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SPACES OF POLYNOMIALS RELATED TO MULTIPLIER MAPS

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Abstract

Let f be a complex polynomial of degree n. We attach to f a polynomial space $W(f)$ which consists of all complex polynomials $p(x)$ of degree at most $n-2$ such that $f(x)$ divides $f''(x)p(x) - f'(x)p'(x)$. The space $W(f)$ arises for its importance in Yuriy G. Zarkhin's solution towards a question posed by Yu. S. Ilyashenko. In this paper, we establish an equivalent condition on $f(x)$ that guarantees $W(f)$ to be nontrivial. Moreover we investigate the dimension of space $W(f)$ using three independent approaches. The first one uses Hermite interpolation, the second one applies Chinese reminder theorem, the third one invokes combinatorial tools and linear algebra.

Table of Contents

List of Tables

Acknowledgments

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Chapter 1
Definitions, notations, and statements

We write $\mathbb{C}$ for the field of complex numbers and $\mathbb{C}[x]$ for the ring of one variable polynomials with complex coefficients. Unless otherwise stated, all vector spaces we shall consider are over the field of complex numbers. First we give a definition of the following polynomial space.

Definition 1.1. *For every* $f(x) \in \mathbb{C}[x]$ *with* $\deg f = n$ *define*

$$W(f) := \Big\{p(x) \in \mathbb{C}[x] : \deg p \leq n-2 \textit{ and } f(x) \textit{ divides } f''(x)p(x) - f'(x)p'(x)\Big\}$$

The space $W(f)$ arises from Zarkhin's computation of the rank of the following map. Let us consider the n-dimensional complex manifold $P_n \subseteq \mathbb{C}^n$ of all monic complex polynomials of degree $n \geq 2$

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

with coefficients $a = (a_0, \ldots, a_{n-1})$ and without multiple roots. We denote roots (in this case simple roots) of $f(x)$ by

$$\alpha = \{\alpha_1, \ldots, \alpha_n\}$$

Locally with respect to a, we may choose each α_i using Implicit Function Theorem as a smooth uni-valued function in a. Further we will try to differentiate these functions with respect to coordinates, with no computation of the roots. And here is our map

$$M : a = (a_0, \ldots, a_{n-1}) \longmapsto f'(\alpha) = (f'(\alpha_1), \ldots, f'(\alpha_n)) \in \mathbb{C}^n$$

By abusing notation, we may assume that M is defined locally on P_n and write $M(f)$ instead of $M(a_0, \ldots, a_{n-1})$. Let $dM : \mathbb{C}^n \to \mathbb{C}^n$ be the corresponding tangent map (at the point $f(x)$). It is convenient to identify the tangent space $\mathbb{C}^n$ with the space of all polynomials $p(x)$ of degree less than or equal to $n-1$. Namely, to a polynomial $p(x) = \sum_{i=0}^{n-1} c_i x^i$, one assigns the tangent vector $(c_0, \ldots, c_{n-1}) \in \mathbb{C}^n$. For example, the derivative $f'(x)$ corresponds to the tangent vector $(a_1, \ldots, (n-1)a_{n-1}, n) \in \mathbb{C}^n$. To emphasize the role of $W(f)$, we briefly outline Zarkhin's proof ([8] Theorem 1.1) that the rank of the tangent map $dM : \mathbb{C}^n \to \mathbb{C}^n$ is $n-1$ at all points of P_n. In fact, Zarkhin shows that the kernel of dM is $W(f) \oplus \mathbb{C} \cdot f'(x)$.

The first question that naturally arises is who to deal with M? We interpret the ordering of the roots as a choice of an isomorphism of commutative semi-simple $\mathbb{C}$-algebras:

$$\psi : \Lambda = \mathbb{C}[x]/f(x)\mathbb{C}[x] \cong \mathbb{C}^n$$
$$u(x) + f(x) \cdot \mathbb{C}[x] \mapsto u(\alpha) := (u(\alpha_1), \ldots, u(\alpha_n))$$

and carry out all the computations, including the differentiation with respect to a, of functions that take values in the algebra Λ, despite of the fact that this algebra does depend on the coefficients a. Of course while differentiating, we will use Leibniz's rule and that $f(x) = 0$ in Λ. In what follows we will often mean under polynomials their images in Λ (i.e. the collection of their values at the roots of $f(x)$, while we try not refer to the roots explicitly). Notice that the absence of multiple roots means that $f'(x)$ is an invertible element of Λ. Also notice that $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$ is the image under ψ of the independent variable x.

The first thing that we want to compute is the derivatives $d\alpha/da_i$. Since $f(\alpha) = 0$, $df(\alpha)/da_i = 0$. So we have

$$\frac{df(\alpha)}{da_i} = \frac{\partial f}{\partial a_i}(\alpha) + f'(\alpha) \cdot \frac{d\alpha}{da_i}$$

Since $\partial f/\partial a_i = x^i$, we obtain that

$$0 = \alpha^i + f'(\alpha) \cdot \frac{d\alpha}{da_i}$$

which gives us

$$\frac{d\alpha}{da_i} = -\frac{\alpha^i}{f'(\alpha)}$$

It follows that for any polynomial $u(x)$ whose coefficients may depend on a,

$$\frac{du(\alpha)}{da_i} = \frac{\partial u}{\partial a_i}(\alpha) + u'(\alpha) \times \frac{d\alpha}{da_i} = \frac{\partial u}{\partial a_i}(\alpha) - u'(\alpha) \times \frac{\alpha^i}{f'(\alpha)}$$

In particular we are interested in the case when

$$u(x) = f'(x) = nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1$$

So we obtain that

$$\frac{df'(\alpha)}{da_i} = i\alpha^{i-1} - \frac{\alpha^i f''(\alpha)}{f'(\alpha)}$$

Actually, the rank of dM at $f(x)$ is the dimension of the subspace of Λ generated by n elements

$$\frac{df'}{da_0}(\alpha), \frac{df'}{da_1}(\alpha), \ldots, \frac{df'}{da_{n-1}}(\alpha)$$

Suppose that a collection of n complex numbers $c_0, \ldots, c_{n-1}$ satisfies

$$\sum_{i=0}^{n-1} c_i \frac{df'}{da_i}(\alpha) = 0 \in \Lambda$$

If we put $p(x) = \sum_{i=0}^{n-1} c_i x^i$, then one may easily observe that $p'(x) = \sum_{i=1}^{n-1} i c_i x^{i-1}$ and in Λ the following equality holds

$$0 = \sum_{i=0}^{n-1} c_i \frac{df'}{da_i}(\alpha) = p'(\alpha) - \frac{p(\alpha) f''(\alpha)}{f'(\alpha)}$$

Without loss of generality, we may multiply this equality by the invertible elements $f'(\alpha)$ to obtain the equivalent condition:

$$f'(\alpha)p'(\alpha) - p(\alpha)f''(\alpha) = 0 \in \Lambda$$

In other words, the polynomial $f'(x)p'(x) - p(x)f''(x)$ is divisible by $f(x)$. Now it is clear that the rank of dM at $f(x)$ equals the codimension of the space of all polynomials $p(x)$ of degree less than or equal to $n-1$ such that $f'(x)p'(x) - p(x)f''(x)$ is divisible by $f(x)$ in $\mathbb{C}^n$. Obviously this space contains nonzero $f'(x)$, which implies that the rank of dM does not exceed $n-1$. Since the degree of $f'(x)$ is $n-1$, it is easy to observe that the kernel of dM at $f(x)$ coincides with the direct some $\mathbb{C} \cdot f'(x) \oplus W(f)$. It follows readily that the rank of dM at $f(x)$ equals

$$(n-1) - \dim[W(f)]$$

Moreover Zakhin uses polynomial algebra to show that $f(x)$ must be divisible by the square of a quadratic polynomial in order for $W(f)$ to be nontrivial ([8] Theorem 1.5). This computes the rank of dM at $f(x)$ as $n-1$ because we assume that $f(x)$ has no multiple roots in the construction of the map M. ($f(x)$ has no multiple roots implies $f(x)$ cannot be divisible by $q^2(x)$ with $q(x) \in \mathbb{C}[x]$ of $\deg q = 2$.)

Besides the important role $W(f)$ plays in computing the rank of dM, we believe that complete understanding of the space $W(f)$ will be helpful to further prove Elmer Rees's conjecture ([1] §2) that the rank of dM at f is equal to the cardinality of the set of simple roots of $f(x)$ for arbitrary complex polynomials $f(x)$ allowing multiple roots. This paper will present the necessary and sufficient condition of $f(x)$ that tells when the space $W(f)$ is non-trivial. Furthermore, we will obtain a dimension formula for the $\mathbb{C}$-vector space $W(f)$ for various $f(x) \in \mathbb{C}[x]$. To complete these tasks, it is essential to group roots of $f(x)$ by different multiplicities and think about how they are going to affect $\dim[W(f)]$ in each case. So, we need to introduce some notations prior to statement of main results.

Notation 1.2. *Let $f(x) \in \mathbb{C}[x]$ with $\deg f = n$. We adopt the following notations for the rest of this paper:*

1. *$R(f)$ is the set of distinct roots of $f(x)$;*
2. *$R_k(f)$ is the set of distinct roots of $f(x)$ with multiplicity exactly k;*
3. *$\alpha = R_1(f), \quad \beta = R_2(f), \quad \gamma = \bigcup_{k\geq 3} R_k(f)$,*
   $\alpha_i, \beta_j, \gamma_s$ are elements in α, β, γ respectively,
   For $\gamma_i \in \gamma, \quad k_i$ denotes its multiplicity;
4. *$n_1 = \#R_1(f), n_2 = \#R_2(f)$, $n_3 = \sum_{k\geq 3} \#R_k(f)$;*
5. *The **kth-part polynomial** of $f(x)$ is defined as $f_k(x) = \prod_{r\in R_k(f)}(x - r)$; and the α, β, γ-part of $f(x)$ are defined similarly.*

Recall Zarkhin's result ([8] Theorem 1.5) that

$$W(f) \text{ is nonzero } \Longrightarrow q^2(x) \text{ divides } f(x) \text{ for some quadratic polynomial } q(x).$$

To study conditions on non-triviality of $W(f)$, Zarkhin proposed questions regarding the converse statement. In other words, if $f(x)$ is divisible by square of a quadratic polynomial, is $W(f)$ nontrivial? Fortunately, the answer is positive as we shall present in §3.

Theorem 1.3 (Non-triviality)**.** *Let $f(x)$ be a complex polynomial. If there exists a quadratic complex polynomial $q(x)$ such that $q^2(x)$ divides $f(x)$, then $W(f)$ is nonzero.*

Knowing what $f(x)$ can produce nontrivial space $W(f)$ is not interesting enough. To obtain more information about $W(f)$, we want to get the dimension of the $\mathbb{C}$-vector space $W(f)$ for general class of $f(x) \in \mathbb{C}[x]$. Following examples give a basic view of $\dim_{\mathbb{C}}[W(f)]$ when $\deg f = 5$ and 6.

Let $q(x)$ be the quadratic polynomial whose square divides $f(x)$. In following calculations we let $h(x) = f(x)/[q(x)]^2$, and for a given $p(x) \in W(f)$ we write $R(x)$ for $f''(x)p(x) - f'(x)p'(x)$. Notice that the relationship $f(x) \mid R(x)$ is preserved under the affine transformation $x \mapsto ax + b$ for any $a, b \in \mathbb{C}, a \neq 0$. This free control of two parameters allows us to consider $q(x)$ only in the following two cases when one computes $W(f)$

- $q(x) = x^2 - 1$ (i.e. when $q(x)$ has distinct roots);
- $q(x) = x^2$ (i.e. when $q(x)$ has multiple roots).

Example 1.4 ($\deg f = 5$)**.** If $\deg(f) = 5$, then $\deg h = \deg f - 2 \cdot \deg q = 1$. So let $h(x) = x - c$ for some constant $c \in \mathbb{C}$. According to the previous remark, we need to compute $W(f)$ only when $q(x) = x^2 - 1$ or x^2.

<u>Case 1: $q(x) = x^2$</u>

(a) If $c \neq 0$, then $f(x)$ has one simple root and one multiple root with multiplicity 4. (i.e. $n_1 = 1, n_2 = 0, n_3 = 1$ with $k_1 = 4$). In this case we have $p(x) \in W(f)$ if and only if $p(x) = x\left(x - \frac{5c}{6}\right)$. So $\dim[W(f)] = 1$.

(b) If $c = 0$, then $f(x)$ has only one multiple root with multiplicity 5 (i.e. $n_1 = n_2 = 0, n_3 = 1$ with $k_1 = 5$). In this case we have $p(x) \in W(f)$ if and only if $p(x)$ is divisible by x^2. So $\dim[W(f)] = 2$.

<u>Case 2: $q(x) = x^2 - 1$</u>

(a) If $c^2 \neq 1$, $f(x)$ has one simple root, and two double roots (i.e. $n_1 = 1, n_2 = 2, n_3 = 0$). In this case we can show that $p(x) \in W(f)$ if and only if $p(x) = (x^2 - 1)(6cx - 5c^2 - 1)$. So $\dim[W(f)] = 1$.

(b) If $c^2 = 1$, $f(x)$ has no simple root, one double root, one root of multiplicity three (i.e. $n_1 = 0, n_2 = 1, n_3 = 1$). In this case, we compute that $p(x) \in W(f)$ if and only if $p(x) = (x^2 - 1)(x - c)$ which shows that $\dim[W(f)] = 1$.

To summarize computation of dimension of the space $W(f)$ for all possible degree five polynomial $f(x)$, we present the following table:

Table 1.1: $\dim[W(f)]$ for all quintic polynomial $f(x)$

n_1	n_2	n_3	$\dim[W(f)]$	$\deg f - 1 - (n_1 + n_2 + 2n_3)$
1	0	1	1	$5 - 1 - (1 + 0 + 2 \cdot 1) = 1$
0	0	1	2	$5 - 1 - (0 + 0 + 2 \cdot 1) = 2$
1	2	0	1	$5 - 1 - (1 + 2 + 2 \cdot 0) = 1$
0	1	1	1	$5 - 1 - (0 + 1 + 2 \cdot 1) = 1$

Similarly, by considering cases whether $q(x), h(x)$ has simple roots or not, we can calculate $\dim[W(f)]$ for all possible polynomials $f(x)$ of degree 6. Table 1.2 is a short summary for all $\deg f = 6$. Both Table 1.1 and Table 1.2 suggests the coincidence between the positive integer

$\deg f - 1 - (n_1 + n_2 + 2n_3)$ and dimension of the space $W(f)$ when $W(f)$ is nontrivial. Computation of $W(f)$ for polynomials $f(x)$ of higher degree also provides support for Conjecture 1.5 formulated here:

Conjecture 1.5. *If $f(x) \in \mathbb{C}[x]$ is divisible by the square of a quadratic polynomial, then*

$$\dim[W(f)] = \deg f - 1 - (n_1 + n_2 + 2n_3)$$

Our main goal in this paper is to prove Conjecture 1.5 in a general case when $f(x)$ does not have too "many" simple roots. What we mean by not having too "many" simple roots in a precise mathematical language is that the number of simple roots n_1 is bounded above by the number $n_2 + \sum_{i=1}^{n_3}(k_i - 2)$, where n_2 is the number of double roots, n_3 is the number of roots with multiplicity at least three, and k_i's are multiplicities of roots $\gamma_1, \ldots, \gamma_{n_3} \in \gamma$.

Table 1.2: $\dim[W(f)]$ for all polynomial $f(x)$ of degree six

n_1	n_2	n_3	$\dim[W(f)]$	$\deg f - 1 - (n_1 + n_2 + 2n_3)$
0	0	1	3	$6 - 1 - (0 + 0 + 2 \cdot 1) = 3$
0	1	1	2	$6 - 1 - (0 + 1 + 2 \cdot 1) = 2$
1	0	1	2	$6 - 1 - (1 + 0 + 2 \cdot 1) = 2$
2	0	1	1	$6 - 1 - (2 + 0 + 2 \cdot 1) = 1$
0	3	0	2	$6 - 1 - (0 + 3 + 2 \cdot 1) = 2$
1	1	1	1	$6 - 1 - (1 + 1 + 2 \cdot 1) = 1$
2	2	0	1	$6 - 1 - (2 + 2 + 2 \cdot 0) = 1$

Structure of the paper

The paper is organized as follows. In §2, we completely characterize $W(f)$ when $f(x)$ does not have any simple roots (i.e. $R_1(f) = \varnothing$). This description of the space $W(f)$ will be used to prove Theorem 1.3 in §3 together with the aid of an important lemma due to Marcin Mazur.

We will restate Conjecture 1.5 in §4 and give motivation of another abstract model $Z(\eta, \omega; s, k)$ in order to analyze $W(f)$. §5 examines basic properties and examples of space $Z(\eta, \omega; s, k)$ that will be used to partially prove Conjecture 1.5. In §6, we will give three different approaches to show Conjecture 1.5 when $f(x)$ does not have "too many" simple roots. The first method essentially combines Hermite interpolation and evaluation homomorphism. The second one directly applies Chinese reminder theorem. The third one computes dimension of $W(f)$ as the rank of a certain (associated) matrix and then uses two identities in finite hypergeometric series to complete an induction argument. Lastly we proposed a plausible way in §7 to handle the case when $f(x)$ has "lots of" simple roots.

Chapter 2
Study of $W(f)$ for f without simple roots

The goal of this section is to prove Conjecture 1.5 assuming $f(x)$ does not have simple roots. We first set up some notations. For $f(x), g(x)$ complex polynomials, we write

$$R(f,g)(x) = f''(x)g(x) - f'(x)g'(x)$$

Suppose k_s are the multiplicity of γ_s for all $1 \le s \le n_3$. Note $k_s \ge 3$ for every $s = 1, 2, \ldots, n_3$ and from Notation 1.2 (4)

$$n = \deg f = n_1 + 2n_2 + \sum_{s=1}^{n_3} k_s \ge n_1 + 2n_2 + 3n_3 \tag{2.1}$$

Also, recall from Notation 1.2 (5) that the α, β, γ-part polynomial of $f(x)$ are defined as

$$f_\alpha(x) = \prod_{i=1}^{n_1}(x-\alpha_i), f_\beta(x) = \prod_{j=1}^{n_2}(x-\beta_j), f_\gamma(x) = \prod_{s=1}^{n_3}(x-\gamma_s)$$

This is also equivalent to $f_\alpha(x) = f_1(x), f_\beta(x) = f_2(x)$. Moreover,

$$f_\gamma(x) = \prod\nolimits_{k\ge 3} f_k(x) \text{ and } f(x) = f_\alpha(x) f_\beta^2(x) \prod\nolimits_{k\ge 3} \left[f_k(x)\right]^k$$

We are interested in following spaces for their deep connection to $W(f)$.

Definition 2.1. *Given $f(x) \in \mathbb{C}[x]$, we define sets*

$$W(f,\alpha) := \left\{p(x) \in \mathbb{C}[x] \mid \deg p \le (n-2), f_\alpha(x) \textit{ divides } R(f,p)(x)\right\}$$
$$W(f,\beta) := \left\{p(x) \in \mathbb{C}[x] \mid \deg p \le (n-2), f_\beta^2(x) \textit{ divides } R(f,p)(x)\right\}$$
$$W(f,\gamma) := \left\{p(x) \in \mathbb{C}[x] \mid \deg p \le (n-2), \widetilde{f}_\gamma(x) = f(x)/[f_\alpha(x) f_\beta^2(x)] \textit{ divides } R(f,p)(x)\right\}$$

Remark 2.2. $W(f,\alpha), W(f,\beta)$ and $W(f,\gamma)$ are finite dimensional vector spaces.

Assume $f(x), p_1(x), p_2(x)$ are polynomials of complex coefficients with $p_1(x), p_2(x) \in W(f, \beta)$. Let $c \in \mathbb{C}$ be given. From definition of $p_1(x), p_2(x) \in W(f, \beta)$, we have $f_\beta^2(x)$ divides $R(f,\ p_1)(x) = f''(x)p_1(x) - f'(x)p_1'(x)$ and $f_\beta^2(x)$ divides $R(f,\ p_2)(x) = f''(x)p_2(x) - f'(x)p_2'(x)$. In particular, $f_\beta^2(x)$ divides

$$\begin{aligned} R(f, p_1)(x) + cR(f, p_2)(x) &= [f''(x)p_1(x) - f'(x)p_1'(x)] + c[f''(x)p_2(x) - f'(x)p_2'(x)] \\ &= f''(x)\left(p_1(x) + cp_2(x)\right) - f'(x)\left(p_1'(x) + cp_2'(x)\right) \\ &= R(f, p_1 + cp_2)(x) \end{aligned}$$

So $f_\beta^2(x) | R(f, p_1 + cp_2)(x) \implies p_1(x) + cp_2(x) \in W(f, \beta)$. Therefore $W(f, \beta)$ is a vector space. One can also check using the exact same technique that $W(f, \gamma)$ and $W(f, \alpha)$ are vector spaces by using $\widetilde{f}_\gamma(x)$ and $f_\alpha(x)$ respectively instead of $f_\beta^2(x)$ from above argument.

Remark 2.3. $W(f) = W(f, \alpha) \cap W(f, \beta) \cap W(f, \gamma)$. In particular if $R_1(f) = \varnothing$ (i.e. $f_\alpha(x) \equiv 1$) then $W(f, \alpha)$ is the space of all polynomial with degree at most $n - 2$ which means

$$W(f) = W(f, \beta) \cap W(f, \gamma)$$

By weakening conditions on $R(f, p)(x)$, we get larger spaces as $W(f, \beta)$ and $W(f, \gamma)$. The advantage of doing this is because spaces of such type are relatively easier to characterize. Following two propositions are common facts in elementary study of single variable polynomials, we are going to use them quite often in proof of preceding lemmas.

Proposition 2.4. *If $f(x) \in \mathbb{C}[x]$, then $r \in R_k(f)$ if and only if*

$$f(r) = f'(r) = \cdots = f^{(k-1)}(r) = 0, \text{ and } f^{(k)}(r) \neq 0$$

where $f^{(i)}(r)$ is the ith derivative of $f(x)$ evaluated at $x = r, i \in \mathbb{Z}_+$.

Proposition 2.5. *If $f(x) \in \mathbb{C}[x]$, then $r \in \bigcup_{j \geq k} R_j(f)$ (i.e. $(x - r)^k$ divides $f(x)$) if and only if $f(r) = f'(r) = \cdots = f^{(k-1)}(r) = 0$.*

Lemma 2.6 (Double Roots)**.** *Given $f(x) \in \mathbb{C}[x], p(x) \in W(f)$ with $\beta \in R_2(f)$, then $(x - \beta)^2$ divides $R(f, p)(x)$ if and only if $(x - \beta)$ divides $p(x)$.*

Proof. Let $x = \beta$ be a double root of $f(x)$, from Proposition 2.4 $f(\beta) = f'(\beta) = 0$ and $f''(\beta) \neq 0$. Since $R(f, p)(x) = f''(x)p(x) - f'(x)p'(x)$, we have

$$\begin{aligned} \frac{d}{dx}\Big[R(f, p)(x)\Big] &= \Big[f'''(x)p(x) + f''(x)p'(x)\Big] - \Big[f''(x)p'(x) + f'(x)p''(x)\Big] \\ &= f'''(x)p(x) - f'(x)p''(x) \end{aligned}$$

So it follows from above formula of $R(f,p)(x)$ and $R'(f,p)(x)$ that

$$R(f,p)(\beta) = f''(\beta)p(\beta),\ R'(f,p)(\beta) = f'''(\beta)p(\beta)$$

Also, from Proposition 2.5

$$(x-\beta)^2 | R(f,p)(x) \iff R(f,p)(\beta) = R'(f,p)(\beta) = 0$$

Because $f''(\beta) \neq 0$

$$R(f,p)(\beta) = 0 \iff p(\beta) = 0$$

Thus combine with $R(f,p)'(\beta) = f'''(\beta)p(\beta)$ we have

$$R(f,p)(\beta) = R(f,p)'(\beta) = 0 \iff p(\beta) = 0$$

Hence using Proposition 2.5, we have $(x-\beta)^2$ divides $R(f,p)(x)$ if and only if $(x-\beta)$ divides $p(x)$ □

Theorem 2.7. *$p(x) \in W(f,\beta)$ if and only if $f_\beta(x)$ divides $p(x)$*

Proof. From definition, $p(x) \in W(f,\beta) \iff f_\beta^2(x) = \prod_{i=1}^{n_2}(x-\beta_i)^2$ divides $R(f,p)(x)$. Because $\beta_i \neq \beta_j$ for all $1 \leq i \neq j \leq n_2$, we know $f_\beta^2(x) = \prod_{i=1}^{n_2}(x-\beta_i)^2$ divides $R(f,p)(x)$ if and only if $(x-\beta_i)^2$ divides $R(f,p)(x)$ for each $1 \leq i \leq n_2$. From Lemma 2.6, for every $1 \leq i \leq n_2$, $(x-\beta_i)^2$ divides $R(f,p)(x) \iff (x-\beta_i)$ divides $p(x)$. By the fact that $(x-\beta_i)$ and $(x-\beta_j)$ are relatively prime whenever $i \neq j$, we have

$$(x-\beta_1)|p(x), (x-\beta_2)|p(x), \ldots, (x-\beta_{n_2})|p(x) \iff f_\beta(x) = \prod\nolimits_{\beta_i \in \beta}(x-\beta_i)|p(x)$$

Therefore, $p(x) \in W(f,\beta)$ if and only if $f_\beta(x) = \prod_{i=1}^{n_2}(x-\beta_i)$ divides $p(x)$. □

Previous theorem tells us exactly what restrictions we should put on $p(x) \in W(f)$ when we consider only the affect of β on $p(x)$. We shall proceed to see a similar result as we switch the case to γ.

Lemma 2.8 (Higher Order Roots)**.** *Given $f(x) \in \mathbb{C}[x], p(x) \in W(f)$ with $\gamma \in R_k(f)$ $(k \geq 3)$, then $(x-\gamma)^k$ divides $R(f,p)(x)$ if and only if $(x-\gamma)^2$ divides $p(x)$.*

Proof. Assume $\gamma \in R_k(f)$ where $k \geq 3$ and $k \in \mathbb{Z}^+$. It follows from Proposition 2.4 that $f(x) = (x-\gamma)^k \widetilde{f}(x)$ where $\widetilde{f}(\gamma) \neq 0$. So, we have the following expressions for $f'(x)$ and $f''(x)$ using $\widetilde{f}(x), \widetilde{f}'(x), \widetilde{f}''(x)$.

$$f'(x) = k(x-\gamma)^{k-1}\widetilde{f}(x) + (x-\gamma)^k \widetilde{f}'(x)$$

$$f''(x) = k(k-1)(x-\gamma)^{k-2}\widetilde{f}(x) + 2k(x-\gamma)^{k-1}\widetilde{f}'(x) + (x-\gamma)^k\widetilde{f}''(x)$$

We denote

$$Q(x) = R(f,p)(x)/(x-\gamma)^{k-2}$$

and substitute formulas of $f'(x)$ and $f''(x)$ into $R(f,p)(x)$. We get an expression of $Q(x)$ in terms of $\widetilde{f}(x)$

$$\begin{aligned} Q(x) = \Big[k(k-1)\widetilde{f}(x) + 2k(x-\gamma)\widetilde{f}'(x) + (x-\gamma)^2\widetilde{f}''(x)\Big]p(x) \\ - (x-\gamma)p'(x)\Big[k\widetilde{f}(x) + (x-\gamma)\widetilde{f}'(x)\Big] \end{aligned}$$

Next, we rearrange $Q(x)$ by grouping terms without $(x-\gamma)$, $(x-\gamma)$, and $(x-\gamma)^2$

$$Q(x) = k(k-1)\widetilde{f}(x)p(x) + k(x-\gamma)\Big[2\widetilde{f}'(x)p(x) - \widetilde{f}(x)p'(x)\Big] + (x-\gamma)^2R(\widetilde{f},\ p)(x)$$

explicit substitution shows that $Q(\gamma) = k(k-1)\widetilde{f}(\gamma)p(\gamma)$. Both k and $k-1$ are not equal to zero because $k \geq 3$. And we also know $\widetilde{f}(\gamma) \neq 0$ from the beginning. So

$$Q(\gamma) = 0 \iff p(\gamma) = 0$$

In addition

$$\begin{aligned} Q'(x) = k(k-1)\Big[\widetilde{f}'(x)p(x) + \widetilde{f}(x)p'(x)\Big] + k\Big[2\widetilde{f}'(x)p(x) - \widetilde{f}(x)p'(x)\Big] \\ + k(x-\gamma)\Big[2\widetilde{f}''(x)p(x) + \widetilde{f}'(x)p'(x) - \widetilde{f}(x)p''(x)\Big] \\ + 2(x-\gamma)R(\widetilde{f},\ p)(x) + (x-\gamma)^2R'(\widetilde{f},\ p)(x) \end{aligned}$$

Substitute $x = \gamma$ into above formula we get

$$Q'(\gamma) = k(k+1)\widetilde{f}'(\gamma)p(\gamma) + k(k-2)\widetilde{f}(\gamma)p'(\gamma)$$

So if $Q(\gamma) = Q'(\gamma) = 0$, we have $p(\gamma) = 0$ and $Q'(\gamma) = k(k-2)\widetilde{f}(\gamma)p'(\gamma) = 0$. Both k and $k-2$ are nonzero because $k \geq 3$. It follows that $p'(\gamma) = 0$ since $\widetilde{f}(\gamma) \neq 0$. Conversely, $p(\gamma) = p'(\gamma) = 0$ also implies $Q(\gamma) = Q'(\gamma) = 0$. So we have shown the following

$$(x-\gamma)^2|Q(x) \iff (x-\gamma)^2|p(x)$$

From construction of $Q(x)$ and Proposition 2.5, $(x-\gamma)^k$ divides $R(f,p)(x)$ if and only if $(x-\gamma)^2$ divides $Q(x)$. So it follows from above argument that $(x-\gamma)^k$ divides $R(f,p)(x)$ if and only if

$(x-\gamma)^2$ divides $p(x)$. □

Theorem 2.9. *$p(x) \in W(f,\gamma)$ if and only if $f_\gamma^2(x)$ divides $p(x)$.*

Proof. From definition, $p(x) \in W(f_\gamma) \iff \prod_{i=1}^{n_3}(x-\gamma_i)^{k_i}$ divides $R(f,p)(x)$. Because $\gamma_i \neq \gamma_j$ for all $1 \leq i \neq j \leq n_3$, we know $\prod_{i=1}^{n_3}(x-\gamma_i)^{k_i}$ divides $R(f,p)(x)$ if and only if $(x-\gamma_i)^{k_i}$ divides $R(f,p)(x)$ for each $1 \leq i \leq n_3$. From Lemma 2.8, for every $1 \leq i \leq n_3$, $(x-\gamma_i)^{k_i}$ divides $R(f,p)(x) \iff (x-\gamma_i)^2$ divides $p(x)$. By the fact that $(x-\gamma_i)^2$ and $(x-\gamma_j)^2$ are relatively prime whenever $i \neq j$, we have

$$(x-\gamma_1)^2|p(x), (x-\gamma_2)^2|p(x), \ldots, (x-\gamma_{n_3})^2|p(x) \iff f_\gamma^2(x) = \prod_{\gamma_i \in \gamma}(x-\gamma_i)^2|p(x)$$

Hence, $p(x) \in W(f,\gamma)$ if and only if $f_\gamma^2(x) = \prod_{i=1}^{n_3}(x-\gamma_i)^2$ divides $p(x)$. □

Since β and γ intersects trivially, $f_\beta(x)$ and $f_\gamma^2(x)$ are relatively prime. So it is an immediate consequence of Theorem 2.7 and Theorem 2.9 that

$$p(x) \in W(f,\beta) \cap W(f,\gamma) \iff f_\beta(x) f_\gamma^2(x) \text{ divides } p(x)$$

In particular, we can prove Theorem 1.3 and Conjecture 1.5 assuming $n_1 = 0$ because from Remark 2.3

$$W(f) = W(f,\beta) \cap W(f,\gamma) = \{p(x) \in \mathbb{C}[x] \mid \deg p \leq n-2, \text{ and } f_\beta f_\gamma^2 \text{ divides } p\}$$

This shows $\dim(W(f)) = n - 1 - (n_2 + 2n_3)$ which agrees with our dimension formula; and the existence follows simply from $\dim(W(f)) > 0$.

Corollary 2.10. *If $R_1(f) = \varnothing$ then $W(f) = \{p(x) \in \mathbb{C}[x] \mid \deg p \leq n-2, \text{ and } f_\beta f_\gamma^2 \text{ divides } p\}$*

Chapter 3
Non-triviality of the space $W(f)$

In this section we prove Theorem 1.3 for arbitrary $f(x) \in \mathbb{C}[x]$. The proof combines Corollary 2.10 and the following lemma due to Marcin Mazur.

Lemma 3.1 (Marcin Mazur)**.** *Let $f(x) \in \mathbb{C}[x]$, $\deg f = n$, $r \in \mathbb{C}$ be a constant such that $f(r) \neq 0$. Suppose $p(x)$ is a nonzero monic polynomial in $W(f)$. If we set $\widetilde{f}(x) = (x-r)f(x)$ and*

$$\widetilde{p}(x) = (x-r)^2 p(x) - \frac{1}{n+1}\widetilde{f}'(x)$$

then $\widetilde{p}(x)$ is a nonzero element in $W(\widetilde{f})$.

Proof. Let $r \in \mathbb{C}$ be given with $f(r) \neq 0$, $\widetilde{f}(x) = (x-r)f(x)$ implies

$$\widetilde{f}'(x) = f(x) + (x-r)f'(x), \quad \widetilde{f}''(x) = 2f'(x) + (x-r)f''(x) \tag{3.1-1}$$

Without loss of generality, we may assume $p(x)$ is a monic polynomial. Since the leading coefficient of $\widetilde{f}'(x)$ is $n+1$, we take $c = 1/(n+1)$ so that $c\widetilde{f}'(x)$ is a monic polynomial. It follows that the term x^n vanishes in $\widetilde{p}(x) = (x-r)^2 p(x) - c\widetilde{f}'(x)$ hence $\deg \widetilde{p}(x) = n-1 = \deg \widetilde{f} - 2$.

From construction $\widetilde{p}(x) \equiv 0$ if and only if $(n+1)(x-r)^2 p(x) = \widetilde{f}'(x)$. Substitute $\widetilde{f}'(x)$ from (3.1-1), we have $(x-r)^2 p(x) = f(x) + (x-r)f'(x)$ which means

$$f(x) = (n+1)(x-r)^2 p(x) - (x-r)f'(x) = (x-r)\big[(n+1)(x-r)p(x) - f'(x)\big]$$

But above expression would imply $f(r) = 0$ contradicts to our assumption that $f(r) \neq 0$. So, we have shown $\widetilde{p}(x)$ is a nonzero polynomial.

Differentiate $\widetilde{p}(x)$ from definition we have

$$\begin{aligned} \widetilde{p'}(x) &= 2(x-r)p(x) + (x-r)^2 p'(x) - c\widetilde{f''}(x) \\ &= 2(x-r)p(x) + (x-r)^2 p'(x) - c\big[2f'(x) + (x-r)f''(x)\big] \end{aligned} \tag{3.1-2}$$

We use the shorthand notation $\widetilde{R}(x)$ for $\widetilde{R}(\widetilde{f},\ \widetilde{p})(x)$ and substitute (3.1-2) into $\widetilde{R}(x) = \widetilde{f''}(x)\widetilde{p}(x) -$

$\widetilde{f}'(x)\widetilde{p}'(x)$

$$\widetilde{R}(x) = \widetilde{f''}(x)\Big[(x-r)^2p(x) - c\widetilde{f}'(x)\Big] - \widetilde{f}'(x)\Big[2(x-r)p(x) + (x-r)^2p'(x) - c\widetilde{f''}(x)\Big]$$

Cancel $c\widetilde{f''}(x)\widetilde{f}'(x)$ according to above expression of $\widetilde{R}(x)$, we get

$$\widetilde{R}(x) = \widetilde{f''}(x)(x-r)^2p(x) - \widetilde{f}(x)\Big[2(x-r)p(x) + (x-r)^2p'(x)\Big] \tag{3.1-3}$$

Now, substitute expressions of $\widetilde{f''}(x)$ and $\widetilde{f}'(x)$ in (3.1-1) into (3.1-3)

$$\begin{aligned}\widetilde{R}(x) &= (x-r)^3\Big[f''(x)p(x) - f'(x)p'(x)\Big] - (x-r)f(x)\Big[p(x) + (x-r)p'(x)\Big] \\ &= (x-r)^3R(f,p)(x) - \widetilde{f}(x)\Big[p(x) + (x-r)p'(x)\Big]\end{aligned}$$

Because $f(x) \in W(f)$, $f(x)$ divides $R(f,p)(x) = f''(x)p(x) - f'(x)p'(x)$. So

$$\widetilde{f}(x) = (x-r)f(x) \text{ divdies } (x-r)R(f,p)(x) \tag{$*$}$$

It follows from $(*)$ that

$$\widetilde{f}(x) \text{ divides } a(x)(x-r)R(f,p)(x) - b(x)\widetilde{f}(x) \text{ for any } a(x), b(x) \in \mathbb{C}[x]$$

In particular, we can say $\widetilde{f}(x)$ divides $\widetilde{R}(x)$ when one takes

$$a(x) = (x-r)^2 \text{ and } b(x) = p(x) + (x-r)p'(x)$$

In short, our $\widetilde{p}(x)$ is a nontrivial polynomial of degree $\deg \widetilde{f} - 2$ such that $\widetilde{f}(x)$ divides $\widetilde{R}(x) = \widetilde{R}(\widetilde{f},\ \widetilde{p})(x)$ which means $\widetilde{p}(x)$ is a nonzero element in $W(\widetilde{f})$. □

Proof of Theorem 1.3

We are ready to prove $W(f)$ is nonzero when $f(x)$ is divisible by the square of a quadratic polynomial. Let $f(x) \in \mathbb{C}[x]$ with $\deg f = n$. We proceed to prove the result by induction on the number of simple roots. To avoid confusion, we point out that polynomials $f_i(x)$s are different from what we defined in Notation 1.2.

***Base Case*:** Put $f_0(x) = f(x)/f_\alpha(x), p_0(x) = f_\beta(x)f_\gamma^2(x)$. Since $f(x)$ is divisible by square of a quadratic polynomial $q(x)$, we know $p_0(x)$ is non-constant for at least $n_2 \geq 2$ or $n_3 \geq 1$.

Because $R_1(f_0) = \varnothing$, we can apply Corollary 2.10 in this case to say $p_0(x) \in W(f_0)$.

***Induction Step*:** For each $1 \le k \le n_1$, we define $f_k(x) = (x - \alpha_k)f_{k-1}(x)$. By induction hypothesis, there exists $p_{k-1}(x)$ nonzero elements in $W(f_{k-1})$. Same analogy from proof of Lemma 3.1 we can pick $c_k = 1/[\deg(f_{k-1}) + 1]$ constant such that

$$p_k(x) := (x - \alpha_k)^2 p_{k-1}(x) - c_k f_k'(x)$$

has degree $\le \deg p_{k-1} + 1 \le \deg f_{k-1} - 2 + 1 = \deg f_k - 2$. (notice $(\deg f_{k-1}) + 1 = \deg f_k$)

Since $\deg p_k \le \deg f_k - 2$, we could treat $f_k(x)$ as $\widetilde{f}_{k-1}(x)$ so that

$$p_k(x) = (x - \alpha_k)^2 p_{k-1}(x) - c_k \widetilde{f}_{k-1}'(x) = \widetilde{p}_{k-1}(x)$$

It follows from Lemma 3.1 that $\widetilde{p}_{k-1}(x) \in W(\widetilde{f}_{k-1}) \Longrightarrow p_k(x) \in W(f_k)$. Repeat this argument for $k = 1, 2, \ldots$ up to $k = n_1$. We can say there exists nonzero polynomial $p_{n_1}(x) \in W(f_{n_1})$. However

$$\begin{aligned} f_{n_1}(x) &= (x - \alpha_{n_1})f_{n_1-1}(x) = (x - \alpha_{n_1})(x - \alpha_{n_1-1})f_{n_1-2}(x) = \ldots \\ &= f_{k-1}(x)\prod_{i=k}^{n_1}(x - \alpha_i) = \cdots = f_0(x)\prod_{i=1}^{n_1}(x - \alpha_i) = f_0(x)f_\alpha(x) = f(x) \end{aligned}$$

So, $f(x) = f_{n_1}(x) \Rightarrow W(f) = W(f_{n_1})$. It follows that $W(f)$ is nonzero because $W(f)$ contains a nonzero polynomial $p_{n_1}(x)$.

Chapter 4
Reformulation of Conjecture 1.5

We continue to show the dimension formula (Conjecture 1.5) when $f(x)$ does not have "too many" simple roots (i.e. $n_1 \leq n_2 + \sum_{i=1}^{n_3}(k_i - 2)$ where k_i is the multiplicity of root $\gamma_i \in \gamma$). For any $p(x) \in W(f)$, we denote $p_\alpha(x) = p(x)/[f_\beta(x)f_\gamma^2(x)]$ and the rational functions $d(x)$ as follows

$$d(x) = \frac{f''_\alpha(x)}{f'_\alpha(x)} + \sum_{i=1}^{n_2} \frac{3}{x - \beta_i} + \sum_{s=1}^{n_3} \frac{2(k_s - 1)}{x - \gamma_s} \tag{4.1}$$

From §2, we only need to consider how simple roots are going to change $\dim(W(f))$. The next theorem, which completely characterizes $W(f)$, is an essential step to obtain the dimension formula.

Theorem 4.1. *Let $f(x) \in \mathbb{C}[x]$ then $p(x) \in W(f)$ if and only if*

(1) $p_\alpha(x) \in \mathbb{C}[x]$ (i.e. $f_\beta(x)f_\gamma^2(x)$ divides $p(x)$);

(2) $d(x)p_\alpha(x) - p'_\alpha(x)$ vanishes at $x = \alpha_i$ for all $i = 1, 2, \ldots, n_1$.

Part (1) of Theorem 4.1 is a restatement of Corollary 2.10 and Part (2) is a direct consequence of the following lemma.

Lemma 4.2. *Let $f(x) \in \mathbb{C}[x]$ and suppose $f_\beta(x)f_\gamma^2(x)$ divides $p(x)$ then $p(x) \in W(f, \alpha)$ if and only if $d(x)p_\alpha(x) - p'_\alpha(x)$ vanishes at $x = \alpha_j$ for all $j = 1, 2, \ldots, n_1$*

Before we proceed to the proof of Lemma 4.2, we reveal one of its crucial consequence which largely reduces the study of $W(f)$ to lower dimensions.

Definition 4.3. *Given $f(x) \in \mathbb{C}[x]$, we define $r = (\deg f - 2) - (n_2 + 2n_3)$ the* ***reduction degree*** *for the polynomial space $W(f)$.*

Recall from (2.1) in §2, we have

$$r = \Big(n_1 + 2n_2 + \sum_{s=1}^{n_3} k_s\Big) - 2 - (n_2 + 2n_3) = n_1 + (n_2 - 2) + \sum_{s=1}^{n_3}(k_s - 2)$$

It is clear from above expression that $r \geq n_1$ since $f(x)$ is divisible by the square of a quadratic polynomial implies either $n_2 \geq 2$ or $n_3 \geq 1$ together with $k_1 \geq 4$. We write $\widetilde{W}(f,\alpha)$ for the space of all polynomials $p(x)$ satisfying condition (2) in Theorem 4.1 with $\deg[p(x)] \leq r$. It follows immediately from Theorem 4.1 that

$$W(f) = (f_\beta f_\gamma^2) \cdot \widetilde{W}(f,\alpha)$$

In particular since $f_\beta f_\gamma^2$ is always nonzero, we have $\dim[\widetilde{W}(f,\alpha)] = \dim W(f)$. The way we describe space $\widetilde{W}(f,\alpha)$ motivates following definition.

Definition 4.4. *Let $\eta = (\eta_1, \ldots, \eta_s), \omega = (\omega_1, \ldots, \omega_s)$ be points in $\mathbb{C}^s$ and suppose $\omega_i \neq \omega_j$ for all $i \neq j$. We define $Z(\eta, \omega; s, k)$ to be the space of all complex polynomial $p(x)$ such that*

(1) $\deg[p(x)] \leq k$

(2) $p'(\omega_i) = \eta_i p(\omega_i)\ \forall\ i$ [*i.e.* $p'(x) \equiv \eta_i p(x) (\mathrm{mod}(x - \omega_i))$]

First note condition (1) implies $Z(\eta, \omega; s, k)$ is always finite dimensional. Notice also condition (2) in Definition 4.4 precisely mimics (2) in Theorem 4.1. The space $W(f)$ is an instance of space $Z(\eta, \omega; s, k)$ when chosen η, ω appropriately. We can restate Lemma 4.2 in the context of $Z(\eta, \omega; s, k)$ which also shows complete understanding on dimension of the space $Z(\eta, \omega; s, k)$ would suffice to prove Conjecture 1.5.

Theorem 4.5. *Let $\alpha = (\alpha_1, \ldots, \alpha_{n_1}), \delta = (d(\alpha_1), \ldots, d(\alpha_{n_1}))$ be points in $\mathbb{C}^{n_1}$ where d is the rational function introduced in (4.1). The map $\phi : W(f) \to Z(\delta, \alpha; n_1, r)$ defined by*

$$p(x) \longmapsto p(x)/f_\beta(x) f_\gamma^2(x)$$

is an $\mathbb{C}$-vector space isomorphism. In particular, $\dim[W(f)] = \dim[Z(\delta, \alpha; n_1, r)]$.

Proof. Notice $Z(\delta, \alpha; n_1, r) = \widetilde{W}(f, \alpha)$. So the remark we made before Definition 4.4 and Lemma 4.2 together shows $\phi : W(f) \to Z(\delta, \alpha; n_1, r)$ and its inverse ϕ^{-1} sending $\widetilde{p}(x)$ to $f_\beta(x) f_\gamma^2(x) \widetilde{p}(x)$ are both well-defined. We only need to check ϕ is one-to-one and also an homomorphism. To claim ϕ is injective, observe

$$\phi(p) = 0 \iff (p/f_\beta f_\gamma) \equiv 0 \iff p \equiv 0$$

To check $\phi : W(f) \to Z(\delta, \alpha; n_1, r)$ is an homomorphism. Let $a, b \in \mathbb{C}$ be constant and $p(x), q(x) \in W(f)$ be given. Then

$$\phi(ap + bq) = \frac{ap(x) + bq(x)}{f_\beta(x) f_\gamma^2(x)} = a\frac{p(x)}{f_\beta(x) f_\gamma^2(x)} + b\frac{q(x)}{f_\beta(x) f_\gamma^2(x)} = a\phi(p) + b\phi(q)$$

Therefore we conclude ϕ is an isomorphism. In particular $\dim[W(f)] = \dim[Z(\delta, \alpha; n_1, r)]$ for $W(f)$ is a finite dimensional $\mathbb{C}$-vector space. $\square$

Proof of Lemma 4.2.

Put

$$\widetilde{f_\gamma}(x) = \frac{f(x)}{f_\alpha(x) f_\beta^2(x)} = \prod_{i=1}^{n_3} (x - \gamma_i)^{k_i}$$

We know from polynomial algebra that for any $g(x) = \prod_{i=1}^{n} (x - \omega_i)$ a polynomial with complex coefficients,

$$\frac{g'(x)}{g(x)} = \sum_{i=1}^{n} \frac{1}{x - \omega_i}$$

Using this fact, we can rewrite $d(x)$ in (4.1) as follows

$$d(x) = \frac{f''_\alpha(x)}{f'_\alpha(x)} + 3\frac{f'_\beta(x)}{f_\beta(x)} + 2\frac{\widetilde{f_\gamma}'(x)}{\widetilde{f_\gamma}(x)} - 2\frac{f'_\gamma(x)}{f_\gamma(x)}$$

We set $\widetilde{f_\beta} = f_\beta^2, p_\gamma = f_\gamma^2$ and rewrite f, p as $f = f_\alpha \cdot \widetilde{f_\beta} \cdot \widetilde{f_\gamma}, p = p_\alpha \cdot f_\beta \cdot p_\gamma$ It follows that

$$\begin{aligned} p' &= p'_\alpha f_\beta p_\gamma + p_\alpha f'_\beta p_\gamma + p_\alpha f_\beta p'_\gamma \\ f' &= f'_\alpha \widetilde{f_\beta}\widetilde{f_\gamma} + f_\alpha(\widetilde{f_\beta}'\widetilde{f_\gamma} + \widetilde{f_\beta}\widetilde{f_\gamma}') \\ f'' &= f''_\alpha \widetilde{f_\beta}\widetilde{f_\gamma} + 2f'_\alpha(\widetilde{f_\beta}'\widetilde{f_\gamma} + \widetilde{f_\beta}\widetilde{f_\gamma}') + f_\alpha(\widetilde{f_\beta}''\widetilde{f_\gamma} + \widetilde{f_\beta}\widetilde{f_\gamma}'') \end{aligned} \tag{4.2-1}$$

Because f_α vanishes for all $x = \alpha_i$, it is clear that $R(f,\ p) = f''p - f'p'$ vanishes for all $x = \alpha_i$ if and only if $R(f, p)(\mathrm{mod}\, f_\alpha)$ as a polynomial vanishes for every $x = \alpha_i$. So we can disregard terms which are of the form $f_\alpha(x)k(x)$ for some $k(x) \in \mathbb{C}[x]$ in the representation of $R(f, p)$ using (4.2-1).

$$\begin{aligned} F &= R(f,p) - f_\alpha\Big[p(\widetilde{f_\beta}''\widetilde{f_\gamma} + \widetilde{f_\beta}\widetilde{f_\gamma}'') - (\widetilde{f_\beta}'\widetilde{f_\gamma} + \widetilde{f_\beta}\widetilde{f_\gamma}')p'\Big] \\ &= \Big[f'' - f_\alpha(\widetilde{f_\beta}''\widetilde{f_\gamma} + \widetilde{f_\beta}\widetilde{f_\gamma}'')\Big]p - \Big[f' - f_\alpha(\widetilde{f_\beta}'\widetilde{f_\gamma} + \widetilde{f_\beta}\widetilde{f_\gamma}')\Big]p' \\ &= \Big[f''_\alpha\widetilde{f_\beta}\widetilde{f_\gamma} + 2f'_\alpha(\widetilde{f_\beta}'\widetilde{f_\gamma} + \widetilde{f_\beta}\widetilde{f_\gamma}')\Big]p_\alpha f_\beta p_\gamma - f'_\alpha\widetilde{f_\beta}\widetilde{f_\gamma}\Big[p'_\alpha f_\beta p_\gamma + p_\alpha f'_\beta p_\gamma + p_\alpha f_\beta p'_\gamma\Big] \end{aligned}$$

As we claimed at the beginning, F vanishes for all $x = \alpha_i$ if and only if $R(f.\ p)$ vanishes for all $x = \alpha_i$. Next, we simplify expression for F by substituting $\widetilde{f_\beta} = f_\beta^2, \widetilde{f_\beta}' = 2f_\beta f_\beta'$.

$$F = \Big[f_\alpha'' f_\beta^2 \widetilde{f_\gamma} + 2f_\alpha'(2f_\beta f_\beta' \widetilde{f_\gamma} + f_\beta^2 \widetilde{f_\gamma}')\Big] p_\alpha f_\beta p_\gamma - f_\alpha' f_\beta^2 \widetilde{f_\gamma}\Big[p_\alpha' f_\beta p_\gamma + p_\alpha f_\beta' p_\gamma + p_\alpha f_\beta p_\gamma'\Big] \tag{4.2-2}$$

Divide $G(x) = f_\alpha'(x) f_\beta^3(x) p_\gamma(x) \widetilde{f_\gamma}(x)$ on both sides of (4.2-2), and denote $\widetilde{F}(x) = F(x)/G(x)$ we get

$$\begin{aligned}
\widetilde{F} &= \Big[f_\alpha'' f_\beta^2 \widetilde{f_\gamma} + 2f_\alpha'(2f_\beta f_\beta' \widetilde{f_\gamma} + f_\beta^2 \widetilde{f_\gamma}')\Big] \frac{p_\alpha}{f_\alpha' f_\beta^2 \widetilde{f_\gamma}} - \frac{1}{f_\beta p_\gamma}\Big[p_\alpha' f_\beta p_\gamma + p_\alpha f_\beta' p_\gamma + p_\alpha f_\beta p_\gamma'\Big] \\
&= \Big[\frac{f_\alpha'' f_\beta^2 \widetilde{f_\gamma}}{f_\alpha' f_\beta^2 \widetilde{f_\gamma}} + \frac{2}{f_\beta^2 \widetilde{f_\gamma}}\Big(2f_\beta f_\beta' \widetilde{f_\gamma} + f_\beta^2 \widetilde{f_\gamma}'\Big)\Big] p_\alpha - \Big[p_\alpha' + p_\alpha\Big(\frac{f_\beta' p_\gamma}{f_\beta p_\gamma} + \frac{p_\gamma' f_\beta}{f_\beta p_\gamma}\Big)\Big] \\
&= \Big[\frac{f_\alpha''}{f_\alpha'} + 2\Big(2\frac{f_\beta'}{f_\beta} + \frac{\widetilde{f_\gamma}'}{\widetilde{f_\gamma}}\Big)\Big] p_\alpha - \Big[p_\alpha' + p_\alpha\Big(\frac{f_\beta'}{f_\beta} + \frac{p_\gamma'}{p_\gamma}\Big)\Big] = \Big[\frac{f_\alpha''}{f_\alpha'} + 3\frac{f_\beta'}{f_\beta} + 2\frac{\widetilde{f_\gamma}'}{\widetilde{f_\gamma}} - \frac{p_\gamma'}{p_\gamma}\Big] p_\alpha - p_\alpha'
\end{aligned}$$

Since $p_\gamma = f_\gamma^2, p_\gamma' = 2f_\gamma f_\gamma' \Longrightarrow p_\gamma'/p_\gamma = 2f_\gamma'/f_\gamma$. It follows from our definition of $d(x)$ that $\widetilde{F}(x) = d(x)p_\alpha(x) - p_\alpha'(x)$. Note G does not vanishes for all $x = \alpha_i$ since $f_\alpha'(x), f_\beta(x), p_\gamma(x)$, and $\widetilde{f_\gamma}(x)$ all do not have factor $(x - \alpha_i)$ in their irreducible factorization. In conclusion, $R(f,p) \equiv \widetilde{F}(x)(\mathrm{mod}(x - \alpha_i))$ for every $i = 1, 2, \ldots, n_1$. Since $\widetilde{F}(x) = d(x)p_\alpha(x) - p_\alpha'(x)$, we are done.

Example 4.6. We shall also see how Theorem 4.1 applies to particular examples.

- Consider $f(x) = (x^2+1)^2(x^{n+1}-1), m, n \in \mathbb{Z}_+$. In this case

$$f_\alpha(x) = x^{n+1} - 1, f_\beta(x) = x^2 + 1, f_\gamma(x) = \widetilde{f_\gamma}(x) \equiv 1$$

We know from Lemma 4.2 that

$$d(x) = \frac{n}{x} + \frac{6x}{x^2+1} = \frac{(n+6)x^2 + n}{x(x^2+1)}$$

So, $p(x) \in W(f)$ if and only if $x^2 + 1$ divides $p(x)$ and $p_\alpha(x) = p(x)/(x^2+1)$ satisifes

$$[(n+6)\alpha_i^2 + n]p_\alpha(\alpha_i) = \alpha_i(\alpha_i^2 + 1)p_\alpha'(\alpha_i) \text{ for all } i = 1, 2, \ldots, s$$

- Consider $f(x) = x^{m+1}(x^{n+1}-1), m, n \in \mathbb{Z}_+, m \geq 3$. In this case

$$f_\alpha(x) = x^{n+1} - 1, f_\beta(x) \equiv 1, f_\gamma(x) = x, \text{ and } \widetilde{f_\gamma}(x) = x^{m+1}$$

From previous results we know $d(x) = (2m+n)/x$. Furthermore, $p(x) \in W(f)$ if and only if x^2 divides $p(x)$ and $(2m+n)p_\alpha(\alpha_i) = \alpha_i p'_\alpha(\alpha_i)$ for every $i = 1, 2, \ldots, n$ where $p_\alpha(x) = p(x)/x^2$.

Chapter 5
Basic properties of the abstract model $Z(\eta, \omega; s, k)$

Our plan for this section carries as follows. We begin with basic properties of the space $Z(\eta, \omega; s, k)$ such as the chain of natural embeddings, invariance under affine change of coordinate $x \mapsto ax + b$, and a theorem which provide lower bound for dimension of space $Z(\eta, \omega; s, k)$. Then we move on to introduce the associated matrix of space $Z(\eta, \omega; s, k)$, the concept of non-degenerate space and derive several equivalent form of Conjecture 1.5.

Proposition 5.1 (Natural Embedding)**.** *Let η, ω be points in $\mathbb{C}^s$ with $\omega_i \neq \omega_j$ for all $i \neq j$ and assume $s' \leq s, k' \leq k$. If $\eta' = (\eta_1, \dots, \eta_{s'}), \omega' = (\omega_1, \dots, \omega_{s'})$ are points in $\mathbb{C}^{s'}$ then*

1. *We have the following chain of vector space embeddings:*

$$Z(\eta, \omega; s, k') \xhookrightarrow{i_{k'k}} Z(\eta, \omega; s, k) \xhookrightarrow{i_{ss'}} Z(\eta', \omega'; s', k')$$

   where $i_{k'k}, i_{ss'}$ are natural inclusion maps.

2. *For any $k'' \geq k$ we have*

$$\dim[Z(\eta, \omega; s, k'')] \leq \dim[Z(\eta, \omega; s, k)] + (k'' - k)$$

Proof.

Part (1) Observe for $Z(\eta, \omega; s, k)$ if we increase k, we are adding more polynomials in the original space so the natural inclusion $i_{kk'} : Z(\eta, \omega; s, k) \to Z(\eta, \omega; s, k')$ is a vector space embedding whenever $k' \geq k$. On the other hand every polynomial $p(x)$ in the space $Z(\eta', \omega'; s', k)$ can be obtained from a polynomial $\widetilde{p}(x)$ in $Z(\eta, \omega; s, k)$ by dropping certain relations on $\widetilde{p}(x)$. Therefore, the natural inclusion $i_{ss'} : Z(\eta, \omega; s, k) \to Z(\eta', \omega'; s', k)$ is also a vector space embedding.

Part (2) Actually, we can say more on the embedding $Z(\eta, \omega; s, k) \hookrightarrow Z(\eta, \omega; s, k + 1)$. Note when we go from subspace $Z(\eta, \omega; s, k)$ to $Z(\eta, \omega; s, k + 1)$, we at most obtain one more basis (some polynomial of degree $k + 1$). Hence we dimension of $Z(\eta, \omega; s, k + 1)$ compare to the subspace $Z(\eta, \omega; s, k)$ increase at most one. So $\dim[Z(\eta, \omega; s, k+1)] \leq \dim[Z(\eta, \omega; s, k)] + 1$. Repeat this inequality consecutively, we get

$$\dim[Z(\eta, \omega; s, k'')] \leq \dim[Z(\eta, \omega; s, k''-1)] + 1 \leq \cdots \leq \dim[Z(\eta, \omega; s, k)] + (k'' - k)$$

□

We proceed to state another useful result which roughly says the space $Z(\eta, \omega; s, k)$ is invariant under a linear change of coordinates on ω.

Notation 5.2. *Let $a, b \in \mathbb{C}$ be constant numbers and for any $P = (P_1, P_2, \ldots, P_n)$ a point in $\mathbb{C}^n$, we denote $aP + b := (aP_1 + b, aP_2 + b, \ldots, aP_n + b)$.*

Given any $a, b \in \mathbb{C}$ constant number with a nonzero, we write $\eta' = a^{-1}\eta, \omega' = a\omega + b$. Observe for any $p(x) \in Z(\eta, \omega; s, k)$ the polynomial $\widetilde{p}(x) = p(a^{-1}(x - b))$ is an element in $Z(\eta', \omega'; s, k)$ since for any $1 \leq i \leq s, p'(\omega_i) = \eta_i p(\omega_i)$ and $\widetilde{p}'(x) = a^{-1}p'(a^{-1}(x-b))$ implies

$$\begin{aligned} \widetilde{p}'(a\omega_i + b) &= a^{-1}p'\left(a^{-1}[(a\omega_i + b) - b]\right) \\ &= a^{-1}p'(\omega_i) = a^{-1}\eta_i p(\omega_i) = a^{-1}\eta_i \widetilde{p}(a\omega_i + b) \end{aligned}$$

So the map $\phi_{a,b} : Z(\eta, \omega; s, k) \to Z(\eta', \omega'; s, k)$ given by $p(x) \mapsto p((x - b)/a)$ is both one-to-one and onto. Moreover, $\phi_{a,b}$ is an isomorphism because it obviously preserves vector addition and scalar multiplication.

Proposition 5.3 (Invariance under Affine Transform)**.** *For $a, b \in \mathbb{C}$ constants with $a \neq 0$, the map $\phi_{a,b} : Z(\eta, \omega; s, k) \to Z(\eta', \omega'; s, k)$ defined by*

$$\phi_{a,b}(p(x)) = p(a^{-1}(x - b))$$

is an vector space isomorphism where $\eta' = a^{-1}\eta, \omega' = a\omega + b$.

Next theorem gives an lower bound for dimension of the polynomial space $Z(\eta, \omega; s, k)$.

Theorem 5.4 (Lower Bound of Dimension)**.** *If $k \geq s - 1$ then $\dim[Z(\eta, \omega; s, k)] \geq k + 1 - s$.*

Proof. Let $p(x) \in Z(\eta, \omega; s, k)$ be given, since $p(x)$ is a complex polynomial of degree at most k, we can write p in its standard monomial representation as follows

$$p(x) = a_k x^k + \cdots + a_1 x + a_0 = \sum_{i=0}^{k} a_i x^i$$

From Definition 4.4, we know $p(x)$ also have to satisfy

$$p'(\omega_1) = \eta_1 p(\omega_1), \quad p'(\omega_2) = \eta_2 p(\omega_2), \quad \ldots \quad , p'(\omega_s) = \eta_s p(\omega_s) \tag{$*$}$$

The system $(*)$ can be treated as homogeneous linear system with s linear equations in $k+1$ unknowns $x = (a_0, a_1, \ldots, a_k) \in \mathbb{C}^{k+1}$. So we would like to write down the matrix A explicitly from the system $(*)$.

$$A = \begin{pmatrix} \eta_1 & \omega_1\eta_1 - 1 & \ldots & \omega_1^k\eta_1 - k\omega_1^{k-1} \\ \eta_2 & \omega_2\eta_2 - 1 & \ldots & \omega_2^k\eta_2 - k\omega_1^{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \eta_s & \omega_s\eta_s - 1 & \ldots & \omega_s^k\eta_s - k\omega_s^{k-1} \end{pmatrix}$$

Since $s \leq k+1$, the number of columns in A is always greater or equal than the number of rows of A. From basic linear algebra, the number of free variables in A is equal to the dimension of the collection of all $p(x) \in Z(\eta, \omega; s, k)$. So,

$$\dim[Z(\eta, \omega; s, k)] = \#\text{ columns of } A - \operatorname{rank} A = (k+1) - \operatorname{rank} A \tag{5.4-1}$$

It is also a fact in linear algebra that

$$\operatorname{rank} A \leq \min\{\#\text{ columns of } A, \#\text{ rows of } A\} = \min\{k+1, s\} = s$$

Hence $\operatorname{rank} A \leq s$ which implies $\dim[Z(\eta, \omega; s, k)] = k + 1 - \operatorname{rank} A \geq k + 1 - s$. □

Definition 5.5. *We define the* ***associated matrix*** A *attached to the polynomial space* $Z(\eta, \omega; s, k)$ *to be the one obtained in proof of Theorem 5.4.*

The associate matrix A is an useful device to study polynomial space $Z(\eta, \omega; s, k)$ for its connection to dimension of the space $Z(\eta, \omega; s, k)$ as stated in the next corollary

Corollary 5.6. *Let A be the associated matrix of the space $Z(\eta, \omega; s, k)$. If $k \geq s - 1$ then*

$$\dim[Z(\eta, \omega; s, k)] = k + 1 - \operatorname{rank} A$$

Proof. See derivation of (5.4-1) in proof of Theorem 5.4. □

We point out this corollary allow us to form another equivalent way to state Conjecture 1.5.

Conjecture 1.5 holds $\iff$ A the associated matrix of $Z(\delta, \alpha; n_1, r)$ attains full rank

Before proceed to examples, we introduce one more definition.

Definition 5.7. *Assume $k \geq s-1$, we say the space $Z(\eta,\omega;s,k)$ is **non-degenerate** if*

$$\dim[Z(\eta,\omega;s,k)] = k+1-s$$

otherwise it's degenerate. We also say the space $W(f)$ is degenerate if and only if its isomorphic image $Z(\delta,\alpha;n_1,r)$ is degenerate.

In fact, Theorem 5.4 tell us immediately that every degenerate space $Z(\eta,\omega;s,k)$ has dimension strictly greater than $k+1-s$. And it follows from our definition that

$$\text{Conjecture 1.5 holds} \iff W(f) \text{ is non-degenerate}$$

because Defintion 5.7 says $W(f)$ is non-degenerate if and only if

$$\begin{aligned}\dim[Z(\delta,\alpha;n_1,r)] &= r+1-n_1 = [n-2-(n_2+2n_3)]+1-n_1\\ &= n-1-(n_1+n_2+2n_3)\end{aligned}$$

So far, we obtain various equivalent form of Conjecture 1.5, we summarize this as the following.

Remark 5.8. *All statements listed below are equivalent to each other*

- *Conjecture 1.5 holds*
- *The space $W(f) \cong Z(\delta,\alpha;n_1,r)$ is non-degenerate*
- *The space $Z(\delta,\alpha;n_1,r)$ has dimension $r+1-n_1$.*
- *The associated matrix A of $Z(\delta,\alpha;n_1,r)$ has full rank.*

It's time to look at some examples to get an intuition for general patterns.

Example 5.9. Let $\eta = 0$ be the origin of $\mathbb{C}^s$, we check $\dim[Z(0,\omega;s,k)] = k+1-s$.
In this case, let $\widetilde{V}(\omega)$ be the matrix obtained by taking the second to the $(s+1)$th columns in the associated matrix of $Z_s^k(0,\omega)$.

$$\widetilde{V}(\omega) = \begin{pmatrix} -1 & -2\omega_1 & \dots & -s\omega_1^{s-1} \\ -1 & -2\omega_2 & \dots & -s\omega_2^{s-1} \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -2\omega_s & \dots & -s\omega_s^{s-1} \end{pmatrix}$$

It's not hard to check $\widetilde{V}(\omega)$ is obtained from the Vandermonde matrix $V(\omega)$ multiplying the jth column by $-j$ for each $1 \leq j \leq s$. Therefore

$$\det \widetilde{V}(\omega) = s!(-1)^s \det V(\omega) = s!(-1)^s v_n(\omega) = s!(-1)^s \prod_{1\leq i<j\leq s} (\omega_j - \omega_i) \neq 0$$

where $v_n = \prod_{1\leq i<j\leq n}(x_j - x_i)$ is the Vandermonde polynomial. Therefore $\operatorname{rank}(\widetilde{V}(\omega)) = s$ implies $\operatorname{rank} A = s$. So $\dim Z(\eta, \omega; s, k) = k + 1 - \operatorname{rank} A = k + 1 - s$.

Example 5.10. We use brutal force calculation to check if $k \geq 3$,

$$\dim[Z(\eta, \omega; 2, k)] = k + 1 - 2 = k - 1$$

Since $k \geq 3$, the associated matrix A has at least four columns. Our plan is proof by contradiction. Suppose to the contrary then Remark 5.8 says A does not have full rank. Let A_1, A_2 be the first and second row of A respectively. Since A is a $2 \times (k + 1)$ matrix

$$A \text{ does not attain full rank} \iff \operatorname{rank} A < 2 \iff A_1, A_2 \text{ are linearly dependent}$$

So, there exists nonzero constant $c \in \mathbb{C}$ such that $A_1 = cA_2$. It follows from the explicit representation of A produced in Theorem 5.4 that

$$\begin{aligned} A_1 &= (\eta_1, \eta_1\omega_1 - 1, \eta_1\omega_1^2 - 2\omega_1, \eta_1\omega_1^3 - 3\omega_1^2, \dots) \\ &= c(\eta_2, \eta_2\omega_2 - 1, \eta_2\omega_2^2 - 2\omega_2, \eta_2\omega_2^3 - 3\omega_2^2, \dots) = cA_2 \end{aligned}$$

Equate the first entry from above expression, we get $\eta_1 = c\eta_2$. Substitute $\eta_1 = c\eta_2$ into the proceeding three entries we have

$$c\eta_2(\omega_1 - \omega_2) = 1 - c \quad (5.10\text{-}1)$$

$$c\eta_2(\omega_1^2 - \omega_2^2) = 2\omega_1 - 2c\omega_2 \quad (5.10\text{-}2)$$

$$c\eta_2(\omega_1^3 - \omega_2^3) = 3\omega_1^2 - 3c\omega_2^2 \quad (5.10\text{-}3)$$

We continue to show (5.10-1) and (5.10-2) implies

$$c = -1,\ \eta_1 + \eta_2 = 0,\ \text{and } \eta_2(\omega_1 - \omega_2) = -2 \quad (5.10\text{-}4)$$

We begin with the right hand side of (5.10-2):

$$2\omega_1 - 2c\omega_2 = 2\omega_1 - 2c\omega_2 + (2\omega_2 - 2\omega_2) = 2(\omega_1 - \omega_2) + 2\omega_2(1 - c)$$

Substitute $1-c$ obtained from (5.10-1), we get

$$2\omega_1 - 2c\omega_2 = 2(\omega_1-\omega_2) + 2\omega_2 c\eta_2(\omega_1-\omega_2) = (\omega_1-\omega_2)(2+2c\eta_2\omega_2)$$

So (5.10-2) is equivalent to the following

$$c\eta_2(\omega_1^2-\omega_2^2) = c\eta_2(\omega_1-\omega_2)(\omega_1+\omega_2) = (\omega_1-\omega_2)(2+2\omega_2 c\eta_2)$$

Cancel $\omega_1-\omega_2$ on both sides because $\omega_1 \neq \omega_2$

$$c\eta_2(\omega_1+\omega_2) = 2+2c\eta_2\omega_2 \Longrightarrow c\eta_2(\omega_1-\omega_2) = 2$$

From (5.10-1), we know $1-c = c\eta_2(\omega_1-\omega_2)$, so $2 = 1-c \Rightarrow c = -1$. Hence $\eta_1 = c\eta_2 \Longrightarrow \eta_1+\eta_2 = 0$ and (5.10-1) implies $\eta_2(\omega_1-\omega_2) = -2$.

We are ready to get a contradiction. From (5.10-4) $c=-1$, so (5.10-3) is equivalent to

$$-\eta_2(\omega_1-\omega_2)(\omega_1^2+\omega_1\omega_2+\omega_2^2) = 3(\omega_2^2+\omega_2^2)$$

From (5.10-4), we can substitute $\eta_2(\omega_1-\omega_2) = -2$ into above expression. We get

$$2(\omega_1^2+\omega_1\omega_2+\omega_2^2) = 3(\omega_1^2+\omega_2^2)$$

Simplify the equation further by moving everything from left hand side to the right hand side,

$$\omega_1^2+\omega_2^2-2\omega_1\omega_2 = 0 \iff (\omega_1-\omega_2)^2 = 0 \iff \omega_1 = \omega_2 \text{ (contradiction)}$$

Note that this example might serve as base case for certain induction arguments.

Example 5.11. We verify $\dim[Z(\eta,\omega;s,k)] = k+1-s$ when $\eta_i\omega_i = 1/2$ for every $i = 1,2,\dots,s$. By Remark 5.8, we just need to show the associated matrix A of $Z(\eta,\omega;s,k)$ has full rank. First we write down A explicitly under the assumption that $\eta_i\omega_i = 1/2$

$$A = \begin{pmatrix} -1 & \omega_1 & 3\omega_1^2 & \dots & (2k-1)\omega_1^{k-1} \\ -1 & \omega_2 & 3\omega_2^2 & \dots & (2k-1)\omega_2^{k-1} \\ -1 & \omega_3 & 3\omega_3^2 & \dots & (2k-1)\omega_3^{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & \omega_s & 3\omega_s^2 & \dots & (2k-1)\omega_s^{k-1} \end{pmatrix}$$

Take $\widetilde{V}(\omega)$ to be the $s \times s$ matrix obtained from the first s column of A, we have

$$\det[\widetilde{V}(\omega)] = (-1) \cdot 3 \cdot 5 \cdot \cdots \cdot (2s-1) v_n(\omega) \neq 0$$

Therefore,

$$\operatorname{rank}[\widetilde{V}(\omega)] = s \Longrightarrow \operatorname{rank} A = s \Longrightarrow \dim Z(\eta, \omega; s, k) = k + 1 - s$$

In general, similar argument tells us that we could assume $\eta_i \omega_i = c$ for any $c \in \mathbb{C}$ constant and obtain the same result. (The case when $c \in \{1, 2, \ldots, s\}$ is subtle).

We have been checked a lot examples where $Z(\eta, \omega; s, k)$ is non-degenerate. However it is essential to point out that there are degenerate spaces as we shall show in the proceeding example.

Example 5.12 (Degenerate Case)**.** Let $\eta = \omega = (1, -1) \in \mathbb{C}^2$, we show $\dim[Z(\eta, \omega; 2, 2)] = 2$.

In this case, $k = s = 2$ and the associated matrix A of $Z(\eta, \omega; 2, 2)$ has size 2×3

$$A = \begin{pmatrix} \eta_1 & \omega_1\eta_1 - 1 & \omega_1(\omega_1\eta_1 - 2) \\ \eta_2 & \omega_2\eta_2 - 1 & \omega_2(\omega_2\eta_2 - 2) \end{pmatrix}$$

Substitute $\eta_1 = \omega_1 = 1$ and $\eta_2 = \omega_2 = -1$ into this expression we get

$$A = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \Longrightarrow \operatorname{rank} A = 1 < 2$$

Remember we have shown from (5.4-1) that

$$\dim[Z(\eta, \omega, 2, 2)] = 2 + 1 - \operatorname{rank} A = 2$$

Because the associated matrix does not attain full rank, we conclude the space $Z(\eta, \omega, 2, 2)$ must degenerate.

Chapter 6
Proof of Conjecture 1.5 when f does not have "too many" simple roots

Throughout this section, we are always going to assume that $n_1 \leq 2r - 2$ where n_1 is the number of simple roots of the given polynomial $f(x)$ and $r = \deg f - 2 - (n_2 + 2n_3)$ is the reduction degree. It is easy to verify this condition is equivalent to $n_1 \leq n_2 + \sum_{i=1}^{n_3}(k_i - 2)$. We present three independent proofs of Conjecture 1.5 under the assumption $n_1 \leq 2r - 2$. The first one uses machinery of the abstract space $Z(\eta, \omega; s, k)$. The second one attacks the problem using the Chinese Reminder Theorem. The third one develops a lemma on reduction of associated matrix then deduces the result from mathematical induction.

6.1 First Approach: Application of Theory on Space $Z(\eta, \omega; s, k)$

We know from Theorem 4.5 that $W(f)$ is isomorphic to $Z(\delta, \alpha; n_1, r)$. Moreover, we also made the remark that Conjecture 1.5 is equivalent to say

$$\dim[Z(\delta, \alpha; n_1, r)] = r + 1 - n_1 \iff Z(\delta, \alpha; n_1, r) \text{ is non-degenerate}$$

So to bring our question into abstract setting of the polynomial space $Z(\eta, \omega; s, k)$ by the replacement

$$(n_1, r, \delta, \alpha) \longrightarrow (s, k, \eta, \omega)$$

We might ask

$$\boxed{\text{Is } Z(\eta, \omega; s, k) \text{ non-degenerate whenever } k \geq 2s - 1?}$$

The answer is positive. We give a general sketch of the proof before proceeds to the forest of details. Assume $k \geq 2s-1$, then we can always use Hermite interpolation to build an epimorphism ev_s from $Z(\eta, \omega; s, k)$ to $\mathbb{C}^s$ where ev_s is the evaluation map $p(x) \mapsto (p(\omega_1), \ldots, p(\omega_s))$. Then by the first isomorphism theorem, $Z(\eta, \omega; s, k)$ factors into two spaces $\mathbb{C}^s$ and $\mathrm{Ker}\,(\mathrm{ev}_s)$ whose dimension can be easily compute.

Theorem 6.1 (Hermite Interpolation)**.** *Let* $k = 2s - 1$ *and* $y = (y_1, y_2, \ldots, y_s)$ *be a point in* $\mathbb{C}^s$ *then there exits a unique* $h(x) \in Z(\eta, \omega; s, k)$ *such that*

$$h(\omega_i) = y_i \text{ and } h'(\omega_i) = \eta_i y_i \text{ for each } i = 1, 2, \ldots, s \qquad (*)$$

The polynomial constructed in Theorem 6.1 is a special case of Hermite interpolation polynomial, which involves construction of polynomial with prescribed value at each point and its derivative up to certain order. See [7] (§4.1.2 Page 136) for details. As a consequence of Theorem 6.1, we can check whenever $k = 2s - 1$, the map $\mathrm{ev}_s : Z(\eta, \omega; s, k) \to \mathbb{C}^s$ given by $h(x) \mapsto (h(\omega_1), h(\omega_2), \ldots, h(\omega_s))^T$ is a well defined surjective map. In fact we can say more about ev_s as the following lemma shows.

Corollary 6.2. *If* $k = 2s - 1$ *then the map* $\mathrm{ev}_s : Z(\eta, \omega; s, k) \to \mathbb{C}^s$ *given by*

$$\mathrm{ev}_s(h) = (h(\omega_1), h(\omega_2), \ldots, h(\omega_s))^T$$

is a well-defined vector space isomorphism.

Proof. Note ev_s is well-defined since for every $h \equiv g \implies h(\omega_i) = g(\omega_i),\ \forall\ 1 \leq i \leq s$ which implies

$$\mathrm{ev}_s(h) = (h(\omega_1), h(\omega_2), \ldots, h(\omega_s)^T = (g(\omega_1), g(\omega_2), \ldots, g(\omega_s))^T = \mathrm{ev}_s(g)$$

Also, ev_s is bijective from the uniqueness and existence of Hermite interpolation.

To check ev_s is a vector space homomorphism, let $h, g \in Z(\eta, \omega; s, k)$ and $c \in \mathbb{C}$ be a constant. Recall, both vector addition and scalar multiplication are defined to be point wise (i.e. $(h + cg)(x) = h(x) + cg(x)$). So from direct calculation,

$$\begin{aligned}\mathrm{ev}_s(h) + c\,\mathrm{ev}_s(g) &= (h(\omega_1), h(\omega_2), \ldots, h(\omega_s))^T + c(g(\omega_1), g(\omega_2), \ldots, g(\omega_s))^T \\ &= (h(\omega_1) + cg(\omega_1), h(\omega_2) + cg(\omega_2), \ldots, h(\omega_s) + cg(\omega_s))^T \\ &= ((h + cg)(\omega_1), (h + cg)(\omega_2), \ldots, (h + cg)(\omega_s))^T = \mathrm{ev}_s(h + cg)\end{aligned}$$

Since the choice of $h(x), g(x), c$ are arbitrary, we can say ev_s is a homomorphism. Therefore ev_s is an vector space isomorphism from $Z(\eta, \omega; s, k)$ to $\mathbb{C}^s$. □

Theorem 6.3. *If* $k \geq 2s - 1$, *then* $\dim[Z(\eta, \omega; s, k)] = k + 1 - s$.

Proof. Suppose $k \geq 2s-1$, By Proposition 5.1 the usual inclusion map

$$i : Z(\eta, \omega; s, 2s-1) \hookrightarrow Z(\eta, \omega; s, k)$$

is a vector space embedding. Same method in proof of Corollary 6.2 can show the map $\mathrm{ev}_s : Z(\eta, \omega; s, k) \to \mathbb{C}^s$ given by $q(x) \mapsto (q(\alpha_1), q(\alpha_2), \ldots, q(\alpha_s))^T$ is a homomorphism. In addition, ev_s is surjective in our case since $Z(\eta, \omega; s, 2s-1) \cong \mathbb{C}^s$ embeds into $Z(\eta, \omega; s, k)$ as a subspace . By the first isomorphism theorem we learned in basic algebra ([3] §3.3. Theorem 16. Page 97),

$$Z(\eta, \omega; s, k)/\mathrm{Ker}\,(\mathrm{ev}_s) \cong \mathbb{C}^s \tag{6.3-1}$$

It follows from (6.3-1) that $Z(\eta, \omega; s, k) \cong \mathrm{Ker}\,(\mathrm{ev}_s) \oplus \mathbb{C}^s$. So,

$$\dim Z(\eta, \omega; s, k) = \dim[\mathrm{Ker}\,(\mathrm{ev}_s)] + \dim \mathbb{C}^s = \dim[\mathrm{Ker}\,(\mathrm{ev}_s)] + s$$

From definition

$$\mathrm{Ker}\,(\mathrm{ev}_s) = \{q(x) \in Z(\eta, \omega; s, k) \mid q(\omega_i) = 0 \text{ for every } 1 \leq i \leq s, i \in \mathbb{Z}_+\}$$

For every $q(x) \in \mathrm{Ker}\,(\mathrm{ev}_s)$, $q(\omega_i) = 0\ \forall\ i = 1, 2, \ldots, s$ implies

$$q'(\omega_i) = \eta_i q(\omega_i) = \eta_i \cdot 0 = 0$$

for each $1 \leq i \leq s, i \in \mathbb{Z}_+$. By Proposition 2.5, $(x-\omega_i)^2$ divides $q(x)$ for all i. Since $\omega_i \neq \omega_j \implies \gcd((x-\omega_i)^2, (x-\omega_j)^2) = 1$ for all $i \neq j$, it follows that $q(x)$ is divisible by $\prod_{i=1}^{s}(x-\omega_i)^2$. Let $\Omega(x) := \prod_{i=1}^{s}(x-\omega_i)$, above argument shows,

$$\mathrm{Ker}\,(\mathrm{ev}_s) = \{g(x)\Omega^2(x) \mid g(x) \in \mathbb{C}[x], \deg g \leq k-2s\}$$

In particular, $\dim[\mathrm{Ker}\,(\mathrm{ev}_s)] = (k-2s)+1$. Therefore,

$$\dim[Z(\eta, \omega; s, k)] = \dim[\mathrm{Ker}\,(\mathrm{ev}_s)] + s = (k-2s+1) + s = k+1-s$$

□

Corollary 6.4. *If* $r \geq 2n_1 - 1$ *then* $\dim[W(f)] = \deg f - 1 - (n_1 + n_2 + 2n_3)$.

Proof. Apply Theorem 6.3 when $s = n_1, k = r, \eta = \delta$ and $\omega = \alpha$, we get

$$\dim[Z(\delta, \alpha; n_1, r)] = r + 1 - n_1$$

So $W(f) \cong Z(\delta, \alpha; n_1, r)$ must be non-degenerate. Thus Conjecture 1.5 holds when $r \geq 2n_1 - 1$. □

6.2 Second Approach: Chinese Reminder Theorem

Instead of developing machinery of $Z(\eta, \omega; s, k)$, we give another proof of Corollary 6.4. This approach stems from comments by Yuri G. Zarkhin who suggests to tackle the problem directly by the Chinese Reminder Theorem.

To begin with, we denote $R = \mathbb{C}[x]$, $I = \langle p(x) \rangle$ the ideal in R generated by polynomial $p(x)$, and define our auxiliary polynomial

$$A_f(x) := f_\alpha(x) f_\beta(x) f_\gamma^2(x)$$

Also we write $I_r = \langle x - r \rangle$ for each $r \in R(f)$. So we can define a quotient space corresponds to A_f

$$V(f) := \prod_{i=1}^{n_1} (R/I_{\alpha_i}) \prod_{j=1}^{n_2} (R/I_{\beta_j}) \prod_{l=1}^{n_3} (R/I_{\gamma_l}^2)$$

Since ideals $I_{\alpha_i}, I_{\beta_j}, I_{\gamma_l}$ are coprime inside the ring R, we can apply Chinese Reminder Theorem to say that

$$V(f) \cong R/\langle f_\alpha \rangle \times R/\langle f_\beta \rangle \times R/\langle f_\gamma \rangle^2 \cong R/\langle A_f \rangle \text{ as } \mathbb{C}\text{-vector spaces.}$$

It follows that

$$\dim[V(f)] = \deg[A_f(x)] = n_1 + n_2 + 2n_3$$

Next, we consider the map $\widetilde{\pi} : R \to V(f)$ given by

$$\widetilde{\pi}(p(x)) = \begin{cases} (d_i p(x) - p'(x))(\operatorname{mod}(x - \alpha_i)) & \text{if } 1 \leq i \leq n_1 \\ p(x)(\operatorname{mod}(x - \beta_j)) & \text{if } 1 \leq j \leq n_2 \\ p(x)(\operatorname{mod}(x - \gamma_k)^2) & \text{if } 1 \leq k \leq n_3 \end{cases}$$

where for all $i = 1, \dots, n_1$

$$d_i = f''(\alpha_i)/f'(\alpha_i)$$

Note each d_i is well-defined since α_i are simple roots of $f(x)$. Besides the map from R to factors of the form R/I_{β_j} and R/I_{γ_l} are canonical projections modulo $(x - \beta_j), (x - \gamma_l)^2$ respectively.

Next theorem shows $\widetilde{\pi}$ $\mathbb{C}$-vector space epimorphism.

Theorem 6.5. *The map $\widetilde{\pi} : R \to V(f)$ defined above is a $\mathbb{C}$-vector space epimorphism.*

Proof. Given $a_i, b_j, c_k \in \mathbb{C}$ constants where $i, j, k \in \mathbb{Z}_+$ with $1 \le i \le n_1, 1 \le j \le n_2, 1 \le k \le n_3$, we want to find a polynomial $p(x) \in R$ such that

$$\begin{cases} d_i p(x) - p'(x) \equiv a_i(\mathrm{mod}(x-\alpha_i)) & \text{for all } 1 \le i \le n_1 \\ p(x) \equiv b_j(\mathrm{mod}(x-\beta_j)) & \text{for all } 1 \le j \le n_2 \\ p(x) \equiv c_k(\mathrm{mod}(x-\gamma_k)) & \text{for all } 1 \le k \le n_3 \end{cases} \tag{$*$}$$

Since ideals $I_{\alpha_i}, I_{\beta_j}, I_{\gamma_l}$ are coprime in the ring R, from the Chinese Reminder Theorem, we can pick $p(x) \in R$ which simultaneously satisfies the following

$$p(x) \equiv \begin{cases} h_i(x)(\mathrm{mod}(x-\alpha_i)^2) & \text{if } 1 \le i \le n_1 \\ b_j(\mathrm{mod}(x-\beta_j)) & \text{if } 1 \le j \le n_2 \\ c_k(\mathrm{mod}(x-\gamma_k)^2) & \text{if } 1 \le k \le n_3 \end{cases} \tag{6.5-1}$$

where the linear polynomial $h_i(x)$ are defined as

$$h_i(x) = \begin{cases} a_i x + \widetilde{a_i} & \text{if } d_i \ne 0 \\ -a_i x & \text{if } d_i = 0 \end{cases}$$

with constants $\widetilde{a_i} \in \mathbb{C}$ constructed from

$$\widetilde{a_i} = \frac{2a_i}{d_i} - \alpha_i a_i \text{ for all } d_i \ne 0, 1 \le i \le n_1 \tag{6.5-2}$$

To check $(*)$ holds, it suffice to prove

$$d_i p(x) - p'(x) \equiv a_i(\mathrm{mod}(x-\alpha_i)) \text{ for each } 1 \le i \le n_1$$

First we proceed the case where $d_i = 0$, under the assumption $d_i p(x) - p'(x) = -p'(x)$. From (6.5-1), we know $p(x) \equiv (-a_i x)(\mathrm{mod}(x-\alpha_i)^2)$. By definition,

$$p(x) = -a_i x + q_i(x)(x-\alpha_i)^2 \text{ for some } q_i(x) \in \mathbb{C}[x]$$

Differentiate both sides with respect to x, we obtain

$$p'(x) = -a_i + [q_i'(x)(x-\alpha_i) + 2q_i(x)](x-\alpha_i)$$

It follows that $-p'(x) \equiv a_i (\mod (x - \alpha_i))$. Thus ($*$) holds for $1 \le i \le n_1$ when $d_i = 0$. Now suppose $d_i \neq 0$, we define $g_i(x) \in \mathbb{C}[x]$ as follows

$$g_i(x) = d_i x - (1 + d_i \alpha_i)$$

So, we immediately know after the definition that

$$g_i'(x) = d_i \text{ and } g_i(\alpha_i) = -1 \tag{6.5-3}$$

Since $p(x) \equiv (a_i x + \widetilde{a}_i)(\mod (x - \alpha_i))$, we can also say

$$g_i(x)p(x) \equiv g_i(x)(a_i x + \widetilde{a}_i)(\mod (x - \alpha_i)^2)$$

Again from the definition,

$$g_i(x)p(x) = g_i(x)(a_i x + \widetilde{a}_i) + q_i(x)(x - \alpha_i)^2 \tag{6.5-4}$$

for some $q_i(x) \in \mathbb{C}[x]$. Because

$$\widetilde{g}_i(x) = \frac{d}{dx}\Big[g_i(x)(a_i x + \widetilde{a}_i)\Big] = g_i'(x)(a_i x + \widetilde{a}_i) + g_i(x)a_i = 2d_i a_i x + \Big[d_i \widetilde{a}_i - a_i(1 + d_i \alpha_i)\Big]$$

we must have

$$\widetilde{g}_i(\alpha_i) = 2d_i a_i \alpha_i + d_i \widetilde{a}_i - a_i - d_i a_i \alpha_i = 2d_i a_i \alpha_i + d_i \left(\frac{2a_i}{d_i} - a_i \alpha_i\right) - a_i - d_i a_i \alpha_i = a_i$$

Take derivative on both sides of (6.5-4) with respect to x we get

$$g_i'(x)p(x) + g_i(x)p'(x) = \widetilde{g}_i(x) + \Big[2q_i(x) + q_i'(x)(x - \alpha_i)\Big](x - \alpha_i)$$

This shows

$$g_i'(\alpha_i)p(x) + g_i(\alpha_i)p'(x) \equiv \widetilde{g}_i(\alpha_i)(\mod (x - \alpha_i))$$

We know $\widetilde{g}_i(\alpha_i) = a_i$ and $g_i'(x) = d_i, g_i(\alpha_i) = -1$ by (6.5-3). Therefore

$$d_i p(x) - p'(x) \equiv a_i (\mod (x - \alpha_i))$$

Finally, it's trivial to check $\widetilde{\pi}$ is an $\mathbb{C}$-vector space homomorphism. □

The result of this theorem allow us to deduce Conjecture 1.5 when $n_1 \ge 2r - 1$. (i.e. Corollary 6.4).

Proof of Corollary 6.4 by Theorem 6.5 Since we have an epimorphism

$$\widetilde{\pi} : R \longrightarrow V(f) \cong R/\langle A_f \rangle$$

Under the assumption that $n_1 \geq 2r - 1$ we have

$$\deg A_f = n_1 + n_2 + 2n_3 \leq n - 1$$

This induces a $\mathbb{C}$-vector space epimorphism in an obvious way

$$\widetilde{\pi}_* : R/\langle x^{n-1} \rangle \longrightarrow V(f)$$

Notice $p(x) \in \text{Ker } \widetilde{\pi}_*$ if and only if $(x - \beta_j)$ divides $p(x)$, $(x - \gamma_k)^2$ divides $p(x)$ and $(x - \alpha_i)$ divides $f''(x)p(x) - f'(x)p'(x)$ since

$$\begin{aligned} R(f,p)(x) = f''(x)p(x) - f'(x)p'(x) &\equiv \big[f''(\alpha_i)p(x) - f'(\alpha_i)p(x)\big](\text{mod}(x - \alpha_i)) \\ &\equiv f'(\alpha_i)\big[d_i p(x) - p'(x)\big](\text{mod}(x - \alpha_i)) \equiv 0(\text{mod}(x - \alpha_i)) \end{aligned}$$

Theorem 4.1 says Ker $\widetilde{\pi}_* = W(f)$. From the first isomorphism theorem,

$$(R/\langle x^{n-1} \rangle)/(\text{Ker } \widetilde{\pi}_*) = (R/\langle x^{n-1} \rangle)/W(f) \cong V(f) \cong R/\langle A_f \rangle$$

In other words

$$R/\langle x^{n-1} \rangle \cong (R/\langle A_f \rangle) \oplus W(f)$$

Therefore

$$\begin{aligned} \dim[W(f)] &= \dim(R/\langle x^{n-1} \rangle) - \dim(R/\langle A_f \rangle) \\ &= \deg(x^{n-1}) - \deg A_f = n - 1 - (n_1 + n_2 + 2n_3) \end{aligned}$$

In conclusion the space $W(f)$ is non-degenerate when $n_1 \geq 2r - 1$.

6.3 Third Approach: Reduction of Associated Matrix

We are going to use associated matrix to investigate properties of degenerate spaces $Z(\eta, \omega; s, k)$ through out this section. Our first remark is that when $k \geq s$, $Z(\eta, \omega; s, k)$ is degenerate if and only if the row space of the associated matrix A is linearly dependent. This is not necessarily true when $k \leq s-1$. Let A_i denote the i-th row of A. So if $k \geq s$ and space $Z(\eta, \omega; s, k)$ is degenerate we know there exists some positive integer $1 \leq i \leq s$ such that A_i can be written as the linear combination of the other rows. For the sake of simplicity, we always assume this i to be the last row unless otherwise stated. The main result of this section is the following.

Lemma 6.6 (Reduction of Associated Matrix)**.** *Assume $k \geq s+1$, let A be the associated matrix of $Z(\eta, \omega; s+1, k)$, and suppose $Z(\eta, \omega; s+1, k)$ degenerates. Then the homogenous linear system $A^T x = 0$ has a nontrivial solution for which we shall denote by $c = (c_1, \dots, c_s) \in \mathbb{C}^s$. Moreover, if $\widetilde{A}$ is the associated matrix of $Z(\widetilde{\eta}, \widetilde{\omega}; s, k-2)$ where $\widetilde{\omega} = (\omega_1, \dots, \omega_s)$, $\widetilde{\eta} = (\widetilde{\eta}_1, \dots, \widetilde{\eta}_s)$ with $\widetilde{\eta}_i$ defined by*

$$\widetilde{\eta}_i = \eta_i - \frac{2}{\omega_i - \omega_{s+1}} \text{ for all } i = 1, \dots, s$$

then the system $\widetilde{A}^T x = 0$ also has a nontrivial solution $\widetilde{c} = (\widetilde{c}_1, \dots, \widetilde{c}_s)$ where $\widetilde{c}_i = (\omega_i - \omega_{s+1})^2 c_i$.

Proof of Lemma 6.6 is rather brutal force. We need the following fact from finite hypergeometric series.

Proposition 6.7. *Let $a, b \in \mathbb{C}$ and $k \in \mathbb{Z}_+$ then*

1. $-(k+1)b^{k+1} + \sum_{l=0}^{k} a^{k+1-l} b^l = (a-b) \sum_{l=0}^{k} \left[(l+1) a^{k-l} b^l\right]$;
2. $k a^{k+1} + k b^{k+1} - 2 \sum_{l=1}^{k} a^{k+1-l} b^l = (a-b)^2 \sum_{l=0}^{k-1} \left[(l+1)(k-l) a^{k-1-l} b^l\right]$.

Example. Both identities in Proposition 6.7 are instances of hypergeometric series. We list obvious examples for these identities when $k = 1, 2, 3$. To check (1) when $k = 1$ and 2

$$\begin{aligned}
-2b^2 + (a^2 + ab) &= (a^2 - b^2) + (ab - b^2) = (a-b)[(a+b) + b] = (a-b)(a+2b) \\
-3b^3 + (a^3 + a^2 b + ab^2) &= (a^3 - b^3) + (a^2 b - b^3) + (ab^2 - b^3) \\
&= (a-b)[(a^2 + ab + b^2) + b(a+b) + b^2] = (a-b)(a^2 + 2ab + 3b^2)
\end{aligned}$$

To check (2) for $k = 2$ and 3, one observes

$$\begin{aligned}
&2a^3 + 2b^3 - 2(a^2 b + ab^2) = 2(a^3 - a^2 b) + 2(b^3 - ab^2) = 2a^2(a-b) - 2b^2(a-b) = (a-b)^2[2a + 2b] \\
&3a^4 + 3b^4 - 2(a^3 b + a^2 b^2 + ab^3) = 3a^4 - 2a^3 b - 2a^2 b^2 - 2ab^3 + 3b^4 \\
&\qquad = 3(a^4 - a^3 b) + (a^3 b - a^2 b^2) - (a^2 b^2 - ab^3) - 3(ab^3 - b^4) = (a-b)[3a^3 + a^2 b - ab^2 - 3b^3] \\
&\qquad = (a-b)[3(a^3 - ab^2) + 4(a^2 b - ab^2) + 3(ab^2 - b^3)] = (a-b)^2(3a^2 + 4ab + 3b^2)
\end{aligned}$$

Proof of Lemma 6.7. Let $n \in \mathbb{Z}_+$, consider the polynomial $f(x,y) = x^{n+1} - y^{n+1} \in \mathbb{C}[x,y]$. As a smooth function,

$$\frac{\partial f}{\partial y} = -(n+1)y^n \tag{6.7-1}$$

On the other hand, we can factor the linear form $x - y$ from $f(x,y)$

$$f(x,y) = (x-y)(x^n + x^{n-1}y + \cdots + y^n) = (x-y)\sum_{i=0}^{n} x^{n-i}y^i \tag{6.7-2}$$

Taking the partial derivative of f with respect to y on both sides of (6.7-2) yields

$$\frac{\partial f}{\partial y} = -\sum_{i=0}^{n} x^{n-i}y^i + (x-y)\sum_{i=1}^{n} ix^{n-i}y^{i-1} \tag{6.7-3}$$

We combine (6.7-1) and (6.7-3) together to get

$$-(n+1)y^n + \sum_{i=0}^{n} x^{n-i}y^i = (x-y)\sum_{i=1}^{n} ix^{n-i}y^{i-1} \tag{6.7-4}$$

The left hand side of (6.7-4) can be simplified as

$$\begin{aligned} -(n+1)y^n + (x^n + x^{n-1}y + \cdots + y^n) &= -(n+1)y^n + y^n + (x^n + x^{n-1}y + \cdots + xy^{n-1}) \\ &= -ny^n + (x^n + x^{n-1}y + \cdots + xy^{n-1}) \\ &= -ny^n + \sum_{i=0}^{n-1} x^{n-i}y^i \end{aligned}$$

Also, by the change of index $i \to i+1$, the right hand side of (6.7-4) is $(x-y)\sum_{i=0}^{n-1}(i+1)x^{n-1-i}y^i$. So equation (6.7-4) is equivalent to

$$-ny^n + \sum_{i=0}^{n-1} x^{n-i}y^i = (x-y)\sum_{i=0}^{n-1}(i+1)x^{n-1-i}y^i \tag{6.7-5}$$

To obtain (1) from (6.7-5), we just consider the substitution

$$(x,y,n) \to (a,b,k+1)$$

Similarly, from identity (6.7-2), it suffices to show

$$n(x^{n+1} + y^{n+1}) - 2\left[x \cdot \frac{f(x,y)}{x-y} - x^{n+1}\right] = (x-y)^2 \frac{\partial^2}{\partial x \partial y}\left[y \cdot \frac{f(x,y)}{x-y}\right] \tag{6.7-6}$$

for (2) just follows from the substitution $(x,y,n) \to (a,b,k)$ into identity (6.7-6). We start with the left hand side of (6.7-6)

$$\text{LHS of (6.7-6)} = n(x^{n+1}+y^{n+1}) - 2x\left[\frac{x^{n+1}-y^{n+1}}{x-y} - x^n\right] = n(x^{n+1}+y^{n+1}) - 2xy\cdot\frac{x^n-y^n}{x-y}$$

To simplify the right hand side of (6.7-6) observe

$$\frac{\partial}{\partial x}\left[y\cdot\frac{x^{n+1}-y^{n+1}}{x-y}\right] = y\cdot\frac{(n+1)x^n(x-y)-(x^{n+1}-y^{n+1})}{(x-y)^2} = y\cdot\frac{nx^{n+1}-(n+1)x^ny+y^{n+1}}{(x-y)^2}$$

It follows that

$$\begin{aligned}(x-y)^2\frac{\partial^2}{\partial x\partial y}\left[y\cdot\frac{f(x,y)}{x-y}\right] &= (x-y)^2\frac{\partial}{\partial y}\left[y\cdot\frac{nx^{n+1}-(n+1)x^ny+y^{n+1}}{(x-y)^2}\right]\\ &= \left[nx^{n+1}-(n+1)x^ny+y^{n+1}\right] + y(x-y)^2\frac{\partial}{\partial y}\left[\frac{nx^{n+1}-(n+1)x^ny+y^{n+1}}{(x-y)^2}\right]\end{aligned}$$

The second term on the right hand side of above equations is

$$\begin{aligned}&y(x-y)^2\frac{\partial}{\partial y}\left[\frac{nx^{n+1}-(n+1)x^ny+y^{n+1}}{(x-y)^2}\right]\\ &= y\cdot\frac{[-(n+1)x^n+(n+1)y^n](x-y)^2-[nx^{n+1}-(n+1)x^ny+y^{n+1}]\cdot 2(y-x)}{(x-y)^2}\\ &= [-(n+1)x^ny+(n+1)y^{n+1}] + \frac{2y\cdot[nx^{n+1}-(n+1)x^ny+y^{n+1}]}{x-y}\end{aligned}$$

So

$$\begin{aligned}\text{RHS of (6.7-6)} &= (x-y)^2\frac{\partial^2}{\partial x\partial y}\left[y\cdot\frac{f(x,y)}{x-y}\right]\\ &= \left[nx^{n+1}-2(n+1)x^ny+(n+2)y^{n+1}\right] + \frac{2y\cdot[nx^{n+1}-(n+1)x^ny+y^{n+1}]}{x-y}\\ &= n(x^{n+1}+y^{n+1}) - 2y[(n+1)x^n-y^n] + \frac{2y\cdot[nx^{n+1}-(n+1)x^ny+y^{n+1}]}{x-y}\\ &= n(x^{n+1}+y^{n+1}) - 2y\cdot\frac{[(n+1)x^n-y^n](x-y)-[nx^{n+1}-(n+1)x^ny+y^{n+1}]}{x-y}\end{aligned}$$

If one compares RHS and LHS of (6.7-6), notice it is enough to show

$$[(n+1)x^n-y^n](x-y)-[nx^{n+1}-(n+1)x^ny+y^{n+1}] = x(x^n-y^n) \qquad (6.7\text{-}7)$$

Indeed

$$\text{LHS of (6.7-7)} = [(n+1)x^{n+1} - (n+1)x^n y - xy^n + y^{n+1}] - [nx^{n+1} - (n+1)x^n y + y^{n+1}]$$
$$= x^{n+1} - xy^n = x(x^n - y^n) = \text{RHS of (6.7-7)}$$

This finishes (2). □

Before we proceed to the proof, let us examine important consequences of Lemma 6.6.

Theorem 6.8. *Given $\eta, \omega \in \mathbb{C}^s$ with $s \geq 2$. If the space $Z(\eta, \omega; s, 2s-2)$ is degenerate then*

$$\eta_i = \sum_{j \neq i}^{s} \frac{2}{\omega_i - \omega_j} \text{ for all } i = 1, 2, \ldots, s$$

Proof. We will prove the result by induction on the number of ω_i. For the base case $s = 2$ you can check Example 5.10. Suppose now that $Z(\eta, \omega; s+1, 2s)$ is degenerate, then from Lemma 6.6 the space $Z(\widetilde{\eta}, \widetilde{\omega}; s, 2s-2)$ also degenerates with

$$\widetilde{\eta}_i = \eta_i - \frac{2}{\omega_i - \omega_{s+1}} \text{ and } \widetilde{\omega}_i = \omega_i$$

for all $i = 1, 2, \ldots, s$. Applying induction hypothesis on the degenerate space $Z(\widetilde{\eta}, \widetilde{\omega}; s, 2s-2)$, we can say for each $i = 1, 2, \ldots, s$

$$\widetilde{\eta}_i = \sum_{j \neq i}^{s} \frac{2}{\omega_i - \omega_j} \Longrightarrow \eta_i = \sum_{j \neq i}^{s} \frac{1}{\omega_i - \omega_j} + \frac{2}{\omega_i - \omega_{s+1}} = \sum_{j \neq i}^{s+1} \frac{2}{\omega_i - \omega_j}$$

This result is deduced from the fact that A_{s+1} is a linear combination of other rows $\sum_{i=1}^{s} c_i A_i$. We can assume without loss of generality that the row A_{s+1} is not identically zero. Then it follows that there exists $c_i \neq 0$. For the sake of simplicity, assume that $c_1 \neq 0$. The exact same argument as above can be applied to show

$$\eta_i = \sum_{j \neq i}^{s+1} \frac{2}{\omega_i - \omega_j} \text{ for all } i = 2, 3, \ldots, s+1$$

This finishes our proof that $\eta_i = g''(\omega_i)/g(\omega_i)$ for all $1 \leq i \leq s+1$. So from induction the proof is complete. □

It follows from this theorem immediately that $W(f)$ is non-degenerate whenever $r \geq 2n_1 - 2$.

Corollary 6.9. *The space $W(f)$ is non-degenerate whenever $r \geq 2n_1 - 2$.*

Proof. Suppose $r = 2n_1 - 2$, remember we have

$$r = n - 2 - (n_2 + 2n_3) \text{ and } n \geq n_1 + 2n_2 + 3n_3$$

We claim first that above relations plus $r < 2n_1 - 1$ imply

$$n_2 + n_3 \leq n_1 \tag{6.9-1}$$

To begin with, we substitute $r = n - 2 - (n_2 + 2n_3)$ into $r = 2n_1 - 2$

$$n - 2 - (n_2 + 2n_3) = 2n_1 - 2 \iff n - (n_2 + 2n_3) = 2n_1$$

Since $n \geq n_1 + 2n_2 + 3n_3$,

$$n_1 + n_2 + n_3 = (n_1 + 2n_2 + 3n_3) - (n_2 + 2n_3) \leq n - (n_2 + 2n_3) \leq 2n_1$$

Cancel n_1 on both sides of above equality, we get (6.9-1).

Next, recall the rational function $d(x)$ defined at the beginning of Section 4. We denote

$$\widetilde{d}(x) := d(x) - \frac{f''_\alpha(x)}{f'_\alpha(x)} = \sum_{i=1}^{n_2} \frac{3}{x - \beta_i} + \sum_{j=1}^{n_3} \frac{2(k_j - 1)}{x - \gamma_j} \tag{6.9-2}$$

Because $\widetilde{d}(x)$ is a rational function, the numerator of $\widetilde{d}(x)$ (in lowest terms), for which we shall denote by $h(x)$, is a complex polynomial with degree at most $n_2 + n_3 - 1$.

Since we only consider nonzero space $W(f)$ (i.e.$n_2 \geq 2$ or $n_3 \geq 1$), $\widetilde{d}(x)$ is not identically zero. So is the polynomial $h(x)$. Then we deduce from

$$\deg[h(x)] \leq n_2 + n_3 - 1 \leq n_1 - 1$$

and the fundamental theorem of algebra that $h(x)$ cannot vanish at more than $n_1 - 1$ points. Now suppose to the contrary that $W(f) \cong Z(\delta, \alpha; n_1, r)$ is degenerate when $r = 2n_1 - 2$. Then it follows from the previous theorem that for every $i = 1, 2, \ldots, n_1$.

$$d(\alpha_i) = \delta_i = \sum_{j \neq i}^{n_1} \frac{2}{\alpha_i - \alpha_j} = \frac{f''_\alpha(\alpha_i)}{f'_\alpha(\alpha_i)} \iff \widetilde{d}(\alpha_i) = 0$$

The fact $\widetilde{d}(\alpha_i)$ vanishes for all $i = 1, \ldots, n_1$ implies polynomial $h(x)$ vanishes for n_1 distinct points $\alpha_1, \ldots, \alpha_{n_1}$. But this is a contradiction. So far we have shown the space $Z(\delta, \alpha; n_1, 2n_1 -$

2) is non-degenerate which is equivalent to say

$$\dim[Z(\delta, \alpha; n_1, 2n_1 - 2)] = (2n_1 - 2) + 1 - n_1 = n_1 - 1$$

Now let $r \geq 2n_1 - 2$, we know from the natural embedding proposition that

$$\begin{aligned}\dim[W(f)] = \dim[Z(\delta, \alpha; n_1, r)] &\leq \dim[Z(\delta, \alpha; n_1, 2n_1 - 2)] + r - (2n_1 - 2) \\ &= (n_1 - 1) + r - (2n_1 - 2) = r + 1 - n_1\end{aligned}$$

We have shown that $r + 1 - n_1$ is an lower bound of $\dim[W(f)]$ by computing the rank of the associated matrix. It follows that

$$\dim[W(f)] = r + 1 - n_1 = n - 1 - (n_1 + n_2 + 2n_3)$$

Therefore the space $W(f)$ is non-degenerate (i.e. Conjecture 1.5 holds) when $r \geq 2n_2 - 2$. □

Proof of Lemma 6.6

Notations such as $\widetilde{\eta}, \widetilde{\omega}$ are same as we stated in Lemma 6.6. Notice it suffices to prove Lemma 6.6 in the case where $k = s + 2$. Assume A is the associated matrix of space $Z(\eta, \omega; s + 1, s + 2)$ and let $c = (c_1, \ldots, c_{s+1})$ be a nontrivial solution of the system $A^T x = 0$. Up to multiplication by scalars we can assume $c_{s+1} = -1$ for simplicity. The matrix equation $A^T c = 0$ is equivalent to

$$A_{s+1} = c_1 A_1 + c_2 A_2 + \cdots + c_s A_s \tag{6.6-1}$$

where A_i are i-th row of A. We want to show

$$\widetilde{c} = \begin{pmatrix} (\omega_1 - \omega_{s+1})^2 c_1 \\ (\omega_2 - \omega_{s+1})^2 c_2 \\ \vdots \\ (\omega_s - \omega_{s+1})^2 c_s \end{pmatrix}$$

solves the system

$$B^T \cdot x = 0 \tag{6.6-3}$$

where B is the associated matrix of $Z(\widetilde{\eta}, \widetilde{\omega}; s, s)$. We point out that B is a $s \times (s+1)$ complex matrix which can be explicitly written as

$$B = \begin{pmatrix} \widetilde{\eta_1} & \widetilde{\eta_1}\omega_1 - 1 & \dots & \widetilde{\eta_1}\omega_1^{s+1} - (s+1)\omega_1^s \\ \widetilde{\eta_2} & \widetilde{\eta_2}\omega_2 - 1 & \dots & \widetilde{\eta_2}\omega_2^{s+1} - (s+1)\omega_2^s \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{\eta_s} & \widetilde{\eta_s}\omega_s - 1 & \dots & \widetilde{\eta_s}\omega_s^{s+1} - (s+1)\omega_s^s \end{pmatrix} \tag{6.6-4}$$

Observe the system (6.6-1) is equivalent to

$$\eta_{s+1}\omega_{s+1}^i - i\omega_{s+1}^{i-1} = \sum_{j=1}^{s} c_j(\eta_j\omega_j^i - i\omega_j^{i-1}) \; \forall \; i = 0, 1, \dots s+3 \tag{6.6-5}$$

Here the i index runs till $s+3$ since A has $s+3$ columns. Put $i = 0$ in (6.6-5), we get

$$\eta_{s+1} = \textstyle\sum_{i=1}^{s} \eta_i c_i$$

Substitute $i = 1$ into the system (6.6-5) and eliminate η_{s+1} using above equation we have

$$-1 + (c_1 + \dots + c_s) = \sum_{i=1}^{s} c_i\eta_i(\omega_i - \omega_{s+1})$$

Consider the right hand side of above equation

$$\sum_{i=1}^{s} c_i\eta_i(\omega_i - \omega_{s+1}) = \sum_{i=1}^{s} c_i[\eta_i(\omega_i - \omega_{s+1}) - 2] + 2\sum_{i=1}^{s} c_i = \sum_{i=1}^{s} c_i(\omega_i - \omega_{s+1})\widetilde{\eta_i} + 2\sum_{i=1}^{s} c_i$$

Move $2\sum_{i=1}^{s} c_i$ to the left hand side, previous equation becomes

$$-(c_1 + c_2 + \dots + c_s + 1) = \sum_{i=1}^{s} \left[c_i(\omega_i - \omega_{s+1})\widetilde{\eta_i}\right] \tag{6.6-6}$$

We are ready to prove that $B^T\widetilde{c} = 0$ when expressed in the same way as (6.6-5) is equivalent to

$$\sum_{i=1}^{s} \widetilde{c_i}(\widetilde{\eta_i}\omega_i^j - j\omega_i^{j-1}) = 0 \; \forall \; j = 0, 1, 2, \dots, s+1 \tag{6.6-7}$$

Our proof of (6.6-7) is by induction on j. For the base case we need to show

$$\textstyle\sum_{i=1}^{s} \widetilde{c_i}\widetilde{\eta_i} = 0$$

First we use $\eta_{s+1} = \sum_{i=1}^{s} c_i\eta_i$ to cancel η_{s+1} in the system (6.6-5) when consider only $i = 2$

$$-2\omega_{s+1} + 2\sum_{i=1}^{s} c_i\omega_i = \sum_{i=1}^{s} c_i\eta_i(\omega_i^2 - \omega_{s+1}^2) \tag{6.6-8}$$

Right hand side of (6.6-8) can be simplified as

$$\begin{aligned}\text{RHS of (6.6-8)} &= \sum_{i=1}^{s} c_i\eta_i(\omega_i - \omega_{s+1})(\omega_i + \omega_{s+1}) \\ &= \sum_{i=1}^{s} c_i[\eta_i(\omega_i - \omega_{s+1}) - 2](\omega_i + \omega_{s+1}) + 2\sum_{i=1}^{s} c_i(\omega_i + \omega_{s+1}) \\ &= \sum_{i=1}^{s} c_i(\omega_i - \omega_{s+1})\widetilde{\eta}_i(\omega_i + \omega_{s+1}) + 2\sum_{i=1}^{s} c_i(\omega_i + \omega_{s+1})\end{aligned}$$

Cancellation with the left hand side of (6.6-8) yields

$$0 = 2\omega_{s+1}(1 + c_1 + c_2 + \cdots + c_s) + \sum_{i=1}^{s} c_i\widetilde{\eta}_i(\omega_i - \omega_{s+1})(\omega_i + \omega_{s+1})$$

Substitute (6.6-6) to replace $c_1 + \cdots + c_s + 1$, we have

$$0 = \sum_{i=1}^{s} c_i\widetilde{\eta}_i(\omega_i^2 - \omega_{s+1}^2) - 2\omega_{s+1}\sum_{i=1}^{s} c_i\widetilde{\eta}_i(\omega_i - \omega_{s+1}) = \sum_{i=1}^{s} c_i\widetilde{\eta}_i(\omega_i - \omega_{s+1})^2 = \sum_{i=1}^{s} \widetilde{c}_i\widetilde{\eta}_i$$

So we verifies (6.6-7) when $j = 0$.

For the induction step, suppose (6.6-7) is true for all $j = 0, 1, 2 \ldots, m$ $(m \in \mathbb{Z}_+, m < s)$, we want to show (6.6-7) for $j = m + 1$. We write down equation $i = m + 3$ in system (6.6-5) first and use $\eta_{s+1} = \sum_{i=1}^{s} c_i\eta_i$ to replace η_{s+1} as before

$$-(m+3)\omega_{s+1}^{m+2} + (m+3)\sum_{i=1}^{s} c_i\omega_i^{m+2} = \sum_{i=1}^{s} c_i\eta_i(\omega_i^{m+3} - \omega_{s+1}^{m+3}) \tag{6.6-9}$$

From $a^k - b^k = (a - b)(a^{k-1} + a^{k-2}b + \cdots + b^{k-1})$, we could simplify the right hand side of (6.6-9) as

$$\begin{aligned}\text{R.H.S. of (6.6-9)} &= \sum_{i=1}^{s}\left(c_i\eta_i(\omega_i - \omega_{s+1})\sum_{l=0}^{m+2}\omega_i^{m+2-l}\omega_{s+1}^l\right) \\ &= \sum_{i=1}^{s}\left(c_i[\eta_i(\omega_i - \omega_{s+1}) - 2]\sum_{l=0}^{m+2}\omega_i^{m+2-l}\omega_{s+1}^l\right) + 2\sum_{i=1}^{s}\left(c_i\sum_{l=0}^{m+2}\omega_i^{m+2-l}\omega_{s+1}^l\right) \\ &= \sum_{i=1}^{s}\sum_{l=0}^{m+2}\left(c_i\widetilde{\eta}_i(\omega_i - \omega_{s+1})[\omega_i^{m+2-l}\omega_{s+1}^l]\right) + 2\sum_{i=1}^{s}\sum_{l=0}^{m+2}\left(c_i\omega_i^{m+2-l}\omega_{s+1}^l\right)\end{aligned}$$

Cancellation with the left hand side of (6.6-9) would give us

$$0 = \sum_{i=1}^{s}\sum_{l=0}^{m+2}\Big(c_i\widetilde{\eta}_i(\omega_i-\omega_{s+1})[\omega_i^{m+2-l}\omega_{s+1}^l]\Big) + (m+3)\omega_{s+1}^{m+2}(1+c_1+\cdots+c_s)$$
$$+\,2\sum_{i=1}^{s}\sum_{l=1}^{m+1}\Big(c_i\omega_i^{m+2-l}\omega_{s+1}^l\Big) - (m+1)\sum_{i=1}^{s}c_i(\omega_i^{m+2}+\omega_{s+1}^{m+2})$$

Substitute equation (6.6-6) to replace $1+\sum_{i=1}^{s}c_i$

$$\begin{aligned}0 &= \sum_{i=1}^{s}\Big(c_i\widetilde{\eta}_i(\omega_i-\omega_{s+1})\sum_{l=0}^{m+2}\omega_i^{m+2-l}\omega_{s+1}^l\Big) - (m+3)\omega_{s+1}^{m+2}\sum_{i=1}^{s}c_i(\omega_i-\omega_{s+1})\widetilde{\eta}_i\\ &\quad + 2\sum_{i=1}^{s}\sum_{l=1}^{m+1}\Big(c_i\omega_i^{m+2-l}\omega_{s+1}^l\Big) - (m+1)\sum_{i=1}^{s}c_i(\omega_i^{m+2}+\omega_{s+1}^{m+2})\\ &= \sum_{i=1}^{s}\Big(c_i\widetilde{\eta}_i(\omega_i-\omega_{s+1})\Big[-(m+2)\omega_{s+1}^{m+1}+\sum_{l=0}^{m+1}\omega_i^{m+2-l}\omega_{s+1}^l\Big]\Big)\\ &\quad + 2\sum_{i=1}^{s}\sum_{l=1}^{m+1}\Big(c_i\omega_i^{m+2-l}\omega_{s+1}^l\Big) - (m+1)\sum_{i=1}^{s}c_i(\omega_i^{m+2}+\omega_{s+1}^{m+2})\end{aligned}$$

For any $1 \le i \le s, i \in \mathbb{Z}_+$ apply Proposition 6.7 for $a=\omega_i, b=\omega_{s+1}$ and $k=m+1$ we get

$$-(m+2)\omega_{s+1}^{m+2} + \sum_{l=0}^{m+1}\omega_i^{m+2-l}\omega_{s+1}^l = (\omega_i-\omega_{s+1})\sum_{l=0}^{m+1}\Big[(l+1)\omega_{i+1}^{m+1-l}\omega_{s+1}^l\Big]$$
$$(m+1)[\omega_i^{m+2}+\omega_{s+1}^{m+2}] - 2\sum_{l=1}^{m+1}\omega_i^{m+2-l}\omega_{s+1}^l = (\omega_i-\omega_{s+1})^2\sum_{l=0}^{m}\Big[(l+1)(m+1-l)\omega_i^{m-l}\omega_{s+1}^l\Big]$$

Plugging this two equation back to the one obtained one step above, we have

$$\begin{aligned}0 &= \sum_{i=0}^{s}\sum_{l=0}^{m+1}\Big[\widetilde{c}_i\widetilde{\eta}_i(l+1)\omega_{i+1}^{m+1-l}\omega_{s+1}^l\Big] - \sum_{i=1}^{s}\sum_{l=0}^{m}\Big[\widetilde{c}_i(l+1)(m+1-l)\omega_i^{m-l}\omega_{s+1}^l\Big]\\ &= (m+2)\omega_{s+1}^{m+1}\sum_{i=0}^{s}\widetilde{c}_i\widetilde{\eta}_i + \sum_{i=1}^{s}\sum_{l=0}^{m}\Big[(l+1)\omega_{s+1}^l\widetilde{c}_i(\widetilde{\eta}_i\omega_i^{m+1-l}-(m+1-l)\omega_i^{m-l})\Big]\\ &= (m+2)\omega_{s+1}^{m+1}\sum_{i=0}^{s}\widetilde{c}_i\widetilde{\eta}_i + \sum_{l=0}^{m}\Big((l+1)\omega_{s+1}^l\sum_{i=1}^{s}\widetilde{c}_i\Big[\widetilde{\eta}_i\omega_i^{m+1-l}-(m+1-l)\omega_i^{m-1}\Big]\Big)\end{aligned}$$

We have shown that $\sum_{i=1}^{s}\widetilde{c}_i\widetilde{\eta}_i = 0$. Moreover, by induction hypothesis

$$\sum_{i=1}^{s}\widetilde{c}_i\Big[\widetilde{\eta}_i\omega_i^{m+1-l}-(m+1-l)\omega_i^{m-l}\Big] = 0 \text{ for all } l=1,2,\ldots,m$$

Therefore all terms vanished in previous equation except the one where $l=0$. This means

$$\sum_{i=1}^{s}\widetilde{c}_i\Big[\widetilde{\eta}_i\omega_i^{m+1}-(m+1)\omega_i^{m}\Big] = 0$$

which is exactly what we want to show for the induction step. Thus we conclude that $B^T\cdot\widetilde{c}=0$.

Since the system has a nonzero solution $\widetilde{c}$, we know B^T cannot attain full rank from linear algebra.

Chapter 7
Future plan

Previous work on $W(f)$ suggests the following idea to approach remaining case of Conjecture 1.5: simple roots $\alpha_1, \ldots, \alpha_{n_1}$ are good parameters for the space $W(f)$. We know that $W(f) = Z(\delta, \alpha; n_1, r)$ is degenerate if and only if the associated matrix A does not attain full rank. This gives a clue to construct counter-examples if one assumes the existence of some degenerate space $W(f)$. More precisely, $\operatorname{rank} A < n_1$ if and only if all the $n_1 \times n_1$ minors vanish. However entries of A are rational functions in $\alpha_1, \ldots, \alpha_{n_1}$. So define $F(\alpha_1, \ldots, \alpha_{n_1}) \in \mathbb{C}[\alpha_1, \ldots, \alpha_{n_1}]$ to be the common zero of all $n_1 \times n_1$ minors in A, if $\dim[W(f)] > (\deg f - 1) - (n_1 + n_2 + 2n_3)$, F must be a non-constant polynomial. (i.e. common zeros of all $n_1 \times n_1$ minors of A cutoff a nonempty set in affine space $\mathbb{A}^{n_1}$). We believe this is the key step to attack last case where one either proves Conjecture 1.5 for $n_1 < r < 2n_1 - 2$ or constructs counter-examples.

Appendix
Computation of dimension using Macaulay2

The program `WSpace.m2` computes dimension of space $Z(\eta, \omega; s, k)$ and $W(f)$ for given polynomials. To begin with, the method `getHMatrix` compute the matrix we introduced at the beginning of Section 6.

```
i2 : eta = {1/2, 3/4, -7/8, 9, 31}; omega = {47, 2, -3, -5/7, -4/11};
i4 : A = getHMatrix(eta, omega, 5)
o4 = | 1/2  45/2    2021/2  90569/2    4049097/2   180548197/2   |
     | 3/4  1/2     -1      -6         -20         -56           |
     | -7/8 13/8    -15/8   -27/8      297/8       -1539/8       |
     | 9    -52/7   295/49  -1650/343  9125/2401   -50000/16807  |
     | 31   -135/11 584/121 -2512/1331 10752/14641 -45824/161051 |
              5        6
o4 : Matrix QQ  <--- QQ
```

To compute the dimension of space $Z(\eta, \omega; s, k)$ from the associated matrix A, we use the method `dimH`. This operation are easily executed internally via the `rank` command for

$$\dim[Z(\eta, \omega; s, k)] = k + 1 - \operatorname{rank} A$$

from Remark 5.4 in previous section. The method `isConjectureHold` checks if dimension of the space $Z(\eta, \omega; s, k)$ is equal to $k + 1 - s$. (i.e. the associated matrix A is full rank or not)

```
i5 : dimH(eta, omega, 5)
o5 = 1
i6 : isConjectureHold(eta, omega, 5)
o6 = true
```

We point out all methods involves $Z(\eta, \omega; s, k)$ has three inputs which corresponds to η, ω, k respectively. Also, among all methods which computes information of $Z(\eta, \omega; s, k)$, failure to provide lists with different length or distinct elements in the second list (i.e. the list of ω) will result in an error unless the `UnSafe` option is `true`.

To proceed on $W(f)$, we start with the construction method `wSpace`. Because all information on $W(f)$ is obtained by factorizing $f(x)$ into products of linear terms, the method `wSpace` ask user to plug in two data sets: a list of roots and their corresponding multiplicities.

```
i2 : roots = {1/2, -3/4, 78, -29, 31/47, 2}
i3 : rootsMultiplicity = {1, 1, 1, 1, 2, 2}
i4 : f = wSpace(roots, rootsMultiplicity)
o4 : WSpace
```

Our object `WSpace` are descended from `HashTable`. Internally, they are `HashTable` where each key is a root of $f(x)$ and each associated value is the corresponding multiplicity. We also stress that unless the `UnSafe` option was set to be `false`, the construction method will always checks if all multiplicities are positive integer and all roots possess the same ambient ring. (for the sake of simplicity, our program set $\mathbb{Q}$ as the ambient ring). Methods like `getRoots`, `getPolynomial` are constructed in order to access internal data and provide computational convenience.

```
i6 : f = wSpace({1, 2, 3, 4}, {1, 1, 2, 2})
o6 : WSpace
i7 : getPolynomial( f )
      6      5       4       3       2
o7 = x  - 17x  + 117x  - 415x  + 794x  - 768x + 288
o7 : QQ[x, y]
i8 : getRoots( f )
o8 = {1, 2, 3, 4}
o8 : List
```

As we have shown above, the method `getPolynomial` returns the polynomial $f(x)$ that corresponds to the space $W(f)$ and the output of `getRoots` is the set of distinct roots of $f(x)$. Next example illustrates the following point: `getPolynomial(f, k)` returns the *k-th part polynomial* of $f(x)$ (see Notation 1.2 of Section 1), and `getRoots(f, k)` returns $R_k(f)$ (i.e. the set of roots whose multiplicity is exactly k).

```
i11 : f = wSpace({1, 3, -3, 2, 9, 4, 13}, {1, 1, 1, 1, 4, 5, 7})
o11 : WSpace
i12 : getPolynomial(f, 1)
```

```
        4      3      2
o12 = x  - 3x  - 7x  + 27x - 18
o12 : QQ[x, y]
i13 : getRoots(f, 1)
o13 = {1, 2, 3, -3}
o13 : List
```

Recall in Theorem 4.1 of Section 4 , we proved that

$$W(f) = (f_\beta f_\gamma^2) \cdot \widetilde{W}(f, \alpha) = (f_\beta f_\gamma^2) \cdot Z(\delta, \alpha; n_1, r)$$

Hence calculating dimension of $W(f)$ essentially boils down to compute dimension of $Z(\delta, \alpha; n_1, r)$. This facts motivates the next method. The command `getHMatrix` with input type `WSpace` returns the associated matrix of the space $Z(\delta, \alpha; n_1, r)$.

```
i2 : f = wSpace({1/2, -3/4, 5/6, 7/12, 9/10}, {1, 1, 1, 2, 2})
o2 : WSpace
i4 : A = getHMatrix( f )
o4 = | -28973/4180 70199/16720 -160437/66880 330831/267520 |
     | -479/10     -499/20     -519/40       -539/80       |
     | -489/19     -853/38     -1485/76      -23225/1368   |
                                3                      4
o4 : Matrix (frac QQ[x, y])  <--- (frac QQ[x, y])
```

The method `dimW` calculates dimension of $W(f)$ by calling the associated matrix of $Z(\delta, \alpha; n_1, r)$ first. If the option `UseFormula` is set up to be `true`, `dimW` will compute the dimension from Conjecture 1.5. Finally `isConjectureHold` checks if the dimension computed by the matrix is same as our dimension formula $\dim[W(f)] = \deg f - 1 - (n_1 + n_2 + 2n_3)$. (i.e. verifying Conjecture 1.5)

```
i15 : roots = {5, 7/8, 2, 3/2, 3/5, 1, 5/6, 3/4, 1/5, 6/5}
i16 : rootsMultiplicity = {1, 1, 1, 1, 1, 2, 2, 3, 5, 7}
i17 : f = wSpace(roots, rootsMultiplicity)
o17 : WSpace
i18 : dimW( f )
o18 = 10
i19 : dimW(f, UseFormula => true)
o19 = 10
```

```
i20 : isConjectureHold( f )
o20 = true
```

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