The dissertation of Karan Govil was reviewed and approved* by the following:

Murat Günaydin  
Professor of Physics  
Dissertation Advisor, Chair of Committee

Martin Bojowald  
Professor of Physics

Richard Robinett  
Director of Graduate Studies  
Department of Physics

Ping Xu  
Professor of Mathematics

*Signatures are on file in the Graduate School.
Abstract

Massless conformal scalar fields in four-dimensional and six-dimensional Minkowski space-time correspond to minimal unitary representations of $SO(4,2) \sim SU(2,2)$ and $SO(6,2) \sim SO^*(8)$, respectively. The Fradkin-Vasiliev type higher spin algebras are defined as the symmetry algebras of massless Klein-Gordon equation for a scalar field which implies that the universal enveloping algebras of the minimal representation are just the conformal higher spin algebras in four and six dimensions, or the $AdS_5$ and $AdS_7$ higher spin algebras, respectively.

In this thesis, using the quasiconformal methods developed by Günaydin, Koepsell, and Nicolai, we formulate the minimal unitary representation for $SU(2,2)$ and $SO^*(8)$ in terms of non-linear twistorial oscillators that transform non-linearly under the respective Lorentz groups, $SL(2,\mathbb{C})$ in four dimensions, and $SL(2,\mathbb{H})$ in six dimensions. Using this formulation, we will define the conformal higher spin algebras $hs(4,2)$ and $hs(6,2)$. We will also define a one-parameter family of continuous deformations, $hs(4,2;\zeta)$, and discrete deformations $hs(6,2;t)$ of these higher spin algebras. Here $\zeta$ is the continuous helicity of massless conformal fields in four dimensions and $t$ is the spin of an $SU(2)$ subgroup of the little group of massless particles, $SO(4)$, in six dimensions. Our results imply the existence of a family of (supersymmetric) HS theories in $AdS_5$ and $AdS_7$ which are dual to free (super)conformal field theories (CFTs) or to interacting but integrable (supersymmetric) CFTs in four and six dimensions, respectively.

Using the quasiconformal methods, we also construct the minimal unitary representations (and its deformations) for the exceptional supergroup $D(2,1;\lambda)$ which is the most general $\mathcal{N} = 4$ superconformal group in one dimension. We shall also review Lorentz covariant twistorial oscillator construction for $SO(3,2) \sim Sp(4,\mathbb{R})$, $SU(2,2)$, and $SO^*(8)$. 
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Dedicated To My Parents and Megan
Chapter 1
Introduction

“The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve.”

– Eugene Wigner, *The Unreasonable Effectiveness of Mathematics in the Natural Sciences* [1]

The role of symmetry in the formulations of laws of nature and invariance principles is ubiquitous in theoretical physics, and group theory is the language of symmetry. Even though symmetry and conservation laws have been around in physics dating back to Newton, Kepler and even Greeks who were fascinated by the symmetries of regular objects and believed that these might be reflected in the principles of nature itself, it was Einstein, in particular, who regarded symmetry principles as fundamental and insisted that the various conservation laws and transformation laws be consequences of symmetry (Noether’s theorem [2] explains the connection between symmetry and conservation laws).

A great triumph for the symmetry principle was the equivalence principle which governs the dynamics of gravity and of space-time itself, and is a statement of invariance of the laws of nature under the local changes of the space-time coordinates.

The development of quantum mechanics in the early 20th century rendered the need for a more mathematically rigorous treatment of symmetry principles using group theoretical ideas. Wigner laid the foundations for the application of group theory in quantum mechanics in a series of seminal papers during 1926-1935. A study of the infinite dimensional unitary irreducible representations of Lorentz group was started by Wigner [3] and Dirac [4] which served as the inspiration for the works of Bargmann [5].

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1 According to George Mackey’s annotations to the first volume of Wigner’s *Collected Works*, it was von Neumann who pointed out to Wigner that if the Hamiltonian of a quantum system is unchanged by some group of symmetries $G$, then the eigenspaces of $H$ span representations for $G$, which was indeed stated by Wigner in the introduction of the second paper in Volume I.
Gel’fand-Naimark [6] and Harish-Chandra [7] who performed a mathematically rigorous and systematic study of the unitary representations of $SL(2, \mathbb{R})$ and $SL(2, \mathbb{C})$. These works had shown the fruitfulness of investigations of unitary representations of groups relevant to physics and they were followed by intense works on representation theory of semisimple Lie groups and their applications in physics by a number of physicists and mathematicians alike.

Quantum field theory, the relativistic version of quantum mechanics, describes the dynamics and interactions of fundamental particles and forces. The Standard Model of elementary particles, discovered by Glashow, Weinberg and Salam [8–10], is a quantum field theory based on the local gauge group $SU(3) \times SU(2) \times U(1)$ and describes all the known elementary particles and their interactions except gravity. It is one of the most precisely tested theory of physics and the Higgs boson, one of the most important pieces of Standard Model, was recently discovered at the Large Hadron Collider at CERN [11,12]. The two most important characteristic energy scales for the standard model are $\Lambda_{\text{QCD}} \sim 200$ MeV, which is the energy scale at which QCD (Quantum Chromodynamics, a gauge theory of strong interactions) becomes strongly coupled, and, $m_{W,Z,H} \sim 100$ GeV, the scale at which electromagnetic and weak interaction becomes unified.

With the Standard Model, we have been able to formulate quantum theories of electromagnetism, weak, and strong interaction; however, serious difficulties arise when one tries to reconcile gravity or general relativity with quantum mechanics. One of the problems with quantum gravity is that it is non-renormalizable in $d \geq 4$, which means that the predictive power of the theory at high energy scales breaks down. General relativity describes gravity classically at large length scales and standard model is only an effective theory which does not take into account quantum gravity effects at high energy scales. The characteristic energy scale at which gravity becomes strong is the Planck scale $M_{\text{Pl}} \sim 10^{19}$ GeV, which is not accessible at the high energy particle accelerators which are currently operating at fifteen orders of magnitude below Planck scale. As a result, the research in quantum gravity is mainly driven by thought experiments and symmetry principles.

The search for unified field theories can be traced back to Weyl [13] where he tried to use gauge invariance to relate gravitation and electricity. Since then we have come a long way and String Theory has emerged as one of most promising candidates for a unified field theory, especially as a theory of quantum gravity. One of the most important aspects of String Theory that has kept physicists busy for over a decade is the anti-de Sitter-conformal field theory, i.e. $\text{AdS/CFT}$ correspondence, which is also known as gauge-gravity duality. It was first proposed by Maldacena in reference [14] where he realized that the type IIB superstring theory on $AdS_5 \times S^5$ background is equivalent
to $\mathcal{N} = 4$ super Yang-Mills theory on the four dimensional boundary. In the large $N$ limit, the $AdS/CFT$ duality is a statement about quantum gravity and strings in an asymptotically $AdS$ background involving black holes being dual to a conformal field theory of the boundary of the $AdS$, which is a non gravitational quantum field theory. Even though we know of many examples of the duality in various dimensions and many checks have been performed, there are still many open questions with the most important being a mathematical proof of the conjecture.

The duality between $AdS_5 \times S^5$ superstring and $\mathcal{N} = 4$ super Yang-Mills is one of the most abundantly studied $AdS/CFT$ dualities but it is far from being the simplest. The model of a duality between large $N$ conformal field theories in $d$ dimensions containing $N$–component vector field and theories of infinite number of higher spin massless gauge fields in $AdS_{d+1}$ was put forward in reference [15]. This turns out to be one of the simplest examples of $AdS/CFT$ duality with the dual conformal field theory being a free theory because of the presence of infinitely many conserved charges. This infinite symmetry is called the higher spin symmetry and forms the basis of the study of higher spin gauge theories in curved backgrounds initiated by Fradkin and Vasiliev in the 1980s [16,17].

There has been an enormous amount of work on higher spin theories by Vasiliev and various collaborators (we refer to [18] and [19] for detailed reviews on higher spin theories). This work has been mainly targeted at higher spin theories in $AdS_4$ where the existence of Lorentz covariant spinorial variables makes it convenient to formulate the higher spin algebra under which the infinite massless higher spin gauge fields transform irreducibly. However, from the $AdS/CFT$ and M/Superstring Theory point of view, $AdS_5$ and $AdS_7$ are also extremely important. The $AdS_d$ higher spin algebras for $d > 4$ are complicated and the progress in these theories has been slow due to the lack of a general formulation of irreducible higher spin algebras that are realized unitarily. It is the aim of this thesis to construct $AdS_{5,7}$ higher spin algebras using the quasiconformal approach (developed first for the exceptional group $E_8$ in references [20,21] and generalized and extended to superalgebras in references [22,24]) and elucidate novel features such as one parameter family of deformations of these algebras. The important ingredients for the construction of these higher spin algebras are group theoretic origins of $AdS/CFT$ correspondence, minimal unitary representations, and higher spin theories of Vasiliev, and we shall spend rest of this chapter on reviewing these concepts.
1.1 On singletons, doubletons, AdS/CFT, and minimal representations

In 1963 [25], Dirac discovered remarkable representations of the four dimensional anti-de Sitter (AdS$_4$) group $SO(3,2)$ which are known to have a singular Poincaré limit [26,27]. These representations are also known as $Di$ and $Rac$ or singleton representations. It was shown in references [26–30] that the singleton tensor products $Di \otimes Di$, $Rac \otimes Rac$ and $Di \otimes Rac$ decompose into an infinite set of massless unitary irreducible representations which have a smooth Poincaré limit in $d = 4$ and go over to the massless representations of the Poincaré group labeled by helicity. The singletons are the only two conformal massless representations in three dimensions where $SO(3,2)$ acts as conformal group. It should be noted that the tensor products of two singletons, which are massless in four dimensional (AdS$_4$) sense are in fact massive when considered as representations of the three dimensional conformal group. The analogues of singletons for AdS$_5$ and AdS$_7$ groups $SO(4,2)$ and $SO(6,2)$ respectively are called doubleton $^2$ representations and were first obtained in reference [32] for $SO(4,2) \sim SU(2,2)$ and in reference [33] for $SO(6,2) \sim SO^*(8)$. A similar story holds for the AdS$_4$ supergroup $OSp(N|4)$ where the tensor product of the singleton supermultiplet decomposes into an infinite set of massless supermultiplets which have a Poincaré limit in four dimensions [28,34–36]. The doubleton supermultiplets in AdS$_5$ ($SU(2,2|N)$) and AdS$_7$ ($OSp(8^*|2N)$) share the same features, i.e. their tensor products decompose into infinitely many massless supermultiplets [32,33,36,39].

The issue of masslessness in AdS is a subtle one because the Poincaré mass operator is not a Casimir invariant of the AdS group. Thus the following definition of a massless representation or a supermultiplet of an AdS supergroup was proposed in reference [40]:

**A representation (or a supermultiplet) of an AdS group (or supergroup) is massless if it occurs in the decomposition of the tensor product of two singleton or two doubleton representations (or supermultiplets).**

The group theoretical origin of AdS$_{d+1}$/CFT$_d$ correspondence lies in the fact the conformal group, $SO(d,2)$, in $d$ Minkowski space-time dimensions is same as the AdS$_d$ isometry group. The hints of duality were already known back in 1984 [32] when the

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$^2$Oscillator methods for constructing unitary representations of non-compact groups and supergroups using bosonic and fermionic oscillators was formulated in references [29,31]. Given a semisimple Lie group $G$ admitting lowest weight representation with maximal compact subgroup of the form $H \times U(1)$ (for compact subgroup $H$), the oscillator method describes unitary irreducible representations by realizing the generators of $G$ as creation and annihilation operators on some Fock space, and have them transform in the (anti-)fundamental representation of $H$. It turns out that for symplectic groups $Sp(2n,R)$, the minimum number of sets of oscillators needed is one and the result is singleton representations whereas for $SU(n,m)$ and $SO^*(2n)$ type groups, one needs a minimum of two sets of bosonic oscillators and such representations were termed as doubletons.
Kaluza-Klein spectrum of type IIB supergravity was first obtained by repeated tensoring of CPT self-conjugate doubleton supermultiplet of $SU(2,2|4)$ (the $\mathcal{N} = 4$ superconformal group in four dimensions) with itself repeatedly and restricting to the CPT self-conjugate short supermultiplets. Moreover, it was pointed out in reference [32] that the CPT self-conjugate doubleton supermultiplet of $SU(2,2|4)$ does not have a Poincaré limit in five dimensions and its field theory lives on the boundary of $AdS_5$, on which $SU(2,2) \sim SO(4,2)$ acts as the conformal group and that the unique candidate for this theory is the conformally invariant $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions. Similarly the Kaluza-Klein spectra of the compactifications of eleven dimensional supergravity over $AdS_4 \times S^7$ and $AdS_7 \times S^4$ were obtained by tensoring the singleton supermultiplet of $OSp(8|4,\mathbb{R})$ [35] and scalar doubleton supermultiplet of $OSp(8^*|4,\mathbb{R})$ [33], respectively. It was also pointed out in reference [33,35] that the field theories of the singleton and scalar doubleton supermultiplets live on the boundaries of $AdS_4$ and $AdS_7$, respectively, as conformal field theories. Thus, within the framework of Kaluza-Klein supergravities, these are some of the earliest works on $AdS/CFT$ correspondence. The modern era of $AdS/CFT$ was started by the extension of these ideas to Superstring and M-theory arena [14,41,42].

From a purely mathematical point of view, these singleton and scalar doubleton representations are very special and are known as minimal unitary representations. The minimal unitary representation (minrep) of a non-compact Lie group is defined as a unitary representation over a Hilbert space of functions with smallest or minimal number of variables possible. The idea of the minrep was introduced by Joseph in reference [43] in the context of determining the least number of degrees of freedom for which a quantum mechanical system admits a given semisimple Lie algebra as a spectrum generating symmetry. The corresponding representations were termed as minimal representations. Such representations are also known in the literature as singular representations [44]. The minreps have a low Gelfand-Kirillov dimension which suggests that there are perhaps some “missing” states in the representation similar to massless gauge bosons such as photon, graviton etc, which lack the longitudinal states. Thus minreps play an important role in the description of massless states in gauge invariant quantum field theories.

The minrep of a Lie algebra exponentiates to a unitary representation of the corresponding non-compact group over a Hilbert space of functions depending on the smallest (minimal) number of variables possible. Joseph presented the minimal realizations of the

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3The unitary dual of a semisimple Lie group contains the tempered representations which enter the Plancherel decomposition of $L^2(G)$ and their complement, “singular representations” [44].

4Let $(\pi,V)$ be an irreducible representation of a Lie algebra $\mathfrak{g}$ on a vector space $V$, $\mathfrak{h}$ be any vector in $V$, and $U^n(\mathfrak{g})$ be the subspace of universal enveloping algebra consisting at most $n$ elements of $\mathfrak{g}$. The Gel'fand-Kirillov dimension [45] then measures the size of $(\pi,V)$ in terms of growth rate of $\dim(U^n(\mathfrak{g}) \cdot \mathfrak{h})$ as $n \to \infty$. 

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complex forms of classical Lie algebras and of $G_2$ in a Cartan-Weil basis. The existence of the minimal unitary representation of $E_{8(8)}$ using Langland’s classification was first shown by Vogan [46]. The minimal unitary representations of simply laced groups were studied by Kazhdan and Savin [47], and Brylinski and Kostant [48,49]. The minimal representations of quaternionic real forms of exceptional Lie groups were later studied by Gross and Wallach [50]. For a review and more complete list of references on the subject in the mathematics literature prior to 2000, we refer to the lectures of Jian-Shu Li [51]. Pioline, Kazhdan and Waldron [52] reformulated the minimal unitary representations of simply laced groups given in reference [47] and gave explicit realizations of the simple root (Chevalley) generators in terms of pseudo-differential operators for the simply laced exceptional groups as well as the spherical vectors necessary for the construction of modular forms.

The construction of the minrep for the exceptional group $E_{8(8)}$ using the quasiconformal methods was first obtained in reference [20]. Quasiconformal realizations exist for different real forms of all non-compact groups as well as for their complex forms [21,53]. The term “quasiconformal” refers to the fact that the geometric action of the generators in this realization leaves invariant a light-cone defined by a quartic distance function in a generalized space time. Remarkably, upon quantization of this geometric realization, we get the minrep of the corresponding group. This was first shown explicitly for the split exceptional group $E_{8(8)}$ with the maximal compact subgroup $SO(16)$ [20]. Consequently, the minrep of the three dimensional $U$-duality group $E_8(-24)$ of the exceptional supergravity [54] was obtained in reference [55].

A unified formulation of the minreps of non-compact groups using the quantization of quasiconformal action was presented in reference [22]. This was also extended to minreps of non-compact supergroups $G$ whose even subgroups are of the form $H \times SL(2,\mathbb{R})$ with $H$ compact. These supergroups include $G(3)$ with even subgroup $G_2 \times SL(2,\mathbb{R})$, $F(4)$ with even subgroup $Spin(7) \times SL(2,\mathbb{R})$, $D(2,1;\sigma)$ with even subgroup $SU(2) \times SU(2) \times SU(1,1)$ and $OSp(N|2,\mathbb{R})$. Inspired by the $AdS/CFT$ applications, the minrep for supergroups of the form $SU(n, m|p + q)$ and $OSp(2N^*|2M)$ was obtained in reference [23,24,56] and applied to conformal superalgebras in four and six dimensions.

An isomorphism between the minrep of $SU(2,2)$ and the scalar doubleton representation was established in reference [23]. It was further shown that the minrep in quasiconformal formulation admits a one parameter family ($\zeta$) of deformations that can

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5 For the largest exceptional group $E_{8(8)}$, the quasiconformal action is the first and only known geometric realization of $E_{8(8)}$ and leaves invariant a generalized light-cone with respect to a quartic distance function in 57 dimensions [21].

6 We shall be mainly working at the level of Lie algebras and Lie superalgebras and be cavalier about using the same symbol to denote a (super)group and its Lie (super)algebra.
be identified with helicity, which can be continuous. The resulting unitary irreducible
representation of $SU(2,2)$ corresponds to a 4d massless conformal field transforming in
$(0, \frac{\zeta}{2})$ (for $\zeta > 0$) or $(-\frac{\zeta}{2}, 0)$ (for $\zeta < 0$) representation of the Lorentz subgroup, $SL(2, \mathbb{C})$.
These deformed minimal representations for integer values of $\zeta$ turn out to be isomorphic
to the doubletons of $SU(2,2)$ \cite{32,37,38}. Similarly the minimal unitary supermultiplet
of $SU(2,2|N)$ turns out to be the CPT self-conjugate (scalar) doubleton supermultiplet,
and for $PSU(2,2|4)$ it is simply the four dimensional $N = 4$ Yang-Mills supermultiplet.
Once again it was shown that there exists a one-parameter family of deformations of
the minimal unitary supermultiplet of $SU(2,2|N)$ and its deformations with integer $\zeta$ are
isomorphic to the unitary doubleton supermultiplets studied in reference \cite{32,37,38}.

Analogous to the $AdS_5$ case, the minrep of $AdS_7$ or 6d conformal group $SO(6,2) =
SO^*(8)$ and its deformations were studied in reference \cite{24}. The minrep was again
shown to be isomorphic to the scalar doubleton representation and the minrep admits
deformations labelled by the eigenvalues of the Casimir of an $SU(2)_T$ subgroup of the little
group, $SO(4)$, of massless particles in six dimensions. These deformed minreps labeled
by spin $t$ of $SU(2)_T$ are positive energy unitary irreducible representations of $SO^*(8)$
that describe massless conformal fields in six dimensions. Quasiconformal construction
of the minimal unitary supermultiplet of $OSp(8^*|2N)$ and its deformations were given
in reference \cite{24,56}. The minimal unitary supermultiplet of $OSp(8^*|4)$ is the massless
conformal $(2,0)$ supermultiplet whose interacting theory is believed to be dual to M-theory
on $AdS_7 \times S^4$. It is isomorphic to the scalar doubleton supermultiplet of $OSp(8^*|4)$ first
constructed in reference \cite{33}.

The minimal unitary representation for symplectic groups $Sp(2N, \mathbb{R})$ constructed using
the covariant oscillator realization coincides with the one obtained from quantization of
its quasiconformal action \cite{22}. This is due to the fact that the quartic invariant operator
that enters the quasiconformal construction vanishes for symplectic groups and hence
the resulting generators involve only bilinears of oscillators. Therefore the minreps of
$Sp(4, \mathbb{R})$ are simply the scalar and spinor singletons that were called the remarkable
representations of anti-de Sitter group by Dirac \cite{25} as discussed earlier.

1.2 Vasiliev’s higher spin theories and their underlying algebras

One of the most important motivations for studying higher spin gauge theories comes from
Superstring Theory where the spectrum includes massive infinite higher spin excitations
of all spins \cite{57}. Although these higher spin excitations are massive, one can contemplate
the existence of a symmetric phase where all these higher spin excitations are massless
and acquire mass by spontaneous symmetry breaking.
The Lagrangians for free massive \[58,59\] and massless \[60,61\] higher spin fields have been known since the 1970s. However there are several no-go theorems (Weinberg \[62\], Aragone-Deser \[63\], Weinberg-Witten \[64\]) for massless fields due to inconsistencies which arise when one turns on the interactions for the higher spin fields by replacing the ordinary derivatives by covariant derivatives. We refer to reference \[65\] for an excellent review on no-go theorems for massless higher spin fields.

Non-trivial consistent cubic interactions in higher spin gauge theories were constructed in references \[66–73\], but these did not include gravitational interactions, which was problematic because of the universal role of gravity. The problems with coupling gravity to higher spin gauge fields were elucidated in references \[63,74,75\] and can be summarized as follows: in order to couple higher spin fields with gravity, we need to covariantize the derivatives, i.e. \(\partial \rightarrow D = \partial - \Gamma\), which breaks the gauge invariance of higher spin fields. The reason is that when one tries to take the variation of the action under the higher spin gauge transformations, the derivatives need to be commuted and the commutator of covariant derivatives is proportional to the Riemann tensor and it is not clear how to cancel these terms by adding terms to the action or modifying the gauge transformations.

This problem was resolved in reference \[76,77\] where it was shown that one needs to consider (anti-) de Sitter background to construct consistent higher spin gravitational interactions because the gauge invariant gravitational interaction term in \(AdS\) contains terms that are inversely proportional to the cosmological constant and consequently diverge in the flat space limit. We should stress the non-analyticity in the cosmological constant is needed for preserving higher spin gauge symmetries, which are expected to be broken in a realistic physical model/phase and thus the expectation value of the cosmological constant will also be modified in such a physical phase. Subsequently, the equations of motion for full non-linear theories of interacting higher spin gauge fields in \(AdS_d\) were developed by Vasiliev in reference \[78\] (we refer to the reviews \[79,80\] and references therein for details) in the so-called unfolded formulation that is a generalization of Cartan’s geometric formulation of Einstein gravity. However, a complete Lagrangian formulation for interacting higher spin gauge fields is still missing.

One of the most important ingredients for the formulation of higher spin gauge theories is the higher spin symmetry algebra, under which the higher spin fields transform irreducibly. The \(AdS_4\) higher spin algebras and superalgebras were constructed in reference \[81,82\] and the underlying principle was the fact that the tensor product of a pair of singletons decomposes into infinite \(AdS_4\) massless representations of \(SO(3, 2) \sim Sp(4, \mathbb{R})\) (Flato-Fronsdal theorem \[27,83\]) played an important role. The extensions of Flato-Fronsdal theorem for \(AdS_5\) and \(AdS_7\) were obtained in reference \[32\] and \[33\], respectively, as discussed in the previous section.
The bosonic $AdS_d$ higher spin algebra used in reference [78] is the infinite dimensional symmetry algebra of the massless Klein-Gordon equation, i.e. symmetry algebra of the massless scalar in $d-1$ dimensions. One can also consider symmetry algebra (bosonic) of a boundary spinor, which for $AdS_4$, turns out to be isomorphic to the scalar symmetry algebra. Based on these algebraic ideas, a higher spin/vector model holographic duality was proposed in reference [15] (see also [84–92]). This holographic duality is regarded as one of the simplest non-trivial examples of $AdS/CFT$ correspondence. It was shown in reference [93] that the spectrum of operators in the vector model (CFT) is not renormalized at infinite $N$ and thus the spectrum of fields in the bulk is simple. The most non-trivial part of this duality thus lies in the exact agreement between the correlation functions on the boundary conformal field theory i.e. the vector model and those of Vasiliev’s higher spin gauge theory in the bulk. This agreement was checked explicitly by direct calculations in the seminal papers of Giombi and Yin [94,95] where they computed the three-point functions of higher spin currents and showed that they matched with those of free and critical $O(N)$ vector models. Since then, a considerable progress has been made in higher spin holography in the last few years including generalizing the conjecture to parity violating Vasiliev theories and Chern-Simons vector models [93,96], structure of correlation functions [97,99], proof of CFT/higher spin version of the Coleman-Mandula theorem [100,101], exact large $N$ computations in Chern-Simons vector models [93,102,103], supersymmetric extension and symmetry breaking, and connection between Vasiliev’s higher spin gauge theory and string theory [96]. Some approaches towards deriving the higher spin/vector model duality from first principles were investigated in reference [104] and [105–108] (see also [109] for relevant earlier work). Recently, a dS/CFT version of the duality was also proposed [110], and further studied in references [111,112]. There has also been exciting development in the $AdS_5/CFT_2$ version of higher spin holography (see [113] and references therein), as well as in higher dimensions [114].

As we discussed in the previous section, there are only two conformally massless representations for $SO(3,2)$, namely $Di$ and $Rac$ or the singletons. For $AdS_{5,7}$, however, we have an infinite number of doubletons and thus one would expect to get a family of higher spin algebras. These families of infinite dimensional higher spin algebras in $AdS_5$ and $AdS_7$ and associated higher spin holographic dualities are the subject of this thesis.

### 1.3 Overview of the Thesis

We have explored various aspects of minimal representations in quasiconformal formalism and higher spin symmetries in preceding sections. This thesis explores algebraic aspects of higher spin symmetry by constructing higher spin algebras and their unitary
representations in $AdS_5$ and $AdS_7$.

In Chapter two, we study the minimal representations of the exceptional supergroup $D(2,1;\lambda)$ which happens to be the most general $\mathcal{N} = 4$ superconformal group in one dimension. This will serve as an exercise in using quasiconformal methods to construct minimal representations and their deformations. We also establish a correspondence between the generators of the minimal representation and certain models of $\mathcal{N} = 4$ superconformal quantum mechanics in harmonic superspace [115].

Chapter three is devoted to the study of $AdS_5$ higher spin algebras $hs(4,2;\zeta)$ or the four dimensional conformal higher spin algebras. We will review the oscillator construction for $SO(3,2)$ and $SO(4,2)$ and then formulate the quasiconformal realization obtained in reference [23] in terms of non-linear twistors that transform non-linearly under the Lorentz group $SO(3,1)$. We shall then proceed to define the higher spin algebras as enveloping algebras of the minimal representations and its deformations.

Similarly in Chapter four, we study $AdS_7$ higher spin algebras $hs(6,2;t)$ or the six dimensional conformal higher spin algebras. We will review the oscillator construction for $SO(6,2)$ and then formulate the quasiconformal realization obtained in references [24,56] in terms of non-linear twistors that transform non-linearly under the Lorentz group $SO(5,1)$. As in Chapter three, we define the higher spin algebras as enveloping algebras of the minimal representations and its deformations.

Our results suggest the existence of a family of (supersymmetric) higher spin theories in $AdS_5$ and $AdS_7$ which are dual to free (super)conformal field theories or to interacting but integrable (supersymmetric) conformal field theories in four and six dimensions, respectively. We conclude in Chapter five by discussing applications of our results to higher spin holography and address some open problems.
Chapter 2  |  Minimal Unitary Representation of $D(2, 1; \lambda)$

In this chapter we will study the minimal unitary supermultiplet for the exceptional supergroup $D(2, 1; \lambda)$ and its $SU(2)$ deformations, and its connection with certain models of superconformal quantum mechanics in harmonic superspace. This chapter will serve as a warm-up to using quasiconformal methods to obtain minreps and studying their deformations. These results were published in [116] which we follow closely in our review in this section.

The plan of this chapter is as follows. We start by discussing the basics of $D(2, 1; \lambda)$ and in section 2.2, we review the minimal unitary realization of $D(2, 1; \sigma)$ as constructed in [22] using quasiconformal techniques. In section 2.3 we construct the $SU(2)$ deformations of the minrep of $D(2, 1; \lambda)$ using bosonic oscillators in the non-compact 5-graded basis. In Section 2.4 we reformulate the results of section 2.3 in the compact 3-graded basis and show that the deformations of the minrep of $D(2, 1; \lambda)$ are positive “energy” (unitary lowest weight) representations of $D(2, 1; \lambda)$. We then present the corresponding unitary supermultiplets. Section 2.5 discusses deformations of the minrep using both bosons and fermions and how the deformed $D(2, 1; \lambda)$ commutes with a non-compact super algebra $OSp(2n^*|2m)$ with the even subgroup $SO^*(2m) \times USp(2n)$ constructed using “deformation” bosons and fermions. In section 2.6 we review some of the results of work on $\mathcal{N} = 4$ superconformal mechanics and show how its symmetry generators and spectrum map into the generators of $D(2, 1; \lambda)$ deformed by a pair of bosonic oscillators and the resulting unitary supermultiplets. We conclude with a brief discussion of our results.
2.1 \( D(2,1;\lambda) \) basics

A complete classification of Lie superalgebras was given by Kac in [117]. The algebras \( D(2,1;\lambda) \) are a one-parameter family of non-isomorphic superalgebras of dimension 17, and is isomorphic to \( OSp(4|2,\mathbb{R}) \) for \( \lambda = -1/2 \). It is the most interesting of the exceptional superalgebras in the sense that there is no Lie algebraic counterpart i.e. there does not exist a family of non-isomorphic simple Lie algebras with the same dimension, and it is also the most general \( \mathcal{N} = 4 \) superconformal algebra in one dimension. The irreducible representations of \( D(2,1;\lambda) \) were first studied in [118]. In the present chapter, we will focus on obtaining the positive energy or lowest weight representations, in particular the minrep, by using the quasiconformal methods.

The even or bosonic part of \( D(2,1;\lambda) \) is \( SU(2) \times SU(2) \times SU(2) \), however among all the possible non-compact real forms, only the real form with even part \( SU(2) \times SU(2) \times SU(1,1) \) admits lowest energy unitary representation that are relevant for physical applications. We will only study this real form in this chapter and label the even subalgebras as \( SU(2)_A \times SU(2)_T \times SU(1,1)_K \) with odd generators transforming in the \((1/2,1/2,12/)\) representation of the even part.

The representation of \( OSp(4|2,\mathbb{R}) \) (i.e. \( D(2,1;-1/2) \)) were studied using twistorial oscillators in [34] and the minrep of \( D(2,1;\lambda) \) in terms of one bosonic and four fermionic coordinates using quasiconformal methods was studied in [22]. We shall reformulate the minrep that was obtained in [22] and obtain a one-parameter family of deformations of the minrep. Note that in [22], the parameter \( \lambda \) was called \( \sigma \) and the two are related as follows:

\[
\sigma = \frac{2\lambda + 1}{3} \tag{2.1.1}
\]

In the next section, we shall first briefly review the construction of [22] and then reformulate it in a manner suitable to obtaining one-parameter deformations of the minrep.

2.2 Minimal unitary representation for \( D(2,1;\lambda) \): A review

Let us start by reviewing the construction of \( D(2,1;\sigma) \) minrep given in [22]. The Lie super algebra of \( D(2,1;\sigma) \) can be given a 5-graded decomposition of the form

\[
D(2,1;\sigma) = \mathfrak{g}^{(-2)} \oplus \mathfrak{g}^{(-1)} \oplus \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(+1)} \oplus \mathfrak{g}^{(+2)} \tag{2.2.1}
\]
where the grade \( \pm 2 \) subspaces are one dimensional and the grade zero subalgebra is

\[
g^{(0)} = \mathfrak{su}(2)_A \oplus \mathfrak{su}(2)_T \oplus \mathfrak{so}(1, 1)_\Delta \tag{2.2.2}
\]

The grade \( \pm 2 \) generators together with \( \Delta \) in \( g^{(0)} \) form the \( \mathfrak{su}(1, 1)_K \) subalgebra. The remaining 8 odd generators transforming in the \((1/2, 1/2)\) representation of \( SU(2)_A \times SU(2)_T \) subgroup belong to \( g^{(\pm 1)} \).

We can label the generators in various subspaces according to how they transform under \( SU(2)_A \times SU(2)_T \times SO(1,1)_\Delta \) subgroup as follows:

\[
D(2,1;\sigma) = (0,0)^{-2} \oplus (1/2,1/2)^{-1} \oplus (\mathfrak{su}(2)_A \oplus \mathfrak{su}(2)_T \oplus \Delta) \oplus (1/2,1/2)^{+1} \oplus (0,0)^{+2} \tag{2.2.3}
\]

\[
D(2,1;\sigma) = E \oplus E^{\alpha,\dot{\alpha}} \oplus (M^{\alpha,\beta}_{(1)} + M^{\dot{\alpha},\dot{\beta}}_{(2)} + \Delta) \oplus F^{\alpha,\dot{\alpha}} \oplus F \tag{2.2.4}
\]

The single bosonic coordinate and its canonical momentum \((x,p)\) satisfy

\[
[x, p] = i \tag{2.2.5}
\]

The four fermionic “coordinates” \( X^{\alpha,\dot{\alpha}} \) satisfy the anti-commutation relations [22]:

\[
\{X^{\alpha,\dot{\alpha}}, X^{\beta,\dot{\beta}}\} = \epsilon^{\alpha\beta,\dot{\alpha}\dot{\beta}} \tag{2.2.6}
\]

where \( \alpha,\dot{\alpha},.. \) denote the spinor indices of \( SU(2)_A \) and \( SU(2)_T \) subgroups and \( \epsilon_{\alpha\beta} \) and \( \epsilon_{\dot{\alpha}\dot{\beta}} \) are the Levi-Civita tensors in the respective spaces. The generators belonging to the negative and zero grade subspaces are realized as bilinears

\[
E = \frac{1}{2} x^2 \quad E^{\alpha,\dot{\alpha}} = x X^{\alpha,\dot{\alpha}} \quad \Delta = \frac{1}{2} (xp + px) \tag{2.2.7}
\]

\[
M^{\alpha,\beta}_{(1)} = \frac{1}{4} \epsilon_{\alpha\beta} \left( X^{\alpha,\dot{\alpha}} X^{\beta,\dot{\beta}} - X^{\beta,\dot{\beta}} X^{\alpha,\dot{\alpha}} \right) \tag{2.2.8}
\]

\[
M^{\dot{\alpha},\dot{\beta}}_{(2)} = \frac{1}{4} \epsilon_{\alpha\beta} \left( X^{\alpha,\dot{\alpha}} X^{\beta,\dot{\beta}} - X^{\beta,\dot{\beta}} X^{\alpha,\dot{\alpha}} \right) \tag{2.2.9}
\]
They satisfy the (super)commutation relations

\[
\begin{align*}
[M_{(1)}^{\alpha,\beta} \cdot M_{(1)}^{\lambda,\mu}] &= \epsilon^{\lambda \beta} M_{(1)}^{\alpha,\mu} + \epsilon^{\mu \alpha} M_{(1)}^{\beta,\lambda} \\
[M_{(2)}^{\alpha,\beta} \cdot M_{(2)}^{\lambda,\mu}] &= \epsilon^{\lambda \beta} M_{(2)}^{\alpha,\mu} + \epsilon^{\mu \alpha} M_{(2)}^{\beta,\lambda} \\
[M_{(1)}^{\alpha,\beta} \cdot M_{(2)}^{\lambda,\mu}] &= 0 \\
\{E^{\alpha,\dot{\alpha}}, E^{\beta,\dot{\beta}}\} &= 2 \epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}} E
\end{align*}
\] (2.2.10)

The quadratic Casimirs of $SU(2)_A$ and $SU(2)_T$ are as follows:

\[
\mathcal{I}_4 = \epsilon_{\alpha\beta\lambda\mu} M_{(1)}^{\alpha\lambda} M_{(1)}^{\beta\mu} \quad \mathcal{J}_4 = \epsilon_{\dot{\alpha}\dot{\beta}\dot{\lambda}\dot{\mu}} M_{(2)}^{\alpha\dot{\lambda}} M_{(2)}^{\beta\dot{\mu}}
\] (2.2.11)

They differ by a c-number $\mathcal{I}_4 + \mathcal{J}_4 = -\frac{3}{2}$ thus we can use either one of them to express the generator $F$ of $g^{+2}$ subspace as

\[
F = \frac{1}{2} p^2 + \frac{\sigma}{x^2} \left( \mathcal{I}_4 + \frac{3}{4} + \frac{9}{8} \sigma \right)
\] (2.2.12)

Using the closure of algebra and the property of graded spaces ($[g^m, g^n] \subset g^{m+n}$), the grade $+1$ generators are given by

\[
F^{\alpha\dot{\alpha}} = -i \left[ E^{\alpha,\dot{\alpha}}, F \right]
\] (2.2.13)

and one finds

\[
\{ F^{\alpha\dot{\alpha}}, F^{\beta\dot{\beta}} \} = 2 \epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}} F \quad [F^{\alpha\dot{\alpha}}, F] = 0
\] (2.2.14)

and

\[
\{ F^{\alpha\dot{\alpha}}, E^{\beta\dot{\beta}} \} = \epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}} \Delta - (1 - 3\sigma) i\epsilon^{\alpha\beta} M_{(2)}^{\alpha\dot{\beta}} - (1 + 3\sigma) i\epsilon^{\dot{\alpha}\dot{\beta}} M_{(1)}^{\beta\dot{\alpha}}
\] (2.2.15)

As mentioned before, for $\sigma = 0$ the superalgebra $D(2, 1, \sigma)$ is isomorphic to $OSp(4|2, \mathbb{R})$ and for the values $\sigma = \pm\frac{1}{3}$ it reduces to

\[SU(1,1|2) \times SU(2)\]

2.3 $SU(2)$ Deformations of the minimal unitary supermultiplet of $D(2, 1; \lambda)$

Although the construction in the previous section is suitable to obtain the minrep, we need to reformulate it in order to deform the minrep. We shall first rewrite the (super)commutation relations of its generators in a split basis in which the $U(1)$ generators
of the two $SU(2)$ subgroups are diagonalized

\[
SU(2)_A \quad \Rightarrow \quad A_+, A_-, A_0
\]

\[
SU(2)_T \quad \Rightarrow \quad T_+, T_-, T_0
\]

and the fermionic “coordinates” $X^{\alpha, \dot{\alpha}}$ are written as fermionic annihilation and creation operators $\alpha(\alpha^\dagger)$ and $\beta(\beta^\dagger)$ with definite values of $U(1)$ charges

\[
\{\alpha, \alpha^\dagger\} = 1 = \{\beta, \beta^\dagger\}
\]

\[
\{\alpha, \beta^\dagger\} = 0 = \{\beta, \alpha^\dagger\}
\]

We can now choose a fermionic Fock vacuum such that

\[
\alpha|0\rangle_F = 0 = \beta|0\rangle_F
\]

The generators of $SU(2)_T$ are then given by the following bilinears of these fermionic oscillators:

\[
T_+ = \alpha^\dagger \beta \quad T_- = \beta^\dagger \alpha \quad T_0 = \frac{1}{2} (N_\alpha - N_\beta)
\]

where $N_\alpha = \alpha^\dagger \alpha$ and $N_\beta = \beta^\dagger \beta$ are the respective number operators. They satisfy the following commutation relations:

\[
[T_+, T_-] = 2T_0 \quad [T_0, T_\pm] = \pm T_\pm
\]

with the Casimir

\[
T^2 = T_0^2 + \frac{1}{2} (T_+ T_- + T_- T_+)
\]

Similarly, the generators of the subalgebra $su(2)_A$ are given by the following bilinears of fermionic oscillators:

\[
A_+ = \alpha^\dagger \beta^\dagger \quad A_- = (A_+)^\dagger = \beta \alpha
\]

\[
A_0 = \frac{1}{2} (N_\alpha + N_\beta - 1)
\]

The quadratic Casimir of the subalgebra $su(2)_A$ is

\[
C_2 [su(2)_A] = A^2 = A_0^2 + \frac{1}{2} (A_+ A_- + A_- A_+) .
\]
As before, the quadratic Casimirs $T^2$ and $A^2$ are not independent and are related to each other as follows:

$$T^2 + A^2 = \frac{3}{4}$$

(2.3.11)

The Fock space of two pairs of fermionic oscillators, $(\alpha, \alpha^\dagger)$ and $(\beta, \beta^\dagger)$, is four dimensional and the states transform in the $(1/2, 1/2)$ representation of $SU(2)_A \times SU(2)_T$. We shall label them by their eigenvalues under $T_0$ and $A_0$ as $|m_t; m_a\rangle$ as follows:

$$T_0|m_t; m_a\rangle = m_t|m_t; m_a\rangle$$

(2.3.12)

$$A_0|m_t; m_a\rangle = m_a|m_t; m_a\rangle$$

(2.3.13)

More explicitly we have

\[
\begin{align*}
|0\rangle_F &= |0, -\frac{1}{2}\rangle_F = |0, \downarrow\rangle_F \\
\alpha^\dagger\beta^\dagger|0\rangle_F &= |0, \frac{1}{2}\rangle_F = |0, \uparrow\rangle_F \\
\alpha^\dagger|0\rangle_F &= |\frac{1}{2}, 0\rangle_F = |\uparrow, 0\rangle_F \\
\beta^\dagger|0\rangle_F &= |-\frac{1}{2}, 0\rangle_F = |\downarrow, 0\rangle_F
\end{align*}
\]

where $\uparrow$ denotes $+1/2$ and $\downarrow$ denotes $-1/2$ eigenvalue of the respective $U(1)$ generator.

The grade -1 generators can then be written as bilinears of the coordinate $x$ with the fermionic oscillators:

$$Q = x \alpha \quad Q^\dagger = x \alpha^\dagger$$

$$S = x \beta \quad S^\dagger = x \beta^\dagger$$

(2.3.14)

They close into $K_-$ under anti-commutation:

$$\{Q, Q^\dagger\} = 2K_-$$

(2.3.15)

The grade zero generator, $\Delta$, that gives the 5-grading, is same as before:

$$\Delta = \frac{1}{2} (xp + px)$$

(2.3.16)

The generator $K_-$ of grade -2 subspace and the 4 supersymmetry generators in the grade -1 subspace form a super Heisenberg algebra and they do not change in going over to the deformed minreps.

One makes an ansatz for grade +2 generator $K_+$ of the form

$$K_+ = \frac{1}{2} p^2 + \frac{1}{x^2} (c_1 T^2 + c_2 A^2 + c_3)$$

(2.3.17)
where the constants $c_1, \ldots c_3$ are to be determined. The grade +1 generators are defined by the commutators:

$$\tilde{Q} = i \{Q, K_+\} \quad \tilde{Q}^\dagger = \tilde{Q}^\dagger = i \{Q^\dagger, K_+\} \quad (2.3.18)$$

Using the Jacobi identities and closure of algebra one determines the four unknown constants in terms of $\lambda$ and obtain

$$K_+ = \frac{1}{2} p^2 + \frac{1}{x^2} \left( 2 \lambda T^2 + \frac{2}{3} (\lambda - 1) A^2 + \frac{3}{8} + \frac{1}{2} \lambda (\lambda - 1) \right)$$

$$K_- = \frac{1}{2} p^2 + \frac{1}{x^2} \left( \frac{2}{3} (2 \lambda + 1) T^2 + \frac{\lambda^2}{2} + 1 \right) \quad (2.3.19)$$

We shall now introduce bosonic oscillators $a_m, b_m$ and their hermitian conjugates $a_m^\dagger = (a_m)^\dagger, b_m^\dagger = (b_m)^\dagger \ (m, n, \ldots = 1, 2)$ that satisfy the commutation relations:

$$[a_m, a^n] = [b_m, b^n] = \delta^n_m \quad [a_m, a_n] = [a_m, b_n] = [b_m, b_n] = 0 \quad (2.3.20)$$

To obtain the deformations of the minrep, we use the bosonic oscillators to introduce an $SU(2)_S$ Lie algebra whose generators are as follows:

$$S_+ = a^m b_m \quad S_- = (S_+)^\dagger = a^m b^m \quad S_0 = \frac{1}{2} (N_a - N_b) \quad (2.3.21)$$

where $N_a = a^m a_m$ and $N_b = b^m b_m$ are the respective number operators. They satisfy:

$$[S_+, S_-] = 2 S_0 \quad [S_0, S_\pm] = \pm S_\pm \quad (2.3.22)$$

The quadratic Casimir of $su(2)_S$ is

$$C_2 [su(2)_S] = S^2 = S_0^2 + \frac{1}{2} (S_+ S_- + S_- S_+)$$

$$= \frac{1}{2} (N_a + N_b) \left[ \frac{1}{2} (N_a + N_b) + 1 \right] - 2 a^m b^n a_{m} b_{n} \quad (2.3.23)$$

where square bracketing $a_{m} b_{n} = \frac{1}{2} (a_m b_n - a_n b_m)$ represents antisymmetrization of weight one. The bilinears $a_{m} b_{n}$ and $a^m b^n$ close into $U(P)$ generated by the bilinears

$$U_m = a^m a_n + b^m b_n \quad (2.3.24)$$

under commutation and all together they form the Lie algebra of non-compact group $SO^*(2P)$ with the maximal compact subgroup $U(P)$. The group $SO^*(2P)$ thus generated
commutes with $SU(2)_S$ as well as with $D(2,1;\lambda)$.

In order to obtain the $SU(2)$ deformations of the minrep of $D(2,1;\lambda)$, we extend the $SU(2)_T$ subalgebra with the diagonal subgroup of $SU(2)_T$ and $SU(2)_S$. We shall call this as $SU(2)_T$ and its generators are given below:

$$su(2)_T \implies su(2)_S \oplus su(2)_T$$

$$\mathcal{T}_+ = S_+ + T_+ = a^m b_m + \alpha^\dagger \beta$$
$$\mathcal{T}_- = S_- + T_- = b^m a_m + \beta^\dagger \alpha$$
$$\mathcal{T}_0 = S_0 + T_0 = \frac{1}{2} (N_a - N_b + N_\alpha - N_\beta)$$

with the quadratic Casimir

$$C_2 [su(2)_T] = T^2 = \mathcal{T}_0^2 + \frac{1}{2} (\mathcal{T}_+ \mathcal{T}_- + \mathcal{T}_- \mathcal{T}_+)$$

The generator $\Delta$ and the negative grade generators defined by it remain unchanged in going over to the deformed minreps.

However as the zero grade subalgebra has changed, we need to make a new ansatz for $K_+$:

$$K_+ = \frac{1}{2} p^2 + \frac{1}{x^2} (c_1 T^2 + c_2 S^2 + c_3 A^2 + c_4)$$

where $c_1, \ldots, c_4$ are some constant parameters. Once again, using the closure of the algebra, we determine these four unknown constants in terms of $\lambda$ and obtain:

$$K_+ = \frac{1}{2} p^2 + \frac{1}{4x^2} \left( 8\lambda T^2 + \frac{8}{3} (\lambda - 1) A^2 + \frac{3}{2} + 8\lambda (\lambda - 1) S^2 + 2\lambda (\lambda - 1) \right)$$

We can now solve for the $+1$ grade generators by using the grading property as follows:

$$\tilde{Q} = i [Q, K_+] \quad \tilde{Q}^\dagger = (\tilde{Q})^\dagger = i [Q^\dagger, K_+]$$
$$\tilde{S} = i [S, K_+] \quad \tilde{S}^\dagger = (\tilde{S})^\dagger = i [S^\dagger, K_+]$$

This gives the following result for the $+1$ grade generators:

$$\tilde{Q} = -p_\alpha + \frac{2i}{x} \left[ \lambda \left\{ \mathcal{T}_0 + \frac{3}{4} \alpha + \mathcal{T}_- \beta \right\} - \frac{\lambda - 1}{3} \left\{ \left( A_0 - \frac{3}{4} \right) \alpha - 2A_- \beta \right\} \right]$$
$$\tilde{Q}^\dagger = -p_\alpha^\dagger - \frac{2i}{x} \left[ \lambda \left\{ \mathcal{T}_0 - \frac{3}{4} \alpha^\dagger + \mathcal{T}_+ \beta^\dagger \right\} - \frac{\lambda - 1}{3} \left\{ \left( A_0 + \frac{3}{4} \right) \alpha^\dagger - 2A_+ \beta \right\} \right]$$
\[
\tilde{S} = -p\beta - \frac{2i}{x} \left[ \lambda \left\{ \left( T_0 - \frac{3}{4} \right) \beta - T_+ \alpha \right\} - \frac{\lambda - 1}{3} \left\{ \left( A_0 + \frac{3}{4} \right) \beta - A_+ \alpha \right\} \right]
\]
\[
\tilde{S}^\dagger = -p\beta^\dagger + \frac{2i}{x} \left[ \lambda \left\{ \left( T_0 + \frac{3}{4} \right) \beta - T_- \alpha \right\} - \frac{\lambda - 1}{3} \left\{ \left( A_0 - \frac{3}{4} \right) \beta^\dagger - A_- \alpha \right\} \right]
\]

(2.3.31)

The complete commutation relations of the algebra are given as follows:

\[
\{ Q, \tilde{Q} \} = -\Delta - 2i\lambda T_0 + i(\lambda + 1)A_0
\]
\[
\{ Q^\dagger, \tilde{Q} \} = -\Delta + 2i\lambda T_0 - i(\lambda + 1)A_0
\]
\[
\{ S, \tilde{S} \} = -\Delta + 2i\lambda T_0 + i(\lambda + 1)A_0
\]
\[
\{ S^\dagger, \tilde{S} \} = -\Delta - 2i\lambda T_0 - i(\lambda + 1)A_0
\]

(2.3.32)

\[
\{ Q, \tilde{S} \} = +2i(\lambda + 1)A_-
\]
\[
\{ Q, \tilde{S}^\dagger \} = -2i\lambda T_- \quad (2.3.33)
\]
\[
\{ Q^\dagger, \tilde{S} \} = -2i(\lambda + 1)A_+
\]
\[
\{ Q^\dagger, \tilde{S}^\dagger \} = +2i\lambda T_+ \quad (2.3.34)
\]
\[
\{ S, \tilde{Q} \} = -2i(\lambda + 1)A_-
\]
\[
\{ S, \tilde{Q}^\dagger \} = -2i\lambda T_+ \quad (2.3.35)
\]
\[
\{ S^\dagger, \tilde{Q} \} = +2i(\lambda + 1)A_+
\]
\[
\{ S^\dagger, \tilde{Q}^\dagger \} = +2i\lambda T_- \quad (2.3.36)
\]
\[
\{ Q, \tilde{Q} \} = \{ Q^\dagger, \tilde{Q}^\dagger \} = \{ S, \tilde{S} \} = \{ S^\dagger, \tilde{S}^\dagger \} = 0 \quad (2.3.37)
\]
\[
[\tilde{Q}, K_-] = iQ \quad [\tilde{Q}^\dagger, K_-] = iQ^\dagger
\]
\[
[\tilde{S}, K_-] = iS \quad [\tilde{S}^\dagger, K_-] = iS^\dagger \quad (2.3.38)
\]

The quadratic Casimir of $\text{su}(1,1)_K$ generated by $K_{\pm 2}$ and $\Delta$ is

\[
C_2 [\text{su}(1,1)_K] = \kappa^2 = \frac{1}{2} (K_+ K_- + K_- K_+) - \frac{1}{4} \Delta^2
\]
\[
= \frac{\lambda T^2 + \frac{\lambda - 1}{3} A^2 + \lambda (\lambda - 1) S^2 + \frac{\lambda (\lambda - 1)}{4}}{4} \quad (2.3.39)
\]

There exists a one parameter family of quadratic Casimir elements $C_2(\mu)$ that commute
with all the generators of $D(2, 1; \lambda)$.

$$C_2(\mu) = \frac{\mu}{4} \lambda^2 - \frac{\lambda}{4}(\mu - 8)T^2 - \frac{1}{12}(16 + 8\lambda + \mu(\lambda - 1))A^2 + \frac{i}{4}f(Q, S)$$

$$= \frac{\lambda}{4}(8 + \mu(\lambda - 1)) \left(S^2 + \frac{1}{4}\right)$$

(2.3.40)

where

$$f(Q, S) = [Q, \tilde{Q}^\dagger] + [Q^\dagger, \tilde{Q}] + [S, \tilde{S}^\dagger] + [S^\dagger, \tilde{S}]$$

(2.3.41)

is the contribution from the odd generators.

The quadratic Casimir $C_2$ depends only on the Casimir $S^2$ of $SU(2)_S$ and hence the eigenvalues $s(s + 1)$ i.e. the spin $s$ of $SU(2)_S$ can be used to label the deformed unitary supermultiplets of $D(2, 1; \lambda)$.

## 2.4 $SU(2)$ deformed minimal unitary representations as positive energy unitary supermultiplets of $D(2, 1; \lambda)$

### 2.4.1 Compact 3-grading

As shown in previous section, the Lie superalgebra $D(2, 1; \lambda)$ admits a 5-graded decomposition where the grading is given by the generator $\Delta$:

$$D(2, 1; \lambda) = \mathcal{g}^{(-2)} \oplus \mathcal{g}^{(-1)} \oplus [su(2)_T \oplus su(2)_A \oplus so(1, 1)_\Delta] \oplus \mathcal{g}^{(+1)} \oplus \mathcal{g}^{(+2)}$$

$$= K_- \oplus [Q, Q^\dagger, S, S^\dagger] \oplus [A_{\pm,0}, T_{\pm,0}, \Delta] \oplus [\tilde{Q}, \tilde{Q}^\dagger, \tilde{S}, \tilde{S}^\dagger] \oplus K_+$$

In addition to the 5-grading, $D(2, 1; \lambda)$ also admits a 3-graded decomposition with respect to its compact subsuperalgebra $osp(2|2) \oplus u(1) = su(2|1) \oplus u(1)$, which we shall refer to as compact 3-grading:

$$D(2, 1; \lambda) = \mathcal{c}^- \oplus \mathcal{c}^0 \oplus \mathcal{c}^+$$

(2.4.1)

$$D(2, 1; \lambda) = (A_-, B_-, \Omega_-, \mathcal{G}_-) \oplus (T_{\pm,0}, H, \Omega_0, \mathcal{G}_0, \mathcal{G}_0) \oplus (A_+, B_+, \Omega_+, \mathcal{G}_+)$$

(2.4.2)

The generators belonging to grade -1 subspace $\mathcal{c}^-$ are as follows:

$$A_- = \beta \alpha$$

(2.4.3)

$$B_- = \frac{i}{2} [\Delta + i(K_+ - K_-)]$$

(2.4.4)
\[
\Omega_- = \frac{1}{2}(Q - i\bar{Q})
\]
\[
\Sigma_- = \frac{1}{2}(S - i\bar{S})
\]
where
\[
L^2 = \lambda T^2 + \frac{1}{3}(\lambda - 1)A^2 + \lambda(\lambda - 1)S^2 + \frac{1}{4}\lambda(\lambda - 1)
\]

The grade +1 generators in \( \mathfrak{c}^+ \) are obtained by Hermitian conjugation of grade −1 generators:

\[
A_+ = \alpha^\dagger \beta^\dagger
\]
\[
B_+ = -\frac{i}{2}[\Delta - i(K_+ - K_-)]
\]
\[
\Omega_+ = (\Omega_-)^\dagger = \frac{1}{2}(Q^\dagger + i\bar{Q}^\dagger)
\]
\[
\Sigma_+ = (\Sigma_-)^\dagger = \frac{1}{2}(S^\dagger + i\bar{S}^\dagger)
\]

The grade 0 fermionic generators in \( \mathfrak{c}^0 \) are given by

\[
\Omega_0 = \frac{1}{2}(Q + i\bar{Q})
\]
$$\begin{align*}
\mathcal{Q}_+ &= \frac{1}{2} \left( \frac{2\lambda+1}{3} \left( G_0 - \frac{3}{4} \right) + \lambda S_0 \right) \alpha + \left\{ \frac{2\lambda+1}{3} G_+ + \lambda S_+ \right\} \beta \\
\mathcal{Q}_0 &\equiv \frac{1}{2} (S + i \tilde{S}) \\
\mathcal{Q}_- &= \frac{1}{2} \left( \frac{2\lambda+1}{3} \left( G_0 - \frac{3}{4} \right) + \lambda S_0 \right) \beta - \left\{ \frac{2\lambda+1}{3} G_+ + \lambda S_+ \right\} \alpha \\
\mathcal{Q}_0^\dagger &= \frac{1}{2} (Q^\dagger - i \tilde{Q}^\dagger) \\
\mathcal{Q}_-^\dagger &= \frac{1}{2} \left( \frac{2\lambda+1}{3} \left( G_0 + \frac{3}{4} \right) + \lambda S_0 \right) \beta^\dagger - \left\{ \frac{2\lambda+1}{3} G_- + \lambda S_- \right\} \alpha^\dagger \\
\mathcal{Q}_0^\dagger &= \frac{1}{2} (S^\dagger - i \tilde{S}^\dagger) \\
\mathcal{Q}_+^\dagger &= \frac{1}{2} \left( \frac{2\lambda+1}{3} \left( G_0 + \frac{3}{4} \right) + \lambda S_0 \right) \beta^\dagger - \left\{ \frac{2\lambda+1}{3} G_- + \lambda S_- \right\} \alpha^\dagger
\end{align*}$$

The grade zero odd generators together with the \textit{SU}(2)\textsubscript{T} generators \(T_+, T_-, T_0\) and \textit{U}(1) generator

\[\mathcal{J} = (\lambda + 1) A_0 + \frac{1}{2} (K_+ + K_-)\]

generate the sub-supergroup \textit{SU}(2\mid 1). They satisfy the anticommutation relations

\[
\begin{align*}
\{ \Omega_0, \Omega_0^\dagger \} &= -\lambda T_0 + \mathcal{J} \\
\{ \Omega_0, \Omega_0 \} &= +\lambda T_0 + \mathcal{J} \\
\{ T_0, \Omega_0 \} &= -\frac{1}{2} \Omega_0 \\
\{ T_0, \Omega_0^\dagger \} &= +\frac{1}{2} \Omega_0^\dagger \\
\{ \mathcal{J}, \Omega_0 \} &= -\frac{i}{2} \Omega_0 \\
\{ \mathcal{J}, \Omega_0^\dagger \} &= +\frac{i}{2} \Omega_0^\dagger \\
\{ T_+, \Omega_0 \} &= -\Omega_0 \\
\{ T_+, \Omega_0^\dagger \} &= +\Omega_0^\dagger \\
\{ T_-, \Omega_0 \} &= -\Omega_0^\dagger \\
\{ T_-, \Omega_0^\dagger \} &= +\Omega_0
\end{align*}
\]
The anticommutation relations of grade zero fermionic generators with grade $\pm 1$ generators in the compact 3-grading are as follows
\[
\{ \Omega_0, \Omega_+ \} = 2B_+ \quad \{ \mathcal{S}_0, \Omega_+ \} = 0
\]
\[
\{ \Omega_0, \mathcal{S}_+ \} = 0 \quad \{ \mathcal{S}_0, \mathcal{S}_+ \} = 2(\lambda + 1)A_+
\]
\[
\{ \Omega_0, \mathcal{S}_- \} = -2(\lambda + 1)A_+ \quad \{ \mathcal{S}_0, \mathcal{S}_- \} = 2(\lambda + 1)A_-
\]
\[
\{ \Omega_0, \Omega_- \} = 0 \quad \{ \mathcal{S}_0, \mathcal{S}_- \} = 0
\]
\[
\{ \Omega_0, \mathcal{S}_- \} = -2(\lambda + 1)A_- \quad \{ \mathcal{S}_0, \mathcal{S}_- \} = 0
\]
\[
\{ \Omega_0, \mathcal{S}_- \} \equiv 2B_- \quad \{ \mathcal{S}_0, \mathcal{S}_- \} \equiv 2B_-
\]

The generator $\mathcal{H}$ that determines the compact three grading is given by
\[
\mathcal{H} = \frac{1}{2} (K_+ + K_-) + A_0
\]
\[
= B_0 + \frac{1}{2} (N_\alpha + N_\beta - 1)
\]

where
\[
B_0 = \frac{1}{4} (p^2 + x^2) + \frac{1}{x^2} \left( L^2 + \frac{3}{16} \right)
\]
\[
= \frac{1}{4} (p^2 + x^2) + \frac{1}{x^2} \left( \lambda T^2 + \frac{1}{3} (\lambda - 1)A^2 + \lambda (\lambda - 1)S^2 + \frac{1}{4} \lambda (\lambda - 1) + \frac{3}{16} \right)
\]

2.4.2 Unitary supermultiplets of $D(2, 1; \lambda)$

The generators $B_-$ and $B_+$ defined above close into $B_0$ under commutation and generate the distinguished $\mathfrak{su}(1, 1)_K$ subalgebra
\[
[B_-, B_+] = 2B_0 \quad \quad [B_0, B_+] = B_+ \quad \quad [B_0, B_-] = -B_-
\]

The generator $B_0$ can be written as follows:
\[
H_{\text{Conf}} = 2B_0 = \frac{1}{2} \left( x^2 + p^2 \right) + \frac{g^2}{x^2}
\]

with $g^2 = (2L^2 + \frac{3}{4})$. This can be interpreted as $H_{\text{Conf}}$, the Hamiltonian for conformal quantum mechanics [119] or of a singular oscillator [120]. The role of coupling constant for the inverse square potential is played by $g^2$.

Since we are looking at $SU(1, 1)_K$ which is a non-compact group, the unitary representations are infinite dimensional. We are interested in the physically relevant positive
energy representations which are bounded from below. These are also known as lowest weight representations and are uniquely determined by a state $|\psi_0^\omega\rangle$ with the lowest eigenvalue of $B_0$ that is annihilated by $B_-:

\begin{equation}
B_-|\psi_0^\omega\rangle = 0
\tag{2.4.23}
\end{equation}

In the coordinate $(x)$ representation its wave function is given by

\begin{equation}
\psi_0^\omega(x) = C_0 x^\omega e^{-x^2/2}
\tag{2.4.24}
\end{equation}

where $C_0$ is the normalization constant, $\omega$ is given by

\begin{equation}
\omega = \frac{1}{2} + \left( \frac{1}{4} + 2\hat{g}^2 \right)^{1/2}
\tag{2.4.25}
\end{equation}

and $\hat{g}^2$ is the eigenvalue of $(2L^2 + \frac{3}{8})$

\begin{equation}
(2L^2 + \frac{3}{8})|\psi_0^\omega\rangle = \hat{g}^2 |\psi_0^\omega\rangle
\tag{2.4.26}
\end{equation}

We shall denote the functions obtained by the repeated action of differential operators $B_+$ on $\psi_0^\omega(x)$ in the coordinate representation as $\psi_n^\omega(x)$ and the corresponding states in the Hilbert space as $|\psi_n^\omega\rangle$:

\begin{equation}
\psi_n^\omega(x) = c_n (B_+)^n \psi_0^\omega(x)
\tag{2.4.27}
\end{equation}

where the normalization constant is given as

\begin{equation}
c_n = \frac{(-1)^n}{2^n} \frac{\sqrt{n!} \Gamma(n + \omega + 1/2)}{\sqrt{\Gamma(n + 1/2) \Gamma(n + \omega + 1/2)}}
\tag{2.4.28}
\end{equation}

The wave functions $\psi_n^\omega(x)$ can be written as

\begin{equation}
\psi_n^\omega(x) = \sqrt{\frac{2(n!)}{\Gamma(n + \omega + 1/2)}} L_n^{(\omega-1/2)}(x^2) x^\omega e^{-x^2/2}
\tag{2.4.29}
\end{equation}

where $L_n^{(\omega-1/2)}(x^2)$ is the generalized Laguerre polynomial.

We shall use $|\Omega\rangle$ to uniquely label the irreducible lowest weight (positive energy) representations of $D(2,1;\lambda)$. Being the lowest weight states, $|\Omega\rangle$ are all annihilated by the generators $B_-, A_-, \Omega_-$ and $\mathcal{G}_-$ in $\mathfrak{c}^-$ and transform irreducibly under $\mathfrak{c}^0$ subalgebra $OSp(2/2) \times U(1)$.

As explained earlier, the positive energy refers to the fact that the spectrum of $\mathcal{H}$ is bounded from below and continuing the same terminology, we shall refer to $\mathcal{H}$ as the
Hamiltonian and its eigenvalues as total energy. Each state in the set $|\Omega\rangle$ is a lowest (conformal) energy state of a positive energy irrep of $SU(1,1)_K$, since they are all annihilated by $B_-$. The conditions

\begin{align*}
B_-|\Omega\rangle &= 0 \\
A_-|\Omega\rangle &= 0 \\
\Omega_-|\Omega\rangle &= 0 \\
\mathcal{G}_-|\Omega\rangle &= 0 \quad (2.4.30)
\end{align*}

imply that the states $|\Omega\rangle$ must be linear combinations of the tensor product states of the form

\begin{equation}
|F\rangle \times |B\rangle \times |\psi^\mu_0\rangle \quad (2.4.31)
\end{equation}

where the state $|F\rangle$ in (2.4.31) is either the fermionic Fock vacuum

\begin{equation}
|0\rangle_F = |m_t = 0; m_a = -1/2\rangle_F = |0, \downarrow\rangle \quad (2.4.32)
\end{equation}

or one of the following $SU(2)_T$ doublet of states:

\begin{align*}
\alpha^\dagger |0\rangle_F &\equiv |m_t = 1/2; m_a = 0\rangle_F = |\uparrow, 0\rangle, \quad (2.4.33) \\
\beta^\dagger |0\rangle_F &\equiv |m_t = -1/2; m_a = 0\rangle_F = |\downarrow, 0\rangle \quad (2.4.34)
\end{align*}

and the state $|B\rangle$ in (2.4.31) is any one of the states

\begin{equation}
a^{m_1} \cdots a^{m_k} b^{m_{k+1}} \cdots b^{m_2} |0\rangle_B \quad (2.4.35)
\end{equation}

where $|0\rangle_B$ is the bosonic Fock vacuum annihilated by the bosonic oscillators $a_m$ and $b_m$ ($m = 1, 2, \cdots, P$). For fixed $n$ the states of the form (2.4.35) transform in the spin $s$ representation of $SU(2)_S$. They also form representations of $SO^*(2P)$ generated by the bilinears

\begin{align*}
U_n^m &= a^m a_n + b^m b_n, \\
U_{mn} &= (a_n b_m - a_m b_n), \\
U^{mn} &= (a^m b^n - a^n b^m) \quad (2.4.36)
\end{align*}

and which commutes with $D(2,1;\lambda)$. Therefore as far as the $SU(2)$ deformations of the minrep of $D(2,1;\lambda)$ are concerned we can restrict our analysis to $P = 1$. Then for $P = 1$ we simply have

\begin{align*}
S_+ &= a^\dagger b \\
S_- &= b^\dagger a
\end{align*}
and
\[ S_0 = \frac{1}{2}(a^\dagger a - b^\dagger b) \]

Then the states \(|B\rangle\) belong to the set
\[ |s, m_s\rangle_B \equiv (a^\dagger)^k (b^\dagger)^{2s-k}|0\rangle_B \quad (2.4.37) \]

where \(m_s = k - s\) (\(k = 0, 1, \ldots, 2s\)) and transform in spin \(s\) representation of \(SU(2)_S\):
\[ S^2|s, m_s\rangle_B = s(s+1)|s, m_s\rangle_B \quad (2.4.38) \]
\[ S_0|s, m_s\rangle_B = m_s|s, m_s\rangle_B \quad (2.4.39) \]

The action of raising operator \(S_+ = a^\dagger b\) and lowering operator \(S_- = b^\dagger a\) on this state is then given as
\[ S_+ |s, m_s = k-s\rangle_B = \sqrt{(k+1)(2s-k)} |s, k-s+1\rangle_B \quad (2.4.40) \]
\[ S_- |s, m_s = k-s\rangle_B = \sqrt{k(2s-k+1)} |s, k-s-1\rangle_B \quad (2.4.41) \]

The eigenvalues \((1/4 + 2\hat{g}^2)\) of \((4L^2 + 1)\) on the above states determine the values of \(\omega\) labeling the eigenstates \(|\psi_\omega^0\rangle\) of \(B_0\) annihilated by \(B_-\):
\[ (4L^2 + 1)|m_t = 0; m_a = -1/2\rangle_F \times |s, m_s\rangle_B = \lambda^2(2s+1)^2|0; -1/2\rangle_F \times |s, m_s\rangle_B \quad (2.4.42) \]
\[ (4L^2 + 1)|m_t = \pm 1/2; m_a = 0\rangle_F \times |s, m_s\rangle_B = |\lambda(2s+1) + 1|^2|\pm 1/2; 0\rangle_F \times |s, m_s\rangle_B \quad (2.4.43) \]

The \(SU(2)\) subalgebra of \(SU(1|2)\) is the diagonal subalgebra \(SU(2)_T\) of \(SU(2)_S\) and \(SU(2)_T\). Therefore we shall work in a basis where \(T^2\) and \(T_0\) are diagonalized and denote the simultaneous eigenstates of \(B_0\), \(T^2\), \(T_0\), \(A^2\) and \(A_0\) as
\[ |\omega; t, m_t; a, m_a\rangle \quad (2.4.44) \]

where \(\omega\) is the eigenvalue of \(B_0\).

The set of states \(|\Omega\rangle\) must be linear combinations of the tensor product states of the form \(|F\rangle \times |B\rangle \times |\psi_\omega^0\rangle\) where the state \(|F\rangle\) could be either \(|0\rangle_F = |0, \downarrow\rangle_F\) or one of the

\footnote{We introduce this notation for the states because the tensor product states of the form \(|F\rangle \times |B\rangle \times |\psi_\omega^0\rangle\) are not always definite eigenstates of \(T^2\) and \(T_0\) but it is easier to understand the structure of supermultiplets in terms of these tensor product states. Thus we will use both notations for states.}
$SU(2)_T$ doublet of states $|\uparrow, 0\rangle_F$ or $|\downarrow, 0\rangle_F$. We will now study the unitary representations for these two cases.

### 2.4.2.1 $|F\rangle = |0\rangle_F$

For states with $t = 0$, we can use equation (2.4.25) to write

$$\omega = \frac{1}{2} \pm \lambda (2s + 1)$$

(2.4.45)

where the sign of the square root is determined by the sign of $\lambda$ and the range of $\lambda$ is determined by the square integrability of the states and the positivity of $\hat{g}^2$. This leads to the following restriction on $\lambda$

$$|\lambda| > \frac{1}{2(2s + 1)}$$

(2.4.46)

Let us first consider the case $s = 0$ and with the positive square root taken in the above equations. Then the lowest energy state $|0, \downarrow\rangle_F \times |0\rangle_B \times \left|\psi_{\lambda+1/2}^0\right\rangle$, annihilated by all the generators in $C^{\perp_1}$, is a singlet of the grade zero super algebra $SU(1\vert 2)$ since

$$Q_0 |0, \downarrow\rangle_F \times |0\rangle_B \times \left|\psi_{\lambda+1/2}^0\right\rangle = 0$$

$$S_0 |0, \downarrow\rangle_F \times |0\rangle_B \times \left|\psi_{\lambda+1/2}^0\right\rangle = 0$$

$$Q^+_0 |0, \downarrow\rangle_F \times |0\rangle_B \times \left|\psi_{\lambda+1/2}^0\right\rangle = 0$$

$$S^+_0 |0, \downarrow\rangle_F \times |0\rangle_B \times \left|\psi_{\lambda+1/2}^0\right\rangle = 0$$

(2.4.47)

States generated by action of grade +1 generators $C^+$ on this lowest weight state $\left|\psi_{\lambda+1/2}^0\right\rangle$

$$B^+ |0, \downarrow\rangle_F \times |0\rangle_B \times \left|\psi_{\lambda+1/2}^0\right\rangle = |0, \downarrow\rangle_F \times |0\rangle_B \times \left|\psi_{\lambda+1/2}^0\right\rangle$$

$$A^+ |0, \downarrow\rangle_F \times |0\rangle_B \times \left|\psi_{\lambda+1/2}^0\right\rangle = |0, \uparrow\rangle_F \times |0\rangle_B \times \left|\psi_{\lambda+1/2}^0\right\rangle$$

$$Q_+ |0, \downarrow\rangle_F \times |0\rangle_B \times \left|\psi_{\lambda+1/2}^0\right\rangle = |\uparrow, 0\rangle_F \times |0\rangle_B \times \left|\psi_{\lambda+3/2}^0\right\rangle$$

$$S_+ |0, \downarrow\rangle_F \times |0\rangle_B \times \left|\psi_{\lambda+1/2}^0\right\rangle = |\downarrow, 0\rangle_F \times |0\rangle_B \times \left|\psi_{\lambda+3/2}^0\right\rangle$$

(2.4.48)

form a supermultiplet transforming in the representation with super tableau \[ \begin{array}{c}
\end{array} \] of $SU(2\vert 1)$. The states $\Omega_+ |0, \downarrow\rangle_F \times |0\rangle_B \times \left|\psi_{\lambda+1/2}^0\right\rangle$ and $\Omega_+ |0, \downarrow\rangle_F \times |0\rangle_B \times \left|\psi_{\lambda+1/2}^0\right\rangle$ are both lowest weight vectors of $SU(1,1)_K$ transforming as a doublet of $SU(2)_T$. The state
$A^+ |0, \downarrow \rangle_F \times |0 \rangle_B \times |\psi_0^{\lambda+1/2} \rangle$ is a lowest weight vector of $SU(1,1)_K$ and together with
$|0, \downarrow \rangle_F \times |0 \rangle_B \times |\psi_0^{\lambda+1/2} \rangle$ form a doublet of $SU(2)_A$. The commutator of two susy generators
in $\mathcal{C}^+$ satisfies

$$[\Omega_+, \mathcal{G}_+] |0, \downarrow \rangle_F \times |0 \rangle_B \times |\psi_0^{\lambda+1/2} \rangle \propto \alpha^+ \beta^+ B^+ (0)_F \times |0 \rangle_B \times |\psi_0^{\lambda+1/2} \rangle$$

Hence one does not generate any new lowest weight vectors of $SU(1,1)_K$ by further actions of grade +1 supersymmetry generators.

The conformal group in $d$ dimensions is $SO(d,2)$ and positive energy unitary or lowest weight representations correspond to conformal fields in $d$ (Minkowski) dimensions with the conformal dimension given by the eigenvalue of the $SO(2)$ generator. In the case of $D(2,1;\lambda)$, we are working in one dimension and the positive energy unitary representations are instead identified with conformal wave functions. We shall denote the conformal wave function associated with a positive energy unitary representation of $SO(2,1)$ with lowest weight vector $|\psi_0^{(\omega)} \rangle$ as $\Psi^{\omega}(x)$. The conformal wave functions transforming in the $(t,a)$ representation of $SU(2)_T \times SU(2)_A$ will then be denoted as

$$\Psi^{\omega}_{(t,a)}(x)$$

Thus the unitary supermultiplet of $D(2,1;\lambda)$ with the lowest weight vector $|0 \rangle_F \times |0 \rangle_B \times |\psi_0^{\lambda+1/2} \rangle$ decomposes as:

$$\Psi^{(\lambda+1)/2}_{(0,1/2)} \oplus \Psi^{(\lambda+2)/2}_{(1/2,0)}$$

This is simply the minimal unitary supermultiplet of $D(2,1;\lambda)$ and for $\lambda = -1/2$ coincides with the singleton supermultiplet of $OSp(4|2,\mathbb{R}) = D(2,1; -1/2)$.

Next we consider the representations for the case $s \neq 0$ with $\lambda > 0$. In this case the lowest energy states

$$|0, \downarrow \rangle_F \times |s, m_s = (k - s) \rangle_B \times |\psi_0^{\omega} \rangle,$$

where $\omega = 1/2 + \lambda(2s + 1)$, and $k = 0, \ldots, 2s$, are not annihilated by all supersymmetry generators of $SU(2|1)$:

$$\Omega_0 |0, \downarrow \rangle_F \times |s, k - s \rangle_B \times |\psi_0^{\omega} \rangle = 0$$
$$\mathcal{G}_0 |0, \downarrow \rangle_F \times |s, k - s \rangle_B \times |\psi_0^{\omega} \rangle = 0$$
$$\Omega_0^+ |0, \downarrow \rangle_F \times |s, k - s \rangle_B \times |\psi_0^{\omega} \rangle = \lambda (2s - k) |\uparrow, 0 \rangle_F \times |s, k - s \rangle_B \times |\psi_0^{\omega-1} \rangle$$
$$- \lambda \sqrt{(k+1)(2s-k)} |\downarrow, 0 \rangle_F \times |s, k - s + 1 \rangle_B \times |\psi_0^{\omega-1} \rangle$$
$$\mathcal{G}_0^+ |0, \downarrow \rangle_F \times |s, k - s \rangle_B \times |\psi_0^{\omega} \rangle = \lambda k |\downarrow, 0 \rangle_F \times |s, k - s \rangle_B \times |\psi_0^{\omega-1} \rangle$$
which decomposes under the even subgroup $SU(2)\tau$ acting on the states with spin $t = s$ one would expect to obtain states with both $t = s \pm 1/2$. However setting $k = 2s$ in the above formulas we find

$$\begin{align*}
\Omega_0^\dagger |0, \downarrow\rangle_F \times |s, s\rangle_B \times |\psi_0^s\rangle &= 0 \\
\Sigma_0^\dagger |0, \downarrow\rangle_F \times |s, s\rangle_B \times |\psi_0^s\rangle &= 2s |0, \downarrow\rangle_F \times |s, s\rangle_B \times |\psi_0^s\rangle \\
&\quad - \frac{\sqrt{2s}}{\lambda} |\uparrow, 0\rangle_F \times |s, s - 1\rangle_B \times |\psi_0^{s-1}\rangle
\end{align*}$$

(2.4.52)

which implies that we only get states with $t = s - 1/2$. Therefore the lowest energy supermultiplets of $SU(2)|1$ that uniquely determine the deformed minimal unitary supermultiplets of $D(2,1;\lambda)$ transform in the representation with the super tableau

\[
\begin{array}{c|c|c|c|c}
\hline
\hline
\hline
\hline
\end{array}
\]

\begin{array}{c}
\hline
\hline
\hline
\hline
\end{array}
\]

which decomposes under the even subgroup $SU(2)\tau \times U(1)\mathcal{J}$ as

$$\begin{align*}
\begin{array}{c|c|c|c|c}
\hline
\hline
\hline
\hline
\end{array} &= ( \begin{array}{c|c|c|c|c}
\hline
\hline
\hline
\hline
\end{array} , 0) \oplus ( \begin{array}{c|c|c|c|c}
\hline
\hline
\hline
\hline
\end{array} , \frac{\lambda}{2})
\end{align*}$$

(2.4.53)

By acting with grade $+1$ generators of the compact 3-grading on these states with $t = s$ and $t = s - 1/2$ to obtain states with $t = s \pm 1/2$ and $t = s$ :

$$\begin{align*}
B^+ |0, \downarrow\rangle_F \times |s, k - s\rangle_B \times |\psi_0^s\rangle &= |0, \downarrow\rangle_F \times |s, k - s\rangle_B \times |\psi_0^s\rangle \\
A^+ |0, \downarrow\rangle_F \times |s, k - s\rangle_B \times |\psi_0^s\rangle &= |0, \uparrow\rangle_F \times |s, k - s\rangle_B \times |\psi_0^s\rangle \\
\Omega_+ |0, \downarrow\rangle_F \times |s, k - s\rangle_B \times |\psi_0^s\rangle &= |\uparrow, 0\rangle_F \times |s, k - s\rangle_B \times |\psi_0^{s+1}\rangle \\
&\quad - \lambda (2s - k) |\uparrow, 0\rangle_F \times |s, k - s\rangle_B \times |\psi_0^{s-1}\rangle \\
&\quad + \lambda \sqrt{(k + 1)(2s - k)} |\downarrow, 0\rangle_F \times |s, k - s + 1\rangle_B \times |\psi_0^{s-1}\rangle \\
\Sigma_+ |0, \downarrow\rangle_F \times |s, k - s\rangle_B \times |\psi_0^s\rangle &= |\downarrow, 0\rangle_F \times |s, k - s\rangle_B \times |\psi_0^{s+1}\rangle \\
&\quad - \lambda (2s - k) |\downarrow, 0\rangle_F \times |s, k - s\rangle_B \times |\psi_0^{s-1}\rangle \\
&\quad + \lambda \sqrt{k(2s - k + 1)} |\uparrow, 0\rangle_F \times |s, k - s - 1\rangle_B \times |\psi_0^{s-1}\rangle
\end{align*}$$

(2.4.54)

The commutator of two supersymmetry generators does not generate any new lowest
weight vector of $SU(1,1)_K$:
\[
[\Omega_+, \Omega_-] |0, \downarrow\rangle_F \times |s, k-s\rangle_B \times |\psi^\alpha_0\rangle \propto |0, \uparrow\rangle_F \times |s, k-s\rangle_B \times |\psi^\alpha_1\rangle = \alpha^\dagger \beta^\dagger |0, \downarrow\rangle_F \times |s, k-s\rangle_B \times |\psi^\omega_0\rangle
\]
(2.4.55)

Thus the complete supermultiplet is simply
\[
\Psi_{(s-1/2,0)} + \Psi^{p+1/2}_{(s,1/2)} + \Psi^{p+1}_{(s+1/2,0)}
\]
(2.4.56)

where $p = \lambda(2s + 1)/2$. We have summarized the deformed supermultiplets for lowest weight states with $t = s$ and $\lambda > 1/(4s + 2)$ in Table 2.1.

**Table 2.1.** Decomposition of $SU(2)$ deformed minimal unitary lowest energy supermultiplets of $D(2,1;\lambda)$ with respect to $SU(2)_T \times SU(2)_A \times SU(1,1)_K$. The conformal wavefunctions transforming in the $(t, a)$ representation of $SU(2)_T \times SU(2)_A$ with conformal energy $\omega$ are denoted as $\Psi^\omega_{(t,a)}$. The first column shows the super tableaux of the lowest energy $SU(2|1)$ supermultiplet, the second column gives the eigenvalue of the $U(1)$ generator $H$. The allowed range of $\lambda$ in this case is $\lambda > 1/(4s + 2)$.

| $SU(2|1)$ l.w.v. | $H$ | $SU(1,1)_K \times SU(2)_T \times SU(2)_A$ |
|------------------|-----|---------------------------------|
| 1                | $\lambda/2$ | $\Psi^{(\lambda+1)/2}_{(0,1/2)} \oplus \Psi^{(\lambda+2)/2}_{(1/2,0)}$ |
|                  | $\lambda$ | $\Psi^\lambda_{(0,0)} \oplus \Psi^{\lambda+1/2}_{(1/2,1/2)} \oplus \Psi^{\lambda+1}_{(1,0)}$ |
|                  | 3$\lambda/2$ | $\Psi^{3\lambda/2}_{(1/2,0)} \oplus \Psi^{(3\lambda+1)/2}_{(1,1/2)} \oplus \Psi^{(3\lambda+2)/2}_{(3/2,0)}$ |
|                  | :          | :                               |
|                  | :          | :                               |
|                  | (2s + 1)$\lambda/2$ | $\Psi^p_{(s-1/2,0)} \oplus \Psi^{p+1/2}_{(s,1/2)} \oplus \Psi^{p+1}_{(s+1/2,0)}$ |
| 2s               |             | $p = (2s + 1)\lambda/2$ |

So far we have considered the representations for $\lambda > 0$ when the lowest weight state
has \( t = s \). Now we take a look at the case when \( \lambda < 0 \) and \( \omega \) is then given as

\[
\omega = \frac{1}{2} - \lambda(2s + 1) \tag{2.4.57}
\]

The action of grade 0 supersymmetry generators on these states produces states with \( t = s + 1/2 \). This is different from the case with \( \lambda > 0 \) where we obtained states with \( t = s - 1/2 \) by the action of grade 0 supersymmetry generators. Thus the the lowest energy supermultiplets of \( SU(2|1) \) that uniquely determine the deformed minimal unitary supermultiplets of \( D(2, 1; \lambda) \) transform in the representation with the super tableau

\[
\begin{array}{cccccc}
\text{2s+1} \\
\text{\ldots} & \text{\ldots} & \text{\ldots} & \text{\ldots} & \text{\ldots} & \text{\ldots} \\
\text{2s} & & & & & \\
\end{array}
\]

which decomposes under the even subgroup \( SU(2)_T \times U(1)_J \) as

\[
\begin{array}{cccccc}
\text{2s+1} \\
\text{\ldots} & \text{\ldots} & \text{\ldots} & \text{\ldots} & \text{\ldots} & \text{\ldots} \\
\text{2s} & & & & & \\
\end{array} = \left( \begin{array}{c} \text{2s+1} \\
\text{\ldots} & \text{\ldots} & \text{\ldots} & \text{\ldots} & \text{\ldots} & \text{\ldots} \\
\text{2s} & & & & & \\
\end{array} \right) \oplus \left( \begin{array}{c} \text{\ldots} & \text{\ldots} & \text{\ldots} & \text{\ldots} & \text{\ldots} & \text{\ldots} \\
\text{2s} & & & & & \\
\end{array} \right) \frac{\lambda}{2} \tag{2.4.58}
\]

By acting with the grade +1 supersymmetry generators on these states, we complete the \( D(2, 1; \lambda) \) supermultiplet given as:

\[
\Psi^p_{t+1/2} \oplus \Psi^p_{t+1/2} \oplus \Psi^p_{t+1/2} \tag{2.4.59}
\]

where \( p = |\lambda|(2s + 1)/2 \). We have summarized the deformed supermultiplets for lowest weight states with \( t = s \) and \( \lambda < -1/(4s + 2) \) in Table 2.2.

2.4.2.2 \( |F\rangle = \left( \begin{array}{c} |\uparrow, 0\rangle_F \\
|\downarrow, 0\rangle_F \end{array} \right) \)

If we choose the doublet of states \( |F\rangle = |\uparrow, 0\rangle_F \) and \( |F\rangle = |\downarrow, 0\rangle_F \) as part of the lowest energy supermultiplet, the states \( |\omega, t, m_t, a, m_a\rangle \) satisfying the conditions given in (2.4.30) will have \( t = s \pm 1/2 \) and can be written as

\[
|\omega(2s + 1)/2, s + 1/2, m_t, m_a, 0, 0\rangle = \frac{1}{\sqrt{2s + 1}} \left\{ \sqrt{s + 1/2 + m_t} |\uparrow, 0\rangle_F \times |s, m_t - 1/2\rangle_B \times |\omega\rangle_0 + \sqrt{s + 1/2 - m_t} |\downarrow, 0\rangle_F \times |s, m_t + 1/2\rangle_B \times |\omega\rangle_0 \right\} \tag{2.4.60}
\]

where \( \omega = -1/2 - \lambda(2s + 1) \) with \( \lambda < 0 \). The range of \( \lambda \) is determined by the square integrability of the states and the positivity of \( \hat{g}^2 \). This leads to the following restriction on \( \lambda \)

\[
\lambda < -\frac{3}{2(2s + 1)} \tag{2.4.61}
\]
Table 2.2. Decomposition of $SU(2)$ deformed minimal unitary lowest energy supermultiplets of $D(2,1;\lambda)$ with respect to $SU(2)_T \times SU(2)_A \times SU(1,1)_K$. The conformal wavefunctions transforming in the $(t,a)$ representation of $SU(2)_T \times SU(2)_A$ with conformal energy $\omega$ are denoted as $\Psi_\omega^{(t,a)}$. The first column shows the super tableaux of the lowest energy $SU(2|1)$ supermultiplet, the second column gives the eigenvalue of the $U(1)$ generator $H$. The allowed range of $\lambda$ in this case is $\lambda < -1/(4s + 2)$.

| $SU(2|1)l.w.v$ | $H$ | $SU(1,1)_K \times SU(2)_T \times SU(2)_A$ |
|----------------|-----|------------------------------------------|
| $\square$      | $|\lambda|/2$ | $\Psi^{3/2}_{(1/2,0)} \oplus \Psi^{1/2}_{(0,1/2)}$ |
| $\bigcirc\bigcirc$ | $|\lambda|$ | $\Psi^{|\lambda|/2}_{(1,0)} \oplus \Psi^{1/2}_{(1/2,1/2)} \oplus \Psi^{1}_{(0,0)}$ |
| $\bigcirc\bigcirc\bigcirc$ | $3|\lambda|/2$ | $\Psi^{3/2}_{(3/2,0)} \oplus \Psi^{1/2}_{(1,1/2)} \oplus \Psi^{1}_{(1/2,0)}$ |
| $\vdots$       | $\vdots$ | $\vdots$ |
| $\vdots$       | $\vdots$ | $\vdots$ |
| $\bigcirc\bigcirc\cdots\bigcirc$ | $(2s + 1)|\lambda|/2$ | $\Psi^{p}_{(s+1/2,0)} \oplus \Psi^{1/2}_{(s,1/2)} \oplus \Psi^{1}_{(s-1/2,0)}$
| $2s+1$        | $p = (2s + 1)|\lambda|/2$ | |

On the other hand for $t = s - 1/2$, we have

$$
|\lambda(2s + 1)/2, s - 1/2, m_t, 0, 0\rangle = \frac{1}{\sqrt{2s + 1}} \left\{ \sqrt{s + 1/2 + m_t} |\downarrow, 0\rangle_F \times |s, m_t + 1/2\rangle_B \times |\psi_0^\prime\rangle \\
- \sqrt{s + 1/2 - m_t} |\uparrow, 0\rangle_F \times |s, m_t - 1/2\rangle_B \times |\psi_0^\prime\rangle \right\}
$$

(2.4.62)

where $\omega = -1/2 + \lambda(2s + 1)$ with $\lambda > 0$. The range of $\lambda$ is determined by the square integrability of the states and the positivity of $\hat{g}^2$. This leads to the following restriction on $\lambda$

$$
\lambda > \frac{3}{2(2s + 1)}
$$

(2.4.63)

Let us now study the simplest case when $s = 0$. The lowest energy states that are annihilated by grade -1 generators are $|\pm1/2, 0\rangle_F \times |0\rangle_B \times |\psi_0^{1/2}\rangle$ where $\lambda < 0$. The
action of grade 0 supersymmetry generators on these states gives:

\[
\mathcal{Q}_0^0 |\uparrow, 0\rangle_F \times |0\rangle_B \times |\psi_0^{\lambda - 1/2}\rangle = |0, \downarrow\rangle_F \times |0\rangle_B \times |\psi_0^{\lambda + 1/2}\rangle \\
\mathcal{S}_0^0 |\uparrow, 0\rangle_F \times |0\rangle_B \times |\psi_0^{\lambda - 1/2}\rangle = 0 \\
\mathcal{Q}_0^1 |\uparrow, 0\rangle_F \times |0\rangle_B \times |\psi_0^{\lambda - 1/2}\rangle = 0 \\
\mathcal{S}_0^1 |\uparrow, 0\rangle_F \times |0\rangle_B \times |\psi_0^{\lambda - 1/2}\rangle = 0 \quad (2.4.64)
\]

\[
\mathcal{O}_0^0 |\downarrow, 0\rangle_F \times |0\rangle_B \times |\psi_0^{\lambda - 1/2}\rangle = 0 \\
\mathcal{S}_0^0 |\downarrow, 0\rangle_F \times |0\rangle_B \times |\psi_0^{\lambda - 1/2}\rangle = |0, \downarrow\rangle_F \times |0\rangle_B \times |\psi_0^{\lambda + 1/2}\rangle \\
\mathcal{Q}_0^0 |\downarrow, 0\rangle_F \times |0\rangle_B \times |\psi_0^{\lambda - 1/2}\rangle = 0 \\
\mathcal{S}_0^0 |\downarrow, 0\rangle_F \times |0\rangle_B \times |\psi_0^{\lambda - 1/2}\rangle = 0 \quad (2.4.65)
\]

Thus the action of grade 0 supersymmetry generators on states with \(t = 1/2\) produce states with \(t = 0\), but not \(t = 1\) as might be expected. Next we examine the action of grade +1 supersymmetry generators on these states.

\[
\mathcal{Q}_+ |\uparrow, 0\rangle_F \times |0\rangle_B \times |\psi_0^{\lambda - 1/2}\rangle = 0 \\
\mathcal{S}_+ |\uparrow, 0\rangle_F \times |0\rangle_B \times |\psi_0^{\lambda - 1/2}\rangle = |0, \uparrow\rangle_F \times |0\rangle_B \times |\psi_0^{\lambda + 1/2}\rangle \quad (2.4.66)
\]

\[
\mathcal{Q}_+ |\downarrow, 0\rangle_F \times |0\rangle_B \times |\psi_0^{\lambda - 1/2}\rangle = |0, \uparrow\rangle_F \times |0\rangle_B \times |\psi_0^{\lambda + 1/2}\rangle \\
\mathcal{S}_+ |\downarrow, 0\rangle_F \times |0\rangle_B \times |\psi_0^{\lambda - 1/2}\rangle = 0 \quad (2.4.67)
\]

Thus the complete supermultiplet in this case is

\[
\Psi_{(1/2,0)}^{\lambda/2} \oplus \Psi_{(0,1/2)}^{(\lambda+1)/2}, \quad \lambda < 0 \quad (2.4.68)
\]

Let us now consider the action of grade 0 and grade+1 supersymmetry generators on states with \(s \neq 0\) given in (2.4.60) and (2.4.62). The action of grade 0 generators on \(t = s + 1/2\) states is given as

\[
\mathcal{O}_0|\lambda\rangle(2s + 1)/2, s + 1/2, m_t, 0, 0\rangle = \sqrt{\frac{s + 1/2 + m_t}{2s + 1}} |0, \downarrow\rangle_F \times |s, m_t - 1/2\rangle_B \times |\psi_0^{s+1}\rangle
\]
\[ \mathcal{G}_0 |\lambda(2s + 1)/2, s + 1/2, m_t, 0, 0\rangle = \sqrt{\frac{s + 1/2 - m_t}{2s + 1}} |0, \downarrow\rangle_F \times |s, m_t + 1/2\rangle_B \times |\psi_0^{\omega+1}\rangle \]

\[ \mathcal{G}_0^\dagger |\lambda(2s + 1)/2, s + 1/2, m_t, 0, 0\rangle = -2\lambda \sqrt{\frac{s + 1/2 - m_t}{2s + 1}} (s + 1/2 + m_t) \times \\
|0, \uparrow\rangle_F \times |s, m_t + 1/2\rangle_B \times |\psi_0^{\omega-1}\rangle \]

\[ \mathcal{G}_0^\dagger |\lambda(2s + 1)/2, s + 1/2, m_t, 0, 0\rangle = -2\lambda \sqrt{\frac{s + 1/2 + m_t}{2s + 1}} (s + 1/2 - m_t) \times \\
|0, \uparrow\rangle_F \times |s, m_t - 1/2\rangle_B \times |\psi_0^{\omega-1}\rangle \]

Let us now evaluate the action of +1 grade supersymmetry generators on the states with \( t = s + 1/2. \)

\[ \mathcal{G}_+ |\lambda(2s + 1)/2, s + 1/2, m_t, 0, 0\rangle = \sqrt{\frac{s + 1/2 - m_t}{2s + 1}} \left\{ |0, \uparrow\rangle_F \times |s, m_t + 1/2\rangle_B \times |\psi_0^{\omega+1}\rangle \right. \\
+ 2\lambda(s + 1/2 + m_t) |0, \uparrow\rangle_F \times |s, m_t + 1/2\rangle_B \times |\psi_0^{\omega-1}\rangle \} \\
= \sqrt{\frac{s + 1/2 - m_t}{2s + 1}} \left\{ |0, \uparrow\rangle_F \times |s, m_t + 1/2\rangle_B \times |\psi_1^{\omega-1}\rangle \right. \\
+ (2m_t - 1) |0, \uparrow\rangle_F \times |s, m_t + 1/2\rangle_B \times |\psi_0^{\omega-1}\rangle \right\} \tag{2.4.70} \]

\[ \mathcal{G}_+ |\lambda(2s + 1)/2, s + 1/2, m_t, 0, 0\rangle = \sqrt{\frac{s + 1/2 + m_t}{2s + 1}} \left\{ |0, \uparrow\rangle_F \times |s, m_t - 1/2\rangle_B \times |\psi_0^{\omega+1}\rangle \right. \\
+ 2\lambda(s + 1/2 - m_t) |0, \uparrow\rangle_F \times |s, m_t - 1/2\rangle_B \times |\psi_0^{\omega-1}\rangle \} \\
= \sqrt{\frac{s + 1/2 + m_t}{2s + 1}} \left\{ |0, \uparrow\rangle_F \times |s, m_t - 1/2\rangle_B \times |\psi_1^{\omega-1}\rangle \right. \\
- (2m_t + 1) |0, \uparrow\rangle_F \times |s, m_t - 1/2\rangle_B \times |\psi_0^{\omega-1}\rangle \right\} \tag{2.4.71} \]

\[ [\mathcal{Q}_+, \mathcal{G}_+] |\lambda(2s + 1)/2, s + 1/2, m_t, 0, 0\rangle = 0 \tag{2.4.72} \]

Next we need to evaluate the action of +1 grade supersymmetry generators on the states obtained in \(2.4.69\) which are of the form \( |0, \downarrow\rangle_F \times |s, m_t \pm 1/2\rangle_B \times |\psi_0^{\omega+1}\rangle. \) From the previous section we would expect states with \( t = s \pm 1/2 \) but the states with \( t = s + 1/2 \) obtained in this fashion are excitations so the only new states we obtain are the states with \( t = s - 1/2. \)

The lowest energy supermultiplet for \( t = s + 1/2 \) corresponds to the following \( SU(2|1) \)
and the resulting unitary supermultiplet of $D(2,1;\lambda)$ decomposes as

$$
\Psi^p_{(s+1/2,0)} \oplus \Psi^{p+1/2}_{(s,1/2)} \oplus \Psi^{p+1}_{s-1/2,0}
$$

where $p = (2s + 1)|\lambda|/2$ with $\lambda < 0$. We have summarized the deformed supermultiplets for lowest weight states with $t = s + 1/2$ in Table 2.3. Note that these occur only for $\lambda < -3/(4s + 2)$.

**Table 2.3.** Decomposition of $SU(2)$ deformed minimal unitary lowest energy supermultiplets of $D(2,1;\lambda)$ with respect to $SU(2)_T \times SU(2)_A \times SU(1,1)_K$. The conformal wavefunctions transforming in the $(t,a)$ representation of $SU(2)_T \times SU(2)_A$ with conformal energy $\omega$ are denoted as $\Psi^{\omega}_{(t,a)}$. The first column shows the super tableaux of the lowest energy $SU(2|1)$ supermultiplet, the second column gives the eigenvalue of the $U(1)$ generator $\mathcal{H}$. The allowed range of $\lambda$ in this case is $\lambda < -3/(4s + 2)$.

| $SU(2|1)l.w.v.$ | $\mathcal{H}$ | $SU(1,1)_K \times SU(2)_T \times SU(2)_A$ |
|-----------------|--------------|------------------------------------------|
| $\square$ | $|\lambda|/2$ | $\Psi^{|\lambda|/2}_{(1/2,0)} \oplus \Psi^{(|\lambda|+1/2)}_{(0,1/2)}$ |
| $\square \square$ | $|\lambda|$ | $\Psi^{|\lambda|}_{(1,0)} \oplus \Psi^{(|\lambda|+1/2)}_{(1/2,1/2)} \oplus \Psi^{(|\lambda|+1)}_{(0,0)}$ |
| $\square \square \square$ | $3|\lambda|/2$ | $\Psi^{3|\lambda|/2}_{(3/2,0)} \oplus \Psi^{3(|\lambda|+1/2)}_{(1,1/2)} \oplus \Psi^{3(|\lambda|+1)}_{(1/2,0)}$ |
| : | : | : |
| : | : | : |
| $\square \square \square \cdots$ | $(2s + 1)|\lambda|/2$ | $\Psi^p_{(s+1/2,0)} \oplus \Psi^{p+1/2}_{(s,1/2)} \oplus \Psi^{p+1}_{s-1/2,0}$ |

Next we look at the states with $t = s - 1/2$. The action of grade 0 generators on these
states is given below:

\[ \Omega_0 |\lambda(2s + 1)/2, s - 1/2, m_t, 0, 0\rangle = -\sqrt{\frac{s + 1/2 - m_t}{2s + 1}} |0, \downarrow\rangle_F \times |s, m_t - 1/2\rangle_B \times |\psi^{+1}_0\rangle \]

\[ \mathcal{G}_0 |\lambda(2s + 1)/2, s - 1/2, m_t, 0, 0\rangle = \sqrt{\frac{s + 1/2 + m_t}{2s + 1}} |0, \downarrow\rangle_F \times |s, m_t + 1/2\rangle_B \times |\psi^{+1}_0\rangle \]

\[ \Omega_0^\dagger |\lambda(2s + 1)/2, s - 1/2, m_t, 0, 0\rangle = 2\lambda \sqrt{\frac{s + 1/2 + m_t}{2s + 1}} (s + 1/2 - m_t) \times \]

\[ |0, \uparrow\rangle_F \times |s, m_t + 1/2\rangle_B \times |\psi^{\omega-1}_0\rangle \]

\[ \mathcal{G}_0^\dagger |\lambda(2s + 1)/2, s - 1/2, m_t, 0, 0\rangle = -2\lambda \sqrt{\frac{s + 1/2 - m_t}{2s + 1}} (s + 1/2 + m_t) \times \]

\[ |0, \uparrow\rangle_F \times |s, m_t - 1/2\rangle_B \times |\psi^{\omega-1}_0\rangle \] (2.4.75)

The action of +1 grade supersymmetry generators on the states with \( t = s - 1/2 \) is given as:

\[ \Omega_+ |\lambda(2s + 1)/2, s - 1/2, m_t, 0, 0\rangle = \sqrt{\frac{s + 1/2 + m_t}{2s + 1}} \left\{ |0, \uparrow\rangle_F \times |s, m_t + 1/2\rangle_B \times |\psi^{+1}_0\rangle \right. \]

\[ -2\lambda(s + 1/2 - m_t) |0, \uparrow\rangle_F \times |s, m_t + 1/2\rangle_B \times |\psi^{\omega-1}_0\rangle \right\} \]

\[ = \sqrt{\frac{s + 1/2 + m_t}{2s + 1}} \left\{ |0, \uparrow\rangle_F \times |s, m_t + 1/2\rangle_B \times |\psi^{\omega-1}_0\rangle \right. \]

\[ + (2m_t - 1)\lambda |0, \uparrow\rangle_F \times |s, m_t + 1/2\rangle_B \times |\psi^{\omega^{-1}}_0\rangle \] (2.4.76)

\[ \mathcal{G}_+ |\lambda(2s + 1)/2, s - 1/2, m_t, 0, 0\rangle = -\sqrt{\frac{s + 1/2 - m_t}{2s + 1}} \left\{ |0, \uparrow\rangle_F \times |s, m_t - 1/2\rangle_B \times |\psi^{+1}_0\rangle \right. \]

\[ -2\lambda(s + 1/2 + m_t) |0, \uparrow\rangle_F \times |s, m_t - 1/2\rangle_B \times |\psi^{\omega-1}_0\rangle \right\} \]

\[ = \sqrt{\frac{s + 1/2 - m_t}{2s + 1}} \left\{ |0, \uparrow\rangle_F \times |s, m_t - 1/2\rangle_B \times |\psi^{\omega^{-1}}_0\rangle \right. \]

\[ - (2m_t + 1)\lambda |0, \uparrow\rangle_F \times |s, m_t - 1/2\rangle_B \times |\psi^{\omega^{-1}}_0\rangle \] (2.4.77)

\[ [\Omega_+, \mathcal{G}_+] |\lambda(2s + 1)/2, s - 1/2, m_t, 0, 0\rangle = 0 \] (2.4.78)

Next we need to evaluate the action of +1 grade supersymmetry generators on the states obtained in (2.4.75) which are of the form \( |0, \downarrow\rangle_F \times |s, m_t \pm 1/2\rangle_B \times |\psi^{+1}_0\rangle \). From the previous section we would expect states with \( t = s \pm 1/2 \) but the states with \( t = s - 1/2 \) obtained in this fashion are excitations so the only new states we obtain are the states with \( t = s + 1/2 \).
The lowest energy super multiplet for \( t = s - 1/2 \) corresponds to the following \( SU(2|1) \) supertableau

\[
\begin{array}{c|c}
\begin{array}{c}
\cdots \\
2s
\end{array} & \begin{array}{c}
\cdots \\
2s
\end{array}
\end{array} = \left( \begin{array}{c|c}
\begin{array}{c}
\cdots \\
2s
\end{array} & 1 \\
\end{array} \right) + \left( \begin{array}{c|c}
\begin{array}{c}
\cdots \\
2s
\end{array} & \square \end{array} \right)
\]

and leads to the supermultiplet

\[
\Psi^p_{(s-1/2,0)} \oplus \Psi^{p+1/2}_{(s,1/2)} \oplus \Psi^{p+1}_{(s+1/2,0)} \tag{2.4.80}
\]

where \( p = (2s + 1)\lambda/2 \) with \( \lambda > 0 \). We have summarized the deformed supermultiplets for lowest weight states with \( t = s - 1/2 \) in Table 2.4. Note that these occur only for \( s > 1/2 \) and \( \lambda > 3/(4s + 2) \).

**Table 2.4.** Decomposition of \( SU(2) \) deformed minimal unitary lowest energy supermultiplets of \( D(2,1;\lambda) \) with respect to \( SU(2)_T \times SU(2)_A \times SU(1,1)_K \). The conformal wavefunctions transforming in the \( (t,a) \) representation of \( SU(2)_T \times SU(2)_A \) with conformal energy \( \omega \) are denoted as \( \Psi^\omega_{(t,a)} \). The first column shows the super tableaux of the lowest energy \( SU(2|1) \) supermultiplet, the second column gives the eigenvalue of the \( U(1) \) generator \( H \). The allowed range of \( \lambda \) in this case is \( \lambda > 3/(4s + 2) \).

| \( SU(2|1)l.w.v \) | \( \mathcal{H} \) | \( SU(1,1)_K \times SU(2)_T \times SU(2)_A \) |
|----------------|-------------|-----------------------------------------------|
| \[
\begin{array}{c|c}
\begin{array}{c}
\cdots \\
2s
\end{array} & \begin{array}{c}
\cdots \\
2s
\end{array}
\end{array}
\] | \( \lambda \) | \( \Psi^\lambda_{(0,0)} \oplus \Psi^{\lambda+1/2}_{(1/2,1/2)} \oplus \Psi^{\lambda+1}_{(1,0)} \) |
| \[
\begin{array}{c|c}
\begin{array}{c}
\cdots \\
2s
\end{array} & \begin{array}{c}
\cdots \\
2s
\end{array}
\end{array}
\] | \( 3\lambda/2 \) | \( \Psi^{3\lambda/2}_{(1/2,0)} \oplus \Psi^{(3\lambda+1)/2}_{(1,1/2)} \oplus \Psi^{3\lambda+1}_{(3/2,0)} \) |
| \[
\begin{array}{c|c}
\begin{array}{c}
\cdots \\
2s
\end{array} & \begin{array}{c}
\cdots \\
2s
\end{array}
\end{array}
\] | \( 2\lambda \) | \( \Psi^{2\lambda}_{(1,0)} \oplus \Psi^{2\lambda+1/2}_{(3/2,1/2)} \oplus \Psi^{2\lambda+1}_{(2,0)} \) |
| \[
\begin{array}{c|c}
\begin{array}{c}
\cdots \\
2s
\end{array} & \begin{array}{c}
\cdots \\
2s
\end{array}
\end{array}
\] | \( (2s + 1)\lambda/2 \) | \( \Psi^p_{(s-1/2,0)} \oplus \Psi^{p+1/2}_{(s,1/2)} \oplus \Psi^{p+1}_{(s+1/2,0)} \) |

We note the similarities of Table 2.1 with Table 2.4 and that of Table 2.2 with Table 2.3. This shows that the supermultiplets obtained for lowest weight states with \( t = s \) and \( t = s - 1/2 \) and \( \lambda > 0 \) are the same and the supermultiplets for lowest weight states with \( t = s \) and \( t = s + 1/2 \) and \( \lambda < 0 \) are same. The difference between these two types of supermultiplets is that the \( SU(1,1)_K \) spin (labeled as \( p \)) gets interchanged between states.
with \( t = s + 1/2 \) and \( t = s - 1/2 \) as we change the sign of \( \lambda \).

### 2.5 SU(2) Deformations of the minimal unitary representation of \( D(2, 1; \lambda) \) using both bosons and fermions and \( OSp(2n^*|2m) \) superalgebras

Above we obtained unitary supermultiplets of \( D(2, 1; \lambda) \) which are \( SU(2) \) deformations of the minimal unitary representation. This was achieved by introducing bosonic oscillators \( a_n \) and \( b_n \) and extending the \( SU(2)_T \) generators to the generators of the diagonal subgroup of \( SU(2)_T \) and \( SU(2)_S \) realized as bilinears of the bosonic oscillators

\[
S_+ = a_n b_n \quad (2.5.1)
\]
\[
S_- = b^* a_n \quad (2.5.2)
\]
\[
S_0 = \frac{1}{2}(N_a - N_b) \quad (2.5.3)
\]

As stated above, the non-compact group \( SO^*(2n) \) generated by the bilinears

\[
A_{mn} = a_m b_n - a_n b_m \quad (2.5.4)
\]
\[
A^{mn} = a^m b^n - a^n b^m \quad (2.5.5)
\]
\[
U_m^n = a^m a_n + b_n b^m \quad (2.5.6)
\]

commutes with the generators of \( D(2, 1; \lambda) \). One can similarly obtain \( SU(2) \) deformations of the minimal unitary supermultiplet of \( D(2, 1; \lambda) \) by introducing fermionic oscillators \( \rho_r \) and \( \sigma_s \) \((r, s, .. = 1, .. n)\) satisfying

\[
\{ \rho_r, \rho^s \} = \{ \sigma_r, \sigma^s \} = \delta^s_r \quad \{ \rho_r, \sigma_s \} = \{ \rho_r, \sigma_s \} = \{ \sigma_r, \sigma_s \} = 0 \quad (2.5.7)
\]

and extend the generators of \( SU(2)_T \) to the generators of the diagonal subgroup of \( SU(2)_T \) and \( SU(2)_F \) generated by

\[
F_+ = \rho^* \sigma_r \quad (2.5.8)
\]
\[
F_- = \sigma^* \rho_r \quad (2.5.9)
\]
\[
F_0 = \frac{1}{2}(\rho^* \rho_r - \sigma^* \sigma_r) \quad (2.5.10)
\]

In this case the compact \( USp(2n) \) generated by the fermion bilinears

\[
S_{rs} = \rho_r \sigma_s + \rho_s \sigma_r \quad (2.5.11)
\]

38
\[ S^{rs} = \sigma^r \rho^s + \sigma^s \rho^r \quad (2.5.12) \]
\[ S^r_s = \rho^r \rho_s - \sigma_s \sigma^r \quad (2.5.13) \]

commute with the generators of \( D(2,1;\lambda) \). One can deform the minimal unitary representation of \( D(2,1;\lambda) \) using fermions and bosons simultaneously. This is achieved by replacing the \( SU(2)_T \) generators by the diagonal generators of \( SU(2)_T \times SU(2)_S \times SU(2)_F \), which we shall denote as \( U_+, U_- \) and \( U_0 \)

\[ U_+ = T_+ + S_+ + F_+ \quad (2.5.14) \]
\[ U_- = T_- + S_- + F_- \quad (2.5.15) \]
\[ U_0 = T_0 + S_0 + F_0 \quad (2.5.16) \]

and substituting the quadratic Casimir of \( SU(2)_T \) in the Ansatz for \( K_- \) with the quadratic Casimir of \( SU(2)_D \). Remarkably, in this case the resulting generators of \( D(2,1;\lambda) \) commute with the generators of the non-compact superalgebra \( OSp(2n^* | 2m) \) generated by the generators of \( SO^* (2n) \) and \( USp(2m) \) given above and the supersymmetry generators:

\[ \Pi_{mr} = a_m \sigma_r - b_m \rho_r, \quad \Pi^{mr} = (\Pi_{mr})^\dagger = a^m \sigma^r - b^m \rho^r \quad (2.5.17) \]
\[ \Sigma^r_m = a_m \rho^r + b_m \sigma^r, \quad \Sigma^m_r = (\Sigma^r_m)^\dagger = a^m \rho^r + b^m \sigma^r \quad (2.5.18) \]

The (anti) commutation relations for the \( OSp(2n^* | 2m) \) algebra are given below:

\[
\begin{align*}
[A^i_j, A^k_l] &= \delta^k_j A^i_l - \delta^l_j A^i_k, & [S^r_s, S^t_u] &= \delta^u_s S^r_t - \delta^r_u S^t_s \\
[A_{ij}, A^k_l] &= \delta^k_j A_{il} - \delta^l_j A_{ik}, & [S_{rs}, S^t_u] &= \delta^t_r S_{su} + \delta^s_u S_{rt} \\
[A^i_j, A^k_l] &= \delta^i_j A^{kl} - \delta^l_j A^{ik}, & [S^r^s, S^t_u] &= -\delta^u_s S^{rt} - \delta^r_u S^{ts} \\
[A_{ij}, A^{kl}] &= -\delta^k_j A^l_i + \delta^l_j A^k_i - \delta^i_l A^k_j + \delta^i_k A^l_j \\
[S_{rs}, S^{tu}] &= -\delta^t_r S^u_s - \delta^u_t S^r_s - \delta^s_u S^t_r - \delta^r_s S^t_u
\end{align*}
\]
\[\{\Pi_{mr}, \Pi^{rs}\} = \delta^s_r \delta^n_m - \delta^n_m \delta_r^s, \quad \{\Sigma^r_m, \Sigma^n_s\} = \delta^r_s \delta^n_m + \delta^s_n \delta^n_m\]

2.6 \[\text{SU}(2)\] deformed minimal unitary supermultiplets of \[D(2, 1; \alpha)\] and \[\mathcal{N} = 4\] superconformal mechanics

2.6.1 \[\mathcal{N} = 4\] Superconformal Quantum Mechanical Models

Conformal quantum mechanical models have been subjects of interest since the seventies and one of the main reason is that they characterize an important integrable (completely solvable) system discovered by Calogero in \[\text{[121,122]}\]. A concrete study of conformal mechanics at both classical and quantum levels was first performed in \[\text{[119]}\]. The interest resurfaced in connection with AdS/CFT correspondence \[\text{[14]}\] after it was shown that the dynamics of a super-particle near the horizon of an extremal Reissner-Nordström black-hole, in the limit of large black-hole mass, is governed by a superconformal quantum mechanical model in \[\text{[123]}\].

An \[\mathcal{N} = 4\] superconformal quantum mechanical model that is invariant under \[D(2, 1; \alpha)\] symmetry algebra was studied by in \[\text{[115,124]}\]. In this section we will review that construction following \[\text{[115]}\].

The on shell component action was shown to take the form \[\text{[115]}\]

\[
S = S_b + S_f, \\
S_b = \int dt \left[ \dot{x}^i \dot{\bar{z}}^k - \frac{\alpha^2 (\bar{z}^k z^k)^2}{4 x^2} - A (\bar{z}^k z^k - c) \right], \\
S_f = -i \int dt \left( \bar{\psi}_k \dot{\psi}^k - \dot{\bar{\psi}}^k \psi^k \right) + 2\alpha \int dt \frac{\psi^i \bar{\psi}^k \bar{z}^k (i \bar{\psi}_k)}{x^2} + \frac{\alpha}{2} (1 + 2\alpha) \int dt \frac{\psi^i \bar{\psi}^k \bar{\psi} (i \bar{\psi}_k)}{x^2}
\]

Here \(x, \bar{z}^i\) and \(\psi^j\) \((i, j = 1, 2)\) are \(d = 1\) bosonic and fermionic “fields”, respectively. The fields \(z^i\) form a complex doublet of the R-symmetry group \(SU(2)\). The last term in \[\text{(2.6.2)}\] represents the constraint

\[
\bar{z}_k z^k = c,
\]

(2.6.4)
and $A$ is the Lagrange multiplier.

Upon quantization of the action given in (2.6.1), the dynamical variables were promoted to quantum mechanical operators with following commutators:

$$[X, P] = i, \quad [Z^i, \bar{Z}^j] = \delta^j_i, \quad \{\Psi^i, \bar{\Psi}_j\} = -\frac{1}{2} \delta^j_i \quad (i, j = 1, 2). \quad (2.6.5)$$

As the action is invariant under the group $D(2,1;\alpha)$, the corresponding symmetry generators can be obtained by the Noether procedure. The results as given in [115] are:

$$Q^i = P\Psi^i + 2i\alpha \frac{Z^i\bar{Z}^k\Psi_k}{X} + i(1 + 2\alpha) \frac{\langle\Psi_k\bar{\Psi}^k\bar{\Psi}^i\rangle}{X}, \quad (2.6.6)$$

$$\bar{Q}_i = P\bar{\Psi}_i - 2i\alpha \frac{Z^i\bar{Z}^k\bar{\Psi}^k}{X} + i(1 + 2\alpha) \frac{\langle\bar{\Psi}^k\bar{\Psi}_k\Psi^i\rangle}{X}, \quad (2.6.7)$$

$$S^i = -2X\Psi^i + tQ^i, \quad \bar{S}_i = -2X\bar{\Psi}_i + t\bar{Q}_i, \quad (2.6.8)$$

$$H = \frac{1}{4}P^2 + \alpha^2 \frac{Z^kZ^k}{4X^2} + 2\alpha \frac{Z^i\bar{Z}^k\Psi_k}{X^2} - \alpha \frac{\langle\Psi_k\bar{\Psi}^k\bar{\Psi}^i\rangle}{X^2} + \frac{(1 + 2\alpha)^2}{16X^2}, \quad (2.6.9)$$

$$K = X^2 - t \left\{X, P\right\} + t^2H, \quad (2.6.10)$$

$$D = -\frac{1}{2} \left\{X, P\right\} + tH, \quad (2.6.11)$$

$$J^{i\bar{k}} = i \left[Z^i\bar{Z}^k + 2\Psi^i\bar{\Psi}^k\right], \quad (2.6.12)$$

$$I^{i'\bar{i}'} = -i\Psi_k\bar{\Psi}^k, \quad I^{d'\bar{d}'} = i\bar{\Psi}^k\bar{\Psi}_k, \quad I^{i'\bar{i}'} = -\frac{i}{2} \left[\Psi_k, \bar{\Psi}^k\right]. \quad (2.6.13)$$

where $t$ is time variable and the symbol $\langle\ldots\rangle$ denotes Weyl ordering:

$$\langle\psi_k\bar{\psi}^k\bar{\psi}^i\rangle = \psi_k\bar{\psi}^k\bar{\psi}^i + \frac{1}{2} \psi^i, \quad \langle\bar{\psi}^k\bar{\psi}_k\psi^i\rangle = \bar{\psi}^k\bar{\psi}_k\psi^i + \frac{1}{2} \bar{\psi}_i$$

and $Q_i = -(Q^i)^+, \quad S_i = -(S^i)^+.$

In the above set of generators $Q^i$ and $S^i$ are supertranslation and superconformal boost generators respectively. The generators $H$, $K$ and $D$ are the Hamiltonian, special conformal transformations and dilatation generators respectively and they form an $su(1,1)$ algebra. The remaining generators $J^{i\bar{k}}$ and $I^{d'\bar{d}'}$ are the generators of two $su(2)$ algebras.
2.6.2 Mapping between the harmonic superspace generators of $\mathcal{N} = 4$ superconformal mechanics and generators of deformed minimal unitary representations of $D(2, 1; \alpha)$

The symmetries of $\mathcal{N} = 2$ supersymmetric sigma models with a quaternionic Kähler background that couple to supergravity in $d = 4$ were studied in harmonic superspace were studied in [125] and a precise correspondence with the minrep of the isometry group was established. It was further suggested in [126] that the correspondence can be extended to the full quantum correspondence on both sides by reducing the $\mathcal{N} = 2$ sigma model to one dimension and quantizing it to get supersymmetric quantum mechanics. The bosonic spectrum of this quantum mechanics should provide the minrep for the symmetry group. The results presented in this chapter and the $D(2, 1; \alpha)$ superconformal quantum mechanics reviewed in previous section [115] provides an opportunity to test this proposal.

The basic quantum mechanical operators in the minrep and their deformations was given in section 2.3 are the coordinate $x$ and its momentum $p$, fermionic oscillators $\alpha^\dagger, \alpha, \beta^\dagger$ and $\beta$ and the bosonic oscillators $a^\dagger, a, b^\dagger$ and $b$ with the following commutation relations:

$$[a, a^\dagger] = 1 = [b, b^\dagger], \quad \{\alpha, \alpha^\dagger\} = 1 = \{\beta, \beta^\dagger\}$$

(2.6.14)

The generators of quantized $\mathcal{N} = 4$ superconformal mechanics in harmonic superspace go over to the generators of minimal unitary realization of $D(2, 1; \lambda)$ deformed by a pair of bosonic oscillators if we make the simple substitutions listed in Table 2.5.

As expected from the results of [125,126] we find a one-to-one correspondence between the symmetry generators of $D(2, 1; \alpha)$ superconformal quantum mechanics and the generators of the minimal unitary representations of $D(2, 1; \alpha)$ deformed by a pair of bosons, which we present in Table 2.6. Using this mapping it is easy to see that the quadratic Casimir obtained in equation (4.26) of [115] is the same as the one obtained by our construction given in (2.3.40) for $\mu = 4$. The quantum spectra of $\mathcal{N} = 4$ superconformal mechanics were also studied in [115] using the realization reviewed in the previous section. To relate the quantum spectra of these models to the minimal unitary realizations of $D(2, 1; \lambda)$ we tabulate the correspondence between the $SU(1, 1), SU(2)_R, SU(2)_L$ quantum numbers of [115] and the $SU(1, 1)_K, SU(2)_T, SU(2)_A$ spins of our construction in Table 2.7. Using this table we see that the superfield contents of the quantum spectra of these models as given in Table 2 of [115] are exactly the same as supermultiplets described in Tables 2.1, 2.2, 2.3 and 2.4 above.
Table 2.5. Below we give the correspondence between the quantum mechanical operators of $\mathcal{N} = 4$ superconformal mechanics and the operators of minimal unitary supermultiplet of $D(2,1;\lambda)$ deformed by a pair of bosons. The $SU(2)$ indices on the left column are raised and lowered by the Levi-Civita tensor $\epsilon_{ij}$ (with $\epsilon_{12} = \epsilon_{21} = 1$).

<table>
<thead>
<tr>
<th>Operators of $\mathcal{N} = 4$ Superconformal Mechanics in Harmonic superspace</th>
<th>Operators of minimal unitary representation of $D(2,1;\lambda)$ deformed by a pair of bosons</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi^1$</td>
<td>$-\frac{i}{\sqrt{2}}\alpha^+$</td>
</tr>
<tr>
<td>$\psi^2$</td>
<td>$-\frac{i}{\sqrt{2}}\alpha^+$</td>
</tr>
<tr>
<td>$\bar{\psi}_1$</td>
<td>$\frac{i}{\sqrt{2}}\alpha$</td>
</tr>
<tr>
<td>$\bar{\psi}_2$</td>
<td>$\frac{i}{\sqrt{2}}\beta$</td>
</tr>
<tr>
<td>$Z^1$</td>
<td>$-ia^+$</td>
</tr>
<tr>
<td>$Z^2$</td>
<td>$-ib^+$</td>
</tr>
<tr>
<td>$\bar{Z}_1$</td>
<td>$ia$</td>
</tr>
<tr>
<td>$\bar{Z}_2$</td>
<td>$ib$</td>
</tr>
</tbody>
</table>

2.7 Discussion

In this chapter we reformulated the minrep of $D(2,1;\lambda)$ and constructed the $SU(2)$ deformed minimal unitary supermultiplets of $D(2,1;\lambda)$ using quasiconformal methods. Similar deformations of the minrep were obtained for the $4d$ and $6d$ superconformal algebras in [23,24,56]. We also established a correspondence between deformations obtained by a pair of bosons and the quantum spectra of the $\mathcal{N} = 4$ superconformal mechanical models studied in [115]. However we found a more general class of deformations involving an arbitrary numbers of pairs of bosons and/or pairs of fermions which begs the question of what kind of superconformal models correspond to these general deformations.

As discussed in previous section, the results of [125,126] indicate these deformations to describe the spectra of superconformal quantum mechanical models with quaternionic Kähler sigma models that descend from $4d$, $N = 2$ supersymmetric sigma models that couple to $N = 2$ supergravity. Another interesting problem is to understand the precise connection between the above results and the $d = 2$, $\mathcal{N} = 4$ supersymmetric gauged WZW models studied in [127,128] that extend the results of [129] on $N = 2$ supersymmetric gauged WZW models.

These gauged WZW models correspond to realizations over spaces of the form

$$\frac{G_c}{H \times SU(2)} \times SU(2) \times U(1)$$

(2.7.1)
where $G_c/(H \times SU(2))$ is a compact quaternionic symmetric space. On the other hand the quaternionic Kähler manifolds that couple to 4d, $N = 2$ supergravity are non-compact.
Table 2.6. Below we give the mapping between the symmetry generators of \( \mathcal{N} = 4 \) superconformal mechanics in harmonic superspace and the minimal unitary representation of \( D(2, 1; \lambda) \) deformed by a pair of bosons. The first column lists the Symmetry generators of \( \mathcal{N} = 4 \) superconformal quantum mechanics in harmonic superspace (\( \mathcal{N} = 4 \) SCQM in HSS) and the second column lists the generators for the minimal unitary realization of \( D(2, 1; \lambda) \) deformed by a pair of bosons.

<table>
<thead>
<tr>
<th>( \mathcal{N} = 4 ) SCQM in HSS</th>
<th>Deformed Minreps of ( D(2, 1; \lambda) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( iI^1'1' )</td>
<td>( A_+ )</td>
</tr>
<tr>
<td>( -iI^2'2' )</td>
<td>( A_- )</td>
</tr>
<tr>
<td>( iI^1'2' )</td>
<td>( A_0 )</td>
</tr>
<tr>
<td>( -iJ^{11} )</td>
<td>( \mathcal{T}_+ )</td>
</tr>
<tr>
<td>( iJ^{22} )</td>
<td>( \mathcal{T}_- )</td>
</tr>
<tr>
<td>( iJ^{12} )</td>
<td>( \mathcal{T}_0 )</td>
</tr>
<tr>
<td>( -\frac{i}{\sqrt{2}} Q^1 )</td>
<td>( \tilde{Q}^1 )</td>
</tr>
<tr>
<td>( -\frac{i}{\sqrt{2}} Q^2 )</td>
<td>( \tilde{S}^1 )</td>
</tr>
<tr>
<td>( -\frac{i}{\sqrt{2}} \tilde{Q}_1 )</td>
<td>( \tilde{Q}_1 )</td>
</tr>
<tr>
<td>( -\frac{i}{\sqrt{2}} \tilde{Q}_2 )</td>
<td>( \tilde{S}_1 )</td>
</tr>
<tr>
<td>( -\frac{i}{\sqrt{2}} S^1 )</td>
<td>( Q^1 )</td>
</tr>
<tr>
<td>( -\frac{i}{\sqrt{2}} S^2 )</td>
<td>( S^1 )</td>
</tr>
<tr>
<td>( -\frac{i}{\sqrt{2}} S_1 )</td>
<td>( Q_1 )</td>
</tr>
<tr>
<td>( -\frac{i}{\sqrt{2}} S_2 )</td>
<td>( S_1 )</td>
</tr>
<tr>
<td>( 2H )</td>
<td>( K_+ )</td>
</tr>
<tr>
<td>( -2D )</td>
<td>( \Delta )</td>
</tr>
<tr>
<td>( \frac{1}{2}K )</td>
<td>( K_- )</td>
</tr>
</tbody>
</table>
Table 2.7. Below we give the mapping between the quantum numbers of the spectrum of $\mathcal{N} = 4$ superconformal mechanics of [115] and the minimal unitary supermultiplets of $D(2, 1; \lambda)$ deformed by a pair of bosons.

<table>
<thead>
<tr>
<th>Quantum spectrum of $\mathcal{N} = 4$ SCQM</th>
<th>Deformed Minreps of $D(2, 1; \lambda)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_0$</td>
<td>$p$</td>
</tr>
<tr>
<td>$j$</td>
<td>$t$</td>
</tr>
<tr>
<td>$i$</td>
<td>$a$</td>
</tr>
<tr>
<td>$c$</td>
<td>$2s$</td>
</tr>
</tbody>
</table>
Chapter 3  
Conformal Higher Spin Algebras in $d = 4$: $hs(4, 2; \zeta)$

3.1 Introduction

Higher spin algebras are defined as Lie algebra of global symmetries of a theory that has higher spin particles in its spectrum. A more mathematically precise definition was given by Eastwood in [130] where a higher spin algebra in $AdS_{d+1}$ space was defined as the universal enveloping algebra $\mathfrak{u}(\mathfrak{so}(d, 2))$ of the conformal group $SO(d, 2)$ in $d$ Minkowski dimensions, quotiented by a two sided ideal $\mathfrak{j}(\mathfrak{so}(d, 2))$, known as the Joseph ideal [43], which is the annihilator of the minrep. In other words, higher spin algebra is the enveloping algebra of the minrep of $SO(d, 2)$. Thus in order to construct the higher spin algebra in any dimension, all we need to do is construct the minrep and its universal enveloping algebra is just the higher spin algebra.

Most of the work on higher spin theories that has been done in the literature is for $AdS_4$ and $AdS_3$ where the structure of higher spin algebras is understood fairly well. However, from the string theory and M-theory point of view, $AdS_5$ and $AdS_7$ higher spin theories are very important and the associated $AdS/CFT$ dualities can provide new insights into higher spin theories in these dimensions. However, the lack of convenient spinor formulations in $AdS_5$ and $AdS_7$ spaces makes it harder to formulate higher spin algebras in these dimensions. In this chapter we will focus on formulating the minrep of the four dimensional conformal group $SO(4, 2) \sim SU(2, 2)$ as bilinears of deformed twistorial oscillators that transform nonlinearly under the Lorentz group $SO(3, 1)$. The minrep (and its deformations) for $SU(2, 2)$ was first obtained in [23] using the quasiconformal method and we shall reformulate their results and show that the highly nonlinear realization can be conveniently rewritten as bilinears of deformed twistorial oscillators which themselves are nonlinear and have nontrivial commutation relations. These results were published
The plan of the chapter is as follows: In section 3.2 we review the covariant twistorial oscillator (singleton) construction of the conformal group in three dimensions $SO(3, 2) \sim Sp(4, \mathbb{R})$ and its superextension $OSp(N|4, \mathbb{R})$. Then we review the covariant twistorial oscillator (doubleton) construction for the four dimensional conformal group $SO(4, 2) \sim SU(2, 2)$ in section 3.3.1. In section 3.3.2 we present the minimal unitary representation of $SU(2, 2)$ obtained by the quasiconformal approach [23] in terms of certain deformed twistorial oscillators that transform nonlinearly under the Lorentz group. We then define a one-parameter family of these deformed twistors, which we call helicity deformed twistorial oscillators and express the generators of a one parameter family of deformations of the minrep given in [23] as bilinears of the helicity deformed twistors. They describe massless conformal fields of arbitrary helicity which can be continuous. In section 3.3.4 we use the deformed twistors to realize the superconformal algebra $PSU(2, 2|4)$ and its deformations in the quasiconformal framework. In section 3.4 we review the Eastwood’s formula for the generator $J$ of the annihilator of the minrep (Joseph ideal) and show by explicit calculations that it vanishes identically for the singletons of $SO(3, 2)$ and the minrep of $SU(2, 2)$ obtained by quasiconformal methods. We then present the generator $J$ of the Joseph ideal in 4d covariant indices and use them to define the deformations $J_{\zeta}$ that are the annihilators of the deformations of the minrep. In section 3.4.4 we use the fact that annihilators vanish identically to identify the $AdS_5/Conf_4$ higher spin algebra (as defined by Eastwood [130]) and define its deformations as the enveloping algebras of the deformations of the minrep within the quasiconformal framework. In section 4.4 we discuss the extension of these results to higher spin superalgebras.

3.2 3d conformal algebra $SO(3, 2) \sim Sp(4, \mathbb{R})$ and its minimal unitary realization

In this section we shall review the twistorial oscillator construction of the unitary representations of the conformal groups $SO(3, 2)$ in $d = 3$ dimensions that correspond to conformally massless fields in $d = 3$ following [35,132]. These representations turn out to be the minimal unitary representations and are also called the singleton (scalar and spinor singleton) representations of Dirac [25]. Since the quartic invariant vanishes identically for the symplectic groups $Sp(2N, \mathbb{R})$, the quasiconformal and covariant oscillator constructions of $Sp(2N, \mathbb{R})$ coincide [22] and thus we will only review the oscillator construction of $Sp(4, \mathbb{R})$ following [35,132].
3.2.1 Twistorial oscillator construction of $SO(3, 2)$

The covering group of the three dimensional conformal group or the AdS$_4$ isometry group $SO(3, 2)$ is isomorphic to the non-compact symplectic group $Sp(4, \mathbb{R})$ with the maximal compact subgroup $U(2)$. Commutation relations of its generators can be written as

$$[M_{AB}, M_{CD}] = i(\eta_{BC}M_{AD} - \eta_{AC}M_{BD} - \eta_{BD}M_{AC} + \eta_{AD}M_{BC})$$

(3.2.1)

where $\eta_{AB} = \text{diag}(-, +, +, -)$ and $A, B = 0, 1, \ldots, 4$. The spinor representation of $SO(3, 2)$ can be realized in terms of four-dimensional gamma matrices $\gamma^\mu$ that satisfy

$$\{\gamma_\mu, \gamma_\nu\} = -2\eta_{\mu\nu}$$

where $\eta_{\mu\nu} = \text{diag}(-, +, +)$ and $\mu, \nu = 0, 1, \ldots, 3$ and $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$ as follows:

$$\Sigma_{\mu\nu} = -\frac{i}{4}[\gamma_\mu, \gamma_\nu], \quad \Sigma_{\mu 4} = -\frac{1}{2}\gamma_\mu \quad (3.2.2)$$

We adopt the following conventions for gamma matrices in four dimensions:

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_m = \begin{pmatrix} 0 & -\sigma_m \\ \sigma_m & 0 \end{pmatrix}, \quad \gamma_5 = i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.2.3)$$

where $\sigma_m (m = 1, 2, 3)$ are Pauli matrices. Consider now a pair of bosonic oscillators $a_i, a_i^\dagger$ ($i = 1, 2$) that satisfy

$$[a_i, a_j^\dagger] = \delta_{ij} \quad (3.2.4)$$

and define a twistorial (Majorana) spinor $\Psi$ and its Dirac conjugate in terms of these oscillators $\overline{\Psi} = \Psi^\dagger\gamma_0$

$$\Psi = \begin{pmatrix} a_1 \\ -ia_2 \\ ia_2^\dagger \\ -a_1^\dagger \end{pmatrix}, \quad \overline{\Psi} = \begin{pmatrix} a_1^\dagger \\ ia_2 \\ i a_2^\dagger \\ a_1 \end{pmatrix} \quad (3.2.5)$$

Then the bilinears $M_{AB} = 2\overline{\Psi}\Sigma_{AB}\Psi$ satisfy the commutation relations (3.2.1) of $SO(3, 2)$ Lie algebra.

The Fock space of these oscillators decompose into two irreducible unitary representations of $Sp(4, \mathbb{R})$ that are simply the two remarkable representations of Dirac [25] which were called Di and Rac in [133]. As representations of AdS$_4$, these representations do not have a Poincaré limit in 4d and their field theories live on the boundary of AdS$_4$ which can be identified with the conformal compactification of three dimensional Minkowski space [134]. It should be noted that Di and Rac (singletons) are conformal massless
representations in three dimensions but not of \( AdS_4 \). The tensor products of two singletons decomposes into infinite irreducible representations which are massless in \( AdS_4 \) sense but massive as representations of three dimensional conformal group. By massless in \( AdS_4 \) we mean that they become massless in the Poincaré limit since the \( P^2 \) is no longer a Casimir invariant for the \( AdS_4 \) group.

### 3.2.2 \( SO(3,2) \) algebra in conformal three-grading and 3d covariant twistors

The conformal algebra in \( d \) dimensions can be given a three graded decomposition with respect to the non-compact dilatation generator \( \Delta \) as follows:

\[
\mathfrak{so}(d,2) = K_\mu \oplus (M_{\mu \nu} \oplus \Delta) \oplus P_\mu \tag{3.2.6}
\]

We shall call this conformal 3-grading. The commutation relations of the algebra in this basis are given as follows:

\[
\begin{align*}
[M_{\mu \nu} , M_{\rho \tau}] &= i (\eta_{\rho \tau} M_{\mu \nu} - \eta_{\mu \nu} M_{\rho \tau} - \eta_{\nu \tau} M_{\mu \rho} + \eta_{\mu \rho} M_{\nu \tau}) \\
[P_\mu, M_{\nu \rho}] &= i (\eta_{\mu \nu} P_\rho - \eta_{\mu \rho} P_\nu) \\
[K_\mu, M_{\nu \rho}] &= i (\eta_{\mu \nu} K_\rho - \eta_{\mu \rho} K_\nu) \\
[\Delta, M_{\mu \nu}] &= [P_\mu, P_\nu] = [K_\mu, K_\nu] = 0 \\
[\Delta, P_\mu] &= +i P_\mu \quad [\Delta, K_\mu] = -i K_\mu \\
[P_\mu, K_\nu] &= 2i (\eta_{\mu \nu} \Delta + M_{\mu \nu})
\end{align*}
\]

where \( M_{\mu \nu} \) (\( \mu, \nu = 0, 1, ..., (d - 1) \)) are the Lorentz groups generators. \( P_\mu \) and \( K_\mu \) are the generators of translations and special conformal transformations.

In \( d = 3 \) dimensions the Greek indices \( \mu, \nu, \ldots \) run over 0, 1, 2 and dilatation generator is simply

\[
D = -M_{34} \tag{3.2.8}
\]

and translations \( P_\mu \) and special conformal transformations \( K_\mu \) are given by:

\[
\begin{align*}
P_\mu &= M_{\mu 4} + M_{\mu 3} \tag{3.2.9} \\
K_\mu &= M_{\mu 4} - M_{\mu 3} \tag{3.2.10}
\end{align*}
\]

In order to make connection with higher spin (super-)algebras it is best to write the algebra in \( SO(2,1) \) covariant spinorial oscillators. Let us now introduce linear combinations
of \(a_i, a_i^\dagger\) which we shall call 3d twistors\(^1\):

\[
\begin{align*}
\kappa_1 &= \frac{i}{2} \left( a_1 + a_2^\dagger + a_1^\dagger + a_2 \right), \\
\mu_1 &= \frac{i}{2} \left( a_1 - a_2^\dagger - a_1^\dagger + a_2 \right) \\
\kappa_2 &= \frac{1}{2} \left( a_1 + a_2^\dagger - a_1^\dagger - a_2 \right), \\
\mu_2 &= \frac{1}{2} \left( a_1 - a_2^\dagger + a_1^\dagger - a_2 \right)
\end{align*}
\]

(3.2.11) (3.2.12)

They satisfy the following commutation relations:

\[
[\kappa_\alpha, \mu^\beta] = \delta_\alpha^\beta
\]

(3.2.13)

Using these we can write (spinor conventions for \(SO(2,1)\) are given in appendix A.1):

\[
\begin{align*}
P_{\alpha\beta} &= (\sigma^\mu P_\mu)_{\alpha\beta} = -\kappa_\alpha \kappa_\beta \\
K^{\alpha\beta} &= (\bar{\sigma}^\mu K_\mu)^{\alpha\beta} = -\mu^\alpha \mu^\beta
\end{align*}
\]

(3.2.14) (3.2.15)

Similarly we can define the Lorentz generators

\[
M^{\alpha\beta} = i (\sigma^\mu \bar{\sigma}^\nu)^{\alpha\beta} M_{\mu\nu}
\]

(3.2.16)

and the dilatation generator

\[
\Delta = -\frac{i}{4} (\kappa_\alpha \mu^\alpha + \mu^\alpha \kappa_\alpha)
\]

(3.2.18)

In this basis the conformal algebra becomes:

\[
\begin{align*}
[M^{\alpha\beta}, M^{\gamma\delta}] &= \delta^{\delta}_{\alpha} M^{\beta}_{\gamma} - \delta^{\beta}_{\alpha} M^{\delta}_{\gamma} \\
[P_{\alpha\beta}, M^{\gamma\delta}] &= 2\delta^{\delta}_{(\alpha} P_{\beta)\gamma} - \delta^{\beta}_{\alpha} P_{\alpha\beta} \\
[K^{\alpha\beta}, M^{\gamma\delta}] &= -2\delta^{\gamma}_{(\alpha} K^{\beta)\delta} + \delta^{\delta}_{\alpha} K^{\alpha\beta} \\
[P_{\alpha\beta}, K^{\gamma\delta}] &= 4\delta^{(\gamma}_{(\alpha} M^{\delta)}_{\beta)} + 4i\delta^{(\gamma}_{(\alpha} \delta^{\beta)}_{\beta)} \Delta
\end{align*}
\]

(3.2.19) (3.2.20) (3.2.21) (3.2.22)

\[
[\Delta, K^{\alpha\beta}] = -i K^{\alpha\beta}, \quad [\Delta, M^{\alpha\beta}] = 0, \quad [\Delta, P_{\alpha\beta}] = i P_{\alpha\beta}
\]

(3.2.23)

The conformal group \(Sp(4,\mathbb{R})\) in three dimensions admits extensions to supergroups \(OSp(N|4,\mathbb{R})\) with even subgroups \(Sp(4,\mathbb{R}) \times O(N)\). We review the minimal unitary realization of \(OSp(N|4,\mathbb{R})\) in Appendix A.2.

\(^1\)Note that the 3d twistor variables defined in [132] look slightly different from the ones defined here because the oscillators \(a_i, a_i^\dagger\) are linear combinations of the ones used in [132].
3.3 Conformal and superconformal algebras in four dimensions

In this section, we present two different realizations of the conformal algebra and its supersymmetric extensions in $d = 4$. We start by reviewing the doubleton oscillator realization \[32,37,38\] and its reformulation in terms of Lorentz covariant twistorial oscillators \[37,132\]. We then present a novel formulation of the quasiconformal realization of the minimal unitary representation and its deformations first studied in \[23\] in terms of deformed twistorial oscillators.

3.3.1 Covariant twistorial oscillator construction of the doubletons of $SO(4,2)$

The covering group of the conformal group $SO(4,2)$ in four dimensions is $SU(2,2)$. Denoting its generators as $M_{AB}$ the commutation relations in the canonical basis are

$$[M_{AB}, M_{CD}] = i(\eta_{BC}M_{AD} - \eta_{AC}M_{BD} - \eta_{BD}M_{AC} + \eta_{AD}M_{BC})$$

where $\eta_{AB} = \text{diag}(-, +, +, +, +, -)$ and $A, B = 0, \ldots, 5$. The spinor representation of $SO(4,2)$ can be realized in terms in four-dimensional gamma matrices $\gamma_\mu$ that satisfy

$$\{\gamma_\mu, \gamma_\nu\} = -2\eta_{\mu\nu}$$

where $\eta_{\mu\nu} = \text{diag}(-, +, +, +, +, +)$ ($\mu, \nu = 0, \ldots, 3$) as follows:

$$\Sigma_{\mu\nu} = -\frac{i}{4}[\gamma_\mu, \gamma_\nu], \quad \Sigma_{\mu4} = -\frac{i}{2}\gamma_\mu\gamma_5, \quad \Sigma_{\mu5} = -\frac{1}{2}\gamma_\mu, \quad \Sigma_{45} = -\frac{1}{2}\gamma_5$$

Consider now two pairs of bosonic oscillators $a_i, a_i^\dagger$ ($i, j = 1, 2$) and $b_r, b_r^\dagger$ ($r, s = 1, 2$) that satisfy

$$[a_i, a_j^\dagger] = \delta_{ij}, \quad [b_r, b_s^\dagger] = \delta_{rs}$$

We form a twistorial Dirac spinor $\Psi$ and its conjugate $\Psi = \Psi^\dagger\gamma_0$ in terms of these oscillators:

$$\Psi = \begin{pmatrix} a_1 \\ a_2 \\ b_1^\dagger \\ -b_1 \\ b_2^\dagger \\ -b_2 \end{pmatrix}, \quad \Psi = \begin{pmatrix} a_1^\dagger a_2^\dagger b_1 b_2 \end{pmatrix}$$

Then the bilinears $M_{AB} = \Psi \Sigma_{AB} \Psi$ ($A, B = 0, \ldots, 5$) generate the Lie algebra of $SO(4,2)$:

$$[\Psi \Sigma_{AB} \Psi, \Psi \Sigma_{CD} \Psi] = \Psi [\Sigma_{AB}, \Sigma_{CD}] \Psi$$
which was called the doubleton realization \cite{32,37,38}.

The Lie algebra of $SU(2, 2)$ can be given a three-grading with respect to the algebra of its maximal compact subgroup $SU(2)_L \times SU(2)_R \times U(1)$

\[
su(2, 2) = L_{ir} \oplus (L^i_j + R^i_j + E) \oplus L^{ir}
\]

which is referred to as the compact three-grading. For the doubleton realizations one has

\[
L^i_j = a^i a_j - \frac{1}{2} \delta^i_j (a^k a_k), \quad R^i_j = b^i b_j - \frac{1}{2} \delta^i_j (b^k b_k) \tag{3.3.7}
\]

\[
L_{ir} = a_i b_r, \quad E = \frac{1}{2} (a^i a_i + b^r b_r), \quad L^{ir} = a^i b^r \tag{3.3.8}
\]

where the creation operators are denoted with upper indices, i.e $a^i_j = a^i$.

Under the $SU(2)_L \times SU(2)_R$ subgroup of $SU(2, 2)$ generated by the bilinears $L^i_j$ and $R^i_j$, the oscillators $a_i(a^i_j)$ and $b_i(b^i_j)$ transform in the $(1/2, 0)$ and $(0, 1/2)$ representation. Contrasting this to the situation in three dimensions where the Fock space of oscillators decomposes into the Di and Rac irreps, the Fock space of these bosonic oscillators decomposes into an infinite set of positive energy unitary irreducible representations (UIRs), called doubletons of $SU(2, 2)$. These UIRs are uniquely determined by a subset of states with the lowest eigenvalue (energy) of the $U(1)$ generator and transform irreducibly under the $SU(2)_L \times SU(2)_R$ subgroup. The possible lowest energy irreps of $SU(2)_L \times SU(2)_R$ for positive energy UIRs of $SU(2, 2)$ are of the form

\[
|a_{i_1} a_{i_2} \cdots a_{i_n} 0 \rangle \Leftrightarrow (j_L, j_R) = \left( \frac{n}{2}, 0 \right) \quad E = 1 + n/2 \tag{3.3.9}
\]

\[
|b_{r_1} b_{r_2} \cdots b_{r_m} 0 \rangle \Leftrightarrow (j_L, j_R) = \left( 0, \frac{m}{2} \right) \quad E = 1 + m/2 \tag{3.3.10}
\]

Analogous to the singletons in three dimensions, doubleton representations are massless in four dimensions and their tensor products decompose into an infinite set of massless spin representations in $AdS_5$ \cite{32,37,38}. The tensoring procedure in the oscillator construction just amounts to taking multiple copies (colors) of oscillators $a_i(\xi), a^i(\xi), b_i(\eta), b^i(\eta)$ where $\xi, \eta = 1, \ldots, P$ for a $P$-fold tensor product. The massless representations correspond to $P = 2$ and for $P > 2$, the tensor product decomposes into an infinite set of massive representations in $AdS_5$ which are also multiplicity free \cite{32,37,38}. We should stress that the resulting representations are multiplicity free and the explicit formulas for the tensor product decompositions of two irreducible doubleton representations were given in \cite{135}.

\footnote{The term doubleton refers to the fact that we are using oscillators that decompose into two irreps under the action of the maximal compact subgroup. For $SU(2, 2)$ that is the minimal set required. For symplectic groups the minimal set consists of oscillators that form a single irrep of their maximal compact subgroups.}
To relate the oscillators transforming covariantly under the maximal compact subgroup $SO(4) \times U(1)$ to twistorial oscillators transforming covariantly with respect to the Lorentz group $SL(2,\mathbb{C})$ with a definite scale dimension one acts with the intertwining operator 

$$T = e^{\frac{\pi}{4} M_{05}}$$ \hspace{1cm} (3.3.11)

which intertwines between the compact and the non-compact pictures

$$\mathcal{M}_a T = T L_a$$
$$\mathcal{N}_a T = T R_a$$
$$D T = T E$$ \hspace{1cm} (3.3.12)

where $L_a$ and $R_a$ denote the generators of $SU(2)_L$ and $SU(2)_R$, respectively. $M_a$ and $N_a$ are the generators of $SU(2)_M$ and $SU(2)_N$ given by the following linear combinations of the Lorentz group generators $M_{\mu\nu}$

$$\mathcal{M}_a = -\frac{1}{2} \left( \frac{1}{2} \epsilon_{abc} M_{bc} + i M_{0a} \right), \quad \mathcal{N}_a = -\frac{1}{2} \left( \frac{1}{2} \epsilon_{abc} M_{bc} - i M_{0a} \right)$$ \hspace{1cm} (3.3.13)

They satisfy

$$[\mathcal{M}_a, \mathcal{M}_b] = i \epsilon_{abc} M_c, \quad [\mathcal{N}_a, \mathcal{N}_b] = i \epsilon_{abc} N_c, \quad [\mathcal{M}_a, \mathcal{N}_b] = 0$$ \hspace{1cm} (3.3.14)

where $a, b, .. = 1, 2, 3$.

Under the action of the intertwining operator $T$, the compact subgroup $SU(2)_L \times SU(2)_R$ gets intertwined with the non-compact Lorentz group $SL(2,\mathbb{C})$. The oscillators $a_i (a^i)$ and $b_i (b^i)$ that transform in the $(1/2, 0)$ and $(0, 1/2)$ representation of $SU(2)_L \times SU(2)_R$ go over to covariant oscillators transforming as Weyl spinors $(1/2, 0)$ and $(0, 1/2)$ of the Lorentz group $SL(2,\mathbb{C})$. Denoting the components of the Weyl spinors with undotted ($\alpha, \beta, .. = 1, 2$) and dotted Greek indices ($\dot{\alpha}, \dot{\beta}, .. = 1, 2$) one finds :

$$\eta^\alpha = T a_i T^{-1} = \frac{1}{\sqrt{2}} (b_i - a^i)$$
$$\lambda_\alpha = T a^i T^{-1} = \frac{1}{\sqrt{2}} (b^i + a_i)$$
$$\tilde{\eta}^{\dot{\alpha}} = T b_i T^{-1} = \frac{1}{\sqrt{2}} (a_i - b^i)$$
$$\tilde{\lambda}_{\dot{\alpha}} = T b^i T^{-1} = \frac{1}{\sqrt{2}} (a^i + b_i)$$ \hspace{1cm} (3.3.15)

where $\alpha, \beta, \dot{\alpha}, \dot{\beta}, .. = 1, 2$ and the covariant indices on the left hand side match the indices
The dilatation generator in terms of covariant twistorial oscillators takes the form:

\[ \eta^a, \lambda_\beta = \delta_\beta^a \]
\[ \bar{\eta}^\dot{a}, \bar{\lambda}_\dot{\beta} = \delta_\dot{\beta}^\dot{a} \]  

(3.3.16)

They lead to the standard twistor relations\(^3\):

\[ P_{\alpha\dot{\beta}} = -(\sigma^\mu P_\mu)_{\alpha\dot{\beta}} = 2\lambda_\alpha \dot{\lambda}_\dot{\beta} = Ta^b b^r T^{-1} \]  
\[ K^{\dot{\alpha}\dot{\beta}} = -i(\bar{\sigma}^\mu \bar{K}_\mu)^{\dot{\alpha}\dot{\beta}} = 2\bar{\eta}^\dot{a} \eta^a = T a^b b^r T^{-1} \]  

(3.3.17) (3.3.18)

The dilatation generator in terms of covariant twistorial oscillators takes the form:

\[ \Delta = \frac{i}{2} (\lambda_\alpha \eta^\alpha + \bar{\eta}^\dot{\alpha} \bar{\lambda}_{\dot{\alpha}}) \]  

(3.3.19)

The Lorentz generators \( M_{\mu\nu} \) in a spinorial basis can also be written as bilinears of Lorentz covariant twistorial oscillators:

\[ M^{\alpha}_{\beta} = -\frac{i}{2} (\sigma^{\mu\nu})^{\alpha}_{\beta} M_{\mu\nu} = \lambda_\alpha \eta^\beta - \frac{1}{2} \delta_{\alpha}^{\beta} \lambda_\gamma \eta^\gamma \]  

(3.3.20)

\[ \bar{M}^{\dot{\alpha}}_{\dot{\beta}} = -\frac{i}{2} (\bar{\sigma}^{\mu\nu})_{\dot{\alpha}}^{\dot{\beta}} M_{\mu\nu} = -\left(\bar{\eta}^{\dot{a}} \dot{\lambda}_{\dot{a}} - \frac{1}{2} \delta_{\dot{a}}^{\dot{a}} \bar{\eta}^{\dot{\gamma}} \lambda_{\dot{\gamma}}\right) \]  

(3.3.21)

In this basis the conformal algebra becomes:

\[ [M^{\alpha}_{\beta}, M^\gamma_{\delta}] = \delta^{\gamma}_{\delta} M^{\alpha}_{\beta} - \delta^{\delta}_{\gamma} M^{\alpha}_{\beta}, \quad [\bar{M}^{\dot{\alpha}}_{\dot{\beta}}, \bar{M}^{\dot{\gamma}}_{\dot{\delta}}] = \delta^{\dot{\gamma}}_{\dot{\delta}} \bar{M}^{\dot{\alpha}}_{\dot{\beta}} - \delta^{\dot{\delta}}_{\dot{\gamma}} \bar{M}^{\dot{\alpha}}_{\dot{\beta}} \]  

(3.3.22)

\[ [P_{\alpha\dot{\beta}}, M^\gamma_{\delta}] = -\delta^\gamma_{\delta} P_{\alpha\dot{\beta}} + \frac{1}{2} \delta^\gamma_{\delta} P_{\alpha\dot{\beta}}, \quad [P_{\alpha\dot{\beta}}, \bar{M}^{\dot{\gamma}}_{\dot{\delta}}] = \delta^{\dot{\gamma}}_{\dot{\delta}} P_{\alpha\dot{\beta}} - \frac{1}{2} \delta^{\dot{\gamma}}_{\dot{\delta}} P_{\alpha\dot{\beta}} \]  

(3.3.23)

\[ [K^{\dot{\alpha}\dot{\beta}}, M^\gamma_{\delta}] = \delta^{\gamma}_{\delta} K^{\dot{\alpha}\dot{\beta}} - \frac{1}{2} \delta^\gamma_{\delta} K^{\dot{\alpha}\dot{\beta}}, \quad [K^{\dot{\alpha}\dot{\beta}}, \bar{M}^{\dot{\gamma}}_{\dot{\delta}}] = -\delta^{\dot{\gamma}}_{\dot{\delta}} K^{\dot{\alpha}\dot{\beta}} + \frac{1}{2} \delta^{\dot{\gamma}}_{\dot{\delta}} K^{\dot{\alpha}\dot{\beta}} \]  

(3.3.24)

\[ [\Delta, K^{\dot{\alpha}\dot{\beta}}] = -i K^{\dot{\alpha}\dot{\beta}}, \quad [\Delta, M^\gamma_{\delta}] = [\Delta, \bar{M}^{\dot{\gamma}}_{\dot{\delta}}] = 0, \quad [\Delta, P_{\alpha\dot{\beta}}] = i P_{\alpha\dot{\beta}} \]  

(3.3.25) (3.3.26)

The linear Casimir operator \( Z = N_a - N_b = a^i a_i - b^r b_r \) when expressed in terms of \( SL(2, \mathbb{C}) \) covariant oscillators becomes

\[ Z = N_a - N_b = \bar{\lambda}_\alpha \bar{\eta}^\alpha - \lambda_\alpha \eta^\alpha \]  

(3.3.27)

which shows that \( \frac{1}{2} Z \) is the helicity operator.

\(^3\) Note the overall minus sign in these expressions compared to \(^{132}\). This is due to the fact that we are using a mostly positive metric in this thesis.
Denoting the lowest energy irreps in the compact basis as $|\Omega(j_L, j_R, E)\rangle$ one can show that the coherent states of the form
\[
e^{-ix^\mu P_\mu T}|\Omega(j_L, j_R, E)\rangle \equiv |\Phi^\ell_{j_M,j_N}(x_\mu)\rangle
\] (3.3.28)
transform exactly like the states created by the action of conformal fields $\Phi^\ell_{j_M,j_N}(x_\mu)$ acting on the vacuum vector $|0\rangle$ with exact numerical coincidence of the compact and the covariant labels $(j_L, j_R, E)$ and $(j_M, j_N, -l)$, respectively, where $l$ is the scale dimension [38]. The doubletons correspond to massless conformal fields transforming in the $(j_L, j_R)$ representation of the Lorentz group $SL(2,C)$ whose conformal (scaling dimension) is $\ell = -E$ where $E$ is the eigenvalue of the $U(1)$ generator which is the conformal Hamiltonian (or $AdS_5$ energy) [32,37,38].

### 3.3.2 Quasiconformal approach to the minimal unitary representation of $SO(4,2)$ and its deformations

The construction of the minrep of the 4d conformal group $SO(4,2)$ by quantization of its quasiconformal realization and its deformations were given in [23], which we shall reformulate in this section in terms of what we call deformed twistorial oscillators which transform nonlinearly under the Lorentz group.

The quasiconformal realization of $SO(4,2)$ leaves invariant a light cone defined by a quartic distance function in five dimensions. Upon quantization, this realization leads to the nonlinear realization of the minrep in terms of two ordinary bosonic oscillators $d, d^\dagger$ and $g, g^\dagger$, and a singlet coordinate $x$, its conjugate momentum $p$ satisfying [23]:
\[
[x, p] = i, \quad [d, d^\dagger] = 1, \quad [g, g^\dagger] = 1
\] (3.3.29)
The nonlinearities can be absorbed into certain “singular” oscillators which are functions of the coordinate $x$, momentum $p$ and the oscillators $d, g, d^\dagger, g^\dagger$:
\[
A_C = a - \frac{\mathcal{L}}{\sqrt{2}x}, \quad A^\dagger_C = a^\dagger - \frac{\mathcal{L}}{\sqrt{2}x}
\] (3.3.30)
where
\[
a = \frac{1}{\sqrt{2}}(x + ip) \quad a^\dagger = \frac{1}{\sqrt{2}}(x - ip)
\]
56
\[ L = N_d - N_g - \frac{1}{2} = d^d - g^g - \frac{1}{2} \]  

(3.3.31)

They satisfy the following commutation relations:

\[ [A_g, A_K] = -\frac{(G - K)}{2x^2} \]

\[ [A^g, A^K] = +\frac{(G - K)}{2x^2} \]  

(3.3.32)

\[ [A_g, A^K] = 1 + \frac{(G + K)}{2x^2} \]

assuming that \([G, K] = 0\).

The realization of the minrep of \(SO(4, 2)\) obtained by the quasiconformal approach is nonlinear and “interacting” in the sense that they involve operators that are cubic or quartic in terms of the oscillators in contrast to the covariant twistorial oscillator realization, reviewed in section 3.3.1 [32], which involves only bilinears. The algebra \(so(4, 2)\) can be given a 3-graded decomposition with respect to the conformal Hamiltonian, which is referred to as the compact 3-grading and the generators in this basis are reproduced in Appendix A.3 following [23].

The Lie algebra of \(SO(4, 2)\) also has a non-compact (conformal) three graded decomposition determined by the dilatation generator \(\Delta\) as well

\[ so(4, 2) = \mathfrak{N}^- \oplus \mathfrak{N}^0 \oplus \mathfrak{N}^+ \]  

(3.3.33)

\[ = K_\mu \oplus (M_{\mu\nu} \oplus \Delta) \oplus P_\mu \]  

(3.3.34)

One can write the generators of quantized quasiconformal action of \(SO(4, 2)\) as bilinears of deformed twistorial oscillators \(Z^\alpha, \tilde{Z}^\dot{\alpha}, Y_\alpha, \tilde{Y}_{\dot{\alpha}} (\alpha, \dot{\alpha} = 1, 2)\) which are defined as:

\[ Z_1 = \frac{A_\ell}{\sqrt{2}} - ig^\dagger, \quad Y^1 = -\frac{A^\dagger_\ell}{\sqrt{2}} + ig \]  

(3.3.35)

\[ \tilde{Z}_1 = \frac{A^\dagger_\ell}{\sqrt{2}} + ig, \quad \tilde{Y}^1 = \frac{A_\ell}{\sqrt{2}} + ig \]  

(3.3.36)

\[ Z_2 = -\frac{A^-_\ell}{\sqrt{2}} - id, \quad Y^2 = \frac{A^-_\ell}{\sqrt{2}} - id \]  

(3.3.37)

\[ \tilde{Z}_2 = -\frac{A^-_\ell}{\sqrt{2}} + id, \quad \tilde{Y}^2 = \frac{A^\dagger_\ell}{\sqrt{2}} - id \]  

(3.3.38)

Using \((\sigma^\mu)_{\alpha\dot{\alpha}} = (1, \sigma)\) and \((\tilde{\sigma}^\mu)^{\dot{\alpha}\alpha} = (-1, \tilde{\sigma})\) one finds that the generators of translations
and special conformal transformations can be written as follows:

\[
P_{\alpha\beta} = (\sigma^\mu P_\mu)_{\alpha\beta} = -Z_\alpha \bar{Z}_\beta
\]

\[
K^{\dot{\alpha}\dot{\beta}} = (\bar{\sigma}^\mu K_\mu)^{\dot{\alpha}\dot{\beta}} = -\bar{Y}^{\dot{\alpha}} Y^{\dot{\beta}}
\]

We see that the operators \(Z\) and \(Y\) in the quasiconformal realization play similar roles as covariant twistorial oscillators \(\lambda\) and \(\eta\) in the doubleton realization. However, they transform nonlinearly under the Lorentz group and their commutations relations are given in Appendix A.4.

The dilatation generator in terms of deformed twistorial oscillators takes the form:

\[
\Delta = \frac{i}{4} \left( Z_\alpha Y^\alpha + \bar{Y}^{\dot{\alpha}} \bar{Z}_{\dot{\alpha}} \right)
\]

The Lorentz group generators \(M_{\mu\nu}\) in a spinorial basis can also be written as bilinears of these deformed twistorial oscillators:

\[
M^\alpha_\beta = -\frac{i}{2} (\sigma^\mu \bar{\sigma}^\nu)_{\alpha\dot{\beta}} M_{\mu\nu} = \frac{1}{2} \left( Z_\alpha Y^\beta - \frac{1}{2} \delta_\alpha^\beta Z_\gamma Y^\gamma \right)
\]

\[
\bar{M}^{\dot{\alpha}}_{\dot{\beta}} = -\frac{i}{2} (\bar{\sigma}^\mu \sigma^\nu)^{\dot{\alpha}}_{\dot{\beta\dot{\gamma}}} \bar{M}_{\mu\nu} = -\frac{1}{2} \left( \bar{Y}^{\dot{\alpha}} \bar{Z}_{\dot{\beta}} - \frac{1}{2} \delta_{\dot{\alpha}}^{\dot{\beta}} \bar{Y}^{\dot{\gamma}} \bar{Z}_{\dot{\gamma}} \right)
\]

We should stress the important point that even though the deformed twistorial oscillators transform nonlinearly under the Lorentz group, their bilinears \(P_{\alpha\beta}, K^{\dot{\alpha}\dot{\beta}}, M^\alpha_\beta\) and \(\bar{M}^{\dot{\alpha}}_{\dot{\beta}}\) transform covariantly and satisfy the commutation relations given in equations 3.3.22.

### 3.3.3 Deformations of the minimal unitary representation of \(SU(2, 2)\)

The minrep corresponds to a conformal massless scalar field but we know from the previous section that there are also doubleton representations corresponding to massless conformal fields with non-zero helicity \(h\). These representations can be recovered in the quasiconformal formalism by introducing a one-parameter family of deformations of the minrep as was done in [23]. The continuous deformation parameter \(\zeta\) is related to the helicity \(h\) of the conformal massless fields as \(h = \frac{\zeta}{2}\).

The generators of the deformed minrep take the same form as given in section 3.3.2 with the simple replacement of the singular oscillators \(A_L\) and \(A_L^\dagger\) by “deformed” singular oscillators:

\[
A_L^\zeta = a - \frac{\mathcal{L}_\zeta}{\sqrt{2}x} \quad \quad \quad A_L^{\dagger\zeta} = a^\dagger - \frac{\mathcal{L}_\zeta}{\sqrt{2}x}
\]

\(^4\)Note that in our conventions \(P^0\) is positive definite.
where
\[ \mathcal{L}_\zeta = \mathcal{L} + \zeta = N_d - N_g + \zeta - \frac{1}{2} \quad (3.3.45) \]

Since \( \zeta/2 \) labels the helicity we define “helicity deformed twistors” as follows:

\[
\begin{align*}
Z_1(\zeta) &= \frac{A_{\zeta}}{\sqrt{2}} - ig^\dagger, & Y^1(\zeta) &= -\frac{A_{\zeta}}{\sqrt{2}} + ig \\
\tilde{Z}_1(\zeta) &= \frac{A_{\zeta}}{\sqrt{2}} + ig, & \tilde{Y}^1(\zeta) &= \frac{A_{\zeta}}{\sqrt{2}} + ig^\dagger \\
Z_2(\zeta) &= -\frac{A_{-\zeta}}{\sqrt{2}} - id, & Y^2(\zeta) &= -\frac{A_{-\zeta}}{\sqrt{2}} - id^\dagger \\
\tilde{Z}_2(\zeta) &= -\frac{A_{-\zeta}}{\sqrt{2}} + id^\dagger, & \tilde{Y}^2(\zeta) &= \frac{A_{-\zeta}}{\sqrt{2}} - id
\end{align*}
\]

The realization of the minimal unitary representation in terms of deformed twistors carry over to realization in terms of helicity deformed twistors:

\[
\begin{align*}
P_{\alpha\dot{\beta}} &= (\sigma^\mu P_\mu)_{\alpha\dot{\beta}} = -Z_\alpha(\zeta)\tilde{Z}_\dot{\beta}(\zeta) \\
K^{\alpha\beta} &= (\bar{\sigma}^\mu K_\mu)^{\alpha\beta} = -\tilde{Y}^{\dot{\alpha}}(\zeta)Y^{\beta}(\zeta)
\end{align*}
\]

The dilatation generator then takes the form:

\[
\Delta = \frac{i}{4} \left( Z_\alpha(\zeta)Y^{\alpha}(\zeta) + \tilde{Y}^{\dot{\alpha}}(\zeta)\tilde{Z}_\dot{\alpha}(\zeta) \right)
\]

and the Lorentz generators \( M_{\mu\nu} \) also take the same form in terms of helicity deformed twistors:

\[
\begin{align*}
M_{\alpha\beta} &= -\frac{i}{2} (\sigma^\mu \sigma^\nu)_{\alpha\beta} M_{\mu\nu} = \frac{1}{2} \left( Z_\alpha(\zeta)Y^{\beta}(\zeta) - \frac{1}{2} \delta^\beta_{\gamma} Z_\gamma(\zeta)Y^{\gamma}(\zeta) \right) \\
\tilde{M}^{\dot{\alpha}\dot{\beta}} &= -\frac{i}{2} (\bar{\sigma}^\mu \bar{\sigma}^\nu)_{\dot{\alpha}\dot{\beta}} M_{\mu\nu} = -\frac{1}{2} \left( \tilde{Y}^{\dot{\alpha}}(\zeta)\tilde{Z}_\dot{\beta}(\zeta) - \frac{1}{2} \delta^{\dot{\gamma}}_{\dot{\beta}} \tilde{Y}^{\dot{\gamma}}(\zeta)\tilde{Z}_\dot{\gamma}(\zeta) \right)
\end{align*}
\]

The realization of the generators of \( SU(2, 2) \) in terms of helicity deformed twistors describe positive energy unitary irreducible representations which can best be seen by going over to the compact three-grading reviewed in Appendix A.3.

Since the quasiconformal realization of the minrep and its deformations are nonlinear the tensoring procedure in quasiconformal framework is a non-trivial and open problem. However since the representations with integer values of \( \zeta \) are isomorphic to the doubleton representations, the result of tensoring is already known and was discussed in section 3.3.1.
3.3.4 Minimal unitary supermultiplet of $SU(2, 2|4)$, its deformations and deformed twistors

The construction of the minimal unitary representations of non-compact Lie algebras by quantization of their quasiconformal realizations extends to non-compact Lie superalgebras $[22, 24, 56]$. In particular, the minimal unitary supermultiplets of $SU(2, 2|N)$ and their deformations were studied in $[23]$ using quasiconformal methods. In this section we shall reformulate the minimal unitary realization of 4d superconformal algebra $SU(2, 2|4)$ and its deformations in terms of deformed twistorial oscillators $^5$.

The superconformal algebra $su(2, 2|4)$ can be given a (non-compact) 5-graded decomposition with respect to the dilatation generator $\Delta$:

$$ su(2, 2|4) = \mathfrak{N}^{-1} \oplus \mathfrak{N}^{-1/2} \oplus \mathfrak{N}^0 \oplus \mathfrak{N}^{1/2} \oplus \mathfrak{N}^1 $$

$$ = K^{\dot{\alpha} \dot{\beta}} \oplus S_I^\alpha, \bar{S}^{I\dot{\alpha}} \oplus (M_\alpha^{\beta} \oplus M^{\dot{\alpha}}_{\dot{\beta}} \oplus \Delta \oplus R^I_{J}) \oplus Q_I^{\alpha}, \bar{Q}_{I\dot{\alpha}} \oplus P_{\alpha \dot{\beta}}, $$

$$(I, J = 1, 2, 3, 4)$$

where the grade zero subspace $\mathfrak{N}^0$ consists of the Lorentz algebra $so(3, 1)$ ($M_\alpha^{\beta}, M^{\dot{\alpha}}_{\dot{\beta}}$), the dilatations ($\Delta$) and R-symmetry $su(4)$ ($R^I_{J}$) generators, grade +1 and −1 subspaces consist of translation ($P_{\alpha \dot{\beta}}$) and special conformal generators ($K^{\dot{\alpha} \dot{\beta}}$) and the grade +1/2 and -1/2 subspaces consist of Poincaré supersymmetry ($Q_I^{\alpha}, \bar{Q}_{I\dot{\alpha}}$) and special conformal supersymmetry generators ($S_I^\alpha, \bar{S}^{I\dot{\alpha}}$) respectively.

The helicity deformed twistors for the superalgebra $SU(2, 2|4)$ are obtained from the deformed twistors of $SU(2, 2)$ by replacing $L_\zeta$ in the corresponding deformed singular oscillators $A_{L_\zeta}$ with $L_\zeta^\times$:

$$ L_\zeta \rightarrow L_\zeta^\times = N_d - N_s + N_\xi + \zeta - \frac{5}{2} $$

where $N_\xi = \xi^I \xi_J$ is the number operator of four fermionic oscillators $\xi_I (\xi^J)$ ($I, J = 1, 2, 3, 4$) that satisfy

$$ \{\xi_I, \xi^J\} = \delta^J_I $$

The expressions for the generators of $SU(2, 2)$ given in $[\text{3.3.2}]$ get modified as follows in going over to $SU(2, 2|4)$:

$$ P_{\alpha \dot{\beta}} = -Z_\alpha^\times(\zeta) \bar{Z}^\times_{\dot{\beta}}(\zeta) $$

$$ K^{\dot{\alpha} \dot{\beta}} = -Y^{s\dot{\alpha}}(\zeta) Y^{s\beta}(\zeta) $$

$^5$ $PSU(2, 2|4)$ is the symmetry superalgebra of $IIB$ supergravity compactified over $AdS_5 \times S^5$ symmetry.
\[ \Delta = \frac{i}{4} \left( Z^a_\alpha(\zeta) Y^{s\alpha}(\zeta) + \bar{Y}^{s\dot{\alpha}}(\zeta) \bar{Z}^a_\dot{\alpha}(\zeta) \right) \]  
(3.3.60)

\[ M_{\alpha}^{\beta} = \frac{1}{2} \left( Z^a_\alpha(\zeta) Y^{s\beta}(\zeta) - \frac{1}{2} \delta_\alpha^\beta Z^a_\gamma(\zeta) Y^{s\gamma}(\zeta) \right) \]  
(3.3.61)

\[ \bar{M}^{\dot{\alpha}}_{\dot{\beta}} = -\frac{1}{2} \left( \bar{Y}^{s\dot{\alpha}}(\zeta) \bar{Z}^a_\beta(\zeta) - \frac{1}{2} \delta^{\dot{\alpha}}_{\dot{\beta}} \bar{Y}^{s\dot{\gamma}}(\zeta) \bar{Z}^a_\gamma(\zeta) \right) \]  
(3.3.62)

where the “supersymmetric” helicity deformed twistors are defined as:

\[ Z^a_\alpha(\zeta) = \frac{A_{\zeta}}{\sqrt{2}} - i g^\alpha, \quad Y^{s1}(\zeta) = -\frac{A^\dagger_{\zeta}}{\sqrt{2}} + i g \]

\[ \bar{Z}^a_\dot{\alpha}(\zeta) = \frac{A^\dagger_{\zeta}}{\sqrt{2}} + i g, \quad \bar{Y}^{s1}(\zeta) = -\frac{A_{\zeta}}{\sqrt{2}} + i g^\dagger \]

\[ Z^a_\alpha(\zeta) = -\frac{A^\dagger_{\zeta}}{\sqrt{2}} - i d, \quad Y^{s2}(\zeta) = -\frac{A_{\zeta}}{\sqrt{2}} - i d^\dagger \]

\[ \bar{Z}^a_\dot{\alpha}(\zeta) = -\frac{A_{\zeta}}{\sqrt{2}} + i d, \quad \bar{Y}^{s2}(\zeta) = -\frac{A^\dagger_{\zeta}}{\sqrt{2}} + i d \]

The supersymmetry generators of \( SU(2,2|4) \) are given by the bilinears of deformed twistorial oscillators and fermionic oscillators:

\[ Q^I_{\cdot \alpha} = Z^a_\alpha(\zeta) \xi^I, \quad \bar{Q}^{\dot{I}}_{\cdot \dot{\alpha}} = -\xi^{\dot{I}} \bar{Z}^a_\dot{\alpha}(\zeta) \]  
(3.3.63)

\[ S^I_{\cdot \alpha} = -\xi^I Y^{s\alpha}(\zeta), \quad \bar{S}^{\dot{I}}_{\cdot \dot{\alpha}} = \bar{Y}^{s\dot{\alpha}}(\zeta) \xi^I \]  
(3.3.64)

The generators \( R^I_{\cdot J} \) of R-symmetry group \( SU(4) \) are given by:

\[ R^I_{\cdot J} = \xi^I \xi^J - \frac{1}{4} \delta^I_{\cdot J} \xi^K \xi^K \]  
(3.3.65)

They satisfy the following anti-commutation relations:

\[ \left\{ Q^I_{\cdot \alpha}, \bar{Q}^J_{\cdot \dot{\beta}} \right\} = \delta^I_J P_{\alpha \beta} \]  
(3.3.66)

\[ \left\{ \bar{S}^{I\dot{\alpha}}, S^J_{\cdot \beta} \right\} = \delta^I_J K^{\dot{\alpha} \beta} \]  
(3.3.67)

\[ \left\{ Q^I_{\cdot \alpha}, S^J_{\cdot \beta} \right\} = -2 \delta^I_J M_{\alpha \beta} + 2 \delta^I_J R^I_{\cdot J} + \delta^I_J \delta^J_{\alpha \beta} (i\Delta + C) \]  
(3.3.68)

\[ \left\{ \bar{S}^{I\dot{\alpha}}, \bar{Q}^J_{\cdot \dot{\beta}} \right\} = 2 \delta^I_J M^{\dot{\alpha} \dot{\beta}} - 2 \delta^I_J R^I_{\cdot J} + \delta^I_J \delta^J_{\dot{\alpha} \dot{\beta}} (i\Delta - C) \]  
(3.3.69)
where \( C = \frac{\zeta}{2} \) is the central charge.

The commutators of conformal group generators with supersymmetry generators are as follows:

\[
\begin{align*}
[P_{a\beta}, S_I^\gamma] &= 2 \delta_\beta^\gamma Q_{I\beta}, \quad [K^{\alpha\beta}, Q_I^\gamma] = -2 \delta_\beta^\gamma \bar{S}^I_{\bar{\alpha}} \\
[P_{a\beta}, \bar{S}^I_{\bar{\gamma}}] &= 2 \delta_\beta^{\bar{\gamma}} Q_a^I, \quad [K^{\bar{\alpha}\bar{\beta}}, \bar{Q}_{I\bar{\gamma}}] = -2 \delta_\beta^{\bar{\gamma}} S_I^\beta
\end{align*}
\] (3.3.70)

\[
\begin{align*}
[M_{\alpha}^\beta, Q_I^\gamma] &= \delta_\beta^\gamma Q_I^\alpha - \frac{1}{2} \delta_\alpha^\beta Q_I^\gamma, \quad [M_{\alpha}^\beta, S_I^\gamma] = -\delta_\alpha^\gamma S_I^\beta + \frac{1}{2} \delta_\beta^\gamma S_I^\gamma \\
[M_{\bar{\alpha}}_{\bar{\beta}}, \bar{Q}_{I\bar{\gamma}}] &= -\delta^{\bar{\gamma}}_\beta Q_{I\bar{\beta}} + \frac{1}{2} \delta^{\bar{\gamma}}_\alpha \bar{Q}_{I\bar{\alpha}}, \quad [M_{\bar{\alpha}}_{\bar{\beta}}, \bar{S}^I_{\bar{\gamma}}] = \delta^{\bar{\gamma}}_{\bar{\beta}} \bar{S}^I_{\bar{\alpha}} - \frac{1}{2} \delta^{\bar{\gamma}}_{\alpha} \bar{Q}^I_{\bar{\gamma}}
\end{align*}
\] (3.3.72)

\[
\begin{align*}
[\Delta, Q_I^\alpha] &= \frac{i}{2} Q_I^\alpha, \quad [\Delta, \bar{Q}_{I\bar{\alpha}}] = \frac{i}{2} \bar{Q}_{I\bar{\alpha}} \\
[\Delta, S_I^\alpha] &= -\frac{i}{2} S_I^\alpha, \quad [\Delta, \bar{S}^I_{\bar{\alpha}}] = -\frac{i}{2} \bar{S}^I_{\bar{\alpha}}
\end{align*}
\] (3.3.74)

The \( \mathfrak{su}(4)_R \) generators satisfy the following commutation relations:

\[
[R^I_J, R^K_L] = \delta^K_J R^I_L - \delta^K_L R^I_J
\] (3.3.76)

They act on the R-symmetry indices \( I, J \) of the supersymmetry generators as follows:

\[
\begin{align*}
[R^I_J, Q^K_{\alpha}] &= \delta^K_J Q_I^\alpha - 4 \delta^K_I Q_{J\alpha}, \quad [R^I_J, \bar{Q}_{K\bar{\alpha}}] = -\delta^K_J \bar{Q}_{I\bar{\alpha}} + 4 \delta^K_I \bar{Q}_{J\bar{\alpha}} \\
[R^I_J, S^K_{\alpha}] &= -\delta^K_J S_I^\alpha + 4 \delta^K_I S_{J\alpha}, \quad [R^I_J, \bar{S}^{K\bar{\alpha}}] = \delta^K_J \bar{S}^{I\bar{\alpha}} - 4 \delta^K_I \bar{S}^{J\bar{\alpha}}
\end{align*}
\] (3.3.77)

The minimal unitary representation of \( PSU(2,2|4) \) is obtained when the deformation parameter \( \zeta \), which is also the central charge, vanishes. The resulting minimal unitary supermultiplet of massless conformal fields in \( d = 4 \) is simply the \( N = 4 \) Yang-Mills supermultiplet \cite{23}. For each value of the deformation parameter \( \zeta \) one obtains an irreducible unitary representation of \( SU(2,2|4) \). For integer values of the deformation parameter these unitary representations are isomorphic to doubleton supermultiplets studied in \cite{32,37,38}.

The unitarity of the representations of \( SU(2,2|N) \) may not be manifest in the Lorentz covariant non-compact five grading. It is however manifestly unitary in compact three grading with respect to the sub-supergroup \( SU(2|N-M) \times SU(2|M) \times U(1) \) as was shown for the the doubletons in \cite{32,37} and for the quasiconformal construction in \cite{23}. The Lie algebra of \( SU(4) \) can be given a 3-graded structure with respect to the Lie algebra of its subgroup \( SU(2) \times SU(2) \times U(1) \). Similarly the Lie superalgebra \( SU(2,2|4) \) can be
given a 3-graded decomposition with respect to its subalgebra $SU(2|2) \times SU(2|2) \times U(1)$. This is the basis that was originally used by Gunaydin and Marcus [32] in constructing the spectrum of $IIB$ supergravity over $AdS_5 \times S^5$ using twistorial oscillators. In this basis, choosing the Fock vacuum as the lowest weight vector leads to CPT-self-conjugate supermultiplets and it is also the preferred basis in applications to integrable spin chains. The corresponding compact 3-grading of the quasiconformal realization of $SU(2,2|4)$ was given in [23], to which we refer for details.

### 3.4 Higher spin (super-)algebras, Joseph ideals and their deformations

In this section we start by reviewing Eastwood’s results [130,136] on defining $HS(g)$ algebras as the quotient of universal enveloping algebra $\mathcal{U}(g)$ by its Joseph ideal $J(g)$. We will then explicitly compute the Joseph ideal for $SO(3,2)$, $SO(4,2)$ and its deformations, using the Eastwood formula [136] and recast it in a Lorentz covariant form.

The universal enveloping algebra $\mathcal{U}(g)$, $g = so(d-1,2)$ is defined as follows:

$$\mathcal{U}(g) = \mathcal{G}/\mathcal{J}$$  \hfill (3.4.1)

where $\mathcal{G}$ is the associative algebra freely generated by elements of $\mathfrak{g}$, and $\mathcal{J}$ is the ideal of $\mathcal{G}$ generated by elements of form $gh - hg - [g, h]$ ($g, h \in \mathfrak{g}$).

The enveloping algebra $\mathcal{U}(g)$ can be decomposed into standard adjoint action of $\mathfrak{g}$ which by Poincare-Birkhoff-Witt theorem is equivalent to computing symmetric products $M_{AB} \sim  \Box$. In particular, $\otimes^2 so(d-1,2)$ decomposes as:

$$\begin{array}{c}
\otimes \\
\oplus \\
\oplus \\
\oplus \\
\bullet
\end{array}$$  \hfill (3.4.2)

where $\bullet$ is the quadratic Casimir $C_2 \sim M_B^A M_A^B$. It was already noted in [80] that the higher spin algebra $HS(g)$ must be a quotient of $\mathcal{U}(g)$ because the higher spin fields in $AdS_d$ are described by traceless two row Young tableaux. Thus the relevant ideal should quotient out all the diagrams except the first one in the above decomposition. This ideal was identified in [130] to be the Joseph ideal or the annihilator of the minimal unitary representation (scalar doubleton). The uniqueness of this quadratic ideal in $\mathcal{U}(g)$ was proved in [136] and an explicit formula for the generator $J_{ABCD}$ of the ideal was given as:

$$J_{ABCD} = M_{AB}M_{CD} - M_{AB} \otimes M_{CD} - \frac{1}{2} [M_{AB}, M_{CD}] + \frac{n-4}{2(n-1)(n-2)} \langle M_{AB}, M_{CD} \rangle$$  \hfill (3.4.3)
\[
\frac{1}{2} M_{AB} \cdot M_{CD} - M_{AB} \odot M_{CD} + \frac{n-4}{4(n-1)(n-2)} \langle M_{AB}, M_{CD} \rangle 1 \quad (3.4.3)
\]

where the dot \( \cdot \) denotes the symmetric product

\[
M_{AB} \cdot M_{CD} \equiv M_{AB}M_{CD} + M_{CD}M_{AB} \quad (3.4.4)
\]

of the generators and \( \langle M_{AB}, M_{CD} \rangle \) is the Killing form of \( SO(n-2, 2) \). \( \eta_{AB} \) is the \( SO(n-2, 2) \) invariant metric and the symbol \( \odot \) denotes the Cartan product of two generators, which for \( SO(n-2, 2) \), can be written in the form \[137\]:

\[
M_{AB} \odot M_{CD} = \frac{1}{3} M_{AB}M_{CD} + \frac{1}{3} M_{DC}M_{BA} + \frac{1}{6} M_{AC}M_{BD}
\]

\[
- \frac{1}{6} M_{AD}M_{BC} + \frac{1}{6} M_{DB}M_{CA} - \frac{1}{6} M_{CB}M_{DA}
\]

\[
- \frac{1}{2(n-2)} (M_{AE}M_{C}^{E} \eta_{BD} - M_{BE}M_{C}^{E} \eta_{AD} + M_{BE}M_{D}^{E} \eta_{AC} - M_{AE}M_{D}^{E} \eta_{BC})
\]

\[
- \frac{1}{2(n-2)} (M_{CE}M_{A}^{E} \eta_{BD} - M_{CE}M_{B}^{E} \eta_{AD} + M_{DE}M_{B}^{E} \eta_{AC} - M_{DE}M_{A}^{E} \delta_{BC})
\]

\[
+ \frac{1}{(n-1)(n-2)} M_{EF}M_{EF} (\eta_{AC}\eta_{BD} - \eta_{BC}\eta_{AD}) \quad (3.4.5)
\]

The Killing term is given by

\[
\langle M_{AB}, M_{CD} \rangle = h M_{EF}M_{GH}(\eta^{EG}\eta^{FH} - \eta^{EH}\eta^{FG})(\eta_{AC}\eta_{BD} - \eta_{AD}\eta_{BC}) \quad (3.4.6)
\]

where \( h = \frac{2(n-2)}{n(4-n)} \) is a c-number fixed by requiring that all possible contractions of \( J_{ABCD} \) with the metric vanish. We shall refer to the operator \( J_{ABCD} \) as the generator of the Joseph ideal.

The generator of the Joseph ideal defined in equation \[3.4.3\] contains exactly the operators that correspond to the “unwanted” diagrams described in equation \[3.4.2\] and quotienting the enveloping algebra by this ideal guarantees that the resulting algebra will contain only two row traceless diagrams and thus correctly describe massless higher spin fields.

In the following sections we will compute the generator \( J_{ABCD} \) for \( d = 3 \) and 4 conformal algebras \( SO(3, 2) \) and \( SO(4, 2) \) in various realizations discussed in previous sections. We shall also decompose the generators of the Joseph ideal of \( SO(3, 2) \) and \( SO(4, 2) \) with respect to the corresponding Lorentz groups \( SO(2,1) \) and \( SO(3,1) \), respectively. The Lorentz covariant decomposition makes the massless nature of the minimal unitary representations explicit along with certain other identities that must be satisfied within the representation in order for it to be annihilated by the Joseph ideal. This will also allow us to define the annihilators of the deformations of the minrep of \( SO(4,2) \) and
the corresponding deformations of the Joseph ideal. These deformations define a one parameter family of $AdS_5/CFT_4$ $HS$ algebras.

### 3.4.1 Joseph ideal for $SO(3, 2)$ singletons

We will now use the twistorial oscillator realization for $SO(3, 2)$ described in section 3.2.1. For $Sp(4, \mathbb{R}) = SO(3, 2)$, the generator $J_{ABCD}$ of the Joseph ideal is

$$J_{ABCD} = M_{AB}M_{CD} - M_{AB} \otimes M_{CD} - \frac{1}{2}[M_{AB}, M_{CD}] - \frac{1}{40}(M_{AB}, M_{CD})$$

Substituting the realization of $Sp(4, \mathbb{R}) = SO(3, 2)$ in terms of a twistorial Majorana spinor $\Psi$ one finds that the operator $J_{ABCD}$ vanishes identically.

Considered as the the three dimensional conformal group the minreps of $Sp(4, \mathbb{R})$ (Di and Rac) correspond to massless scalar and spinor fields which are known to be the only massless representations of the Poincaré group in three dimensions [44]. If instead of a twistorial Majorana spinor one considers a twistorial Dirac spinor corresponding to taking two copies (colors) of the Majorana spinor one finds that the generators $J_{ABCD}$ of the Joseph ideal do not vanish identically and hence they do not correspond to minimal unitary representations. The corresponding Fock space decomposes into an infinite set of irreducible unitary representations of $Sp(4, \mathbb{R})$, which correspond to the massless fields in $AdS_4$ [134]. Taking more than two colors in the realization of the Lie algebra of $Sp(4, \mathbb{R})$ as bilinears of oscillators leads to representations corresponding to massive fields in $AdS_4$ [35].

#### 3.4.1.1 Joseph ideal of $SO(3, 2)$ in Lorentz covariant basis

To get a more physical picture of what the vanishing of the Joseph ideal means we shall go to the conformal 3-graded basis defined in equation 3.2.6. Evaluating the Joseph ideal in this basis, we find that the vanishing ideal is equivalent to the linear combinations of certain quadratic identities, full set of which hold only in the singleton realization. First we have the masslessness conditions:

$$P^2 = P^\mu P_\mu = 0 \quad , \quad K^2 = K^\mu K_\mu = 0 \quad (3.4.8)$$

The remaining set of quadratic relations that define the Joseph ideal are

$$6\Delta \cdot \Delta + 2M^{\mu \nu} \cdot M_{\mu \nu} + P^\mu \cdot K_\mu = 0 \quad (3.4.9)$$

$$P^\mu \cdot (M_{\mu \nu} + \eta_{\mu \nu} \Delta) = 0 \quad (3.4.10)$$
\[ K^{\mu} \cdot (M_{\nu\mu} + \eta_{\nu\rho}\Delta) = 0 \]  \hspace{1cm} (3.4.11)

\[ \eta^{\mu\nu}M_{\mu\rho} \cdot M_{\nu\sigma} - P_{\rho} \cdot K_{\sigma} + \eta_{\rho\sigma} = 0 \]  \hspace{1cm} (3.4.12)

\[ \Delta \cdot M_{\mu\nu} + P_{[\mu} \cdot K_{\nu]} = 0 \]  \hspace{1cm} (3.4.13)

\[ M_{[\mu\nu} \cdot P_{\rho]} = 0 \]  \hspace{1cm} (3.4.14)

\[ M_{[\mu\nu} \cdot K_{\rho]} = 0 \]  \hspace{1cm} (3.4.15)

The ideal generated by these relations is completely equivalent to equation 3.4.7 but it sheds light on the massless nature of these representations. The scalar and spinor singleton modules for SO(3, 2) are the only minreps and there are no other deformations. This is a general phenomenon for all symplectic groups Sp(2n, R) \((n > 2)\) and within the quasiconformal approach it can be explained by the fact that the corresponding quartic invariant that enters the minimal unitary realization vanishes for symplectic groups.

The Casimir invariants for the singleton or the minrep of SO(3, 2) are as follows:

\[ C_2 = M_B^A M_A^B = \frac{5}{2} \]  \hspace{1cm} (3.4.16)

\[ C_4 = M_B^A M_C^B M_D^C M_A^D = -\frac{35}{8} \]  \hspace{1cm} (3.4.17)

Computing the products of the generators of the above singleton realization corresponding to the Young tableaux \(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\) and \(\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\) one finds that they vanish identically and the resulting enveloping algebra contains only the operators whose Young tableaux have two rows.

### 3.4.2 Joseph ideal of SO(4, 2)

In this subsection we shall first evaluate the generator \(J_{ABCD}\) of the Joseph ideal of SO(4, 2) in the covariant twistorial operator realization and then in the quasiconformal realization of its minrep to highlight the essential differences.

#### 3.4.2.1 Joseph ideal in the covariant twistorial oscillator or doubleton realization

We will use the doubleton realization reviewed in section 3.3.1 to compute the generator \(J_{ABCD}\) of the Joseph ideal for SO(4, 2):

\[ J_{ABCD} = M_{AB}M_{CD} - M_{AB} \otimes M_{CD} - \frac{1}{2} [M_{AB}, M_{CD}] - \frac{1}{60} (M_{AB}, M_{CD}) \]

\[ = \frac{1}{2} M_{AB} \cdot M_{CD} - M_{AB} \otimes M_{CD} - \frac{1}{60} (M_{AB}, M_{CD}) \]  \hspace{1cm} (3.4.18)
Substituting the expressions for the generators in the covariant twistorial realization one finds that it does not vanish identically as an operator in contrast to the situation with the singletonic realization of SO(3, 2). However one finds that $J_{ABCD}$ has only 15 independent non-vanishing components which turn out to be equal to one of the following expressions (up to an overall sign):

\[
(a_1 b_2 + a_2 b_1) \pm (a_1^i b_2^1 + a_1^1 b_2^i) Z
\]

\[
(a_1 b_2 - a_2 b_1) \pm (a_1^i b_2^1 - a_1^1 b_2^i) Z
\]

\[
(a_1 b_1 + a_2 b_2) \pm (a_2^1 b_1^i + a_1^i b_1^1) Z
\]

\[
(a_1 b_1 - a_2 b_2) \pm (a_2^1 b_1^i - a_1^i b_1^1) Z
\]

\[
(a_1 a_2 + a_1 a_2^1 \pm b_1 b_2 \pm b_1 b_2^1) Z
\]

\[
(a_1 a_2 - a_1 a_2^1 \pm b_1 b_2 \mp b_1 b_2^1) Z
\]

\[
(N_{a_1} - N_{a_2}) \pm (N_{b_1} - N_{b_2}) Z
\]

\[
(N_a + N_b + 2) Z
\]

Similarly, computing the products of the generators corresponding to the the Young tableaux explicitly in the covariant twistorial realization one finds that they do not vanish but we have the following relation:

\[
= \; Z
\]

Thus all non-vanishing four-row diagrams can be dualized to two-row diagrams and the resulting generators of the universal enveloping algebra in doubleton realization are described by two-row diagrams.

The operator $Z = (N_a - N_b)$ commutes with all the generators of $SU(2, 2)$ and its eigenvalues label the helicity of the corresponding massless representation of the conformal group [23,32,37]. All the components of the generator $J_{ABCD}$ of Joseph ideal vanish on the states that form the basis of an UIR of $SU(2, 2)$ whose lowest weight vector is the Fock vacuum $|0\rangle$ since

\[
Z|0\rangle = 0
\]

The corresponding unitary representation describes a conformal scalar field in four dimensions (zero helicity) and is the true minrep of $SU(2, 2)$ annihilated by the Joseph ideal [23].
The Casimir invariants for $SO(4,2)$ in the doubleton representation are as follows:

\[ C_2 = M_B^A M_A^B = \frac{3}{2} (4 - Z^2) \] (3.4.29)

\[ C_3' = \epsilon^{ABCDEF} M_{AB} M_{CD} M_{EF} = 4 Z C_2 = 8 C_2 \sqrt{1 - \frac{C_2}{6}} \] (3.4.30)

\[ C_4 = M_B^A M_C^B M_D^C M_A^D = \frac{C_2^2}{6} - 4 C_2 \] (3.4.31)

Thus we see that all the higher order Casimir invariants are functions of the quadratic Casimir $C_2$ which itself is given in terms of $Z = N_a - N_b$.

### 3.4.2.2 Joseph ideal and the quasiconformal realization of the minrep of $SO(4,2)$

To apply Eastwood’s formula to the generator of Joseph ideal in the quasiconformal realization it is convenient to go from the conformal 3-graded basis to the $SO(4,2)$ covariant canonical basis where the generators $M_{AB}$ satisfy the following commutation relations:

\[ [M_{AB}, M_{CD}] = i (\eta_{BC} M_{AD} - \eta_{AC} M_{BD} - \eta_{BD} M_{AC} + \eta_{AD} M_{BC}) \] (3.4.32)

where the metric $\eta_{AB} = \text{diag}(-, +, +, +, +, -)$ is used to raise and lower the indices $A, B = 0, 1, \ldots, 5$ etc. In addition to Lorentz generators $M_{\mu\nu} \ (\mu, \nu, \ldots = 0, 1, 2, 3)$, we have the following linear relations between the generators in the canonical basis and the conformal 3-graded basis

\[ M_{\mu 4} = \frac{1}{2} (P_{\mu} - K_{\mu}) \] (3.4.33)

\[ M_{\mu 5} = \frac{1}{2} (P_{\mu} + K_{\mu}) \] (3.4.34)

\[ M_{45} = -\Delta \] (3.4.35)

Substituting the expressions for the quasiconformal realization of the generators of $SO(4,2)$ given in subsection 3.3.2 into the generator of the Joseph ideal in the canonical basis:

\[ J_{ABCD} = \frac{1}{2} M_{AB} \cdot M_{CD} - M_{AB} \odot M_{CD} - \frac{1}{60} \langle M_{AB}, M_{CD} \rangle \] (3.4.36)

one finds that it vanishes identically as an operator showing that the corresponding unitary representation is indeed the minimal unitary representation.

We should stress the important point that the tensor product of the Fock spaces of the two oscillators $d$ and $g$ with the state space of the singular oscillator $A_L (A_L^*)$ form the
basis of a single UIR which is the minrep. In contrast, the Fock space of the covariant
twistorial oscillators reviewed in section 3.3.1 decomposes into infinitely many UIRs
doubletons of which only the irreducible representation whose lowest weight vector is
the Fock vacuum, which is annihilated by $J_{ABCD}$, is the minimal unitary representation
of $SO(4,2)$.

3.4.2.3 4d Lorentz Covariant Formulation of the Joseph ideal of $SO(4,2)$

Above we showed that the generator of the Joseph ideal given in equation (3.4.36)
vanishes identically as an operator for the quasiconformal realization of the minrep of
$SO(4,2)$ in the canonical basis. To get a more physical picture of what the vanishing
of generator $J_{ABCD}$ means we shall go to the conformal three grading defined by the
dilatation generator $\Delta$. Evaluating the generator of the Joseph ideal in this basis, we
find that the vanishing condition is equivalent to linear combinations of certain quadratic
identities, full set of which hold only in the quasiconformal realization of the minrep.
First we have the conditions:

$$P^2 = P\mu P_\mu = 0 \quad , \quad K^2 = K\mu K_\mu = 0$$  \hspace{1cm} (3.4.37)

which hold also for the twistorial oscillator realization given in section 3.3.1. The remaining
set of quadratic relations that define the Joseph ideal are

$$4\Delta \cdot \Delta + M_{\mu\nu} \cdot M_{\mu\nu} + P^\mu \cdot K_\mu = 0$$  \hspace{1cm} (3.4.38)

$$P^\mu \cdot (M_{\mu\nu} + \eta_{\mu\nu}\Delta) = 0$$  \hspace{1cm} (3.4.39)

$$K^\mu \cdot (M_{\mu\nu} + \eta_{\nu\mu}\Delta) = 0$$  \hspace{1cm} (3.4.40)

$$\eta^{\mu\nu} M_{\rho\sigma} \cdot M_{\mu\sigma} - P_{(\rho} \cdot K_{\sigma)} + 2\eta_{\rho\sigma} = 0$$  \hspace{1cm} (3.4.41)

$$M_{\mu\nu} \cdot M_{\rho\sigma} + M_{\mu\sigma} \cdot M_{\nu\rho} + M_{\mu\rho} \cdot M_{\nu\sigma} = 0$$  \hspace{1cm} (3.4.42)

$$\Delta \cdot M_{\mu\nu} + P_{[\mu} \cdot K_{\nu]} = 0$$  \hspace{1cm} (3.4.43)

$$M_{[\mu\nu} \cdot P_{\rho]} = 0, \quad M_{[\mu\nu} \cdot K_{\rho]} = 0$$  \hspace{1cm} (3.4.44)

In four dimensions, using the Levi-Civita tensor one can define the Pauli-Lubanski vector,
$W^\mu$ and its conformal analogue, $V^\mu$ as follows:

$$W^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma}, \quad V^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} K_\nu M_{\rho\sigma}$$  \hspace{1cm} (3.4.45)
where $\epsilon_{0123} = +1, \epsilon^{0123} = -1$ and the indices are raised and lowered by the Minkowski metric. For massless fields, $W^\mu$ and $V^\mu$ are proportional to $P^\mu$ and $K^\mu$ respectively with the proportionality constant related to helicity of the fields [138,139]. Equations (3.4.44) imply that for the minrep both the $W^\mu$ and $V^\mu$ vanish implying that it describes a zero helicity (scalar) massless field [3].

Computing the products of the generators in the above quasiconformal realization corresponding to Young tableaux \[ \begin{array}{c} \hline \hline \end{array} \] \[ \begin{array}{c} \hline \hline \end{array} \] explicitly one finds that they vanish identically and the resulting enveloping algebra contains only operators with two row Young tableaux.

The Casimir operators of the minrep of $SO(4,2)$ are as take on the following values (using the definitions given in equations (3.4.29) - (3.4.31)):

\[ C_2 = 6, \quad C_3 = 0 \quad C_4 = -18. \] (3.4.46)

### 3.4.3 Deformations of the minrep of $SO(4,2)$ and their associated ideals

As was shown in reference [23], the minimal unitary representation of $SU(2,2)$ that corresponds to a conformal scalar field admits a one-parameter ($\zeta$) family of deformations corresponding to massless conformal fields of helicity $\frac{\zeta}{2}$ in four dimensions, which can be continuous. For non-integer values of the deformation parameter $\zeta$ they correspond, in general, to unitary representations of an infinite covering of the conformal group [7].

The generators of the deformed minreps were reformulated in terms of deformed twistorial oscillators in section [3.3.3]. Substituting the expressions for the generators of the deformed minreps of $SO(4,2)$ into the generator $J_{ABCD}$ of the Joseph ideal one finds that it does not vanish for non-zero values of the deformation parameter $\zeta$. One might therefore ask if there exists deformations of the Joseph ideal that annihilate the deformed minimal unitary representations labelled by the deformation parameter $\zeta$. Remarkably, this is indeed the case. The quadratic identities that define the Joseph ideal in the conformal basis discussed in the previous section go over to identities involving the deformation parameter $\zeta$ and define the deformations of the Joseph ideal. One finds that the helicity conditions are modified as follows:

\[ \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} M_{\nu\rho} \cdot P_\sigma = \zeta P_\mu \] (3.4.47)

---

\* We should note that a similar set of identities (constraints) were discussed in [139] in the context of deriving field equations for particles of all spins (acting on field strengths) where they arise as conformally covariant forms of massless particles.

\* Recently, such a continuous helicity parameter was introduced as a spectral parameter for scattering amplitudes in N=4 super Yang-Mills theory in [140].
\[
\frac{1}{2} \epsilon^\mu\nu\rho\sigma M_\nu \cdot K_\sigma = -\zeta K^\mu
\] (3.4.48)

The identities (3.4.41), (3.4.42) and (3.4.43) get also modified as follows:

\[
\eta^{\mu\nu} M_\mu \cdot M_\nu - P_{[\nu} \cdot K_{\sigma]} + 2\eta_{\rho\sigma} = \frac{\zeta^2}{2} \eta_{\rho\sigma}
\] (3.4.49)

\[
M_\mu \cdot M_{\rho\sigma} + M_{\mu\sigma} \cdot M_{\nu\rho} + M_{\mu\rho} \cdot M_{\sigma\nu} = \zeta \epsilon_{\mu\nu\rho\sigma} \Delta
\] (3.4.50)

\[
\Delta \cdot M_\mu + P_{[\mu} \cdot K_{\nu]} = -\frac{\zeta}{2} \epsilon_{\mu\nu\rho\sigma} M^{\rho\sigma}
\] (3.4.51)

The other quadratic identities remain unchanged in going over to the deformed minimal unitary representations.

The Casimir invariants for the deformations of the minrep of \(SO(4,2)\) depend only the deformation parameter \(\zeta\) and are given as follows (using the definitions given in equations 3.4.29 - 3.4.31):

\[
C_2 = 6 - \frac{3\zeta^2}{2}
\] (3.4.52)

\[
C'_3 = 6\zeta (\zeta^2 - 4) = -8C_2 \sqrt{1 - \frac{C_2}{6}}
\] (3.4.53)

\[
C_4 = \frac{3}{8} (\zeta^4 + 8\zeta^2 - 48) = \frac{C_2^2}{6} - 4C_2
\] (3.4.54)

The quartic Casimir in [141] is defined using the epsilon tensor as follows:

\[
C_4^{\epsilon} = \epsilon^{ABCDEF} M_{CD} M_{E} M_{F} C^{CD} D^{E} E^{F} M^{C} D^{E} E^{F}
\]

\[
= -24\zeta^2 (\zeta^2 - 4)
\] (3.4.55)

which is just a linear combination of \(C_4\) and \(C_4\) defined above.

The products of generators corresponding to the Young tableaux do not vanish for the deformed minimal unitary representations and depend on the deformation parameter \(\zeta\) as follows:

\[
\begin{array}{c}
= \zeta
\end{array}
\] (3.4.56)

Hence all non-vanishing four-row diagrams can be dualized to two-row diagrams and the resulting generators of the universal enveloping algebra in deformed realization are described by two-row diagrams. For \(\zeta = 0\) all four row diagrams vanish and one obtains the standard high spin algebra of Vasiliev type whose generators transform in representations of the underlying \(AdS\) group corresponding to Young tableaux containing
only two row traceless diagrams. For non-zero $\zeta$ the corresponding enveloping algebras describe deformed higher spin algebras as discussed in the next subsection.

We saw earlier in section 3.4.2.1 that the generator $J_{ABCD}$ of the Joseph ideal did not vanish identically as an operator for the covariant twistorial oscillator realization of $SO(4,2)$. It annihilates only the states belonging to the subspace that form the basis of the true minrep of $SO(4,2)$. By going to the conformal three grading, one finds that the generator $J_{ABCD}$ of the Joseph ideal can be written in a form similar to the deformed quadratic identities above with the deformation parameter replaced by the linear Casimir operator $Z = N_a - N_b$

$$\eta^{\mu\nu} M_{\mu\rho} \cdot M_{\nu\sigma} - P_{[\rho} \cdot K_{\sigma]} + 2\eta_{\mu\sigma} = -Z^2/2 \eta_{\rho\sigma}$$

$$M_{\mu\nu} \cdot M_{\rho\sigma} + M_{\mu\sigma} \cdot M_{\nu\rho} + M_{\mu\rho} \cdot M_{\nu\sigma} = -Z \epsilon_{\mu\nu\rho\sigma} \Delta$$

$$\Delta \cdot M_{\mu\nu} + P_{[\mu} \cdot K_{\nu]} = Z^2 \epsilon_{\mu\nu\rho\sigma} M^{\rho\sigma}$$

$$\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} M_{\nu\rho} \cdot P_{\sigma} = -Z P^\mu$$

$$\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} M_{\nu\rho} \cdot K_{\sigma} = Z K^\mu$$

The Fock space of the oscillators decompose into an infinite set of unitary irreducible representations of $SU(2,2)$ corresponding to massless conformal fields of all integer and half-integer helicities labelled by the eigenvalues of $Z/2$.

### 3.4.4 Higher spin algebras and superalgebras and their deformations

We shall adopt the definition of the higher spin $AdS_{(d+1)}/CFT_d$ algebra as the quotient of the enveloping algebra $\mathfrak{u}(SO(d,2))$ of $SO(d,2)$ by the Joseph ideal $\mathfrak{j}(SO(d,2))$ and denote it as $HS(d,2)$ [130]:

$$HS(d,2) = \frac{\mathfrak{u}(SO(d,2))}{\mathfrak{j}(SO(d,2))}$$

We shall however extend it to define deformed higher spin algebras as the enveloping algebras of the deformations of the minreps of the corresponding $AdS_{d+1}/Con_{f_d}$ algebras. For these deformed higher spin algebras the corresponding deformations of the Joseph ideal vanish identically as operators in the quasiconformal realization as we showed explicitly above for the conformal group in four dimensions. We expect a given deformed higher spin algebra to be the unique infinite dimensional quotient of the universal enveloping algebra of an appropriate covering* of the conformal group by the deformed ideal as was shown for the undeformed minrep in [142]. Similarly, we define the higher spin

*In $d = 4$ deformed minreps describe massless fields with the helicity $\frac{\zeta}{2}$. For non-integer values of $\zeta$ one has to go to an infinite covering of the $4d$ conformal group.
superalgebras and their deformations as the enveloping algebras of the minimal unitary realizations of the underlying superalgebras and their deformations, respectively.\footnote{We should note that the universal enveloping algebra of a Lie group as defined in the mathematics literature is an associative algebra with unit element. Under the commutator product inherited from the underlying Lie algebra it becomes a Lie algebra.}

In four dimensions \( (d = 4) \) we have a one parameter family of higher spin algebras labelled by the helicity \( \zeta/2 \):

\[
HS(4, 2; \zeta) = \frac{\mathfrak{u}(SO(4, 2))}{\mathfrak{j}_\zeta(SO(4, 2))} \tag{3.4.63}
\]

where \( \mathfrak{j}_\zeta(SO(4, 2)) \) denotes the deformed Joseph ideal of \( SO(4, 2) \) defined in section \[3.4.3\].\footnote{After the main results of this chapter was announced at the GGI Conference on higher spin theories in May 2013, we became aware of the work of \[143\] where possible deformations of purely bosonic higher spin algebras in arbitrary dimensions were studied and it was shown that the deformations can depend at most on one parameter. In a subsequent work \[144\] it was shown that the \( AdS_4 \) higher spin algebra is unique in \( d = 4 \) and \( d > 6 \) under their assumptions on the spectrum of generators. They also imply that the one parameter family of deformations of \[143\] must be the same as the one parameter family discussed in this chapter which is based on earlier work \[23\] that they cite. The results of \[143\] are based on Young tableaux analysis of gauge fields in \( AdS_5 \). Whether and how their deformation parameter is related to helicity in \( 4d \) is not known at this point.}

On the \( AdS_{d+1} \) side the generators of the higher spin algebras correspond to higher spin gauge fields, while on the \( Conf_d \) side they are related to conserved tensors including the conserved stress-energy tensor. The charges associated with the generators of conformal algebra \( SO(d, 2) \) are defined by conserved currents constructed by contracting the stress-energy tensor with conformal Killing vectors. Similarly, the higher conserved currents are obtained by contracting the conformal Killing tensors with the stress-energy tensor. These higher conformal Killing tensors are obtained simply by tensoring the conformal Killing vectors with themselves. Though implicit in previous work on the subject \[87,145\], explicit use of the language of conformal Killing tensors in describing higher spin algebras seems to have first appeared in the paper of Mikhailov \[91\]. This connection was put on a rigorous foundation by Eastwood in his study of the higher symmetries of the Laplacian \[130\], who showed that the undeformed higher spin algebra can be obtained as the quotient of the enveloping algebra of \( SO(d, 2) \) generated by the conformal Killing tensors quotiented by the Joseph ideal.

The connection between the doubleton realization of \( SO(4, 2) \) in terms of covariant twistorial oscillators and the corresponding conformal Killing tensors was also studied by Mikhailov \[91\]. He pointed out that the higher conformal Killing tensors correspond to the products of bilinears of oscillators that generate \( SO(4, 2) \) in the doubleton realization \[9\]. The generators of the higher spin algebra \( shs^E(8|4) \) of reference \[146\] in terms of Killing spinors in \( AdS_4 \) were obtained in the work of \[147\] on higher spin \( N = 8 \) supergravity in \( d = 4 \).
of $[32,37,38]$, which are elements of the enveloping algebra. The undeformed higher spin algebra $HS(4, 2; \zeta = 0)$ was also studied by the authors of [89] who used the doubletonic construction as well. The undeformed higher spin algebra corresponding to $\zeta = 0$ describes fields of all spins which are multiplicity-free. The model studied in detail in [89] is based on the minimal infinite dimensional subalgebra of $HS(4, 2; \zeta = 0)$ that describes only even spins. For the purely bosonic higher spin algebras in $AdS_d$ the factorization of the ideal generated by the bilinear operators corresponding to the last three irreps in $3.4.2$ was discussed in great detail, at the level of abstract enveloping algebra, in [148] without reference to explicit oscillator realization. They corroborated that in $d = 5$ there is an equivalence between factoring out this ideal and the oscillator construction in [89].

The supersymmetric extension of the higher spin algebras $HS(4, 2; \zeta)$ is given by the enveloping algebra of the deformed minimal unitary realization of the N-extended conformal superalgebras $SU(2, 2|N)$ with the even subalgebras $SU(2, 2) \oplus U(N)$. We shall denote the resulting higher spin algebra as $HS[SU(2, 2|N); \zeta]$. These supersymmetric extensions involve odd powers of the deformed twistorial oscillators and the identities that define the Joseph ideal get extended to a supermultiplet of identities obtained by the repeated actions of $Q$ and $S$ supersymmetry generators on the generators of Joseph ideal.

On the conformal side the odd generators correspond to the products of the conformal Killing spinors with conformal Killing vectors and tensors. If we denote the resulting deformed super-ideal as $\mathfrak{J}[SU(2, 2|N)]$ we can formally write

$$HS[SU(2, 2|N); \zeta] = \frac{\mathfrak{U}(SU(2, 2|N))}{\mathfrak{J}[SU(2, 2|N)]} \quad (3.4.64)$$

As discussed in previous sections, the bosonic higher spin gauge fields are described by two-row Young tableaux of $SO(4, 2)$. However in order to extend the $SO(4, 2)$ Young tableaux to super Young tableaux, one needs to represent the corresponding representations in terms of $SU(2, 2)$ Young tableaux since the relevant superconformal algebras are $SU(2, 2|N)$ with the even subalgebra $SU(2, 2) \oplus U(N)$ and not superalgebras of the orthosymplectic type. The identifications of Young tableaux can be easily checked by calculating the dimensions of corresponding irreps. In going from $SU(2, 2)$ to $SU(2, 2|N)$ one simply replaces the Young tableaux of $SU(2, 2)$ by super Young tableaux of $SU(2, 2|N)$.

The Young diagram of the adjoint representation of $SO(4, 2) = SU(2, 2)$ goes over to the following super tableau of $SU(2, 2|N)$ which involves one dotted and one undotted

---

12 We should note that the infinite dimensional ideal that appears in the work of [89] is not the Joseph ideal.

13 For a review of super Young tableaux we refer to [149] and a study of the relations between Super young tableaux and Kac-Dynkin labelling see [150].
“superboxes”:

\[
\begin{array}{c}
\text{SO}(4,2) \leftrightarrow \text{SU}(2,2) \xrightarrow{\text{supersymmetrize}} \text{SU}(2,2|N)
\end{array}
\]  \hspace{1cm} (3.4.65)

Notice that the \(\square\) of \(\text{SU}(2,2)\) is represented as \(\bullet\). This is because supersymmetric extensions of \(\text{SU}(2,2)\) do not have an invariant super Levi-Civita tensor. The adjoint representation \(\mathbf{\square}\) of \(\text{SU}(2,2|N)\) can be decomposed with respect to \(\text{SU}(2,2) \times U(N)\) as follows:

\[
\mathbf{\square} = (\mathbf{\bullet \cdot}, 1) \oplus (\mathbf{\bullet}, \mathbf{\square}) \oplus (\mathbf{\square}, \mathbf{\bullet}) \oplus (1, \mathbf{\bullet \cdot}) \oplus (1, 1)
\]  \hspace{1cm} (3.4.66)

The window diagram appearing in the tensor product of two adjoint representations can be supersymmetrized as follows:

\[
\begin{array}{c}
\text{SO}(4,2) \leftrightarrow \text{SU}(2,2) \xrightarrow{\text{supersymmetrize}} \text{SU}(2,2|N)
\end{array}
\]  \hspace{1cm} (3.4.67)

Thus the higher spin gauge fields are described by the following \(\text{SU}(2,2)\) diagram and can be consequently supersymmetrized in a straightforward manner:

\[
\begin{array}{c}
\text{two-row diagram} \leftrightarrow \text{SU}(2,2) \xrightarrow{\text{supersymmetrize}} \text{SU}(2,2|N)
\end{array}
\]  \hspace{1cm} (3.4.68)

Thus the generators of the supersymmetric extension of the undeformed \((\zeta = 0)\) bosonic higher spin algebra decompose as

\[
\text{HS}[\text{SU}(2,2|N); 0] = \sum_{\oplus} \mathbf{\bullet \cdot \cdot \cdot \bullet \cdot \cdot \cdot}
\]  \hspace{1cm} (3.4.69)

For the decomposition of the deformed higher spin algebras for integer values of the deformation parameter one can also use the super tableaux to represent their generators. However for non-integer values of the deformation parameter it may be necessary to use Kac-Dynkin or other labellings. We should stress the important point that the maximal finite dimensional subalgebra of \(\text{HS}[\text{SU}(2,2|N); \zeta]\) is the superconformal algebra \(\text{SU}(2,2|N)_\zeta\) with \(\zeta\) labelling the central charge. The decomposition of the generators of
the higher spin superalgebra with respect to the $SU(2, 2|N)\zeta$ subalgebra is multiplicity free.

Furthermore, the minimal unitary realization of the supersymmetric extensions $SU(2, 2|N)$ of $SU(2, 2)$ the enveloping algebra of $SU(2, 2|N)$ has as subalgebras the enveloping algebras of different unitary representations of $SU(2, 2)$ that form the minimal unitary supermultiplet modded out by the corresponding deformed Joseph ideals. These different irreps of $SU(2, 2)$ are deformations of the minrep where the deformation is driven by the fermionic oscillators and the deformation parameter $\zeta$ is given by the eigenvalue of the number operator of these fermionic oscillators $N_\xi$ in the minimal unitary supermultiplet. In the deformations of the minimal unitary supermultiplet labelled by $\zeta$ the enveloping algebra of $SU(2, 2|N)$ has a similar decomposition in which the irreps making of the supermultiplet are labelled by a linear combination of $\zeta$ and the eigenvalue of $N_\xi$.

We should note that the undeformed super algebra $HS[SU(2, 2|4); \zeta = 0]$ for the particular value of $N = 4$ was studied in [151] by using the doubletonic realization. As in the purely bosonic case the higher spin theory studied in [151] is based on a subalgebra of $HS[SU(2, 2|4); \zeta = 0]$ and the resulting linearized gauging was shown to correspond to massless $PSU(2, 2|4)$ multiplets whose maximal spins are even integers $2, 4, ...$.

The situation is much simpler for $AdS_4/Conf_3$ higher spin algebras. There are only two minreps of $SO(3, 2)$, namely the scalar and spinor singletons. The higher spin algebra $HS(3, 2)$ is simply given by the enveloping algebra of the singletonic realization of $Sp(4, \mathbb{R})$ [40,146]. Singletonic realization of the Lie algebra of $Sp(4, \mathbb{R})$ describes both the scalar and spinor singletons. They form a single irreducible supermultiplet of $OSp(1|4, \mathbb{R})$ generated by taking the twistorial oscillators as the odd generators [40]. The odd generators correspond to conformal Killing spinors, which together with conformal Killing vectors in $d = 3$, generate the Lie super-algebra of $OSp(1|4, \mathbb{R})$. Its enveloping algebra leads to the higher spin super-algebra of Fradkin-Vasiliev type involving all integer and half integer spin fields in $AdS_4$. One can also construct N-extended higher spin superalgebras in $AdS_4$ as enveloping algebras of the singletonic realization of $OSp(N|4, \mathbb{R})$ [40,146].

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14 We should stress again that the infinite dimensional ideal that appears in the work of [151] is not the Joseph ideal.
Chapter 4 | 
\(AdS_7/CFT_6\) Higher spin algebras in \(d = 6\): \(hs(6, 2; t)\)

4.1 Introduction

The conformal group in six dimensions is \(SO^*(8) \sim SO(6, 2)\) and its minrep describes a conformal massless scalar field in six dimensions. The “discrete” deformations of the minrep describe massless conformal fields that are symmetric tensors in the spinorial representation of the six dimensional Lorentz group, \(SO(5, 1) \sim Sl(2, \mathbb{H})\). The minrep and its deformations for \(SO(6, 2)\) were first studied in [24,56] using the quasiconformal methods. Following the approach of the previous chapter, we shall reformulate the minrep and deformations thereof in terms of deformed twistorial oscillators such that the generators are bilinears of twistors that transform nonlinearly under the six dimensional Lorentz group. Subsequently we show that the Joseph ideal vanishes explicitly for the minrep and we can define a discrete one-parameter family of ideals. The universal enveloping algebra of \(SO(6, 2)\) in evaluated in the minrep is then the \(AdS_7\) higher spin algebra \(hs(6, 2)\) and the corresponding enveloping algebra evaluated in the deformations of the minrep is the one-parameter discrete family of deformations, \(hs(6, 2; t)\) of the \(AdS_7\) higher spin algebra.

At this time we should explain the use of the term “deformations” to label a discrete family of higher spin algebras. As we saw in the previous chapter, there exists an infinite family of inequivalent higher spin algebras in four dimensions which are labeled by the continuous helicity of conformal massless fields in four dimensions. It was established that this helicity is nothing but the eigenvalue of the little group \(U(1)\) of massless particles in four dimensions which is Abelian and takes on continuous values. However, the little group in six dimensions is \(SO(4) = SU(2)_T \times SU(2)_A\) whose representations are labeled by eigenvalues \((j_T, j_A)\) which are discrete. It turns out that for unitary conformal massless representations, the representations are of the form \((j_T, 0)\) or \((0, j_A)\). This means that the
conformally massless representations in six dimensions are labeled by a discrete parameter in contrast with the continuous helicity labels for the analogous conformally massless representations in four dimensions. Hence we refer to the higher spin algebras based on these representations as discrete deformations of $hs(6,2)$. It should be noted that these discrete deformations admit supersymmetric extensions for arbitrary number of supersymmetry generators. These results were published in [152] which we follow closely in our review in this section.

The plan of the chapter is as follows: We start by reviewing the covariant twistorial (doubleton) construction in section 4.2.1 following [33,39,153] and its reformulation in terms of Lorentz covariant twistorial oscillators [132,153]. Next in sections 4.2.2-4.2.4 we present a novel reformulation of the quasiconformal realization of the minimal unitary representation (minrep) of $SO(6,2)$ and its supersymmetric extensions and their deformations [24,56] in terms of deformed twistors that transform nonlinearly under the 6d Lorentz group. In section 4.3 we review the Eastwood’s formula for the generators $J$ of the annihilator of the minrep (Joseph ideal) and show by explicit calculations that it vanishes identically as an operator for the quasiconformal realization of the minrep. Then in section 4.3.2 we use the 6d Lorentz covariant formulation of the Joseph ideal to identify the deformed generators $J_t$ that are the annihilators of the deformations of the minrep. These discrete deformations are labelled by the eigenvalues of an $SU(2)_G$ symmetry realized as bilinears of fermionic oscillators. Interestingly, the 6d analog of the Pauli-Lubansky vector does not vanish for the deformed minreps and becomes an (anti-)self-dual operator of rank three. Next we compare the generators of the Joseph ideal computed for doubleton realization and identify the analog of the deformation $SU(2)_G$ in the quasiconformal realization. For the doubleton realization the generators of the Joseph ideal do not vanish as operators. They annihilate only the subspace of the Fock space of the covariant oscillators corresponding to the minrep. In section 4.3.4 we define the $AdS_7/CFT_6$ higher spin algebra and its deformations as the enveloping algebra of the minrep and its deformations in the quasiconformal framework, respectively. We also extend these results to corresponding higher spin superalgebras.
4.2 Realizations of the 6d conformal algebra $SO(6, 2) \sim SO^*(8)$ and its supersymmetric extension $OSp(8^*|4)$

4.2.1 Covariant twistorial oscillator construction of the massless representations (doubletons) of 6d conformal group $SO(6, 2)$

In this subsection we shall review the construction of positive energy unitary representations of $SO(6, 2)$ that correspond to massless conformal fields in six dimensions following [33, 39, 153]. The Lie algebra of the conformal group $SO(6, 2)$ in six dimensions is isomorphic to that of $SO^*(8)$ with the maximal compact subgroup $U(4)$. Commutation relations of the generators $M_{AB}$ of $SO(6, 2)$ in the canonical basis have the form

$$[M_{AB}, M_{CD}] = i(\eta_{BC} M_{AD} - \eta_{AC} M_{BD} - \eta_{BD} M_{AC} + \eta_{AD} M_{BC})$$ (4.2.1)

where $\eta_{AB} = \text{diag}(-, +, +, +, +, +, -)$ and $A, B = 0, \ldots, 7$. Chiral spinor representation of $SO(6, 2)$ can be written in terms in six-dimensional gamma matrices $\Gamma_\mu$ that satisfy

$$\{\Gamma_\mu, \Gamma_\nu\} = -2\eta_{\mu\nu}$$

where $\eta_{\mu\nu} = \text{diag}(-, +, +, +, +, +, +)$ and $\mu, \nu = 0, \ldots, 5$ as follows\[^1\]

$$\Sigma_{\mu\nu} := -\frac{i}{4} [\Gamma_\mu, \Gamma_\nu], \quad \Sigma_{\mu 6} := \frac{1}{2} \Gamma_\mu \Gamma_7, \quad \Sigma_{\mu 7} := -\frac{1}{2} \Gamma_\mu, \quad \Sigma_{67} := -\frac{1}{2} \gamma_7$$ (4.2.2)

We adopt the conventions of [153] for gamma matrices:

$$\begin{align*}
\Gamma_0 &= \sigma_3 \otimes I_2 \otimes I_2 \\
\Gamma_1 &= i\sigma_1 \otimes \sigma_2 \otimes I_2 \\
\Gamma_2 &= i\sigma_1 \otimes \sigma_1 \otimes \sigma_2 \\
\Gamma_3 &= i\sigma_1 \otimes \sigma_3 \otimes \sigma_2 \\
\Gamma_4 &= i\sigma_2 \otimes I_2 \otimes \sigma_2 \\
\Gamma_5 &= i\sigma_2 \otimes \sigma_2 \otimes \sigma_1 \\
\Gamma_7 &= -\Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 \Gamma_5
\end{align*}$$ (4.2.3)

Consider the bosonic oscillators $c_i, d_j$ and their hermitian conjugates $c^i, d^j$ respectively ($i, j = 1, 2, 3, 4$) that satisfy

$$[c_i, c^j] = \delta_i^j, \quad [d_i, d^j] = \delta_i^j$$ (4.2.4)

\[^1\text{Opposite chirality spinor representation is obtained by taking } \Sigma_{\mu 7} := +\frac{1}{2} \Gamma_\mu \text{ and } \Sigma_{67} := +\frac{1}{2} \gamma_7.\]
and form a twistorial spinor operator $\Psi$ and its Dirac conjugate $\overline{\Psi} = \Psi^\dagger \Gamma_0$ as:

$$\Psi = \begin{pmatrix} c_i \\ d^i \end{pmatrix}, \quad \overline{\Psi} = \begin{pmatrix} c^i \\ -d_i \end{pmatrix}$$

(4.2.5)

Then the bilinears $M_{AB} = \overline{\Psi} \Sigma_{AB} \Psi$ provide a realization of the Lie algebra of $SO(6,2)$:

$$[\overline{\Psi} \Sigma_{AB} \Psi, \overline{\Psi} \Sigma_{CD} \Psi] = \overline{\Psi} [\Sigma_{AB}, \Sigma_{CD}] \Psi$$

(4.2.6)

and the Fock space of the oscillators decompose into an infinite set of positive energy unitary irreducible representations of $SO(6,2)$ corresponding to massless conformal fields in six dimensions. The resulting representations for one pair (color) of oscillators were called doubletons of $SO(6,2)$ in [33,39,153].

The Lie algebra of the conformal group $SO(6,2)$ has a three-graded decomposition (referred to as compact three-grading) with respect to its maximal compact subalgebra $\mathcal{L}^0 = SU(4) \times U(1)_E$,

$$SO(6,2) = \mathcal{L}^- \oplus \mathcal{L}^0 \oplus \mathcal{L}^+,$$

(4.2.7)

where the three-grading is determined by the conformal Hamiltonian $E = \frac{1}{2}(P_0 + K_0)$. To construct positive energy unitary representations of $SO^*(8)$, one realizes the generators as following bilinears:

$$A_{ij} = c_i d_j - c_j d_i \in \mathcal{L}^-, \quad A^{ij} = c^i d^j - d^j d^i \in \mathcal{L}^+$$

$$M^i_j = c^i c_j + d_j d^i \in \mathcal{L}^0$$

(4.2.8)

where $i, j = 1, 2, 3, 4$.

$M^i_j$ generate the maximal compact subgroup $U(4)$. The conformal Hamiltonian is given by the trace $M^i_i$,

$$Q_B = \frac{1}{2} M^i_i = \frac{1}{2} (N_B + 4),$$

(4.2.9)

where $N_B \equiv c^i c_i + d^i d_i$ is the bosonic number operator. We shall denote the eigenvalues of $Q_B$ as $E$. The hermitian linear combinations of $A_{ij}$ and $A^{ij}$ are the non-compact generators of $SO(6,2)$ [33,39,153]. Each lowest weight (positive energy) UIR is uniquely determined by a set of states transforming in the lowest energy irreducible representation $|\Omega\rangle$ of $SU(4) \times U(1)_E$ that are annihilated by all the elements of $\mathcal{L}^-$. The possible lowest weight vectors for one pair of oscillators (doubletons) in this compact basis have

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\footnote{Equivalently, the lowest weight vector of the lowest energy irreducible representation of $SU(4)$ determines the UIR. Hence, by an abuse of terminology, we shall use interchangeably the terms “lowest weight vector” and “lowest energy irreducible representation”.

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SU(4) Young tableaux with one row [39], They are of the form

\[ |0\rangle, \]
\[ c^1 |0\rangle = |\square\rangle, \]
\[ c^{(i_1,c^2)} |0\rangle = |\square\rangle, \]
\[ \vdots \]
\[ c^{(i_1,c^2,\ldots,c^n)} |0\rangle = |\begin{array}{c}
\hline
\vdots \\
\hline
\end{array}\rangle, \quad (4.2.10) \]

plus those obtained by interchanging \(c\)-type oscillators with \(d\)-type oscillators and the state

\[ c^{(i,d^j)} |0\rangle = |\square\rangle. \quad (4.2.11) \]

The positive energy UIR’s of \(SO^*(8)\) can be identified with conformal fields in six dimensions transforming covariantly under the six-dimensional Lorentz group with definite conformal dimension. The Lorentz covariant spinorial oscillators are obtained from the \(SU(4)\) covariant oscillators by the action of an intertwining operator \(T = e^{\frac{\pi}{2}M_{06}}\). We will use Greek letters for the Lorentz group \(SO(5,1) \sim SU^*(4)\) spinorial indices – \(\alpha, \beta = 1, 2, 3, 4\). Without convention of gamma matrices, the Lorentz covariant spinorial oscillators \(\lambda^i, \eta^{\alpha j}\) are given by:

\[
\lambda^1 = \begin{pmatrix}
    c^3 + d_1 \\
    -c^4 + d_2 \\
    -c^1 + d_3 \\
    c^2 + d_4
\end{pmatrix}, \quad \lambda^2 = \begin{pmatrix}
    -d^3 + c_1 \\
    d^4 + c_2 \\
    d^1 + c_3 \\
    -d^2 + c_4
\end{pmatrix} \quad (4.2.12)
\]
\[
\eta^{\alpha 1} = \begin{pmatrix}
    -c^1 - d_3 \\
    -c^2 + d_4 \\
    -c^3 + d_1 \\
    -c^4 - d_2
\end{pmatrix}, \quad \eta^{\alpha 2} = \begin{pmatrix}
    d^1 - c_3 \\
    d^2 + c_4 \\
    d^3 + c_1 \\
    d^4 - c_2
\end{pmatrix} \quad (4.2.13)
\]

They satisfy the following commutation relations:

\[
[\eta^{\alpha i}, \lambda^j] = -2\delta^\alpha_\beta \epsilon^{ij} \quad (4.2.14)
\]

where \(i, j = 1, 2\) (\(\epsilon_{12} = \epsilon^{21} = +1\)) and \(\alpha, \beta = 1, 2, 3, 4\). One finds that [132]

\[
(\Sigma^\mu P_\mu)_{\alpha \beta} = P_{\alpha \beta} = \lambda_{\alpha i} \lambda^j_{\beta} \epsilon_{ij}, \quad (\bar{\Sigma}^\mu K_\mu)^{\alpha \beta} = K^{\alpha \beta} = -\eta^{\alpha i} \eta^j_{\beta} \epsilon_{ij} \quad (4.2.15)
\]
where $\Sigma$-matrices in $d = 6$ are the analogs of Pauli matrices $\sigma_\mu$ in $d = 4$. The explicit form of $\Sigma^\mu, \Sigma^\nu$ is given in Appendix B.1. Note that the form of $\lambda_\alpha^i, \eta^{\alpha j}$ is slightly different from $\lambda_\alpha^i, \eta^{\alpha j}$ as we are in the mostly positive signature and also the form of intertwining operator is slightly different.

Similarly we can define Lorentz generators with spinor indices as follows:

$$M_\alpha^\beta = \frac{i}{2} \left( \Sigma^\mu \Sigma^\nu \right)_\alpha^\beta M_{\mu\nu}$$

(4.2.16)

In terms of spinors $\lambda_\alpha^i, \eta^{\alpha j}$, they are given as follows:

$$M_\alpha^\beta = -\frac{1}{2} \left( \lambda_\alpha^i \eta^{\beta j} - \frac{1}{4} \delta_\alpha^\beta \lambda_\alpha^i \eta^{\beta j} \right) \epsilon_{ij}$$

(4.2.17)

The dilatation generator is given by:

$$\Delta = \frac{i}{8} \left( \eta^{\alpha j} \lambda_\alpha^i - \lambda_\alpha^i \eta^{\alpha j} \right) \epsilon_{ij}$$

(4.2.18)

The conformal algebra in terms of these generators is as follows:

$$[M_\alpha^\beta, M_\gamma^\delta] = \delta_\gamma^\beta M_\alpha^\delta - \delta_\delta^\beta M_\alpha^\gamma$$

(4.2.19)

$$[P_{\alpha\beta}, M_\gamma^\delta] = 2 \delta_\alpha^\beta P_{\beta\gamma} + \frac{1}{2} \delta_\gamma^\delta P_{\alpha\beta}, \quad [K^{\alpha\beta}, M_\gamma^\delta] = -2 \delta_\delta^\alpha K^{\beta\gamma} - \frac{1}{2} \delta_\gamma^\delta K^{\alpha\beta}$$

(4.2.20)

$$[P_{\alpha\beta}, K^{\gamma\delta}] = 16 \left( \delta_\alpha^\gamma M_\beta^\delta - \frac{i}{2} \delta_\beta^\delta M_\alpha^\gamma \right) \Delta$$

(4.2.21)

$$[\Delta, P_{\alpha\beta}] = i P_{\alpha\beta}, \quad [\Delta, K^{\alpha\beta}] = -i K^{\alpha\beta}, \quad [\Delta, M_\alpha^\beta] = 0$$

(4.2.22)

The doubletons correspond to massless conformal fields in six dimensions that transform as symmetric tensors $\Psi_{\alpha\beta\gamma...} \equiv \Psi_{(\alpha\beta\gamma...)}$ in their spinor indices.

### 4.2.2 Quasiconformal approach to minimal unitary representation of $SO(6, 2)$

The construction of the minimal unitary representation of the $6d$ conformal group $SO(6, 2)$ by quantization of its quasiconformal action and its deformations were given in [24, 56].

In this section we will reformulate the generators of these representations in terms of deformed twistorial oscillators as was done for $4d$ conformal group $SU(2, 2)$ in [3]

The group $SO(6, 2) \sim SO^+(8)$ can be realized as a quasiconformal group that leaves light-like separations with respect to a quartic distance function in nine dimensions.
invariant. The quantization of this geometric action leads to a nonlinear realization of the generators of $SO(6,2)$ in terms of a singlet coordinate $x$, its conjugate momentum $p$ and four bosonic oscillators $a_m, a^m$ and $b_m, b^m$, $(m, n = 1, 2)$ satisfying \[ [x, p] = i, \quad [a_m, a^n] = \delta^n_m, \quad [b_m, b^n] = \delta^n_m \] (4.2.23)

The semisimple component of the little group of massless particles in six dimensions is $SO(4)$ which can be written as $SU(2)_S \times SU(2)_A$. Its normalizer inside $SO(6,2)$ is $SO(2,2)$ which also decomposes as $SU(1,1)_K \times SU(1,1)_N$. The generators of $SU(2)_S$ and of $SU(2)_A$ subgroups of $SO^*(8)$ are realized as bilinears of $a$ and $b$ type oscillators within the quasiconformal approach as follows:

\[
S_+ = a^m b_m, \quad S_- = b^m a_m, \quad S_0 = \frac{1}{2} (N_a - N_b) \quad (4.2.24)
\]

\[
A_+ = a^i a_2 + b^i b_2, \quad A_- = a_1 a^2 + b_1 b^2, \quad A_0 = \frac{1}{2} (a^i a_1 - a^2 a_2 + b^i b_1 - b_2 b_2) \quad (4.2.25)
\]

where $N_a = a^m a_m$ and $N_b = b^m b_m$ are the respective number operators. They satisfy

\[
[S_0, S_{\pm}] = \pm S_{\pm}, \quad [S_+, S_-] = 2S_0 \quad (4.2.26)
\]

\[
[A_0, A_{\pm}] = \pm A_{\pm}, \quad [A_+, A_-] = 2A_0 \quad (4.2.27)
\]

In the previous section we followed conventions given in [153] for the 6$d$ sigma matrices $(\Sigma^\mu, \bar{\Sigma}^\mu)$ within the covariant twistorial construction of doubletons. In order to make contact with the spinor-helicity formalism used in the amplitudes literature, we will follow the Clifford algebra conventions of [154], summarized in Appendix B.2, for the minimal unitary representation and its deformations within the quasiconformal approach. To avoid confusion, we will denote these 6$d$ sigma matrices as $\hat{\sigma}^\mu, \bar{\hat{\sigma}}^\mu$ ($\mu, \nu = 0, 1, \ldots 5$)

To express the momentum generators of $SO(6,2)$ of the minrep we shall introduce two sets of deformed twistors $Z_\alpha^i$ and $\bar{Z}_\alpha^i$ ($\alpha, \beta = 1, 2, 3, 4, i, j = 1, 2$) that transform nonlinearly under the Lorentz group (their commutation relations are given in Appendix B.4):

\[
Z_1^1 = b_1 - \frac{1}{2} (x - ip) + \frac{1}{x} \left( S_0 + \frac{3}{4} \right), \quad Z_1^2 = a_1 - \frac{S_-}{x} \quad (4.2.28)
\]

\[
Z_2^1 = b_2 - \frac{S_+}{x}, \quad Z_2^2 = a_2 - \frac{1}{2} (x - ip) - \frac{1}{x} \left( S_0 - \frac{3}{4} \right) \quad (4.2.29)
\]

\[
Z_3^1 = -a^2 + \frac{1}{2} (x + ip) + \frac{1}{x} \left( S_0 + \frac{3}{4} \right), \quad Z_3^2 = b^2 - \frac{S_-}{x} \quad (4.2.30)
\]

\[
Z_4^1 = a^1 - \frac{S_+}{x}, \quad Z_4^2 = -b^1 + \frac{1}{2} (x + ip) - \frac{1}{x} \left( S_0 - \frac{3}{4} \right) \quad (4.2.31)
\]
\[ \tilde{Z}_1^1 = b_1 - \frac{1}{2} (x - ip) + \frac{1}{x} \left( S_0 - \frac{3}{4} \right) , \quad \tilde{Z}_1^2 = a_1 - \frac{S_-}{x} \] (4.2.32)

\[ \tilde{Z}_2^1 = b_2 - \frac{S_+}{x} , \quad \tilde{Z}_2^2 = a_2 - \frac{1}{2} (x - ip) - \frac{1}{x} \left( S_0 + \frac{3}{4} \right) \] (4.2.33)

\[ \tilde{Z}_3^1 = -a_2 + \frac{1}{2} (x + ip) + \frac{1}{x} \left( S_0 - \frac{3}{4} \right) , \quad \tilde{Z}_3^2 = b_2 - \frac{S_-}{x} \] (4.2.34)

\[ \tilde{Z}_4^1 = a_1 - \frac{S_+}{x} , \quad \tilde{Z}_4^2 = -b_1 + \frac{1}{2} (x + ip) - \frac{1}{x} \left( S_0 + \frac{3}{4} \right) \] (4.2.35)

In terms of these deformed twistors the momentum generators can then be written as bilinears:

\[ P_{\alpha\beta} = Z_\alpha^i \tilde{Z}_\beta^j \epsilon_{ij} \] (4.2.36)

We should stress the fact that even though the above deformed twistors transform nonlinearly under the Lorentz group the bilinears \( P_{\alpha\beta} \) transform covariantly as antisymmetric tensors in spinorial indices.

In order to realize the special conformal generators, we need another set of deformed twistors \( Y^{\alpha i} \) and \( \tilde{Y}^{\alpha i} \) (their commutation relations are given in Appendix B.4):

\[ Y^{11} = a_1 + \frac{S_+}{x} , \quad Y^{12} = -b_1 - \frac{1}{2} (x + ip) + \frac{1}{x} \left( S_0 - \frac{3}{4} \right) \] (4.2.37)

\[ Y^{21} = a_2 + \frac{1}{2} (x + ip) + \frac{1}{x} \left( S_0 + \frac{3}{4} \right) , \quad Y^{22} = -b^2 - \frac{S_-}{x} \] (4.2.38)

\[ Y^{31} = -b_2 - \frac{S_+}{x} , \quad Y^{32} = -a_2 - \frac{1}{2} (x - ip) - \frac{1}{x} \left( S_0 - \frac{3}{4} \right) \] (4.2.39)

\[ Y^{41} = b_1 + \frac{1}{2} (x - ip) - \frac{1}{x} \left( S_0 + \frac{3}{4} \right) , \quad Y^{42} = a_1 + \frac{S_-}{x} \] (4.2.40)

\[ \tilde{Y}^{11} = a_1 + \frac{S_+}{x} , \quad \tilde{Y}^{12} = -b_1 - \frac{1}{2} (x + ip) + \frac{1}{x} \left( S_0 + \frac{3}{4} \right) \] (4.2.41)

\[ \tilde{Y}^{21} = a_2 + \frac{1}{2} (x + ip) + \frac{1}{x} \left( S_0 - \frac{3}{4} \right) , \quad \tilde{Y}^{22} = -b^2 - \frac{S_-}{x} \] (4.2.42)

\[ \tilde{Y}^{31} = -b_2 - \frac{S_+}{x} , \quad \tilde{Y}^{32} = -a_2 - \frac{1}{2} (x - ip) - \frac{1}{x} \left( S_0 + \frac{3}{4} \right) \] (4.2.43)

\[ \tilde{Y}^{41} = b_1 + \frac{1}{2} (x - ip) - \frac{1}{x} \left( S_0 - \frac{3}{4} \right) , \quad \tilde{Y}^{42} = a_1 + \frac{S_-}{x} \] (4.2.44)

that transform nonlinearly under the Lorentz group. The special conformal generators can then be written as bilinears:

\[ K^{\alpha\beta} = Y^{\alpha i} \tilde{Y}^{\beta j} \epsilon_{ij} \] (4.2.45)

which transform covariantly under the Lorentz group.
The Lorentz subgroup $SO(5,1) \sim SU^*(4) \sim SL(2, \mathbb{H})$ generators of the minrep of $SO(6,2)$ with spinorial indices take the form

$$M_{\alpha}^{\beta} = -\frac{i}{2} (\hat{\sigma}^\mu \hat{\bar{\sigma}}^\nu)^\alpha_{\beta} M_{\mu\nu}$$

(4.2.46)

which in terms of deformed twistorial oscillators $Y, Z$ can be written as:

$$M_{\alpha}^{\beta} = -\frac{1}{2} \left( Z_{\alpha}^{i} \bar{Y}^{\beta j} - \frac{1}{4} \delta_{\alpha}^{\gamma} Z_{\gamma}^{i} \bar{Y}^{\gamma j} \right) \epsilon_{ij}$$

(4.2.47)

$$= \frac{1}{2} \left( Y^{\beta i} \bar{Z}_{\alpha}^{j} - \frac{1}{4} \delta_{\alpha}^{\gamma} Y^{\gamma i} \bar{Z}_{\gamma}^{j} \right) \epsilon_{ij}$$

(4.2.48)

The dilatation generator $\Delta$ takes the form:

$$\Delta = \frac{i}{8} \left( Z_{\alpha}^{i} \bar{Y}^{\alpha j} - Y^{\alpha i} \bar{Z}_{\alpha}^{j} \right) \epsilon_{ij}$$

(4.2.49)

The commutation relations of the generators of the minrep of the conformal algebra $SO(6,2)$ given above are as follows:

$$[M_{\alpha}^{\beta}, M_{\gamma}^{\delta}] = \delta_{\alpha}^{\delta} M_{\gamma}^{\beta} - \delta_{\beta}^{\delta} M_{\alpha}^{\gamma}$$

(4.2.50)

$$[P_{\alpha\beta}, M_{\gamma}^{\delta}] = -2\delta_{\alpha}^{\delta} P_{\beta\gamma} - \frac{1}{2} \delta_{\beta}^{\delta} P_{\alpha\beta}, \quad [K^{\alpha\beta}, M_{\gamma}^{\delta}] = 2\delta_{\alpha}^{\gamma} K^{\beta\delta} + \frac{1}{2} \delta_{\beta}^{\gamma} K^{\alpha\delta}$$

(4.2.51)

$$[P_{\alpha\beta}, K^{\gamma\delta}] = 16 \left( \delta_{[\alpha}^{[\gamma} M_{\beta]}^{\delta]} + \frac{i}{2} \delta_{[\alpha}^{[\gamma} \delta_{\beta]}^{\delta]} \Delta \right)$$

(4.2.52)

$$[\Delta, P_{\alpha\beta}] = i P_{\alpha\beta}, \quad [\Delta, K^{\alpha\beta}] = -i K^{\alpha\beta}, \quad [\Delta, M_{\alpha}^{\beta}] = 0$$

(4.2.53)

The algebra $so(6,2)$ can be given a 3-graded decomposition with respect to the conformal Hamiltonian, which is referred to as the compact 3-grading and the generators in this basis are reproduced in Appendix B.3 following [24].

### 4.2.3 Deformations of the minimal unitary representation of $SO(6,2)$

It was shown in [24] that the minrep of $SO^*(8)$ that was studied in previous section is simply isomorphic to the scalar doubleton representation that describes a conformal massless scalar field in six dimensions. However we have seen in section [4.2.1] that $SO^*(8)$ admits infinitely many doubleton representations corresponding to massless conformal fields transforming as symmetric tensors in the spinorial indices. As in the case of 4d conformal group $SU(2,2)$, it was shown in [24] that there exists a discrete infinity of
deformations to the minrep of $SO(6,2)$ labeled by the spin $t$ of an $SU(2)$ symmetry group which is the $6d$ analog of helicity in $4d$. Allowing this spin $t$ to take all possible values, one obtains a discretely infinite set of deformations of the minrep which are isomorphic to the doubleton representations. In this section we will show that the generators of these representations can be recast as bilinears of deformed twistorial operators as was done in the previous subsection for the true minrep that corresponds to $t = 0$.

Following [24], let us introduce an arbitrary number $P$ pairs of fermionic oscillators $\rho_x$ and $\chi_x$ and their hermitian conjugates $\rho^x = (\rho_x)\dagger$ and $\chi^x = (\chi_x)\dagger$, $(x = 1, 2, \ldots, P)$ that satisfy the anti-commutation relations:

\[
\{\rho_x, \rho^y\} = \{\chi_x, \chi^y\} = \delta^y_x, \quad \{\rho_x, \rho_y\} = \{\rho_x, \chi_y\} = \{\chi_x, \chi_y\} = 0 \quad (4.2.54)
\]

and refer to them as “deformation fermions”. The following bilinears of these fermionic oscillators

\[
G_+ = \rho^x\chi_x \quad G_- = \chi^x\rho_x \quad G_0 = \frac{1}{2} (N_\rho - N_\chi) \quad (4.2.55)
\]

,where $N_\rho = \rho^x\rho_x$ and $N_\chi = \chi^x\chi_x$ are the respective number operators, generate an $su(2)_G$ algebra:

\[
[G_+, G_-] = 2G_0, \quad [G_0, G_\pm] = \pm G_0 \quad (4.2.56)
\]

The fermionic oscillators $\rho^x$ and $\chi^x$ form a doublet of $SU(2)_G$. We choose the Fock vacuum of these fermionic oscillators such that

\[
\rho_x |0\rangle = \chi_x |0\rangle = 0 \quad (4.2.57)
\]

for all $x = 1, 2, \ldots, P$. The states of the form

\[
\chi^{x_1\chi^{x_2} \cdots \rho^{(n-1)}\rho^{x_n}} |0\rangle
\]

with definite eigenvalue $n \leq P$ of the total number operator $N_T = N_\chi + N_\rho$ transform irreducibly in the spin $j = n/2$ representation of $SU(2)_G$ \[3\]. Among the irreducible representations of $SU(2)_G$ in the Fock space of $P$ pairs of deformation fermions the multiplicity of the highest spin ($j = P/2$) representation is one.

Now to deform the minimal unitary realization of $so^*(8)$, one extends the subalgebra $su(2)_S$ to the diagonal subalgebra $su(2)_T$ of $su(2)_S$ and $su(2)_G$ \[24\]. In other words, the generators of $su(2)_S$ receive contributions from the $\rho$- and $\chi$-type fermionic oscillators as

\[3\] Note that square bracketing of indices implies complete anti-symmetrization of weight one.
follows:

\[ T_+ = S_+ + G_+ = a^m b_m + \rho^x \chi_x \]
\[ T_- = S_- + G_- = b^m a_m + \chi^x \rho_x \]
\[ T_0 = S_0 + G_0 = \frac{1}{2} (N_a - N_b + N_\rho - N_\chi) \]

The quadratic Casimir of this subalgebra \( \text{su}(2)_T \) is given by

\[ C_2 [\text{su}(2)_T] = T^2 = T_0 T_0 + \frac{1}{2} (T_+ T_- + T_- T_+) . \] (4.2.59)

In order to obtain the generators for the deformations of the minrep all we need to do is replace \( S_0, S_\pm \) by \( T_0, T_\pm \) respectively in equations (4.2.28) - (4.2.35) and equations (4.2.37) - (4.2.44). We will denote the resulting deformed twistors as \((Z_t)_\alpha^i, (\bar{Z}_t)_\alpha^i\) and \((Y_t)^\alpha_i, (\bar{Y}_t)^\alpha_i\).

The generators can then be written as:

\[ P_{\alpha\beta} = (Z_t)_\alpha^i (\bar{Z}_t)^j_\beta \epsilon_{ij}, \quad K^{\alpha\beta} = (Y_t)^\alpha (\bar{Y}_t)^\beta \epsilon_{ij} \] (4.2.60)

\[ M^\beta_\alpha = -\frac{1}{2} \left( (Z_t)_\alpha^i (\bar{Y}_t)^j_\beta - \frac{1}{4} \delta^\beta_\gamma (Z_t)_\gamma^i (\bar{Y}_t)^j_\gamma \right) \epsilon_{ij} \] (4.2.61)
\[ = \frac{1}{2} \left( (Y_t)^j_\beta (\bar{Z}_t)_\alpha^i - \frac{1}{4} \delta^\beta_\gamma (Y_t)^j_\gamma (\bar{Z}_t)_\alpha^i \right) \epsilon_{ij} \] (4.2.62)

\[ \Delta = \frac{i}{8} \left( (Z_t)_\alpha^i (\bar{Y}_t)^j_\alpha - (Y_t)^\alpha (\bar{Z}_t)_\alpha^i \right) \epsilon_{ij} \] (4.2.63)

The Casimir invariants for \( SO(6,2) \) for the deformed minreps depend only on the quadratic Casimir of \( SU(2)_G \) involving deformation fermions. For one set of fermions \( (P = 1) \) one finds:

\[ C_2 = M^A_B M^B_A = 16 - 6 (N_\rho - N_\chi)^2 \] (4.2.64)
\[ C_4 = M^A_B M^B_C M^C_D M^D_A = -\frac{69}{12} C_2 - 20 \] (4.2.65)
\[ C_4' = \epsilon^{ABCD\bar{E}FGH} M_{AB} M_{CD} M_{EF} M_{G\bar{H}} = 60 (C_2 - 16) \] (4.2.66)
\[ C_6 = M^A_B M^B_C M^C_D M^D_E M^E_F M^F_A = \frac{63}{48} C_2 + 475 \] (4.2.67)

which shows clearly that the deformations are driven by fermionic oscillators.

**4.2.4 Minimal unitary supermultiplet of \( OSp(8^*|4) \) and its deformations**

The construction of the minimal unitary representations of non-compact Lie algebras by quantization of their quasiconformal realizations extends to non-compact Lie superalgebras
In this section we will reformulate the minimal unitary realization of 6d superconformal algebra $\text{OSp}(8^*|4)$ with the even subgroup $SO^*(8) \times USp(4)$ given in [56] in terms of deformed twistors. Extension to general superalgebras $\text{OSp}(8^*|2N)$ is straightforward.

Consider the superconformal (non-compact) 5-graded decomposition of the Lie superalgebra $\mathfrak{osp}(8^*|4)$ with respect to the dilatation generator $\Delta$:

$$
\mathfrak{osp}(8^*|4) = \mathfrak{N}^{-1} \oplus \mathfrak{N}^{-1/2} \oplus \mathfrak{N}^0 \oplus \mathfrak{N}^{1/2} \oplus \mathfrak{N}^1
$$

where the grade zero space consists of the Lorentz algebra $\mathfrak{so}(5,1)$ ($M_{\alpha\beta}$), the dilatations ($\Delta$) and R-symmetry algebra $\mathfrak{usp}(4)$ ($U^{ab}$), grade $+1$ and $-1$ spaces consist of translations ($P_{a\beta}$) and special conformal transformations ($K_{\alpha\beta}$) respectively, and the $+1/2$ and $-1/2$ spaces consist of Poincaré supersymmetries ($Q_a^\alpha$) and conformal supersymmetries ($S_{\alpha a}$) respectively.

We introduce fermionic oscillators $\xi_{ai}$ where $a, b = 1, 2, 3, 4$ are the $USp(4) \sim SO(5)$ indices which are raised and lowered by the antisymmetric symplectic metric

$$
\Omega^{ab} = \begin{pmatrix}
0 & 1_2 \\
-1_2 & 0
\end{pmatrix}
$$

and $i, j = 1, 2$ are the $SU(2)$ indices raised and lowered by $\epsilon_{ij}$ ($\epsilon_{12} = \epsilon^{21} = +1$). These oscillators satisfy:

$$
\{\xi^a_i, \xi^b_j\} = \Omega^{ab} \epsilon^{ij}
$$

and they will be referred to as supersymmetry fermions. The following bilinears of these fermions

$$
F_+ = \frac{1}{2} \xi^a \xi^b \Omega_{ba}, \quad F_- = \frac{1}{2} \xi^a \xi^b \Omega_{ab}, \quad F_0 = \frac{1}{2} (\xi^a \xi^b + \xi^b \xi^a) \Omega_{ba}
$$

generate a $\mathfrak{su}(2)_F$ algebra:

$$
[F_+, F_-] = 2F_0, \quad [F_0, F_0] = \pm F_0
$$

To obtain the supersymmetric extensions of the deformations of the minrep of $SO^*(8)$, one extends the $\mathfrak{su}(2)_T$ subalgebra to $\mathfrak{su}(2)_T$ which is the diagonal subalgebra of $\mathfrak{su}(2)_T$ and $\mathfrak{su}(2)_F$. The generators of $\mathfrak{su}(2)_T$ are then given by:

$$
T_+ = S_+ + G_+ + F_+ = a^i b_i + \rho^x \chi_x + \frac{1}{2} \xi^a \xi^b \Omega_{ba}
$$
\[ T_- = S_- + G_- + F_- = b^i a_i + \chi^\alpha \rho_\alpha + \frac{1}{2} \xi^{a_2} \xi^{b_2} \Omega_{ab} \]  
(4.2.73)

\[ T_0 = S_0 + G_0 + F_0 = \frac{1}{2} (N_a - N_b + N_\rho - N_\chi) + \frac{1}{4} (\xi^{a_1} \xi^{b_2} + \xi^{a_2} \xi^{b_1}) \Omega_{ba} \]  
(4.2.74)

The generators of the even subgroup \( SO^*(8) \) of the deformations of the minrep of \( OSp(8^*|4) \) are then obtained simply by replacing \( S_0, S_\pm \) by \( T_0, T_\pm \) respectively in equations 4.2.28 - 4.2.35 and equations 4.2.37 - 4.2.44. We will denote the resulting deformed twistors as \( (Z\alpha^i)^a, (Z\bar{\alpha}^i)^a \) and \( (Y^a)^{\alpha a}, (Y\bar{\alpha})^{a a} \). The generators of \( SO^*(8) \) can then be written as bilinears of these deformed twistors:

\[ P_{\alpha\beta} = (Z\alpha^i)^a (Z\bar{\beta}^j)^a \epsilon_{ij}, \quad K^{\alpha\beta} = (Y^a)^{\alpha a} (Y\bar{\beta})^{a a} \epsilon_{ij} \]  
(4.2.75)

\[ M_{\alpha}^\beta = -\frac{1}{2} \left( (Z\alpha^i)^a (Y\bar{\beta})^{a a} j - \frac{1}{4} \delta^j_{\alpha} (Z\alpha^i)^a (Y\bar{\beta})^{a a} j \right) \epsilon_{ij} \]  
(4.2.76)

\[ = \frac{1}{2} \left( (Y^a)^{\alpha a} (Z\bar{\beta}^j)^a - \frac{1}{4} \delta^j_{\alpha} (Y^a)^{\alpha a} (Z\bar{\beta}^j)^a \right) \epsilon_{ij} \]  
(4.2.77)

\[ \Delta = \frac{i}{8} \left( (Z\alpha^i)^a (Y\bar{\beta})^{a a} j - (Y^a)^{\alpha a} (Z\bar{\beta}^j)^a \right) \epsilon_{ij} \]  
(4.2.78)

The supersymmetry generators \( Q_{\alpha}^a, S^{\alpha a} \) can similarly be realized simply as bilinears of ordinary fermionic oscillators and deformed twistors as follows:

\[ Q_{\alpha}^a = (Z\alpha^i)^a \xi^a j \epsilon_{ij} = \xi^a (Z\bar{\beta}^j)^a \epsilon_{ij} \]  
(4.2.79)

\[ S^{\alpha a} = (Y^a)^{\alpha a} \xi^a j \epsilon_{ij} = \xi^a (Y\bar{\beta})^{a a} \epsilon_{ij} \]  
(4.2.80)

The supersymmetry generators satisfy

\[ \{ Q_{\alpha}^a, Q_{\beta}^b \} = -\Omega^{ab} P_{\alpha\beta}, \quad \{ S^{\alpha a}, S^{\beta b} \} = -\Omega^{ab} K^{\alpha\beta} \]  
(4.2.81)

\[ \{ S^{\alpha a}, Q_{\beta}^b \} = -2\Omega^{ab} M_{\beta}^\alpha - i\delta^a_{\beta} \Omega^{ab} \Delta - 2\delta^a_{\beta} U^{ab} \]  
(4.2.82)

The \( R \)-symmetry group \( USp(4) \sim SO(5) \) can realized as bilinears of fermionic oscillators as follows:

\[ U^{ab} = \left( \xi^{a i} \xi^{b j} - \frac{1}{4} \Omega^{ab} \xi^c \xi^d \right) \epsilon_{ij} \]  
(4.2.83)

\(^4\)To obtain the generators for the true minimal unitary supermultiplet of \( OSp(8^*|4) \) one needs only to drop the deformation fermions from the generators of \( su(2)_\tau \) and the corresponding deformed twistors will be denoted as \( (Z\alpha^i)^a, (Z\bar{\alpha})^i)^a \) and \( (Y^a)^{\alpha a}, (Y\bar{\alpha})^{a a} \).
They satisfy the following commutation relations:

\[ [U^{ab}, U^{cd}] = 2\Omega^a(cU^d)b + 2\Omega^b(cU^d)a \] (4.2.84)

The commutators of \( SO^*(8) \) generators with the supersymmetry generators are as follows:

\[ [M_\alpha^\beta, Q_\gamma^a] = -\delta_\beta^\gamma Q_\alpha^a + \frac{1}{4} \delta_\alpha^\beta Q_\gamma^a, \quad [M_\alpha^\beta, S^\gamma^a] = \delta_\beta^\gamma S_\alpha^a - \frac{1}{4} \delta_\alpha^\beta S^\gamma^a \] (4.2.85)

\[ [K^{\alpha \beta}, Q_\gamma^a] = -4\delta^{[\alpha}_\beta S^{]\gamma^a}, \quad [P_{\alpha \beta}, S^\gamma^a] = -4\delta^{\gamma^a}_{[\beta} Q^{\alpha]} \] (4.2.86)

\[ [\Delta, Q_\gamma^a] = \frac{i}{2} Q_\alpha^a, \quad [\Delta, S^\alpha^a] = -\frac{i}{2} S_\alpha^a \] (4.2.87)

The \( R \)-symmetry generators act on \( USp(4) \) indices of supersymmetry generators as follows:

\[ [U^{ab}, Q_\alpha^c] = -2\Omega^c(aQ^b_\gamma^\gamma), \quad [U^{ab}, S^{\alpha^a}^c] = -2\Omega^c(aS^b_\gamma^\gamma) \] (4.2.88)

The generators given above transform covariantly with respect to the subgroup \( SU^*(4) \times SO(1,1) \times Usp(4) \). Unitarity and positive energy nature of the resulting representations are made manifest by going to the compact three grading of \( OSp(8*|4) \) with respect to the compact sub-superalgebra \( SU(4|2) \times U(1) \) [24].

### 4.3 AdS\(_7\)/CFT\(_6\) higher spin (super-)algebras, Joseph ideals and their deformations

As reviewed in Chapter 3, the standard \( AdS_d/CFT_{(d-1)} \) higher spin algebra of Fradkin-Vasiliev type is simply given by the quotient of the universal enveloping algebra of \( SO(d,2) \) by a two-sided ideal [30,80,130,136]. This two-sided ideal is the Joseph ideal that annihilates the minimal unitary representation. Denoting the higher spin algebra as \( HS(g) \) with \( g = so(d - 1,2) \) and the universal enveloping algebra as \( \mathfrak{u}(g) \) we have

\[ HS(g) = \frac{\mathfrak{u}(g)}{\mathfrak{j}(g)} \] (4.3.1)

where \( \mathfrak{j}(g) \) denotes the Joseph ideal.

The uniqueness of the Joseph ideal was proved in [136] and an explicit formula for the generators of this ideal for \( SO(n-2,2) \) was given as:

\[
J_{ABCD} = M_{AB}M_{CD} - M_{AB} \otimes M_{CD} - \frac{1}{2} [M_{AB} , M_{CD}] + \frac{n-4}{4(n-1)(n-2)} \langle M_{AB}, M_{CD} \rangle \mathbf{1} \\
= \frac{1}{2} M_{AB} \cdot M_{CD} - M_{AB} \otimes M_{CD} + \frac{n-4}{4(n-1)(n-2)} \langle M_{AB}, M_{CD} \rangle \mathbf{1} \] (4.3.2)
where the dot · denotes the symmetric product

\[ M_{AB} \cdot M_{CD} \equiv M_{AB} M_{CD} + M_{CD} M_{AB} \quad (4.3.3) \]

\[ \langle M_{AB}, M_{CD} \rangle \] is the Killing form of \( SO(n-2,2) \) given by

\[ \langle M_{AB}, M_{CD} \rangle = h M_{EF} M_{GH} (\eta^{EG} \eta^{FH} - \eta^{EH} \eta^{FG}) (\eta_{AC} \eta_{BD} - \eta_{AD} \eta_{BC}) \quad (4.3.4) \]

where \( h = \frac{2(n-2)}{n(4-n)} \) chosen such that all possible contractions of \( J_{ABCD} \) with the metric vanish. The symbol \( \otimes \) denotes the Cartan product of two generators [137]:

\[
\begin{align*}
M_{AB} \otimes M_{CD} &= \frac{1}{3} M_{AB} M_{CD} + \frac{1}{3} M_{DC} M_{BA} + \frac{1}{6} M_{AC} M_{BD} \\
&- \frac{1}{6} M_{AD} M_{BC} + \frac{1}{6} M_{DB} M_{CA} - \frac{1}{6} M_{CB} M_{DA} \\
&- \frac{1}{2(n-2)} (M_{AE} M_{C}^{E} \eta_{BD} - M_{BE} M_{C}^{E} \eta_{AD} + M_{BE} M_{D}^{E} \eta_{AC} - M_{AE} M_{D}^{E} \eta_{BC}) \\
&- \frac{1}{2(n-2)} (M_{CE} M_{A}^{E} \eta_{BD} - M_{CE} M_{B}^{E} \eta_{AD} + M_{DE} M_{B}^{E} \eta_{AC} - M_{DE} M_{A}^{E} \delta_{BC}) \\
&+ \frac{1}{(n-1)(n-2)} M_{EF} M_{EF} (\eta_{AC} \eta_{BD} - \eta_{BC} \eta_{AD}) 
\end{align*}
\]

(4.3.5)

We shall refer to the operator \( J_{ABCD} \) as the generator of the Joseph ideal which for \( SO(6,2) \) takes the form:

\[ J_{ABCD} = \frac{1}{2} M_{AB} \cdot M_{CD} - M_{AB} \otimes M_{CD} - \frac{1}{112} \langle M_{AB}, M_{CD} \rangle \quad (4.3.6) \]

The enveloping algebra \( \mathfrak{u}(\mathfrak{g}) \) of a Lie algebra \( \mathfrak{g} \) can be decomposed with respect to the adjoint action of \( \mathfrak{g} \). By Poincare-Birkhoff-Witt theorem this is equivalent to computing symmetric products of the generators \( M_{AB} \sim \) of \( \mathfrak{g} \). For \( \mathfrak{so}(6,2) \) the symmetric product of the adjoint action decomposes as:

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higher spin algebra $HS(6,2)$.

### 4.3.1 Joseph ideal for minimal unitary representation of $SO(6,2)$

Since the Eastwood formula for Joseph ideal is in the canonical basis we define

$$M_{\mu 6} = \frac{1}{2} (P_\mu - K_\mu)$$

$$M_{\mu 7} = \frac{1}{2} (P_\mu + K_\mu)$$

$$M_{67} = -\Delta$$

which together with the Lorentz group generators $M_{\mu \nu}$ form the canonical basis $M_{AB}$ ($A,B,.. = 0,1,...7$). Substituting the expressions for the generators $M_{AB}$ of the minrep of $SO(6,2)$ from the quasiconformal realization into the generator of Joseph ideal (equation 4.3.6) one finds that it vanishes identically as an operator. To get a better insight into the physical meaning of the vanishing of the Joseph ideal we write $J_{ABCD}$ in the Lorentz covariant conformal basis ($K_\mu, M_{\mu \nu}, \Delta, P_\mu$), which is equivalent to certain quadratic identities. In addition to the conditions:

$$P_\mu P_\mu = K_\mu K_\mu = 0 \quad \text{or,} \quad P^2 = K^2 = 0$$

one finds the following identities:

$$6\Delta \cdot \Delta + M^{\mu \nu} \cdot M_{\mu \nu} + 2P_\mu \cdot K_\mu = 0$$

$$P_\mu \cdot (M_{\mu \nu} + \eta_{\mu \nu} \Delta) = 0$$

$$K_\mu \cdot (M_{\nu \mu} + \eta_{\nu \mu} \Delta) = 0$$

$$\eta^{\mu \nu} M_{\mu \rho} \cdot M_{\nu \sigma} - P_{(\rho} \cdot K_{\sigma)} + 4\eta_{\mu \sigma} = 0$$

$$M_{\mu \nu} \cdot M_{\rho \sigma} + M_{\mu \sigma} \cdot M_{\nu \rho} + M_{\mu \rho} \cdot M_{\sigma \nu} = 0$$

$$\Delta \cdot M_{\mu \nu} + P_{[\mu} \cdot K_{\nu]} = 0$$

$$M_{[\mu \nu} \cdot P_{\rho]} = 0$$

$$M_{[\mu \nu} \cdot K_{\rho]} = 0$$

Defining the generalized Pauli-Lubanski tensor and its conformal analogue in six dimensions as:

$$A_{\mu \nu \rho} = \frac{1}{3!} \epsilon_{\mu \nu \rho \sigma \tau} M^{\sigma \delta} \cdot P^{\tau}$$

$$B_{\mu \nu \rho} = \frac{1}{3!} \epsilon_{\mu \nu \rho \sigma \tau} M^{\sigma \delta} \cdot K^{\tau}$$

(4.3.20)
we find that they vanish identically for the minrep given above

\[ A_{\mu\nu\rho} = 0 \quad B_{\mu\nu\rho} = 0 \]  

(4.3.21)

Computing the products of the generators of the above minimal unitary realization corresponding to the Young tableaux \( \begin{array}{cccc}
  & & & \\
  & & & \\
  & & & \\
\end{array} \) and \( \begin{array}{cccc}
  & & & \\
  & & & \\
  & & & \\
\end{array} \) one finds that they vanish identically and the resulting enveloping algebra contains only the operators whose Young tableaux have two rows:

\[ \begin{array}{cccc}
  & & & \\
  & & & \\
  & & & \\
\end{array} \]

\[ n \text{ boxes} \]

4.3.2 Deformations of the minimal unitary representation of SO(6,2) and the Joseph ideal

As we saw above the generator \( J_{ABCD} \) of the Joseph ideal for \( SO^*(8) \) vanishes identically as an operator for the minrep obtained by quasiconformal techniques. However when one substitutes the generators of the deformed minreps one finds that \( J_{ABCD} \) does not vanish identically. However as we will show the generators of the deformed minreps satisfy certain quadratic identities which correspond to deformations of the Joseph ideal.

To exhibit the quadratic identities corresponding to deformations of the Joseph ideal we decompose the components of \( J_{ABCD} \) \( (A,B,.. = 0,1,2,...,7) \) in terms of Lorentz \( (SU^*(4) ) \) covariant indices. We find that the totally antisymmetric tensors \( A_{\mu\nu\rho} \) and \( B_{\mu\nu\rho} \) \( (\mu,\nu,... = 0,1,...,5) \) defined in the previous section that vanished identically for the minrep do not vanish for the deformed minreps. Remarkably they become self-dual and anti-self-dual tensorial operators, respectively:

\[ A_{\mu\nu\rho} = \tilde{A}_{\mu\nu\rho} \]

\[ B_{\mu\nu\rho} = -\tilde{B}_{\mu\nu\rho} \]

(4.3.22)

The identities \( (4.3.16) \) and \( (4.3.19) \) no longer hold separately but they combine and the following identity holds true for deformed generators:

\[ M_{\mu\nu} \cdot M_{\rho\sigma} + M_{\mu\sigma} \cdot M_{\nu\rho} + M_{\mu\rho} \cdot M_{\sigma\nu} = \epsilon_{\mu\nu\rho\sigma} \delta^\tau (P_\delta \cdot K_\tau] + M_\delta \cdot \Delta) \]

(4.3.23)
The deformation of the identity (4.3.15) is as follows:

$$\eta^\mu\nu M_{\mu\rho} \cdot M_{\nu\sigma} - P_{\rho} \cdot K_{\sigma} + 4\eta_{\rho\sigma} = 2G^2\eta_{\rho\sigma}$$ \hspace{1cm} (4.3.24)

where $G^2$ is quadratic Casimir of $su(2)_G$ defined in equation 4.2.55 and involves only the deformation fermions. Note that the quadratic Casimir operator $G^2$ of $su(2)_G$ is related to the quadratic Casimir operator of the deformed minrep of $SO^*(8)$ as follows [24]:

$$C_2[so^*(8)]_{deformed} = 8\left(2 - G^2\right)$$ \hspace{1cm} (4.3.25)

The eigenvalues $t(t+1)$ ($t = 0, 1/2, 1, 3/2, ...$) of $G^2$ label the deformations of the minrep of $SO^*(8)$. This is to be contrasted with the deformations of the minrep of $4d$ conformal group $SO(4,2)$ which are labelled by a continuous parameter that enters the quadratic identities explicitly in the form of continuous helicity [3]. Since the minrep and its deformations obtained by quasiconformal methods correspond to massless conformal fields we do not expect any continuous deformations in six dimensions since the little group of massless particles is $SO(4) = SU(2)_T \times SU(2)_A$ whose unitary representations are discretely labelled $(j_T, j_A)$ where $j_A$ and $j_T$ are non-negative integers or half integers. Furthermore for conformally massless representations either $j_T$ or $j_A$ (or both) vanishes. However we do not have a proof that continuous deformations do not exist.

For the discrete deformations of the minrep the operators corresponding to the symmetric tensor with Young Tableau appearing in the symmetric product of the generators as shown in 3.4.2 still vanish On the other hand the operators with the Young tableau

\[ \begin{array}{c} \hline \end{array} \]

do not vanish. These operators satisfy an eight dimensional self-duality condition which corresponds to a three form gauge field with a self-dual field strength. In $AdS_7$ they correspond to three form gauge fields that satisfy odd dimensional self-duality condition just like the three form field that descends from eleven dimensional supergravity on $AdS_7 \times S^4$. In six dimensions they correspond to conformal two form fields with a self-dual field strength, which is simply the tensor field that appears in the $(2,0)$ conformal supermultiplet whose interacting theory is believed to be dual to M-theory over $AdS_7 \times S^4$.

The symmetric tensor products of the generators of the discrete deformations of $SO(6,2)$ leads to a $AdS_7/CFT_6$ higher spin algebra whose generators include Young tableaux of the form
This suggests that the theories based on discrete deformations of the minrep describe higher spin theories of Fradkin-Vasiliev type in $AdS_7$ coupled to tensor fields that satisfy self-duality conditions and their higher extensions corresponding to the Young tableaux

\begin{align*}
\begin{array}{|c|c|c|c|c|c|c|}
\hline
& & & & & & \\
\hline
& & & & & & \\
\hline
& & & & & & \\
\hline
& & & & & & \\
\hline
& & & & & & \\
\hline
& & & & & & \\
\hline
\end{array}
\end{align*}

The study of higher spin theories based on discrete deformations of the minrep that extend Fradkin-Vasiliev type higher spin theories will be the subject of a separate study.

### 4.3.3 Comparison with the covariant twistorial oscillator realization

Substituting the generators $M_{AB}$ of $SO^*(8)$ realized as bilinears of covariant twistorial oscillators (doubletons) (section 4.2.1) in equation 4.3.6 to compute the generator of the Joseph ideal one finds that it does not vanish identically. However the non-vanishing components of the generator $J_{ABCD}$ of the Joseph ideal factorize in a similar fashion as in the doubleton realization of $SO(4,2)$\[^3\] Symbolically this factorization takes the form

\begin{equation}
J_{ABCD} = (\ldots)B_a
\end{equation}

where $B_a (a = 1, 2, 3)$ is a generator of an $SU(2)_B$ algebra that commutes with $SO^*(8)$. In terms of the covariant twistorial oscillators the generators of $SU(2)_B$ are

\begin{equation}
B_- = d^i c_i, \quad B_+ = c^i d_i, \quad B_0 = \frac{1}{2}(c^i c_i - d^i d_i)
\end{equation}

with the quadratic Casimir

\begin{equation}
B^2 = B_0^2 + \frac{1}{2}(B_+ B_- + B_- B_+)
\end{equation}

Acting on the subspace of the Fock space of covariant twistorial oscillators that is $SU(2)_B$ singlet the generator $J_{ABCD}$ vanishes. This subspace corresponds to the true minrep of $SO^*(8)$ and describes a conformal scalar field in six dimensions. In fact the authors of [155] studied a purely bosonic higher spin algebra in $AdS_7$ using the doubletonic realization
of $SO^*(8)$. After imposing an infinite set of constraints, restricting to an $SU(2)$ singlet sector and modding out by an infinite ideal containing all the traces they obtain an higher spin algebra. They also state that their results can not be extended to $SU(2)$ non-singlet sectors.

The algebra $su(2)_B$ for the doubleton representations is the analog of $su(2)_G$ for the deformed minreps studied above. The Casimir operator $B^2$ of $SU(2)_B$ is related to the quadratic Casimir $C_2$ of the doubletonic realization of $SO^*(8)$ in terms of covariant twistorial oscillators:

$$C_2 [so^*(8)]_{\text{doubleton}} = 8(2 - B^2) \quad (4.3.29)$$

which reflects the fact that $su(2)_B$ and $so^*(8)_{\text{doubleton}}$ form a reductive dual pair inside $sp(16, \mathbb{R})$. However this is not the case with the deformed minreps since $su(2)_G$ does not commute with the generators of $so^*(8)_{\text{deformed}}$. Another critical difference is the fact that the possible eigenvalues $b(b + 1)$ of $SU(2)_B$ span the entire set of irreps, i.e $b = 0, 1/2, 1, 3/2, ...$. On the other hand, for a given number $P$ pairs of deformation Fermions possible eigenvalues $j(j + 1)$ of $SU(2)_G$ is $j = 0, 1/2, 1, ..., P/2$

For the doubleton realization the quadratic identities satisfied by the generators satisfy formally the same identities as the deformed minrep given in the previous subsection with $G^2$ replaced by $B^2$:

$$\eta^\mu\nu M_{\mu\rho} \cdot M_{\nu\sigma} - P_{(\rho} K_{\sigma)} + 4\eta_{\rho\sigma} = 2B^2 \eta_{\rho\sigma} \quad (4.3.30)$$

The Casimir invariants for $SO(6, 2)$ in the doubleton representation are as follows:

$$C_2 = 8(2 - B^2) \quad (4.3.31)$$
$$C_4 = \frac{C_2^2}{8} - 9C_2 \quad (4.3.32)$$
$$C_4' = 96C_2 - 6C_2^2 \quad (4.3.33)$$
$$C_6 = \frac{C_2^3}{64} - \frac{27}{8} C_2^2 + 81C_2 \quad (4.3.34)$$

4.3.4 $AdS_7/CFT_6$ Higher spin algebras and superalgebras and their deformations

Following [130] and Chapter 3, we will use the following definition for the standart higher spin algebra in six dimensions:

$$HS(6, 2) = \frac{\mathfrak{u}(so(6, 2))}{\mathfrak{b}(so(6, 2))} \quad (4.3.35)$$
where \( \mathfrak{u}(SO(6, 2)) \) is the universal enveloping algebra and \( \mathfrak{j}(\mathfrak{so}(6, 2)) \) denotes the Joseph ideal of \( \mathfrak{so}(6, 2) \). The Joseph ideal vanishes identically for the quasiconformal realization of the minrep. Therefore to construct \( HS(6, 2) \) one needs simply take the enveloping algebra of the minrep in the quasiconformal construction. Since the minrep of \( \mathfrak{so}(6, 2) \) admits deformations we define deformed \( AdS_7/CFT_6 \) higher spin algebras \( HS(6, 2; t) \) as the enveloping algebras of \( \mathfrak{so}(6, 2) \) quotiented by the deformed Joseph ideal \( \mathfrak{j}_t(\mathfrak{so}(6, 2)) \)

\[
HS(6, 2; t) = \mathfrak{u}(\mathfrak{so}(6, 2)) / \mathfrak{j}_t(\mathfrak{so}(6, 2))
\]

(4.3.36)

For these deformed high spin algebras the corresponding deformed Joseph ideal vanishes identically as operator as we showed explicitly above for the conformal group in six dimensions. Deformed minreps describe massless conformal fields of higher spin labeled by the spin \( t \) of the \( SU(2)_G \) subgroup which is the analogue of helicity in \( d = 4 \).

We saw in section 4.2.3 that deformations of the minrep are driven by fermionic oscillators. For \( P \) pairs of deformation fermions the Fock space decomposes as the direct sum of the two spinor representations of \( SO(4P) \) generated by all the bilinears of the oscillators. The centralizer of \( SU(2)_G \) inside \( SO(4P) \) is \( USp(2P) \). Under \( USp(2P) \times SU(2)_G \) the Fermionic Fock space decomposes as

\[
2^{2P} = \sum_{r=0}^{P} (R^r, t = (P - r)/2)
\]

(4.3.37)

where \( R^r \) is the symplectic traceless tensor of rank \( r \) of \( USp(2P) \) and \( t \) is the spin of \( SU(2)_G \). The \( USp(2P) \) invariant (singlet) subspace transforms in the spin \( t = P/2 \) representation of \( SU(2)_G \). Therefore restricting to this invariant subspace we get a deformed minrep corresponding to a 6d massless conformal field transforming as a totally symmetric tensor of rank \( P \) in the spinor indices with respect to the Lorentz group \( SU^*(4) \). This way one can construct all conformally massless representations of \( SO(6, 2) \) as deformations of the minrep by choosing \( P = 0, 1, 2, ... \). The enveloping algebras of these deformed irreducible irreps define then a discrete infinity of higher spin algebras labelled by \( t = P/2 \). Equivalently one can simply substitute \( (P + 1) \times (P + 1) \) irreducible representation matrices for generators of \( SU(2)_G \) in place of the bilinears of deformation fermions. The latter is useful for writing down the irreducible infinite higher spin algebra without reference to its action on a representation space.

We shall define the \( 2N \) extended \( AdS_7/CFT_6 \) higher spin superalgebra as the enveloping algebra of the minimal unitary realization of the super algebra \( OSp(8^*|2N) \) obtained via the quasiconformal approach. For this algebra there are only \( 2N \) supersymmetry fermions and the minimal supermultiplet of \( OSp(8^*|2N) \) consists of the following massless
conformal fields:

\[ \Phi[A_1 A_2 \ldots A_N] \oplus \Psi_{\alpha}[A_1 A_2 \ldots A_{N-1}] \oplus \phi[[A_1 A_2 \ldots A_{N-2}] \oplus \ldots \]  \hspace{1cm} (4.3.38)

where \( \alpha, \beta, \ldots \) are the spinor indices of the 6d Lorentz group \( SU^*(4) \) and \( A_i \) denote the \( USp(2N) \) indices. The generator \( J_{ABCD} \) of the Joseph ideal vanishes when acting on the conformal scalars that are part of the minimal unitary supermultiplet. On the other fields of the minimal unitary supermultiplet deformed quadratic identities \( 4.3.2 \) involving supersymmetry fermions are satisfied.

Deformed \( AdS_7/CFT_6 \) higher spin superalgebras \( HS(6,2|2N;t) \) algebras are defined simply as enveloping algebras of the deformed minimal unitary realizations of the super algebras \( OSP(8^*|2N) \) with the even subalgebra \( SO^*(8) \oplus USp(2N) \) involving deformation fermions \( 4.2.4 \). As explained above restricting to \( USp(2P) \) invariant sector of the Fock space of \( 2P \) deformation fermions one gets a deformed minimal unitary realization of \( OSP(8^*|2N)_t \) for \( t = P/2 \). Equivalently one can simply substitute the \( (P + 1) \times (P + 1) \) representation matrices of \( SU(2)_G \) in place of the bilinears of deformation fermions. Their enveloping algebras define a discrete infinite family of higher spin superalgebras labeled by \( t = 0, 1/2, 1, 3/2, 2, \ldots \).
Chapter 5  
Discussion

5.1 $AdS_5/CFT_4$ higher spin holography

The existence of a one-parameter family of $AdS_5/\text{Conf}_4$ higher spin algebras and superalgebras raises the question as to their physical meaning. Before discussing the situation in four dimensions, let us summarize what is known for $AdS_4/\text{Conf}_3$ higher spin algebras. In 3$d$, there is no deformation of the higher spin algebra except for the super extension corresponding to $Sp(4; \mathbb{R}) \rightarrow OSp(N|4; \mathbb{R})$. For the bosonic $AdS_4$ higher spin algebras one finds that the higher spin theories of Vasiliev are dual to certain conformally invariant vector-scalar/spinor models in 3$d$ [15,84] in the large $N$ limit. More recently, Maldacena and Zhiboedov [100] studied the constraints imposed on a conformal field theory in three dimensions by the existence of a single conserved higher spin current. They found that this implies the existence of an infinite number of conserved higher spin currents. This corresponds simply to the fact that the generators of $SO(3,2)$ get extended to an infinite spin algebra when one takes their commutators with an operator which is bilinear or higher order in the generators, except for those operators that correspond to the Casimir elements. They also showed that the correlation functions of the stress tensor and the conserved currents are those of a free field theory in three dimensions, either a theory of $N$ free bosons or a theory of $N$ free fermions [100], which are simply the scalar and spinor singletons.

The distinguishing feature of 3$d$ is the fact that there exists only two minimal unitary representations corresponding to massless conformal fields which are simply the $Di$ and $Rac$ representations of Dirac. However in 4$d$, we have a one-parameter, $\zeta$, family of deformations of the minimal unitary representation of the conformal group corresponding to massless conformal fields of helicity $\zeta/2$. The same holds true for the minimal unitary supermultiplet of $SU(2,2|N)$. From M/superstring point of view the most important interacting and supersymmetric CFT in $d = 4$ is the $N = 4$ super Yang-Mills theory.
It was argued in references [85,89,156] that the holographic dual of $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $SU(N)$ at $g^2_{YM}N = 0$ for $N \to \infty$ should be a free gauge invariant theory in $AdS_5$ with massless fields of arbitrarily high spin and this was supported by calculations in [91]. Moreover the scalar sector of $\mathcal{N} = 4$ super Yang-Mills theory at $g^2_{YM}N = 0$ with $N \to \infty$ should be dual to bosonic higher spin theories in $AdS_5$ which provides a non-trivial extension of $AdS/CFT$ correspondence in superstring theory [14,41] to non-supersymmetric large $N$ field theories. The Kaluza-Klein spectrum of IIB supergravity over $AdS_5 \times S^5$ was first obtained by tensoring of the minimal unitary supermultiplet (scalar doubleton) of $SU(2,2|4)$ with itself repeatedly and restricting to CPT self-conjugate sector [32]. The massless graviton supermultiplet in $AdS_5$ sits at the bottom of this infinite tower. In fact all the unitary representations corresponding to massless fields in $AdS_5$ can be obtained by tensoring of two doubleton representations of $SU(2,2)$ which describe massless conformal fields on the boundary of $AdS_5$ [32,37,38,40]. As was argued by Mikhailov [91], in the large $N$ limit the correlation functions in the CFT side become products of two point functions which correspond to products of two doubletons. As such they correspond to massless fields in the $AdS_5$ bulk. At the level of correlation functions the same arguments suggest that corresponding to a one parameter family of deformations of the $N=4$ Yang-Mills supermultiplet there must exists a family of supersymmetric massless higher spin theories in $AdS_5$. Turning on the gauge coupling constant on the Yang-Mills side leads to interactions in the bulk and most of the higher spin fields become massive.

The fact that the quasi-conformal realization of the minrep of $SU(2,2|4)$ is nonlinear implies that the corresponding higher spin theory in the bulk must be interacting. Since the same minrep can also be obtained by using the doubletonic realization [32], which corresponds to free field realization, suggests that the interacting supersymmetric higher spin theory may be integrable in the sense that its holographic dual is an integrable conformal field theory with infinitely many conserved currents, just like the classical $N = 4$ super Yang-Mills theory in four dimensions [157]. There are other deformed higher spin algebras corresponding to non-CPT self-conjugate supermultiplets of $SU(2,2|4)$ that contain scalar fields and are deformations of the minrep. The above arguments suggest that they too should correspond to interacting but integrable supersymmetric higher spin theories in $AdS_5$. One solid piece of the evidence for this is provided by the fact that the symmetry superalgebras of interacting (nonlinear) superconformal quantum mechanical models of [115] furnish a one parameter family of deformations of the minimal unitary representation of the $N = 4$ superconformal algebra $D(2,1; \alpha)$ in one dimension. This was predicted in [125] and shown explicitly in [116].

Most of the work on higher spin algebras until now have utilized the realizations of
underlying Lie (super)algebras as bilinears of oscillators which correspond to free field realizations. The quasiconformal approach allows one to give a natural definition of super Joseph ideal and leads directly to the interacting realizations of the superextensions of higher spin algebras. The next step in this approach is to reformulate these interacting quasiconformal realizations in terms of covariant gauge fields and construct Vasiliev type nonlinear theories of interacting higher spins in $AdS_5$.

Another application of our results will be to reformulate the spin chain models associated with $N = 4$ super Yang-Mills theory in terms of deformed twistorial oscillators and study the integrability of corresponding spin chains non-perturbatively. In fact, a spectral parameter related to helicity and central charge was introduced recently for scattering amplitudes in $N = 4$ super Yang-Mills theory \cite{158}. This spectral parameter corresponds to our deformation parameter which is helicity and appears as a central charge in the quasiconformal realization of the super algebra $SU(2,2|4)$.

5.2 $AdS_7/CFT_6$ higher spin holography

The main result of Chapter four is the reformulation of the minimal unitary representation of $SO^*(8)$ and its deformations in terms of deformed twistors that transform nonlinearly under the Lorentz group in six dimensions. Their enveloping algebras lead to a discrete infinite family of $AdS_7/CFT_6$ higher spin algebras labelled by the spin of an $SU(2)$ symmetry for which certain deformations of the Joseph ideal vanish. Remarkably these deformations involve (anti-)self-duality of the $6d$ tensorial operator which is the analog of Pauli-Lubanski vector in four dimensions. These results carry to superalgebras $OSp(8^*|2N)$ and one finds a discrete infinity of $AdS_7/CFT_6$ higher spin superalgebras. As we argued for $AdS_5/CFT_4$ algebras, our results suggest the existence of a family of (supersymmetric) higher spin theories in $AdS_7$ that are dual to free (super) CFT’s or to interacting but integrable (supersymmetric) CFT’s in six dimensions.

Of particular interest are the higher spin superalgebras based on $OSp(8^*|4)$ and $OSp(8^*|8)$ whose minimal unitary supermultiplets reduce to $N = 4$ Yang-Mills supermultiplet and $N = 8$ supergravity multiplet under dimensional reduction to four dimensions \cite{132}. The minimal unitary supermultiplet of $OSp(8^*|4)$ is the $6d$ $(2,0)$ conformal tensor multiplet \cite{33,39} whose interacting theory is believed to be dual to M-theory over $AdS_7 \times S^4$ \cite{14}. Whether there exists a limit of this interacting theory that is dual to a higher spin theory in $AdS_7$ is an interesting open problem. Our results in section 3.2 suggest that such a limit should exist. On the other hand it is not known if there exists an interacting non-metric $(4,0)$ supergravity theory based on the minimal unitary supermultiplet of $OSp(8^*|8)$ \cite{132,159}. 

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Appendix A

A.1 Spinor conventions for $SO(2, 1)$

We follow [160] for the spinor conventions in $d = 3$ and thus all the 3d spinors are Majorana with $\eta_{\mu\nu} = \text{diag}(-, +, +)$. The gamma-matrices in Majorana representation terms of the Pauli matrices $\sigma^i$ are as follows:

$$\gamma^0 = -i\sigma^2, \quad \gamma^1 = \sigma^1, \quad \gamma^2 = \sigma^3$$

(A.1.1)

and they satisfy

$$\{\gamma^\mu, \gamma^\nu\}_\alpha^\beta = 2\eta^\mu\nu\delta_\alpha^\beta .$$

(A.1.2)

Thus the matrices $(\gamma^\mu)^\alpha_\beta$ are real and the Majorana condition on spinors imply that they are real two component spinors. Spinor indices are raised/lowered by the epsilon symbols with $\epsilon^{12} = \epsilon_{12} = 1$ and choosing NW-SE conventions

$$\epsilon^{\alpha\gamma}\epsilon_{\beta\gamma} = \delta_\alpha^\beta, \quad \lambda^\alpha := \epsilon^{\alpha\beta}\lambda_\beta \Leftrightarrow \lambda_\beta = \lambda^\alpha\epsilon_{\alpha\beta} .$$

(A.1.3)

Introducing the real symmetric matrices $(\sigma^\mu)_{\alpha\beta} := (\gamma^\mu)^\rho_\beta \epsilon_{\rho\alpha}$ and $(\bar{\sigma}^\mu)^{\alpha\beta} := (\epsilon\cdot\sigma^\mu\cdot\epsilon)^{\alpha\beta} = -\epsilon^{\beta\rho} (\gamma^\mu)^\alpha_\rho$, a three vector in spinor notation writes as a symmetric real matrix as

$$V_{\alpha\beta} := (\sigma^\mu V_\mu)_{\alpha\beta} \Rightarrow V^\mu = \frac{1}{2} (\bar{\sigma}^\mu)^{\alpha\beta} V_{\alpha\beta} .$$

(A.1.4)

A.2 Minimal unitary supermultiplet of $OSp(N|4, \mathbb{R})$

In this section we will formulate the minimal unitary representation of $OSp(N|4, \mathbb{R})$ which is the superconformal algebra with $N$ supersymmetries in three dimensions. The superalgebra $osp(N|4)$ can be given a five graded decomposition with respect to the
noncompact dilatation generator $\Delta$ as follows:

$$\text{osp}(N|4) = K^{\alpha\beta} \oplus S^\alpha \oplus (O^I_J \oplus M^\alpha_J \oplus \Delta) \oplus Q^I_\alpha \oplus P_{\alpha\beta}$$ (A.2.1)

We shall call this superconformal 5-grading. The bosonic conformal generators are the same as given in previous section. In order to realize the $R$-symmetry algebra $SO(N)$ and supersymmetry generators, we introduce Euclidean Dirac gamma matrices $\gamma^I$ ($I, J = 1, 2, \ldots, N$) which satisfy

$$\{\gamma^I, \gamma^J\} = \delta^{IJ}$$ (A.2.2)

The $R$-symmetry generators are then simply given as follows:

$$O^{IJ} = \frac{1}{2} (\gamma^I \gamma^J - \gamma^J \gamma^I)$$ (A.2.3)

and supersymmetry generators are the bilinears of 3d twistorial oscillators and $\gamma^I$:

$$Q^I_\alpha = \kappa_\alpha \gamma^I, \quad S^I_\alpha = \gamma^I \mu^\alpha$$ (A.2.4)

They satisfy the following commutation relations:

$$\{Q^I_\alpha, Q^J_\beta\} = \delta^{IJ} P_{\alpha\beta}, \quad \{S^I_\alpha, S^J_\beta\} = \delta^{IJ} K^{\alpha\beta}$$ (A.2.5)

$$\{Q^I_\alpha, S^J_\beta\} = M^\beta_\alpha \delta^{IJ} + 2O^{IJ} \delta^\beta_\alpha + i \left(\Delta - \frac{i}{2}\right) \delta^{IJ} \delta^\beta_\alpha$$ (A.2.6)

The action of conformal group generators on supersymmetry generators is as follows:

$$[M^\alpha_\beta, Q^I_\gamma] = -\delta^\beta_\gamma Q^I_\alpha + \frac{1}{2} Q^I_\gamma, \quad [M^\alpha_\beta, S^I_\gamma] = \delta^\alpha_\gamma S^I_\beta - \frac{1}{2} S^I_\gamma$$ (A.2.7)

$$[\Delta, Q^I_\alpha] = \frac{i}{2} Q^I_\alpha, \quad [\Delta, S^I_\alpha] = -\frac{i}{2} S^I_\alpha$$ (A.2.8)

$$[P_{\alpha\beta}, S^I_\gamma] = -2\delta^\gamma_\beta Q^I_\alpha, \quad [K^{\alpha\beta}, Q^I_\gamma] = 2\delta^\alpha_\gamma S^\beta_I$$ (A.2.9)

$$[P_{\alpha\beta}, Q^I_\gamma] = [K^{\alpha\beta}, S^I_\gamma] = 0$$ (A.2.10)

The $R$-symmetry generators act only on the $I,J$ indices and rotate them as follows:

$$[Q^I_\alpha, O^{JK}] = \frac{1}{2} \delta^{[I,J} Q^K_{\alpha]}, \quad [S^I_\alpha, O^{JK}] = \frac{1}{2} \delta^{[I,J} S^{K]_\alpha}$$ (A.2.11)
For even $N$ the singleton supermultiplet consists of conformal scalars transforming in
a chiral spinor representation of $SO(N)$ and conformal space-time spinors transforming
in the opposite chirality spinor representation of $SO(N)$. There exists another
singleton supermultiplet in which the roles of two spinor representations of $SO(N)$ are
interchanged. For odd $N$ both the conformal scalars and conformal space-time spinors
transform in the same spinor representation of $SO(N)$.

A.3 The quasiconformal realization of the minimal unitary representation of $SO(4, 2)$ in compact three-grading

In this appendix we provide the formulas for the quasiconformal realization of generators
of $SO(4, 2)$ in compact 3-grading following [23]. Consider the compact three graded
decomposition of the Lie algebra of $SU(2, 2)$ determined by the conformal Hamiltonian

$$so(4, 2) = \mathfrak{c}^- \oplus \mathfrak{c}^0 \oplus \mathfrak{c}^+$$

where $\mathfrak{c}^0 = so(4) \oplus so(2)$. Following [23] we shall label the generators in $\mathfrak{c}^\pm$ and $\mathfrak{c}^0$ subspaces as follows:

$$(B_1, B_2, B_3, B_4) \in \mathfrak{c}^- \quad (A.3.1)$$

$$L_{\pm,0} \oplus H \oplus R_{\pm,0} \in \mathfrak{c}^0 \quad (A.3.2)$$

$$(B^1, B^2, B^3, B^4) \in \mathfrak{c}^+ \quad (A.3.3)$$

where $L_{\pm,0}$ and $R_{\pm,0}$ denote the generators of $SU(2)_L \times SU(2)_R$ and $H$ is the $U(1)$ generator.

The generators of $so(4, 2)$ in the compact 3-grading take on very simple forms when
expressed in terms of the singular oscillators introduced in section 3.3.2.

$$H = \frac{1}{2} \left( A_{-\mathcal{L}+1}^1 A_{\mathcal{L}+1} + \mathcal{L} + \frac{1}{2} (N_d + N_g) + \frac{5}{2} \right) \quad (A.3.4)$$

$$L_+ = -\frac{i}{2} A_{\mathcal{L}} d_+^\dagger, \quad L_- = \frac{i}{2} d A_{\mathcal{L}}^\dagger, \quad L_3 = N_d - \frac{1}{2} (H - 1) \quad (A.3.5)$$

$$R_+ = \frac{i}{2} g A_{-\mathcal{L}}^\dagger, \quad R_- = -\frac{i}{2} A_{-\mathcal{L}}^\dagger g, \quad R_3 = N_g - \frac{1}{2} (H + 1) \quad (A.3.6)$$

$$B_1 = -i A_{\mathcal{L}} A_{-\mathcal{L}}, \quad B_2 = -i\sqrt{2} d A_{-\mathcal{L}}, \quad B_3 = -i\sqrt{2} A_{\mathcal{L}} g, \quad B_4 = -2igd \quad (A.3.7)$$
\[ B^1 = iA^+_{-c}A^c_{-\ell}, \quad B^2 = i\sqrt{2}A^+_{-c}d^1, \quad B^3 = i\sqrt{2}g^1A^c_{-\ell}, \quad B^4 = 2ig^1d^1 \] (A.3.8)

### A.4 Commutation relations of deformed twistorial oscillators for \( SO(4,2) \)

\[
\begin{align*}
[Y^1, \bar{Y}^1] &= \frac{1}{2x}(Y^1 - \bar{Y}^1), & [Y^2, \bar{Y}^2] &= -\frac{1}{2x}(Y^2 - \bar{Y}^2) \quad (A.4.1) \\
[Y^1, Y^2] &= \frac{1}{2x}(Y^1 + Y^2), & \bar{Y}^1, \bar{Y}^2 &= \frac{1}{2x}(\bar{Y}^1 + \bar{Y}^2) \quad (A.4.2) \\
[Y^1, \bar{Y}^2] &= -\frac{1}{2x}(Y^1 + \bar{Y}^2), & [\bar{Y}^1, Y^2] &= -\frac{1}{2x}(\bar{Y}^1 + Y^2) \quad (A.4.3)
\end{align*}
\]

\[
\begin{align*}
[Z_1, \bar{Z}_1] &= -\frac{1}{2x}(Z_1 + \bar{Z}_1), & [Z_2, \bar{Z}_2] &= -\frac{1}{2x}(Z_2 + \bar{Z}_2) \quad (A.4.4) \\
[Z_1, Z_2] &= -\frac{1}{2x}(Z_1 - Z_2), & [\bar{Z}_1, \bar{Z}_2] &= \frac{1}{2x}(\bar{Z}_1 - \bar{Z}_2) \quad (A.4.5) \\
[Z_1, \bar{Z}_2] &= -\frac{1}{2x}(Z_1 + \bar{Z}_2), & [\bar{Z}_1, Z_2] &= \frac{1}{2x}(\bar{Z}_1 + Z_2) \quad (A.4.6)
\end{align*}
\]

\[
\begin{align*}
[Y^1, Z_1] &= 2 + \frac{1}{2x}(Y^1 - Z_1), & [\bar{Y}^1, \bar{Z}_1] &= 2 - \frac{1}{2x}(\bar{Y}^1 + \bar{Z}_1) \quad (A.4.7) \\
[Y^1, Z_2] &= \frac{1}{2x}(Y^1 - Z_2), & [\bar{Y}^1, \bar{Z}_2] &= -\frac{1}{2x}(\bar{Y}^1 + \bar{Z}_2) \quad (A.4.8) \\
[Y^1, \bar{Z}_1] &= \frac{1}{2x}(Y^1 + \bar{Z}_1), & [\bar{Y}^1, Z_1] &= -\frac{1}{2x}(\bar{Y}^1 - Z_1) \quad (A.4.9) \\
[Y^1, \bar{Z}_2] &= \frac{1}{2x}(Y^1 + \bar{Z}_2), & [\bar{Y}^1, Z_2] &= -\frac{1}{2x}(\bar{Y}^1 - Z_2) \quad (A.4.10)
\end{align*}
\]

\[
\begin{align*}
[Y^2, Z_1] &= \frac{1}{2x}(Y^2 + Z_1), & [\bar{Y}^2, \bar{Z}_1] &= -\frac{1}{2x}(\bar{Y}^2 - \bar{Z}_1) \quad (A.4.11) \\
[Y^2, Z_2] &= 2 + \frac{1}{2x}(Y^2 + Z_2), & [\bar{Y}^2, \bar{Z}_2] &= 2 - \frac{1}{2x}(\bar{Y}^2 - \bar{Z}_2) \quad (A.4.12) \\
[Y^2, \bar{Z}_1] &= \frac{1}{2x}(Y^2 - \bar{Z}_1), & [\bar{Y}^2, Z_1] &= -\frac{1}{2x}(\bar{Y}^2 + Z_1) \quad (A.4.13) \\
[Y^2, \bar{Z}_2] &= \frac{1}{2x}(Y^2 - \bar{Z}_2), & [\bar{Y}^2, Z_2] &= -\frac{1}{2x}(\bar{Y}^2 + Z_2) \quad (A.4.14)
\end{align*}
\]
Appendix B

B.1 Clifford algebra conventions for doubleton realization

In this appendix, we will give the 6d analogs of $\sigma_\mu$ matrices (in mostly positive metric) used in section 4.2.1 as defined in [39] for the doubleton representations of $SO^*(8)$. Pauli matrices

$$
\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

satisfy

$$
\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu = 2\eta^{\mu\nu}.
$$

Their 6d counterparts are defined as [39]

$$
\Sigma^0 = -i\sigma_2 \otimes \sigma_3, \quad \bar{\Sigma}^0 = -i\sigma_2 \otimes \sigma_3
$$

$$
\Sigma^1 = i\sigma_2 \otimes \sigma_0, \quad \bar{\Sigma}^1 = -i\sigma_2 \otimes \sigma_0
$$

$$
\Sigma^2 = i\sigma_1 \otimes \sigma_2, \quad \bar{\Sigma}^2 = -i\sigma_1 \otimes \sigma_2
$$

$$
\Sigma^3 = i\sigma_3 \otimes \sigma_2, \quad \bar{\Sigma}^3 = -i\sigma_3 \otimes \sigma_2
$$

$$
\Sigma^4 = \sigma_0 \otimes \sigma_2, \quad \bar{\Sigma}^4 = \sigma_0 \otimes \sigma_2
$$

$$
\Sigma^5 = \sigma_2 \otimes \sigma_1, \quad \bar{\Sigma}^5 = \sigma_2 \otimes \sigma_1.
$$

with the convention that the six dimensional $\Sigma^\mu$ have lower spinorial indices while the $\bar{\Sigma}^\mu$ have upper spinorial indices.
B.2 Clifford algebra conventions for deformed twistors

In order to make contact with the spinor helicity literature in 6d, we will use the mostly positive metric and follow the conventions of [154] for 6d analogs of Pauli matrices in the formulation of deformed minreps in terms of deformed twistors. We use a hat over the 6d sigma matrices in order to avoid confusion with the standard Pauli matrices.

\[
\hat{\sigma}^0 = i\sigma_1 \otimes \sigma_2 \quad \hat{\sigma}^0 = -i\sigma_1 \otimes \sigma_2 \quad \text{(B.2.1a)}
\]

\[
\hat{\sigma}^1 = i\sigma_2 \otimes \sigma_3 \quad \hat{\sigma}^1 = i\sigma_2 \otimes \sigma_3 \quad \text{(B.2.1b)}
\]

\[
\hat{\sigma}^2 = -\sigma_2 \otimes \sigma_0 \quad \hat{\sigma}^2 = \sigma_2 \otimes \sigma_0 \quad \text{(B.2.1c)}
\]

\[
\hat{\sigma}^3 = -i\sigma_2 \otimes \sigma_1 \quad \hat{\sigma}^3 = -i\sigma_2 \otimes \sigma_1 \quad \text{(B.2.1d)}
\]

\[
\hat{\sigma}^4 = -\sigma_3 \otimes \sigma_2 \quad \hat{\sigma}^4 = \sigma_3 \otimes \sigma_2 \quad \text{(B.2.1e)}
\]

\[
\hat{\sigma}^5 = i\sigma_0 \otimes \sigma_2 \quad \hat{\sigma}^5 = i\sigma_0 \otimes \sigma_2. \quad \text{(B.2.1f)}
\]

They satisfy :

\[
\hat{\sigma}^\mu \hat{\sigma}^\nu + \hat{\sigma}^\nu \hat{\sigma}^\mu = -2\eta^{\mu\nu}. \quad \text{(B.2.2)}
\]

Again we adopt the convention that the six dimensional \( \hat{\sigma}^\mu \) have lower spinorial indices while the \( \hat{\sigma}^\mu \) have upper spinorial indices. With these conventions, we define:

\[
P_{\alpha\beta} = (\hat{\sigma}^\mu P_\mu)_{\alpha\beta} = \begin{pmatrix}
0 & iP_4 + P_5 & P_1 + iP_2 & P_0 - P_3 \\
-iP_4 - P_5 & 0 & -P_0 - P_3 & -P_1 + iP_2 \\
-P_1 - iP_2 & P_0 + P_3 & 0 & -iP_4 + P_5 \\
-P_0 + P_3 & P_1 - iP_2 & iP_4 - P_5 & 0
\end{pmatrix} \quad \text{(B.2.3)}
\]

\[
K^{\alpha\beta} = (\hat{\sigma}^\mu K_\mu)^{\alpha\beta} = \begin{pmatrix}
0 & -iK_4 + K_5 & K_1 - iK_2 & -K_0 - K_3 \\
iK_4 - K_5 & 0 & K_0 - K_3 & -K_1 - iK_2 \\
-K_1 + iK_2 & -K_0 + K_3 & 0 & iK_4 + K_5 \\
K_0 + K_3 & K_1 + iK_2 & -iK_4 - K_5 & 0
\end{pmatrix} \quad \text{(B.2.4)}
\]

B.3 The quasiconformal realization of the minimal unitary representation of \( SO(6, 2) \) in compact 3-grading

In this appendix we provide the formulas for the quasiconformal realization of generators of \( SO(6, 2) \) and their deformations in compact 3-grading following [24]. We shall give the formulas with the deformation fermions included. The generators for the true minrep
can be obtained simply by setting the deformation fermions to zero.

Consider the compact three graded decomposition of the Lie algebra of $SO^*(8)$ determined by the conformal Hamiltonian $H$

$$so^*(8) = \mathfrak{c}^- \oplus \mathfrak{c}^0 \oplus \mathfrak{c}^+ \quad (B.3.1)$$

where $\mathfrak{c}^0 = su(4) \oplus u(1)$. We shall label the generators in $\mathfrak{c}^\pm$ and $\mathfrak{c}^0$ as follows:

\begin{align*}
(W_m, X_m, N_-, B_-) &\in \mathfrak{c}^- \quad (B.3.2) \\
(D_m, E_m, D^m, E^m, T_{\pm,0}, A_{\pm,0}, J, H) &\in \mathfrak{c}^0 \quad (B.3.3) \\
(W^m, X^m, N_+, B_+) &\in \mathfrak{c}^+ \quad (B.3.4)
\end{align*}

where $m, n, \ldots = 1, 2$. The generators of $su(4)$ algebra in $\mathfrak{c}^0$ subspace has a 3-graded decomposition with respect to its $su(2)_T \oplus su(2)_A \oplus u(1)_J$ subalgebra where the $U(1)_J$ generator $J$ determines the 3-grading of $su(4)$. The generators for $su(2)_T$ were given in equation 4.2.58 (for the true minrep without deformation fermions $su(2)_T$ reduces simply to $su(2)_S$ whose generators were given in equation 4.2.24) and those of $su(2)_A$ are as follows:

$$A_+ = a_1a_2 + b_1b_2, \quad A_- = a_1a^2 + b_1b^2, \quad A_0 = \frac{1}{2} \left( a_1^2 a_1 - a_2^2 a_2 + b_1^2 b_1 - b_2^2 b_2 \right)$$

and they satisfy:

$$[A_+, A_-] = 2A_0, \quad [A_0, A_\pm] = \pm A_\pm \quad (B.3.5)$$

The generators $D_m, E_m, D^m, E^m$ belonging to the coset $SU(4)/SU(2) \times SU(2) \times U(1)$ are realized as bilinears of the oscillators $a_m, b_m$ and the following “singular” oscillators:

$$A_\mathcal{L}_\pm = \frac{1}{\sqrt{2}} \left( x + ip - \frac{\mathcal{L}_\pm}{x} \right), \quad A_\mathcal{K}_\pm = \frac{1}{\sqrt{2}} \left( x + ip - \frac{\mathcal{K}_\pm}{x} \right) \quad (B.3.6)$$

where

$$\mathcal{L}_\pm = 2 \left( T_0 \pm T_- - \frac{3}{4} \right), \quad \mathcal{K}_\pm = -2 \left( T_0 \pm T_+ + \frac{3}{4} \right) \quad (B.3.7)$$

They satisfy the commutation relations:

$$[\mathcal{L}_+, \mathcal{L}_-] = 2(\mathcal{L}_+ - \mathcal{L}_-), \quad [\mathcal{K}_+, \mathcal{K}_-] = 2(\mathcal{K}_+ - \mathcal{K}_-) \quad (B.3.8)$$

$$[\mathcal{L}_\pm, \mathcal{K}_{\pm}] = 2(\mathcal{L}_\pm - \mathcal{K}_{\pm}), \quad [\mathcal{L}_{\pm}, \mathcal{K}_{\mp}] = -2(\mathcal{L}_{\pm} - \mathcal{K}_{\mp}) \quad (B.3.9)$$

In general for two singular oscillators defined in terms of operators $\mathcal{L}_1$ and $\mathcal{L}_2$ that commute
with the singlet coordinate $x$ but not with each other we have we have

$$[A_{L_1}, A_{L_2}] = \frac{1}{2} \left( \frac{L_2 - L_1}{x^2} + \frac{[L_1, L_2]}{x^2} \right) \quad (B.3.10)$$

$$[A_{L_1}^\dagger, A_{L_2}^\dagger] = \frac{1}{2} \left( \frac{L_1^\dagger - L_2^\dagger}{x^2} + \frac{[L_1^\dagger, L_2^\dagger]}{x^2} \right) \quad (B.3.11)$$

$$[A_{L_1}, A_{L_2}^\dagger] = 1 + \frac{1}{2} \left( \frac{L_1 + L_2^\dagger}{x^2} + \frac{[L_1, L_2]}{x^2} \right) \quad (B.3.12)$$

where $A_{L_\pm}^\dagger = \frac{1}{\sqrt{2}} \left( x + ip - \frac{L^\dagger_\pm}{x} \right)$. The commutation relations $[B.3.9]$ lead to the commutation relations for $A_{L_\pm}$ and $A_{K_\pm}$ as follows:

$$[A_{L_+}, A_{L_-}] = \frac{(L_+ - L_-)}{2x^2}, \quad [A_{K_+}, A_{K_-}] = \frac{(K_+ - K_-)}{2x^2} \quad (B.3.13)$$

$$[A_{L_+}, A_{K_-}] = \frac{3(L_+ - K_-)}{2x^2}, \quad [A_{L_-}, A_{K_+}] = -\frac{3(L_- - K_+)}{2x^2} \quad (B.3.14)$$

The generator that determines the compact 3-grading of $SO^*(8)$ is given as follows:

$$H = H_a + H_b + H_\odot \quad (B.3.15)$$

where

$$H_\odot = \frac{1}{4} \left( A_{L_-} A_{L_-}^\dagger + A_{K_+} A_{K_+}^\dagger + L_- + K_+ - 1 \right) \quad (B.3.16)$$

and

$$H_a = \frac{1}{2} (N_a + 2), \quad H_b = \frac{1}{2} (N_b + 2) \quad (B.3.17)$$

are simply the Hamiltonians of standard bosonic oscillators of $a$- and $b$-type ($N_a = a^1a_1 + a^2a_2, N_b = b^1b_1 + b^2b_2$). This $u(1)$ generator is the AdS energy or the conformal Hamiltonian when $SO^*(8) \simeq SO(6,2)$ is taken as the seven dimensional AdS group or the six dimensional conformal group, respectively.

The generator that determines the 3-grading of $su(4)$ is as follows:

$$J = H_a + H_b - H_\odot \quad (B.3.18)$$

The $SU(4)/SU(2)_S \times SU(2)_A \times U(1)_J$ coset generators can be written as

$$D_m = \frac{1}{\sqrt{2}} \left( a_m A_{L_+}^\dagger + b_m A_{L_-}^\dagger \right), \quad D^m = \frac{1}{\sqrt{2}} \left( A_{L_+} a^m + A_{K_+} b^m \right) \quad (B.3.19)$$

$$E_m = \frac{1}{\sqrt{2}} \left( a_m A_{K_+}^\dagger - b_m A_{K_-}^\dagger \right), \quad E^m = \frac{1}{\sqrt{2}} \left( A_{L_-} a^m - A_{K_-} b^m \right) \quad (B.3.20)$$
where $m, n = 1, 2$. They close into the generators of the subgroup $SU(2)_S \times SU(2)_A \times U(1)_J$ given in (1.2.58) and (B.3.5), which do not involve singular oscillators. Then the su(4) algebra can be rewritten in a fully $SU(2)_S \times SU(2)_A$ covariant form

\[
\begin{align*}
[S_{n'}^m, S_{n''}^{m'}] &= \delta_{n'}^{n''} S_{n'}^{m'} - \delta_{n''}^{n'} S_{n''}^{m'} , \\
[A_{n'}^m, A_{n''}^k] &= \delta_{n'}^k A_{n'}^m - \delta_{n''}^m A_{n''}^k \\
[C_{n'n''}^{m'}, C_{n'n''}^{m''}] &= \delta_{m'}^{m''} S_{n'}^{m'} + \delta_{m''}^{m'} A_{n'}^m + \delta_{m'}^{m''} \delta_{n'}^{n''} J \\
[S_{n'}^{m'}, C_{n'n''}^{k'm} &= \delta_{m'}^{k''} C_{n'}^{m'm} - \frac{1}{2} \delta_{m'}^{m''} C_{n'n''}^{k'm} , \\
[A_{n'}^m, C_{n'n''}^{m'k}] &= \delta_{n'}^k C_{n'}^{m'm} - \frac{1}{2} \delta_{n'}^{m''} C_{n'n''}^{m'k} .
\end{align*}
\]

where we have labeled the generators of $su(2)_S$ and $su(2)_A$ as $S_{n'}^{m'}$ $(m', n' = 1, 2)$ and $A_{n'}^m$ respectively:

\[
\begin{align*}
S_1^1 = -S_2^2 = S_0 & \quad S_1^2 = S_+ \\
A_1^1 = -A_2^2 = A_0 & \quad A_1^2 = A_+ \\
& \quad S_1^1 = (S_1^2)^\dagger = S_- \\
& \quad A_1^1 = (A_1^2)^\dagger = A_- \\
\end{align*}
\]

and defined

\[
\begin{align*}
C_{1m} &= D_m + E_m , \quad C_{2m} = D_m - E_m \\
C_{1m}^1 &= D_m + E_m , \quad C_{2m}^1 = D_m - E_m \\
\end{align*}
\]

The generators belonging to $\mathfrak{c}^-$ are given as follows:

\[
W_m = \left( A_{K^+} a_m + A_{\mathcal{L}_-} b_m \right) , \quad X_m = \frac{1}{\sqrt{2}} \left( A_{K^-} a_m - A_{\mathcal{L}_+} b_m \right) \\
N_- = a_1 b_2 - a_2 b_1 , \quad B_- = \frac{1}{4} \left( A_{K^+} A_{\mathcal{K}_-} + A_{\mathcal{L}_-} A_{\mathcal{L}_+} \right) \\
\]

and the generators in $\mathfrak{c}^+$ are given by their hermitian conjugates:

\[
W^m = \left( a_m A_{K^+}^\dagger + b_m A_{\mathcal{L}_-}^\dagger \right) , \quad X^m = \frac{1}{\sqrt{2}} \left( a_m A_{K^-}^\dagger - b_m A_{\mathcal{L}_+}^\dagger \right) \\
N_+ = a_1 b_2 - a_2 b_1 , \quad B_+ = \frac{1}{4} \left( A_{K^+}^\dagger A_{\mathcal{K}_+} + A_{\mathcal{L}_-}^\dagger A_{\mathcal{L}_+}^\dagger \right) \\
\]

\[110\]
The commutators $[\mathcal{C}^-, \mathcal{C}^+]$ close into $\mathcal{C}^0$:

\[
\begin{align*}
[W_m, W^n] &= 2 (\delta^m_n H + A^m_n) + \delta^m_n (T_- + T_+) \\
[X_m, X^n] &= 2 (\delta^m_n H + A^m_n) - \delta^m_n (T_- + T_+) \\
[W_m, X^n] &= \delta^m_n (2T_0 - T_- + T_+) \\
[X_m, W^n] &= \delta^m_n (2T_0 + T_- - T_+) \\
[W_m, N_+] &= \epsilon_{mn} E^n \\
[X_m, N_+] &= \epsilon_{mn} D^n \\
[W_m, B_+] &= D_m \\
[X_m, B_+] &= E_m \\
[N_-, N_+] &= H + J \\
[B_-, B_+] &= H - J
\end{align*}
\]

(B.3.29)

**B.4 Commutation relations of deformed twistorial oscillators for $SO(6, 2)$**

We should note that the deformed twistorial operators transform nonlinearly under the Lorentz group. However their bilinears that enter the generators of the $SO^*(8)$ transform covariantly with respect to the Lorentz group $SU^*(4)$. We give below some of the commutators of deformed twistorial oscillators:

\[
\begin{align*}
[Z_1^1, Z_2^1] &= -\frac{3}{2x} Z_2^2, \\
[Z_1^1, Z_2^1] &= \frac{1}{2x} Z_2^1, \\
[Z_1^1, Z_2^2] &= \frac{1}{2x} (Z_1^1 - Z_2^2), \\
[Z_1^1, Z_3^1] &= -\frac{1}{2x} (Z_1^1 - Z_3^1), \\
[Z_1^1, Z_3^2] &= -\frac{1}{2x} (2Z_1^2 + Z_3^2), \\
[Z_1^1, Z_4^1] &= \frac{1}{2x} Z_4^1, \\
[Z_1^1, Z_4^2] &= \frac{1}{2x} (Z_1^1 - Z_4^2), \\
\end{align*}
\]

\[
\begin{align*}
[Z_1^1, \bar{Z}_1^1] &= \frac{1}{2x} (Z_1^1 - \bar{Z}_1^1), \\
[Z_1^1, \bar{Z}_1^2] &= -\frac{3}{2x} \bar{Z}_1^2, \\
[Z_1^1, \bar{Z}_2^1] &= \frac{1}{2x} \bar{Z}_2^1, \\
[Z_1^1, \bar{Z}_2^2] &= \frac{1}{2x} (Z_1^1 - \bar{Z}_2^2), \\
[Z_1^1, \bar{Z}_3^1] &= -\frac{1}{2x} (\bar{Z}_1^1 - Z_3^1), \\
[Z_1^1, \bar{Z}_3^2] &= -\frac{1}{2x} (2\bar{Z}_1^2 + \bar{Z}_3^2), \\
[Z_1^1, \bar{Z}_4^1] &= \frac{1}{2x} \bar{Z}_4^1, \\
[Z_1^1, \bar{Z}_4^2] &= \frac{1}{2x} (Z_1^1 - \bar{Z}_4^2)
\end{align*}
\]

(B.4.1)

111
\[ Z_{11}, Y_{11} = \frac{1}{2x} Y_{11}, \]
\[ Z_{11}, Y_{12} = -\frac{1}{2x} (Z_{11}^1 + Y_{12}), \]
\[ Z_{11}, Y_{21} = -\frac{1}{2x} (Z_{11}^1 - Y_{21}), \]
\[ Z_{11}, Y_{22} = -\frac{1}{2x} (2Z_{11}^2 + Y_{22}), \]
\[ Z_{11}, Y_{31} = \frac{1}{2x} Y_{31}, \]
\[ Z_{11}, Y_{32} = \frac{1}{2x} (Z_{11}^1 - Y_{32}), \]
\[ Z_{11}, Y_{41} = \frac{1}{2x} (Z_{11}^1 + Y_{41}), \]
\[ Z_{11}, Y_{42} = \frac{1}{2x} (2Z_{11}^2 - Y_{42}), \]
\[ Y_{11}, Y_{12} = -\frac{3}{2x} Y_{11}, \]
\[ Y_{11}, Y_{21} = -\frac{1}{2x} Y_{11}, \]
\[ Y_{11}, Y_{22} = -\frac{1}{x} (Y_{12} + Y_{21}), \]
\[ Y_{11}, Y_{31} = 0, \]
\[ Y_{11}, Y_{32} = \frac{1}{2x} (Y_{11} - 2Y_{31}), \]
\[ Y_{11}, Y_{41} = \frac{1}{2x} Y_{11}, \]
\[ Y_{11}, Y_{42} = \frac{1}{x} (Y_{12} - Y_{41}), \]
\[ Y_{11}, \tilde{Y}_{11} = 0, \]
\[ Y_{11}, \tilde{Y}_{12} = -\frac{3}{2x} \tilde{Y}_{11}, \]
\[ Y_{11}, \tilde{Y}_{21} = -\frac{1}{2x} \tilde{Y}_{11}, \]
\[ Y_{11}, \tilde{Y}_{22} = -\frac{1}{x} (\tilde{Y}_{12} + \tilde{Y}_{21}), \]
\[ Y_{11}, \tilde{Y}_{31} = 0, \]
\[ Y_{11}, \tilde{Y}_{32} = \frac{1}{2x} (\tilde{Y}_{11} - 2\tilde{Y}_{31}), \]
\[ Y_{11}, \tilde{Y}_{41} = \frac{1}{2x} \tilde{Y}_{11}, \]
\[ Y_{11}, \tilde{Y}_{42} = \frac{1}{x} (\tilde{Y}_{12} - \tilde{Y}_{41}), \]
Bibliography


[40] Gunaydin, M. “Singleton and doubleton supermultiplets of space-time supergroups and infinite spin superalgebras,” Invited talk given at Trieste Conf. on Supermembranes and Physics in (2+1)-Dimensions, Trieste, Italy, Jul 17-21, 1989.


Vita

Karan Govil

Born in Gurgaon, India, Karan Govil received the Bachelor and Master of Technology degrees in Aerospace Engineering from the Indian Institute of Technology Madras, Chennai, India, in July of 2009. He enrolled in the Ph.D. program at the Pennsylvania State University in August 2009.