The Pennsylvania State University
The Graduate School
Eberly College of Science

ROKHLIN DIMENSION FOR C^*-CORRESPONDENCES

A Dissertation in
Mathematics
by
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Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Doctor of Philosophy

August 2015

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Abstract

The notion of nuclear dimension for C^*-algebras was defined by Winter and Zacharias in [70] as a noncommutative analog of covering dimension for topological spaces. In recent years nuclear dimension has generated a great deal of interest, not only due to its connection to other important structural properties of C^*-algebras such as Jiang-Su stability and strict comparison, but also because it seems to be a unifying principle in the classification program of nuclear C^*-algebras using K-theory. As such, much work as been done to understand how nuclear dimension behaves for various C^*-constructions. Along these lines, Hirshberg, Winter, and Zacharias proved in [27] that if A is a C^*-algebra having finite nuclear dimension and $\alpha \in \mathrm{Aut}(A)$ is an automorphism having finite Rokhlin dimension, then the associated crossed product $A \rtimes_\alpha \mathbb{Z}$ has finite nuclear dimension. This thesis substantially generalizes this result. Indeed, since a crossed product by $\mathbb{Z}$ can be regarded as a Cuntz-Pimsner algebra associated to a singly-generated C^*-correspondence, we propose a definition of Rokhlin dimension for arbitrary C^*-correspondences that agrees with the traditional one in the singly-generated case. We then show that in many cases (such as for finitely generated projective correspondences), finiteness of nuclear dimension for Pimsner algebras is preserved in the presence of finite Rokhlin dimension. We conclude by using these results to prove that certain types of amalgamated free products have finite nuclear dimension.

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Acknowledgments

This work would not have been possible without the guidance of my advisor Nate Brown. His endless supply of patience, support, and encouragement were unwavering during the last five years. Nate's confidence in my ability to persevere was a necessary source of strength and optimism during the seemingly insurmountable task of learning and contributing to mathematics. Thank you for everything. I also want to thank my extended mathematical family. José Carrión, Hung-Chang Liao, and my C^*-friends abroad, the many conversations we had over the years were immeasurably helpful. You are all brilliant and your futures are bright. Last but certainly not least, I want to thank the mathematics department at Penn State. The teaching and learning experiences I had during my time here were more enriching than I could have imagined.

Dedication

This thesis is dedicated to my family and friends. To my mom, dad, and brother, you have kept me whole. To Sara, your love and encouragement has been a wonderful constant during the turbulence of graduate school. To my pets, may your unconditional devotion and purity of heart serve as a model for all my future endeavors.

Chapter 1 Introduction

The theory of C^*-algebras is often regarded as noncommutative topology. This is justified not only by Gelfand's theorem, which asserts that every commutative C^*-algebra is of the form $C_0(X)$ for some locally compact Hausdorff space X, but also because many important concepts and tools from algebraic topology (such as K-theory) have been carried over to the C^*-setting with tremendous success. This raises a natural and often fruitful question: to what extent does a given topological property have a noncommutative analog? More specifically, in formulating a property of a space X in terms of its associated algebra $C_0(X)$, do we obtain anything interesting and useful if we apply the same definition to arbitrary C^*-algebras?

An important example of such a generalization has been within the context of dimension theories. If X is a topological space, there are several ways to define the dimension of X. One such definition is the (Lebesgue) covering dimension, which is the smallest integer n such that every open cover of X has a refinement whose order does not exceed n. In familiar examples such $\mathbb{R}^k$, or more generally any separable metric space, covering dimension not only agrees with other notions of dimension (such as vector-space dimension and inductive dimension), but also serves as a powerful numerical invariant with which one can distinguish spaces.

Although the definition given above is in terms of open covers, there are several equivalent formulations of covering dimension in terms of approximate factorizations of the identity map $\mathrm{id} : C_0(X) \to C_0(X)$. In the commutative setting these formulations coincide, but for general C^*-algebras one obtains quite distinct definitions. Among others, these include stable rank, real rank, completely positive rank, decomposition rank, and nuclear dimension (see [4, 34, 47, 66, 70]). While these are all interesting and important in their own right, the last several decades have revealed that nuclear dimension (together with decomposition rank) is most relevant for uncovering the structure of a large class of C^*-algebras. Along these lines, the highlight of nuclear dimension is the Toms-Winter conjecture. It asserts that for simple nuclear C^*-algebras, finite nuclear dimension is equivalent to certain structural properties that are analogous to those exploited by Connes when he proved uniqueness of the injective

II_1 factor in [9]. In recent years, the Toms-Winter conjecture was has been confirmed for many important examples (see [3]), and it now it seems likely that a complete verification is within reach.

Another important feature of nuclear dimension is that it appears to be precisely the right context in which to formulate the classification program for nuclear C^*-algebras using K-theoretic invariants. In the 1970s, Elliott proved that AF algebras are completely classified by their ordered K_0 groups. This generalized earlier results by Glimm, which showed that UHF algebras are classified by their associated supernatural numbers. After Elliott's success, there was an exploration into larger classes for which a similar kind of classification might also be established. To this end, positive results have been obtained both for stably finite and purely infinite C^*-algebras. For example, results by Elliott in [18] show that simple $A\mathbb{T}$ algebras (which include the noncommutative tori) are classified by K-theory and traces. More generally, Elliott, Gong, and Li showed in [20] that an analogous result holds for simple unital AH algebras having bounded dimension. On the purely infinite side, a remarkable classification theorem of Kirchberg and Phillips shows that all Kirchberg algebras in the UCT class are classified by their K-theory.

Given the importance of nuclear dimension, it is natural to ask for which examples it is finite. For instance, AF algebras are characterized by having nuclear dimension equal to zero. In [40], it was shown that all Kirchberg algebras have finite nuclear dimension (in fact, they have nuclear dimension one - see [3]). In [57], Szabó proved that for free actions of $\mathbb{Z}^n$ on a compact metric space X, the associated crossed product $C(X) \rtimes \mathbb{Z}^n$ has finite nuclear dimension whenever X has finite covering dimension. Szabó's work was inspired by (and depended on) results of Hirshberg, Winter, and Zacharias in [27], where the classical Rokhlin property for automorphisms was extended (and generalized) to the realm of C^*-algebras in the form of *Rokhlin dimension*. Roughly speaking, an automorphism has finite Rokhlin dimension if it approximately permutes a certain kind of partition of unity in a cyclical manner.

A fundamental result in [27] is the following: if A is a C^*-algebra with finite nuclear dimension and α is an automorphism having finite Rokhlin dimension, then the associated crossed-product $A \rtimes_\alpha \mathbb{Z}$ also has finite nuclear dimension. Moreover, for certain C^*-algebras A, automorphisms having finite Rokhlin dimension are generic in $\mathrm{Aut}(A)$. This shows not only that Rokhlin dimension is intricately connected with nuclear dimension, but also that its presence is ubiquitous. As such, further generalizations have been considered in several distinct directions. For example, Rokhlin dimension for actions of residually finite groups was defined by Szabó, Wu, and Zacharias in [58], where a similar permanence property regarding the nuclear dimension of crossed products was proven. For yet another approach, see [22] for the definition of Rokhlin dimension for actions of compact groups.

An interesting feature of crossed products by $\mathbb{Z}$ is that they can be realized as certain kinds of Cuntz-Pimsner algebras. As such, one can ask if the notion of Rokhlin dimension can be extended to C^*-correspondences, and whether the associated Pimsner algebras have finite nuclear dimension. This is the main contribution of this thesis. Indeed, we propose a definition of Rokhlin dimension for (countably generated) C^*-correspondences in Definition 6.4.1, and then show that in many cases (such as for finitely generated projective correspondences), it implies having finite nuclear dimension. Our main result (Theorem 6.7.1) is as follows.

Theorem 1.0.1. Suppose that $\mathcal{H}$ is a countably generated C^*-correspondence over a separable unital C^*-algebra A. Assume further that $\mathcal{H}$ is quasidiagonal. Then

$$\dim_{\text{nuc}}(\mathcal{T}(H)) \leq 2(\dim_{\text{nuc}}(A) + 1)(\dim_{\text{Rok}}(\mathcal{H}) + 1) - 1.$$

The main technical difficulty is adapting the proof used in [27] to work first for free modules, and then using a theorem of Kasparov for arbitrary modules.

Here is an outline of what follows. In Chapter 2, we review some preliminary definitions, examples, and constructions. Chapter 3 focuses primarily on Hilbert C^*-modules, C^*-correspondences, and their associated Pimsner algebras. Chapter 4 contains a brief review of K-theory for C^*-algebras, and mentions some of the highlights from the classification program prior to the advent of nuclear dimension. We end this chapter by stating the Elliott conjecture (together with the Elliott invariant) for simple nuclear C^*-algebras. Chapter 5 is devoted to nuclear dimension; we mention how it relates to classification (by way of $\mathcal{Z}$-stability and strict comparison), and then state the Toms-Winter conjecture. We also review some recent progress in its verification. In Chapter 6 we define Rokhlin dimension for C^*-correspondences, and outline how in the singly generated case it implies permanence of finite nuclear dimension for the associated Pimsner algebras. We then resolve the technical obstacles necessary for adapting the singly-generated methods to countably generated modules. As an application, we conclude by showing that Theorem 1.0.1 implies finite nuclear dimension for certain kinds of reduced amalgamated free products.

Chapter 2
C^*-Algebras

All vector spaces are assumed to be over the field of complex numbers $\mathbb{C}$. If x and y are two elements in a Banach space and $\epsilon > 0$, we write $x \approx_\epsilon y$ whenever $\|x - y\| < \epsilon$.

2.1 Preliminary Definitions and Examples

The following definitions, examples, and facts come from standard texts such as [41], [5], and [50], where much more can be found.

Let A be a Banach algebra. An *involution* on A is a conjugate-linear map $a \mapsto a^*$ such that $a^{**} = a$ and $(ab)^* = b^*a^*$ for all $a, b \in A$. The pair $(A, *)$ is called a Banach *-algebra. We say that A is a *C^*-algebra* if

$$\|a^*a\| = \|a\|^2$$

for every $a \in A$.

Remark 2.1.1. All C^*-algebras in this thesis will be assumed separable unless otherwise noted (or obviously false).

If $b \in A$, we say that b is

- *normal* if $b^*b = bb^*$,
- *self-adjoint* if $b = b^*$,
- *positive* if $b = a^*a$ for some $a \in A$,
- a *projection* if $b = b^* = b^2$,
- a *partial isometry* if b^*b is a projection.

If A is unital, we say that b is

- *invertible* if there exists $b^{-1} \in A$ satisfying $b^{-1}b = bb^{-1} = 1$,

- *unitary* if b is invertible and $b^{-1} = b^*$,
- an *isometry* if $b^*b = 1$.

Let $\sigma(b)$ and $r(b)$ denote the spectrum and spectral radius of b, respectively. A *C^*-subalgebra* of A is a subset $B \subseteq A$ that is also a C^*-algebra. The *commutant of B in A* is the set $B' = \{a \in A \ : \ ab = ba \text{ for every } b \in B\}$. The *center* of A is the C^*-subalgebra $\mathcal{Z}(A) = \{a \in A \ : \ ab = ba \text{ for all } b \in A\}$, so that in particular $\mathcal{Z}(A)' = A$. If $S \subseteq A$ is any subset, the C^*-algebra *generated by S* is the smallest C^*-subalgebra of A containing S, denoted by $C^*(S)$. A *hereditary subalgebra of A* is a C^*-subalgebra $B \subseteq A$ such that there is some positive element $b \in B$ satisfying $B = \overline{bAb}$. An *ideal* of A is a C^*-subalgebra $J \trianglelefteq A$ satisfying $aj \in J$ and $ja \in J$ for every $a \in A$ and $j \in J$. The *quotient* A/J is a C^*-algebra with multiplication and adjoints given by $(a + J)(b + J) = ab + J$ and $(a + J)^* = a^* + J$ for $a, b \in A$. The norm on A/J is given by $\|a + J\| = \inf_{j \in J} \|a + j\|$. The *quotient map* $A \to A/J$ is given by $a \mapsto a + J$. If A has no proper nontrivial ideals, we say A is *simple*.

Let A_+, $\mathcal{P}(A)$, and $\mathcal{U}(A)$ denote the set of positive elements, projections, and unitary elements in A, respectively. If $a, b \in A_+$, write $a \leq b$ if $b - a \in A_+$, write $a \perp b$ if $ab = 0$, and write $a \perp_\delta b$ if $ab \approx_\delta 0$ for some $\delta > 0$. An *approximate unit* for A is an increasing sequence $(v_n)_{n \in \mathbb{N}}$ of positive elements in the closed unit ball of A such that $a = \lim_n av_n = \lim_n v_n a$ for every $a \in A$. An approximate unit $(v_n)_{n \in \mathbb{N}}$ for an ideal $J \trianglelefteq A$ is called *quasicentral in A* if $\lim_n \|av_n - v_n a\| = 0$ for every $a \in A$.

There are three equivalence relations we consider on $\mathcal{P}(A)$. If $p, q \in \mathcal{P}(A)$, write

1. $p \sim_h q$ if there is a continuous map $\gamma : [0, 1] \to \mathcal{P}(A)$ satisfying $\gamma(0) = p$ and $\gamma(1) = q$.
2. $p \sim_u q$ if there a unitary element $u \in \mathcal{U}(\widetilde{A})$ with $q = upu^*$ (see 2.2.1 for the definition of $\widetilde{A}$).
3. $p \sim q$ if there exists a partial isometry $v \in A$ with $p = v^*v$ and $q = vv^*$.

The relations $\sim_h, \sim_u$, and $\sim$ are known as *homotopy equivalence, unitary equivalence, and Murray-von Neumann equivalence*, respectively. One can show that homotopy equivalence implies unitary equivalence, and that unitary equivalence implies Murray-von Neumann equivalence. Although the converse statements are not true in $\mathcal{P}(A)$, they become true if we allow amplifications. More specifically, if $p, q \in \mathcal{P}(A)$, we can consider the projections $p \oplus 0 = \operatorname{diag}(p, 0)$ and $q \oplus 0 = \operatorname{diag}(q, 0)$ in $\mathcal{P}(M_2(A))$. Then $p \sim q$ implies $p \oplus 0 \sim_u q \oplus 0$, and $p \sim_u q$ implies $p \oplus 0 \sim_h q \oplus 0$ (see [50, Proposition 2.2.28], for example).

Definition 2.1.2. Let $p \in \mathcal{P}(A)$. We say p is *infinite* if there exists $q \in \mathcal{P}(A)$ satisfying $q < p \sim q$. If p is not infinite, it is *finite*. We say p is *properly infinite* if it is non-zero and there exist $q, q' \in \mathcal{P}(A)$ satisfying $q + q' \leq p \sim q \sim q'$. The C^*-algebra A is *infinite* if it contains an infinite projection; it is *finite* otherwise. We say A is *stably finite* if $M_n(A)$ is finite for every $n \in \mathbb{N}$. If A is unital, it is *properly infinite* if its unit is a properly infinite projection. If A is simple, it is *purely infinite* if every (non-zero) hereditary C^*-subalgebra of A is infinite. We say A is *stably projectionless* if $A \otimes \mathbb{K}$ contains no projections (see Remark 2.1.5 for the definition of $\mathbb{K}$, and 2.3.1 for the definition of $\otimes$).

Definition 2.1.3. Let A be a C^*-algebra.

1. A has *real rank zero*, written $\mathrm{RR}(A) = 0$, if every self-adjoint element in A is the norm limit of self-adjoint elements with finite spectrum.

2. If A is unital, it has *stable rank one*, written $\mathrm{sr}(A) = 1$, if the set of invertible elements in A is dense in A. If A is non-unital, it has stable rank one if $\mathrm{sr}(\widetilde{A}) = 1$.

Example 2.1.4.

1. Let H be a Hilbert space and denote by $\mathbb{B}(H)$ the set of all bounded linear operators on H. The norm on H induces a norm on $\mathbb{B}(H)$ making it into a Banach algebra. By the Riesz representation theorem, each element $T \in \mathbb{B}(H)$ has an adjoint T^* characterized by $\langle Th, k\rangle = \langle h, T^*k\rangle$ for every $h, k \in H$. The adjoint operation is an involution that turns $\mathbb{B}(H)$ into a C^*-algebra. If $H = \mathbb{C}^n$, then $\mathbb{B}(H)$ is the algebra $M_n(\mathbb{C})$ of $n \times n$ complex matrices. The adjoint in this case is given by the complex conjugate transpose: $(a_{ij})^*_{i,j} = (\overline{a}_{ji})_{i,j}$.

2. Within $\mathbb{B}(H)$ there is a distinguished C^*-subalgebra. For each $x, y \in H$, let $e_{x,y} : H \to H$ be the *rank-one operator* given by $z \mapsto \langle z, y\rangle x$. The *compact operators* $\mathbb{K}(H)$ on H is the C^*-algebra generated by the set $\{e_{x,y} \; : \; x, y \in H\}$. One checks that $Se_{x,y}T = e_{Sx,T^*y}$ for each $S, T \in \mathbb{B}(H)$, so $\mathbb{K}(H)$ is in fact an ideal. If H is finite dimensional, then every bounded operator is compact and hence $\mathbb{K}(H) \simeq M_{\dim(H)}(\mathbb{C}) \simeq \mathbb{B}(H)$.

3. The quotient $\mathbb{B}(H)/\mathbb{K}(H)$ is called the *Calkin algebra* associated to H. If H is infinite dimensional, the Calkin algebra is a highly nontrivial (and somewhat mysterious) object.

4. Let X be a locally compact Hausdorff space. We say that a function $f : X \to \mathbb{C}$ *vanishes at infinity* if for every $\epsilon > 0$, the set $\{x \in X \; : \; |f(x)| \geq \epsilon\}$ is compact. Denote the set of such functions by $C_0(X)$. It is a Banach algebra under pointwise operations and the supremum norm $\|f\| = \sup_{x\in X}|f(x)|$. Under the involution $f \mapsto \overline{f}$, $C_0(X)$ becomes a C^*-algebra. If X is compact then $C_0(X) = C(X)$, the C^*-algebra of all continuous complex-valued function on X.

5. If A is a C^*-algebra and $n \in \mathbb{N}$, then the set $M_n(A)$ of $n \times n$ matrices with entries in A is a C^*-algebra. Multiplication is given by ordinary matrix multiplication and the involution is given by $(a_{ij})^*_{i,j} = (a^*_{ji})_{i,j}$. The norm of an element $a \in M_n(A)$ is given by $\|a\| = r(a^*a)^{1/2}$.

6. Let ω be a free ultrafilter on $\mathbb{N}$ and let A be a C^*-algebra. Denote a sequence $\mathbb{N} \to A$ in A by (a_n), where a_n is the image of n. Let

$$\ell^\infty(A) = \{(a_n) : \mathbb{N} \to A \;:\; \sup\|a_n\| < \infty\},$$

and let

$$c_\omega(A) = \{(a_n) \in \ell^\infty(A) \;:\; \lim_{n\to\omega}\|a_n\| = 0\} \subset \ell^\infty(A).$$

It is easily checked that $\ell^\infty(A)$ is a C^*-algebra under pointwise operations and the supremum norm. Furthermore, $c_\omega(A)$ is an ideal in $\ell^\infty(A)$; denote the quotient $\ell^\infty(A)/c_\omega(A)$ by A_ω and let $\pi_\omega : \ell^\infty(A) \to A_\omega$ be the quotient map. The C^*-algebra A_ω is called the *ultrapower of A (associated to ω).* If $i : A \to \ell^\infty(A)$ is the map sending $a \in A$ to the constant sequence (a), let i_ω be the composite map $\pi_\omega \circ i$. Finally, let $F(A)$ be the C^*-algebra $A_\omega \cap i_\omega(A)'$. Although a bit involved, this construction is very useful since we can often translate asymptotic properties of A into properties of $F(A)$. For example, if (a_n) is an asymptotically central sequence in A, then $\pi_\omega(a_n)$ is a central element in $F(A)$.

Remark 2.1.5. The separable Hilbert space $\ell^2(\mathbb{N})$ is unique in the sense that it is isomorphic to any other separable infinite dimensional Hilbert space. To simplify notation, we denote $\mathbb{K}(\ell^2(\mathbb{N}))$ by $\mathbb{K}$.

Let A and B be C^*-algebras. A **-homomorphism* from A to B is a linear map $\pi : A \to B$ satisfying $\pi(ab) = \pi(a)\pi(b)$ and $\pi(a^*) = \pi(a)^*$ for every $a, b \in A$. If A and B are unital, we say π is *unital* if $\pi(1) = 1$. If π is bijective, it is called a **-isomorphism.* If $A = B$, we say π is an *endomorphism.* An *automorphism* is a bijective endomorphism. If $\pi : A \to B$ is a *-homomorphism, then $\ker(\varphi)$ is an ideal in A. It is an interesting feature of C^*-algebras that *-homomorphisms are automatically contractive, and they are isometric if and only if they are injective. As a result, $\mathrm{im}(\pi)$ is a C^*-subalgebra of B and we have $A/\ker(\pi) \simeq \mathrm{im}(\pi)$.

A **-representation* of a C^*-algebra A is a pair (H, π) consisting of a Hilbert space H and a *-homomorphism $\pi : A \to \mathbb{B}(H)$. The representation is called *nondegenerate* if for every vector $\xi \in H$, there is some $a \in A$ such that $\pi(a)\xi \neq 0$; it is called *faithful* if π is injective. A representation of the form $\pi : A \to \mathbb{B}(\mathbb{C})$ is called a *character.*

An important property of *-homomorphisms is that they preserve positivity. Indeed, if $\pi : A \to B$ is a *-homomorphism and $a \in A_+$ satisfies $a = b^*b$ for some $b \in A$, then $\pi(a) = \pi(b)^*\pi(b) \in B_+$. Isolating this property leads to the definition of a positive map.

Definition 2.1.6. A linear map $\varphi : A \to B$ is said to be *positive* if $\varphi(a) \in B_+$ whenever $a \in A_+$. We say φ is *completely positive* if the *inflation map* $\varphi^n : M_n(A) \to M_n(B)$ given by $\varphi^n((a_{ij})_{i,j}) = (\varphi(a_{ij}))_{i,j}$ is positive for every $n \in \mathbb{N}$. We use the abbreviations c.p., c.p.c., and u.c.p. to refer to completely positive, completely positive contractive, and completely positive unital, respectively.

Note that a *-homomorphism π is c.p.c. since the inflations π^n are also *-homomorphisms. More generally, if $\pi : A \to \mathbb{B}(H)$ is a *-homomorphism and $V \in \mathbb{B}(H)$, then the map $\varphi : A \to \mathbb{B}(H)$ given by $a \mapsto V^*\pi(a)V$ is completely positive. The following theorem of Stinespring shows that all c.p. maps are essentially of this form.

Theorem 2.1.7 (Stinespring)**.** Let A be a unital C^*-algebra and $\varphi : A \to B(H)$ be a c.p. map. Then there exists a Hilbert space K, a *-representation $\pi : A \to B(K)$, and an operator $V : H \to K$ such that

$$\varphi(a) = V^*\pi(a)V$$

for every $a \in A$. In particular, $\|\varphi\| = \|V^*V\| = \|\varphi(1)\|$. If φ is unital, then V is an isometry. The projection $VV^* \in B(K)$ is called the *Stinespring projection*.

Remark 2.1.8. For certain classes of C^*-algebras, the notion of a positive and completely positive maps coincide. For example, if either A or B is an abelian C^*-algebra, then any positive map $A \to B$ is completely positive (see [43, Theorem 3.9] for example). However, this is not true in general: the transpose map on $M_n(\mathbb{C})$ is positive but not completely positive.

Definition 2.1.9. A *trace* on a C^*-algebra A is a linear function $\tau : A \to \mathbb{C}$ that satisfies the *trace condition*: $\tau(ab) = \tau(ba)$ for every $a, b \in A$. If τ is positive (and hence completely positive by Remark 2.1.8) and satisfies $\|\tau\| = 1$, then τ is called a *tracial state*. Denote the set of tracial states on A by $T(A)$.

If A is unital, then $T(A)$ is a compact convex set when equipped with the relative weak-* topology in the dual space A^*. Moreover, we have by [54, Theorem 3.1.18] that $T(A)$ is a metrizable Choquet simplex whenever A is separable.

Example 2.1.10. The *un-normalized* trace on $M_n(\mathbb{C})$ is the function $\mathrm{Tr}_n(a_{ij})_{i,j} = \sum_{i=1}^n a_{i,i}$. The *normalized* trace on $M_n(\mathbb{C})$ is the tracial state $\mathrm{tr}_n = \frac{1}{n}\mathrm{Tr}_n$. One can show that the set $T(M_n(\mathbb{C}))$ is the singleton set $\{\mathrm{tr}_n\}$.

The following definition adds an orthogonality condition to c.p. maps.

Definition 2.1.11. Let A and B be C^*-algebras and $\psi : A \to B$ a c.p. map. We say ψ has *order zero* if it preserves orthogonality: for every pair of positive elements $a, b \in A$,

$$a \perp b = 0 \Rightarrow \psi(a) \perp \psi(b).$$

In [69], Winter and Zacharias proved the following structure theorem for order zero maps. In light of Stinespring's theorem, it roughly states that they are "in-between" c.p. maps and *-homomorphisms.

Theorem 2.1.12. Let $\psi : A \to B$ be a c.p. order zero map, and let C the C^*-algebra in B generated by the image of ψ. Then there is a positive element $h \in \mathcal{M}(C) \cap C'$ satisfying $\|h\| = \|\psi\|$ and a *-homomorphism $\pi_\psi : A \to \mathcal{M}(C) \cap \{h\}'$ such that

$$\psi(a) = \pi_\psi(a)h \text{ for all } a \in A.$$

If A is unital, then $h = \psi(1) \in C$.

Weakening the assumption of preserving orthogonality, an approximate version of the order zero property is defined as follows.

Definition 2.1.13. Let A be a C^*-algebra, K a finite-dimensional C^*-algebra, and $\delta > 0$. A c.p.c. map $\psi : K \to A$ is $\delta-$*order zero* if it preserves orthogonality up to δ: for every pair of positive contractions $a, b \in A$,

$$a \perp b \Rightarrow \psi(a) \perp_\delta \psi(b)$$

In [34], Kirchberg and Winter proved the following useful stability property for order zero maps coming from finite-dimensional C^*-algebras.

Proposition 2.1.14. Fix a finite-dimensional C^*-algebra K. Then for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $\psi : K \to A$ is a δ-order zero map into any C^*-algebra A, there is an order zero map $\psi' : K \to A$ such that $\|\psi - \psi'\| < \epsilon$.

2.2 Constructions

2.2.1 Unitizations

Given a non-unital C^*-algebra, there are various ways of embedding it in a unital C^*-algebra; such a process is called *unitization*. We review two canonical unitization constructions.

Let A be a C^*-algebra, with or without unit. Set $\widetilde{A} = \{(a, \lambda) \ : \ a \in A, \ \lambda \in \mathbb{C}\}$. Addition and scalar multiplication on $\widetilde{A}$ are defined coordinate-wise; multiplication and involution are given by $(a, \lambda)(b, \mu) = (ab + \mu a + \lambda b, \lambda\mu)$ and $(a, \lambda)^* = (a^*, \overline{\lambda})$. Define a norm on $\widetilde{A}$ by

$$\|(a, \lambda)\| = \max\{\sup\{\|ba\| \ : \ b \in A, \ \|b\| \leq 1\}, \ |\lambda|\}.$$

This gives $\widetilde{A}$ the structure of a unital C^*-algebra with unit $(0, 1)$. Let $i : A \to \widetilde{A}$ and $\pi : \widetilde{A} \to \mathbb{C}$ be given by $i(a) = (a, 0)$ and $\pi(a, \lambda) = \lambda$. Then i and π are injective and surjective *-homomorphisms, respectively. We can (and do) identify A with its image under i; it is easily seen that A is an ideal in $\widetilde{A}$. Moreover, $\widetilde{A} \simeq A \oplus \mathbb{C}$ if and only if A is unital.

Next, we define a *double centralizer* on A to be a pair (L, R) of bounded linear maps on A satisfying

$$L(ab) = L(a)b, \quad R(ab) = aR(b), \quad R(a)b = aL(b)$$

for every $a, b \in A$. For each $c \in A$, there is an associated double centralizer (L_c, R_c) defined by $L_c(a) = ca$ and $R_c(a) = ac$. Denote the set of all double centralizers on A by $\mathcal{M}(A)$. Addition and scalar multiplication in $\mathcal{M}(A)$ are defined coordinate-wise: $\alpha(L, R) + \beta(L', R') = (\alpha L + \beta L', \alpha R + \beta R')$. Multiplication and involution are given by

$$(L, R)(L', R') = (LL', R'R) \qquad \text{and} \qquad (L, R)^* = (R^*, L^*),$$

where the notation T^* for a bounded map $T : A \to A$ is defined by $T^*(a) = (T(a^*))^*$. If $(L, R) \in \mathcal{M}(A)$, we have by [41, Lemma 2.1.4] that $\|L\| = \|R\|$; let $\|(L, R)\|$ denote this common value. This gives $\mathcal{M}(A)$ the structure of a unital C^*-algebra with unit $(\mathrm{id}_A, \mathrm{id}_A)$. Furthermore, the map $A \to \mathcal{M}(A)$ given by $c \mapsto (L_c, R_c)$ is an injective *-homomorphism. We can (and do) identify A with its image under this map, and as before A is an ideal in $\mathcal{M}(A)$. We call $\mathcal{M}(A)$ the *multiplier algebra* of A. Note that if A is unital, then the inclusion $A \subseteq \mathcal{M}(A)$ is unital (and hence $A = \mathcal{M}(A)$). In contrast, the unit of A is always a proper projection in $\widetilde{A}$.

Example 2.2.1. Let X be a locally compact Hausdorff space. Denote by αX and βX the one-point and Stone-Čech compactifications of X, respectively. Then

$$\widetilde{C_0(X)} \simeq C(\alpha X) \qquad \text{and} \qquad \mathcal{M}(C_0(X)) \simeq C(\beta X).$$

Having introduced multiplier algebras, we can define various equivalences among *-homomorphisms. Two *-homomorphisms $\varphi, \psi : A \to B$ are:

- *unitarily equivalent* (written $\varphi \sim_u \psi$) if there exists a unitary $u \in \mathcal{U}(\mathcal{M}(B))$ such that $\psi(\cdot) = u\varphi(\cdot)u^*$,
- *approximately unitarily equivalent* (written $\varphi \approx_u \psi$) if for every finite subset $F \subset A$ and $\epsilon > 0$, there exists a unitary $u \in \mathcal{U}(\mathcal{M}(B))$ such that $\|\psi(x) - u\varphi(x)u^*\| < \epsilon$ for every $x \in F$,
- *homotopic* (written $\varphi \sim_h \psi$) if there is a path of *-homomorphisms $\varphi_t : A \to B$, with $t \in [0, 1]$, such that $t \mapsto \varphi_t(a)$ is continuous for each $a \in A$, $\varphi_0 = \varphi$, and $\varphi_1 = \psi$,
- *asymptotically unitarily equivalent* (written $\varphi \approx_{uh} \psi$) if there is a norm continuous $\{u_t\}_{t \in [0,\infty)}$ of unitaries in $\mathcal{U}(\mathcal{M}(B))$ such that $\psi = \lim_{t\to\infty} u_t \varphi u_t^*$.

2.2.2 Gelfand's Theorem

For each locally compact Hausdorff space X, the C^*-algebra $C_0(X)$ is abelian. An important theorem of Gelfand shows that the converse is true. Gelfand's theorem for abelian C^*-algebra justifies the term *noncommutative topology*, which is often used to describe the theory of C^*-algebras.

Let A be an abelian C^*-algebra and let $\Omega(A)$ denote the collection of nonzero characters $\tau : A \to \mathbb{C}$. One checks that $\Omega(A)$ is contained in the closed unit ball of the dual space A^*, so it inherits a relative weak* topology. The topological space $\Omega(A)$ is called the *(Gelfand) spectrum* of A. The spectrum is a locally compact Hausdorff space; it is compact if and only if A is unital. If $a \in A$, the (continuous) function $\hat{a} : \Omega(A) \to \mathbb{C}$ given by $\tau \mapsto \tau(a)$ is called the *Gelfand transform of* a. For each $\epsilon > 0$, the set $\{\tau \in \Omega(A) \ : \ |\tau(a)| \geq \epsilon\}$ is weak* closed (by the Banach-Alaoglu theorem). This shows $\hat{a} \in C_0(\Omega(A))$.

Theorem 2.2.2 (Gelfand)**.** If A is an abelian C^*-algebra, then the map $a \mapsto \hat{a}$ implements an isomorphism

$$A \simeq C_0(\Omega(A)).$$

Example 2.2.3.

1. Let X be a locally compact Hausdorff space. Then the Gelfand spectrum of $C_0(X)$ is homeomorphic to X. This shows that the construction above gives nothing new if we start with functions on a space.

2. Let A be any unital C^*-algebra and suppose that $a \in A$ is a normal element. Then the C^*-algebra $C^*(a,1)$ generated by a and 1 is an abelian C^*-algebra. Furthermore, the Gelfand spectrum $\Omega(C^*(a,1))$ is homeomorphic to the spectrum $\sigma(a)$. In other words, $C^*(a,1) \simeq C(\sigma(a))$.

Gelfand's theorem leads to one of the most useful tools in all of C^*-algebra theory, commonly referred to as the *functional calculus.* If $a \in A$ is normal and $f : \sigma(a) \to \mathbb{C}$ is a continuous function, then the following result allows us to make sense of the expression $f(a)$ as an element in A.

Theorem 2.2.4. Let a be a normal element of a unital C^*-algebra A. Denote by z the inclusion map $z : \sigma(a) \to \mathbb{C}$. Then there is a unique unital injective *-homomorphism $\pi : C(\sigma(a)) \to A$ such that $\pi(z) = a$. Moreover, $\mathrm{im}(\pi) = C^*(a,1)$.

2.2.3 Direct Sums and the GNS Construction

A prototypical example of a C^*-algebra is $\mathbb{B}(H)$. In a sense, this is the most important example; a construction due to Gelfand, Naimark, and Segal shows that every C^*-algebra is (isomorphic to) a subalgebra of $\mathbb{B}(H)$ for an appropriate Hilbert space H. Such a concrete realization does not exist for arbitrary Banach algebras, and it is partly for this reason that the theory of C^*-algebras is more accessible.

If $(A_i)_{i \in I}$ is a collection of C^*-algebras, then the direct sum $\bigoplus A_i$ is a C^*-algebra under pointwise operations. We record the following important theorem that characterizes finite-dimensional C^*-algebras.

Theorem 2.2.5. If A is a non-zero finite-dimensional C^*-algebra, then there are natural numbers $n_1, \ldots, n_k$ such that $A \simeq M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$.

If $(H_i, \pi_i)_{i \in I}$ is a collection of representations of a C^*-algebra A, their *direct sum* is the representation $(\bigoplus_{i \in I} H_i, \oplus \pi_i)$, where $\oplus \pi_i(a)(x_i)_{i \in I} = (\pi_i(a)(x_i))_{i \in I}$. Notice that if, for each nonzero $a \in A$, there is a representation (H_a, π_a) of A satisfying $\pi_a(a) \neq 0$, then the direct sum $(\bigoplus_{a \in A} H_a, \oplus \pi_a)$ is faithful.

Definition 2.2.6. A *state* on a C^*-algebra A is a positive linear functional on A of norm one. Let $S(A)$ denote the set of states of A.

If $\tau \in S(A)$, let $N_\tau = \{a \in A \; : \; \tau(a^* a) = 0\}$. One checks that N_τ is a closed left ideal of A, and that the map $\langle \cdot, \cdot \rangle_\tau : A/N_\tau \times A/N_\tau \to \mathbb{C}$ given by $\langle a + N_\tau, b + N_\tau \rangle_\tau = \tau(b^* a)$ is a well defined inner product on the quotient A/N_τ. Let H_τ denote the Hilbert space completion of A/N_τ with respect to $\langle \cdot, \cdot \rangle_\tau$. For each $a \in A$, let $\pi_\tau(a) : A/N_\tau \to A/N_\tau$ be given by $\pi_\tau(a)(b + N_\tau) = ab + N_\tau$. Since $\|\pi_\tau(a)\| \leq \|a\|$, the operator $\pi_\tau(a)$ has a unique extension to a bounded operator on H_τ; we denote this extension by $\pi_\tau(a)$ as well. It is easy to check that $a \mapsto \pi_\tau(a)$ is a *-representation of A on H_τ; it is known as the *GNS representation* associated to τ. The *universal representation* of a C^*-algebra A is the direct sum $(\bigoplus_{\tau \in S(A)} H_\tau, \oplus \pi_\tau)$.

Theorem 2.2.7 (Gelfand-Naimark)**.** If A is a C^*-algebra, then its universal representation is faithful.

Example 2.2.8. Let X be a locally compact Hausdorff space. By Riesz's theorem each state $\tau \in S(C_0(X))$ corresponds to a Borel probability measure μ_τ on X satisfying

$$\tau(f) = \int_X f \, d\mu_\tau.$$

The space N_τ consists of those functions $f \in C_0(X)$ satisfying $\int |f|^2 d\mu_\tau = 0$. It is not difficult to check that $H_\tau = \ell^2(X, \mu_\tau)$, the Hilbert space of μ_τ-square-integrable functions on X. Moreover, the *-representation $\pi_\tau : C_0(X) \to \mathbb{B}(\ell^2(X, \mu_\tau))$ is given by multiplication: $\pi_\tau(f)(g) = fg$ for $f \in C_0(X)$ and $g \in \ell^2(X, \mu_\tau)$. This example shows that an element of an abelian C^*-algebra can be regarded as a *diagonal* (or *multiplication*) *operator* on a Hilbert space.

Remark 2.2.9. The GNS construction shows that we can always regard an abstract C^*-algebra as being a C^*-subalgebra of $\mathbb{B}(H)$. If A is separable, H can be taken to be separable. When we write $A \subset \mathbb{B}(H)$ without qualification, we are using the GNS construction.

2.2.4 Inductive Limits

If $\{A_i\}_{i\in I}$ is a collection C^*-algebras, let $\prod A_i$ be the set of all functions $a : I \to \bigcup_I A_i$ such that $a(i) \in A_i$ for each $i \in I$ and

$$\|a\| = \sup\{\|a(i)\|_{A_i} \;:\; i \in I\} < \infty.$$

Under pointwise operations, $\prod A_i$ is a C^*-algebra. Within $\prod A_i$, let $\sum A_i$ denote the C^*-subalgebra generated by the set

$$\{a \in \prod A_i \;:\; a(i) = 0 \text{ for all but finitely many } i \in I\}.$$

It is easily checked that $\sum A_i$ is an ideal in $\prod A_i$; let $\pi : \prod A_i \to \prod A_i / \sum A_i$ be the quotient map.

An *inductive sequence* of C^*-algebras is a sequence $\{A_n\}_{n\in\mathbb{N}}$ of C^*-algebras and a sequence $\{\varphi_n : A_n \to A_{n+1}\}_{n\in\mathbb{N}}$ of $*$-homomorphisms called *connecting maps*. We often write the sequence as

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots. \tag{2.1}$$

If $m > n$, let $\varphi_{m,n} = \varphi_{m-1} \circ \cdots \circ \varphi_n : A_n \to A_m$. Define $\varphi_{n,n} = \mathrm{id}_{A_n}$ and $\varphi_{m,n} = 0$ whenever $m < n$. An *inductive limit* of the sequence (2.1) is a pair $(A, \{\mu_n\}_{n\in\mathbb{N}})$ consisting of a C^*-algebra A and a sequence $\{\mu_n : A_n \to A\}_{n\in\mathbb{N}}$ of $*$-homomorphisms such that the following two conditions hold.

1. For each $n \in \mathbb{N}$, the following diagram commutes:

$$\begin{array}{ccc} A_n & \xrightarrow{\varphi_n} & A_{n+1} \\ & {\scriptstyle\mu_n}\searrow \quad \swarrow{\scriptstyle\mu_{n+1}} & \\ & A & \end{array}$$

2. If $(B, \{\nu_n\}_{n\in\mathbb{N}})$ is a pair consisting of a C^*-algebra B and a sequence of $*$- homomorphisms $\nu_n : A_n \to B$ satisfying $\nu_n = \nu_{n+1} \circ \varphi_n$ for each $n \in \mathbb{N}$, then there is a unique $*$-homomorphism $\lambda : A \to B$ making the following diagram commute:

$$\begin{array}{ccc} & A_n & \\ & {\scriptstyle\mu_n}\swarrow \quad \searrow{\scriptstyle\nu_n} & \\ A & \xrightarrow[\lambda]{} & B \end{array}$$

If an inductive limit exists, it is unique. Indeed, it follows from the definition that if $(A, \{\mu_n\}_{n\in\mathbb{N}})$ and $(B, \{\nu_n\}_{n\in\mathbb{N}})$ are both inductive limits of (2.1), then there is a unique

*-isomorphism $\lambda : A \to B$ satisfying $\nu_n = \lambda \circ \mu_n$ for each $n \in \mathbb{N}$. To show it exists, let $\bar{\mu}_n : A_n \to \prod A_m$ denote the *-homomorphism given by $a \mapsto (\varphi_{m,n}(a))_{m\in\mathbb{N}}$. Let $\mu_n = \pi \circ \bar{\mu}_n$. One checks that $\mu_{n+1} \circ \varphi_n = \mu_n$, so $\{\mu_n(A_n)\}_{n\in\mathbb{N}}$ is an increasing sequence of C^*-algebras. Let

$$A = \overline{\bigcup_{n=1}^{\infty} \mu_n(A_n)} \subseteq \prod A_m / \sum A_m.$$

Then $(A, \{\mu_n\}_{n\in\mathbb{N}})$ is an (the) inductive limit of (2.1); we denote it by $\varinjlim A_n$.

Definition 2.2.10. Let A be the inductive limit of (2.1).

1. A is an *Approximately Finite-dimensional* C^*-algebra (or *AF algebra* for short) if each A_n is finite dimensional. If the connecting maps φ_n are unital and $A_n \simeq M_{k_n}(\mathbb{C})$ for each $n \in \mathbb{N}$, we say A is a *Uniformly Hyperfinite C^*-algebra*, or *UHF algebra* for short.

2. A is an *$A\mathbb{T}$ algebra* if each A_n is of the form $C(\mathbb{T}) \otimes F_n$ for some finite dimensional C^*-algebras F_n.

3. A is an *Approximately Homogeneous* C^*-algebra (or *AH algebra* for short) if each A_n is of the form
$$A_n = \bigoplus_{i=1}^{r_n} p_{n,i} M_{k_{n,i}}(C(X_{n,i})) p_{n,i} \tag{2.2}$$
for some $r_n, k_{n,i} \in \mathbb{N}$, some compact Hausdorff spaces $X_{n,i}$, and some projections $p_{n,i} \in M_{k_{n,i}}(C(X_{n,i}))$. We will also assume (without loss of generality) that each connecting map is unital, and that the map $X_{n,i} \to \mathbb{Z}$ given by $x \mapsto \dim(p_{n,i}(x))$ is constant for all n and i. Denote this constant number by $d_{n,i}$. The sequence has *slow dimension growth* if
$$\lim_{n\to\infty} \max \left\{ \frac{\dim(X_{n,i})}{d_{n,i}} \ : \ i = 1, \ldots, r_n \right\} = 0.$$
It has *bounded dimension* if $\sup \dim(X_{n,i}) < \infty$. An AH algebra has *slow dimension growth* (resp. *bounded dimension*) if it is the inductive limit of a sequence with slow dimension growth (resp. bounded dimension).

4. A is an *Approximately Subhomogenous* C^*-algebra (or *ASH algebra* for short) if each A_n is a C^*-subalgebra of $M_{k_n}(C_0(X_n))$ for some $k_n \in \mathbb{N}$ and some locally compact Hausdorff space X_n. The algebras A_n are called *subhomogenous* C^*-algebras.

Remark 2.2.11. Note that it is possible for an AH algebra to be the inductive limit of two different sequences, one with slow dimension growth and one without. Furthermore, every unital AH algebra arises as an inductive limit of algebras of the form (2.2) with each space $X_{n,i}$ a finite CW-complex (see [1]).

Example 2.2.12.

1. Let B be a C^*-algebra and let $\{A_n\}_{n\in\mathbb{N}}$ be an increasing sequence of C^*-subalgebras of D. Put
$$A = \overline{\bigcup_{n\in\mathbb{N}} A_n},$$
and for each n let $i_n : A_n \to A$ be the inclusion map. Then $(A, \{i_n\}_{n\in\mathbb{N}})$ is the inductive limit of the sequence $A_1 \hookrightarrow A_2 \hookrightarrow A_3 \hookrightarrow \cdots$, where the connecting maps are inclusions.

2. Consider the sequence
$$\mathbb{C} \xrightarrow{\varphi_1} M_2(\mathbb{C}) \xrightarrow{\varphi_2} M_3(\mathbb{C}) \xrightarrow{\varphi_3} \cdots, \qquad \varphi_n(x) \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}.$$
The inductive limit of this sequence is isomorphic to the compact operators $\mathbb{K}$. In particular, $\mathbb{K}$ is an AF algebra.

2.2.5 Group C^*-Algebras and Crossed Products

Throughout this section, let G be a countable discrete group with identity element e.

The *complex group ring* of G is the ring of finitely supported complex-valued functions on G, denoted by $\mathbb{C}[G]$. Addition in $\mathbb{C}[G]$ is defined pointwise and multiplication is given by convolution:
$$ab(h) = \sum_{g\in G} a(g)b(g^{-1}h)$$
for every $a, b \in \mathbb{C}[G]$ and $h \in G$. We can furthermore define an involution on $\mathbb{C}[G]$ by setting $a^*(g) = \overline{a(g^{-1})}$ for every $a \in \mathbb{C}[G]$ and $g \in G$. This turns $\mathbb{C}[G]$ into a *-algebra. Now form the Hilbert space $\ell^2(G) = \{f : G \to \mathbb{C} \ : \ \sum_{g\in G} |f(g)|^2 < \infty\}$, and let $\{\delta_g \ : \ g \in G\} \subset \ell^2(G)$ be the canonical orthonormal basis. The *left-regular representation* of G is the homomorphism $\lambda : G \to \mathcal{U}(\mathbb{B}(\ell^2(G)))$ defined on basis vectors by $\lambda_g(\delta_h) = \delta_{gh}$ for every $g, h \in G$. The left-regular representation can be extended to an injective *-homomorphism (which we also denote by λ)
$$\lambda : \mathbb{C}[G] \to \mathbb{B}(\ell^2(G)) \qquad a \mapsto \sum_{g\in G} a(g)\lambda_g.$$

Definition 2.2.13 (Reduced group C^*-algebra)**.** The *reduced group C^*-algebra of G*, denoted by $C^*_r(G)$, is the completion of $\mathbb{C}[G]$ with respect to the norm
$$\|a\|_r = \|\lambda(a)\|_{\mathbb{B}(\ell^2(G))}.$$

Example 2.2.14.

1. If G is finite, then $C_r^*(G) \simeq \bigoplus_{i=1}^m M_{n_i}(\mathbb{C})$. Here m is the number of conjugacy classes of G, n_i are natural numbers that divide $|G|$, and $n_1^2 + \cdots + n_m^2 = |G|$ (see [52, Chapter 8] for example).

2. If G is abelian, then its Pontryagin dual $\hat{G}$ is a compact Hausdorff space. The C^*-algebra $C_r^*(G)$ is abelian, and there is a homeomorphism between $\Omega(C_r^*(G))$ and $\hat{G}$. In particular, we get
$$C_r^*(G) \simeq C(\hat{G}).$$
In the specific case $G = \mathbb{Z}$, the Pontryagin dual is the circle group $\mathbb{T}$. This gives an isomorphism $C_r^*(\mathbb{Z}) \simeq C(\mathbb{T})$.

In general, there are multiple ways to complete $\mathbb{C}[G]$ to get a C^*-algebra. Here is another one.

Definition 2.2.15. The *universal* group C^*-algebra of G, denoted by $C^*(G)$, is the completion of $\mathbb{C}[G]$ with respect to the norm

$$\|a\|_u = \sup\{\|\pi(a)\| \ : \ \pi : \mathbb{C}[G] \to \mathbb{B}(H) \text{ is a (cyclic) *-homomorphism}\}$$

It is evident that $C^*(G)$ surjects onto $C_r^*(G)$, but in general $C_r^*(G) \not\simeq C^*(G)$. For example, let $\mathbb{F}_2$ be the free noncommutative group on two generators. Then $C^*(\mathbb{F}_2)$ has a character, but Powers showed in [46] that $C_r^*(\mathbb{F}_2)$ is simple. However, for a certain nice class of groups, the reduced and universal group algebras coincide.

Definition 2.2.16. We say that G is *amenable* if, for every finite subset $E \subset G$ and $\epsilon > 0$, there exists a finite subset $F \subset G$ satisfying

$$\max_{g \in E} \frac{|gF \vartriangle F|}{|F|} < \epsilon,$$

where $\vartriangle$ denotes the symmetric difference between two sets.

Example 2.2.17. Finite groups and abelian groups are amenable. Moreover, the class of amenable groups is closed under taking subgroups, extensions, quotients, and inductive limits. The smallest class of groups containing finite groups and abelian groups, and which is closed under these four operations, is called the class of *elementary amenable groups.* Unfortunately, not every amenable group is elementary amenable.

Among the many characterizations of amenability, we have the following theorem.

Theorem 2.2.18. G is amenable if and only if $C_r^*(G) \simeq C^*(G)$.

We now review crossed products. These are generalizations of group C^*-algebras, but involve not only the group but also a C^*-algebra on which the group acts. In a sense, the crossed product captures the structure of the group, the C^*-algebra, and the action.

Definition 2.2.19. A *C^*-dynamical system* is a triple (A, G, α) consisting of a C^*-algebra A and a group homomorphism $\alpha : G \to \mathrm{Aut}(A)$.

Remark 2.2.20. If α is a single automorphism of A, there is an associated dynamical system with $G = \mathbb{Z}$. The homomorphism $\mathbb{Z} \to \mathrm{Aut}(A)$ is given by $n \mapsto \alpha^n$. Conversely, a dynamical system of the form $(A, \mathbb{Z}, \alpha)$ comes from a single automorphism given by $a \mapsto \alpha(1)(a)$ for $a \in A$.

Let $A[G]$ denote the set of finitely supported A-valued functions on G. Addition in $A[G]$ can be defined pointwise; and multiplication is given by an "α-twisted" convolution:

$$(ff')(h) = \sum_{g \in G} f(g)\alpha_g(f'(g^{-1}h))$$

for every $f, f' \in A[G]$ and $h \in G$. We can furthermore define f^* by setting $f^*(g) = \alpha_g(f(g^{-1})^*)$ for each $g \in G$ and $f \in A[G]$. This turns $A[G]$ into a *-algebra. Notice that if $A = \mathbb{C}$ (and α is trivial), then $A[G]$ is the complex group ring $\mathbb{C}[G]$.

Let H be a Hilbert space. A *covariant representation* of (A, G, α) on H is a pair (π, u) consisting of a *-representation $\pi : A \to \mathbb{B}(H)$ and a unitary representation $u : G \to \mathcal{U}(\mathbb{B}(H))$ satisfying $u_g \pi(a) u_g^* = \pi(\alpha_g(a))$ for every $a \in A$ and $g \in G$. Associated to (π, u) is a *-representation $\pi \rtimes u : A[G] \to \mathbb{B}(H)$, called its *integrated form*, given by

$$\pi \rtimes u(f) = \sum_{g \in G} \pi(f(g)) u_g$$

for each $f \in A[G]$. For a given H, the collection of nondegenerate *- representations of $A[G] \to \mathbb{B}(H)$ is in one-to-one correspondence with covariant representations of $A[G]$ on H. The integrated form of (π, u) gives one direction of this correspondence. To see the other direction in the unital case, define for each $a \in A$ and $g \in G$ the functions $d_a, d_g : G \to A$ in $A[G]$ by

$$d_a(h) = \begin{cases} a & \text{if } h = e \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad d_g(h) = \begin{cases} 1 & \text{if } h = g \\ 0 & \text{otherwise} \end{cases}.$$

If $\rho : A[G] \to \mathbb{B}(H)$ is a *-homomorphism, let $\pi_\rho : A \to \mathbb{B}(H)$ and $u_\rho : G \to \mathcal{U}(\mathbb{B}(H))$ be given by $\pi_\rho(a) = \rho(d_a)$ and $u_\rho(g) = \rho(d_g)$. One checks that $\rho = \pi_\rho \rtimes u_\rho$. The non-unital case is similar.

To put a C^*-norm on $A[G]$, we proceed similarly to the group algebra case.

Definition 2.2.21. Let H be a Hilbert space such that $A \subset \mathbb{B}(H)$, and define the covariant representation (π, λ) of (A, G, α) on $\ell^2(G) \otimes H$ by

$$\pi(a)(\delta_h \otimes x) = \delta_h \otimes \alpha_{h^{-1}}(a)x \qquad \text{and} \qquad \lambda_g(\delta_h \otimes x) = \delta_{gh} \otimes x.$$

for $x \in H$, $g, h \in G$, and $a \in A$. The *reduced crossed product* of (A, G, α), denoted by $A \rtimes_{\alpha,r} G$, is the norm closure of the image of $A[G]$ under the integrated form $\pi \rtimes \lambda : A[G] \to \mathbb{B}(\ell^2(G) \otimes H)$.

Remark 2.2.22. Although it is not immediately obvious, the C^*-algebra $A \rtimes_{\alpha,r} G$ is independent of the faithful representation $A \subset \mathbb{B}(H)$ (see [5, Proposition 4.1.5] for example).

Example 2.2.23.

1. Let $\theta \in \mathbb{R}$ and consider the homeomorphism $h_\theta : \mathbb{T} \to \mathbb{T}$ given by $h_\theta(z) = e^{-2\pi i\theta} z$ for $z \in \mathbb{T}$. There is an induced automorphism α_θ of the C^*-algebra $C(\mathbb{T})$ given by $\alpha(f) = f \circ h^{-1}$ for $f \in C(\mathbb{T})$. The crossed product $C(\mathbb{T}) \rtimes_{\alpha_\theta,r} \mathbb{Z}$ is called a *rotation algebra*, and is often denoted by A_θ. One can show that $A_0 \simeq C(\mathbb{T}^2)$, and that $A_\theta \simeq A_{\theta'}$ if and only if $\theta_1 + \theta_2 \in \mathbb{Z}$ or $\theta_1 - \theta_2 \in \mathbb{Z}$. The algebras A_θ are sometimes called *noncommutative tori*.

2. More generally, suppose X is a compact Hausdorff space and let $h : X \to X$ be a homeomorphism. The pair (X, h) is called a *classical dynamical system*. Associated to (X, h) is C^*-dynamical system coming from the automorphism α_h of $C(X)$ given by $\alpha_h(f) = f \circ h^{-1}$ for $f \in C(X)$. In principle, the crossed product $C(X) \rtimes_{\alpha_h,r} \mathbb{Z}$ captures information about the dynamical system. For example, (X, h) is said to be *minimal* if every orbit is dense in X. If X is infinite, then

$$h \text{ is minimal} \Leftrightarrow C(X) \rtimes_{\alpha_h,r} \mathbb{Z} \text{ is simple.}$$

   See [12, Theorem VIII.3.9] for a proof.

As with group algebras, there are multiple ways to complete $A[G]$ to get a C^*-algebra.

Definition 2.2.24. The *universal crossed product* of the C^*-dynamical system (A, G, α), denoted by $A \rtimes_\alpha G$, is the completion of $A[G]$ with respect to the norm

$$\|x\|_u = \sup\{\|\pi(x)\| \ : \ \pi : A[G] \to \mathbb{B}(H) \text{ is a (cyclic) *-homomorphism}\}.$$

Remark 2.2.25. An easy consequence of Definition 2.2.24 is that universal crossed products have the following property: for every covariant representation (π, u) of (A, G, α) on a Hilbert space H, there is a *-homomorphism $\rho : A \rtimes_\alpha G \to \mathbb{B}(H)$ such that $\rho(f) = \pi \rtimes u(f)$ for each $f \in A[G]$. In other words, $A \rtimes_\alpha \mathbb{Z}$ *-homomorphically surjects onto any other C^*-completion of $A[G]$.

Similar to the group algebra case, crossed products by amenable groups behave well with respect to norms. In particular, we have the following result.

Theorem 2.2.26. Let G be an amenable group and let (A, G, α) be a C^*-dynamical system. Then $A \rtimes_{\alpha,r} G \simeq A \rtimes_\alpha G$.

2.3 Nuclearity

Throughout this section, let A and B be C^*-algebras.

2.3.1 Tensor Products

The *-algebra $A \odot B$ is the algebraic tensor product of A and B with multiplication and involution given by

$$\left(\sum_i a_i \otimes b_i\right)\left(\sum_j c_j \otimes d_j\right) = \sum_{i,j} a_i c_j \otimes b_i d_j \quad \text{and} \quad \left(\sum_i a_i \otimes b_i\right)^* = \sum_i a_i^* \otimes b_i^*$$

for every $a_i, c_j \in A$ and $b_i, d_j \in B$. A *C^*-norm* $\|\cdot\|_\alpha$ on $A \odot B$ is a norm satisfying $\|xy\|_\alpha \leq \|x\|_\alpha \|y\|_\alpha$, $\|x\|_\alpha = \|x^*\|_\alpha$, and $\|x^*x\|_\alpha = \|x\|_\alpha^2$ for every $x, y \in A \odot B$. Denote by $A \otimes_\alpha B$ the completion of $A \odot B$ with respect to $\|\cdot\|_\alpha$.

Suppose now that there are Hilbert spaces H and K such that $A \subset \mathbb{B}(H)$ and $B \subset \mathbb{B}(K)$. If $a \in A$ and $b \in B$, the map $a \otimes_\sigma b : H \otimes K \to H \otimes K$ given by $\sum_i h_i \otimes k_i \mapsto \sum_i a(h_i) \otimes b(k_i)$ belongs to $\mathbb{B}(H \otimes K)$. The notation here is suggestive: the map defined by sending the elementary tensor $a \otimes b$ to the operator $a \otimes_\sigma b$ can be extended to an injective *-homomorphism $\sigma : A \odot B \to \mathbb{B}(H \otimes K)$. This means we can take a C^*-closure of $A \odot B$ with respect to the norm $\|x\|_\sigma = \|\sigma(x)\|_{\mathbb{B}(H \otimes K)}$ for each $x \in A \odot B$.

Definition 2.3.1. The *spatial tensor product* of A and B is the C^*-algebra $A \otimes_\sigma B$.

Remark 2.3.2. Just like with crossed products, the C^*-algebra $A \otimes_\sigma B$ is independent of the faithful representations $A \subset \mathbb{B}(H)$ and $B \subset \mathbb{B}(K)$ (see [5, Proposition 3.3.11] for example).

As before, there are various ways to complete $A \odot B$ to get a C^*-algebra.

Definition 2.3.3. The *maximal C^*-norm* on $A \odot B$ is given by

$$\|x\|_{\max} = \sup\{\|\pi(x)\| \ : \ \pi : A \odot B \to \mathbb{B}(H) \text{ is a (cyclic) *-homomorphism}\}$$

for $x \in A \odot B$. Denote by $A \otimes_{\max} B$ the completion of $A \odot B$ with respect to $\|\cdot\|_{\max}$.

By construction, it is obvious that $\|x\|_\alpha \leq \|x\|_{\max}$ for any C^*-norm $\|\cdot\|_\alpha$ and every $x \in A \odot B$. Furthermore, a theorem of Takesaki (see [59, Theorem 2]) implies that $\|x\|_\sigma \leq \|x\|_\alpha$ for every $x \in A \odot B$. In other words, every C^*-norm is "sandwiched" between the spatial and maximal tensor norms. Although $A \otimes_{\max} B$ always surjects onto $A \otimes_\sigma B$, in general there will be a non-trivial kernel. For example, a result of Junge and Pisier (see [29]) shows that $\mathbb{B}(\ell^2(\mathbb{N})) \otimes_\sigma \mathbb{B}(\ell^2(\mathbb{N})) \not\cong \mathbb{B}(\ell^2(\mathbb{N})) \otimes_{\max} \mathbb{B}(\ell^2(\mathbb{N}))$.

Definition 2.3.4. A C^*-algebra A is said to be *nuclear* if for any other C^*-algebra B, there is a unique C^*-norm on the *-algebra $A \odot B$. In other words, A is nuclear if and only if $A \otimes_\sigma B \simeq A \otimes_{\max} B$ for every B. If A is nuclear, we will write $A \otimes B$ since there is no ambiguity.

As for permanence properties, we have the following. See [5] for proofs.

Proposition 2.3.5. Nuclearity is preserved under direct sums, tensor products, hereditary subalgebras, quotients, extensions, and inductive limits.

Examples 2.3.6. The class of nuclear C^*-algebras contains many important examples.

1. Every abelian C^*-algebra is nuclear. In fact, nuclearity is often regarded as the noncommutative analogue of having a partition of unity, since the major ingredient in proving $C_0(X)$ is nuclear is the existence of partitions of unity.

2. Finite-dimensional C^*-algebras are nuclear. More generally, inductive limits of nuclear C^*-algebras are nuclear, so in particular $\mathbb{K}$ is nuclear. A C^*-algebra A is called *stable* if $A \otimes \mathbb{K} \simeq A$.

3. If G is a discrete group, then G is amenable if and only if $C^*(G)$ (and hence $C_r^*(G)$) is nuclear. More generally, if A is a nuclear C^*-algebra and (A, G, α) is a C^*-dynamical system with an amenable group G, then $A \rtimes_\alpha G$ (and hence $A \rtimes_{\alpha,r} G$) is nuclear. We will see a proof in the case $G = \mathbb{Z}$ in 6.1, but a very general result was proven by Green in [25, Proposition 14].

2.3.2 The Completely Positive Approximation Property

Definition 2.3.7. A map $\theta : A \to B$ is called *nuclear* if, for any finite subset $F \subset A$ and any $\epsilon > 0$, there exist c.p.c. maps $\varphi : A \to M_n(\mathbb{C})$ and $\psi : M_n(\mathbb{C}) \to B$ such that

$$\|\psi \circ \varphi(a) - \theta(a)\| < \epsilon$$

for every $a \in F$. Nuclearity is often called the *completely positive approximation property* (or CPAP for short).

Remark 2.3.8. Loosening the norm requirement on the map ψ and asking only for a bound on $\|\psi\|$ that is independent of F and ϵ yields an equivalent definition of a nuclear map.

To justify the terminology, we have the following essential theorem.

Theorem 2.3.9 (Choi, Effros, Kirchberg)**.** Let A be a C^*-algebra. The following statements are equivalent:

1. A is a nuclear.

2. The identity map $\mathrm{id}_A : A \to A$ is nuclear.

We end this section with a classical result. If $J \trianglelefteq B$ is an ideal, a c.p.c. map $\theta : A \to B/J$ is called *liftable* if there is a c.p.c. map $\bar{\theta} : A \to B$ such that $\theta = Q \circ \bar{\theta}$ where $Q : B \to B/J$ is the quotient map. In [7], Choi and Effros proved the following essential theorem.

Theorem 2.3.10 (Choi-Effros Lifting Theorem)**.** Every nuclear c.p.c. map from a separable C^*-algebra A into a quotient C^*-algebra B/J is liftable. In particular, every c.p.c. map from a separable nuclear C^*-algebra is liftable.

Remark 2.3.11. The separability assumption in Theorem 2.3.10 is essential, as there is no bounded linear lifting from ℓ^∞/c_0 to ℓ^∞.

2.4 Quasidiagonality

Definition 2.4.1. A C^*-algebra A is *quasidiagonal* (QD for short) if there exists a sequence of c.p.c. maps $\varphi_n : A \to M_{k_n}(\mathbb{C})$ which are asymptotically multiplicative and isometric:

$$\lim_{n\to\infty} \|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \|\varphi_n(a)\| = \|a\|$$

for every $a, b \in A$.

C^*-subalgebras of QD C^*-algebras are QD. If $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of QD C^*-algebras, then $\prod A_n$ (and hence $\sum A_n$) is QD. Quasidiagonality passes to inductive limits if the connecting maps are injective. The spatial tensor product of QD algebras is QD, but it is currently unknown if the same holds for maximal tensor products.

The historic roots of quasidiagonality are in single operator theory. An operator $T \in \mathbb{B}(H)$ is said to be *quasidiagonal* if there an orthonormal basis of H with respect to which T is a block-diagonal operator plus a compact operator. More generally, a (norm separable) subset $\Omega \subset \mathbb{B}(H)$ is *quasidiagonal* if there exists an increasing sequence of compact projections P_n that converge strongly to the identity (meaning $P_n\xi \to \xi$ for every $\xi \in H$) and satisfy $\|P_nT - TP_n\| = 0$ for every $T \in \Omega$. A representation $\pi : A \to \mathbb{B}(H)$ of a (separable) C^*-algebra is *quasidiagonal* if the image $\pi(A) \subset \mathbb{B}(H)$ is a quasidiagonal set of operators. We say that A is *strongly quasidiagonal* if every representation of A on a separable Hilbert space is quasidiagonal. The following theorem of Voiculescu (see [63]) proves that this notion coincides with Definition 2.4.1.

Theorem 2.4.2. Let A be a separable C^*-algebra. Then the following statements are equivalent:

1. A is QD.

2. A has a faithful quasidiagonal representation on a separable Hilbert space.

3. Every faithful unital representation $\pi : A \to \mathbb{B}(H)$ on a separable Hilbert space that satisfies $\pi(A) \cap \mathbb{K}(H) = \{0\}$ is quasidiagonal.

A routine computation shows that QD C^*-algebras are stably finite. However, it is an important (and seemingly very difficult) open problem to determine for which class of C^*-algebras does stable-finiteness imply quasidiagonality. In general, it is not true that every stably finite C^*-algebra is QD, but the counterexamples are non-nuclear. In [2, Question 7.3.1], Blackadar and Kirchberg ask whether every (separable) stably finite nuclear C^*-algebra is QD.

A remarkable aspect of the quasidiagonal condition is that it is invariant under homotopy equivalence. Proven by Voiculescu in [63], this topological feature is partly the reason why QD algebras behave differently from other classes, and why their structure is not very well understood. A C^*-algebra A is said to *homotopically dominate* a C^*-algebra B if there are *-homomorphisms $\varphi : B \to A$ and $\psi : A \to B$ such that $\psi \circ \varphi \sim_h \mathrm{id}_B$. If furthermore $\varphi \circ \psi \sim_h \mathrm{id}_A$, then A and B are called *homotopically equivalent*.

Theorem 2.4.3 (Voiculescu)**.** If A homotopically dominates B and A is QD, then B is also QD. In particular, quasidiagonality is a homotopy-equivalence invariant.

Example 2.4.4.

1. Finite dimensional C^*-algebras are obviously QD. If $A \simeq C_0(\Omega(A))$, then the direct sums of point evaluations give the requisite sequence of maps from A to finite-dimensional C^*-algebras. Hence, abelian C^*-algebras are QD.

2. If A is any C^*-algebra, the *cone over* A is the defined as $CA = C_0(0,1] \otimes A$. Since CA is homotopic to the zero C^*-algebra, it is QD.

Chapter 3
C^*-Correspondences

3.1 Hilbert Modules

Definition 3.1.1. Let A be a C^*-algebra. An *inner product A–module* is a complex vector space $\mathcal{H}$ which is a right A–module (with compatible scalar multiplication: $\lambda(x.a) = (\lambda x).a = x.\lambda a$ for $x \in \mathcal{H}$, $a \in A$, and $\lambda \in \mathbb{C}$), together with a map $\langle\ ,\ \rangle : \mathcal{H} \times \mathcal{H} \to A$ such that for every $x, y, z \in \mathcal{H}$, $\alpha, \beta \in \mathbb{C}$, and $a \in A$,

1. $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$,
2. $\langle x, y.a\rangle = \langle x, y\rangle a$,
3. $\langle y, x\rangle^* = \langle x, y\rangle$,
4. $\langle x, x\rangle \geq 0$; if $\langle x, x\rangle = 0$ then $x = 0$.

If $x \in \mathcal{H}$, we can define $\|x\|$ to be $\|\langle x, x\rangle\|^{1/2}$. The conditions above guarantee that $\|\cdot\|$ is a norm on $\mathcal{H}$, and we say that $\mathcal{H}$ is a *Hilbert A–module* if it is complete with respect to $\|\cdot\|$. Two Hilbert A–modules $\mathcal{H}$ and $\mathcal{K}$ are *isomorphic* if there is a linear bijection $\psi : \mathcal{H} \to \mathcal{K}$ such that $\langle\psi(x), \psi(y)\rangle_{\mathcal{K}} = \langle x, y\rangle_{\mathcal{H}}$ for every $x, y \in \mathcal{H}$. A set $E \subset \mathcal{H}$ is said to *generate* $\mathcal{H}$ when the submodule $\sum_A E$ of finite A–linear sums is a dense set in $\mathcal{H}$. We say $\mathcal{H}$ is *free* if it has an orthonormal set of generators; we say $\mathcal{H}$ is *finitely generated projective* if it is a direct summand of a finitely generated free module.

Definition 3.1.2. Suppose that $\mathcal{H}$ and $\mathcal{K}$ are Hilbert A–modules. We define $\mathbb{B}(\mathcal{H}, \mathcal{K})$ to be the set of all maps $T : \mathcal{H} \to \mathcal{K}$ for which there is a map $T^* : \mathcal{K} \to \mathcal{H}$ such that

$$\langle Tx, y\rangle = \langle x, T^* y\rangle \quad \forall x \in \mathcal{H}, y \in \mathcal{K}.$$

If $x \in \mathcal{H}$ and $y \in \mathcal{K}$, define $e_{y,x} \in \mathbb{B}(\mathcal{H}, \mathcal{K})$ by

$$e_{y,x}(z) = y.\langle x, z\rangle.$$

Denote by $\mathbb{K}(\mathcal{H},\mathcal{K})$ the closed linear subspace of $\mathbb{B}(\mathcal{H},\mathcal{K})$ spanned by $\{e_{y,x} \mid x \in \mathcal{H},\ y \in \mathcal{K}\}$. If $\mathcal{H} = \mathcal{K}$, we will write $\mathbb{B}(\mathcal{H})$ and $\mathbb{K}(\mathcal{H})$. It is an important fact that $\mathbb{B}(\mathcal{H})$ and $\mathbb{K}(\mathcal{H})$ are C^*-algebras; we call them *adjointable* and *compact* operators on $\mathcal{H}$, respectively.

Definition 3.1.3. The *strict topology* on $\mathbb{B}(\mathcal{H},\mathcal{K})$ is defined by the seminorms $T \mapsto \|Tx\|$ (for $x \in \mathcal{H}$) and $T \mapsto \|T^*y\|$ (for $y \in \mathcal{K}$). In other words, $T_n \to T$ strictly in $\mathbb{B}(\mathcal{H},\mathcal{K})$ if and only if

$$\lim_n \|(T_n - T)(x)\|_{\mathcal{K}} = 0 \quad \text{and} \quad \lim_n \|(T_n^* - T^*)(y)\|_{\mathcal{H}} = 0$$

for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$.

Remark 3.1.4. In [30], Kasparov proved a substantial generalization of the fact that $\mathcal{M}(\mathbb{K}) \simeq \mathbb{B}(\ell^2(\mathbb{N}))$. He showed that if $\mathcal{H}$ Hilbert A-module, then $\mathcal{M}(\mathbb{K}(\mathcal{H})) \simeq \mathbb{B}(\mathcal{H})$.

Here is an important class of examples.

Examples 3.1.5. If A is a C^*-algebra, then A itself is a Hilbert A–module if we define $\langle a, b\rangle = a^*b$. If A is unital, then $A \simeq \mathbb{K}(A) = \mathbb{B}(A)$; if A is not unital, then $A \simeq \mathbb{K}(A) \subset \mathbb{B}(A) \simeq \mathcal{M}(A)$. More generally, if $\{\mathcal{H}_i\}_{i\in I}$ is a countable collection of Hilbert A–modules, we can form their direct sum $\bigoplus_i \mathcal{H}_i$ by considering all formal sums $\sum_i i.x_i$ such that $x_i \in \mathcal{H}_i$ and $\sum_i \langle x_i, x_i\rangle$ converges in A. If $x = \sum_i i.x_i$ and $y = \sum_i i.y_i$, we define $\langle x, y\rangle$ to be $\sum_i \langle x_i, y_i\rangle$. In the case where $\mathcal{H}_i = A$ for every i, it can easily be shown that $\mathbb{K}(\bigoplus_I A)$ is isomorphic to the C^*-tensor product $A \otimes \mathbb{K}$.

Remark 3.1.6. Notice that if $\mathcal{H}$ is a countably generated free Hilbert A–module with free generators I, then $\mathcal{H} \simeq \bigoplus_I A$.

The following essential theorem of Kasparov (proven in [30]) shows that all countably generated Hilbert modules are direct summands in a free module. This is automatic for Hilbert spaces (in fact they are free by the Gram-Schmidt procedure), but is nontrivial for Hilbert modules.

Theorem 3.1.7 (Kasparov's Stabilization Theorem)**.** If $\mathcal{H}$ is a countably generated Hilbert A–module, then there is a countably generated free Hilbert A-module $\mathcal{H}'$ satisfying

$$\mathcal{H} \oplus \mathcal{H}' \simeq \mathcal{H}'.$$

Definition 3.1.8. Let A be a C^*-algebra and $\mathcal{H}$ a Hilbert A-module. We call $\mathcal{H}$ a *C^*-correspondence* over A if there is a unital injective *-homomorphism $\omega : A \to \mathbb{B}(\mathcal{H})$. The map ω is called the left action of A on $\mathcal{H}$, and we write $\omega_a(x)$ as $a.x$ when there is no confusion. If $\{\mathcal{H}_i\}_{i\in I}$ is a collection of C^*-correspondences over A, then the direct sum $\bigoplus \mathcal{H}_i$ is naturally a C^*-correspondence over A with the left action given by $\omega_{\bigoplus \mathcal{H}_i} = \bigoplus \omega_{\mathcal{H}_i}$. The simplest C^*-correspondence is the *identity correspondence*, which is the Hilbert A-module A with the left action given by multiplication from the left.

Remark 3.1.9. C^*-correspondences are defined by some authors for non-unital C^*-algebras and without any assumptions on $\omega(1)$ and $\ker\omega$. In all that follows, we stick with Definition 3.1.8.

Definition 3.1.10. If $\mathcal{H}, \mathcal{K}$ are C^*-correspondences over A, the algebraic tensor product $\mathcal{H} \odot \mathcal{K}$ is naturally a right A–module with an A–valued semi-inner product given by

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle y_1, \langle x_1, x_2 \rangle . y_2 \rangle.$$

Denote by $\mathcal{H} \otimes \mathcal{K}$ the Hilbert A-module obtained from $\mathcal{H} \odot \mathcal{K}$ by separation and completion. It is easily checked that $x.a \otimes y = x \otimes a.y$ for every $x, \in \mathcal{H}$, $y \in \mathcal{K}$, and $a \in A$. There is a natural faithful *-representation $\mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{H} \otimes \mathcal{K})$ given by $T \mapsto T \otimes 1$, where $(T \otimes 1)(x \otimes y) = Tx \otimes y$. In particular, $\mathcal{H} \otimes \mathcal{K}$ is a C^*-correspondence over A. We call this correspondence the *interior tensor product* of $\mathcal{H}$ and $\mathcal{K}$. Elements of the form $h \otimes k \in \mathcal{H} \otimes \mathcal{K}$ are called *elementary tensors*. We say a C^*-correspondence is *free* (resp. *finitely generated projective*) if the underlying Hilbert module is free (resp. finitely generated projective).

Remarks 3.1.11.

1. In the context of Hilbert modules, there are two important notions of tensor product: *interior* and *exterior*. Definition 3.1.10 defines the interior tensor product; we will not discuss exterior tensor products aside from this remark.

2. Note that there are natural identifications $\mathcal{H} \otimes A \simeq \mathcal{H}$ and $A \otimes \mathcal{H} \simeq \mathcal{H}$.

Lemma 3.1.12. Let $\mathcal{H}$ and $\mathcal{K}$ be as in Definition 3.1.10. Regarding A as a subalgebra of $\mathbb{B}(\mathcal{K})$, there is a natural *-representation

$$A' \cap \mathbb{B}(\mathcal{K}) \to \mathbb{B}(\mathcal{H} \otimes \mathcal{K})$$

given by $T \mapsto 1 \otimes T$.

Definition 3.1.13. Let $\mathcal{H}$ be a C^*-correspondence over A. A *representation of $\mathcal{H}$ on a C^*-algebra B* is a pair (π, τ) consisting of a *-homomorphism $\pi : A \to B$ and a linear map $\tau : \mathcal{H} \to B$ satisfying

$$\tau(a.x.b) = \pi(a)\tau(x)\pi(b) \quad \text{and} \quad \tau(x)^*\tau(y) = \pi(\langle x, y\rangle).$$

We denote by $C^*(\pi, \tau)$ the C^*-subalgebra of B generated by $\pi(A)$ and $\tau(\mathcal{H})$. A representation $(\tilde{\pi}, \tilde{\tau})$ of $\mathcal{H}$ is *universal* if for any other representation (π, τ) of $\mathcal{H}$, there is a *-homomorphism from $C^*(\tilde{\pi}, \tilde{\tau})$ onto $C^*(\pi, \tau)$ sending $\tilde{\pi}(a)$ to $\pi(a)$ and $\tilde{\tau}(\xi)$ to $\tau(\xi)$.

Definition 3.1.14. If (π, τ) is a representation of $\mathcal{H}$ on B, then the map

$$e_{x,y} \mapsto \tau(x)\tau(y)^*$$

extends to a *-homomorphism from $\mathbb{K}(\mathcal{H})$ to B. We say that (π, τ) is *covariant* if $\pi(a) = \sigma_\tau(a)$ for every $a \in A \cap \mathbb{K}(\mathcal{H})$. A covariant representation $(\tilde{\pi}, \tilde{\tau})$ of $\mathcal{H}$ is *universal* if for any other covariant representation (π, τ) of $\mathcal{H}$, there is a *-homomorphism from $C^*(\tilde{\pi}, \tilde{\tau})$ onto $C^*(\pi, \tau)$ sending $\tilde{\pi}(a)$ to $\pi(a)$ and $\tilde{\tau}(\xi)$ to $\tau(\xi)$.

Definition 3.1.15. A representation (π, τ) *admits a gauge action* if there is an action β of $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ on $C^*(\pi, \tau)$ such that

$$\beta_z(\pi(a)) = \pi(a) \quad \text{and} \quad \beta_z(\tau(\xi)) = z\tau(\xi).$$

In [21], Fowler, Muhly, and Raeburn proved the *gauge invariance uniqueness* theorem.

Theorem 3.1.16. Let $\mathcal{H}$ be a C^*-correspondence over A and (π, τ) a representation of $\mathcal{H}$ on B that admits a gauge action. Assume furthermore that π is faithful. Then the following is true.

1. If $\pi(A) \cap \overline{\text{span}}\{\tau(x)\tau(y)^* : x, y \in \mathcal{H}\} = \{0\}$, then (π, τ) is universal.
2. If (π, τ) is covariant, then it is universal.

3.2 Toeplitz-Pimsner and Cuntz-Pimsner Algebras

Definition 3.2.1. Let $\mathcal{H}$ be a C^*-correspondence over A. Set $\mathcal{H}^{\otimes 0} = A$, the identity correspondence, and $\mathcal{H}^{\otimes k} = \mathcal{H} \otimes \cdots \otimes \mathcal{H}$, the k–fold tensor product. The *full Fock space* over $\mathcal{H}$ is the C^*-correspondence $\mathcal{F}(\mathcal{H})$ over A defined by

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{k=0}^{\infty} \mathcal{H}^{\otimes k}.$$

For $p \in \mathbb{N}$, denote by $\mathcal{F}_p(\mathcal{H})$ the p^{th}*-cutoff Fock space* $\bigoplus_{k=0}^{p-1} \mathcal{H}^{\otimes k}$. We say an elementary tensor $x = x_1 \otimes \cdots \otimes x_k \in \mathcal{H}^{\otimes k} \subset \mathcal{F}(\mathcal{H})$ has *length* k and write $|x| = k$. Note that elementary tensors of length zero correspond to elements in (the Hilbert module) A.

The left action of A on $\mathcal{F}(\mathcal{H})$ is given by

$$a.(x_1 \otimes \cdots \otimes x_k) = (a.x_1) \otimes \cdots \otimes x_k.$$

For each $x \in \mathcal{H}$ we define $T_x \in \mathbb{B}(\mathcal{F}(\mathcal{H}))$ by

$$T_x(a) = x.a \quad \text{and} \quad T_x(x_1 \otimes \cdots \otimes x_n) = x \otimes x_1 \otimes \cdots \otimes x_n.$$

The operators T_x are called *creation operators*. For $a \in \mathcal{H}^{\otimes 0}$, set $T_a = a$; for $x = x_1 \otimes \cdots \otimes x_k \in \mathcal{H}^{\otimes k}$ set $T_x = T_{x_1} \cdots T_{x_k}$. In [45], Pimsner considered the C^*-algebra generated by A and creation operators.

Definition 3.2.2. Let $\mathcal{H}$ be a C^*-correspondence over A. The *Toeplitz-Pimsner algebra* $\mathcal{T}(\mathcal{H})$ is the C^*-subalgebra of $\mathbb{B}(\mathcal{F}(\mathcal{H}))$ generated by A and $\{T_x \mid x \in \mathcal{H}\}$.

The following was proven by Pimsner in [45].

Theorem 3.2.3. Let $\mathcal{T}(\mathcal{H})$ be the Toeplitz-Pimsner algebra of a C^*-correspondence $\mathcal{H}$ over A.

1. For every $\alpha \in \mathbb{C}$, $x, y \in \mathcal{H}$, and $a, b \in A$, the creation operators satisfy

$$T_{\alpha x + y} = \alpha T_x + T_y, \quad T_{a.x.b} = a T_x b, \quad T_x^* T_y = \langle x, y \rangle.$$

   In particular, $\mathcal{T}(\mathcal{H}) = \overline{\operatorname{span}}\{T_x T_y^* \mid x \in \mathcal{H}^{\otimes k},\ y \in \mathcal{H}^{\otimes l},\ \text{and } k, l \geq 0\}$.

2. There exists a non-degenerate conditional expectation $E_{\mathcal{H}}$ from $\mathcal{T}(\mathcal{H})$ onto A such that $E_{\mathcal{H}}(T_x T_y^*) = 0$ for every $x \in \mathcal{H}^{\otimes k}$ and $y \in \mathcal{H}^{\otimes l}$ with $(k, l) \neq (0, 0)$.

3. The representation $(a \mapsto a, x \mapsto T_x)$ of $\mathcal{H}$ on $\mathcal{T}(\mathcal{H})$ admits a gauge action.

Combining Theorems 3.2.3 and 3.1.16 shows the following.

Theorem 3.2.4. The representation $(a \mapsto a, x \mapsto T_x)$ of $\mathcal{H}$ on $\mathcal{T}(\mathcal{H})$ is universal.

Remark 3.2.5. Using "rank-one" operators $e_{x,y} \in \mathbb{K}(\mathcal{H}^{\otimes |x|}, \mathcal{H}^{\otimes |y|})$ associated to elementary tensors in $\mathcal{F}(\mathcal{H})$, we have the Fock space representation

$$T_x T_y^* = \sum_{k=0}^{\infty} e_{x,y} \otimes 1_{\mathcal{H}^{\otimes k}}, \tag{3.1}$$

where convergence is understood in the strict topology of $\mathbb{B}(\mathcal{F}(\mathcal{H}))$.

The Toeplitz-Pimsner algebra $\mathcal{T}(\mathcal{H})$ is too large for many purposes, so we define the Cuntz-Pimsner algebra $\mathcal{O}(\mathcal{H})$ to be a natural quotient of $\mathcal{T}(\mathcal{H})$. Denote by $J(\mathcal{H})$ the C^*-algebra generated in $\mathbb{B}(\mathcal{F}(\mathcal{H}))$ by

$$\mathbb{B}\Big(\bigoplus_{k=0}^{\text{finite}} \mathcal{H}^{\otimes k}\Big).$$

The multiplier algebra $\mathcal{M}(J(\mathcal{H}))$ can be identified with all $T \in \mathbb{B}(\mathcal{F}(\mathcal{H}))$ satisfying both $TJ(\mathcal{H}) \subset J(\mathcal{H})$ and $J(\mathcal{H})T \subset J(\mathcal{H})$. In particular, there is an inclusion $\mathcal{T}(\mathcal{H}) \subset \mathcal{M}(J(\mathcal{H}))$.

Definition 3.2.6. The *Cuntz-Pimsner algebra* $\mathcal{O}(\mathcal{H})$ is the C^*-algebra $Q(\mathcal{T}(\mathcal{H}))$, where $Q : \mathcal{M}(J(\mathcal{H})) \to \mathcal{M}(J(\mathcal{H}))/J(\mathcal{H})$ is the quotient map. We denote by S_x the image of the creation operator T_x under Q.

There is another description of the Cuntz-Pimsner algebra, proven by Pimsner in [45]. Let $I_{\mathcal{H}} = A \cap \mathbb{K}(\mathcal{H}) \subset \mathbb{B}(\mathcal{H})$. Since $I_{\mathcal{H}}$ is an ideal in A and $\mathcal{F}(\mathcal{H})I_{\mathcal{H}}$ is a $\mathbb{B}(\mathcal{F}(\mathcal{H}))$-invariant C^*-subcorrespondence of $\mathcal{F}(\mathcal{H})$, the C^*-algebra

$$\mathbb{K}(\mathcal{F}(\mathcal{H})I_{\mathcal{H}}) = \overline{\operatorname{span}}\{e_{x,y} \mid x, y \in \mathcal{F}(\mathcal{H})I_{\mathcal{H}}\}$$

is an ideal in $\mathbb{B}(\mathcal{F}(\mathcal{H}))$.

Proposition 3.2.7. $\mathbb{K}(\mathcal{F}(\mathcal{H})I_{\mathcal{H}}) \subset \mathcal{T}(\mathcal{H})$, and in particular $\mathbb{K}(\mathcal{F}(\mathcal{H})I_{\mathcal{H}}) = \ker Q|_{\mathcal{T}(\mathcal{H})}$. Hence,

$$\mathcal{O}(\mathcal{H}) \simeq \mathcal{T}(\mathcal{H})/\mathbb{K}(\mathcal{F}(\mathcal{H})I_{\mathcal{H}}).$$

Proposition 3.2.8. Let $\mathcal{O}(\mathcal{H})$ be the Cuntz-Pimsner algebra of a C^*-correspondence $\mathcal{H}$ over A.

1. For every $\alpha \in \mathbb{C}, x, y \in \mathcal{H}$, and $a, b \in A$, the following holds:

$$S_{\alpha x+y} = \alpha S_x + S_y, \quad S_{a.x.b} = aS_x b, \quad S_x^* S_y = \langle x, y\rangle.$$

2. The representation $(a \mapsto a, x \mapsto S_x)$ of $\mathcal{H}$ on $\mathcal{O}(\mathcal{H})$ is covariant and universal.

Remarks 3.2.9.

1. If $\mathcal{H}$ is a finitely generated projective C^*-correspondence, then $\mathbb{K}(\mathcal{H}) = \mathbb{B}(\mathcal{H})$ (see [65]) and hence $A \cap \mathbb{K}(\mathcal{H}) = A$. Since A is unital, $\mathbb{K}(\mathcal{F}(\mathcal{H})I_{\mathcal{H}}) = \mathbb{K}(\mathcal{F}(\mathcal{H}))$. This shows $\mathcal{T}(\mathcal{H})$ contains all of $\mathbb{K}(\mathcal{F}(\mathcal{H}))$.

2. On the other extreme, if $A \cap \mathbb{K}(\mathcal{H}) = \{0\}$, the kernel of Q is trivial and hence there is a *-isomorphism $\mathcal{O}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ sending a to a and S_x to T_x.

Example 3.2.10. Viewing $\mathbb{C}^n$ as a C^*-correspondence over $\mathbb{C}$, the Toeplitz-Pimsner algebra $\mathcal{T}(\mathbb{C}^n)$ is generated by isometries $T_1, \ldots, T_n$ such that $1 - \sum_{i=1}^n T_i T_i^*$ is a rank-one projection. It is easily checked that $I_{\mathbb{C}} = \mathbb{C}$, so $\mathcal{O}(\mathbb{C}^n)$ is generated by isometries $S_1, \ldots, S_n$ satisfying $\sum_{i=1}^n S_i S_i^* = 1$. If $n = 1$, then $\mathcal{T}(\mathbb{C})$ is the classical Toeplitz algebra generated by the unilateral shift on $\ell^2(\mathbb{N})$ and $\mathcal{O}(\mathbb{C})$ is the algebra of continuous functions on the unit circle $\mathbb{T}$. For $n > 1$, the algebras $\mathcal{O}(\mathbb{C}^n)$ are the Cuntz algebras $\mathcal{O}_n$ from [10].

Here is an important permanence property, originally proven by Dykema and Shlyakhtenko in [13].

Theorem 3.2.11. Let $\mathcal{H}$ be a C^*-correspondence over A. Then the Toeplitz-Pimsner algebra $\mathcal{T}(\mathcal{H})$ is nuclear (resp. exact) if and only if A is.

Chapter 4
Classification of Simple Nuclear C^*-Algebras

4.1 K-Theory for C^*-Algebras

We now give a brief overview of K-theory for C^*-algebras. The following exposition essentially follows [50].

4.1.1 Ordered Abelian Groups

Definition 4.1.1.

1. A pair (G, G^+) is called a *preordered abelian group* if G is an abelian group and $G^+ \subseteq G$ is a subset satisfying $0 \in G^+$ and $G^+ + G^+ \subseteq G^+$. If $g, h \in G$, write $g \le h$ if $h - g \in G^+$ and $g < h$ if $h - g \in G^+ \backslash \{0\}$. An element $u \in G^+$ is an *order unit* if for every $g \in G$, there exists $n \in \mathbb{N}$ such that $-nu \le g \le nu$. We say (G, G^+) is *simple* if every element in G^+ is an order unit.

2. The *state space* of (G, G^+, u), denoted by $S(G, G^+, u)$, is the set of all homomorphisms $f : G \to \mathbb{R}$ such that $f(G^+) \subset \mathbb{R}^+$ and $f(u) = 1$. We write $S(G)$ in place of $S(G, G^+, u)$ if G^+ and u are understood from context. When equipped with the weak*-topology, $S(G)$ is a compact convex set.

3. An *ordered abelian group* is a preordered abelian group (G, G^+) that also satisfies $G^+ - G^+ = G$ and $G^+ \cap (-G^+) = \{0\}$. A triple (G, G^+, u) consisting of an ordered abelian group (G, G^+) and an order unit u is called an *ordered abelian group with a distinguished order unit.*

4. Let (G, G^+) and (H, H^+) be ordered abelian groups. A homomorphism $\alpha : G \to H$ is *positive* if $\alpha(G^+) \subseteq H^+$; it is an *order isomorphism* if α is an isomorphism and $\alpha(G^+) = H^+$. If (G, G^+, u) and (H, H^+, v) are ordered abelian groups with distinguished

order units, a positive homomorphism $\alpha : G \to H$ is *order unit preserving* if $\alpha(u) = v$. The triples (G, G^+, u) and (H, H^+, v) are *isomorphic* (written $(G, G^+, u) \simeq (H, H^+, v)$) if there is an order unit preserving order isomorphism $\alpha : G \to H$.

5. An ordered abelian group (G, G^+) is *unperforated* if, for any $g \in G$, we have $g \geq 0$ whenever $ng \geq 0$ for some $n \in \mathbb{N}$. If (G, G^+) is simple and $g \geq 0$ whenever $ng > 0$ for some $n \in \mathbb{N}$, we say (G, G^+) is *weakly unperforated.* (G, G^+) has the *Riesz interpolation property* if, whenever $g_1, g_2, h_1, h_2 \in G$ satisfy $g_i \leq h_j$ for $i, j = 1, 2$, there exists $x \in G$ satisfying $g_i \leq x \leq h_j$ for $i, j = 1, 2$.

Example 4.1.2.

1. Let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. The ordered abelian group $(\mathbb{Z}, \mathbb{Z}_+)$ is simple.
2. Consider the group G of all sequences $\{x_n\}_{n\in\mathbb{N}}$ of integers x_n such that $x_n = 0$ for all but finitely many n. Let G^+ consist of those sequences $\{x_n\}_{n\in\mathbb{N}}$ for which $x_n \geq 0$ for all $n \in \mathbb{N}$. Then (G, G^+) is an ordered abelian group with no order unit.

Consider a sequence

$$G_1 \xrightarrow{\alpha_1} G_2 \xrightarrow{\alpha_2} G_3 \xrightarrow{\alpha_3} \cdots \tag{4.1}$$

of abelian groups with homomorphisms $\alpha_n : G_n \to G_{n+1}$. An *inductive limit* of this sequence is a pair $(G, \{\beta_n\}_{n\in\mathbb{N}})$ consisting of an abelian group G and a sequence of homomorphisms $\beta_n : G_n \to G$ that satisfy the following two properties. First, $\beta_{n+1} \circ \alpha_n = \beta_n$ for every $n \in \mathbb{N}$. Second, if $(H, \{\gamma_n\}_{n\in\mathbb{N}})$ is another pair consisting of an abelian group H and a sequence of homomorphisms $\gamma_n : G_n \to H$ satisfying $\gamma_{n+1} \circ \alpha_n = \gamma_n$ for every $n \in \mathbb{N}$, then there is a unique homomorphism $\lambda : G \to H$ such that $\lambda \circ \beta_n = \gamma_n$ for every $n \in \mathbb{N}$. If the groups G_n happen to be ordered abelian groups and $\{\alpha_n : G_n \to G_{n+1}\}_{n\in\mathbb{N}}$ is a sequence of positive homomorphisms, then $(G, \{\beta_n\}_{n\in\mathbb{N}})$ is an *inductive limit* if the same two conditions hold, with the additional requirement that G and H as above are ordered abelian groups, and the maps β_n, γ_n and λ are positive homomorphisms.

As in the case of C^*-algebras, inductive limits of (ordered) abelian groups exist and are unique. Uniqueness follows from the definition, and to show existence we proceed similarly to the C^*-case. If we have a sequence as in (4.1), let $\prod G_m$ be the set of all infinite sequences $(g_m)_{m\in\mathbb{Z}}$ such that $g_m \in G_m$ for all $m \in \mathbb{N}$. We can turn G into a group with entry-wise addition. Let $\sum G_m$ be the subgroup consisting of those sequences for which $g_m = 0$ for all but finitely many m, and let $q : \prod G_m / \sum G_m$ be the quotient map. For each $n \in \mathbb{N}$, let $\bar{\beta}_n : G_n \to \prod G_m$ denote the homomorphism given by $g \mapsto (\alpha_{m,n}(g))_{m\in\mathbb{N}}$, where as before $\alpha_{m,n} = \alpha_{m-1} \circ \cdots \circ \alpha_n$ if $m > n$, $\alpha_{n,n} = \mathrm{id}_{G_n}$, and $\alpha_{m,n} = 0$ if $m < n$. Let $\beta_n = q \circ \bar{\beta}_n$ and put

$$G = \bigcup_{n=1}^{\infty} \beta_n(G_n) \subseteq \prod G_m / \sum G_m.$$

Then $(G, \{\beta_n\}_{n\in\mathbb{N}})$ is the inductive of the sequence (4.1), and we denote it by $\varinjlim G_n$.

If the sequence (4.1) consists of ordered abelian groups G_n with positive homomorphisms $\alpha_n : G_n \to G_{n+1}$, we can regard the sequence without any order structure and obtain the inductive limit $(G, \{\beta_n\}_{n\in\mathbb{N}})$ as above. Now define G^+ to be $\bigcup_{n=1}^{\infty} \beta_n(G_n^+)$. Then (G, G^+) is an ordered abelian group, β_n is a positive homomorphism for each $n \in \mathbb{N}$, and $((G, G^+), \{\beta_n\}_{n\in\mathbb{N}})$ is the inductive limit.

4.1.2 $K_0(A)$

Let A be a C^*-algebra and as before let $\mathcal{P}(A)$ denote the set of projections in A. Set

$$\mathcal{P}_n(A) = \mathcal{P}(M_n(A)) \quad \text{and} \quad P_\infty(A) = \bigcup_{n=1}^{\infty} P_n(A).$$

Let $p \in \mathcal{P}_n(A)$ and $q \in \mathcal{P}_m(A)$. There is a binary operation $\oplus$ on $\mathcal{P}_\infty(A)$ given by $p \oplus q = \mathrm{diag}(p, q) \in \mathcal{P}_{n+m}(A)$. There is also an equivalence relation $\sim_0$ on $\mathcal{P}_\infty(A)$ defined by declaring $p \sim_0 q$ whenever there is an element $v \in M_{m,n}(A)$ with $p = v^*v$ and $q = vv^*$. Set $\mathcal{D}(A) = \mathcal{P}_\infty(A)/\sim_0$ and denote by $[p]_{\mathcal{D}} \in \mathcal{D}(A)$ the equivalence class containing the projection $p \in \mathcal{P}_\infty(A)$. Using $\oplus$, there binary operation $+$ on $\mathcal{D}(A)$ given by $[p]_{\mathcal{D}} + [q]_{\mathcal{D}} = [p \oplus q]_{\mathcal{D}}$. One checks that $+$ is well defined, and in fact turns $\mathcal{D}(A)$ into an abelian semigroup.

We now use a construction due to Grothendieck that creates an abelian group from an abelian semigroup. Let $(S, +)$ be an arbitrary abelian semigroup. Define an equivalence relation on $S \times S$ by declaring $(x_1, y_1) \sim (x_2, y_2)$ if there exists $z \in S$ such that $x_1 + y_2 + z = x_2 + y_1 + z$. Set $G(S) = (S \times S)/\sim$ and denote by $\langle x, y\rangle$ the equivalence relation containing (x, y). The binary operation $+$ on $G(S)$ given by $\langle x_1, y_1\rangle + \langle x_2, y_2\rangle = \langle x_1 + x_2, y_1 + y_2\rangle$ is well defined and turns $G(S)$ into a group, known as the *Grothendieck group* of S. For any $y \in S$, the map $\gamma_S : S \to G(S)$ given by $x \mapsto \langle x + y, y\rangle$ is additive and independent of y. It is called the *Grothendieck map.*

Definition 4.1.3. Let A be a unital C^*-algebra. Define $K_0(A)$ to be the Grothendieck group of $\mathcal{D}(A)$:

$$K_0(A) = G(\mathcal{D}(A)).$$

Define $[\cdot]_0 : \mathcal{P}_\infty(A) \to K_0(A)$ by setting $[p]_0 = \gamma([p]_{\mathcal{D}})$ for each $p \in \mathcal{P}(A)$, where $\gamma : \mathcal{D}(A) \to K_0(A)$ is the Grothendieck map.

Remark 4.1.4. There is a so-called *standard picture* for $K_0(A)$ in the unital case, which asserts that

$$K_0(A) = \{[p]_0 - [q]_0 \ : \ p, q \in \mathcal{P}_\infty(A)\} = \{[p]_0 - [q]_0 \ : \ p, q \in \mathcal{P}_n(A)\}.$$

Although this follows immediately from Grothendieck's construction, it is very useful since it gives a concrete way to work with elements in $K_0(A)$.

In the non-unital case, the construction is a bit more involved. First, a general fact. Suppose that A and B are unital C*-algebras. Let $\varphi : A \to B$ be a unital *-homomorphism and for each $n \in \mathbb{N}$ let $\varphi^n : M_n(A) \to M_n(B)$ be the inflation map $(a_{ij})_{i,j} \mapsto (\varphi(a_{ij}))_{i,j}$. Regarding $\varphi(p)$ as $\varphi^n(p)$ when $p \in \mathcal{P}_n(A)$, the induced map $K_0(\varphi) : K_0(A) \to K_0(B)$ given by $K_0(\varphi)([p]_0) = [\varphi(p)]_0$ is a homomorphism. Now suppose that A is non-unital and consider the associated short-exact sequence

$$0 \longrightarrow A \xrightarrow{i} \widetilde{A} \xrightarrow{\pi} \mathbb{C} \longrightarrow 0.$$

We *define* $K_0(A)$ to be the kernel of the induced homomorphism $K_0(\pi) : K_0(\widetilde{A}) \to K_0(\mathbb{C})$. Since $A \subset \widetilde{A}$, we can regard a projection $p \in \mathcal{P}_\infty(A)$ as an element in $\mathcal{P}_\infty(\widetilde{A})$. Since $K_0(\pi)([p]_0) = [\pi(p)]_0 = 0$, we get $[p]_0 \in \ker K_0(\pi)$. This means $[p]_0 \in K_0(A)$, so as before there is a map $[\cdot]_0 : \mathcal{P}_\infty(A) \to K_0(A)$.

Note that if A is any C*-algebra (unital or not), there is a short exact sequence

$$0 \longrightarrow K_0(A) \longrightarrow K_0(\widetilde{A}) \xrightarrow{K_0(\pi)} K_0(\mathbb{C}) \longrightarrow 0.$$

If A is unital, then the map $K_0(A) \to K_0(\widetilde{A})$ is $K_0(i)$. If not, it is the inclusion map. Using this we can make sense of $K_0(\varphi)$ in the non-unital case, where $\varphi : A \to B$ is any *-homomorphism. Indeed, let $\widetilde{\varphi} : \widetilde{A} \to \widetilde{B}$ be its unitization and consider the induced diagram

$$\begin{array}{ccccc} K_0(A) & \longrightarrow & K_0(\widetilde{A}) & \xrightarrow{K_0(\pi_A)} & K_0(\mathbb{C}) \\ \Big\downarrow & & \Big\downarrow K_0(\widetilde{\varphi}) & & \Big\downarrow K_0(\mathrm{id}_{\mathbb{C}}). \\ K_0(B) & \longrightarrow & K_0(\widetilde{B}) & \xrightarrow[K_0(\pi_B)]{} & K_0(\mathbb{C}) \end{array}$$

There is a unique homomorphism (indicated with the dashed arrow) that makes it commute; it is necessarily the restriction of $K_0(\widetilde{\varphi})$ to $K_0(A)$. We *define* this map to be $K_0(\varphi)$.

The construction of $K_0(A)$ is functorial in the sense that if A, B, C are C*-algebras, then $K_0(\mathrm{id}_A) = \mathrm{id}_{K_0(A)}$ and $K_0(\psi \circ \varphi) = K_0(\psi) \circ K_0(\varphi)$ for any *-homomorphisms $\varphi : A \to B$ and $\psi : B \to C$. We also have, $K_0(A \oplus B) \simeq K_0(A) \oplus K_0(B)$. Moreover, one can show that applying the non-unital construction to a unital C*-algebra yields the same K_0 group.

Example 4.1.5.

1. Let H be a (separable) Hilbert space. Suppose first $\dim(H) < \infty$ so that $\mathbb{B}(H) = \mathbb{K}(H) \simeq M_n(\mathbb{C})$ for some $n \in \mathbb{N}$. If Tr is the standard trace on $M_n(\mathbb{C})$, then $K_0(\mathrm{Tr}) : K_0(M_n(\mathbb{C})) \to \mathbb{Z}$ is an isomorphism. More specifically, $K_0(M_n(\mathbb{C}))$ is generated by $[e]_0$, where e is any one-dimensional projection in $M_n(\mathbb{C})$. If $\dim(H) = \infty$, then a

slight modification of this argument shows $K_0(\mathbb{K}(H)) \simeq \mathbb{Z}$ (see [50, Corollary 6.4.2]). For $K_0(\mathbb{B}(H))$, there is a semigroup isomorphism $\mathcal{D}(\mathbb{B}(H)) \to \{0,1,\ldots,\infty\}$ given by $[p]_{\mathcal{D}} \mapsto \dim(p)$. Since the Grothendieck group of $\{0,1,\ldots,\infty\}$ is trivial (see [50, Example 3.1.3(ii)]), we get $K_0(\mathbb{B}(H)) = 0$.

2. Let X be a topological space and let $\mathrm{Vect}(X)$ denote the set of complex vector bundles on X. Two vector bundles (E,p,X) and (E',p',X) are *isomorphic* if there is a homeomorphism $g : E \to E'$ such that $p' \circ g = p$, and for each $x \in X$ the restriction of g to E_x is a vector space isomorphism $E_x \to E'_x$. If (E,p,X) and (E',p',X) are two vector bundles on X, their *direct sum* $(E,p,X) \oplus (E',p',X)$ is defined to be the triple (F,q,X), where $F = \{(e,e') \in E \times E' \ : \ p(e) = p(e')\}$ and $q(e,e') = p(e) = p'(e')$. The fibers F_x are given by the (vector space) direct sum $E_x \oplus E'_x$. Let $V(X)$ denote the set of isomorphism classes of vector bundles on X, and denote the equivalence class containing (E,p,X) by $\langle E,p,X\rangle$. Define the operation $+$ on $V(X)$ by $\langle E,p,X\rangle + \langle E',p',X\rangle = \langle (E,p,X) \oplus (E',p',X)\rangle$. One checks that $+$ is well-defined, and in fact turns $V(X)$ into an abelian semigroup. Let $K^0(X)$ be the Grothendieck group of $(V(X),+)$. A beautiful result shows that if X is a compact Hausdorff space, then $K_0(C(X)) \simeq K^0(X)$ (see [50, 3.3.7], for example).

The group $K_0(A)$ comes equipped with a (pre-)order structure. Indeed, let $K_0(A)^+ \subseteq K_0(A)$ be the set $\{[p]_0 \ : \ p \in \mathcal{P}_\infty(A)\}$. Then $(K_0(A), K_0(A)^+)$ is a preordered abelian group. If A is unital, then $K_0(A)^+ - K_0(A)^+ = K_0(A)$; if A is stably finite, then $K_0(A)^+ \cap (-K_0(A)^+) = \{0\}$. If A is unital and stably finite, then $(K_0(A), K_0(A)^+, [1]_0)$ is an ordered abelian group with a distinguished order unit. Note that these conditions are sufficient, but not always necessary. For example, if A is an AF algebra (unital or not), then $(K_0(A), K_0(A)^+)$ is an ordered abelian group. The *dimension range* of a C^*-algebra is defined to be the set $\mathcal{D}_0(A) = \{[p]_0 \ : \ p \in \mathcal{P}(A)\} \subseteq K_0(A)^+$.

If $\varphi : A \to B$ is a *-homomorphism, then it is easy to check that $K_0(\varphi)(K_0(A)^+) \subseteq K_0(B)^+$. We summarize this by saying that $K_0(\varphi)$ is a positive homomorphism, even in the case that $(K_0(A), K_0(A)^+)$ is not an ordered abelian group. If φ is an isomorphism, then $K_0(\varphi)$ is an order isomorphism. If A is unital, then $K_0(\varphi)$ is an order unit preserving order isomorphism.

We conclude this section with a result that is often summarized by saying K_0 is a continuous functor.

Proposition 4.1.6. Suppose $\{A_n, \varphi_n\}_{n\in\mathbb{N}}$ is an inductive sequence of C^*-algebras, and obtain an inductive sequence $\{K_0(A_n), K_0(\varphi_n\}_{n\in\mathbb{N}}$ of preordered abelian groups. Then $K_0(\varinjlim A_n) \simeq \varinjlim K_0(A_n)$.

4.1.3 The Pairing Between K_0 and Traces

Let A be a unital C^*-algebra. Using the standard picture for $K_0(A)$, we can define for each $\tau \in T(A)$ a map $K_0(A) : K_0(A) \to \mathbb{R}$ by

$$K_0(\tau)([p]_0 - [q]_0) = \tau(p) - \tau(q).$$

It is easily checked that $K_0(\tau) \in S(K_0(A))$, so we can define the (continuous and affine) map $r_A : T(A) \to S(K_0(A))$ by $\tau \mapsto K_0(\tau)$. If A is nuclear, then r_A is surjective; if projections separate traces in $A \otimes \mathbb{K}$, then r_A is injective (see [48, Theorem 1.1.11 and Proposition 1.1.12]). Moreover, r_A induces a pairing

$$K_0(A) \times T(A) \to \mathbb{R} \qquad (g, \tau) \mapsto r_A(\tau)(g).$$

If A and B are unital C^*-algebras, any unital $*$-homomorphism $\varphi : A \to B$ induces affine continuous maps $T(\varphi) : T(B) \to T(A)$ and $\widehat{K_0(\varphi)} : S(K_0(B)) \to S(K_0(A))$ given by $\tau \mapsto \tau \circ \varphi$ and $f \mapsto f \circ K_0(\varphi)$, respectively. In turn, these give rise to the following commutative diagram:

$$\begin{array}{ccc} T(B) & \xrightarrow{\;T(\varphi)\;} & T(A) \\ {\scriptstyle r_B}\big\downarrow & & \big\downarrow{\scriptstyle r_A} \\ S(K_0(B)) & \xrightarrow[\widehat{K_0(\varphi)}]{} & S(K_0(A)). \end{array}$$

4.1.4 $K_1(A)$

Let A be a unital C^*-algebra and as before let $\mathcal{U}(A)$ denote the set of unitary elements in A. Set

$$\mathcal{U}_n(A) = \mathcal{U}(M_n(A)) \quad \text{and} \quad \mathcal{U}_\infty(A) = \bigcup_{n=1}^{\infty} \mathcal{U}_n(A).$$

Let $u \in \mathcal{U}_n(A)$ and $v \in \mathcal{U}_m(A)$. There is a binary operation $\oplus$ on $\mathcal{U}_\infty(A)$ given by $u \oplus v = \operatorname{diag}(u, v) \in \mathcal{U}_{n+m}(A)$. There is also an equivalence relation $\sim_1$ on $\mathcal{U}_\infty(A)$ defined by declaring $u \sim_1 v$ whenever there exists a natural number $k \geq \max\{m, n\}$ such that there is a continuous map $\gamma : [0, 1] \to \mathcal{U}_k(A)$ satisfying $\gamma(0) = u \oplus 1_{k-n}$ and $\gamma(1) = v \oplus 1_{k-m}$. Here we are regarding $w \oplus 1_0$ as w for each $w \in \mathcal{U}_\infty(A)$.

Definition 4.1.7. For each C^*-algebra A, define

$$K_1(A) = \mathcal{U}_\infty(\widetilde{A}) / \sim_1 .$$

Denote by $[u]_1 \in K_1(A)$ the equivalence class containing $u \in \mathcal{U}_\infty(A)$, and define a binary operation + on $K_1(A)$ by $[u]_1 + [v]_1 = [u \oplus v]$.

If A is any C^*-algebra, then $(K_1(A), +)$ is an abelian group, where the inverse of $[u]_1$ is $[u^*]_1$ for every $u \in \mathcal{U}_\infty(A)$. If A is a unital, then $\mathcal{U}_\infty(A)/\sim_1$ is an abelian group isomorphic to $K_1(A)$. In other words, $K_1(A) \simeq K_1(\widetilde{A})$ for every A.

Suppose A and B are C^*-algebras and let $\varphi : A \to B$ be a *-homomorphism and with inflation maps $\varphi^n : M_n(A) \to M_n(B)$. If $u \in \mathcal{U}_n(\widetilde{A})$, we regard $\widetilde{\varphi}(u)$ as $\widetilde{\varphi}^n(u) \in \mathcal{U}_n(\widetilde{B}))$. Then the map $K_1(\varphi) : K_1(A) \to K_1(B)$ given by $K_1(\varphi)([u]_1) = [\widetilde{\varphi}(u)]_1$ is a homomorphism.

Remark 4.1.8. If we have a short-exact sequence $0 \to I \xrightarrow{\varphi} A \xrightarrow{\psi} B \to 0$ of C^*-algebras, there is a homomorphism (called the *index map*) $\delta_1 : K_1(B) \to K_0(I)$ that makes the sequence

$$\begin{array}{ccccc} K_1(I) & \xrightarrow{K_1(\varphi)} & K_1(A) & \xrightarrow{K_1(\psi)} & K_1(B) \\ & & & & \big\downarrow \delta_1 \\ K_0(B) & \xleftarrow[K_0(\psi)]{} & K_0(A) & \xleftarrow[K_0(\varphi)]{} & K_0(I). \end{array}$$

exact. The index map measures the obstruction to lifting unitary elements in (matrix algebras over) $\widetilde{B}$ to a unitary elements in (matrix algebras over) $\widetilde{A}$. See [48, Chapter 9] for the construction.

Example 4.1.9. If H is a Hilbert space of any dimension, then $K_1(\mathbb{B}(H)) \simeq K_1(\mathbb{K}(H)) \simeq 0$. This follows from the fact that $\mathcal{U}_n(\mathbb{B}(H))$ is connected for every $n \in \mathbb{N}$. If H is infinite dimensional, then $K_1(\mathbb{B}(H)/\mathbb{K}(H)) \simeq \mathbb{Z}$. This follows from the short-exact sequence $0 \to \mathbb{K}(H) \to \mathbb{B}(H) \to \mathbb{B}(H)/\mathbb{K}(H) \to 0$, and the induced exact sequence from Remark 4.1.8.

Like $K_0(A)$, the construction of $K_1(A)$ is also functorial: if A, B, C are C^*-algebras, then $K_1(\mathrm{id}_A) = \mathrm{id}_{K_1(A)}$ and $K_1(\psi \circ \varphi) = K_1(\psi) \circ K_1(\varphi)$ for any *-homomorphisms $\varphi : A \to B$ and $\psi : B \to C$. We also have $K_1(A \oplus B) \simeq K_1(A) \oplus K_1(B)$. Moreover, K_1 is continuous in the sense that $K_1(\varinjlim A_n) \simeq \varinjlim K_1(A_n)$.

We end this section with a canonical way to combine $K_0(A)$ and $K_1(A)$. Let $K_*(A)$ denote the *graded K-group* $K_0(A) \oplus K_1(A)$. The *graded dimension range* is defined as

$$\mathcal{D}_*(A) = \{([p]_0, [u]_1) \; : \; p \in \mathcal{P}(A), \; u \in \mathcal{U}(pAp)\} \subseteq K_*(A).$$

A map $\alpha : K_*(A) \to K_*(B)$ is a *graded homomorphism* if it is a homomorphism that satisfying $\alpha(K_i(A)) \subseteq K_i(B)$ for $i = 0, 1$. Here $K_0(A)$ and $K_1(A)$ are being regarded as subgroups of $K_*(A)$.

4.1.5 KK-Theory and the Universal Coefficient Theorem

We now give a (very) brief introduction to those parts of KK-theory that are needed to state the relevant classification results in the following sections. This picture of KK-theory is one of several, and is due to Cuntz in [11]. The following exposition essentially follows [48, Section 2.4].

Let A and B be C^*-algebras. A *quasi-homomorphism from A to B* is a pair of *-homomorphisms $(\varphi_+, \varphi_-) : A \to \mathcal{M}(B \otimes \mathbb{K})$ satisfying $\varphi_+(a) - \varphi_-(a) \in B \otimes \mathbb{K}$ for all $a \in A$. Two quasi-homomorphisms (φ_+, φ_-) and (ψ_+, ψ_-) are *homotopic* if they are connected by a path of quasi-homomorphisms (α_+^t, α_-^t), where $t \mapsto \alpha_\pm^t(a)$ are strictly continuous in $\mathcal{M}(B \otimes \mathbb{K})$ (see Definition 3.1.3) and $t \mapsto \alpha_+^t(a) - \alpha_-^t(a)$ is norm continuous in $B \otimes \mathbb{K}$ for every $a \in A$. Let $KK(A,B)$ denote the set of homotopy equivalence classes of quasi-homomorphisms from A to B, and let $[(\varphi_+, \varphi_-)]$ denote the equivalence class containing (φ_+, φ_-). If $n \in \mathbb{N}$, each *-homomorphism $\varphi : A \to M_n(B)$ defines a quasi-homomorphism $(\varphi, 0)$ since we can regard $M_n(B)$ as a C^*-subalgebra of $A \otimes \mathbb{K} \subset \mathcal{M}(A \otimes \mathbb{K})$; set $KK(\varphi) = [(\varphi, 0)] \in KK(A,B)$. If two homomorphisms $\varphi, \psi : A \to M_n(B)$ are asymptotically unitarily equivalent, then $KK(\varphi) = KK(\psi)$.

Let s_1, s_2 be two isometries in $\mathcal{M}(B \otimes \mathbb{K})$ satisfying $s_1 s_1^* + s_2 s_2^* = 1$. If (φ_+, φ_-) and (ψ_+, ψ_-) are two quasi-homomorphisms from A to B, let

$$\lambda_\pm(a) = s_1 \varphi_\pm(a) s_1^* + s_2 \psi_\pm(a) s_2^*$$

for each $a \in A$. Then (λ_+, λ_-) is also a quasi-homomorphism from A to B, and we *define* $[(\varphi_+, \varphi_-)] + [(\psi_+, \psi_-)]$ to be $[(\lambda_+, \lambda_-)]$. The operation + is well defined, and in fact turns $(KK(A,B), +)$ into an abelian group. Moreover, $[(\varphi, \varphi)] = 0$ for every *-homomorphism $\varphi : A \to \mathcal{M}(B \otimes \mathbb{K})$, and $-[(\varphi_+, \varphi_-)] = [(\varphi_-, \varphi_+)]$ for each quasi-homomorphism (φ_+, φ_-).

Theorem 4.1.10 (Kasparov)**.** Let A, B, C be C^*-algebras. Then there exists a bi-additive map (called the *Kasparov product*) $KK(A,B) \times KK(B,C) \to KK(A,C)$, written $(x,y) \mapsto x \cdot y$, satisfying the following properties:

1. If $x \in KK(A,B)$, $y \in KK(B,C)$, and $z \in KK(C,D)$, then $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
2. If $\varphi : A \to B$ and $\psi : B \to C$ are *-homomorphisms, then $KK(\varphi) \cdot KK(\psi) = KK(\psi \circ \varphi)$.
3. $(KK(A,A), +, \cdot)$ is a ring with unit, denoted by $1_{KK(A)}$.

An element $x \in KK(A,B)$ is *invertible* if there exists $y \in KK(B,A)$ such that $x \cdot y = 1_{KK(A)}$ and $y \cdot x = 1_{KK(B)}$. If there exists an invertible element in $KK(A,B)$, then A and B are said to be *KK-equivalent*.

Definition 4.1.11. The *UCT class $\mathcal{N}$* is defined to be the family of all separable C^*-algebras that are KK-equivalent to an abelian C^*-algebra.

We have the following beautiful relationship between K-theory and KK-theory: for every C^*-algebra A,

$$K_0(A) \simeq KK(\mathbb{C}, A) \qquad \text{and} \qquad K_1(A) \simeq KK(C_0(\mathbb{R}), A).$$

Using this we can, for $i = 0, 1$, define *-homomorphisms

$$\gamma_i : KK(A, B) \to \mathrm{Hom}(K_i(A), K_i(B))$$

by $\gamma_0(x)(z_0) = z_0 \cdot x$ and $\gamma_1(x)(z_1) = z_1 \cdot x$, where $z_0 \in K_0(A) \simeq KK(\mathbb{C}, A)$ and $z_1 \in K_1(A) \simeq KK(C_0(\mathbb{R}), A)$. The following theorem follows from results of Rosenberg and Schochet in [51]; it allows us to pass from KK-equivalence to K-equivalence for C^*-algebras in $\mathcal{N}$.

Theorem 4.1.12 (The Universal Coefficient Theorem)**.** Let A and B be C^*-algebras.

1. The homomorphism $\gamma_0 \oplus \gamma_1 : KK(A, B) \to \bigoplus_{i=0}^{1} \mathrm{Hom}(K_i(A), K_i(B))$ is surjective for each C^*-algebra A in $\mathcal{N}$ and for each separable C^*-algebra B.

2. If both A and B are in $\mathcal{N}$, then $x \in KK(A, B)$ is invertible if and only if $\gamma_i(x) : K_i(A) \to K_i(B)$ are invertible maps for $i = 0, 1$.

4.2 Classification by K-Theory

4.2.1 AF Algebras

The classification of C^*-algebras using K-theory began with the following theorem of Elliott in [15].

Theorem 4.2.1. Let A and B be AF algebras. If there is an isomorphism $\alpha : K_0(A) \to K_0(B)$ such that $\alpha(\mathcal{D}_0(A)) = \mathcal{D}_0(B)$, then there is a *-isomorphism $\varphi : A \to B$ satisfying $K_0(\varphi) = \alpha$. If A and B are unital and $\alpha : K_0(A) \to K_0(B)$ is an order unit preserving order isomorphism, then the same result holds.

Remark 4.2.2. Note that K_1 plays no role in the classification of AF algebras. Indeed, we saw in Example 4.1.9 that if B is a finite dimensional C^*-algebra, then $K_1(B) = 0$. By continuity, we conclude that $K_1(A)$ is trivial for every AF algebra A.

A *dimension group* is an ordered abelian group which is isomorphic to the inductive limit of the sequence of ordered abelian groups

$$\mathbb{Z}^{n_1} \xrightarrow{\alpha_1} \mathbb{Z}^{n_2} \xrightarrow{\alpha_2} \mathbb{Z}^{n_3} \xrightarrow{\alpha_3} \cdots$$

for positive integers n_j and positive group homomorphisms α_j. Here the order structure on $\mathbb{Z}^n$ is the usual one: $(\mathbb{Z}^n)^+ = \{(x_1, \ldots, x_n) : x_j \geq 0\}$. In [14], Effros, Handelman, and Shen showed that a countable ordered abelian group is a dimension group if and only if it is unperforated and has the Riesz interpolation property.

The importance of dimension groups in the classification program is that they constitute the range of the K_0 functor for AF algebras. Indeed, if A is AF, then $(K_0(A), K_0(A)^+)$ is a dimension group. Conversely, every dimension group arises as the ordered K_0 group of some AF algebra (see [48, Section 1.4]).

4.2.1.1 UHF Algebras and Supernatural Numbers

Prior to Elliott's theorem for AF algebras, Glimm proved a classification result for the subclass of UHF algebras using a different (but related) invariant. Before moving on, we review his result. Recall that a UHF algebra is a C^*-algebra isomorphic to the inductive limit of a sequence of the form

$$M_{k_1}(\mathbb{C}) \xrightarrow{\varphi_1} M_{k_2}(\mathbb{C}) \xrightarrow{\varphi_2} M_{k_3}(\mathbb{C}) \xrightarrow{\varphi_3} \cdots, \tag{4.2}$$

where each $k_n \in \mathbb{N}$ and each φ_n is a unital *-homomorphism. The natural numbers k_n necessarily satisfy $k_n | k_{n+1}$ for every $n \in \mathbb{N}$, since there exists a unital *-homomorphism $\varphi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ if and only if $m = dn$, where $d = \mathrm{Tr}(\varphi(e))$ for any minimal projection in $M_n(\mathbb{C})$.

Let p_j denote the j^{th} prime number. A *supernatural number* is a formal product of the form

$$n = \prod_{j=1}^{\infty} p_j^{n_j}, \tag{4.3}$$

where each n_j belongs to the set $\{0, 1, 2 \ldots, \infty\}$. If n is a supernatural number, let $Q(n)$ be the additive subgroup of $(\mathbb{Q}, +)$ consisting of all rational numbers x/y where $x \in \mathbb{Z}$ and $y = \prod_{j=1}^{\infty} p_j^{m_j}$ for some non-negative integers $m_j \leq n_j$ satisfying $m_j = 0$ for all but finitely many j. Note that $1 \in Q(n)$ for each supernatural number n. Conversely, if G is a subgroup of $(\mathbb{Q}, +)$ such that $1 \in G$, then there is a supernatural number n such that $G \simeq Q(n)$. See [50, Proposition 7.4.3] for a construction.

If n, n' are two supernatural numbers, write $(Q(n), 1) \simeq (Q(n'), 1)$ if there is an isomorphism $\alpha : Q(n) \to Q(n')$ satisfying $\alpha(1) = 1$. We have that $(Q(n), 1) \simeq (Q(n'), 1)$ if and only if $n = n'$, and more generally $Q(n) \simeq Q(n')$ if and only if there are integers m, m' such that $mn = m'n'$.

If A is isomorphic to the inductive limit of (4.2), let $n_A = \prod_j p_j^{n_j}$ be the supernatural number with

$$n_j = \sup\{n : p_j^n | k_i \text{ for some } i\}.$$

In [24], Glimm showed that n_A forms a complete invariant for A. More precisely, if A and B

are UHF algebras and $n_A = n_B$, then $A \simeq B$. Additionally, for any supernatural number n there exists a UHF algebra A with $n_A = n$. In other words, the set of mutually non-isomorphic UHF algebras is in one-to-one correspondence with the set of supernatural numbers, and hence is uncountable. To put this slight diversion into the context of K-theory, there is an isomorphism $(K_0(A), [1]_0) \simeq (Q(n_A), 1)$ for every UHF algebra A (see [50, Lemma 7.4.4] for example).

4.2.2 AH Algebras

After Elliott's success with AF algebras, larger classes were considered. A natural direction is to look at AH algebras, since they are assembled from very manageable pieces. A particularly nice subclass are the A$\mathbb{T}$ algebras.

Example 4.2.3.

1. In [19], Elliott and Evans showed that if $\theta \in \mathbb{R}$ is irrational, then the rotation algebra A_θ from Example 2.2.23 is a simple A$\mathbb{T}$ algebra of real rank zero.

2. Let H be a (separable) infinite dimensional Hilbert space and choose an orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$. An operator $T \in \mathbb{B}(H)$ is called a *shift operator* if there is a bounded sequence $\{a_n\}_{n\in\mathbb{Z}}$ of complex numbers with $T(e_n) = \alpha_n e_{n+1}$. We say that T has *period* p if $\alpha_{n+p} = \alpha_n$ for every $n \in \mathbb{Z}$. If $\pi : \mathbb{B}(H) \to \mathbb{B}(H)/\mathbb{K}(H)$ is the quotient map, let B denote the C^*-subalgebra of $\mathbb{B}(H)/\mathbb{K}(H)$ generated by the set of elements $\pi(T)$, where T is a shift operator of period 2^k for some k. In [6], Bunce and Deddens show that B is the inductive limit of the sequence

$$C(\mathbb{T}) \xrightarrow{\varphi_1} C(\mathbb{T}) \otimes M_2(\mathbb{C}) \xrightarrow{\varphi_2} C(\mathbb{T}) \otimes M_4(\mathbb{C}) \xrightarrow{\varphi_3} \cdots,$$

   where if u denotes the canonical generator of $C(\mathbb{T})$, the connecting maps are given by

$$\varphi_1(u) = \begin{pmatrix} 0 & u \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \varphi_{n+1} = \varphi_1 \otimes \mathrm{id}_{M_{2^n}(\mathbb{C})}.$$

   Due to its construction, B is often referred to as the *Bunce-Deddens algebra of type* 2^∞. Bunce and Deddens also showed that B is simple and not AF.

In [18], Elliott proved the following classification theorem for simple A$\mathbb{T}$ algebras.

Theorem 4.2.4. Let A and B be simple unital A$\mathbb{T}$ algebras and suppose there is an order unit preserving order isomorphism $\alpha_0 : K_0(A) \to K_0(B)$, an isomorphism $\alpha_1 : K_1(A) \to K_1(B)$, and an affine homeomorphism $\gamma : T(B) \to T(A)$ making the the diagram

$$\begin{array}{ccc} T(B) & \xrightarrow{\gamma} & T(A) \\ \downarrow r_B & & \downarrow r_A \\ S(K_0(B)) & \xrightarrow[\widehat{\alpha_0}]{} & S(K_0(A)) \end{array}$$

commute, where $\widehat{\alpha_0}(f) = f \circ \alpha$. Then there is an isomorphism $\varphi : A \to B$ satisfying $K_0(\varphi) = \alpha_0$, $K_1(\varphi) = \alpha_1$, and $T(\varphi) = \gamma$.

Remark 4.2.5. There are several variants of Theorem 4.2.4. If A and B are $A\mathbb{T}$ algebras having real rank zero, Elliott proved in [16] that $A \simeq B$ if and only if there is a graded group isomorphism $\alpha : K_*(A) \to K_*(B)$ such that $\alpha(\mathcal{D}_*(A)) = \mathcal{D}_*(B)$. Moreover, for each such α there is an isomorphism $\varphi : A \to B$ satisfying $K_*(\varphi) = \alpha$. When A and B are also simple and unital, the classification theorem reduces to the following: $A \simeq B$ if and only if there is an order unit preserving order isomorphism $\alpha_0 : K_0(A) \to K_0(B)$ and an isomorphism $\alpha_1 : K_1(A) \to K_1(B)$.

Going beyond $A\mathbb{T}$ algebras, one can consider AH algebras with slow dimension growth or bounded dimension.

Example 4.2.6.

1. Let X be a compact Hausdorff space. Suppose we have sequences $\{k_n\}_{n\in\mathbb{N}}$ and $\{l_n\}_{n\in\mathbb{N}}$ such that $k_n | k_{n+1}$ and $l_n < k_{n+1}/k_n$ for each $n \in \mathbb{N}$. Take points $\{x_{n,i} : n \in \mathbb{N}, i = 1, \dots, l_n\} \subset X$ and let $F_n = \{x_{n,1}, \dots, x_{n,l_n}\}$. Associated to this data is the inductive sequence
$$C(X, M_{k_1}(\mathbb{C})) \xrightarrow{\varphi_1} C(X, M_{k_2}(\mathbb{C})) \xrightarrow{\varphi_2} C(X, M_{k_3}(\mathbb{C})) \xrightarrow{\varphi_3} \cdots,$$
where φ_n is the unital *-homomorphism given by
$$\varphi_n(f)(x) = \mathrm{diag}(f(x_{n,1}), \dots, f(x_{n,l_n}), f(x), \dots, f(x)),$$
for $x \in X$ and $f \in C(X, M_{k_n}(\mathbb{C}))$. Let A be the inductive limit of this sequence, so that A is a unital AH algebra. One can show that A is simple if and only if the set $\bigcup_{n=k}^{\infty} F_n$ is dense in X for each k. A simple C^*-algebra arising in this way is called a *Goodearl algebra*. Although their construction seems a bit strange, this class of C^*-algebras is important because it demonstrates the relevance of the trace space in the classification program. We will review this point in 4.2.4.

2. In [62], Villadsen constructed an important class of simple AH algebras. These provided counterexamples to several outstanding conjectures. In particular, he showed there exists a simple unital AH algebra whose K_0 group is not weakly unperforated, thereby showing that not every simple unital AH algebras has bounded dimension (see [48, Proposition 3.3.7]). We will revisit this example when discussing the Jiang-Su algebra in 5.3.1.

In [20], Elliott, Gong, and Li proved a substantial generalization of Theorem 4.2.4. They showed that for simple unital AH algebras of bounded dimension, the 6-tuple

$$(K_0(A), K_0(A)^+, [1]_0, K_1(A), T(A), r_A)$$

forms a complete invariant. As in the A$\mathbb{T}$ case, the situation is considerably simpler when the algebras have real rank zero. Indeed, Dadarlat, Elliott, and Gong proved that if A and B are simple unital AH algebras of slow dimension growth and real rank zero, then $A \simeq B$ if and only if there is an order unit preserving order isomorphism $\alpha_0 : K_0(A) \to K_0(B)$ and an isomorphism $\alpha_1 : K_1(A) \to K_1(B)$.

4.2.3 Kirchberg Algebras

Definition 4.2.7. A *Kirchberg algebra* is a nuclear, simple, separable, purely infinite C^*-algebra.

Example 4.2.8.

1. Fix $n \in \mathbb{N}$ not equal to 1. The *Cuntz algebra* $\mathcal{O}_n$ is the universal C^*-algebra generated by isometries $s_1, \ldots, s_n$ satisfying $s_1 s_1^* + \cdots + s_n s_n^* = 1$. Define $\mathcal{O}_\infty$ to be the universal unital C^*-algebra generated by an infinite sequence of isometries $s_1, s_2, \ldots$ such that the projections $s_j s_j^*$ are mutually orthogonal. In [10], Cuntz proved that the C^*-algebras $\mathcal{O}_n$, for $2 \le n \le \infty$ are Kirchberg algebras. We also have

$$K_0(\mathcal{O}_n) = \begin{cases} \mathbb{Z}/(n-1)\mathbb{Z} & 2 \le n < \infty \\ \mathbb{Z} & n = \infty \end{cases} \qquad \text{and} \qquad K_1(\mathcal{O}_n) = 0 \quad 2 \le n \le \infty.$$

2. Let $E = (E^0, E^1)$ be an oriented graph with vertex set E^0, edge set E^1, and range and source maps $r, s : E^1 \to E^0$. If $|s^{-1}(v)| < \infty$ for every $v \in E^0$, we say E is *row-finite*; if both $|s^{-1}(v)| < \infty$ and $|r^{-1}(v)| < \infty$ for every $v \in E^0$, we say E is *locally finite*. If $v \in E^0$ is such that $s^{-1}(v) = \varnothing$, we say v is a *sink*. We say E is *cofinal* if for every $v \in E^0$ and every infinite path γ in E, there is a finite path from v to some vertex in γ.

   Define $C^*(E)$ to be the universal C^*-algebra generated by mutually orthogonal projections $\{p_v\}_{v\in E^0}$ and partial isometries $\{s_e\}_{e\in E^1}$ satisfying $s_e^* s_e = p_{r(e)}$ for all $e \in E^1$ and $\sum_{s(e)=v} s_e s_e^* = p_v$ for all $v \in E^0$ that are not sinks. If E has no sinks, then $C^*(E)$ is generated by partial isometries $\{s_e\}_{e\in E^1}$ with mutually orthogonal range projections $s_e s_e^*$ that also satisfy

$$s_e^* s_e = \sum_{f\in E^1} A_E(e,f) s_f s_f^*, \qquad e \in E^1,$$

where $(A(e,f))_{e,f\in E^1}$ is the edge matrix of E defined by $A_E(e,f) = 1$ if $r(e) = s(f)$, and 0 otherwise. It was shown in [37] and [36] that if E is locally finite, has no sinks, is cofinal, has at least one loop, and has the property that every loop in E has an exit, then $C^*(E)$ is a Kirchberg algebra. The K-theory of $C^*(E)$ for a row finite directed graph E is computed in [37, Corollary 6.12] to be

$$K_0(C^*(E)) \simeq \operatorname{coker}(I - A_E^T) \qquad \text{and} \qquad K_1(C^*(E)) \simeq \ker(I - A_E^T).$$

Using this, Neubuser found in [42] a construction of an oriented graph E such that $C^*(E)$ is a Kirchberg algebra with $K_0(C^*(E)) \simeq 0$ and $K_1(C^*(E)) \simeq \mathbb{Z}$.

In [31] and [44], Kirchberg and Phillips proved the following classification result for Kirchberg algebras.

Theorem 4.2.9. Let A and B be Kirchberg algebras. If A and B are unital, and if there is an invertible element $x \in KK(A,B)$ satisfying $\gamma_0(x)([1_A]_0) = [1_B]_0$, then there is an isomorphism $\varphi : A \to B$ with $KK(\varphi) = x$. If A and B are stable, and if there is an invertible element $x \in KK(A,B)$, then there is an isomorphism $\varphi : A \to B$ with $KK(\varphi) = x$.

Remark 4.2.10.

1. If we restrict ourselves to the realm of Kirchberg algebras belonging to $\mathcal{N}$, then the UCT gives a classification in terms of K_0 and K_1.

2. It is currently unknown if every Kirchberg algebra belongs to $\mathcal{N}$. Note that Kirchberg showed in [31] that every separable nuclear C^*-algebra is KK-equivalent to a Kirchberg algebra.

4.2.4 The Elliott Conjecture

If A is a simple unital and stably finite C^*-algebra, its *Elliott invariant* is the 6-tuple

$$\operatorname{Ell}(A) = \big(K_0(A), K_0(A)^+, [1_A]_0, K_1(A), T(A), r_A\big).$$

On the other hand if we have a 6-tuple $(G_0, G_0^+, g_0, G_1, \Delta, \lambda)$ consisting of a (countable) ordered abelian group (G_0, G_0^+, g_0) with distinguished order unit g_0, a (countable) abelian group G_1, a (metrizable) Choquet simplex Δ, and a continuous affine surjection $\lambda : \Delta \to S(G_0)$, we say that it is the Elliott invariant of a (separable) C^*-algebra A if

$$\big(G_0, G_0^+, g_0, G_1, \Delta, \lambda\big) \simeq \big(K_0(A), K_0(A)^+, [1_A]_0, K_1(A), T(A), r_A\big),$$

in the sense that there is a order unit preserving order isomorphism $\alpha_0 : K_0(A) \to G_0$, an isomorphism $\alpha_1 : K_1(A) \to G_1$, and an affine homeomorphism $\gamma : \Delta \to T(A)$ such that the diagram

$$\begin{CD}
\Delta @>{\gamma}>> T(A) \\
@V{\lambda}VV @VV{r_A}V \\
S(G_0) @>>{\widehat{\alpha_0}}> S(K_0(A))
\end{CD}$$

commutes, where $\widehat{\alpha_0}(f) = f \circ \alpha_0$ whenever $f \in S(G_0)$.

The following is known as the *Elliott conjecture*. It asserts that Elliott invariant forms a complete invariant for a natural class of C^*-algebras.

Conjecture 4.2.11. Let A and B be nuclear, simple, separable, unital, infinite dimensional C^*-algebras. Assume there is an order unit preserving order isomorphism $\alpha_0 : K_0(A) \to K_0(B)$, an isomorphism $\alpha_1 : K_1(A) \to K_1(B)$, and an affine homeomorphism $\gamma : T(B) \to T(A)$ such that the diagram

$$\begin{CD}
T(B) @>{\gamma}>> T(A) \\
@V{r_B}VV @VV{r_A}V \\
S(K_0(B)) @>>{\widehat{\alpha_0}}> S(K_0(A))
\end{CD}$$

commutes. Then there is an isomorphism $\varphi : A \to B$ satisfying $K_0(\varphi) = \alpha_0$, $K_1(\varphi) = \alpha_1$, and $T(\varphi) = \gamma$.

Remark 4.2.12.

1. As it is written, Conjecture 4.2.11 is false. To make it a valid conjecture, an additional assumption needs to be added (or the invariant changed). Doing so will require a few extra definitions, so we hold off until the next section to introduce the details.

2. One may wonder why the trace space $T(A)$ and the map r_A are included in the invariant. To explain this, let X be a connected compact Hausdorff space, put $k_n = (n!)^2$, put $l_n = 1$, and let $\{x_n\}_{n\in\mathbb{N}}$ be a dense sequence in X. Let A be the Goodearl algebra with this data (Example 4.2.6). Goodearl showed that $T(A)$ is affinely homeomorphic to the simplex of probability measures on X, and the set of extreme points of $T(A)$ is homeomorphic to X. Moreover, the state space $S(K_0(A))$ is a singleton set, and

$$(K_0(A), K_0(A)^+, [1_A]_0, K_1(A)) \simeq (K_0(\mathcal{Q}), K_0(\mathcal{Q})^+, [1_{\mathcal{Q}}]_0, K_1(\mathcal{Q})),$$

where $\mathcal{Q}$ is the UHF algebra satisfying $K_0(\mathcal{Q}) \simeq \mathbb{Q}$. Since there are uncountably many distinct choices for X, there are uncountably many non-isomorphic Goodearl algebras whose Elliott invariants differ only by their trace space.

3. If A has real rank zero, then $r_A : T(A) \to S(K_0(A))$ is injective and hence an affine homeomorphism. The Elliott invariant for a simple, unital, stably finite C^*-algebra A of real rank zero thus reduces to the 4-tuple

$$(K_0(A), K_0(A)^+, [1_A]_0, K_1(A)).$$

4. In the infinite case, the invariant is further reduced to $K_0(A)$ and $K_1(A)$ as abelian groups with no additional structure. If A is a simple and unital (resp. stable) C^*-algebra such that $A \otimes \mathbb{K}$ contains an infinite projection, its *Elliott invariant* is the triple

$$(K_0(A), [1_A]_0, K_1(A)) \qquad \text{(resp. the pair } (K_0(A), K_1(A))).$$

5. As the previous section indicated, the Elliott conjecture is confirmed for simple unital AH algebras of bounded dimension, and for Kirchberg algebras in $\mathscr{N}$.

We conclude by reviewing the range of the Elliott invariant for ASH algebras and for Kirchberg algebras in the UCT class $\mathscr{N}$. This explains why these classes are central to the classification program.

In [17] Elliott proved that if we have a 6-tuple as described above with the added assumptions that G_0 and G_1 are countable, (G_0, G_0^+, g_0) is simple and weakly unperforated, and Δ is metrizable, then there is a simple unital ASH algebra A whose Elliott invariant is isomorphic to $(G_0, G_0^+, g_0, G_1, \Delta, \lambda)$. Moreover, this ASH algebra can be taken to be the inductive limit of direct sums of building blocks B, which fit into a short-exact sequence of the form

$$0 \to C_0(\mathbb{R}) \to B \to M_n(\mathbb{C}) \otimes C(\mathbb{T}) \otimes I_m \to 0,$$

where I_m is the so-called *(interval) dimension drop* C^*-algebra, defined to be the set of elements in $C([0,1], M_m(\mathbb{C}))$ satisfying $f(0) = 0$ and $f(1) = \alpha I$ for some $\alpha \in \mathbb{C}$. The fact that B is a subhomogenous C^*-algebra follows from the standard permanence properties for such algebras.

Remark 4.2.13. If we assume also that (G_0, G_0^+, g_0) has the Reisz interpolation property, λ is extreme point preserving, and G_0 modulo its torsion subgroup is not isomorphic to $\mathbb{Z}$, then [20] contains the result (by Villadsen) that A can be taken to be a simple unital AH algebra.

Now let G_0 and G_1 be countable abelian groups and let $g_0 \in G_0$. Then there is a unital Kirchberg algebra A in $\mathscr{N}$ whose Elliott invariant satisfies $(K_0(A), [1]_0, K_1(A)) \simeq (G_0, g_0, G_1)$. The construction of A comes from an inductive sequence for the form

$$B_1 \otimes C(\mathbb{T}) \xrightarrow{\varphi_1} B_2 \otimes C(\mathbb{T}) \xrightarrow{\varphi_2} B_3 \otimes C(\mathbb{T}) \xrightarrow{\varphi_3} \cdots,$$

such that for each $n \in \mathbb{N}$ the connecting map φ_n is a unital *-homomorphism, and

$$B_n = M_{k_1}(\mathcal{O}_{n_1}) \oplus \cdots \oplus M_{k_r}(\mathcal{O}_{n_r})$$

for some natural numbers $k_1, \ldots, k_r$ and some $n_1, \ldots, n_r \in \{2, 3, \ldots, \infty\}$. See [48, Proposition 4.3.4] for a proof.

Chapter 5
Nuclear Dimension

5.1 From Kirchberg's Third Approximation Theorem to Nuclear Dimension

We begin this section with [26, Theorem 1.4], an important theorem of Kirchberg (together with Hirshberg and White). Roughly speaking, it says the CPAP implies a stronger version of itself.

Theorem 5.1.1. If A is a nuclear C^*-algebra, then for any finite subset $F \subset A$ and $\epsilon > 0$, there is an integer $n \geq 0$, a finite-dimensional C^*-algebra $K = K^{(0)} \oplus \cdots \oplus K^{(n)}$, a c.p.c. map $\varphi : A \to K$, and a c.p. map $\psi : K \to A$ satisfying

1. $\|\psi \circ \varphi(a) - a\| < \epsilon$ for all $a \in F$.
2. For each $i = 0, \ldots, n$, the restriction of ψ to $K^{(i)}$ is contractive and order zero.

Remark 5.1.2. The theorem proved in [26] was actually stronger. Indeed, the map ψ can actually be chosen to be a convex combination of n contractive order zero maps. However, for the purposes of this thesis this strengthened version will not be considered.

Theorem 5.1.1 suggests a very natural subclass of nuclear C^*-algebras to consider: those that admit a local factorization as in Theorem 5.1.1, but with the added assumption that the number n is independent of ϵ and K. The following definition was given by Winter and Zacharias in [70].

Definition 5.1.3. A C^*-algebra A has *nuclear dimension* at most n if for every finite subset $F \subset A$ and $\epsilon > 0$, there is a finite-dimensional C^*-algebra $K = K^{(0)} \oplus \cdots \oplus K^{(n)}$, a c.p.c. map $\varphi : A \to K$, and a c.p. map $\psi : K \to A$ satisfying

1. For each $i = 0, \ldots, n$, the restriction of ψ to $K^{(i)}$ is contractive and order zero.
2. $\|\psi \circ \varphi(a) - a\| < \epsilon$ for every $a \in F$.

We write $\dim_{\mathrm{nuc}}(A) \le n$; if n is the smallest such natural number we write $\dim_{\mathrm{nuc}}(A) = n$. For notational convenience, let $\dim_{\mathrm{nuc}}^{+1}(A) = \dim_{\mathrm{nuc}}(A) + 1$.

Remark 5.1.4.

1. A notion related to nuclear dimension is that of that of *decomposition rank*. A C^*-algebra A has decomposition rank at most n, written $\mathrm{dr}(A) \le n$, if $\dim_{\mathrm{nuc}}(A) \le n$ and the map ψ in Definition 5.1.3 can be chosen to be contractive. It is immediate that $\dim_{\mathrm{nuc}}(A) \le \mathrm{dr}(A)$ for any C^*-algebra A, and that $\dim_{\mathrm{nuc}}(A) < \infty$ implies A is nuclear (by Remark 2.3.8). It is also evident that $\dim_{\mathrm{nuc}}(A) = 0$ if and only if $\mathrm{dr}(A) = 0$. However, this is very specialized and in general it is not true that nuclear dimension coincides with decomposition rank. For example, Kirchberg and Winter prove that having finite decomposition rank implies quasidiagonality (see [34, Proposition 5.1]), but we will see below every Kirchberg algebra has finite nuclear dimension.

2. Nuclear dimension appears to be the "correct" notion of dimension for the class of nuclear C^*-algebras. As the next several sections will show, it covers large classes of examples and is related to the C^*-analogue of Connes' theorem from [9], which shows that an injective II_1 factor is hyperfinite. Moreover, it may be precisely the right context for the classification program using the Elliott invariant.

The following permanence properties are proven in [70, Propositions 2.3, 2.5].

Proposition 5.1.5. Let A, B, C, D, and E be C^*-algebras; suppose $a \in A_+$, $C = \varinjlim C_n$ is an inductive limit of C^*-algebras, and D is a quotient of E. Then

1. $\dim_{\mathrm{nuc}}(\overline{aAa}) \le \dim_{\mathrm{nuc}}(A)$,

2. $\dim_{\mathrm{nuc}}(A \oplus B) = \max\{\dim_{\mathrm{nuc}}(A), \dim_{\mathrm{nuc}}(B)\}$,

3. $\dim_{\mathrm{nuc}}^{+1}(A \otimes B) \le \dim_{\mathrm{nuc}}^{+1}(A) \dim_{\mathrm{nuc}}^{+1}(B)$,

4. $\dim_{\mathrm{nuc}}(C) \le \liminf_n \dim_{\mathrm{nuc}}(C_n)$,

5. $\dim_{\mathrm{nuc}}(D) \le \dim_{\mathrm{nuc}}(E)$.

If we replace nuclear dimension with decomposition rank in the permanence properties above, the same results are true.

Remark 5.1.6. If $\mathcal{H}$ is a Hilbert A-module, then by Theorem 3.1.7 the compacts $\mathbb{K}(\mathcal{H})$ form a hereditary subalgebra of $A \otimes \mathbb{K}$. Using the fact that finite nuclear dimension is preserved for hereditary subalgebras, we obtain the inequality

$$\dim_{\mathrm{nuc}}(\mathbb{K}(\mathcal{H})) \le \dim_{\mathrm{nuc}}(A).$$

As for extensions, nuclear dimension and decomposition rank tend to behave differently. If we have a short-exact sequence

$$0 \to J \to E \to A \to 0$$

of C^*-algebras, then by [70, Proposition 2.9] we have $\dim_{\mathrm{nuc}}(E) \le \dim_{\mathrm{nuc}}(J) + \dim_{\mathrm{nuc}}(A) + 1$. In general the same cannot be said about $\mathrm{dr}(E)$, but we have the following proposition. The statement about decomposition rank is [34, Proposition 6.1]; the statement about nuclear dimension has an identical proof.

Proposition 5.1.7. Assume that $J \trianglelefteq E$ has a quasicentral approximate unit consisting of projections. Then $\mathrm{dr}(E) = \max\{\mathrm{dr}(J), \mathrm{dr}(A)\}$ and $\dim_{\mathrm{nuc}}(E) = \max\{\dim_{\mathrm{nuc}}(J), \dim_{\mathrm{nuc}}(A)\}$.

To show this requirement is necessary, consider the following example.

Example 5.1.8. Let $\mathcal{T}$ denote the *Toeplitz algebra*, which is the C^*-algebra generated by the unilateral shift v on $\ell^2(\mathbb{N})$. Since v is a proper isometry, it follows that $\mathcal{T}$ cannot be quasidiagonal and hence has infinite decomposition rank. However, one can show that there is a short-exact sequence of the form $0 \to \mathbb{K} \to \mathcal{T} \to C(\mathbb{T}) \to 0$, so $\dim_{\mathrm{nuc}}(\mathcal{T}) \le 2$. Note that $\dim_{\mathrm{nuc}}(\mathcal{T}) > 0$ since $\mathcal{T}$ is not AF, but it is currently unknown if $\dim_{\mathrm{nuc}}(\mathcal{T})$ equals 1 or 2.

We end this section with a useful characterization of finite nuclear dimension, which follows immediately from Proposition 2.1.14.

Proposition 5.1.9. A C^*-algebra A has nuclear dimension at most n if and only if for any finite set $F \subset A$ and $\epsilon > 0$, there is a finite-dimensional C^*-algebra $K = K^{(0)} \oplus \cdots \oplus K^{(n)}$ such that for any $\delta > 0$, there is a c.p.c. map $\varphi : A \to K$ and a c.p. map $\psi : K \to A$ satisfying

1. $\|\psi \circ \varphi(a) - a\| < \epsilon$ for all $a \in F$,
2. For each $i = 0, \dots, n$, the restriction of ψ to $K^{(i)}$ is c.p.c. and δ-order zero.

5.2 Examples

5.2.1 Abelian C^*-Algebras

Definition 5.2.1. Let X be a topological space. The *order* of a collection $\{U_i\}_{i \in I}$ of subsets of X does not exceed n if, for any $n+2$ distinct indices $i_0, \dots, i_{n+1}$, the intersection of $U_0, \dots, U_{n+1}$ is empty. The *covering dimension* of X does not exceed n if every finite open cover of X has an open refinement whose order does not exceed n. If this is the case, we write $\dim_{\mathrm{top}}(X) \le n$; if n is the smallest such integer, we write $\dim_{\mathrm{top}}(X) = n$.

Example 5.2.2.

1. If X is a discrete topological space, then $\dim_{\text{top}}(X) = 0$.

2. Let $X = (0,1)$. Every finite open cover of X has a refinement of the form

$$\{(0,b_0),(a_1,b_1),\dots,(a_{k-1},b_{k-1}),(a_k,1)\}$$

   for some $k \geq 0$ and such that $a_{i+1} < b_i < b_{i+1}$ for each i. This shows $\dim_{\text{top}}(X) \leq 1$. Since the open cover $\{(0,\frac{3}{4}),(\frac{1}{4},1)\}$ has no refinement whose order does not exceed 0, we get $\dim_{\text{top}}(X) = 1$.

3. More generally, if X is any space homeomorphic to $\mathbb{R}^n$, then $\dim_{\text{top}}(X) = n$.

We say that a collection of subsets $\{U_i\}_{i\in I}$ of X is *n-decomposable* if there is a partition $I = I_0 \cup \cdots \cup I_n$ such that $U_i \cap U_{i'} = \varnothing$ whenever $i \neq i' \in I_j$. If X is locally compact and Hausdorff, then $\dim_{\text{top}}(X) \leq n$ if and only if every finite open cover of X has an n-decomposable open refinement (see [34, Proposition 1.5]). Using this, we can show that for abelian C^*-algebras, finite covering dimension implies a refinement of the completely positive approximation property.

Proposition 5.2.3. Suppose that X is a compact Hausdorff space. Then

$$\dim_{\text{nuc}}(C(X)) \leq \dim_{\text{top}}(X).$$

Proof. If $\dim_{\text{top}}(X) = \infty$ there is nothing to show, so assume $\dim_{\text{top}}(X) \leq n$. Choose a finite subset $F \subset C(X)$ and fix $\epsilon > 0$.

Since F is finite, there is a finite open cover $\{V_1,\dots,V_m\}$ of X such that for any $f \in F$, any $i = 0,\dots,m$, and any pair of points $x, y \in V_i$, we have

$$|f(x) - f(y)| < \epsilon.$$

By the assumption, there is an open refinement $\{U_1,\dots,U_k\}$ of $\{V_1,\dots,V_m\}$ and a partition $\{1,\dots,k\} = I_0 \cup \cdots \cup I_n$ such that $U_i \cap U_{i'} = \varnothing$ whenever $i \neq i' \in I_j$. Now write $\mathbb{C}^k$ as $K^{(0)} \oplus \cdots \oplus K^{(n)}$, where $K^{(j)} = \bigoplus_{I_j} \mathbb{C}$.

For each $i = 1,\dots,k$ let x_i be an arbitrary element in U_i. Define $\varphi : C(X) \to \mathbb{C}^k$ by

$$f \mapsto (f(x_1),\dots,f(x_k)).$$

It is easy to see that φ is a *-homomorphism, so in particular it is c.p.c.. Next let $\{\sigma_1,\dots,\sigma_k\}$ be a partition of unity subordinate to $\{U_0,\dots,U_k\}$. Define $\psi : \mathbb{C}^k \to C(X)$ by

$$(x_1,\dots,x_k) \mapsto \sum_{i=1}^{k} x_i \sigma_i.$$

This map is obviously positive, so it is completely positive since both the domain and range are abelian (Remark 2.1.8). If $x, y \in K_+^{(j)}$ satisfy $xy = 0$, then $\psi(x)\psi(y) = 0$ since the the supports of the functions in the set $\{\sigma_i \ : \ i \in I_j\}$ are disjoint. Lastly,

$$\|\psi \circ \varphi(f) - f\| = \|\sum_{i=1}^{k} f(x_i)\sigma_i - f(\sum_{i=1}^{k}\sigma_i)\| = \|\sum_{i=1}^{k}(f(x_i)1 - f)\sigma_i\| < \epsilon.$$

□

Remark 5.2.4.

1. Notice that the only place in the proof of Proposition 5.2.3 where we used $\dim_{\text{top}}(X) \le n$ was in showing that (the C^*-algebra) $\mathbb{C}^k$ can be written as a direct sum of $n+1$ finite-dimensional C^*-algebras, and that on each summand ψ restricts to a c.p.c. order zero map. Aside from this, the proof above is typically used to prove that $C(X)$ is nuclear.

2. Although we did not prove it here, the relationship between the dimension of X and the nuclear dimension of its associated C^*-algebra is even stronger. Indeed, if X is any locally compact Hausdorff space we have (by [34, Proposition 3.3] and [70, Proposition 2.4]) the following equality:

$$\dim_{\text{nuc}}(C_0(X)) = \dim_{\text{top}}(X).$$

   Moreover, both of these values equal $\text{dr}(C_0(X))$.

5.2.2 AF Algebras

If B is a finite-dimensional C^*-algebra, it is obvious that $\dim_{\text{nuc}}(B) = 0$. Passing to inductive limits, we obtain $\dim_{\text{nuc}}(A) = 0$ for any AF algebra A. Interestingly enough, the converse is also true:

$$\dim_{\text{nuc}}(A) = 0 \Leftrightarrow A \text{ is AF}.$$

We provide a sketch of the proof in the unital case. For this, we need the following useful characterization of AF algebras.

Proposition 5.2.5 (Glimm-Bratelli)**.** A C^*-algebra A is AF if and only if for every finite subset $\{a_1, \dots, a_n\}$ and $\epsilon > 0$, there exists a finite dimensional C^*-subalgebra B of A and elements $b_1, \dots, b_n$ in B satisfying $\|a_i - b_i\| < \epsilon$ for each $i = 1, \dots, n$.

Suppose now A is a unital C^*-algebra satisfying $\dim_{\text{nuc}}(A) = 0$. Fix a finite subset $F = \{a_1, \dots, a_n\}$ and $\epsilon > 0$. Find a factorization $\varphi : A \to K$ and $\psi : K \to A$ as in the definition of nuclear dimension with respect to the pair $(F \cup \{1\}, \epsilon/2)$. Using [34, Remark 4.3.ii], we may assume that φ is unital. By Theorem 2.1.12, there is a *-homomorphism

$$\pi_\psi : K \to \mathcal{M}(C^*(\psi(K))) \cap \{\psi(1)\}'$$

such that $\psi(a) = \pi_\psi(a)\psi(1)$ for all $a \in A$. Note that $\psi(1) = \psi(\varphi(1)) \approx_{\epsilon/2} 1$, so without loss of generality we can assume that $\psi(1)$ is invertible. This means π_ψ takes values in A, and in particular $B = \pi_\psi(K) \subseteq A$ is a finite-dimensional C^*-subalgebra. Taking $b_i = \pi_\psi(\varphi(a_i)) \in B$ gives

$$a_i \approx_{\epsilon/2} \psi(\varphi(a_i)) = \pi_\psi(\varphi(a_i))\psi(1) \approx_{\epsilon/2} b_i.$$

By Proposition 5.2.5, A is AF.

5.2.3 Kirchberg Algebras

Let A be a Kirchberg algebra in $\mathcal{N}$. In [70], Winter and Zacharias used a key result of Kirchberg regarding the uniqueness (up to approximate unitary equivalence) of endomorphisms of Cuntz algebras to prove that $\dim_{\mathrm{nuc}}(\mathcal{O}_n) = 1$ and $\dim_{\mathrm{nuc}}(\mathcal{O}_\infty) \le 2$. Combining this with the fact that A is an inductive limit of C^*-algebras of the form

$$M_{k_1}(\mathcal{O}_{n_1}) \oplus \cdots \oplus M_{k_r}(\mathcal{O}_{n_r}) \otimes C(\mathbb{T}),$$

for some natural numbers $k_1, \ldots, k_r$ and some $n_1, \ldots, n_r \in \{2, 3, \ldots, \infty\}$, they proved that A has nuclear dimension at most 5 (see [70, Theorem 7.5]). Ruiz, Sims, and Sørensen strengthened this result in [53, Theorem 6.6] and showed in fact that $\dim_{\mathrm{nuc}}(A) = 1$.

Now let B be any Kirchberg algebra, without any assumptions on belonging to $\mathcal{N}$. In [40, Theorem 7.1], Matui and Sato proved $\dim_{\mathrm{nuc}}(B) \le 3$. To conclude the story of nuclear dimension of Kirchberg algebras, it was shown in [3] that $\dim_{\mathrm{nuc}}(B) = 1$.

5.2.4 Minimal Actions

Although nuclear dimension behaves fairly well with respect to some C^*-constructions (such as direct sums, inductive limits, quotients, etc.), it is currently somewhat mysterious in the context of crossed products by general (discrete) groups. However, for actions of $\mathbb{Z}$ (or more generally $\mathbb{Z}^n$) we can say more.

Suppose that X is a finite dimensional compact metrizable space. If $h : X \to X$ is a minimal homeomorphism of X, results by Hirshberg, Winter, and Zacharias in [27] (and refined by Szabó in [57]), show that

$$\dim_{\mathrm{nuc}}(C(X) \rtimes_{\alpha_h} \mathbb{Z}) < \infty,$$

where α_h is the associated action of $\mathbb{Z}$ on $C(X)$. Their results rely on a certain property for automorphisms, which happens to be automatic for α_h. Namely, they show that such actions have finite *Rokhlin dimension*. This is the main topic of the next chapter, but for now we simply state how the presence of finite Rokhlin dimension for an automorphism influences the nuclear dimension of the associated crossed product. Specifically, if $(A, \mathbb{Z}, \alpha)$ is a dynamical system with $\dim_{\text{nuc}}(A) < \infty$, then

$$\dim_{\text{nuc}}(A \rtimes_\alpha \mathbb{Z}) < \infty$$

whenever α has finite Rokhlin dimension.

5.3 The Toms-Winter Conjecture

5.3.1 The Jiang-Su Algebra $\mathcal{Z}$

Suppose that $(G_0, G_0^+, u) = (\mathbb{Z}, \mathbb{Z}^+, 1)$, $G_1 = 0$, and Δ is a singleton set. It is not difficult to see that the C^*-algebra $\mathbb{C}$ has this as its Elliott invariant. In [28], Jiang and Su proved that there is an infinite dimensional, simple, unital ASH C^*-algebra $\mathcal{Z}$, called the *Jiang-Su algebra*, that also has this data as its Elliott invariant. This is the reason why Conjecture 4.2.11 has the assumption of infinite dimensionality. This assumption separates $\mathcal{Z}$ from $\mathbb{C}$.

If A is a simple unital C^*-algebra and if $K_0(A)$ is weakly unperforated, then by [23, Theorem 1], A and $A \otimes \mathcal{Z}$ have isomorphic Elliott invariants. We say that A is *$\mathcal{Z}$-stable* if $A \otimes \mathcal{Z} \simeq A$. It was shown in [28] that AF algebras and Kirchberg algebras are $\mathcal{Z}$-stable. Gong, Jiang, and Su proved that all simple unital AH algebras of bounded dimension are $\mathcal{Z}$-stable. In [23], the same authors proved that Villadsen's simple AH algebra from Example 4.2.6 is not $\mathcal{Z}$-stable.

Here is the punchline. In [60], Toms provided an example of a simple, unital, nuclear, separable, infinite-dimensional, and stably finite C^*-algebra A satisfying

$$A \not\simeq A \otimes \mathcal{Z} \qquad \text{and} \qquad \text{Ell}(A) \simeq \text{Ell}(A \otimes \mathcal{Z}).$$

The significance of is that if we wish to keep the Elliott invariant unchanged, we must add $\mathcal{Z}$-stability as an assumption to the Elliott conjecture. As such, understanding $\mathcal{Z}$-stability has become an important part of the classification program. In particular, the investigation into other structural properties of C^*-algebras that relate to $\mathcal{Z}$-stability has culminated in a conjecture of Toms and Winter. In the following sections we state and then give the current status of the Toms-Winter conjecture. However, before moving on we review one particular construction of $\mathcal{Z}$.

Let $k, m, n \in \mathbb{N}$ be such that both k and m divide n. The *dimension drop algebra* is the

C^*-algebra

$$I(k,m,n) = \{f : [0,1] \to M_n(\mathbb{C}) \;:\; f(0) \in M_k(\mathbb{C}) \otimes 1_{n/k} \text{ and } f(1) \in 1_{n/m} \otimes M_m(\mathbb{C})\}.$$

If k and m are relatively prime and $n = km$, we write $I(k,m)$ in place of $I(k,m,km)$ and refer to $I(k,m)$ as a *prime dimension drop algebra.* Define a map $\pi : (k,m,n) \to M_k(\mathbb{C})$ by $f \mapsto f(0)$. It is easy to check that π is a surjective *-homomorphism whose kernel is equal to $C_0((0,1]) \otimes 1_{n/m} \otimes M_m(\mathbb{C})$. Since both $M_k(\mathbb{C})$ and $\ker(\pi)$ are nuclear, we know that $I(k,m,n)$ is nuclear. Jiang and Su showed that there exists an inductive sequence

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots,$$

with unital connecting maps φ_n and prime dimension drop algebras A_n, such that $\mathcal{Z} := \varinjlim A_n$ is unital, simple, infinite dimensional, projectionless, has a unique tracial state, and has the same K-theory as $\mathbb{C}$. Their construction is unique in the following sense. For any inductive sequence

$$B_1 \xrightarrow{\phi_1} B_2 \xrightarrow{\phi_2} B_3 \xrightarrow{\phi_3} \cdots,$$

with unital connecting maps ϕ_n and prime dimension drop algebras B_n, we have $\varinjlim B_n \simeq \mathcal{Z}$ if and only if $\varinjlim B_n$ is simple and has a unique tracial state.

Remark 5.3.1.

1. In [32, Theorem 2.4], Kirchberg and Rørdam show that if A is a unital separable C^*-algebra, then A is $\mathcal{Z}$-stable if and only if there is a unital embedding of $\mathcal{Z}$ into $F(A)$ (from Example 2.1.4).

2. The Jiang-Su algebra $\mathcal{Z}$ is *strongly self absorbent* in the the sense that the *-homomorphisms $\psi_1, \psi_2 : \mathcal{Z} \to \mathcal{Z} \otimes \mathcal{Z}$ given by $\psi_1(a) = a \otimes 1$ and $\psi_2(a) = 1 \otimes a$ are approximately unitarily equivalent. In particular, $\mathcal{Z}$ is $\mathcal{Z}$-stable. Note that this is completely analogous to the situation for von-Neumann algebras; the (unique) hyperfinite II_1 factor $\mathcal{R}$ satisfies this property. From this, Connes was able to show in [9] that an injective II_1 factor is hyperfinite.

5.3.2 The Cuntz Semigroup and Strict Comparison

Let A be a C^*-algebra, and let $M_\infty(A)$ denote the algebraic limit of the system

$$A \xrightarrow{\varphi_1} M_2(A) \xrightarrow{\varphi_2} M_3(A) \xrightarrow{\varphi_3} \cdots \qquad \varphi_n(a) = \mathrm{diag}(a,0).$$

Denote by $M_\infty(A)_+$ denote the positive elements in $M_\infty(A)$. If $a, b\ M_\infty(A)_+$, let $a \oplus b = \mathrm{diag}(a,b) \in M_\infty(A)_+$. We say that a is *Cuntz subequivalent* to b, and write $a \precsim b$, whenever there is a sequence $\{v_n\}_{n\in\mathbb{N}}$ of elements in $M_\infty(A)$ such that $\lim_{n\to\infty}\|v_n b v_n^* - a\| = 0$. We say a and b are *Cuntz equivalent*, and write $a \sim b$, if both $a \precsim b$ and $b \precsim a$. The Cuntz equivalence $\sim$ is an equivalence relation. Furthermore, if $\langle a\rangle$ denotes the equivalence class containing $a \in M_\infty(A)$, the set

$$W(A) = M_\infty(A)/\sim$$

is a positively ordered abelian semigroup under the operations $\langle a\rangle + \langle b\rangle = \langle a \oplus b\rangle$ and $\langle a\rangle \le \langle b\rangle \Leftrightarrow a \precsim b$. $W(A)$ is known as the *Cuntz semigroup* of A. The Grothendieck group of $W(A)$ is denoted by $K_0^*(A)$.

An interesting feature of the Cuntz semigroup is that it serves as a potential alternative to the Elliott invariant. Indeed, if B is the counterexample to Elliott's conjecture provided by Toms, then B is distinguished from $B \otimes \mathcal{Z}$ by its Cuntz semigroup (see [60, Corollary 1.1]). In other words, another possible direction in the classification program is to add $W(A)$ to $\mathrm{Ell}(A)$ without assuming anything extra about the structure of A.

Let $\mathrm{Tr}_k : M_k(\mathbb{C}) \to \mathbb{C}$ denote the un-normalized trace on $M_k(\mathbb{C})$. If $\tau \in T(A)$, the *dimension function* associated to τ is the map $d_\tau : M_\infty(A)_+ \to [0,\infty)$ defined by

$$d_\tau(a) = \lim_{n\to\infty}(\mathrm{Tr}_k \otimes \tau)(a^{1/n}) \qquad a \in M_k(A)_+.$$

Definition 5.3.2. If A is a simple, unital, and nuclear C^*-algebra, we say A has *strict comparison for positive elements* if, for all $a, b \in M_\infty(A)_+$, the following implication holds:

$$d_\tau(a) < d_\tau(b)\ \forall \tau \in T(A) \Rightarrow a \precsim b.$$

Remark 5.3.3. Note that the strict comparison property is analogous to the situation for II_1 factors in von-Neumann algebra theory. Indeed, if $\mathcal{M}$ is a II_1 factor, then $\mathcal{M}$ contains a unique trace τ, and $p \sim q \in \mathcal{P}(\mathcal{M})$ if and only if $\tau(p) = \tau(q)$.

5.3.3 The Toms-Winter Conjecture

Conjecture 5.3.4. Let A be a simple, unital, separable, stably finite, nuclear, infinite-dimensional C^*-algebra. Then the following are equivalent.

1. A has finite nuclear dimension.

2. A is $\mathcal{Z}$-stable.

3. A has strict comparison of positive elements.

If A is stably finite, then a revised version of the conjecture states that the first condition can be replaced with $\mathrm{dr}(A) < \infty$.

Here is the current status of the conjecture. In [67] and [68], Winter proved that finite nuclear dimension implies $\mathcal{Z}$-stability. In [49] Rørdam proved that $\mathcal{Z}$-stability implies strict comparison. If A has strict comparison and $T(A)$ is a Bauer simplex satisfying $\dim_{\text{top}}(\partial T(A)) < \infty$, a collection of results by Kirchberg and Rørdam in [33], Sato in [55], and Toms, White, and Winter in [61] shows that A is $\mathcal{Z}$-stable. These extended work by Matui and Sato in [39] which handled the case where $|\partial T(A)| < \infty$. If A is $\mathcal{Z}$ stable and has a unique trace, then results by Matui and Sato in [40] and Sato, White, and Winter in [56] shows that A has finite nuclear dimension. Recently, this was improved by Bosa, Brown, Sato, Tikuisis, White, and Winter in [3] to handle the case that $T(A)$ is a Bauer simplex. To summarize, we have the following theorem.

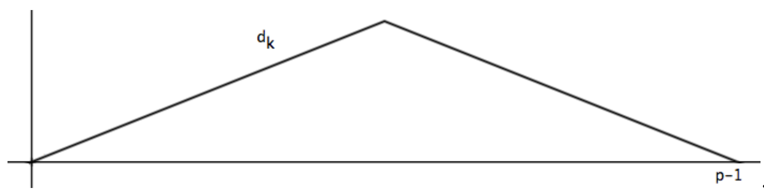
Theorem 5.3.5. Let A be a simple, unital, separable, stably finite, nuclear, infinite-dimensional C^*-algebra. If $T(A)$ is a Bauer simplex and satisfies $\dim_{\text{top}}(\partial T(A)) < \infty$, then then Toms-Winter conjecture holds for A.

Remark 5.3.6. Let $\mathscr{C}$ the isomorphism classes of simple ASH algebras with no dimension growth. It is well-known that $\mathscr{C}$ exhausts the Elliott invariant, so if the Toms-Winter conjecture holds and if A is as in the conjecture, then $A \in \mathscr{C}$. We say that A is *homotopy rigid* if $A \in \mathscr{C}$ whenever A is homotopic to an element in $\mathscr{C}$. In [40], Matui and Sato proved (among other things) that for simple, unital, and separable C^*-algebras having finite nuclear dimension, homotopy rigidity follows from having a unique trace.

Chapter 6
Rokhlin Dimension

The following lemma will be used several times in this chapter. We prove it now to get it out of the way.

Lemma 6.0.7. Let A be a C^*-algebra and $p \in \mathbb{N}$ an odd integer. For $k = 0, \ldots, p-1$ let $d_k = 1 - \frac{|p-1-2k|}{p-1}$:



.

If $f_0, \ldots f_{p-1}$ are positive contractions in A, then for any $0 \leq N \leq p-1$,

$$\sum_{k=N}^{p-1} d_k \left(f_k + f_{\frac{p-1}{2}+k} \right) \approx_{\frac{4N^2}{p-1}} \sum_{k=0}^{p-1} f_k. \tag{6.1}$$

Proof. It's easy to verify that

$$d_k + d_{\frac{p-1}{2}+k} = 1 \text{ if } k = 0, \ldots, \frac{p-1}{2} \quad \text{and} \quad d_k + d_{k-\frac{p-1}{2}} = 1 \text{ if } k = \frac{p-1}{2}, \ldots, p-1,$$

so that

$$\sum_{k=0}^{p-1} d_k \left(f_k + f_{\frac{p-1}{2}+k} \right) = \sum_{k=0}^{\frac{p-1}{2}} \left(d_k + d_{k+\frac{p-1}{2}} \right) f_k + \sum_{k=\frac{p-1}{2}}^{p-1} \left(d_k + d_{k-\frac{p-1}{2}} \right) f_k = \sum_{k=0}^{p-1} f_k.$$

Moreover,

$$\| \sum_{k=0}^{N-1} d_k \left(f_k + f_{\frac{p-1}{2}+k} \right) \| \leq 2 \sum_{k=0}^{N-1} d_k \leq 2N \left(1 - \frac{p-1-2N}{p-1} \right) = \frac{4N^2}{p-1},$$

so the result follows. □

6.1 Nuclearity for Crossed Products by $\mathbb{Z}$

Let us recall some basic facts regarding crossed products by $\mathbb{Z}$. Let A be a C^*-algebra acting faithfully on a Hilbert space H. If $\alpha \in \mathrm{Aut}(A)$, define $\pi : A \to \mathbb{B}(\ell^2(\mathbb{Z}) \otimes H)$ and $\lambda : \mathbb{Z} \to \mathbb{B}(\ell^2(\mathbb{Z}) \otimes H)$ by

$$\pi(a)(i \otimes \xi) = i \otimes \alpha^{-i}(a)\xi \quad \text{and} \quad \lambda_j(i \otimes \xi) = (i + j) \otimes \xi.$$

The pair (π, λ) is covariant (in the sense that $\lambda_1 \pi(a) = \pi(\alpha(a))\lambda_1$ for every $a \in A$), and the reduced crossed-product $A \rtimes_\alpha \mathbb{Z}$ is the C^*-subalgebra of $\mathbb{B}(\ell^2(\mathbb{Z}) \otimes H)$ generated by $\{\pi(a) \mid a \in A\}$ and the unitary $U = \lambda_1$. We write $\pi(a)U^n$ as aU^n. Since $\mathbb{Z}$ is an amenable group, the following is a reiteration of Remark 2.2.25 and Theorem 2.2.26.

Theorem 6.1.1. Let B be a unital C^* algebra generated by a copy of A and a unitary W satisfying $Wa = \alpha(a)W$. Then there is a *-homomorphism from $A \rtimes_\alpha \mathbb{Z}$ onto B sending a to a and U to W.

Next we have an important permanence property. The argument given below outlines the germ of the ideas needed to prove Theorem 6.7.1. The details are omitted, but the main ingredients are made explicit.

Theorem 6.1.2. Let A be a C^*-algebra and $\alpha \in \mathrm{Aut}(A)$. If A is nuclear, then $A \rtimes_\alpha \mathbb{Z}$ is also nuclear.

Proof. Choose $a \in A, n \in \mathbb{Z}$, and $p \in \mathbb{N}$. Compression of the crossed product by the orthogonal projection P onto $\ell^2(\{0, \dots, p-1\}) \otimes H \subset \ell^2(\mathbb{Z}) \otimes H$ provides a c.p.c. outgoing map $\varphi : A \rtimes_\alpha \mathbb{Z} \to M_p(A)$. The tricky part is to find a c.p. incoming map $\psi : M_p(A) \to A \rtimes_\alpha \mathbb{Z}$. Let $R \in M_{1,p}(A \rtimes_\alpha \mathbb{Z})$ be the row vector

$$R = \begin{bmatrix} \frac{1}{\sqrt{p}}U^0 & \frac{1}{\sqrt{p}}U^1 & \cdots & \frac{1}{\sqrt{p}}U^{p-1} \end{bmatrix}.$$

If we regard $M_p(A)$ as a subalgebra of $M_p(A \rtimes_\alpha \mathbb{Z})$, we can define $\psi : M_p(A) \to A \rtimes_\alpha \mathbb{Z}$ by $x \mapsto RxR^*$. By Theorem 2.1.7, ψ is completely positive. Moreover, it is readily checked that

$$\psi : e_{i,j} \otimes a \mapsto \frac{1}{p} U^i a U^{*j}.$$

Since $\varphi(aU^n) = \sum_{k=n}^{p-1} e_{k,k-n} \otimes \alpha^{-k}(a)$, we have

$$\psi \circ \varphi(aU^n) = \frac{1}{p} \sum_{k=n}^{p-1} U^k \alpha^{-k}(a) U^{*k-n} = \frac{p-n}{p} aU^n.$$

For a fixed value of n, we can make p large enough so that $\psi \circ \varphi(aU^n)$ is within any prescribed tolerance of aU^n. The result follows because $A \rtimes_\alpha \mathbb{Z} = \overline{\text{span}}\{aU^n \mid a \in A, n \in \mathbb{Z}\}$, and $M_n(A)$ is nuclear whenever A is. □

Remark 6.1.3. In general, it is not known when finite nuclear dimension is preserved under crossed products by $\mathbb{Z}$. In the proof above, the map from $\psi : M_p(A) \to A \rtimes_\alpha \mathbb{Z}$ was c.p.c., but it did not have order zero. For example,

$$(1 \otimes e_{0,0}) \perp (1 \otimes e_{1,1}) \quad \text{but} \quad \psi(1 \otimes e_{0,0})\psi(1 \otimes e_{1,1}) = 1.$$

6.2 Measurable Dynamics and the Rokhlin Lemma

In this section, we take a brief digression to motivate some upcoming definitions. Let $(X, \mathcal{B})$ be a measurable space. We say a set $A \in \mathcal{B}$ is an *atom* if $\mu(A) > 0$, and for any $B \in \mathcal{B}$ we have $\mu(B) = 0 \Leftrightarrow \mu(B) < \mu(A)$. A measure which has no atoms is called *non-atomic*.

Let $(X, \mathcal{B}, \mu)$ be a probability space and $T : X \to X$ a measurable and measure preserving map (i.e. $T^{-1}A \in \mathcal{B}$ and $\mu(T^{-1}A) = \mu(A)$ for every $A \in \mathcal{B}$). The 4-tuple $(X, \mathcal{B}, \mu, T)$ is called a *measure preserving dynamical system*. If T is invertible and T^{-1} is measurable, we say the system is *invertible*. A measurable set $A \subseteq X$ is *invariant under* T if $T^{-1}(A) = A$; the system is called *ergodic* if every invariant set has measure 0 or 1. Ergodicity is a minimality condition: A is invariant under T if and only if $X \backslash A$ is invariant under T, so the system is ergodic whenever it cannot be split into two nontrivial pieces.

Example 6.2.1. If G is a compact group with normalized Haar measure μ, then for each $g \in G$ the map $T_g : G \to G$ given by $T(h) = gh$ is measurable and measure preserving.

The following is a classical result in the theory of measurably dynamics, originally proven by Rokhlin and Kakutani. Roughly speaking, it states that invertible ergodic transformations on X behave as shift operators on arbitrarily large portions of X.

Theorem 6.2.2 (Rokhlin Lemma)**.** Let $(X, \mathcal{B}, \mu, T)$ be an invertible ergodic measure preserving system and assume that μ is non-atomic. Then for any $n \in \mathbb{N}$ and $\epsilon > 0$, there is a set $B \in \mathcal{B}$ with the property that $B, T(B), \ldots, T^{n-1}B$ are disjoint sets, and $\mu(B \cup T(B) \cup \cdots \cup T^{n-1}(B)) > 1 - \epsilon$.

In the noncommutative setting, Connes proved in [8] that an analogous result holds for certain types of automorphisms on von Neumann algebras. If N is a von Neumann algebra and $\tau \in T(N)$, denote by $\|x\|_2$ the number $\tau(x^*x)^{1/2}$ for $x \in N$.

Theorem 6.2.3. Let N be a finite von Neumann algebra, τ a faithful normal trace on N, $\tau(1) = 1$, and θ an aperiodic automorphism of N which preserves τ. Then for any integer n

and any $\epsilon > 0$, there exists a partition of unity $(F_j)_{j=1,\ldots,n}$ in N such that

$$\|\theta(F_1) - F_2\|_2 \le \epsilon,\ \|\theta(F_2) - F_3\|_2 \le \epsilon\ ,\ldots,\ \|\theta(F_n) - F_1\|_2 \le \epsilon.$$

6.3 Rokhlin Dimension for Automorphisms

Back in the realm of C^*-algebras, the notion of Rokhlin dimension for automorphisms was introduced by Hirshberg, Winter, and Zacharias in [27] as a generalization of the Rokhlin property (see [35] for example).

Definition 6.3.1. Let A be a C^*-algebra and $\alpha \in \mathrm{Aut}(A)$. We say that α has *Rokhlin dimension at most d*, $\dim_{\mathrm{Rok}}(\alpha) \le d$, if the following holds.

For any finite subset $F \subset A$, any $p \in \mathbb{N}$, and any $\epsilon > 0$, there are positive (Rokhlin) contractions

$$\left\{f_k^l\right\}_{k=0,\ldots,p-1}^{l=0,\ldots,d} \subset A$$

satisfying

1. $\|f_k^l f_{k'}^l\| < \epsilon$ whenever $k \ne k'$,
2. $\|\sum_{k,l} f_k^l - 1\| < \epsilon$,
3. $\|\alpha(f_k^l) - f_{k+1}^l\| < \epsilon$ for all k, l,
4. $\|[f_k^l, a]\| < \epsilon$ for all k, l and $a \in F$.

Remark 6.3.2. In [27], Definition 6.3.1 is known as Rokhlin dimension *with single towers*. This distinction is important for the Rokhlin property but it is relatively mild for Rokhlin dimension (see [27, Proposition 2.8]).

Example 6.3.3. Examples of finite Rokhlin dimension are abundant. Indeed, we have by [27, Theorem 3.4] that if A is a separable unital and $\mathcal{Z}$-stable C^*-algebra, then a dense G_δ set of automorphisms of A have Rokhlin dimension at most one. In the same paper, it was shown that for a minimal homeomorphism φ of a compact metric space X, there is an inequality

$$\dim_{\mathrm{Rok}}(\varphi^*) \le 2\dim_{\mathrm{top}}(X) + 1.$$

In particular, irrational rotations on $\mathbb{T}$ have Rokhlin dimension at most three.

Next we review [27, Theorem 4.1], which relates the Rokhlin dimension of an automorphism and the nuclear dimension of the associated crossed-product. As with Theorem 6.1.2, we highlight only those components of the proof that will be needed to motivate key technical constructions in subsequent sections. In particular, we omit explicit estimates.

Theorem 6.3.4. Let A be a C^*-algebra and $\alpha \in \mathrm{Aut}(A)$. If $\dim_{\mathrm{nuc}}(A)$ and $\dim_{\mathrm{Rok}}(\alpha)$ are finite, then $\dim_{\mathrm{nuc}}(A \rtimes_\alpha \mathbb{Z})$ is finite.

Proof. The idea is to use finite Rokhlin dimension to circumvent the obstacle addressed in Remark 6.1.3.

Choose $aU^n \in A \rtimes_\alpha \mathbb{Z}$ and let $p \in \mathbb{N}$ be an odd integer. Let d_k be as in Lemma 6.0.7 and define the scalar matrix $\Delta = \mathrm{diag}(d_0, \dots, d_{p-1})$. If $\varphi : A \rtimes_\alpha \mathbb{Z} \to M_p(A)$ is compression by $\sqrt{\Delta}P$, then since $d_{k-n} \approx d_k$ we have

$$\varphi(aU^n) \approx \sum_{k=n}^{p-1} d_k e_{k,k-n} \otimes \alpha^{-k}(a).$$

The incoming maps in the proof of nuclearity did not have order zero, so we need replacements. To this end, suppose (for the moment) that $q_0, \dots, q_{p-1} \in \mathcal{Z}(A)$ are mutually orthogonal projections satisfying

$$q_0 + \cdots + q_{p-1} = 1 \quad \text{and} \quad \alpha(q_k) = q_{k+1}.$$

Let $R \in M_{1,p}(A \rtimes_\alpha \mathbb{Z})$ be the row vector

$$R = \begin{bmatrix} q_0 U^0 & q_1 U^1 & \cdots & q_{p-1} U^{p-1} \end{bmatrix},$$

and note that $R^* R = \mathrm{diag}(q_0, \dots, q_0) \in \mathcal{Z}(M_p(A))$. This means the map $\sigma : M_p(A) \to A \rtimes_\alpha \mathbb{Z}$ given by $x \mapsto RxR^*$ has order zero, and taking x to be $\varphi(aU^n)$ shows that

$$\sigma \circ \varphi(aU^n) \approx \sum_{k=n}^{p-1} d_k q_k a U^n.$$

Now let $\hat{R} \in M_{1,p}(A \rtimes_\alpha \mathbb{Z})$ be the row vector

$$\hat{R} = \begin{bmatrix} q_{\frac{p-1}{2}+0} U^0 & q_{\frac{p-1}{2}+1} U^1 & \cdots & q_{\frac{p-1}{2}+p-1} U^{p-1} \end{bmatrix},$$

and let $\hat{\sigma} : M_p(A) \to A \rtimes_\alpha \mathbb{Z}$ be given by $x \mapsto \hat{R} x \hat{R}^*$. It can similarly be shown that $\hat{\sigma}$ has order zero and that

$$\hat{\sigma} \circ \varphi(aU^n) \approx \sum_{k=n}^{p-1} d_k q_{\frac{p-1}{2}+k} a U^n.$$

If $\psi = \sigma + \hat{\sigma}$, then by Lemma 6.0.7 we have

$$\psi \circ \varphi(aU^n) \approx aU^n.$$

The result follows because $A \rtimes_\alpha \mathbb{Z} = \overline{\mathrm{span}}\{aU^n \mid a \in A, n \in \mathbb{N}\}$, and $M_p(A)$ has finite nuclear dimension whenever A does.

The key extra ingredient in this argument is the existence of central projections $q_0, \dots, q_{p-1}$

that are cyclically permuted by α. Finite Rokhlin dimension does not in general give such projections, but it does provide the next best thing. Indeed, suppose for the moment that $\dim_{\mathrm{Rok}}(\alpha) = 0$ and use (the square-root of) Rokhlin contractions $(f_0)^{1/2}, \ldots, (f_{p-1})^{1/2}$ in place of $q_0, \ldots, q_{p-1}$. This creates two holes in the argument above.

1. The row vector $R = \begin{bmatrix}(f_0)^{1/2}U^0 & \cdots & (f_{p-1})^{1/2}U^{p-1}\end{bmatrix}$ does not satisfy the property $R^*R \in \mathcal{Z}(M_p(A))$, but this can be fixed. Since $M_p(A)$ has finite nuclear dimension, find an approximate factorization
$$M_p(A) \xrightarrow{\Phi} K \xrightarrow{\Psi} M_p(A)$$
such that $\dim(K) < \infty$ and Ψ can be decomposed into a sum of c.p.c. maps having order zero. By Proposition 5.1.9, we only need $\sigma \circ \Psi$ and $\hat{\sigma} \circ \Psi$ to be approximately contractive and approximately order zero. To arrange this, choose the Rokhlin contractions so that $f_0 + \cdots + f_{p-1}$ approximately equals 1 and R^*R approximately commutes with the norm compact set $\bigcup_i \Psi(\mathrm{Ball}_1 K^{(i)}) \subset M_p(A)$.

2. The Rokhlin contractions are only approximately permuted by α. This adds some error to the estimates, but it does not invalidate the argument.

In general, we have $d+1$ collections $\{f_k^l\}_{k=0,\ldots,p-1}^{l=0,\ldots,d}$. This further adds some error to the estimates and increases $\dim_{\mathrm{nuc}}(A \rtimes_\alpha \mathbb{Z})$, but finiteness of nuclear dimension still holds. □

6.4 Rokhlin Dimension for C^*-Correspondences

To motivate the definition of Rokhlin dimension for C^*-correspondences, let us consider the crossed product from a different perspective. Recall that if A is a C^*-algebra and $\alpha \in \mathrm{Aut}(A)$, then $A \rtimes_\alpha \mathbb{Z}$ is the universal C^*-algebra generated by a copy of A and a unitary U satisfying $UaU^* = \alpha(a)$. We can regard $A^\alpha = A$ as a singly generated free C^*-correspondence over A with the left action given by $a.b = \alpha(a)b$. For each $k \geq 0$ the k-fold tensor product $(A^\alpha)^{\otimes k}$ is also singly generated and free; denote the generator $1 \otimes \cdots \otimes 1$ of $(A^\alpha)^{\otimes k}$ by k. The left action is given by $a.k = k.\alpha^k(a)$, and the creation operator $T = T_1$ is an isometry sending k to $k+1$. If $S = Q(T)$, it is easily checked that S is unitary satisfying $S^*aS = \alpha(a)$. By Theorem 6.1.1, there is a *-homomorphism $A \rtimes_\alpha \mathbb{Z} \to \mathcal{O}(A^\alpha)$ sending a to a and U to S^*. Now consider the covariant representation (π, τ) of A^α on $A \rtimes_\alpha \mathbb{Z}$ given by $\pi(a) = a$ and $\tau(1) = U^*$. By universality of the Cuntz-Pimsner algebra (Theorem 3.2.8), there is a *-homomorphism $\mathcal{O}(A^\alpha) \to A \rtimes_\alpha \mathbb{Z}$ sending a to a and S to U^*. These two maps are obviously inverses, so

$$\mathcal{O}(A^\alpha) \simeq A \rtimes_\alpha \mathbb{Z}.$$

This identification suggests that various dynamical properties of α may be recast in the language of correspondences. In this spirit, we give the following definition and briefly review the proof of Theorem 6.3.4 from the point of view of a Toeplitz-Pimsner algebra acting on a Fock space.

Definition 6.4.1. Let $\mathcal{H}$ be a countably generated C^*-correspondence over A. We say that $\mathcal{H}$ has *Rokhlin dimension at most* d, $\dim_{\mathrm{Rok}}(\mathcal{H}) \leq d$, if the following holds: for any $\epsilon > 0$, any $p \in \mathbb{N}$, any finite set $F \subset A$, and any finite set $\mathcal{V} \subset \mathcal{H}$, there exist positive contractions

$$\{f_k^l\}_{k=0,\dots,p-1}^{l=0,\dots,d} \subset A$$

satisfying

1. $\|f_k^l f_{k'}^l\| < \epsilon$ when $k \neq k'$ and all l.
2. $\|\sum_{k,l} f_k^l - 1\| < \epsilon$.
3. $\|z.\, f_k^l - f_{k+1}^l.\, z\| < \epsilon$ for all k, l, and $z \in \mathcal{V}$.
4. $\|[f_k^l, a]\| < \epsilon$ for all k, l and $a \in F$.

Remark 6.4.2. We can express Definition 6.4.1 without explicitly mentioning approximate centrality of the Rokhlin contractions. Indeed, $\dim_{\mathrm{Rok}}(\mathcal{H}) \leq d$ if and only if for any $\epsilon > 0$, any $p \in \mathbb{N}$, and any finite set $\mathcal{V}$ of elementary tensors in $\mathcal{F}_p(\mathcal{H})$, there exist positive contractions

$$\{f_k^l\}_{k=0,\dots,p-1}^{l=0,\dots,d} \subset A$$

satisfying

1. $\|f_k^l f_{k'}^l\| < \epsilon$ when $k \neq k'$ and all l.
2. $\|\sum_{k,l} f_k^l - 1\| < \epsilon$.
3. $\|z.\, f_k^l - f_{k+|z|}^l.\, z\| < \epsilon$ for all k, l, and $z \in \mathcal{V}$.

Since elements in A correspond to elementary tensors of length zero, the third condition above adequately replaces the last two conditions in Definition 6.4.1.

Theorem 6.4.3. Let A be a C^*-algebra and $\alpha \in \mathrm{Aut}(A)$. If $\dim_{\mathrm{nuc}}(A)$ and $\dim_{\mathrm{Rok}}(A^\alpha)$ are finite, then $\dim_{\mathrm{nuc}}(\mathcal{O}(A^\alpha))$ is finite.

Sketch of Proof. Fix $u, v \in \{0\} \cup \mathbb{N}$ and $a \in A$. Assume for the moment that $u \geq v$. For an odd integer $p \in \mathbb{N}$ let P denote the projection onto the p^{th} cutoff Fock space $\mathcal{F}_p(A^\alpha) = \bigoplus_{k=0}^{p-1}(A^\alpha)^{\otimes k}$. There is a $*$-isomorphism $\mathbb{B}(\mathcal{F}_p(A^\alpha)) \simeq M_p(A)$ given by $e_{i.a,j} \mapsto e_{i,j} \otimes a$. If $\Delta = \mathrm{diag}(d_0, \dots, d_k)$ is as in Theorem 6.3.4, let $\phi : \mathcal{T}(A^\alpha) \to M_p(A)$ be compression by $\sqrt{\Delta}P$. We have

$$\phi(T_u a T_v^*) \approx \sum_{k=0}^{p-1-u} d_k e_{u+k,v+k} \otimes \alpha^{-k}(a).$$

If we regard $M_p(A)$ as a subalgebra of $M_p(\mathcal{T}(A^\alpha))$, we can define incoming maps as follows. For each l, let $R^l \in M_{1,p}(\mathcal{T}(A^\alpha))$ be the $1 \times p$ row vector

$$R^l = \left[(f_0^l)^{1/2} T_0 \quad (f_1^l)^{1/2} T_1 \quad \cdots \quad (f_{p-1}^l)^{1/2} T_{p-1} \right],$$

and let $\sigma^l : M_p(A) \to \mathcal{T}(A^\alpha)$ be compression by R^l. We have

$$\begin{aligned} \sigma^l \circ \phi(T_u a T_v^*) &= \sum_{k=0}^{p-1-u} d_k (f_{u+k}^l)^{1/2} T_{u+k} \alpha^{-k}(a) T_{v+k}^* (f_{v+k}^l)^{1/2} \\ &\approx (\sum_{k=u}^{p-1} d_k f_k^l) T_u a T_v^* + \mathbb{K}(\mathcal{F}(A^\alpha)). \end{aligned}$$

Moreover, $R^{l*} R^l \approx \operatorname{diag}(f_0^l, \ldots, f_{p-1}^l)$. Now define $\hat{R}^l$ to be the row vector

$$\hat{R}^l = \left[(f_{\frac{p-1}{2}+0}^l)^{1/2} T_0 \quad (f_{\frac{p-1}{2}+1}^l)^{1/2} T_1 \quad \cdots \quad (f_{\frac{p-1}{2}+p-1}^l)^{1/2} T_{p-1} \right],$$

and let $\hat{\sigma}^l$ be compression by $\hat{R}^l$. It can similarly be shown that

$$\hat{\sigma}^l \circ \phi(T_u a T_v^*) \approx (\sum_{k=u}^{p-1} d_k f_{\frac{p-1}{2}+k}^l) T_u a T_v^* + \mathbb{K}(\mathcal{F}(A^\alpha)),$$

and that $\hat{R}^{l*} \hat{R}^l \approx \operatorname{diag}(f_{\frac{p-1}{2}}^l, \ldots, f_{\frac{p-1}{2}}^l)$. Adding σ^l and $\hat{\sigma}^l$, applying (6.1), summing over $l = 0, \ldots, d$, and applying the quotient map Q, we have

$$Q \circ \sum_{l=0}^{d} (\sigma^l + \hat{\sigma}^l) \circ \phi(T_u a T_v^*) \approx Q(T_u a T_v^*).$$

The outgoing map ϕ takes values in $M_p(A)$, so we can find a factorization

$$M_p(A) \xrightarrow{\Phi} K \xrightarrow{\Psi} M_p(A)$$

such that $\dim(K) < \infty$ and Ψ can be decomposed into a sum of c.p.c. maps having order zero. We can arrange the Rokhlin contractions so that $\operatorname{diag}(f_0^l, \ldots, f_0^l)$ and $\operatorname{diag}(f_{\frac{p-1}{2}}^l, \ldots, f_{\frac{p-1}{2}}^l)$ both approximately commute with the norm compact set

$$\bigcup_i \Psi(\mathrm{Ball}_1 K^{(i)}) \subset M_p(A).$$

This means we can arrange for $\sigma^l \circ \Psi$ and $\hat{\sigma}^l \circ \Psi$ to be approximately order zero. The result follows because there is a c.p.c. lift $V : \mathcal{O}(A^\alpha) \to \mathcal{T}(A^\alpha)$ (Theorems 2.3.10 and 3.2.11), the algebra $M_p(A)$ has finite nuclear dimension whenever A does, and $\mathcal{O}(A^\alpha) =$

$\overline{\text{span}}\{Q(T_u a T_v^*)\}$. □

In the general case of a C^*-correspondence with finite Rokhlin dimension, the argument above is incomplete in two ways:

1. The range of the outgoing map ϕ will in general not be a matrix algebra over A. It is therefore necessary to check that the outgoing map takes values in a C^*-algebra with finite nuclear dimension.

2. In the case $\mathcal{H} = A^\alpha$, the incoming map is defined by first embedding $M_p(A)$ into $M_p(\mathcal{T}(A^\alpha))$ and then using row vectors in $M_{1,p}(\mathcal{T}(A^\alpha))$ to compress. In the general case, we need to find suitable replacements for these row vectors.

In the next two sections, we resolve these issues.

6.5 Range of Outgoing Maps

Let $\mathcal{H}$ be a countably generated C^*-correspondence over A. For each $p \in \mathbb{N}$, form $\mathcal{F}_p(\mathcal{H}) = \bigoplus_{k=0}^{p-1} \mathcal{H}^{\otimes k}$. Let $D_p(\mathcal{H}) \subset \mathbb{B}(\mathcal{F}_p(\mathcal{H}))$ be the C^*-algebra generated by elements of the form $e_{x,y} \otimes 1_{\mathcal{H}^{\otimes k}} \in \mathbb{B}(\mathcal{F}_p(\mathcal{H}))$ where x and y are elementary tensors in $\mathcal{F}_p(\mathcal{H})$ and $k \geq 0$ satisfies $0 \leq \max(|x|, |y|) + k < p$. Evidently, $\mathbb{K}(\mathcal{F}_p(\mathcal{H}))$ is an ideal in $D_p(\mathcal{H})$.

Definition 6.5.1. We say that $\mathcal{H}$ is *quasidiagonal* if for every $p \in \mathbb{N}$, there is an approximate unit consisting of projections in $\mathbb{K}(\mathcal{F}_p(\mathcal{H}))$ that are quasicentral in $D_p(\mathcal{H})$.

Example 6.5.2. If $\mathcal{H}$ is a finitely generated projective C^*-correspondence, then so is $\mathcal{H}^{\otimes k}$ for any $k \geq 0$ (c.f. Proposition 4.7 in [38]). Hence, $\mathbb{K}(\mathcal{F}_p(\mathcal{H}))$ is unital for any $p \in \mathbb{N}$ and we get

$$\mathbb{K}(\mathcal{F}_p(\mathcal{H})) = D_p(\mathcal{H}) = \mathbb{B}(\mathcal{F}_p(\mathcal{H})),$$

so clearly $\mathcal{H}$ is quasidiagonal.

Lemma 6.5.3. If $\mathcal{H}$ is quasidiagonal, then for every $p \in \mathbb{N}$ we have

$$\dim_{\text{nuc}}(D_p(\mathcal{H})) \leq \dim_{\text{nuc}}(A).$$

Proof. It is not hard to see that $D_1(\mathcal{H}) \simeq A$. Suppose the result holds for D_i for $i = 1, \ldots, p$. Let

$$\tilde{D}_p = \overline{\text{span}}\{(e_{x,y} \otimes 1_{\mathcal{H}^{\otimes k}}) \otimes 1_{\mathcal{H}} \ : \ 0 \leq \max(|x|, |y|) + k < p\} \subset D_{p+1}(\mathcal{H}).$$

It's clear that $\tilde{D}_p \simeq D_p(\mathcal{H})$ and that

$$D_{p+1}(\mathcal{H}) = C^*(\mathbb{K}(\mathcal{F}_{p+1}(\mathcal{H})) \cup \tilde{D}_p) \qquad \text{and} \qquad \mathbb{K}(\mathcal{F}_{p+1}(\mathcal{H})) \cap \tilde{D}_p = \{0\}.$$

This shows that $D_{p+1}(\mathcal{H})$ is an extension of (an algebra isomorphic to) $D_p(\mathcal{H})$ by the compacts $\mathbb{K}(\mathcal{F}_{p+1}(\mathcal{H}))$. The result follows by induction, Proposition 5.1.7, and Remark 5.1.6. □

6.6 Defining Incoming Maps

Throughout this section, let $\mathcal{K}$ be a countably generated free C^*-correspondence over A with generating set W_1. Let W denote the set of all words in the generators. For $k \geq 0$, the free generators of $\mathcal{K}^{\otimes k}$ correspond to words of length k in W. Denote these by W_k and let $W_{<p} = \bigcup_{k=0}^{p-1} W_k$. We have

$$\mathcal{K}^{\otimes k} \simeq \bigoplus_{W_k} A, \quad \mathcal{F}_p(\mathcal{K}) \simeq \bigoplus_{W_{<p}} A, \quad \mathcal{F}(\mathcal{K}) \simeq \bigoplus_{W} A.$$

With this notation at hand, here is another example of a quasidiagonal C^*-correspondence.

Example 6.6.1. Let $\mathcal{K}$ be countably generated free Hilbert A-module with generating set W_1. If $\{\alpha_i\}_{i\in W_1} \subset \mathrm{Aut}(A)$ is a countable set of automorphisms of A, then the correspondence obtained by defining a left action as $a.\,j = j.\,\alpha_j(a)$ is quasidiagonal.

Proof. If $\mu = i_1 \cdots i_k \in W_k$, denote by α_μ the automorphism given by $a \mapsto \alpha_{i_k} \circ \cdots \circ \alpha_{i_1}(a)$. Let $W^n_{<p}$ denote the finite set of all words of length at most $p-1$ in the letters $\{1, \dots, n\}$. Finite rank projections $q_n = \sum_{\zeta\in W^n_{<p}} e_{\zeta,\zeta}$ are quasicentral in $D_p(\mathcal{K})$, since for sufficiently large n we have

$$\begin{aligned}
q_n e_{\mu.a,\nu} \otimes 1_{\mathcal{K}^{\otimes k}} &= \sum_{\zeta\in W^n_{<p}} e_{\zeta,\zeta} \sum_{\eta\in W_k} e_{\mu\otimes\eta.\alpha_\eta(a),\nu\otimes\eta} \\
&= \sum_{\zeta\in W^n_{<p}} \sum_{\eta\in W_k} e_{\zeta.\langle\zeta,\mu\otimes\eta\rangle\alpha_\eta(a),\nu\otimes\eta} \\
&= \sum_{\eta\in W^n_k} e_{\mu\otimes\eta.\alpha_\eta(a),\nu\otimes\eta} \\
&= \sum_{\eta\in W_k} \sum_{\zeta\in W^n_{<p}} e_{\mu\otimes\eta.\alpha_\eta(a),\zeta.\langle\zeta,\nu\otimes\eta\rangle} \\
&= \sum_{\eta\in W_k} e_{\mu\otimes\eta.\alpha_\eta(a),\nu\otimes\eta} \sum_{\zeta\in W^n_{<p}} e_{\zeta,\zeta} \\
&= e_{\mu.a,\nu} \otimes 1_{\mathcal{K}^{\otimes k}} q_n.
\end{aligned}$$

□

6.6.1 Countably Generated Free Modules

Let $\mathscr{H}$ be an arbitrary Hilbert A-module and let I be an arbitrary countable set.

Lemma 6.6.2. There is an inclusion $\mathbb{B}(\bigoplus_I A) \hookrightarrow \mathbb{B}(\bigoplus_I \mathscr{H})$. More specifically, an operator $S \in \mathbb{B}(\bigoplus_I A)$ acts adjointably on $\bigoplus_I \mathscr{H}$ via

$$i.\, x \mapsto \sum_{j\in I} j.\,(\langle j, Si\rangle .\, x).$$

Proof. The map $a \otimes x \mapsto a.\, x$ implements an isomorphism between $A \otimes \mathscr{H}$ and $\mathscr{H}$, so the map $m : i.\, a \otimes x \mapsto i.\,(a.\, x)$ ought to extend to an isomorphism between $(\bigoplus_I A) \otimes \mathscr{H}$ and $\bigoplus_I \mathscr{H}$. Indeed

$$\langle i.\, a \otimes x, j.\, b \otimes y\rangle = \langle a.\, x, \langle i, j\rangle .\, b.\, y\rangle = \langle i.\,(a.\, x), j.\,(b.\, y)\rangle = \langle m(i.\, a \otimes x), m(j.\, b \otimes y)\rangle,$$

so m can be extended to the tensor product $(\bigoplus_I A) \otimes \mathscr{H}$. Moreover, the image of m contains the dense set of all elements in $\bigoplus_I \mathscr{H}$ for which all but finitely many terms are zero, so m is a unitary operator. The result follows since $T \mapsto mTm^{-1}$ is a *-isomorphism between $\mathbb{B}((\bigoplus_I A) \otimes \mathscr{H})$ and $\mathbb{B}(\bigoplus_I \mathscr{H})$, and there is a natural embedding $\mathbb{B}(\bigoplus_I A) \hookrightarrow \mathbb{B}((\bigoplus_I A) \otimes \mathscr{H})$. □

Corollary 6.6.3. For each $p \in \mathbb{N}$, there is an inclusion $\mathbb{B}(\mathcal{F}_p(\mathcal{K})) \hookrightarrow \mathbb{B}(\bigoplus_{W_{<p}} \mathcal{F}(\mathcal{K}))$ that sends $e_{\mu.a,\nu} \otimes 1_{\mathcal{H}^{\otimes k}}$ to the operator

$$\sum_{\zeta\in W_{<p}} \zeta .\, x_\zeta \mapsto \sum_{\zeta,\zeta'\in W_k} \mu \otimes \zeta' .\,(\langle \zeta', a.\, \zeta\rangle .\, x_{\nu\otimes\zeta}),$$

and a (as an operator in $\mathbb{B}(\mathcal{F}_p(\mathcal{K}))$) to the operator

$$\sum_{\zeta\in W_{<p}} \zeta .\, x_\zeta \mapsto \sum_{\zeta,\zeta'\in W_{<p}} \zeta' .\,(\langle \zeta', a.\, \zeta\rangle .\, x_{\zeta}).$$

Proof. Replace $\mathscr{H}$ with $\mathcal{F}(\mathcal{K})$ and I with $W_{<p}$ in Lemma 6.6.2. □

Lemma 6.6.4. If $\{T_i\}_{i\in I}$ is a collection of isometries in $\mathbb{B}(\mathscr{H})$ with orthogonal ranges and such that $\sum_i T_i T_i^*$ converges strictly in $\mathbb{B}(\mathscr{H})$, then the map $[T_i]_I : \bigoplus_I \mathscr{H} \to \mathscr{H}$ given by $\sum_i i.\, x_i \mapsto \sum_i T_i x_i$ is an adjointable operator with adjoint given by $x \mapsto \sum_i i.\, T_i^* x$. Moreover, $[T_i]_I$ is an isometry.

Proof. We first prove the following claim: if $\sum_i i.\, x_i \in \bigoplus_I \mathscr{H}$ satisfies $\langle x_i, x_j\rangle = \delta_{i,j} 1_A$, then $\sum_i x_i$ converges in $\mathscr{H}$. By completeness, it suffices to show the net of finite partial sums is Cauchy. Let $\mathscr{F}$ be the poset of finite subsets of I and fix $\epsilon > 0$. Since $\sum_{i\in I} \langle x_i, x_i\rangle$ converges in A, find $F \in \mathscr{F}$ such that for all $\bar{F} \in \mathscr{F}$ disjoint from F, the inequality $\|\sum_{i\in\bar{F}} \langle x_i, x_i\rangle\| < \epsilon^2$ holds. If $F', F'' \in \mathscr{F}$ dominate F, then $F'\Delta F'' \cap F = \varnothing$ and hence

$$\|\sum_{i\in F'} x_i - \sum_{i\in F''} x_i\|^2 = \|\sum_{i\in F'\Delta F''} \langle x_i, x_i\rangle\| < \epsilon^2.$$

The claim follows by taking square-roots.

Using the claim and the fact that the images of T_i are orthogonal, we get that $[T_i]_I$ is a map from $\bigoplus_i \mathscr{H}$ to $\mathscr{H}$. Hence, $[T_i]_I$ will be an adjointable operator if we show $\sum_i i.T_i^* x$ is in $\bigoplus_I \mathscr{H}$. To this end, fix $\epsilon > 0$ and find a finite set $F \in \mathscr{F}$ such that whenever $\bar{F} \in \mathscr{F}$ is disjoint from F, the inequality $\|\sum_{i\in\bar{F}} T_i T_i^* x\| < \epsilon \|x\|^{-1}$ holds. If $F', F'' \in \mathscr{F}$ dominate F, then

$$\|\sum_{i\in F'} \langle T_i^* x, T_i^* x\rangle - \sum_{i\in F''} \langle T_i^* x, T_i^* x\rangle\| \le \|\sum_{i\in F'\Delta F''} T_i T_i^* x\| \|x\| < \epsilon.$$

This shows that the net of finite partial sums of $\sum_{i\in I} \langle T_i^* x, T_i^* x\rangle$ is Cauchy in A, and hence converges. Lastly, $[T_i]_I$ is an isometry since

$$[T_i]_I^* [T_i]_I (\sum_i i.x_i) = \sum_j j. \sum_i T_j^* T_i x_i = \sum_i i.x_i.$$

□

Remark 6.6.5. An assumption we made in Lemma 6.6.4 was that $\sum_i T_i T_i^*$ converges strictly in $\mathbb{B}(\mathscr{H})$. Although this is automatic for Hilbert spaces, it relies on the fact that the bounded operators on a Hilbert space form a von Neumann algebra. In general, $\mathbb{B}(\mathscr{H})$ is not a von Neumann algebra so it is not automatically true that every bounded and monotonically increasing sequence of positive elements strictly converges.

Corollary 6.6.6. For any $k \ge 0$, there is an isometry $[T_\eta]_k \in \mathbb{B}(\bigoplus_{W_k} \mathcal{F}(\mathcal{K}), \mathcal{F}(\mathcal{K}))$ given by

$$\sum_{\eta\in W_k} \eta. x_\eta \mapsto \sum_{\eta\in W_k} T_\eta(x_\eta).$$

Proof. Since $\sum_{\eta\in W_k} T_\eta T_\eta^*$ converges strictly to $1_{\mathcal{F}(\mathcal{K})} - 1_{\mathcal{F}_k(\mathcal{K})}$ in $\mathbb{B}(\mathcal{F}(\mathcal{K}))$, replace I with W_k and $\mathscr{H}$ with $\mathcal{F}(\mathcal{K})$ in Lemma 6.6.4. □

Remark 6.6.7. For each $p \in \mathbb{N}$ and $0 \le k < p$, we can regard $[T_\eta]_k$ as being an element in $\mathbb{B}(\bigoplus_{W_{<p}} \mathcal{F}(\mathcal{K}), \mathcal{F}(\mathcal{K}))$ by identifying $\bigoplus_{W_{<p}} \mathcal{F}(\mathcal{K})$ with $\bigoplus_{W_0} \mathcal{F}(\mathcal{K}) \oplus \cdots \oplus \bigoplus_{W_{p-1}} \mathcal{F}(\mathcal{K})$ in the obvious way, and then defining $[T_\eta]_k$ to be zero on $\bigoplus_{W_{k'}} \mathcal{F}(\mathcal{K})$ whenever $k' \ne k$.

Lemma 6.6.8. Let $p \in \mathbb{N}$ and let $\mathcal{G} = (g_0, \ldots, g_{p-1})$ be an (ordered) set of positive contractions in A. Then the map defined for elementary tensors x, y in $\mathcal{F}_p(\mathcal{K})$ by

$$e_{x,y} \otimes 1_{\mathcal{K}^{\otimes k}} \mapsto g_{k+|x|} S_x S_y^* g_{k+|y|}$$

extends to a completely positive map $\rho_{\mathcal{G}} : D_p(\mathcal{K}) \to \mathcal{O}(\mathcal{K})$.

Proof. Fix $a \in A$, two generators $\mu, \nu \in W_{<p}$, and $k \ge 0$ such that $e_{\mu.a,\nu} \otimes 1_{\mathcal{K}^{\otimes k}} \in D_p(\mathcal{K})$. Let $R = \sum_{k=0}^{p-1} g_k [T_\eta]_k \in \mathbb{B}(\bigoplus_{W_{<p}} \mathcal{F}(\mathcal{K}), \mathcal{F}(\mathcal{K}))$, and let $\sigma : D_p(\mathcal{K}) \to \mathcal{M}(J(\mathcal{K}))$ be compression by

R. If $z \in \mathcal{F}(\mathcal{K})$, we have

$$\begin{aligned}
(Re_{\mu.a,\nu} \otimes 1_{\mathcal{K}^{\otimes k}} R^*)(z) &= Re_{\mu.a,\nu} \otimes 1_{\mathcal{K}^{\otimes k}} (\sum_{\eta \in W_{<p}} \eta. T_\eta^* g_{|\eta|}. z) \\
&= R(\sum_{\eta \in W_k} \mu. a \otimes \eta. T_\eta^* T_\nu^* g_{k+|\nu|}. z) \\
&= R(\sum_{\eta,\eta' \in W_k} \mu \otimes \eta'. \langle \eta', a. \eta \rangle T_\eta^* T_\nu^* g_{k+|\nu|}. z) \\
&= g_{k+|\mu|} T_\mu \sum_{\eta,\eta' \in W_k} T_{\eta'} \langle \eta', a. \eta \rangle T_\eta^* T_\nu^* g_{k+|\nu|}. z \\
&= g_{k+|\mu|} T_\mu a (\sum_{\eta \in W_k} T_\eta T_\eta^*) T_\nu^* g_{k+|\nu|}. z \\
&= (g_{k+|\mu|} T_\mu a (1_{\mathcal{F}(\mathcal{K})} - 1_{\mathcal{F}_k(\mathcal{K})}) T_\nu^* g_{k+|\nu|})(z).
\end{aligned}$$

Applying the quotient map $Q : \mathcal{M}(J(\mathcal{K})) \to \mathcal{O}(\mathcal{K})$ shows that $\rho_{\mathcal{G}}$ agrees with $Q \circ \sigma$ on elements in $D_p(\mathcal{K})$ of the form $e_{\mu.a,\nu} \otimes 1_{\mathcal{K}^{\otimes k}}$ where μ and ν are generators in $W_{<p}$. By density, $\rho_{\mathcal{G}} = Q \circ \sigma$. □

Lemma 6.6.9. Let $\mathcal{G} = (g_0, \dots, g_{p-1})$ and R be as in Lemma 6.6.8 and let δ be the maximum value of $\|g_i g_j\|$, $i \neq j$.

1. Regarding $g_k^2|_{\mathcal{K}^{\otimes k}}$ as an element of $\mathbb{B}(\bigoplus_{W_{<p}} \mathcal{F}(\mathcal{K}))$, we have

$$R^* R \approx_{p^2 \delta} g_0^2|_{\mathcal{K}^{\otimes 0}} + \dots + g_{p-1}^2|_{\mathcal{K}^{\otimes p-1}}.$$

2. Let w and z be elementary tensors in $\mathcal{F}_p(\mathcal{K})$ satisfying $w. g_l \approx_\delta g_{l+|w|}. w$ and $z. g_l \approx_\delta g_{l+|z|}. z$. If $k \geq 0$ is such that $0 \leq \max(|w|, |z|) + k < p$, then

$$\|[R^* R, e_{w,z} \otimes 1_{\mathcal{K}^{\otimes k}}]\| < 2(p^2 + 2)\delta.$$

Proof. The estimate on $R^* R$ follows from the fact that $[T_\eta]_0, \dots, [T_\eta]_{p-1}$ are isometries and the computation

$$\begin{aligned}
[T_\eta]_k^* g_k^2 [T_\eta]_k (\zeta. x) &= [T_\eta]_k^* (g_k^2. \zeta \otimes x) \\
&= \sum_{\eta \in W_k} \eta. T_\eta^* (g_k^2. \zeta \otimes x) \\
&= \sum_{\eta \in W_k} \eta. (\langle \eta, g_k^2. \zeta \rangle. x) \\
&= g_k^2 (\zeta. x),
\end{aligned}$$

where $\zeta \in W_k$ and $x \in \mathcal{F}(\mathcal{K})$. For the estimate on the commutator, we have

$$\begin{aligned}
(e_{w,z} \otimes 1_{\mathcal{K}^{\otimes k}})(R^*R) &\approx_{p^2\delta} (e_{w,z} \otimes 1_{\mathcal{K}^{\otimes k}})(g_0^2|_{\mathcal{K}^{\otimes 0}} + \cdots + g_{p-1}^2|_{\mathcal{K}^{\otimes p-1}}) \\
&= (e_{w,z} \otimes 1_{\mathcal{K}^{\otimes k}})(g_{k+|z|}^2|_{\mathcal{K}^{\otimes k+|z|}}) \\
&= e_{w,g_{k+|z|}^2 . z} \otimes 1_{\mathcal{K}^{\otimes k}} \\
&\approx_{2\delta} e_{w.g_k^2,z} \otimes 1_{\mathcal{K}^{\otimes k}} \\
&\approx_{2\delta} e_{g_{k+|w|}^2 . w,z} \otimes 1_{\mathcal{K}^{\otimes k}} \\
&= (g_{k+|w|}^2|_{\mathcal{K}^{\otimes k+|w|}})(e_{w,z} \otimes 1_{\mathcal{K}^{\otimes k}}) \\
&= (g_0^2|_{\mathcal{K}^{\otimes 0}} + \cdots + g_{p-1}^2|_{\mathcal{K}^{\otimes p-1}}) e_{w,z} \otimes 1_{\mathcal{K}^{\otimes k}} \\
&\approx_{p^2\delta} (R^*R)(e_{w,z} \otimes 1_{\mathcal{K}^{\otimes k}}).
\end{aligned}$$

□

6.6.2 Countably Generated Modules

Let $\mathcal{H}$ be a countably generated C^*-correspondence over A. By Theorem 3.1.7, there is a countably generated free Hilbert A-module $\mathcal{H}'$ such that their direct sum $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}'$ is free. Choose a left action of A on $\mathcal{H}'$ such that $A \cap \mathbb{K}(\mathcal{H}') = \{0\}$. The diagonal action of A turns $\mathcal{K}$ into a C^*-correspondence. The orthogonal projection $P_{\mathcal{H}} \in \mathbb{B}(\mathcal{K})$ onto $\mathcal{H} \oplus 0$ commutes with the image of the left action of A on $\mathcal{K}$, so for $k > 0$ the map

$$P_{\mathcal{H}}^{\otimes k} : x_1 \otimes \cdots \otimes x_k \mapsto P_{\mathcal{H}} x_1 \otimes \cdots \otimes P_{\mathcal{H}} x_k$$

is an orthogonal projection in $\mathbb{B}(\mathcal{K}^{\otimes k})$.

Lemma 6.6.10. Defining $P_{\mathcal{H}}^{\otimes 0}$ to be the identity map on A, we have

1. $\mathcal{H}^{\otimes k} = P_{\mathcal{H}}^{\otimes k}(\mathcal{K}^{\otimes k})$,
2. $\mathcal{F}_p(\mathcal{H}) = (\sum_{k=0}^{p-1} P_{\mathcal{H}}^{\otimes k})(\mathcal{F}_p(\mathcal{K}))$, and
3. $\mathcal{F}(\mathcal{H}) = (\sum_{k \geq 0} P_{\mathcal{H}}^{\otimes k})(\mathcal{F}(\mathcal{K}))$.

Proof. For the first part, the map $V_k : x_1 \otimes \cdots \otimes x_k \mapsto (x_1, 0) \otimes \cdots \otimes (x_k, 0)$ implements an isomorphism between the k-fold tensor product $\mathcal{H}^{\otimes k}$ and the image of $P_{\mathcal{H}}^{\otimes k}$ in $\mathcal{K}^{\otimes k}$. The second part follows easily from the first since the sums are finite. Lastly, it is clear that the series $\sum_{k \geq 0} P_{\mathcal{H}}^{\otimes k}$ converges strictly to a projection in $\mathbb{B}(\mathcal{F}(\mathcal{K}))$ with the desired property. □

Denote the orthogonal projections $\sum_{k=0}^{p-1} P_{\mathcal{H}}^{\otimes k}$ and $\sum_{k \geq 0} P_{\mathcal{H}}^{\otimes k}$ given above by $P_{\mathcal{F}_p(\mathcal{H})}$ and $P_{\mathcal{F}(\mathcal{H})}$, respectively. If $z = (x_1, y_1) \otimes \cdots \otimes (x_k, y_k) \in \mathcal{K}^{\otimes k}$ is an elementary tensor, we say that z is an $\mathcal{H}$-*elementary tensor* if $y_1 = \cdots = y_k = 0$. Equivalently, z is $\mathcal{H}$-elementary if $P_{\mathcal{H}}^{\otimes k}(z) = z$.

Lemma 6.6.11. Suppose $\dim_{\mathrm{Rok}}(\mathcal{H}) = d$. Then for any $\epsilon > 0$, any $p \in \mathbb{N}$, and any finite set $\mathcal{V}$ of $\mathcal{H}$-elementary tensors in $\mathcal{F}_p(\mathcal{K})$, there exist positive contractions

$$\{f_k^l\}_{k=0,\dots,p-1}^{l=0,\dots,d} \subset A$$

satisfying

1. $\|f_k^l f_{k'}^l\| < \epsilon$ when $k \neq k'$ and all l.
2. $\|\sum_{k,l} f_k^l - 1\| < \epsilon$.
3. $\|z.\, f_k^l - f_{k+|z|}^l.\, z\| < \epsilon$ for all k, l, and $z \in \mathcal{V}$.

Proof. A finite set of $\mathcal{H}$-elementary tensors in $\mathcal{F}_p(\mathcal{K})$ corresponds to a finite set of elementary tensors in $\mathcal{F}_p(\mathcal{H})$. This gives positive contractions $\{f_k^l\}_{k=0,\dots,p-1}^{l=0,\dots,d}$ in A satisfying the first two conditions of Remark 6.4.2. Since the left action of A on $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}'$ is a direct sum, the third condition is automatic. □

Proposition 6.6.12. Let (π, τ) be the representation of $\mathcal{H}$ on $\mathcal{O}(\mathcal{K})$ given by $\pi : a \mapsto a$ and $\tau : x \mapsto S_{(x,0)}$. Then there is a *-isomorphism $\theta : C^*(\pi, \tau) \to \mathcal{T}(\mathcal{H})$ sending a to a and $S_{(x,0)}$ to T_x.

Proof. Consider first the representation $(\tilde{\pi}, \tilde{\tau})$ of $\mathcal{H}$ on $\mathcal{T}(\mathcal{K})$ given by $\tilde{\pi} : a \mapsto a$ and $\tilde{\tau} : x \mapsto T_{(x,0)}$. It is clear that

$$\tilde{\pi}(A) \cap \overline{\mathrm{span}}\{\tilde{\tau}(x)\tilde{\tau}(y)^* \ : \ x, y \in \mathcal{H}\} = \{0\}.$$

If we restrict the gauge action on $\mathcal{T}(\mathcal{K})$ to $C^*(\pi, \tau)$, Theorem 3.1.16 gives a *-isomorphism $\tilde{\theta} : C^*(\tilde{\pi}, \tilde{\tau}) \to \mathcal{T}(\mathcal{H})$ sending a to a and $T_{(x,0)}$ to T_x. Now, since the left action of A on $\mathcal{H}'$ was defined so that $A \cap \mathbb{K}(\mathcal{H}') = \{0\}$, each $a \in A \subset \mathbb{B}(\mathcal{K})$ differs from any finite sum of rank-one operators by at least $\|a\|$. Therefore, $A \cap \mathbb{K}(\mathcal{K}) = \{0\}$ and so by Remark 3.2.9 there is a *-isomorphism $\bar{\theta} : \mathcal{O}(\mathcal{K}) \to \mathcal{T}(\mathcal{K})$ sending a to a and $S_{(x,y)}$ to $T_{(x,y)}$. Moreover, it's clear that $\bar{\theta}(C^*(\pi, \tau)) = C^*(\tilde{\pi}, \tilde{\tau})$ so taking $\theta = \tilde{\theta} \circ \bar{\theta}|_{C^*(\pi,\tau)}$ completes the proof. Here is the diagram.

$$\begin{array}{ccc}
(A, \mathcal{H}) & \xrightarrow{\quad (\pi,\tau) \quad} & C^*(\pi,\tau) \subset \mathcal{O}(\mathcal{K}) \\
{\scriptstyle (a \mapsto a,\, x \mapsto T_x)} \Big\downarrow & \searrow^{(\tilde{\pi},\tilde{\tau})} & \Big\downarrow {\scriptstyle \bar{\theta}|_{C^*(\pi,\tau)}} \\
\mathcal{T}(\mathcal{H}) & \xleftarrow[\quad \tilde{\theta} \quad]{} & C^*(\tilde{\pi},\tilde{\tau}) \subset \mathcal{T}(\mathcal{K})
\end{array}$$

□

For the remainder of this section, fix $p \in \mathbb{N}$.

Lemma 6.6.13. Let $D_p^0(\mathcal{K})$ be the C^*-subalgebra of $D_p(\mathcal{K})$ generated by elements of the form $e_{x,y} \otimes 1_{\mathcal{K}^{\otimes k}}$ where x and y are $\mathcal{H}$-elementary tensors. Then the map

$$\gamma : e_{x_1 \otimes \cdots \otimes x_k, y_1 \otimes \cdots \otimes y_l} \mapsto e_{(x_1,0)\otimes \cdots \otimes (x_k,0),(y_1,0)\otimes \cdots \otimes (y_l,0)}$$

extends to an isomorphism $\gamma : D_p(\mathcal{H}) \to D_p^0(\mathcal{K})$. In particular if $\mathcal{H}$ is quasidiagonal, then $\dim_{\mathrm{nuc}}(D_p^0(\mathcal{K})) \le \dim_{\mathrm{nuc}}(A)$.

Proof. The map $\bar{V} = \sum_{k=0}^{p-1} V_k$ sending a to a and $x_1 \otimes \cdots \otimes x_k$ to $(x_1, 0) \otimes \cdots \otimes (x_k, 0)$ extends to an unitary from $\mathcal{F}_p(\mathcal{H})$ to $P_{\mathcal{F}_p(\mathcal{H})}(\mathcal{F}_p(\mathcal{K}))$. Let $l, m, k \ge 0$ satisfy $0 \le \max(l, m) + k < p$ and let $x, y, w, z \in \mathcal{F}_p(\mathcal{H})$ be elementary tensors of lengths l, m, m, and k, respectively. We have

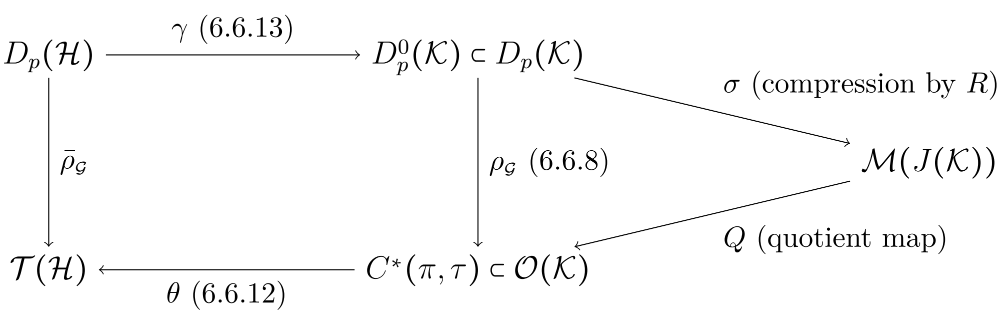
$$\begin{aligned} \bar{V}^* e_{\bar{V}x,\bar{V}y} \otimes 1_{\mathcal{K}^{\otimes k}} \bar{V}(w \otimes z) &= \bar{V}^*(\bar{V}x.\langle \bar{V}y, \bar{V}w\rangle \otimes \bar{V}z) \\ &= \bar{V}^*(\bar{V}x.\langle y, w\rangle \otimes \bar{V}z) \\ &= e_{x,y} \otimes 1_{\mathcal{H}^{\otimes k}}(w \otimes z), \end{aligned}$$

so γ is implemented by unitary conjugation and the result follows. □

Corollary 6.6.14. Let $\mathcal{G} = (g_0, \ldots, g_{p-1})$ be an ordered set of positive contractions in A. Then the map defined for elementary tensors x, y in $\mathcal{F}_p(\mathcal{H})$ by

$$e_{x,y} \otimes 1_{\mathcal{H}^{\otimes k}} \mapsto g_{k+|x|} T_x T_y^* g_{k+|y|}$$

extends to a completely positive map $\bar{\rho}_{\mathcal{G}} : D_p(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ such that the following diagram commutes:



Proof. By Lemma 6.6.13, there is a *-isomorphism $\gamma : D_p(\mathcal{H}) \to D_p^0(\mathcal{K})$ given by

$$e_{x_1 \otimes \cdots \otimes x_l, y_1 \otimes \cdots \otimes y_m} \otimes 1_{\mathcal{H}^{\otimes k}} \mapsto e_{(x_1,0)\otimes \cdots \otimes (x_l,0),(y_1,0)\otimes \cdots \otimes (y_m,0)} \otimes 1_{\mathcal{K}^{\otimes k}}.$$

By Lemma 6.6.8, there is a completely positive map $\rho_{\mathcal{G}} : D_p^0(\mathcal{K}) \to \mathcal{O}(\mathcal{K})$ given by

$$e_{(x_1,0)\otimes \cdots \otimes (x_l,0),(y_1,0)\otimes \cdots \otimes (y_m,0)} \otimes 1_{\mathcal{K}^{\otimes k}} \mapsto g_{k+l} S_{(x_1,0)\otimes \cdots \otimes (x_l,0)} S^*_{(y_1,0)\otimes \cdots \otimes (y_m,0)} g_{k+m}.$$

Notice that the image of this map is in $C^*(\pi, \tau)$. By Proposition 6.6.12, there is a *-isomorphism $\theta : C^*(\pi, \tau) \to \mathcal{T}(\mathcal{H})$ sending a to a and $S_{(x,0)}$ to T_x. Hence,

$$\theta : g_{k+l} S_{(x_1,0)\otimes\cdots\otimes(x_l,0)} S^*_{(y_1,0)\otimes\cdots\otimes(y_m,0)} g_{k+m} \mapsto g_{k+l} T_{x_1\otimes\cdots\otimes x_l} T^*_{y_1\otimes\cdots\otimes y_m} g_{k+m},$$

and the result follows. □

Corollary 6.6.15. Let $\mathcal{C} \subset D_p(\mathcal{H})$ be a norm compact subset consisting of contractions. For every $\delta, \bar{\delta} > 0$, there exists a finite subset $\mathcal{V}$ of elementary tensors in $\mathcal{F}_p(\mathcal{H})$ and $0 < \delta' < \delta$ such that if $\mathcal{G} = (g_0, \ldots, g_{p-1})$ is a set of contractions in A satisfying

1. $\|g_k g_{k'}\| < \delta'$ whenever $k \neq k'$,
2. $\|\sum_k g_k^2 - 1\| < \delta'$,
3. $\|z.\, g_k - g_{k+|z|}.\, z\| < \delta'$ for all k and $z \in \mathcal{V}$,

then the c.p. map $\bar{\rho}_{\mathcal{G}}$ obtained in Corollary 6.6.14 is $\bar{\delta}$-contractive and satisfies

$$x \perp y \in \mathcal{C} \Rightarrow \bar{\rho}_{\mathcal{G}}(x) \perp_\delta \bar{\rho}_{\mathcal{G}}(y).$$

Proof. By compactness, we can find a finite subset $\mathcal{V}$ of elementary tensors in $\mathcal{F}_p(\mathcal{H})$ satisfying

$$\mathcal{C} \subset_{\delta/4} \{ \sum_{w,z\in\mathcal{V}} \alpha_{w,z,k} e_{w,z} \otimes 1_{\mathcal{H}^{\otimes k}} \ : \ \alpha_{w,z,k} = 0 \text{ or } 1 \}.$$

Let $\delta' > 0$ be small enough so that

$$2|\mathcal{V}|^2(p^2+2)\delta' < \delta/2 \quad \text{and} \quad p^2\delta' < \bar{\delta}. \tag{6.2}$$

Let $\mathcal{V}^0$ denote the set of $\mathcal{H}$-elementary tensors in $\mathcal{F}_p(\mathcal{K})$ corresponding to $\mathcal{V}$. We have

$$\gamma(\mathcal{C}) \subset_{\delta/4} \{ \sum_{w,z\in\mathcal{V}^0} \alpha_{w,z,k} e_{w,z} \otimes 1_{\mathcal{K}^{\otimes k}} \ : \ \alpha_{w,z,k} = 0 \text{ or } 1 \}.$$

Since γ and θ are *-isomorphisms, it suffices to show that $\rho_{\mathcal{G}}(\gamma(x)) \perp_\delta \rho_{\mathcal{G}}(\gamma(y))$. Furthermore, the proof of Lemma 6.6.8 demonstrated that $\rho_{\mathcal{G}} = Q \circ \sigma$, where Q was the canonical quotient map $Q : \mathcal{T}(\mathcal{K}) \to \mathcal{O}(\mathcal{K})$ and σ was compression by R. This means it suffices to check that $\|[R^*R, \gamma(x)]\| < \delta$. By Lemma 6.6.9, we have

$$\|[R^*R, e_{w,z} \otimes 1_{\mathcal{K}^{\otimes k}}]\| < 2(p^2+2)\delta'$$

for every $w, z \in \mathcal{V}^0$. This implies

$$\|[R^*R, \gamma(x)]\| < 2|\mathcal{V}|^2(p^2+2)\delta' + \delta/2 < \delta,$$

which shows that $\bar{\rho}_{\mathcal{G}}$ is δ-order zero on $\mathcal{C}$. To show it's $\bar{\delta}$-contractive, we have by Lemma 6.6.9

$$\|R^*R\| \approx_{p^2\delta'} 1,$$

and the result follows by (6.2). □

Remark 6.6.16. If $\bar{\rho}_{\mathcal{G}}$ is not contractive, we can replace it with $\tilde{\rho}_{\mathcal{G}} = \frac{\bar{\rho}_{\mathcal{G}}}{\|\bar{\rho}_{\mathcal{G}}\|}$. Then $\tilde{\rho}_{\mathcal{G}}$ is a c.p.c. map (still from $D_p(\mathcal{H})$ to $\mathcal{T}(\mathcal{H})$) and satisfies

1. $\tilde{\rho}_{\mathcal{G}}(e_{x,y} \otimes 1_{\mathcal{H}^{\otimes k}}) \approx_{\bar{\delta}} g_{k+|x|}T_xT_y^*g_{k+|y|}$
2. $x \perp y \in \mathcal{C} \Rightarrow \tilde{\rho}_{\mathcal{G}}(x) \perp_{\delta+2\bar{\delta}} \tilde{\rho}_{\mathcal{G}}(y)$

Since δ and $\bar{\delta}$ were arbitrary, we will assume without loss of generality that $\bar{\rho}_{\mathcal{G}}$ is contractive.

The following proposition is the main technical result of this section.

Proposition 6.6.17. Suppose $\dim_{\mathrm{Rok}}(\mathcal{H}) = d$. Let $\mathcal{W} \subset \mathcal{F}_p(\mathcal{H})$ be a finite subset of elementary tensors, let K be a finite-dimensional C^*-algebra, and let $\psi : K \to D_p(\mathcal{H})$ be a c.p.c. order zero map.

For any $\delta > 0$, there are positive contractions $\{f_k^l\}_{k=0,\dots,p-1}^{l=0,\dots,d}$ in A such that the following holds:

1. The maps $\bar{\rho}_{\mathcal{G}^l}$ and $\bar{\rho}_{\hat{\mathcal{G}}^l}$ defined using Corollary 6.6.14 with respect to

$$\mathcal{G}^l = ((f_0^l)^{1/2}, \dots, (f_{p-1}^l)^{1/2}) \quad \text{and} \quad \hat{\mathcal{G}}^l = ((f^l_{\frac{p-1}{2}+0})^{1/2}, \dots, (f^l_{\frac{p-1}{2}+p-1})^{1/2})$$

   are c.p.c. and δ-order zero,
2. $\|\sum_{k,l} f_k^l - 1\| < \delta$,
3. $\|z.(f_k^l)^{1/2} - (f_{k+|z|}^l)^{1/2}.z\| < \delta$ for every $z \in \mathcal{W}$.

Proof. Let $\mathcal{C}$ be the norm compact set $\psi(\mathrm{Ball}_1 K)$. By Corollary 6.6.15, obtain $\mathcal{V}$ and $\delta' < \delta$. Now find Rokhlin contractions $\{f_k^l\}_{k=0,\dots,p-1}^{l=0,\dots,d}$ satisfying

1. $\|(f_k^l)^{1/2}(f_{k'}^l)^{1/2}\| < \delta'$ for all l and $k \neq k'$,
2. $\|\sum_{k,l} f_k^l - 1\| < \delta'$,
3. $\|z.(f_k^l)^{1/2} - (f_{k+|z|}^l)^{1/2}.z\| < \delta'$ for all k, l, and $z \in \mathcal{V} \cup \mathcal{W}$.

The result follows by using Corollary 6.6.15 applied to $g_k^l = (f_k^l)^{1/2}$ and $g_k^l = (f^l_{\frac{p-1}{2}+k})^{1/2}$. □

6.7 Nuclear Dimension of Pimsner Algebras

We now have the ingredients to prove the main result of this thesis.

Theorem 6.7.1. Suppose that $\mathcal{H}$ is a countably generated C^*-correspondence over a separable unital C^*-algebra A. Assume further that $\mathcal{H}$ is quasidiagonal. Then

$$\dim_{\mathrm{nuc}}^{+1}(\mathcal{T}(\mathcal{H})) \leq 2\dim_{\mathrm{nuc}}^{+1}(A)\dim_{\mathrm{Rok}}^{+1}(\mathcal{H}).$$

Proof. If either $\dim_{\mathrm{nuc}}(A)$ or $\dim_{\mathrm{Rok}}(\mathcal{H}, A) = \infty$ there is nothing to show. Otherwise, let $\dim_{\mathrm{nuc}}(A) = n$ and $\dim_{\mathrm{Rok}}(\mathcal{H}) = d$.

Let F be a finite subset of $\mathcal{T}(\mathcal{H})$ and fix $\epsilon > 0$. By Proposition 5.1.9, it will suffice to show that there is a finite dimensional C^*-algebra $K = K^{(0)} \oplus \cdots \oplus K^{(n)}$ such that for any $\delta > 0$, there is a c.p.c. map $\varphi : \mathcal{T}(\mathcal{H}) \to K$ and a c.p. map $\psi : K \to \mathcal{T}(\mathcal{H})$ satisfying

1. $\|\psi \circ \varphi(a) - a\| < \epsilon$ for every $a \in F$,

2. For each $I = 0, \ldots, n$, the restriction of ψ to $K^{(i)}$ is c.p.c. and δ-order zero.

By density, we can assume there is an integer $N \in \mathbb{N}$ and a finite subset $\mathcal{W}$ consisting of elementary tensors in $\mathcal{F}_{N+1}(\mathcal{H})$ such that

$$F \subset \{ \sum_{x,y \in \mathcal{W}} \alpha_{x,y} T_x T_y^* \ : \alpha_{x,y} = 0 \text{ or } 1\}.$$

Let $p \in \mathbb{N}$ be large enough so that

$$\frac{(d+1)(\sqrt{8N(p-1)} + 4N^2)}{p-1} < \frac{\epsilon}{3|\mathcal{W}|^2}, \tag{6.3}$$

and let $P \in \mathbb{B}(\mathcal{F}(\mathcal{H}))$ be the projection onto the p^{th}-cutoff Fock space $\mathcal{F}_p(\mathcal{H})$. Let $\Delta = \operatorname{diag}(d_0, \ldots, d_{p-1})$ be defined as in Theorem 6.3.4 and let $\phi : \mathcal{T}(\mathcal{H}) \to D_p(\mathcal{H})$ be compression by $\sqrt{\Delta}P$. Pick a typical summand $T_x T_y^*$ and (for the moment) assume $|x| \geq |y|$. Applying ϕ to $T_x T_y^*$, using (3.1), and noting that $\sqrt{d_{k+|x|}} \approx_{\sqrt{2N/(p-1)}} \sqrt{d_{k+|y|}}$ for $k = 0, \ldots, p-1$ we have

$$\begin{aligned}\phi(T_x T_y^*) &= \sum_{k=0}^{p-1-|x|} \sqrt{d_{k+|x|} d_{k+|y|}} e_{x,y} \otimes 1_{\mathcal{H}^{\otimes k}} \\ &\approx_{\sqrt{2N/(p-1)}} \sum_{k=0}^{p-1-|x|} d_{k+|x|} e_{x,y} \otimes 1_{\mathcal{H}^{\otimes k}}.\end{aligned} \tag{6.4}$$

Let $\epsilon' > 0$ be small enough so that

$$2(d+1)\epsilon' < \frac{\epsilon}{3|\mathcal{W}|^2}, \tag{6.5}$$

and use $\dim_{\mathrm{nuc}}(D_p(\mathcal{H})) \leq n$ to obtain a factorization

$$D_p(\mathcal{H}) \xrightarrow{\Phi} K = K^{(0)} \oplus \cdots \oplus K^{(n)} \xrightarrow{\Psi} D_p(\mathcal{H})$$

such that each $K^{(i)}$ is a finite dimensional C^*-algebra, Φ is c.p.c., Ψ is c.p.c. order zero on each $K^{(i)}$, and $\|\Psi \circ \Phi \circ \phi(T_x T_y^*) - \phi(T_x T_y^*)\| < \epsilon'$ for each $x, y \in \mathcal{W}$.

Let $\delta > 0$ be arbitrary and let $\bar{\delta} > 0$ be small enough so that

$$(4d+5)\bar{\delta} < \frac{\epsilon}{3|\mathcal{W}|^2} \quad \text{and} \quad \bar{\delta} < \delta. \tag{6.6}$$

Use Proposition 6.6.17 (with respect to $\bar{\delta}$) to obtain positive contractions $\{f_k^l\}_{k=0,\ldots,p-1}^{l=0,\ldots,d}$ in A such that the following holds:

1. For each l, the maps $\bar{\rho}_{\mathcal{G}^l} \circ \psi$ and $\bar{\rho}_{\hat{\mathcal{G}}^l} \circ \psi$ are c.p.c. and δ-order zero,
2. $\|\sum_{k,l} f_k^l - 1\| < \bar{\delta}$,
3. $\|z.(f_k^l)^{1/2} - (f_{k+|z|}^l)^{1/2}.z\| < \bar{\delta}$ for every $z \in \mathcal{W}$.

To simplify notation, let ρ^l and $\hat{\rho}^l$ denote $\bar{\rho}_{\mathcal{G}^l}$ and $\bar{\rho}_{\hat{\mathcal{G}}^l}$, respectively. Applying ρ^l and $\hat{\rho}^l$ to $\phi(T_x T_y^*)$ and using (6.4), we get

$$\rho^l \circ \phi(T_x T_y^*) \approx_{\sqrt{2N/(p-1)}} \sum_{k=0}^{p-1-|x|} d_{k+|x|}(f_{k+|x|}^l)^{1/2} T_x T_y^* (f_{k+|y|}^l)^{1/2} \approx_{2\bar{\delta}} (\sum_{k=|x|}^{p-1} d_k f_k^l) T_x T_y^*, \text{ and}$$

$$\hat{\rho}^l \circ \phi(T_x T_y^*) \approx_{\sqrt{2N/(p-1)}} \sum_{k=0}^{p-1-|x|} d_{k+|x|}(f_{\frac{p-1}{2}+k+|x|}^l)^{1/2} T_x T_y^* (f_{\frac{p-1}{2}+k+|y|}^l)^{1/2} \approx_{2\bar{\delta}} (\sum_{k=|x|}^{p-1} d_k f_{\frac{p-1}{2}+k}^l) T_x T_y^*.$$

This means

$$\rho^l \circ \Psi \circ \Phi \circ \phi(T_x T_y^*) \approx_{\epsilon' + \sqrt{2N/(p-1)} + 2\bar{\delta}} (\sum_{k=|x|}^{p-1} d_k f_k^l) T_x T_y^*, \text{ and}$$

$$\hat{\rho}^l \circ \Psi \circ \Phi \circ \phi(T_x T_y^*) \approx_{\epsilon' + \sqrt{2N/(p-1)} + 2\bar{\delta}} (\sum_{k=|x|}^{p-1} d_k f_{\frac{p-1}{2}+k}^l) T_x T_y^*.$$

Adding these, applying Lemma 6.0.7, and summing over l gives

$$\sum_{l=0}^{d} (\rho^l + \hat{\rho}^l) \circ \Psi \circ \Phi \circ \phi(T_x T_y^*) \approx_{(d+1)\left(2\epsilon' + \sqrt{8N/(p-1)} + 4\bar{\delta} + 4N^2/(p-1)\right) + \bar{\delta}} T_x T_y^*.$$

A nearly identical argument shows the same estimates in the case $|x| < |y|$. By (6.3), (6.5), and (6.6) we have

$$\frac{(d+1)(\sqrt{8N(p-1)} + 4N^2)}{p-1} + 2(d+1)\epsilon' + (4d+5)\bar{\delta} < \frac{\epsilon}{|\mathcal{W}|^2},$$

and since any $a \in F$ is the sum of at most $|\mathcal{W}|^2$ terms of the form $T_x T_y^*$, we get

$$\sum_{l=0}^{d} (\rho^l + \hat{\rho}^l) \circ \Psi \circ \Phi \circ \phi(a) \approx_\epsilon a$$

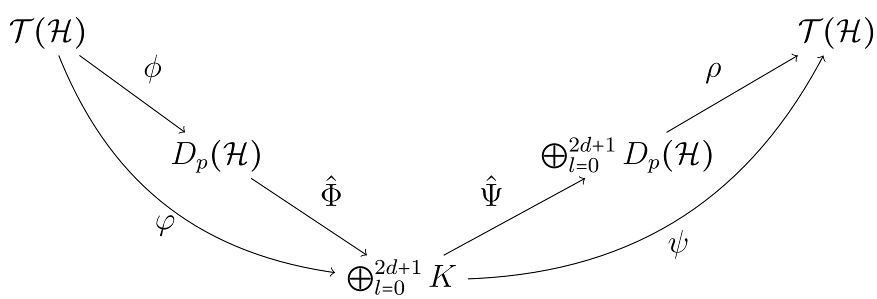
for all $a \in F$. Now consider the following maps:

$\phi : \mathcal{T}(\mathcal{H}) \to D_p(\mathcal{H})$	compression by $\sqrt{\Delta} P$
$\hat{\Phi} : D_p(\mathcal{H}) \to \bigoplus_{l=0}^{2d+1} K$	$x \mapsto \Phi(x) \oplus \cdots \oplus \Phi(x)$
$\hat{\Psi} : \bigoplus_{l=0}^{2d+1} K \to \bigoplus_{l=0}^{2d+1} D_p(\mathcal{H})$	$\hat{\Psi} = \bigoplus_{l=0}^{2d+1} \Psi$
$\rho : \bigoplus_{l=0}^{2d+1} D_p(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$	$x_0 \oplus \cdots \oplus x_{2d+1} \mapsto \sum_{l=0}^{d} \rho^l(x_l) + \hat{\rho}^l(x_{l+d+1})$.

Defining $\varphi : \mathcal{T}(\mathcal{H}) \to \bigoplus_{l=0}^{2d+1} K$ and $\psi : \bigoplus_{l=0}^{2d+1} K \to \mathcal{T}(\mathcal{H})$ by

$$\varphi = \hat{\Phi} \circ \phi \quad \text{and} \quad \psi = \rho \circ \hat{\Psi}$$

finishes the proof. Here is a diagram outlining the maps.



□

Corollary 6.7.2. Under the same hypotheses as the previous theorem,

$$\dim_{\text{nuc}}^{+1}(\mathcal{O}(\mathcal{H})) \le 2 \dim_{\text{nuc}}^{+1}(A) \dim_{\text{Rok}}^{+1}(\mathcal{H}).$$

Proof. This follows from Proposition 5.1.5 and the fact that $\mathcal{O}(\mathcal{H})$ is a quotient of $\mathcal{T}(\mathcal{H})$ by the ideal $\mathbb{K}(\mathcal{F}(\mathcal{H})I_{\mathcal{H}})$. $\square$

6.8 Amalgamated Free Products

We now use the results given in the previous sections to deduce finite nuclear dimension of a certain class of reduced amalgamated free products. We start with a brief overview of the free product construction. For a more comprehensive treatment, see the monograph [64].

Definition 6.8.1. Let $1 \in D \subset A$ be unital C^*-algebras with a conditional expectation E from A to D. In other words, E is a D–bimodule c.p.c. projection. Then A is naturally a right D–module and $\langle a, b\rangle = E(a^*b)$ is a D–valued semi-inner product on A. Denote by $L^2(A, E)$ the Hilbert D–module obtained from A by separation and completion and by $\widehat{a} \in L^2(A, E)$ the vector corresponding to $a \in A$. Left multiplication by elements in A defines a $*$-representation $\pi_E : A \to \mathbb{B}(L^2(A, E))$, and if $\xi_E = \widehat{1}$, then

$$E(a) = \langle \xi_E, \pi_E(a)\xi_E\rangle \quad \forall a \in A.$$

We call $(\pi_E, L^2(A, E), \xi_E)$ the *GNS representation* for (A, E); we say the conditional expectation E is nondegenerate if π_E is faithful.

Note that E defines a projection $\widehat{E} \in \mathbb{B}(L^2(A, E))$ via $\widehat{a} \mapsto \widehat{E(a)}$. Indeed, if $E(a^*a) = 0$ then the Cauchy-Schwarz inequality for 2–positive maps (c.f. [43]) implies $E(E(a)^*E(a)) = 0$. This shows $\widehat{E}$ is well-defined. Moreover,

$$\langle \widehat{b}, \widehat{E(a)}\rangle = E(b^*E(a)) = E(b^*)E(a) = E(E(b^*)a) = \langle \widehat{E(b)}, \widehat{a}\rangle.$$

The image of $\widehat{E}$ is $\xi_E D$, and $\widehat{\ker(E)}$ is dense in $\ker(\widehat{E})$:

$$x \in \ker(\widehat{E}) \text{ and } \widehat{x_n} \to x \text{ and } x_n \in A \quad \Rightarrow \quad y_n := x_n - E(x_n) \in \ker(E) \text{ and } \widehat{y_n} \to x.$$

Denote by $L^2(A, E)^\circ$ the kernel of $\widehat{E}$, so that $L^2(A, E) \simeq \xi_E D \oplus L^2(A, E)^\circ$. Since $L^2(A, E)^\circ$ is invariant under $\pi_E(D)$, we may regard $L^2(A, E)^\circ$ as a C^*-correspondence over D.

Now let I be a set and $\{A_i\}_{i\in I}$ be a collection of unital C^*-algebras, each of which contains a unital copy of some fixed C^*-algebra D. Assume there exist nondegenerate conditional expectations $E_i : A_i \to D$, and denote by $(\pi_i, L^2(A_i, E_i), \xi_i)$ the GNS representation for (A_i, E_i). Define the *free product Hilbert D–module* $(\mathcal{H}, \xi) = *(L^2(A_i, E_i), \xi_i)$ by

$$\mathcal{H} = D\xi \oplus \bigoplus_{n\geq 1} \bigoplus_{i_1 \neq \cdots \neq i_n} L^2(A_{i_1}, E_{i_1})^\circ \otimes \cdots \otimes L^2(A_{i_n}, E_{i_n})^\circ.$$

Here, ξD is the trivial Hilbert D–module D with $\xi = \widehat{1}$ and the notation $i_1 \neq \cdots \neq i_n$ means that $i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{n-1} \neq i_n$. For each i, let

$$\mathcal{H}_i = \xi D \oplus \bigoplus_{n\geq 1} \bigoplus_{\substack{i_1 \neq \cdots \neq i_n \\ i_1 \neq i}} L^2(A_{i_1}, E_{i_1})^\circ \otimes \cdots \otimes L^2(A_{i_n}, E_{i_n})^\circ.$$

The map $U : L^2(A_i, E_i) \otimes \mathcal{H}_i \to \mathcal{H}$ given by

$$\begin{aligned}
\xi_i D \otimes \xi D &\to \xi D \\
L^2(A_i, E_i)^\circ \otimes \xi D &\to L^2(A_i, E_i)^\circ \\
\xi_i D \otimes \big(L^2(A_{i_1}, E_{i_1})^\circ \otimes \cdots \otimes L^2(A_{i_n}, E_{i_n})^\circ\big) &\to L^2(A_{i_1}, E_{i_1})^\circ \otimes \cdots \otimes L^2(A_{i_n}, E_{i_n})^\circ \\
L^2(A_i, E_i)^\circ \otimes \big(L^2(A_{i_1}, E_{i_1})^\circ \otimes \cdots \otimes L^2(A_{i_n}, E_{i_n})^\circ\big) &\to L^2(A_i, E_i)^\circ \otimes L^2(A_{i_1}, E_{i_1})^\circ \otimes \cdots \\
&\quad \cdots \otimes L^2(A_{i_n}, E_{i_n})^\circ.
\end{aligned}$$

is an isomorphism of Hilbert D−modules. We define a *-representation $\lambda_i : A_i \to \mathbb{B}(\mathcal{H})$ by

$$\lambda_i(x) = U_i(\pi_i \otimes 1)(x)U_i^*.$$

Definition 6.8.2. The *reduced amalgamated free product* $(A, E) = *_D(A_i, E_i)$ is the C^*-subalgebra of $\mathbb{B}(\mathcal{H})$ generated by $\bigcup_i \in I\lambda_i(A_i)$, together with the conditional expectation from A onto D given by $E(x) = \langle \xi, x\xi \rangle$.

The construction above is commutative and associative. Direct calculation shows that $E_i = E \circ \lambda_i$ on A_i. Also, λ_i is injective because E_i is nondegenerate; hence we will often omit λ_i and view A_i as a subalgebra of A. Here are the main properties of the reduced amalgamated free product.

Proposition 6.8.3. Let $1 \in D \subset A_i$ be unital C^*-algebras with nondegenerate conditional expectations $E_i : A_i \to D$, and let $(A, E) = *_D(A_i, E_i)$.

1. There is an inclusion $1 \in D \subset A$ and a nondegenerate conditional expectation $E : A \to D$.
2. There are inclusions $D \subset A_i \subset A$ which are compatible on D, and A is generated by $\bigcup_I A_i$ as a C^*-algebra.
3. One has $E|_{A_i} = E_i$ for every i, and the C^*-subalgebras A_i are free over D in (A, E). Namely,
$$E(a_1 \cdots a_n) = 0$$
for every $a_j \in \ker(E_{i_j})$ with $i_1 \neq \cdots \neq i_n$.

Moreover, the above conditions uniquely characterize the reduced amalgamated free product (A, E).

Example 6.8.4. There is a *-isomorphism

$$C(\mathbb{T}) * C(\mathbb{T}) \simeq C_r^*(\mathbb{F}_2).$$

Here the free product is being taken with respect to the state $f \mapsto \int_{\mathbb{T}} f(z)dz$ on $C(\mathbb{T})$.

The free group $\mathbb{F}_2$ on two generators is not amenable, so the reduced group C^*-algebra is not nuclear and consequently has infinite nuclear dimension. What this shows is that unlike other canonical C^*-constructions, finite nuclear dimension is not in general preserved under the reduced amalgamated free product construction, even for abelian C^*-algebras. However, the next result by Speicher shows that there are exceptions.

Proposition 6.8.5. Let $\mathcal{H}_i$ be C^*-correspondences over A. Then

$$(\mathcal{T}(\bigoplus_I \mathcal{H}_i), E_{\oplus \mathcal{H}_i}) \simeq *_A(\mathcal{T}(\mathcal{H}_i), E_{\mathcal{H}_i}).$$

In light of Theorem 6.7.1, we obtain the following.

Theorem 6.8.6. Let $\mathcal{H}$ be a finitely generated projective C^*-correspondence over A. Suppose that the nuclear dimension of A and the Rokhlin dimension of $\mathcal{H}$ are both finite. Then the amalgamated free product of any finite number of copies of copies of $\mathcal{T}(\mathcal{H})$ (with respect to the canonical expectation $E_{\mathcal{H}}$) has finite nuclear dimension.

Proof. Let I be a finite set. By Proposition 6.8.5, it suffices to show that $\bigoplus_I \mathcal{H}$ has finite Rokhlin dimension. Note first that the left action of A on $\bigoplus_I \mathcal{H}$ is given by $a.(\sum_{i \in I} i.x_i) = \sum_{i \in I} i.(a.x_i)$. Let $\epsilon > 0$, $p \in \mathbb{N}$, $\mathcal{V}$ a finite subset of elementary tensors in $\mathcal{F}_p(\bigoplus_I \mathcal{H})$. If $x \in \mathcal{V}$, there is some $n = 0, \ldots, p-1$ and elements $x_{i,j} \in \mathcal{H}$ for $j = 1, \ldots, n$ and $i \in I$ such that

$$x = x_1 \otimes \cdots \otimes x_n \qquad x_j = \sum_{i \in I} i.x_{i,j}.$$

Now let $\delta > 0$ be small enough so that

$$\delta < \frac{\epsilon}{4n}.$$

Find a finite set $F \subset I$ such that for each $j = 0, \ldots, n$ we have

$$x_j \approx_\delta \sum_{i \in F} i.x_{i,j}.$$

Let $\mathcal{V}_F = \{x_{i,j} \ : \ j = 1, \ldots, n, \text{ and } i \in F\}$ and let $\bar{\delta} > 0$ be small enough so that

$$\bar{\delta} < \frac{\epsilon}{2n|F|}.$$

Find Rokhlin contractions $\{f_k^l\}_{k=0,\dots,p}^{l=0,\dots,d} \in A$ with respect to $(\bar{\delta}, p, \mathcal{V}_F)$. For each k, l we have

$$\begin{aligned} x. f_k^l &\approx_{n\delta} (\sum_{i\in F} i. x_{i,1}) \otimes \cdots \otimes (\sum_{i\in F} i. x_{i,n-1}) \otimes (\sum_{i\in F} i. (x_{i,n}. f_k^l)) \\ &\approx_{|F|\bar{\delta}} (\sum_{i\in F} i. x_{i,1}) \otimes \cdots \otimes (\sum_{i\in F} i. (x_{i,n-1}. f_{k+1}^l)) \otimes (\sum_{i\in F} i. x_{i,n}) \\ &\quad \vdots \\ &\approx_{|F|\bar{\delta}} (\sum_{i\in F} i. (x_{i,1}. f_{k+n-1}^l)) \otimes \cdots \otimes (\sum_{i\in F} i. x_{i,n-1}) \otimes (\sum_{i\in F} i. x_{i,n}) \\ &\approx_{|F|\bar{\delta}} (f_{k+n}^l. \sum_{i\in F} i. x_{i,1}) \otimes \cdots \otimes (\sum_{i\in F} i. x_{i,n-1}) \otimes (\sum_{i\in F} i. x_{i,n}) \\ &\approx_{n\delta} f_{k+n}^l. x. \end{aligned}$$

By construction, we conclude $x. f_k^l \approx_\epsilon f_{k+|x|}. x$ and the result follows since the other requisite properties of the Rokhlin contractions are automatic. □

In comparison with Example 6.8.4, the following demonstrates an interesting consequence of Theorem 6.8.6.

Example 6.8.7. If φ is a minimal homeomorphism of $\mathbb{T}$, then $\dim_{\mathrm{Rok}}(\varphi^*)$ is finite. By Theorem 6.8.6, the nuclear dimension of

$$\mathcal{T}(C(\mathbb{T})^{\varphi^*}) * \mathcal{T}(C(\mathbb{T})^{\varphi*}) \simeq \mathcal{T}(C(\mathbb{T}) \oplus C(\mathbb{T}))$$

is also finite. Here the free product is being taken with respect to the usual conditional expectation $E_{C(\mathbb{T})\oplus C(\mathbb{T})}$.

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