraints. Note that the constraint that causes the infeasibility at $d_{i_{ub}+1}$ is an indication of how the transmit power needs to change after $d_m$. Due to Corollary 2.1, this determines what the battery state should be at $d_m$. A constant transmission with power $p_1$ either over-depletes or overflows the battery at $d_{i_{ub}+1}$. The former case implies that the transmit power after $d_m$ needs to decrease, and the latter implies that the transmit power needs to increase in order to conform to feasibility conditions beyond $d_{i_{ub}}$. This can be verified by calculating the values of $p_e[i]$ and $p_f[i]$ in the shifted problem after the first step of the policy is determined.

Figure 2.3 shows how the boundaries of the interval $P[i]$ for the shifted problem, shown in red for the three possible positions with respect to $p_1$, move away from the chosen transmit power $p_1$ when re-calculated starting from time $d_m$. This indicates that
if an interval $P[i]$ falls above or below a chosen transmit power $p_1$, then the corresponding intervals of the shifted problem will continue to fall above or below $p_1$, requiring the feasible transmit power $p_2$ to be smaller or larger than $p_1$, respectively. Due to Corollary 2.1, an increase or decrease in power can occur only when the battery depleted or full, respectively. Therefore, the violated constraint at $d_{i_{ub}}+1$ determines whether $p_1$ should deplete or fill the battery at $d_m$, i.e., be equal to $p_e[m]$ or $p_f[m]$.

![Fig. 2.3: Illustration of the power intervals $P[i]$ and their updates.](image)

The observation in Figure 2.3 provides useful insights about the optimal power policy. Based on these, we present Algorithm 2.1 in the next page, and show that it determines the throughput maximizing power policy for an energy harvesting transmitter node with energy harvests $E_i$ at times $d_i$, battery size $E^{max}$ and a deadline $D$. 
Algorithm 2.1 Throughput Maximization Algorithm

1. Calculate $i_{ub}$ from (2.8). If $i_{ub} = N + 1$, assign $p_1 = p_e[N + 1]$ and $\bar{d}_2 = d_{N+1}$, i.e., transmit with constant power $p_1$ until the end of transmission, and terminate.

2. Determine whether $P[i_{ub} + 1]$ falls below or above $\bigcap_{n=2}^{i_{ub}} P[n]$.

3. if $P[i_{ub} + 1] \geq \bigcap_{n=2}^{i_{ub}} P[n]$, transmit with

$$\bar{d}_2 = d_m, \quad p_1 = p_e[m], \quad m = \max \left\{ i \left| p_e[i] \in \bigcap_{n=2}^{i} P[n] \right. \right\},$$

(2.9)

else (i.e., if $P[i_{ub} + 1] < \bigcap_{n=1}^{i_{ub}} P[n]$), transmit with

$$\bar{d}_2 = d_m, \quad p_1 = p_f[m], \quad m = \max \left\{ i \left| p_f[i] \in \bigcap_{n=2}^{i} P[n] \right. \right\},$$

(2.10)

4. Repeat algorithm for the shifted problem with modified parameters

$$E'_1 = \sum_{n=1}^{m} E_n - \bar{d}_2 p_1, \quad N' = N - m, \quad D' = D - \bar{d}_2,$$

$$d'_i = d_{i+m} - d_m, \quad E'_i = E_{i+m}, \quad i = 1, \ldots, N' + 1.$$  

(2.11)
A graphical description of the algorithm is provided in Figure 2.4, which shows the feasible energy tunnel for the energy harvesting transmitter T1. The upper staircase-shaped boundary represents the cumulative energy harvested by the node up to a particular point in time. The lower staircase-shaped boundary, which is the upper boundary shifted down by $E^{\text{max}}$, represents the minimum amount of energy to consume in order to avoid an energy overflow. Together, the two boundaries define the feasible energy tunnel. The energy consumed by the transmitter forms a continuous non-decreasing curve on this plot, which must stay within the feasible energy tunnel to conform to the energy feasibility conditions. In particular, a power policy that extends above the tunnel spends more energy than available, and one that extends below the tunnel causes a battery overflow. Therefore, the set of energy-feasible power policies $\mathcal{P}$ is comprised of curves that remain within the energy tunnel. Note that this graphical depiction is analogous to that in [126], where a similar tunnel formed by received data packets constrains the data transferred by a node. In contrast, here, the tunnel is formed by harvested energy, and constrains the power policy of the node.

In this visual representation, the values $p_e[i]$ and $p_f[i]$ correspond to the slopes of the straight lines that extend from the origin to each corner points of the tunnel, as shown with dotted and dashed lines, respectively, in Figure 2.4. The interval $\mathbf{P}[i]$ represents the collection of slopes of the lines that are in the tunnel at $d_i$, and is marked with an arc on the figure for $i = 2, 3$. The first step of Algorithm 2.1 determines the constant power transmission policy that stays within this tunnel for the longest time. The second step determines whether this policy crosses the upper boundary or the lower boundary first beyond $i_{ub}$. This is accomplished by comparing the first unreachable interval $\mathbf{P}[i_{ub} + 1]$
with the intersection of the earlier ones. Finally, the third step selects the longest feasible constant power policy from one of the sets \( \{ p_e[i] \} \) and \( \{ p_f[i] \} \), allowing a new transmit power in the next epoch that may extend beyond \( i_{ub} \).

**Theorem 2.1.** Algorithm 2.1 yields the throughput maximizing power policy.

**Proof:** We shall prove the optimality of Algorithm 2.1 by analyzing the two options in step 3 and the termination condition in step 1. First, assume that \( P[i_{ub}+1] < \cap_{n=2}^{i_{ub}} P[n] \), an example of which is shown in Figure 2.5a. In this case, Algorithm 2.1 suggests that the optimal transmit power for the duration \( d_m \) is \( p_f[m] \). This can be shown by contradiction. Take any time \( \tau, 0 < \tau < d_m \), and assume that an alternative power policy consumes energy \( E_{alt}(\tau) \neq E_{opt}(\tau) = p_f[m] \tau \), where \( E(t) \) denotes the cumulative energy consumed by a policy at time \( t \). Note that by definition, \( E_{alt}(t) \) and \( E_{opt}(t) \) are continuous and increasing. Due to the energy feasibility constraints in \( (2.3) \), the following
two inequalities must hold for the cumulative energy of the alternative policy:

\[ E_{alt}(d_m) \geq \sum_{n=1}^{m} E_n - E_{max} = E_{opt}(d_m) \quad (2.12) \]

\[ E_{alt}(d_{iub+1}) \leq \sum_{n=1}^{i_{ub}} E_n < p_f[m]d_{iub+1} \quad (2.13) \]

where the second inequality in (2.13) follows from the assumption \( P[i_{ub+1}] < \cap_{n=2}^{i_{ub}} P[n] \).

If \( E_{alt}(\tau) < E_{opt}(\tau) \), i.e., if \( E_{alt}(\tau) \) falls in region II of Figure 2.5a, due to (2.12) and continuity, \( E_{alt}(t) \) and \( E_{opt}(t) \) must intersect somewhere in \([\tau, d_m]\). At this intersection point, consumed energies are equal and the constant power policy departs more bits due to Lemma 2.1. Otherwise, i.e., if \( E_{alt}(\tau) > E_{opt}(\tau) \), \( E_{alt}(\tau) \) falls in region I of Figure 2.5a. In this case, \( E_{alt}(t) \) and the linear extension of \( E_{opt}(t) \), shown with a dashed line in Figure 2.5a, must intersect somewhere in \([\tau, d_{iub+1}]\) due to (2.13), at which point the same Lemma proves the alternative policy to be suboptimal.

An analogous statement applies to the other case in step 3 where \( P[i_{ub+1}] \geq \cap_{n=2}^{i_{ub}} P[n] \). This case is shown in Figure 2.5b, and a transmission of duration \( d_m \) with power \( p_e[m] \) is suggested to be optimal. This can be shown by a similar contradiction.

As in the first case, an alternative policy that extends into region III in Figure 2.5b has to intersect the solid line, and an alternative policy that extends into region IV has to intersect either the solid line or the dashed line. At these intersection points, the total energy consumed by the power policies are equal, and since the algorithm’s allocation or its extension departs strictly more bits, and the alternative policy is suboptimal.
Fig. 2.5: The optimal constant power transmission policy up to $d_m$, and the resulting suboptimal regions.
The termination condition in step 1 of Algorithm 2.1 suggests that if a constant transmit power that depletes the battery at time $D$ is feasible up to time $D$, then it is optimal. Since constant power transmission gives the unique global maxima due to Lemma 2.1, and the battery should be depleted at time $D$ due to Lemma 2.5, it is the unique solution to the problem whenever feasible. ■

With the algorithm for finding the throughput maximizing power policy outlined, we next consider the transmission completion time minimization problem of [120], showing that the two problems are related and can be solved with similar algorithms.

### 2.4 Transmission Completion Time Minimization Problem

An interesting problem previously studied for an energy harvesting transmitter is the transmission completion time minimization problem [120]. This problem focuses on determining an energy-feasible power policy $p(t) \in \mathcal{P}$ that, given the total number of bits to send as $B$, finalizes the transmission in the shortest time possible. This problem has the total transmission duration $D$ as the objective to be minimized, and the throughput of the system as its constraint, and can be expressed as

$$\min_{p(t)} D, \quad \text{s.t.} \quad B - \int_0^D r(p(t)) dt \leq 0, \quad p(t) \in \mathcal{P}. \quad (2.14)$$

Reference [120] solves this problem by employing lemmas similar to those in Section 2.3.1, without the battery constraints. It is in fact possible to extend the algorithm in [120] to the finite-sized battery case by individually extending the lemmas and conditions for optimality. Instead, in this section, we show that the throughput maximization and
transmission completion time minimization problems are closely related, and leverage Algorithm 2.1 to solve the problem in (2.14). In Theorem 2.2 we state how the two problems are related.

**Theorem 2.2.** For a given energy harvesting scenario \( \{E_i, d_i\} \), the throughput maximization problem and the transmission completion time minimization problem yield identical power policies for matching transmission duration and transmitted data values. In other words, if the maximum-throughput policy for a deadline \( D \) departs \( B \) bits in total, then the minimum-time policy for \( B \) bits completes the transmission at time \( D \), and the two optimal power policies are identical.

**Proof:** We first perform Lagrangian relaxation on the throughput constraint, and write the Lagrangian dual problem of the time minimization problem in (2.14) as

\[
\max_{c \geq 0} \left( \min_{p(t) \in \mathcal{P}, D} \left( D + c \left( B - \int_0^D r(p(t)) dt \right) \right) \right). \tag{2.15}
\]

By problem formulation, we know that \( D \) defines the support of \( p(t) \). For each \( D \), \( p(t) \) can be solved, i.e., (2.15) is equivalent to

\[
\max_{c \geq 0} \left( \min_D \min_{p(t) \in \mathcal{P}} \left( D + c \left( B - \int_0^D r(p(t)) dt \right) \right) \right), \tag{2.16}
\]

which then can be restated as

\[
\max_{c \geq 0} \left( \min_D \left( D + cB - c \max_{p(t) \in \mathcal{P}} \int_0^D r(p(t)) dt \right) \right). \tag{2.17}
\]
It can now be readily observed that the optimal power policy $p(t)$ is the solution of the maximization problem nested in (2.17) for some optimal completion time $D^*$, which is identical to the throughput maximization problem (2.4). Therefore the solution of the completion time minimization problem is identical to the solution of the throughput maximization problem where the time constraint is the minimum transmission completion time $D^*$. ■

Remark 2.2. An alternative proof follows from the monotonicity of the two problems in deadline $D$ and bits $B$. Namely, the transmission time minimization problem yields a strictly larger completion time for more bits, and the throughput maximization problem departs strictly more bits for a larger deadline, due to the strict concavity of the power-rate function. Assume that the maximum-throughput policy for interval $[0, D]$ departs a total of $B$ bits but the minimum-time policy for $B$ bits completes the transmission at time $D' \neq D$. Consider the two cases $D' > D$ and $D' < D$. The contradiction in the former case is trivial as the maximum-throughput policy departs the same number of bits in a shorter time; and thus the minimum-time policy cannot be optimal. In the latter case, minimum-time policy achieves the same throughput in a shorter time $D' < D$ indicating that strictly more than $B$ bits can be sent in the larger time interval $[0, D]$ and thus the suggested maximum-throughput policy cannot be optimal.

Theorem 2.2 states that the solution to the former and latter problems are in fact identical. Therefore Lemmas 2.1-2.5 that characterize the optimal power policy also apply to the completion time minimization problem. We make use of this relationship to develop a modified algorithm that yields a throughput maximizing power policy.
while departing exactly the desired number of bits, thus solving the completion time
minimization problem.

The throughput maximizing algorithm terminates at a certain fixed time, whereas
the completion time minimizing algorithm terminates when a certain number of bits have
been transmitted. In the absence of a fixed deadline, the definitions in (2.6)-(2.7) need
to be updated. First, a lower bound $D_{lb}$ for the transmission completion time is found
by solving

$$B = D_{lb} \left( \frac{\sum_{k} d_{nk} < D_{lb} E_k}{D_{lb}} \right)$$

(2.18)

for $D_{lb}$. This is the time at which, if all energy harvested up to $D_{lb}$ is used to transmit
with a constant power, departs exactly $B$ bits. Clearly, this power policy is not always
feasible, but provides a lower bound on $D$ due to Lemma 2.1. Let $N$ be the number of
energy harvests up to time $D_{lb}$, and let $d_{N+1} = D_{lb}$ with $E_{N+1} = 0$; i.e., we introduce
a dummy energy harvest at time $D_{lb}$ which carries zero energy. This is analogous to the
dummy harvest in Section 2.3.2, and can be interpreted as a candidate point for end of
transmission, if feasible. We then update the definition of $P[i]$ of Section 2.3.2 as

$$P[i] = \{ p \mid p_f[i] \leq p \leq p_e[i] \}, \quad i = 2, \ldots, N,$$

(2.19)

$$P[N + 1] = \{ p_e[N + 1] \},$$

(2.20)

where $p_e[i]$ and $p_f[i]$ are defined in (2.6). Note that if $P[N + 1]$ lies within $P[i]$ for $i \leq N$,
then a constant power transmission departing $B$ bits is feasible, and is therefore optimal.
Otherwise, the first constant power section of the optimal policy cannot extend past $d_N$, therefore energy harvests beyond $d_N$ are not required to calculate the first section of the optimal policy. With the updated parameters we now provide the modified algorithm that yields the minimum transmission completion time, and prove that the algorithm yields the optimal solution.

THEOREM 2.3. Algorithm 2.2 yields the transmission completion time minimizing power policy.

Proof: We utilize the relation between the two problems to simplify the proof. If the suggested power policy with completion time $D^*$ is identical to the throughput maximizing allocation with a time constraint $D^*$, Theorem 2.2 states that the power policy is the transmission completion time minimizing policy.

Algorithm 2.2 differs from Algorithm 2.1 at only the termination condition determined by (2.18) and the update in (2.23). Here, (2.23) only affects the termination condition through the calculation of $D_{lb}$ in (2.18). Consequently the two allocations found using the two algorithms are identical until either one reaches a termination step. The termination step of the time minimization algorithm is reached when there exists a feasible constant power transmission step that departs all the remaining bits by $D^*$. The feasibility of this policy implies that the same constant power step is feasible in the corresponding throughput maximization problem with deadline $D^*$, which leads to its termination. Conversely, if there does not exist a feasible constant power departing all remaining bits at $D^*$, then the same applies to the throughput maximization problem for deadline $D^*$. As the iterations of the two algorithms up to the termination step are
Algorithm 2.2 Transmission Completion Time Minimization Algorithm

1. Calculate $D_{lb}$ from (2.18), let $d_{N+1} = D_{lb}$, and calculate $i_{ub}$ from (2.8). If $i_{ub} = N + 1$, assign $p_1 = p_c[N + 1]$ and $d_2 = D_{lb}$, i.e., transmit with constant power $p_1$ until the end of transmission, and terminate.

2. Determine whether $P[i_{ub} + 1]$ falls below or above $\bigcap_{n=2}^{i_{ub}} P[n]$.

3. If $P[i_{ub} + 1] \geq \bigcap_{n=2}^{i_{ub}} P[n]$, transmit with

$$\tilde{d}_2 = d_m, \quad p_1 = p_c[m], \quad m = \max \left\{ i \mid p_c[i] \in \bigcap_{n=2}^{i} P[n] \right\},$$

(2.21)

else (i.e., if $P[i_{ub} + 1] < \bigcap_{n=2}^{i_{ub}} P[n]$), transmit with

$$\tilde{d}_2 = d_m, \quad p_1 = p_f[m], \quad m = \max \left\{ i \mid p_f[i] \in \bigcap_{n=2}^{i} P[n] \right\}.$$

(2.22)

4. Repeat algorithm for the shifted problem with modified parameters

$$E'_1 = \sum_{n=1}^{m} E_n - \tilde{d}_2 p_1, \quad B' = B - r(p_1)d_m,$$

$$d'_n = d_{n+m} - d_m, \quad E'_i = E_{i+m}, \quad i = 2, 3, \ldots.$$

(2.23)
identical, the energy remaining for the termination step are equal. Consequently, we can state that the two algorithms terminating at the same time $D^*$ departing $B$ bits perform identical termination steps. Hence, the two resulting power policies are identical, and by Theorem 2.2, the output of Algorithm 2.2 is optimal. ■

2.5 Directional Water-Filling (DWF) in the Fading Channel

A relevant and insightful extension of our model and results to the fading channel is provided in [71], which we briefly introduce in this section. Reference [71] considers a fading communication channel with the power-rate function $r(p, h) = \frac{1}{2} \log(1 + hp)$, where $h$ is the power gain of the channel. We define an epoch as the time between two consecutive events, where an event may be an energy harvest or a change in the power gain. Let $\tau_i = d_{i+1} - d_i$ be the duration of the $i$th epoch, and $h_i$ be the constant power gain within the $i$th epoch. For an energy harvesting transmitter that adheres to the energy causality and no-battery-overflow constraints in Definitions 2.1 and 2.2, the throughput maximization problem can be written as

$$\max_{\{p_i\}} \sum_{i=1}^{N} \tau_i r(p_i, h_i)$$

s.t. $0 \leq \sum_{n=1}^{i} (E_n - \tau_n p_n) \leq E_{i+1} - E_i$, $i = 1, \ldots, N$,

$$p_i \geq 0, \quad i = 1, \ldots, N,$$
where $p_i$ is the transmit power allocated to epoch $i$. To find the optimal power policy for this convex problem, one can compute the Lagrangian of (2.24) as

$$
L = \sum_{i=1}^{N} \tau_i \frac{1}{2} \log (1 + h_i p_i) - \sum_{i=1}^{N} \lambda_i \left( \sum_{n=1}^{i} (\tau_n p_n - E_n) \right) - \sum_{i=1}^{N} \beta_i \left( \sum_{n=1}^{i} (E_n - \tau_n p_n) - E_{max} \right) - \sum_{i=1}^{N} \mu_i p_i,
$$

(2.25)

with the Lagrangian multipliers $\{\lambda_i\}$, $\{\beta_i\}$, and $\{\mu_i\}$. Applying the Karush-Kuhn-Tucker (KKT) conditions for optimality yields the optimal power policy [71]

$$
P^*_i = \left[ \nu_i - \frac{1}{h_i} \right]^+, \quad \nu_i = \frac{1}{\sum_{n=i}^{N} (\lambda_n - \beta_n)},
$$

(2.26)

where $\nu_i$ is referred to as the water level at the $i$th time slot. The KKT complementary slackness conditions indicate that $\lambda_i$ and $\beta_i$ are positive only when the battery is empty and full, respectively, and zero otherwise; implying that water level $\nu_i$ increases and decreases only when the corresponding constraint is satisfied with equality. This defines a directional water-filling (DWF) algorithm, which determines $\nu_i$, and therefore $p_i$ values that satisfy all KKT conditions. The algorithm operates as follows: The $i$th time slot is assigned the base level $1/h_i$, and the initial water levels are set as $\nu_i = E_i/\tau$. Then, water flow is allowed only in the forward direction, i.e., from time slot $i$ to $i + 1$, not exceeding $E_{max}$, while $\nu_i > \nu_{i+1}$. These flow dynamics ensure that increasing and decreasing water levels only occur when the corresponding constraint is active, and that the water level is equalized elsewhere, thus solving the power allocation problem for a single user in a Gaussian fading channel [71].
Note that the DWF algorithm and Algorithm 2.1 show notable similarities. In particular, the water level in the DWF algorithm remains constant until the battery is full or depleted, at which point it may only change as dictated by the state of the battery. This is similar to the properties in Lemma 2.1 and 2.4, but with the properties acting on the water levels \( \{ \nu_i \} \) rather than the transmit powers \( \{ p_i \} \). For a static channel with \( h_1 = \cdots = h_N \), the directional water-filling algorithm and Algorithm 2.1 can immediately be seen to yield the same power policy.

### 2.6 Numerical Results

In this section, we present numerical results to demonstrate the behavior and the performance of the algorithms. First, we have a sample run of Algorithm 2.1 presented in Section 2.3.2 for a node with \( E_{\text{max}} = 10 \) mJ and energy harvests and times

\[
E_i = [2, 1, 6, 4, 8, 1] \text{ mJ}, \quad d_i = [0, 2, 4, 5, 7, 11] \text{ s}.
\]  

(2.27)

We refer to the sequence of energy harvests \( \{ E_i \} \) and their times \( \{ d_i \} \) as the *energy harvesting scenario*. The scenario in (2.27) is depicted in Figure 2.6a. The feasible energy tunnel corresponding to this harvesting scenario is shown in Figure 2.6b along with the result of the throughput maximization algorithm for a deadline constraint of \( D = 12 \) s.

At the first step of the algorithm, we calculate the sets \( p_e[i] \) and \( p_f[i] \) and determine the corresponding intervals \( P[i] \). These intervals are depicted in Figure 2.6c with vertical lines, positioned at the times of the corresponding energy harvests. We find the
Fig. 2.6: Depiction of the (a) harvesting scenario, (b) algorithm output and (c) algorithm's first step for the sample run.

Based on the position of $P[5]$, we identify the longest feasible transmission with power in $p_{ei}[i]$ at $m = 3$, and assign the first constant power transmission as $p_1 = p_{ei}[3]$ with length $\bar{d}_2 = d_3$. When the algorithm is repeated for the shifted problem, we observe the opposite behavior in the new intervals beyond $d_m$, and pick a transmit power within the set $p_{fi}[i]$ instead. For the final shifted problem, we find that a constant transmit power is feasible until $D$, and the algorithm is terminated.

Note that no other power policy within the feasible energy tunnel can perform better than the one in Figure 2.6b. This can be shown by observing that any feasible policy passing through the origin and the end point at $D$ must cross the dashed line in Figure 2.6b at least once, at which point it consumes the same amount of energy while departing strictly less bits.

Remark 2.3. The optimal policy in Figure 2.6b is also the shortest path in the feasible energy tunnel extending from the origin to the point $(D, \sum_{n=1}^{N} E_n)$. This is a consequence of the length of a continuous curve being a concave function of its derivative. In particular, let $E(t)$ be the cumulative energy consumed by the node up to time $t$, and therefore $p(t) = \frac{dE(t)}{dt}$. Then, the length of the curve $E(t)$ on $t \in [0, D]$ could be expressed as

$$\text{length} \left( E(t) \bigg|_0^D \right) = \int_0^D \sqrt{1 + p^2(t)} dt = \int_0^D -r(p(t)) dt \quad (2.28)$$

Note that minimizing the length is equivalent to maximizing its additive inverse, and that the function $r(p) = -\sqrt{1 + p^2}$ is strictly concave in $p$. Thus, the problem of minimizing
the length of a curve $E(t)$ subject to the constraints in (2.3) falls within the class of problems solved by Algorithm 2.1. Since Algorithm 2.1 is independent of $r(p)$, the solution to (2.4) for any concave power-rate function is the shortest path through the energy tunnel.

For the harvesting scenario in (2.27), we next compare the results of Algorithm 2.1 and Algorithm 2.2 for a range of deadlines $D$ and packet sizes $B$ in Figure 2.7. As implied by Theorem 2.2, the curves coincide at every point, verifying the relationship between the two problems. A point $(D_1, B_1)$ on this curve corresponds to a power policy that solves both of the problems in consideration simultaneously: The throughput maximization problem for a given deadline $D_1$ seconds is solved by departing a maximum of $B_1$ bits, while the transmission completion time minimization problem for a given packet size of $B_1$ bits is solved by completing the transmission in $D_1$ seconds with an identical power policy. Another noteworthy observation from Figure 2.7 is the strictly increasing nature of both of the problems with respect to their parameters. As a consequence, more time is required to depart a longer packet, or similarly more bits can be transmitted when given a more lenient deadline.

After demonstrating the algorithms and the optimal policy on a smaller time scale, we next perform simulations for longer realizations in order to evaluate the average long term performance of the algorithms. We assume an additive white Gaussian noise (AWGN) channel, for which the power-rate function is given by

$$ r(p) = \frac{B}{2} \log \left( 1 + \frac{hp}{BN_0} \right). \quad (2.29) $$
Fig. 2.7: Overlaid plot of deadline versus throughput and completion time versus packet size for the energy harvesting scenario in (2.27).

Fig. 2.8: Throughput comparison of the optimal offline power policy, the constant power policy, and a non-energy-harvesting upper bound.
Here, $h = -90$ dB is the power gain of the channel, $N_0 = 10^{-19}$ W/Hz is the spectral noise power density at the receiver, and $B = 1$ MHz is the allocated bandwidth. For a battery size of $E_{\text{max}} = 10$ mJ, we generate energy harvests randomly, with energy packet size $E_i$ distributed uniformly and independently on $[0, E_{\text{max}}]$, and inter-harvest times $d_i - d_{i-1}$ distributed exponentially with an average of 5 s. We choose a uniform distribution for the energy packet size in order to better reflect the extreme cases of receiving very little or very large energy packets, thereby maximizing the entropy over the finite support of feasible energy packet sizes, $[0, E_{\text{max}}]$. For this scenario, Figure 2.8 compares the throughput of the optimal power policy found using Algorithm 2.1 with two alternatives for a deadline of $D = 10^4$ s. The first is obtained by providing all harvested energy, namely $\sum_{n=1}^{N} E_n$, to the transmitter at time $t = 0$. This results in a constant power transmission throughout the transmission duration, without regard to the energy constraints introduced by energy harvesting. As such, this is the performance of a “traditional” transmitter with no energy harvesting, and is presented as an upper bound for our model where we are bound to conform to energy feasibility and battery constraints. The second is an alternative algorithmic approach named the constant power algorithm. It is based on the fact that for a strictly concave power-rate relationship, constant power transmission is the most energy-efficient. In this policy, the node transmits with a predetermined constant power when energy is available, and shuts off when energy is depleted. The constant transmit power is chosen as the average energy harvesting rate, which is proven to be optimal for an infinitely long transmission duration and an infinite storage capacity [73].
Figure 2.8 reveals that the energy constraints of the problem result in a notable performance loss with respect to having the energy without any constraints. However, a major portion of this loss can be recovered by the use of variable power transmission, and by adapting to the variations in the energy harvesting scenario. The optimal offline algorithm we present in this section provides this remedy, and significantly outperforms the constant power algorithm.

Throughout further simulations involving various battery sizes and harvesting statistics, we observe that the optimal offline algorithm in Algorithm 2.1 provides notable improvements over simpler policies such as the constant power transmission algorithm above. The improvements are particularly significant for small battery sizes $E^{max}$ with respect to $E_i$, and for large standard deviations on the energy harvesting process. Figures 2.9 and 2.10 present simulations with varying battery sizes and harvest randomness, respectively, for the AWGN channel defined above. In Figure 2.9, harvested energy is distributed uniformly in $[0,E^{har}]$. In Figure 2.10, $E_i$ is distributed uniformly with mean $\mu_E = E^{max}/2$ and variance $\sigma_E$, and $d_i - d_{i-1}$ is distributed uniformly with mean $\mu_d = 5$ s and variance $\sigma_d = \sigma_E * \mu_E / \mu_d$. For the purpose of comparison, throughput of the optimal policy is normalized with the throughput of the constant power transmission policy. It can be observed that with increasing battery size and decreasing energy harvest variance, the optimal performance approaches the constant power performance. Therefore employing optimal power policy is more beneficial for systems with finite-sized batteries, such as small scale transmitters with limited battery room, or energy harvests with large variations due to the nature of the harvested resource.
Fig. 2.9: Throughput of the optimal policy normalized by that of the constant power policy as a function of normalized battery size.

Fig. 2.10: Throughput of the optimal policy normalized by that of the constant power policy as a function of the variance of harvested energy.
2.7 Chapter Summary

In this chapter, we solved the throughput maximization problem for a single-link setup with an energy harvesting transmitter, and a finite-sized battery. Furthermore, we showed that obtaining the maximum amount of data transferred by a given deadline is equivalent to solving for the minimum completion time (the deadline) given this amount of data. We proposed algorithms that yield the optimal solution of both problems and proved their optimality. We observed that the solution to the throughput maximization problem has a shortest path interpretation, where the cumulative energy consumed by the node is the shortest path that connects the origin and the deadline while remaining in the feasible energy tunnel defined by the energy harvesting scenario. Through simulations, we observed that the optimal offline policy provides a significant throughput increase compared to a traditional transmission policy without power control. This increase was more pronounced when the battery size is small, and when the harvested energy is less predictable. Overall, these results provided insights on developing optimal transmission policies for transmitters that have some notion of when and how much energy they can harvest.
Chapter 3

The Energy Harvesting Transmitter with Inefficient Energy Storage

3.1 Introduction

In this chapter, we consider a practical extension of the energy harvesting communication model in Chapter 2 where the battery suffers from charging/discharging efficiency. We determine the throughput maximizing power policies for single-user and broadcast channel models, and for static and fading channels, with finite-sized and infinite-sized batteries. We also formulate the online version of the power allocation problem as a Markov decision process, and solve it via dynamic programming.

Energy storage/retrieval imperfections can manifest themselves in many different ways [90, 91], for instance: imperfections in energy conversion from one technology to another, charging/discharging imperfections (only a portion of the available energy can be saved in the battery at the time of charging), energy leakage over time (saved energy is leaked and lost over time), battery size degradation (battery capacity size gets smaller at every recharge), etc. An earlier work to formulate a form of practical energy storage inefficiency in the context of offline throughput maximization is [20]. In [20], two forms of storage imperfections are considered: leakage of saved energy over time and battery degradation. The major effect of such imperfections on the throughput maximization problem is that they modify the feasible energy tunnel in Chapter 2,
which is the tunnel that is formed by the energy causality upper staircase and the no-energy-overflow lower staircase. Reference [20] demonstrates that, in the cases of energy leakage and battery degradation, the energy feasibility tunnel gets narrower with the lower staircase increasing over time, and developed the optimum offline power policy that maximizes the throughput.

The imperfections studied in [20] are long-term effects on energy storage, which affect communications in durations much larger than typical packet transmission duration. In this chapter, we study another class of energy storage inefficiency, which occurs at the time of energy storage almost instantaneously. In particular, we consider the inefficiency (loss) that occurs at the time of charging/discharging: when $E$ units of energy is to be stored in the battery, only $\eta E$ units is saved for future use and $(1 - \eta)E$ is lost instantaneously due to charging/discharging inefficiency, where $0 \leq \eta \leq 1$ represents this efficiency. Depending on the technology used in energy storage, $\eta$ can be as low as 66% [90, 91, 43, 38]. Such losses have been considered in communications in [80, 81] for duty-cycling with constant transmission rate under energy neutrality conditions, but not in the context of offline throughput maximization.

In this chapter, we consider the offline throughput maximization problem under such losses and determine the corresponding optimum energy management policies. The effects of imperfections at charging/discharging considered in this chapter are significantly different than leakage/degradation imperfections studied in [20]. In particular, while leakage/degradation imperfections affect the shape of the feasible energy tunnel, in our case, the energy feasibility tunnel is unaffected. Instead, in our case, we need to
determine what portion of the incoming energy to store despite storage losses, and how to use the stored energy.

Another consideration in power allocation for energy harvesting networks is developing power policies that do not require offline knowledge of energy harvests and channel states. Termed online policies, these policies determine transmit power based on past energy harvests and channel states only. This problem was first considered in [51] for a single transmitter with a Markov energy replenishment model. Optimal online policies in energy harvesting networks are subsequently considered for the point-to-point fading channel [71, 13], the multiple access channel [14], and the two-hop channel [3] in the literature. The common approach is to pose the problem as a Markov decision process (MDP) [10] and use dynamic programming to find the optimal policy [71, 14, 3]. In cases where the statistics of the harvesting process are also unknown, [13] takes a learning theoretic approach to estimate and adapt to its environment. In addition to the offline optimal policy, in this chapter, we formulate the problem of finding the optimal online power policy as a Markov decision process, and present its output as well as its performance in comparison to the optimal offline power policy. This work was published in [105].

3.2 System Model

We adopt the time-slotted model in Figure 3.1 in this chapter, and in the chapters that follow, for simplicity. Namely, we consider the case where energy harvests take place at regular intervals of $\tau$, and the time intervals of length $\tau$ between consecutive energy harvests are termed time slots. While this is a special case of the model in Chapter 2
with \( d_i = (i - 1)\tau \), it can approximate the general model arbitrarily well by choosing a sufficiently small \( \tau \). Moreover, a constant transmit power within each time slot is considered without loss of optimality as a consequence of Lemma 2.1. Hence, this model serves to provide insights about the optimal power policies without the added intricacy of arbitrary energy harvest times and continuous-time power policies.

We further consider \( \tau = 1 \) s, i.e., unit length time slots, for ease of presentation. The results extend trivially to any positive slot length \( \tau > 0 \). The communication session consists of \( N \) time slots. At the beginning of the \( i \)th time slot, the transmitter harvests an energy \( E_i \geq 0 \), which is known non-causally at the transmitter prior to the transmission session. It also retrieves an additional \( \rho_i \geq 0 \) units of energy from the battery, and allocates \( \gamma_i \geq 0 \) for storage in the battery. The resulting transmit power \( p_i \) within time slot \( i \) with length \( \tau = 1 \) is

\[
p_i = E_i - \gamma_i + \rho_i. \quad (3.1)
\]

We note that energy harvests, storing, and retrieving are not instantaneous in reality, but take place throughout the respective time slot. Since energy consumption is also
distributed over the time slot, it is possible, and mathematically convenient, to express energy values as arriving to the node at the beginning of the slot.

The battery has a storing efficiency of $0 \leq \eta \leq 1$: when $\gamma_i$ units of energy is allocated for storing, only $\eta \gamma_i$ units can be stored for future use and $(1 - \eta) \gamma_i$ units of energy is lost due to storage inefficiency. Similarly, a loss may occur when energy is retrieved from the battery. In this work, these two losses are combined in the model in $\eta$, which effectively represents the fraction of energy that can be retrieved from the battery per unit energy stored. The system model is shown in Figure 3.2.

The power policy of the node consists of energy values chosen for storage and retrieval, namely $\gamma_i$ for storage and $\rho_i$ for retrieval, respectively, at time slots $i = 1, \ldots, N$. Since $p_i$ is a deterministic function of $\gamma_i$ and $\rho_i$, it is not included in the power policy. Note that by definition, $\gamma_i \geq 0$, $\rho_i \geq 0$ and $p_i \geq 0$ for $i = 1, \ldots, N$. From (3.1), the last
condition imposes that $E_i + \rho_i - \gamma_i \geq 0$ on $\rho_i$ and $\gamma_i$. Furthermore, the energy causality and no-battery-overflow constraints in Definitions 2.1 and 2.2 apply to this model as well. Let the initial charge of the battery be $E_0$. Denoting the amount of energy in the battery at time slot $i$ as $S_i$, the energy causality constraints in (2.1) become

$$S_i = E_0 + \sum_{n=1}^{i} (\eta \gamma_n - \rho_n) \geq 0, \quad i = 1, \ldots, N,$$

(3.2)

and the no-battery-overflow constraints in (2.2) for a maximum storage capacity of $E_{\text{max}}$ become

$$S_i = E_0 + \sum_{n=1}^{i} (\eta \gamma_n - \rho_n) \leq E_{\text{max}}, \quad i = 1, \ldots, N.$$

(3.3)

Together, the constraints (3.2) and (3.3) ensure that the energy allocated for retrieval, i.e., $\rho_i$, does not exceed the amount of energy stored in the battery for $i = 1, \ldots, N$.

In this chapter, we consider an additive white Gaussian noise communication channel with a power gain of $h_i$ at time slot $i$. The power-rate function for this model with transmit power $p$, power gain $h$, bandwidth $B$ and receiver noise spectral density $N_0$ is given by (2.29). Since $B$ and $N_0$ in (2.29) are constants, we redefine $h$ as $h/BN_0$, and omit the constant factor $B/2$, thus using the simpler form

$$r(p) = \log (1 + hp)$$

(3.4)

without loss of generality. Under this channel model, we consider the problem of maximizing the throughput of this system, which is tantamount to maximizing the average of $r(p_i)$ over a duration of $N$ time slots, by choosing the optimal power policy \{(\gamma_i; \rho_i)\}_{i=1}^{N}.$
We will begin by considering the infinite-sized battery and non-fading channel case, and subsequently extend our results to the finite-sized battery and fading cases.

### 3.3 Optimal Transmission Policy for an Infinite-Sized Battery

For simplicity, we first consider a non-fading channel, i.e., \( h_i = h \) for \( i = 1, \ldots, N \), and an infinite-sized battery, \( E_{\text{max}} = \infty \). The throughput maximization problem for the model in Figure 3.2 over \( N \) time slots is expressed as

\[
\max \left\{ \gamma_i, \rho_i \right\} \sum_{i=1}^{N} r(E_i - \gamma_i + \rho_i),
\]

subject to

\[
E_0 + \sum_{n=1}^{i} (\eta \gamma_n - \rho_n) \geq 0, \quad i = 1, \ldots, N,
\]

\[
E_i - \gamma_i + \rho_i \geq 0, \quad i = 1, \ldots, N,
\]

\[
\gamma_i \geq 0, \quad \rho_i \geq 0, \quad i = 1, \ldots, N,
\]

where \( r(p) \) is given in (3.4). We first present the following lemma which states that it is sub-optimal to store and retrieve energy simultaneously in the same time slot for \( \eta < 1 \).

The \( \eta = 1 \) case is omitted since it is equivalent to the problem considered in Chapter 2.

**Lemma 3.1.** For \( \eta < 1 \), the solution to (3.5) satisfies \( \gamma_i \rho_i = 0 \) for \( i = 1, \ldots, N \), i.e., the optimal policy never stores and retrieves energy simultaneously.

**Proof:** Let \( \left\{ \left( \gamma_i, \rho_i \right) \right\}_{i=1}^{N} \) be a feasible power policy with \( \gamma_n \rho_n > 0 \) for some \( n \). Let

\[
\bar{\gamma}_n = \left[ \gamma_n - \rho_n / \eta \right]^+, \quad \bar{\rho}_n = \left[ \rho_n - \eta \gamma_n \right]^+,
\]

(3.6)
where $[x]^+ = \max\{0, x\}$. For all $i \neq n$, let $\gamma_i = \gamma_i$ and $\rho_i = \rho_i$. Note that the battery dynamics in (3.2) are unaffected by this change, since $\eta \gamma_i - \rho_i = \eta \gamma_i - \rho_i$, for all $i$. Therefore, the policy \{$(\gamma_i, \rho_i)$\}$^N_{i=1}$ is feasible. On the other hand, the resulting transmit power $\bar{p}_n$ at time slot $n$ becomes

$$\bar{p}_n = E_n - \gamma_n + \rho_n = \begin{cases} E_n - \gamma_n + \rho_n/\eta, & \text{if } \eta \gamma_n \geq \rho_n, \\ E_n - \eta \gamma_n + \rho_n, & \text{otherwise,} \end{cases}$$

and consequently $\bar{p}_n > p_n$ due to $\eta < 1$, $\gamma_n > 0$ and $\rho_n > 0$. Since the rate $r(p)$ is increasing in $p$, we have $r(\bar{p}_n) > r(p_n)$, while $r(\bar{p}_i) = r(p_i)$ for $i \neq n$. Hence, the power policy \{$(\gamma_i, \rho_i)$\}$^N_{i=1}$ achieves a larger throughput than \{$(\gamma_i, \rho_i)$\}$^N_{i=1}$, and the latter policy cannot be optimal.

Lemma 3.1 shows that we can restrict our search for the optimal policy to those that do not store and retrieve energy simultaneously at any time. We remark that simultaneously charging and discharging a battery may or may not be physically possible, but through Lemma 3.1, we show that it is mathematically sub-optimal for our problem.

### 3.3.1 The Double-Threshold Policy

We next observe a property of the optimal power policy. Since all constraints are linear and $r(p)$ is strictly concave in $p$, the problem in (3.1) is a convex optimization problem. Hence, Karush-Kuhn-Tucker (KKT) conditions are necessary and sufficient for
optimality. The Lagrangian of (3.5) is

\[
\mathcal{L} = \sum_{i=1}^{N} \left( r(E_i - \gamma_i + \rho_i) + \lambda_i \left( E_0 + \sum_{n=1}^{i} (\eta \gamma_n - \rho_n) \right) + \mu_i (E_i - \gamma_i + \rho_i) + \phi_i \gamma_i + \psi_i \rho_i \right),
\]

(3.8)

where \( \lambda_i, \mu_i, \phi_i \) and \( \psi_i, i = 1, \ldots, N \) are non-negative Lagrange multipliers corresponding to the energy causality, non-negativity of power, and non-negativity of stored and retrieved energy, respectively. The KKT optimality conditions are found by taking the derivatives with respect to \( \gamma_i \) and \( \rho_i \) for \( i = 1, \ldots, N \) as

\[
-\frac{h}{1 + hp_i} + \eta \sum_{n=i}^{N} \lambda_n - \mu_i + \phi_i = 0, \quad i = 1, \ldots, N,
\]

(3.9)

\[
\frac{h}{1 + hp_i} - \sum_{n=i}^{N} \lambda_n + \mu_i + \psi_i = 0, \quad i = 1, \ldots, N,
\]

(3.10)

with the complementary slackness conditions

\[
\lambda_i \left( E_0 + \sum_{n=1}^{i} (\eta \gamma_n - \rho_n) \right) = 0, \quad i = 1, \ldots, N,
\]

(3.11a)

\[
\mu_i (E_i - \gamma_i + \rho_i) = 0, \quad i = 1, \ldots, N,
\]

(3.11b)

\[
\phi_i \gamma_i = 0, \quad \psi_i \rho_i = 0, \quad i = 1, \ldots, N.
\]

(3.11c)

From (3.9) and (3.10), we find the optimal transmit powers \( p_i \) as

\[
p_i = \frac{1}{\eta \sum_{n=i}^{N} \lambda_n - \mu_i + \phi_i} - \frac{1}{h}
\]
\[
= \frac{1}{\sum_{n=i}^{N} \lambda_n - \mu_i - \psi_i} - \frac{1}{h}, \quad i = 1, \ldots, N. \tag{3.12}
\]

We define two sets of thresholds, \( p_{\gamma,i} \) and \( p_{\rho,i} \), as

\[
p_{\gamma,i} = \frac{1}{\eta \sum_{n=i}^{N} \lambda_n} - \frac{1}{h}, \quad i = 1, \ldots, N, \tag{3.13a}
\]
\[
p_{\rho,i} = \frac{1}{\sum_{n=i}^{N} \lambda_n} - \frac{1}{h}, \quad i = 1, \ldots, N. \tag{3.13b}
\]

Note that these thresholds satisfy

\[
p_{\gamma,i} \geq p_{\rho,i}, \quad i = 1, \ldots, N, \tag{3.14}
\]

and are related as

\[
\frac{1 + hp_{\rho,i}}{1 + hp_{\gamma,i}} = \eta, \quad i = 1, \ldots, N. \tag{3.15}
\]

We also note that whenever \( p_i > 0 \), we have \( \mu_i = 0 \) from (3.11b). Then, from the first equality in (3.12), since \( \phi_i \geq 0 \), we have \( p_i \leq p_{\gamma,i} \). Similarly, from the second equality in (3.12), since \( \psi_i \geq 0 \), we have \( p_i \geq p_{\rho,i} \). Therefore, for \( p_i > 0 \), we have

\[
p_{\gamma,i} \geq p_i \geq p_{\rho,i}. \tag{3.16}
\]

Hence, we refer to \( p_{\gamma,i} \) and \( p_{\rho,i} \) as thresholds: when transmit power \( p_i > 0 \), it must be larger than the lower threshold \( p_{\rho,i} \), and smaller than the upper threshold \( p_{\gamma,i} \). In
the following lemma, we show that charging and discharging are also related to these thresholds in the optimal policy.

**Lemma 3.2.** Whenever the battery is being charged, i.e., $\gamma_i > 0$, a non-zero transmit power must satisfy $p_i = p_{\gamma,i}$. Conversely, whenever the battery is being discharged, i.e., $\rho_i > 0$, a non-zero transmit power must satisfy $p_i = p_{\rho,i}$.

**Proof:** For a non-zero transmit power $p_i > 0$, due to (3.11b) we have $\mu_i = 0$. When the battery is being charged, i.e., $\gamma_i > 0$, from (3.11c), we get $\phi_i = 0$. Substituting this in the first equality in (3.12) yields $p_i = p_{\gamma,i}$. When the battery is being discharged, i.e., $\rho_i > 0$, from (3.11c), we get $\psi_i = 0$. Substituting this in the second equality in (3.12) yields $p_i = p_{\rho,i}$. ■

Due to Lemma 3.2, we call $p_{\gamma,i}$ the *storing threshold* and $p_{\rho,i}$ the *retrieving threshold*. We observe from Lemma 3.1 that we have either $\gamma_i > 0$ and $\rho_i = 0$, or $\gamma_i = 0$ and $\rho_i > 0$, or $\gamma_i = 0$ and $\rho_i = 0$. When $\gamma_i = \rho_i = 0$, from (3.1), we have $p_i = E_i$, which must satisfy (3.16). These conditions show that there is a double-threshold policy on $p_i$. Specifically, when the battery is being charged, the transmit power equals the storing threshold $p_{\gamma,i}$; and when the battery is being discharged, the transmit power equals the retrieving threshold $p_{\rho,i}$. If the battery is neither being charged or discharged, i.e., the battery is passive, then $p_i = E_i$, i.e., the transmitter uses all the harvested energy in the current slot.

**Theorem 3.1.** The power policy solving the problem in (3.5) has the following double-threshold structure:
a. If $E_i > p_{\gamma,i}$, then $p_i = [p_{\gamma,i}]^+$. Consequently, $\gamma_i = E_i - [p_{\gamma,i}]^+ > 0$ and $\rho_i = 0$ (storing).

b. If $E_i < p_{\rho,i}$, then $p_i = p_{\rho,i}$. Consequently, $\gamma_i = 0$ and $\rho_i = p_{\rho,i} - E_i > 0$ (retrieving).

c. If $p_{\gamma,i} \geq E_i \geq p_{\rho,i}$, then $\gamma_i = \rho_i = 0$ and $p_i = E_i$ (passive).

**Proof:** We prove each case separately:

a) Consider the case $E_i > p_{\gamma,i}$. From (3.1) and Lemma 3.1, we have $\gamma_i \leq E_i$. We consider the three distinct cases $\gamma_i = 0$, $0 < \gamma_i < E_i$, and $\gamma_i = E_i$ as follows. When $\gamma_i = 0$, from (3.1), we get $p_i \geq E_i$. Together with $E_i > p_{\gamma,i}$, this contradicts (3.16). Therefore, this case cannot be optimal. When $E_i > \gamma_i > 0$, from (3.1) we get $p_i > 0$ and from Lemma 3.2 we have $p_i = p_{\gamma,i}$. Finally, when $\gamma_i = E_i$ and therefore $\rho_i = 0$, (3.1) yields $p_i = 0$. Since $\phi_i = 0$ from (3.11c), substituting in (3.12) gives $p_{\gamma,i} \leq p_i = 0$. Hence, for all possible sub-cases in this case, $p_i = [p_{\gamma,i}]^+$.

b) Consider the case $E_i < p_{\rho,i}$. We consider the three distinct cases, $\rho_i = 0$ and $p_i > 0$, $\rho_i = 0$ and $p_i = 0$, and $\rho_i > 0$ as follows. When $\rho_i = 0$ and $p_i > 0$, from (3.1), we get $p_i \leq E_i$. Together with $E_i < p_{\rho,i}$, this contradicts (3.16), and therefore this case cannot be optimal. When $\rho_i = 0$ and $p_i = 0$, from (3.1) we have $\gamma_i = E_i$, implying that $\phi_i = 0$ due to (3.11c). From (3.12) and (3.14), we get $p_{\rho,i} \leq p_i = 0$, which contradicts $E_i < p_{\rho,i}$. Therefore, this case cannot be optimal. Finally, when $\rho_i > 0$, this implies $p_i > 0$ due to (3.1) and Lemma 3.1. From Lemma 3.2 we have $p_i = p_{\rho,i}$. Hence, for the $E_i < p_{\rho,i}$ case, the only possible transmit power is $p_i = p_{\rho,i}$.
c) Consider the case \( p_{\gamma,i} \geq E_i \geq p_{\rho,i} \). When \( \gamma_i > 0 \), then from (3.1) and Lemma 3.1, we get \( p_i < E_i \). Due to \( p_{\gamma,i} \geq E_i \), this contradicts Lemma 3.2. Therefore, this case cannot be optimal. On the other hand, when \( \rho_i > 0 \), then (3.1) and Lemma 3.1 yield \( p_i > E_i \). Due to \( E_i \geq p_{\rho,i} \), this contradicts Lemma 3.2, and therefore this case cannot be optimal. Hence, \( \gamma_i = \rho_i = 0 \) in this case, yielding \( p_i = E_i \) from (3.1).

Theorem 3.1 shows that the optimal policy is a double-threshold policy that can be expressed as

\[
p_i = \min \left\{ \max \left\{ E_i, p_{\rho,i} \right\}, [p_{\gamma,i}]^+ \right\}, \tag{3.17a}
\]

\[
\gamma_i = [E_i - p_i]^+, \quad \rho_i = [p_i - E_i]^+ . \tag{3.17b}
\]

To find the entire policy, it remains to find the thresholds \( p_{\gamma,i} \) and \( p_{\rho,i} \) for \( i = 1, \ldots, N \), which we describe next.

### 3.3.2 Finding the Thresholds

To determine the thresholds defined in (3.13), we make the observations stated in the following two lemmas: Lemma 3.3 states that these thresholds are non-decreasing in general, and remain constant in stretches of time slots when there is energy in the battery. Therefore, they only potentially increase when the battery is depleted. Lemma 3.4 states that the battery must be depleted by the end of the communication session, otherwise the throughput can be increased by retrieving and using the remaining energy in the battery in the last \( (N\text{th}) \) time slot.
**Lemma 3.3.** The thresholds $p_{\gamma,i}$ and $p_{\rho,i}$ in (3.13) are non-decreasing, and remain constant unless the battery is depleted, i.e., $p_{\gamma,i+1} = p_{\gamma,i}$ and $p_{\rho,i+1} = p_{\rho,i}$ for all $i$ when $S_i > 0$.

**Proof:** The non-decreasing property follows from $\lambda_i \geq 0$ in (3.13a)-(3.13b). The second property in the lemma is a consequence of the complementary slackness condition in (3.11a), which implies that when $S_i > 0$ we have $\lambda_i = 0$ and $p_{\gamma,i}$ and $p_{\rho,i}$ remain constant from (3.13a)-(3.13b). ■

**Lemma 3.4.** In the optimal policy, the battery is depleted at the end of the session, i.e., $S_N = 0$.

**Proof:** The proof is by contradiction. Let $\{(\gamma_i,\rho_i)\}_{i=1}^N$ be a feasible policy with $S_N > 0$. Let $\bar{\gamma}_i = \gamma_i$ for $i = 1, \ldots, N$, $\bar{\rho}_i = \rho_i$ for $i = 1, \ldots, N-1$, and $\bar{\rho}_N = \rho_N + S_N$. Note that $\{\{(\bar{\gamma}_i,\bar{\rho}_i)\}_{i=1}^N$ is a feasible policy. For this new policy, we have $r(\bar{\rho}_i) = r(p_i)$ for $i = 1, \ldots, N-1$ and $r(\bar{\rho}_N) > r(p_N)$, yielding a larger throughput. Hence, $\{(\gamma_i,\rho_i)\}_{i=1}^N$ cannot be optimal. ■

Note that these lemmas are generalizations of Lemmas 2.1, 2.3, 2.4, and 2.5 to the inefficient storage case with an infinite-sized battery. The main difference is that in Lemma 3.3, the structural properties apply to the thresholds rather than the transmit powers. Hence, the design insights of Algorithm 2.1 are largely in effect on the thresholds of problem as well.

In light of Lemmas 3.3 and 3.4, we seek a set of thresholds that are non-decreasing for all $i$, only increasing when $S_i = 0$, and depleting the battery at the end of the
transmission. Note that it suffices to find $p_{\gamma,i}$ values only, and $p_{\rho,i}$ can be calculated from the fixed relationship in (3.15). Based on the insights obtained from Algorithm 2.1, we propose Algorithm 3.1 to find the optimal thresholds.

**Algorithm 3.1** Finding the Optimal Thresholds for an EH Transmitter with Inefficient Energy Storage

1. Let $n = 1$.
2. Find the largest threshold $p_{\gamma} = p_{\gamma,n} = p_{\gamma,n+1} = \cdots = p_{\gamma,N} \geq 0$, and the corresponding $\{p_{\rho}\}$ from (3.15), for which $\{p_i\}$ given by (3.17) is feasible in $i = n, \ldots, N$.
3. Find the smallest $m > n$ such that $S_m = 0$, and assign optimal thresholds $p_{\gamma}$ and $p_{\rho}$ to time slots $i = n, \ldots, m$.
4. If $m < N$, repeat steps 2-4 for $n = m + 1$.

The procedure in Algorithm 3.1 ensures that the resulting thresholds are non-decreasing and remain constant while the battery is not depleted, as required by Lemma 3.3. The non-decreasing property can be seen as follows: At a step starting from time slot $n$, the previous threshold $p_{\gamma,n-1}$ is feasible in $i = n, \ldots, N$ by construction. Hence, the new threshold $p_{\gamma,n} \geq p_{\gamma,n-1}$. Next, we prove the optimality of these thresholds.

**Theorem 3.2.** The policy in (3.17) with thresholds $\left\{(p_{\gamma,i}^*, p_{\rho,i}^*)\right\}_{i=1}^N$ found using Algorithm 3.1 is the solution to (3.5).
Proof: We show that using $p^*_{\gamma,i}$ and $p^*_{\rho,i}$, $i = 1, \ldots, N$, a set of Lagrange multipliers satisfying all KKT conditions in (3.9)-(3.11) can be found. Note that $p^*_{\gamma,i}$ and $p^*_{\rho,i}$ are non-decreasing. Let

$$\lambda_i = \frac{1}{\eta(p^*_i + 1/h)} - \frac{1}{\eta(p^*_{i+1} + 1/h)}$$

for $i = 1, \ldots, N$, with $p^*_{\gamma,N+1} = \infty$ by definition. This satisfies $\lambda_i \geq 0$ since $p^*_{\gamma,i}$ is non-decreasing, and satisfies (3.11a) since $p^*_{\gamma,i}$ only changes when $S_i = 0$.

Next, let $\mu_i = 0$, $i = 1, \ldots, N$, which satisfy (3.11b). Since $p^*_{\gamma,i} \geq 0$ and $p^*_{\rho,i} \geq 0$ by construction, from (3.17a) we get $p^*_{\gamma,i} \geq p^*_i \geq p^*_{\rho,i}$ for all $i$. Calculate $\phi_i$ and $\psi_i$ from (3.12) as

$$\phi_i = \frac{h}{1 + hp^*_i} - \frac{h}{1 + hp^*_{\gamma,i}}, \quad \psi_i = \frac{h}{1 + hp^*_{\rho,i}} - \frac{h}{1 + hp^*_i}.$$  

Note that these values are non-negative since $p^*_{\gamma,i} \geq p^*_i \geq p^*_{\rho,i}$. Furthermore, they satisfy (3.11c) as follows: when $\gamma_i > 0$, (3.17b) implies $E_i > p^*_i$. Thus, from (3.17a), we have $p^*_i = [p^*_{\gamma,i}]^+ = p^*_{\gamma,i}$, and therefore (3.19) yields $\phi_i = 0$. Similarly, when $\rho_i > 0$, (3.17b) implies $E_i < p^*_i$, and therefore (3.17a) implies $p^*_i = p^*_{\rho,i}$. Hence, (3.19) yields $\psi_i = 0$. □

An example run of Algorithm 3.1 and the resulting optimal transmission policy is shown in Figure 3.3. The example is over $N = 5$ time slots with storage efficiency $\eta = 0.5$, energy harvests $E_i = [9, 4, 2, 13, 4]$, initial charge $E_0 = 0$, and $h = 1$. Starting from $n = 1$, the largest feasible thresholds satisfying (3.15) are found as $p^*_\gamma = 7$ and $p^*_\rho = 3$, depleting the battery at the end of time slot $i = 3$. Setting these thresholds for
the second set of thresholds starting from \( n = 4 \) are found as \( p_\gamma = 11 \) and \( p_\rho = 5 \), depleting the battery at the end of time slot \( i = 5 = N \). With these thresholds, the optimal transmit powers \( p_i^* \) are shown in red. Energy stored at the first time slot, marked as I, is retrieved at the third time slot, marked as II. Similarly, energy stored in the fourth time slot, marked as III, is retrieved and consumed entirely in the fifth time slot, marked as IV. Note that since \( p_{\gamma,i}^* \geq E_i \geq p_{\rho,i}^* \) at \( i = 2 \), no charging or discharging takes place. In this slot, transmitter is in the passive state in Theorem 3.1, and uses only the incoming energy for transmission, i.e., \( p_i^* = E_i \).

The optimal policy derived in this section can be shown to converge to the results of [120] when energy storage is ideal. This is when \( \eta = 1 \), and (3.15) yields \( p_{\gamma,i} = p_{\rho,i} \).
for all $i$. Due to (3.17a), this implies $p_i = [p_{\gamma,i}]^+$ at all times. Consequently, the optimal policy consists of piecewise constant power transmissions, with the transmit power increasing only at instances of empty battery due to Lemma 3.3. This coincides with the result with ideal battery in [120] which finds the longest constant power stretches, and changes the power only when the battery is depleted. As battery efficiency $\eta$ decreases, the two thresholds $p_{\gamma,i}$ and $p_{\rho,i}$ separate, yielding a larger region for the passive state. Our optimal policy describes the transition from the constant power policy in one extreme, $\eta = 1$, to the spend what you get policy without storage in the other, $\eta = 0$.

3.4 Optimal Transmission Policy for a Finite-Sized Battery

In this section, we extend the infinite-sized battery problem in (3.5) to the case of a finite-sized battery by including the additional no-energy-overflow constraint in (3.3). For a battery of size $E_{max}$, the throughput maximization problem becomes

$$\max_{\{\gamma_i, \rho_i\}} \sum_{i=1}^{N} r \left( E_i - \gamma_i + \rho_i \right),$$  

$$\text{s.t. } 0 \leq E_0 + \sum_{n=1}^{i} (\eta_n \gamma_n - \rho_n) \leq E_{max}, \quad i = 1, \ldots, N,$$

$$E_i - \gamma_i + \rho_i \geq 0, \quad i = 1, \ldots, N,$$

$$\gamma_i \geq 0, \quad \rho_i \geq 0, \quad i = 1, \ldots, N,$$

where $r(p)$ is defined in (3.4). The Lagrangian of (3.20) is

$$\mathcal{L} = \sum_{i=1}^{N} \left( r(E_i - \gamma_i + \rho_i) + \lambda_i \left( E_0 + \sum_{n=1}^{i} (\eta_n \gamma_n - \rho_n) \right) \right)$$
\[
- \beta_i \left( E_0 + \sum_{n=1}^i (\eta \gamma_n - \rho_n) - E^{max} \right)
\]
\[
+ \mu_i (E_i - \gamma_i + \rho_i) + \phi_i \gamma_i + \psi_i \rho_i \right), \quad (3.21)
\]

where $\beta_i$, $i = 1, \ldots, N$ are the non-negative Lagrange multipliers for the no-energy-overflow constraints in (3.3). The KKT optimality conditions are

\[
- \frac{h}{1 + hp_i} + \eta \sum_{n=i}^N (\lambda_n - \beta_n) - \mu_i + \phi_i = 0, \quad i = 1, \ldots, N, \quad (3.22)
\]
\[
\frac{h}{1 + hp_i} - \sum_{n=i}^N (\lambda_n - \beta_n) + \mu_i + \psi_i = 0, \quad i = 1, \ldots, N, \quad (3.23)
\]

and the complementary slackness conditions corresponding to $\beta_i$ are

\[
\beta_i \left( E_0 + \sum_{n=1}^i (\eta \gamma_n - \rho_n) - E^{max} \right) = 0, \quad i = 1, \ldots, N, \quad (3.24)
\]

which, together with those listed in (3.11), constitute the complementary slackness conditions for the problem in (3.20). From (3.22) and (3.23), we find the optimal transmit powers $p_i$ as

\[
p_i = \frac{1}{\eta \sum_{n=i}^N (\lambda_n - \beta_n) - \mu_i + \phi_i} - \frac{1}{h}, \quad i = 1, \ldots, N.
\]

\[
= \frac{1}{\sum_{n=i}^N (\lambda_n - \beta_n) - \mu_i - \psi_i} - \frac{1}{h}, \quad i = 1, \ldots, N. \quad (3.25)
\]
In view of the new multipliers $\beta_i$, we update the definition of $p_{\gamma,i}$ and $p_{\rho,i}$, as

\begin{align}
  p_{\gamma,i} &= \frac{1}{\eta \sum_{n=i}^{N} (\lambda_n - \beta_n)} - \frac{1}{h}, \quad i = 1, \ldots, N, \\
  p_{\rho,i} &= \frac{1}{\sum_{n=i}^{N} (\lambda_n - \beta_n)} - \frac{1}{h}, \quad i = 1, \ldots, N,
\end{align}

(3.26a) (3.26b)

which satisfy (3.14) and (3.15). Observing that $\mu_i = 0$ when $p_i > 0$, (3.16) must also hold for the optimal policy.

### 3.4.1 Finding the Thresholds for a Finite-Sized Battery

In the finite-sized battery case, Lemmas 3.1, 3.2 and 3.4 continue to hold, i.e.,

- $\gamma_i \rho_i = 0$ (simultaneous storing and retrieval is sub-optimal), if $\gamma_i > 0$ then $p_i = p_{\gamma,i}$ (when storing, the power must be equal to the storing threshold), if $\rho_i > 0$ then $p_i = p_{\rho,i}$ (when retrieving, the power must be equal to the retrieving threshold), and $S_N = 0$ (the battery must be depleted at the end of the communication session). However, due to $\beta_i$, the new thresholds in (3.26) no longer satisfy Lemma 3.3, i.e., the new thresholds are no longer monotone. Instead, they satisfy the property stated in the following lemma.

**Lemma 3.5.** The thresholds $p_{\gamma,i}$ and $p_{\rho,i}$ in (3.26) are non-decreasing while $S_i < E_{\text{max}}$, and non-increasing while $S_i > 0$. Consequently, they remain constant if the battery is not depleted or full, i.e., $p_{\gamma,i+1} = p_{\gamma,i}$ and $p_{\rho,i+1} = p_{\rho,i}$ for all $i$ while $0 < S_i < E_{\text{max}}$.

**Proof:** For $S_i < E_{\text{max}}$, (3.24) gives $\beta_i = 0$. Substituting in (3.26), this implies that $p_{\gamma,i}$ and $p_{\rho,i}$ are non-decreasing. Similarly, for $S_i > 0$, (3.11a) gives $\lambda_i = 0$. Substituting in (3.26), this implies that $p_{\gamma,i}$ and $p_{\rho,i}$ are non-increasing. Finally, for $0 < S_i < E_{\text{max}}$, ...
(3.11a) and (3.24) give \( \lambda_i = 0 \) and \( \beta_i = 0 \), which yields \( p_{\gamma,i+1} = p_{\gamma,i} \) and \( p_{\rho,i+1} = p_{\rho,i} \) from (3.26), which implies that \( p_{\gamma,i} \) and \( p_{\rho,i} \) remain constant in \( i \). ■

Note that this is merely the generalization of Lemma 2.4 to the inefficient storage case. We are looking for a feasible set of thresholds satisfying Lemmas 3.1, 3.2, 3.4 and 3.5. As an extension of Algorithm 3.1 based on Algorithm 2.1, we propose Algorithm 3.2 to find the optimal policy in the finite-sized battery case.

**Algorithm 3.2** Finding the Optimal Thresholds for an EH Transmitter with Finite and Inefficient Energy Storage

1. Let \( n = 1 \).

2. Find the largest threshold \( p_{\gamma} = p_{\gamma,n} = p_{\gamma,n+1} = \cdots = p_{\gamma,N} \geq 0 \), and the corresponding \( \{p_{\rho}\} \) from (3.15), for which \( \{p_{\gamma}\} \) given by (3.17) does not violate (3.2) first, i.e., either the policy is feasible for \( i = n, \ldots, N \), or (3.3) is violated before (3.2).

3. Find the smallest \( m > n \) such that \( S_m = 0 \) or \( S_m = E_{max} \), and assign optimal thresholds \( p_{\gamma} \) and \( p_{\rho} \) to time slots \( i = n, \ldots, m \).

4. If \( m < N \), repeat steps 2-4 for \( n = m + 1 \).

**Lemma 3.6.** The thresholds found by Algorithm 3.2 satisfy the conditions in Lemma 3.5.

**Proof:** Starting from some \( n \), let steps 2-3 of the algorithm output \( p_{\gamma,n} \), \( p_{\rho,n} \) and \( m \). Consider the case \( S_m = 0 \). Then, the constant threshold \( p_{\gamma,n} \) must yield a full battery at some \( k > m \), or be feasible until \( i = N \), since otherwise a smaller \( p_{\gamma,n} \) would have been chosen by the algorithm. Hence, starting from time slot \( m \), the next threshold cannot be
less than \( p_{\gamma,n} \). Now, consider the case \( S_m = E_{\text{max}} \). Then, the constant threshold \( p_{\gamma,n} \) violates (3.2) or depletes the battery at some \( k > m \) by construction. Hence, starting from time slot \( m \), the next threshold cannot be greater than \( p_{\gamma,n} \). ■

As a result of Lemma 3.6, we have that the thresholds found by Algorithm 3.2 are non-decreasing if \( S_i = 0 \), non-increasing if \( S_i = E_{\text{max}} \), and by construction constant in between. Next, we prove the optimality of the resulting policy.

**Theorem 3.3.** The policy in (3.17) with thresholds \( \{(p_{\gamma,i}^*, p_{\rho,i}^*)\}_{i=1}^N \) found using Algorithm 3.2 is the solution to (3.20).

**Proof:** We show that using \( p_{\gamma,i}^* \) and \( p_{\rho,i}^* \), \( i = 1, \ldots, N \), a set of Lagrange multipliers satisfying all KKT conditions in (3.11) and (3.22)-(3.24) can be found. First, note that \( p_{\gamma,i}^* \) and \( p_{\rho,i}^* \) are constant unless the battery is depleted or full, non-decreasing if \( S_i = 0 \) and non-increasing if \( S_i = E_{\text{max}} \), as shown in Lemma 3.6. Let

\[
\lambda_i = \left[ \frac{1}{\eta(p_{\gamma,i}^* + 1/h)} - \frac{1}{\eta(p_{\gamma,i+1}^* + 1/h)} \right]^+, \tag{3.27a}
\]

\[
\beta_i = \left[ \frac{1}{\eta(p_{\gamma,i+1}^* + 1/h)} - \frac{1}{\eta(p_{\gamma,i}^* + 1/h)} \right]^+, \tag{3.27b}
\]

for \( i = 1, \ldots, N \), with \( p_{\gamma,N+1}^* = \infty \) by definition. These satisfy (3.11a) and (3.24) due to Lemma 3.6. The rest of the Lagrangian multipliers are found as in the proof of Theorem 3.2, by replacing (3.12) with (3.25). ■

The policy for the finite-sized battery case in this section converges to the previous results for the ideal battery case studied in Chapter 2 for \( \eta = 1 \). In this case, the thresholds are equal and thus the optimal policy is a constant power policy.
3.5 Optimal Transmission Policy for the Fading Channel

We next consider a fading channel, where the power gain $h_i, i = 1, \ldots, N$, is constant throughout time slot $i$, but varies among time slots. The power gains are known non-causally at the transmitter and the receiver. This is an extension of [71] to the inefficient energy storage case; see also [41]. The instantaneous rate in slot $i$ is given in (3.4), which we will denote as $r(p_i, h_i)$ in this section to emphasize its dependence on the channel gain $h_i$.

The throughput maximization problem in a fading channel for a transmitter with a finite-sized and inefficient battery becomes

$$
\max_{\{\gamma_i, \rho_i\}} \sum_{i=1}^{N} r(E_i - \gamma_i + \rho_i, h_i)
$$

s.t. $0 \leq E_0 + \sum_{n=1}^{i} (\eta \gamma_n - \rho_n) \leq E_{\text{max}}, \quad i = 1, \ldots, N,$

$$
E_i - \gamma_i + \rho_i \geq 0, \quad i = 1, \ldots, N,
$$

$$
\gamma_i \geq 0, \quad \rho_i \geq 0, \quad i = 1, \ldots, N,
$$

yielding the KKT optimality conditions

$$
-\frac{h_i}{1 + h_ip_i} + \eta \sum_{n=i}^{N} (\lambda_n - \beta_n) - \mu_i + \phi_i = 0, \quad i = 1, \ldots, N,
$$

$$
\frac{h_i}{1 + h_ip_i} - \sum_{n=i}^{N} (\lambda_n - \beta_n) + \mu_i + \psi_i = 0, \quad i = 1, \ldots, N,
$$
and the complementary slackness conditions in (3.11) and (3.24). We note that Lemma 3.1 holds for the fading case as well, since it only depends on the rate function $r(p_i, h_i)$ being non-decreasing in $p_i$.

Recall from Section 2.5 that the properties of optimal transmit powers $\{p_i\}$ in the non-fading channel resurface as properties of optimal water levels $\{\nu_i\}$ in the fading channel, yielding a directional water-filling algorithm. Hence, one might expect that the transmit power thresholds $\{p_{\gamma,i}\}$ and $\{p_{\rho,i}\}$ in Sections 3.3-3.4 would have water level counterparts for the fading channel. Based on this insight, we define the following water level thresholds,

$$
\nu_{\gamma,i} = \frac{1}{\eta \sum_{n=i}^{N} (\lambda_n - \beta_n)}, \quad \nu_{\rho,i} = \frac{1}{\sum_{n=i}^{N} (\lambda_n - \beta_n)},
$$

which satisfy

$$
\nu_{\rho,i} = \eta \nu_{\gamma,i}, \quad i = 1, \ldots, N.
$$

With these definitions, we observe that a positive transmit power, $p_i > 0$, (3.11b) yields $\mu_i = 0$. Therefore, if the battery is being charged, i.e., $\gamma_i > 0$, from (3.11c) and (3.29) we have $p_i = \nu_{\gamma,i} - 1/h_i$. Similarly, if the battery is being discharged, i.e., $\rho_i > 0$, from (3.11c) and (3.30) we have $p_i = \nu_{\rho,i} - 1/h_i$. If $p_i = 0$, then $\gamma_i = E_i$ is being stored, and hence $\phi_i = 0$. Since $\mu_i \geq 0$ by definition, (3.29) gives $\nu_{\gamma,i} - 1/h_i < 0$. Hence, for the storing and retrieving cases, the optimal transmit powers are

$$
p_i = \left[\nu_{\gamma,i} - \frac{1}{h_i}\right]^+ \text{ (storing)}, \quad p_i = \left[\nu_{\rho,i} - \frac{1}{h_i}\right]^+ \text{ (retrieving)}.
$$
Note that the thresholds $\nu_{\gamma,i}$ and $\nu_{\rho,i}$ no longer equal transmit powers directly in these cases, as in Sections 3.3 and 3.4, but set the water levels over which water-filling is to be performed. In particular, if we can find water levels $\nu_{\gamma,i}$ and $\nu_{\rho,i}$ satisfying (3.32) such that the power policy

$$p_i = \min \left\{ \max \left\{ E_i, \nu_{\rho,i} - \frac{1}{h_i} \right\}, \left[ \nu_{\gamma,i} - \frac{1}{h_i} \right]^+ \right\}$$

is feasible, and all KKT conditions are satisfied, then this policy is optimal. Finding these water level thresholds is possible exactly as in Algorithm 3.2 after replacing the power policy in (3.17) with (3.34) and the thresholds $p_{\gamma,i}$ and $p_{\rho,i}$ with $\nu_{\gamma,i}$ and $\nu_{\rho,i}$, respectively.

An example of directional water-filling with thresholds is given in Figure 3.4 for a storage efficiency of $\eta = 0.5$ and $N = 5$. Power gains levels and the harvested energy for $i = 1, \ldots, 5$ are shown in Figure 3.4a in gray and blue, respectively. In particular, the height of the grey area represents $1/h_i$, and the height of the blue area represents $p_i$ in each time slot. The battery size is sufficiently large to store all harvested energy in this example. Two pairs of thresholds satisfying (3.32) are found such that the battery is empty at the end of time slots 3 and 5. Consequently, the thresholds only change at the end of the third time slot. Energy in the areas marked as I and III are stored, and later retrieved and consumed in the areas marked as II and IV, respectively.
Fig. 3.4: Directional water-filling with energy storage and retrieval thresholds: (a) The initial water levels with $p_i = E_i$, and resulting thresholds, and (b) resulting water levels and optimal transmit powers.
3.6 Optimal Transmission Policy for the Broadcast Channel

We next consider a Gaussian broadcast channel, which consists of an energy harvesting transmitter with an inefficient battery, and two receivers, as shown in Figure 3.5. At time slot \(i\), the transmitter allocates power \(p_i\) for transmission, achieving a rate pair \((r_{1,i}, r_{2,i}) \in \mathcal{R}(p_i)\). Here, \(\mathcal{R}(p_i)\) is the capacity region for transmit power \(p_i\), given by

\[
\mathcal{R}(p_i) = \left\{ (r_{1,i}, r_{2,i}) \mid r_{1,i} \leq \frac{B}{2} \log \left( 1 + \frac{\theta h_{1,i} p_i}{BN_{0,1}} \right), \\
r_{2,i} \leq \frac{B}{2} \log \left( 1 + \frac{(1 - \theta) h_{2,i} p_i}{\theta h_{2,i} p_i + BN_{0,2}} \right), \quad 0 \leq \theta \leq 1 \right\},
\]

for bandwidth \(B\), receiver noise spectral densities \(N_{0,1}\) and \(N_{0,2}\), and power gains \(h_1\) and \(h_2\), \(h_1 N_{0,2} \geq h_2 N_{0,1}\) for receivers 1 and 2, respectively [18]. As in (3.4), we omit the constant terms, and redefine \(h_k = h_k / BN_{0,k}\) for \(k = 1, 2\) to obtain

\[
\mathcal{R}(p_i) = \left\{ (r_{1,i}, r_{2,i}) \mid r_{1,i} \leq \log \left( 1 + \theta h_{1,i} p_i \right), \\
r_{2,i} \leq \log \left( 1 + \frac{(1 - \theta) h_{2,i} p_i}{\theta h_{2,i} p_i + 1} \right), \quad 0 \leq \theta \leq 1 \right\},
\]

As a property that is common to all capacity regions, the region in (3.36) is convex.

For this channel, we characterize the maximum throughput region \(\mathcal{R}_{EH}\) as the set of achievable throughput pairs under the energy harvesting constraints in (3.2) and (3.3). This is the extension of the maximum departure region in [75, Defn. 1] to the case
of inefficient storage. Specifically, we write

\[ \mathcal{R}_{EH} = \left\{ \left( \sum_{i=1}^{N} r_{1,i}^i \sum_{i=1}^{N} r_{2,i}^i \right) \bigg| (r_{1,i}^i, r_{2,i}^i) \in \mathcal{R}(E_i - \gamma_i + \rho_i), \gamma_i, \rho_i \geq 0, E_i - \gamma_i + \rho_i \geq 0, (3.2), (3.3) \right\}. \]  

We first present the following result, which is an extension of [116, Lemma 2] to the case of inefficient energy storage.

**Lemma 3.7.** The throughput region \( \mathcal{R}_{EH} \) is convex.

**Proof:** Let \( \{ (\gamma_i^i, \rho_i^i) \}_{1}^{N} \) and \( \{ (\gamma_i^{i'}, \rho_i^{i'}) \}_{1}^{N} \), be two feasible policies yielding transmit powers \( \{ p_i \}_{1}^{N} \) and \( \{ p_i^{i'} \}_{1}^{N} \), and achieving rate pairs \( \{ (r_{1,i}^i, r_{2,i}^i) \}_{1}^{N} \) and \( \{ (r_{1,i}^{i'}, r_{2,i}^{i'}) \}_{1}^{N} \), respectively. Let \( \tilde{\gamma}_i = \theta \gamma_i + (1 - \theta) \gamma_i^{i'} \) and \( \tilde{\rho}_i = \theta \rho_i + (1 - \theta) \rho_i^{i'}, i = 1, \ldots, N \), which yields \( \tilde{p}_i = \theta p_i + (1 - \theta) p_i^{i'}, i = 1, \ldots, N \). Then, due to the convexity of \( \mathcal{R}(p) \), \( \{ (\tilde{\gamma}_i, \tilde{\rho}_i) \}_{1}^{N} \) can achieve the rates \( \tilde{r}_{k,i} \geq \theta r_{k,i} + (1 - \theta) r_{k,i}^{i'}, k = 1, 2 \). Furthermore, the policy \( \{ (\tilde{\gamma}_i, \tilde{\rho}_i) \}_{1}^{N} \)
is feasible since (3.2) and (3.3) are linear in $\gamma_i$ and $\rho_i$. Hence, $\{(\bar{\rho}_{k,i})\}_{1}^{N}$ is achievable, and therefore $\mathcal{R}_{EH}$ is convex. ■

As a result of Lemma 3.7, the boundary of $\mathcal{R}_{EH}$ can be traced by solving a weighted sum-throughput maximization problem. In particular, we solve the following problem for weights $\omega \geq 0$:

\[
\max \left\{ r_{1,i}, r_{2,i} \right\} \quad \omega \sum_{i=1}^{N} r_{1,i} + \sum_{i=1}^{N} r_{2,i},
\]

s.t. \( \left( \sum_{i=1}^{N} r_{1,i}, \sum_{i=1}^{N} r_{2,i} \right) \in \mathcal{R}_{EH}. \) (3.38a)

By substituting (3.37) in (3.38), and separating the maximization over \( \{\gamma_i, \rho_i\} \) and \( \{r_{1,i}, r_{2,i}\} \), we rewrite the weighted sum-throughput maximization problem as

\[
\max \left\{ \gamma_i, \rho_i \right\} \quad \sum_{i=1}^{N} r_{\omega}(E_i - \gamma_i + \rho_i),
\]

s.t. \( E_i - \gamma_i + \rho_i \geq 0, \quad i = 1, \ldots, N, \) (3.39b)

\[
\gamma_i \geq 0, \quad \rho_i \geq 0, \quad (3.2), \quad (3.3), \quad i = 1, \ldots, N,
\]

(3.39c)

where $r_{\omega}(p)$ is the maximum weighted sum-rate function defined as

\[
r_{\omega}(p) = \max_{\{r_1, r_2\}} \omega r_1 + r_2 \quad \text{s.t.} \quad (r_1, r_2) \in \mathcal{R}(p). \quad (3.40)
\]

We next show the concavity of $r_{\omega}(p)$ in the following lemma\(^1\):

\(^1\)We remark that the concavity property in Lemma 3.8 is also shown in [75, Lemma 2] specifically for a Gaussian broadcast channel with $K \geq 2$ receivers. In fact, the weighted sum-rate
Lemma 3.8. The maximum weighted sum-rate function \( r_\omega(p) \) in (3.40) is concave in \( p \).

Proof: Let \( r_\omega(p) = \omega r_1 + r_2 \) and \( r_\omega(p') = \omega r_1' + r_2' \), with \( (r_1, r_2) \in \mathcal{R}(p) \) and \( (r_1', r_2') \in \mathcal{R}(p') \). By concavity of \( \mathcal{R} \), we have \( (\theta r_1 + (1 - \theta)r_1', \theta r_2 + (1 - \theta)r_2') \in \mathcal{R}(\theta p + (1 - \theta)p') \).

From the definition in (3.40), we can write

\[
r_\omega(\theta p + (1 - \theta)p') \geq \omega(\theta r_1 + (1 - \theta)r_1') + \theta r_2 + (1 - \theta)r_2'
\]  

(3.41)

\[
= \theta r_\omega(p) + (1 - \theta)r_\omega(p'),
\]

(3.42)

which implies the concavity of \( r_\omega(p) \) in \( p \). □

With a concave objective (3.39a) and linear constraints (3.39b)-(3.39c), (3.39) is a convex program. This problem differs from that in (3.20) only in the objective. Finding the respective KKT conditions, the relation between the thresholds \( p_{\gamma,i} \) and \( p_{\rho,i} \), given for the single-link setup in (3.15), becomes

\[
\left. \frac{dr_\omega(p)}{dp} \right|_{p_{\gamma,i}} = \eta \left. \frac{dr_\omega(p)}{dp} \right|_{p_{\rho,i}}, \quad i = 1, \ldots, N,
\]

(3.43)

where \( p_{\gamma,i} \) and \( p_{\rho,i} \) are defined as the solutions to

\[
\left. \frac{dr_\omega(p)}{dp} \right|_{p_{\gamma,i}} = \eta \sum_{n=i}^{N} (\lambda_n - \beta_n), \quad i = 1, \ldots, N,
\]

(3.44a)

\[
\left. \frac{dr_\omega(p)}{dp} \right|_{p_{\rho,i}} = \sum_{n=i}^{N} (\lambda_n - \beta_n), \quad i = 1, \ldots, N.
\]

(3.44b)

\( r_\omega(p) \) for any setting is concave due to the possibility of time-sharing between different transmit powers. Hence, these results can be generalized to a larger class of channels.
By construction, the properties in Lemmas 3.2, 3.4 and 3.5 extend to this case. The optimal power policy is therefore found as in Algorithm 3.2 by substituting (3.15) with (3.43). The resulting thresholds satisfy (3.16) and Lemma 3.6 by construction, and therefore yield valid Lagrange multipliers through (3.19) and (3.27).

We remark that the optimal policy conforms to the double-threshold structure defined in Theorem 3.1, regardless of what the weight $\omega$ is. However, unlike the efficient storage case in [75, Lemma 3], the power policies are no longer identical for all weights $\omega$. In particular, the relationship in (3.43) depends on the weight $\omega$, and hence the thresholds depleting or filling the battery in Algorithm 3.2 are affected by the weight. This insight applies to other channel models, and the single-link setup, as well: in [120] and Chapter 2, the optimal power policy is identical for all concave power-rate functions $r(p)$. However, in the inefficient storage case, the derivative $dr_\omega(p)/dp$ affects how the thresholds are related, and thus the optimal policy. As a conclusion, unlike models with ideal batteries previously considered, the rate function plays a direct role in determining the optimal policy in the inefficient storage case.

### 3.7 Online Transmission Policies

The previous sections, as well as Chapter 2, derive optimal policies when the harvesting process over the duration of the session, i.e., $E_i$, $i = 1, \ldots, N$, is known before the session starts. This approach provides a benchmark solution as well as insights for efficient power allocation, and is applicable in scenarios where the harvested energy is controlled or predictable [28]. For other applications where such information may not
be available non-causally, in this section, we develop policies that only require causal knowledge of the harvested energy.

We refer to those transmission policies where the transmitter chooses its power value based on the energy harvested up to that point in time, i.e., with causal information, as online policies. In particular, in an online policy, the transmitter only has access to the realizations of $E_1, \ldots, E_i$ and $h_1, \ldots, h_i$ when choosing the stored and retrieved energy $\gamma_i$ and $\rho_i$ in the $i$th time slot. We remark that simultaneous storage and retrieval of energy is sub-optimal in the online case as well, i.e., Lemma 3.1 extends to online policies. Thus, the transmitter choosing its transmit power $p_i$ is sufficient to find the corresponding $\gamma_i$ and $\rho_i$ values from (3.1). We formalize this choice as the action of the transmitter,

$$p_i = \pi_i(E^i, h^i),$$  \hspace{1cm} \text{(3.45)}$$

which is a function of past energy harvests $E^i$, channel power gains $h^i$, and the time index $i$. In this action, the transmitter needs to simultaneously consider achieving a high throughput in the current time slot, and saving energy for future future time slots. Since these two are contradicting goals by nature, finding the best action is not a trivial task.

3.7.1 Optimal Online Policy

We define the optimal online policy as the action $\pi_i$ that maximizes the expected throughput of the transmitter in $N$ time slots, where the expectation is taken over the energy harvests $E^N$. At time slot $i$, the system can be considered at state $(E^i, h^i)$ which
is known by the transmitter. Upon taking the action $\pi_i(E_i, h_i)$, the system transitions to state $(E_{i+1}, h_{i+1})$ with probability $p(E_{i+1}, h_{i+1}|E_i, h_i)$. In each time slot, the resulting utility is the throughput $r(\pi_i(E_i, h_i))$. By definition, this is a Markov Decision Process [10], the solution to which can be found using dynamic programming, namely value or policy iteration, which we formulate next.

The value function $J_i(E_i, h_i)$ is the achieved throughput in time slot $i$ and the expected future throughput of the system after time slot $i$. This can be found recursively from the Bellman equation,

$$J_i(E_i, h_i) = \max_{\pi_i} \left( r(\pi_i(E_i, h_i), h_i) + \mathbb{E} \left[ \sum_{n=i+1}^{N} r(\pi_n(E_n, h_n), h_n) \right] \right)$$

$$= \max_{\pi_i} \left( r(\pi_i(E_i, h_i), h_i) + \mathbb{E} \left[ J_{i+1}(E_{i+1}, h_{i+1}) \right] \right),$$

where the expectation in (3.46b) is taken over the distribution of the harvesting energy $E_i$ and power gain $h_i$, and $J_{N+1}(\cdot) = 0$. The optimal online power policy $\pi_i^*(\cdot)$ is the maximizer of the Bellman equation in (3.46) [10]. Note that the dimension of the action $\pi_i$ increases with $i$. Thus, solving this problem through value iteration has exponential complexity, and is intractable for large $N$.

It is possible to simplify the problem if the harvested energy $E_i$ and power gains $h_i$ are first order Markov processes. Such harvesting processes are considered previously in [58, 51, 41], and recent work with empirical solar and wind harvesting data confirms that a Markov process is a good model for harvested energy [40]. Finite state Markov channels are also known to be good models for Rayleigh fading channels [110, 109, 127]. Hence, we consider harvested energy values $E_i$ and power gains $h_i$ to be i.i.d. or first
order Markov processes. In this case, only the battery state $S_{i-1}$ and last harvest $E_i$ impose a constraint on the transmit power $p_i$, and only $h_i$ affects the rate, in time slot $i$. Furthermore, future realizations of $E_n$ and $h_n$ for $n > i$ are independent of the past values $E^{i-1}$ and $h^{i-1}$ given their current values $E_i$ and $h_i$. Hence, having different actions for different values of $E_i$ and $h_i$ is not necessary for optimality, which allows us to simplify the actions as

$$\pi_i(E^i, h^i) = \pi_i(S_{i-1}, E_i, h_i).$$ (3.47)

This yields the simplified Bellman equation for first order Markovian harvests and fading,

$$J_i(S_{i-1}, E_i, h_i) = \max_{\pi_i} \left\{ r_i(\pi_i(S_{i-1}, E_i, h_i), h_i) + \mathbb{E}\left[ J_{i+1}(S_{i+1}, E_{i+1}, h_{i+1}) \right] \right\},$$ (3.48)

which can be easily computed using value iteration. Namely, starting from $i = N$ and choosing $J_{N+1}(S_N, E_{N+1}, h_{N+1}) = 0$, optimal actions $\pi_N(S_{N-1}, E_N, h_N)$ and value functions $J_N(S_{N-1}, E_N, h_N)$ are calculated. These values are then used to calculate the optimal actions and value functions at $i = N - 1$ from (3.48), and the process is repeated for $i = N, N - 1, \ldots, 1$.

Finally, we consider the infinite-horizon problem, i.e., $N \to \infty$, and find the optimal online policy via a discounted approach by introducing a discount factor $\Theta \leq 1$. This is also the policy that the value iteration algorithm on (3.48) converges to for very large $N$. We denote this policy with $\pi^*(S_{i-1}, E_i, h_i)$, and express the discounted
Bellman equation as

\[ J(S_{i-1}, E_i, h_i) = \max_\pi r\left( \pi \left(S_{i-1}, E_i, h_i\right), h_i\right) + \Theta \mathbb{E}\left[ J\left(S_i, E_{i+1}, h_{i+1}\right)\right]. \]  

(3.49)

Starting from arbitrary actions and \( J(S_{i-1}, E_i, h_i) = 0 \), iterating (3.49) converges to a stationary value function and the corresponding optimal policy \( \pi^*(S_{i-1}, E_i, h_i) \). As the discount factor \( \Theta \to 1 \), the resulting policy approaches the optimal infinite horizon policy.

Figure 3.6 shows the optimal infinite horizon policy for a non-fading Gaussian channel with power gain \( h_i = -100 \) dB, bandwidth \( B = 1 \) MHz, noise spectral density \( N_0 = 10^{-19} \) W/Hz, a finite-sized battery \( E_{max} = 100 \) mJ, and i.i.d. uniform energy harvests in \([0, 20]\) mJ. Note that for a fixed stored energy, the optimal online policy exhibits a double-threshold structure similar to that in Theorem 3.1, e.g., the bold line for \( S_{i-1} = 60 \) mJ in the figure. The optimal transmit power is equal to the harvested energy for a range of \( S_{i-1} \) and \( E_i \) values, marked as region I. Regions II and III are separated from region I by a set of thresholds, indicated with dashed lines. Within these regions, the transmit powers vary only slightly with harvested energy rate \( E_i \). The two thresholds separating region I from regions II and III are observed to satisfy the relationship in (3.15) for each \( S_{i-1} \). The thresholds, however, change with \( S_{i-1} \): For small \( S_{i-1} \), the thresholds are lower, with \( p_{\rho,i} = 0 \) at \( S_{i-1} = 0 \) since retrieving energy is not feasible. For large \( S_{i-1} \), the thresholds are higher, reaching to \( p_{\gamma,i} > 20 \) mW at \( S_{i-1} = E_{max} \), since storing energy is not feasible.
Fig. 3.6: Optimal online transmit powers for the i.i.d. energy harvesting model.
For harvesting processes with memory, we consider two scenarios with Markovian energy harvests in Figure 3.7 and Figure 3.8. In Figure 3.7, harvesting is a bursty process where the next energy harvest remains the same, i.e., $E_{i+1} = E_i$, with probability $1/2$, and a new value that is uniform in $[0, 20]$ mJ is generated with probability $1/2$. Hence, the process consists of bursts of constant rate harvests. As seen in Figure 3.7, this harvesting model also yields a double-threshold policy that resembles the i.i.d. case in Figure 3.6. Figure 3.8 shows the optimal policy when the harvested energy $E_i$ performs a random walk on $[0, 20]$ mJ over the time slots, where it increases or decreases 1 mJ with probability $2/5$ each, and remains the same with probability $1/5$. In this case, the optimal policy is to consume all harvested energy, which does not show a threshold characteristic. This is intuitive, because a high or low harvest rate is sustained for extended periods of time in this model, and consistently storing or retrieving would likely overflow or deplete the battery.

3.7.2 Proposed Online Policy

Sections 3.3-3.6 show that the optimal policy has a double-threshold structure where the thresholds are related for all $i = 1, \ldots, N$. The infinite-horizon optimal online policies found in Section 3.7.1 for i.i.d. and bursty Markov harvests also exhibit a similar double-threshold structure. With these insights in mind, in this section, we propose a simpler online double-threshold policy by assigning constant thresholds throughout the communication session.

We first consider a non-fading channel and Markovian harvested energy values $E_i$ with stationary probability distribution $p_E(e)$. We propose finding constant thresholds
Fig. 3.7: Optimal online transmit powers for the bursty energy harvesting model.

Fig. 3.8: Optimal online transmit power for the random walk energy harvesting model.
\( p_{\gamma,i} = p_{\gamma} \) and \( p_{\rho,i} = p_{\rho} \), \( i = 1, \ldots, N \), that satisfy (3.15) and

\[
\eta \int_{p_{\gamma}}^{\infty} (e - p_{\gamma}) p_{E}(e) \, de - \int_{0}^{p_{\rho}} (p_{\rho} - e) p_{E}(e) \, de = 0.
\] (3.50)

This equation can be interpreted as an energy stability condition, since it implies that the expected energy stored in and retrieved from the battery are equal. Thus, neither the energy storage is underutilized, nor an excessive amount of energy is stored without utility. Note that at \( \eta = 1 \), this reduces to a constant power policy that preserves energy-neutrality, and resembles the best-effort transmission scheme of [73] which is optimal for and infinite length transmission. On the other hand, as \( \eta \to 0 \), (3.50) is satisfied with \( p_{\rho} \to 0 \) and \( p_{\gamma} \to \infty \). This means that no energy is stored, i.e., \( p_{i} = E_{i} \), which is optimal since storing energy is clearly not a desirable option when \( \eta = 0 \).

The above policy can be readily extended to a fading channel with Markovian power gains \( h_{i} \) and joint stationary distribution \( p_{E,H}(e,h) \) by finding water level thresholds \( \nu_{\gamma,i} = \nu_{\gamma} \) and \( \nu_{\rho,i} = \nu_{\rho} \), \( i = 1, \ldots, N \), that satisfy (3.32) and

\[
\int_{0}^{\infty} \int_{0}^{\infty} \eta \left[ e - \left[ \nu_{\gamma} - \frac{1}{h} \right]^{+} \right]^{+} - \left[ \nu_{\rho} - \frac{1}{h} - e \right]^{+} p_{E,H}(e,h) \, de \, dh = 0,
\] (3.51)

where (3.51) is the fading equivalent of the energy stability condition in (3.50).

### 3.8 Numerical Results

In this section, we provide numerical results on the performances of the optimal offline policy and the online policies. We simulate communication sessions consisting of
$N = 10^4$ time slots, with a slot length of $\tau = 10$ ms. Since the model in Section 3.2 assumed a unit slot length, the optimal policies in this case are found by scaling transmit powers and consumed energy values accordingly. We consider an energy harvesting transmitter node equipped with a battery of size $E^{\text{max}} = 1$ mJ and initial charge $E_0 = 0$. We have a Gaussian noise spectral density of $N_0 = 10^{-19}$ W/Hz at the receiver, and a bandwidth of $B = 1$ MHz. The path loss between the transmitter and receiver is $h = -100$ dB.

For the purpose of comparison, we introduce two algorithms. The first is the directional water-filling (DWF) algorithm of [71], which is indifferent to the storage efficiency $\eta$, and a feasible policy is obtained as in [71]. The second is the efficiency-adaptive directional water-filling algorithm, which is obtained by forcing the two thresholds of the optimal offline algorithm in Section 3.3 to be equal, thus resembling DWF in [71]. However, it accounts for the storage efficiency $\eta$ when choosing its constant water levels, and therefore is a near-optimal heuristic derived from the insights obtained in this chapter.

We consider the single-user setting with a finite-sized battery in Section 3.4. Figure 3.9 shows the throughput for the offline and online policies versus storage efficiency $\eta$ when the harvested energy $E_i$ in each time slot is generated in an i.i.d. fashion, distributed uniformly in $[0, 200] \mu$J. This corresponds to an average energy harvesting rate of 10 mW. Simulations are repeated for bursty and random walk harvesting models of Section 3.7.1 in Figure 3.10 and Figure 3.11, respectively, for a harvested energy range of $[0, 200] \mu$J. We observe that the performance of DWF degrades rapidly with decreasing $\eta$, since it does not adapt to storage efficiency. Efficiency-adaptive DWF performs reasonably well for high storage efficiency, but worse at low storage efficiency since it also
Fig. 3.9: Throughput for a static channel with i.i.d. energy harvests and an average harvesting rate of 10 mW.

Fig. 3.10: Throughput for a static channel with Markov (bursty) energy harvests and an average harvesting rate of 10 mW.
Fig. 3.11: Throughput for a static channel with Markov (random walk) energy harvests and an average harvesting rate of 10 mW.

Fig. 3.12: Throughput for a Rayleigh fading channel with i.i.d. energy harvests and an average harvesting rate of 10 mW.
relies on frequently storing and retrieving energy. Moreover, in all cases, the proposed online policy performs very close to the optimal online policy, both providing a notable improvement over DWF and efficiency-adaptive DWF.

In Figure 3.12, we compare the throughput of offline and online policies for a fading channel. We consider Rayleigh fading with $\mathbb{E}[h_i] = -100$ dB, and the remaining parameters are unchanged from those in Figure 3.9. Here, the optimal offline and online policies compare similar to the non-fading case. We observe that efficiency-adaptive DWF performs close to optimal for high storage efficiency, while DWF rapidly departs from the optimal as $\eta$ decreases. The proposed online policy is notably close to the optimal for all storage efficiency values.

We next consider an average harvesting rate of 80 $\mu$W, which is more realistic for small-sized energy harvesting sensor nodes with limited access to ambient energy. We generate energy harvests accordingly, while the remaining parameters are unchanged. Figures 3.13-3.16 present the throughput for the offline and online policies versus storage efficiency $\eta$ for i.i.d., bursty, and random walk energy harvests in a static channel, and i.i.d. energy harvests in a Rayleigh fading channel, respectively. The significance of the double-threshold policy is more pronounced in this low-power scenario, as the performance of both DWF and efficiency-adaptive DWF quickly depart from that of the optimal as $\eta$ decreases.

We plot the throughput of the above policies relative to the optimal offline policy, i.e., scaled by the optimal throughput, as a function of average energy harvesting rate in Figure 3.17, for i.i.d. energy harvests in a static channel with the same parameters. We observe that while the performance of the optimal online and proposed online algorithms
Fig. 3.13: Throughput for a static channel with i.i.d. energy harvests and an average harvesting rate of 80 µW.

Fig. 3.14: Throughput for a static channel with Markov (bursty) energy harvests and an average harvesting rate of 80 µW.
Fig. 3.15: Throughput for a static channel with Markov (random walk) energy harvests and an average harvesting rate of 80 $\mu$W.

Fig. 3.16: Throughput for a Rayleigh fading channel with i.i.d. energy harvests and an average harvesting rate of 80 $\mu$W.
Fig. 3.17: Throughput of efficiency-adaptive DWF, DWF, and online policies relative to optimal throughput.
are virtually identical to that of the optimal offline policy, the performance of efficiency-adaptive DWF and DWF falls to as low as 90% and 67% of the optimal throughput, respectively, at low harvesting rates.

Finally, to examine the dynamics of the policies further, we present a smaller numerical example with $N = 5$ time slots of duration $\tau = 10$ ms, storage efficiency $\eta = 0.66$, battery size $E^{max} = 20 \mu J$, and energy harvests $E = [18, 20, 2, 9, 4] \mu J$. In this scenario, the optimal power policy sets thresholds $p_\gamma = 1.43 \text{ mW}$ and $p_\rho = 0.61 \text{ mW}$, and yields the transmit powers $p = [1.43, 1.43, 0.61, 0.90, 0.61] \text{ mW}$ with an average throughput of 0.4861 bits/sec/Hz. In comparison, efficiency-adaptive DWF yields $p = [0.93, 0.93, 0.93, 0.93, 0.93] \text{ mW}$ with average throughput 0.4733 bits/sec/Hz, and DWF yields $p = [0.70, 0.70, 0.70, 0.70, 0.70] \text{ mW}$ with average throughput 0.3825 bits/sec/Hz. Note that the optimal offline policy consumes more energy in the first two time slots, and does not insist on equalizing the powers $p_2$ and $p_3$. This is to its benefit, because for these transmit powers, energy storage loss overcomes the advantage of constant power transmission. Meanwhile, being aware of the storage inefficiency, the efficiency-adaptive DWF algorithm chooses a constant transmit power of 0.93 mW, while the DWF algorithm chooses a constant transmit power of 0.70 mW. As a result, the average throughput of the optimal offline algorithm is significantly better than that of DWF, and is only approached by efficiency-adaptive DWF.

3.9 Chapter Summary

In this chapter, we showed that the optimum power policy for a transmitter with energy storage inefficiency is a double-threshold policy. Specifically, we showed storing
energy in the battery only when the harvested energy is above an upper threshold, retrieving energy from the battery only when the harvested energy is below a lower threshold, and using harvested energy in its entirety when the harvested energy is in between these two thresholds to be optimal. We observed that the two thresholds remain constant unless the battery is depleted or full. We provided an algorithm to determine the sequence of optimum thresholds. For the case with fading, we developed a directional water-filling algorithm which has a double-threshold structure. For a broadcast channel with an energy harvesting transmitter, we obtained the throughput region by maximizing weighted sum-throughput for all weights, where the optimal policy in each case is a double-threshold policy. Finally, we formulated the online problem by posing it as a Markov decision process and using dynamic programming, and numerically observed that the online policy exhibits a double-threshold structure as well.

An insight drawn from these results was that when battery inefficiency is taken into consideration, the optimal power policy is no longer piecewise constant as was the case with ideal batteries [120, 99, 71, 41, 116, 6, 75]. Instead, two thresholds emerged in both the offline and online optimal policies, between which harvested energy is consumed immediately, i.e., without energy storage or retrieval. When battery was set to be lossless, these two thresholds were equal, and the policies converged to previous results. In essence, it can be said that double-threshold policies result from the inefficiency of the battery, and introduce an interval within which the losses due to inefficiency outweigh the benefits of storage. In addition, we observed that the conventional directional water-filling algorithm, which does not adapt to storage inefficiency, incurred a significant performance loss as storage efficiency decreases.
Chapter 4

The Energy Harvesting Interference Channel

4.1 Introduction

As previous chapters suggest, the design principles of energy harvesting wireless networks are fundamentally different than their traditional counterparts: In order to utilize the wireless and the energy resources in the best possible way, the network needs to be optimized subject to the constraints on the instantaneously available energy. A particularly important network structure is one where multiple energy harvesting nodes share the wireless medium to communicate to multiple destinations. In this setting, the utilities and thus the power policies of the energy harvesting nodes are intertwined due to the shared medium. Therefore, each node needs not only to adapt to its own energy availability, but also to adapt to that of the other nodes. In this chapter, we consider a multiterminal energy harvesting network, namely the energy harvesting interference channel, in which two of the network are energy harvesting.

Power allocation in energy harvesting multiterminal setups span over a variety of channel models. A broadcast channel with one energy harvesting transmitter and two receivers is studied in [6, 116, 75], which utilize iterative algorithms to find the optimal policy, as well as in Section 3.6. This model is extended to a fading broadcast channel in [76]. For a multiple access channel with two energy harvesting transmitters and one receiver, the generalized iterative backward water-filling algorithm is introduced
in [119] as an extension of the directional water-filling algorithm of [71]. Other multiuser energy harvesting models include the interference channel [100], the two-hop channel with a single relay [34, 68, 67, 69, 3, 55, 42] or multiple relays [2], and the two-way relay channel [97].

The interference channel is a fundamental building block for multiterminal wireless networks. Consequently, identifying the optimal transmission policies under energy harvesting scenario for this channel will furnish us with insights needed for energy harvesting wireless ad-hoc network design. A critical issue is the lack of conclusive results on the capacity of the interference channel. For the Gaussian two-user interference channel the strong interference capacity region was characterized in [82]. Relatively recent results with respect to the capacity region and weak interference sum-capacity have been obtained in [84, 61]. The known sum-capacity results point out the fact that the capacity is notably influenced by the interaction of the transmitters, and how interference is processed at the receivers [84]. Considering that the energy availability of energy harvesting nodes are varying, the problem of optimal power allocation in this setting becomes an interesting one to tackle.

The focus of this chapter is sum-throughput maximization in a two-user Gaussian interference channel with energy harvesting transmitters. For this purpose, we first formulate and solve the sum-throughput maximization problem, showing that an iterative generalized directional water-filling algorithm gives the optimal power policy. For this model, we also present a distributed power policy where nodes are only aware of their own energy harvests, and formulate the optimal online power policy as a Markov decision process. This work was published in [100].
4.2 System Model

The two-user Gaussian interference channel with energy harvesting transmitters is shown in Figure 4.1. Transmitters T1 and T2 have independent data packets addressed to receivers R1 and R2, respectively. At each channel use, the outputs $Y_k$ at node Rk, $k = 1, 2$, are given by

\[
Y_1 = \sqrt{h_{1,1}} X_1 + \sqrt{h_{2,1}} X_2 + N_1, \tag{4.1a}
\]

\[
Y_2 = \sqrt{h_{1,2}} X_1 + \sqrt{h_{2,2}} X_2 + N_2, \tag{4.1b}
\]

where $X_k$ is the channel input of Tk, $N_k$ is the receiver noise at Rk, and $h_{k,j}$ is the power gain from transmitter Tk to receiver Rj. The receiver noise $N_k$ has a Gaussian distribution with zero mean and $\sigma_k^2$ variance.

Following the energy harvesting model in Section 3.2, we consider $N$ time slots of duration $\tau = 1$ s, indexed as $i = 1, \ldots, N$. The energy harvested at node Tk, $k = 1, 2$, at the beginning of time slot $i$ is denoted as $E_{k,i}$, with the initial battery state included in $E_{k,1}$. The energy harvests are stored in the battery of the node, which has a finite energy storage capacity of $E_k^{max}$. Any harvest in excess of the battery size is lost, and thus energy harvests are truncated in the model accordingly. We first consider the offline power allocation problem in Sections 4.3-4.6, where the energy harvesting scenario is known non-causally by both transmitters before transmission. We then briefly study the optimal online algorithm and near-optimal non-centralized algorithms in Section 4.7.

For simplicity, we normalize the power gains $h_{1,1}, h_{2,2}$ and the noise powers by scaling channel inputs, outputs, transmit powers, harvested energy values, and battery
Fig. 4.1: The interference channel with energy harvesting transmitters and data arrivals.
sizes. In particular, we define $\bar{Y}_k = Y_k/\sigma_k$, $\bar{X}_k = \sqrt{h_{k,k}}X_k/\sigma_k$, and $\bar{N}_k = N_k/\sigma_k \sim \mathcal{N}(0,1)$, which yield

$$\bar{Y}_1 = \bar{X}_1 + \sqrt{a}\bar{X}_2 + \bar{N}_1, \quad \bar{Y}_2 = \sqrt{b}\bar{X}_1 + \bar{X}_2 + \bar{N}_1,$$

(4.2)

where

$$a = \frac{\sigma^2 h_{2,1}}{\sigma^2 h_{2,2}}, \quad b = \frac{\sigma^2 h_{1,2}}{\sigma^2 h_{1,1}},$$

(4.3)

are the normalized power gains. Since this normalization requires scaling the channel inputs of $T_k$ by $\sqrt{h_{k,k}/\sigma_k}$, we also scale the energy harvests and battery sizes of $T_k$ by $h_{k,k}/\sigma_k^2$. For the normalized channel in (4.2), the sum-capacity for the Gaussian interference channel for various ranges for $a$ and $b$ is summarized in [84, Table 1]. We adopt these capacity values as the sum-rate function $r(p_1, p_2)$, and denote the corresponding achievable rate for the $T_k$-R link as $r_k(p_1, p_2)$. Specific regions of $a$ and $b$ will be denoted as interference regions, and their sum-rates will be indicated with a superscript when needed. Note that except for the interference region where the capacity is unknown, the sum-rates are jointly concave in $p_1$ and $p_2$ by definition. As in Chapter 2, it is assumed for the sake of simplicity that transmission is the dominant source of energy expenditure in the system, and other factors such as base power or processing power are ignored.

The energy causality and no-battery-overflow constraints in Definitions 2.1 and 2.2 apply to transmitters T1 and T2 as well. Namely, denoting the transmit power of
Tk in time slot $i$ as $p_{k,i}$, we have

$$
\sum_{n=1}^{i} E_{k,n} - \sum_{n=1}^{i} p_{k,n} \geq 0, \quad k = 1, 2, \tag{4.4}
$$

$$
E_{k}^{\text{max}} - \sum_{n=1}^{i+1} E_{k,n} + \sum_{n=1}^{i} p_{k,n} \geq 0, \quad k = 1, 2, \tag{4.5}
$$

Note that the no-battery-overflow constraint is employed without loss of optimality due to an extension of Lemma 2.2 which is based on the power-rate function $r(p_1, p_2)$ being increasing in $p_1$ and $p_2$.

We define the problem of maximizing the total number of bits sent by the transmitter in $N$ time slots, as the sum-throughput maximization, which can be expressed as follows:

$$
\max_{p_1, p_2} \sum_{i=1}^{N} r(p_{1,i}, p_{2,i}) \tag{4.6a}
$$

s.t. $\sum_{n=1}^{i} \left( E_{k,n} - p_{k,n} \right) \geq 0, \quad i = 1, \ldots, N, \ k = 1, 2, \tag{4.6b}$

$$
E_{k}^{\text{max}} - \sum_{n=1}^{i} \left( E_{k,n} - p_{k,n} \right) - E_{k,i+1} \geq 0, \quad i = 1, \ldots, N, \ k = 1, 2, \tag{4.6c}
$$

$$
p_k \geq 0, \quad k = 1, 2. \tag{4.6d}
$$

Here, $p_k = \{p_{k,1}, p_{k,2}, \ldots, p_{k,N}\}$ is a vector representing the collection of transmit powers of Tk, and will be referred to as the power policy of Tk. Constraints (4.6b) and (4.6c) correspond to energy causality and no-energy-overflow constraints, respectively.
The expression $p_k \geq 0$ implies component-wise non-negativity of the transmit power policy.

4.3 Alternating Maximization in the Interference Channel

Since (4.6) has linear constraints and a jointly concave objective, it is a convex program and can be solved using standard convex optimization tools [11]. However, the dimension of the problem is $2N$, since each power policy $p_k$ has $N$ components. Hence, solving the problem takes significantly longer time as $N$ increases. In this section, we propose an alternative approach using alternating maximization, where we decompose (4.6) into two $N$-dimensional problems, which we solve iteratively and efficiently using a generalized version of the directional water-filling algorithm in [71].

Formally, we start from an arbitrary initial feasible pair $(p_1^{(0)}, p_2^{(0)})$, and perform the following update on the $m$th iteration,

$$p_1^{(m)} = \arg\max_{p_1} \sum_{i=1}^{N} r \left( p_{1,i}, p_{2,i}^{(m-1)} \right)$$  \hspace{1cm} (4.7a)

subject to

$$\sum_{n=1}^{i} \left( E_{1,n} - p_{1,n} \right) \geq 0, \quad p_1 \geq 0, \quad i = 1, \ldots, N,$$  \hspace{1cm} (4.7b)

$$\sum_{n=1}^{i} \left( p_{1,n} - E_{1,n} \right) + E_{1}^{\text{max}} - E_{1,i+1} \geq 0, \quad i = 1, \ldots, N,$$  \hspace{1cm} (4.7c)

$$p_2^{(m)} = \arg\max_{p_2} \sum_{i=1}^{N} r \left( p_{1,i}^{(m)}, p_{2,i} \right)$$  \hspace{1cm} (4.8a)
\[
s.t. \quad \sum_{n=1}^{i} (E_{2,n} - p_{2,n}) \geq 0, \quad p_2 \geq 0, \quad i = 1, \ldots, N, \quad (4.8b)\]
\[
\sum_{n=1}^{i} (p_{2,n} - E_{2,n}) + E_{2,i+1}^{max} - E_{2,i} \geq 0, \quad i = 1, \ldots, N, \quad (4.8c)\]

for \( m = 1, \ldots, \) where the superscript \((m)\) denotes the iteration index. In the case of multiple solutions in a problem, we choose the power policy \( p_k^{(m)} \) closest in Euclidian distance to \( p_k^{(m-1)} \), which we refer to as the minimum displacement rule. Note that the energy constraints for only the optimized power policy is included in each problem. This is possible since the constraints in (4.6b)-(4.6d) are separable in \( k \), and thus changing \( p_k \) while keeping \( p_j, j \neq k \) fixed cannot violate the constraints on \( p_j \). We next show that the alternating maximization in (4.7)-(4.8) converges to the global optimum of (4.6).

**Theorem 4.1.** The iterative algorithm given in (4.7) and (4.8) converges to the global optimum of (4.6).

**Proof:** The alternating maximization approach among the variables of the problem, also referred to as the block coordinate descent method, is known to converge for a problem in the form of
\[
\max \quad f(x_1, x_2, \ldots, x_n) \quad s.t. \quad x \in \mathcal{X} \quad (4.9)
\]
when the objective function \( f \) is continuously differentiable over \( \mathcal{X} \), and the feasible set \( \mathcal{X} \) can be expressed as the Cartesian product of convex sets \( \mathcal{X}_1, \ldots, \mathcal{X}_n \). Furthermore, it is required for the objective function to yield a unique maximum in all variables \( x_i \), i.e.,
\[
\max_{\xi \in \mathcal{X}_i} f(x_1, x_2, \ldots, x_{i-1}, \xi, x_{i+1}, \ldots, x_n) \quad (4.10)
\]
needs to have a unique $\xi$ solving this problem [11, Prop. 2.7.1].

In (4.7) and (4.8) we perform the iterations over the power policies of the two users, partitioning the variables into two, namely $p_1$ and $p_2$. Since the two nodes harvest and consume the energy independently, the set of constraints on $p_1$ and $p_2$ can be separated. The two constraint sets are also convex, since the individual constraints are linear in their respective elements $p_{k,i}$. Thus the constraint sets satisfy the requirements for convergence.

For the objective function, we can verify immediately that the interference channel sum-capacity expressions in [84, Table 1] are concave and continuously differentiable. Thus, we only need to ensure that the property in (4.10) is satisfied. Although this is trivial for a strictly concave objective function on a convex set $X_i$, in the case of an interference channel, the sum-capacity is not necessarily strictly concave. This is overcome by introducing two auxiliary vectors $c_1$ and $c_2$ and restating the maximization problem with the objective function

$$g(p_1, p_2, c_1, c_2) = r(p_1, p_2) - \epsilon \|p_1 - c_1\|^2 - \epsilon \|p_2 - c_2\|^2 \quad (4.11)$$

to replace $r(p_1, p_2)$ in (4.6). This is the proximal point modification of the block coordinate descent method, and is shown to converge for an arbitrary positive scalar $\epsilon > 0$ for a convex [12] or non-convex [30] sum-rate function $r(p_1, p_2)$. For the convex case, this follows immediately from the modified objective $g(p_1, p_2, c_1, c_2)$ being strictly concave. This algorithm cycles between $p_1, p_2, c_1$ and $c_2$ in its iterations. The iterations on $c_1$ and $c_2$ trivially yield $c_1 = p_1$ and $c_2 = p_2$ to minimize the Euclidean distance in (4.11).
On the other hand, the iterations on $p_1$ and $p_2$ requires considering the additional terms in (4.11) which increases as the power policy $p_k$ moves away from $c_k$. In essence, this modification employs a penalty on moving away from the previous value of the power policy. For $\epsilon$ sufficiently small, this algorithm can be thought of choosing the closest maximizer in Euclidean distance whenever the maximum is not unique. Since the block coordinate descent method for this strictly concave cost function converges for an arbitrarily small $\epsilon$ [11], so does the proposed iterative algorithm in (4.7) and (4.8), provided that in case of multiple maximizers the one closer to the previous power vector is favored. This is ensured by the minimum displacement rule described below (4.8).

With the convergence of the alternating maximization algorithm verified, we move on to solving the iterations (4.7) and (4.8) efficiently. These problems involve single-user power policy optimization of a sum of concave functions over a linear set of constraints. A similar problem was solved in [48] by a generalized water-filling algorithm. In order to conform to energy causality and no-battery-overflow constraints, we enhance this algorithm with the directional water flow insights in Section 2.5 and [71].

### 4.4 Generalized Directional Water-Filling (GDWF)

In this section, we build upon the directional water-filling algorithm of [71], summarized in Section 2.5, to solve the problems in (4.7)-(4.8). We first consider (4.7), and then extend the solution to (4.8) which is mathematically identical to (4.7) after
swapping the variables of \( r(p_1, p_2) \). We begin by finding the Lagrangian of (4.7) as

\[
\mathcal{L} = \sum_{i=1}^{N} r(p_{1,i}, p_{2,i}^{(m-1)}) - \sum_{i=1}^{N} \sum_{n=1}^{i} (p_{1,n} - E_{1,n})
\]

\[
- \sum_{i=1}^{N} \beta_{1,i} \left( \sum_{n=1}^{i} (E_{1,n} - p_{1,n}) + E_{1,i+1} - E_{1}^{max} \right) + \sum_{i=1}^{N} \mu_{1,i} p_{1,i},
\]

(4.12)

where \( \lambda_{1,i}, \beta_{1,i}, \) and \( \mu_{1,i} \) are the Lagrange multipliers for the energy causality, no-energy-overflow, and non-negativity constraints, respectively. The KKT stationarity condition implies

\[
\frac{\partial}{\partial p_1} r(p_{1,i}, p_{2,i}^{(m-1)}) - \sum_{n=1}^{N} (\lambda_{1,n} - \beta_{1,n}) + \mu_{1,i} = 0
\]

(4.13)

holds in the \( i \)th time slot for \( i = 1, \ldots, N \). The complementary slackness and dual feasibility conditions are

\[
\lambda_{1,i} \left( \sum_{n=1}^{i} E_{1,n} - \sum_{n=1}^{i} p_{1,n} \right) = 0, \quad \lambda_{1,i} \geq 0, \quad (4.14a)
\]

\[
\beta_{1,i} \left( \sum_{n=1}^{i} p_{1,n} + E_{1}^{max} - \sum_{n=1}^{i+1} E_{1,n} \right) = 0, \quad \beta_{1,i} \geq 0, \quad (4.14b)
\]

\[
\mu_{i} p_{1,i} = 0, \quad \mu_{i} \geq 0, \quad i = 1, \ldots, N. \quad (4.14c)
\]

Note that \( \lambda_{1,i}, \beta_{1,i}, \) and \( \mu_{1,i} \) are positive only when their respective constraint is active, i.e., when the battery is empty, full, or transmit power is zero, respectively.
We define the generalized water level for node T1, $\nu_{1,i}$, as

$$
\nu_{1,i} = \left( \frac{\partial}{\partial p_{1,i}} r(p_{1,i},p_{2,i}) \right)^{-1} = \left( \sum_{n=i}^{N} (\lambda_{1,n} - \beta_{1,n}) - \mu_{1,i} \right)^{-1}, \quad i = 1, \ldots, N,
$$

(4.15)

where the second inequality is due to (4.13). To solve (4.8), we follow the steps in (4.12)-(4.15) and define the generalized water level

$$
\nu_{2,i} = \left( \frac{\partial}{\partial p_{2,i}} r(p_{1,i},p_{2,i}) \right)^{-1} = \left( \sum_{n=i}^{N} (\lambda_{2,n} - \beta_{2,n}) - \mu_{2,i} \right)^{-1}, \quad i = 1, \ldots, N.
$$

(4.16)

In this case, $p_{1,i}^{(m)}$ is kept constant and the water level can be shown to exhibit the same properties.

The generalized water level $\nu_{k,i}$, $k = 1, 2$, remains constant in $i$ over the transmission period unless one of the Lagrange multipliers, $\lambda_{k,i}$, $\beta_{k,i}$ or $\mu_{k,i}$, is positive. The effect of each case, either increasing or decreasing the generalized water level and thus $p_{k,i}$, depends on the sign of the corresponding multiplier in (4.13). In particular, we observe the following properties:

1. A positive $\lambda_{k,n}$ results in a decrease in the water level for $i = 1, \ldots, n$. For an increasing and jointly concave power-rate function $r(p_{1},p_{2})$, this corresponds to a lower $p_{k,i}$ for $i = 1, \ldots, n$. Thus, we see that whenever the battery is empty at time slot $n$, $\lambda_{k,n}$ may be positive, and future transmit powers $p_{k,i}$ for $i = n + 1, \ldots, N$ may be larger than $p_{k,n}$.
2. A positive $\beta_{k,n}$ has the opposite effect to $\lambda_{k,n} > 0$, i.e., it results in an increase in the water level for $i = k, \ldots, n$, which corresponds to a higher $p_{k,i}$ for $i = 1, \ldots, n$. Thus, we see that whenever the battery is full at time slot $n$, $\beta_{k,n}$ may be positive, and future transmit powers $p_{k,i}$ for $i = n + 1, \ldots, N$ may be smaller than $p_{k,n}$.

3. While $\lambda_{k,i}$ and $\beta_{k,i}$ mark points of change in transmit power, the effect of $\mu_{k,i}$ is contained only to $p_{k,i}$. When $\mu_{k,i} > 0$, only the water level $\nu_{k,i}$ increases, which requires $p_{k,i}$ to increase. Since $\mu_{k,i} > 0$ is only possible when $p_{k,i} = 0$, this only occurs when $\mu_{k,i} = 0$ yields a negative $p_{k,i}$. Hence, if a non-negative solution to

$$\frac{\partial r(p_{1,i}, p_{2,i})}{\partial p_{k,i}} = \sum_{n=i}^{N} (\lambda_{k,n} - \beta_{k,n}) - \mu_{k,i}$$

(4.17)

does not exist for $\mu_{k,i} = 0$, then $p_{k,i} = 0$ and $\mu_{k,i} > 0$ is the solution to (4.17) at $p_{k,i} = 0$.

The properties of the generalized water levels resemble those of the water levels in directional water-filling [71]. In fact, the fading channel of [71] is a special case of this problem when $p_{2,i}$ is considered as a substitute for $h_i$, i.e., $r(p_{1,i}, p_{2,i}) = r(p_{1,i}, h_i) = \frac{1}{2} \log(1 + h_ip_{1,i})$. In this case, the generalized water level simplifies to $\nu_{1,i} = p_{1,i} + 1/h_i$, thus yielding the well-known water-filling interpretation of [26]. In the presence of energy causality and no-energy-overflow constraints, the directional water-filling algorithm [71] can be used to solve (4.7), provided that the water level expression is replaced with the generalized water level in (4.15). This formalization is adopted from the generalized
water-filling approach followed in [48], and a similar solution is proposed in [118] for the multiple access channel with energy harvesters.

Generalized directional water-filling starts from an initial power policy $p_k$, such as the one consuming all harvested energy in the same time slot, i.e., $p_{k,n} = E_{k,n}$. The transmit powers $p_{k,i}$ and the water levels $\nu_{k,i}$ are related through (4.15)-(4.16). Energy flow, and therefore water flow, is allowed only in the forward direction, and the total energy flow into time slot $i$ is limited to $E_{k,\text{max}} - E_{k,i}$. The flow between two neighboring time slots occur while the generalized water level of the earlier slot is larger than that of the later slot. Water levels stabilize when no water flow can take place, at which point the transmit powers are optimal. Similar to Theorem 3.2, this is due to stable water levels providing a set of Lagrange multipliers that satisfy the KKT conditions in (4.13)-(4.14). The algorithm is given in Algorithm 4.1

**Algorithm 4.1 Generalized Directional Water-Filling (GDWF) Algorithm for node Tk**

1. Let $p_{k,i} = E_{k,i}$ for $k = 1, 2, n = 1, \ldots, N$, and fix $p_{j,i}$ for $j \neq k$. Let flows $\Delta_i = E_{k,i}$ for $i = 2, \ldots, N$.

2. for $i = 2, \ldots, N$, do
   
   if $\nu_{k,i-1} > \nu_{k,i}$, then
   
   Find $c$, $p_{k,i-1} \geq c \geq 0$, which satisfies either $c = E_{k,\text{max}} - \Delta_i$ or $\nu_{k,i-1} = \nu_{k,i}$ after energy flow $c$ from slot $i-1$ to slot $i$.
   
   Update $p_{k,i-1} = p_{k,i-1} - c$, $p_{k,i} = p_{k,i} + c$, $\Delta_i = \Delta_i + c$.

3. Repeat steps 2-3 while $c > \epsilon$ for all $i$ in step 2.
An illustration of generalized directional water-filling is shown in Figure 4.2 for $N = 8$ time slots. First, energy harvested by the node in each time slot is placed in the respective slot, i.e., $p_{k,i} = E_{k,i}$, and the initial water levels $\nu_{k,i}$ are found as in Figure 4.2a. Next, the optimal water levels $\nu^*_k, i = 1, \ldots, N$ are found as in Figure 4.2b by allowing water to flow only forward in time, until water levels among slots are equalized. In order to satisfy the no-battery-overflow constraints, the total flow from one time slot to the next is limited to $E_{k,\text{max}}^k$. Non-zero flows are depicted with green taps, and flows limited by the no-battery-overflow constraint are denoted with red taps. Note that the battery of the node is full at the end of the fifth time slot, preventing further energy flow into the sixth time slot. As such, that two consecutive slots with different water levels may only be optimal if either the energy causality or the no-battery-overflow constraint is active between the two slots [71].

### 4.5 Iterative Generalized Directional Water-Filling (IGDWF) in the Interference Channel

The *Iterative* Generalized Directional Water-Filling (IGDWF) algorithm employs generalized directional water-filling sequentially for each user until all power levels $p_k, k = 1, 2$, converge. From Theorem 4.1, we know that this alternating maximization algorithm converges to the optimal solution. Although the optimization is carried on separately for each node at each iteration, the transmit powers of the two nodes interact through the generalized water levels in (4.15)-(4.16). The iterative algorithm is given in Algorithm 4.2.
Algorithm 4.2 Iterative Generalized Directional Water-Filling (IGDWF) Algorithm

1. Let \( p_{k,i}^{(0)} = E_{k,i} \) for \( k = 1, 2, n = 1, \ldots, N \), and let \( m = 1 \).

2. Fix \( p_{2,i}^{(m-1)} \) and calculate \( p_{1,i}^{(m)} \) from (4.7) using Algorithm 4.1.

3. Fix \( p_{1,i}^{(m)} \) and calculate \( p_{2,i}^{(m)} \) from (4.8) using Algorithm 4.1.

4. if \( \| p_{k,i}^{(m)} - p_{k,i}^{(m-1)} \| > \epsilon \) for \( k = 1 \) or \( k = 2 \), then
   
   Repeat steps 2-4 for \( m = m + 1 \).
In this section, we study some interference regions where parts of the IGDWF reduce to simpler algorithms, such as the directional water-filling algorithm of [71] or the throughput maximization algorithm in Section 2.3.2.

4.5.1 Asymmetric Interference with $ab > 1$

The asymmetric interference region is where one transmitter has a strong normalized power gain and the other has a weak one. It covers the two symmetric cases $a \leq 1$, $b \geq 1$ and $a \geq 1$, $b \leq 1$. We shall assume the former case in this section, with the result easily applicable to the latter case by swapping the nodes.

The sum-rate function for the asymmetric interference case with $ab > 1$ is given by its sum-capacity [84]

$$r_A(p_1, p_2) = \frac{1}{2} \log \left( 1 + \frac{p_1}{1 + ap_2} \right) + \frac{1}{2} \log \left( 1 + p_2 \right). \quad (4.18)$$

This rate is achieved by decoding the interference caused by T1 at R2 and treating interference as noise at R1. Given $ab > 1$, R2 is guaranteed to be able to decode the interference.

We obtain the generalized water levels $\nu_{1,i}$ from (4.15) by substituting (4.18) as

$$\nu_{1,i} = 2(1 + p_{1,i} + ap_{2,i}). \quad (4.19)$$

The direct implication of this form is treating the term $1 + ap_{2,i}$ as the virtual power gain $1/h_i$ and using the directional water-filling algorithm of [71]. Notice that since the noise power is normalized in the channel model, the term $1 + ap_{2,i}$ is in fact the total
interference and noise power. On the other hand, calculating the generalized water levels \( \nu_{2,i} \) from (4.16) by substituting (4.18) gives

\[
\nu_{2,i} = \left( \frac{1}{2(1+p_2)} - \frac{ap_1}{2(1+p_1+ap_2)(1+ap_2)} \right)^{-1},
\]

(4.20)

which does not have an immediate interpretation as in (4.19). Hence, (4.20) serves as the generalized water level, which Algorithm 4.2 equalizes throughout the transmission.

### 4.5.2 Asymmetric Interference with \( ab \leq 1 \)

We next consider the complementary asymmetric interference region to Subsection 4.5.1 for which power gains satisfy \( a \leq 1, \ b \geq 1 \) and \( ab \leq 1 \). The sum-rate function for this interference region is given by

\[
r_B(p_1,p_2) = \min \left\{ \frac{1}{2} \log \left( 1 + \frac{p_1}{1+ap_2} \right) + \frac{1}{2} \log \left( 1 + p_2 \right), \frac{1}{2} \log \left( 1 + bp_1 + p_2 \right) \right\},
\]

(4.21)

Note that the second term can also be expressed as \( \frac{1}{2} \log \left( 1 + \frac{bp_1}{1+p_2} \right) + \frac{1}{2} \log \left( 1 + p_2 \right) \).

This rate is achieved similar to the \( ab \leq 1 \) case by decoding the interference caused by T1 at R2 and treating interference as noise at R1. Since transmission of T1 needs to be decoded at both receivers in this scheme, the minimum operation decides which receiver will limit the transmission rate of T1.

We first tackle the single-user problem for T1. It can be seen that the value of \( p_2 \) is sufficient to determine which of the two terms comes out of the minimum in (4.21).
Specifically, the following threshold condition

\[ p_2 \leq \frac{b - 1}{1 - ab} \triangleq p_c \tag{4.22} \]

implies the dominance of the first term for the sum-rate in (4.21). Therefore, given a fixed transmission policy for the second transmitter, \( p_2 \), the minimum term dominating in the rate expression is known regardless of the value of the optimization variable \( p_1 \).

As a result, the generalized water level \( \nu_{1,i} \) can be written as

\[ \nu_{1,i} = 2(p_{1,i} + 1/h_i), \tag{4.23} \]

where

\[
\frac{1}{h_i} = \begin{cases} 
1 + ap_{2,i}, & p_{2,i} < p_c; \\
\frac{1 + p_{2,i}}{b}, & p_{2,i} \geq p_c; 
\end{cases} \tag{4.24}
\]

and can be solved via directional water-filling of [71] with the base level defined in (4.24).

For node T2, the generalized water levels \( \nu_{2,i} \) are found as

\[
\nu_{2,i} = \begin{cases} 
\left( \frac{1}{2(1+p_{2,i})} - \frac{ap_{1,i}}{2(1+p_{1,i}+ap_{2,i})(1+ap_{2,i})} \right)^{-1}, & p_{2,i} < p_c; \\
2(1 + bp_{1,i} + p_{2,i}), & p_{2,i} \geq p_c; 
\end{cases} \tag{4.25}
\]

and can be solved in a similar manner to its counterpart in Subsection 4.5.1 using the generalized directional water-filling algorithm.
4.5.3 Very Strong Interference

The very strong interference case was identified in [15] as the case when the normalized power gains are large enough to ensure that the interfering signals can be decoded at both receivers. The rate for each user is therefore the single-link Gaussian channel capacity, yielding the sum-rate function

\[ r_C(p_1, p_2) = \frac{1}{2} \log (1 + p_1) + \frac{1}{2} \log (1 + p_2), \]  

(4.26)

when \( a > 1 + p_1 \) and \( b > 1 + p_2 \) are satisfied. Calculating the generalized water levels in this case yields

\[ \nu_{1,i} = 2p_{1,i}, \quad \nu_{2,i} = 2p_{2,i}. \]  

(4.27)

The iterative algorithm for such a rate function is trivial: for both users, a single-link throughput maximization as in Chapter 2 is to be followed in each iteration. Since the two subproblems are independent, a single iteration for each user suffices to reach the optimal policy. However, the nature of the problem suggests that the transmit powers of the two users are varying, and thus the requirements following (4.26) might not hold throughout the transmission. This approach is justified when the resulting power policies obey \( a > 1 + p_{1,i} \) and \( b > 1 + p_{2,i} \) for all \( i \), which can easily be verified by comparing \( a \) and \( b \) with the maximum power of the two transmitters.
4.6 Extension to Data Arrivals

The problems considered so far assume an infinite backlog of data at the transmitter, and do not impose any restrictions on how much data the transmitter can transmit. However, in some applications such as sensor networks, the amount of data at the transmitter is varying as well, with new data generated throughout the transmission. Hence, it is of practical interest to tailor the power policy of the nodes to data arrivals. In this section, we discuss the extension of the energy harvesting interference channel model in Figure 4.1 to the case where data is not available before transmission, but instead arrives throughout the transmission with the size of arrivals in each time slot known to the transmitters beforehand. In this setting, the transmitter aims to make the best effort to send as many of the arriving bits as possible. This problem is particularly relevant when data collection or arrival process shows a significant variation.

Let the amount of data arriving at transmitter $T_k$ at time slot $i$ be $A_{k,i}$, as shown in Figure 4.1. Arriving data packets are stored in an infinite data buffer, i.e., we do not consider data buffer overflows. The data arrivals introduce a new kind of causality constraint, named the data causality constraint, which is defined as follows:

**Definition 4.1.** A power policy $\{p_{k,i}\}$, $k = 1, 2$, satisfies the data causality constraint if it does not depart more data in the first $i$ time slots than has arrived in the first $i$ time slots, for $i = 1, \ldots, N$ and $k = 1, 2$, i.e.,

$$
\sum_{n=1}^{i} A_{k,n} - \sum_{n=1}^{i} r_k(p_{1,n}, p_{2,n}) \geq 0, \quad k = 1, 2, \quad i = 1, \ldots, N,
$$

(4.28)
where \( r_k(p_{1,i}, p_{2,i}) \) is the achievable rate for the Tk-Rk link for the transmit powers \( p_{1,i} \) and \( p_{2,i} \).

The problem with data arrivals is expressed in (4.6) with the additional data causality constraint in (4.28). Data causality substantially affects the problem by imposing a joint constraint for the two users, as the individual rate achieved by each user is a function of both transmit powers. These constraints are not necessarily convex either, since a convex combination of two power policies tend to increase the average achieved rate, potentially violating the data causality constraint. Thus, the convexity and Cartesian product form properties of the constraints cease to hold, challenging the convergence of iterative algorithms for this setting.

A simple example to why a direct implementation of iterative algorithms might fail to converge to the optimal policy is presented in Figure 4.3. Consider two transmitters T1 and T2 with the former having a large amount of data available in time slot \( i = 1 \), and the latter having only a few bits to send in \( i = 1 \), with more data arriving in \( i = 2 \), as depicted with the arrows. Let the harvested energy of these two nodes be so that the initial water levels are as shown in the figure, with T2 sending exactly \( A_{2,1} \) bits at the first time slot with this assignment. Notice that without data causality constraints, T1 would favor equalizing its water levels in \( i = 1, 2 \). However, an attempt to do so in an iteration step decreases the interference on the T2-R2 link, violating its data causality constraint at the end of the first time slot. Thus an iterative algorithm is stuck at these water levels. On the other hand, if T1 were to equalize its water levels, T2 would also
benefit from this since it would require less power for the first $A_{2,1}$ bits, and both users would have more power in the second time slot, resulting in a better performance.

As observed in the example above, the data causality constraint may cause an iterative algorithm to not converge to the global optimum. However, without this constraint, we are assured the convergence of the algorithm due to Theorem 4.1. One approach to resolve this issue is to replace the data causality constraint with penalty functions [11]. In particular, we employ a quadratic penalty for the data constraints, the coefficient of which starts at zero and is allowed to grow indefinitely with iterations. The problem

Fig. 4.3: A scenario where the iterative algorithm converges to a suboptimal point due to non-convex data constraints.
with a quadratic penalty function is expressed as

$$\max_{\mathbf{p}_1 \geq 0, \mathbf{p}_2 \geq 0} \sum_{i=1}^{N} r(p_{1,i}, p_{2,i}) - \zeta_m \sum_{i=1}^{N} \|C_i\|^2$$

s.t.

$$\sum_{n=1}^{i} \left( E_{k,n} - p_{k,n} \right) \geq 0, \quad (4.29a)$$

$$\sum_{n=1}^{i} \left( p_{k,n} - E_{k,n} \right) + E_{k,n}^{\text{max}} - E_{k,i+1} \geq 0, \quad k = 1, 2, \quad i = 1, \ldots, N,$$

where $C_i$ is the amount of data causality violation at time slot $i$, expressed as

$$C_i = \sum_{k=1}^{2} \sum_{n=1}^{i} \max \left\{ 0, r_k(p_{1,n}, p_{2,n}) - A_{k,n} \right\}, \quad (4.30)$$

and $\zeta_m$ is the penalty parameter for the $m$th iteration, increasing unboundedly with $m$.

When the problem in (4.29) is solved iteratively, the effect of data causality constraints is relatively small for small $m$, allowing the nodes to explore the otherwise infeasible regions of the space of power policies. As the penalty coefficient $\zeta_m$ increases with $m$, data causality constraints gradually force the algorithm to converge to a power vector conforming to data causality.

For a continuous objective function and closed constraint set, the penalty function method is observed to have good convergence properties in practice [11], making it a good candidate for the energy harvesting problem with data constraints. The addition of the penalty term to the objective affects the water-filling algorithm for each user by creating an offset term in the water level expression whenever a data causality constraint
is violated. This can be visualized as a *pump* element between time slots in the water-filling interpretation, forcing water flow in either direction until the difference in water levels matches the offset term. As \( \zeta_m \to \infty \), any non-zero offset term grows indefinitely, requiring the solution to comply with data causality for all \( i \). The offset term is increasing in a user’s power if the corresponding constraint is of the same user, and decreasing otherwise. Thus, the pump element forces water flow in the forward direction when data causality for the user in consideration is violated, or backward when data causality for the other user is violated, until the difference in water level matches the offset term.

An example algorithm run is demonstrated in Figure 4.4 for a single transmitter with \( N = 5 \), energy harvests \( E = [1, 0, 1, 0.5, 0] \), \( E_{max} = 1 \), data arrivals \( A = [0, 1.5, 0, 0.3, 0.7] \) and a linear power-rate function \( r(p) = p \) for simplicity. The taps above with forward facing arrows correspond to the no-battery-overflow constraint, and resist water flow after a total flow of \( E_{max} \) including the energy harvest to the next time slot. The bidirectional pumps below relate to the data causality constraint, and activate in forward direction when more bits are being departed than received by the transmitter. Elements corresponding to active constraints are shown green, and a red tap implies a tight no-battery-overflow constraint. The algorithm starts with the received energies placed in respective slots in Figure 4.4b. When water flow is performed, the resulting policy and remaining active constraints are shown in Figure 4.4c. This corresponds to the water distribution for which any increase in water level is due to a forward pump or energy causality, and any decrease in water level is due to a reverse pump or a no-battery-overflow tap.
Fig. 4.4: (a) Energy harvest and data arrival scenario, (b) initial water levels and (c) final water levels for the directional water-filling algorithm with pumps.
An interesting outcome of the pump modification to the water-filling algorithm is the possible tension between the flow elements. It is possible in some extreme cases that the forward pump for data causality and the tap for no-battery-overflow are active simultaneously. An example of this is given in Figure 4.5. Consider a single-user problem in two time slots with $E_1 = E_2 = E_{\text{max}}$, $A_1 = 0$ and $A_2 > 0$. Since there is no data to send in the first time slot, any transmission would violate data causality and therefore the pump between slots 1 and 2 in Figure 4.5b stays active in the forward direction until there is no water in slot 1 as $\zeta_m \to \infty$. On the other hand, since the second slot receives an energy of $E_2 = E_{\text{max}}$, the no-energy-overflow tap is also active, not allowing any water to be pumped into the slot. It is trivial in this example that the first energy harvest has no use and will inevitably be lost, which is what the contradiction between algorithm elements also implies. In such cases, the water causing the contradiction is removed without any reduction in the performance of the optimal policy shown in Figure 4.5c.

In specific cases, the rate achieved by one user is independent of the transmit power of the other user. Some examples to these cases are the very strong interference in Subsection 4.5.3, or the asymmetric interference in Subsections 4.5.1 and 4.5.2 where the individual rate of T2 is independent of the transmit power of T1. In such cases, the cross dependence of the data causality constraints vanishes, and the directional water-filling algorithm with the pump element is applicable without the backward-pumps. Note that in this manner, the pump extension is also a solution to the single-link optimal power allocation problem with energy harvests, no-battery overflow, and data causality constraints.
Fig. 4.5: (a) Energy harvest and data arrival scenario, (b) contradiction between the tap and pump elements, and (c) resolution of contradiction by the removal of excess energy.
4.7 Distributed and Online Transmission Policies

The optimal policies calculated using the iterative algorithms proposed so far in this chapter require the knowledge of energy harvests of both transmitters at a centralized controller prior to the transmission. In practice, such information may not be available or may not be desired to be shared. In this section, we propose near-optimal policies that require less information and thus are more realistic using the insight gained from the optimal iterative solution.

We begin with distributed policies, where each transmitter only has information about its own energy harvests. We observed in Section 4.3 that an iteration between the single-user problems converges to the optimal policy. In some cases demonstrated in Section 4.5, the single-user subproblems can further simplify or be independent of the power policy of the other user. This indicates that when the system is restricted to local information when choosing a power policy, a reasonable approach is to solve the single-link problem while assuming expected values for the unknown parameters. In very strong interference case, this policy matches the optimal offline policy. This simplified approach performs remarkably well, as demonstrated through simulations in Section 4.8.

Online policies are needed when the energy harvests are not known by the transmitters prior to the transmission, and transmitters are to choose power policies as energy is harvested and data packets arrive. This is the interference channel extension of the problem in Section 3.7. When energy harvests are independent in time, it is sufficient to represent the state of the system at any time using the battery state as in Section 3.7. The action taken by a transmitter at time slot $i$ is then a function of the states available
to said transmitter, which depend on whether or not it is feasible to share the battery state information with the other transmitter. The optimal online policy in this formulation can be computed using dynamic programming as in Section 3.7. Denoting the battery states of user \( k \) at time slot \( i \) as \( S_{k,i} \), and assuming all states are available to all transmitters, the Bellman equation for the problem is given by

\[
J_i(S_{1,i}, S_{2,i}) = \max_{\pi_{k,i}} r(p_{1,i}, p_{2,i}) + \mathbb{E} \left[ J_{i+1}(S_{1,i} - p_{1,i} + E_{1,i+1}, S_{2,i} - p_{2,i} + E_{2,i+1}) \right],
\]

where \( J_i(S_{1,i}, S_{2,i}) \) is the value function at time slot \( i \) and \( \pi_{k,i}(\cdot) \) is the action of \( T_k \) at time slot \( i \) given the current states as its parameters. The expectation is taken over the energy harvests \( E_{1,i+1} \) and \( E_{2,i+1} \). As in Section 3.7, for \( N \) time slots, the dynamic program is initialized with \( J_{N+1}(\cdot) = 0 \), and is solved in reverse, i.e., for \( i = N, N-1, \ldots, 1 \) to get the optimal actions. For an infinite transmission duration, the fixed point of (4.31) with a discount factor \( \Theta \) can be found, for which the optimal actions \( \pi_{k} \) give the optimal online transmission policy.

### 4.8 Numerical Results

In this section, we present simulation results of the power policies obtained by the iterative algorithms proposed in this chapter. We start by comparing the outputs of single-user directional water-filling and two user iterative directional water-filling in a short time scale. We consider the throughput maximization problem in the asymmetric interference region with \( ab < 1 \) as in Section 4.5.1. We consider a communication session
of \( N = 20 \) time slots with length \( \tau = 1 \) s each, and battery sizes \( E^{max}_1 = E^{max}_2 = 10 \) mJ. The normalized power gains are chosen as \( a = 0.9 \) and \( b = 2 \), derived from receiver noise spectral densities \( N_{0,1} = N_{0,2} = 10^{-19} \) W/Hz, bandwidth \( B = 1 \) MHz, and power gains \( h_{1,1} = h_{2,2} = -100 \) dB, \( h_{1,2} = -97 \) dB and \( h_{2,1} = -100.5 \) dB. The sum-capacity for this parameter region is given by (4.18) after normalization. We assume sufficient number of bits is available at both transmitters prior to transmission, and the randomly generated energy harvest vectors

\[
E_1 = [5, 0, 0, 0, 3, 0, 0, 0, 0, 0, 7, 0, 0, 0, 4, 0, 0, 0, 6, 0] \text{ mJ},
\]

\[
E_2 = [10, 0, 7, 0, 0, 0, 0, 9, 0, 0, 5, 0, 8, 0, 5, 0, 0, 0, 0] \text{ mJ},
\]

as indicated in Figure 4.6 with corresponding numbers in mJ for T1 on the left and T2 on the right. When single-user directional water-filling is performed for both transmitters independently, the resulting power policy is shown in Figure 4.6. When the iterative algorithm is utilized, the resulting optimal power policies are shown as in Figure 4.7. Note that for T1, this is the result of directional water-filling over base levels determined from \( p_2 \). This is in accordance with Section 4.5.1, which suggests directional water-filling with base level \( \frac{1}{7} = 1 + p_2 \) for T1, as seen in Figure 4.7; and generalized directional water-filling for T2. The interaction of the two power policies can be observed in the optimal policy, such as when T1 remains silent while T2 has high power in time slots \( i = 1, 2 \), or when T2 significantly reduces transmit power when T1 is strongly interfering in time slots \( i = 19, 20 \). The effects of such interactions is notable even in this two user model, and would be even more critical for a higher number of users sharing the same medium.
Fig. 4.6: Energy harvesting scenarios and power policies with single-link directional water-filling for T1 (left) and T2 (right).

Fig. 4.7: Optimal power policies with iterative directional water-filling for T1 (left) and T2 (right).
Next we compare the performance of the optimal power policy, the distributed directional water-filling algorithm suggested in Section 4.7, and the constant transmit power policy described in Section 2.6 employed at both transmitters. In particular, in the constant power policy, the nodes attempt constant power transmission with \( p_i = \mathbb{E}[E_i] \) if sufficient energy is available, and transmit with all remaining energy otherwise. We assume a Gaussian interference channel with receiver noise spectral density \( N_{0,1} = N_{0,2} = 10^{-19} \) W/Hz, bandwidth \( B = 1 \) MHz and power gains \( h_{1,1} = h_{2,2} = -100 \) dB, \( h_{1,2} = -101.55 \) dB, and \( h_{2,1} = -93.01 \) dB, yielding normalized power gains \( a = 0.7 \) and \( b = 5 \). These parameters fall within the asymmetric interference region of Section 4.5.1. For battery sizes \( E_{1,\text{max}} = E_{2,\text{max}} = 10 \) mJ, we generate energy harvests with energy distributed uniformly in \([0, E_{\text{max}}]\) and inter-harvest times distributed exponentially with mean 5 s, quantized to time slots of duration \( \tau = 1 \) s. For this setting, the cumulative departures of these algorithms are plotted in Figure 4.8. It is apparent that the water-filling algorithms provide notable performance increase over the naïve approach. Moreover, it is observed in this simulation as well as others with different parameters that the single-user directional water-filling performs very close to optimal, making it a favorable candidate for practical applications.

4.9 Chapter Summary

In this chapter, we formulated the sum-throughput maximization problem for a two-user Gaussian interference channel with energy harvesting nodes, and solved it with an iterative algorithm. We observed that in the asymmetric interference and very strong interference cases, the resulting generalized iterative water-filling algorithm reduced to
Fig. 4.8: Simulation of the optimal power policy, distributed power policy, and constant power policy in asymmetric interference setting.
modified versions of the single-user directional water-filling algorithm. Furthermore, we extended the model and the related algorithm to the scenario with data arrivals by introducing a pump element to the directional water-filling algorithm through a penalty for data causality violation. With the insights obtained from the optimal offline policy, we suggested computationally simpler near-optimal alternatives for online and distributed versions of the problem. We verified the performance of the suggested iterative directional water-filling algorithm and its distributed near-optimal counterpart through simulations, showing a notable performance boost over naïve algorithms.
Chapter 5

Transmission Policies for
General Energy Harvesting Channel Models

5.1 Introduction

So far in this thesis, we have focused on specific channel models with energy harvesting transmitters, and ignored other computational or operational power costs. However, the insights obtained from our work is neither restricted to these channel models, nor to transmit power consumption. In this chapter, we provide a general solution to the collection of offline optimization problems maximizing the sum of any instantaneous utility, e.g., the instantaneous throughput functions in previous chapters.

It is proposed in [33] that receiver decoding power can be significant in specific applications, and that there is a fundamental trade-off between transmitter and decoding power. Arguably receiver power consumption is even more critical in the energy harvesting setting. The formalization in this chapter enables handling possible transmitter-receiver power tradeoffs, as well as transmitter or receiver side practicalities including storage inefficiency and practical transmission schemes, e.g., adaptive modulation techniques [25, 79]. This is achieved by leveraging the iterative generalized directional water-filling algorithm in Section 4.5. We demonstrate this fact using an energy harvesting transmitter and an energy harvesting receiver as an example. This work was presented in [98].
5.2 General System Model and Problem Formulation

We consider $K$ energy harvesting network nodes, each harvesting and storing their own energy throughout the operation of the network, as shown in Figure 5.1. These nodes can be sensors, transmitters, relays, and receivers, each operating within their own energy budget to improve the utility of the network. The state of the entire channel is represented by the vector $h_i$ in time slot $i$. This vector can, for instance, correspond to the fading state, in which case our assumption would require that the coherence time of the channel is longer than a time slot. We study a system with an information delivery deadline of $N$ time slots.

As the setting suggests, the energy harvesting process is varying and different for each user. We adopt the time-slotted model in the previous chapters, with energy harvests of size $E_{k,i}$ arriving to node $Tk$ at the beginning of time slot $i$, and stored in a battery of capacity $E_{k}^{max}$. Each energy harvesting node utilizes its energy towards
the network objective, aiming to improve the overall utility of the network. The rate of energy consumption at each node is limited by its harvesting process and storage capacity. Denoting the average power consumption of node $k$ in time slot $i$ as $p_{k,i}$, the cumulative energy consumed by node $k$ in time slot $i$ is bounded by the energy causality and no-battery-overflow constraints in (4.4) and (4.5), respectively.

We wish to formulate the problem in a general fashion in order to cover a larger set of optimization problems. In previous chapters, the utility to be optimized has been the throughput of the system. Here, we define a general utility function,

$$
u(p_{1,i}, p_{2,i}, \ldots, p_{K,i}, h_i), \quad (5.1)$$

which yields the instantaneous utility of the network given the instantaneous powers of nodes as $p_{k,i}$, $k = 1, \ldots, K$ and the related channel parameter as the vector $h_i$. This utility can be replaced with any instantaneous and additive performance metric depending on the application, some examples of which are instantaneous rate, average distortion, or average successful transmission probability. With the instantaneous utility and energy constraints defined above, the average utility maximization problem is expressed as

$$\max_{\substack{p_{k,i} \geq 0, \ k = 1, \ldots, K}} \sum_{i=1}^{N} u(p_{1,i}, p_{2,i}, \ldots, p_{K,i}, h_i) \quad (5.2a)$$

subject to (4.4), (4.5), $k = 1, \ldots, K, \ i = 1, \ldots, N. \quad (5.2b)$

Note that this problem is a generalized version of the problems in Chapters 2 and 4.
5.3 Decoupled Problem and Analysis

In Section 4.5, we utilize the constraints being decoupled among transmitters and the concavity of the rate function to decompose the joint maximization problem into single-user problems, and solve each problem using generalized directional water-filling. Note that despite the arbitrary number of nodes $K$ or their different functions in the channel, the constraints in (5.2) are also separable among the nodes. Hence, we only require a jointly concave utility function $u(p_1, p_2, \ldots, p_K, h)$ to be able to employ the IGDWF algorithm in this problem. For this purpose, we present the following lemma:

**Lemma 5.1.** Given any instantaneous utility function $u(p_1, p_2, \ldots, p_K, h)$, one can construct a concave instantaneous utility function $u_c(p_1, p_2, \ldots, p_K, h)$ by time-sharing between different power policies within a time slot.

**Proof:** We expressing $u_c(p_1, p_2, \ldots, p_K, h)$, which we refer to as the concavified utility function as follows:

$$u_c(p_1, p_2, \ldots, p_K, h) = \max \left\{ \sum_m \theta_m u(p_{1,m}, p_{2,m}, \ldots, p_{K,m}, h) \right\}$$

$$\left| \sum_m \theta_m \leq 1, \sum_m \theta_m p_{k,m} \leq p_k, p_{k,m} \geq 0, k = 1, \ldots, K \right\}$$

(5.3)

Here $\theta_m$ are time-sharing fractions and $p_{k,m}$ are the corresponding powers. As such, this is the maximum of all possible utilities achieved by time-sharing. Note that $u_c(.)$ is jointly concave in $p_1, p_2, \ldots, p_K$ by definition. This follows from the constraints in the right hand side of (5.3) being linear, and hence any convex combination of utilities being
achieved by a convex combination of the respective powers and time-sharing parameters.

As a result of Lemma 5.1, (5.2) can be solved by solving the single-user problem

\[
\max_{p_{k,i} \geq 0} \sum_{i=1}^{N} u(p_{1,i}, p_{2,i}, \ldots, p_{K,i}, h_i) \quad (5.4a)
\]

s.t. (4.4), (4.5), \quad i = 1, \ldots, N. \quad (5.4b)

iteratively for \( k = 1, 2, \ldots, K \) until all transmit powers converge. The solution to (5.4) itself can be found using the generalized directional water-filling algorithm with water levels

\[
\nu_{k,i} = \frac{\partial}{\partial p_k} u_{c}(p_1, \ldots, p_K, h_i) \Big|_{p_{k,i}} \quad (5.5)
\]

where \( u_{c}(p_1, \ldots, p_K, h_i) \) is the concavified utility function in Lemma 5.1.

5.4 Example: Energy Harvesting Transmitter and Receiver

For demonstration, we consider the energy harvesting transmitter and an energy harvesting receiver pair shown in Figure 5.2. Transmitter T1 harvests energy \( E_{1,i} \) and receiver R2 harvests energy \( E_{2,i} \) in time slot \( i \). Both nodes are equipped with a finite capacity battery of size \( E_k^{\max} \) for \( k = 1, 2 \). In contrast to the models of previous chapters where the rate of the system depends on transmit power alone, we adopt a rate function \( r(p_1, p_2) \) which depends on both the transmit power \( p_1 \) of T1, and the power \( p_2 \) consumed by R2. We remark that the power consumed at the receiver side may corresponds to various costs related to receiving data, e.g., the cost of operating the receiver circuitry
such as filters and amplifiers, the cost of processing for the purpose of decoding, or the cost of storing the data received.

![Diagram of a single link energy harvesting transmitter and receiver model.]

Fig. 5.2: A single link energy harvesting transmitter and receiver model.

The throughput maximization problem for this setup can be expressed as

\[
\max_{p_1, p_2} \sum_{i=1}^{N} r(p_{1,i}, p_{2,i}) \tag{5.6a}
\]

subject to

\[
\sum_{n=1}^{i} (E_{k,n} - p_{k,n}) \geq 0, \quad i = 1, \ldots, N, \quad k = 1, 2, \tag{5.6b}
\]

\[
E_{k}^{\text{max}} - \sum_{n=1}^{i} (E_{k,n} - p_{k,n}) - E_{k,i+1} \geq 0, \quad i = 1, \ldots, N, \quad k = 1, 2, \tag{5.6c}
\]

\[
p_k \geq 0, \quad k = 1, 2. \tag{5.6d}
\]

where (5.6b) are the energy causality constraints, (5.6c) are the no-battery-overflow constraints, and (5.6d) are the non-negativity constraints.
In practice, the rate function \( r(p_1, p_2) \) is determined by various factors, and thereby shows different characteristics particularly on the receiver side [33, 31, 122]. However, it is always possible to find a jointly concave rate function by employing time-sharing between transmitter and receiver powers, as shown in Lemma 5.1. As an example, consider a wireless sensor as the transmitter and a data center as the receiver in a static AWGN channel. Consider the following power requirements by the nodes for successful operation: For device operation, let a constant power of \( p_1^{on} \) and \( p_2^{on} \) be needed by the transmitter and receiver, respectively, to transmit or receive at any nonzero rate. Otherwise the node is assumed to be in a sleep state with negligible power consumption producing zero utility. To successfully depart bits at rate \( r \), let the transmitter require a transmission power \( p_1 \) satisfying \( r = \log(1 + hp_1) \), where \( h \) depends on received noise at the receiver and the path loss of the channel. Let the receiver require a decoding power exponential in the desired rate, and a circuit power linear in the rate of processing and storage. The resulting transmitter rate function \( r_T(p_1) \) and receiver rate function \( r_R(p_2) \) with some arbitrary parameters is shown in Figure 5.3 in blue and red, respectively, along with the achieved rate-based utility \( r(p_1, p_2) \) given as the minimum of the two achieved rates, i.e., \( r(p_1, p_2) = \min\{r_T(p_1), r_R(p_2)\} \).

Note that \( r(p_1, p_2) \) in Figure 5.3 is not jointly concave. However, by time-sharing between the sleep state, i.e., with \( p_1 = p_2 = 0 \), and a nonzero rate, the rates in Figure 5.4 can be achieved. In this figure, the bold region indicates rates that are achieved directly, and the light colored region indicates rates that are achieved via time-sharing. When the allocated energy values in a time slot falls on the light colored region, the nodes consume more power for some fraction of the time slot, and sleep in the remaining
Fig. 5.3: The rate function $r(p_1, p_2)$ for the EH transmitter - EH receiver setup.

Fig. 5.4: The concavified rate function $r_c(p_1, p_2)$ for the EH transmitter - EH receiver setup obtained via time-sharing.
fraction to achieve the rate in Figure 5.4. Note that the two water-filling algorithms in the IGDWF interact through the concavified rate function $r_c(p_1, p_2)$, which depends on the transmitter and receiver energy consumption models.

### 5.5 Numerical Results

We next provide simulation results to account for the performance of the optimal and some alternative suboptimal transmitter and receiver policies. We simulate an energy harvesting transmitter receiver pair in a static AWGN channel for $N = 1000$ time slots and a slot length of $\tau = 1$ s. Both the transmitter and the receiver require a constant operating power of $p_{1}^{on} = p_{2}^{on} = 50 \text{ mW}$ for being operational. Additionally, the transmitter performs adaptive modulation for power control [25, 79] with a rectangular M-QAM constellation, achieving a discrete set of rates; while the receiver is required to process and store the received data, modeled with a constant energy cost of 5 mJ/bit, yielding the following individual rates achieved:

$$
\begin{align*}
    r_T(p) &= \begin{cases} 
        0 \text{ bits/s/Hz}, & p - p_T^{on} < 1 \text{ mW (OFF)} \\
        1 \text{ bits/s/Hz}, & p - p_T^{on} \in [1, 2) \text{ mW (BPSK)} \\
        2 \text{ bits/s/Hz}, & p - p_T^{on} \in [2, 6) \text{ mW (QAM)} \\
        3 \text{ bits/s/Hz}, & p - p_T^{on} \in [6, 10) \text{ mW (8-QAM)} \\
        \vdots & \vdots
    \end{cases} \\
    r_R(p) &= \left[ \frac{p - 50 \text{ mW}}{5} \right]^{+} \text{ bits/s/Hz.}
\end{align*}
$$

(5.7)
We define the system utility as $\min(r_T, r_R)$, i.e., the utility of this system is the total number of successfully transmitted, processed and stored bits. The energy harvesting process is assumed to be an i.i.d. process, with $E_{1,i}$ and $E_{2,i}$ distributed uniformly in $[0, 100]$ mJ, yielding an average harvesting rate of 50 mW per node.

![Graph](image)

**Fig. 5.5:** Average throughput achieved by IGDWF and heuristic policies for varying energy storage capacity.

The system is simulated with various power allocation policies, and the average utility achieved against the storage capacity of the nodes ($E_{1}^{max} = E_{2}^{max}$) is plotted in Figure 5.5. The blue lines represent hasty power policies which consume the harvested energy as it is received. The red lines represent constant power policies that waits for an energy threshold and transmits with the optimal constant power whenever threshold is reached. Additionally, the two dashed plots marked as concavified term are allowed
to use the concavified rates, i.e., $r_c(.)$ instead of $r(.)$. The throughput achieved by the IGDWF algorithm is shown in green. Finally, an upper-bound on average rate derived using a non-energy harvesting model with an average power of 50 mW per node is plotted for comparison.

As expected, policies without optimal power allocation in each time slot, i.e., without utilization of the time-sharing results of the energy efficiency problem, perform particularly worse for smaller storage capacities, since they can barely turn on the circuitry with the average harvested energy. When these algorithms start employing optimal time-sharing within the slots, shown in dashed lines, the performance is significantly increased. The optimal algorithm, using both time-sharing and optimal energy allocation with iterative water-filling, performs notably better than all alternatives, allowing the system to yield a utility close to the upper-bound, especially when the storage is limited.

5.6 Chapter Summary

This chapter provides a framework for optimizing energy harvesting networks using the iterative generalized directional water-filling (IGDWF) algorithm of Section 4.5. This approach yields a solution that can be extended to any network topology with an instantaneous, i.e., memoryless, utility function. As an example, an energy harvesting transmitter - energy harvesting receiver pair is analyzed. In this case, numerical results reveal that the optimal policy performs notably closer to the upper bound, which is the throughput of the non-energy harvesting system, i.e., when the nodes have the total energy available to them at the commencement of the session.
Energy Harvesting Networks with Energy Cooperation

6.1 Introduction

Since energy harvesting networks may experience energy deprivation when sufficient energy is not available to harvest, they can benefit from the recent advances in wireless energy transfer. Short range energy transfer is already present in today’s RFID systems [111]. Energy cooperation is a viable option in mid-range as well, with wireless energy transfer efficiency values reaching up to 40% using coupled magnetic resonance [50, 47]. This provides the possibility of energy cooperation, allowing networks to have additional control over the energy available at each node, as proposed in [35, 36, 37]. In essence, energy cooperation introduces a new dimension for power management in energy harvesting networks.

The problem of optimizing energy consumption for data transmission and energy transfer is first introduced in [35], where a two-hop network with an energy harvesting transmitter and relay is considered, and the source can transfer energy to the relay. It is shown that throughput can be improved with respect to energy harvesting alone [34], even with uni-directional energy transfer. Two-way and multiple access channels with uni-directional energy cooperation are also studied by the same authors in [36], proposing a two-dimensional water-filling algorithm to find the optimal policy. These
studies assume an infinite-sized battery size for the transmitters. A different line of work studies transferring energy and information jointly [32, 77, 65, 60, 16].

In this chapter, we follow the model of transferring energy and data separately as in [35, 36, 37]. We generalize this set up to energy harvesting nodes all of which are capable of transferring energy to one another, i.e., in any direction. As communication models, we consider those in [35, 36, 37], i.e., two-way, two-hop, and multiple access channels, allowing unrestricted energy transfers between all nodes. We also extend these models to the case where all nodes have finite-sized battery. We will see that allowing unrestricted energy transfers and limited batteries require a careful solution methodology and bring on new design insights.

In the following sections, we identify the jointly optimal transmit power and energy transfer policies that maximize sum-throughput for the two-way, two-hop, and multiple access channels. We show for nodes with infinite-sized batteries that delaying energy transfers until energy is needed immediately at the receiving node is sum-throughput optimal. Focusing on such procrastinating policies without loss of optimality, we decompose the stated joint optimization problem into energy transfer and consumed energy allocation problems which are solved in tandem. This decomposition is shown to hold for the finite-sized battery case as well, using partially procrastinating policies that avoid battery overflows. For the two-hop channel, we observe that the optimal policy has a two fluid water-filling interpretation, and for the multiple access channel, it reduces to a single transmitter problem with aggregate energy harvests. We demonstrate the throughput improvement with bi-directional energy cooperation over no cooperation.
and uni-directional cooperation through numerical results. This work was presented in [101, 102, 103] and is under review for publication in [104].

6.2 The Energy Harvesting and Energy Cooperating (EHEC) Two-Way Channel (TWC)

Consider the Gaussian two-way channel (TWC) [87] with two energy harvesting and energy cooperating (EHEC) nodes, T1 and T2, as shown in Figure 6.1. Denoting the channel inputs by $X_k$ and the power gains by $h_k$, $k = 1, 2$, the channel outputs at nodes T1 and T2 for the Gaussian two-way channel are given by

$$
Y_1 = X_1 + \sqrt{h_2} X_2 + N_1, 
$$

$$
Y_2 = X_2 + \sqrt{h_1} X_1 + N_2,
$$

(6.1)

(6.2)

where $N_k$ is Gaussian noise with variance $\sigma_k^2$ at node Tk, $k = 1, 2$. In this setting, each node is able to cancel out its own contribution to the channel output, i.e., Tk can subtract $X_k$ from $Y_k$, thus reducing the model to two parallel additive white Gaussian noise (AWGN) channels with power gains $h_1$ and $h_2$.

We consider the time-slotted model in Section 3.2 with slot length $\tau = 1$ s, and $N$ time slots indexed by $i = 1, \ldots, N$. In time slot $i$, node Tk, $k = 1, 2$, harvests $E_{k,i}$ units of energy, which it stores in its battery of size $E_{k,\text{max}}$. Within this time slot, Tk transmits with average power $p_{k,i}$, thus consuming $p_{k,i}$ units of energy due to unit slot length. In addition to harvesting energy, the nodes are also capable of transferring energy to each other. In time slot $i$, Tk transfers $\delta_{k,i}$ units of energy to Tj, $j \neq k$. This transfer has an
end-to-end efficiency of $\alpha_k \leq 1$, and Tj receives $\alpha_k \delta_{k,i}$ units of energy as a result. The power policy of the network is defined as the collection of transmit powers and transferred energy values $\{p_{k,i}, \delta_{k,i}\}$ for $k = 1, 2$ and $i = 1, \ldots, N$.

As in previous chapters, we consider the offline setting where the energy harvests $E_{k,i}$ throughout the session are known at the beginning of the communication session, and the power policy of the network is constrained by the energy available to each node in each time slot. However, in this case, the energy remaining in the battery of node Tk depends on both the energy harvested at node Tk, and the energy transferred to and received from node Tj for $j \neq k$. The battery state $S_{k,i}$, i.e., the energy stored in the battery of Tk at the end of time slot $i$, evolves as

$$S_{k,i} = \min \left\{ E_{k,i}^{\max}, S_{k,i-1} + E_{k,i} - p_{k,i} - \delta_{k,i} + \alpha_j \delta_{j,i} \right\}. \tag{6.3}$$
for $k = 1, 2$, with $S_{k,0} = 0$. The initial charge of the batteries are introduced to the model through the energy harvests $E_{1,1}$ and $E_{2,1}$ in the first time slot. We express the energy causality constraints in Definition 2.1 as

$$p_{k,i} + \delta_{k,i} \leq S_{k,i-1} + E_{k,i} + \alpha_j \delta_{j,i},$$

(6.4)

for $k = 1, 2$, $i = 1, \ldots, N$, which can equivalently be expressed as $S_{k,i} \geq 0$ for $k = 1, 2$ and $i = 1, \ldots, N$. We will first consider the infinite-sized battery case, and will introduce the no-battery-overflow constraint in Section 6.7.

For transmit powers $p_{1,i}$ and $p_{2,i}$ in time slot $i$, the maximum sum-rate for the Gaussian two-way channel in Figure 6.1 is given by

$$r^{TWC}(p_{1,i}, p_{2,i}) = \frac{1}{2} \log \left( 1 + \frac{h_1 p_{1,i}}{\sigma_2^2} \right) + \frac{1}{2} \log \left( 1 + \frac{h_2 p_{2,i}}{\sigma_1^2} \right).$$

(6.5)

The corresponding EHEC sum-throughput maximization problem for the TWC over transmit powers $p_{k,i}$ and energy transfers $\delta_{k,i}$, throughout a communication session of $N$ time slots is expressed as

$$\max_{\{p_{k,i}, \delta_{k,i}\}} \sum_{i=1}^{N} r^{TWC}(p_{1,i}, p_{2,i})$$

(6.6a)

s.t. $S_{k,i} \geq 0, \quad k = 1, 2, \quad i = 1, \ldots, N,$

(6.6b)

$p_{k,i} \geq 0, \quad \delta_{k,i} \geq 0, \quad k = 1, 2, \quad i = 1, \ldots, N,$

(6.6c)
where (6.6b) are the energy causality constraints, and (6.6c) are the non-negativity constraints for transmit power and transferred energy.

6.3 Properties of Optimal Policies for Infinite-Sized Batteries

We begin with the infinite-sized battery case, $E_k^{\text{max}} = \infty$, for ease of exposition. For this case, the battery state in (6.3) can be rewritten as

$$S_{k,i} = \sum_{n=1}^i \left( E_{k,n} - p_{k,n} + \alpha_j \delta_{j,n} - \delta_{k,n} \right),$$

(6.7)

for $k, j = 1, 2$ and $j \neq k$.

The problem in (6.6) involves the joint optimization of transferred energy and transmit powers of the two nodes over $N$ time slots, i.e., $4N$ variables in total. In this section, we identify some properties of optimal policies which help us eliminate the additional complexity introduced by energy cooperation.

6.3.1 Procrastinating Policies

We first show that a subset of power policies, named procrastinating policies, includes at least one optimal policy.

**Definition 6.1.** A power policy $\{p_{k,i}, \delta_{k,i}\}$ is a procrastinating policy if it satisfies

$$p_{k,i} - \alpha_j \delta_{j,i} \geq 0, \quad j, k = 1, 2, \ j \neq k, \ i = 1, \ldots, N.$$  

(6.8)
In each time slot, a procrastinating policy transfers energy from one node to the other only if all of the transferred energy is to be consumed for transmission immediately. This can be interpreted as the energy transferring nodes delaying energy transfers until the time slot they are immediately needed at the receiving end, hence the name procrastinating policies. In a procrastinating policy, energy transfers that are necessary for the feasibility of \( p_{k,i} \) are postponed until the conditions in (6.8) are satisfied. The following lemma establishes the optimality of procrastinating policies.

**Lemma 6.1.** There exists at least one procrastinating policy that is a solution of (6.6).

**Proof:** Let \( \{p^*_{k,i}, \delta^*_{k,i}\} \) be an optimal policy which is not procrastinating, i.e., there exists \( p^*_{k,n} < \alpha_j \delta^*_{j,n} \) for some \( n, j \) and \( k, j \neq k \). Starting from \( i = 1 \), if \( p^*_{k,i} < \alpha_j \delta^*_{j,i} \), set \( \theta = p^*_{k,i}/(\alpha_j \delta^*_{j,i}) \), and update \( \delta^*_{j,i+1} = \delta^*_{j,i+1} + (1-\theta)\delta^*_{j,i} \) and \( \delta^*_{j,i} = \theta \delta^*_{j,i} \). This postpones excess transferred energy to the next time slot whenever (6.8) is violated. Note that the update in time slot \( i \) only affects \( S_{k,i} \) and \( S_{j,i} \) in (6.6b), decreasing the former by \( \alpha_j \delta^*_{j,i} - p^*_{k,i} \) and increasing the latter by \( \delta^*_{j,i} - p^*_{k,i}/\alpha_j \). However, since \( S_{k,i} \geq 0 \) in the original policy and \( \delta^*_{k,i} = 0 \) from Lemma 6.2, this change does not violate (6.6b). Repeating the updates for \( i = 2, \ldots, N \) and \( k = 1, 2 \) yields a feasible procrastinating policy. Meanwhile, since \( p^*_{k,i} \) is unchanged, the objective in (6.6a) is unchanged, and therefore the resulting procrastinating policy is also optimal. □

Lemma 6.1 shows that by delaying energy transfers unless immediately required for transmission, any feasible transmit power policy \( \{p_{k,i}\} \) can be realized with a procrastinating policy. Next, we utilize this property to decompose (6.6) into two subproblems regarding the energy harvesting and energy cooperation aspects of the original problem.
6.3.2 Decomposition to Energy Transfer and Power Allocation Problems

We define the consumed power \( \bar{p}_{k,i} \) as the power drawn from the battery of node Tk, taking both transmission and transfers in time slot \( i \) into consideration. This term is expressed as

\[
\bar{p}_{k,i} = p_{k,i} + \delta_{k,i} - \alpha_j \delta_{j,i}.
\]  

(6.9)

Note that by definition, consumed power can be negative. However, a procrastinating policy, as defined in Definition 6.1, satisfies \( \bar{p}_{k,i} \geq 0 \) for all \( i = 1, \ldots, N \) and \( k = 1, 2 \).

We first present the following lemma.

**Lemma 6.2.** There exists an optimal policy which satisfies \( \delta_{k,i} \delta_{j,i} = 0 \) for all \( k,j = 1, 2, j \neq k, i = 1, \ldots, N \), i.e., energy transfer is never in both directions in a given time slot.

**Proof:** Let \( \{p_{k,i}^*, \delta_{k,i}^*\} \) be an optimal policy. Define \( \tilde{\delta}_{k,i} = \max\{\delta_{k,i}^* - \delta_{j,i}^*, 0\} \) for all \( k,j = 1, 2, j \neq k, \) and \( n = 1, \ldots, N \), which satisfies \( \tilde{\delta}_{k,i} \tilde{\delta}_{j,i} = 0 \). With these energy transfers, \( S_{k,i} \) in (6.7) increases for all \( k \) and \( i \), and therefore the procrastinating policy \( \{p_{k,i}^*, \tilde{\delta}_{j,i}^*\} \) is feasible. Since \( p_{k,i}^* \) are unchanged, it also yields the same objective as \( \{p_{k,i}^*, \delta_{j,i}^*\} \) and is therefore optimal. ■

Lemma 6.2 is a natural consequence of energy transfer being lossy, and is intuitively pleasing. As a consequence of the lemma, we can restrict our attention to policies satisfying the lemma without loss of optimality. Hence, for procrastinating policies satisfying the lemma, the non-negativity constraints \( p_{k,i} \geq 0 \) in (6.6c) are equivalent to \( \bar{p}_{k,i} \geq \delta_{k,i}, i = 1, \ldots, N, k = 1, 2 \). Restricting the feasible set of (6.6) to procrastinating...
policies satisfying Lemma 6.2, without loss of optimality, we rewrite (6.6) as

\[
\max \{ \bar{p}_{k,i}, \delta_{k,i} \} \sum_{i=1}^{N} r^{TWC} \left( \left[ \bar{p}_{k,i} + \alpha_j \delta_{j,i} - \delta_{k,i} \right] \right) \tag{6.10a}
\]

s.t. \[ \sum_{n=1}^{i} \left( E_{k,n} - \bar{p}_{k,n} \right) \geq 0, \quad k = 1, 2, \quad i = 1, \ldots, N, \tag{6.10b} \]

\[ \bar{p}_{k,i} \geq \delta_{k,i}, \quad \delta_{k,i} \geq 0, \quad k = 1, 2, \quad i = 1, \ldots, N. \tag{6.10c} \]

Here, \([p_{k,i}] = (p_{1,i}, p_{2,i})\) denotes both parameters of \(r^{TWC}\), which are found by substituting \(k = 1, 2\). Note that the constraints in (6.6b), which include both energy transfers \(\delta_{k,i}\) and transmit powers \(p_{k,i}\), are replaced with (6.10b)-(6.10c) where energy transfers and consumed powers are now decoupled. Furthermore, the \(i\)th summation term in the objective (6.10a) depends only on the variables for the respective time slot \(i\). Hence, (6.10) can be decomposed as

\[
\max \{ \bar{p}_{k,i} \} \sum_{i=1}^{N} r^{S}(\bar{p}_{1,i}, \bar{p}_{2,i}) \tag{6.11a}
\]

s.t. \[ \sum_{n=1}^{i} \left( E_{k,n} - \bar{p}_{k,n} \right) \geq 0, \quad k = 1, 2, \quad i = 1, \ldots, N, \tag{6.11b} \]

\[ \bar{p}_{k,i} \geq 0, \quad k = 1, 2, \quad i = 1, \ldots, N. \tag{6.11c} \]

where \(r^{S}(\bar{p}_{1,i}, \bar{p}_{2,i})\) is the per-slot sum-rate for consumed powers \(\bar{p}_{1,i}\) and \(\bar{p}_{2,i}\), defined as

\[
r^{S}(\bar{p}_{1,i}, \bar{p}_{2,i}) = \max_{\delta_{1,i}, \delta_{2,i}} r^{TWC} \left( \left[ \bar{p}_{k,i} + \alpha_j \delta_{j,i} - \delta_{k,i} \right] \right) \tag{6.12a}
\]
\begin{equation}
\text{s.t. } \bar{p}_{k,i} \geq \delta_{k,i}, \quad \delta_{k,i} \geq 0, \quad k = 1, 2. \tag{6.12b}
\end{equation}

Note that (6.12) yields the optimal energy transfers $\delta_{k,i}$ within a single time slot $i$ for a fixed pair of consumed powers $(\bar{p}_{1,i}, \bar{p}_{2,i})$. Being separated from $\delta_{k,i}$, (6.11) finds the optimal allocation of consumed powers $\bar{p}_{k,i}$, $i = 1, \ldots, N$ throughout the transmission. This decomposition implies that the power transfer optimization can be performed separately and in a slot-by-slot basis, i.e., the optimal energy transfers $\delta_{k,i}$ can be found using only the consumed powers $\bar{p}_{k,i}$ in the same time slot.

**Lemma 6.3.** $r_S(p_1, p_2)$ is jointly concave in $p_1$ and $p_2$.

**Proof:** The lemma follows from the constraints (6.12b) being linear, and the sum-rate function $r^{TW C}$ in (6.12a) being jointly concave in its arguments. Specifically, let the solution to (6.12) for $[p_k]$ and $[\bar{p}_k]$ be $[\delta_k]$ and $[\bar{\delta}_k]$, respectively. For consumed powers $[\theta p_k + (1 - \theta)\bar{p}_k]$ with $0 \leq \theta \leq 1$, the energy transfers $[\theta \delta_k + (1 - \theta)\bar{\delta}_k]$ are feasible, and yield

\begin{equation}
    r_S([\theta p_k + (1 - \theta)\bar{p}_k]) > \theta r_S([p_k]) + (1 - \theta)r_S([\bar{p}_k]), \quad (6.13)
\end{equation}

due to the joint concavity of $r^{TW C}(p_1, p_2)$ in $p_1$ and $p_2$. ■

As a result of Lemma 6.3, the consumed power allocation problem in (6.11) is a convex program. Furthermore, the constraints in (6.11b) and (6.11c) are separable among transmitters $k = 1, 2$, and hence a block coordinate descent (alternating maximization) algorithm that alternates between $\{\bar{p}_{1,i}\}$ and $\{\bar{p}_{2,i}\}$ converges to the optimal
policy [11]. In particular, at each iteration, a single transmitter problem with a strictly concave objective function and linear energy causality constraints is solved. The iterations evolve, alternating over the optimized variables, until the policies converge. Namely, we solve

$$\max_{\{p_{k,i}\}} \sum_{i=1}^{N} r_S(p_{1,i}, p_{2,i})$$

(6.14a)

s.t. \( \sum_{n=1}^{i} \left( E_{k,n} - \bar{p}_{k,n} \right) \geq 0, \quad i = 1, \ldots, N, \) (6.14b)

$$\bar{p}_{k,i} \geq 0, \quad i = 1, \ldots, N,$$

(6.14c)

for a fixed \( k \) at each iteration, alternating between \( k = 1 \) and \( k = 2 \), while \( \{p_{j,i}\}, j \neq k \), is held constant. Note that (6.14) differs from its counterpart without energy cooperation [120] only in the rate function \( r_S(p_{1,i}, p_{2,i}) \). However, \( \bar{p}_{j,i}, j \neq k \), may change in time, and hence the solution to each iteration step in (6.14) is not the constant power policy in [120].

6.4 Optimal Policy for the EHEC-TWC with Infinite-Sized Batteries

The decomposition in (6.11)-(6.12) simplifies the analysis of the problem by separating the power allocation problem from energy transfer variables \( \{\delta_{k,i}\} \), and calculating optimal energy transfers in a slot-by-slot basis. We first solve the energy transfer problem within a single slot, i.e., (6.12), which we then substitute in (6.14) to solve (6.11).
6.4.1 Optimal Energy Transfers for the EHEC-TWC

Consider time slot \( i \) first. We focus on the two subsets of the feasible space of (6.12), namely those satisfying \( \delta_{1,i} = 0 \) and \( \delta_{2,i} = 0 \), one of which contains an optimal policy as implied by Lemma 6.2. We solve (6.12) for these subsets, and choose the maximum of the two. For the policies satisfying \( \delta_{j,i} = 0 \), the solution to (6.12) is found as

\[
\delta_{k,i}^* = \min \left\{ \tilde{p}_{k,i}, \frac{1}{2} \left[ \left( \frac{\sigma_j^2}{h_k} + \tilde{p}_{k,i} \right) - \frac{1}{\alpha_k} \left( \frac{\sigma_k^2}{h_j} + \tilde{p}_{j,i} \right) \right]^{+} \right\},
\]

(6.15)

for \( k \neq j \), where \([x]^+\) denotes \( \max\{0, x\}\). This yields two optimal transfer candidates, \( \delta_{1,i}^* \) and \( \delta_{2,i}^* \), each requiring the other to be zero. Note that the case where both candidates are positive, i.e., \( \delta_{1,i}^* > 0 \) and \( \delta_{2,i}^* > 0 \), requires \( \alpha_1 \alpha_2 > 1 \), which is not possible since \( \alpha_1, \alpha_2 \leq 1 \) by definition. Hence, at least one of the two candidates is always zero, and (6.15) immediately gives the solution to (6.12). The per-slot sum-rate achieved by the optimal energy transfer policy, corresponding to \( r_S (\tilde{p}_{1,i}, \tilde{p}_{2,i}) \) in (6.11) and (6.12), is then expressed as in (6.16).

6.4.2 Optimal Power Allocation for the EHEC-TWC

Substituting (6.16) in (6.14), it remains to solve for the optimal \( \{\tilde{p}_{k,i}\} \) by iterating between \( \tilde{p}_{1,i} \) and \( \tilde{p}_{2,i} \). We show that the solution to each iteration admits a generalized directional water-filling interpretation as in Section 4.4, and consequently (6.6) can be solved using the iterative generalized directional water-filling algorithm in Section 4.5.
particular, using (4.15)-(4.16) with \( r_S(\bar{p}_{1,i}, \bar{p}_{2,i}) \), we obtain the generalized water levels in (6.17) and employ Algorithm 4.1 in each iteration.

For this setup, we present an example of the directional water-filling algorithm for \( N = 4 \) time slots of length \( \tau = 1 \) s and \( \alpha_1 = \alpha_2 = 0.5 \) in Figure 6.2. Energy harvests at the two nodes are \( E_1 = [2, 5, 0, 0] \) mJ and \( E_2 = [0, 4, 0, 7] \) mJ. The final (equilibrium) water levels are shown in blue for node T1, and in green for node T2, for \( h_1 = h_2 = -100 \) dB, and \( \sigma_1^2 = \sigma_2^2 = 10^{-13} \) W for a bandwidth of \( B = 1 \) MHz. Observe that in the first time slot, \( \bar{p}_{1,1} = 2 \) mW and \( \bar{p}_{2,1} = 0 \) yields the optimal energy transfers \( \delta_{1,1}^* = 1 \) mJ and \( \delta_{2,1}^* = 0 \), i.e., node T1 transfers 1 mJ of energy to node T2. The energy transfer candidates for time slots \( i = 2, 3 \) are zero, and no energy is transferred. In the last time slot, the optimal energy transfer rate is found as \( \delta_{1,4}^* = 0 \) and \( \delta_{2,4}^* = 2 \) mJ, i.e., the energy transfer is from node T2 to node T1. With the final water levels in the figure, no further water flow is feasible for either node.
\[ \nu_{k,i} = \begin{cases} 
2 \left( \frac{\sigma_j^2}{\eta_k} + \bar{p}_{k,i} \right), & \delta_{k,i} = \delta_{j,i} = 0, \\
\left( \frac{\sigma_j^2}{\eta_k} + \bar{p}_{k,i} \right) + \frac{1}{\alpha_k} \left( \frac{\sigma_k^2}{\eta_j} + \bar{p}_{j,i} \right), & 0 < \delta_k^* < \bar{p}_k, \\
\left( \frac{\sigma_j^2}{\eta_k} + \bar{p}_{k,i} \right) + \alpha_j \left( \frac{\sigma_k^2}{\eta_j} + \bar{p}_{j,i} \right), & 0 < \delta_j^* < \bar{p}_j, \\
2 \left( \bar{p}_{k,i} + \frac{1}{\alpha_k} \left( \frac{\sigma_j^2}{\eta_j} + \bar{p}_{j,i} \right) \right), & 0 < \delta_k^* = \bar{p}_k, \\
2 \left( \frac{\sigma_j^2}{\eta_k} + \bar{p}_{k,i} + \alpha_j \bar{p}_{j,i} \right), & 0 < \delta_j^* = \bar{p}_j.
\end{cases} \quad (6.17) \]

Fig. 6.2: Iterative generalized directional water-filling example for the EHEC-TWC.
6.5 The EHEC Two-Hop Channel (THC)

We next consider the two-hop channel (THC) as a simple example of a multi-hop setting as done in [35], extended to bi-directional energy transfers. The channel model is as shown in Figure 6.3. For this case, we denote the transmit power of the source node by \( p_{1,i} \), the relay node by \( p_{2,i} \), and the source-relay and relay-destination power gains by \( h_1 \) and \( h_2 \), respectively. As in Section 6.4, we consider the infinite-sized battery case. The messages are delay constrained, and the relay node T2 needs to forwards all received messages immediately to the destination. The source and the relay are both capable of energy transfer. The sum-rate function for this channel is given by the minimum of the capacities of its two links,

\[
\begin{align*}
\mathcal{R}^{THC}(p_{1,i}, p_{2,i}) &= \min \left\{ \frac{1}{2} \log \left( 1 + \frac{h_1 p_{1,i}}{\sigma_2^2} \right), \frac{1}{2} \log \left( 1 + \frac{h_2 p_{2,i}}{\sigma_1^2} \right) \right\}.
\end{align*}
\] (6.18)

Note that as in the two-way channel, \( \mathcal{R}^{THC} \) is jointly concave in \( p_{1,i} \) and \( p_{2,i} \) since it is the minimum of two jointly concave functions. Hence, the optimization problem in this case also satisfies Lemma 6.1, and allows the decomposition in (6.11)-(6.12).

6.5.1 Optimal Energy Transfers for the EHEC-THC

Given the rate function in (6.18), the two-hop version of (6.12) can be written as

\[
\begin{align*}
\max_{\delta_{1,i}, \delta_{2,i}} & \quad \min \left\{ \frac{1}{2} \log \left( 1 + \frac{h_1 p_{1,i}}{\sigma_2^2} \right), \frac{1}{2} \log \left( 1 + \frac{h_2 p_{2,i}}{\sigma_1^2} \right) \right\} \\
\text{s.t.} & \quad \bar{p}_{k,i} \geq \delta_{k,i}, \quad \delta_{k,i} \geq 0, \quad k = 1, 2.
\end{align*}
\] (6.19a)
The objective is the minimum of two linear functions, and the two terms of the minimum change in opposite directions with $\delta_{1,i}$ or $\delta_{2,i}$. Hence, the optimal is attained when the two terms are equal, if feasible. Solving (6.19) for $\delta_{1,i}$ and $\delta_{2,i}$ satisfying

$$\frac{h_1}{\sigma_2^2} \left( \bar{p}_{1,i} + \alpha_2 \delta_{2,i} - \delta_{1,i} \right) = \frac{h_2}{\sigma_1^2} \left( \bar{p}_{2,i} + \alpha_1 \delta_{1,i} - \delta_{2,i} \right)$$

(6.20)

yields the energy transfers

$$\delta_{1,i} = \left[ \frac{\sigma_1^2 h_1 \bar{p}_{1,i} - \sigma_2^2 h_2 \bar{p}_{2,i}}{\alpha_1 \sigma_2^2 h_2 + \sigma_1^2 h_1} \right]^+, \quad \delta_{2,i} = \left[ \frac{\sigma_2^2 h_2 \bar{p}_{2,i} - \sigma_1^2 h_1 \bar{p}_{1,i}}{\alpha_2 \sigma_1^2 h_1 + \sigma_2^2 h_2} \right]^+, \quad (6.21)$$

where $[x]^+$ denotes $\max\{0, x\}$. Since $\alpha_1, \alpha_2 \geq 0$, these energy transfer values are feasible and therefore optimal. Observe that due to (6.21), the difference of received powers, i.e.,

$$\frac{h_1}{\sigma_2^2} \bar{p}_{1,i} - \frac{h_2}{\sigma_1^2} \bar{p}_{2,i}$$

determines the direction of energy transfer, and the transferred energy is non-zero unless the two received powers are equal.
6.5.2 Optimal Power Allocation for the EHEC-THC

Substituting the optimal values in (6.21) into the power allocation problem in (6.11) yields

\[
\max_{\bar{p}_{k,i}} \sum_{i=1}^{N} \log \left( 1 + h_1 h_2 \min \left \{ \frac{\bar{p}_{1,i} + \alpha_2 \bar{p}_{2,i}}{\alpha_2 \sigma_1^2 h_1 + \sigma_2^2 h_2}, \frac{\alpha_1 \bar{p}_{1,i} + \bar{p}_{2,i}}{\sigma_1^2 h_1 + \alpha_1 \sigma_2^2 h_2} \right \} \right)
\]

(6.22a)

s.t. \[ \sum_{n=1}^{i} \left( E_{k,n} - \bar{p}_{k,n} \right) \geq 0, \bar{p}_{k,i} \geq 0, \ k = 1, 2, \ i = 1, \ldots, N. \]

(6.22b)

Due to the convexity of the problem, the iterative generalized directional water-filling algorithm in Section 4.5 can also be used for (6.22) by solving (6.22) iteratively in \( \{\bar{p}_{1,i}\} \) and \( \{\bar{p}_{2,i}\} \). In this model, the generalized water levels are found for the \( \{\bar{p}_{k,i}\} \) iteration, keeping \( \{\bar{p}_{j,i}\}, j \neq k \) constant, as

\[
\nu_{k,i} = \begin{cases} 
\bar{p}_{k,i} + \alpha_j \bar{p}_{j,i} + \left( \frac{\sigma_j^2}{h_j} + \frac{\alpha_j \sigma_k^2}{h_k} \right), & \sigma_k^2 h_k \bar{p}_{k,i} < \sigma_j^2 h_j \bar{p}_{j,i}, \\
\bar{p}_{k,i} + \frac{\bar{p}_{j,i}}{\alpha_k} + \left( \frac{\sigma_j^2}{h_j} + \frac{\sigma_k^2}{\alpha_k h_j} \right), & \sigma_k^2 h_k \bar{p}_{k,i} \geq \sigma_j^2 h_j \bar{p}_{j,i}.
\end{cases}
\]

(6.23)

We remark that the water levels are linear in consumed powers, and therefore the algorithm resembles conventional water-filling [26]. In the iteration for \( \{\bar{p}_{k,i}\} \), the consumed powers \( \{\bar{p}_{j,i}\}, j \neq k \) are kept constant, which introduces a base level over which water-filling is performed. The first terms in (6.23) are consumed powers, the second terms are base levels due to the other transmitter, and the third terms are constant.

We further remark that water levels \( \nu_{1,i} \) and \( \nu_{2,i} \) are linearly related, with ratio \( \alpha_2 \) or \( \alpha_1 \) depending on the direction of energy transfer. As a consequence, unless \( \alpha_1 = \)
\( \alpha_2 = 1 \), if the water levels in two consecutive time slots are equal for both transmitters, the direction of energy transfer must remain the same as well. An insight that can be drawn from this observation is that the direction of energy transfer remains unchanged in time unless the water levels change for one of the nodes, which only occurs when the respective node is out of energy.

Combining these two remarks, we observe that the generalized directional water-filling algorithm has an intuitive two-fluid interpretation for the two-hop channel. Namely, we can solve (6.22) by considering \( \bar{p}_{1,i} \) and \( \bar{p}_{2,i} \) as levels of \textit{two immiscible fluids}, and scaling these fluids appropriately in each iteration while performing directional water-filling based on the total water level. The analogy is even more apparent in the uni-directional energy transfer case, where \( \alpha_2 = 0 \). This gives \( \nu_{2,i} = \infty \) whenever \( \sigma_1^2 h_1 \bar{p}_{1,i} < \sigma_2^2 h_2 \bar{p}_{2,i} \), thus restricting the solution to \( \sigma_1^2 h_1 \bar{p}_{1,i} \geq \sigma_2^2 h_2 \bar{p}_{2,i} \). With this restriction, we have \( \nu_{2,i} = \alpha_1 \nu_{1,i} \), and therefore no iteration is necessary. The optimal consumed powers are found as the resulting water levels when both fluids are allowed to flow while satisfying the condition \( \sigma_1^2 h_1 \bar{p}_{1,i} \geq \sigma_2^2 h_2 \bar{p}_{2,i} \). An example to this two-fluid water-filling is illustrated in Figure 6.4 for \( \alpha_1 = 0.5, \alpha_2 = 0, E_{1,i} = [4, 0, 2, 6] \) mJ, \( E_{2,i} = [0, 3, 0, 0] \) mJ, and the same channel parameters in Figure 6.2. Here, the blue, green and orange areas correspond to source consumption \( \bar{p}_1 \), relay consumption \( \bar{p}_2 \) and the constant term in (6.23), respectively. Note that in this example, water flow for node T2 (green) from \( i = 2 \) to \( i = 3 \) occurs, even against the water level gradient, until the condition \( \sigma_1^2 h_1 \bar{p}_{1,2} \geq \sigma_2^2 h_2 \bar{p}_{2,2} \) is satisfied. The resulting optimal consumed powers are \( \bar{p}_1 = [3, 1, 2, 6] \) mW and \( \bar{p}_2 = [0, 1, 2, 0] \) mW, with \( \bar{p}_2 \) scaled by \( \alpha_1 \) in Figure 6.4.
Fig. 6.4: Directional water-filling (a) initial levels and (b) levels after water flow, for an example setting with $N = 4$. 
6.6 The EHEC Multiple Access Channel (MAC)

In this section, we extend the results in Sections 6.3 and 6.4 to the Gaussian multiple access channel (MAC) with infinite-sized batteries, shown in Figure 6.5.

The sum-rate function for this channel is given by

\[ r_{MAC}(p_{1,i}, p_{2,i}) = \frac{1}{2} \log \left( 1 + \frac{h_1 p_{1,i}}{\sigma_2^2} + \frac{h_2 p_{2,i}}{\sigma_1^2} \right) , \]  

(6.24)

and the corresponding sum-throughput maximization problem is expressed as

\[ \max_{\{p_{k,i}, \delta_{k,i}\}} \sum_{i=1}^{N} r_{MAC}(p_{1,i}, p_{2,i}) \]  

(6.25a)

\[ \text{s.t. } S_{k,i} \geq 0, \ p_{k,i} \geq 0, \ \delta_{k,i} \geq 0, \ k = 1, 2, \ i = 1, \ldots, N. \]  

(6.25b)
Since \( r^{MAC} \) is also jointly concave in \( p_{1,i} \) and \( p_{2,i} \), the MAC sum-throughput maximization problem also satisfies Lemma 6.1, yielding the decomposition in (6.11)-(6.12).

### 6.6.1 Optimal Energy Transfers for the EHEC-MAC

Substituting the consumed powers in (6.9) into (6.25) yields

\[
\max_{\{\bar{p}_{k,i}\}} \sum_{i=1}^{N} r^{MAC} \left( \bar{p}_{k,i} + \alpha_j \delta_{j,i} - \delta_{k,i} \right)
\]  

\[(6.26a)\]

subject to

\[
\sum_{n=1}^{i} \left( E_{k,n} - \bar{p}_{k,n} \right) \geq 0, \quad k = 1, 2, \quad i = 1, \ldots, N,
\]  

\[(6.26b)\]

where \( \delta_{1,i} \) and \( \delta_{2,i} \) are found as the solution to the energy transfer problem

\[
\max \delta_{1,i} \left( \alpha_1 \frac{h_2}{\sigma_1^2} - \frac{h_1}{\sigma_2^2} \right) + \delta_{2,i} \left( \alpha_2 \frac{h_1}{\sigma_2^2} - \frac{h_2}{\sigma_1^2} \right)
\]  

\[(6.27a)\]

subject to

\[
0 \leq \delta_{k,i} \leq \bar{p}_{k,i}, \quad k = 1, 2.
\]  

\[(6.27b)\]

Note that (6.27) is a linear program, with the optimal achieved at a corner of the rectangle defined by (6.27b). The optimal policy is to choose \( \delta_{k,i} = \bar{p}_{k,i} \) if \( \alpha_k \sigma_j^2 h_j > \sigma_k^2 h_k \), and choose \( \delta_{k,i} = 0 \) otherwise. Consequently, the allocated power at Tk is entirely transferred to Tj if \( \alpha_k \sigma_j^2 h_j > \sigma_k^2 h_k \), or is entirely used for transmission if \( \alpha_k \sigma_j^2 h_j \leq \sigma_k^2 h_k \). This also implies that energy transfers only depend on the channel parameters, and hence the optimal energy transfer direction remains the same throughout the transmission.
6.6.2 Optimal Power Allocation for the EHEC-MAC

The analysis in 6.6.1 reveals that in the optimal policy, either no energy transfer occurs, or one node transfers all of its energy to the other. In the former case, we get an energy harvesting MAC without energy transfers, the sum-capacity of which was found in [119]. The problem is solved by combining harvested energy in a single pool, thus reducing the problem to the single-link power allocation problem in [120]. In the latter case, let $\alpha_2 \sigma_1^2 h_1 > \sigma_2^2 h_2$ without loss of generality. Then, the optimal energy transfers in Section 6.6.1 yield the water levels

$$\nu_{1,i} = \frac{\sigma_2^2}{h_1} + \bar{p}_{1,i} + \alpha_2 \bar{p}_{2,i}, \quad \nu_{2,i} = \frac{\sigma_2^2}{h_1 \alpha_2} + \frac{\bar{p}_{1,i}}{\alpha_2} + \bar{p}_{2,i}. \quad (6.28)$$

Note that $\nu_{1,i} = \alpha_2 \nu_{2,i}$. In this case, we can equivalently consider the policy of node T1 only, and transfer all energy harvested by node T2 immediately to node T1. We establish this by scaling $\{E_{2,i}\}$ with the end-to-end efficiency of the transfer, $\alpha_2$, and adding them to the harvests of the transmitting node, $\{E_{1,i}\}$. The resulting problem consists of a single energy harvesting link, which can be solved as in [120]. Therefore, in both cases, the power allocation problem reduces to that of a single-link setup. In order to generalize the solution to all cases, we define

$$\alpha_k^* = \max \left\{ 1, \frac{\alpha_k \sigma_j^2 h_j}{\sigma_k^2 h_k} \right\}, \quad (6.29)$$
and find the optimal power policy as the solution to the single-user problem with equivalent energy harvests

\[ \bar{E}_i = \alpha_1^* E_{1,i} + \alpha_2^* E_{2,i}. \] (6.30)

The solution is a piecewise constant, non-decreasing sum-power policy, in which the sum-power only changes when all batteries are depleted [120].

**Remark 6.1.** Optimality of procrastinating policies also extends to channels with more than two transmitters, such as the K-user MAC, as shown in [102]. In this case, Lemma 6.2 extends to not transferring and receiving energy simultaneously, regardless of the direction [102, Lemma 1]. The procrastination condition extends to the sum of energy transfers arriving to a node [102, Eqn 12]. This allows the iterative directional water-filling algorithm to be used for such models, by iterating over all transmitters.

### 6.7 Optimal Policies for Nodes with Finite-Sized Batteries

We now extend our model, properties, and solution to nodes with finite-sized batteries. In Lemma 6.1, it is shown that an optimal procrastinating policy exists if the batteries have infinite storage capacity. This is justified by always being able to postpone energy transfers which are not consumed within the same time slot, i.e., which do not satisfy (6.8). In the case of finite-sized batteries, this argument is no longer sufficient, since postponing energy transfers from Tk to Tj may yield a battery overflow at Tk that the original energy transfer policy would have avoided. In this section, we provide a class
of policies that procrastinate to the point they can avoid such overflows, and show that they are optimal policies for the EHEC two-way channel with finite-sized batteries.

Consider the finite-sized battery version of the two-way channel model in Section 6.2, i.e., $E_{k}^{\text{max}} < \infty$ in (6.3). We first reiterate that the optimal policy should not cause any battery overflows. This is an extension of Lemma 2.2, which states that a power policy that yields a battery overflow is suboptimal. In particular, energy overflow in time slot $i$ can be avoided by consuming more energy in time slot $i - 1$. This strictly increases the sum-throughput in time slot $i - 1$, and does not affect the battery state $S_{k,j}$ for $j = i, \ldots, N$. Therefore, without loss of optimality, we restrict our attention to policies that do not cause energy overflows. We use (6.7), while imposing the constraint $S_{k,i} \leq E_{k}^{\text{max}}$ in (6.6). The resulting sum-throughput maximization problem for a TWC with finite-sized batteries is

$$
\max_{\{p_{k,i}, \delta_{k,i}\}} \sum_{i=1}^{N} r_{TWC}(p_{1,i}, p_{2,i}) 
$$

subject to

$$
E_{k}^{\text{max}} \geq S_{k,i} \geq 0, \quad k = 1, 2, \quad i = 1, \ldots, N, 
$$

$$
p_{k,i} \geq 0, \quad \delta_{k,i} \geq 0, \quad k = 1, 2, \quad i = 1, \ldots, N,
$$

where $S_{k,i}$ is given by (6.7).
6.7.1 Partially Procrastinating Policies

We next modify the set of procrastinating policies to prevent energy overflows. We begin by splitting $\delta_{k,i}$ into two components, $\gamma_{k,i} \geq 0$ and $\rho_{k,i} \geq 0$, as

$$\delta_{k,i} = \gamma_{k,i} + \rho_{k,i}.$$  

(6.32)

These components represent the portion of the transferred energy that is consumed immediately, and the excess portion that is stored for future use, respectively. Clearly, power policies defined as $\{p_{k,i}, \gamma_{k,i}, \rho_{k,i}\}$ include all feasible power policies for (6.31). Based on these variables, we define *partially procrastinating policies*, which are an extension of procrastinating policies in Section 6.3.1, as follows:

**Definition 6.2.** A power policy $\{p_{k,i}, \gamma_{k,i}, \rho_{k,i}\}$ is a partially procrastinating policy if it satisfies

$$p_{k,i} - \alpha_j \gamma_{j,i} \geq 0, \quad k, j = 1, 2, \ j \neq k, \ i = 1, \ldots, N,$$  

(6.33)

$$\gamma_1, i \gamma_2, i = 0, \quad i = 1, \ldots, N,$$  

(6.34)

$$\rho_{k,i} \left( E_k^{max} - S_{k,i} \right) = 0, \quad k = 1, 2, \ i = 1, \ldots, N.$$  

(6.35)

In a *partially procrastinating* policy, the condition for procrastination, i.e., (6.8), is restricted to the immediately consumed component $\gamma_{k,i}$, as seen in (6.33). Meanwhile, the excess component $\rho_{k,i}$ can only be non-zero if the battery of $T_k$ is full, i.e., $S_{k,i} = E_k^{max}$, as dictated by (6.35). This component allows transferring the excess energy that
would otherwise be lost due to battery overflows. We next show that there exists at least one optimal policy that is partially procrastinating.

**Lemma 6.4.** There exists a partially procrastinating policy \( \{ p_{k,i}, \gamma_{k,i}, \rho_{k,i} \} \) such that the transferred energy values \( \{ \delta_{k,i} \} \) calculated from (6.32) and the transmit powers \( \{ p_{k,i} \} \) solve (6.31).

**Proof:** Let \( \{ p_{k,i}^*, \delta_{k,i}^* \}, k = 1, 2, i = 1, \ldots, N \), be a solution to (6.31). We will construct a partially procrastinating policy \( \{ p_{k,i}^*, \gamma_{k,i}^*, \rho_{k,i}^* \} \) that is feasible. Let \( \delta_{k,1} = \delta_{k,1}^*, k = 1, 2 \).

Starting from \( i = 1 \), we calculate

\[
\gamma_{k,i} = \min \left\{ \delta_{k,i}, \frac{p_{j,i}^*}{\alpha_k} \right\}, \tag{6.36}
\]

\[
\delta_{k,i+1} = \delta_{k,i+1}^* + \delta_{k,i}^* - \gamma_{k,i}, \tag{6.37}
\]

for \( k, j = 1, 2, j \neq k \), and \( i = 1, \ldots, N \). Note that \( \{ \gamma_{k,i} \} \) in (6.36) satisfy (6.33) by definition. Next, for \( i = 1, \ldots, N \), let

\[
\gamma_{k,i}^* = \max \{ 0, \gamma_{k,i} - \gamma_{j,i} \}, \quad k, j = 1, 2, j \neq k. \tag{6.38}
\]

This yields \( \{ \gamma_{k,i}^* \} \) that satisfy both (6.33) and (6.34). Let \( \rho_{k,i}^* = 0 \) for \( k, j = 1, 2, j \neq k \), and \( i = 1, \ldots, N \). Starting from \( i = 1 \), we recalculate \( \delta_{k,i} \) from (6.32) using \( \{ \gamma_{k,i}^*, \rho_{k,i}^* \} \), and calculate \( S_{k,i} \) using the recalculated \( \delta_{k,i} \), i.e.,

\[
S_{k,i} = S_{k,i-1} + E_{k,i} - p_{k,i} - \delta_{k,i} + \alpha_j \delta_{j,i}^*, \tag{6.39}
\]
for \(k,j = 1,2\) and \(j \neq k\), while updating the optimal stored component \(\rho_{k,i}^*\) as

\[
\rho_{k,i}^* = \max\{0, S_{k,i} - E_{k}^{max}\}.
\] (6.40)

Note that this immediately satisfies (6.35) for all \(k,j = 1,2\) and \(i = 1,\ldots,N\).

The process outlined above postpones energy transfers that are not immediately needed via (6.36)-(6.37), eliminates cases of simultaneous bi-directional energy transfer via (6.38), and transfers excess energy that is overflowing via (6.40). The resulting policy, \(\{p_{k,i}^*, \gamma_{k,i}^*, \rho_{k,i}^*\}\), is a partially procrastinating policy. Given that the original policy \(\{p_{k,i}, \gamma_{k,i}, \delta_{k,i}\}\) is feasible, \(\{p_{k,i}^*, \gamma_{k,i}^*, \rho_{k,i}^*\}\) is also feasible by construction. This policy is also optimal since the objective of (6.31) depends only on the transmit powers \(\{p_{k,i}\}\), and the transmit powers \(\{p_{k,i}^*\}\) are equal in both policies. ■

### 6.7.2 Finding the Optimal Transmission Policy

We update the definition of consumed powers in (6.9) as

\[
\bar{p}_{k,i} = p_{k,i} - \alpha_j \gamma_{j,i} + \gamma_{k,i},
\] (6.41)

for \(k,j = 1,2\), \(j \neq k\), and \(i = 1,\ldots,N\). Substituting in (6.7), this yields

\[
S_{k,i} = \sum_{n=1}^{i} \left( E_{k,n} + \alpha_j \rho_{j,n} - \rho_{k,n} - \bar{p}_{k,n} \right).
\] (6.42)
We next rewrite (6.31) in terms of $\bar{p}_{k,i}$, $\gamma_{k,i}$, and $\rho_{k,i}$ as

$$
\max_{\{\bar{p}_{k,i}, \gamma_{k,i}, \rho_{k,i}\}} \sum_{i=1}^{N} r \cdot TWC \left( \left[ \bar{p}_{k,i} + \alpha_{j} \gamma_{j,i} - \gamma_{k,i} \right] \right) \quad (6.43a)
$$

s.t. $E_{k}^{max} \geq S_{k,i} \geq 0$, $k = 1, 2, \ i = 1, \ldots, N$, \quad (6.43b)

$\bar{p}_{k,i} \geq \gamma_{k,i} \geq 0$, $\gamma_{1,i} \gamma_{2,i} = 0$, $k = 1, 2, \ i = 1, \ldots, N$, \quad (6.43c)

$\bar{p}_{k,i} \geq 0$, $\rho_{k,i} \geq 0$, $k = 1, 2, \ i = 1, \ldots, N$. \quad (6.43d)

In (6.43), we have selectively imposed the conditions (6.33) and (6.34) without loss of optimality due to Lemma 6.4. Note that (6.43a) and (6.43c) are independent of $\rho_{k,i}$, while (6.43b) and (6.43d) are independent of $\gamma_{k,i}$. Moreover, the $i$th term in (6.43a) depends on $\bar{p}_{k,i}$, $\gamma_{k,i}$, and $\rho_{k,i}$ only, allowing us to decompose (6.43) as

$$
\max_{\{\bar{p}_{1,i}, \bar{p}_{2,i}\}} \sum_{i=1}^{N} r_{S}(\bar{p}_{1,i}, \bar{p}_{2,i}) \quad (6.44a)
$$

s.t. $E_{k}^{max} \geq S_{k,i} \geq 0$, $k = 1, 2, \ i = 1, \ldots, N$, \quad (6.44b)

$\bar{p}_{k,i} \geq 0$, $\rho_{k,i} \geq 0$, $k = 1, 2, \ i = 1, \ldots, N$, \quad (6.44c)

where $r_{S}(\bar{p}_{1,i}, \bar{p}_{2,i})$ is the per-slot sum-rate, given by

$$
r_{S}(\bar{p}_{1,i}, \bar{p}_{2,i}) = \max_{\gamma_{1,i}, \gamma_{2,i}} r \cdot TWC \left( \left[ \bar{p}_{k,i} + \alpha_{j} \gamma_{j,i} - \gamma_{k,i} \right] \right) \quad (6.45a)
$$

s.t. $\bar{p}_{k,i} \geq \gamma_{k,i} \geq 0$, $k = 1, 2$, \quad (6.45b)

$\gamma_{1,i} \gamma_{2,i} = 0$. \quad (6.45c)
We remark that (6.44) and (6.45) are the finite-sized battery extensions of the power allocation and energy transfer problems in (6.11) and (6.12), respectively. As in Section 6.4.1, the solution to (6.45) is given by (6.15)-(6.16). It remains to solve (6.44) and identify the optimal \( \{ \bar{p}_{k,i} \} \) and \( \{ \rho_{k,i} \} \). Observing that (6.44) is a convex program, we write the KKT optimality conditions

\[
-\frac{\partial r_S(\bar{p}_{1,i}, \bar{p}_{2,i})}{\partial \bar{p}_{k,i}} + \sum_{n=i}^{N} (\lambda_{k,n} - \beta_{k,n}) - \mu_{k,i} = 0,
\]

(6.46)

\[
\sum_{n=i}^{N} (\lambda_{k,n} - \beta_{k,n}) - \alpha_k \sum_{n=i}^{N} (\lambda_{j,n} - \beta_{j,n}) - \kappa_{k,i} = 0,
\]

(6.47)

\[
\lambda_{k,i} S_{k,i} = 0, \quad \beta_{k,i} (S_{k,i} - E_{k}^{\text{max}}) = 0,
\]

(6.48)

\[
\mu_{k,i} \bar{p}_{k,i} = 0, \quad \kappa_{k,i} \rho_{k,i} = 0,
\]

(6.49)

for \( k, j = 1, 2, j \neq k \), and \( i = 1, \ldots, N \). Here, \( \lambda_{k,i} \geq 0 \) and \( \beta_{k,i} \geq 0 \) are the Lagrange multipliers for the constraints in (6.44b), and \( \mu_{k,i} \geq 0 \) and \( \kappa_{k,i} \geq 0 \) are the Lagrange multipliers for the constraints in (6.44c). We adopt the generalized water levels in (4.15)-(4.16). For \( \rho_{k,i} > 0 \), the optimal water levels \( \nu_{k,i} \) may increase only when the battery is empty, \( S_{k,i} = 0 \), and decrease only when the battery is full, \( S_{k,i} = E_{k}^{\text{max}} \). Omitting (6.47), this would be the finite-sized battery extension of (4.13)-(4.14), and could be solved by the iterative generalized directional water-filling algorithm in Algorithm 4.2.

However, energy cooperation over \( \rho_{k,i} \) introduces the possibility of energy (and hence water) flow between the two nodes in addition to between two time slots. This flow dimension was first considered in the two-dimensional directional water-filling algorithm (2D-DWF) in [37, Alg. 1] for nodes with infinite batteries. Here, we extend it to
the finite-sized battery case, and simplify the algorithm significantly by observing the structure of optimal energy transfers \( \{ \rho_{k,i} \} \).

**Lemma 6.5.** If the generalized directional water-filling algorithm with finite-sized batteries yields \( \bar{p}_{j,i} > 0 \) for some \( i \), then \( \rho_{k,i} = 0 \) is optimal, i.e., energy transfer from \( T_k \) to \( T_j \) with the purpose of storage is not necessary in the optimal policy.

**Proof:** Since \( \{ \bar{p}_{k,i} \} \) is the output of the generalized directional water-filling algorithm, there exists \( \lambda_{k,i}, \beta_{k,i}, \) and \( \mu_{k,i} \) that satisfy (6.46), (6.48), and (6.49). Given \( \bar{p}_{j,i} > 0 \), the second sum term in (6.47) is equal to \( \nu_{j,i}^{-1} \), while the first sum term in (6.47) is greater than or equal to \( \nu_{k,i}^{-1} \) due to \( \mu_{k,i} \geq 0 \). By (6.17), water levels always satisfy \( \nu_{k,i} \geq \alpha_j \nu_{j,i} \geq \alpha_k \alpha_j \nu_{k,i} \), and hence (6.47) can be satisfied for some \( \kappa_{k,i} > 0 \) by choosing \( \rho_{k,i} = 0 \). ■

The Lemma implies that it is sufficient to have a non-zero \( \rho_{k,i} \) only when \( \bar{p}_{j,i} = 0 \). We combine this insight with the condition in (6.35), which implies that it is sufficient to have a non-zero \( \rho_{k,i} \) only when the battery of \( T_k \) is full, and propose the **2D-DWF algorithm with restricted transfers**. In this implementation, we modify the 2D-DWF algorithm by allowing water flow from \( T_k \) to \( T_j \) only if the battery of \( T_k \) is full and \( \bar{p}_{j,i} = 0 \). We allow water flow until (6.47) is satisfied for \( \kappa_{k,i} = 0 \). In accordance with the no-battery-overflow constraint, we also limit the water flow among neighboring time slots to \( E_k^{max} \).

The 2D-DWF algorithm with restricted transfers is demonstrated in Figure 6.6. Initially, the entire harvested energy \( E_{k,i} \) is allocated to transmission, i.e., \( p_{k,i} = E_{k,i} \), and water levels are obtained from (6.17). This state is depicted in Figure 6.6a. Next,
directional water flow is allowed for each user individually and in time only, i.e., flow in the vertical direction is not allowed. The taps marked with right facing arrows limit water flow to a maximum of \( E_k^{\text{max}} \) between time slots. The resulting water levels are shown in Figure 6.6b. Finally, vertical water flow is allowed only in the time slots ending with a full battery, e.g., at \( i = 2 \) from T1 to T2, by turning on the taps marked with vertical arrows. Water flow from T1 to T2 continues until (6.47) is satisfied, yielding the optimal water levels in Figure 6.6c. Note that water flow from Tk to Tj is not allowed unless the battery of Tk is full, as seen at \( i = 2 \) in Figure 6.6b.

Fig. 6.6: Two-dimensional directional water-filling with restricted transfers, with (a) initial water levels, (b) water levels after flow within each node, and (c) water levels after flow between the two nodes.
Recall that water flow from Tk to Tj, which represents \( \rho_{k,i} \), is only one component of transferred energy. Energy transfer \( \delta_{k,i} \) may be taking place at \( i = 1, 3 \) via the immediately consumed component \( \gamma_{k,i} \), which are found using (6.15).

**Remark 6.2.** The proof of Lemma 6.4 is independent of the objective function of (6.31). Hence, the optimality of partially procrastinating policies, shown in Lemma 6.4, immediately extends to the two-hop and multiple access models in Sections 6.5 and 6.6. As a result, the 2D-DWF algorithm with restricted transfers can be used for these models as well, provided that the water levels are updated as in (6.23) for the THC and (6.28) for the MAC.

### 6.8 Numerical Results

In our simulations of the three channel models, we consider a transmission period of \( N = 100 \) time slots, a noise spectral density of \( N_0 = 10^{-19} \) W/Hz and a bandwidth of \( B = 1 \) MHz for both nodes. Battery sizes are \( E_{1}^{\max} = E_{2}^{\max} = 10 \) mJ. Unless otherwise stated, the energy harvests are generated uniformly and independently in \([0, 10]\) mJ, the power gains are \( h_1 = h_2 = -100 \) dB, and the energy transfer efficiency values are \( \alpha_1 = \alpha_2 = 0.5 \). For the purpose of comparison with conventional power policies, we also evaluate the performance of a constant power policy. As in previous chapters, nodes employing the constant power policy attempt transmission with a transmit power equal to their average energy harvesting rate, i.e., \( p_{k,i} = \min\{S_{k,i}, \mathbb{E}[E_{k,i}]\} \), whenever they are not in an energy outage. To verify that the difference in performance is not solely due to energy cooperation, we additionally allow energy cooperation between nodes employing
the constant power policy for their consumed powers, i.e., \( \tilde{p}_{k,i} = \min\{S_{k,i}, \mathbb{E}[E_{k,i}]\} \), while allowing the optimal instantaneous energy transfers given by (6.15), (6.21) and below (6.27).

### 6.8.1 EHEC Two-Way Channel

For the two-way channel, the average sum-throughput values are plotted in Figure 6.7 for the optimal policy with two-way energy cooperation, optimal policy without energy cooperation, and the two constant power policies. The plots are obtained by varying the peak harvest rate of T1, referred to as \( E_h \), in \([0, 10]\) mJ, while \( E_{1,i} \) is distributed uniformly on \([0, E_h]\). \( E_{2,i} \) is distributed uniformly on \([0, 10]\) mJ. It can be observed that energy cooperation yields a significant increase in performance, particularly at low \( E_h \), i.e., when T1 is energy deprived compared to T2. In other evaluations not shown here, a similar insight is observed to hold when one node has a notably worse channel. Constant power policies, on the other hand, perform consistently worse than the respective optimal policies found via generalized directional water-filling.

In most cases, one node is clearly at a disadvantage in terms of energy, and the direction of optimal energy transfer is usually fixed, i.e., uni-directional energy transfer usually suffices to achieve the maximum throughput with energy cooperation. However, when nodes have similar parameters, cases where bi-directional transfer outperforms uni-directional transfer in either direction are noted. An example is shown in Figure 6.8 for the same channel parameters in Figure 6.7 but with an energy harvesting scenario where T1 is energy deprived for one half of the transmission, and T2 is energy deprived for the other. Note that for low or high harvest rates for node T1, the uni-directional
Fig. 6.7: Sum-throughput versus peak harvested energy $E_{1,i}$ for the two-way channel with and without energy transfer, compared with the heuristic constant power policy.

Fig. 6.8: Sum-throughput for the two-way channel with one-way energy transfer, and without excess energy transfers $\rho_{k,i}$. 
energy transfer performs better in different directions, and both directions are essential to achieve the optimal throughput.

Figure 6.8 also shows the no excess energy transfers policy, where the excess energy transfers $\rho_{k,i}$ are forced to be zero while instantaneously consumed energy transfers, $\gamma_{k,i}$ are chosen freely. The departure of this policy from the full cooperation case indicates that the excess energy transfers are necessary to find the optimal policy, whereas their impact on throughput is not as significant as the impact of instantaneously consumed energy transfers. This departure can be seen more clearly in Figure 6.9, in which $E_h = 10$ mJ is fixed, and the transfer efficiency $\alpha_1$ is varied in $[0, 1/2]$. We also remark that below a certain energy transfer efficiency, energy cooperation in the direction from $T_1$ to $T_2$ is not necessary to achieve the optimal throughput. However, as energy transfer becomes more efficient, i.e., for $\alpha_1 > 0.1$, the optimal throughput increases for both the uni-directional and bi-directional energy cooperation cases.

6.8.2 EHEC Two-Hop Channel

We next provide numerical results for the two-hop channel by varying the peak harvest rate of $T_1$ in $[0, 10]$ mJ in Figure 6.10. Due to the nature of this channel, $T_1$ being energy deprived significantly hinders the performance in the absence of energy cooperation. Hence, in this case, energy cooperation is observed to be very useful for low $E_h$. Meanwhile, the performance of constant power policies are significantly worse.
Energy transfer efficiency from T1 to T2 ($\alpha_1$)

Sum-throughput per Hz (bits/s/Hz)

Full energy cooperation
No excess energy transfers
No energy cooperation
One-way EC from T1 to T2

Fig. 6.9: Sum-throughput versus transfer efficiency $\alpha_1$ for the two-way channel.

Peak harvested energy per slot (mJ)

Sum-throughput per Hz (bits/s/Hz)

Full energy cooperation
No energy cooperation
Constant power policy with optimal EC
Constant power policy without EC

Fig. 6.10: Sum-throughput versus peak harvested energy $E_{1,i}$ for the two-hop channel with and without energy transfer, compared with the heuristic constant power policy.
6.8.3 EHEC Multiple Access Channel

Finally, we present numerical results for the two-user multiple access channel, varying the peak harvest rate of T1 in [0, 10] mJ. The channel gains for users 1 and 2 are $h_1 = -100$ dB and $h_2 = -110$ dB, respectively. Performance of the optimal policies with and without energy transfer, and constant power policies, are shown in Figure 6.11. Once again, it can be seen that when T1 is energy deprived, energy cooperation significantly increases the throughput.

Fig. 6.11: Sum-throughput versus peak harvested energy $E_{1,i}$ for the multiple access channel with and without energy transfer, compared with the heuristic constant power policy.
6.9 Chapter Summary

In this chapter, we identified the jointly optimal transmit power and energy cooperation policies of energy harvesting channels for maximizing sum-throughput. We first considered the two-way channel with infinite-sized batteries at the transmitters, and proved that a subset of feasible policies, composed of those which postpone energy transfers until immediately needed, includes an optimal policy. Named procrastinating policies, this subset allowed a decomposition of the joint optimization problem into separate energy transfer allocation and consumed power allocation problems. We subsequently showed that the separation extends to two-hop and multiple access channels. We demonstrated that generalized directional water-filling solves the problem for the two-way channel, while a two-fluid water-filling algorithm suffices for the two-hop channel, and a single-user policy as in [119], with scaled aggregate energy harvests, suffices for the multiple access channel. We extended our approach to transmitters with finite-sized batteries, and showed that partially procrastinating policies are optimal. By leveraging a simplified version of the two-dimensional water-filling algorithm in [35], we solved the throughput optimization problem with joint energy transfer and transmit powers. Through numerical results, we demonstrated the advantage of bi-directional energy cooperation in energy harvesting networks over no cooperation [34, 120] and uni-directional cooperation [35, 36, 37].
Chapter 7

The Binary Energy Harvesting Channel (BEHC)

7.1 Introduction

Throughout Chapters 2-6, the energy harvesting transmitter is considered from a power management perspective. Namely, a concave communication rate equal to the capacity of the underlying channel model with an average transmit power is assumed, and the problem of power allocation among time slots is solved. Although this is a valid and insightful approach when each time slot consists of a large number of transmitted symbols, it does not consider the impact of energy harvesting on individual symbols and therefore coding. In this chapter, we shift our attention from transmission policies at the time slot level to coding schemes at the symbol level.

References [73, 72, 57, 21, 45] study the capacity of channels with energy harvesting transmitters with an infinite-sized battery [73], with no battery [72], and with a finite-sized battery [57, 21, 45]. Reference [73] shows that the capacity with an infinite-sized battery is equal to the capacity with an average power constraint equal to the average recharge rate. This reference proposes save-and-transmit and best-effort-transmit schemes, both of which are capacity achieving when the battery size is unbounded. At the other extreme, [72] studies the case with no battery, and shows that this is equivalent to a time-varying stochastic amplitude-constrained channel. Reference [72] views harvested energy as a causally known state, combines the results of Shannon on channels
with causal state at the transmitter [86] and Smith on amplitude constrained channels [89], and argues that the capacity achieving input distribution is discrete as in the case of [89]. More recent work [57, 21, 45] consider the case with a finite-sized battery. Reference [57] provides a multi-letter capacity expression that is difficult to evaluate since it requires optimizing multi-letter Shannon strategies [86] for each channel use. The authors conjecture that instantaneous Shannon strategies are optimal for this case, i.e., strategies that only observe the current battery state to determine the channel input are sufficient to achieve the capacity. Reference [21] finds approximations to the capacity of the energy harvesting channel within a constant gap of 2.58 bits/channel use. For a deterministic energy harvesting profile, [45] provides a lower bound on the capacity by exploiting the volume of energy-feasible input vectors.

In this chapter, we introduce the binary energy harvesting channel (BEHC), where each transmitted binary symbol is individually constrained by the available energy. Namely, the transmitter harvests energy in each channel use, stores it in a finite-sized battery, and puts a symbol to the channel which is feasible in terms of available energy. Consequently, available energy can be viewed as the state of this channel, which is naturally known causally at the encoder, but unknown at the decoder. Due to energy storage, this state is correlated over time, and is driven by the exogenous energy harvesting process, energy storage capacity of the battery, and the past channel inputs. As such, this channel model introduces unprecedented constraints on the channel input, departing from traditional channels with average or peak power constraints, and requires new approaches to determine its capacity.
For channels with state available causally at the transmitter, the capacity is achieved using Shannon strategies [86] if the state is independent and identically distributed (i.i.d.) over time, and is independent of the channel inputs. However, in the BEHC, the battery state has memory since the battery stores the energy through channel uses. Further, the evolution of the battery state depends on the past channel inputs since different symbols consume different amounts of energy. Therefore, Shannon strategies of [86] are not necessarily optimal for this channel. This channel model resembles the model of reference [112] with action dependent states, where the encoder controls the state of the channel through its own actions. However, different from [112], in the case of BEHC, actions and channel inputs are equal, i.e., the two cannot be chosen independently. This yields a conflict between choosing inputs with the purpose of communicating, and with the purpose of controlling the state.

In this chapter, we consider a special case of the BEHC with no channel noise. Even in this special case, finding the capacity is challenging due to the memory in the state, the lack of battery state information at the receiver, and the inter-dependence of the battery state and the channel inputs. In essence, the uncertainty in this model is not due to the communication channel, but due to the random energy harvests and the battery state that impose intricate constraints on the channel inputs. For this case, we propose upper bounds, achievable rates, and find the capacity of the binary channel within 0.03 bits/channel use. This work was presented in [93, 95] and is under review for publication in [94].
7.2 Channel Model

We consider a single-user communication setup with an energy harvesting encoder that has a finite-sized battery, as shown in Figure 7.1. In each channel use, the encoder harvests energy that is a multiple of a fixed unit, and stores it in a battery which has a capacity that is also a multiple of this unit. Each channel input then consumes an integer number of units of energy. In this chapter, we consider the binary version of this setting, which we refer to as the binary energy harvesting channel (BEHC). In a BEHC, energy is harvested in binary amounts (0 or 1 unit), and the battery has a storage capacity of one unit. The channel inputs are binary, and sending a 1 through the channel requires one unit of energy per channel use, while sending a zero is free in terms of energy. Hence, the encoder may only send a 1 when it has the required energy in the battery, while it can send a 0 anytime. A similar abstraction of communicating with energy packets over an interactive link can be found in [77].

Fig. 7.1: The binary energy harvesting channel (BEHC) with an energy harvesting encoder and a finite-sized battery.
The encoder harvests an energy unit with probability $q_h$ in each channel use, i.e., $E_i$ is Bernoulli($q_h$), and stores it in its battery of size $E^{max} = 1$ unit. The harvests are i.i.d. over time. The encoder transmits a symbol $X_i \in \{0, 1\}$ in channel use $i$, which is constrained by the energy available in the battery at that channel use. Hence, for the transmitter to send an $X_i = 1$, it must have a unit of energy in the battery. In this chapter, we focus on the case of a noiseless channel, i.e., the channel output $Y_i = X_i$.

If the battery is full, harvested energy is lost, i.e., $E_i$ cannot be used immediately in the same time slot without storing. We refer to this particular sequence of events within a channel use as the transmit first model, since the encoder first sends $X_i$ and then harvests energy $E_i$. The battery state $S_i$ denotes the number of energy units available in the battery at the beginning of channel use $i$, and evolves as

\[
S_{i+1} = \min\{S_i - X_i + E_i, E^{max}\},
\]  

where $X_i = 0$ if $S_i = 0$ due to the energy constraint. The encoder knows the battery state $S_i$ causally, i.e., at the beginning of time slot $i$, but does not know what $E_i$ or $S_{i+1}$ will be until after sending $X_i$. The decoder is unaware of the energy harvests at the encoder, and therefore the battery state. As seen from (7.1), the battery state $S_i$ has memory, is affected by the channel inputs $X_n$ for $n \leq i$, and imposes a constraint on the channel input $X_i$. 
7.3 Achievable Rates with Instantaneous Shannon Strategies

For a channel with i.i.d. and causally known states at the transmitter, Shannon shows in [86] that the capacity is achieved using the now so-called Shannon strategies. In particular, the codebook consists of i.i.d. strategies $U_i \in \mathcal{U}$, which are functions from channel state $S_i$ to channel input $X_i$. In channel use $i$, the encoder observes $S_i$ and puts $X_i = U_i(S_i)$ into the channel. The capacity of this channel is given by

$$C_{CSIT} = \max_{p_U} I(U; Y),$$

(7.2)

where $p_U$ is the distribution of $U$ over all functions from $S_i$ to $X_i$.

In the BEHC, the state of the channel, i.e., the battery state of the encoder, is not i.i.d. over time. Therefore, (7.2) does not give the capacity for this system. To overcome the memory in the state, [57] uses strategies that are functions of all past battery states to express the capacity in a multi-letter form. However, since the dimension of such strategies grow exponentially with the number of channel uses, this approach is intractable. Alternatively, it is possible to use results of [86] to develop encoding schemes based on Shannon strategies and obtain achievable rates. One tractable such scheme is obtained when strategies are functions of the current battery state only, which is proposed as an achievable rate in [57] and [93]; and is conjectured to be capacity achieving in [57]. In this section, we consider such encoding schemes.

For the $E_{\text{max}}^{\text{max}} = 1$ case, we have two states, $S_i \in \{0, 1\}$. We denote a strategy $U$ as $U = (X, X')$, where $U(0) = X$ and $U(1) = X'$, i.e., $X$ is the channel input when $S = 0$ and $X'$ is the channel input when $S = 1$. Due to the inherent energy constraint
of the BEHC, \( X = 1 \) requires \( S = 1 \), and thus, we have two feasible strategies, namely \((0,0)\) and \((0,1)\).

We first construct a codebook by choosing \( U_i \) i.i.d. for each codeword and channel use. Let the probability of choosing \( U_i = (0,1) \) be \( q_u \) for all \( i \) and all codewords. We will consider two alternative approaches to decoding the message. First, note that the i.i.d. codebook construction yields an ergodic battery state process for any message, with the transition probabilities

\[
\Pr[S_{i+1} = 1 | S_i = 0] = q_h, \quad \Pr[S_{i+1} = 0 | S_i = 1] = q_u(1 - q_h),
\]

yielding the stationary probability

\[
\Pr[S = 1] = \frac{q_h}{q_u + q_h - q_u q_h}.
\]

The receiver can ignore the memory in the model, consider a channel with i.i.d. states with the state probability given in (7.4), and perform joint typicality decoding. This is similar to the approach used in [77] for a communication scenario with energy exchange. Denoting \( U = (0,0) \) as 0 and \( U = (0,1) \) as 1, this channel is expressed as

\[
p(y|u) = \Pr[S = 1] \delta(y - u) + \Pr[S = 0] \delta(y),
\]

where \( \delta(u) \) is 1 at \( u = 0 \), and zero elsewhere. Since the channel is memoryless, its capacity is given by (7.2). Note that this is an achievable rate, but it is not the capacity of the BEHC, since the decoder treats the channel as if it was memoryless. Hence, we
refer to this scheme as the naïve i.i.d. Shannon strategy (NIID). The best achievable rate for the NIID scheme is given by

\[
R_{NIID} = \max_{q_u \in [0,1]} H_2\left(\frac{q_u q_h}{q_u + q_h - q_u q_h}\right) - q_u H_2\left(\frac{q_h}{q_u + q_h - q_u q_h}\right),
\]

(7.6)

where \(H_2(q_u) = -q_u \log(q_u) - (1 - q_u) \log(1 - q_u)\) is the binary entropy function.

While the NIID scheme permits an easy analysis, it fails to make use of the memory in the channel. Instead, the decoder can exploit the memory by using the \(n\)-letter joint probability \(p(u^n, y^n)\) when performing joint typicality decoding. Since this is the best that can be done for an i.i.d. codebook, we will refer to this scheme as the optimal i.i.d. Shannon strategy (OIID), which yields the achievable rate

\[
R_{OIID} = \max_{q_u \in [0,1]} \lim_{n \to \infty} \frac{1}{n} I(U^n; Y^n).
\]

(7.7)

The challenge with this scheme is in calculating the limit of the \(n\)-letter mutual information \(I(U^n; Y^n)\). To this end, we use the message passing algorithm proposed in [7]. This algorithm requires that the joint probability \(p(y_i, u_i, s_{i+1}, s_i)\) is independent of the channel index \(i\). In our case, we have independent \(U_i\), which yields

\[
p(y_i, u_i, s_{i+1}, s_i) = p(y_i, s_{i+1}, u_i, s_i)p(u_i),
\]

(7.8)

where \(p(y_i, s_{i+1}, u_i, s_i)\) is independent of \(i\) by the definition of the channel. Thus, we can use the algorithm in [7] to exhaustively search \(U\) and solve (7.7).
It is possible to further improve such achievable rates by constructing more involved codebooks. For example, reference [57] considers generating codewords with Markov processes, which introduces additional memory to the system through the codewords. This approach improves the achievable rate as shown in [57] at the cost of increased computational complexity in the Markov order of the codebook. We evaluate and compare these achievable rates in Section 7.9.

### 7.4 Timing Representation of the BEHC

We next propose an alternative representation of the BEHC, which yields a simpler analysis via a single-letter expression for the capacity. In particular, we equivalently represent channel outputs $Y_i$ with the number of channel uses between instances of $Y_i = 1$. We show that this transformation eliminates the memory in the state of the system, and allows constructing tractable achievable rates and upper bounds for the BEHC.

The input sequence $X^n$ and the output sequence $Y^n$ of the noiseless BEHC are both binary sequences. Let $T_1 \in \{1, 2, \ldots\}$ be defined as the number of channel uses before the first instance of output $Y = 1$, and $T_k \in \{1, 2, \ldots\}$ for $k = 2, \ldots, m$ be defined as the number of channel uses between the $(k - 1)$th instance of output $Y = 1$ and the $k$th instance of output $Y = 1$. In other words, the sequence $T^m$ represents the differences between the channel uses where 1s are observed at the output of the channel. Clearly, $T^m$ and $Y^n$ are equivalent since there is a unique sequence $T^m$ corresponding to each $Y^n$ and vice versa.
When a 1 is transmitted in the $i$th channel use, the entire energy stored in the unit-sized battery of the encoder is consumed. Hence, the encoder cannot transmit another 1 until another energy unit is harvested. We define the idle time $Z_k \in \{0, 1, \ldots\}$ of the encoder as the number of channel uses the encoder waits for energy after the $(k-1)$th 1 is transmitted. Since the probability of harvesting an energy unit is distributed i.i.d. with Bernoulli($q_h$), $Z_k$ is also i.i.d. and distributed Geometric($q_h$) on $\{0, 1, \ldots\}$. Note that during the idle period, the encoder cannot send any 1s. Once the energy is harvested, the encoder observes $Z_k$ and chooses to wait $V_k \in \{1, 2, \ldots\}$ channel uses before sending the next 1. Hence, we have a timing channel with causally known additive state $Z_k$, channel input $V_k$, and channel output $T_k$, satisfying
\[ T_k = V_k + Z_k. \quad (7.9) \]

We illustrate the variables $T_k$, $V_k$, and $Z_k$ in Figure 7.2. In slots representing one use of the BEHC, an energy harvest, i.e., $E_i = 1$, is marked with a circle and sending a 1, i.e., $X_i = 1$, is marked with a triangle. Note that since energy is harvested immediately after sending a 1, we have $Z_3 = 0$, and the one use of the timing channel spans $T$ uses of the BEHC.

![Fig. 7.2: Graphical representation of timing channel parameters $T_k$, $V_k$ and $Z_k$.](image-url)
We remark that the timing channel constructed from the time difference between consecutive 1s resembles the noiseless channel with symbols of varying durations [85]. The symbol durations in [85] are fixed, while the symbol durations in our model depend on the energy harvesting process, and therefore may change each time a symbol is sent. Hence, while [85] studies the problem of packing the most information within a given block length, our problem is also concerned with the randomness introduced by energy harvesting. In this sense, the timing channel defined here is analogous to the telephone signaling channel in [5] and its discrete-time counterpart in [78], with the exception of causal knowledge of $Z_k$ at the encoder in our model.

### 7.4.1 Equivalence of the BEHC and the Timing Channel

In the timing channel, the decoder observes $T^m$, which can be used to calculate the BEHC output sequence $Y^n$. The encoder observes $Z^m$ causally, which can be combined with past timing channel inputs $V^{m-1}$ to obtain the state sequence $S^n$ causally. Hence, any encoding/decoding scheme for the BEHC can be implemented in the timing channel, and vice versa, implying that the two channels are equivalent. However, note that in the timing channel, the $k$th channel use consists of $T_k$ uses of the BEHC. To take the time cost of each timing channel use into consideration, we define the timing channel capacity $C_T$ as the maximum achievable message rate per use of the BEHC channel. In particular, given a timing channel codebook consisting of $M$ codewords of length $m$, sending a codeword takes $n = m\mathbb{E}[T]$ uses of the BEHC on average, and the corresponding rate is
defined as

$$R = \frac{\log M}{m \mathbb{E}[T]} = \frac{\log M}{n}. \quad (7.10)$$

We remark that this definition is a variation of the rate of the telephone signaling channel introduced in [5, Definition 5]. With both rates defined per use of the binary channel, the timing channel and the BEHC have the same capacity. This is due to the encoders and decoders of these channels having different but equivalent representations of the same channel. We state this fact as a lemma.

**Lemma 7.1.** The timing channel capacity with additive causally known state at the encoder, $C_T$, and the BEHC capacity, $C_{BEHC}$, are equal, i.e., $C_{BEHC} = C_T$.

### 7.4.2 Capacity of the Timing Channel

The timing channel defined in (7.9) is memoryless since $Z_k$ are independent. For such channels, the capacity is given by (7.2), or more explicitly by the following expression [86]

$$C_{CSIT} = \max_{p(u), t(u, z)} I(U; T), \quad (7.11)$$

where $U$ is an auxiliary random variable that represents the Shannon strategies, and $v(U, Z)$ is a mapping from auxiliary $U$ and state $Z$ to the channel input $V$. The cardinality bound on the auxiliary random variable is $|U| \leq \min\{(|V| - 1)|Z| + 1, |T|\}$.

As stated in [22, Theorem 7.2], a deterministic $v(u, z)$ can be assumed without losing optimality. Hence, solving (7.11) requires finding the optimal distribution for $U$, $p(u)$, and the optimal deterministic mapping $v(u, z)$. 

Due to Lemma 7.1, we are interested in $C_T$, which is defined per use of the binary channel, i.e., with a time cost of $T_k$ binary channel uses for the $k$th timing channel use. To this end, we combine the approaches in [86] for channels with causal state information at the transmitter, and [5] for timing channels, to state the following theorem.

**Theorem 7.1.** The capacity of the timing channel with additive causally known state, $C_T$, is

$$C_T = \max_{p(u),v(u,z)} \frac{I(U;T)}{E[T]},$$  \hspace{1cm} (7.12)

**Proof:** Let $W$ denote the message which is uniform on $\{1, \ldots, M\}$. Let $n$ be the maximum number of binary channel uses, averaged over the energy harvests $E_k$, to send a message $W$. We note that by definition, we have

$$\sum_{k=1}^{m} E[T_k] \leq n, \hspace{1cm} (7.13)$$

where the expectation is over the energy harvest sequence $E_i$ and the message $W$.

For the converse proof, we define $U_k = (W, T^{k-1})$. Since $E_i$ is an i.i.d. random process, $Z_k$ is independent of $W$ and $T^{k-1}$, and therefore $U_k$. We write

$$\log(M) - H(W|T^m) = H(W) - H(W|Y^m) \hspace{1cm} (7.14)$$

$$= I(W; T^m) \hspace{1cm} (7.15)$$

$$= \sum_{k=1}^{m} I(W; T_k | T^{k-1}) \hspace{1cm} (7.16)$$

$$\leq \sum_{k=1}^{m} I(W, T^{k-1}; T_k) \hspace{1cm} (7.17)$$
\[
\sum_{k=1}^{m} I(U_k; T_k) = \sum_{k=1}^{m} I(U_k; T_k) \leq \frac{n}{\sum_{k=1}^{m} E[T_k]} \sum_{k=1}^{m} I(U_k; T_k) \leq n \sup_{U} \frac{I(U; T)}{E[T]} = nC_T,
\]

where (7.19) follows from (7.13), and (7.20) follows from \( U_i \) being independent of \( Z_i \) and the inequality \( \sum_i a_i/b_i \leq \max_i a_i/b_i \) for \( a_i, b_i > 0 \). When \( m \to \infty \), if the probability of error goes to zero, then Fano’s inequality implies \( H(W|T^m) \to 0 \). Combining this with (7.10) and (7.20), we get \( \frac{\log(M)}{n} = R \leq C_T \), which completes the converse proof.

For the achievability of this rate, we use the encoding scheme in [86]. In particular, the message rate \( I(U; T) \) per use of the timing channel is achievable with a randomly generated codebook consisting of strategies \( U_k \) [86]. Therefore, as \( m \to \infty \), we have \( n = mE[T] \), and the message rate \( R = \frac{I(U; T)}{E[T]} \) per use of the BEHC is achievable, completing the achievability proof.

We noted in Section 7.3 that the optimal distribution over Shannon strategies can be found numerically for the BEHC. This is due to the fact that for a binary input \( X_i \) and binary state \( S_i \), there are only two feasible Shannon strategies. However, for the timing channel, both the input \( V_k \in \{1, 2, \ldots \} \) and the state \( Z_k \in \{0, 1, \ldots \} \) have infinite cardinality. This also implies that the cardinality bound on \( U \) is infinite. Therefore, although (7.12) is a single-letter expression, it is difficult to evaluate explicitly. In the following sections, we first develop upper bounds for the capacity using a genie-aided method and using a method that quantifies the leakage of the state information to the
receiver; and then develop lower bounds (explicit achievable schemes) by certain specific selections for \( p(u) \) and \( v(u, z) \); and compare these achievable rates and the upper bounds.

### 7.5 Upper Bounds on the Capacity of the BEHC

#### 7.5.1 Genie Upper Bound

We first provide the timing channel state \( Z_k \) to the decoder as genie information. This yields an upper bound since the decoder can choose to ignore \( Z_k \) in decoding. However, with the knowledge of \( Z_k \), the decoder can calculate \( V_k = T_k - Z_k \), and thus we obtain the upper bound

\[
C_{UB}^{\text{genie}} = \max_{p(v)} \frac{H(V)}{E[V] + E[Z]} \quad \text{(7.21)}
\]

\[
= \max_{\mu \geq 1} \frac{1}{\mu + E[Z]} \max_{E[V] \leq \mu} H(V). \quad \text{(7.22)}
\]

Note that in (7.22), we partition the maximization into choosing the optimal \( E[V] = \mu \) and choosing the optimal distribution of \( V \) with \( E[V] \leq \mu \). The equality in (7.22) holds since the term \( (\mu + E[Z])^{-1} \) is decreasing in \( \mu \), and therefore the optimal \( \mu \) equals the expectation of the optimal \( V \). The second maximization in (7.22) involves finding the entropy maximizing probability distribution over the discrete support set \( Z^+ = \{1, 2, \ldots\} \) with the constraint \( E[V] \leq \mu \). The solution to this problem is a geometric distributed \( V \) with parameter \( q_u = \frac{1}{\mu} \). Its entropy is given by \( H(V) = \mu \frac{H_2(q_u)}{q_u} \), where \( H_2(q_u) \) is the binary entropy function. Noting that \( Z \) is also geometrically distributed
with parameter $q_h$, the genie upper bound reduces to

$$C^\text{genie}_{UB} = \max_{q_u \in [0,1]} \frac{H_2(q_u)}{q_u} = \max_{q_u \in [0,1]} \frac{q_h H_2(q_u)}{q_h + q_u (1 - q_h)}.$$  \hfill (7.23)

The genie upper bound in (7.23) overcomes the state dependence of the timing channel by effectively removing the state $Z_k$ from the channel. Although this neglects the main challenges of our model, we will show in Section 7.6.2 that this is a useful upper bound which in fact is asymptotically tight as $q_h \to 0$.

### 7.5.2 State Leakage Upper Bound

Another approach to obtain an upper bound is to quantify the minimum amount of information $T^m$ carries about $Z^m$. Since $Z^m$ is independent of the message, information leaked about it via $T^m$ reduces the potential information that $T^m$ can carry about the message. Following this intuition, in this subsection, we find an upper bound on $H(Z|T = t, U = u)$, which yields the state leakage upper bound for the timing channel capacity.

An example that relates to this idea can be found in [92]. This reference considers communicating through a queue with a single packet buffer, where the encoding is performed over harvest times to the buffer. The decoder recovers the message by observing the buffer departure times of packets, which have suffered random delays through the buffer. What this example suggests is that it is possible to achieve a positive message rate through a buffer that causes random delays. In a similar manner, we can consider timing channel input $V$ as random delay, and achieve a positive rate between the harvesting
process and the decoder in addition to the message rate of the timing channel. Since the total message rate is limited to $H(Y)$ or $H(T)/E[T]$ by the cutset bound, quantifying this non-zero rate between the harvesting process and the decoder is useful in finding an upper bound.

We first present the following lemma, where we provide an upper bound for $H(Z|T = t, U = u)$. This conditional entropy represents the amount of uncertainty remaining in $Z$ after the decoder receives $T$ and successfully decodes $U$.

**Lemma 7.2.** For the timing channel $T = V + Z$, where $Z$ is geometric with parameter $q_h$, and $V = v(U, Z)$ with the auxiliary random variable $U$ independent of $Z$, we have

$$H(Z|T = t, U = u) \leq H(Z_t),$$

(7.24)

where $Z_t$ is a truncated geometric random variable on $\{0, 1, \ldots, t-1\}$ with the probability mass function

$$p_{Z_t}(z) = \begin{cases} q_h(1-q_h)^z, & \text{if } z < t, \\ 1-(1-q_h)^t, & \text{otherwise.} \end{cases}$$

(7.25)

**Proof:** We first examine the joint distribution $p(z, t|u)$ resulting from a deterministic $v(U, Z)$, which is depicted as a two-dimensional matrix in Figure 7.3. Given $Z = z$ and $U = u$, the output of the channel is $T = v(u, z) + z$. Therefore, each row of $p(z, t|u)$ in the figure contains one non-zero term. We also have

$$p(z, t|u) = 0, \quad z \geq t,$$

(7.26)
since \( v(u,z) \) is positive by definition. This is denoted by the shaded area in the figure. Moreover, we write

\[
p(z, v(u,z) + z|u) = \sum_{t=1}^{\infty} p(z,t|u) = p(z|u) = p(z),
\]

(7.28)

implying that the non-zero term in row \( z \) is equal to \( \Pr[Z = z] \). Here, the second equality in (7.28) follows from the independence of \( U \) and \( Z \).

\[\begin{array}{cccccc}
& T = 1 & T = 2 & \ldots & T = t & \ldots \\
Z = 0 & 0 & 0 & \ldots & 0 & 0 & p_0 & 0 \\
Z = 1 & & & & p_1 & 0 & 0 & 0 \\
& \vdots & & & & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots \\
Z = t - 1 & & & & & p_t & 0 & 0 & 0 \\
& \vdots & & & & & & 0 & 0 & 0 \\
\end{array}\]

Fig. 7.3: The joint probability matrix \( p(z,t|u) \) for a fixed strategy \( u \).

Note that to find \( H(Z|T = t, U = u) \), only column \( t \) of the probability matrix \( p(z,t|u) \), which is marked with a bold rectangle in the figure, is required. Let \( \mathcal{A} \subset \{0,1,\ldots,t-1\} \) denote the set of indices \( z \in \{0,1,\ldots,t-1\} \) for which \( p(z,t|u) = p(z) \).
As such, we can write \( p(z|t, u) \) as

\[
p_A(z) = p(z|t, u) = \frac{p(z, t|u)}{\sum_{t=1}^{\infty} p(z, t|u)} \quad (7.29)
\]

\[
= \begin{cases} 
q_h (1-q_h)^{z}, & \text{if } z \in A, \\
\sum_{a \in A} q_h (1-q_h)^a, & \text{otherwise.}
\end{cases} 
\quad (7.30)
\]

We next prove that \( H(Z|T = t, U = u) \) is maximized when \( A^* = \{0, 1, \ldots, t-1\} \), i.e., when all terms in the bold rectangle in Figure 7.3 are non-zero. To this end, we show that the distribution \( p_{A^*}(z) \) is majorized by \( p_A(z) \) for all index sets \( A = \{a_0, a_1, \ldots, a_{k-1}\} \subset \{0, 1, \ldots, t-1\}, \ k \leq t \). Without loss of generality, we assume that \( a_0 < a_1 < \ldots < a_{k-1} \), which implies the ordering

\[
p_A(a_0) > p_A(a_1) > \ldots > p_A(a_{k-1}). \quad (7.31)
\]

for any \( A \). For \( 0 \leq i \leq k - 1 \), we write

\[
\sum_{j=0}^{i} p_{A}(a_j) = \frac{\sum_{j=0}^{i} q_h (1-q_h)^{a_j}}{\sum_{j=0}^{k-1} q_h (1-q_h)^{a_j}} \quad (7.32)
\]

\[
\geq \frac{\sum_{j=0}^{i} (1-q_h)^{a_j+j-i}}{\sum_{j=0}^{k-1} (1-q_h)^{a_j+j-i} + \sum_{j=i+1}^{k-1} (1-q_h)^{a_j}} \quad (7.33)
\]

\[
\geq \frac{\sum_{j=0}^{i} (1-q_h)^{a_j+j-i}}{\sum_{j=0}^{k-1} (1-q_h)^{a_j+j-i}} \quad (7.34)
\]

\[
\geq \frac{\sum_{j=0}^{i} (1-q_h)^{j}}{\sum_{j=0}^{k-1} (1-q_h)^{j}} = \sum_{j=0}^{i} p_{A^*}(j), \quad (7.35)
\]
where we obtain (7.33) by subtracting

$$
\sum_{j=0}^{i} (1 - q_h)^{a_j} - \sum_{j=0}^{i} (1 - q_h)^{a_i + j - i}
$$

(7.36)

from both the numerator and the denominator, and we obtain (7.34) by adding

$$
\sum_{j=i+1}^{k-1} (1 - q_h)^{a_i + j - i} - \sum_{j=i+1}^{k-1} (1 - q_h)^{a_j}
$$

(7.37)

to the denominator. Note that both (7.36) and (7.37) are non-negative since \(a_i - a_j \geq i - j\), for \(i \geq j\). Finally, (7.35) follows from \(k \leq t\).

Due to the concavity of \(f(x) = -x \log(x)\), and since the set \(\mathcal{A}\) is finite, the majorization shown in (7.32)-(7.35) implies that \(H(Z|T = t, U = u)\) is maximized for \(\mathcal{A}^* = \{0, 1, \ldots, t - 1\}\). In this case, the conditional distribution of \(Z\) given \(t\) and \(u\) is truncated geometric. Hence, for any \(v(U, Z)\), \(H(Z|T = t, U = u)\) is upper bounded by the entropy of a truncated geometric random variable, \(H(Z_t)\).

Using the bound obtained in Lemma 7.2, we next present the leakage upper bound on the timing channel capacity \(C_T\).

**Theorem 7.2.** The capacity of the timing channel and therefore the BEHC is upper bounded by

$$
C_{UB}^{\text{leakage}} = \max_{p_T(t) \in \mathcal{P}} \frac{H(T) - \sum_{t=1}^{\infty} \frac{H_2((1-q_h)^t)}{1-(1-q_h)} p(t)}{\mathbb{E}[T]},
$$

(7.38)
where

$$\mathcal{P} = \left\{ p_T(t) \left| \sum_{t=1}^{c} p(t) \leq 1 - (1 - q_{h})^c, \; c = 1, 2, \ldots \right. \right\}. \quad (7.39)$$

**Proof:** Using the chain rule of mutual information, we write the numerator of (7.12) as

$$I(U; T) = I(U, Z; T) - I(Z; T|U) \quad (7.40)$$

$$= H(T) - H(T|U, Z) - I(Z; T|U) \quad (7.41)$$

$$= H(T) - I(Z; T|U), \quad (7.42)$$

where the last equality follows since $T = v(U, Z) + Z$ is a deterministic function of $U$ and $Z$. Note that the $I(Z; T|U)$ term in (7.42) quantifies the information leaked to the decoder about the energy harvesting process $Z$. We lower bound this term as

$$I(Z; T|U) = H(Z|U) - H(Z|T, U) \quad (7.43)$$

$$= H(Z) - H(Z|T, U) \quad (7.44)$$

$$\geq \sum_{t=1}^{\infty} \sum_{u} p(t, u) \left[ H(Z) - H(Z|T = t, U = u) \right] \quad (7.45)$$

$$\geq \sum_{t=1}^{\infty} [H(Z) - H(Z_t)] \sum_{u} p(t, u) \quad (7.46)$$

$$= \sum_{t=1}^{\infty} [H(Z) - H(Z_t)] p(t), \quad (7.47)$$
where (7.44) is due to the independence of $Z$ and $U$, and (7.46) is due to Lemma 7.2.

Substituting (7.42) and (7.47) in (7.12), we get

$$C_T \leq \max_{p(u), v(u,z)} \frac{H(T) - \sum_{t=1}^{\infty} [H(Z) - H(Z_t)]p(t)}{\mathbb{E}[T]}.$$  

(7.48)

Note that the objective is a function of $p_T(t)$ only. Therefore, without loss of generality, we can perform the maximization over distributions $p_T(t)$ that are achievable by some auxiliary $p_U(u)$ and function $v(U, Z)$. Since $T > Z$ by definition, such a distribution must satisfy

$$\sum_{t=1}^{c} p(t) \leq \sum_{z=0}^{c-1} p(z) = 1 - (1 - q_h)^c, \quad c = 1, 2, \ldots .$$  

(7.49)

As a result, the distribution $p_T(t)$ induced by any $p_U(u)$ and $v(U, Z)$ lies in the set of distributions $\mathcal{P}$ defined in (7.39). We finally note that for geometrically distributed $Z$ and truncated geometric distributed $Z_t$, we have

$$H(Z) - H(Z_t) = \frac{H_2((1 - q_h)^t)}{1 - (1 - q_h)^t}.$$  

(7.50)

Substituting (7.49) and (7.50) in (7.48), we arrive at the upper bound in (7.38).

### 7.5.3 Computing the State Leakage Upper Bound

Solving (7.38) requires finding the optimal $p(t)$ distribution in $\mathcal{P}$. We next find the properties of the optimal distribution $p^*(t)$ to simplify its calculation. We begin by
rewriting the maximization problem in (7.38) as

\[ C_{UB}^{\text{leakage}} = \max_{\mu} \frac{1}{\mu} \max_{p_T(t) \in \mathcal{P}, \mathbb{E}[T] \leq \mu} H(T) - \sum_{t=1}^{\infty} \Gamma_t p(t), \]  

(7.51)

where we have defined \( \Gamma_t = \frac{H_2((1-q_h)^t)}{1-(1-q_h)^t} \). The inner maximization in (7.51) is a convex program since it has a concave objective and linear constraints. For this problem, we write the KKT optimality conditions\[11\] as

\[ p(t) = \exp\left( -\beta t - \Gamma_t + \lambda_t - \sum_{c=1}^{t} \psi_c - \phi - 1 \right), \quad t = 1, 2, \ldots , \]  

(7.52)

\[ \lambda_t p(t) = 0, \quad \lambda_t \geq 0, \]  

(7.53)

\[ \psi_t \left( \sum_{c=1}^{t} p_T(c) - 1 + (1-q_h)^t \right) = 0, \quad \psi_t \geq 0, \]  

(7.54)

\[ \beta \left( \mathbb{E}[T] - \mu \right) = 0, \quad \beta \geq 0, \]  

(7.55)

\[ \phi \left( \sum_{c=1}^{\infty} p_T(c) - 1 \right) = 0, \]  

(7.56)

where \( \beta, \lambda_t, \psi_t, \) and \( \phi \) are the Lagrange multipliers for the constraints \( \mathbb{E}[T] \leq \mu, p(t) \geq 0, \sum_{c=1}^{t} p_T(c) \leq 1 - (1-q_h)^t \), and \( \sum_{c=1}^{\infty} p_T(c) = 1 \), respectively.

In order to have \( p(t) = 0 \) for some \( t \), we need the exponent term in (7.52) to go to \(-\infty\). This makes \( \lambda_t \) in the expression of \( p(t) \) redundant due to (7.53). Hence, we assign \( \lambda_t = 0 \) for all \( t \), and obtain

\[ p^*(t) = \Lambda \exp \left( -\beta t - \Gamma t \sum_{n=1}^{t} \psi_n \right), \]  

(7.57)
where we have defined $\Lambda = e^{-\phi - 1}$. We find $\Lambda$ from (7.56) for all $\beta \geq 0$ and $\psi_i$ as

$$\Lambda = \left( \sum_{t=1}^{\infty} e^{-\beta t - \Gamma t - \sum_{n=1}^{t} \psi_n} \right)^{-1}, \quad (7.58)$$

which, together with (7.57), gives us a class of distributions with parameters $\psi_i$ and $\beta$. In addition, from (7.54), we know that $\psi_t$ is positive only when the constraint in (7.49) is satisfied with equality. As a result, for each $\mu$, we can find the optimal distribution $p^*(t)$ numerically by searching the class of distributions in (7.57) for the optimal $\psi_t$ and $\beta$ satisfying the above conditions.

7.6 Achievable Rates for the BEHC

In this section, we propose two choices for the auxiliary random variable $U$ and the mapping $v(u, z)$ in (7.12) and find lower bounds on the timing channel capacity and hence the BEHC capacity.

7.6.1 Modulo Encoding with a Finite Cardinality Auxiliary

Let $U$ be distributed over the finite support set $\{0, 1, \ldots, N-1\}$, where $N$ is a parameter to be optimized. We choose the mapping

$$v(U, Z) = (U - Z \mod N) + 1, \quad (7.59)$$
which gives a channel input $V = v(U, Z)$ in $\{1, 2, \ldots, N\}$. The output of the timing channel becomes $T = V + Z = (U - Z \mod N) + 1 + Z$. The decoder calculates

$$T' = (T - 1 \mod N) = ((U - Z \mod N) + Z \mod N) \tag{7.60}$$

$$= U \mod N = U, \tag{7.61}$$

and therefore perfectly recovers $U$ in each channel use. Hence, the achievable rate for this scheme with parameter $N$ is

$$R_A^{(N)} = \max_{p(u), U \in \{0, \ldots, N-1\}} \frac{H(U)}{\mathbb{E}[V + Z]}. \tag{7.62}$$

We then find the best rate achievable with this scheme by optimizing over $N$ as

$$R_A^{\text{mod}} = \max_N R_A^{(N)}. \tag{7.63}$$

This encoding scheme has the following interpretation for the BEHC: Consider that after each instance of $X_i = 1$, future channel uses are indexed cyclically with the numbers $\{0, 1, \ldots, N-1\}$, as illustrated in Figure 7.4 for $N = 4$. These indices are known to both the encoder and the decoder since the channel is noiseless. The encoder can then convey any symbol $U \in \{0, 1, \ldots, N - 1\}$ to the decoder by sending a 1 in a channel use indexed with $U$. This is performed at the earliest possible such channel use in which the required energy is available. For example, $U_1 = 2$ in the figure is conveyed in the first channel use indexed with a 2 (in the first frame of $N$ channel uses) as the energy becomes available for that transmission. However, $U_2 = 1$ in the figure is conveyed in
the second channel use indexed with a 1 (in the second frame of $N$ channel uses), since energy is not yet harvested in the first channel use indexed by a 1 (in the first frame of $N$ channel uses). As such, in this coding scheme, the encoder partitions future channel uses into frames of length $N$, and uses the earliest feasible frame to convey its symbol $U_k$ to the decoder.

This encoding scheme resembles the idea of concentration proposed by Willems in [113, 114] for Gaussian channels with causal state information. In particular, part of the channel input in [113, 114] is used to concentrate the channel state onto a set of values so that it can be decoded and eliminated at the decoder. Here, by waiting for the next frame of length $N$ when necessary, the effective state $Z_k$ is concentrated onto the lattice of the integer multiples of $N$. The concentrated state is then removed by the decoder with the modulo operation when calculating $T'$. Hence, this encoding scheme can also be interpreted as lattice coding in the timing channel.
We next show that the modulo encoding scheme proposed in Section 7.6.1 is asymptotically optimal as the harvest rate $q_h \to 0$. We establish this by comparing the achievable rate of the modulo encoding scheme in (7.62)-(7.63) with the genie-aided upper bound in (7.23).

**Theorem 7.3.** The modulo encoding scheme for the timing channel with auxiliary $U \in \{0, 1, \ldots, N-1\}$ and the channel input given in (7.59) is asymptotically optimal as energy harvest rate $q_h \to 0$.

**Proof:** We show that the upper bound $C_{UB}^{\text{genie}}$ and the achievable rate $R_A^{\text{mod}}$ scale with the same rate as $q_h$ goes to zero, i.e.,

$$\lim_{q_h \to 0} \frac{C_{UB}}{R_A} = 1. \quad (7.64)$$

For fixed $q_h$, the problem in (7.23) is convex since the objective is concave in $q_u$. Therefore, the optimal $q_u^*$ solving (7.23) is the solution of

$$\frac{q_h (\log(1 - q_u^*) - q_h \log(q_u^*))}{(q_u^* + q_h - q_u^* q_h)^2} = 0, \quad (7.65)$$

which reduces to

$$q_h = \frac{\log(1 - q_u^*)}{\log(q_u^*)}, \quad (7.66)$$
for $q_h > 0$. Consequently, there exists an optimal $0 < q_u^* \leq 0.5$ for all harvest rates $0 < q_h \leq 1$, which approaches zero with $q_h$, i.e.,

$$\lim_{q_h \to 0} q_u^* = 0. \quad (7.67)$$

We choose the parameters of the encoding scheme as $N = \left\lceil \frac{1}{q_u^*} \right\rceil$, and $p(u) = 1/N$ for $0 \leq u \leq N - 1$, i.e., $U$ is uniformly distributed. Note that $q_u^* \leq 0.5$ implies $N \geq 2$. Since $U$ is uniform and independent of $Z$, from (7.59), we observe that $V$ is distributed uniformly on $\{1, 2, \ldots, N\}$. This gives $\mathbb{E}[V] = (N + 1)/2$, and the achievable rate for this scheme becomes

$$R_A^{\text{mod}} = \frac{H(U)}{\mathbb{E}[V] + \mathbb{E}[Z]} = \frac{\log(N)}{N+1} + \frac{1-q_h}{2q_h} \geq \frac{q_h \log(N)}{Nq_h + 1 - q_h}, \quad (7.68)$$

where $\mathbb{E}[Z] = (1 - q_h)/q_h$. Observing that the last term in (7.68) is increasing in $N$ within the interval $\left[\frac{1}{q_u^*}, \frac{1}{\lceil q_u^* \rceil} \right]$, we further lower bound $R_A^{\text{mod}}$ as

$$R_A^{\text{mod}} \geq \frac{q_h \log(N)}{Nq_h + 1 - q_h} \geq \frac{-q_h q_u^* \log(q_u^*)}{q_h + q_u^*(1 - q_h)} = \bar{R}_A, \quad (7.69)$$

and upper bound the left hand side of (7.64) as

$$\lim_{q_h \to 0} \frac{C_{\text{genie}}^{\text{UB}}}{R_A^{\text{mod}}} \leq \lim_{q_h \to 0} \frac{C_{\text{UB}}^{\text{genie}}}{R_A} \quad (7.70)$$

$$= \lim_{q_h \to 0} \frac{q_h H(q_u^*)}{q_h + q_u^*(1 - q_h)} \cdot \frac{q_h + q_u^*(1 - q_h)}{-q_h q_u^* \log(q_u^*)} \quad (7.71)$$
\[ q_u^* \rightarrow 0 \Rightarrow \lim_{q_u^* \rightarrow 0} \frac{(1 - q_u^*) \log(1 - q_u^*)}{q_u^* \log(q_u^*)} = 1. \] (7.72)

Since \( C_{UB}^{\text{genie}} \geq R_A^{\text{mod}} \) by definition, this proves (7.64) and thus the theorem. ■

Theorem 7.3 states that as \( q_h \rightarrow 0 \), the capacity achieving encoding scheme approaches a uniformly distributed \( U \) over \( \{0, \ldots, N-1\} \), where \( N \rightarrow \infty \). This gives us a simple and asymptotically optimal encoding scheme for scenarios with very low energy harvesting rates.

### 7.6.3 Extended Modulo Encoding

To improve the rates achievable with modulo encoding of Section 7.6.1, we propose an extended version of the scheme with \( U \in \{0, 1, \ldots\} \) and

\[
v(U, Z) = \begin{cases} 
U - Z + 1, & U \geq Z, \\
(U - Z \mod N) + 1, & U < Z.
\end{cases}
\] (7.73)

The interpretation of this encoding scheme for the BEHC is given in Figure 7.5 for \( N = 4 \). Unlike modulo encoding, we index channel uses with \( \{0, 1, \ldots\} \) in this case. If the required energy is harvested by the channel use indexed with \( U_k \), then the encoder sends a 1 in that channel use, as is the case for \( U_1 \) in the figure. However, if the intended channel use is missed due to lack of energy, the encoder sends a 1 within \( N \) channel uses after harvesting energy, such that the channel index and \( U_k \) are equal in modulo \( N \). An
example is $U_2$ in the figure, where the channel index and $U_2$ are equal in modulo $N$, i.e.,

$$(T - 1) \mod N = U \mod N.$$ (7.74)

The achievable rate for this scheme is calculated by solving

$$R_A^{ext} = \max_N \max_{p(u), U \in \{0, 1, \ldots\}} \frac{I(U; Y)}{\mathbb{E}[V + Z]}$$ (7.75)

numerically by searching distributions of $U$. Although this problem is more difficult than that in (7.63), it is more tractable than (7.12) since the function $v(U, Z)$ is fixed.

We note that this scheme is an extended version of the modulo encoding scheme in Section 7.6.1, where $U$ is not restricted to be within $[0, N - 1]$. Therefore, the extended modulo scheme also includes the modulo scheme as a special case when $p(u) = 0$ for $u \geq N$. In fact, this scheme can be interpreted as a combination of modulo encoding and a best effort encoding scheme where the closest feasible symbol is transmitted. As an example, consider two random variables $Q_1 \in \{0, 1, \ldots, N-1\}$ and $Q_2 \in \{0, 1, \ldots\}$, and
let $U = Q_1 + Q_2 N$. Then, the $Q_1$ component is always perfectly recovered at the decoder as $(T - 1) \mod N$, as in modulo encoding. On the other hand, the $Q_2$ component is estimated as $\lfloor (T - 1)/N \rfloor$, which is as close to $Q_2$ as can be given $Z_k$.

As a final remark, we note that the Shannon strategies that consider only the current state, i.e., those presented in Section 7.3, can also be represented in the timing channel. For example, if the binary Shannon strategies are chosen i.i.d. with $\Pr[U = (0,1)] = q_u$, then a geometric distributed timing input $V$ with parameter $q_u$ yields the same channel input distribution and thus the same rate. Similarly, if binary Shannon strategies are chosen by a first order Markov process, an i.i.d. timing input strategy $U$ that yields the same input distribution can be constructed. Hence, encoding schemes for the timing channel include the Shannon strategy schemes of Section 7.3. However, for codebooks generated with higher order Markov processes, it is necessary to have timing auxiliary sequences $U^n$ with memory, and a function $v_k(U_k, Z^k)$ that utilizes the history of the states.

\section*{7.7 Capacity with Infinite-Sized Battery and No Battery}

For the purposes of comparison, in this section, we present two extreme cases, the case of no energy storage, and the case of infinite-sized energy storage.

\subsection*{7.7.1 Capacity with No Battery}

We first consider an encoder without energy storage capability. That is, we allow a non-zero channel input $X_i = 1$ only if energy is harvested within that channel use, i.e., $E_i = 1$. We note that this is slightly different than the \textit{transmit first} model described in
Section 7.2, where the channel input is sent before energy is harvested in each channel use. In contrast, here we consider a *harvest first* model. For this model, $E_i$ can be considered as an i.i.d. channel state known at the encoder [72], for which the capacity is given in (7.11). Using the Shannon strategies $U_1 = (0, 0)$ and $U_2 = (0, 1)$, with $\Pr[U_2] = q_u$, the capacity in this case becomes

$$C_{ZS} = \max_{q_u} H_2(q_u q_h) - q_u H_2(q_h), \quad (7.76)$$

where $H_2(q_u)$ is the binary entropy function.

### 7.7.2 Capacity with Infinite-Sized Battery

Next, we consider the case with an infinite-sized battery at the encoder. Reference [73] studies the Gaussian counterpart of this channel, showing that the save-and-transmit scheme is optimal. A similar argument applies for the binary case, implying that a rate of $H(X)$ can be achieved, where $X$ is constrained as $\mathbb{E}[X] \leq q_h$. Hence, the capacity of the channel with an infinite-sized storage is

$$C_{IS} = \begin{cases} H_2(q_h), & q_h \leq \frac{1}{2}, \\ 1, & q_h > \frac{1}{2}. \end{cases} \quad (7.77)$$

### 7.8 Extension to the Ternary Channel

The equivalence of the energy harvesting channel and the timing channel extends beyond binary input channels. As an example, in this section, we present results for
the ternary energy harvesting channel (TEHC). The TEHC has three input and output
symbols, \( X, Y \in \{-1, 0, 1\} \), and both \( X = -1 \) and \( X = 1 \) require one unit of energy to
be transmitted. This extension can further be generalized to \( K \)-ary channels, with each
symbol consuming either 0 or 1 unit of energy.

7.8.1 Achievable Rates with Shannon Strategies

We first consider achievable rates with Shannon strategies. As in the BEHC case,
we only have two states, \( S_i \in \{0, 1\} \). A strategy \( U \) is in the form \( U = (X, X') \), where
\( U(0) = X \) and \( U(1) = X' \). Note that \( X = 1 \) or \( X = -1 \) is possible only when \( S = 1 \),
and thus we only have three feasible strategies, namely \((0, 0)\), \((0, -1)\) and \((0, 1)\).

We first consider codebooks generated by choosing \( U_i \) i.i.d. for each codeword and
channel use. Let the probability of choosing \( U_i = (0, -1) \) and \( U_i = (0, 1) \) be \( q_{u,2} \) and
\( q_{u,3} \), respectively, for all \( i \) and all codewords. First, note that this construction yields
an ergodic battery state, with the transition probabilities

\[
\Pr[S_{i+1} = 1 | S_i = 0] = q_h, \quad \Pr[S_{i+1} = 0 | S_i = 1] = (q_{u,2} + q_{u,3})(1 - q_h),
\]

yielding the stationary probability

\[
\Pr[S = 1] = \frac{q_h}{q_{u,2} + q_{u,3} + q_h - (q_{u,2} + q_{u,3})q_h}
\]

Note that the stationary probability is a function of \( q_{u,2} + q_{u,3} \), rather than \( q_{u,2} \) and
\( q_{u,3} \) individually. Denoting \( U = (0, 0) \) as 0, \( U = (0, -1) \) as -1 and \( U = (0, 1) \) as 1, the
channel in the case of naïve Shannon strategies is expressed as

\[ p(y|u) = \Pr[S = 1] \delta(y - u) + \Pr[S = 0] \delta(y). \tag{7.80} \]

The best achievable rate with this scheme is given by

\[ R_{NIID} = \max_{q_{u,2}, q_{u,3} \in [0,1]} H(Y) - (q_{u,2} + p_3) H_2 \left( \frac{q_h}{q_{u,2} + q_{u,3} + q_h - (q_{u,2} + q_{u,3}) q_h} \right), \tag{7.81} \]

where \( H_2(q_u) \) is the binary entropy function. We observe that whenever \( q_{u,2} + q_{u,3} \) is kept constant, the channel in (7.80) and the term \( (q_{u,2} + q_{u,3}) H_2 \left( \frac{q_h}{q_{u,2} + q_{u,3} + q_h - (q_{u,2} + q_{u,3}) q_h} \right) \) in (7.81) remain unchanged. On the other hand, \( H(Y) \) is a concave function of the distribution of \( Y \). Hence, by Jensen’s inequality, when we fix \( q_{u,2} + q_{u,3} = 2q_u \), selecting \( q_{u,2} = q_{u,3} = q_u \) yields the highest rate in (7.81). Therefore, the optimum selection is \( q_{u,2} = q_{u,3} = q_u \), and we obtain the following simpler rate expression:

\[ R_{NIID} = \max_{q_u \in [0,1]} H(Y) - 2q_u H_2 \left( \frac{q_h}{2q_u + q_h - 2q_u q_h} \right). \tag{7.82} \]

Similar to the BEHC case, the decoder can exploit the memory by using the \( n \)-letter joint probability \( p(u^n, y^n) \) for the channel and obtain the optimal i.i.d. Shannon strategy (OID). This achieves the rate

\[ R_{OID} = \max_{q_u \in [0,1]} \lim_{n \to \infty} \frac{1}{n} I(U^n; Y^n), \tag{7.83} \]
where again \( q_{u,2} = q_{u,3} = q_u \), the optimality of which follows from similar arguments as before. Calculating the limit of the \( n \)-letter mutual information rate \( \frac{1}{n} I(U^n; Y^n) \) is possible by using the algorithm in [7]. Moreover, we can further improve such achievable rates by constructing codebooks with Markovian Shannon strategies. We evaluate and compare these achievable rates in Section 7.9.

### 7.8.2 Timing Equivalence and Related Bounds

In order to find a timing equivalent for the TEHC, we represent the channel output \( Y^n \in \{-1, 0, 1\} \) with two sequences, \( T_m \in \{1, 2, \ldots\}^m \) and \( L_m \in \{-1, 1\}^m \). Here, \( T_k \) is the duration between the \((k-1)\)st and the \( k \)th non-zero outputs in \( Y^n \), and \( L_k \) is the sign of the \( k \)th non-zero output. As in the binary case, \((T_m, L_m)\) and \( Y^n \) are different and complete representations of the same channel output, and are therefore equivalent.

The timing equivalent of the TEHC consists of two parallel channels, namely a timing channel and a sign channel, expressed as

\[
T_k = V_k + Z_k, \quad L_k = Q_k, \quad (7.84)
\]

where \( Q_k \) is the sign of the \( k \)th non-zero input. Extending Lemma 7.1 to include the sign channel, we observe that the sum-capacity of the two independent channels in (7.84) is equal to the capacity of the TEHC. The capacity of the noiseless sign channel is \( \log_2 |L| = 1 \) bit per channel use. One use of the sign channel also requires \( \mathbb{E}[T] \) uses of the TEHC on average. Considering this, the capacity of the TEHC is given in the following theorem.
Theorem 7.4. The capacity of the ternary energy harvesting channel is

$$C_{TEHC} = \max_{p(u), v(u,s)} \frac{I(U; T) + 1}{E[T]}.$$  \hspace{1cm} (7.85)

This result is parallel to those in reference [5] on queues with information-bearing packets. In the timing equivalent of the TEHC, each non-zero channel input can be interpreted as a packet bearing one bit of information. Hence, as in [5], coding for the two channels in (7.84) is performed independently, yielding the capacity in (7.85).

The upper and lower bounds for the BEHC immediately extend to the TEHC, since the capacity for the sign channel is simple. The two upper bounds on $C_{TEHC}$ become

$$C_{UB}^{\text{genie}} = \max_{q_u \geq 0} \frac{H_2(q_u)/q_u + 1}{q_u + \frac{1-q_h}{q_h}} = \max_{q_u \geq 0} \frac{q_h H_2(q_u) + q_u q_h}{q_h + q_u (1-q_h)},$$  \hspace{1cm} (7.86)

$$C_{UB}^{\text{leakage}} = \max_{p_T(t) \in \mathcal{P}} \frac{H(T) - \sum_{t=1}^{\infty} \frac{H_h((1-q_h)^t)}{1-(1-q_h)^t} p(t) + 1}{E[T]},$$  \hspace{1cm} (7.87)

where $\mathcal{P}$ is given in (7.39), and the two achievable rates become

$$R_A^{\text{mod}} = \max_N \max_{p(u), U \in \{0,1,...,N-1\}} \frac{H(U) + 1}{E[V + Z]},$$  \hspace{1cm} (7.88)

$$R_A^{\text{ext}} = \max_N \max_{p(u), U \in \{0,1,...\}} \frac{I(U; Y) + 1}{E[V + Z]},$$  \hspace{1cm} (7.89)

with $v(U,Z)$ is given in (7.59) for the modulo encoding scheme, and in (7.73) for the extended modulo encoding scheme.
7.8.3 Capacity with Infinite-Sized Battery and No Battery

We first consider the capacity with zero energy storage. That is, we allow a non-zero channel input \( X_i = 1 \) or \( X_i = -1 \) only when energy is harvested in that channel use, i.e., \( E_i = 1 \). Using the Shannon strategies \( U_1 = (0, 0), U_2 = (0, -1) \) and \( U_3 = (0, 1) \), with \( \Pr[U_2] = q_{u,2} \) and \( \Pr[U_3] = q_{u,3} \), the capacity becomes

\[
C_{ZS} = \max_{q_{u,2}, q_{u,3}} H(Y) - (q_{u,2} + q_{u,3})H_2(q_h),
\]

where \( Y \) has the ternary distribution \((q_{u,2}q_h, 1 - (q_{u,2} + q_{u,3})q_h, q_{u,3}q_h)\) and \( H_2(q_u) \) is the binary entropy function. Since \( H(Y) \) is a concave function of the distribution of \( Y \), when \( q_{u,2} + q_{u,3} \) is fixed, by Jensen’s inequality \( q_u = q_{u,2} = q_{u,3} \) is the optimal selection. Therefore, we get

\[
C_{ZS} = \max_{q_u} H(Y) - 2q_u H_2(q_h),
\]

where \( Y \) has the distribution \((q_uq_h, 1 - 2q_uq_h, q_uq_h)\).

Next, we consider the capacity with an infinite-sized battery. Similar to the binary case, a rate of \( H(X) \) can be achieved, where \( X \) is a ternary variable that is constrained as \( \mathbb{E}[X^2] \leq q_h \). Hence, the capacity of the channel with infinite-sized storage is

\[
C_{IS} = \begin{cases} 
H(q_h/2, 1 - q_h, q_h/2), & q_h \leq \frac{2}{3}, \\
\log_2(3), & q_h > \frac{2}{3}, 
\end{cases}
\]
where $H(q_h/2, 1 - q_h, q_h/2)$ denotes the entropy of the ternary distribution $(q_h/2, 1 - q_h, q_h/2)$.

### 7.9 Numerical Results

In this section, we compare the timing channel upper bounds and achievable rates in Sections 7.5 and 7.6, Shannon strategy based achievable rates in Section 7.3, and capacity results for extreme cases in Section 7.7 for the BEHC, followed by the results in Section 7.8 for the TEHC. The upper bounds and achievable rates for the BEHC evaluated at $q_h \in \{0, 0.1, \ldots, 1\}$ are given in Table 7.1.

<table>
<thead>
<tr>
<th>Harvest prob. ($q_h$)</th>
<th>$C_{UB}^{\text{genie}}$</th>
<th>$C_{UB}^{\text{leakage}}$</th>
<th>$R_A^{ext}$</th>
<th>$R_A^{mod}$</th>
<th>$R_{M2}$</th>
<th>$R_{M1}$</th>
<th>$R_{OIID}$</th>
</tr>
</thead>
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<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
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<td>0.2600</td>
<td>0.2516</td>
<td>0.2317</td>
<td>0.2313</td>
<td>0.2199</td>
<td>0.2188</td>
<td>0.2178</td>
</tr>
<tr>
<td>0.2</td>
<td>0.4056</td>
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<td>0.3529</td>
<td>0.3415</td>
<td>0.3384</td>
<td>0.3351</td>
</tr>
<tr>
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<td>0.4487</td>
<td>0.4451</td>
<td>0.4364</td>
<td>0.4320</td>
<td>0.4301</td>
</tr>
<tr>
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<td>0.5485</td>
<td>0.5297</td>
<td>0.5230</td>
<td>0.5178</td>
<td>0.5130</td>
<td>0.5115</td>
</tr>
<tr>
<td>0.5</td>
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<td>0.6164</td>
<td>0.6033</td>
<td>0.5914</td>
<td>0.5890</td>
<td>0.5880</td>
<td>0.5861</td>
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<td>0.6729</td>
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<td>0.6617</td>
<td>0.6591</td>
<td>0.6555</td>
</tr>
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<td>0.7403</td>
<td>0.7205</td>
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<tr>
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<td>0.8808</td>
<td>0.8807</td>
<td>0.8797</td>
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<tr>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 7.1: Upper bounds and achievable rates for the BEHC in bits/ch. use.

Figure 7.6 shows the genie upper bound $C_{UB}^{\text{genie}}$ in (7.23), the leakage upper bound $C_{UB}^{\text{leakage}}$ in (7.38), the modulo encoding achievable rate $R_A^{mod}$ in (7.63), and the extended encoding achievable rate $R_A^{ext}$ in (7.75) in comparison with the zero storage capacity $C_{ZS}$ in (7.76) and the infinite-sized storage capacity $C_{IS}$ in (7.77). All of these
quantities are zero at \( q_h = 0 \), because in this case, no energy is harvested, and thus no communication is possible. Moreover, they are all equal to 1 at \( q_h = 1 \), because in this case, the battery is always full, and the channel is equivalent to a binary noiseless discrete memoryless channel without any energy constraints.

From Figure 7.6, we first observe that the leakage upper bound, \( C_{UB}^{\text{leakage}} \), and the achievable rate with the extended encoding scheme, \( R_{A}^{\text{ext}} \), provide a gap smaller than 0.03 bits per channel use for the capacity, for all harvesting rates \( q_h \). For small \( q_h \), both upper bounds and both achievable rates get very close, as expected from the asymptotic optimality of \( R_{A}^{\text{mod}} \) as \( q_h \to 0 \). On the other hand, for large \( q_h \), we observe that the genie upper bound \( C_{UB}^{\text{genie}} \) is looser compared to the leakage upper bound \( C_{UB}^{\text{leakage}} \). This implies that the correlation between the harvesting process and the channel outputs is high in this regime. Finally, we note that although the gap between the infinite storage capacity \( C_{IS} \) and the zero storage capacity \( C_{ZS} \) is large, a unit-sized energy storage device recovers a significant amount of this difference. This demonstrates that even the smallest sized energy storage device can be very beneficial in energy harvesting communication systems.

We next compare the modulo and extended achievable rates, \( R_{A}^{\text{mod}} \) and \( R_{A}^{\text{ext}} \), with the Shannon strategy based achievable rates described in Section 7.3. We remind that the schemes in Section 7.3, which are also studied in [57], only observe the instantaneous battery state in each channel use. Thus, we have simple Shannon strategies, but we allow a Markovian dependence over time in the codewords. Figure 7.7 shows \( R_{A}^{\text{mod}} \) and \( R_{A}^{\text{ext}} \) along with the optimal i.i.d. Shannon strategy rate \( R_{\text{OIID}} \) in (7.7) and the optimal 1st and 2nd order Markov Shannon strategy rates \( R_{M1} \) and \( R_{M2} \). We observe that
Fig. 7.6: Upper bounds and achievable rates for the BEHC.
although $R_A^{\text{mod}}$ outperforms $R_{\text{OIID}}$ for all $q_h$, the 1st and 2nd order Markov Shannon strategies outperform $R_A^{\text{mod}}$ for large $q_h$, as seen in the inset in Figure 7.7. However, the extended encoding rate $R_A^{\text{ext}}$ outperforms both $R_{M1}$ and $R_{M2}$, for all harvesting rates $q_h$. These can also be observed partially (for harvesting rates $q_h \in \{0, 0.1, \ldots, 1\}$) from Table 7.1. We note that the increase in the achievable rate with the Markov order of the input seems to be small. However, due to the exponential increase in the computational complexity with the Markov order, it was not tractable to simulate and compare inputs of higher Markov orders, i.e., 3rd and higher Markov orders.

A parameter of interest is the optimal frame length $N$ for the modulo encoding scheme in Section 7.6.1, which we plot in Figure 7.8. The larger $N$ is, the larger the support of $U$ is, and more information can be packed into a single use of the timing channel. However, as $N$ increases, so does $\mathbb{E}[T]$, and thus each symbol takes more time, and more harvested energy is potentially wasted. Thus, for small harvest rates, e.g., $q_h \leq 0.7$, optimal $N$ decreases with increasing $q_h$ so that less harvested energy is wasted. On the other hand, for $q_h > 0.7$, the node is receiving excessive energy, and thus the optimal $N$ increases to pack more information in each timing channel use.

Finally, we present the upper bounds and the achievable rates for the ternary channel, given in (7.86)-(7.89), together with the capacities for the no battery and infinite-sized battery cases $C_{ZS}$ and $C_{IS}$ given in (7.91)-(7.92), in Figure 7.9. We also compare the achievable rates in Section 7.6 with the optimal i.i.d. and the 1st order Markov Shannon strategies for the ternary channel in Figure 7.10. Note that in the ternary channel, the $q_h = 1$ case corresponds to a ternary noiseless discrete memoryless channel, and thus has a capacity of $\log_2(3) = 1.58$ bits per channel use. We observe that similar to the
Fig. 7.7: Achievable rates with timing encoding compared with instantaneous Shannon strategies for the BEHC.
Fig. 7.8: Optimal frame length $N$ for the modulo encoding scheme.
binary case, the leakage upper bound $C_{UB}^{\text{leakage}}$ and the extended encoding rate $R_A^{ext}$ approximate the capacity within 0.05 bits per channel use, and the extended encoding rate outperforms the i.i.d. and the 1st order Markov Shannon strategies, for all harvesting rates $q_h$.

7.10 Chapter Summary

We considered a binary energy harvesting communication channel with a finite-sized battery at the transmitter. In this model, the channel input is constrained by the available energy at each channel use, which is driven by an external energy harvesting process, the size of the battery, and the previous channel inputs. We found an equivalent representation for this channel based on the timings of the symbols, and determined a single letter capacity expression for the resulting equivalent timing channel via an auxiliary random variable. We proposed achievable rates based on certain selections of this auxiliary random variable which resemble lattice coding for the timing channel. We developed upper bounds for the capacity by using a genie-aided method, and also by quantifying the leakage rate of the state information to the receiver. We showed that the proposed achievable rates are asymptotically capacity achieving for small energy harvesting rates. We extended the results to the case of ternary channel inputs. Our achievable rates provided the capacity of the binary channel within 0.03 bits/channel use, the ternary channel within 0.05 bits/channel use, and outperformed basic Shannon strategies that only consider instantaneous battery states.
Fig. 7.9: Upper bounds and achievable rates for the TEHC.
Fig. 7.10: Achievable rates with timing encoding compared with instantaneous Shannon strategies for the TEHC.
Chapter 8

State Amplification and State Masking in the BEHC

8.1 Introduction

In the energy harvesting channel introduced in Chapter 7, the harvested energy is a random process that is revealed to the transmitter causally throughout the transmission. Since the transmitted symbols are constrained by the energy available at the encoder, the decoder obtains some information about the energy harvesting process in addition to the intended message. Depending on the application, it may be desirable to maximize or minimize this side information about the harvested energy, e.g., it may facilitate smart scheduling based on energy harvesting rates, but may also reveal the position or energy source of a wireless node. These scenarios result in the problems of state amplification [49, 17] and state masking [59], respectively.

The state amplification problem, i.e., sending information about system state along with the message, is first studied in [49]. This reference quantifies the side information revealed to the receiver as the mutual information between the output sequence and the state sequence. This problem is later considered in terms of the distortion in channel state estimation in [17]. The state masking counterpart, i.e., concealing channel state as much as possible while sending a message, is introduced in [59] for both
causal and non-causal state information at the encoder. In an energy harvesting setting, the state amplification problem is first studied in [74] for an AWGN channel with infinite-sized battery and no battery at the encoder.

In this chapter, we evaluate state amplification and state masking regions, which outline the set of achievable message rates and state revealing rates for the binary energy harvesting channel. Our goal is to investigate the trade-off between the energy harvest information that the receiver can extract from the communication and the message transmission rate. We consider binary energy harvests as the state of the channel, where each energy unit corresponds to the energy required to send a 1 through the binary channel once. We consider the no battery case and the infinite-sized battery case, for which the channel capacities are known, and the unit-sized battery case, for which we derive inner bounds using the achievable rates in Chapter 7. This work was presented in [96].

8.2 Channel Model and Problem Formulation

We consider the binary energy harvesting channel in Figure 7.1, with energy harvests $E_i$ in time slot $i$. However, in this chapter, we lift the noiseless channel assumption for the no battery and infinite-sized battery cases, and consider a binary symmetric communication channel (BSC) with output

$$Y_i = \begin{cases} 
X_i, & \text{w.p. } 1 - q_e, \\
1 - X_i, & \text{w.p. } q_e, 
\end{cases} \quad (8.1)$$
where $q_c \leq \frac{1}{2}$ is the crossover probability of the communication channel. The battery dynamics are as in (7.1).

We are interested in how much the decoder can learn about the energy harvesting process $E^n$ by observing $Y^n$. From the decoder’s perspective, there are $2^{H(E^n)}$ possible energy harvest sequences, since $2^{H(E^n)}$ is the size of the typical set for $E^n$. Upon receiving $Y^n$, the decoder can reduce the size of this list to those that are possible given $Y^n$, which has a size of $2^{H(E^n | Y^n)}$. Hence, the reduction in the entropy of $E^n$ for the decoder can be expressed as

$$
\Delta = \frac{1}{n} \left( H(E^n) - H(E^n | Y^n) \right) = \frac{1}{n} I(E^n; Y^n). \quad (8.2)
$$

Note that the value of $\Delta$ is related strongly to the encoding scheme adopted by the encoder. For example, the encoder can choose to send $X_i = 0$ for all $i$, thus achieving no message rate, but obtaining $\Delta = 0$. On the other extreme, the encoder can choose $X_i = E_i$, once again achieving no message rate, but obtaining $\Delta = H(E^n)$. For a non-zero rate, different encoding schemes achieving the same rate may yield different values for $\Delta$.

### 8.2.1 The State Amplification Problem

In the state amplification problem, the encoder intends the decoder to obtain as much information as possible about the energy harvesting process $E^n$, i.e., maximize $\Delta$, while reliably conveying a message with some rate $R$. This problem is first considered in [49], where the achievable message rates and state amplification rates are shown to
satisfy

\[ R \leq I(U;Y), \quad (8.3) \]
\[ \Delta \leq H(S), \quad (8.4) \]
\[ R + \Delta \leq I(X,S;Y), \quad (8.5) \]

for a memoryless channel with state \( S \) known causally at the transmitter. Here, \( U \) is an auxiliary random variable yielding the joint distribution \( p(s)p(u)p(x|u,s)p(y|x,s) \).

### 8.2.2 The State Masking Problem

The state masking problem, studied first in [59], finds a lower bound on \( \Delta \) for any given message rate \( R \). Hence, it indicates the minimum amount of information \( \Delta \) that must be revealed to the decoder about the state in order to achieve some rate \( R \).

The achievable \((R, \Delta)\) regions are obtained by the union of the regions

\[ R \leq I(U;Y), \quad (8.6) \]
\[ \Delta \geq I(S;Y|U), \quad (8.7) \]

for causally available state at the encoder [59]. Note that (8.3) and (8.6) are identical, and (8.7) provides a lower bound on \( \Delta \) while (8.4) and (8.5) provide upper bounds.

In the remainder of this chapter, we consider the state amplification and state masking problems individually for the cases \( E_{\text{max}} = 0, E_{\text{max}} = \infty \), and \( E_{\text{max}} = 1 \).
8.3 No Battery Case: $E^{max} = 0$

For $E^{max} = 0$, the energy available at channel use $i$ is $E_i$, which is i.i.d., and therefore the state of the channel is memoryless. Hence, the results of [49] extend directly to this case. Given the two states, $E_i \in \{0, 1\}$, the two inputs $X_i \in \{0, 1\}$, and the restriction $X_i \leq E_i$, there are two input strategies. As in Section 7.3, we denote these strategies as $U = (0, 0)$ and $U = (0, 1)$, corresponding to always transmitting a zero and attempting to send a 1, respectively. For an encoding scheme with $Pr[U = (0, 1)] = q_u$, the exact $(R, \Delta)$ region for state amplification is obtained as

$$R \leq H(q_u q_h q_e) - pH(q_h q_e) - (1 - q_u)H(q_e), \quad (8.8)$$

$$\Delta \leq H(q_h), \quad (8.9)$$

$$R + \Delta \leq H(q_u q_h q_e) - H(q_e), \quad (8.10)$$

where $H(a)$ denotes the binary entropy function, and $q_u q_h = q_u (1 - q_h) + (1 - q_u)q_h$.

Next, we utilize the results of [59], and characterize the exact $(R, \Delta)$ region for state masking as (8.8) and

$$\Delta \geq q_u H(q_h q_e) - q_u H(q_e). \quad (8.11)$$

We remark that for a noiseless binary channel, i.e., $q_e = 0$, the bounds for $\Delta$ in (8.8)-(8.10) and (8.11) match only at

$$R = H(q_u q_h) - q_u H(q_h), \quad \Delta = q_u H(q_h). \quad (8.12)$$
8.4 Infinite-Sized Battery Case: $E^{max} = \infty$

We next consider the infinite-sized battery case, i.e., $E^{max} = \infty$. For this case, [73] showed that a save-and-transmit scheme achieves the AWGN capacity with average transmit power. This scheme first saves energy for a negligible duration of the transmission, and then encodes as if constrained by an average power constraint only. The save-and-transmit scheme can be extended to the noiseless binary channel as in Section 7.7.2, and to the binary symmetric channel, yielding the capacity

$$C_{BSC} = \begin{cases} 
H(q_h + q_e) - H(q_e), & q_h \leq \frac{1}{2}, \\
1 - H(q_e), & q_h > \frac{1}{2},
\end{cases} \quad (8.13)$$

with the channel input distribution $Pr[X = 1] = \min\left\{q_h, \frac{1}{2}\right\}$. Note that this is also the capacity of a BSC with an input constraint $E[X] \leq q_h$, as is the case in [74] for the AWGN channel. With this observation, we present the state amplification region for this channel in the following lemma.

**Lemma 8.1.** The exact $(R, \Delta)$ region for the binary EH channel with an infinite-sized battery at the transmitter satisfies

$$R + \Delta \leq C_{BSC}, \quad 0 \leq \Delta \leq H(q_h). \quad (8.14)$$

**Proof:** We first show the achievability of these $(R, \Delta)$ pairs. Clearly, the rate $R = C_{BSC}$ is achievable with the save-and-transmit scheme, which we assume to yield $\Delta = 0$. Furthermore, by compressing the $E^n$ sequence and sending it as a part of the message,
the encoder can trade any portion of the message rate $R$ with $\Delta$, provided that this portion does not exceed $H(q_h)$ for $q_h \leq 0.5$ and 1 for $q_h > 0.5$. Due to the causal availability of $E_i$, this is performed in a block Markov fashion. For the converse, we write

$$I(X^n;Y^n) = I(X^n, E^n, W; Y^n)$$

$$\geq I(E^n, W; Y^n)$$

$$= I(E^n; Y^n) + H(W) - H(E^n|Y^n, E^n)$$

$$\geq I(E^n; Y^n) + H(W) - H(\epsilon) - \epsilon \log(nR)$$

$$= n\Delta + nR - H(\epsilon) - \epsilon \log(nR),$$

where $W$ is the message and $\epsilon$ is the decoding error probability. Here, (8.15) follows from the Markov chain $(W, E^n) - X^n - Y^n$, (8.17) is due to the independence of $W$ and $E^n$, and (8.18) follows from Fano’s inequality. Hence, whenever the decoding error probability $\epsilon$ goes to zero as $n \to \infty$, we have

$$\Delta + R \leq \lim_{n \to \infty} \frac{1}{n} I(X^n; Y^n) \leq C_{BSC},$$

which concludes the converse.

For the masking problem, as emphasized in [74], since $(R, \Delta) = (C_{BSC}, 0)$ is achievable, perfect masking of the state $E^n_i$ is possible using the save-and-transmit scheme. Hence, we have $\Delta \geq 0$ as the masking lower bound.
8.5 Unit-Sized Battery Case: $E^{max} = 1$

So far in this chapter, we considered cases for which the channel capacity is known. Since this is not the case for $E^{max} = 1$, we are not able to determine the entire $(R, \Delta)$ region. In this section, we utilize the two encoding schemes in Sections 7.3 and 7.6.1 to find inner bounds on the $(R, \Delta)$ region for the binary noiseless channel with $q_e = 0$.

8.5.1 Instantaneous Shannon Strategies

We start with instantaneous Shannon strategies, described in Section 7.3, for which the best i.i.d. achievable rate is given in (7.7). To find the corresponding $\Delta$ for this encoding scheme, we first define the random variable

$$\psi_{i,j} = \begin{cases} 0, & E_k = 0, \ i \leq k < j, \\ 1, & \text{otherwise,} \end{cases}$$

which is an indicator of whether a unit of energy has arrived or not between the $i$th and $j$th channel uses. We then define the set $\Psi(u^n) = \{\psi_{i_1^1, i_2^1}, \psi_{i_2^2, i_3^2}, \ldots\}$ as the collection of mutually exclusive indicators $\psi_{i_k^{i_k+1}}$, where $i_1 = 1$, and $i_k, k = 2, 3, \ldots$ are the channel indices that satisfy $u_{i_k^n} = (0, 1)$. In other words, $\Psi(u^n)$ is the set of indicators that show whether energy is available or not for each attempt of sending a 1 given the strategy sequence $u^n$. We then write

$$H(E^n|Y^n) - H(E^n|Y^n, U^n) = I(E^n; U^n|Y^n) \leq H(U^n|Y^n)$$

(8.22)
\[
\leq H(W|Y^n) 
\]  \quad (8.24)

\[
\leq H(\epsilon) - \epsilon \log(nR),
\]  \quad (8.25)

where \(\epsilon\) is the probability of a decoding error. Here, (8.24) is due to \(U^n\) being a function of message \(W\), and (8.25) is due to Fano’s inequality. Hence, whenever the error probability \(\epsilon\) goes to zero as \(n \to \infty\), we have

\[
\lim_{n \to \infty} \frac{1}{n} H(E^n|Y^n) = \lim_{n \to \infty} \frac{1}{n} H(E^n|Y^n, U^n). 
\]  \quad (8.26)

With this observation, we write \(\Delta\) as

\[
\lim_{n \to \infty} \frac{1}{n} I(E^n; Y^n) = \lim_{n \to \infty} \frac{1}{n} \left( H(E^n) - H(E^n|Y^n, U^n) \right) 
\]  \quad (8.27)

\[
= \lim_{n \to \infty} \frac{1}{n} \left( H(E^n) - H(E^n|Y^n, U^n, \Psi(U^n)) \right) 
\]  \quad (8.28)

\[
= \lim_{n \to \infty} \frac{1}{n} \left( H(E^n) - H(E^n|\Psi(U^n), U^n) \right) 
\]  \quad (8.29)

\[
= \lim_{n \to \infty} \frac{1}{n} H(\Psi(U^n)|U^n). 
\]  \quad (8.30)

Here, (8.27) follows from (8.26), and (8.28) holds as \(\Psi(U^n)\) can be obtained from \(Y^n\) and \(U^n\). Similarly, (8.29) follows since \(Y^n\) can be obtained from \(U^n\) and \(\Psi(U^n)\). Finally, (8.30) holds since \(H(\Psi(U^n)|U^n, E^n) = 0\), and \(U^n\) is independent of \(E^n\).

What the series of equalities in (8.27)-(8.30) imply is that, observing \(Y^n\), the decoder learns the indicators \(\Psi(U^n)\) about \(E^n\), and nothing more. Note that the intervals \((i_k, i_{k+1})\) are disjoint, and therefore the elements of the set \(\Psi(u^n)\) are independent. Since \(Pr[\psi_i^j = 0] = (1-q_h)^{j-i}\) and \(u_i\) are generated i.i.d. with probability \(Pr[U = (0,1)] = q_u\),
(8.30) can be further simplified as

$$
\Delta = \lim_{n \to \infty} \frac{1}{n} \sum_{u} p(u^n) \sum_{k} H \left( (1 - q_h)^{d_k} \right),
$$

(8.31)

where $d_k = i_{k+1} - i_k$ denotes the number of channel uses between the $k$th and $(k+1)$th use of the channel. $u = (0, 1)$ in $u^n$, and is distributed i.i.d. geometric with parameter $q_u$. As $n \to \infty$, due to the law of large numbers, the set $\Psi(u^n)$ has $n q_u$ elements, yielding

$$
\Delta = \lim_{n \to \infty} \frac{1}{n} I(E^n; Y^n) = q_u^2 \sum_{d=1}^{\infty} (1 - q_u)^{d-1} H \left( (1 - q_h)^d \right).
$$

(8.32)

**Lemma 8.2.** For the i.i.d. encoding scheme with Shannon strategies in the noiseless channel, the decrease in entropy of $E^n$ upon observing $Y^n$, i.e., $\Delta$, is equal to (8.32).

### 8.5.2 Timing-Based Modulo Encoding

We next consider the timing channel-based encoding scheme of Section 7.6.1. For a frame length of $N$, this scheme uses Shannon strategies $U \in \{0, 1, \ldots, N-1\}$ to generate an i.i.d. codebook, and the decoder can obtain $U_i = T_i - 1 \mod N$ without error. For this encoding scheme, we first calculate $\Delta$ as a function of the strategy distribution $p(u)$. For a given output sequence $T^m = t^m$, we define $a_i = \sum_{j=1}^{i} t_j$. Then, the $i$th use of the timing channel lies on the $(a_i + 1)$th to $(a_i + t_i)$th uses of the binary channel. For this interval, the decoder can infer the following: For $t_i \leq N$, we have $z_i \leq t_i$ from (7.9). Using the definition in (8.21), this implies $\psi_{a_i}^{a_i+t_i} = 1$. Otherwise, for $t_i > N$, we have $z_i \geq t_i - N$ since $v_i \leq N$ by definition. This implies that $\psi_{a_i}^{a_i+t_i-N} = 0$ and...
Based on these two cases, we define the sets

\[
\bar{\Psi}_0(t^m) = \bigcup_{t_i > N} \psi_{a_i + t_i}^{a_i + t_i - N},
\]  

(8.33)

\[
\bar{\Psi}_1(t^m) = \left( \bigcup_{t_i \leq N} \psi_{a_i + t_i}^{a_i + t_i} \right) \cup \left( \bigcup_{t_i > N} \psi_{a_i + t_i}^{a_i + t_i - N} \right),
\]  

(8.34)

and write

\[
H(E^n|T^m) = \sum_{t^m} p(t^m) H(E^n|T^m = t^m)
\]  

(8.35)

\[
= \sum_{t^m} p(t^m) H(E^n|T^m = t^m, U^m = u^m, \bar{\Psi}_0(t^m) = 0, \bar{\Psi}_1(t^m) = 1)
\]  

(8.36)

\[
= \sum_{t^m} p(t^m) H(E^n|\bar{\Psi}_0(t^m) = 0, \bar{\Psi}_1(t^m) = 1),
\]  

(8.37)

where \(\bar{\Psi}_0(t^m) = 0\) denotes element-wise equality for all elements of the set \(\bar{\Psi}_0(t^m)\). Here, (8.36) holds since \(u^m\) can be obtained from \(t^m\), which also reveals that \(\bar{\Psi}_0(t^m) = 0\) and \(\bar{\Psi}_1(t^m) = 1\). Note that (8.37) is the entropy of \(E^n\) given that parts of \(E^n\) are zero and parts include at least one non-zero energy harvest. Calculating and averaging over \(t^m\), we get

\[
\frac{1}{n} I(E^n; Y^n) = H(q_h) - \frac{1}{\mathbb{E}[T]} \mathbb{E} \left[ \frac{dH(q_h) - H((1 - q_h)^d)}{1 - (1 - q_h)^d} \right],
\]  

(8.38)
where $d$ is the length of $\psi$ terms in $\bar{\Psi}_1(t^n)$, and is distributed as

$$d = \begin{cases} 
  k, & 1 \leq k < N, \text{ w.p. } p_U(k-1)(1-(1-q_h)^k), \\
  N, & \text{ w.p. } \sum_{u=0}^{N-1} p_U(u)(1-q_h)^{u+1}.
\end{cases}$$

(8.39)

**Lemma 8.3.** For the encoding scheme in [93], the decrease in entropy of $E^n$ upon observing $Y^n$, i.e., $\Delta$, for a specific auxiliary distribution $p_U(u)$ is equal to (8.38).

Finally, we obtain $(R, \Delta)$ pairs for this encoding scheme by exhaustively searching $p_U(u)$ and using (7.62)-(7.63) and (8.38).

### 8.6 Numerical Results

For comparison, we evaluate the state amplification and state masking regions for the $E_{max} = 0$ and $E_{max} = \infty$ cases, and the maximum and minimum achievable values of $\Delta$ with respect to some message rate $R$ for the $E_{max} = 1$ case, in a noiseless channel with $q_h = 0.5$. The state amplification results are plotted in Figure 8.1, and the state masking results are plotted in Figure 8.2.

In Figure 8.1, we observe that the instantaneous Shannon encoding scheme in Section 8.5.1 performs state amplification almost as good as the ideal case of Section 8.4 for low message rates. As message rate approaches the best achievable rate, state amplification is sacrificed. Moreover, we note that for the most part, instantaneous Shannon strategies of Section 7.3 provide more state information than the timing channel based encoding scheme of Section 7.6.1.
Fig. 8.1: The maximum $\Delta$ values with respect to message rate $R$, i.e., state amplification boundaries, for $q_h = 0.5$ and $q_e = 0$.

Fig. 8.2: The minimum $\Delta$ values with respect to message rate $R$, i.e., state masking boundaries, for $q_h = 0.5$ and $q_e = 0$. 
For the state masking problem in Figure 8.2, we observe that for low rates, $E^{\text{max}} = 1$ provides significantly better state masking compared to the $E^{\text{max}} = 0$ case. Similar to the state amplification case, the timing-based encoding scheme delivers less state information than instantaneous Shannon encoding, although this is desirable for the state masking case. Hence, we conclude that timing-based encoding outperforms instantaneous Shannon encoding in state masking, while the reverse is true in state amplification.

8.7 Chapter Summary

In this chapter, we considered the problems of state amplification and state masking in an energy harvesting binary symmetric channel. We focused on the no battery, infinite-sized battery, and unit-sized battery cases. For the no battery case, we obtained the regions using previous results. For the infinite-sized battery case, we found that perfect state amplification and perfect state masking are possible, in the sense that message and state information rates add up to the capacity of the channel in the case of state amplification. For the unit-sized battery case, we compared the instantaneous Shannon strategy encoding scheme and the timing-based encoding scheme in the noiseless case, and observed that the former provides better state amplification while the latter provides better state masking.
Chapter 9

Conclusion

9.1 Thesis Summary

In this thesis, optimal transmit power policies in energy harvesting networks and information theoretic capacity of communication channels with an energy harvesting encoder were considered.

First, in Chapter 2, an energy harvesting transmitter equipped with a finite-sized battery was studied in a single-link communication setup. For this basic model, the energy causality and no-battery-overflow constraints which govern the power policy of energy harvesting nodes were identified. For the offline throughput maximization problem with a deadline, necessary conditions for optimality were derived, and an algorithm yielding the optimal policy was proposed. Visualizing the energy constraints as an energy tunnel, the optimal policy was observed to follow the shortest path through the tunnel. Noting the relationship between the throughput maximization and the transmission completion time minimization problems, an algorithm for the latter was also proposed based on the results for the former. Numerical results showed a notable increase in throughput when the optimal policy was employed, compared to a constant power transmission.

The effects of having an inefficient battery was considered next in Chapter 3. In particular, a fraction of stored energy was assumed to be lost, resulting in a trade-off between immediately consuming and storing harvested energy. This trade-off was
shown to yield storing and retrieving thresholds on harvested energy, between which the optimal power policy is to consume all harvested energy immediately without storing or retrieving. Its extension to the fading channel revealed a double-threshold variation of directional water-filling as the algorithm to find the optimal policy, and the online power allocation problem formulated as a Markov decision process also exhibited a double-threshold structure in its solution.

As a building block for multi-transmitter models, the interference channel was considered in Chapter 4. For this channel, it was shown that the problem of jointly optimizing the transmit powers could be solved via alternating maximization. Namely, the single-user problem could be solved for each user iteratively, while keeping the power policy of the other user constant. This led to the iterative generalized directional water-filling (IGDWF) algorithm, where each single-user iteration was solved using a generalized version of directional water-filling. For some interference regions, it was shown that the single-user iterations could be simplified to directional water-filling, with the transmit powers of the other user providing a base level for water-filling. The IGDWF algorithm was subsequently shown to extend to a general communication model in Chapter 5, and an energy harvesting transmitter-energy harvesting receiver pair was considered as an example.

A new dimension in power allocation was explored via energy cooperation in Chapter 6, which involves energy harvesting nodes wirelessly transferring energy to each other as needed. First, the optimality of a class of power policies, named procrastinating policies, was shown. Using such policies, the problem of energy cooperation in a time
slot and the problem of power allocation among time slots was separated. The solution to the former problem was then used as the utility in the latter problem, which was solved using iterative generalized directional water-filling. It was shown that for the energy cooperating two-hop channel, the solution had a two-fluid water-filling interpretation, while for the energy cooperating multiple access channel, a single run of the single-user algorithm in Chapter 2 was sufficient. The benefit of energy cooperation was demonstrated through simulations.

Switching gears to the information theory of energy harvesting transmitters, the binary energy harvesting channel was introduced in Chapter 7. In particular, an encoder whose channel inputs were individually constrained by the available energy was studied. It was shown that this channel, which has an input dependent state that exhibits memory, is equivalent to a timing channel without memory for the unit-sized battery and noiseless channel case. Using this equivalence, a single-letter expression for its capacity was found, and upper bounds and achievable rates were derived. As a result, the capacity of this channel was found within 0.03 bits per channel use.

The rate of revealing the energy harvesting process to the decoder in the binary energy harvesting channel was studied in Chapter 8. Namely, the problems of state amplification and state masking were formulated for the cases of maximizing and minimizing revealed harvesting information, respectively. It was shown that encoding based on instantaneous Shannon strategies resulted in better state amplification, whereas encoding based on the timing channel of Chapter 7 resulted in better state masking.
9.2 Future Directions

While our work on the power allocation problem provides fundamental insights and an initial path forward towards deployment of energy harvesting communication networks, a complete understanding of the benefits and trade-offs of energy harvesting for wireless communications require much further study. As well, there are a number of practical concerns that need to be considered before these systems become ubiquitous. These range from energy issues related to sensing and processing power, battery limitations, and power amplifier efficiency, to communication issues such as synchronization, and to networking issues such as medium access control and routing. Although some of these problems have been considered from an energy-efficient or an energy-aware perspective [121, 62, 53], the energy harvesting setting offers a new point of view and new insights that come with it. For example, recently, the impact of circuit power has been considered in [70, 115, 29, 8], where the transmitter incurs an energy loss by being on, i.e., when the transmit power is non-zero. This setting disfavors long and constant stretches of transmit powers, revealing a new trade-off between efficient low and constant power transmission and bursty transmission with less circuit power consumption. Therefore, facilitating interdisciplinary research to bring theoretical insights and practical realities together is currently the most crucial step in realizing energy harvesting networks.

The power allocation problems in this thesis consider energy harvests and channel gains that are known perfectly by the transmitters. This approach provides valuable insights about optimal policies, which later extend to online scenarios where these parameters are unknown. The robustness and sensitivity of transmission policies for energy
harvesting systems with imperfect knowledge about these parameters remain an interesting future direction. To fully assess the benefit of optimal policies, it would be beneficial to further analyze the sensitivity of the solutions to system parameters such as harvested energy and channel gain. Additionally, one can also pose robust optimization problems which embody the uncertainty in these parameters. Future work in this bearing is particularly important when only partial or noisy information is available about the environment, as is the case in most practical applications.

On the information theoretic capacity side, the field is in its infancy and much remains to be considered. The capacity of even the simplest binary energy harvesting channel with finite battery is unknown, even when the communication channel is noiseless. The variability in harvested energy and the memory in energy state due to energy storage proves to be a challenging combination, rendering a complete understanding of channels possessing these traits rather involved. Clearly a most important future direction in this field is the full characterization of the capacity of a finite-sized battery energy harvesting channel both for discrete memoryless and Gaussian channels. This requires advancing results related to channels with causal state, channels with memory, and channels with input dependent state, and promises a broad understanding of currently open theoretic Shannon problems.
References


Vita

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Selected Publications


