THE MATHEMATICAL REASONING OF TEACHERS
DURING THE DESIGN AND DELIVERY OF INSTRUCTION

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by
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ABSTRACT

Much attention in recent years has been given to the nature of the mathematical knowledge of teachers. Teaching in ways that foster rich, connected student understandings of mathematics places multifaceted demands on the mathematical knowledge of teachers. A meaningful articulation of the nature of the mathematical knowledge needed for teaching can support the work of researchers, professional developers, and teacher educators. This study sought to add to the existing understandings of the mathematics education community by focusing on the use of mathematical knowledge and reasoning by teachers during the design and delivery of instruction. The study centered on the classroom practice of four secondary mathematics teachers and their responses during stimulated recall interviews. The stimulated recall interviews centered on the video recordings of the three or four observed classes of each participant and afforded the researcher the opportunity to explore the teachers’ uses of the mathematical knowledge and reasoning in the design of instruction as well as the in-the-moment decision-making during the delivery of instruction. The analysis of data suggested something more than the accumulated, mathematical knowledge of the teacher was used during the design and delivery of instruction. The hypothesized construct of pedagogical content reasoning captured the unique, dual nature of this mathematical reasoning as teachers sought to make instructional moves while maintaining a triune focus on the mathematics of the learning goal(s), the mathematics of the students, and the mathematical path from one to the other.
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Chapter 1:

The Problem of Mathematical Knowledge for Teaching
Adding It Up: Helping Children Learn Mathematics (National Research Council, 2001) represents efforts to find more common ground and move towards a more holistic and widely embraced view of our goals for students and a broader understanding of its implications for teachers and the classroom. One of the cornerstones of their work is the framework defining the five interwoven and interdependent strands of mathematical proficiency:

- **conceptual understanding**—comprehension of mathematical concepts, operations, and relations
- **procedural fluency**—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- **strategic competence**—ability to formulate, represent, and solve mathematical problems
- **adaptive reasoning**—capacity for logical thought, reflection, explanation, and justification
- **productive disposition**—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy.

(National Research Council, 2001, p. 5)

This image of what it means to learn mathematics captures many of the goals for student learning that emerged from the Standards documents (National Council of Teachers of Mathematics, 1989, 2000) and the vision of school mathematics presented in them while also embracing the traditional importance of proficiency with routines and procedures and the role of justification. While developing the understanding of what it means to know mathematics in this way, the mathematics education community also must come to understand more fully what kinds of instruction will build these competencies and how teachers can be equipped to provide that kind of instruction.

Teaching for mathematical proficiency imposes certain responsibilities on the teacher. Among other responsibilities, NCTM’s Professional Teaching Standards (National Council of Teachers of Mathematics, 1991) suggests that teachers must: choose appropriate tasks; manage classroom discourse; maintain supportive classroom environments; and make decisions based on
student learning, the tasks in which they are engaged, and the context in which they are learning. Meeting these responsibilities is challenging. A teacher must not only choose appropriate tasks but also implement them in ways that position the student as decision-maker and meaning-maker. A teacher must not only encourage a discourse but also skillfully use student contributions to build understanding. A teacher must not just be supportive when students struggle but do so in a way that equips students with greater mathematical power and autonomy. All of these require teachers to constantly assess what students understand and to make instructional decisions accordingly.

Factors Impacting Instructional Decisions

If the aforementioned are the instructional approaches that lead to the development of the types of understandings we want for our students, how can we better understand the knowledge and skills with which we need to equip teachers to teach in this way? The answer to that question is as multifaceted as the factors that impact a teacher’s instructional decisions. As Shavelson puts it, “Any teaching act is the result of a decision, either conscious or unconscious” (1973, p. 144). While many have interpreted the view of teacher as decision-maker as too limited because the act of teaching is not seen as deliberate as perhaps originally conceived by Shavelson, the conceptions, beliefs, and knowledge of teachers and their impact on instruction have been actively researched over the last 20 years (Calderhead, 1996; Fennema & Franke, 1992; Pajares, 1992; Thompson, 1992). For this study, the researcher adopts Shavelson’s view of any teaching act being a result of a decision made by the teacher, whether consciously realized or not consciously realized.
At many important moments in the teaching process, a teacher must analyze a broad set of data and synthesize it into a choice of action. There are external factors such as the goals for students; the nature of the task at hand; and inferences about where students are emotionally, socially, and mathematically. But there are also internal factors that determine how a teacher might make sense of a given set of data: their knowledge of students (Fennema, Carpenter, Franke, & Carey, 1993a; Fennema et al., 1996; Fennema, Franke, Carpenter, & Carey, 1993; Hill, Ball, & Schilling, 2008; Peterson, Fennema, & Carpenter, 1991), of mathematics (Fennema & Franke, 1992; Hill et al., 2008; Hill & Charalambous, 2012; Morris, Hiebert, & Spitzer, 2009; Shechtman, Roschelle, Haertel, & Knudsen, 2010; Wilhelm, 2014; Wilkins, 2002; M. R. Wilson, 1994), of curriculum (Charalambous & Hill, 2012; Fennema, Carpenter, et al., 1993a; Knuth, 2002b; Stein & Kaufman, 2010), and of teaching (Baumert et al., 2010; Even & Tirosh, 1995), as well as their beliefs about all of these (Borko et al., 1992; Frykholm, 1999; Knuth, 2002b; Peterson, Fennema, Carpenter, & Loef, 1989; Stipek, Givvin, Salmon, & MacGyvers, 2001; Thompson, 1984, 1992), what they think a mathematics class should look like (Fernandez, 1997; Thompson, 1984), and even how they were taught.

Intuitively however, nothing would seem to matter more than a teacher’s knowledge of the subject matter. Clearly, a teacher cannot expect to create and manage classroom experiences in ways that develop meaningful mathematical knowledge within students unless he or she possesses that type of knowledge. In fact, a number of recent studies on the mathematical knowledge of teachers have demonstrated how teachers tend to represent mathematics to their students in the ways they understand it themselves. Stein and her colleagues (Stein, Baxter, & Leinhardt, 1990) characterized student conceptions as being limited by the teacher’s conceptions
in an overemphasis on rules. Sanchez and Llinares (2003) found the conceptions of function the
four student teachers had, the representations each teacher privileged, and the importance each
teacher ascribed to function all manifested themselves in the ways they organized material to be
presented and the salient features they were planning to highlight.

Kendal and Stacey (2001) found distinct differences in conceptions of mathematics, the
privileging of representations, the time allotted to understanding connections between
representations, and the nature of the use of a CAS system in the teaching of calculus by two
teachers. These differences in instruction were linked to differences in student performance
among the students of each teacher. The students of the teacher who privileged symbolic
representations and the symbolic manipulation of the CAS system outperformed the other class
on measures of CAS use for symbolic differentiation. The students of the teacher who privileged
making connections between representations and understanding underlying concepts performed
better on what Kendal and Stacey termed interpretation competencies.

Charalambous (2010) found a similar dynamic with the two teachers in his study.
Whether the teacher possessed rich conceptions or narrow ones, those same features of
mathematical knowledge characterized their interaction with students around mathematics. One
teacher’s focus on the symbolic representation and correct answers permeated her dealing with
unanticipated student responses and alternative solution methods leading to translations of other
representations into the symbolic to assess their validity. Conversely, the other teacher’s rich and
flexible understanding of representations and confidence in exploring mathematical ideas
mirrored her approach with student ideas and solution strategies. Not surprisingly, these findings
suggest that the nature of students’ mathematical knowledge is going to be shaped by their
teachers’ mathematical knowledge. This study aims to enhance our understanding of the role that the mathematical knowledge of the teacher plays in shaping instruction.

The Mathematical Knowledge of Teachers and Classroom Practice

In spite of the basic understanding that the mathematical knowledge of the teacher shapes the students’ opportunities to learn, many questions concerning the mathematical knowledge of teachers are unresolved. What is the impact of teacher knowledge on classroom practice and student achievement? What do teachers really know about the content they are teaching? What are effective ways to conceptualize and quantify teacher knowledge? Is there more to understanding the impact of mathematical knowledge on classroom practice than simply understanding the nature of their mathematical knowledge? These are not simple questions. In order to understand how this study supports the efforts of the field to answer these questions, it is important to understand some of the ways the field has undertaken to answer them.

Consideration of Large Scale Studies

Understanding how best to prepare teachers is an essential question of research, and when characteristics of preparation can be linked to student achievement, the argument for that kind of teacher development becomes more compelling. Initial efforts to make these links used proxies for teacher knowledge and preparation such as the number of courses taken in subject area and education, whether a teacher held a subject area major or certification, the GPA in major, and scores on various tests of (presumably) teacher knowledge. The conclusions from this research were far from compelling.
Many of these studies were able to use large data sets such as data from the National Educational Longitudinal Study (NELS) (Goldhaber & Brewer, 1996; Rowan, Chiang, & Miller, 1997a), the National Study of American Youth (Monk, 1994), or a multisite data set developed by the researchers (Begle, 1972; Eisenberg, 1977; The National Center for Research on Teacher Learning, 1991). A variety of teacher variables chosen as indicators of teacher subject matter knowledge were chosen: scores on the math quiz from the NELS (Rowan, Chiang, & Miller, 1997b), courses taken or degree level in mathematics (Goldhaber & Brewer, 1996; Monk, 1994; Rowan, et al., 1997a), GPA in major (Ferguson & Womack, 1993) or scores on assessments designed to measure teacher knowledge (Begle, 1972; Eisenberg, 1977; Ferguson & Womack, 1993).

Wilson, Floden, and Ferrini-Mundy (2001) reviewed research that was published in the 1980s and 1990s and that they deemed to be empirical and rigorous. They focused on answering five questions related to teacher preparation (irrespective of subject area). The first was the question of teacher subject-matter preparation. Their review concludes that the research supports the notion that there is a positive, but not strong, correlation between subject matter preparation and its impact in the classroom. They also found evidence to support the benefit of subject-specific education courses.

Similarly findings were suggested by Allen’s (2003) review and synthesis. His examination included all but five of the studies from the Wilson, Floden and Ferrini-Mundy review and added to it a number of relevant research reports from larger reports and books. Allen was more systematic about classifying the nature of the evidential support categorizing evidence as strong support, moderate support, limited support, or inconclusive.
of subject matter preparation, Allen’s findings confirm and extend the conclusions of others, namely, there is moderate support for the importance of subject matter knowledge, but it is unclear about how much.

To illustrate the point, consider the findings of a few of the studies found in these reviews. Both Goldhaber and Brewer (1996) and Rowan, Chiang, and Miller (1997a) found positive associations between teachers with Bachelor’s or Master’s degrees in mathematics and student achievement. Monk (1994) found that undergraduate and graduate mathematics education coursework had a positive impact on student performance, as did the number of undergraduate mathematics courses in a teacher’s background. However, Monk found the impact of mathematics coursework on student’s mathematical performance seemed to diminish after the fifth course. Furthermore, there is some indication that, after the first five mathematics courses, mathematics education coursework has a larger effect on student achievement than mathematics coursework. Ferguson and Womack (1993), in their study across all subject areas, found contributions from education coursework accounted for 16.5% of the variance in teaching performance while GPA in major and scores on an assessment of teacher knowledge accounted for a total of only 4% of the variance.

The findings from the National Center for Research on Teacher Learning (The National Center for Research on Teacher Learning, 1991) provide some insight into the limited difference education and subject matter coursework might make in mathematics. In their study of teacher education programs, these researchers found a large percentage of preservice teachers who could not represent concepts or explain the mathematical reasons for procedures at the beginning of the programs were still unsuccessful on similar tasks at the end of the program.
The findings of these large-scale efforts to link measures of teacher knowledge with student achievement demonstrate that the effect of mathematical knowledge of teachers is important, but questions remain about the nature of that mathematical knowledge needed for teaching. These studies point to the fact that it is not simply a case of requiring teachers to have a certain major. Instead, the mathematical knowledge needed for teaching and its relationship to practice warrants more careful scrutiny. A more in-depth examination of this relationship can be found in the efforts of a number of researchers who undertook qualitative studies examining mathematical knowledge of teachers in a given content area. These studies produced descriptive accounts of the breadth and depth of knowledge these teachers possessed about a concept and provided a picture of strengths, weaknesses, and in some cases, implications for their teaching of these concepts. These studies provide support for the importance for teachers to possess rich, flexible, connected understandings of mathematical topics, but they provide limited insight into the sufficiency of that knowledge for mathematics teaching that supports our goals for student mathematics learning.

Understanding Weaknesses in Mathematical Knowledge

A number of studies at both the elementary and secondary level have been conducted with the goal of understanding the nature of in-service and preservice teachers’ knowledge of the mathematics they are teaching. At the elementary level, research on teachers’ knowledge of mathematics has focused on number and operations, place value, and fractions with fractions having received the bulk of the attention (Ball, Lubienski, & Mewborn, 2001; Hill, Sleep, Lewis, & Ball, 2007). At the secondary level, these efforts have focused on teachers’ understanding of
functions, proof, algebra, proportionality, geometry, and measurement. These studies reveal gaps and inadequacies in teachers’ mathematical knowledge.

**Teacher misunderstandings.** Data from several studies reveal a portion of teachers who make mistakes in solutions and explanation. These weaknesses exist even within the population of mathematics teachers who hold mathematics degrees. McDiarmid and Wilson (1991) explored the mathematical understandings of alternate-route-certified teachers as a subset of the data from the Teacher Education and Learning to Teach (TELT) project. That project involved the study of eleven elementary and secondary teacher development programs over a four-year period, and this particular study analyzed data from 55 participants in either one of two alternate-route certification programs. All 55 held mathematics degrees. Yet, in their examination of the questionnaire and interview data around the topics of division, the nature of zero, operations with integers, proportion, and slope, the researchers found some disturbing weaknesses. Nearly 40% of the participants could not choose a correct equation for representing: *One quantity is 50% more than another*. Almost half did not identify a correct conception of slope opting instead for an incorrect expression for the formula to calculate slope. Over 40 percent of these participants could not explain whether an unorthodox method for a calculation would work every time.

Li (2007), in her study of teacher knowledge of algebraic equations in 72 middle school and high school teachers, found a large portion of teachers who thought one would always get balanced equations if he or she did the same thing on both sides without regard to the nature of the transformation. Half of the participants in this study did not understand that a functional approach for solving a linear equation in one variable and a graphical approach for a system of linear equations produced two different kinds of solutions (a single value of a variable for
solution to the linear equation versus an ordered pair for the solution to the system). The teachers in this study were all teachers of first-year algebra classes.

Other examples are embedded in a number of the studies described in greater detail subsequently. Thirty percent of the time, the teachers in the study conducted by Ward and Thomas (2006) gave an incorrect answer to problems involving fractions, operations, and proportions. The teachers in Fisher’s study (1988) had trouble answering questions involving inverse proportion. Even and Tirosh (1995) found limitations in teacher knowledge characterized as “knowing that,” the most basic level of knowledge for a teacher. They described one case in detail in which the teacher knew an accurate definition of function and the vertical line test but wrongly classified a circle and an ellipse.

These studies illustrate the presence of fundamental misunderstandings of teachers of the mathematics they are expected to teach. While not a majority and certainly not definitive for all teachers, these mistakes are nevertheless troublesome. Results of these studies and others like them raise the question of whether the deficits in the mathematical knowledge of teachers and its impact in the classroom could be easily addressed by continuing to seek ways to bolster the common content knowledge of teachers. However, as the review of literature reported in subsequent sections reveals, the problem of the mathematical knowledge of teachers and the impact on the classroom is more complex.

**Weak understandings of concepts and procedures.** Although there is a small, yet significant, portion of teachers in many of these studies who make mathematical errors or express misunderstandings, there are equally troubling concerns about the depth of the mathematical knowledge of teachers who *can* provide the correct answers to problems. For
those teachers, their understanding was often limited to one method or representation and seemed to impact their ability to provide mathematically rich explanations.

For example, in a study of a stratified random sample of 20 secondary mathematics teachers, three-fourths of whom had more than 5 years experience, Fisher (1988) found minimal levels of competency in a large portion of the teachers. While almost all were able to successfully work problems involving directly proportional relationships (they had trouble with inverse relationships), almost half identified the use of an algebraic proportion as the only truly proportional strategy from among several mathematically equivalent ones. Only seven of the teachers correctly identified all of the proportional strategies as such.

This focus on the formula for a proportion and the fact that the teachers’ identified student difficulty with setting up the formula as the most frequently encountered student error led Fisher to characterize these teachers’ conceptions of proportionality as rule-based. Given that the researcher also found that 80% of the teachers suggested that the students use the same strategies as they did in solving these problems, it would not be surprising to find rule-based conceptions of proportionality in the students of these teachers, though the study did not examine this aspect.

Perhaps the most striking examples of a narrow, rule-based conception of mathematics involved the study participants in studies of teachers’ understanding of function (Even, 1993; Even & Tirosh, 1995; Stein, et al., 1990). In each of these cases, researchers found limited knowledge on the part of participating teachers, which led to instruction, in their view, that was mathematically inadequate. Stein and her colleagues (1990) studied a fifth-grade teacher with 18 years of experience as they examined his knowledge of functions and how he engaged his
students in a elementary study of function concepts. The researchers found that the teacher’s limited conception of function led to instruction that overgeneralized limited truths, failed to make connections between concepts and representations, and missed opportunities to lay a foundation for future study. Rules were emphasized throughout the lesson as instruction focused on superficial features of equations, graphs, and functions.

Even (1993) found three trends in the preservice teachers’ conceptions of function:

(1) Functions are (or always can be represented as) equations or formulas;
(2) Graphs of functions should be ‘nice’; and
(3) Functions are ‘known.’

(Even, 1993, p. 111)

These findings parallel the findings of Even and Tirosh (1995) in their study of 10 prospective secondary teachers in the United States and 33 practicing Israeli teachers. These researchers made a distinction in two types of subject matter knowledge: “knowing that” and “knowing why.” Limitations in teachers knowing that and knowing why impacted their responses to student questions, student-generated hypotheses, and incorrect answers. For these teachers, a lack of knowing why led to an overemphasis on mathematical rules, and when students responded incorrectly, these teachers made little attempt to examine student thinking. In short, the weaknesses in mathematical knowledge made it difficult for these teachers to explore and understand student thinking and use it to further the learning for each individual and the group.

Similarly, the teachers in Ward and Thomas (2006) had a narrow knowledge base of key concepts of fractions, operations, and proportions. These 44 middle grades teachers were asked to provide the mathematical solution to various problems in a given scenario and then identify key understandings involved in the scenario or the teaching actions required. While they were able to provide the correct solution 70% of the time, they were able to identify key
understandings or appropriate teaching moves only one third of the time. While there are questions about whether the items in this study arose out of practice and truly reflected literature-based understandings of teaching and learning of these topics, the analysis revealed clear differences in what mathematics teachers could do and their potential for making meaningful instructional decisions. The researchers understood this difference as a reflection of weak conceptual knowledge of the mathematics in roughly one third of these practicing teachers. However, understanding the disconnect between the knowledge of content and the application of that knowledge to the classroom represents an important aspect of the questions motivating this study.

**Weak conceptions of mathematics as a discipline.** If mathematics is not understood and valued by teachers as a logically connected set of understandings and justifiable relationships, it will be difficult for those teachers to develop those appreciations in their students. Several studies make this point as they bring into question how teachers view mathematics as a system of thought (Knuth, 2002a; W. G. Martin & Harel, 1989; McDiarmid & Wilson, 1991).

The most likely place to find teachers who recognize mathematics as logical and useful would be in the population of teachers with mathematics degrees. For the alternate-route teachers in the TELT study, the conceptions of mathematics as sensible, useful, and worthwhile were strong (McDiarmid & Wilson, 1991). Yet, even though these teachers held degrees in mathematics, only 33% could correctly identify a story to represent the division of fractions. In the subsequent interviews, all expressed the difficulty of this story task and one teacher even questioned the application of division by fractions in real life. Additionally, several of those interviewed alluded to the notion of mathematics having some things one just has to memorize.
In another section on the survey, participants were asked if each one of a list of mathematical facts could be explained. Two facts of particular interest were: *Multiplying two negatives produces a positive* and *Any nonzero number to the zero power is 1*. Over a quarter of these teachers who held mathematics degrees, stated that either they were not sure one could explain or one just had to accept these facts as true. That means that over 25% of these teachers did not view mathematics as being a coherent, connected body of knowledge. It is important to note that the teachers in this study were not asked to produce a story or explain the mathematics. The questions were more basic than that. They were asked to *identify* a story as involving the application of division of fractions, and they were asked only whether the mathematical facts *could* be explained. The limited nature of the demands of the questions brings additional questions about the strength of the participants’ conceptions of mathematics as a discipline.

Similarly, in Even’s (1993) study of questionnaire data \(n = 152\) and interview data \(n = 10\) from prospective secondary teachers about their conception of function, the researcher found 8 out of these 10 teachers unable to provide a reason for the univalence criterion in the definition of function even though they had an appreciation for its importance. Furthermore, of the teachers who had a strong sense of the modern definition of function (univalent and arbitrary), two-thirds dropped these notions when discussing how they would define functions for students. Interviews revealed that many thought this was beyond the level of the students, so instead of developing a representation that embodied these essential mathematical notions and was accessible to students, these teachers chose to drop an important conceptualization.

The centerpiece of mathematics as a system of thought is the notion of justification, whether it is in the form of proof or explanation. In Knuth’s (2002a) examination of teachers’
conceptions of the role of proof, what constitutes proof, and what counts as a convincing argument, Knuth found that a limited conception of proof prevailed. Teachers had difficulty identifying nonproofs and many believed that proofs were fallible (in other words, a counterexample might exist). Additionally, the researcher found that what teachers found convincing had more to do with form rather than mathematical substance—the amount of detail in the argument, the teachers’ familiarity with the argument, and the method used to construct the argument. While these teachers expressed a general appreciation for the role of proof, they gave no indication that they believed proof could promote understanding.

The work of Martin and Harel (1989) suggested a similar weakness in the broader area of justifications. Even though this study involved prospective elementary teachers, it is nevertheless informative in light of Knuth’s work. Participants in Martin and Harel’s study were asked to assess the mathematical validity of verifications of familiar and unfamiliar generalizations. These verifications varied along a number of dimensions, most notably: inductive–deductive, valid–invalid, generalizations or examples and nonexamples. Eighty percent of the participants gave a high rating of mathematical proof to inductive arguments. Fewer than 10% of participants gave all inductive examples a low rating of mathematical proof. When asked to consider a verification in a deductive form but using a particular value for a variable, a significant number of those rated as High Deductive by the researchers counted this verification as a valid proof. Furthermore, many prospective teachers who correctly identified a deductive proof as valid also rated an invalid proof in a similar form as valid. The researchers attributed this misconception to students’ perceptions of the form of the proof (i.e., it looked like a deductive argument).
While not all of these studies associated the mathematical knowledge of teachers with practice, the weakness in many teachers’ conceptions of mathematics is clear. With these weaknesses in understanding of proof and justification, it is difficult to imagine these teachers using these notions effectively in the classroom in any form. Without the mathematical norms of justification, explanation, and verification, the development of student understandings and appreciations of mathematics as a discipline seems unlikely if not impossible in these classrooms.

**Summary.** The studies reviewed in this section establish the shortcomings of the mathematical knowledge of teachers. Like the mathematical knowledge of many students, the mathematical knowledge of many prospective and in-service teachers is too often narrow, unconnected, rule-based, and filled with misunderstandings and lacking in depth. A few studies have attempted to connect the deficits in a teacher’s knowledge with his or her instructional decisions or emphases. In light of the efforts of this study to understand the nature and use of the mathematical knowledge during instruction, it is especially important to examine studies that link knowledge and practice.

**The Impact of Content Knowledge on Practice**

A few studies illustrate how the richer conceptions of mathematical concepts on the part of the teacher shape the teacher’s choices and ultimately the experience of students with the mathematical ideas under study. In the studies of Charalambous (2010) and Li (2007), broader measures of teacher knowledge are used as the teaching moves of teachers with strong mathematical knowledge are contrasted with those with weaker mathematical knowledge. Other
studies associate rich teaching practice with stronger content knowledge (Ebert, 1993; Fernandez, 1997; Lloyd & Wilson, 1998; Swafford, Jones, & Thornton, 1997).

Charalambous (2010), in a case study of two elementary teachers, focused on understanding the relationship between the mathematical knowledge of the teacher and the choice, presentation, and implementation of tasks. Focusing on two teachers with significantly different levels of MKT as measured by the assessment used by Ball and her colleagues (Ball, Thames, & Phelps, 2008), Charalambous captured a picture of poignant differences in the ways these two teachers represented mathematics for students and how they engaged students in consideration of mathematical ideas. The teacher with stronger scores on the MKT assessment used representations flexibly to fit the situation and reasoned through the mathematical implications of unusual methods and unfamiliar situations. In this teacher’s interactions with students, representations were used to further the understanding of students, students were expected to justify answers and reason through implications, and more higher demand mathematical tasks were chosen, presented and implemented at a higher level. In contrast, the teacher with the lower scores on the MKT assessment was also able to understand various representations of mathematics but seemed to translate those into a symbolic form before using them to solve problems. Likewise in her teaching, she represented ideas in different ways but did not use those representations to advance student thinking. Furthermore, her consideration of student methods was focused almost exclusively on whether the answer was correct.

In Li’s (2007) study mentioned previously, Li noticed a pattern among teachers with stronger mathematical knowledge as measured by the questionnaire and subsequent interviews. Namely, those teachers used their mathematical knowledge to understand the mathematical
limitations of physical models or metaphors for mathematical procedures. Consequently, they were able to choose more powerful representations. Those teachers with weaker mathematical knowledge chose poor metaphors and failed to recognize the mathematical inadequacy of them.

Other studies demonstrate an association between strong mathematical knowledge and meaningful instructional practices as well. The nine secondary teachers in Fernandez’s study (1997) used their mathematical knowledge to use unanticipated student responses to further understanding of the individual and the class. The teachers demonstrated four distinct strategies for doing so: following through a student thought to a logical conclusion, introducing a counterexample for student consideration, working a simpler or related problem to model thinking, and incorporating a student’s alternative solution to deepen understandings.

In a study of 49 in-service teachers in Grades 4 through 8 participating in a three-year program designed to enrich teacher knowledge of geometry, researchers found significant gains in content knowledge of teachers and meaningful changes to instruction (Swafford, et al., 1997). The program included a content component in which participants engaged in the development of geometric understandings as well as a research component in which they engaged in consideration of student thinking and developed an understanding of the van Hiele levels of geometric thinking (Level 0: Visualization, Level 1: Analysis, Level 2: Abstraction, Level 3: Deduction, and Level 4: Rigor). Pre- and postassessments included measures of content knowledge and knowledge of students, while other data included an analysis of lesson planning, classroom observation data and interview data (n = 8). The results revealed that teachers demonstrated improved content knowledge and planned lessons that asked for less recall (van Hiele Level 1) and more analysis (van Hiele Level 2). Also, 66% of teachers made substantial
additions to the lesson as outlined in the teacher’s edition—most involving substantive student exploration. On the second lesson plans, researchers also noted a slight increase in the number of activities requiring students to explain their thinking and more frequent suggestions of a preassessment to determine what students already know. Furthermore, during the classroom observations, researchers noted a pattern of instruction in which, among other benefits, teachers were more confident in their ability to engage students in higher level thinking. The teachers attributed this increase to greater content knowledge and greater research-based knowledge of students.

Ebert (1993) found similar results in her study of 11 prospective secondary teachers’ mathematical and pedagogical knowledge of functions. Using a variety of data from a written assessment of function knowledge to responses to classroom vignettes, unit plans, and stimulated recall interviews to assess pedagogical knowledge of functions, this study focused on the transformation of knowledge into practice. The researcher looked at the features of pedagogical content knowledge that were displayed through explanations, analogies, examples, representations, and demonstrations. The three teachers with the strongest subject-matter knowledge also had the most powerful explanations, the most conceptual view of mathematics, the strongest belief in students’ ability to do mathematical thinking, and the most student-centered pedagogy.

In Lloyd and Wilson’s (1998) case study, they examined the understandings of function of a fourteen-year veteran high school teacher and his implementation of a six-week functions unit from a reformed curriculum. Mr. Allen’s flexible and multifaceted understanding of functions enabled students to develop similar facility. The students were encouraged to use a
variety of representations and to make connections among them. The questions Mr. Allen asked of his students led them to focus on the nature of relationships between real world quantities and among representations. His technique forced students to articulate those understandings without directing them to particularly salient features—an important feature of instruction that keeps the learner as the meaning maker.

Other studies have worked with assessments of mathematical knowledge for teaching (MKT) and sought to correlate the performance of teachers on those assessments with indicators of the quality and nature of instruction. As part of the California Mathematics Project Professional Development Institutes, Ball, Hill, and colleagues developed tools to measure the MKT of teachers and developed conceptualizations of the mathematical knowledge for teaching that are discussed in subsequent sections (Ball et al, 2008). Several studies emerged from this work involving the researchers in efforts to understand the impact of MKT on instructional practice. Hill, Blunk, and colleagues (2008) analyzed data from this project to evaluate correlations between MKT and the mathematical quality of instruction (MQI). They collected video recordings of classroom practice, conducted interviews, and used that data as they assessed the MQI of the observed lessons by examining mathematical errors, inappropriate responses to students (misinterpretation or failure to respond), connectedness of classroom activity to mathematics, richness of the mathematics (including multiple representations, making connections, providing explanations and justification), appropriate responses to students, and the use of mathematical language. The researchers used a paper-and-pencil assessment of MKT involving items designed to measure the common content knowledge of teachers as well as the specialized content knowledge for teaching mathematics (Ball et al, 2008).
In the 90 lessons taught by 10 teachers that were analyzed for this study, Hill and colleagues found a substantial link between the level of MKT and the mathematical quality of instruction. Teachers with stronger MKT made fewer mathematical errors, responded more appropriately to students, and chose examples that helped students construct meaning. Teachers with weaker MKT were less successful at selecting and sequencing examples, at presenting and elaborating on definitions provided by the textbook, and at using multiple representations to help students construct meaning. The researchers note, however, that these findings were not consistent across all cases and posit that other factors such as a teacher’s belief and views on mathematics and mathematics teaching could have mediated the results (Hill, Blunk, et al., 2008).

Similarly, Hill and Charalambous (2012) in their cross-case analysis of four case studies of teachers with varying levels of MKT and professional support to enact a reformed curriculum found that MKT contributed to the MQI for teachers in this study. High MKT teachers more consistently used rich and precise mathematical language while low-MKT and mid-MKT teachers used less. They also noted that stronger MKT provided more meaningful explanations, more connections across ideas and representations, and more connected lesson segments and task sequences.

To summarize, the findings of the studies discussed in this section would suggest that rich conceptions of content, connected understandings, flexible use of representations, and other dimensions of strong content knowledge frequently accompany instruction that is similarly rich. Teachers with higher levels of mathematical knowledge chose, used, and sequenced representations and examples flexibly and in ways that help students construct meaning.
(Charalambous, 2010; Ebert, 1993; Hill et al., 2008; Hill & Charalambous, 2012). They tended to choose higher demand tasks (Charalambous, 2010) and maintain the demand of tasks through implementation (Charalambous, 2010; Wilhelm, 2014). They are less inclined to ask for recall, and they elicit the articulation of mathematical relationships from students (Lloyd & Wilson, 1998; Swafford et al., 1997). They are able to use unanticipated student responses to further the understanding of students by following through to a logical conclusion, introducing a counterexample, working a simpler or related problem, or incorporating an alternative solution in ways that deepen understanding (Fernandez, 1997). They make fewer mathematical errors and use more dense and precise mathematical language (Charalamous & Hill, 2012; Hill et al., 2008; Hill & Charalambous, 2012). These findings provide strong evidence of the correlation between the mathematical knowledge of the teacher and these instructional practices, and they provide a picture of what teachers with stronger levels of MKT can do. However, they do not provide an explanation of how mathematical knowledge influences instructional moves. Understanding the relationship between these instructional practices and the use of mathematical knowledge of the teacher represents an important aspect of the motivations for this study.

**Questions about the Impact of Mathematical Knowledge on Practice**

In spite of the strength of this evidence, a number of studies suggest that strong content knowledge is not a sufficient condition for teaching that is conceptually rich for students. Two groups of studies are of particular interest and provide a poignant contrast to the studies just discussed.

The first group of studies illustrates that even though teachers possessed solid understandings of mathematics, they were not always able to implement instruction that fosters
the building of rich, connected understandings of mathematics. In Wilson’s (1994) study, a prospective teacher’s knowledge of functions grew over the course of a semester into a more connected and flexible understanding, yet she described her goals for students as procedural and without the need for the development of underlying meaning. Even (1993) found that teachers with seemingly robust modern definitions of functions choosing not to include the key notions of univalence and arbitrariness in their explanations provided to their students. Speer and Wagner (2009) highlighted the struggles of an experienced college professor in teaching an inquiry-based differential equations course. His subject matter knowledge was unquestionably strong, yet he repeatedly missed opportunities to use student responses to develop deeper understandings. Leikin and Winicki-Landman (2000) found that the considerations teachers used to establish the mathematical equivalency of definitions extended well beyond mathematical justifications. Their considerations of students and teaching overlaid, interacted with, and sometimes trumped the mathematical truths. What is most noteworthy is that these teachers were engaged in a purely mathematical discussion yet used other domains of knowledge to make the judgments. Morris, Hiebert, and Spitzer (2009) found that the prospective teachers in their study possessed the mathematical knowledge and ability to understand and unpack the learning goals for students yet failed to use many of those understandings to evaluate instruction, assess student understanding, and design instruction. In each of these studies, robust understandings of mathematics did not translate into classroom teaching that reflected the rich, connected understandings possessed by the teacher.

The second group of studies incorporated the same instrument to measure MKT as some of the studies affirming the relationship between the level of MKT and the mathematical quality
of instruction. Using the instrument developed by Ball, Hill, and their colleagues (Hill, Schilling, & Ball, 2004) each of these studies examined the relationship between the level of MKT (and in some cases, other measures of teacher knowledge) and some element of instruction. Wilhelm (2014) found a significant relationship between the level of MKT and the maintenance of cognitive demand in the implementation of tasks but not in the selection of tasks of high demand for the 213 middle school mathematics teachers in her study. Stein & Kaufman (2010) used a measure of teacher capacity involving level of education, experience, and mathematical knowledge to examine the relationship between student capacity and the quality of implementation of two different reformed mathematics curricula as determined by the maintenance of high levels of cognitive demand. The teacher knowledge component was assessed using items from the MKT assessment. These researchers found no significant correlation between teacher capacity and quality of implementation of the curricula.

One other study presents significant findings that suggest limited correlation between the mathematical knowledge of teachers and instructional practice. Shechtman and her colleagues (2010) studied 181 middle school mathematics teachers to explore the relationships between teachers’ mathematical knowledge, teachers’ classroom decision-making, and student achievement outcomes on rate, proportionality, and linear functions. They developed their own measure of mathematical knowledge. The researchers found a lack of correlation between teachers’ mathematical knowledge and instructional decision-making. The researchers posit that other factors mediate the effect and richer models of mathematical knowledge for teaching and how it impacts instruction are warranted.
As suggested, the findings of these studies demonstrate that strong mathematical knowledge on the part of the teacher does not always lead to mathematically enriched instruction. Along with the two previous discussions in which some studies associated weaker mathematical knowledge with less rich teaching practices and other studies associated stronger mathematical knowledge with teaching practices fostering mathematical proficiency, and these studies suggesting that strong content knowledge is not the sole answer, the puzzle of the impact of mathematical knowledge on instruction remains unsolved. These mixed results also suggest that existing conceptualizations of the domains of mathematical knowledge for teaching might need to be refined further.

**Conceptualizations of the Domain of Mathematical Knowledge for Teaching**

Shulman (1986) recognized the complexity of the nature of mathematical knowledge needed for teaching and made some distinctions within the domain of mathematical knowledge. He conceived of a separation of mathematical knowledge into two categories: *content knowledge* and *pedagogical content knowledge*. This distinction captured the view that there is indeed something unique about the mathematical knowledge for teaching. A number of efforts have been made along these lines to conceptualize the domains of teacher knowledge in ways that provide insight into the impact of each domain on the classroom. In particular, the work of Ball, Hill and colleagues (Ball, et al., 2008) has involved refining notions of domains of mathematical and pedagogical knowledge with the ultimate goal of creating valid measures of the mathematical knowledge needed for teaching (MKT).
Although the work of Ball, Hill and colleagues in developing items for assessments shows promise for identifying domains of knowledge and measuring the mathematical knowledge needed for teaching (MKT) at scale, they acknowledge that weaknesses in their findings suggest the need for continued refinement of conceptualizations and measures of MKT and in particular *knowledge of content and students* (Hill, Ball, et al., 2008) – a construct that will be explicated in the next section. As was the case with Even in her conceptualization of the facets of mathematical knowledge of a topic (Even, 1990), careful reading of the descriptions of the conceptualization of MKT reveals a focus on describing the domains of *accumulated* knowledge. The definition of MKT provided by these researchers reinforces the point. They define the mathematical knowledge for teaching as “the mathematical knowledge that teachers use in classrooms to produce instruction and student growth” (Hill, Ball, et al., 2008, p. 374). The construct clearly refers to the mathematical knowledge as distinct from its use, but because the knowledge base is defined by the context in which it is used (teaching), the distinction between the knowledge and its use becomes somewhat blurred.

Mason and Spence (1999) describe this use of knowledge as knowing-to act in the moment. In developing this construct, they make an important distinction between *knowing-to* and other forms of knowledge described by a number of researchers in various ways, in particular Ryle (1949), Skemp (1976), Ball (1991) and Even and Tirosh (1995). Mason and Spence contend that knowing-that (factual knowledge), knowing-how (operational or procedural knowledge), knowing-why (propositional knowledge) and knowing-about (epistemic knowledge) are types of accumulated knowledge. Knowing-to act in the moment requires relevant knowledge to be recognized as such and acted upon at the right time. The other domains of
knowledge alone are not sufficient for knowing to act. Knowing to act requires awareness. That awareness can come spontaneously or as a result of experience, and as it comes, it restructures attention allowing other forms of knowing to be brought to bear on understanding the situation and how to act. “Once the moment of knowing-to takes place, then knowing-how takes over to exploit the fresh idea” (Mason & Spence, 1999, p. 146). Mason and Spence conclude that existing conceptualizations of the domains of mathematical knowledge have missed one important component of teacher knowledge: knowing-to act in the moment.

Beginning with a careful explication of the domains of mathematical knowledge conceptualized by Ball and her colleagues, drawing distinctions between the accumulated knowledge of teachers and the use of that knowledge base in practice serves as an important step in understanding the opportunities to advance our understanding of the conceptualizations of the domain that this study aims to explore.

The Domains of Mathematical Knowledge for Teaching

From the beginning, Ball and her colleagues recognized a demand on the use of teacher knowledge in practice and distinguished the knowledge base from its use.

Because one big challenge of teaching is to integrate across many kinds of knowledge in the context of particular situations, the fact that there are patterns in and predictability to what students might think and that there are well-tried approaches to develop certain mathematical ideas, can help manage this challenge. However, a body of such bundled knowledge may not always equip the teacher with the flexibility needed to manage the complexity of practice. (Ball & Bass, 2000, p. 88)

However, their work focused attention on the conceptualization of that bundled knowledge rather than the use of that knowledge in practice. Nevertheless, the conceptualizations of the mathematical knowledge for teaching (MKT) (Hill, Ball, et al., 2008) serve as an important tool in making sense of the use of mathematical knowledge in practice.
For this reason, this framework is discussed with an emphasis on making these distinctions between knowledge and its use in practice. The domains of MKT as conceived of by Ball, Hill, and colleagues are captured in schematic form through Figure 1.1.

*Figure 1.1. Mathematical Knowledge for Teaching Domain Map.* (Hill, Ball, et al., 2008)

The domain of MKT is divided into two large areas: subject matter knowledge and pedagogical content knowledge. Subject Matter Knowledge has been subdivided into three areas: *common content knowledge, specialized content knowledge, and knowledge at the mathematical horizon.* Pedagogical Content Knowledge has been subdivided into *knowledge of content and students, knowledge of content and teaching, and knowledge of curriculum.* Each of the dimensions identified in the domain map is discussed subsequently. Whenever possible, direct quotes and examples used by the authors to describe the domain will be used to highlight the need for greater clarity. Refinement of these definitions follows.
Mathematical Knowledge for Teaching (MKT) is defined as “the mathematical knowledge that teachers use in classrooms to produce instruction and student growth” (Hill, Ball, et al., 2008, p. 374). It is important to note that the authors make an implicit distinction between knowledge and its use. For this study, MKT will refer exclusively to the knowledge separate from its use.

Common Content Knowledge (CCK) is defined as “knowledge that is used in the work of teaching in ways in common with how it is used in many other professions or occupations that also use mathematics” (Hill, Ball, et al., 2008, p. 377). Again, an implicit distinction between the knowledge and its use is made. For this study, CCK will refer exclusively to the knowledge of content as distinct from how it is used.

Specialized Content Knowledge (SCK) is defined as “the mathematical knowledge that allows teachers to engage in particular teaching tasks including how to accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures, and examine and understand unusual solution methods to problems” (Hill, Ball, et al., 2008, pp. 377-378). In this characterization of SCK, a distinction between accumulated knowledge and its use begins to blur. For example, explanations for common rules and procedures are likely contained in the knowledge base of a teacher with the richest content knowledge. In contrast, examining and understanding unusual solution methods involves an application of that knowledge and mathematical reasoning assuming that those explanations of unusual solution methods are not part of the accumulated knowledge of the teacher. Even though this dimension of subject matter knowledge is used to understand student methods, it is not intended to include that use in the definition. The lack of clarity in this distinction between specialized content knowledge and its
use could explain some of the inconsistencies in the findings from studies on the mathematical knowledge of teachers and its impact on practice. For this study, this distinction is an important one.

*Knowledge of Content and Students* (KCS) is defined as “Content knowledge intertwined with knowledge of how students think about, know, or learn this particular content. KCS is used in the tasks of teaching that involve attending to both the specific content and something particular about learners” (Hill, Ball, et al., 2008, p. 375). While definition of this subdomain is unambiguous with respect to the distinction between knowledge and the application of it, the examples the authors chose to illustrate this construct bring the distinction into question. They provide two examples of KCS. One involves knowing typical approaches, mistakes, and misconceptions about which a teacher would want to be aware as they are teaching that content—this is the accumulated knowledge. The other example involves teacher thinking about a student’s approach to a problem and drawing on this knowledge—knowledge in use. Once again, the second example could be interpreted as referring to the knowledge used, but the reference to thinking blurs the distinction. Similar ambiguity exists in descriptions in other articles (Ball, et al., 2008). For the sake of this study, the definition as written will serve our purposes.

*Knowledge of Content and Teaching* (KCT) is defined rather indirectly following a series of examples. “Knowledge of teaching and content is an amalgam, involving a particular mathematical idea or procedure and familiarity with pedagogical principles for teaching that particular content” (Ball, et al., 2008, p. 402). Again, the ambiguity about the distinction between knowledge and knowledge use comes in the presentation of this construct. The initial
examples discuss instructional decisions and considerations a teacher must make. Those decisions are based in part on the teacher’s KCT, but those decisions are made through the application of KCT to the situation. For the sake of this study, KCT will be defined as the knowledge of ways to represent, explore, model, and demonstrate mathematical topics and the pedagogical advantages and disadvantages of each. With the clarity about the distinction added, this definition is intended to be consistent with the definition of Ball and her colleagues.

While no published definitions were found for the other elements of this domain, knowledge at the mathematical horizon and knowledge of content and curriculum, these domains will be defined here with the best effort to do so in terms of the original intent of Ball, Hill and colleagues. For the purposes of this study, knowledge at the mathematical horizon will be defined as knowledge of mathematical ideas whose formal or informal foundations can be found in the mathematics under study. In other words, if a sixth-grade teacher is engaged in a task with her students that includes a consideration of the rate of growth of a quantity, that teacher would possess knowledge at the mathematical horizon if she recognized the connection to slope, rate of change, and possibly derivative.

Knowledge of Content and Curriculum is the knowledge of the progression of understandings represented in the curriculum over the course of a year and over multiple years. This type of knowledge would involve the teacher knowing at what grades students encounter particular topics and in what ways and understanding prerequisite topics for a given topic.

These distinctions among the various domains of the mathematical knowledge for teaching can explain some of the question-raising results in studies in which strong mathematical knowledge was present yet rich instructional practices were not. For example, Speer and
Wagner (2009) were able to explain the professor’s struggles in terms of weaknesses in the knowledge of content and students and in specialized content knowledge even though he possessed strong subject matter knowledge. Izsak, Caglayan, and Olive (2009) observed the teacher who could not effectively convey his more advanced considerations of the best choice of representation to his students who were focused on approaching a problem in the easiest way for them to solve it. These difficulties can be readily understood in terms of the teacher’s lack of knowledge of content and students. Perhaps the prospective teachers in the study of Morris, Hiebert, and Spitzer (2009) who could not use their knowledge to unpack lesson goals to assess and adjust instruction had weaker knowledge of content and teaching.

Unanswered Questions

In addition to their explanatory power, these conceptualizations have proven useful in the development of assessment measures for the various domains of mathematical knowledge needed for teaching. The factor analyses completed in conjunction with the assessment work of Ball, Hill, and colleagues (Ball, et al., 2008; Hill, Ball, et al., 2008; Hill, et al., 2004) provide support for the conceptualizations of MKT as described by these researchers, but the results are not definitive. Those results statistically established the presence of three distinct domains: knowledge of content in number concepts and operations, knowledge of content in patterns, functions, and algebra, and knowledge of content and students in number concepts and operations. Further analysis suggested that knowledge of content and knowledge of students and content were distinguishable factors and suggested that common content knowledge and specialized content knowledge were related but not completely equivalent. Even these researchers acknowledge that much work remains in the refinement of these conceptualizations.
and the construction of assessment items, and the results of some of their own studies found inconsistent patterns across all teachers (Hill, Blunk et al, 2008).

Other questions emerge from a deeper consideration of a study by Baumert and colleagues (2010). Like others, these researchers sought to understand the extent to which content knowledge of mathematics and pedagogical content knowledge influenced instructional quality. They defined instructional quality as the provision of cognitively activating learning opportunities, curricular level of tasks, the individual learning support, and classroom management. Using a paper-and-pencil test developed for this study, they measured mathematics content knowledge and three dimensions of pedagogical content knowledge: task dimension (identifying multiple solution paths), student dimension (recognizing student misconceptions, difficulties, and solution strategies) and the instruction dimension (knowledge of various representations and explanations). Their analysis found that 39% of the variance in student achievement was explained by the pedagogical content knowledge while only 4.6% was explained by the content knowledge measures.

It is important to note, however, that for Baumert and his colleagues, many of the elements of teacher knowledge they classified as pedagogical content knowledge are categorized as specialized content knowledge by Hill, Ball, and their colleagues—for example, knowledge of various representations and explanations. It is unclear whether the intersection is equivalent to Ball’s specialized content knowledge. Furthermore, it is impossible to determine whether these apparent differences in results are simply the result of differences in the definitions of the domains under study or do they point to a dimension of mathematical and pedagogical knowledge that is not captured by either of the conceptualizations of the mathematical and
pedagogical knowledge of teachers put forth by Ball and by Baumert. Nevertheless, the results from this study suggest that conceptualizations and measurements of the domains of the mathematical knowledge for teaching still need enhancements and point to the need for more clarity about the relationship between specialized content knowledge and its use in practice.

In addition to the less-than-clear accounting for the conceptualization of mathematical knowledge in action, the study of the act of teaching, as it happens, has not been widely used as a context for understanding teacher knowledge and the impact on practice in either of the lines of research described here. Many studies have chosen to examine how the mathematical knowledge of teachers impacts practice in a clinical setting (e.g. Ball & Wilson, 1990; Even, 1993; Fisher, 1988; Li, 2007; Morris, et al., 2009; Ward & Thomas, 2006) or through examination of teacher planning (e.g. Sanchez & Llinares, 2003; Swafford, et al., 1997). Studies frequently involve subjects (often preservice teachers) responding to surveys or questionnaires that include written descriptions of student responses or classroom scenarios. While the advantage of more control is clear, the in-the-moment decision-making of teachers can be obscured in these clinical situations and warrants additional attention.

A few researchers have made concerted efforts to include these dynamics of knowledge-in-action in their work. In particular, they have sought to study the activation of teacher knowledge in the moment with respect to the important instructional practices discussed previously. For example, Speer and Wagner (2009) examined the knowledge required to use student conceptions to advance the mathematical understanding of the class in their consideration of the practice of analytic scaffolding. Charalambous (2010) examined how the mathematical knowledge of teachers impacts the choice, presentation, and implementation of tasks. Zodik and
Zaslavsky (2008) considered the knowledge teachers need to choose powerful examples and use them powerfully and paid particular attention to spontaneous examples. Fernandez (1997) studied how teachers use their knowledge in response to unanticipated student responses.

These studies provide insight into the dynamic activation and application of knowledge in the moment, during instruction, yet more needs to be understood about how knowledge is used in practice—and in particular, practice that fosters the kind of rich, connected understandings for which we strive. Thus, the literature on the mathematical knowledge needed for teaching gives rise to a number of questions that fall into three distinct categories:

- Are current frameworks for thinking of the knowledge of teachers in general and the mathematical knowledge for teaching robust enough? Do they make useful distinctions between domains of knowledge? Do these distinctions hold in a secondary setting?
- Do we understand enough about the role of mathematical knowledge during instruction? What aspects of a teacher’s knowledge and how it is used in the moment are not captured by the clinical settings in which teacher knowledge has been assessed? Have we examined experienced teachers enough?
- What role does a teacher’s mathematical reasoning play in interactive decision making? How is this reasoning different from and dependent on the various domains of teacher knowledge, particularly specialized content knowledge? What role does the reasoning of a teacher play in building teacher knowledge in various domains?

In an effort to add to our current understanding of the nature of mathematical knowledge needed for teaching, further analysis of how teachers use their mathematical knowledge during instruction is warranted. More studies situated in secondary classrooms with experienced teachers can add to the knowledge base.

**The Research Question**

This study was situated at the intersection of these questions and gaps by focusing attention on the use of mathematical knowledge in the instructional practice of secondary
teachers. In order to maximize the potential insight to be gained by this study, it was important to conceptualize a framework for understanding mathematical knowledge use in practice and for that framework to incorporate a useful conception of the domains of mathematical knowledge for teaching and how it is used in practice. A second important element of this study involved focusing on the knowledge use of teachers while engaging in teaching practices that have theoretical and empirical support for their effectiveness in developing the types of understandings for which we strive—namely, rich, connected understandings. Finally, it was equally important to situate this study in practice to examine teaching in the moment and the interactive decision-making on the part of the teacher. Thus, this study was designed to answer the question:

*How do experienced, secondary teachers use their mathematical knowledge during instruction? What mechanisms are used to translate that knowledge into practice?*

The next chapter aims to more clearly explicate the domains under study as well as the rationale for the structure of the study by focusing on the following questions:

- How can we identify important instructional moments in the teaching of mathematics? What are the moments at which demands are placed on a teacher’s mathematical knowledge as they make instructional decisions and implement instructional moves? What are the theoretical and empirical justifications for their significance?
- What are the approaches that distinguish productive and supportive teaching practices at these moments (ones that support the development of rich, connected understandings) from potentially nonsupportive ones?
- What do we already understand about the active use of mathematical knowledge in practice?
Chapter 2:

The Theoretically- and Empirically- Based Framework
In order to properly direct attention during this study, it was important to identify focal points during the design and delivery of instruction that afforded the researcher the opportunity to explore the teachers’ use of mathematical knowledge and reasoning during instruction. These focal points needed to be readily identifiable in the data and associated with teaching practices that supported the development of the rich, connected understandings for which we strive. It is important to note that while the instructional moves and teacher action at these focal points may or may not lead to the development of rich, connected understandings on the part of students, these focal points needed to offer the teacher the opportunity to design and deliver instruction that could potentially develop these understandings.

Through a review of existing literature and theoretical perspectives, the researcher developed the construct of critical instructional moments to identify these types of focal points for the study. Critical instructional moments are defined to be those times during the design and delivery of instruction, through a teacher’s actions, are likely to shape the nature of the mathematical activity of the student(s). They are critically important to the design and delivery of instruction that has the potential to develop rich, connected understandings in students. They involve students expression of their mathematical thinking or the teacher’s consideration of it. And most importantly for this study, these critical instructional moments are likely to place demands on the mathematical knowledge of teachers during the design and delivery of instruction because of the nature of the decisions they require teachers to make in these moments.
By contrast, it is important to note what elements of a classroom experience would not be considered to be critical instructional moments. These items are important elements of effective classrooms and teaching, but do not fit the criteria outlined in the definition of critical instructional moments just described. For example, administrative functions of the teacher—attendance, grades, and other recordkeeping—would not be considered critical instructional moments. These and other routine tasks of the classroom like reviewing answers to homework problems do not directly shape the mathematical activity of the student. Providing students with a demonstration of a mathematical procedure through direct instruction, giving tests or other summative assessments, monitoring students while they work independently on projects or problems or completing entry tasks with technology would not be considered critical instructional moments because they do not involve an active consideration of students’ mathematical thinking. And finally, rapport-building activities such as casual conversation, encouragement and critical feedback about non-mathematical aspects of a students’ performance like attitude and effort, and classroom management would not be considered critical instructional moments because they do not directly involve the mathematical activity of the students.

In the first part of this chapter, theoretical considerations of how students learn and the role of teachers in supporting the mathematical activity of students are used to support the identification of four types of critical instructional moments.

In the second part of this chapter, teaching approaches during critical instructional moments will be described. This discussion will focus on the nature of teaching practices for which there is theoretical and empirical evidence to suggest these practices support concept-building activity on the part of students—in other words, the development of rich, connected
understandings. These approaches emerge out of a consideration of the theoretical perspectives on learning and teaching outlined in the first part of the chapter and empirical studies from a variety of research perspectives. These approaches further focus attention on the elements of planning and implementing instruction which serve our ultimate goals for mathematical understanding for students and provide further evidence of the importance of those moments identified as critical.

In the final portion of the chapter, refinements to the conceptualization of MKT (Ball, Hill, & Bass, 2005; Ball, et al., 2008; Hill, Ball, et al., 2008) and a definition for a new construct, pedagogical content reasoning, will be developed out of theoretical and empirical considerations. The carefully refined notion of MKT and the conceptualization of pedagogical content reasoning form a basis for the consideration of a teacher’s use of mathematical knowledge during instruction.

**Part I: Theoretical Basis for Identifying Critical Instructional Moments**

The essence of mathematical proficiency as defined in *Adding It Up* (National Research Council, 2001) is the development of rich, connected understandings of the relationships among mathematical ideas, their application to phenomena in the world and their place in the structure of mathematics as a discipline. In order to equip teachers with the knowledge and skills to provide students with opportunities to develop these understandings, we must first consider how students learn and extract teaching approaches and practices from that understanding of students. Through this consideration of how students learn, types of critical instructional moments can be identified. These moments are points at which the teacher has opportunities to shape the
mathematical activity of the students and ultimately their opportunities to develop rich, connected understandings. It is on these types of critical instructional moments that this study will focus attention in an effort to understand how teachers use their mathematical knowledge at these pivotal points.

**Understanding Teaching**

The nature of the role of tasks in the learning process places a particular set of demands on teachers. In order to know what questions to ask students, what representations to engage them in considering, what counterexamples to pose in response to a student’s statement, or what problem to get students to solve, a teacher must consider the learning goal(s) of the lesson and the nature of current conceptions of the student(s). This dual focus requires continual attention throughout instruction. Understanding the conceptions of the students and their relation to the learning goals of instruction shapes the choice of task, the implementation or possible modifications of the task, the nature of support provided to students, and the ways in which the reflection by each student is facilitated.

This dual focus on the mathematics of students and the mathematics of the learning goals is best understood in terms of Simon’s model of the Mathematics Teaching Cycle (Simon, 1995, 1997). The Mathematics Teaching Cycle captures the dynamic interplay between a teacher’s mathematical knowledge, learning goals for students, anticipated learning, and observed learning (inferred from observations). A schematic representing the Mathematics Teaching Cycle (Simon, 1997) is provided in Figure 2.1. Given the ongoing cycle of observation, modification, implementation, and reflection that occurs during teaching, the model has no clear starting point. For the sake of discussion, start with a learning goal for students. From that goal, as Simon
describes, the teacher must construct a Hypothetical Learning Trajectory (HLT) that involves the activities that will potentially produce that learning. This HLT is constructed based on the teacher’s knowledge, conceptions, and beliefs of or about mathematics, pedagogy, and students (in general) as well as the teacher’s model of the students’ existing knowledge—that is, how the students currently understand the mathematics at hand. That HLT might involve problem posing or particular moves in the facilitating of discourse such as questions, examples, counterexamples, justification, and the like. Almost continually, the teacher is engaged in a process of re-articulation of the HLT as new understandings of student thinking emerge. Throughout this process, a teacher has to balance consideration of and response to the

Figure 2.1. Simon’s Mathematics Teaching Cycle, expanded. The arrows indicate direction of influence.
mathematics of students (as understood by the teacher) and the use of problems and discourse to
direct student attention towards the mathematical goals of the lesson. This model provides a
framework for understanding the considerations of a teacher as he or she seeks to design
instruction to foster a conceptual advance.

Since the Mathematics Teaching Cycle identifies several elements of instruction designed
to shape the mathematical activity of the student and explicitly identifies a teacher’s
consideration of the mathematical thinking of students—two central aspects of critical
instructional moments—the elements of teaching reflected in the model provide a basis for
identifying some critical instructional moments. Namely, the model suggests the critical
importance of the identification of learning goals, the selection of mathematical activities for
students, the interaction between the teacher and students around those activities, and the
adjustments made by the teacher in response to those interactions. For the teacher, task
selection becomes one of the primary mechanisms through which he or she influences the
mathematical activity of the students and as such warrants further discussion.

**Understanding the Importance of Tasks**

Learning is a complex phenomenon with a cognitive dimension and a social dimension,
and we must understand both the interactions in the classroom and the impact on the individual
learner to understand learning in ways that support the identification of critical instructional
moments. In this light, it is natural to examine the student’s opportunities for learning from an
interactionist perspective. One such perspective is found in activity theory, and given its focus
on the activity of the learner (both internal and external), it provides a sound perspective through
which we can account for the learning that occurs in classrooms and develop some insight into
the nature of the experiences that foster that learning. It also provides a theoretical basis for identifying the selection of tasks as a critical instructional moment.

Although tasks can take many forms such as a question to be answered, a contradiction to resolve, a representation to understand, or literally a problem in the traditional sense, consider first the ways in which well-chosen tasks support learning. Tasks mediate the activity of the students (Jonassen, 2002). For Davydov (1988a), the task sets the goal and the conditions for the attainment of that goal. That goal and those conditions give rise to a perceived need (Leont'ev cited by Davydov, 1988a) that provides the motivation for the cognitive activity of the student (Hershkowitz, Schwarz, & Dreyfus, 2001; Hung, 2001). In other words, tasks, through their definition of goals and initial conditions, can create a need on the part of students for the construction of a concept, the creation or novel application of a strategy, or the development of a modified understanding. Students’ efforts to meet this need become directed and intentional (Jonassen, 2002). These intentions arise out of differences a student observes in his or her environment or between what he or she believes is known and what he or she believes is needed in order to accomplish the goal. This intentionality of student activity serves an important role in the learning process as students engage in making sense of their activity and the directedness of their activity (towards the resolving the problematic nature of the task) leads them in the direction of the mathematical goals of the teacher who chose the task to support those goals.

**Understanding the Importance of Task Implementation**

While the selection of tasks plays a crucial role in shaping the mathematical activity of the students, the implementation of the tasks by the teacher can also play a significant role. The work of researchers in the QUASAR project provides empirical support for the importance of
task implementation. These researchers characterized differences in the demands tasks and teachers placed on students in the completion of those tasks in terms of levels of cognitive demand (Silver & Stein, 1996; Stein & Lane, 1996). These distinctions arose out of refinements made to the distinctions described by Doyle (1983, 1988). The level of cognitive demand is a useful way to discuss differences in the nature of support given to students in the completion of tasks.

The QUASAR researchers developed a framework for assessing the level of cognitive demand implicit in tasks by identifying four types of tasks: memorization tasks, procedures without connections tasks, procedures with connections tasks, and doing mathematics tasks. Memorization tasks were those that required students only to recall facts, definitions, or properties in order to complete them and had no connection to the underlying meaning of what was being reproduced. Procedures without connections tasks were those tasks that are generally algorithmic in nature. These tasks require students to apply a procedure in a predictable and easily recognizable setting with no connection to the underlying significance of the algorithm or appreciation for why it works. Each of these two lower levels of tasks emphasizes the importance of the answer rather than the development of any mathematical understandings of the process of getting the answer.

In contrast, procedures with connections tasks have features that afford students the opportunity to use multiple approaches, consider the conceptual underpinnings of more general procedures and encourage the development of connections between representations and methods of solution. Doing mathematics tasks are those tasks for which there are no routinized strategies and thus students are required to choose from multiple possibilities or combine existing
understandings to complete the task (Stein, Smith, Henningsen, & Silver, 2000). In short, tasks placing lower levels of cognitive demand on the students focus on the answer, can be completed with a readily identifiable way to proceed, and involve the application of recalled knowledge. Tasks placing higher levels of cognitive demand on students focus on the development of rich, connected understandings, might be completed using multiple strategies, and require important mathematical decisions to be made.

While the mathematical activity of the student is shaped by the particular tasks they are asked to complete, it is not merely dependent on the features of the task. Rather, their mathematical activity is also shaped by how these tasks are presented and implemented by the teacher (Stein, et al., 2000). For example, task features that might make the task able to be completed using multiple strategies might be ignored in the presentation leading students to see the task as an application of one particular strategy. Likewise, a task might afford students the opportunity to make connections between representations but those opportunities might be missed if the teacher does not make the consideration of those connections visible. Even tasks that do not have features that lend themselves to multiple strategies or representations might create opportunities for students to engage in demanding tasks such as reasoning, conjecturing, or interpreting.¹ The studies associated with the QUASAR project provide strong evidence for the impact on student learning of these kinds of emphases during instruction.

These studies found specific evidence of differences in the nature of the achievement among students and linked those differences with differences in the implementation of tasks. More specifically, students in classes with teachers participating in the training associated with

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¹ Borasi’s (1994) account of using student errors as springboards for inquiry provides a model for how that might be accomplished and is discussed in more detail in the section on the consideration of student responses
the project outperformed a comparable group from the National Assessment of Educational Progress study in the areas of conceptual understanding and problem solving (Silver & Lane, 1995). Furthermore, not only did the QUASAR students consistently enhance their conceptual understanding, critical thinking and communication skills—as demonstrated by their performance on pre- and posttests—but the researchers also observed the greatest gains in student performance in classes that had the largest percentage of tasks of high cognitive demand and the highest percentage of those tasks that were implemented in ways that required students to engage in high levels of cognitive processing (Stein & Lane, 1996). The findings of these studies suggest task implementation is a critical feature of instruction.

**Identifying Critical Instructional Moments**

As discussed, the theoretical perspectives provided by activity theory and the Mathematics Teaching Cycle along with the empirical findings of the QUASAR project provide a foundation for the identification of four types of critical instructional moments:

- Identification of learning goals
- Selection of tasks
- Implementation of tasks
- Observable student responses

A brief discussion of each of these components of instruction can adequately characterize the identification of critical instructional moments in a classroom setting.

While *learning goals* can involve the development of broader understandings over the course of the year, a unit, or a series of lessons, ultimately, they need to be focused on a single lesson or even a single exchange with a single student in order to be translated into instructional moves. These learning goals can be established in the planning in advance of instruction or even during the course of instruction. They might be explicitly articulated or tacitly presumed. In
either case, it is presumed that these learning goals can be brought to light through analysis of teacher cognition.

Tasks, as discussed previously, are defined broadly as questions to answer, examples or counterexamples to generate or consider, representations to create or interpret, or problems to solve. They, too, can be preplanned or spontaneous, recalled or created as they are applied to serve the identified or emerging learning goal. The implementation of tasks involves the presentation and any reframing of the tasks, the support provided in completion of the task, and the directing of attention to salient features during or after the completion of the task. Although there is ambiguity in distinguishing new tasks from adjustments involved in the implementation of tasks, the ambiguity was resolved through an iterative process in the data analysis phase of the study and is described in the methodology chapter.

The final category of critical moments involves observable student responses relating to mathematical ideas, problems, questions, examples, demonstrations, and the like. Student responses can take a variety of forms: verbal or nonverbal communication, written or oral, performance-based or conception-based, and questions or observations. Essentially, events in this category are any observable behavior that is a response to the mathematics under study. Examples include student answers to questions, student solutions to problems, questions asked by the student, processes described, explained, or justified, or some form of nonverbal communication such as pointing or drawing. These responses are potential indicators of the nature of student conceptions and that is what makes them critical moments during instruction. It is this set of observations that a teacher must consider to determine if goals are being met and
what modifications to goals or the nature of tasks must be made to move students toward desired understandings.

Summary

In this section, a theoretical and empirical consideration of how students learn through tasks and the role teachers can play in providing opportunities to students led to the identification of types of critical instructional moments: identification of learning goals, selection of tasks, implementation of tasks, and observable student responses were identified and described. They are critical because they are important elements of the design and delivery of instruction that are likely to shape the nature of the mathematical activity of the students in ways that support the development of rich, connected understandings. The critical nature of these moments is derived from the opportunity they present. The teaching approaches and instructional decisions that support the development of these rich, connected understandings at these critical instructional moments are discussed in the next section. It is the teacher’s thinking about these moments, the conclusions he or she might draw, and the use of mathematical knowledge to understand these moments and determine subsequent teaching moves that is of interest in this study.

Part II: The Critical Features of Instruction at Critical Instructional Moments

Identifying critical instructional moments during the course of instruction is an essential first step in the analysis of teachers’ use of mathematical knowledge during instruction. However, the critical instructional moments described in the previous section do not include distinctions among teacher actions that might support the development of rich, connected
understandings on the part of students and those that do not. Characterizing these distinctions establishes an important foundation for this study in two ways. First, consideration of the nature of these teaching practices leads to the formulation of an initial understanding about the types of demands that might be placed on a teacher during instruction. Second, the descriptions of these teaching practices will afford the researcher the opportunity to identify these approaches in the data and focus on the use of mathematical knowledge in those instances. The study aims to understand the use of mathematical knowledge in practice when that practice is supportive of the development of rich, connected understandings. When possible, comparisons with the use of knowledge in instructional practice that does not support rich, connected understandings will be made.

This section examines each type of critical instructional moment and outlines an approach, supported by the literature, for teachers to take to maximize the opportunities for students to develop rich, connected understandings of mathematics. Throughout these approaches, a common theme emerges. As teachers determine a course of action at each of these critical instructional moments, they are faced with a triune consideration of the existing understandings of the students, the mathematical learning goals of the teacher, and the potential path of mathematical development from one to the other. Throughout the analysis of the data, the researcher looked for evidence of this triune focus.

**Identifying Learning Goals**

A rich, connected understanding of mathematics involves making connections among mathematical ideas, among various solution strategies, and between existing understandings and new ideas. Connections among mathematical ideas might involve understanding how the
properties of some mathematical entity, such as linearity, manifest themselves in different representations—in numerical, graphical and symbolic forms. It might involve recognizing and making generalizations about the structural similarities among situations in which specific algorithms apply. This type of generalization might lead to a general understanding of notions like equivalence or what it means to solve an equation rather than simply knowing specific steps coming to a solution to particular types of problems. Connections among various solution strategies might involve recognition of how different strategies are similar and how they are different as well as why each one produces a valid result. Connections between existing understandings and new ideas might involve making an elaboration or enhancement to existing conceptions (assimilation) such as recognizing an arithmetic sequence as a discrete representation of a linear relationship. In many situations, connections between existing understandings and new ideas might involve a reconceptualization of how a student thought of the mathematical ideas prior to the new experience (accommodation). This reconceptualization occurs when existing conceptual structures are not robust enough to incorporate a new experience such as when students are asked to reconceive of the trigonometric ratios in right triangles as trigonometric functions.

While helping students establish these rich, connected understandings, a teacher must translate these broader goals into unit- and lesson-level goals as these notions are applied to particular portions of the content. Even with these goals translated into manageable chunks for a given class period, the identification of the set of component or prerequisite understandings for lesson goals is an essential step in the process of choosing or constructing tasks for students specifically designed to promote the development of those understandings. Each of these
translations from course goals to unit goals to lesson goals to component understandings requires the teacher to unpack the mathematics involved. This notion of unpacking learning goals was identified by Hiebert and his colleagues as an essential teaching skill (Hiebert, Morris, Berk, & Jansen, 2007; Hiebert, Morris, & Glass, 2003; Morris, et al., 2009). Others describe it as decompressing (McCrory, Floden, Ferrini-Mundy, Reckase, & Senk, 2012). For these researchers, well-specified goals lay a foundation for a careful examination of one’s teaching including assessing whether students are developing the desired understandings. Underspecified goals leave ambiguity when it comes to assessing the value of an activity or lesson and make it difficult for teachers to learn from instruction (Hiebert, et al., 2007).

As these goals are unpacked and specified to a level that allows them to be used for the selection or construction of tasks designed to promote those specific understandings, the teacher must simultaneously attend to the salient features of any concepts or procedures, the existing understandings of students, and the possible advantages to various sequencing options. For this study, the level of mathematical specificity of learning goals will be used as a distinguishing characteristic of teaching at the critical instructional moment of identifying learning goals.

Selecting Tasks

Once a learning goal is specified, the selection of a task becomes the next critical instructional moment to consider. Tasks that support the development of the types of connected understandings for which we strive have been identified by a variety of mathematics educators from different perspectives—curriculum developers such as those espousing the Japanese model of the Open-Ended Approach (Hashimoto, 1998) or the Dutch tradition of Realistic Mathematics Education (Gravemeijer, 1994) and researchers involved in large-scale research programs such
as Cognitively Guided Instruction (CGI) (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Fennema, Franke, et al., 1993) and in individual research projects (Ball, 1993; Doyle, 1983, 1988; Hiebert & Wearne, 1993; Simon, et al., 2004). Each of these studies provides insight into the nature of tasks that produce rich, connected understandings, enhance problem-solving and reasoning abilities, and develop conceptual understandings. A few of these efforts provide additional insight into how a teacher might choose tasks.

In her effort to articulate the nature of instruction and the role of the teacher in building connected understandings of mathematics, Ball (1993) identifies the selection of the representational context as one of the more important ways a teacher determines the medium through which the mathematics is learned. For Ball, the representational context includes the domain of the investigation (e.g., the problem, the representation or example, and the mathematical concepts they embody) as well as the meaning and discourse associated with that investigation. She identifies two initial considerations in choosing or constructing a rich representational context for student learning. First, a representational context should bring forth salient features of the concept(s) under study, not just the superficial or procedural ones. Second, the selection of a representational context should take into account existing student conceptions and anticipated student learning.

The impact of this dual consideration is best understood in terms of the notion of progressive mathematization found in the discussion of the Dutch approach called Realistic Mathematics Education. Students engage in resolving problematic situations that embody the key mathematical features and can support the development of increasingly formal mathematical conceptions. The students’ informal ways of reasoning about the situation are supported by
models of the specific context. Models of the situation are representations of the mathematical relationships in the problem but they are closely linked to the context of the problem. They support the solving of the problem and lay an informal foundation for more formal understandings. As the process of guided reinvention occurs, these models of the situation evolve and get replaced by models for the mathematics. Models for the mathematics are more mathematically sophisticated representations, are more removed from the context in which they were derived, and support the students’ ways of reasoning in a more abstracted form. In this way, models for support the development of more formal mathematical conceptions. In this sense, guided reinvention of the mathematics takes place as students move through a process of progressively developing more mathematically rich conceptions—progressive mathematization. It is with an eye towards that process of reinvention that teachers can design contexts for the learning of mathematics by asking the question, “Given existing understandings, how could a student have invented this?” (Gravemeijer & Doorman, 1999). This question embodies the triune consideration of the mathematics of the student, the learning goals of the teacher, and the anticipated path to the mathematical learning goals and provides teachers a useful way to conceive of task selection.

Thus, at the critical instructional moment of choosing a task for students, the teacher must envision a task that will engage students in ways that will lead to the desired mathematical development while considering what will be accessible to students. Choosing rich representational contexts that embody the essential elements to understand the concept and envisioning a path for students to move from existing conceptions to new, more connected understandings leads to the selection of tasks that will facilitate the attainment of our learning
goals for students. In short, well-chosen tasks must possess three critical features: They must be problematic, conceptually oriented, and accessible.

**Well-chosen tasks must be problematic.** Well-chosen tasks also must create a perceived need on the part of the learner and this is also part of the problematic nature of them. Regardless of the form of the task—a question, a problem, or an activity—the goal for the student is to complete it. Any initial lack of clarity about the path of resolution represents the difference between what the student believes he or she knows and what he or she believes is needed in order to solve the problem. This awareness forces the student to encounter choices to be made about paths of analysis or solution strategies, and these choices create issues to be resolved by the student. The cognitive activity of the students gets directed towards meeting this perceived need and stimulates the student to consider aspects of the mathematical relationships as he or she currently conceives of them. As the learner explores and works to meet this need, his or her activity must be modified to account for the constraints of the context, the relationships between the elements of the problem context, the apparent contradictions, and other properties existing independently of the problem itself—that is, to fill the gap between what the student believes he or she knows and what he or she believes she needs to know in order to solve the problem. According to Leont’ev, students must adapt their activity (test, and if necessary, modify their conceptions) to account for these conflicts (Davydov, 1988a).

**Well-chosen tasks must be conceptually oriented.** The work of Simon and his colleagues (Simon, Tzur, Heinz, & Kinzel, 2000, 2004) helps us understand how these adaptations result in new understandings and point to another characteristic of well-chosen tasks. As described by these researchers, new conceptions evolve out of a learner’s “goal-directed
activity and natural processes of reflection” (Simon, et al., 2004, p.318). As learners engage in a mental activity sequence intended to achieve some goal (the resolution of a well-chosen task), they make judgments about the result and effects of those activities. As the learner reflects on these actions and their effects, an abstraction of the activity-effect relationships can occur, which coordinates two existing conceptions and forms the basis for new conceptions. Thus, well-chosen tasks are designed to produce the desired conceptual advance on the part of the learner.²

Well-chosen tasks must be accessible. Finally, because student activity is inherently dependent on students’ existing ways of knowing (Davydov, 1988a; van der Veer & Valsiner, 1991; Vygotsky, 1978), well-chosen tasks must be accessible to students through existing understandings. Without this accessibility, the task would never appear problematic or allow students to engage in activity sequences leading to a conceptual advance. Other theoretical perspectives provide a rationale for the importance of the accessibility of tasks as well. In the tradition of another interactionist perspective, enactivism, Pirie and Kieren (1994) posit the foundational nature of what they refer to as a student’s primitive knowing in the growth of mathematical understanding. If the prior knowledge of the student does not match the primitive knowing required to engage in the activity, the student cannot make the intended conceptual advances.

Pirie and Kieren’s model has been used by researchers as a way to map the growth of student understanding (Borgen & Stan Manu, 2002; Kieren, 1990; Kieren, Clavert, Reid, & Simmt, 1995; Pirie & Kieren, 1994; Pirie & Martin, 1997; Towers & Davis, 2002). As such, these studies provide rich descriptions of the growth of student understanding including the

² While Simon and his colleagues would not align themselves definitively with activity theorists, there are notable similarities in their approaches to understanding learning, their views of the importance of goal-directed activity arising out of the student’s existing conceptions, and their account of learning through tasks.
importance of primitive knowing. For example, in describing the growth of student understanding of linear equations, Pirie and Martin (1997) emphasize how the teacher had to make explicit the primitive knowing required to work productively in the domain of the activity about solving linear equations. He did this by providing an accessible metaphor for the students and by referring them to the goals of an activity from the previous day. Their primitive knowing becomes the basis for the students’ engagement with the activity and the expression of that activity that leads to the growth of mathematical understandings. Even if the primitive knowing has to be made explicit by the teacher, well-chosen tasks are ones for which students possess the necessary primitive knowing to engage in the task.

**Summary of the nature of well-chosen tasks.** With such a pivotal role, tasks must possess three critical features to support the development of rich, connected understandings. First, tasks must be mathematically rich, yet problematic. Second, as tasks set the goal and conditions for the attainment of that goal, they must engage students in activity sequences specifically designed to promote a particular conceptual advance. And third, tasks must be designed to encourage students to use their existing understandings to engage in the completion of the tasks. In short, learners must appreciate what makes the tasks problematic thus forming the basis of the perceived need, and they must be able to engage in activity sequences for which they have existing conceptions. With these characteristics in place, rich, connected understandings of mathematics can emerge. For this study, the degree to which a task is selected with direct consideration of the learning goal, the accessibility of the tasks to students, and the potential for the task to engage students in the mathematical activity that will lead to the
conceptual advance will be used as distinguishing characteristics of teaching at the critical instructional moment of selecting tasks.

**Implementing Tasks**

Once a task is chosen, a teacher must implement the task in ways that continue to position the student as the sense-maker and the mathematical decision-maker. That implementation might involve a decision not to intervene at a given moment as much as it might involve a particular comment, question, or other overt response. Whatever the case, in order for conceptual advances to occur, the learner needs to understand what is problematic about the task, work to resolve the discontinuities with existing understandings, and reflect on his or her activity and its impact. The necessary action on the part of the student defines, in large measure the principles guiding task implementation for a teacher: (a) directing student attention (if necessary) to what makes a situation problematic, (b) supporting students’ engagement in the mathematical activity of the task in ways that lead to conceptual advances, and (c) facilitating reflection on the activity and how to resolve the problem. Each of these involves the consideration of existing conceptions of students and the mathematical learning goals.

**Directing attention to what is problematic.** If students do not see what is problematic about a situation, then there is limited opportunity for their activity to be motivated by the perceived need. Although a well-chosen task typically makes the problem clear to students, there are times in the course of completion of the task that a student might need attention redirected to continue to advance mathematical development. Reiser (2004) describes the value of making student work more problematic (to the student) by directing student attention to
features of their work, the problem, or the mathematics that represents a mathematical inadequacy in their thinking or would lead to some mathematical discontinuity or error. These features require students to make choices or at least give consideration to particular aspects of the problem. By directing student attention to those features, the teacher (or technological tool, or fellow student, or manipulative, etc.) is introducing greater complexity than the student would have otherwise faced, which actually increases the potential for conceptual advance by broadening the scope of concepts under consideration and requiring possible accommodation.

The potential benefits of problematizing student work can also be understood in terms of two of the theoretical perspectives discussed previously. First, problematizing student work directs student attention towards contradictions or elements of a problem or its solution that need to be reconciled—that is, creating a perceived need. These contradictions form the basis of the student’s activity (Davydov, 1988a). Without recognition of contradictions, conceptual development is limited. Second, the construct of folding back as originally introduced by Pirie and Kieren (1994) can be used to understand the potential value of problematizing student work. Pirie and Kieren describe this feature of their model as one of the key mechanisms by which students make conceptual advances. As a student who is operating at a higher level of understanding encounters a problem that cannot be solved given the current nature of their conceptions, that student must re-engage in activity at a lower level in light of this limitation of their knowledge. If the student does not recognize the need to reconsider his or her conceptions, then the teacher must direct student attention to the limitations—that is, problematize student work (L. Martin, 2000). In doing so, a student’s more general conceptions of action, results, and properties can be re-formed to account for this prior limitation. Martin emphasizes the need for
teachers to use language that allows for a variety of responses and encourages the student to 
explore as opposed to being prescriptive about what the student needs to consider. In this way, 
the teacher is facilitating the process of connecting new understandings to existing ones through 
the process of folding back and problematizing student work.

How a teacher provides direction to students who may not be engaged in mathematically 
productive activity is a critical feature of instruction during task implementation. Problematizing 
student work encourages mathematical activity on the part of the student or students and 
provides a useful way to distinguish among teaching moves in this study.

**Supporting student activity.** A second critical dimension of implementing tasks 
involves providing support to students who cannot complete the task independently. In spite of 
the teacher’s effort to provide students with tasks that can be completed using existing 
understandings, students sometimes will require assistance to complete the task. By definition, 
when a student is working in his or her zone of proximal development (van der Veer & Valsiner, 
1991; Vygotsky, 1978), the student will need assistance. How that assistance is provided 
determines, in large measure, the mathematical development of the student at that moment. Two 
distinct lenses provide insight into how to provide that assistance in productive ways.

The first such lens is a form of scaffolding. *Scaffolding* (Wood, Bruner, & Ross, 1976) 
has frequently been used as a metaphor for describing the role of knowledgeable others in 
providing assistance as efforts are made to connect existing understandings to new ones. It 
serves as a particularly useful metaphor for discussing task implementation that supports 
conceptual advances, and it is important to distinguish between forms that foster mathematical 
development and those that potentially limit it.
Wood and his colleagues (1976) provide a description of six types of assistance: recruitment of the child’s interest, reduction of the degrees of freedom, direction maintenance, marking critical features, frustration control, and demonstration. While recruitment of the child’s interest and frustration control involve affective consideration primarily, the other four types of assistance can be classified as forms of structuring and could easily lead to instruction that potentially limits the conceptual development of students. For example, as a teacher reduces the degrees of freedom for students, he or she is eliminating decisions that the student has to make. If those choices are made for the student in an effort to reduce the degrees of freedom, this reduction leads to fewer mathematical decisions on the part of the student and therefore, fewer potential opportunities to recognize a need to assimilate new understandings with existing ones or to modify existing conceptions to accommodate new ones. The same could be said for marking critical features of the task and demonstration. If too much direction is provided for students to attend to particular features of the task or to perform particular routines, then it is unclear whether the student has the conceptual structure to understand the critical dimensions of the task. Successful completion of the task might merely be a function of the student’s ability to mimic the actions of the teacher in some sort of mechanical way and would not necessarily lead to a conceptual advance on the part of the student.

In contrast, Reiser (2004) provides a way to think about structuring that more readily supports learners and their learning. Like Wood and his colleagues, structuring involves supporting the activity of students by simplifying the problem by providing assistance in planning, organizing, or monitoring, but this assistance does not have to lead to limited opportunities for conceptual advancement. Consider the distinction between empirical and
theoretical learning made by Davydov (1988b). As described by Karpov and Bransford (1995), Davydov’s distinction implies that empirical learning involves the student making generalizations from specific examples, whereas theoretical learning involves the students being provided the generalizations and then applying it to specific examples. While empirical learning would seem to foster conceptual development, involve less structuring on the part of the teacher, and more active construction of knowledge on the part of the learner, a closer look suggests quite the opposite and provides a picture of structuring that supports the development of connected understandings.

Suppose an algebra teacher is teaching his or her students about solving equations. In the case of empirical learning, the students would be provided procedures for solving various types of problems—for example, solving linear equations, or solving quadratic equations. From those specific procedures, the teacher would hope or even expect that the student would develop an understanding of why each procedure worked, when it is appropriate to use it, how to interpret the results of the procedure—that is, making generalizations that apply to the category of things known as solving equations. However, knowledge of specific algorithms may or may not help the student understand how to handle situations that are mathematically similar—for example, solving rational equations. In the case of theoretical learning, the teacher might provide general principles to be used in solving equations such as maintaining equivalence or finding values of the variable(s) that make the statement true. These more general principles are essentially the concepts behind the procedures and provide students a structure for understanding solutions to all types of equations. Theoretical learning amounts to structuring to support the sense making on the part of the student.
Providing appropriate structure to students is a critical mechanism for providing support for task completion. To provide this kind of structure in ways that supports the students’ conceptual development requires an appreciation of the existing understandings of students and a consideration of those understandings relative to the learning goals.

A second lens through which to view a teacher’s support of a student’s mathematical activity is that of cognitive demand. The QUASAR researchers identified several features of instruction in which high demand tasks were implemented in high demand ways. Their findings suggest that if high-level tasks are implemented in these ways, then the demand level of those tasks will be maintained through instruction. These researchers found that connections to prior knowledge, scaffolding without reducing complexity, sufficient time allowed, modeling of high performance, and sustained (teacher) press for explanation and meaning were each present in over 75% of the tasks when high level was maintained throughout the implementation (Henningsen & Stein, 1997). The most often-cited factors associated with the decline in demand were the removal of complexity of the problem and a focus on correctness and completeness rather than understanding (Henningsen & Stein, 1997). These findings serve as a guide for the types of support a teacher can provide in order to engage students in the higher level thinking required to build rich, connected understandings.

Supporting student activity in the context of well-chosen tasks is an essential responsibility of a teacher at the critical instructional moment of task implementation. The nature of that support can change the nature of the engagement of the student and the potential for conceptual advances. Providing that support through scaffolding that does not reduce the complexity of the problem, modeling high performance in ways that do not routinize the
solution, and making connections to prior knowledge afford the student the best opportunity for mathematical development and require continued focus on the existing understandings of students, the mathematical goals of the task and the interventions that will foster the mathematical development of the student in the direction of those goals. For this study, these characteristics of support will be used as a distinguishing characteristic of teaching at the critical instructional moment of task implementation when students need the help of the teacher.

**Using reflection and justification.** The third and final critical dimension of task implementation involves the use of reflection. As previously discussed, reflection on one’s activity and its effect is an important component of conceptual development. Students may not spontaneously engage in reflection and it is essential for teachers to work to ensure they do so. This effort can take many forms. When teachers ask students to express, explain, and justify their thinking, they are asking students to reflect on what they did and why they did it and reconcile those actions within the mathematical understandings of the class. This required justification serves multiple purposes: Students are directed to reflect on their activity (Simon, et al., 2004); to construct mathematical arguments and connect explanations to existing understandings (Henningsen & Stein, 1997); to develop a sense of the standards of the discipline through the norms established by what is accepted as valid (Yackel and Cobb, 1996); and to use student ideas as “springboards for inquiry” (Borasi, 1994, p. 167). In fact, the researchers from the QUASAR project identified the sustained press for justification or explanation as a key feature of teaching when tasks were implemented in cognitively demanding ways for students (Henningsen & Stein, 1997). Clearly, requiring students to justify and explain their thinking shapes the nature of the students’ mathematical activity.
One can understand the learning that can occur as a result by considering the mental activity associated with this sharing of ideas in this way. As the teacher directs students to express or justify their thinking, the focus of the activity shifts from the response to the task to the strategy used to resolve the problem associated with the task. When justifying one’s work, the perceived need shifts from choosing a path to establishing an argument for that path. When multiple solution paths are considered, the perceived need shifts from resolving discontinuities arising out of one’s own work to resolving any inconsistencies (contradictions) or curious results from others’ work, and to reconciling the multiple successful strategies. Out of these perceived needs, goal-directed activity arises and conceptual advances can be made as the student resolves those conflicts.

These conceptual advances cannot happen without reflection. In order to engage in the reflection on activity-effect relationship, the student must be faced with the choice to act, to recognize the conditions under which that action makes sense, and reflect on the results of his or her action under those conditions. If a student is not engaged in this reflection independently, then he or she must be guided to do so. If the guidance is too directive, the reflection process is circumvented and the learner might miss out on the opportunity to make a conceptual advance. On the other hand, if the learner is reflecting but not attending to aspects of the task and his or her action that can lead to productive understandings, then the learner can flounder needlessly. Thus, the teacher must maintain a bifocal perspective on the conceptions of the student and the mathematical goals of the lesson even in the process of encouraging reflection. For this study, the manner in which a teacher uses reflection and justification during instruction will be a
distinguishing characteristic of instruction and a focal point for understanding the use of mathematical knowledge in these moments.

**Considering Observable Student Responses**

As a critical instructional moment, observable student responses involve students’ multi-modal responses to questions posed, examples presented, counterexamples considered, problems posed, and observations made. Consideration of these responses and the inferences a teacher might make about the understandings of the students is central to the teacher’s consideration of each of the other three domains of critical moments. This consideration forms the basis for how learning goals are operationalized in a given lesson, what makes a task well chosen and how best to intervene with students. It is at the heart of a productive discourse in the classroom. A number of studies provide support for the importance of consideration of student responses and how a teacher might use student responses constructively in the course of instruction. This description will serve as a foundation for making distinctions in knowledge use during the analysis phase of this study.

The most extensive and conclusive body of research related to student thinking and its potential for impacting classroom practices and ultimately student learning is the work of the Cognitively Guided Instruction (CGI) project. This project was designed to examine the impact of teacher knowledge of student thinking on their instructional practice and their students’ achievement (Carpenter & Fennema, 1992; Carpenter, Fennema, Peterson, & Carey, 1988; Carpenter, et al., 1989; Fennema, Carpenter, Franke, & Carey, 1993b; Fennema, et al., 1996; Fennema, Franke, et al., 1993; Peterson, Carpenter, & Fennema, 1989; Peterson & et al., 1991;
Peterson, et al., 1991). The project included an intensive summer professional development program, assessments of teacher knowledge of student solutions to addition and subtraction word problems before and after this summer program, and follow-up observations, interviews, and other interactions during subsequent school years.

While the focus of the project was on developing the teachers’ knowledge of student thinking and examining changes in classroom instruction and student achievement, the case studies associated with the project (Carpenter & Fennema, 1992; Fennema, Carpenter, et al., 1993b; Fennema, Franke, et al., 1993; Peterson, Carpenter, et al., 1989) and a comparative study of CGI teachers and their peers reveal important differences in the ways in which teachers taught the addition and subtraction word problems and the concepts associated with them as well as significant differences in student outcomes.

The first set of important differences involves the nature of the content and structure of the CGI classrooms compared to their non-CGI peers (Carpenter, et al., 1989). The study of these 40 teachers showed that CGI teachers changed their instructional practice in significant ways during the year following the summer course. They more frequently posed problems for students, encouraged multiple solution strategies, and elicited student explanations of solution strategies and processes used. By carefully choosing problems, teachers required students to make decisions about what method or manipulative aid to use to help them solve the problem as they used existing understandings and ways of knowing to work with the problem context and to produce solutions. By asking students to explain their solution strategies, teachers were holding students accountable to making these higher levels of cognitive effort and provided students opportunities to make connections between various methods and approaches. As the researchers
evaluated the student performance in these classes, the students in CGI classrooms outperformed their non-CGI counterparts in the areas of number facts, problem solving, reported understanding, and confidence. The performance in number facts is especially enlightening given that the non-CGI teachers spent more time on number facts and review than did the CGI teachers. These findings establish the importance of consideration of student thinking.

The second set of differences the researchers observed pertained to how the CGI teachers used their research-based understandings of student conceptions of these operations concepts. Even though the CGI teachers consistently posed more problems and elicited student explanations of their thinking more frequently than their non-CGI counterparts, the researchers noticed differences in the ways in which CGI teachers used those student-generated solutions and the subsequent performance of students on problem-solving measures. These differences occurred in the context of similar problems being used by these teachers and provide insight into how a teacher can use student responses productively.

For some CGI teachers, after listening carefully to student explanations of solution strategies, they imposed their preferred strategy without allowing students the opportunity to make sense of the connections between various strategies. In these cases, the teacher directed the students to make sense of particular aspects of the solution strategies and relieved students of the burden of doing the mathematical sense-making of why one strategy might be more efficient or effective than another. The students of these teachers performed less favorably on the problem-solving measures (Fennema, Carpenter, et al., 1993b).

For other CGI teachers, the students were encouraged to present multiple strategies and given opportunities to make connections among those strategies and to other mathematical ideas
such as place value, multiplication, and partitioning. These students were engaged in solving these problems and allowed to use their own conceptions to build new ones. The students were encouraged (and allowed) to make sense of the various methods and this fostered the development of the mathematical connections. These students performed better on the problem-solving measures (Fennema, Franke, et al., 1993).

Other studies provide additional insight into specific ways to use student responses during instruction to foster rich, connected understandings. Borasi (1994) examined the ways in which teachers used student errors as learning opportunities. Through her observations of classroom interactions, she examined the ways in which teachers used student errors as launching points for whole-class investigations. The teachers in this study turned the responsibility of understanding limitations of their thinking to the students as they analyzed incorrect definitions, debugged errors on homework questions, and followed an errant theorem to its logical conclusion. Perhaps more uniquely, these teachers also used errant thinking to pursue mathematics beyond the scope of the original problem as the students dealt with an unresolvable contradiction. While this exploration did not support the learning goals for that class, it presented an opportunity for the teacher to have students pursue mathematical relationships that supported other goals such as an understanding of mathematics as logical system of thought.

Facilitating a consideration on the part of students to understand what about a student’s work is “in error” is at the heart of making student work problematic because it involves an insistence that students reconcile this error with existing understandings. The students of the teachers in Borasi’s (1994) study engaged in constructive doubt, challenging problem solving, and mathematical exploration which in turn led them to consider mathematics as a discipline,
experience the need for justification, and verbalize their mathematical ideas. As this study demonstrates, putting the responsibility of reconciling student errors on students, as opposed to having the teacher provide the explanation, creates greater opportunities for students to connect these new understandings with existing ones and build stronger conceptions of mathematics as a discipline.

Parallel conclusions can be drawn from the work of Fernandez (1997). In her case study of teaching episodes of nine secondary teachers, she considered alternative solution strategies and student errors, difficulties, or misinterpretations in the category of unanticipated student responses. In her analysis, she identified four distinct ways these teachers used their knowledge of mathematics to engage students in a discourse about the mathematics of their response. The researcher saw patterns of response to students taking the following forms: generating counterexamples, following through to a logical conclusion, considering a simpler or related problem, and incorporating a student’s method. In each of these approaches, the teacher used the student response as a springboard for inquiry (Borasi, 1994) into the mathematics providing students an opportunity to develop deeper, more connected understandings of concepts and procedures and to reflect, explain, and justify their work.

These research studies reinforce the status of observable student responses as a critical instructional moment and provide some insight into how teachers should consider those responses in the design of instruction. Whether student responses are recognized as typical and predicted in planning or spontaneously revealed and unanticipated, they can be used not just to further the understanding of the individual who gave the response but can also be made visible and used to potentially further the understanding of the entire class. To do so, a teacher must
recognize productive errors, insights, representations, and methods teased out of student responses. The extent to which teachers recognize and use student responses as opportunities to develop rich, connected understandings will be examined among the teachers in the study. The use of mathematical knowledge on the part of the teacher to understand these student responses and construct useful approaches to using those responses to deepen the understanding of students will also be a focal point of the analysis.

**Summary—the Image of Instruction**

This section has outlined a number of teaching practices associated with using critical instructional moments in ways that support the development of rich, connected understandings on the part of students. These features of instruction are what make critical instructional moments critical.

Not only do teachers need to identify learning goals and select tasks to support those learning goals, they need to specify lesson-level learning goals to a degree that allows them to assess the nature of student understanding relative to that goal and to identify tasks to support the attainment of that goal. The identification of a specific, detailed and unpacked set of learning goals can form the foundation a teacher uses to guide instructional decisions and shape the mathematical activity of students in ways which support the development of rich, connected understandings. In contrast, identified goals that are superficial, procedural, or non-specific provide more limited guidance to teachers during the design and delivery of instruction and potentially leading to instruction that supports few opportunities for students to develop rich, connected understandings.
Likewise, selecting well-chosen tasks, ones that are accessible to students, problematic, and conceptually oriented towards the attainment of those learning goals requires a triune focus on the mathematics of the students, the mathematics of the learning goals, and the mathematics of the task as a path from one to the other. Tasks selected in this way orient the mathematical activity of students towards the building of these rich, connected understandings by creating a perceived need that motivates the formation of new concepts and connects new ideas to existing ones. In contrast, tasks that are not accessible to students using existing understandings offer limited opportunity for students to connect their existing understandings to emerging ones. If selected tasks are not sufficiently problematic or conceptually-oriented, the process of completing the task is at risk of becoming more about practicing a routine procedure than about making sense of the mathematics.

Furthermore, implementing tasks in ways that foster the mathematical development of students involves eliciting explanation, reflection, and justification from them, problematizing their work, directing student attention productively, supporting student work by structuring their thinking and maintaining the cognitive demand of the tasks. Each of these approaches requires a continual consideration of the emerging understandings of students, the learning goals of the activity, and potential paths of development from one to the other. Even if tasks are well-chosen, if they are not implemented in ways that maintain the position of the student as the mathematical decision-maker, then the opportunities a student has to develop these rich, connected understandings are limited.

And finally, when the responses of students are considered with this same triune focus—on their mathematics, the mathematical learning goal, and the potential path from one to the
other—and used to make instructional decisions, teachers support the development of rich, connected understandings on the part of the student. However, if student responses are not used as an opportunity to develop an understanding of their mathematical conceptions or they are not used to shape the instructional decisions, then developing the students’ rich and connected understandings of mathematics would seem less likely. These types of instructional approaches and decisions based on the triune focus on the mathematics of the student, the mathematics of the learning goal, and the potential path from one to the other form the basis for identifying these instructional moments as critical.

Herbst (2003) identified this triune focus as tensions facing each teacher. The navigation of these tensions shapes instructional decisions and provides a basis for understanding instructional decisions (Herbst, 2003). This simultaneous consideration of the mathematics of the student and the mathematics of the lesson places significant demands on the mathematical knowledge and reasoning of the teacher during instruction. Understanding how a teacher uses his or her mathematical knowledge and mathematical reasoning to make sense of student thinking, mathematical tasks, and mathematical goals at these critical instructional moments represents an important dimension of developing these skills and understandings in teachers and the focus of the analysis of this study.

**Part III: The Construct of Pedagogical Content Reasoning**

As discussed previously, existing conceptions of the mathematical knowledge for teaching do not take into account the use of the knowledge in practice. Mason and Spence identified this type of knowing as knowing to act in the moment (1999). The image of
instruction derived from the literature requires a multifaceted set of active, in-the-moment considerations—the mathematics of the student, the mathematics of the learning goals, and the potential path from one to the other—during the planning and implementation of instruction. Maintaining this triune focus and making instructional decisions involving task selection, task implementation, and use of student responses primarily requires the application of what Ball and her colleagues would call specialized content knowledge. How that knowledge gets used in practice is of primary interest in this study.

In the context of teaching outlined in the previous section, a teacher must come to understand how a student or students are understanding the material, what essential understandings are missing, what sequence of understandings would most likely support the desired mathematical development, and what tasks (from questions to problems) might engage students in the activity that will develop these understandings. Each component of this process involves the triune attention to and working with the mathematics of the students, the mathematics of the learning goals, and potential paths of mathematical development. It is hypothesized for the purposes of this study that some of these demands are met by drawing on existing knowledge, whereas others require the teacher to reason his or her way to an instructional understanding. This reasoning takes place in the planning of instruction and in the spontaneous adjustments that might be made during the course of instruction. Some of this reasoning involves pedagogical thinking, while a significant portion involves reasoning mathematically about student understandings and learning goals.

The researcher has identified this mathematical reasoning to make instructional decisions as pedagogical content reasoning. It is a way to understand how a teacher reasons
mathematically during the planning for and implementation of instruction. The term *pedagogical content reasoning* captures the unique nature of this reasoning due to its blending of mathematical and pedagogical considerations, knowledge, and reasoning. It is a parallel construct to Shulman’s pedagogical content knowledge, which categorizes the accumulated understandings of teachers for the teaching of mathematics. Understanding the nature of various dimensions of pedagogical content reasoning serves as a primary focus of this study and characterizations of this domain of mathematical knowledge for teaching should necessarily evolve as the data are collected and analyzed. As a starting point, the researcher has hypothesized the role of pedagogical content reasoning through careful consideration of research, theory, and practice.

Mathematical reasoning is the mental process by which principles of logic, mathematical justification and mathematics are used to identify patterns, generate hypothesis, produce examples and counter-examples, produce supporting arguments, and evaluate conclusions. Pedagogical content reasoning for mathematics instruction is the mental process by which mathematical reasoning is used to understand the mathematical thinking of students, the component mathematical understandings of a mathematical concept, or the ways in which particular mathematical understandings might be used to solve problems or complete tasks. Pedagogical content reasoning for mathematics instruction also involves the application of those understandings to instructional decisions and the development of instructional moves. As such, it involves mathematical reasoning about students, mathematics, and tasks and the application of those conclusions to the design and implementation of instruction. It also hypothesized to be closely related to the accumulated specialized content knowledge as well as a mechanism for
building specialized content knowledge. The model of teacher knowledge in action provided in
Figure 2.2 represents the role of pedagogical content reasoning plays in converting mathematical
knowledge for teaching and observations and data from the classroom into instructional moves.
Figure 2.2. A model of teacher knowledge in action.
The Model Explained

The model represents the cycle of mathematical considerations of teachers during the planning and delivery of instruction. It is difficult to define a starting point, but it seems reasonable to assume that instruction begins with the establishment of a learning goal. As discussed previously, the formation of that goal involves the consideration of the context of the learning as well as the mathematical understandings of the students and the mathematical learning goals of the teacher. Each of these considerations is shaped by the knowledge and belief base of the teacher, and when taken together, they require some reasoning on the part of the teacher to be transformed into instructional practice. As instruction takes place—that is, the teacher and the students interact around mathematics content, the teacher receives additional information from those interactions. The knowledge, conceptions, and beliefs of a teacher act as a filter through which the collection of data from the classroom is processed, assimilated, and privileged. At this point, the teacher needs to make mathematical sense of the observations from the class—for example, a student’s explanation of an alternative method—and this mathematical analysis would require the use of mathematical knowledge and/or mathematical reasoning. Once mathematical sense is made of those observations as they relate to the conceptions of students, the mathematical learning goals of the lesson and potential paths to mathematical development, then these understandings get translated into instructional practice and the cycle begins anew. As a teacher reflects on instructional interventions and their results, the knowledge, conceptions, and beliefs can grow, evolve, and change.

In this model, pedagogical content reasoning is the content-oriented reasoning about one’s knowledge and the observations gathered from interactions with students and the
application of that reasoning to the design and delivery of instruction. In short, it is the process of operationalizing knowledge into practice. The following tasks of teaching are hypothesized to involve pedagogical content reasoning:

- identifying learning goals and subgoals
- selecting an appropriate task from known alternatives
- designing prompts for tasks
- sequencing of tasks
- knowing when to act
- determining features of a task to which to direct student attention
- determining what to make problematic when a student is not engaging in productive mathematical activity
- reframing a task to make it more accessible to a student without diminishing the mathematical activity in which he or she will engage
- building mental representations of student(s) mathematical conceptions
- making mathematical sense of a student’s solution method, alternative representation, or explanation
- mapping a mathematical progression from existing understandings of students to desired understandings
- constructing or modifying tasks to support mathematical goals
- unpacking a task—what mathematical thinking is required of students, what features of a task accentuate what features of concepts, what is the mathematical significance of the task relative to the learning goal(s)
- unpacking a learning goal—what are the component understandings inherent in the goal, what are prerequisite understandings for the learning goal, how should component understandings be sequenced

These activities involve the active application of mathematical knowledge and reasoning to problems of practice and the design and delivery of instruction.

There are some initial distinctions between the mathematical knowledge for teaching that might be involved at each of the critical instructional moments and the nature of the pedagogical content reasoning that can occur at those moments. These distinctions form the basis for identifying pedagogical content reasoning in the analysis of the data. They were derived from a consideration of the domains of mathematical knowledge for teaching as accumulated knowledge and the demands on a teacher’s knowledge as he or she uses that knowledge to make
instructional decisions at each critical instructional moment. Distinctions are made between the types of mathematical knowledge for teaching a teacher might possess and draw on, and the types of mathematical activity that might be required to design and deliver instruction aligned with the image of instruction presented in the last section. Each critical instructional moment is considered separately.

**Distinctions between Knowledge and Reasoning while Identifying Goals**

In Table 2.1, the knowledge and reasoning potentially involved in the establishment of learning goals for an activity sequence is detailed. An activity sequence could span multiple class periods, a single class period, or some portion of a class period. Suppose a teacher (suppose it is a female teacher) is planning an activity sequence to develop rich, connected understandings of single-variable, absolute value, linear inequalities. To establish the learning goals for the lesson, the teacher might draw on several forms of mathematical knowledge. In no particular order, she would need to consider the concept and her own understandings of it. She might have previously engaged in such an analysis of the concept or she might use the resources in her curriculum to identify component understandings. She might ask herself, “What does it mean to understand single-variable, absolute value, linear inequalities?” and add to the list of component understandings. She might consider how methods for solving single-variable, absolute value, linear inequalities relate to methods for solving other equations and inequalities. She would need to consider her understanding of the existing understandings of students—asking, “What do the students already know about essential aspects such as solving and graphing single-variable, linear equations and inequalities?” She might consider particular aspects of the topic that have historically given students difficulty, like distinguishing between absolute value
statements that are described by a union of two sets and those that are described by an intersection of two sets. All of these considerations would support the development and sequencing of component understandings of the learning goals for the lesson sequence. This work would involve drawing on the existing, accumulated knowledge of the teacher while some of it would require the active application of mathematical reasoning to produce a well-sequenced set of learning goals.

**Table 2.1.** Knowledge and reasoning involved at the critical instructional moment of establishing learning goals.

<table>
<thead>
<tr>
<th>Establishing mathematical learning goals for an activity sequence—nested within the context of broader goals for lesson, unit and course or possibly another activity sequence.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Knowledge Involved (potentially):</strong></td>
</tr>
<tr>
<td>Knowing the mathematical concept and its connection to others (CCK, SCK)</td>
</tr>
<tr>
<td>Knowing existing student conceptions (KCS)</td>
</tr>
<tr>
<td>Knowing component understandings of broader goals (SCK)</td>
</tr>
<tr>
<td>Knowing typical ways a student might develop desired conceptions (KCS)</td>
</tr>
<tr>
<td>Knowing various sequencing options and potential benefits (KCT)</td>
</tr>
<tr>
<td>Knowing how current concepts related to study of future topics (KMH)</td>
</tr>
<tr>
<td>Knowing how current lesson and unit are situated within the curriculum (KCC)</td>
</tr>
<tr>
<td><strong>Pedagogical Content Reasoning</strong></td>
</tr>
<tr>
<td>Integrating these various forms of knowledge to articulate a specific goal for this group of students in this context at this time.</td>
</tr>
<tr>
<td>Unpacking learning goals to identify the set of component understandings (if set is not readily recalled from accumulated knowledge)</td>
</tr>
<tr>
<td>Mapping anticipated path from existing student understandings to desired understandings</td>
</tr>
</tbody>
</table>
Distinctions between Knowledge and Reasoning while Selecting Tasks

With the component understandings identified and specified, the teacher would need to select or design tasks to support the development of these component understandings. Table 2.2 details the ways in which the task selection process might draw on the accumulated mathematical knowledge for teaching and the ways mathematical reasoning might be involved in the design or selection of tasks. Continuing with the example involving single-variable, absolute-value, linear inequalities, a teacher presumably may have an array of examples that she has come to recognize as illustrating important principles, properties, or relationships involving single-variable, absolute-value, linear inequalities—what Even (Even, 1990) referred to as the basic repertoire. This repertoire might include examples that differ in various ways that change the nature of the solution process such as:

\begin{align*}
|x| &\leq 3 \\
|3x| &\leq 3 \\
|4x - 1| &\geq 3 \\
|-2x + 1| &\leq 3 \\
\end{align*}

Or it may include examples that present dilemmas to students that require them to think conceptually about the task:

\begin{align*}
|2x - 1| &\geq -3 \\
|2x - 1| &\leq -3 \\
\end{align*}

It might include various representations of these types of relationships beyond the symbolic such as real-world scenarios that can be represented by a single-variable, absolute value, linear inequality as well as other nonstandard representations. Drawing on this basic repertoire, she would need to select tasks that would engage the existing understandings of students in goal-
directed ways that would encourage mathematical activity on their part that could lead to the development of the component understandings identified previously. This task selection requires the teacher to consider the mathematics of the students, the mathematics involved in completing the task, and the potential for supporting the development of component understandings. It requires the teacher to consider the mathematics a student is likely to draw on to complete the task. If the teacher’s repertoire of examples is insufficient to provide such a task, then she would have to modify a familiar task or construct a new task designed to support the development of the desired, component understanding of the concept. Furthermore, the teacher would need to determine an appropriate sequencing of tasks that would identify an order of tasks that would allow for the understandings developed through the completion of one task to be used in the completion of other tasks.

The existing repertoire of examples represents a portion of the accumulated knowledge of the teacher. Making instructional decisions about which examples to use while considering the mathematics of the students, the mathematics of the learning goals, and how the mathematics of the task supports the development of the component understandings involves mathematical and pedagogical reasoning. If the teacher has to construct or modify a task in light of her considerations of the mathematical context—the existing understandings of the student as she views them, the likely ways a student is going to engage in a task, and the desired mathematical understanding—then the teacher will have to use mathematical and pedagogical reasoning to do so. Even the process of understanding how a given example or representation for the repertoire of a teacher would support the learning goals involves mathematical reasoning about the task and how a student might engage in mathematical activity to complete the task. Table 2.2 summarizes
and categorizes these types of considerations and uses of mathematical knowledge for teaching and pedagogical content reasoning at the critical instructional moment of task selection.

**Table 2.2.** Knowledge and reasoning involved at the critical instructional moment of choosing a task.

<table>
<thead>
<tr>
<th><strong>Task Choice</strong>—Choosing a question to ask, a problem to pose, a counterexample to present, a representation to consider</th>
<th><strong>Pedagogical Content Reasoning:</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Knowledge Involved (potentially):</strong></td>
<td><strong>Selecting appropriate example, problem, or representation to fit a given situation (student understanding, context, goals, mathematics on the horizon)</strong></td>
</tr>
<tr>
<td>Knowing various examples, representations, problems involving the learning goal (SCK)</td>
<td>Creating or modifying a task to address a student conception, question, or line of reasoning</td>
</tr>
<tr>
<td>Knowing features of the topic at hand that might be accentuated by various tasks (KCT)</td>
<td>Considering how various tasks relate to the learning goal (i.e., what features of the topic at hand might they accentuate)</td>
</tr>
<tr>
<td>Knowing productive ways to sequence tasks to accomplish the learning goal (KCT)</td>
<td></td>
</tr>
<tr>
<td>Knowing existing student conceptions related to the learning goal (KCS)</td>
<td></td>
</tr>
<tr>
<td>Knowing typical student conceptions or ways of reasoning about the topic as it relates to the learning goal (KCS)</td>
<td></td>
</tr>
<tr>
<td>Knowing the tasks that relate to this learning goal found in the textbook (KCC)</td>
<td></td>
</tr>
</tbody>
</table>

**Distinctions between Knowledge and Reasoning while Implementing Tasks**

As the selected or constructed tasks are implemented, a teacher might draw on existing understandings of how a student might engage in a task or an awareness of particular indicators
of understanding, much like the teachers in the CGI project used their newly developed understandings of patterns of student conceptualization of number and operations concepts to recognize when a student was prepared to make a conceptual advance (Fennema, Franke, et al., 1993). She might need to maintain an awareness of the mathematics of the task and to which problematic aspect students should attend while completing the task. There might be a repertoire of instructional moves—questions to ask, other examples to use and contrast with the work of the students, modifications of the task—that the teacher can draw on that provide support and scaffolding for students in ways that do not diminish the level of cognitive demand. However, determining an instructional move based on an active consideration of the mathematical activity of the students and the mathematics of the learning goals requires an active application of mathematical knowledge and reasoning. Suppose students were asked to graph the set of values of air temperature for which a heating and air unit would run if the thermostat were set at 72 degrees and the room temperature was expected to get no more than 2 degrees above or below the thermostat setting. Such a task might support a connection between the graphical representation of a union of two sets and a meaningful context, and making sense of that representation might be an important component understanding of single-variable, absolute value, linear inequalities. Students might experience difficulty with going directly from the verbal description to the graph. In such a case, a teacher would need to be prepared to offer some support that would not diminish the cognitive demand on students, perhaps suggesting that the student find the actual temperature ranges and then graph. A teacher would need to craft questions and modifications to the task in ways that not only helped the student to solve this particular problem but also supported the development of understandings that would contribute
to the student’s understanding of single-variable, absolute value, linear inequalities. This kind of support might involve asking the student to represent the numerical solution with inequalities or to represent the two inequalities with a single inequality. The effectiveness of these choices depends on the interpretation of the student’s conceptions based on the observed responses to the task.

These questions and task modifications could involve a teacher drawing on her past experiences with students and determining which intervention might provide the right support for a student or constructing new responses on the spot. In either case, there are instructional decisions to be made that require a teacher to use the collection of accumulated knowledge and the observations from the classroom to determine the best course of action for this student relative to a particular set of learning goals. This kind of reasoning is both mathematical and pedagogical and represents what this researcher refers to as pedagogical content reasoning. The same is true for the interpretation of student responses. A student might express his or her conceptions in limited or imprecise ways. The teacher must not only interpret those expressions to build some understanding of the student’s understanding but also weigh those relative to the learning goals. If the teacher possesses a rich knowledge of content and students, she might have some sense of students’ typical ways of conceiving of the concept under study and could draw on this knowledge to build her understanding of a particular student’s understanding. There may even be some typical responses to students that make sense in light of this interpretation. In this way, the instructional intervention leans more heavily on the existing, accumulated knowledge of the teacher rather than the active reasoning to make an instructional decision. In either case, the implementation of instruction and dealing with observable student responses involve both
accessing accumulated knowledge and engaging in an active mathematical reasoning about the mathematics of the student, the mathematics of the learning goal, and the potential path from one to the other. The nature of types of mathematical knowledge and reasoning used during the implementation of tasks and in response to observable student responses is summarized in Table 2.3.

**Table 2.3.** Knowledge and reasoning involved at the critical instructional moments of implementing a task and observing student responses.

<table>
<thead>
<tr>
<th>Implementing Tasks and Considering Observable Student Responses</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Knowledge Involved (potentially):</strong></td>
</tr>
<tr>
<td>Knowing what needs to be recognized as problematic about a given task (SCK or KCT)</td>
</tr>
<tr>
<td>Knowing instructional moves that provide support for students to complete these kinds of tasks (KCT)</td>
</tr>
<tr>
<td>Knowing the mathematical activity of the student as they are engaging in the task (KCS)</td>
</tr>
<tr>
<td>Knowing connections between mathematical elements of the task and other mathematics (SCK)</td>
</tr>
<tr>
<td>Knowing typical student conceptual difficulties or errors (KCS)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Constructing responses to student questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constructing a mental representation of the</td>
</tr>
<tr>
<td>understanding of the student</td>
</tr>
<tr>
<td>-----------------------------</td>
</tr>
<tr>
<td>Comparing the robustness of emerging conceptions of student relative to learning goals</td>
</tr>
<tr>
<td>Considering the logical and mathematical implications of student responses (i.e. evaluating the validity or plausibility of a claim)</td>
</tr>
</tbody>
</table>

It is important to understand that this description of pedagogical content reasoning was an initial conceptualization of the notion. This study was designed to explore the usefulness of the distinctions inherent in its definition as the use of mathematical knowledge at critical instructional moments was explored. Refinements to the definition were made as patterns emerged from the data. These refinements are discussed in the next chapter.

**Summary of Theoretical Framework**

Several elements of the theoretical lens that underpins this study were worth highlighting and summarizing at this point. First, this study is designed to examine the use of mathematical and pedagogical knowledge during instruction that is oriented towards developing rich, connected understandings of mathematics in students. A rich, connected understanding of mathematics involves making connections among mathematical ideas, among various solution strategies, and between existing understandings and new ideas. Connections among mathematical ideas might involve understanding how the properties of some mathematical entity manifest themselves in different representations. It might involve recognizing and making generalizations about the structural similarities among situations in which specific algorithms
Connections among various solution strategies might involve recognition of how different strategies are similar and how they are different as well as why each one produces a valid result. Connections between existing understandings and new ideas might involve modifying existing understandings to create new, more powerful ones. This study is designed to deepen our understanding of how mathematical knowledge and reasoning are used to design and deliver instruction oriented towards developing these are the types of understandings in students.

A second important element of the theoretical lens through which the data will be analyzed involves the elements of instructional practice that can be identified as pivot points for the nature of instruction. These elements, identified as critical instructional moments, are moments during the design and delivery of instruction that are likely to place demands on the mathematical knowledge of teachers. Efforts were made to identify these important moments based on theoretically and empirically based understandings of what aspects of instruction matter. However, it should be noted, that the researcher recognizes that these moments may not be the only ones that matter. The contention is that there is strong support for the selection of these moments as pivot points for instruction. The initial conceptualization of these moments identified four such moments: the identification of learning goals, the selection or development of tasks, the implementation of tasks and observable student responses. One of the first tasks of the analysis of data for this study is to establish the viability of these characterizations and make refinements to the definitions and delineations as dictated by the analysis.

A third important element of the theoretical lens involves the image of instruction derived from the theoretical and empirical literature. This image of instruction captures the instructional
practices at each of the critical instructional moments that have theoretical and empirical support for supporting the development of rich, connected understandings of students. The research-based image of instruction suggests that not only are clear learning goals important but they also should be sufficiently unpacked to support the design and delivery of instruction. It suggests that tasks should be selected that are problematic and designed to foster goal-directed mathematical activity in students, conceptually oriented towards those learning goals, and accessible to students using their existing understandings. The image of instruction also identifies approaches to task implementation designed to foster the development of rich, connected understandings in students. Particular emphasis is placed on the nature of support provided to students. Tasks should be presented and scaffolded in ways that maintain the level of cognitive demand on students. Finally, student responses to tasks should shape instruction. In other words, student responses of all types—questions, hypotheses, solutions, alternative approaches, and so on—should be used to further the collective understanding of the class. Instructional interventions should be designed to use the expressed understandings of students to support the development of new ones. This study aims to understand how the teacher’s mathematical and pedagogical knowledge is used at critical instructional moments to design and deliver instruction likely to support the development of rich, connected understandings in students.

And finally, the theoretical framework hypothesizes that the construct of pedagogical content reasoning serves as the mechanism through which mathematical knowledge for teaching gets operationalized into practice. Not only do teachers draw on their accumulated mathematical knowledge for teaching when making instructional decisions, but they also have to reason mathematically about the mathematics of the student, the mathematics of the learning goal, and
the potential path from one to the other. The study is designed to explore this contention during the design and delivery of instruction intended to produce rich, connected understandings of mathematics.
Chapter 3:

Methodology
In light of the focus of this study on the use of mathematical knowledge and mathematical reasoning by teachers during instruction, it makes sense that this research project would involve many of the methods associated with a naturalistic paradigm (Moschkovich & Brenner, 2000). These methods were easily combined with methods of other approaches to ensure a rich data set and thorough analysis while overcoming some of the potential difficulties associated with trying to understand teacher thinking in action.

As outlined by Moschkovich and Brenner (2000), research using the naturalistic approach centers on three guiding principles:

1) The research considers multiple points of view of events.
2) The research seeks to connect theory verification with theory building.
3) The research involves studying cognitive activity in context.

To varying degrees, the design of this study applied all three of these guiding principles to the process of data collection and analysis. First, the question of how a teacher uses mathematical knowledge and reasoning during instruction was considered primarily through the perspectives of the teacher and researcher but it is impossible to do so without consideration of the perceived impact on students as inferred through their observable responses to the work of the class. In terms of theory building, this work began with an initial conceptualization of critical instructional moments and pedagogical content reasoning (as outlined in the previous chapter), and those notions were refined through a cyclical approach to data collection and analysis. Finally, situating this study in the practice of teachers through classroom observation is motivated by the need to examine how teachers use mathematical knowledge and reasoning during instruction. This naturalistic approach not only served as a reasonable methodology for
the given question, it addressed an area of limited attention within the research base outlined previously, namely, the limited research of mathematics teachers in practice.

Naturalistic methodologies include natural observation and conversational, as opposed to task-based, interviews. Both were a part of this study. However, naturalistic methods can be combined with other approaches as was the case with this study (Moschkovich & Brenner, 2000). Thus, the underlying approaches to this research study represent a combination of sociolinguistic and naturalistic approaches. This research was sociolinguistic (Carlsen, 1991) in that it focused in large measure on the naturally occurring discourse in the classroom around these critical instructional moments. The significance of these moments for the teachers and their interpretation of the significance of the moments to the student were examined through the language and nonverbal communication as well as through interactions during the class and through the stimulated recall interviews. With these goals and foci, data collection methods are described in this chapter and included background interviews and baseline observations, video and audio recordings of classroom observations, field notes, stimulated recall interviews, and video recordings of those interviews. Each of these data sources was analyzed through the cyclical and nested process also outlined in this chapter.

**Participants**

The goal of this study was to understand the use of a teacher’s mathematical knowledge in practice and to understand the role of pedagogical content reasoning in that process. In particular, this research focused on the use of mathematical knowledge and reasoning at critical instructional moments during the planning and delivery of instruction designed to support the development of rich, connected understandings. Thus, the choice of subjects for this study
addressed a number of potential issues with this research. Those issues are described in the subsequent paragraphs and that general description of the issues is followed by a discussion of the selection process and the participants who were specifically involved in this study.

Three considerations shaped the selection of participants in this study—the broad goals teachers had for students, the nature of their educational experience, and the interest the teacher had in sharing his or her thinking about instructional practice. First, participants in this study needed to include the goal of the development of rich, connected understandings in students among their goals for students. As the researcher examined the use of knowledge at critical instructional moments, it was important to examine teaching practice that supported the development of rich, connected understandings—making connections among mathematical ideas, among various solution strategies, and between existing understandings and new ideas.

Second, participants in this study needed to have extensive experience in the mathematics classroom. The researcher chose not to include a measure of the accumulated mathematical knowledge of the participants in the study because the study was designed to focus on the use of mathematical knowledge in the design and delivery of instruction. Although it is not essential for this study to have an objective measure of the mathematical knowledge of the teacher, the richer this knowledge base, the richer the data for this study would be. If a teacher had a limited knowledge base for teaching, it would be difficult to know if instructional choices were a function of those limitations of knowledge or a limitation in how the teacher was using the knowledge during instruction. Selecting participants with several years of experience in mathematics classrooms and a minimum of a master’s degree in mathematics or mathematics
education would ensure a level of mathematics and pedagogical knowledge that would support the opportunity for a rich data set.

Finally, participants in this study needed to be comfortable discussing their thinking during the stimulated recall interviews. They needed to be articulate and comfortable in their discussion of mathematics teaching and learning and willing to share the reasons behind their decisions. While it is difficult to develop an instrument or some other proxy for this comfort, it was nevertheless an important consideration in the selection of participants.

One other dimension of the qualities of the participants in this study pertained to the degree of alignment of their instructional practice with the image of instruction outlined in the previous chapter. If differences emerged in the nature of the use of mathematical knowledge and reasoning at the critical instructional moments, it would be important to understand if these differences were associated with differences in the nature of instruction at critical instructional moments. An understanding of the alignment with the image of instruction was initially expected to be gained from the use of the Reformed Teaching Observation Protocol (Piburn et al., 2000). The RTOP (see Appendix A) was designed to assess the degree to which teaching practice matches the principles of reformed teaching as outlined in Standards publications (National Council of Teachers of Mathematics, 1991, 2000). The validity and reliability of this instrument had been previously established through a rigorous process (Piburn, et al., 2000) and training in its use was completed by this researcher.

However, during piloting of this tool, a number of factors led to abandoning the use of it for the purposes of this study. First, the protocol contains 25 items related to classroom practice with each rated on a scale of 0 to 4. The ratings served to quantify the nature of the classroom
activity as it compared to the degree of alignment with the 25 descriptors. While an overall rating of a teacher’s classroom practice was useful for the original purposes of the protocol, a small number of participants would not produce a large enough pool to determine how large an overall rating would satisfy the needs of the study. More importantly, the comparison of scores among the participants was also of limited use since a total score masked the range of scores relative to each individual descriptor. Two of the five sets of descriptors involved the classroom culture and communication skills of the teacher, a higher overall score may not mean that one teacher aligned his or her instruction on the content and pedagogical descriptors more closely than another. In other words, it was possible for a teacher to earn ratings on the two relational sections high enough to surpass another teacher’s rating who actually scored better on the content and pedagogical descriptors. These limitations made the RTOP ill-suited for the purposes of this study. The researcher relied on other data to determine the suitability of participants for the study. Those methods are described in the following section.

In light of these considerations and limitations, it seemed unlikely that a random selection of participants would yield the qualities among participants that would address these concerns. In fact, some familiarity with the participants, the nature of their goals for students, and an awareness of their instructional practice would be advantageous in selecting an appropriate participant pool. With over 20 years experience in mathematics education in secondary schools at the time of this study, the researcher had familiarity with over 60 secondary teachers through both structured and casual classroom observations (not in a supervisory capacity), professional dialogue in workshops, department meetings, and individual conversations, professional development workshops as a participant and a provider, collaborative lesson planning and
curriculum development, and direct observation of work with students outside the classroom typically unplanned. From this pool of teachers, the researcher identified six teachers who potentially met the criteria for inclusion: goals for students that involved the development of rich, connected understandings, extensive experience in the mathematics classroom including an advanced degree in mathematics or mathematics education, a demonstrated willingness and capacity for discussing his or her instructional practice, and instructional practice that seemed to be well aligned with the image of instruction. The first four teachers in this group agreed to participate in the study.

These teachers were initially identified in light of the researcher’s pre-existing understandings of the professional traits, experiences, and pedagogical thinking of each potential participant, in light of this awareness, the researcher had reason to believe that each of these teachers fit the profile outlined previously. Each of the participants in this study had more than 10 years of experience and three of the four had over 20 years of experience. Three of the four participants had spent at least 10 years in his or her current school while the fourth had recently changed schools to become the mathematics department chair after a 10-year career at his former school. None of the participants had a history of difficulty with classroom management.

Second, their knowledge of mathematics for teaching appeared strong. Each of the participants held a master’s degree in Mathematics or Mathematics Education at the time of this study. One participant was in the late stages of a PhD program in Mathematics Education. Another had held the Department Chair position at a highly respected independent school for over 12 years while another had just taken a department chair position at a new school. The other two participants had been teaching AP Calculus for over 10 years.
Third, and again, based on past experiences with these teachers involving classroom observations, shared professional development experiences, and numerous conversations about the mathematics teaching and learning, the researcher believed that each of these teachers wanted their students to develop rich, connected understandings of mathematics and that their instructional approach aligned with the image of instruction derived from the literature in Chapter Two. As a former colleague to three of the four participants, the researcher was aware of that each of these teachers routinely discussed, with their colleagues, problems and activities from their classes, the nature of student thinking and performance on those tasks, and ideas they had for making the experience better. This prior experience made these teachers candidates for the study.

In essence, the combination of the traits and experiences of these teachers increased the likelihood that data from classroom interactions of these teachers with students would likely yield a rich data set of critical instructional moments and teacher thinking around those moments. The fact that the researcher had a rapport with each of these teachers and the researcher believed that each would be comfortable sharing his or her thinking about the instruction added to their potential suitability for the purposes of this study. The researcher inquired of their interest and willingness to participate in the study. In the initial request to participate in the study, none expressed discomfort with being video recorded or any hesitancy about the stimulated recall interview. Each embraced the professional opportunity and potential for growth through the process.

Once the commitments to participate and appropriate permissions were secured, the researcher conducted background interviews to confirm the suitability of each participant for the
The plan for the semi-structured interview is in Appendix B. These interviews occurred prior to beginning classroom observations and provided some basic biographical information and explored the nature of the teacher’s thinking about his or her goals for students, his or her approach to lesson planning, and the prevalence of discourse in the classroom. The questions were designed to elicit the teacher’s articulation of broad goals for students as well as the lesson level goals to determine if they were consistent with the development of rich, connected understandings. Once those goals were articulated, the teachers were asked to discuss how they planned lessons to support the attainment of those goals, how they responded to unexpected outcomes or to students who were unable to complete a task successfully to determine if their image of instruction was generally aligned with the image of instruction outlined in the previous chapter. However, other questions were also asked to probe for further clarity and elaboration as needed. Interviews were audio recorded, transcribed, and analyzed. A discussion of the findings of these interviews, as it relates to the inclusion of each teacher in the study, follows.

Duncan, a teacher with over 20 years experience, expressed a desire for his students to understand how mathematics models our world and for them to understand the logic and structure of mathematics. As he put it, “I try to share with them how we can use mathematics to describe our world and model it, how it is structured, the underpinnings of mathematics, and how it allows us to think in a logical manner” (Duncan Background Interview, Lines 39—41). He went on to describe his desire for students to feel comfortable taking mathematical risks and to try to find more than one way to solve a problem. When asked about the nature of student understanding required to take those types of risks, he stated, “It is conceptual. They need to understand what the problem is asking.” (Duncan Background Interview, Lines 83—84). These
descriptions of his goals for students suggest that Duncan strives for the same types of rich, connected understandings of mathematics as described in the literature review—understandings that involve making sense of mathematics by understanding the structure, logic and the relationships among the underlying concepts. He also expressed a belief in the application of mathematics in the real world.

Duncan went on to describe his ideal lesson—one that involved a student-centric approach that would include student exploration and investigation, a robust mathematical discourse, and the active construction of understanding on the part of the students.

A lot depends on the subject, the particular topic. In an ideal world, I would like to do everything in discovery-based in which I can present a topic and let the kids play with it and they debate among themselves and ask questions of me and not questions that would lead them. And that would be ideal, everything would be discovery based because in my experience, if they discover an idea, they are going to retain it as opposed to me standing and presenting an idea. For some topics, this is easier to do than others. (Duncan Background Interview, Lines 95—101)

His references to “discovery-based” lessons, student exploration of a topic, active mathematical discourse, and the avoidance of leading questions suggested that the classroom teaching of Duncan was well aligned with the image of instruction derived from the literature. It also suggested that the interactions of Duncan with his students would provide a robust collection of critical instructional moments for analysis.

Susan, a teacher of over 30 years of experience and 12 years as the Department Chair in her current school, described mathematics as the study of relationships and stated that she actively tries to convey that understanding by “emphasizing multiple representations, emphasizing connections, and when I am aware of applications they might not think of, I mention those” (Susan Background Interview, Lines 20—21). When asked to elaborate on the
understandings that are most important for her students to develop, Susan responded by saying, “exactly those connections, an understanding of concepts—to be able to explain [the concept] in one’s own words, to be able to apply it, and be able to recognize it, its applications” (Susan Background Interview, Lines 47—51). Susan’s goals for students as expressed in the background interview also suggested a desire to develop rich, connected understandings on the part of students. She acknowledged the need for the students to build those understandings and the importance of representing mathematical ideas in multiple ways to build connected understandings. Like Duncan, but to a lesser degree, she acknowledged the need for students to apply the mathematics.

In planning instruction, Susan described considering both what she needs to accomplish and what they are prepared to handle. “I look for connections with what they have done in the past. I think about some of their skill sets that might be used, and also think a little bit about where it might be going” (Susan Background Interview, Lines 106—108). Also in her interview, Susan describes the importance of student feedback. When asked how she handles an unexpected or erroneous response, she said,

Actually, I like that. Because, if it was an answer I was not expecting, that makes me think, and it is also that in the moment, I am trying to analyze: Where did this come from? Could this be a misconception that other students have? How do I clarify that misconception besides just saying, ‘No, that’s wrong?’ Sometimes I get great insights from kids’ own answers if they are willing to let me into their thinking.

In this quote, Susan makes direct reference to her efforts to understand the conceptions of students and to construct responses to those expressions of understanding that clarify the conceptions for the students. In this account as well as in her description of her goals for students, Susan described a thoughtful consideration of student thinking during the planning and
delivery of instruction. This kind of consideration of student thinking is of particular interest in this study and along with her goals for students and her image of instruction made Susan a suitable participant.

Harold, a teacher with over 10 years experience, was in the process of completing his PhD in Mathematics Education at the time of this study. Harold’s goals for students can be summarized by a three-part definition of what it means to understand mathematical ideas. He shared, “If they can explain what they are doing, why they are doing it, and how they can justify that it is correct mathematically—if they can do all three of those things, then they might understand, assuming all three of those things are correct” (Harold Background Interview, Lines 93–96). He described mathematics as a “socially constructed set of concepts that you can interpret as ways to make sense of the world” (Harold Background Interview, Lines 3–4). As such, he emphasized the importance of developing an understanding of each student’s construction of mathematics.

I ask students to, in multiple ways, show me how they are constructing mathematics. So that can be anything from them showing me their answer, to explaining their thinking to me, to them explaining their thinking to each other, seeing if they have the ability to listen to another student’s thinking and recognize that the other student’s thinking is the other student’s construction of the same concept and are they able to take their construction and recognize it in other student’s. That includes being able to create models of what they have constructed, being able to communicate algebraically as well as in typical English words their construction, to be able to present their ideas orally as well as draw and manipulate physical models. So, I use the combination of all of that to evaluate their understanding and then, and frankly this is where I think the real part of teaching comes in, is trying to figure out how to take them from where they are to get to where I recognize they need to be. (Harold Background Interview, Lines 40–52).

This extensive description of what he asks of students suggested that Harold also seeks to develop rich, connected understandings in his students. At the same time, he emphasized his desire to understand what the students understand so that he can, “figure out how to take them
from where they are to where I recognize they need to be.” The description of the teaching process for him as captured by this entire quote was quite consistent with the triune focus outlined in the literature review—a focus on the mathematics of the learning goal, the mathematics of the student, and a potential path from one to the other—and suggested that Harold was also a participant who might yield a rich data set for analysis.

For Jackie, also a teacher with over 30 years of experience, mathematics was described as the study of patterns. She summarized her approach to teaching “as [trying] to get students to discover what a pattern looks like to them or what looks like an actual consequence to them and work together to establish or debunk it, which is really how historically, mathematics happened” (Jackie Background Interview, Lines 9–12). Twice in this description, Jackie made explicit reference to what the pattern or consequence looks like “to them” which suggested a clear focus on the work with and the development of their understanding. When asked how to clarify further, several statements seem to have captured her approach:

I ask a lot of questions and a lot of them start with, “What would you think would happen if...?” I create things, on an overhead or whatever makes patterns more obvious so that they will see them. I try to get them to see them.

(Jackie Background Interview, Lines 30–31)

Jackie emphasized her desire to get the students to develop the understanding—that is, to “see” the ideas by using “whatever makes the pattern more obvious.” Once again, her emphasis on the development of the students’ conceptions was implicit in this description and as she described, her instructional choices directly reflected her thinking about how they could see those patterns more readily.

According to her, Jackie’s effort often involved the use of examples the students generate. “I use examples that they come up with as much as I can.” (Jackie Background
Interview, Line 36). The following quote captured her expression of the thought process she uses when she chooses examples or tasks for students:

I want [an example] to have an element of familiarity or review… an element of review, an element of applying the new stuff that we are just finishing, and an element of the unknown to see if they can take it a step further.

(Jackie Background Interview, Lines 62–74)

This description suggested that when she chooses examples or tasks for students to complete, she considers her understanding of the existing understandings of students as well as how the example might be structured to provide them with an opportunity to take a productive step forward in their understandings. This triune focus mirrors that of Harold and the image of instruction derived from the literature. As such, these descriptions of her approach to her teaching suggested that Jackie would also be a suitable participant in the study and the data from her classes and subsequent interviews would provide rich examples of her work with students and her decision-making at critical instructional moments.

From these descriptions of each participant’s goals for students and approach to teaching, the researcher had confidence in the potential richness of the teaching episodes in providing ample opportunities to explore the demands on teachers’ mathematical knowledge during instruction focused on developing the kind of rich, conceptual understandings for which we strive. Each participant expressed, in various ways, a desire to develop rich, connected understandings on the part of the student. Each acknowledged the need for the students to develop those understandings. Three of the four (Jackie, Harold, and Susan) explicitly described an active consideration of student thinking in the planning and delivery of instruction. These descriptions of their instructional approach suggested these participants would yield a robust data set for the study.
Data Collection

Even with an effective participant selection process, the collection of a rich data set that adequately revealed indicators of teacher reasoning presented a challenge for this study. A pilot study was conducted to ensure that data collection techniques would capture relevant classroom episodes and allow for the reliable identification of critical instructional moments. The pilot study also allowed the researcher to hone stimulated recall interview questions to elicit teacher thinking about their instructional moves. The results of the pilot study gave the researcher confidence in the viability of all facets of the data collection. All teacher and student utterances were captured by the classroom recording. The positioning of the camera for the stimulated recall interview was designed to capture the interviewer, the teacher, and all references to the video during the discussion. Critical instructional moments during the class seemed to be readily identifiable and teacher decision-making at those moments seemed to be revealed through the stimulated recall interview. A discussion of each data source and the nature of the data collection follows.

Lesson Selection

Prior to the beginning of observations, the researcher conducted a screening interview (via phone, typically) with each participant in order to identify lessons suitable for inclusion in the study. The researcher explained to each participant the desire to observe a lesson in which new material was introduced to the students. The researcher did not specify whether the topic needed to be completely new to the students or if it could be an extension of previously studied material. However, an emphasis was placed on identifying lessons for which the teacher had a
goal of developing a new understanding or set of understandings of students. The participants discussed various options with the researcher, and the researcher determined the lessons to be recorded for the study based on which classes seemed likely to yield rich data with respect to the teacher’s use of mathematical and pedagogical knowledge during instruction.

**Video Recording of Classroom Observations**

A number of considerations were made regarding the video recordings of classroom observations in order to capture teacher moves and student–teacher interactions around critical instructional moments. Two cameras, one stationary to capture whole-class interaction and the other handheld to capture student–teacher interactions in small groups or individually, were used to capture the events of the class. Each camera provided a time stamp of events and afforded the researcher ample opportunity to capture pictures of classroom artifacts by moving around to capture individual student work as needed. The built-in audio on the cameras was used and the pilot study confirmed that it adequately captured student comments made during whole class interactions. Each teacher used computer projection, overhead projection, or the whiteboard for instruction given to the entire class thus providing ample opportunity for the researcher to capture the presentation of tasks, the responses of students, and the responses of the teacher to those student responses through the video recording.

Multiple observations of each teacher provided an adequate collection of each type of critical instructional moment for each teacher and across teachers. Over 250 critical instructional moments were identified in the data. These multiple occurrences afforded the researcher opportunity to develop conceptions of how each teacher used mathematical knowledge and reasoning at critical instructional moments and to examine patterns across teachers.
Each class was video recorded in its entirety. During the class, the researcher identified critical instructional moments in real time and recorded the time stamp in his field notes. Questions about instructional moves were also included in the field notes serving as a first-pass analysis of the lesson and initial planning for the stimulated recall interview. For each participant, the researcher recorded and observed multiple lessons varying along several dimensions as described in the Table 3.1.

**Table 3.1.** Observed lessons taught by each participant.

<table>
<thead>
<tr>
<th>Name</th>
<th>Lesson Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jackie</td>
<td>Three lessons in the same Geometry class were recorded. Classes were designed and sequenced to help students move from an understanding of triangle trigonometry to develop and connect to the foundations of circle trigonometry and polar coordinates. Classes observed were not consecutive, but in sequence as student project presentations broke up the string of lessons designed to develop the ideas.</td>
</tr>
<tr>
<td>Harold</td>
<td>Four lessons were recorded. A two-lesson sequence in Math 6 was designed to develop student understanding of division of fractions. A two-lesson sequence in Algebra I introduced systems of linear inequalities. Each pair of lessons was taught on consecutive days.</td>
</tr>
<tr>
<td>Susan</td>
<td>Three lessons were recorded. The first two lessons were the same lesson taught to two different Algebra II classes on rational functions. The third observed and recorded lesson was the next lesson in the sequence taught to one of those classes.</td>
</tr>
<tr>
<td>Duncan</td>
<td>Three consecutive lessons taught to a Precalculus class during an introductory unit on parametric equations were recorded.</td>
</tr>
</tbody>
</table>
Notes on Preliminary Lesson Analysis

As soon as possible following each lesson observation, the researcher undertook a preliminary analysis while reviewing the artifacts of the lesson. This initial analysis occurred within no more than 2 days of the classroom observation. In many cases, the analysis occurred on the same day. Through this analysis, critical instructional moments were identified by time stamps and probing questions were designed to elicit the teacher’s reasoning and use of mathematical knowledge at each of the identified critical instructional moments. Teacher moves and student responses immediately before, during, and immediately after each critical instructional moment were also noted. From these notes, a specific set of interview questions was constructed for each lesson. See Appendix D for a general guideline for these specific interviews.

Stimulated Recall Interviews

The goal of the stimulated recall interviews was to elicit the teacher’s thinking about each lesson segment containing a critical instructional moment. This involved developing an understanding of the teacher’s intent in a given moment, how he or she understood the student responses in that moment, how the teacher used that understanding to make an instructional move, and what the teacher hoped to elicit from students through that instructional move.

During these interviews, the researcher and participant teacher watched video episodes from the class and the researcher asked probing questions developed to elicit the teacher’s thinking during the episode. To capture the substance of each interview, a camera was positioned behind the researcher in such a way to capture the interaction of the teacher with the researcher, any references to the video of the classroom episode, and the video of the lesson.
With this camera angle, the researcher was able to produce verbatim and annotated transcripts aligning the transcript of the classroom episode of a critical instructional moment with the stimulated recall discussion of that episode.

In considering the use of stimulated recall interviews, the potential for interactive effects of the researcher on teacher thinking is significant. Yinger (1986) raised a more fundamental question about the use of stimulated recall. He questioned whether the teacher is actually recalling his or her cognition at the time or making inferences about what he or she might have been thinking in light of the evidence from the video. As Hall (2000) points out, video recordings can capture an experience that no participant in the study actually had. For Yinger (1986), a stimulated recall interview might be more about how the teacher thinks about this new experience (what he or she sees in the video) than about what he or she was actually thinking at the time.

With this study, the researcher mitigated this concern within each interview and in the course of the analysis of the data. During the interview, the researcher deliberately asked questions requiring the teacher to make an explicit distinction between what he or she was recalling about his or her thinking at the time and what he or she was considering in hindsight. The effort made some of the interactive effect of the researcher more visible as participants (unprompted) often qualified remarks by making references to what they were recalling and what they were anticipating about their thinking. Triangulation of data also served to guard against the potential ambiguity about the origin of the expressed thinking of the teacher. With sufficient richness of data, background and preobservation interviews, multiple instances of each of the critical instructional moments, and inferences about teaching thinking from the lesson itself
provided ample opportunity to weigh the validity of a teacher’s response by triangulating data from these multiple sources.

Also worthy of consideration with stimulated recall interviews was the potential increased awareness and more focused reflection a teacher might experience as a result of the interview process. Given the same teacher will be observed and interviewed multiple times, it was conceivable that a teacher would be more likely to consider student thinking and be more deliberate in his or her thinking after considering his or her teaching in the interview setting. In fact, there was some evidence of this effect with one of the teachers. Again, through the background interviews, the initial observations, and the first stimulated recall interview, a baseline of understanding of teacher practice and his or her use of mathematical knowledge and reasoning during instruction was established. Analysis of this data was done separately from the analysis of subsequent stimulated recall interviews to provide the researcher with a basis of comparison to consider the effect of the researcher. A summary of the data sources and types is found in Table 3.2.
Table 3.2. Summary of data sources and types collected.

<table>
<thead>
<tr>
<th>Data source</th>
<th>Data type</th>
</tr>
</thead>
<tbody>
<tr>
<td>Background interview with each teacher (see interview protocol in Appendix B)</td>
<td>Field notes and audio recordings</td>
</tr>
<tr>
<td>Pre-observation interview with each teacher</td>
<td>Field notes and video recordings</td>
</tr>
<tr>
<td>Classroom observation for stimulated recall interview (3–4 observations for each teacher)</td>
<td>Field notes of lesson with identification of critical instructional moments and questions for recall interview Video recordings of each lesson Annotated, verbatim transcript of complete class proceedings</td>
</tr>
<tr>
<td>Stimulated recall interviews with each teacher</td>
<td>Video recordings of each interview Two-column annotated, verbatim transcripts of each interview alongside annotated lesson segments</td>
</tr>
</tbody>
</table>

Data Analysis Process

Audio recordings from the background interviews were transcribed. Along with notes from the preobservation conference, information from these background interviews was reviewed prior to the classroom observations in order to identify potential critical moments, areas of particular interest during the class, and to plan for video recording. The content of these background interviews was also used to understand each teacher’s goals for students and triangulate the researcher’s understanding of these goals as they emerged during the classroom observations or were discussed during the stimulated recall interviews. This data also provided
indicators of changes in plans during instruction, which were important moments to understand in terms of the reasoning of the teacher.

The first layer of analysis began with the field notes and initial review of each lesson video as the researcher used the theoretically based definitions of critical instructional moments to identify those moments in the lesson and prepare questions for the stimulated recall interview. Through this process, particular lesson segments were marked for further exploration—moments of interest included the presentation of a task to students, responses to those tasks by students, and teacher responses to those student responses. Subsequent to the completion of the stimulated recall interviews, verbatim and annotated transcripts of classroom lessons and stimulated recall interviews were produced and integrated into a single, color-coded transcript capturing classroom action as well as teacher commentary from the stimulated recall interview.

Each integrated transcript was then coded to identify critical instructional moments and to mark instances of the use of mathematical knowledge or reasoning of the teacher. Not all critical instructional moments were identified and discussed during the stimulated recall interviews. For those moments for which a teacher’s reasoning was not explored during the stimulated recall interview, inferences were made based on the teacher’s actions during the class. These inferences were triangulated by the explicit expressions of thinking that were captured in the background interviews or during the stimulated recall interviews.

As patterns emerged (as described in the next section), refinements were made to the identification of critical instructional moments and additional cycles of analysis were initiated to test the viability of the refinements. Once the refinements to the construct of critical instructional moments were solidified and critical instructional moments marked, analysis began
on both the nature of instruction at these critical instructional moments and the nature of the
demands on the teacher’s mathematical knowledge and reasoning at these critical instructional
moments.

For each participant in the study and for one type of critical instructional moment at a
time, the researcher examined all occurrences of that type of critical instructional moment found
in the data—background interviews, classroom observations, field notes, and stimulated recall
interviews. The analysis involved a cyclical process of identifying patterns in the nature of
instruction for a given teacher at that type of critical instructional moment. As patterns emerged,
the researcher reviewed the data in light of this pattern and refined the articulation of the pattern
as appropriate. For example, in the initial analysis of the classroom data from Harold at the
critical instructional moment of interpretation of student responses, several instances were noted
in which student responses were identified by Harold as wrong. This interpretation seemed
inconsistent with his expressed desire (in the background interview) for students to construct
their own understandings. Through repeated analysis and review of the data, the researcher’s
came to understand these instances in terms of Harold’s use of strict indicators of understanding
and the use of precise language he associated with those indicators. Patterns like these within
teachers and across teachers were explored as the constructs were further refined. This process
continued until the descriptions and refinements to these patterns stabilized.
Critical Instructional Moments Defined and Refined

As discussed previously, the model of teaching outlined in chapter 2 directs us to critical instructional moments. They occur in the planning of instruction as well as during instruction. Existing research suggests there are four types of critical moments that can make a difference in the development of rich, connected understandings on the part of the students: The identification of goals, the selection of tasks, the implementation of tasks, and the responses of students. The first task of analysis was to establish that these critical instructional moments were consistently identifiable during the course of instruction. Beginning with an initial definition from the literature, the researcher identified occurrences of critical instructional moments. As anomalies to the definition arose, the definition was refined and data re-examined until the definition stabilized.

The first phase of analysis involved the identification of critical instructional moments during the recorded lessons. Using the initial analysis (the one done prior to the stimulated recall interviews) as a starting point, the lessons were analyzed and critical instructional moments were marked. While the identification of goals could be readily identified in the interviews with teachers, ambiguity first emerged in the classification of the critical instructional moments of selection of tasks and implementation of tasks.

This ambiguity arose due to multiple facets of the definitions related to critical instructional moments. Tasks, as previously defined, vary broadly—questions to answer, examples or counterexamples to consider, representations to understand, or problems to solve. They can be preplanned or spontaneous. Some tasks were part of the preplanning of the teacher and were designed (or chosen) to initiate the mathematical activity of the students. For example,
Jackie’s first observed lesson, she asked students to find the length of a diagonal of a parallelogram given two sides and the included angle. She presented the students with the following diagram in Figure 3.1 and asked them how to find the diagonal of the parallelogram.

![Figure 3.1. Diagonal of a parallelogram problem](image)

Essentially, Jackie planned to use this task or at least one like it—the numbers she chose and the size of the included angle were chosen on the spot. She used it as one task in the sequence of tasks designed to develop what she identified as the component understandings that could lead a student to understand the foundations of the unit circle, trigonometric functions, and polar coordinates. Ultimately, her goal was to free students from the constraints of the triangle with the use of the trigonometric ratios to think of a more dependent relationship between the coordinates of a point on the coordinate plane and the angle formed with the positive x-axis.

Based on what Jackie believed the students’ existing understandings to be, this task was designed (or chosen) to engage the existing understandings of students and ways of doing this problem to develop other, more productive understandings relative to her goal. Jackie’s expressed learning goals for students and the learning trajectory she defined will be discussed in detail in the next
The point to emphasize here is that this task served as an example of a task she planned to use to motivate the mathematical activity of the students.

In the course of implementing this task, other tasks were provided to students in response to their work on this given task. For example, after one student asked about finding the other diagonal (an unproductive approach) and another student focused on the properties of a parallelogram that would allow students to determine the other angle of the parallelogram (a productive approach), the teacher asked, “What do I know about the sides of the parallelogram?” [Jackie, Ob #1, Lines 540–541]. This somewhat leading question is a task as defined for this study, yet because it directed student attention towards a productive approach to completing the diagonal-of-the-parallelogram task, it is connected to the completion of another task. It was not preplanned, but rather, given to students in response to their work to complete the parallelogram task. This example could be classified as a critical instructional moment involving the selection of a task or as a critical instructional moment involving the implementation of a task since it represents a task related to the completion of another task. Tasks such as these created ambiguity with respect to the classification of critical instructional moments as they were originally defined.

A number of potentially useful redefinitions were explored with the data to establish a way to categorize lesson segments that more clearly delineated the differences among critical instructional moments while also creating categories that could be reliably applied. A task, strictly defined, is any activity that directs a student to the completion of a goal. A task will make clear the goal and the parameters guiding work to complete the task. This notion remained unchanged. However, this operational definition of tasks served only part of the needs
of the study. Tasks served different purposes during instruction and many supported the implementation of other tasks. Thus, further clarification was needed and emerged through the use of the construct of *lead tasks*. Lead tasks carry a number of traits. They are initiated by the teacher at the beginning of a lesson segment, usually have related tasks or modifications associated with them, are marked by a noted shift in teacher language indicating a shift in the lesson, are often planned in advance of the lesson, and are typically tied directly to one of the learning goals for the lesson.

The first task from Jackie mentioned above was classified as a lead task. It represented the beginning of a lesson segment in which the primary mathematical work of the class centered around the completion of this task and was planned to a large extent prior to the lesson (Note: the teacher acknowledged in the stimulated recall interview that the numbers in the problem were chosen at the moment, but the interview reflected prelesson planning of the basic structure of the example). It was also set apart in the lesson by a marked shift in the lesson. Prior to this task being given, the teacher reviewed a bonus problem from the quiz the previous day and reviewed some homework problems from the textbook. The teacher introduced this task with a clean overhead projector sheet and said, “All right, a new thing” [Jackie, Ob #1, Line 467]. This initial statement reflects a distinct shift in the mathematical attention towards a new task. Along with the previous discussion regarding how this task was connected to the learning progression described by the teacher, these markers demonstrate the consistency of this example with the definition of *lead task*. With this new operational definition of a lead task, each lesson observed could be divided into a small number of segments in which the work of the class was closely associated with the lead tasks.
As lead tasks were identified in the data, a need for additional distinctions and refinements to the initial definitions of the other critical instructional moments emerged regarding the implementation of those tasks. Any refinement to the definitions would need to account for the fact that not all tasks given by the teacher during instruction were lead tasks. As lead tasks were implemented, a teacher often constructed or chose other tasks to facilitate the student mathematical work on the lead task or their understanding of the mathematics related to the lead task. The selection of these tasks also represented a critical instructional moment as they were often provided to students in response to their work on the lead task. The second task identified above, a task which happens to be in the form of a question, represents Jackie’s attempt to direct students’ attention to a productive approach to the lead task of finding the diagonal of the parallelogram. As discussed, it was given to the students in response to their response to the lead task. Again, ambiguity existed in determining whether this critical instructional moment should be classified as a selection of a task or an implementation of a task.

The following example captures another dimension of the ambiguity. In the case of the lead task involving finding the diagonal of the parallelogram, one student asked about finding the other diagonal.

Student: Which diagonal are we looking at?
Jackie: The one I drew.
Student: How does 35…is it the entire angle or just the…?
Jackie: It is the entire angle and I would like the length of diagonal.
Student: Can you draw like the opposite diagonal?
Jackie: The other diagonal? We could, except I don’t want that one. I have a reason, but we could find that one.
Student: Wouldn’t that be easier, though?
Jackie: Well, yes.

[Jackie, Ob #1, Lines 521–529]
The student suggested that the teacher draw the other diagonal. Finding the length of the other diagonal did not support the mathematical activity in which the teacher intended for the students to engage. She declined to pursue the student’s thinking and redirected attention back to the diagonal she had drawn. This response to the student’s response to the lead task was also a part of her implementation of the lead task, but it was not a task. Again, the initial definitions of critical instructional moments created ambiguity in the work of classifying them as this lesson segment could be classified as the implementation of the task or as a response of the student. Any refinement to the definitions of critical instructional moments needed to account for this ambiguity as well.

These ambiguities and weaknesses in the initial identification of critical instructional moments were addressed through a series of adjustments to the definitions and categories. As described previously, the construct of lead tasks was identified as a way to distinguish the selection of tasks that were designed to serve as the focal point of the mathematical activity of the students for that lesson segment and the selection of other tasks that were designed to support the completion of the lead tasks. These latter tasks were identified as associated tasks. Since most lead tasks were preplanned, a useful distinction emerged. It became possible to separate the planning and design of instruction from the implementation of it with two critical instructional moments of interest occurring in the planning and design of instruction: the identification of learning goals and the selection of lead tasks.

With this separation of the critical instructional moments found during the planning of instruction from those found during the implementation of instruction, the focus of the analysis shifted to exploring a reliable way to classify the selection and implementation of tasks during
instruction. In doing so, additional ambiguity emerged in the consideration and identification of student responses. Based on the existing literature, the responses of students and a consideration of those responses represent critical instructional moments. However, the initial identification of the category of observable student responses, lacked the power to reliably capture the critical instructional moments related to the responses of students. While the responses of students are certainly critical to instruction, the identification of critical instructional moments in the planning of instruction focused on the work of the teacher. The teacher identified and unpacked the learning goals. The teacher selected or designed the lead tasks. Thus, an inherent inconsistency existed with the inclusion of observable student responses as a critical instructional moment for this study. Of most interest to the researcher and to this study was the work of the teacher regarding the responses of students. In order to consider the demands on a teacher’s mathematical knowledge during instruction, it was more important to focus attention on the responses of teachers to student responses. Therefore, any redefinition of critical instructional moments during instruction needed to account for the selection of associated tasks, the selection of lead tasks that were not preplanned, the implementation of those tasks which did not involve selection of tasks, and teachers’ responses to the responses of students.

Through repeated cyclical analysis and review of the analyzed data, two distinct categories of critical instructional moments during instruction emerged. As students responded to lead tasks and engaged in the mathematical work of them, the teacher had to interpret the response of the student or students and often would construct a response to the student response. Even though the interpretation of the student response was often linked to the construction of a response to a student response, it was not always connected to the construction of a response.
Recall the previous example about the suggestion from the student to draw the other diagonal. The teacher did not pursue it. Both the interpretation of and the construction of a response to a student response play primary roles in the implementation of tasks. Associated tasks created or chosen to support the completion of a lead task represented one type of construction of a response to student responses. Lead tasks created during instruction were constructed in response to a student response—often the inability to respond productively to the preplanned lead task. The focus of the analysis shifted to identifying the interpretation of student responses and the construction of responses to student responses during instruction. This categorization brought clarity to the classification of critical instructional moments by resolving each of the potential ambiguities that emerged from the initial definition of critical instructional moments.

To summarize, these modifications to the framework and the distinctions made within the definition of critical instructional moments served to advance the identification process in a way that allowed for further analysis of the use of mathematical knowledge and reasoning as a teacher encountered critical instructional moments. The initial categories of critical instructional moments were: Identification of learning goals, selection of tasks, implementation of tasks, and observable student responses. Through multiple iterations of analysis and identification of critical instructional moments, the refinements to the framework stabilized to the following:

**Planning of instruction:**
- Identification of Learning Goals
- Selection or Development of Lead Task(s)

**Implementation of lead tasks:**
- Interpretation of student responses
- Construction of responses to student responses
The formulation of these distinctions resolved the ambiguity in classification of critical instructional moments as defined from existing literature and supported the continued analysis of the mathematical knowledge and reasoning the teacher used during instruction at these critical instructional moments. Lesson segments from classroom data, portions of stimulated recall interviews, and excerpts from preobservation conferences were coded according to this framework. These moments became the sites for the examination of the demands on and use of mathematical knowledge and reasoning on the part of each teacher.

**Analysis**

At each occurrence of each type of critical moments, the researcher sought to understand the ways a teacher’s mathematical knowledge is used to plan or implement instruction (i.e. the demands on a teacher’s knowledge or what is required of the teacher to deliver instruction). Of particular interest was the reasoning that a teacher must do in these moments in order to deliver instruction that is consistent with the practices that promote the mathematical activity of students and the development of rich, connected understandings.

The following questions served to guide the analysis at each type of critical moment:

- **Identification of Learning Goals:**
  - What are the teacher’s mathematical goals?
  - How did he or she decide on these? How did he or she think about them?
  - How did the teacher unpack the goals? What does the teacher understand to be the component understandings?
- **Lead Task Selection:**
  - What was the chosen lead task?
  - Why was it chosen? What did the teacher expect the students to do with it?
  - What learning goal did this task support? How did the teacher think about that?
• How will the teacher know the students are getting what he or she hoped they would get from the task?
• How did the teacher decide to present it? What thought did they give to the presentation?
  o Interpretation of Student Responses:
    ▪ What student responses did the teacher deem significant?
    ▪ How did the teacher understand the student responses?
    ▪ In his or her eyes, how were the students thinking about the mathematics under consideration?
  o Responses to Student Responses
    ▪ How did the teacher respond? What is the nature of that response? Did it maintain the cognitive demand? Was it asking for justification? Clarification? Was it asking students to consider a counterexample? Did the teacher modify the task or direct student attention to one or more salient features of the task or the response?
    ▪ Why did he or she respond that way? What thinking did the teacher have to do to understand the student response and to construct his or her response?

The data provided ample opportunity to explore answers to these questions. The responses of teachers to stimulated recall interview questions provided the most direct insights into the nature of teacher’s use of mathematical knowledge and reasoning during instruction. Once the critical instructional moments were identified in the data, the researcher examined the classroom interactions around those moments and teacher comments and responses during the stimulated recall interviews about that particular moment to explore patterns within and across teachers at critical instructional moments. The observations derived from this data were triangulated with those observations from field notes, responses to background interview questions, teacher interactions at other moments, and a teacher’s responses to other stimulated recall questions. Through this analysis, a portrait of a teacher’s instruction at critical instructional moments and the mathematical thinking he or she did at those moments emerged.
Further analysis of the instructional approaches of each teacher at critical instructional moments revealed differences in the nature of instruction across the four teachers in the study. Although these differences are discussed in the next chapter, it is important to note here the nature of those differences and how they shaped the continued analysis of the data.

Through this analysis, patterns within the approach of each teacher emerged and differences across teachers were noted and organized around each critical instructional moment. In Part II of chapter 2, critical features of instruction were explicated and led to the articulation of an image of instruction. These features provided a research basis for the identification of critical instructional moments and formed an organizing structure to the analysis of the data. A summary is provided below as well as a description of the dimensions along which distinctions were made among the teaching approaches observed in this study.

*Identifying learning goals* involves translating the broader goal of developing a rich, connected understanding of mathematics into lesson-level goals. As established from the literature in chapter 2, the identification of the set of component or prerequisite understandings which support the development of rich, connected understandings of a given mathematical idea or concept requires the teacher to unpack the mathematics involved in developing an understanding of the concept or idea under study. It also requires the teacher to simultaneously attend to the salient features of any concepts or procedures, the existing understandings of students, and the possible advantages to various sequencing options. The analysis of instruction and teacher thinking at the critical instructional moment of identifying learning goals revealed differences in the level of mathematical specificity of the lesson-level learning goals.
The selection of lead tasks, in ways that are aligned with the image of instruction, considers a number of factors. A well-chosen task is accessible yet problematic to students and embodies the key mathematical features of the concepts under study. A task must also facilitate the mathematical activity of the student in ways that support the development of the desired understandings. Selection of tasks that serve these purposes involves a consideration of a potential path for students to move from perceived, existing conceptions to new, more connected ones. The analysis of instruction and teacher thinking at the critical instructional moment of the selection of lead tasks revealed differences in the degree to which a task was selected with direct consideration of the learning goal, the accessibility of the tasks to students, and/or the potential for the task to engage students in the mathematical activity that will lead to the conceptual advance.

Using student responses to shape instruction. Implementing tasks in ways that foster conceptual advances necessitates attention to the work and thinking of students. How this thinking is used to shape instruction in large measure determines the nature of the conceptual development on the part of the student. When students are asked to present and understand multiple strategies, make connections among successful strategies, use student errors as learning opportunities, and justify responses, they engage in more problem solving, consider mathematics as a discipline, and practice verbalizing and justifying their mathematical ideas. When teachers use a student response as a basis of inquiry by generating counterexamples, following through to a logical conclusion,
considering a simpler or related problem, and incorporating a student’s method, students are provided an opportunity to develop richer and more connected understandings of concepts and procedures. The analysis of instruction and teacher thinking at the critical instructional moments of interpreting and responding to student responses revealed differences across teachers in the nature and depth of interpretation of student responses and in the degree to which student responses were used to shape instruction.

An examination of these differences required another cycle of analysis to explore patterns and determine what might be understood about the use of teacher knowledge and reasoning during instruction.

The need for further analysis of these differences led to an examination of the use of mathematical knowledge and reasoning during these critical instructional moments. Several questions guided this analysis:

- To what extent do differences in the nature of the use of mathematical knowledge and reasoning explain the differences in instructional practice at those moments?
- What characterizes the nature of the use of mathematical knowledge and reasoning during these moments?
- What characterizes the differences among teachers in the use of mathematical knowledge and reasoning during these moments?

To seek answers to these questions, the researcher sought to identify the nature of mathematical knowledge and reasoning at critical instructional moments and to explore potential differences in the use of mathematical knowledge and reasoning across teachers. The results of this analysis are discussed in chapter 5.
Chapter 4:

Differences in Instruction at Critical Instructional Moments
The goal of this study is to deepen our understanding of how, by focusing on the use of mathematical knowledge and mathematical reasoning during instruction, the mathematical knowledge of teachers impacts instruction. This exploration of the use of mathematical knowledge during instruction necessitated the development of a focused look at critical moments of instruction. As discussed previously, critical instructional moments were initially identified and defined through a review of literature and their definition was subsequently refined through this research. In conjunction with an articulation of the teaching practices that support the development of rich, connected understandings of mathematics, these critical instructional moments provided a platform for this research. They afforded the opportunity to construct a useful refinement to our understanding of the mathematical knowledge for teaching by explicating the use of a teacher’s mathematical knowledge at these critical instructional moments.

The results of this study are discussed in this chapter and the next. Those results, emerging from the data, contributed the following:

- Refinements of and distinctions made to the construct of critical instructional moments that provide clarity that supports subsequent research to be conducted around these moments (discussed previously),

- A research-based description of the demands on the use of a teacher’s mathematical knowledge during these critical moments including the development of the construct of pedagogical content reasoning—that is, mathematical reasoning done by a teacher during instruction or the planning of instruction, and
An initial articulation of distinctions among the demands on the mathematical knowledge of teachers for teachers with varying degrees of fidelity in their implementation of instruction that produces rich, connected understandings of mathematics.

With critical instructional moments identified for each of the four teachers in the study, the researcher sought to understand the demands placed on a teacher’s mathematical knowledge by a triune focus on the mathematical goals for students, the mathematics of the students and the mathematics of the task. The nature and scope of mathematical knowledge needed for teaching is still emerging, and the demands on mathematical knowledge during instruction need to be understood more fully.

Of particular interest in this study was the nature of the demands on mathematical knowledge at critical instructional moments. As the teacher thinking and instructional moves around these moments were analyzed, differences were noticed across the teachers in the study in nature of their instruction at critical instructional moments. Two questions arose along the lines of these differences. What was the nature of these differences? And, what role did the use of their mathematical knowledge or reasoning play in those differences? This chapter explicates the nature of the differences observed in instructional practices at each type of critical instructional moment among the four teachers in the study.

Variation existed across the four teachers in the study in the focus, extent, and specificity of the articulation of the learning goals for the lesson, in the mathematical unpacking of those goals, in the consideration of existing student understandings, and in the selection and sequencing of tasks designed to support the development of the understandings inherent in the
learning goals. Variation also existed in the degree to which the teachers sought to understand
student conceptions as they emerged during classroom activity and the ways in which that
understanding was used to shape the mathematical work of the class, the selection of associated
tasks, and the responses to student responses.

A detailed discussion of the nature of those differences along the following dimensions of
instruction is provided in this chapter.

- Articulation and unpacking of learning goals
- The nature of lead task selection
- The elicitation and interpretation of student responses
- The use of student responses to shape instruction

**Differences Among Teachers in the Articulation and Unpacking of Learning Goals**

The work of Jackie and Harold represented the most extensive unpacking of learning
goals of the four teachers in the study. Both specifically identified the understanding(s) each
sought to develop in students, and both articulated a progression of understandings each expected
students to develop as a potential mathematical path from the students’ existing understandings
(as understood by the teacher) to the desired understanding(s). Susan and Duncan did not
articulate as extensive an unpacking of the desired understandings. There was less specificity to
the description of the mathematical goal(s) for students and limited articulation of a potential
mathematical path (a sequence of understandings) to develop the desired understandings.
Jackie’s Articulation and Unpacking of Learning Goals

The classroom and the stimulated recall data from Jackie revealed a thoroughly developed set of learning goals, a consideration of student thinking and potential reasoning, and a selection and sequencing of tasks that represented this consideration. Throughout the planning of the lesson, Jackie remained keenly focused on connecting existing understandings of students with the understandings she intended for them to develop, looking for a mathematical path from one to the other. This focus resulted in well-chosen tasks—accessible and problematic to students with the potential for developing the desired understandings.

The three lessons observed worked in concert to develop foundational understandings of the unit circle and trigonometric functions. During the interview about the second observation, Jackie described her goal for the sequence of lessons.

Jackie: …my long-term goal is I want to look at angles greater than 180 and negative angles. So, once I can get rid of the triangles completely and say, “So really if we just knew the ordered pair and drew this line, could we find everything we wanted to know about that angle including the size of it?” And that’s going to be a yes. So, well then we don’t need a triangle at all. Then can this ordered pair be over here (teacher points to second quadrant). What would our ordered pair be?

[Jackie Ob #2, Lines 46–53]

In this excerpt, Jackie expressed a clear and specific image of what she wanted students to know and be able to do at the end of this lesson sequence. Namely, she wanted students to understand the relationship between the ordered pair, the length of the vector, and the angle (as a measure of the amount of rotation). Jackie also recognized that the triangle represented a conceptual restraint for angles greater than 180 degrees and that understanding this relationship without the constraints of the triangle formed a foundation for understanding trigonometric functions and
polar coordinates. In the third interview, Jackie expressed her goals for students and the potential power of this conceptual shift in this way:

Jackie: And then I want them to see, once we leave the triangle behind, now we can talk about angles like full rotations, many rotations and then we can sort of see the pattern and then, not this year, but next year, go onto the function trigonometry, $y = \sin x$ and if they understand the transition I’ve set up for them, then that should be another easy transition and then they see the pattern of basically, any sinusoidal phenomenon. Interviewer: So, what is the transition you set up for them today or the last couple of days.
Jackie: Well, what they are going to do tonight, is just take lots of ordered pairs, most of them I gave them the ordered pair and at the end, I gave them the length of the diagonal and the ordered pair minus one of the coordinates. And we will do variations on that and eventually, we will get to just by way of exploration, what if the radius was one in a circle, how does that change the ordered pair. Then we will observe that the $x$-coordinate is the same as the cosine of the angle and then we will try to figure out why that would be. And then we will revisit the equations that we looked at today: $\tan x$ equals this over that and see how we can solve that for the $x$ and $y$ coordinate. (pause) And then, it would be kind of really fun to move into a tiny bit of polar coordinates

[Jackie Ob #3, Lines 21–42]

In this description, she implicitly identified the constraining nature of the triangle with her statement, “once we leave the triangle behind.” She also made two explicit references to the conceptual link between understanding this relationship between the ordered pair and the angle and the understanding of the unit circle, of function trigonometry, and polar coordinates. As she expressed, she understood that there is a progression from triangle trigonometry to circle trigonometry to function trigonometry. In short, through these comments, Jackie demonstrated her understanding of the mathematics of the student, the mathematics of the learning goal, and a potential path to get from one to the other.

However, understanding the mathematics of the learning goal is not enough to design instruction. As discussed previously, the mathematical goal must be unpacked with consideration given to the mathematics of the student and how they might move from their
current understanding to make the conceptual advance. Jackie developed a sequence of learning activities designed to move students from existing understandings to understanding of vectors to a direct relationship between the coordinates of a point on the ray and the trigonometric values of the angle. Developing this progression required her to think about her mathematical goal, analyze the goal to identify component understandings, and work back from the goal to construct mathematical experiences that would be accessible to students while supporting the development of the component understandings. The elements of the progression were as follows:

- Finding all angle measures and lengths in a kite with given information,
- Finding the angle and length of a diagonal of a parallelogram using triangle trigonometry,
- Defining the concept of a vector and turning the diagonal problem into a vector problem,
- Changing the problem context to the coordinate plane, and
- Removing the parallelogram and triangle altogether and getting students to reflect on a potentially direct association of the angle and length to the coordinates of the tip of the vector.

The sequence of activities and their implementation served to scaffold new understandings based on existing ones in ways that supported the foundational understandings for which she strived. It also provided evidence of the work Jackie did to unpack the mathematics of the learning goal as well as her consideration of the existing understandings of students and the ways they might approach the tasks.

**Harold’s Articulation and Unpacking of Learning Goals**

Harold also gave thought to the progression of understandings that would support the attainment of the desired understandings for his Math 6 students. Harold designed the two-lesson sequence to develop student understanding of the division of fractions. In the preobservation interview, Harold articulated the goal and the learning progression to be the development of an understanding of the division of fractions through the modeling of a series of
mathematical relationships. He presented the following analysis of that goal referring to the division problem, $\frac{a}{b} \div \frac{x}{y}$:

Harold: We are going to divide by $x$ which they are going to recognize as the same as multiplying by $\frac{1}{x}$ and we are going to divide by $\frac{1}{y}$ which is where I was hoping to go next, but really never got there. But we’ve been over that [dividing by $\frac{1}{x}$] before and frankly, I think they are a little bit better with that one than this one [dividing by $x$ is the same as multiplying by $\frac{1}{x}$], but they know that dividing by $\frac{1}{y}$ is the same as multiplying by $y$. So we are going to take dividing by $x$ over $y$ and convert it to dividing by $x$ times $\frac{1}{y}$, $\left(\frac{a}{b} \div \left(x \cdot \frac{1}{y}\right)\right)$. Convert that to dividing by $x$ and then taking all of that and dividing by $\frac{1}{y}$, $\left[\left(\frac{a}{b} + x\right) + \frac{1}{y}\right]$. Convert that to this times $\frac{1}{x}$ times $y$, $\left(\frac{a}{b} \cdot \frac{1}{x} \cdot y\right)$. That’s the hoped-for progression.

[Harold, Ob #1, Lines 105–115]

As expressed in this excerpt, the development of understanding of division of fractions involves the development of the following progressive understandings:

— Since $\frac{x}{y}$ is the same as $x$ times $\frac{1}{y}$, the division of $\frac{a}{b}$ by $\frac{x}{y}$ can considered in two parts—that is, dividing by $x$ and then dividing by $\frac{1}{y}$.

— Dividing by $x$ is the same as multiplying by $\frac{1}{x}$.

— Dividing by $\frac{1}{y}$ is the same as multiplying by $y$.

For Harold, these relationships represented the component understandings required for an understanding of division of fractions. They represented a detailed unpacking of the learning goal and established a progression from existing understandings to desired ones.

In the Algebra 1 classes, Harold asked students to solve a system of inequalities. They had not previously been asked to solve a system of inequalities or even graph an inequality. They had experience with solving a system of equations and graphing equations, and, from Harold’s perspective, the students had the mathematical tools to solve the system of inequalities. The mathematical work of the students during the two-lesson sequence centered on solving a
single system of inequalities. Data from the stimulated recall interview and classroom observations suggest that Harold gave consideration to what he concluded were the existing understandings of students and the ways he anticipated they would use those understandings to complete the task.

Harold expressed goals for the lesson sequence as well as broader goals for the students in that class. The goals themselves reflected his effort to facilitate the development of relational understandings (Skemp, 1976) on the part of the students. For example, throughout the two-lesson sequence, Harold emphasized a definition of a solution that had broad applicability to all types of mathematical statements. When asked about the question he posed to students, “What makes something a solution to a system?” [Harold, Ob #2, Lines 55–56], Harold responded:

Harold: So, if they memorize, “Solving the system means finding the solutions,” well, that is only useful if they know what a solution is. And again, all this is trying to get them to reiterate that a solution is a value or set of values that makes the statement true. I use “statement” so that we could cover equalities and inequalities.

Interviewer: So what is useful about that? Let’s say they understand it, what is useful about that?

Harold: So again, what they are looking for are a way to show, to represent all the values or sets of values that make EVERY inequality, in this case, every statement in the system true. So, if they have to apply something like this to the real world, they want to recognize, “Hey, I am looking for two numbers so that twice one number plus three times the other…” They are looking for specific things and they have specific things they want to make true.

[Harold Ob #2, Lines 57–74]

Two points are worth noting here—one about the specific goal Harold had for students related to this topic and one about how his work with students supported his broader goals for students. To the first point, Harold emphasized a general definition of a solution as the value or set of values that make a statement true. Rather than defining solution to an equation in one way and a solution to an inequality in a separate way, he emphasized this general notion that applies in both
cases and used the word “statement” instead of equation or inequality. Additionally, he made a distinction between a solution and a representation of the set of solutions. When a student responded to his question about a solution to a system, the student used the phrase, “To show all of the solutions to the system” [Harold, Ob #2, Line 37]. Harold directed the attention of the class to the response:

Harold: Let me ask something real quick. What do you guys think of the word, “Show?”
Student: To represent.
Student: To define.
Harold: What do you guys think of the word, “Show?” I like the word, “Show” especially for this, because what is the only way to find all of the solutions?
Student: To show them
Harold: You would have to represent them and what is the only way to represent them in this sort of system?
…
Harold: You have to show them graphically. [Harold, Ob #2, Lines 38–52]

Harold’s emphasis on the general definition of a solution and the distinction he made between the solution and the representation of it suggests a careful consideration of his learning goal for students and in particular, how the students’ work with this system of inequalities could have reinforced and extended existing understandings of solutions in general as well as other systems.

The second point worth noting about Harold’s responses to students and to the researcher about the question, “What makes something a solution to a system?” involved another dimension of his broader goals for students. His responses reflected his expectation that students understand the underlying mathematical meaning rather than simply memorize a process. This goal for students is manifested as much in what he did not do with students as in what he did. For example, at no point in the two-lesson sequence did Harold provide students with a procedure for graphing an inequality or for solving the system of inequalities. He made repeated
and persistent reference to this generalized definition of the solution to a system. For example, when a group of students expressed difficulty with knowing how to show the solutions to an inequality, Harold directed their attention to the notion of solutions.

Harold: So what we are trying to learn here: How do we know what to shade, when to shade, why to shade…You’ve got to recognize that it is all about looking for solutions. Looking for sets of values that make that statement [pointing to the inequality] true.

[Harold, Ob #2, Lines 383–386]

In another portion of the same lesson, Harold worked with the class to show the graph of the solution of one of the inequalities of the system. Harold had sketched the graph of the line, $x=2$.

Harold: What do you know about every point on that line?
Student: The $x$-coordinate is two. [H writes $x=2$]
Harold: Does that make sense for the equation of that line?
Harold: [H interrupts the video during the stimulated recall interview]
You know I am asking them about the equation of a line. I have never talked to them about how to get the equation of a line. So really, I am introducing this idea, so I was actually pretty pleased. I guess we talked about that last class.
Interviewer: [restarts video again]
Student: Yea
Harold: So how do we show all of the points that make that statement true [H writes $x \geq 2$]?  
Student: Because it is greater than or equal to two, it has to be everything that is greater than or equal to two.

[Harold, Ob #2, Lines 503–519]

Again, Harold directed student attention to the notion of a solution and asked students to apply it to graph the solution set of both the equation and the inequality.

These exchanges with students illustrated a broader goal Harold had for his students. Essentially, he wanted students to understand and be able to apply what they know. Harold emphasized this point early in the second observed lesson.

Harold: And so, the way I get them out of that habit of always thinking about these steps they are supposed to execute. Instead have them always think about what I am trying to do. “What is the task here?” So hopefully, everything they are doing is geared
toward that task. So, okay, “I am trying to find values that make it true. I am trying to find solutions.” Instead of, “Okay, so first I add six to both sides, I distribute…” Basically trying to always make sure they have in their mind, “This is my goal.” Cause if they don’t have a goal in mind, then they are doing stuff without any reason, and they can’t possibly understand what they are doing.

[Harold Ob #2, Lines 18–27]

In this response, Harold made explicit reference to avoiding having students look for the “steps they are supposed to execute.” Instead, he wants them to think first in terms of what they are trying to accomplish. In other words, Harold wanted students to understand the context in which they were applying the mathematics and why it made sense given the problem they are trying to solve. He expressed it a bit differently in the background interview.

It is three questions a student can ask themselves to find out if they understand something. If they can explain what they are doing, why they are doing it, and how they can justify that it is correct mathematically—if they can do all three of those things, then they might understand, assuming all three of those things are correct.

[Harold, Background Interview, Lines 91–96]

In Harold’s view, the ability to answer these three questions constituted mathematical understanding and served as broadly applicable goals for his work with students.

In summary, these broader goals were reflected in Harold’s approach throughout the observed lessons in his emphasis on the application of the generalized definition of a solution to solving the system of inequalities and in the lack of the establishment of a routine set of steps for students to follow to solve these types of problems. Harold’s consistent emphasis on the application of this definition of solution throughout the two-lesson sequence suggested clarity on Harold’s part, about the learning goals for the lesson. Furthermore, the repeated references both during class and the stimulated recall interviews to the approach students could have taken to solve this system of inequalities suggested that Harold carefully considered what he understood
to be the existing understandings of students in the establishment of the learning goals and the articulation of a potential solution path using those understandings.

**Susan’s Articulation and Unpacking of Learning Goals**

Susan expressed a broad set of goals for students identifying the mathematical ideas she wanted her students to understand about rational functions. However, the data from the observations of and interviews with Susan revealed limited expression of the specific conceptions a student should develop to support these broader goals. This revealed a lack of specificity in her articulation and unpacking of the learning goals for the lessons. Susan’s learning goals were expressed in broad and wide-ranging terms, and the mathematical understandings to achieve those goals were not unpacked in a way that supported the selection and sequencing of tasks designed to develop those understandings.

The two-lesson sequence centered on understanding rational functions with a primary focus on the relationships between the equations of rational functions and their graphs. When articulating the goal of the first lesson of the sequence (this lesson was observed being taught to two different sections), Susan gave a nonspecific response to the researcher when asked about her goals for the lesson sequence.

> Exposure to rational expressions; trying to understand the definition of a rational function and to see how it fits in initially, an introduction to how it fits into our study of functions.  
> [Susan, Interview #1, Lines 16–18]

When asked for further elaboration and specification, Susan replied:

> So specifically, the fact that it involves a ratio, that it is a ratio of algebraic expressions, not just integers. That from our previous understandings, that comes with some complications, such as zero denominators. And also, we just finished polynomials, our overview of polynomials. And actually, for some of them when we studied quadratics, they had some difficulty with manipulating quadratic expressions so I thought there
would be for some, more comfort level, than others in recognizing those connections within those polynomials. So I did hope that they would see connections with the power functions that we have studied and the connections with asymptotes, the characteristics of functions like end-behavior, and it would be nice if we also had some time to get into the manipulation to be able to determine if an expression, just because it is written as a ratio of polynomials is necessarily a rational expression or not. In other words, does it simplify to something that is rational or not.

Susan, Interview #1, Lines 27–39

This excerpt revealed several goals Susan had for students related to developing an understanding of rational functions: that rational functions involve a ratio of algebraic expressions; that there are complications such as zero denominators; that there are connections with power functions, in particular, asymptotes; the characteristics of rational functions such as end-behavior; determining whether a rational expression is indeed rational. Her response made it unclear as to the specific understandings she sought to develop. Unlike Jackie’s articulation of goals in which she clearly identified a goal and a progression of understandings to achieve that goal, the elaboration of Susan’s initial response contained a number of specific items she identified as goals—“involves a ratio,” “complications such as zero denominators,” “connections with polynomials,” “connections with power functions,” and “connections with asymptotes,” but limited articulation of a set of component understandings or a progression of understandings to attain her goals. For example, she identified “connections with asymptotes” as one of her goals. Understanding asymptotes involves a number of component understandings: understanding the numerical behavior of a function at an asymptote, understanding the graphical behavior at an asymptote; understanding the relationship between the equation of the asymptote and the restrictions on the domain of the function, and understanding algebraically when a restriction yields an asymptote and when it does not. This type of detailed analysis of what it means to understand the “connections with asymptotes” are not identified in Susan’s statement of goals. An analysis of
data involving Susan’s description of goals or implementation of instruction for the lesson sequence revealed limited additional evidence of further specification of learning goals and articulation of what it means to understand rational functions, asymptotes, and the relationship between the algebraic and graphical representations of rational functions.

An example of Susan’s lack of specificity of the learning goal manifested itself in the third observed lesson involving the long-run behavior of rational functions. The goal for the lesson was for students to understand how to determine long-run behavior of rational functions by re-expressing the function using polynomial long division, but Susan did not explicitly articulate the component understandings. Several aspects of the lesson provided a picture of the level of consideration Susan gave to the articulation of the learning goals and the component, mathematical understandings of those goals.

In the first part of the lesson, Susan drew parallels for students between the process of long division within arithmetic and the procedure for polynomial long division. After students completed a numerical long division problem, the class, with Susan’s direction, identified the four steps of the process. Susan had unpacked the goal of understanding polynomial long division and identified the parallels that could be drawn with the existing student understanding of the long division algorithm for whole numbers. While it represented a meaningful consideration of the existing understandings of students and how they might be used to build new understandings, a potentially problematic aspect related to Susan’s unpacking of the learning goal emerged—namely, that the two processes were not as “identical” as Susan presented them to be.
The imprecision of the parallel Susan intended to draw between the two processes revealed itself throughout the lesson as Susan attempted to explain the steps in the polynomial long division process in terms of the long division for whole numbers. She asked one student, Ellen, to present her work on the long division algorithm for 17 divided into 3820 and to describe her process. Susan then proposed a polynomial long division problem and urged students to apply the same four steps to it. She presented the problem this way:

Susan: Now we are going to apply the same process, but to algebra. So, I want to look at a division with polynomials. Same four step process. Suppose I have (Susan refers to the rational expression shown in Figure 4.1):

\[
\frac{3x^2-5x+4}{x+2}
\]

Susan, Ob #3, Lines 189–191

\textit{Figure 4.1. The rational expression used for polynomial long division.}

After writing the ratio as a long division problem, Susan directed students to consider the “guzzinta” step for the polynomial long division—which reference was her word for “goes into,” the first step in the numerical long division algorithm. Susan asked Ellen about the numerical long division process she followed,

Susan: Notice, when Ellen was saying 17 goes into 80, how did she figure that out? How did you figure that out, Ellen?
Student: Just find the closest multiplier to 80.
Susan: Did you do a lot of multiplications by 17 or did you just approximate it?
Student: Well, I kind of like guessed. Two times 17 is 34 and 4 times 17 is 68.
Susan: Do you understand her approximation?
Susan: 17 is close to 20 and 20 will go into 80 four times so that was a very reasonable guess for her. Okay? We are going to do the same thing. We are not going to figure out all of this. We are not going to use all of \( x \) plus 2, we are going to use that critical part, the first \( x \). So \( x \) goes into \( 3x^2 \) squared, how many times?
This excerpt revealed three different potential difficulties with the parallel she attempted to draw between the first steps of the two processes. The first difficulty involved the way Susan characterized the work of the student. When describing the first step in the polynomial long division process, Susan asked, “Do you understand her approximation?” The student had described a process in which she kept finding multiples of 17 until she got close enough to 80. She said, “Well, I kind of like guessed.” However, Susan described the work of the student as an approximation and explicitly stated a different approach than the student presented, “17 is close to 20 and 20 will go into 80 four times, so that is a very reasonable guess for her.” The student did not make an approximation to determine the multiplier in the way Susan described and Susan’s misrepresentation of her work was used throughout her efforts to explain the first step in the polynomial long division process. This misrepresentation seemed to add to the confusion of students as they expressed difficulty in seeing the parallels.

A second difficulty this excerpt revealed was Susan’s insistence that the two processes were exactly the same as she described the “guzzinta” step for the polynomial long division.

We are going to do the same thing. We are not going to figure out all of this. We are not going to use all of x plus 2, we are going to use that critical part, the first x.

The processes are parallel, but they are not the same. In the long division algorithm with whole numbers, the estimate for the first digit of the quotient comes from determining the number of times the divisor can go into a truncated version of the dividend. In the case of the example used by Susan, 17 divided into 3820, this first step would involve determining the largest multiple of 17 that would be less than 3820 and that can be estimated by dividing 17 into 3800. Depending
on the whole numbers involved, it could also involve truncating the divisor to facilitate the estimation of the first digit of the quotient, but that truncation does not make sense with a divisor of 17. For polynomials, the divisor is “truncated” in a sense as you determine the partial quotient by considering the first term of the binomial and divide it into the first term of the dividend. In short, the processes are not the same as Susan asserted, but there are useful parallels to draw as she did. This imprecision in the parallels between the two processes combined with the misinterpretation of what the student did and the repeated assertion throughout the lesson that the processes for numerical and polynomial long division were the same suggested a limited unpacking of the mathematics of an understanding of polynomial long division.

Further evidence of the inadequacy of this analogy was found in the lengthy discussion in the class involving multiple students and their confusion over Susan’s application of the “goes into” rule when she just considered how many times $x$ goes into $3x^2$. The students asked multiple questions about why the entire expression, $x+2$ was not used.

Student: What happened to the two?
Susan: I’m sorry.
Student: [With teacher pointing to the two in $x$ plus two] On the left there is a two. What happened to that up top?
Susan: Nothing yet.
Student: Why? Because shouldn’t… that doesn’t make any sense.
Susan: [Going back to the screen with the numerical example] When I look at the arithmetic problem, I only considered, two times 17 should give me something close, right? So I do the multiplication of two times 17.

[Susan, Ob #3, Lines 336–344]

Susan continued to draw the parallel with the numerical example based on the approximation idea in which 17 divided into 80 is approximated by thinking about how many times 20 goes into 80. In actuality, the student who worked through the numerical example on the board described a different numerical process, “Well, I kind of like guessed. Two times 17 is 34 and 4 times 17
is 68.” [Susan, Ob #3, Line 204]. A more extensive consideration of the specific component understandings of the steps of the process could have supported Susan’s work during the class as she interpreted the students’ work and emphasized the parallels between the processes.

Susan continued to reinforce the notion that the two processes were identical throughout the lesson as students experienced difficulty with the process of polynomial long division. For example, after a student expressed further confusion about the determination of the multiplier in polynomial long division and the actual multiplication step, Susan stated, “It looks much more complicated with the algebra, but the process is identical.” [Susan, Ob #3, Lines 384–385]. The surprise she expressed in the stimulated recall interview regarding the difficulty of the students in understanding polynomial long division through their understanding of numerical long division suggests that this difficulty was unanticipated.

In summary, Susan demonstrated a lack of specificity in her expression of the learning goals when directly asked. She expressed those goals—“involves a ratio,” “complications such as zero denominators,” “connections with polynomials,” “connections with power functions,” and “connections with asymptotes,”—without expressing a careful analysis of the complications or connections to which she alluded. During the third observed lesson in which her goal was more clearly defined—developing an understanding of polynomial long division and re-expressing rational functions using long division to determine end-behavior—Susan drew an imprecise analogy between the processes for numerical and polynomial long division. The inattention to the differences in the processes represented a level of consideration of the component understandings that caused unanticipated difficulties for students.
Duncan’s Articulation and Unpacking of Learning Goals

Duncan’s learning goals involved introducing students to parametric equations and developing their skill in working with various aspects of them: converting parametric equations into rectangular form and constructing equations of sinusoidal parametric equations so that the modeling of the motion would begin at a particular point. However, these goals and Duncan’s articulation of them were limited in their description of the conceptions and understandings a student needed to have to understand parametric equations.

The three lessons observed and taught by Duncan involved applications of parametric equations. The lessons were somewhat separate from one another as understandings from one day were not explicitly needed the next. In each case, the focus of the lesson was introductory in nature or emphasized procedural skill development with limited depth in the consideration of underlying concepts. A discussion of his expressed goals and the unpacking of those goals follows:

In his introduction to the class, Duncan outlined what the first class would be about:

Duncan: What I am going to do today is introduce parametric equations, show you how we use them, show you how to put them on your calculator, cause there is a parametric mode on your calculator and we will look at a couple of examples. One is just kind of mathematical. You may not see much usefulness for it, but then the next two, you will definitely that they are kind of more real life stuff. I want you guys to see and hopefully explain why these guys are important.

[Duncan Obs#1, Lines 6–12]

The primary goal was introductory in nature. In other words, there was limited consideration of concepts to be understood and developed. Rather, the teacher used words like “see” and “do” to describe what he wanted students to gain from the experience. The subsequent class discussion
of the first example illustrated the limited consideration of the underlying concepts and relationships. In the example, the teacher constructed a table of values devoid of context.

Duncan:  Now, I just want to give you a basic example because we are going to look at things separately, horizontally and vertically. [Duncan writes a table of values with three columns on the board].

![Figure 4.2. Duncan’s table introducing a parametric relationship.](image)

Even in this initial presentation of the problem, the reference to looking at “things separately, horizontally, and vertically” suggested an imprecise consideration of the use of parametric equations to describe and explore relationships between quantities. The representation of the relationships between these quantities captures how each quantity relates to time (how it changes over time as well as the value of that quantity at any given time) as well as how the two quantities (x and y) are related to each other—their covariation.

After the presentation of the table, Duncan proceeded to lead the class to the plotting of x versus t, y versus t, and x versus y and then referenced those graphs in the following teacher–led discussion.
Duncan: So if you notice, make sure you see this, we have this table of three columns.
Student: t and x
Duncan: Yea, the first two columns. This one [pointing to the y-t graph] represents which two columns?
Student: t and y
Duncan: And this one [pointing to the third graph] represents…
Student: x and y
Duncan: And you are just plotting points. This [first graph] is going to represent our horizontal motion. This [second graph] is going to represent our vertical motion, over 4 seconds for both of them, and when you put them together [pointing to the third graph] our horizontal and vertical motion together, they are actually traveling in this rectangular shape. Sometimes we are going to want to look at what is happening horizontally and vertically, together and you have to get back to x-y.

Initially, Duncan described the graphs as representing the columns rather than a relationship. Even when he referenced the graphs as representing vertical and horizontal motion in the last part of the excerpt, this reference alluded only to the “motion together” rather than what the table or graphs might reveal about the relationship between the two quantities—in other words, how one changes with respect to the other. His description, which paralleled his focus with the other problems discussed during the first observation, referenced looking at “what is happening vertically and horizontally together.” While true, this description neglected to emphasize that the x-y graph shows how y changes with respect to x. This phrasing and the use of the term “together” was repeated throughout and suggested the limited depth to which Duncan considered what it means to understand these graphs—his expressed learning goal for the lesson.

Duncan’s discussion of his choice of the second example from the first observation involving equations representing the populations of foxes and rabbits further illustrated the nature of his consideration of the underlying relationships and the way he considered using the existing understandings of students.
Interviewer: What’s the purpose of the real world connections for you?
Duncan: Well, a little background on these two equations (referring to the second example), and these are from our textbook. When I look at them, I say, great, I can go back and get some trig review, which they haven’t done in weeks, go back and kind of pull that in and see how much they remember some of the characteristics. That’s why I spent some time writing out period, amplitude and so forth. But I also wanted them to realize, “Okay, there are a lot of things,” and we talked about this when we did trigonometry, about populations, you know, how they can cycle because of prey-predator relationships and so I wanted them to see that example again, because some of these kids will go on to take Environmental Science or some other classes and just to make that connection cross-curricularly. And secondly, here’s a good example of what these things look like separately and then putting it together, I think they were surprised that it was elliptical. I think I heard some kids say it was the circle of life kind of thing. Essentially, that is what it is although it is not circular. But that is the purpose to getting them to see, yea, we can model something that happens in nature, pretty easily mathematically, and then from this mathematical model, we can actually look at how these two animals are related to each other and what that looks like graphically.

[Duncan Ob #1, Lines 157–181]

In this description, Duncan highlighted his intent to integrate review of previously developed understandings or skills as well as his emphasis on making real-world connections. He made a reference to the review of trigonometry with this example and a cross-curricular connection to Environmental Science. However, the real-world connection was described nonspecifically, “we can actually look at how these two animals are related to each other and what that looks like graphically.” The populations of the two animals are related and one can represent that relationship with the rectangular graph of the one population versus the other. The parametric equations and the corresponding graphs represent the populations of each over time. Yet Duncan does not express this level of mathematical specificity in describing the mathematical relationships in his thinking during the stimulated recall interview nor did he do so during the class. In the class, Duncan summarized his goal for this example in this way:

Duncan: So please, everyone understand what I want you to see. The rabbit population goes like this, it’s sinusoidal. The fox population also goes like this, but it trails it a
bit. But, when you put them together, those two sinusoidal functions when you put them together parametrically is actually modeled as an ellipse. And we know, every 12 months we are going to repeat our values.

[Duncan, Ob #1, Lines 620–624]

The emphasis is on what he wanted students to “see.” Duncan wanted students to see mathematically what it looked like when “you put them together,” rather than using the mathematics to explain the relationship between the two quantities.

Duncan’s inattention to a careful consideration of what the mathematical model represented generated some confusion for students. One student expressed that confusion explicitly, but Duncan’s response further reinforced the nature of his definition and specification of his learning goals. When a student said, “I don’t understand what this is?” [Duncan, Ob #1, Line 652], Duncan used an example of a person’s height and width changing over time and made this statement:

Duncan: Sometimes we are looking at just one thing at a time or with parametrics, we can actually look at them together. We can look at them separately and then we can put them together. So here, what are the rabbits doing (motions sinusoidally). What are the foxes doing (motions sinusoidally). What do those things look like together?  

[Duncan, Ob #1, Lines 663–668]

Again, Duncan described the value of parametrics in terms of “looking” at the quantities. “We can look at them separately and then we can put them together.” This description, repeated throughout the first lesson, revealed the surface-level understanding Duncan sought to develop in students. Duncan’s consideration of what students need to understand about parametric equations began with the simple notion of “look[ing] at them separately and then [putting] them together,” and it did not progress much beyond this in the first lesson. These goals for the lesson reflected a limited view of what is mathematically important for students to understand by omitting a reference to the relationships represented by parametric equations, the rectangular
equation, the graphs, and the tables of values and what these various representations reveal about those relationships.

Further evidence of the nature of his goals for students was found in Duncan’s third observed lesson. As he explained to students in the following quote, his goal was straightforward skill development as students practiced the skill of converting from parametric equations to rectangular ones.

Duncan: [to the class] I want to take a look at taking parametric equations and converting them back to rectangular equations cause sometimes it is going to be easier to look at them in rectangular form or parametric form.

He provided students with sets of parametric equations and asked them to combine them to produce an equation in rectangular form.

Duncan: Now today, let me pull up what we are going to do today. [D goes to computer and projects the equations listed below]. Okay, here are some equations.

4. \( x = \frac{3t}{4}, y = 2t - 1 \)
5. \( x = 2t, y = \frac{4}{t} \)
6. \( x = t^2 - 3, y = t^2 + 1 \)
7. \( x = ln(t), y = e^{ln(t)} \)
8. \( x = sec(\theta), y = cos(\theta) \)
9. \( x = 2cos(\beta), y = 2sin(\beta) \)

When asked about his reason for choosing these examples, Duncan revealed a potential influence of the researcher on his thinking, however, as the response suggested, his choice of examples mirrored his thinking from previous classes.

Interviewer: So here you are converting parametric to rectangular. How did you choose these?
Duncan: I knew you were going to ask that question. I purposely chose the first two to be…well the first one to be a very simple way in that you could isolate \( t \) in either equation and I am glad in the class that you taped that John decided to do it in the \( y \) equation to show that it took a little bit longer, but I knew it was something that they could isolate pretty quickly and convert it to rectangular. And the next one, I wanted to do a little bit of review with exponential, you know, let’s look at exponential functions, how do we undo that, logarithms. Most kids remembered logarithms and so forth. Six, seven, and eight were purposefully written so that you did not have to solve for the parameter. There was an easier way to eliminate the parameter than actually solving for \( t \). Although, I did see a couple of kids who said I’m just going to end of solving for \( t \) and squaring a square root and all that kind of stuff that they didn’t need to do necessarily. But six, seven, and eight were purposefully set up so that there was a…actually…yea, those three were set up so that they could get through without eliminating. And then nine and ten was just once again a review of how I could take a circle or an ellipse and put it back into rectangular form hoping that they would recognize that those two equations were those two shapes.

This discussion of his reasons for choosing these examples captured his focus on the development of procedural proficiency of his students. He also expressed an understanding of the scope of different situations the students might encounter in the problem sets and included examples that embody those differences. His selection of examples for this lesson as well as the other lessons revealed a pattern of integrating review of previously studied functions and relationships into the sample problems designed to demonstrate and build new skills. However, for Duncan, the goals for the lessons were largely exposure to routines or introductory notions of “putting things together,” and as such lacked a specific characterization of the understandings he sought to develop.

**Summary of Differences in the Articulation and Unpacking of Learning Goals**

An examination of the articulation of the learning goals for each of the observed lessons and an analysis of the unpacking of the mathematics underlying those learning goals revealed
important differences among the four teachers in the study. The work of Jackie and Harold represented the most extensive unpacking of learning goals of the four teachers in the study. Both specifically identified the understanding(s) each sought to develop in students, and both articulated a progression of understandings each expected students to develop as a potential mathematical path from the students’ existing understandings (as understood by the teacher) to the desired understanding(s). In contrast, neither Susan nor Duncan articulated the learning goals for students in ways that would support the conceptual development of students. For Susan, a broad set of understandings were identified, but they were not unpacked into a set of well-sequenced component understandings that could be used to design instruction. Evidence existed that she did not give careful thought to the potential path from what she perceived to be the existing understandings of students to these desired understandings. For Duncan, the expressed goals remained focused on skill development and superficial features of the representations rather than on the mathematical relationships represented. Understanding these differences in the articulation and unpacking of learning goals along with the differences outlined in the next three sections will be the focus of next chapter.

**Differences Among Teachers in the Nature of Lead Task Selection**

Well-chosen tasks, as previously defined, are accessible to the students using existing understandings and problematic in the sense that they engage those existing understandings in ways that can support the development of new understandings. As previously discussed, variation existed across the four teachers in the study in the nature and degree of specificity of the learning goal or goals for the lessons as well as the extent to which the teachers unpacked the
component understandings of the mathematical goals. Not surprisingly, these differences contributed to variable approaches to the selection of lead tasks both within and among the four teachers. Each of the teachers selected lead tasks with an eye towards the perceived, existing understandings of students. However, differences emerged in the nature of the tasks selected and the degree to which the sequencing of tasks supported what the teacher envisioned as the progression of desired understandings. These differences are discussed in this section. [Note: All lead tasks for each participant are found in Appendix E].

**Jackie’s Lead Task Selection**

The lead tasks chosen by Jackie during the three-lesson sequence reflected intentional consideration of the mathematics of the students, the mathematics of the learning goals, and the mathematical path from one to the other. From the bonus question on the quiz the day before to the last task in the third lesson, Jackie designed tasks for students which were accessible using their existing understandings while motivating the need for the kind of mathematical thinking she needed them to do related to the learning goal. Collectively, the set of tasks mirrored the incremental nature of the steps of the learning progression Jackie developed from unpacking the learning goal. Each task used elements of the previous task, providing an element of continuity among the tasks that was not found in the work of other teachers. This continuity added to the accessibility of the tasks for students as they were asked to apply existing understandings to solve slightly different problems. As Jackie described it, she wanted them to draw a “new conclusion” [Jackie, Ob #1, Line 493] from familiar work.
Jackie’s efforts to build the desired understandings began with a bonus question from a quiz on the previous material. She presented to students a kite with the given information shown in Figure 4.3:

![Figure 4.3. Jackie’s initial lead task.](image)

She described her reasoning behind the selection of this task in this way:

Interviewer: Tell me what your goals were for this part of the lesson?
Jackie: Every time they have a quiz, there is a bonus question because they work at different paces. They hand their quiz in and then they are sitting there doing nothing. So I always give them a bonus question and as much as I can, it leads into the next thing. It also reinforces other things. So we had a quiz on the law of sines and [law of] cosines, which we had just done. We had not talked about kites in a long time. They had lots of theorems about kites. So, they could use the law of cosines here to get one diagonal. They could use the Pythagorean Theorem to get the other. They could use whatever—the rules for kites and I wanted to see how much of that they remembered on the one hand and whether they could put the law of sines and [law of] cosines to use in this sort of unusual setting for them.

Two aspects of her response were worth noting.

First, Jackie chose this task with an explicit consideration of what she perceived the existing understandings of students to be. She referred directly to the law of sines and the law of
cosines, properties of kites, and the Pythagorean Theorem. With this consideration, she expected students to possess the mathematical tools and understandings to successfully complete the task—in other words, the problem was accessible to students. In addition, Jackie intentionally chose to provide students with a kite with two right angles to provide an opportunity to reinforce an understanding she was not sure they had internalized.

Interviewer: So did you pick this to have right angles on purpose?
Jackie: Cause I wanted to see…I wanted them to see that now just because you know the law of sines you don’t have to use it when you have a right triangle. It’s stupid to use it. And we talked about that more today. Inevitably, once they have the law of sines, they want to use it on everything. So they are doing the sine of 90 over something and then they tell me, “Sine of 90? That’s weird.” You know, anyway in our discussion of that, I want to be able to point out stuff like that

[Jackie, Ob #1, Lines 47–55]

In this explanation of her choice of a kite with right angles, she made a specific reference to her consideration of students using the laws of sines and cosines when you have a right triangle and her desire to “point out stuff like that.” This knowledge of the inefficiencies in the application of their mathematical knowledge further reflected her careful consideration of the existing understandings of students in the construction or selection of tasks.

The second aspect of her response worth noting was her explicit reference to her desire to choose a task that can “lead to the next thing.” While it was not particularly problematic for them, the given task involved the applied synthesis of a number of understandings and tools they had developed and activated their prior knowledge as they began to pursue the larger goals for the three-lesson sequence. Throughout the three-lesson sequence, students applied properties of quadrilaterals, used right triangle trigonometric ratios, and applied the law of sines and the law of cosines. Reinforcing each of these in this initial task provided a foundation for the work of the next three lessons.
The next lead task in the sequence involved a problem in which two sides and the included angle of a parallelogram are given as shown in Figure 4.4 and the students are asked to find the longer diagonal.

![Figure 4.4. Jackie's second lead task.](image)

The researcher asked Jackie about the selection of this task in the stimulated recall interview.

Interviewer: Tell me why you chose this example.
Jackie: You mean those particular numbers? No reason.
Interviewer: No, the idea of the parallelogram.
Jackie: Oh, because I am working up to vectors. So, we have already done, “Find the diagonal of a parallelogram.” Just to use the law of sines and [law of] cosines. But in my ulterior motive kind of way, doing it here and then going into vectors, first I want them to tell me the diagonal is not the same for every parallelogram with the same length of sides—it depends on the angle. Both classes saw that real easily. And so, then if we are going to find the length of the diagonal, then we are going to have to use the angle. Not a profound thing.

[Jackie Ob #1, Lines 472–483]

Jackie downplays the power of the choice of this example as it fits into the learning progression. However, two aspects of the presentation of the problem reinforced the thought she put into the choice.

First, Jackie introduced the problem by sketching two versions of the parallelogram with the same lengths and different included angles. Anecdotally and at least informally, this process established some dependence between the length of the diagonal and the included angle.
Secondly, she very intentionally focused student attention on finding the longer diagonal—a point easily made when a student asked about finding the other diagonal.

Student: Can you draw like the opposite diagonal?
Jackie: The other diagonal? We could, except I don’t want that one. I have a reason, but we could find that one.
Student: Wouldn’t that be easier, though?
Jackie: Well, yes.

Jackie, Ob #1, Lines 525–529

Focusing on the longer diagonal in a familiar context provided a direct link from what she understood as the existing understandings of students to the resultant vector concept she planned to introduce next. Jackie described it as trying to “draw a new conclusion from the diagonal.”

Jackie: I just wanted something simple and straightforward.
Interviewer: You also knew that they had done the problem before and all of the mathematics was the same.
Jackie: Right. So what we are trying to do is draw a new conclusion from the diagonal. So we got this length and this angle. ‘Isn’t this a coincidence’ type of thing.

Jackie, Ob #1, Lines 488–494

Jackie explicitly makes this link between this parallelogram problem and the vector problem during the class.

Jackie (to the students): Now how is this useful? How many parallelograms do we need the diagonal, the measure of in this way? Maybe not so many, right? But, imagine this. We are trying to move a piano. You and I. I am assuming you are stronger than I am. We are standing here (teacher points to the vertex of the parallelogram originally formed by the sides labeled 5 and 9), where the piano is. It is not a big piano, so we are really not able to get where we can both be behind it pushing at the same angle. We are trying, but inadvertently, you are pushing that way and I am pushing this way. The angle between what we are pushing is 35 degrees. I am able to push with a force of 5 pounds. You being stronger are pushing with a force of 9 pounds. So I am here and you are here and we are pushing the piano… Guess what AC represents… What do you think 13 represents?

Jackie, Ob #1, Lines 594–609

Jackie makes this link to vectors changing the context of the problem without changing the mathematics of it. She used the identical magnitudes and angle for the vectors as she used with
the parallelogram. As she described, “all the mathematics was the same...so what we are trying to do is draw a new conclusion from the diagonal.”

Jackie used a similar approach in the selection of the next lead task in the sequence. Once vectors were defined and students worked through a few examples without the parallelogram, the next lead task in the sequence involved putting a two-vector problem (similar setup to the parallelogram) on the coordinate plane. Jackie presented the task to students in the second observed lesson.

J (to students): I had vectors of 6 and 10.5 and an angle of 70 and I want to put it on the coordinate plane.

![Jackie's lead task with vectors.](image)

*Figure 4.5. Jackie's lead task with vectors.*

[Jackie, Ob #2, Lines 4–6]

The setup of the problem is mathematically parallel to the vector problems with the primary exception that it will be placed on the coordinate plane. With only a change in mathematical context, the visual familiarity of the diagram and the mathematical familiarity of the problem seemed to afford students an opportunity to apply their existing understandings in a new context. In explaining her rationale, Jackie said:

J (to the interviewer): I wanted them to see another way of doing it. I always like to relate things to the coordinate plane whenever possible. If I say, “Let’s put this triangle in our favorite location,” they’ll all say coordinate plane. So, I just wanted to do that
again. But I also am going to look at the unit circle and so what I arrived at here, was once I got that pair of coordinates, I could figure out the length, sine and cosine—was it this class or the other class that said to drop a perpendicular?

Interviewer: They dropped a perpendicular.

Jackie: Okay, they might have said the same thing in the other class. So now they have a right triangle with those coordinates, but they don’t really need the right triangle and we will get to the point where we will just erase it and just use the coordinates and the length. So, it is sort of setting it up for the unit circle. It won’t be the unit circle at first, it will just be a random radius and I will arrive at, “What if the radius is one?” But that is next week.

[Jackie Ob #2, Lines 19–34]

Jackie articulates her reasons as twofold: to strengthen student understanding and work with the mathematics of the coordinate plane and to lay a foundation for understanding the unit circle.

This explanation reinforces her continual eye towards the learning goal and her thinking about how students can get from their current understandings to it. She views this problem choice as an important step in the progression.

The shift from the work with the parallelogram to the work with vectors to the work on the coordinate plane represents Jackie’s envisioned progression towards the foundational understandings of the unit circle. That progression continued with the tasks selected during the third observed lesson. Jackie began with a review of a similar task completed previously in which the coordinates of the endpoint of a resultant vector were determined given two vectors and the included angle between them as shown in Figure 4.6.

Jackie: I kind of want to go back to where we were the other day. So this is a different situation, but we had some vectors and we put the vectors on the coordinate plane. Do you remember that? And then we found the ordered pair for the end of the vector, where we would have our arrow, up here.
After reviewing the work of the class to find the coordinates of the endpoint of the resultant vector, Jackie shifted the student attention away from vector addition context.

Jackie (to the students): Now I want us to look at this triangle that I am going to call ABC in a minute (teacher traces the triangle with a squiggly line). And I want you to get the angle here (see Figure 4.7).

Student: The angle?
Jackie: So, I am going to trace this (teacher overlays a transparency to trace triangle). There is a lot of stuff on here that we do not need. We need this 12.6 and 3.9. So what is the length of this side?
Students readily found this angle and Jackie introduced the next task in the sequence.

Jackie (to the students): Excellent. Let’s see. I want to try an experiment. What if there was no triangle (teacher draws Figure 4.9 on overhead)

This task represents the first attempt by Jackie to get students to establish a relationship between the coordinates of the endpoint of the vector and the trigonometric values associated with the angle. In the stimulated recall interview, Jackie explained the importance of getting away from
the triangle providing further clarity about her mathematical goals and the learning progression she identified as a path to reaching them.

Jackie (to the interviewer): Yea. I mean I don’t really care what the ordered pair is, but I care that we very gradually move away from that triangle. So for example, dropping that first (teacher draws a parallelogram on a set of coordinate axes with one side on the positive x-axis and drops a perpendicular from one vertex to the x-axis). I am okay with that, but I don’t want to think about it anymore. And then over here, the ones that wanted to drop that (teacher drops a perpendicular from the right most vertex to the x-axis) and I am okay with that too. I think it was the class that you saw that someone mentioned that they were congruent (pointing to the two right triangles formed by the perpendiculars and the sides of the parallelogram). Which was good, but eventually, like you said, that’s just noise and I don’t want to use law of sines and cosines. They are very nice, but I would rather not have to use them. So this gives me a tool to not use them. But all because I am leading towards that association between the angle and the coordinates of the points on the line.

[Jackie Ob #3, Lines 273–289]

Each of the lead tasks during the three observed lessons was designed to support one of the primary goals for the lesson sequence, which is to “move gradually away from that triangle” and “towards that association between the angle and the coordinates of the points on the line.”

Throughout her discussions about her choice of tasks, Jackie demonstrated her continual effort to maintain a consideration of what she perceived to be the students’ existing understandings and ways of reasoning as she determined tasks that would provide opportunities for students to advance their understandings towards her learning goals. In her selection of tasks, Jackie also demonstrated a careful and nuanced thought process about the understandings she sought to advance with each choice of task. Her selection of tasks with structural similarity or identical calculations to previous tasks limited the variability from task to task and seemed to foster the ability of students to complete the new tasks and as Jackie put it, “draw new conclusions.” This incremental nature of the task progression characterized her task choice throughout the three observed lessons. In selecting tasks in this way, Jackie focused student
attention on the conceptual shift she had identified at each step in the sequence. Collectively, these interactions and presentations of the tasks provided evidence of Jackie’s triune focus on the mathematics of the students, the mathematics of the learning goal, and potential each task held for moving students from existing understandings towards new understandings.

**Harold’s Lead Task Selection**

Variation existed in the nature of Harold’s lead task selection. While there were similarities in the nature of the task selection and the student engagement in the first Math 6 lesson and the Algebra I lessons, a marked difference existed with the selection of the lead task in the second Math 6 lesson. These differences will be discussed in this section as well as similarities and differences with Jackie’s selection of tasks.

As discussed previously, the two observed Algebra I classes involved student work on solving a system of linear inequalities. Harold presented the system shown in Figure 4.10 to students:

![Figure 4.10. Harold’s lead task for Algebra I.](image)

The selection (or design) of this task reflected both a consideration of what Harold perceived to be the existing understandings of students and a potential path from those existing
understandings to new understandings. To explore these aspects of his mathematical thinking a bit further, it is important to understand the context of this lesson. Specifically for this lesson, students had not previously encountered a system of inequalities nor had they had extensive experience with graphing. However, they were given a system of inequalities to solve and were expected to do so through graphing. Harold expected them to successfully complete the given task by applying the general definition of a solution—the value or set of values that make the statement or statements true. The following provided a sense of Harold’s expected approach.

Harold: And they didn’t really have a process for graphing the equation either cause, again, for them, other than finding solutions. I am really looking for them to think, again, “we want to find a point where its values make that statement true, so I need a point where the value of $y$ is greater than or equal to the value of $x$ plus six over 2 or whatever it is.

Interviewer: Did they have any basis for understanding that all the points on one side of the line are going to behave the same way?

Harold: No, they do not, and I kind of deliberately avoid that. What I will focus on is, “so, this point, the $y$ is just equal to what it needs to be to make this statement true. If I make $y$ greater, that’s going to also make this statement true. So I am deliberately starting at a solution to the equation and going UP, because if the $y$ value increases, it also makes this guy cause this statement is $y$ is greater than or equal to. So I am very deliberately shading up from each point on the line. But we are not there yet, so I am not under the impression that we have gotten there yet.

[Harold Ob #2, Lines 293–310]

In this excerpt, Harold acknowledged that the students “didn’t really have a process for graphing…other than finding solutions.” Furthermore, he described the process he expected the students to apply to graph a single inequality. Rather than memorizing a shading approach, he expected students to “start with a solution to the equation,” in other words, a point on the line and consider whether $y$ values vertically below or above that point satisfied the inequality. This specific description of the expected approach revealed his consideration of what he anticipated
their existing understandings were as well as the way the students could apply those to complete the task.

Harold’s anticipation of the student approach to the given task was grounded in a number of prior experiences students had in the course. In multiple portions of the stimulated recall interviews, Harold described various aspects of the class work the students experienced earlier in the year and articulated how he expected them to draw on these presumed understandings to successfully complete this task. As the following discussions revealed, Harold viewed the approach to solving the system of inequalities in this way as a natural extension of the work the class had undertaken all year.

Harold: We did another problem…another problem that we worked on was solving a system of equations where…Oh no, not solving a system of equations. When we did solving an inequality, I had them do it graphically.

Interviewer: One variable inequality?
Harold: Single variable inequality. I had them solve it graphically and this was a pretty in-depth graded assignment where they had to do that full, “Here’s what I am doing, here’s why I am doing it, here’s how I know it is right”. And the big thing that we got out there was that graphically, when you are looking for…so they graphed the left side of an equation and the right side of an equation. Let’s say this is the right side and that is the left side. So here, the values of the left side are below the values of the right side. I very deliberately hit that to get them into that habit of thinking less than and greater than representing vertical change if you are looking at y values. So I am really hoping that they will be able to build off of that on this. That’s why I think they should be able to go above and below. And think of it that way. From that graded assignment.

[Harold Ob #2, Lines 350–369]

In this excerpt, Harold described the work the students had done previously with single variable inequalities. Harold approached that task in a particular way “to get them into the habit of thinking less than and greater than representing vertical change, if you are looking at y values.” He even explicitly states that he hopes they will “be able to build off of that on this,” as the reason he believes they have the existing understandings to complete this task.
In another discussion with the researcher, Harold discussed some of the prior experiences students had had with finding solutions as he described, “it is something that we worked on all year.”

Harold: For this task, it is something that we worked on all year. I feel like we have really built up from when they were first asked to solve a real simple equation. The way we started was, “Hey, I’m thinking of a number. If you triple my number and subtract seven, you get eight. What number am I thinking of?” Way back then, they were trying to find the number that made the thing that I said about the number, true. So ever since then, they’ve been looking for solutions, or at least they should have been looking for solutions based on the definition that they are looking for a value, or in this case, sets of values, that make statements true, so for this system, they are looking for every, for the whole set of all values that make every statement, in this case inequalities, in that system true.

[Harold Ob #3, Lines 34–47]

In this discussion, Harold traced the consideration and application of the definition of solutions to “when they were first asked to solve a real simple equation.” He continued, “So ever since then, they’ve been looking for solutions.” Harold considered these experiences as sufficient for successful completion of the task using the approach of finding solutions. These comments also reflected his consideration of the prior experiences of students as well as how those experiences could have been used to solve this system without any direct instruction on how to do so.

In light of these considerations, the task appeared to be well chosen—accessible to students and problematic. As evidenced by these excerpts, Harold expressed confidence in the background knowledge and skills he expected students to possess, and, given this background, one could have reasonably presumed they could complete the task successfully. The task also appeared sufficiently problematic to students since there was no rehearsed routine for them to follow to a solution path.
However, two factors seemed to diminish the potential success of the lesson sequence. First, Harold might have overestimated the power of the existing understandings of students. Clearly he carefully considered a number of prior experiences in the selection of this lead task. The concept of solution had been previously applied to a system of equations, and equations had been graphed by focusing on solutions to the equation. Throughout the class, students were able to recite the appropriate definition of a solution. Yet, the students were not able to successfully complete the task in spite of working on it for two full class periods during the observations. This struggle suggested that the existing understandings were not as robust as Harold might have anticipated nor were they sufficient for completing the task in the way it was presented.

A second factor that may have diminished the success of this task involved the presentation of the problem. The task, as it was presented, did not provide a sufficient motivation for students to engage their existing understandings. In other words, the lead task that Harold had selected, while problematic and accessible, was not designed in a way that would prompt students to engage existing understandings in an incremental way. The instructions, “Solve the system,” provided the single motivation for drawing on prior knowledge. In contrast to the work of Jackie in selecting tasks, Harold presented the task to students without breaking down the work of the students into more incremental steps affording them more opportunity to scaffold new understandings to existing ones.

Like the lead task selection in the Algebra I classes, Harold gave consideration to what he expected the existing understandings of students to be as he chose the tasks for his Math 6 classes. However, the differences in the nature of lead task selection for the two Math 6 lessons
suggested that the consideration of the existing understandings of students was applied in significantly different ways from one class to the next.

As described in the previous section, Harold sought to develop student understanding of the division of fractions in his Math 6 lessons. He expressed a sequence of component understandings he identified as a potential path to developing those understandings.

- Since \( \frac{x}{y} \) is the same as \( x \times \frac{1}{y} \), the division of \( \frac{a}{b} \) by \( \frac{x}{y} \) can be considered in two parts—that is, dividing by \( x \) and then dividing by \( \frac{1}{y} \).
- Dividing by \( x \) is the same as multiplying by \( \frac{1}{x} \).
- Dividing by \( \frac{1}{y} \) is the same as multiplying by \( y \).

In the first Math 6 class, Harold presented the lead task to students as shown below:

\[
\frac{a}{b} = \frac{a}{b} =
\]

Harold: Guys, I want you to know that all you are doing at this point is writing the same thing in a different way. Are we okay with that? … Do we need to prove this [pointing to the equation]? This means that [pointing to the two sides of the equation]… I want you to tell me something that is equal to both of these that is different.

Essentially the task was the first part of the first component understanding listed above. In the interview, the researcher asked him about the response he was anticipating:

Interviewer: What were you looking for from them there?
Harold: \( a \times \frac{1}{b} \) or \( \frac{1}{b} \times a \). I would be good with either.

[Harold Ob #1, Lines 59–99]

His response suggested that Harold intended to use this task as a way to activate prior knowledge and lay a foundation for the development of the remaining part of the first component understanding. As several comments suggested, he did not anticipate the students would struggle with the task. In fact, he really anticipated focusing on the second part of the first
component understanding above: *The division of \( a \) over \( b \) by \( x \) over \( y \) can be considered in two parts, that is, dividing by \( x \) and then dividing by \( 1 \) over \( y \).* He referenced his focus on that second part in the following excerpt:

Harold: (to the interviewer): \( x \) divided by \( a \) times \( b \) equals \( x \) divided by \( a \), parenthesis, divided by \( b \).
Interviewer: Okay.
Harold: And that’s the one I was really going to develop… cause that’s something where it is so easy to say, “Hey if I want to divide something by six, then I can divide by two and then divide by three, can you guys show me how you would write that algebraically?

[Harold Ob #1, Lines 121–128]

Harold’s statement, “And that’s the one I was really going to develop,” suggested that this aspect of the component understanding was the intended primary focus of the first part of the lesson. However, the students struggled throughout the first class to demonstrate an understanding that satisfied the teacher and the remainder of the class involved a series of associated tasks designed to support the completion of this lead task. The discussion never advanced to the primary focus of the lesson.

The second observed Math 6 lesson occurred during the next class meeting with the same group of students. Harold’s goal for the lesson remained focused on developing an understanding of the division of fractions and the sequence of understandings outlined previously. However, the lead task was quite different. For this class, Harold provided students with a task that involved a meaningful context—namely, one that was experientially real to students.

Harold: Twelve dollars needs to be distributed evenly among four brothers. My sister has three boys, but let’s say she has four of them. The four boys need to distribute $12 evenly among themselves.

[Harold, Ob #4, Lines 14–15]
The students were asked to construct a visual and numerical model to represent this situation.

When asked about the choice of task, Harold explained:

H (to the interviewer): I knew I could start off with dollars and then move that to quarters pretty easily and the symbols would change significantly and yet they would still be able to recognize that we were talking about fractions of a whole…I knew this would be a very real-world situation that they could easily model.

[Harold Ob #4, Lines 23–29]

Harold’s description reflected careful thought about his goals for the students and how this task might have supported the development of the key understandings in light of their existing ones. Unlike the lead task in the first lesson, which asked students to complete an algebraic representation of the relationship, \( a \rightarrow b = \frac{a}{b} \) in abstract form, the lead task in the second class was problematic and accessible in a way that engaged their existing understandings and motivated the need to model the component understanding in multiple ways. Students were universally successful in completing this task.

The selection of lead tasks in both classes reflected variability in the nature of Harold’s consideration of existing student understandings and how to engage them through problematic tasks. In the first lesson, the lead task was presented in an abstract form, devoid of context. It directly represented one of the component understandings in algebraic form. Harold expressed surprise and frustration during the class and in the interview at the difficulty students had in demonstrating a successful understanding of the task and its completion. He expressed confidence in what he anticipated the existing understandings to be and how they could be used to complete the algebraic model. These statements suggested that Harold had considered these anticipated understandings of students in the construction of the task, but as was the case in the Algebra I lesson, the lead tasks did not sufficiently engage those existing understandings in the
completion of the task. The second Math 6 lesson stands in contrast to the other three lessons.
The lead task for that lesson embedded the component understandings in a familiar and accessible context, and the universally successful completion of the task provided a stark contrast with the work of the students in the other classes. Understanding the differences in Harold’s approach to task selection in these lessons is an important facet of this study. This analysis will be provided in the next chapter.

**Susan’s Lead Task Selection**

The nature of Susan’s task selection provided some contrasts with the task selection of Jackie and Harold. Even with a nonspecific articulation of the learning goals and a limited unpacking of the mathematical understandings for which she strived, Susan selected lead tasks that held potential for fostering a connection to the existing understandings of students while laying a foundation for new understandings. However, as will be discussed in this and later sections, these tasks were not presented in a problematic way that engaged students in goal-directed, mathematical activity that would potentially foster the development of rich, connected understandings of rational functions. In some cases, the tasks relied on imprecise parallels Susan attempted to draw with existing understandings. In other cases, the tasks were not framed in a way that engaged the existing understandings of students while motivating the need for new ones. Each of these cases will be discussed in this section.

In the first observed lesson, the primary mathematical activity centered on an exploration of a rational function. Susan gave the students the symbolic rule for a function, \( f \):

\[
f(x) = \frac{1}{x - 3}
\]

and asked them to explore several characteristics including domain and range, end behavior, and
the numerical behavior of the graph in general and specifically around $x = 3$. This task was a potentially useful task for engaging the existing understandings of students in light of their prior work determining these characteristics of other functions and their experience with asymptotes in their work with negative power functions such as $f(x) = (x + 1)^{-3}$. The students were successful in completing the task. However, while the task engaged the existing understandings of students, their mathematical activity was not directed in ways that encouraged the development of new understandings. For this task, the students were asked to complete a table of values for the rational function. Susan provided the $x$-values in the table. As she acknowledged in class and in the stimulated recall interview, Susan did not construct the table in a way that focused student attention on critical questions and characteristics. She presented the task to students in this way:

Okay, but today, we are going to expand your thinking to the algebraic of rational, rational functions [passing out the worksheet]. I produced this worksheet rather quickly and I did not setup the table in the most advantageous way. So get out your calculators. The first thing I have asked you to do is to come up with a table of values. What is the most convenient way for you to get a large number of values in a table of values from a function in your calculator?

[Susan, Ob #2, Lines 48–54]

With the values of $x$ provided for them in the table, the initial task amounted to a calculator exercise. After reviewing some calculator-based methods for completing the table, she added, “I would like you to extend the table and put in three more values of your choice” [Susan, Ob #2, Line 73]. The task held potential for revealing patterns in the numerical behavior of the function near the asymptote, but Susan did not provide the students with a table or a sufficient prompt for the task to foster this kind of mathematical consideration in the completion of the task.
The implementation of this task could have transformed it into a more problematic task, but the subsequent work of the class did not reveal such a transformation. The class continued with students completing the table and answering the questions about the domain and range and end-behavior of the function. After students had a chance to discuss their work with a small group of peers, Susan brought the class together for a discussion.

Susan: So conclusions…What is familiar about this activity?
Student: Asymptote
Susan: It’s…You see an asymptote when you look at the graph. What kind of asymptote? Do you see one asymptote or two asymptotes?
Student: One
Student: One
Student: Is there a pair?
Student: It doesn’t cross the (inaudible) axis
Susan: What two?

Several aspects of this exchange are worth noting. Susan initiated the discussion with an open-ended question that invited a range of responses, “What is familiar about this activity?”. A student identified “asymptote” as a familiar aspect—a useful response in that it served to advance the discussion in line with the goals for the lesson. However, rather than insisting the student elaborate, Susan completed the thought and asked a leading question, “Do you see one or two asymptotes?” Four different students responded to her with only one student suggesting that there could be two asymptotes before Susan asked another leading question, “What two?” These leading questions diminished the mathematical thinking required of the students and made it unclear whether the students successfully completed the task on their own.
After a brief discussion about the asymptotes and a connection to the negative power functions the class studied previously, Susan shifted the discussion to the other portions of the worksheet.

Susan: All right, because of these asymptotes, we are going to have some restrictions when we talk about domain and range. Domain for our function of interest is what, all real numbers?

Again, Susan asked a rather leading question. Even with the leading nature of the question, it took a few exchanges with students to arrive at the correct answer that $x$ cannot equal three.

Susan closed the discussion and moved onto the other questions.

Susan: Three. Can it be any other value?
Student: Yes
Susan: Yes. What about range? All real numbers except zero. What about end behavior? Right end behavior? Over here on the right side. That means we are getting way far out here [T is beyond where the positive $x$ axis ends on the graph]. $x$ is getting really, really, really large [T writes “$x \to \infty$”] $x$ approaches some infinitely large number, what happens to the function value?

In this case, Susan answered the question about the range of the function for the students and gave a detailed explanation of right-end behavior before giving the students a chance to answer. Again, Susan presented a potentially meaningful task to students, yet her implementation of the task undermined its potential effectiveness. Like the other parts of this lesson segment, the students were given tasks that exposed them to important characteristics of the graphs of rational functions through a straightforward description of these traits and a highly teacher-centric discussion of the responses. For this portion of the lesson, the mathematical activity of the students on the lead task held some potential for connecting new understandings to prior ones,
the lead task was not presented in a goal-directed, problematic way that engaged those existing understandings and motivated the need for new ones.

The final portion of the first observed lesson provided another example of a lead task that fell short of engaging existing understandings to resolve a problematic situation in ways that could build new understandings. This task involved a consideration of the definition of a rational function and the algebraic representation. The lead task for this portion of the class involved the question below:

Why is \( \frac{x^2 + 2x - 3}{x+3} \) not a rational function?

[Susan, Ob #2, Line 242]

The goal for this portion of the lesson was loosely described as knowing the difference between holes and asymptotes, and the goal for this task was to expose students to holes in graphs. Earlier in the lesson, the students reviewed a definition of rational functions copied from the textbook, which seemed to provide the background for answering this question. Of note, it was also the first time students had encountered a graph in which there was a hole. Thus, this lead task represented a problematic situation for students without a rehearsed routine for approaching it. Yet, beyond the application of the definition of rational functions, which they had just seen for the first time, the task did not engage their existing understandings in ways that could build an understanding of the relationship between the algebraic and graphical representations of rational functions.

As the class worked to complete this lead task, Susan led students through a teacher-centric simplification of the algebraic expression and focused their attention on the nonequivalent nature of the graph of the simplified function and the original function.
Susan: My question is why is that not a rational function? Why does that not fit the definition of a rational function? It looks like one. We’ve got a polynomial divided by a polynomial.

Student: Because it is not a polynomial in $x$?

Susan: Well, the numerator is a polynomial in $x$ and the denominator is a polynomial in $x$. And here that is a generalization in the definition. It could be in any variable.

Student: Can you not divide by $x$?

Susan: I can divide by a variable, but it will be undefined if that variable gives me a zero denominator.

The first two student responses to the initial posing of the task were wildly inaccurate. One student suggested, “Because it is not a polynomial in $x$?” and a second student posed, “Can you not divide by $x$?” These responses suggested the inadequacy of the existing understandings of the students to complete the task or the inadequacy of the task to engage those existing understandings. Without student-initiated progress, Susan had to suggest that students manipulate the expression in some way.

Susan: Can you manipulate that expression any?

Student: You could factor.

Susan: Ooh. There’s an idea. Go for it. [Students work independently and the teacher walks around].

Student: When you factor out $x$ plus three times $x$ minus one and put that all over $x$ plus three, you can divide out and you are just left with $x$ minus one (as shown in Figure 4.11).

![Figure 4.11. The simplification of the first rational expression.](image)

Susan: Ah ha. It is true that this is a simplification of the original one. That’s true, but they are not equivalent. The original function had what restrictions? Well, this is written as an expression, but it could be a function. But what are the restrictions to this?

Student: negative three.
After the students successfully completed the simplification, she made the statement that the two expressions are “not equivalent”, and without giving students a chance to resolve this counterintuitive notion, she proceeded to lead the students to a comparison of the restrictions on each functional expression and to answer the question of whether the original function was rational. These leading questions related to the lead task were necessary because the task did not sufficiently activate the prior knowledge of the students and engage them in the mathematical activity that could have led them to an understanding of the differences between these two expressions, an appreciation of the difference between algebraic equivalence and functional equivalence, and an emerging recognition of the connection between the algebraic and graphic representations of rational functions.

A similar dynamic continued through the presentation of the final lead task in the first two observed lessons. In both classes following the previous question, Susan asked the following question:

\[ \text{Is } \frac{x+3}{x^2+2x-3} \text{ a rational function? Why?} \]  

[Susan Ob #2, Line 312]

Like the previous task, this task also presented a problematic situation for students to resolve but focused student attention on the classification of the function as rational or not rational rather than on the more essential understanding of the algebraic relationships that produce holes and asymptotes—the actual learning goal for this portion of the lesson. Again, Susan led the students through the analysis of the restrictions and answered the question she originally posed for them.

Susan: What are the restrictions on the original?
Student: \( x \) cannot equal one.
Susan: \( x \) cannot equal one. Anything else?
Student: Negative three.
Susan: Okay, $x$ cannot equal negative three. Look at the original function. If I let $x$ equal negative three, I’ll have what? Zero over nine minus six, minus three which is zero over zero. Is that defined? What’s zero over zero?

Student: undefined

Susan: yea. What are the restrictions over here? [pointing to the simplified version]

Student: It can’t be one.

Susan: $x$ cannot equal one. So these are not equivalent. For the original one, I cannot allow $x$ to equal negative three. For the simplified one, I can let $x$ equal negative three. They have different restrictions. But, the question was, “Is this,” it should say “a rational expression.” And the answer is yes, because it keeps the rational form. They are not equivalent still. Okay?

[Susan Ob #2, Lines 325–339]

Just as Susan did with the first question, she directed students to the differences in restrictions and concluded for them that the two expressions were not equivalent. Students were left to make sense out of this lack of equivalence of the functions in the face of the algebraic equivalence of the expressions.

Each of these lead tasks had a problematic aspect in that they looked like rational expressions but did not behave in the same way. In the first case, the expression simplified to a linear expression and in the second one, one of the two factors in the denominator divided out. The direction of student attention to examples along with the construction of examples that vary in the nature of holes and asymptotes held the potential for students to develop an understanding of the relationship between the algebraic representation and the presence of holes and asymptotes on the graph of the function. However, Susan did not present the task to students in a way that would engage their existing understandings, nor did she give them a chance to explore the answer to the initial question about holes and asymptotes with some sort of goal-directed activity. A task in which students were asked to explore the graphical equivalence of the original function and the simplified function could have supported the development of understanding of the connection between the algebraic and graphical representations and engage the students’
existing understandings of graphing and domain restrictions. Instead, Susan’s lead tasks did not consistently present students with problematic situations that engaged existing understandings and initiated mathematical activity that provided students the opportunity to connect new understandings to prior ones.

One additional example of this type of treatment in her lead task selection can be found in the third observed lesson. In this lesson, the learning goal was for students to understand how to determine long-run behavior of rational functions and the teacher had identified component understandings: understanding polynomial long division and understanding how to interpret a re-expressed function to describe long-run behavior. The lead task for the second part of the lesson involved using long division to re-express the function in order to more readily determine the long-run behavior. Susan initiated the lead task with the following:

Susan: All right, why are we doing this? We are going to look at what is called long-run behavior for rational functions. [Teacher projects the following on the overhead]

\[
Use \text{ the division algorithm, then describe the long-run behavior.}
\]

\[
(a) \ y = \frac{2x-6}{x-4}
\]

Susan: This division is pretty straight forward compared to some of the examples we just did. Here, I am doing long division to see another way of expressing this function. [Susan, Ob #3, Lines 632–636]

The framing question, “Why are we doing this?” was asked and answered by Susan without giving the students a chance to put the long division process to which they were just introduced in a relevant context with the work they have been doing with the graphs of rational functions. She also provided the students with direction to use long division to re-express the function without first giving them a chance to make that mathematical decision. In the subsequent class discussion, students did not readily understand what was meant by long-run behavior and Susan
had to explain that concept to them. Lacking this understanding, it would have been difficult for the students to complete the task as it was presented without this direction from the teacher. In other words, Susan did not present this task in a way that supported the students’ use of their existing understandings and their access of those understandings to build new ones.

It is important to understand potential alternative approaches to the tasks Susan selected as a way to provide context for this assessment of her task selection. One can envision an alternative approach in which this entire section of the class could have begun with the goal in mind—for example, by presenting a rational function and asking the students about the end-behavior. Using language that had already been reviewed the previous day, instead of using the term, “long-run,” students could have experienced potential difficulty in determining end-behavior. This difficulty, with the function in its given form, would have potentially motivated the need for the new tool of long division to re-express the function. The students also could have come up with options of how to rewrite the rational function (factoring, dividing) even though they do not have any idea how to divide, which would have connected the work of the previous day with the work with determining long-run behavior. In this case, Susan presented students with a task that was somewhat problematic but did not engage existing understandings in ways that would support the development of new understandings in the completion of the task. As previously discussed, these characteristics of her lead task selection emerged throughout the three observed lessons.

There was also another pattern that emerged from two of her choices for lead task. During the third observed lesson, the lead task on long division designed by Susan involved a direct activation of existing understandings to build new ones. As discussed previously, Susan began
the lesson with a simple long division problem. The steps of the solution were provided by a student and Susan wrote them on the board as shown in Figure 4.12.

![Figure 4.12. The long division with whole numbers problem.](image)

Using the steps provided by the student as a starting point, Susan led the students through a process of generalizing the steps of the arithmetic algorithm (“Guzzinta” (goes into), multiply, subtract, bring down) before applying those steps to a polynomial long division problem. However, even though the steps of the process of long division and polynomial long division were the same, the students struggled to understand the differences in the execution of those steps in the two contexts. The following exchange illustrated one of the sticking points for the students. The teacher had just completed the first step of dividing $x + 2$ into $3x^2 - 5x + 4$.

Student: Then why do you say, “How many times with just $x$?”
Susan: I am approximating. Yea, so I am saying, how many times does $x$ go into three $x$ squared. It’s answering this question. Pointing to the previous figure.
Student: Yes, but that doesn’t include the plus two.
Susan: Exactly. Just like before when Ellen tried dividing by 17, twenty was not the same thing as 17, just a good approximation.

In this excerpt, the student expressed confusion over the determination of the multiplier by looking only at how many times $x$, the first term of the divisor, would go into the first term of the
dividend. She asked Susan why she only divided $x$ into the dividend. This confusion persisted and several other similar examples of the students’ difficulty with this concept were found throughout the lesson.

Student: The first time you did this, you got three $x$, what happened to the…
Student: [Another student completing the question] What happened to the two?
Susan: I’m sorry.
Student: [With teacher pointing to the two in $x$ plus two] On the left there is a two. What happened to that up top? [The teacher had written $3x$ on top]
Susan: Nothing yet.
Student: Why? Because shouldn’t… that doesn’t make any sense.
Susan: [Going back to the screen with the numerical example] When I look at the arithmetic problem, I only considered two times 17 should give me something close, right? So I do the multiplication of two times 17.

This student asked the same question as the student in the first excerpt and the teacher directed students’ attention back to the numerical example. Even though the teacher tried to move the discussion to the next step in the process, a student maintained the discussion on this omission of the two in the dividing step. The teacher was seemingly able to advance the student’s understanding past this confusion with this exchange by emphasizing that she did not “consider” the two.

Student: (same student who responded previously) You didn’t divide by two so you…
Susan: [interrupting] I didn’t consider [emphasizing “consider”] two in estimating what my quotient would be.
Student: I gotcha.
Susan: But now I am ready to multiply.

However, confusion lingered.

Susan: Whatever I put up here [pointing to the quotient, $3x$] times the divisor.
Student: Right.
Susan: How do you do that multiplication?
Student: You distribute it. Three $x$ squared plus six $x$. [Teacher writes it on the board]
Susan: Exactly
Student: So why bring the two back down?
Susan: I am still not understanding what you mean by bring it back down.
Student: You divided by three x when you started and didn’t you also just bring the two back in when you multiplied?

[Susan Obs#3, Lines 371–379]

Without resolving her confusion over what the student meant, Susan attempted to clarify their understanding by re-emphasizing some connections to the numerical long division.

Susan: When Ellen came up with the four here [pointing to the four in the quotient from the long division as shown in Figure 4.13]

![Figure 4.13. Susan pointing to the four.](image)

Susan: [continuing] She all of a sudden said, it’s not four times 20 (the estimated quotient) it is four times seventeen. Okay? It looks much more complicated with the algebra, but the process is identical.

[Susan Obs#3, Lines 380–385]

Her reference the process being “identical” suggested that she did not fully appreciate the differences in the two processes that caused confusion on the part of the students. The repeated exchanges and the lack of anticipation on the part of the teacher also suggested the parallels were not carefully reasoned in advance. Even after prompting in the interview, she did not express a complete understanding of the difficulty of the students in drawing a precise parallel.

Interviewer: But, I wonder what are the numbers that would force them to do that [the researcher was referring to getting the students to think about the numerical approach in a more closely parallel way to the polynomial long division].

Susan: Well if I had made it 67 instead of 17, it would have been less ready to just double 67 and triple 67. It doesn’t take much for these kids to shy away from the arithmetic.

Interviewer: What is it that you really want them to be doing here to carry over to the polynomial division?
Susan: To realize that they need an approximation; that they don’t have to get a precise multiplication; so I want them to estimate using their awareness of the relationships between the quantities or polynomials. Yea, it would have been better if I had used 87 or some other value.

[Susan Obs#3, Lines 211–222]

The researcher prompted Susan to consider the potential value in choosing a different divisor for her long division example. Susan proposed a larger divisor so that the students would have to estimate. There was no recognition of a difference between a divisor that would round up to estimate and one that was truncated when estimating the partial quotient. Instead, she remained focused on the idea of estimation. That estimation—rounding up to 20 to approximate the partial quotient—was not part of the thinking of the student who did the numerical long division problem on the board nor was it a useful parallel to draw. In the steps of the polynomial long division, the divisor is truncated to estimate the partial quotient yet in her chosen example, she emphasized this notion of rounding up to estimate the partial quotient. Susan did not get to this level of depth in her consideration of the parallels of the two processes and the potential difficulties students might encounter due to the differences. She saw the processes as identical, seeing the parallels and the potential for connections but not the differences in the two processes as she presented them. This lack of depth of consideration of the mathematics of the lead task and the mathematics of the students shaped the nature of the choice of task and the implementation of that task.

Elements of each lead task selected by Susan held potential for advancing the understandings of students. She identified salient features of rational functions and developed or selected tasks that directed student attention to those features. For the most part, they were accessible to students using their existing understandings, and Susan sought to draw parallels
with those understandings and apply them to rational functions. However, the lead tasks chosen by Susan were not always presented in a problematic way that engaged their existing understandings in goal-directed activity that fostered the use of those existing understandings to connect to and build the new, desired understandings. When Susan chose tasks that did engage existing understandings in these ways, the parallels drawn between those existing understandings and the new, desired understandings were imprecise and caused confusion for the students. Efforts to explain these shortcomings in lead task selection are described in the next chapter.

**Duncan’s Lead Task Selection**

The three lessons observed and taught by Duncan involved applications of parametric equations. The lessons were somewhat separate from one another as understandings from one day were not explicitly needed the next. As discussed previously, his goals for the lessons were to introduce students to parametric equations and to develop their skill in working with them. Several features of his lead task selection were noteworthy. First, his lead task selection supported those goals by introducing to students the range of representations and some of the connections between them while also exposing them to a range of applications with parametric equations including a number of different previously studied functions. While some of it may have been a function of the topic under study, Duncan integrated review of previously studied functions more adeptly than any other teacher in the study. Second, the lead tasks were more often presented to students as tasks to complete rather than problematic situations to resolve. This approach was consistent with Duncan’s goal of procedural skill development. However, it created limited opportunities for students to consider some of the underlying concepts and
relationships and use their existing understandings in goal-directed activity designed to build understandings of those underlying concepts.

As discussed in the previous section on the articulation of learning goals, Duncan’s response when asked about his choice of examples for the first observed lesson revealed the introductory nature of his goals:

Duncan: I wanted just an introduction into looking at horizontal and vertical components separately, and then how we can put those together. And I think I did that in all three of my examples. Which was kind of the theme I wanted so they could see things separately and what they are like together.

[Duncan Obs#1, Lines 55–67]

In other words, he was not focused on the concepts to be understood and developed. The reference to “see things separately and what they are like together” illustrated a recurring theme in his discussions of his goals and purposes behind his choice of tasks. The statement also illustrated the absence of a discussion of relationships being represented by parametric equations and what each graph or equation might be telling us about how one quantity changes with respect to another.

Duncan’s discussion of his choice of the second example from the first observation involving equations representing the populations of foxes and rabbits illustrated his thinking.

Interviewer: What’s the purpose of the real-world connections for you?
Duncan: Well, a little background on these two equations (referring to the second example), and these are from our textbook. When I look at them, I say, great, I can go back and get some trig review, which they haven’t done in weeks, go back and kind of pull that in and see how much they remember some of the characteristics. That’s why I spent some time writing out period, amplitude and so forth. But I also wanted them to realize, “Okay, there are a lot of things,” and we talked about this when we did trigonometry, about populations, you know, how they can cycle because of prey-predator relationships and so I wanted them to see that example again, because some of these kids will go on to take Environmental Science or some other classes and just
to make that connection cross-curricularly. And secondly, here’s a good example of what these things look like separately and then putting it together, I think they were surprised that it was elliptical. I think I heard some kids say it was the circle of life kind of thing. Essentially, that is what it is although it is not circular. But that is the purpose to getting them to see, yea, we can model something that happens in nature, pretty easily mathematically, and then from this mathematical model, we can actually look at how these two animals are related to each other and what that looks like graphically.

[Duncan Ob #1, Lines 157–181]

This description of his reasoning for this choice of tasks revealed a twofold purpose. First, he discussed his interest in providing a review of trigonometry “which they haven’t done in weeks,” and through the task, students were reminded of many of the ways of working with trigonometric functions, their equations, and their graphs. Second, Duncan described his rationale for this choice of example as “a good example of what these things look like separately and then putting it together,” and “getting them to see that we can model something that happens in nature.” These statements along with the nature of the required work of students with these examples suggested that Duncan expected students to develop a basic understanding of the graphs of parametric equations using a real world example while reviewing some of their previous work with trigonometric functions. This approach and choice of lead task seemed to offer limited opportunity for students to develop relational understandings of parametric equations and their power to explain phenomena beyond simply demonstrating that relationship graphically or numerically. They were simply given the parametric equations, asked to generate the characteristics of the sinusoids, and graph the functions on the calculator.

In the second observed lesson, Duncan’s primary focus was getting students to construct parametric equations to represent a given situation including ones in which the circle or ellipse needed to “start” at a particular point. He gave the students two warm-up problems.
Duncan: Now today, get your calculators out. What we are going to do first is just a quick review from yesterday. Two problems. [D shows two problems on the screen.]

1. Find the parametric equations that represent a circle, centered at (-2,3), radius of 4 and starts at the point (-6,3).

2. Find the parametric equations that represent an ellipse whose major vertices are (4,10) and (4,0) and whose minor vertices are (2,5) and (-6,5).

[Duncan, Ob #2, Lines 7–14]

He discussed his selection of tasks this way:

Duncan: Well, the big problem I wanted them to focus on was the clock problem yesterday and I knew I was going to ask them to start it at a different point. This warm-up problem was essentially dealing with a circle starting at a different point. And the second one, I think, was an ellipse. I just wanted to emphasize how the coefficients are different and how we create an ellipse instead of a circle

[Duncan Ob #2, Lines 29–35]

Duncan’s choice of tasks supported these stated goals by requiring students to construct of set of parametric equations that modeled the motion and the specified starting point. They also laid a foundation for the clock problem given to students later in the class. Also of note in his discussion of the task choice as well as in his treatment of these problems, Duncan’s focus was mechanical and skill-based rather than conceptually oriented. Two aspects of this focus were referenced in this excerpt. First, the application of sinusoidal parametric equations to the modeling of phenomena requires a facility with manipulating the equations to “start” at a different point rather than the 0 angle. Duncan identified this skill as one of his goals and dedicated the second observed lesson to the development of it. However, he used the term, “start” repeatedly, and he delayed making any reference to the connection between this “starting point” and the values of x and y at t = 0 until much later in the class. His approach at that time is discussed later in this chapter. As a result of this choice, students used a trial-and-error approach rather than a phase shift to model the behavior.
The second reference in this excerpt which pointed to Duncan’s procedural focus was his stated emphasis on “how the coefficients are different and how we create an ellipse instead of a circle.” Through the second warm-up task, Duncan provided another opportunity for students to recognize that when the leading coefficients of the sinusoidal parametric equations are different, the graph is elliptical. The task did not provide students an opportunity to make this connection on a more conceptual level. As is discussed in a subsequent section, the connection Duncan expected students to make was limited to the connection between the features of the equation and the graph.

Unlike the clock problem, which asked students to model the motion of the hands of the clock at particular times, the warm-up activities were devoid of context and provided no natural rationale for the combined graph to begin plotting at a particular point. Thus, the lack of context provided limited motivation for students to understand the need for an adjustment in the model. He explained his thinking in this way:

Duncan: Yea, the first one, centered at a particular point, that didn’t have really relevance and the radius didn’t, but I picked the point where it started so that it was a circle but it started on the left side of the circle, halfway. I think I asked…I didn’t even say if it should go clockwise or counterclockwise, but I wanted them to start at that point… So we have some options about whether they are going to use negative leading coefficients or negative values inside so I just wanted to see how they would work with that. That was the only thing about that problem that was specifically picked.

[Duncan Ob #2, Lines 41–55]

In this excerpt, Duncan acknowledged the limited scope of his thinking about the choice of these problems. He recognized the lack of a real-world context that could have provided some meaning to the task, and he specifically indicated that the only aspect about which he was intentional was the selection of a problem that would require the use of negative leading
coefficients or negative values inside—a phase shift. The work with these review problems and
the shift of the starting point was designed to prepare students to model the following situation
given to the students about two-thirds of the way through the class. It required the students to
construct the parametric equations so that the motion modeled by those equations began at a
specific point on the circle and rotated clockwise. This lead task represented the most
problematic scenario Duncan presented to students during the three-lesson sequence. However,
the work of the students on this task continued to remain at a mechanical level even when
Duncan suggested a more conceptually oriented approach.

Duncan: Because here is the next problem. This is the one I am interested in for today.
This is the focus of my lesson for today. [D shows the problem on the overhead]
A circular clock has a minute hand that is 12 inches long and an hour hand that is
9 inches long. Use parametric equations to model the movement of both hands as
they move from 12 o’clock to 1 o’clock.

[Duncan Ob #2, Lines 483–488]

After approximately 15 minutes of students working in groups (a little more than three-fourths of
the way through the class), Duncan reviewed the work of one student with the class and shared
the following idea:

Duncan: All right, one suggestion I have for you guys when you do these is…I think you
guys realized the radius was 12 so the leading coefficient was 12, but how can I get it
to start at the top instead of where we usually start the unit circle which is at 3 o’clock
position? There are a couple of ways to do this, but I like the idea of giving you guys
a table and at time zero, I want to be at the position (0,12). [D constructs a table on the
board with three columns: t, x, and y and fills in the first line with 0, 0, and 12]. If I
am centered at the origin, I want to be straight up at twelve so you guys have to figure
out how that is going to work.

[Duncan Ob #2, Lines 551–558]

The reference to the use of a table and the consideration of the location at time 0 is the only
reference to the conceptual basis for the “starting point.” When asked about this table
representation, Duncan replied:
Duncan: None of them are taking my advice. Even when I was looking at them today, they’re not… I want them just to be able to think about… You know the question is do we use sine or cosine [pointing to the parametric equations for $x(t)$ and $y(t)$]. They are used to thinking of $x$ in terms of cosine and $y$ in terms of sine, but they told me to write them that way. So I wanted them to say why and if somebody said I just tried it that way on my calculator and it worked. Well, why did it work? Can you think about those values? But you know, they were not latching onto that idea, that concept.

[Duncan Ob #2, Lines 602–611]

Duncan re-expressed his desire for students to demonstrate an understanding of why a particular use of sine or cosine functions works, and he acknowledged that the consideration of the location at time 0 and a look at the table of values would serve that purpose. However, he also noted that none of the students were “taking my advice.” The mechanical and procedural focus of the initial review tasks given at the beginning of the class did not provide the foundation or motivation for students to think about the “starting point” in this conceptually grounded way. Furthermore, this clock problem directed students to model the motion of the hands. It did not require them to understand their model or motivate their mathematical activity in a way that would potentially help them develop their understanding. It only required a workable model and that could be completed with a trial-and-error approach. The foundation provided by the warm-up problems only made a trial-and-error approach more likely.

The pattern of the selection of tasks with a focus on procedural or mechanical understandings continued with Duncan’s choice of lead task for the third observed lesson.

Duncan: Now today, let me pull up what we are going to do today. [D goes to computer]. Okay, here are some equations [D projects the following equations on the board].

4. $x = \frac{3t}{4}, y = 2t - 1$
5. $x = 2^t, y = \frac{4}{t}$
6. $x = t^2 - 3, y = t^2 + 1$
7. $x = ln(t), y = e^{ln(t)}$
8. $x = sec(\theta), y = cos(\theta)$
9. $x = 2cos(\beta), y = 2sin(\beta)$

After a brief discussion of the nature of the graphs determined by these parametric equations, Duncan described the task:

Duncan: Here’s what we are going to do today. I am going to show you how to convert from parametric equations, which those are, to rectangular. Now, I am not sure I have used the word rectangular much in describing functions, but you know when you guys graph, isn’t our graph paper rectangular? Well, it is actually squares, but squares are special rectangles. So when I say rectangular, I am talking about an x-y axis. So we do things in terms of x and y. That’s what you guys have been doing ever since you guys have been graphing functions. Since you started doing stuff in terms of x and y. But there’s other…there’s polar graph which we will talk about next week which is actually circular graph paper. And you can do some things on that. So what we are going to look at is how can I change these parametric equations to rectangular?

The activity of the students was focused primarily on the mechanical process, including the use of a variety of methods, of converting these parametric equations into rectangular ones. When asked about why he chose these examples, Duncan responded with a sense of pride:

Duncan: I knew you were going to ask that question. I purposely chose the first two to be…well the first one to be a very simple way in that you could isolate t in either equation and I am glad in the class that you taped that John decided to do it in the y equation to show that it took a little bit longer, but I knew it was something that they could isolate pretty quickly and convert it to rectangular. And the next one, I wanted to do a little bit of review with exponential, you know, let’s look at exponential functions, how do we undo that, logarithms. Most kids remembered logarithms and so forth. Six, seven, and eight were purposefully written so that you did not have to solve for the parameter. There was an easier way to eliminate the parameter than actually solving for t. Although, I did see a couple of kids who said I’m just going to end of solving for t and squaring a square root and all that kind of stuff that they didn’t need to do necessarily. But six, seven, and eight were purposefully set up so that there was a...actually...yea, those three were set up so that they could get through without eliminating. And then nine and ten was just once again a review of how I could take a
circle or an ellipse and put it back into rectangular form hoping that they would recognize that those two equations were those two shapes.

Interviewer: Okay.
Duncan: I patted myself on the back for those examples

[Duncan Ob #3, Lines 101–125]

The sample problems Duncan chose throughout his three lessons reflected this kind of thinking. He understood the scope of different situations the students might encounter in the problem sets and included examples that embodied those differences. He also integrated review of previously studied functions and relationships into the sample problems designed to demonstrate and build new skills. The approach seemed largely successful in that regard as students experience a high level of success in completing the tasks during the observed lessons. Yet, the focus of the mathematical work of the students through these examples was largely procedural and the extent to which students’ drew on their existing skills and understandings to complete the task resulted essentially in a review of previously learned properties and skills rather than serving as a connecting point for new understandings. The tasks, as they were constructed, were limited in their problematic nature in light of the focus on the mechanical skill and the teacher-led establishment of a routine for finding the rectangular form.

The one element of this choice of tasks that would seem to be more conceptually oriented is the range of the examples and the differences among them. When asked about these differences in the choice of task, Duncan discussed his desire to avoid a routinized way of approaching these problems:

Duncan: Because so many of the kids are…even though they are mostly juniors and a couple of seniors, they still want to latch onto some type of procedural way to solve a problem so I wanted them to…Here’s the default procedure that they could do, but I also wanted them to see that that procedure can be quite inefficient so before you just jump in to reaching for a formula or something, can you take a look at the problem and
maybe come up with a more effective or efficient way to do it. That was my thought behind all of that.

[Duncan Ob #3, Lines 128–137]

In Duncan’s thinking, by diversifying the set of examples to incorporate some alternative approaches to the standard, algebraic approach using substitution, the students might be more inclined to think about a more “effective or efficient way to do it.” These alternative approaches involved the use of some other relationships such as between logarithms and exponentials, trigonometric identities, and nonlinear substitutions. By applying these relationships, students were able to consider multiple strategies and determine the most efficient one for the task. While the focus is still on the mechanical execution of a process, these elements introduced some depth and fostered more connected understandings of those procedures.

In line with his stated goals for the three-lesson sequence, Duncan’s choice of lead tasks can be characterized as largely procedurally focused and introductory in nature. The tasks given to students required a largely mechanical approach to complete and did not often direct students’ attention to the relationships underlying those procedures. Duncan routinely incorporated previously studied functions and relationships as an exercise in the review of knowledge and skills rather than as a pathway to engage existing understandings to build new ones in the process of resolving some problematic scenario.

Summary of Differences in Lead Task Selection Among Teachers

Across the four teachers, the nature of task selection varied significantly in their accessibility to students, in their engagement of the perceived existing understandings of students, in the problematic nature of the tasks, and in the nature of the mathematical activity
they evoked in students. The nature of lead task selection within the data of each teacher is discussed below.

Jackie selected lead tasks that supported her learning goals for the three-lesson sequence by initiating mathematical activity in students that fostered a progression of understandings she identified in support of those goals. The tasks were accessible to students using their existing understandings without leading questions from the teacher. Jackie designed a series of tasks making only slight modifications from task to task to support the development of the desired, progressive understandings in students in incremental ways. Through her selection of tasks, Jackie demonstrated a triune focus on the mathematics of the students, the mathematics of the learning goal, and the potential path from one to the other through the chosen task.

The lead tasks selected by Harold reflected his consideration of the existing understandings of students and how those understandings could be used to support the development of new understandings. The lead task selected for the second observed Math 6 lesson represented Harold’s effort to provide students with an accessible and problematic task that was experientially real to students. That task, involving the distribution of money to four people, engaged students in the desired mathematical thinking through the modeling of that task and represented a triune focus like that of Jackie—on the mathematics of the students, the mathematics of the learning goal, and a potential path from one to the other. It motivated and directed the mathematical activity of the students in ways that engaged the existing understandings and supported the attainment of his learning goals. However, Harold’s task selection in the other observed lessons was not as attentive to this triune focus. In the Algebra I lessons, Harold provided a problematic task that could have been completed with what he
understood to be the existing understandings of students. Yet, those understandings did not appear to be as robust as he had anticipated nor did Harold provide enough scaffolding to students to support their use of their existing understandings.

Lead tasks selected by Susan held some potential for the development of the desired understandings of students. They reflected a consideration of the perceived existing understandings of students on the part of Susan with the attempt to draw parallels between arithmetic concepts and their counterpart in the world of rational functions. While the tasks embodied the essential aspects of rational functions and exposed students to those aspects of her learning goals for students, the tasks Susan selected did not reflect a consideration of a set of progressive understandings carefully sequenced to move students from existing understandings to the desired ones. The lead tasks chosen did not present problematic scenarios to students in which their existing understandings were engaged in a goal-directed way that could lead to the development of new understandings.

Like the other teachers in the study, the lead tasks selected by Duncan were also closely connected to his learning goals and involved a consideration of the existing understandings of students. Each of the tasks involved an intentional review of previously explored knowledge and skills. This approach gave students familiarity with some aspects of the tasks, but the tasks were not designed to link those existing understandings or use those skills to build the desired understandings of parametric equations. As was the case with Susan, the tasks Duncan selected exposed students to many essential aspects of procedural work with parametric equations, but they did not reflect a consideration of a set of the concepts and relationships underlying the mathematics of the learning goal nor did the tasks reflect a consideration of the potential
progression from perceived existing understandings to the desired ones. The lead tasks chosen by Duncan also neglected to provide students with problematic scenarios in which their existing understandings were engaged in a goal-directed way that could lead to the development of new understandings. Instead, the focus was largely on procedural execution and the lead task selection reflected that focus.

**Differences Among Teachers in the Elicitation and Interpretation of Student Responses**

Interpretation of student responses often involves developing an understanding of a student’s conception of the mathematics under study and determining whether that thinking is a productive step towards the learning goal, whether it could be used as a productive step, or whether it would be unproductive to pursue it in light of the learning goal. Differences emerged among the teachers in this study in the basis of their interpretation of student responses. These differences were revealed through the degree of tolerance for an imprecise expression of a mathematical idea or an inadequate or incomplete explanation and the degree to which student thinking was interpreted relative to the learning goals. In order to have the opportunity to interpret and possibly use student thinking, a teacher must elicit student thinking in one way or another. Students must be asked to articulate or elaborate their thinking, justify mathematical assertions, or clarify expressed ideas. Differences in the extent to which student thinking was elicited were noted across the four teachers in the study and will be discussed in this section along with the patterns in the interpretation of student responses. Due to the often closely connected nature of interpretation of student responses and responses to student responses, some
aspects of the responses to student responses will also be discussed in this section. However, these differences will be analyzed primarily in the next section.

**Jackie’s Elicitation and Interpretation of Student Responses**

Throughout the three lessons, a clear pattern emerged in Jackie’s implementation of tasks in two distinct ways. First, Jackie routinely elicited student thinking. She asked students to elaborate, clarify, and justify their expressed thinking while rarely adding her own explanation to the student’s response. At times, she asked other students to assess the validity of a solution or an approach to solving a problem provided by one student. Almost without fail, she asked students to justify or explain their reasoning behind both correct and incorrect responses. The second pattern in Jackie’s interactions with students involved how she interpreted their responses. As she elicited student responses, she consistently interpreted expressed student thinking relative to the learning goals she had for the lesson and the class using it as a foundation for instructional moves. She continued to ask students for clarification, elaboration, and justification until the student response revealed a productive understanding relative to the progressive understandings she identified as benchmarks in working towards her learning goal for students. In other words, the well-articulated learning progression was used to interpret student responses and determine the need for elaboration, clarification, or justification. Examples of these two patterns are discussed in this section.

Instances of the elicitation of student thinking can be found throughout the three observed lessons. The very first exchange between a student and Jackie during the first observed lesson illustrated the pattern. As described previously, Jackie initiated a review of a bonus question from the quiz the day before in which students were asked to find the measures of all angles and
lengths of a given kite. A student offered a measure of one of the angles and Jackie asked for justification.

Student: Angle A is 54 degrees.
Jackie: So how do you know that?
Student: It would have to be because angle B and angle D are congruent and then it would have to equal 360 degrees.
Jackie: Quadrilateral sum theorem?
Student: Uh-hm
Jackie: So how do you know B and D are congruent?
Student: It’s a kite.
Jackie: Okay, so the Kite Theorem. More facts.

Even though the student provided a correct answer of 54 degrees, Jackie insisted the student justify his response mathematically, “So how do you know that?” Even though Jackie provided the names of the two theorems the student applied in this case, the student provided the mathematical reasoning and the exchanged served to ensure the student fully understood both the correct solution and the mathematical justification for that conclusion.

As the exploration of this kite problem continued, a similar exchange followed as Jackie probed the students for more measurements they could find.

Jackie: Our goal here is to find all of the sides and the angles. We’ve got the angles.
Student: The first thing I did is that I bisected the kite.
Jackie: (interrupting) In what way?
Student: So angle CBA and CDA were in tact.
Jackie: Ah Ha. So you drew a diagonal. (teacher draws the diagonal joining the ends of the kite, AC in red)
Student: And then I drew…(teacher interrupts)
Jackie: What do you know about that diagonal?
Student: Well, I know that now there are two right triangles. And they are right triangles that I know all of the angle measures too.
Jackie: How do you know them all?
Student: Because it’s a kite, they are dividing angle BCD and angle DAB. So, if you divide BCD by two that would be 63 on each side and on the bottom it would be 27 on each side. So you know all of the angles and one of the sides, so at that point it would be pretty easy to figure out all of the rest of them.
Jackie: Good.

When a student explained that he “bisected the kite,” Jackie immediately asked, “In what way?” Even though the student used a mathematical word imprecisely, she asked for clarification so that she could fully understand the student’s thinking. Again, Jackie only provided the correct mathematical vocabulary when she identified the student’s work as drawing a diagonal. The student provided the explanation. When the student claimed to “know all of the angle measures, too,” Jackie again insisted that the student provide his mathematical reasoning. These initial exchanges with students during the implementation of the first lead task illustrated the pattern of exchanges with students throughout the three lessons as Jackie probed students for clarification and elaboration of their thinking, and mathematical justification of their conclusions.

At times, the pattern of elicitation of student thinking involved drawing other students into the consideration of the mathematical efficacy of an approach. During a discussion of how to find another missing length in the kite problem and after one student provided a correct method, Jackie asked the students for an alternative approach.

Jackie: Does anyone have another way?
Kiki: Yes, I have a faster way.
Jackie: Do you? What is that?
Kiki: If you just do CA over 4 equals tangent 63.
Jackie: What do you think of that idea folks?
Student: You have to put the x over the 4 because that is what would be the opposite side for tangent.
Jackie: What do you think Kiki?
Kiki: Yea.
Jackie: What is tangent?
Kiki: I used…
Jackie: What is CA in the triangle? I am going to hide the other triangle.
Kiki: CA is the hypotenuse. Oh, so it is supposed to be cosine.

[Jackie Ob #1, Lines 370–382]
When Kiki offered an errant approach, Jackie asked the class to consider her proposed approach, “What do you think of that idea folks?” A student provided a correction to Kiki’s approach that provided a correct solution path for finding another side of the triangle, $x$, and Jackie redirected Kiki’s attention to the definition of tangent and what side CA was in the right triangle.

In another example of drawing other students into the consideration of the mathematical validity of a response, Jackie asked students to assess the reasonableness of a student’s numerical response.

Jackie: What did you get Alex?
Alex: 179.7
Jackie: 179? Do you believe it?
Student: (another student) No
Alex: Oh, 13…
Jackie: Do we believe 13?
Student: yes
Jackie: Why do we believe 13? (pause) What does it have to be between? What and what?
Student: 9 and something
Student: 9 and a bigger number
Jackie: (whispers) Triangle sum theorem. I mean Triangle Inequality theorem. Between 9 and?
Student: 14
Jackie: So 170 something…Naa…”

[Jackie Ob #1, Lines 581–594]

In this exchange, a student offered a relatively large answer for the length of a side of the triangle with which the students were working. Jackie asked the class if they believed the answer and Alex reconsidered his response. Even when he provided a more mathematically reasonable response, Jackie continued to insist that he justify its reasonableness using the Triangle Inequality Theorem. In these two instances as well as others, Jackie elicited student thinking by engaging the entire class in the consideration of the mathematical validity or efficacy of a
student’s response. These instances represent the pattern Jackie consistently demonstrated in asking students to clarify their expressed thinking or justify their responses.

As students fully expressed their thinking, Jackie demonstrated a second pattern in her work with student responses as she considered the expressed student thinking of students. In order to determine when and in what ways to probe for more explanation from students, Jackie consistently interpreted expressed student thinking in terms of the learning goals she had for students—both lesson-specific and discipline-specific learning goals. In other words, she measured the expressed understandings of students relative to the progressive understandings she identified in her lesson planning. At the same time, she maintained a commitment to the discipline-specific goals for students—using multiple representations, connecting understandings, and empowering students to think mathematically and solve problems. She did so primarily by embracing alternative approaches while consistently emphasizing the approaches that would support the lesson-specific learning goals. The work of the class during the third observed lesson illustrated some examples of this interpretative lens and the importance of that focus relative to the learning progression.

One of the lead tasks of the third observed lesson involved students finding the coordinates of the endpoint of the diagonal of a parallelogram. Jackie introduced the problem as shown in Figure 4.14 in this way:

Jackie: I kind of want to go back to where we were the other day. So this is a different situation, but we had some vectors and we put the vectors on the coordinate plane. Do you remember that? And then we found the ordered pair for the end of the vector, where we would have our arrow, up here.
Figure 4.14. Jackie’s parallelogram task on the coordinate plane.

Jackie: And then we used that ordered pair to get the angle of direction. So we are calling this angle alpha (teacher labels the angle formed by the vector and the horizontal axis). So let’s catch up with this diagram. We have a vector of 6 and one of 8 and I want to get that ordered pair [J points to the upper right vertex of the parallelogram]. Can you remember how we did that?  

[Jackie Ob #3, Lines 45–53]

The students had worked with a similar diagram the day before and Jackie initiated this lesson with a review of their previous work. The sides of the parallelogram are 6 and 8 and the included angle is 40 degrees and the students were asked to find the coordinates of the diagonal Jackie is pointing to in the figure above. The students went through series of steps to find those coordinates.

The first step involved finding the coordinates of the endpoint of the side of length 6. As was the case at several other points during the three observed lessons, a student offered the use of the law of sines to find the missing sides even though it was a right triangle. As was the case in each instance, Jackie acknowledged the use of the law of sines as a valid approach but she also highlighted the value of using the definitions of the trigonometric ratios within the right triangle.

Jackie: Okay, what did you have?  
Student: Sine 40 over a equals sine 90 over 6.  
Jackie: What method are you using?
Student: Law of cosines…law of sines
Jackie: Law of sines. Right, what kind of a triangle do we have?
Student: Right.
Jackie: When do we need the law of sines?
Student: When we don’t have a right triangle.
Jackie: Yea. Is it wrong to use the law of sines when we have a right triangle?
Student: No
Jackie: Could you use the law of cosines when you have a right triangle? Sure. Is there an easier way to get the equation we need? Is there an equation that has less complexity? What do you have?
Student: Sine 40 equals \(x\) over 6.

[Jackie Ob #3, Lines 81–94]

In this exchange, Jackie asked the student to clarify the method he was using, reinforced the validity of that method, and emphasized the value of using a simpler alternative—the trigonometric ratio for sine.

The emphasis on using the trigonometric definitions served two purposes as expressed by Jackie. First, the ratios were a more efficient way of solving for the missing lengths or angles, and getting students to understand the most efficient ways of finding solutions was one of the points of emphasis in her courses—a broader, discipline-specific goal. In the stimulated recall interview for the first lesson, Jackie described these larger goals in this way:

What I want them to do is not only think about what they are doing and is it valid, but also is there a good alternative. Right now at this point in the year, there is always more ways than one to do something.

[Jackie Ob #1, Lines 261–264]

As discussed previously, Jackie wants students to understand multiple solution methods as well as which method is most efficient in a given situation. The trigonometric ratios are the most efficient way to find the missing lengths and angles in a right triangle. The second purpose of emphasizing the use of the trigonometric ratios in a right triangle involved the importance of this thinking to the lesson-specific learning goals. In order for students to begin to see the
relationship between the coordinates of a point on the plane to the measure of the angle formed with the positive $x$-axis, students must be thinking in terms of the trigonometric ratios. Jackie consistently interpreted the responses of the students with this ultimate goal in mind. The significance of this thinking to the learning goal can be illustrated by another excerpt from the third observed lesson.

At this point in the work with this problem, students had found the coordinates of the endpoint of the diagonal of the parallelogram, and Jackie directed student attention to finding the angle that diagonal forms with the side of the parallelogram which extended along the positive $x$-axis as shown in Figure 4.15.

Jackie: Now I want us to look at this triangle that I am going to call ABC in a minute [teacher traces the triangle with a squiggly line]. And I want you to get the angle here [see angle BAC in the figure below].

![Figure 4.15. Jackie’s working diagram with the parallelogram task on the coordinate plane.](image)

Student: The angle?
Jackie: So, I am going to trace this (teacher overlays a transparency to trace triangle). There is a lot of stuff on here that we do not need. We need this 12.6 and 3.9. So what is the length of this side?
Initially, Jackie highlighted the triangle of interest with a squiggly line and then she redrew the triangle separately (as shown in Figure 4.16) before asking the students to find the angle formed by the hypotenuse of that triangle and the positive x-axis. The students completed this work correctly and then Jackie restated the task without the triangle.

Jackie: Excellent. Let’s see. I want to try an experiment. What if there was no triangle (teacher draws Figure 4.17 on the overhead)
Jackie: And I wanted to find alpha.  

[Jackie Ob #3, Lines 248–251]

This shift to developing a process for connecting the coordinates of the endpoint with the measure of the angle is one of the essential, desired understandings of the three-lesson sequence. The shift can only occur if students are working with the simple relationships of the trigonometric definitions, and Jackie maintained her efforts to keep students focused on this aspect throughout their previous work with this problem and the tasks of the first two lessons in the sequence. During the stimulated recall interview during the third observed lesson, Jackie expressed her thinking about both the embracing of alternative approaches and the importance of a particular approach relative to the learning progression.

I don’t really care what the ordered pair is, but I care that we very gradually move away from that triangle. So for example, dropping that first (teacher draws a parallelogram on a set of coordinate axes with one side on the positive $x$-axis and drops a perpendicular from one vertex to the $x$-axis). I am okay with that, but I don’t want to think about it anymore. And then over here, the ones that wanted to drop that (teacher drops a perpendicular from the right most vertex to the $x$-axis) and I am okay with that too. I think it was the class that you saw that someone mentioned that they were congruent (pointing to the two right triangles formed by the perpendiculars and the sides of the parallelogram). Which was good, but eventually, like you said, that’s just noise and I don’t want to use law of sines and cosines. They are very nice, but I would rather not have to use them. So this gives me a tool to not use them. But all because I am leading towards that association between the angle and the coordinates of the points on the line.  

[Jackie Ob #3, Lines 273–289]

This statement was made in response to the portion of the lesson in which Jackie first attempted to get students to think about the relationship between the angle and the coordinates of the points on the vector. The last statement reflects the clarity with which she engaged students in their work with this sequence of problems over the three lessons as she sought to get students to shift
their work from finding the diagonal of a parallelogram and the coordinates of its endpoint to just thinking of that diagonal without a parallelogram or a triangle.

Jackie’s focus on this central understanding continued as students responded to her question about finding the angle without a triangle.

Student: You could make a triangle.
Jackie: I don’t want to make a triangle. I am tired of making triangles. What did Marlee do that she could have done without a triangle?
Student: Well, if you had a triangle, you could tell what the vertices would be because of the coordinates.
Jackie: What do you mean ‘because of the coordinates’?
Student: Well, you know that one side is 12.6 and the other side is going to be 3.9…
Jackie: So you are still thinking about a triangle.
Student: yes.
Jackie: So if there was a triangle there, those would be the right sides, but I don’t know that you have to draw it.
Student: Yea.
Student: That’s assuming we have a right triangle.
Jackie: Well we have control over that because our imaginary triangle is whatever we want, right? Okay, could we take this a step further. I want to make myself clear. I want you to look at this lovely equation (teacher boxes in the equation involving tangent from the previous problem) and I want you to tell me if there is a way to get that equation up here (pointing to the simplified diagram) without creating an imaginary triangle? Instead of thinking of 3.9 as a triangle side, can you think of 3.9 as coming from somewhere else?

[Jackie Ob #3, Lines 341–362]

The first student offered to make a triangle and Jackie redirected the student to think about what could have been done without a triangle. Another student persisted with the notion of a triangle but included a reference to the coordinates. Recognizing this as potentially useful, Jackie asked for clarification, “What do you mean ‘because of the coordinates’?” The student’s response still involved an implied triangle and Jackie maintained her insistence on students thinking about the relationship without a triangle. Her last rephrasing of the task helped students shift their thinking away from the triangle.
Student: The height?
Jackie: Yes, the height. But you are still kind of talking about a line segment, aren’t you?
Student: Can you do tangent lambda…or alpha equals height over length?
Jackie: Yes I could. But the height is represented by what up there?
Student: The y-value.
Jackie: The y-value. And the length is represented by what?
Student: The x.
Jackie: Rick, what are they saying? Rewrite this thing in the language Rita just used or Kiki. 3.9 is really the blank
Student: 3.9 is the…
Student: You do y over x
Jackie: Rick?
Student: 3.9 over 12.6
Jackie: That’s what we had here, but they are saying that 3.9 is really the…
Rick: y
Jackie: coordinate (teacher writes the equation as shown)

\[
\tan \alpha = \frac{y - \text{coordinate}}{x - \text{coordinate}}
\]

Jackie: If you think of it that way, do you need the triangle at all?
Student: No.

[Jackie Ob #3, Lines 363–381]

Even when the student responded with “the height,” Jackie continued to press students for a conception that was not associated with a line segment and a student was able to replace the concepts of height and length with the y- and x-coordinates. This exchange illustrated her careful consideration of student responses relative to the learning progression and her continual press for clarification and elaboration until the students conceived of the ideas in ways that supported the learning progression.

This quality manifested itself throughout the three lessons in her interpretation and elicitation of student responses. Throughout the three lessons, Jackie maintained a consistent effort to elicit the thinking of students. She routinely asked for justification of correct and incorrect responses. When a student’s expressed thinking was not clearly stated, she asked for
clarification. When a student’s explanation was incomplete, she asked for elaboration. As Jackie interpreted the validity or productive nature of these responses, she used the clearly articulated lesson-specific and discipline-specific learning goals. By eliciting the student thinking consistently, she was able to maintain a focus on the mathematics of the students and by interpreting these responses in light of the learning goals, she was able to maintain a focus on the mathematics of the learning goal and determine a potential path from one to the other as she constructed responses to student responses.

**Harold’s Elicitation and Interpretation of Student Responses**

Like Jackie, Harold routinely elicited student thinking and interpreted the expressed understandings of students in terms of his learning goals for students. However, the elicitation of student thinking and the interpretation of student responses recorded in the observations and interviews with Harold revealed a consistent application of a rigid set of indicators of understanding and a narrow set of acceptable conceptions and approaches to problems. While Jackie was also clear about what she wanted students to understand, Harold frequently equated a student’s proper use of specific mathematical language with understanding, whereas Jackie demonstrated a greater tolerance of mathematical understanding expressed with imprecise language. This pattern of applying strict indicators of understanding in his interpretation of student responses manifested itself in many of Harold’s interactions with students in each of the observed lessons and led to the following patterns in his elicitation and interpretation of student responses:

- a focus more often on what is wrong with a student’s approach, conception, or explanation than on what is correct about the student’s thinking,

- missed opportunities to elicit and explore student thinking,
• limitations to the ways students could express their thinking, and
• occasionally, illumination of a weak conception on the part of a student.

Exchanges from both the Algebra I and Math 6 lessons illustrated the point.

The two observed Algebra I classes involved student work on solving a system of linear inequalities:

\[
\begin{cases}
  y \leq \frac{x+6}{2} \\
  2x + 3y \leq 12 \\
  x \geq 2 \\
  y \geq 1
\end{cases}
\]

Throughout the two lessons, Harold emphasized proper use of vocabulary and broadly applicable definitions such as what it means to find a solution of any inequality or equation. This emphasis focused Harold’s attention on the precision of the language students used to respond to questions. For example, Harold began each of the two observed classes by asking students to reiterate what it means to solve a system. In the first observed lesson, the following exchange occurred:

Student: To show all of the solutions to the system.
Harold: Let me ask something real quick. What do you guys think of the word, “Show?”
Student: To represent.
Student: To define.
Harold: What do you guys think of the word, “show?” I like the word, “show” especially for this, because what is the only way to find all of the solutions?
Student: To show them
Harold: You would have to represent them and what is the only way to represent them in this sort of system? [Pause] You have to show them graphically. I like the word show. Keep going Tom. You said it and I kind of cut you off. I think you said quite well.
Student: To show…what did I say?
Harold: To show all of the solutions to the system. What makes something a solution to a system, Allen?
At the beginning of this exchange, a student provided a definition using the word, “show.”

Harold quickly focused student attention on the use of the word “show” and its power. However, as he directed student attention to the use of the word, he asked a leading question, “I like the word, ‘show’, especially for this because what is the only way to find all of the solutions?” He continued by answering his own question and limiting the opportunity for students to fully express their understandings.

Harold immediately followed up this exchange with a discussion of what it takes for something to be a solution to a system. A student offered a failed attempt to provide a proper definition before Harold redirected him.

Student: To make both equations true
Harold: Now you can memorize that it makes both equations true. Are you going to sound silly?
Student: Yes.
Harold: [H pauses the video and addresses the interviewer]. Alex’s response was… and you know that’s that classic where we’ve done solving systems of equations so a classic response is “you got to find the values that make both equations in the system true.” It has nothing to do with the problem we are studying. [H laughs]. So that’s kind of one of the things I was looking for and hoping not to get.
Harold: Try again. We don’t have two and they aren’t equations so we know your definition can’t make sense, right?

While the response, “to make both equations true,” is not an accurate response, there are several aspects of this exchange and Harold’s reflection on it that are worth noting. First, Harold interpreted the initial response of the student as a reflection of what he had memorized about systems of equations. Getting students to think about how to apply the mathematics rather than
memorizing a process was one of his expressed broader goals for his class and this demonstrates his interpretation of student responses in light of the goals he has for them. Second, Harold seemed to be looking for evidence of memorization as he expressed to the interviewer, “So that’s kind of one of the things I was looking for and hoping not to get.” He anticipated the imprecision in thought and focused on that, which limited his opportunity and effort to understand more of the student’s thinking. His statement to the interviewer that the student’s response had “nothing to do with the problem we are studying” further revealed his interpretation of the student’s response and reflected a strict indicator of understanding.

This disposition towards finding what is wrong with a student’s response and the emphasis on the precision of language manifested itself in other ways as well. At a point during the second observed Algebra I class, Harold asked the class what it meant to graph. A student responded by saying, “To show the equation on the graph,” [Harold Ob #3, Line 278]. In the interview, Harold made a point of stopping the video to interject this perspective.

Harold: That student made…I would say that I am listening for that mistake when they tell me they are going to show me the equation on the coordinate plane. I think I went on to actually draw the equation on the coordinate plane. But I think that is actually a classic mistake, especially when a kid is viewing, you know, like all of were taught the slope-intercept thing. And what I noticed was that students were reading that almost like directions for plotting a line, just like the coordinates of a point are directions for plotting a point. But the relationship is that these are an infinite number of points, each of which has coordinates that make the statement true. None of that was there. That the process of putting a line on a coordinate plane, was identical to the process of plotting a point to students, so I always look for that [pointing to the screen], for a kid to say something like, “we are putting the equation on there” as if you can put the equation on the graph.

[Harold Ob #3, Lines 288–304]

Again, Harold’s comments demonstrated his anticipation of this expression of student thinking. In his words, “I would say that I am listening for that mistake.” This anticipation illustrated his
attention to what is wrong with their use of language rather than seeking to use what is right about their thinking. Furthermore, his comment, “None of that was there,” suggested that Harold attached a lot more weakness to the student’s understanding than the statement, “show the equation on the graph,” directly implied. This disposition and the focus on the deficits in a student’s expressed thinking seemed to inhibit Harold’s efforts to help students build more robust understandings from their existing ones—even those inadequately expressed.

Another byproduct of Harold’s tendency to anticipate mistakes in student work as reflected in the precision of their language was the potential for him to miss opportunities to understand their mathematics more fully. In observing independent student work on graphing one of the inequalities written in standard form, Harold noticed that one student had solved it for $y$ in terms of $x$.

Harold: Why did you do it?
Student: To get $y$ by itself
Harold: Okay, why did you isolate $y$?
Student: So I could plug in.

[Harold Ob #2, Lines 403–406]

Immediately following this exchange and in response to this student, Harold initiated a whole class discussion about the use of the substitution method and emphasized the fact that it is a property of equality. Upon returning to the student, the student explained that he wanted to solve for $y$ so that he could plug it into the graphing calculator. Harold provided a recap of the exchange in the interview:

Harold: What he told me was that he had isolated $y$ in one of the inequalities and what I thought he was saying, so he could plug it in, and I recognized that we haven’t really talked about substitution being a property of equality. So right away I recognized that, “Ooh, I haven’t really gone over that yet,” and it’s important. It might really apply here. But he was actually asking about plugging it into the calculator. [laughs]. So he was actually going to answer it totally correctly.
Unlike as was the case in the work of Jackie, Harold did not continue to press the student to elaborate or clarify his thinking about “plugging in.” Harold assumed that the student intended to solve one inequality for $y$ and substitute that expression for $y$ in another inequality. In other words, he assumed a key point of understanding was lacking (that substitution is a property of equality) for this student. The student actually intended to solve the inequality for $y$ so that he could input the function into the graphing calculator to graph it. Harold’s misinterpretation led to an unnecessary explanation for the student.

Further evidence of Harold’s strict indicators of understanding and the impact they had on his elicitation and interpretation of student responses can be found as the students were exploring the graph of the solutions of each inequality. Harold wanted to emphasize the shading process as a point-by-point process in which the equation of the line, written in slope-intercept form, revealed whether the points directly above or below a given point should be included in the solution set. He explained it to students this way:

Harold: Think about what that looks like graphically. Here, [H points to a point on the line], $y$ is four. What about all of these [H makes dots directly below (2,4) on the graph as shown in Figure 4.18]
Student: Yea, those are true.
Harold: Does that make sense?
Student: Yea, it does make sense.
Harold: Guys, that’s why you shade below. From each of these points, you are saying, “All right, this point makes y exactly equal so that equation [he meant inequality] is barely true. But, could y be below that? [H points to the dotted line he drew].
Student: Yea.

[Harold Ob #3, Lines 742–751]

He discussed it in the interview this way:

Harold: I am trying to make sure they recognize this [pointing to a point on the plane] makes that statement [pointing to the inequality] true because it is underneath that guy [pointing to the line] which made that statement true [pointing to the equation] so there is a direct reliance on less than.

[Harold Ob #3, Lines 760–765]

In this explanation, Harold relates this pointwise emphasis on the shading process to his focus on the solutions to individual statements within the system and to the system as a whole. Two points are worth noting here. First, Harold again provided students with the explanation rather than continuing to elicit their thinking. He provided the conclusion, “that’s why you shade below,” and proceeded to reiterate the relationship. Second, even though the students had no previous experience with graphing the solution to a system of inequalities, this approach outlined the way Harold expected the students could have used their existing understandings to solve the system in this singular, mathematically precise way. Any student expression of an approach that deviated from this approach was not viewed as valid.

The lack of validity in alternative expressions or conceptions of shading became apparent in an exchange with a student during the first observed lesson. A student had accurately graphed the boundary line for one of the inequalities, and Harold asked how she would graph the
solution to the inequality. The student responded by indicating that she would shade on the lower side of the graph and Harold asked why.

Student: You shade the thing. This side of the graph [The student points to the region below the line as shown in Figure 4.19].

Figure 4.19. A student shares her reasoning about shading.

Harold: Why?
Student: Because all of these little dots can make the equation true. On this side, it can’t [pointing to the opposite side of the line].
Harold: Not equation. Watch your language.
Student: Inequality.
Harold: Random guesses. You can always try it. Does the point (0,0)…Is that a solution to that [pointing to the inequality]?

[Harold Ob #2, Lines 313–375]

In this excerpt, the student correctly articulated that the graph represented the “values of the variable that make the statement true” [Harold Ob #2, Line 272]. She also expressed (accurately) that the solutions to the inequality would be represented by shading below the line. When Harold asked her “Why?”, she responded appropriately saying “because all of these little dots can make the equation [Harold corrected her to say, ‘inequality’] true” and that it is not the case for the points on the other side of the line. In spite of the accuracy of her work, Harold questioned the mathematical validity of her reasoning. He described his thinking in the stimulated recall interview.
Harold: I think she is saying it, but I don’t think she knows it. I don’t think she actually knows cause I can see that they are shading directly away from the line. Whenever students do that, they are not thinking about the inequality, they are thinking about shade left or right or above or below.

[Harold Ob #2, Lines 326–331]

He did not believe that she understood how to shade to represent the solutions to the inequality in the way that he wanted them to be thinking about it. He referenced her “shading directly away from the line” as the basis for concluding that she did not understand the mathematical reasoning. Her work was correct. Her expressed reasoning included a statement that the coordinates of the points on that side of the line make the inequality true, demonstrating a focus on using the definition of solutions to justify the shading. Yet, for Harold, her reasoning did not match the specific approach and reasoning he constructed. He wanted students to strictly adhere to a point-by-point consideration of solutions to the equation and a point-by-point determination of which values of y will satisfy the inequality for each value of x for points on the boundary line.

This emphasis focused Harold’s attention on what was wrong with the student’s thinking rather than what was productive about it. After the student expressed her reasoning about her approach to shading as described, Harold’s response was to label the approach as “random guesses.” The interpretation of the student thinking as random represents an unfair characterization of it and left the student with ambiguity about what aspects of her thinking were correct. The student was left to ask, “So what do I do next?” [Harold, Ob #2, Line 387].

A third example from the Algebra I lessons illustrated further Harold’s focus on strict indicators of understanding and his tendency to emphasize and focus student attention on his
preferred approach rather than working with the expressed existing understandings of students to help them develop more robust ones.

Student: The first equation, all of these numbers work and they are going that way and that way [S points to the x values and the y values she has listed on her paper representing the solutions as shown in Figure 4.20.]

Figure 4.20. A student shares her reasoning about solutions to one inequality.

Harold: So you have recognized a pattern. Okay.
Student: Okay, number 2, from this number line [S points at the second set of values]
Harold: Oh wait, let’s do number one first.
Student: I want to show you something.
Harold: I am pretty sure that all you have done is find solutions to the equation. Right?
   We haven’t done the inequality and that’s actually a really important discussion we have to have. And technically, you’ve only found, one, two, three, four, five, six, seven, eight…[H counts the solutions she has listed.]
Student: I just didn’t write all of them.
Harold: No, No, that’s fine, but technically only eleven solutions, right? How many solutions does it have?
Student: Infinite.
Harold: So you have found one over an infinite number, isn’t that like zero? So we haven’t really found much of a fraction of all of our solutions, right? Very, very small. A very small percentage of them, right? So what we need to do, well hopefully as a class we are going to make sure we understand how to get all of them graphed.

While this exchange focused on an important idea related to the solutions, it also represented Harold’s continued focus on what was wrong with a student’s thinking. The student began the exchange by pointing to the solutions she had found for the first equation and then began to
describe the pattern of solutions to the second equation. She seemed to have noticed a pattern in
the coordinates of the solutions to the equation, and this recognition could have represented a
considerable amount of productive thinking. She also had a particular observation to share with
the teacher and even expressed, “I want to show you something.” However, instead of letting
her fully express her thinking, Harold interrupted her explanation and turned her attention
towards the limited number of solutions she had found. Even though the student was able to
express that there was an infinite number of solutions to the equation, Harold made a point of
emphasizing the small portion of all of the solutions she actually found. This redirection of
attention curtailed his effort to understand other aspects of her existing, demonstrated
understandings and illustrated his tendency to focus on what was wrong with student thinking
rather than working with what was productive in a student’s thinking. After the exchange
focused on the limited number of solutions she found, Harold never offered her a chance to ask
her question or show him what she was thinking beyond what he chose to focus on.

A final series of exchanges from Harold’s second observed Algebra I classes punctuated
the point. As students worked independently during the class on the solution to the same system
of inequalities, one of the students expressed some confusion about where to shade to represent
the solutions to the inequalities. The student or group had successfully graphed the line and
continued to have a question about showing the solutions to the inequality.

Student: I am trying to figure out where to shade all the, uh, values.
Harold: What have you done so far? What have you put on your graph so far?
Student: Well, we graphed the ones that we got for this [S points to the equation, \( y = \frac{x+6}{2} \]
written on her paper].
Harold: Yea, what are those things? Those things that you graphed?
Student: the values
H That do what?
Student: That make this statement [S points to the equation again] true.
Harold: And what is the relationship between that statement [H points to the equation, \( y = \frac{x+6}{2} \) written on her paper] and that statement [H points to the inequality, \( y \geq \frac{x+6}{2} \) written on her paper]?
Student: That one’s an equation and that one is an inequality.
Harold: So, these things [H pointing to the graph of the line], what are those?
Student: The graphs of the sets of values for the equation
Harold: And what do you mean the “sets of values for the equation?”
Student: That make this equation true

[Harold, Ob #3, Lines 246–273]

This exchange reinforced the pattern revealed in other portions of the classes. The student initiated the interaction with a question about shading to represent the inequalities. In response, Harold asked a series of questions to get the student to accurately express that the ordered pairs of the points on the line are sets of values of the variables that make the equation true. The group had graphed the solutions to the equation correctly yet Harold pressed for a precise expression of that understanding using mathematical language. While this represented a reasonable emphasis of mathematical communication, the emphasis became a distraction to the essence of the question the student had about the shading. The student adequately expressed this understanding twice in this exchange, but before moving to helping her understand how to shade to represent the solutions to the inequality, Harold moved to a full class discussion about what it means to graph an equation. It is unclear what prompted Harold to shift to the whole-class focus.

Harold: Hey, a question for everybody, what’s it mean to graph? If I ask you to graph an equation or graph an inequality, what am I asking you to do?
Student: To show the equation on the graph

[Harold, Ob #3, Lines 275–278]

Harold immediately focused the attention of the class on this imprecise notion of “showing the equation on the graph.” In response, Harold initiated a full class, teacher-centric lecture on what it really means to graph an equation. This focus on this student’s response put further distance
between the focus of Harold’s explanation and the original question the student had about the shading. His primary effort to advance the student’s understanding revolved around making her language, and the language of the class, more mathematically precise. That focus led to a missed opportunity for Harold to understand the thinking of the student around the solutions to the inequality and to advance that understanding in an effort to attain his learning goals.

Interactions from the Math 6 lessons also reflected Harold’s rigid conceptions and expectations of student expression of those conceptions. These strict indicators of understanding manifested themselves primarily in the limited ways in which students could express adequate mathematical understanding from Harold’s perspective. During the first Math 6 lesson and as previously discussed, the students struggled (from Harold’s perspective) to demonstrate an understanding of the initial lead task which involved representing \( a \div b \) as \( a \cdot \frac{1}{b} \). As the lesson unfolded and with the students unable to respond to the initial task as successfully as he had anticipated, Harold was prompted to rephrase the task numerically as shown in Figure 4.21:

Harold: What is another way to do 6 divided by 2. You know it is three, you know what it should equal, but I am not asking that. I am asking what is another way to express it.

![Figure 4.21. The numerical rephrasing of the lead task.](image)

Student: Six times 1 over 2
Harold: [H writes the student response on the board as shown in Figure 4.21]

![Figure 4.22. The numerical rephrasing of the lead task with the student response.](image)

Harold: Is that true?

[Harold Ob #1, Lines 161–168]
In the discussion about this move to a numerical example, Harold described what he thought the students understood at this point in the lesson.

Interviewer: Did you think they all understand that [pointing to Figure 4.22]?
Harold: [Long Pause]. I did. But, I don’t think they understood it as six halves added together. I think they understood it as half of 6. I think they are pretty good with dividing something by 2 is the same as taking half of it. I am convinced they are good with that, but I have really not done much work with taking six halves and adding them together because frankly I thought it was easier and I didn’t think it would be a problem. But clearly, it was.

[Harold Ob #1, Lines 176–184]

This discussion provided evidence of careful thought about the existing understandings of students as he presented this task and suggested that the numerical example was chosen as a result of this consideration of the understandings of these particular students. However, it also revealed the specific conception for which Harold was looking for students to express. The lack of context for this example made it difficult for students to express this conception in any other way besides that numerical expression. In other words, Harold had a particular understanding he wanted students to develop, but there was only one way to express it in this numerical example.

With a successful response to this numerical example, Harold returned the students’ attention to the original task asking them to express the same idea with variables. Even though a student provided a correct response, Harold was not convinced many of the students possessed the understanding of that response. With the unconvincing response of the class to this student’s correct response, Harold directed their attention to a different numerical example: “What is a fourth of 12?” [Harold Ob #1, Line 279]. After some discussion about the meaning of multiplication, Harold modified the task: “What would 12 groups of a fourth look like?” [Harold Ob #1, Line 313] and a student started to ask a question:
Student: [Student is asking a question while he is talking] When you say a fourth of twelve or a fourth of a whole…?
Harold: Here’s a whole. [Harold draws the picture as shown in Figure 4.23]

![Figure 4.23. Harold’s model of a whole used to represent a fourth.](image)

Harold: Here’s one fourth of a whole, right? [Harold shades the picture as shown in Figure 4.24]

![Figure 4.24. Harold’s model of a fourth of the whole.](image)

Student: Yes.
Harold: Let’s say we have twelve of these [Harold begins to draw multiple versions of the model as shown in Figure 4.25]. What’s that equal?

![Figure 4.25. Harold’s model of multiple fourths.](image)

[Harold Ob #1, Lines 331–340]

The student did not complete his initial question before Harold interrupted concluding that the student was not conceptualizing a fourth of a whole. He then drew the diagram to model the task and reiterated the question, “What’s that equal?” This exchange represented another situation in which Harold had a particular response in mind. When he did not hear the student express the
conception in the way Harold expected, Harold launched into additional modeling of the situation. In this case, Harold’s narrow conception of what constituted an adequate response curtailed his efforts to elicit a complete expression of the student’s thinking.

While many students were offering various guesses, one of the students offered a response of 12 fourths, and the researcher asked Harold about it:

Interviewer: Did you hear that person say “twelve fourths”? Didn’t they just say that?
Harold: I think so. I heard several things that…what I didn’t hear was…You know I am looking for them to recognize that each four quarters is a whole. So I wasn’t seeing that. I wasn’t seeing that thought process.

[Harold Ob #1, Lines 345–349]

Harold acknowledged a particular conception for which he was listening, “that each four quarters is a whole.” Within the interaction with students during this portion of the lesson, students expressed valid mathematical conceptions. One of those, “twelve fourths,” was a productive line of thinking and could have been used to help students construct the conception Harold hoped they would have expressed. In many ways, this episode was indicative of many portions of his lessons. Here, Harold was initially looking for a particular response from the students, and he focused on the fact that they were not giving him what he was looking for rather than listening to how they were thinking of this and working from their expressed conceptions to build the desired understandings as was the case for much of Jackie’s work with her students.

In both the Math 6 and the Algebra I lessons, Harold’s strict indicators of understanding placed limitations on the students’ expression of their thinking. His focus also led to missed opportunities for Harold to understand their thinking and use it to develop new understandings. In spite of these limitations, at times, Harold’s focus on strict indicators of understanding revealed some weak conceptions of students and laid a foundation for richer, more powerful
conceptions. A few examples from the observed classes illustrated his productive application of strict indicators of understanding.

In the Algebra I class, Harold’s interaction with students around correct responses revealed his careful consideration of mathematical understandings and goals beyond the response to the task at hand. In other words, he directed students towards different ways of conceiving of the solution to a task that held the potential to strengthen existing understandings. The first example dealt with a discussion of slope. Harold had intentionally avoided the use of the concept of slope in his work leading up to this class. However, as students began to notice the patterns in their independent work while producing the table of values, he took advantage of the opportunity to develop the concept with a class discussion. After producing a table of values (with contributions from the students) and ordering the list by the x-values, Harold asked about the pattern.

Student: As x rises by two, [H begins to write]
Harold: Now we are going to be more technical [H writes “increases” instead of “rises”]
Student: increases by 2
Harold: And instead of saying, “increases by two”, let’s just say, “As x increases, y increases.” Is that true?
Student: Yes
Harold: Guys, this question is crazy important, “At what rate does y increase?”
Student: [various responses]
Harold: [writing on board] Is that true? As x increases, does y increase by half as much?
Student: Yes
Harold: When x goes up by two, [drawing an arrow from x=0 to x=2 in the table of values], does y go up one?
Student: Yes
Harold: When x goes up by four, does y go up by half as much?
Student: Yes
Harold: Now, what if x only increased by 0.1? [H writes 6.1 in the x-column on the table and draws an arrow from 6 to 6.1].
Student: point zero five
Harold: Would you increase by point zero five? [H writes 6.05 in the y-column on the table and draws an arrow from 6 to 6.05].
Of particular interest here was the recognition on Harold’s part that the student’s initial response, while potentially correct, was not going to provide the foundational understanding for which Harold strived. First, the student used the word “rises” and Harold replaced it with the word, “increases.” Second, he translated the student’s response, “increases by two” into, “As x increases, y increases.” This distinction kept the student from expressing the slope in terms of the amount of increase in $x$ in relation to the amount of increase in $y$. Instead, Harold directed student attention to a formulation and articulation of the slope in terms of the rate of increase, “At what rate does $y$, increase?” He wanted students to think of the change of $y$ relative to the change in $x$. Even though he expressed it for them, “When $x$ goes up by four, does $y$ go up by half as much?”, this particular conception of slope served a particular purpose for Harold and the students. Judging from the quick, correct responses they provided to his specific changes in $x$, it appeared that they understood his conception.

The power of this approach was not quite evident until a subsequent discussion dealing with the rationale for connecting the points on the graph when plotting solutions.

Harold: This little argument right here, guys, [H pointing to the 6 to 6.05 on the y-values in the table]. This is what allows us to connect the dots. So you have recognized that (0,3) is a solution as well as (2,4) is a solution [H plots these points on the graph]. Is there going to be a solution exactly half way in between those [Harold plots the point between the first two as shown in Figure 4.26]?
Figure 4.26. Harold’s point-plotting demonstrating the continuity of a line.

Student: Yep
Harold: How come?
Student: Because y...no, because of the slope.
Harold: [H draws a horizontal line and then a vertical line from (0,3) to the new point he plotted]. If x increases by this much, is y going to go up by half as much? In fact, this particular point that we are talking about, x went from zero to one, and y went from three to three point five. This was a solution, wasn’t it? In fact, some of you guys found that solution. Guys, once you understand this idea of slope, that’s what allows you to recognize that there are going to be an infinite number of points between these guys that are also solutions. That’s why we are allowed to connect the dots. [H sketches the graph of the line.]

[Harold, Ob #3, Lines 636–651]

By conceiving of the change in y relative to the change in x in this way, Harold provided students with a way to apply their conception of slope to justify the continuity of the line. While largely teacher led, in this sequence of exchanges, a student response revealed, to some degree, his or her current conceptions of the underlying mathematics of the task. The student’s approach held some merit but also demonstrated some weakness. Harold’s strict adherence to a particular way of expressing the slope relationship laid a foundation for students to understand other mathematical concepts—in this case, the continuity of the line.
Throughout the observed lessons, Harold held stringent indicators of understanding and defined those in terms of the students’ ability to articulate and represent mathematical relationships using precise language and by making careful statements about the relationships. At times for Harold, these stringent indicators of understanding truncated his effort to understand student thinking more thoroughly leaving the impression that he was more focused on pointing out what was imprecise about a student’s articulation of understanding rather than considering the aspects of a student’s understanding that were productive. At other times, these strict indicators of understanding supported Harold’s work with students to refine their understandings.

**Susan’s Elicitation and Interpretation of Student Responses**

As previously discussed, the three Algebra II lessons observed were designed to introduce students to rational functions and to develop an understanding of the domain restrictions, the connections between the algebraic, numeric, and graphic representations, and the end-behavior of rational functions using polynomial long division. An analysis of the elicitation and interpretation of student responses across the three lessons revealed variability in the consistency with which student thinking was elicited by Susan and the degree to which the student responses were considered and used to advance student thinking. This section will illustrate this variability by focusing on two distinct contrasts that emerged in the data.

The first contrast involved the elicitation of student thinking. Although Susan maintained an environment for much of the observed classes in which students seemed comfortable expressing confusion, asking questions, and sharing ideas, she repeatedly missed potentially productive opportunities to explore their thinking further. Throughout the three lessons, students offered responses to tasks, asked questions when confused, and made conjectures about
relationships when called upon to do so. Susan used this expressed thinking to shape instruction, as is discussed in the section on her responses to student responses. However, Susan also frequently chose not to ask students to provide additional explanation, justification, elaboration or clarification. In some cases, she provided the additional mathematical clarity herself and in other cases, she left a limited student response or conception unexplored.

The second contrast involved the interpretation of student responses. Although Susan often seemed to effectively interpret student understandings through their responses, which in turn led to the construction of appropriate interventions and explanations, there were some notable exceptions in the classroom data. Evidence of the nature of her interpretation of student thinking involved counter-examples to student hypothesis, attention to salient features of the task or mathematical property, and teacher-generated justification, elaboration, or clarification of a student response. In some cases, Susan demonstrated an ability to understand what might be productive about a student’s mathematical thinking even when it was imprecisely or inadequately expressed. However, at other times, Susan’s responses revealed an inadequate consideration and interpretation of student thinking. Examples of each are discussed in this section.

During the three observed lessons, students routinely asked questions, expressed confusion, and offered responses to questions posed to the class. Examples of each appeared throughout the three observed lessons. At times, Susan recognized the need to elicit more explanation from a student. For example, during a discussion about one of the questions designed to introduce students to the algebraic differences between holes and asymptotes, the students were asked to consider the reciprocal of a previous expression.
Susan: What about the next example? [Teacher refers to the following example on the projector]

Is \( \frac{x+3}{x^2+2x-3} \) a rational function? Why?

Susan: Is that a rational function? [long pause] It bears a lot of similarity to the one above. What do you think, Devin?

Student: I think it is.

Susan: Tell us more.

Student: Doesn’t it factor out to be one over \( x \) minus 1?

Susan: Do you agree? Did everybody get that I hope? It is the same polynomials only now the denominator factors into \( x \) plus three times \( x \) minus one so we can divide out the common \( x \) plus three, but when we do that, look at what we end up with. An expression that is still a rational expression. Do we get \( x \) minus one again? [pause]

Student: We get one over \( x \) minus one.

In this interaction, a student responded to her initial question by asserting, “I think it is.” Susan elicited more from the student even though his response was correct. In essence, she asked for a justification of his correct response and the student provided it. However, Susan elaborated extensively on the student’s response rather than asking the student to fully explain his reasoning.

Another occurrence of recognizing the need and value of justification occurred when Susan attempted to help students connect the algebraic and graphical representations of the rational functions.

Susan: Vertical asymptotes, you’ve seen, are often features of rational functions. What causes…in other words if you look at it algebraically, what’s our clue that we might have a vertical asymptote? [short pause] At the beginning of class, you figured out that this function had a vertical asymptote.

Student: An undefined variable…eh, an undefined solution.

Susan: Undefined solution or output. As a result of what? What makes it undefined?

Student: Zero in the denominator.
Again, the student provided a correct response when he offered, “an undefined solution.” And again, Susan did not accept a correct response without asking the student to explain the reason why the solution was undefined.

A third exchange during the third observed lesson also illustrated Susan’s effort to elicit more of a student’s thinking. After a lengthy discussion of the process of polynomial long division, Susan initiated a consideration of the notation for rewriting a quotient with the remainder and reinforced that notation using multiplication to check the division. She once again used the numerical long division in parallel with the polynomial long division to connect new understandings to existing ones.

Susan: How can we use multiplication to check a division problem?
Student: Multiply by the reciprocal.
Susan: Uh, I am not sure what you mean by that. So let’s look at this one. I would like to use multiplication to check that I have done this division correctly.
Student: You take the divisor and multiply it by the answer you got.
Susan: So tell me more. [writing on board] 17 times [T opens parentheses]
Student: [T writing on board as shown in Figure 4.27] 224 plus [T stops writing on board] 12 over 17

![Figure 4.27. Susan re-expresses the 3820 using results from numerical long division.](image)

Susan: Plus twelve over seventeen [T writes the fraction 12/17]. I am pretty sure when I do this I’m gonna get twelve-seventeenths as part of the answer. What am I hoping to see here? That when I check my multiplication, I’ll get what value?
Student: 3820
Susan: So there’s something not quite right here.

[Susan Ob #3, Lines 540–551]

In this exchange, Susan engaged students to share more of their thinking at three different points. First, she told the student that she was “not sure what you mean[t] by that.” She then explicitly
asked a student to explain more of her thinking, “Tell me more”. Finally, instead of telling
students what to do to fix the response, she directed their attention to what was not quite right
and waited for them to correct the response. While these are rather routine examples, they
reinforce the value Susan placed on student responses and her ability to recognize the value of a
justification or elaboration of a response.

In contrast, there were other examples in the data in which Susan neglected to ask
students to justify, clarify, or elaborate their thinking. Of the 55 critical instructional moments
identified as responses to student responses 16 were situations in which Susan chose not to or
neglected to pursue a student’s response further. In some cases, Susan chose not to elicit student
thinking behind an errant response. For example, during the first observed lesson, the students
produced a table of values for the function, \( f(x) = \frac{1}{x-3} \). Susan asked the students about the
output values for the function:

Susan: The table is kind of interesting in peculiar ways. What did you see for your
decimal values?
Student: They were complex.
Susan: Complex meaning?
Student: Long
Susan: Long, yea. Very long. Did they seem to get very high or very low? [pause] In
other words, was there anything in the table that would give you some clues about the
domain?
Student: They all stayed between zero and one.
Susan: The domain is what? What is the definition of domain?

[Susan, Ob #1, Lines 184–193]

Initially, a student described the values as “complex,” and Susan asked the student to elaborate.
She also followed up the student response with a question, “Did they seem to get very high or
very low?” Before allowing a student to respond, she asked a second question directing their
attention to the domain of the function. When a student responded, “They all stayed between
zero and one.” Susan did not ask the student to elaborate on his or her thinking. Instead, she restated the question about domain and asked student to provide the definition of domain. In light of her question about the definition of domain, it seemed likely that Susan interpreted the student to be thinking about the asymptotic behavior of the function as $x$ increased—in other words, thinking about range rather than domain. However, Susan did not ask the student to clarify or elaborate on his thinking leaving it unexplored.

An exchange from the second observed lesson exemplified many of the instances in which Susan neglected to ask students to justify, clarify, or elaborate their expressed thinking. The exchange occurred after Susan presented one of the lead tasks previously discussed.

Susan: So my question is, and you have this fraction written down on your paper, Why is $\frac{x^2+2x-3}{x+3}$ not a rational function?

Student: Because it is not a polynomial in $x$?

Susan: My question is why is that not a rational function? Why does that not fit the definition of a rational function? It looks like one. We’ve got a polynomial divided by a polynomial.

Student: Can you not divide by $x$?

Susan: I can divide by a variable, but it will be undefined if that variable gives me a zero denominator. Can you manipulate that expression any?

Student: You could factor.

Susan: Ooh. There’s an idea. Go for it. [Students work independently and the teacher walks around].

[Susan Ob #2, Lines 241–254]

In this exchange, a student offered a reason (in the form of a question) that the function is not a rational function, “Can you not divide by $x$?” Instead of asking the student to justify or explain her uncertain response, Susan quoted a general rule without forcing the student to articulate his or her thinking more fully. Susan then rephrased the task leading the student to consider if he or she could manipulate the expression any. Even when the third student provided a productive
response, “You could factor,” Susan acknowledged it as a good idea and had the students execute it without asking the student to explain why that would be potentially useful. The students proceeded to factor the expression without necessarily knowing why they should. Following the independent work time and a successfully factored and simplified expression, Susan provided the additional mathematical reasoning for students to begin to make sense of the significance of the factoring and simplification.

Susan: It is true that this is a simplification of the original one. That’s true, but they are not equivalent. The original function had what restrictions?

This response by Susan occurred following the presentation of the factored and simplified version of the expression and was unprompted by a student comment. She introduced the notion that the two expressions are not equivalent, and without any elicitation of student thinking on that observation, Susan directed student attention towards a consideration of the restrictions. A consideration of the concept of algebraic equivalence as opposed to functional or graphical equivalence could have generated rich mathematical thinking on the part of students. Susan could have introduced the notion with a question to elicit more thinking on the part of the students. Instead, she introduced it as an observation and connected it to the restrictions on the variable.

These exchanges illustrated the variability observed in Susan’s elicitation of student thinking. At times, she was persistent in asking students to clarify, elaborate, and justify their thinking. At other times, she chose not to elicit student thinking by either leaving additional explanations un-pursued or providing the clarification, elaboration, or justification herself. Instances of teacher-provided explanations also abound in the data and will be discussed along
with other responses to student responses in the next section. The frequency of these types of exchanges along with others in which she expected students to justify and clarify their responses revealed an inconsistency in her approach to the elicitation of student thinking. A similar pattern of variability emerged in her interpretation of student responses. At times, Susan demonstrated a tolerance for student thinking that was imprecisely or inadequately conveyed and used that expressed thinking to advance the class discussions. Yet, at other times, Susan made unproductive assumptions about student thinking, which yielded what seemed to be unresolved confusion on the part of students. Examples of each are discussed subsequently.

One of the most compelling examples of Susan’s work with inadequately or imprecisely conveyed student thinking occurred in the first observed lesson. Students were asked to describe the end behavior of a rational function using a comparison with the end behavior of the basic negative odd power function \( f(x) = \left( g(x) \right)^{-a} \), where \( a \) is an odd natural number. Negative odd power functions had been studied earlier in the year. The rational function under consideration in this class was the function \( f(x) = \frac{1}{x^3} \), and students were asked to reason about the end behavior of a rational function by thinking of it as a translation of the basic negative odd power function.
Susan: For the basic function, the one in blue [T references a colored graph on the board as shown in Figure 4.28], how would we describe the end behavior, left end and right end? [T retraces the left and right ends of the graph, making them darker, as she speaks].

Student: Left end is parallel to the $x$-axis and the right end is parallel to the $y$-axis.

Susan: Parallel means equal distance apart. Is that what you want to say? [Pause] So let’s find another way to word it. Right end behavior means, as $x$ goes way out to the right. I am going to say, as $x$ approaches a really big positive number—positive infinity. What happens to the output?

Student: It gets closer and closer to zero.

[Susan, Ob #1, Lines 271–282]

Susan assumed that the student simply chose the word “parallel” inappropriately and prompted the student to express his or her understanding more accurately. Also of note, without seeming to lower the level of cognitive demand, Susan defined her “right end” to address the student’s reference to the right end being parallel to the $y$-axis. Susan understood the student to be referring to the portion of the graph to the left of the vertical asymptote in her description of the left end and right end behavior. The correct response of the student, “It gets closer and closer to zero,” suggested that Susan’s interpretation of the student thinking adequately addressed the student’s imprecision in her thinking.

This exchange highlighted two important features of Susan’s interpretation of student responses. First, in her response, she demonstrated a tolerance for the imprecision in the expression of the student’s thinking. The student’s use of the word, “parallel,” while not mathematically appropriate, conveyed some level of understanding, and Susan did not make an assumption about the accuracy of the student’s understanding based on the how she was expressing it. Instead, she emphasized the definition of “parallel” and suggested using an alternative by asking the question, “Is that what you want to say?” The second important feature of Susan’s interpretation of this student’s response was the way she tacitly acknowledged what
was right about the student’s thinking while redirecting her to the right end portion of the graph. By defining “right end” for the student and giving her a hint at the right words to use to describe the behavior, the student was immediately able to apply her correct thinking to the appropriate part of the graph and to express her thinking accurately.

A similar approach characterized Susan’s interactions in the following excerpt from the second observed lesson. The class was asked the question, “Why is \( \frac{x^2 + 2x - 3}{x + 3} \) not a rational function?” (according to Susan’s definition). The class had factored the numerator and simplified the rational expression, and Susan directed student attention to the restrictions on the original expression and the simplified version:

Susan: But what are the restrictions to this? [referring to the un-simplified version]
Student: negative three.
Susan: \( x \) cannot be negative three. What are the restrictions here? [pointing to the simplified expression]. Any restrictions? [long pause]
Student: Can’t be one.
Susan: What would happen if it did? If \( x = 1 \), then what would the whole thing equal?
Student: Then the whole thing would be zero.
Susan: Then the whole expression equals zero. Is that okay? Is that a problem?

[Susan, Ob #2, Lines 260–268]

When the student responded, “Can’t be one,” Susan assumed that the student was looking at the simplified expression \( (x - 1) \) and based his response on what would make that 0. With that understanding, she asked a question to direct the student’s attention to the value of the expression if \( x = 1 \) and then challenged the student to determine if that was problematic. Again, Susan tacitly acknowledged the potential value of the student’s approach of looking at what makes a factor 0 while she redirected his thinking to a consideration of the reason 0 factors really matter.

Perhaps the most prolonged exchange in which Susan demonstrated her commitment to understanding student thinking and constructing responses to it also revealed some inadequacy in
her efforts. Throughout much of the third Algebra II lesson, Susan drew parallels between the numerical long division and polynomial long division algorithms. The explication of polynomial long division began after students completed a numerical long division problem and then as a class and with Susan’s direction, identified the four steps of the process. Susan proposed a polynomial long division problem and urged students to apply the same four steps to it. She presented the problem this way:

Susan: Now we are going to apply the same process, but to algebra. So, I want to look at a division with polynomials. Same four step process. Suppose I have [Susan writes the expression shown in Figure 4.29 on the board]:

![Figure 4.29. The rational expression used for long division task.](image)

After writing the ratio as a long division problem, Susan directed students to consider the “guzzinta” step—this reference was her word for “goes into,” and she asked a student about her numerical long division process:

Susan: Notice, when Ellen was saying 17 goes into 80, how did she figure that out? How did you figure that out, Ellen?
Student: Just find the closest multiplier to 80.
Susan: Did you do a lot of multiplications by 17 or did you just approximate it?
Student: Well, I kind of like guessed. Two times 17 is 34 and 4 times 17 is 68.
Susan: Do you understand her approximation? 17 is close to 20 and 20 will go into 80 four times so that was a very reasonable guess for her. Okay? We are going to do the same thing. We are not going to figure out all of this. We are not going to use all of $x$ plus 2, we are going to use that critical part, the first $x$. So $x$ goes into 3 $x$ squared, how many times?
Even though the student described a trial and error approach, Susan described her approach as using the approximation, 20, for the divisor, 17. Not only did this exchange represent an imprecise interpretation of the expressed work of the student, the approximation idea was also used as a reference throughout the explication of the process and provided an imprecise parallel between the two processes on which Susan attempted to draw throughout the lesson.

As the discussion of polynomial long division unfolded, several questions from students prompted her to interpret student thinking and construct responses to it. Each exchange involved confusion over the imprecise parallel between the dividing-into step of the two algorithms. In this excerpt, the student asked about the divisor, $x+2$. He was confused by Susan’s use of only $x$ to determine the multiplier.

Student: For this one, wouldn’t it be simpler to do separate and do $x$, divisor sign three $x$ squared minus five $x$ plus 4 and then to 2 divisor sign and three $x$ squared minus five $x$ plus 4.

Susan: That’s similar, if I am understanding your question correctly, to saying, “would this [referring to the previously worked numerical example] just be easier to consider this just as dividing by ten and dividing by seven. You see, except I am dividing by 10 plus seven.

[Susan Ob #3, Lines 248–254]

Susan understood the student’s question and the nature of his confusion and provided a numerical parallel. However, she did not prompt the student to express his understanding of that parallel example and apply it to the algebraic situation. In the interview, the researcher asked Susan about this exchange.

Interviewer: Do you think he recognized that dividing by 10 and dividing by 7 is not the same thing as dividing by 17?

Susan: Well, that’s why I said…No, I don’t think he did.

[Susan Ob #3, Lines 255–257]
Susan seemed to begin to defend her response and then interrupted herself to conclude that she did not think he understood. This hesitancy and the fact that she did not intervene in the moment suggested that her doubt about his understanding came in the moment of the stimulated recall interview. In either case, the fact remains that she did not elicit the student’s thinking in that moment to clarify what she understood his understanding to be.

The confusion about this aspect of the polynomial long division persisted as the classroom conversation continued:

Student: Then why do you say, “How many times with just \(x\)?”
Susan: I am approximating. Yea, so I am saying, how many times does \(x\) go into three \(x^2\). It’s answering this question. [Pointing to Figure 4.30].

![Figure 4.30. The algebraic restatement of a part of the long division task.](image)

Student: Yes, but that doesn’t include the plus two.
Susan: Exactly. Just like before when Ellen tried dividing by 17, twenty was not the same thing as 17, just a good approximation. So now we are going to do it exactly. Now we are going to see what happens when we do the next step which is multiply. So we have 3\(x\) times \(x\) plus 2 which is? Three \(x\) squared plus six \(x\).

[Susan, Ob #3, Lines 319–326]

Again, the student expressed confusion regarding the use of \(x\) rather than the entire divisor, \(x+2\). Susan attempted to alleviate that confusion by drawing a parallel, albeit an imprecise one, with the work of the student on the numerical long division problem. Susan referenced the student’s use of 20 as an approximation in her long division example, yet as discussed previously, the student did not approximate the multiplier by approximating 17 with 20. The parallel seemed lost on the students. While Susan understood that the source of the student confusion was the exclusion of two when considering the dividing step of the process, she seemed to lack a
complete appreciation for why her explanation did not make sense to the students. She expressed her surprise at their difficulty on numerous occasions throughout the interview about this episode.

As the progression of the example of polynomial long division advanced to the multiplication step, a student asked, “What happened to the two?” [Susan, Ob #3, Line 336]. Susan interpreted that question to initially represent confusion over the multiplication step, but as the discussion evolved, it became clear that this question was still a result of the confusion over the dividing-into step and the use of only $x$ from $x + 2$. At one point as Susan attempted to proceed with the multiplication step and a student interrupted:

Student: You didn’t divide by two so you…
Susan: [interrupting] I didn’t consider two in estimating what my quotient would be.
Student: I gotcha.

While this exchange suggested that the confusion is resolved, the students continued to raise questions about these steps in the process—ignoring the two when determining the partial quotient, but multiplying the two by the quotient before subtracting.

Student: So why bring the two back down?
Susan: I am still not understanding what you mean by bring it back down.
Student: You divided by three $x$ when you started and didn’t you also just bring the two back in when you multiplied?
Susan: When Ellen came up with the four here [pointing to the four in the quotient from the long division as shown in Figure 4.31]

Figure 4.31. The numerical long division task results.
Susan: [continuing] She all of a sudden said, it’s not four times 20 [the approximated divisor] it is four times seventeen. Okay? It looks much more complicated with the algebra, but the process is identical.

Again, Susan used the numerical model to draw the parallels and made reference to the estimated quotient, 20. The repeated reference to the approximation, which was not really referenced by the student explanation of the numerical process, and the emphasis on the process being “identical” suggested that Susan did not fully appreciate the nature of the difficulty students were experiencing. This difficulty could be a function of the nature of the unpacking of the student thinking in the selection or construction of the lead task or of the learning goals for the lesson. Ultimately, the students and teacher resorted to a mechanical approach focusing on some aspect of the process that would be easy to remember.

Student: But how come you only put $x$ into the divisor instead of two?
Susan: Because the dominant determination for how many times it goes is that $x$. The “plus 2” is just going to affect what I get here [pointing to the result of the multiplication step].
Student: So that’s the only time you use two.
Susan: Yep, when I actually do the multiplication.

While the confusion of the students persisted, it is noteworthy that Susan was able to respond to each question with a slightly different way to think about the process. This suggested an active processing of their responses. It also implied reasoning on the part of the teacher to develop these responses—a feature discussed in the section on her responses to student responses.

However, it is unclear that Susan developed an understanding of the fundamental obstacle to the students’ understanding. Late in the discussion, she still contended that the processes were
identical, yet for the students, there was an essential difference they could not reconcile. Susan did not express an appreciation for the basis of the confusion in the follow-up interview.

Even with the limited understanding of the difficulties of students in this series of exchanges, Susan demonstrated a commitment throughout this particular lesson to elicit and attend to the expressed thinking of students. Throughout the three lessons, and like Jackie, Susan tolerated imprecision in the expression of student thinking and demonstrated an ability to look past imprecision in language and thinking to acknowledge what was productive in a student’s thinking. The variability in her successful interpretation of student thinking matched the inconsistency with which she elicited student thinking throughout the three lessons. At times she insisted a student justify, clarify, or elaborate his or her thinking. At other times, Susan provided the mathematical clarity and justification. Still, at others, Susan moved past errant or correct student responses without any clarification, elaboration, or justification.

**Duncan’s Elicitation and Interpretation of Student Responses**

In contrast to the work of Jackie, Harold, and Susan, Duncan did not elicit student thinking as consistently throughout the observed three-lesson sequence. In light of the procedural and somewhat mechanical emphasis Duncan maintained during the three observed lessons, students were primarily asked questions that required them to execute a skill or perform segments of a larger modeling task as directed by the teacher. Other work often involved unconnected execution of skill with review of previously studied functions. When students asked a question or provided responses that might have revealed something about their mathematical conceptions, Duncan typically provided the answer rather than responding in a
way that would have helped them develop new understandings, and sometimes, he chose not to pursue them. Several examples from his classes illustrated this dynamic.

As previously discussed, the focus of the observed lessons tended to be on exposure to the uses of parametric equations and procedural aspects of their use including work on the graphing calculator rather than on the underlying concepts and the ways in which parametric equations represent relationships. During the first lesson, Duncan introduced the students to parametric equations through three examples. The mathematical work with these examples and the interactions between Duncan and his students revealed the pattern that emerged over the course of the three observed lessons.

The first of the three examples was contrived and involved a table of $t$, $x$, and $y$ values introduced without context. It was designed to provide students an introduction to parametric equations at a basic level. Duncan provided the table and then had the students graph the three sets of points: $x$ versus $t$, $y$ versus $t$, and $x$ versus $y$. After plotting $x$ versus $t$ for the class [as shown in Figure 4.32], Duncan asked the class to interpret the graph.

![Figure 4.32. Duncan’s first lead task.](image)
He introduced a context at this point:

   Duncan: So if you picked an object, something that moves, let’s say your pet at home. Over 4 seconds, the first second it moved a foot (traces the graph), and the second second, what did it do (teacher points back and forth from 1 to 2 on the t axis)? Did it move?

   [Duncan Ob #1, Lines 101–106]

In this excerpt, Duncan asked students to interpret the graph. Without giving them a chance to do so, he provided explanation of how the object moved during the first second. The subsequent question about the movement during the second second was quickly replaced by the leading question, “Did it move?” Each of these instructional moves lowered the level of demand on these students without Duncan eliciting any initial response from them, and this approach characterized many of his interactions with students.

   After the class graphed x versus t and the students independently graphed y versus t, Duncan asked a question about what the graph might represent. A student responded in a limited way, and Duncan proceeded to provide additional context:

   Duncan: And what do you suppose this is going to represent?
   Student: The change in the y
   Duncan: Yea, potentially the vertical change. So maybe your animal will jump in the air? Stay in the air? That’s what I said, this one is not THAT realistic, but it’s pretty straightforward and that is why I wanted to start with it. (pause) Draw your arrows too, that will help, I think, as you plot your points. (teacher plots the points as shown in Figure 4.33)

   \[\text{Figure 4.33. } y \text{ vs. } t \text{ in Duncan’s first lead task.}\]  

   [Duncan Ob #1, Line 121–127]
This exchange captured each aspect of the pattern revealed throughout the three classes. Duncan asked a question that could have revealed something about the student conceptions, and the student offered a limited response. Instead of asking the student to elaborate and make more of her thinking transparent, Duncan rephrased the student’s response to “the vertical change,” provided an unrealistic context, plotted the graph on the board, and moved onto the third graph without eliciting more input from any student. The context used—associating the change in $y$ with the vertical motion of a pet—not only made it difficult for students to relate the $y$ versus $t$ graph and data to the real-world context, it also made it difficult to relate what the $x$ versus $y$ graph and data represented.

As the class continued the work with this first example and after students plotted the $x$ versus $y$ graph, Duncan summarized the work of this portion of the lesson segment:

Duncan: Did you guys get a rectangle? I should have made my scale bigger, cause my $x$’s are only going out to one and my $y$’s are going up to two. But you get that rectangle. Okay? So if you notice, make sure you see this, we have this table of three columns. This graph (the $x$-$t$ graph) represents which two columns?
Student: $t$ and $x$
Duncan: Yea, the first two columns. This one (pointing to the $y$-$t$ graph) represents which two columns?
Student: $t$ and $y$
Duncan: And this one (pointing to the third graph) represents…
Student: $x$ and $y$
Duncan: And you are just plotting points. This (first graph) is going to represent our horizontal motion. This (second graph) is going to represent our vertical motion, over 4 seconds for both of them, and when you put them together (pointing to the third graph) our horizontal and vertical motion together, they are actually traveling in this rectangular shape.

This summary emphasized the mechanical nature of the mathematical work for this segment. Duncan described the graphs as representing the columns rather than the relationships. He referred to the process as “just plotting points,” and he described the third graph as “put[ting]
them together.” He also attached some additional meaning to the third graph when he interpreted the graph as showing the object “traveling in this rectangular shape.” The graph was only a picture of the motion if the x- and y- coordinates represented the distances from perpendicular surfaces. Each of these aspects of this exchange demonstrated Duncan’s emphasis on the mechanical and somewhat superficial treatment of the material as well as the limited demands he placed on students in terms of their responses.

The second example in the first observed lesson revealed another aspect of the student responses during the classes. This example involved the modeling of the population of rabbits and the population of foxes. The parametric equations were trigonometric. Duncan reviewed the connections between the graphs and the equations, and the students graphed the parametric equations on their graphing calculators. Again, Duncan placed limited demands on the mathematical thinking of the students through highly teacher-directed discussions. Duncan began the introduction of this example by writing on the board the trigonometric functions modeling the population of rabbits and the population of foxes over time and initiating the following discussion:

Duncan: All right, quick question, if we look at rabbit populations and fox populations over time, do you think they are related?
Student: No.
Student: Yes.
Student: (inaudible)
Student: Cause foxes eat rabbits.
Duncan: Cause foxes eat rabbits. So we know, foxes eat rabbits. So, let me ask you intuitively for a minute, if the rabbit population’s climbing (motions upward with hand) what do you suppose the fox population is going to do?
Student: Decline
Student: (multiple responses speaking over top of one another)
Duncan: David?
Student: Increase
Duncan: why?
Student: Food.
Duncan: There’s more food. So, if the rabbit population is going up, here comes the fox population and as the fox population goes up, what do you think is going to start to happen to the rabbit population?
Student: Go down
Duncan: It is going to go down (motions a curving downward with hand). And as that food source goes down, what do you think is going to happen to the fox population?
Student: Go down
Student: Level off
Duncan: Go down, level off perhaps and then go down. And do you guys see how they could cycle back and forth? (D motions with hands up and down—sinusoidally).
What type of equation have we seen like this before?
Student: Sine and cosine

This discussion gave students some initial concept of the relationships among the various quantities in this example. However, the exchange represented a largely teacher-driven and teacher-produced explanation of those basic relationships. Much of the dialogue was generated from the teacher and even though the teacher elicited some thinking, students had to offer only limited responses to somewhat leading questions. Duncan became more directive in the last exchange with his nonverbal cues. As he was telling students that the population could cycle back and forth, he was motioning with his hands in a sinusoidal pattern. Given that the trigonometric equations for these populations were already on the board, the entire exchange placed low-level demands on the students.

Duncan revisited his efforts to solidify the meaning of these equations for students later in the class as he asked students to think specifically about the behavior of the population of the foxes and rabbits over time. Initially, questions were of high demand, and demand was maintained. However, in the last exchange, he took over the mathematical thinking and gave them the shape of the curve:
Duncan: You guys look. Here’s my table, 0 to 12 (pointing to the table), I’ve got 1000 rabbits. I’ve got 200 foxes. A couple of months later (pointing to the entry at 2 on the table), I’ve got, we’ll round that to 567 rabbits and 175. So somebody tell me what’s going on for the first three months in the rabbit population?

Student: It’s decreasing.

Duncan: It’s decreasing exactly how, Stella?

Student: At an increasing…decreasing rate.

Duncan: Which one?

Student: Decreasing.

Duncan: Decreasing at a decreasing rate. What’s happening to my fox population for the first three months?

Student: Decreasing

Duncan: It’s also decreasing. It’s actually decreasing at an increasing rate. It’s actually like this (teacher traces an accentuated, concave down curve). It is hard to see because it is just so flat. Now, look at the next three months. The rabbit population is starting to go up and the fox population is still going down. The rabbit population is starting to come up because there are less and less fox to eat them. And then things will come back and you will see that here (pointing to the graph), the foxes are starting to go up and there are a lot of rabbits so the foxes are happy.

Asking about the variability of the populations and the rate of that variation represent cognitively demanding questions. That demand was maintained during the first part of the discussion.

However, during the last segment of the dialogue, Duncan discontinued his effort to elicit the thinking from the students and instead shared with them additional teacher-generated mathematical thinking as he interpreted the graphs in terms of the context. Not only did Duncan describe the behavior of the population of the foxes in more detail than the students gave, he also made the connection between that verbal description (decreasing at an increasing rate) and the behavior of the graph (concave down). He went on to interpret the remaining portions of the graph that had been produced by the graphing calculator.

This attempt by Duncan to connect the mathematical models with some meaningful real-world behaviors did not seem to work for all of the students. After a lengthy class discussion about the graphs of these parametric equations, a student asked, “I don’t understand what this
is?" [Duncan, Ob #1, Line 652]. While it was not entirely clear if she was asking about the parametric graphs or the rectangular one, Duncan interpreted this question as “Why do we do this?” [Duncan, Ob #1, Line 653] and proceeded to give the following explanation to the student.

Because, Natalie, sometimes we want to look at one thing over time. For instance, you know I said earlier, you guys have changed a lot from when you were born. Are you 18 yet? Okay, so you probably have changed a lot from zero to 18. Your parents, probably like most parents, every year or every birthday, they put a little mark like on a doorway or whatever to see how tall you’ve gotten. So they are measuring your vertical change… We don’t always look at things together… But, that is what we are looking at. Sometimes we are looking at just one thing at a time or with parametrics, we can actually look at them together. We can look at them separately and then we can put them together. So here, what are the rabbits doing (motions sinusoidally). What are the foxes doing (motions sinusoidally). What do those things look like together?

[Duncan, Ob #1, Lines 654–668]

Regardless of whether this question was about the parametric graphs or the rectangular graph, this student’s question revealed confusion about the meaning of these graphs not necessarily about why we graph. Instead of focusing her attention on what the three graphs represented and getting the student to do the mathematical thinking about what these graphs might tell us about those relationships, Duncan provided another context and attempted to provide an explanation of the application of parametric equations using that additional, real-world context. It is also worth noting that Duncan did the mathematical work of interpreting the rectangular graph and developing a parallel example. As discussed in the section on the planning of instruction, his response emphasized this idea of “putting them together” rather than the relationship that the rectangular graph represents. The student’s question suggested that she wanted to understand the concept and its usefulness. Opportunities to develop the conceptual understandings of students presented themselves at times throughout the three lessons, yet Duncan typically focused on the procedural aspects of the problems.
Duncan’s response in the stimulated recall interview revealed more about his interpretation of her question in the moment.

Interviewer: Do you think her question was about the $R$ versus $T$ and the $F$ versus $T$ or about the $R$ versus $F$?
Duncan: I think it was both. I think she’s comfortable with $R$ versus $F$ cause that is really $x$ versus $y$ in her mind. And the girl that asked about the width, height and girth. She asks a lot of questions in every class and she is pretty concrete. She’s a senior, so she is on kind of a bit of a slower track. But she works really hard and her confidence this year has been really good. And she is not afraid to ask a lot of questions, yet, when she asked me the question, I got the impression that she did not completely understand why is the time piece necessary?

[Duncan Ob #1, Lines 698–709]

Duncan believed that the student’s question was about the $R$ versus $T$ and the $F$ versus $T$ graphs and he constructed his response to provide a rationale for looking at those graphs and the change of those quantities over time. However, immediately after his explanation to the student, the following exchange occurred with that same student:

Student: Is it always going to be an ellipse?
Duncan: No, just in this example, it happened to be an ellipse. We can get all kinds of different shapes.
Student: So could they just not be related?
Duncan: Yea, I could say the relationship between you getting taller and how dark the green grass is outside. There is no relationship between those two things.

[Duncan Ob #1, Lines 727–732]

These two questions suggested that the student’s confusion was over the $R$ versus $F$ graph—a suggestion Duncan agreed with in the interview:

Interviewer: I really thought her question was about the third graph.
Duncan: $R$ vs. $F$
Interviewer: Yes.
Duncan: It could have been. I should follow up and see.
Interviewer: Because it makes sense…the population of the rabbits versus time, that’s a combination that makes sense. The population of the foxes versus time, that makes sense. It’s looking at the population of rabbits versus population of foxes, which may or may not be related, but you are going to relate them on a graph. I don’t know.

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This student’s questions revealed a depth of thought that Duncan did not fully appreciate in the moment. She expressed questions about the relationship of the two quantities, but Duncan’s response focused on “looking at them separately” and “looking at them together” with no reference to or discussion of the relationships being modeled and what the graph might tell us about those how one quantity might change in relation to another.

These examples from the first observed lessons revealed the pattern evident in other portions of the three lessons. Often, student responses were not elicited as Duncan provided much of the mathematical explanation for them. At other times, the focus of the class was more mechanical or procedural which limited the opportunity for students to respond in ways that revealed their mathematical conceptions. When students offered responses that could have potentially revealed something about their conceptions, Duncan did not consistently pursue them and when he did, he often did so in a highly teacher-centric way.

**Summary of Differences in the Elicitation and Interpretation of Student Responses**

Among the teachers in this study, Jackie was most consistent in eliciting student thinking and interpreting their thinking in relation to the lesson-specific and discipline-specific learning goals. With the clarity and specificity she had articulated the learning goals and the progression of understandings she identified previously, she readily recognized inadequate thinking on the part of students relative to those goals and often asked for elaboration, clarification, and justification to ensure she fully understood their thinking. When a student gave an imprecise or
partially correct response, she recognized the potentially productive thinking and used it to build deeper understanding. When a student proposed an alternative approach, she readily acknowledged it and assessed its vitality relative to the learning goal and learning progression.

Like Jackie, Harold routinely and consistently elicited student thinking in ways that maintained the cognitive demand on them. He also readily recognized inadequate thinking on the part of the students relative to his learning goals. However, in the four observed lessons, Harold held to strict indicators of understanding and often focused on what was wrong with a student’s thinking than working with what was right. While students were expected to express their thinking, the disposition towards what was wrong often seemed to curtail any additional elicitation of elaboration, clarification, and justification on the part the student and led to a low tolerance for imprecise expressions of mathematical ideas.

In contrast to Jackie and to Harold, Susan’s elicitation and interpretation of student responses varied. At times, she asked students to clarify their thinking through elaboration or justification. Like Jackie, she demonstrated a focus on what was right about a student’s thinking and asked clarifying questions to clear up any imprecision in language or thought. At other times, Susan truncated her elicitation and interpretation of student responses. In some cases, this truncation led her to an insufficient interpretation of student thinking and continued confusion on the part of students. In other cases, she provided students with a more didactic explanation taking the mathematical thinking burden off of them.

Unlike Jackie, Harold, and Susan, Duncan’s interactions with students rarely afforded the students the opportunity to respond in ways that revealed something about their mathematical thinking. Instead, the questions students were asked primarily were ones that required them to
execute a skill or perform segments of a larger modeling task as directed by the teacher. When students asked a question or provided responses that might have revealed something about their mathematical conceptions, Duncan typically provided the answer and did the mathematical thinking, rather than responding in a way that would have helped the students develop new understandings. Duncan rarely required students to elaborate, clarify, or justify their thinking.

**Differences Among Teachers in the Responses to Student Responses**

The fourth critical instructional moment under consideration in this study involved the responses to student response on the part of the teacher. Naturally, these responses were often closely associated with the elicitation and interpretation of student responses. Thus, some overlap of excerpts discussed in the previous section will also be analyzed in this section. Predictably, when teachers had well-articulated learning goals, a well-sequence learning progression, and interpreted student thinking in light of those learning goals, their responses to student responses were often designed to use the expressed conceptions of students to develop understandings aligned with those learning goals. In cases in which the learning goals and the sequencing of tasks was less clear, responses to student responses tended to be less clearly aligned with learning goals and more teacher-centric. Differences along this continuum emerged among the four teachers in the study and are discussed in this section.

**Jackie’s Responses to Student Responses**

More consistently than any of the other participants in the study and throughout the three-lesson sequence, Jackie exhibited careful consideration of the responses of students, made intentional efforts to understand their conceptions, and designed instruction in response to those conceptions. As the previously described excerpts illustrated, Jackie sought to understand
student thinking, regardless of the precision of their language, pursued solution paths offered by students to their logical conclusion, recognized the need for clarification and justification, and used student responses to support the attainment of her mathematical goals on multiple levels in a variety of ways.

Three different types of situations highlighted Jackie’s efforts to understand student thinking and use it to design instruction. In some cases, those efforts involved embracing alternative approaches, exploring those approaches to their logical conclusion, and contrasting those approaches with her preferred methods. In other cases, her efforts to use student thinking to design instruction involved recognizing what was correct or productive about a student’s expressed thinking and designing interventions to help the student refine his or her conceptions or work. Still other cases involved Jackie’s responses to errant approaches or unproductive conceptions as she developed associated tasks to scaffold reconstructed conceptions to existing ones.

The first instance of the introduction of an alternative approach on the part of a student occurred in the first observed lesson. Jackie led the class through the work on the kite problem in which students were asked to use the given information to find the remaining lengths and angle measures as shown in Figure 4.34.
After some students offered some initial measurements of angles and sides, one student explained that she drew diagonal BD. In the stimulated recall interview, Jackie offered her thinking about the suggestion.

Interviewer: At this point, I think it was Marlee, that suggested that you draw BD.
Jackie: I didn’t want to draw BD yet, that’s why I put the new plastic on top but I didn’t want to discourage her either so I followed her train of thought and then I took that away so that we could come back to the more logical sequence which was to find this diagonal (teacher points to the vertex diagonal, AC) first and then use it to get the other angles.
Interviewer: Now, you pursued her line of reasoning to the bitter end.
Jackie: Pretty much.

In this excerpt, Jackie acknowledged that she did not want to approach the problem in this way. Her response also reflected her desire to develop a sense of mathematical empowerment in her students and explained, “I didn’t want to discourage her either.” Jackie embraced alternative approaches and in her background interview, even alluded to her desire for students to consider alternative approaches and make connections between them whenever possible. Jackie resolved
these potentially conflicting learning goals by pursuing the student’s thinking and coming back to her diagram in its current form. She did so by placing a second overhead sheet on top of problem so that the work related to drawing BD could be kept isolated allowing her to come back to the diagram in its existing form. In this way, Jackie was able to embrace the alternative approach as a means to support her broader goals for the course while still supporting her lesson-level goals.

A set of exchanges occurring in each of three observed lessons across three different problems further illustrated Jackie’s approach to alternative, but valid approaches and her design of instruction in relation to those approaches and her learning goals for students. In each case, a student suggested the use of the law of sines or law of cosines in a right triangle when the definitions of the trigonometric ratios would suffice. However, these alternative approaches worked against the learning goals for the three-lesson sequence. Jackie’s goal for students was to help them understanding the association between the coordinates of a point on the plane (and ultimately on the unit circle) and the angle of the vector drawn from the origin to that point. That goal required students to think initially in terms of the right triangle trigonometric ratios. In other words, her broader goals of getting students to consider and understand alternative approaches worked against her lesson-specific goals. In each case, Jackie constructed a response emphasizing mathematical efficiency as a way to support the two distinct learning goals for students—acknowledging the validity of the alternative approach and focusing student attention on the use of the trigonometric ratios. The following exchanges illustrated the point.

In the first lesson within the discussion of the kite problem, a student suggested they use the law of sines or the law of cosines. Jackie shared these thoughts with the researcher.
Jackie: Again, Rats! I think we are still on Marlee’s thing [a reference to Drawing BD] and we have a right triangle and I don’t think we should have to use the law of sines and if we did, we would be doing way too much work….We shouldn’t be using the law of sines. We should be using sine, cosine, or tangent.

[Jackie Ob #1, Lines 174–188]

Jackie recognized the solution could be achieved through using the laws of sines but did not want to use it. In her words, “we would be doing way too much work.” She also identified her preferred method, the use of the trigonometric ratios.

In response to this student’s suggestion, Jackie first established, through student input, that there was an alternative to using the law of sines.

Jackie: Let’s follow up on Marlee’s idea. Sue?
Sue: I would just use sine, cosine and tangent.
Jackie: Marlee, can I just do that?
Student: Yea.
Jackie: Why?
Student: Cause (inaudible)
Jackie: Kinda
Student: It's a right triangle.
Jackie: So would you have gotten the right answer?
Marlee: Yes
Jackie: Would you have done way too much work?
Marlee: Yes.

[Jackie Ob #1, Lines 218–231]

Jackie initiated this exchange by asking students to consider Marlee’s idea of using the law of sines. Another student offered the use of the right triangle trigonometry definitions, and Jackie forced the student to explain why this was a valid approach. In doing so, Jackie acknowledged the correctness of the student’s approach using the law of sines. She followed this affirmation with a question about efficiency and directed student attention to the idea that the use of the right triangle trigonometric definitions were less work. Using the idea of mathematical efficiency
served Jackie’s dual purposes—allowing the class to consider alternative approaches and supporting the learning goal for the three-lesson sequence.

In the second observed lesson, another discussion emerged involving the choice to use the definitions of the trigonometric ratios or either the law of sines or the law of cosines. The problem involved finding the length of a resultant vector and its angle of direction. One student suggested drawing a line to complete a parallelogram while another suggested dropping a perpendicular.

Jackie: So those are the two kind of obvious ones. Parallel—I could just count over how far?
Student: 10.5
Jackie: 10.5. Or straight down, in which case I have a right angle [Jackie shows both segments on the diagram as shown in Figure 4.35].

![Figure 4.35. Jackie's vector task from second observed lesson.](image)

The first point to note about Jackie’s initial response to the student suggestions of two different lines was that she restated the two suggestions without privileging one over the other. Dropping the perpendicular supported the learning progression Jackie had determined to support the goal...
of understanding the association between the coordinates of a point on a vector and the angle.

However, as previously discussed, she also embraced alternative approaches. In this case, she made the mathematical choices in both approaches more visible to the class pointing out that you could get the length of the parallel segment by “counting over.” She then directed student attention to the goal of the task and elicited the mathematical methods for finding the angle and length of the diagonal for each choice.

Jackie: Let’s think about it. What am I trying to find?
Student: The length of the diagonal.
Jackie: Yes. That’s one. What’s the other thing?
Student: The angle
Jackie: Yes. Which would help me get there: the segment that completes the parallelogram or the segment perpendicular to the x-axis? (pause) If I have a parallelogram, what method am I going to use to get the length x, let’s not call it x. Let’s call it p.
Student: Law of sines and cosines
Jackie: Law of sines and cosines. If I take Sue’s advice, and go perpendicular, what will I use.
Student: Pythagorean Theorem
Jackie: Pythagorean Theorem or?
Student: Well, just like the regular sines
Jackie: The regular sines, cosines and tangents. So which is easier? In order of difficulty of execution. And remember that means I am probably going to make more mistakes. Law of sines and cosines, Pythagorean Theorem, sine, cosine and tangent ratios.

[Jackie, Ob #2, Lines 392–406]

Again, she framed the choice in terms of mathematical efficiency, “In order of difficulty of execution.” This approach allowed her to embrace the alternative approach, affirm the mathematical thinking of the students, and direct student attention to the approach that would more readily support her lesson-level learning goals. In response to this last question, a student maintained the use of the law of cosines would be easier. Jackie constructed another response to explicitly reinforce her preferred method:
Student: The law of cosines is probably easiest
Jackie: Seriously? Here’s what we are going to do: We are going to do a race…

[Jackie Ob #2, Lines 407–408]

With the race, Jackie engaged students in working the problem using each of the two proposed methods. The students using the trigonometric ratios and the Pythagorean theorem finished well before the other students which allowed Jackie to reinforce the preference for this method.

Throughout this exchange, Jackie maintained the validity of alternative approaches—drawing a parallel line as opposed to dropping a perpendicular and using the law of cosines as opposed to the trigonometric ratios. She made the mathematical choices in both approaches more visible to the class, directed their attention to the potential for mistakes and the efficiency of methods. She anticipated that this juxtaposition would provide an obvious contrast to the two approaches in light of the ease of one over the other. When that contrasts did not lead students to conclude that the use of the trigonometric ratios as the preferred method, Jackie proposed and initiated a race in which students solved the problem in each way. The results of the race clearly suggested the simplicity and efficiency of using the trigonometric ratios. In doing so, Jackie supported the attainment of her course-specific and lesson-level goals.

In the third lesson, a student once again suggested using the law of sines in a right triangle. In this instance, Jackie elected to prove the equivalency of the two equations.

Jackie: So what I want you to see as we have before and that is to see that Alex’s equation [shown below] is basically the same thing. What is the sine of 90? Did anybody remember that from the other day? It’s one. So Alex, your equation boils down to this [Jackie writes the application of the law of sines on the board as shown in Figure 4.36]:
Figure 4.36. Using the law of sines in the right triangle

Jackie: Is that the same as Rita’s equation?  
Student: Yes it is.  
Jackie: How do you know?  
Student: Because when you cross multiply…well when you solve that you get \( 6 \sin 40 \) equals \( x \). And when you cross multiply that one you get \( a \) equals \( 6 \sin 40 \).  

The first equation was derived from the application of the law of sines to the right triangle. In this excerpt, this equation was compared to Rita’s equation (not pictured) that came from the application of the definition of the right triangle trigonometric ratio. Jackie reminded students of the value of the sine of 90 degrees and simply asked if the two equations were the same. The response served multiple purposes. First, it provided another example of the way Jackie dealt with alternative approaches. She acknowledged the validity of the use of the law of sines, but through focusing on its equivalence with the use of the trigonometric ratio, she also reinforced the additional steps involved in its use. The exchange also served to reinforce the importance of justification. When the student asserted the equivalency, Jackie insisted asked, “How do you know?” insisting that the student explain how she knew they were equivalent. Ensuring students know why approaches are valid or relationships are true was another dimension of Jackie’s course-specific goals for students. Again, Jackie developed a response to alternative methods that supported her lesson-level and course-specific goals for students.
These exchanges illustrated several important points. First, Jackie routinely embraced alternative approaches to solving problems, and this approach encouraged students to offer their own mathematical ideas. Jackie often used alternative approaches as opportunities to support some of her broader goals for the class—developing a connected understanding, providing opportunities for and expecting students to justify their work, and developing an appreciation of the history, structure, and logic of mathematics as a discipline. Her work with the students around the use of these various tools of triangle trigonometry contained elements of each of these. Secondly, Jackie kept the mathematical goals of the lesson in the forefront of her thinking while considering the existing understandings of students. For Jackie, the foundational understandings of trigonometric functions that she sought to develop hinged in many ways on students staying focused on the use of the trigonometric ratios in the solution to this problem. In fact, the next step in the learning progression was to push students to think about how to get the angle and length of the vector from only the coordinates of the points—without thinking of the right triangle or any other triangle for that matter. This sequence of exchanges illustrated Jackie’s efforts to remain focused on her learning goals for students—discipline-specific, course-specific, and lesson-specific goals—while considering the expressed understandings and ways of reasoning of students, and seeking to determine a path from those existing understandings in the direction of her learning goals. Her interpretation of student responses and her responses to student responses were determined by this thinking.

Another interesting example of Jackie’s handling of an alternative but valid approach involved a variable assignment during the third observed lesson. Jackie presented a vector problem on the coordinate plane asking students to find the coordinates of the endpoint of the
resultant vector. One intermediate step in the problem was to find the ordered pair for the endpoint of a vector of length 6 at an angle of 70 degrees. Curiously, several students in the class assigned $x$ to the vertical side of the triangle and $y$ to the horizontal side. During her observation of independent student work, she realized one student’s choice.

Jackie: Let me see. (teacher looks at student work and at overhead) Good. Do you mean $x$? Which one is $x$? (student shows her) Okay, that’s kind of…Okay, that’s fine. Normally, we would use $x$ on the $x$-axis (motions horizontally), but not always. As long as you have it labeled as you have it labeled, you’re fine.

[Jackie Ob #3, Lines 74–77]

A few minutes later during the class discussion, Jackie remarked,

Jackie: So usually, we label things on the horizontal as $x$ and on the vertical, $y$. Ya’ll reversed it and it is kind of curious that so many of you did. It is not a problem. It is just kind of interesting.

[Jackie Ob #3, Lines 102–104]

In this case, the teacher recognized the mathematical validity of this variable assignment, but also anticipated the potential confusion resulting from its use. She did not force the correction here. Rather, she made a point of drawing student attention to the arbitrary nature of the variable assignment and the conventional way in which it is done using key words like “normally” and “usually” to suggest the conventional way. In doing so, she potentially reduced the need to translate and interpret the results at the end of the problem when the students were expressing the coordinates of the vector. She also highlighted the mathematical validity of what the students did in assigning the variables in a unconventional way. Given that the learning goal for the lesson sequence involved student associating the coordinates of a point with the angle, the conventional use of the $x$ and $y$ was important and could have foreshadowed the work later in the lesson sequence. However, by making the distinction between the student’s choice and the
conventional way, Jackie acknowledged the validity of the approach without undermining her lesson-level goals for students.

In each of these episodes, Jackie’s work with students when they offered alternative, but valid approaches revealed responses that acknowledged the validity of the approach while directing student attention to the approaches that supported the development of the learning goals for the lesson. Her emphasis on efficiency, simplicity, and convention served this dual purpose. In each of these instances of alternative, but valid approaches, Jackie demonstrated her efforts to understand the thinking of students relative to her discipline-specific and lesson-specific learning goals. She used those understandings and approaches to construct responses to students that attempted to build understandings aligned with those goals.

Other instances of Jackie’s work to understand student thinking and use it to design instruction involved her work with students who expressed a mathematically inadequate understanding. In these cases, Jackie had to recognize what was correct about a student’s response and design associated tasks to support the refinement of the student’s conceptions. Three episodes illustrated her approach to responses to these types of student responses.

In the first observed lesson, a student suggested the use of the tangent ratio as an alternative to the law of sines.

Kiki: If you just do CA over 4 equals tangent 63.
Jackie: What do you think of that idea folks?
Student: You have to put the x over the 4 because that is what would be the opposite side for tangent.
Jackie: What do you think Kiki?
Kiki: Yea.
Jackie: What is tangent?
Kiki: I used…
Jackie: What is CA in the triangle? I am going to hide the other triangle.
Kiki: CA is the hypotenuse. Oh, so it is supposed to be cosine.
Jackie: Do we put the hypotenuse in the numerator, ever?
Student: No
Jackie: Not yet, but remember there are those three other mystery trig functions we are not talking about.
Kiki: I can fix that.
Jackie: Please do.
Kiki: It is cosine of 63 equals 4 over CA.

[Jackie Ob #1, Lines 373–388]

There were two errors in Kiki’s suggestion to do “CA over 4 equals tangent 63.” She used the tangent ratio and placed the hypotenuse of the triangle in the numerator. In spite of these errors, Jackie recognized that the student was thinking in terms of the trigonometric ratios and that dimension of her thinking was productive. In response, Jackie asked two critical questions: “What is tangent?” and “What is CA in the triangle?” These two questions provided the necessary scaffolding by directing Kiki’s attention to her salient, existing understandings. Interestingly, the student applied those to her work and applied the correct trigonometric definition without the teacher directing her to change the trigonometric ratio definition she was using. It was also noteworthy that Jackie made reference to three other trigonometric ratios that the students had not yet studied and qualified her instruction in light of those ratios. The exchange illustrated Jackie’s efforts to design associated tasks that would support student efforts to draw on existing understandings to solve problems and build new understandings.

At other times, dealing with partially correct approaches involved understanding what is right with a student’s thinking in spite of the presence of errors or imprecision in language. In one episode, Jackie asked the students to find the longer diagonal of a parallelogram given two sides and the small angle as shown in Figure 4.37.
A student offered an insight to determining the measure of the other angles of the parallelogram.

Student: Aren’t these two angles and these two angles congruent in a parallelogram so that angle is 35 and you can use the quadrilateral sum theorem and it has to equal 360 so 35 plus 35 equals 70 and 360 minus 70 is 90 so each of the other sides is 45. Does that make sense?

Jackie: Your reasoning is correct, your arithmetic is a little off. Now Marlee just flew, and I mean do mean flew, through a theorem about parallelograms that we proved. What about this angle (teacher points to the 35 degree angle) and that one (teacher points to the consecutive angle of the parallelogram) did she just run through the proof of?

Student: They are supplementary so it would be 145.

Jackie: Ah, you just reproved that in a marvelous way. All right, so now we have that.

The student described her thinking as she determined the measure of one angle of a parallelogram given another angle. The determination could have easily been done by applying one of the properties of a parallelogram—consecutive angles are supplementary. However, the student, instead, offered a reconstruction of the proof of that property albeit with some errors in arithmetic. In spite of these errors and the unnecessary nature of her extensive reasoning, Jackie recognized the structural validity of the student’s response in terms of it serving as a proof of the property that consecutive angles of a parallelogram are supplementary. Once again, she acknowledged the validity of the student’s thinking and used it as an opportunity to reinforce the
students’ consideration of justification, of this particular theorem, and the application of that theorem in this case.

In another example of Jackie’s responses to a student’s thinking that was imprecisely conveyed, Jackie used the work of one student to correct an error in the thinking observed in other students. This excerpt from the work of the class on the kite problem in the first observed lessons captures several aspects of Jackie’s approach. Jackie elicited ideas from students about how they found various angles and sides in the kite.

Student: The first thing I did is that I bisected the kite.
Jackie: (interrupting) In what way?
Student: So angle CBA and CDA were in tact.
Jackie: Ah Ha. So you drew a diagonal. (teacher draws the diagonal joining the ends of the kite)
Student: And then I drew…(teacher interrupts)
Jackie: What do you know about that diagonal?
Student: Well, I know that now there are two right triangles. And they are right triangles that I know all of the angle measures too.
Jackie: How do you know them all?
Student: Because it’s a kite, they are dividing angle BCD and angle DAB. So, if you divide BCD by two that would be 63 on each side and on the bottom it would be 27 on each side. So you know all of the angles and one of the sides, so at that point it would be pretty easy to figure out all of the rest of them.
Jackie: Good. When I was walking around yesterday, some of you were drawing diagonal BD and assuming that was a bisector of the angles B and D. Is it?
Student: (in unison) no

[Jackie Ob #1, Lines 78–93]

When the student initially stated that she “bisected the kite,” Jackie interrupted her explanation to clarify her non-mathematical use of the word, “bisect.” She then forced the student to consider that diagonal and asked the student, “What do you know about that diagonal?” In response, the student explained that she knew that the diagonal divided the triangle into two right triangles and that she knew the angles of those triangles. At this point, Jackie pressed the student for justification, “How do you know them all?” and the student provided a complete response
that Jackie affirmed. Her last statement revealed the rationale behind the pursuit of this exchange as she used it to correct an error she observed in student work the previous day.

Several elements of these responses to the student’s responses are worth noting. The student initially used the term “bisect” and Jackie asked for clarification about her meaning. In light of the error that she observed the previous day regarding the assumption that diagonal BD bisected the angles, Jackie might have thought that this student was going to make the same error. Regardless, Jackie’s response, “In what way?” required the student to clarify her thinking. When the student asserted that she knew all of the angles, Jackie again asked her to clarify her thinking. In each case, the thinking of the student was valid and productive. Even though she did not express the same error in her thinking that Jackie had observed the day before, her questioning made the valid thinking visible, reinforced the value and the expectation that responses should be justified, and provided Jackie with an opportunity to use the thinking to address the error in thinking of other students.

In each of the instances in which a student expressed partially valid or inadequately expressed mathematical thinking, Jackie developed responses designed to more fully reveal the thinking of the student. In addition, she asked questions that directed the students’ attention to their existing understandings in an effort to scaffold new understandings to them. In the last example, student attention was directed to the valid thinking of another student to redirect the thinking of students away from an observed misapplication of an understanding. In these ways, Jackie used the expressed thinking of students to design instruction.

The third and final type of student response that illustrates the nature of Jackie’s responses to student responses involved situations in which student thinking was errant or
unproductive. Three examples from the classes observed illustrate three different approaches to these situations.

In the first episode, a student presented a solution to a problem in which a triangle was formed by points on two parallel lines. Students were asked to consider what happens to the angles of the triangle as a point, P, moved along one of the two parallel lines. The student asserted that the angle at point P would remain the same.

Jackie: So P moves along. Can you draw us another setting with a different color there so we can see what you are talking about? (student draws another example. She makes a mistake neglecting to keep AB the same and then corrects it) So, does that verify what Kiki is saying. Does the angle at B get larger?

Student: yes
Jackie: Did the angle at A get smaller?
Student: yes
Jackie: Did the angle at P stay the same?
Student: yes
(Kiki sits down and Jackie returns to the overhead)
Jackie: It would be interesting to do this on Sketchpad. I want to draw another iteration of this in yet another color. (she adds in the narrow obtuse triangle as shown in Figure 4.38)

![Figure 4.38. Parallel line problem with first of Jackie’s triangles.](image)

Jackie: Did P still stay the same?
Student: It kind of looks like it got smaller.
Student: I think it stayed the same.
Initially, Jackie attempted to get the student to produce her own counterexample. She asked the student directly, “Can you draw us another setting?” When that failed to generate a compelling counterexample and a change in the thinking of the student, Jackie constructed two more counterexamples without directing student attention to any feature. In doing so, the student recognized the power of the counterexample and drew an appropriate conclusion, “It gets smaller.” Jackie’s use of the counterexample directed the student’s attention to the inadequacy of her thinking without any direct instruction from Jackie.

In another situation in which a student expressed an errant or unproductive conception, a student asked a question about using the trigonometric ratio for cosine with the 90-degree angle.
The class had been working through finding the legs of a right triangle given the hypotenuse and one of the acute angles.

Student: I have a question on the cosine thing. When you are talking about the adjacent side over the hypotenuse, what if you were trying to find like the ninety degree angle or the side opposite that?

[Jackie Ob #2 Lines 115–117]

Initially, the teacher did not understand what the student was thinking. She paraphrased the question as, “What if we needed the hypotenuse?” [Jackie Ob #2, Line 118] and created a new task in which the hypotenuse of a right triangle was the missing length. The student reiterated her confusion.

Student: I was thinking what if you had something like cosine 90 equals and you would have to have the adjacent over the hypotenuse. How would you do that because [the hypotenuse] would be the hypotenuse and the opposite.

[Jackie Ob #2 Lines 176–178]

The student appeared to be trying to write an equation for the cosine of 90 using the definitions of the trigonometric ratios. After another attempt by Jackie to understand the student’s question, another student offered her interpretation.

Student: She’s saying since sine 90 is one, it would be sine 90 equals c over c because the c would be opposite and the hypotenuse.

Jackie: Ohho. Those ratios only work when you are standing at one of the not right angles. The law of sines and cosines, you could be standing anywhere. Generally, we don’t even use those when we have a right angle. But if I wanted to use the opposite over the hypotenuse, I am talking about angle A or angle B, I am not talking about angle C. I just totally misunderstood your question. That was your question too, right? I can put myself anywhere on the triangle I want, and I am going to stand on a vertex that is not the vertex of a right angle.

[Jackie Ob #2, Lines 192–200]

Even though the student’s interpretation of the student’s thinking involved the sine of 90 and writing that using the definitions of the trigonometric ratios, the student rightly interpreted the
other student’s question being about using the trigonometric ratios with the 90-degree angle. In spite of needing the additional interpretation of the student’s thinking, Jackie repeatedly pursued an understanding of the student’s question throughout the exchange. In the stimulated recall interview, she described the student’s question as bizarre and acknowledged the difficulty she had in making sense of it. Nevertheless, Jackie demonstrated her desire to help the student reconcile her understanding. She made three different attempts to understand the student’s thinking and to design instruction accordingly. And finally, once Jackie understood the student’s thinking, she directed her attention to the conditions of the definition of the trigonometric ratios thus resolving the confusion.

A third exchange illustrated another approach Jackie employed to address an errant or unproductive conception. Early in the second observed lesson, Jackie presented a problem in which students were asked to find the endpoints of a vector on the coordinate plane. The students had initiated work on this problem the night before, and a student responded to Jackie’s re-introduction of the problem with the following statement:

Student: I didn’t know (inaudible) about the right triangle so I just put a different angle there.
Jackie: Didn’t I? If you are on the coordinate plane, and you are moving to the right and then up, does there have to be a right angle?
Student: no
Jackie: Really? If you are plotting an ordered pair (teacher sketches a set of coordinate axes) and I would like to plot the ordered pair, (2,3), how do I do it?
Student: Go over two and up three.
Jackie: Plotting the points, over two and up three (going on a diagonal line).
Student: Straight up
Jackie: Straight up, so what kind of angle?
Student: Right angle.
Jackie: So there is a right angle here. So even though it wasn’t stated, because you are on the coordinate plane

[Jackie Ob #2, Line 14–77]
Jackie believed the student was missing the rectangular nature of the coordinate plane, but she determined the need for intervention by asking this question, “Didn’t I?...Doesn’t there have to be a right angle?” The student responded, “No” and Jackie immediately produced a task designed to force him to specify his existing understandings in a way that allowed him to answer his own question and resolve the confusion. As the student engaged with the task, Jackie used the student’s own language—“over two and up three” and “straight up”—to direct the student’s attention to the resolution of his problem. In doing so, Jackie directed the student’s attention to the salient feature of his existing understanding of the coordinate plane—an understanding he was not applying to the task.

In each of these instances in which a student response revealed an errant or unproductive approach, Jackie provided associated tasks in the form of questions or modified tasks to help the student scaffold the new understandings to existing ones. She used a counterexample, a modified task, and one student’s own language to direct student attention to the salient features of their existing understandings. The tasks, as implemented, kept the student as the mathematical decision maker. When her initial effort did not generate a reconstructed concept or valid approach, she used those responses to modify the task further until the student was able to provide evidence of a proper conception.

Jackie’s approach to student responses throughout the three-lesson sequence demonstrated her efforts to use those expressed understandings to strengthen existing understandings, to build new understandings, and to reconstruct unproductive understandings. She maintained an intentional effort to elicit student thinking and used those expressed understandings to advance the class discussion and to meet her lesson-specific and broader
course goals. She consistently interpreted student responses in terms of these goals and demonstrated her ability to craft responses that supported the attainment of those goals. Jackie’s approach will be contrasted with the approaches of the other participants in the remaining portions of this section.

**Harold’s Responses to Student Responses**

As discussed previously, Harold’s interpretation of student responses reflected a narrow set of indicators of understanding. This interpretation shaped his responses to student responses in both the Math 6 lessons and the Algebra I lessons, albeit in different ways. In Math 6 lessons, Harold moved readily among numerical, algebraic, and visual models of the relationship as he rephrased, reformulated, and re-stated the tasks in response to student responses. Throughout, the focus remained on students demonstrating a specific set of understandings through their work on these tasks. In the Algebra I lessons, Harold’s efforts to construct a response to students’ responses tended to focus on pointing out the inadequacy of their conceptions or the expression of those conceptions. Often, his responses involved a series of questions designed to re-emphasize definitions and basic principles and to lead students to express those conceptions in a specific way. These questions emphasized distinctions to be made between ways of thinking that worked in a limited context and those that would apply in the broadest of contexts. Throughout his work with students, Harold placed a heavy emphasis on the precision of the language the students used to express those understandings rather than the development of an intervening task that engaged the existing, productive understandings of students while supporting the development of the desired understandings. When students expressed confusion, Harold’s responses typically involved the re-expression of these broadly applicable definitions and
conceptions. Examples of how Harold’s narrow focus shaped his responses to students in both the Math 6 and Algebra I lessons are discussed in this section.

During the first Math 6 lesson and as previously discussed, the students struggled to demonstrate (from Harold’s perspective) an understanding of the initial lead task that involved representing \(a ÷ b\) as \(a \cdot \frac{1}{b}\). The two primary features of the pattern in Harold’s responses to students’ responses can be most clearly emphasized by looking at the sequence of associated tasks he used during the lesson in response to student responses. Each task was constructed in response to the difficulty the students demonstrated in expressing the specific, component understanding Harold hoped to help students develop, and the set of tasks demonstrated his emphasis on, the use of, and the facility with multiple representations. The tasks Harold offered were, in sequence, the following:

- “What’s another way to write a divided by b?” [Haro #1, Line 55]
- “Tell me something that is equal to both of these that is different.” [Haro #1, Line 96]
- “What is another way to do six divided by 2?” [Haro #1, Lines 161]
- “So how would you express this idea with variables?” [Haro #1, Line 220]
- “What’s a fourth of twelve?” [Haro #1, Line 261].
- “What would twelve groups of a fourth look like?” [Haro #1, Line 313]
- “[Drawing a rectangle] Here’s a whole…[Dividing it into four parts and shading one part] Here’s one-fourth of a whole, right?...Let’s say we have twelve of these. What’s that equal?” [Haro #1, Lines 333–339]
- “On your own sheet of paper, model twelve times a fourth. Model it and figure out what it equals.” [Haro #1, Lines 459–460]
- “[Holding up a quarter coin] This is a fourth of a dollar. How can I use these to model this [pointing to twelve times one-fourth]? [Haro #1, Lines 545–546]
- “I want you to multiply three times two using dollars. Model that now.” [Haro #1, Line 603]

In this sequence of associated tasks, the initial task involved a purely algebraic representation of the mathematical relationship devoid of any real-world context. As students’ demonstrated
difficulty or weakness in their expression of their response, Harold rephrased, reformulated, and restated the tasks presented to students. Initially, he rephrased the task to direct students to “tell me something that is equal to these but different,” then he moved from the algebraic model to a numerical context asking students to express the same relationship with a specific, numerical example, six divided by two. After students successfully completed that basic whole number model, he moved back to the algebraic one, “So how would you express this idea with variables?” With unconvincing success on the part of the students, Harold shifted from the simple, whole number representation to one involving fractions asking students, “What’s a fourth of twelve?” Next, he offered a visual model for students with the drawing representing one-fourth of a rectangle and offered a model using money when students failed to express the specific understanding he was looking for. Identifying the weakness in their thinking as related to the repeated addition model of multiplication, he returned to a simple numerical example to focus their thinking on the meaning of multiplication when he asked the students to model three times two using dollars.

Each reformulation of the task represented a shift in the structure or complexity of the task in an effort to scaffold the task to existing understandings. The shift from the algebraic to the numerical model removed the abstractness of the variables and allowed the students to express the desired understanding correctly. When his attempt to get them to translate that understanding into the algebraic understanding failed, he introduced another numerical example with a greater degree of complexity by involving a fraction. As students struggled to express his desired understandings, he provided students with a visual model and then a model with money. With the students’ continued difficulty in modeling the relationship, Harold identified one
element of the relationship, the repeated addition model of multiplication, and developed a numerically simple, monetary example to focus their attention on that dimension of the relationship.

These shifts in structure and complexity were each made in response to the students’ difficulty in expressing the specific relationship Harold had identified as a component understanding for the division of fractions. This sequence of associated tasks and the noted differences in structure and complexity also demonstrated Harold’s ability to construct multiple versions of the original task using numerical, algebraic, visual, and real-world models of the relationship. It also illustrated his emphasis in getting students to work across multiple representations and to connect their understandings of the mathematical ideas across those representations.

Just as Harold’s responses in the Math 6 lessons were shaped by his focus on some specific ways he expected students to represent their understanding, Harold’s responses in the Algebra I lessons were shaped by the specific ways he expected students to express their understandings of the concepts and processes involved in solving a system of inequalities. In particular, Harold’s attention was consistently directed towards conceptions and approaches that would apply in the broadest of contexts, and he expected the ways the students expressed their thinking to be strictly aligned with those conceptions and approaches. Consequently, his responses to students often involved pointing out the inadequacy of their thinking or the expression of their thinking as it related to these broadly applicable ways of conceiving of the mathematical ideas. Often his responses amounted to a series of questions and the use of
counterexamples to direct student attention to the specific mathematical meaning of a term, alternatives for that term, and to the use of particular terms.

Examples of this focus on broadly applicable conceptions and language that reflected this understanding were found throughout the two Algebra I lessons. Each of the two observed lessons in Algebra I began with a series of questions to focus attention on the careful phrasing of what it means to solve a system and all that entails. Two excerpts from the first observed lesson illustrated several dimensions of the pattern of responses to student responses observed in the data.

Harold: What does it mean to solve a system?
Student: To show all of the solutions to the system.
Harold: Let me ask something real quick. What do you guys think of the word, “Show?”
Student: To represent.
Student: To define.
Harold: What do you guys think of the word, “Show?” I like the word, “Show” especially for this, because what is the only way to find all of the solutions?
Student: To show them
Harold: You would have to represent them and what is the only way to represent them in this sort of system?
Harold: You have to show them graphically. I like the word show. [Harold Ob #2, Lines 6–52]

Through the series of questions from asking the students what it meant to solve a system to asking them what was the only way to find all the solutions, Harold directed student attention to this broadly applicable understanding of what it means to solve a solution. He placed special emphasis on the use of the word “show” as an accurate description of the purpose of using a graph.

Harold continued the questioning and the focus on constructing a broadly applicable definition by asking, “What makes something a solution to a system?” [Harold, Ob #2, Lines 55–56]. A student responded.
Student: To make both equations true
Harold: Now you can memorize that it makes both equations true. Are you going to sound silly?
Student: Yes.
Harold: Try again. We don’t have two and they aren’t equations so we know your definition can’t make sense, right?
Student: To show the solution we would find all solutions that make all inequalities true.
Harold: Could you have an equation thrown in here?
Student: Yes, you could
Harold: Do you guys know the word I use to describe equations and inequalities?
Student: Statements
Harold: Statements. I use it very deliberately. By saying statements, you can cover whether it is an equation or an inequality.

[Harold Ob #2, Lines 126–153]

The student’s response to Harold’s question about the solution to a system suggested he was thinking only about the two-equation systems they had previously studied. The subsequent exchange reflects two aspects of the pattern of responses to student responses.

First, Harold responded to the student by reiterating what is wrong with the student’s response rather than working with what was right. He implied the student was sounding silly, and he pointed out that “we don’t have two and they aren’t equations.” Alternatively, Harold could have asked the student, “For what kind of system would your definition of a solution work?” In this way, Harold could reinforce what is accurate about the student’s conception while also directing his attention to the limitations of it. However, his rigid expectation of a correct response focused his attention on the inadequacies of the response.

Second, this series of questions continued to direct to students a specific articulation of the concept of what it means to solve a system in ways that make mathematical sense regardless of the statements within the system or the method of solution. For this student, his experience with a system had been limited to two-equation systems and his response reflected a limited conception of the system and how to find the solution. When pressed by Harold, he rephrased
his response to reference making the inequalities true. In spite of this response being appropriate for the system under consideration, Harold presented students with the concept of a system with equations and inequalities and introduced the word “statement.” In this entire series of exchanges to begin the class, Harold emphasized the concept of solution as a value or set of values of the variable that make the statement true and asked students to articulate how that definition applies to a system of equations or inequalities.

In an effort to further reinforce the power of this broadly applicable articulation of what it means to solve a system, Harold provided an example of a system with equations and inequalities by adding two equations to the system of inequalities under consideration of the class.

Harold: Here, let me show you something. I could give you this system [Harold points to the system of inequalities shown in Figure 4.40]

![Figure 4.40. The system of inequalities lead task from Harold’s Algebra I.](image)

Harold: And add this to it [Harold adds the two equations shown in Figure 4.41]
Harold: What is the only possible solution to it?
Student: (3,4)
Harold: That’s it. If you are supposed to show all the solutions that make every statement in the system true, they have to make this true. [pointing to \( x=3 \) and \( y=4 \)] that means \( x \) has to be 3 and \( y \) has to be 4. Does that work here? [pointing to \( 2x+3y \leq 12 \)]
Student: No
Harold: No solution. You’re done.

[Harold Ob #2, Lines 179–206]

Harold’s implementation of this associated task is largely teacher-centric with him asking leading questions and drawing the mathematical conclusions. For example, after writing the system on the board, he asks, “What is the only possible solution to it?” In doing so, he asserted the mathematical conclusion that there was only one possible solution and left the students to simply figure out what it was. Furthermore, he continued, explaining why (3,4) is the only possible solution and why it is not a solution without eliciting any of that mathematical reasoning from the students. Nevertheless, the example serves to reinforce the importance of the use of the word “statements” and the power of the definition of the solution to a system he has emphasized.
As these exchanges illustrate, Harold presses students for a precise use of mathematical language—breaking down each mathematically significant word—in ways that make the approach to (and conception of) solving a system of inequalities an approach (and conception of) that will work for any system of equations and inequalities. He questions any expression of understanding that lacks that broader applicability. Every word matters to him. The precise meaning of every word matters to him. The associated tasks (in the form of questions and two examples) were designed to bring greater precision to the language of students and to reinforce these more broadly applicable conceptions.

Harold’s focus on the precision of language shaped his responses to students in other ways as well. In multiple instances in which a student expressed difficulty or confusion, Harold responded by redirecting the students back to the basic principles and this careful articulation of the definition of a solution to a system. Several exchanges occurred as students worked to determine how to represent the solutions to a single inequality in the system. In one such instance in which Harold’s focus on strict indicators of understanding was discussed previously, a student had indicated some confusion about how to shade to represent the solutions to the inequality. Harold initiated some questions to ensure that she understood the graph of the solutions to one of the equations in the system.

Harold: Now that you have the graph of… Is that the first one? Now that you have the graph of this…So this is the graph of y=(x+6)/2 [Harold refers to the graph shown in Figure 4.42]
Figure 4.42. Student graph of $y = (x+6)/2$.  

Harold: How do you get the graph of $y$ is less than or equal? So, on this graph [pointing to the line] are a bunch of points.
Student: Yea
Harold: Every point on this line, what do you know about it?
Student: It is a solution to that equation.
Harold: To the EQUATION [Harold emphasized the word]. So for every point on here, the $y$ coordinate will equal the $x$-coordinate plus six divided by two. So how do you get the solution to that [pointing to the inequality].

[Harold Ob #2, Lines 277–287]

The series of questions illustrated once again Harold’s insistence on a specific expression of understanding on the part of the student. In the subsequent exchange, the student responded correctly to the question about shading by indicating that she would shade on the lower side of the graph.

Student: You shade the thing. This side of the graph [Student points to the region below the line graphed in Figure 4.43].
Harold: Why?
Student: Because all of these little dots can make the equation true. On this side, it can’t [pointing to the opposite side of the line].
Harold: Not equation. Watch your language.
Student: Inequality.

Again, the student responded with a correct response, albeit with a careless use of the word *equation*. However, as Harold expressed in the stimulated recall interview, he did not believe that she understood how to shade to represent the solutions to the inequality in the way that he wanted them to be thinking about it. For Harold, he wanted students to strictly adhere to a point-by-point consideration of solutions to the equation and a point-by-point determination of which values of $y$ will satisfy the inequality for each value of $x$ for points on the boundary line. In response, Harold focused her attention on the definition of solution.

Harold: Remember, do we care much about the answer?  
Student: No  
Harold: So what we are trying to learn here: How do we know what to shade, when to shade…You’ve got to recognize that it is all about looking for solutions. Looking for sets of values that make that statement [pointing to the inequality] true.  
Student: So what do I do next?  
Harold: You have got to show all of the solutions…you have got to solve that [pointing to the inequality]. And notice, that is just one inequality.
Judging from the responses of the student during this exchange and the work she produced, the student understood these definitions and conceptions of solutions of equations and inequalities. Harold’s intervention offered her limited direction on how to justify or conceptualize the shading to represent the solutions to the inequalities in the way that he conceived of them.

Like the previous excerpts, this episode represents Harold’s focus on the precise expression of the mathematical ideas associated with the solution of a system. It also illustrates how this focus shaped his direct interventions when students had questions or expressed difficulty. In these cases, he often directed their attention to these generalized conceptions using the precise language as the tool they can use to answer their own question without providing the student with a task that would have directed her to apply these conceptions to the representation of the solutions to the inequality.

In some instances, Harold’s press for precision of language and his direction of student attention to the definitions revealed weak conceptions. At the beginning of the second observed Algebra I lesson, Harold reviewed what it means to solve a system, reminded students of the system with which they were working (the same one from the previous day), and asked students to articulate what the problem was asking of them.

Harold: What are we trying to do on this problem?  
Student: We are trying to make all of these statements true?  
Harold: Are all of these statements true?  
Student: [several at once] yes  
Harold: Ooh  
Student: They can’t be  
Harold: Ouch. If all of them were true, would we have to do much to them to make them true?  

[Harold Ob #3, Lines 95–102]
In response to the student’s correct response to his initial question about the problem, Harold asked a question that challenged the students to think beyond the mantra of the definition they have heard repeatedly. The question, “Are all of these statements true?” reveals some weakness in the conceptions of students. Harold worked to strengthen those conceptions by once again revisiting what it means to solve the system.

Harold: What’s that mean?
Student: [inaudible]
Harold: Make what statement true?
Student: [several start to answer, one continues] the same set of values that make the system—all the inequalities true.
Harold: Okay, you can call these statements or inequalities but guys you keep confusing making the system true, which technically doesn’t make that much sense, because the system is this whole set of inequalities, could be a set of equations, could be a set of a combination of the two. Solving the system, means that you got to make every statement true. The reason why I say these statements can’t always be true, can \(x\) and \(y\) be zero?
Student: No
Student: Yes
Harold: It could be, but if they were, is the system true?
Student: No
Harold: We know right away that zero-zero is not a solution to this system [Pause] [Harold Ob #3, Lines 106–121]

During this exchange, a student used the phrase, “make the system true.” He corrected himself to say, “all the inequalities true,” but Harold focused his explanation on the idea of making the system true. In an attempt to illustrate the concept that the statements in the system are not necessarily always true, Harold asked students to consider the ordered pair \((0, 0)\). He first asked can \(x\) and \(y\) be 0. He got varied responses but he continues by providing the correct response, “It could be,” and stating that \((0, 0)\) is not a solution to the system.

Again, Harold insisted on precision of language. He asked students to consider a question to ensure the students had not simply memorized the right words. In this exchange, this
press for precision of language and explanation revealed a weak conception regarding whether the statements in the system are true. In response, he generated a counterexample that offered an ordered pair that was not a solution to the system.

This exchange and the others discussed in this section revealed a consistent pattern of response to student responses on the part of Harold. In both the Math 6 lessons and the Algebra I lessons, his focus on strict indicators of understanding and precise use of language shaped the nature of his responses to student responses. Because he often equated understanding with a particular expression of the mathematical concept on the part of the student, Harold developed associated tasks designed to direct the students to those ways of conceiving and expression their conceptions. In the Math 6 lessons, the associated tasks most often involved a change of representation as Harold shifted between numerical, algebraic, visual, and real-world models. In the Algebra I lessons, the associated tasks most often took the form of questions directing students back to the careful articulation of the basic definitions and principles.

Susan’s Responses to Student Responses

Across the three observed lessons, Susan’s responses to student responses fell into three categories:

- situations in which Susan provided the mathematical elaboration, clarification, or justification related to a student response,
- situations in which Susan directed attention to the salient features of the mathematics of the task or the student’s thinking, and
- situations in which Susan used a student response to launch some type of full class consideration or inquiry.
Almost all of the critical instructional moments identified as responses to student responses could be categorized in this way. Examples of each are discussed in this section.

Notably, 24 of the 45 critical instructional moments identified as responses to student responses involved teacher-provided mathematical explanation. In these instances, Susan asked and answered her own questions, asked leading questions or suggested solution paths, verbalized or directed students to connections with previously studied content, provided counterexamples and the accompanying rationale to students, and drew mathematical conclusions. In these instances, the elicitation of the mathematical thinking of students was infrequent and the teacher-provided explanations appeared to demand limited mathematical thinking on the part of students.

One of the examples of Susan’s responses to students appeared in the first observed lesson. In this portion of the lesson, the class began to explore the graph of the function, \( f(x) = \frac{x+3}{(x+3)(x-1)} \). On three distinct occasions during this discussion, Susan asked and answered her own question. Susan initiated one portion of the investigation by having students graph the function on their graphing calculator and asking, “Do you see two vertical asymptotes?” [Susan Ob #1, Line 479]. A student indicated that he saw two asymptotes.

Susan: Oh, okay. [T projects the graph] There’s the standard window. Two vertical asymptotes? Where do I think it might be? Based on the equation, I would expect it to perhaps be where there is a restriction? [Susan Ob #1, Lines 483–485]

One student had made an error while inputting the function into his graphing calculator, which gave him a graph with two possible asymptotes. In her effort to guide the student through an analysis of the graph, Susan asked him where the asymptotes might be located and then provided the answer to that question by directing attention to the restrictions. Once this student’s confusion was resolved, the class discussion continued.
Susan: How many vertical asymptotes does it appear I have?
Student: One
Susan: Where is it?
Student: [varied, indistinguishable responses]
Susan: [Talking over some of the responses] Well, once again, I think I will try TRACE. There appears to be one, where—at one? [T enters x=1 on the calculator]. All right, when I ask the calculator to tell me what y goes with an x of 1, what happens?

[Susan Ob #1, Lines 487–493]

Again, the teacher asked the students to locate the asymptote, “Where is it?” After some varied and indistinguishable responses from students, she provided the answer, “There appears to be one, where—at one?” She then provided a method for determining the validity of the answer, “All right, when I ask the calculator to tell me what y goes with an x of one, what happens?” After a student provided a correct response of “nothing,” the teacher provided the remaining analysis and reasoning to support the presence of asymptotic behavior at $x = 1$.

Susan: Nothing. Ah, but what if I move ever so slightly to the right? I let the calculator decide how much I would move to the right. If I move just a little tiny bit, [T enters 1.001 for x], that’s slightly to the right, y is 1000. If I move slightly to the left of one, say .999, y is negative 1000. So I see that vertical asymptote. I can see that sort of game of chicken behavior going on for this particular function.

[Susan Ob #1, Lines 498–502]

Instead of providing the students with an associated task to confirm the asymptotic behavior, Susan outlined the process and completed the work without contributions from the students. She suggested looking for a value “slightly to the right” and “slightly to the left.” She also interpreted the significance of the large values of the function at each of those inputs on either side of $x = 1$, and even provided a visual image for the students in the form of the analogy she had used earlier—the chicken graph. In these exchanges, Susan asked and answered questions, suggested approaches for answering questions, and completed the mathematical analysis for the students rather than eliciting that kind of mathematical thinking from the students.
Instances of this pattern of responses to students also took the form of leading questions in which the solution path was suggested or connections to previously studied material were made. Two examples of such a response occurred during the implementation of one of the lead tasks in the first observed lesson. As described previously, Susan asked students to consider:

Why is \( \frac{x^2+2x-3}{x+3} \) not a rational function? [Susan Ob #1, Line 376]. At the onset of the work, she gave students a few moments to think about the task. However, before eliciting any response from students, she provided a hint. “Hint: Is there any way I might simplify this?” [Susan Ob #1, Line 383]. In doing so, Susan suggests an approach to completing the task without any input from students. After the students successfully factored and simplified the expression, Susan introduced the notion that the two expressions were not exactly equivalent to one another even though the students had done the algebraic work correctly.

Susan: Except it isn’t exactly equivalent to it. It isn’t exactly equivalent to it. [pause] Why? Why is this [pointing to \( x-1 \)] not exactly equivalent to what we started with?
Student: [student initiated an inaudible response but did not finish]
Susan: Think about this: are there any restrictions to what \( x \) can equal [pointing to the original function]? [Susan Ob #1, Lines 411–416]

After asking the question about why these two expressions are not equivalent, Susan interrupted a student response to direct their attention towards the restrictions on \( x \). In each of these examples, Susan engaged students in a consideration of important aspects of the algebraic and graphical representations of rational functions. However, in presenting the task to students she asked a question that directed students down a particular mathematical path towards a solution without giving the students a chance to express their mathematical ideas about how to answer her question.
A third way in which Susan supplied the mathematical thinking involved instances for which she provided a mathematical explanation of the expressed thinking of a student. One such instance occurred during the second observed lesson. As discussed previously, students were asked to explore the characteristics of a rational function, \( f(x) = \frac{1}{x-3} \) using the basic negative odd power function \( f(x) = (g(x))^{-a} \), where \( a \) is an odd natural number, which had been previously studied. Students were asked to reason about the various characteristics such as domain, range, and end behavior of this rational function by thinking of it as a translation of the basic negative odd power function. A student offered the following when asked about the domain of the function.

Student: \( x \) cannot equal zero.
Susan: \( x \) cannot equal zero. Well, [T begins to sketch graph]
Student: Wait are we talking about…
Susan: This one [Susan points to the graph shown in Figure 4.44]. The transformed one. Can \( x \) equal zero on this?

![Figure 4.44. The graph of \( y = \frac{1}{x-3} \).](image)

Susan: Right here [labeling the \( y \) intercept], there is a point where \( x = 0 \). The \( y \)-intercept. Now for our basic function, there wasn’t, but for this, there is one. So, \( x \) cannot equal…

[Susan Ob #2, Lines 116–123]

The student seemed to realize that he was not sure which graph was under consideration, and Susan directed him to consider the transformed function and asked, “Can \( x \) equal zero on this?”.
However, without giving the student a chance correct his response, she identified the point on the transformed graph at which $x = 0$ and continued to make the distinction between the two graphs. Since Susan provided the full explanation and correction of the student’s response, it is unclear if the student’s initial errant response was simply of function of the student being focused on the wrong graph or some deeper conceptual issue.

Each of these three types of instances of teacher-provided mathematical thinking—asking and answering questions, leading questions, and teacher-provided explanations—occurred multiple times in the data and represented a large portion of Susan’s responses to student responses. At times however, Susan diverged from this pattern and responded to student responses by directing their attention to a salient feature of the mathematics of the task or the mathematics of the student. In the initial exploration of the rational function described above, $f(x) = \frac{1}{x-3}$, Susan had asked the students to graph the function on their graphing calculator, and one student asked about the apparent line connecting the two parts of the graph. Susan developed an associated task to direct student attention to the asymptotic behavior of the function as $x$ approached three.

Susan: This blue piece is not connected to that blue piece. Why? There is an asymptote in the way. So that’s a little different. The calculator just connects pixels. So one of the clues as to what’s going on might be to hit TRACE. If we hit trace [T does so on the projected calculator], you do it on yours, we will move along the curve, but where will it get interesting?
Student: Three
Susan: As we get close to three. How close would you like to get?
Student: As close as possible
Susan: Uh, give me an idea of that. Right now, I am at 3.1914. A number that is not convenient. What would you like to trace to?

[Susan Ob #1, Lines 310–319]
In this excerpt, the teacher initially asked the student why the two pieces of the graph were not supposed to be connected and then asked the students to examine the behavior of the function for $x$-values very close to three. Such an exploration directed student attention to the large values of the function for inputs slightly larger than three and afforded the students an opportunity to examine asymptotic behavior in a numerical way.

Another instance of Susan directing student attention to a salient feature of the mathematical thinking of a student occurred during the third observed lesson. As was previously discussed, Susan used an example from numerical long division to help students understand the steps of polynomial long division. The work of the class centered on the polynomial division shown in Figure 4.45.

![Figure 4.45. The rational expression for polynomial long division.](Image)

The students expressed confusion around the process of determining the multiplier for each step using only the $x$ term of the binomial divisor. The repeatedly expressed confusion was discussed in a previous section, but one exchange provided an example of Susan directing student attention to one of the inadequate aspects of the student’s thinking.

Student: For this one, wouldn’t it be simpler to do separate and do $x$, divisor sign three $x$ squared minus five $x$ plus 4 and then to 2 divisor sign and three $x$ squared minus five $x$ plus 4.

Susan: That’s similar, if I am understanding your question correctly, to saying, “Would this [referring to the previously worked numerically example] just be easier to consider this just as dividing by ten and dividing by seven?” You see, except I am dividing by 10 plus seven.

[Susan Ob #3, Lines 248–254]
In this excerpt, the student suggested dividing $x$ into $3x^2 - 5x + 4$ and then dividing 2 into $3x^2 - 5x + 4$. In response, Susan directed the student to consider what would happen if they applied the same reasoning to the numerical example dividing by 10 and then dividing by 7. In essence, Susan provided a numerical counterexample to help the student understand the mathematical fallacy in his thinking. These examples illustrate the efforts Susan made to direct student attention to the mathematically salient features of a task or a student’s thinking about a task through associated tasks or counterexamples. These efforts afforded students the opportunity to strengthen weaknesses in their conceptions and make connections to previously studied concepts and procedures.

The third type of response to student responses found in the data involved Susan’s use of student thinking to engage some type of full class consideration. Several instances stand out in the data. During the first observed lesson as students were trying to determine if a given function was a rational function, a student asked, “Wait, are you saying that a rational function has to have restrictions?” [Susan Ob #1, Line 460]. Susan restated the question to the class and asked the students to consider the response. After one student provided a limited and incorrect response, another student offered a circular argument stating that it would not be rational if it did not have restrictions. Susan followed these responses with her own explanation. While her response in this case required limited exploration on the part of the students and her explanation left a number of dimensions of this question unexplored, it is important to note that she made the student’s question something for the class to consider.

A similar use of student thinking occurred during the third observed lesson. During this lesson, a student began to ask a question about a polynomial long division problem in which the
divisor was a quadratic polynomial. Susan did not initially understand the question and asked him to put the problem on the board.

Susan: Oh, as a divisor. Notice, that’s a good question. Notice this. I am expecting that my dividend will be of a higher degree than the divisor. Shall I repeat that? I expect that my dividend will be a higher degree than my divisor. Paul’s example doesn’t have that. It’s sort of like in arithmetic. I expect that, if I am going to do long division, that my numerator will be bigger than the denominator. Here, I expect that if my denominator is second degree, I expect that my numerator is of a higher degree. At least as high as the denominator. So, x squared or higher because when I say x squared goes into whatever this is, six thousand, eight hundred and ninety four, I’ll have to say, “no times”.

The student happened to write a dividend that was linear and Susan used that example to emphasize the need for the degree of the dividend to be higher than the degree of the divisor. As was the case in other responses to student responses, Susan provided the mathematical explanation rather than designing an associated task that would allow the students a chance to explore the mathematics of the task on their own. However, this example represented another case of Susan turning the response of a student into a point of inquiry for the class.

One final example of Susan’s use of student thinking involved the consideration of a student hypothesis and Susan’s effort to generate a counterexample to the hypothesis. The example occurred during the first observed lesson and as previously discussed involved determining whether two rational expressions satisfied the Susan’s definition of a rational function. In each case, a common factor in the numerator and the denominator cancelled out and Susan asked students to consider what happened graphically at the restriction. For the second function, \( f(x) = \frac{x+3}{x^2+2x-3} \), Susan led the class in a consideration of the numerical behavior of the function around \( x = -3 \) and around \( x = 1 \). Around \( x = 1 \), the calculator produced large positive and large negative function values demonstrating the asymptotic behavior of the function, and
around \( x = -3 \), Susan attempted to illustrate how the asymptotic behavior was nonexistent. In each case, she entered \( x \)-values close to the restriction and examined the output values with the class. After these two miniexplorations, Susan introduced the notion of a hole and asked students to conjecture under what circumstances each would appear.

Susan: Okay, so [T moves back to the worksheet] sometimes you’ll get holes, sometimes you’ll get asymptotes. How can you tell which? [pause] How can you tell when you will get a hole and when you will get an asymptote?
Student: When you have more than one restriction
Susan: What happened for this original problem? [T refers to the first function]. We are getting ahead of ourselves because we looked at this graphically too much. So what might you think would determine a hole versus an asymptote? Let me have you just consider, [writing on board] \( x \) plus one divided by \( x \) plus one squared…eh, no, let’s not make it…\( x \) plus one times \( x \) minus two.

When the student offered a hypothesis that you get a hole when you have more than one restriction, Susan directed the student to consider the original problem. This problem had two restrictions and one produced a hole in the graph, so this counterexample did not refute the student’s hypothesis. Susan then attempted to develop another counterexample, but again, she produced one that had two restrictions with one producing a hole in the graph. As students considered this example, one student offered a different hypothesis in response to a question from Susan.

Susan: We know there are two restrictions. Is there a vertical asymptote at \( x = -1 \)? At \( x = 2 \)?
Student: Would it be the larger restriction?
Susan: Okay, so we have a conjecture here that the larger number is where you would have the…
Student: [inaudible response]
Susan: Because it is the larger number?
Student: Yes
Susan: So whenever you get more than one restriction, the larger restriction gives you a hole?
Student: I am saying that based on the last problem. The larger restriction was the hole.
The student offered an idea that the larger restriction would be where you find the hole. In response to this student hypothesis, Susan directed the students to examine the graph of the new function and the behavior of the function around the restrictions. Without directly addressing the hypothesis of the student, another student offered a third hypothesis that the positive restriction would produce the asymptote and the negative one would produce the hole. With her unsuccessful attempts to get the students to make a hypothesis based on the nature of the simplification, Susan directed student attention to the simplification.

Susan: Again, Alvin is trying to latch onto some ideas here. Could it be something about if it is positive or negative, if it’s larger or smaller. Look more closely at how this was formed. [T points to the factored version]. It really doesn’t have to do with what I end up with. How did I simplify this?

This exchange illustrated Susan’s responsiveness to student thinking. In this instance, she attended to the expressed thinking of students and attempted to engage the class in a consideration of the student’s hypothesis by directing their attention to other functions and the pattern of the occurrence of an asymptote. The associated tasks or counterexamples Susan produced did not refute the hypotheses and seemed unsuccessful in directing students towards more productive ways of thinking.

Throughout the three observed lessons, Susan’s demonstrated consistent responsiveness to the expressed thinking of students using that thinking to shape instruction. Her responses to student responses could be characterized as largely teacher-centric. In more than half of the critical instructional moments identified as responses to student responses, Susan provided the mathematical thinking directly through leading questions, by asking and answering questions,
and by providing the mathematical explanation related to a student response. When she attempted to offer student thinking for full class consideration, she often provided students with tasks that directed their attention to the graphical, algebraic, and numerical manifestation of essential characteristics of rational functions. However, the implementation of those tasks often provided the students with limited opportunity to express their own thinking before Susan provided the mathematical explanations.

**Duncan’s Responses to Student Responses**

As referenced in the discussion of Duncan’s elicitation and interpretation of student responses, Duncan did not consistently elicit student responses that revealed something about their mathematical thinking. At times, the focus of the class was more mechanical or procedural which limited the opportunity for students to respond in ways that revealed their mathematical conceptions. When students offered responses that could have potentially revealed something about their conceptions, Duncan did not consistently pursue them and when he did, he often did so in a highly teacher-centric way. However, there were some notable exceptions to this. Several excerpts from the observed lessons revealed his pattern of responses to student responses and the exceptions to that pattern.

Towards the beginning of the second observed lesson, Duncan asked students to find the parametric equations that represented a circle with a given set of characteristics and starting at a particular point. The phase shift was \( \pi \) radians, so the students could have constructed the parametric equations by multiplying \( x \) by \(-1\). Two different students asked about the possibility of being given a starting point that was not as easy to accommodate for.

Student: Are we ever going to have to start at a point that is in the middle somewhere like diagonally?
Duncan: Ah, that’s a good question. Someone else asked that earlier. If it is going to be at one of the 3 o’clock, 9 o’clock, 12 o’clock, or 6 o’clock positions, then you guys can kind of cheat by making one of the variables negative, but by making it start somewhere in the middle, say, of quadrant one, whatever, then we are going to do that horizontal shift in parenthesis.

The question demonstrated a thoughtful consideration of possibilities on the part of the student as she asked about other potential starting points. It also presented an opportunity to use student thinking to initiate a classroom inquiry, but Duncan’s response diminished the mathematical thinking the students had to do by providing the students with the instances for which the equations were easiest to construct. This direct explanation decreased the opportunity they had to take a conceptual step forward through some goal-directed activity related to answering their question. Rather than apply his mathematical knowledge and reasoning to developing an activity that made sense to the students, engaged their existing understandings, and held the potential for developing new understandings, Duncan applied his thinking directly to answering the question. He did the mathematical work for them.

Later in the second observed lesson, Duncan conducted a class discussion about a clock problem.

A circular clock has a minute hand that is 12 inches long and an hour hand that is 9 inches long. Use parametric equations to model the movement of both hands as they move from 12 o’clock to 1 o’clock.

The students had time to work independently and in groups while Duncan moved around the room checking their work and giving feedback. A student shared his parametric equations for the minute hand and Duncan wanted to emphasize the ways these equations could be adjusted to “start at the top”—Duncan’s reference to the values of $x$ and $y$ at $t = 0$ that would locate the
point in the 12 o’clock position relative to the origin. The following excerpt was discussed previously regarding lead task selection, but it also served as a suitable example regarding his responses to student responses.

Duncan: Alvin, what did you have for your equations? Look up here if you didn’t get it. For the minute hand…

Student: \[x(t) = 12 \sin t\] and then \[y(t) = 12 \cos t\]

Duncan: All right, one suggestion I have for you guys when you do these is…I think you guys realized the radius was 12 so the leading coefficient was 12, but how can I get it to start at the top instead of where we usually start the unit circle which is at 3 o’clock position? There are a couple of ways to do this, but I like the idea of giving you guys a table and at time zero, I want to be at the position (0,12). [D constructs a table on the board with three columns: \(t\), \(x\), and \(y\) and fills in the first line with 0, 0, and 12]. If I am centered at the origin, I want to be straight up at twelve so you guys have to figure out how that is going to work. [Pointing to the equations on the board, line 315] That does work for these equations, so Alvin is good.

[Duncan, Ob #2, Lines 548–559]

In this discussion, Duncan explained the connection between the starting point idea and the values of the variables that the starting point represents. Even though this was the third time in the two observed lessons that a problem involved a different starting point other than the standard 3 o’clock position, this response to students was the first time Duncan has attempted to develop an understanding of the mathematical relationships involved. However, rather than developing a task that required students to think about the values of the variable or a table of values and potentially providing them with a conceptual foundation for “starting at the top,” Duncan simply told the students about the relationship. A few minutes later, Duncan presented them with a potentially productive task:

Duncan: Okay, now, my second question is…I thought you guys would do it this way [D points to the set of equations just discussed and provided by Alvin above]. You would switch sine and cosine to get it to start at noon. But, if I tell you that you have to do it so that \(x\) has to be cosine and \(y\) has to be sine, see if you can do this…and I am going to give you a hint. [D writes the two sets of equations below on the board].

\[
\text{minutes } x(t) = 12 \cos(t)
\]
\[
\begin{align*}
\text{hours} & \quad y(t) = 12 \sin(\_\_\_) \\
\text{} & \quad x(t) = 9 \cos(\_\_\_) \\
\text{} & \quad y(t) = 9 \sin(\_\_\_)
\end{align*}
\]

Duncan: You are going to have to add something to the parentheses. And the same thing down here. You are going to have to do this for minutes and for hours.

[Duncan, Ob #2, Lines 632–639]

In many ways, asking students to reconstruct their response while assigning a cosine function to \(x\) and a sine function to \(y\) seemed to be a well-chosen task—accessible and problematic to students and conceptually oriented towards the learning goals. The students were assumed to have the background knowledge to understand phase shifts as they had just completed a unit on trigonometric functions recently, and the additional restriction to the task directed their efforts towards the understanding for which Duncan was striving—using a phase shift to model the movement of the hands of the clock starting at the 12 o’clock position. However, closer consideration of the task and its presentation revealed some teacher-directed aspects of it. First, immediately after the presentation of the task, Duncan provided the students a hint. He wrote the problem with the parentheses spaced to invite the kind of expression that belonged there and he specifically directed the students to “add something to the parentheses.” Second, the implied justification for the phase shift approach was that it was Duncan’s preferred method. When he first suggested the students consider the values of \(x\) and \(y\) for \(t = 0\), he stated, “I like the idea of giving you guys a table and at time zero, I want to be at the position (0,12).” The construction of the task to direct the mathematical activity of the students seems to successfully focus their work on the phase shift, but the rationale he provided seems to have omitted the mathematical motivation for the use of the phase shift to model the motion—namely, that the phase shift is useful in more situations.
An episode from the third observed lesson also illustrated Duncan’s tendency to implement tasks in more directive ways. Immediately following a productive exploration of the conversion of parametric equations into rectangular form, Duncan initiated a task in which students were expected to compare the graph produced using the parametric equations with graph the produced using the rectangular form. He set up a table with values of \( t \) and a column for \( x \) and one for \( y \) and then asked a question.

Duncan: What am I going to do to get the values of \( x \)? Anybody? You guys have been doing this for a long time.

Student: Plug in values for \( t \).

Duncan: So [pointing to the original parametric equation for \( x \)] I get zero. If I plug in one, I get .75. I get 1.5, 2.25, and 3.

Student: Then you plug in the \( x \) values into…

Duncan: Hold on, I want to get the \( y \) values from my \( t \) values [pointing to the parametric equation for \( y \)].

Student: negative one. One.

Duncan: [writing those values as shown in Figure 4.46] and then it goes up by two.

![Figure 4.46. The table of values for the parametric equations of a line.](image)

Duncan: All right, here is what I want you guys to do. Take these two equations [D circles the original two parametric equations]. Put those in your calculator in parametric mode because they are parametric equations. And then hit graph. [Duncan, Ob #3, Lines 298–311]

The parametric graph produced a ray beginning at (0, -1) and extending in the positive direction.

Duncan then asked, “It looks like a line, but why is it not a full line?” [Duncan, Ob #3, Line 333]. The question sparked some potentially useful contributions from the students:
Student: It’s not continuous. Not infinite values.
Duncan: How come?
Student: Because it’s just a range of values from zero to four.
Duncan: All right, that’s just my table. I just picked values for my table. I could have kept getting more, but why is the graph only part of a line? How many of you got just that part of a line on your calculator? And that’s correct. I am not saying it is wrong.
But I want to know why it only gave me part of a line?
Student: Maybe the other part is undefined.
Duncan: Good thought.
Student: The inputs?
Duncan: Yea. If you guys go back…everybody hit window on your calculator. What’s your t min?

The productive thinking on the part of the students could have led to some more student-initiated investigation. Each of these comments could have been probed to more clearly reveal the student’s thinking and an associated task could have been developed to engage the students in the consideration of the reasoning behind the graph the calculator provided. However, after helping a few students produce the graph, Duncan put all of the pieces together for the students.

Duncan: Hey guys, here’s why you did not get a full line: Your calculator is saying start t at zero and go to four so it is only giving you from here to here [pointing to the table of values as shown in Figure 4.46] or from here to here for your x’s and y’s. Now, if you graph this one on your calculator [D boxes in the rectangular equation]. If you graph \( \frac{8}{3}x – 1 \) on your calculator, this is what it looks like. [D draws the dotted portion of the graph as shown in Figure 4.47.]

|Figure 4.47. The graph of the parametric equations of a line.|

Student: Cause there’s no t.
Duncan: Right. It’s the same line, isn’t it? This [pointing to the rectangular equation] is the same as this [pointing to the parametric equations] cause we converted it from parametric to rectangular, but the reason this [the rectangular equation] goes on forever and ever, there is no time restriction so it doesn’t know to go from zero to four. So sometimes we need to be careful with domain issues. This equation [the rectangular one] is correct, but then we have to say that $x$ only goes from zero to three. And the range only goes from negative one to seven. Does everybody see that?

[Duncan Ob #3, Lines 352–367]

Duncan did not give the students a chance to figure out the answer to his question about why the graph is not a full line. Without a meaningful context for the graph, the student had limited opportunity to connect the constraints of a given situation in the real world with the differences they were observing in the two graphs. The comment, “maybe the other part is undefined” made by a student was a potentially productive line of thinking. Duncan recognized it as such but did not pursue it. Instead, he proceeded to fill in the gaps for the students. Similarly, the student’s observation, “Cause there’s no $t$,” also could have provided an opportunity for students to formalize their thinking. Given Duncan’s awareness of the existing understandings, his apparent command of the subject matter, and the elementary nature of finding values and plotting the functions on the graphing calculator, Duncan could have constructed an associated task that would have led them to produce the graphs and make the connections with the input values and the limitations to the graph. Instead, he led them through the process, step by step, requiring them to make few of the mathematical decisions.

These examples revealed a pattern in Duncan’s responses to student responses and his use of associated tasks. Often, his responses involved providing students with the mathematical explanations and making the mathematical connections for them rather than positioning the students as the mathematical decision- and meaning-makers. At several points throughout the three observed lessons, students asked questions or offered hypothesis that opened an
opportunity for further investigation. Rather than using those student observations to initiate a student-centric, mathematical investigation, Duncan often chose not to pursue those points of inquiry. Several excerpts illustrate this pattern.

After constructing the equation of the ellipse produced by two parametric equations (the second example in the first observed lesson), a student asked about determining the major axis.

Student: How did you get 500 for the major radius?
Duncan: Good question. This ellipse is longer this way so the major axis is horizontal.
Student: So where did you get the 500.
Duncan: I know it goes from 500 to 1500 so how far is this?
Student: Oh
Duncan: Okay.
Student: So does that have to do with our amplitude and midline?
Duncan: [Directed to the student] Yea, kind of. [Directed to the whole class] Now, quick question: What if the amplitudes were the same? What if they were both 500?
Student: You would get a circle

[Duncan, Ob #1, Lines 629–638]

In this excerpt, a student asked a question about the major radius of the ellipse that had been produced by the parametric equations. First, Duncan clarified that the major axis was the longer one and pointed to it on the diagram. The student still did not know where the 500 came from and asked a follow-up question. Duncan provided the answer rather than responding in a way that would have kept the burden of the mathematical thinking on the student. Another student asked a different question about the connection to the amplitude and midline to the length of the axis. Not only is this connection central to the connection between the parametric and rectangular representations when the two sinusoidal functions have the same period, but it could have also been used as way to reinforce the student’s understanding who asked the question about the major radius. Using this question as a springboard for inquiry could have led to a deepening of the conceptual understanding of this relationship in general as well as in the
specific case of sinusoidal parametric equations. Instead, Duncan gave modest consideration to
the student’s question with a quiet, “yea, kind of.” He then immediately transitioned (with a
change of tone and volume with no reference to the student’s question) to the full class in a way
that suggested the question he posed to the entire class was independent of the student’s
question. During the interview, the researcher asked about the student’s response to the last
question.

Interviewer: How do they know that?
Duncan: I think because they had made the connection of how the amplitudes came here
(pointing to the denominators of the equation of the ellipse: \( \frac{(x-1000)^2}{500^2} + \frac{(y-150)^2}{50^2} = 1 \))
and affected where the major and minor vertices are (pointing to the graph of the
eLLipse) and the shape of the graph. So if they are going to be the same, and we had just
talked about this last week, that if they are going to write the equation in this form, if
the denominators are both 500 squared or 50 squared that your horizontal and vertical
stretch are equal so it remains a circle.

[Duncan, Ob #1, Lines 639–648]

Duncan’s response suggested that he believed they were making some connection between the
amplitudes of the parametric sinusoids and the shape of the rectangular graph based on their
recent and previous work of the class. This reference represents Duncan’s consideration of the
existing understandings of students in his construction of responses to student responses.

However, the connection, made through the equation of the ellipse, seemed limited to the
appearance of the equation and where the amplitudes manifested themselves in the equation of
the ellipse. In light of the procedural and introductory nature of Duncan’s goals for students,
this approach makes sense. Yet, the student’s response presented an opportunity to consider
the underlying mathematical relationship between the sinusoidal behavior of \( x \) and \( y \) and how those
two quantities related to each other. This underlying relationship appeared to be at the heart of
the student’s question about amplitude and midline and could have served to develop a rich, connected understanding of these relationships.

Subsequent to this exchange in which Duncan believed the students were making meaningful connections, additional evidence suggested that some students still were unable to make the connection. In a discussion of a different question, a student asked, “Is it always going to be an ellipse?” [Duncan, Ob #1, Line 727]. This question suggested a lack of understanding about the connection between the equation of the ellipse and the amplitudes of the sinusoids. In response to this question, Duncan and the student had the following exchange, which revealed another potential opportunity to deepen the conceptual understandings of the students.

Duncan: No, just in this example, it happened to be an ellipse. We can get all kinds of different shapes.
Student: So could they just not be related?
Duncan: Yea, I could say the relationship between you getting taller and how dark the green grass is outside. There is no relationship between those two things.

[Duncan, Ob #1, Lines 728–732]

Duncan recognized the student’s question, “So could they just not be related,” as an opportunity to make an important point about the nature of parametrically related quantities. In response, he provided a useful example of two quantities that were not dependently or causally related. However, he neglected to explore what the student fully meant by “not being related,” nor did he reinforce the notion that one could still graph the way one changed with respect to the other. In other words, this student response provided an opening into deepening her understanding if her thinking had been used as a springboard for inquiry.

While the prevailing pattern of Duncan’s interactions was one that could be characterized as teacher-centric in which the teacher bore the burden of the mathematical thinking and students did not have to fully express their mathematical thinking, not all exchanges with students...
represented that dynamic. Some interactions illustrated an approach by Duncan that was more consistent with the model of instruction presented in chapter 2. These exchanges primarily occurred during the third observed lesson. The class was organized around the conversion of parametric equations to rectangular ones. The introductory exchange for the set of problems is captured below:

Duncan: So what we are going to look at is how can I change these parametric equations to rectangular? Here’s the first example [D writes the following on the board]

\[ x = \frac{3t}{4}, y = 2t - 1 \]

Duncan: So here’s my question. I want it just in terms of \( x \) and \( y \). That’s what rectangular means. I don’t want it in terms of \( t \), in terms of time.

Duncan: Anybody have a suggestion?

The students began to offer some suggestions for possible approaches, and Duncan engaged the class in some public consideration of those ideas without being overly directive or unnecessarily leading:

Student: Set \( t \) to 0 and make sure \( x \) is in the equation

Duncan: Well if I add some \( x \), then I am going to have \( t \) and \( x \) together and \( t \) and \( x \) together [D points to the two questions]. It doesn’t make a whole lot of sense. John?

Student: Is it kind of like when you have a system of equations when you put one of them in terms of one of the variables and then put it in the other equation?

Duncan: Perhaps. Does that seem like a good idea? Tom, is that what you were thinking? Bill, is that what you were thinking?

Student: I was just thinking use the inverse (inaudible) and set \( t \) equal to zero.

Duncan: Well if you set \( t = 0 \), then that is just going to give you one value. For instance, if \( t \) is zero, it is going to give me zero here [D points to the equation for \( x \)] and negative one here [D points to the equation for \( y \)]. That will give me a point, and I can get a few values. That is not a bad idea.

In this excerpt, Duncan elicited three different ideas from students and handled them in three different ways. For the initial idea offered by a student, Duncan seemed to understand the
unorthodox suggestion, constructed a dismissive reply, and moved onto to another student. The
next student offered the suggestion of treating the situation like a system of equations. Without
overtly acknowledging the viability of this approach, Duncan continued to keep the dialogue
open for other students to contribute. A third student offered the idea of setting \( t \) equal to 0.
Although Duncan provided the student with an idea for pursuing this approach, his thinking
revealed more consideration of the student’s response than was initially apparent.

Interviewer: So this idea… you got the idea from a student about “Is it like a
system of equations?...put one in terms of the other…and then you got this idea about
Duncan: Picking a point
Interviewer: Picking a point or a value for \( t \)...you ended up pursuing that for a
second instead of continuing with “it is like solving a system. So, talk about
that just a little bit.
Duncan: My thought was that if they were going to pick a value for \( t \), that they
may try to plot a few points in terms of \( x \) and \( y \), okay let \( t \) be one, let \( t \) be two,
and create a table of values and then from there they might be able to recognize
a pattern in those inputs and outputs and then from there come up with an
equation. But I got the impression that once I said you will get one and
negative one, then they weren’t sure what to do with that. I didn’t hear
anybody say, “well, let’s get another point. Let’s get three or four points.” And
then John came back in and saying, “No, I think we can do this with [interview
interrupted momentarily]... that he was still wrestling with his system of
equations idea and I wanted to give him some kudos for thinking. Sure, we are
trying to solve the system here.

As Duncan explained in this excerpt from the interview, his response to the student who
proposed setting \( t = 0 \) and finding a point was intended to see if students understood how finding
points could be a viable approach. When the students gave no indication of recognizing the need
for finding other points, Duncan decided to return to the student’s idea about treating it like a
system.

Student: Just make it equal to \( t \) and plug that in
Student: So with the y equation, you would add one and basically isolate $t$ and get $y$ where you had all of the other stuff and then…
Duncan: Okay, so tell me what you are doing [D points to the equation for y]…

This approach provided students with multiple opportunities to share their thinking, and even though portions of the mathematical thinking were largely provided by the teacher, this exchange represented one of the least didactic exchanges in the three observed lessons.

Duncan’s use of student thinking continued as the student proposed solving the equation of the function $y(t)$, for $t$. Duncan recognized that this suggestion not only would require an additional step to solve for $y$ in terms of $x$ in the rectangular form, but it also involved a complex fraction. By following this approach to completion in spite of these less than ideal characteristics, Duncan demonstrated his understanding of the validity of this approach and allowed the students to reflect on a more mathematically efficient alternative. With students dictating each step along the way, Duncan required them to make the mathematical decisions.

At the end of that work, Duncan posed a question about efficiency.

Duncan: Could we have done this a little simpler? John, you’re right by the way, that was a good job.
Student: Could you do the same thing except get $t$ from the $x$ equation?
Duncan: Okay, so Karen is saying…help me out. Go ahead, what are you saying Karen?
Student: Do the same thing, but instead of getting $t$ equals from the $y$ equation, you would get it from the $x$.

With continued guidance provided by the students, Duncan completed this second approach and asked the students which was more efficient, gave them his assessment, and moved on to the next task. This entire exchange centered around the students’ initial experience with expressing parametric equations in rectangular form demonstrates Duncan’s ability to elicit student thinking,
reason about the mathematical implications of various approaches, and create associated tasks that direct student attention towards the mathematically significant understandings for which he strived.

In these examples from the third observed lesson, Duncan elicited student thinking, used that thinking to guide instruction, and maintained the students as decision-makers. However, even with these productive exchanges, the prevailing pattern of responses to student responses throughout the observed lessons was characterized by his tendency to do the mathematical thinking for the students and to neglect the pursuit of student thinking as points of inquiry or to gain clarity about their conceptions. Associated tasks designed to address student confusion or investigate a student hypothesis or question were not often provided to students, and when they were, the questions were often asked and answered by Duncan. In short, Duncan provided limited opportunity for students to engage in a goal-directed activity designed to explore the mathematical ideas under consideration, resolve the cognitive dissonance created by a problematic scenario, and make connections to existing understandings.

**Summary of Differences in the Responses to Student Responses**

The four teachers in this study varied in the consistency with which they used student thinking to shape instruction, constructed associated tasks to direct student attention to the salient features of the mathematics, and implemented those associated tasks in ways that maintained the student as meaning-maker. Jackie’s approach to student responses throughout the three-lesson sequence demonstrated her efforts to use the expressed understandings to strengthen existing understandings, to build new understandings, and to reconstruct unproductive understandings. She maintained an intentional effort to elicit student thinking and used those expressed
understandings to advance the class discussion and to meet her lesson-specific and broader course goals, even when they were in conflict. She consistently interpreted student responses in terms of these goals and demonstrated her ability to craft responses that supported the attainment of those goals. Harold also consistently used the expressed understanding of student to shape instruction. His focus on strict indicators of understanding and precise use of language led to the use of associated tasks designed to direct the students to particular ways of conceiving the mathematical ideas and approaches that he deemed most powerful. In his responses to students, he demonstrated his ability to shift the structure and complexity of tasks to fit the expressed understandings of students changing representations, numerical complexity, representational context, and abstract generalizations. Susan also demonstrated a consistent responsiveness to the expressed thinking of students and used that thinking to shape instruction. She also demonstrated a capacity to move between representations. Her responses to student responses were largely teacher-centric as she often provided the mathematical explanation, clarification or elaboration. Like Harold, she identified opportunities to use student thinking for whole-class inquiry and she prompted the class with an associated task to do so. However, the implementation of those tasks mirrored the implementation of tasks during other parts of the lesson as she directed student attention to the graphical, algebraic, and numerical manifestation of essential characteristics of rational functions. On some occasions, Duncan elicited student thinking, used that thinking to guide instruction, and maintained the students as decision-makers. More often, his responses to student responses were characterized by his tendency to do the mathematical thinking for the students. Associated tasks were typically consistent with his goals.
of introduction and procedural competence and his responses to students typically were designed
to address this type of student confusion.

**Summary of Instructional Differences Among the Four Teachers**

Of the four teachers, Jackie’s observed instruction was most consistent with the image of
instruction derived from existing literature and outlined in chapter two. Her learning goals for
the three-lesson sequence were clearly articulated and the mathematics of those goals was
unpacked in ways that allowed her to use them during instruction. Her lead task selection
consistently focused on establishing a conceptual foundation for understanding new ideas while
building on her perceived existing understandings of students. The tasks were designed (or
chosen) to be problematic for the students, and they supported the connections to existing
knowledge and the establishment of new understandings. Jackie routinely elicited student
thinking throughout the observed lessons, created instructional responses based upon student
responses, modeled and held students to the establishment of productive sociomathematical
norms: justification, understanding multiple representations, and reconciling alternative
strategies. She often made student conjectures, strategies, or errors a basis of productive, whole-
class inquiry. Her use of counterexamples, justification, and reflection supported student
conceptual development and maintained the student as sense-maker and decision-maker.

The design and delivery of Harold’s instruction also involved clearly articulated learning
goals. These goals appeared to be adequately unpacked to facilitate the design or choosing of
tasks. In choosing the lead tasks for both the Algebra I class and the Math 6 class, Harold gave
consideration to both the mathematical learning goals and the anticipated, existing
understandings of the students. With some exceptions, Harold presented tasks to students that
were sufficiently problematic to generate the thinking that could lead to conceptual advances. In other words, the tasks had no obvious or routine solution path, yet they could be solved by a student using ideas and mathematical tools previously encountered in the course. These chosen tasks also embodied the key mathematical ideas under study. During implementation of these tasks, Harold consistently maintained the level of cognitive demand and paid attention to student thinking in most instances. He structured the activity of the students by emphasizing representations, modeling internal dialogue, and using relational definitions that captured the essence of the concept rather than the instrumental use. Harold’s responses to student responses were less well-aligned with the image of instruction. Even though Harold chose sufficiently problematic tasks, elicited student responses, and used student thinking as a basis for instruction, frequently, his responses to student responses did not involve an attempt to scaffold emerging understandings to existing ones.

In Susan’s observed lessons, her instructional practice differed from the model of instruction derived from the literature in several ways. Differences were first noted in the degree of specificity of the learning goals and in the level of unpacking of the mathematics of them. This lack of specificity and articulation manifested itself directly through her descriptions of her goals for students and indirectly through her selection and sequencing of tasks. Although she gave careful consideration to students’ existing understandings and prior experiences in both the selection of tasks and in their implementation, her lead tasks did not reflect a progressive or developmental sequencing of ideas and understandings. The selected tasks were not often presented in a problematic way. During instruction, Susan gave careful attention to student thinking and often allowed it to shape instruction: considering alternative ideas and strategies
generated by students, pursuing conjectures to their logical conclusion, and generating counterexamples or drawing attention to the problematic nature of a student response. However, during a significant amount of the instructional time, Susan’s responses to student responses did not maintain the student as mathematical decision-maker.

Duncan’s instruction was well organized and involved the use of multiple representations that would seem to facilitate a connected understanding of the mathematical ideas under study. However, his observed lessons were the least consistent (compared to the other participants) with the image of instruction derived from the existing literature in chapter two. His learning goals were not specified in a conceptual sense nor were component or prerequisite understandings articulated. Tasks that Duncan chose offered students opportunities to practice demonstrated routines rather than engaging existing understandings to solve a problem or answer a question. Tasks were chosen with some consideration of students’ existing understandings, but that consideration was largely limited to a focus on the types of functions with which students were familiar and which could be included in the problems as review. Because the mathematical activity was largely procedural in the observed lessons, Duncan directed the student activity to a large degree, and the dialogue in class rarely focused on student conceptions.

Many of the differences among the instructional practice of these teachers at critical instructional moments can be understood in terms of the application of each teacher’s mathematical knowledge and mathematical reasoning during the planning and implementation of instruction. An explication of the role of mathematical knowledge and reasoning occurs in the next chapter.
Chapter 5:
Understanding the Use of Mathematical Knowledge and Reasoning
at Critical Instructional Moments
In an effort to understand these differences and how they might be related to the use of the mathematical knowledge and reasoning of the teacher, the researcher explored and analyzed the mathematical activity of the teacher as expressed in the classroom, background interview, and stimulated recall data. As explained previously, each of the participants in the study was highly experienced and well trained. Each expressed a desire to develop rich, connected understandings in their students in the background interviews. Thus, it was important to understand the differences in their work with students and the nature of their use of mathematical knowledge at critical instructional moments.

There was evidence to suggest that something more than accumulated mathematical knowledge for teaching in all its forms was at work during instruction—namely, the contextualization and application of that knowledge as it applies to the particular group of students engaging in mathematical activity focused on developing a specific set of understandings. That contextualization and application of knowledge for instruction requires a form of mathematical reasoning. In the planning of instruction, a teacher might identify learning goals for students based on his or her understanding or accumulated knowledge of mathematics and select tasks for students in support of that learning goal. However, carefully unpacking those learning goals into the component understandings and selecting and sequencing tasks in relation to those learning goals and the anticipated existing understandings of students requires mathematical reasoning. In the implementation of instruction, the application of mathematical knowledge and reasoning is even more dynamic. As students engage in mathematical activity and are asked to express their mathematical understandings, they often do so in inadequate, imprecise and incomplete ways. A teacher must interpret the expressed and demonstrated
student conceptions, consider their relation to the learning goals and desired mathematical understandings, and design instructional interventions accordingly. As discussed previously, the researcher has used the construct of *pedagogical content reasoning* to capture the unique nature of mathematical reasoning applied during the planning and implementation of instruction. It is both pedagogical and mathematical in nature. It involves mathematical reasoning with classroom data including student responses and learning goals to make pedagogical choices about the nature of the instructional interventions.

In this chapter, the explanatory power of the construct of pedagogical content reasoning will be established in three distinct ways. First, examples from the data with respect to the planning of instruction as well as the implementation of instruction will be used to establish the existence and significance of the application of mathematical knowledge and reasoning at critical instructional moments. Second, the construct of pedagogical content reasoning will be used to explain the differences in the nature of instruction at critical instructional moments among the teachers in the study. Third, a summary description of the role of pedagogical content reasoning at critical instructional moments will be explicated.

**The Importance of Pedagogical Content Reasoning During the Planning of Instruction**

Analysis of the data revealed a number of examples that suggested a significant role for pedagogical content reasoning in the planning of instruction. The differences in the nature of the identification of learning goals and in the selection and sequencing of tasks identified and discussed in the previous chapter cannot be fully explained by differences in the mathematical knowledge for teaching. Further analysis of the identification of learning goals and in the
selection and sequencing of tasks revealed differences in the extent to which mathematical
reasoning was applied to the identification of learning goals and the selection of tasks among the
four teachers in the study.

In the cases of Jackie and Harold, analysis of the data revealed an extensive unpacking of
the learning goals and the component understandings identified as central to the attainment of
those goals. In contrast, the analysis of the work of Susan revealed less careful consideration of
the component understandings related to the learning goals and in the case of Duncan, limited
evidence existed related to the unpacking of the learning goals. These differences suggested a
correspondence with the differences in the application of mathematical knowledge and reasoning
in the planning of instruction.

**Jackie’s Reasoning During the Planning of Instruction**

The primary learning goal for Jackie was to develop an understanding of the relationship
between the angle a vector makes with the positive x-axis and coordinates of any point on the
vector. Jackie situated the learning goal for the three lessons in relation to the historical
development of the discipline and the future mathematical concepts to be studied. She discussed
her thinking in the following excerpt from the stimulated recall interview.

Interviewer: What are those key things you want to try to develop in them, in their
understanding as you are leading up to the unit circle?
Jackie: Um, well, the trigonometry is really divided into some categories and they are
matched up pretty well with the historical development. So the triangle trigonometry,
which the oldest is the right triangle trigonometry and so I want them to see the value
of that and how you can find the missing sides and angles and such. Now the Law of
Sines and Cosines which came later, and everything else really falls from that having
to take hundreds of years to fall, but none the less. And so, I kind of want them to
have an appreciation for the history and the kind of breakthrough each one represents.  
[Jackie, Ob #3, Lines 6–17]
In this response, Jackie made a specific reference to the historical development of trigonometry from right triangle trigonometric ratios to the laws of sines and cosines and to the ways in which her approach in class has mirrored that development. During instruction, she described the laws of sines and cosines as allowing us to “break away from the constraints” of the right triangle. As her description continued, she expressed her learning goals for this lesson sequence.

Jackie: And then I want them to see, once we leave the triangle behind, now we can talk about angles like full rotations, many rotations and then we can sort of see the pattern and then, not this year, but next year, go onto the function trigonometry, \( y = \sin x \) and if they understand the transition I’ve set up for them, then that should be another easy transition and then they see the pattern of basically, any sinusoidal phenomenon.

[Jackie, Ob #3, Lines 20–26]

As she described the shift away from the triangle, she emphasized the mathematical opportunity that shift provided—to explore angles as full rotations and lay a foundation for the study of trigonometric functions. The “transition” away from the triangle is essentially the learning goal for the three-lesson sequence. Jackie described it this way:

Jackie: So this gives me a tool to not use them [referring to the triangle trigonometry formulas], but all because I am leading towards that association between the angle and the coordinates of the points on the line.

[Jackie, Ob #3, 287–289]

This set of descriptions of Jackie’s learning goals for students situated the goals for the lesson sequence in the historical development of understandings and in relation to concepts to be encountered in future math courses. It illustrated what Ball and her colleagues refer to as common content knowledge—in this case, the knowledge of trigonometry—as well as knowledge of the mathematical horizon—in this case, the awareness of the power of defining angles and trigonometric ratios outside the confines of a triangle.
However, the identification of these learning goals and the understanding of their relationship to future concepts to be encountered does not necessarily inform the identification of lesson-level goals and the unpacking of those goals in relation to the mathematics of the students. To do so requires the teacher to consider the potential existing understandings of students and how they are likely to develop the desired mathematical understandings and apply that understanding to the context in which the learning will occur—these particular students in this mathematics class. Each selection of lead task revealed Jackie’s active consideration of the mathematics of the students and the component understandings of her learning goal.

As discussed previously, this pedagogical content reasoning is apparent in the choice of the kite problem with right angles as bonus from the previous quiz. Jackie intentionally chose a problem in which students would be faced with the decision to use the law of sines or the right triangle trigonometric ratios. In her words, “I wanted them to see that now just because you know the law of sines you don’t have to use it when you have a right triangle” [Jackie, Ob #1, Lines 48–49]. This rationale revealed Jackie’s consideration of the mathematics of the student in relation to her learning goals for them. She anticipated that some students would use each method, and she wanted to create the opportunity to reinforce the appropriateness of using the ratios in right triangles. Not only did the task choice afford her the opportunity to reinforce the notion of the more efficient method, but it also represented her initial effort to focus the students on using the right triangle trigonometric ratios—a foundational component to her goals for the lesson sequence.

More evidence of Jackie’s pedagogical content reasoning related to the mathematics of the students and the mathematical learning goal for the lessons was found with each choice of
lead task. With each task, she expressed a well-reasoned mathematical path for the completion of the given task with a consideration of the anticipated reasoning of students. Simultaneously, she also considered the component understandings needed to transition students away from conceiving of the triangle in order to use the trigonometric relationships and towards the functional relationship between the angle and the coordinates of a point on the coordinate plane. Jackie’s choice of the parallelogram problem from the first observed lesson further illustrated the point.

As discussed previously, Jackie presented students with a parallelogram problem following the kite problem. She provided the students with two sides and the included angle and asked them to find the longer diagonal. When asked about the choice of the parallelogram, Jackie replied,

Because I am working up to vectors. So, we have already done, “Find the diagonal of a parallelogram.” Just to use the Law of Cosines and Sines. But in my ulterior motive kind of way, doing it here and then going into vectors, first I want them to tell me the diagonal is not the same for every parallelogram with the same length of sides—it depends on the angle. Both classes saw that real easily. And so, then if we are going to find the length of the angle, then we are going to have to use the angle.

[Jackie, Ob #1, Lines 476–483]

In this description of her thinking, Jackie expressed her consideration of the existing understandings of students and the likely approach they will take using the laws of sines and cosines. She also referenced her plan to move into vectors and her desire to help students recognize the dependent relationship between the length of the diagonal and the included angle of the parallelogram first. One of the component understandings she identified as related to her learning goal for the lesson sequence.
This excerpt illustrated Jackie’s focus on the mathematics of the learning goal and her understanding of the existing understanding of the students, the careful unpacking of each, and the consideration of their relationship. The clearly identified and sequenced progression of understandings she hoped to develop over the three-lesson sequence served to inform her choice of task. Likewise, a simultaneous consideration of the existing understandings of students and the anticipated approaches they might take to complete the tasks also informed the choice of task. Applying her mathematical understanding of the learning goal and her contextual knowledge of her students and their mathematics in choosing each lead task required an active reasoning on her part to understand a mathematical path from one to the other in order to identify a task that can support that development.

The data from Jackie’s observations and interviews vividly illustrated the goals she set for students and how she structured and sequenced classroom activities to meet and reinforce those goals. Evidence suggested she used her mathematical knowledge and reasoning to plan the sequence of activities that would prompt students to use their anticipated understandings to develop an understanding of the relationship between angles and coordinates on the coordinate plane—an essential for understanding the unit circle. This active application of her mathematical knowledge and reasoning in the planning of instruction represented the pedagogical content reasoning she used.

**Harold’s Reasoning During the Planning of Instruction**

Evidence of the role of pedagogical content reasoning in the planning of instruction also existed in the two-lesson sequence taught to sixth graders by Harold. An analysis of several lesson segments revealed his use of mathematical knowledge and reasoning in the identification
of learning goals and in the selection and sequencing of tasks. The data from Harold’s observations also provided a unique opportunity to compare his use of his knowledge and reasoning in two different classes as he maintained the same goal in each of the two lessons but chose different lead tasks for each. Understanding the differences in these approaches served to deepen the researcher’s understanding of the role of pedagogical content reasoning in the planning of instruction.

Harold’s use of mathematical knowledge and reasoning was evident in the careful articulation of his learning goal for the two lessons. He identified the goal in the preobservation interview as the development of an understanding of the division of fractions through the modeling of a series of component understandings. He presented the following analysis of that goal:

Harold: We are going to divide by $x$ which they are going to recognize as the same as multiplying by $1$ over $x$ and we are going to divide by $1$ over $y$ which where I was hoping to go next, but really never got there. But we’ve been over that before and frankly, I think they are a little bit better with that one than this one, but they know that dividing by $1$ over $y$ is the same as multiplying by $y$. So we are going to take dividing by $x$ over $y$ and convert it to dividing by $x$ times $1$ over $y$. Convert that to dividing by $x$ and then taking all of that and dividing by $1$ over $y$. Convert that to this times $1$ over $x$ times $1$ over $y$. That’s the hoped for progression.

[Harold, Ob #1, Lines 105–115]

In this description, Harold identified the component understandings he sought to develop.

— Dividing by $x$ is the same as multiplying by $1$ over $x$.

— Dividing by $1$ over $y$ is the same as multiplying by $y$.

— Since $x$ over $y$ is the same as $x$ times $1$ over $y$, the division of $a$ over $b$ by $x$ over $y$ can considered in two parts—namely, dividing by $x$ and then dividing by $1$ over $y$. 
Such an analysis of the understanding of the division of fractions into its component understandings could be reflective of an accumulated set of understandings derived from the teacher’s prior experience or mathematics education coursework. As such, it would represent what Ball (2008) refers to as knowledge of content and students. Alternatively, this sequence of component understandings could also be a product of an application of mathematical reasoning applied to answering the question: What are the sequence of understandings that link the current understandings of students to an understanding of division of fractions? While conclusive evidence was not collected, anecdotal evidence from unrecorded conversations with the teacher suggested that it was a product of his mathematical reasoning.

Nevertheless, a consideration of the two distinct approaches Harold used in the two Math 6 lessons in an effort to achieve this goal provided more compelling evidence of the role of actively applied mathematical knowledge and reasoning in the planning of instruction, in general, and in the selection of tasks, in particular. In the first observed lesson, the lead task for students involved completing the following statement: \( a \div b = \frac{a}{b} = \, \, \text{_______}. \) Harold introduced the task in this way:

Harold: Guys, I want you to know that all you are doing at this point is writing the same thing in a different way. Are we okay with that? [Harold Ob #1, Lines 60–61]

Harold: Do we need to prove this [pointing to the equation]? This means that [pointing to the two sides of the equation]. Does everybody understand the difference?
Student: yep.
Harold: I want you to tell me something that is equal to both of these that is different \[Harold Ob #1, Lines 92–96\]

In this presentation of the lead task, Harold emphasized the way in which \( \frac{a}{b} \) is just writing \( a \div b \) in a different way. He wanted the students to produce a new expression that is equal to both of
these but different. In the interview, the researcher asked him about the response he was anticipating:

Interviewer: What were you looking for from them there?
Harold: $a$ times $1$ over $b$ or $1$ over $b$ times $a$. I would be good with either.

[Harold Ob #1, Lines 97–99]

As discussed previously, Harold’s response suggested that he intended to use this task as a way to activate prior knowledge and lay a foundation for the development of another relationship. He thought they would have little difficulty completing the statement in the way he wanted.

Harold gave several indications in the class as well as in the stimulated recall interviews that the difficulty of students demonstrating their understanding of this initial component understanding was unanticipated. Early in the first observed lesson, Harold commented to the class, “I am not expecting us to have problems here” [Harold, Ob #1, Lines 159–160]. This comment came after their initial difficulty in completing the task. In the stimulated recall interview he asserted, “Yes, but because they had done it before, I didn’t think they would have as much trouble as they did” [Harold, Ob #1, Lines189–190]. His statements represented some consideration of the existing understandings of students and some degree of reasoning about how they could engage with the task successfully. However, as the lesson unfolded, the struggle of the students to respond to the initial task successfully suggested that the existing understandings of students were not as robust as Harold had anticipated or the task, devoid of context, did not engage the existing understandings successfully. More careful pedagogical content reasoning in the selection of this lead task would have involved the development of a task that was experientially real to students, directed the mathematical activity of the students towards the
learning goal, and engaged the existing understandings to do so. Harold’s lead task selection for the second Math 6 lesson demonstrated this kind of thinking.

Between the first and second observed lessons, Harold acknowledged thinking carefully about the first lesson, and even during the interview about the first lesson, expressed his thinking about the struggles of the students, their conceptions, and what he could have done differently to help them progress their understandings in the direction of the learning goals. Several passages throughout the interview about the first lesson provided evidence of his reflection on the lesson and his consideration of alternative approaches. For example, when he introduced an associated task that asked students to represent one-fourth of twelve, Harold used a bar model, sectioned into fourths and shaded one of those fourths. See Figure 5.1.

![Figure 5.1. Harold’s bar model to represent one fourth](figure5_1.png)

In the interview, he remarked:

> And frankly that model was not a good one to use. It would have been better for me to use like a quarter of a circle, to use a region model. Where I like literally draw a pie, a quarter of the circle, but I don’t draw the rest of the circle. Then I could literally add the quarters together. That would be better. Then say, “Okay, how many pies do we have? How many pizzas do we have?” We’ve done that sort of problem before.

[Harold, Ob #1, Lines 413–420]

His reference to a consideration of other models suggested reflection after the lesson rather than his thinking during instruction, but nevertheless demonstrated his careful thought and reasoning about the selected task, its presentation, and the response of the students. Other passages throughout the interview reflected his thinking after the lesson and before the interview.
Again, I’ve got to be honest, if I had written $1 \over b$ times $a$ we might have been good to go. The lesson really might not have broken down the way it did. That being said, it was really good that it broke down the way it did because now I recognized that I have to do a lot more work with that component.

In this excerpt, Harold expressed his consideration of the way he presented the lead task for the first lesson and the potential value of switching the order. While it is unclear from the data why he thought the switch would have impacted the success of the students, Harold’s expressed thinking reflected deeper consideration of the first lesson and the student thinking during the lesson.

The construction of lead tasks for the second observed lesson and Harold’s discussion of his thinking demonstrated a more careful consideration of the existing understandings of students (as he understood them after the first class), the learning goals for the lesson, and a potential task that could move them from one to the other. This task selection for the second class was not a function of an increase in his mathematical knowledge for teaching, but rather, it reflected a more intentional triune focus on the mathematics of the learning goal, the mathematics of the students, and the mathematics of a task. It required a mathematical and a pedagogical reasoning about what conceptions students needed to develop and in what mathematical activity could they engage to support the development of these conceptions.

This reasoning supported the selection of the lead task in the second lesson that was more experientially real to students, more accessible to them using their existing understandings, and more successful in engaging them in productive, goal-directed mathematical activity. The task was presented to students in the following way.
Harold: Twelve dollars needs to be distributed evenly among four brothers. My sister has three boys, but let’s say she has four of them. The four boys need to distribute $12 evenly among themselves.

[Harold, Ob #4, Lines 14–15]

The students were asked to construct a visual and numerical model to represent this situation. Students were universally successful in completing this first task. Seeing this success, Harold modified the task (as he had planned) in order to shift the mathematical activity of the students to support the development of the next part of the desired understandings.

Harold: So, I am going to do it a little bit differently. I want you guys to come up with the numerical statement for the way I divide it up…Same problem but I am going to do it totally differently.

[Harold, Ob #4, Lines 274–278]

Harold went on to describe the situation as his sister having $12 in quarters and that she was going to give one quarter to each son until she did not have anymore. He finished his explanation with the question, “How much money would each kid get?” [Harold, Ob #4, Lines 336–338]. Unlike the lead task in the first lesson in which Harold asked students to complete an algebraic representation of the relationship, the lead tasks in the second class were problematic and accessible in ways that engaged their existing understandings and motivated the need to model the relationship in multiple ways.

In order to select such a task, Harold had to consider the mathematics of the students, the mathematics of the learning goal, and the path to get from one to the other. He described his thinking in the interview following this class.

Harold: Here I am able to come up with a reasonable…a very legitimate real world situation where they can see…you know, coming up with a model for this, you know, you can easily envision the four brothers. You can easily envision the 12 dollars, and I knew I could start off with dollars and then move that to quarters pretty easily and the symbols would change significantly and yet they would still recognize that we were
talking about fractions of a whole. I didn’t think they would have trouble with that part. So, I knew this would be a very real world situation that they could model easily. [Harold, Ob #4, Lines 19–26]

In his description of his thinking, Harold emphasized the real-world nature of the problem, the ease with which students could envision the scenario, the conceptions of “fractions of a whole” that could be maintained, and the anticipated success of the students. He also referenced the shift to distributing quarters instead of dollar bills in the same context. In the interview, Harold was asked about the move to quarters.

Harold: So they were all assuming I had twelve dollar bills which I kind of wanted them to make that assumption we you could have so easily just had a bunch of quarters. And if you have a bunch of quarters, it makes total sense to give a quarter to each kid. Especially if you don’t know how much money you have in total. That’s actually the way you would divvy it up. So I thought that model would make total sense to them. So each kid is going to get four quarters…or twelve quarters. So I thought that they would recognize that they are taking twelve times a quarter pretty easily to get the amount that each kid is getting. So here, [points to the equation: 12 ÷ 4 = 3] they have a very clear numeric expression that corresponds to a specific model [points to the model found in Figure 5.2] that we are going to show is equal to a completely different model and completely different numeric expression that are actually equivalent.

[Harold, Ob #4, Lines 290–305]

![Figure 5.2. Harold’s model of the distribution of twelve dollars to four sons.](image)

Each of these passages suggested three distinct ways Harold had to apply his mathematical knowledge and reasoning to develop these tasks. First, he had to develop a context that supported the modeling of both concepts in his component understanding: $a$ divided by $x$ is the same as $a$ times 1 over $x$. By first designing a task that involved the distribution of $12$ to four
people, he provided a context for the first part of the relationship, \( a \) divided by \( x \). With the modification to quarters, he created a context that involved the second part of the relationship, \( a \) times \( \frac{1}{x} \). He also had to choose the numbers for the context that supported the development of the conceptions he desired. In the first presentation of the problem, he acknowledged that his sister only had three sons, yet he consciously chose the number four. This choice allowed him to use quarters for the second part of the task—a monetary context that made sense to the students. Each element of the design of these two tasks required him to reason about the mathematics of task, the mathematics of the learning goal, and the mathematics of the student.

Secondly, in choosing each of the two tasks, Harold had to anticipate how the students would reason mathematically about the problem. In the response above, Harold indicated his consideration of student thinking with comments such as, “I thought that model would make total sense to them,” and “I thought they would recognize…pretty easily.” He considered what would be experientially real to them and how a student would likely attempt to model the situation. In doing so, he had to reason mathematically as his students would in light of what he understood their existing understandings to be and his observations about their application of those understandings to completing tasks.

And thirdly, he had to set up the situation to naturally move students from what he perceived to be their existing understandings in the direction of these conceptions he wanted them to develop. In other words, he had to choose tasks that would support the development of the mathematical understandings for which he strived. As he explained in the second quote, “a specific model that we are going to show is equal to a completely different model and completely
different numerical expression that are actual equivalent.” This equivalency was one of the component understandings he identified as key to an understanding of the division of fractions. Also of importance, Harold designed the second task as a limited modification of the first. The students were still dealing with distributing 12 dollars to four people, but they were asked to complete the same task in a different way—using quarters instead of whole dollars. The parallel structure and similarity of the two tasks seemed to serve as an effective tool for focusing student attention on the differences and building a natural understanding of the equivalency of the two results. It also demonstrated Harold’s effective anticipation of an accessible path to the completion of the task. Constructing the tasks in ways that supported the development of the desired conceptions required careful reasoning about the mathematics of the learning goal, the mathematical approaches of the students, and the potential path from one to the other.

The differences in the selection of lead tasks in the two Math 6 lessons led to differences in the success of students in completing the tasks. In the case of the lead task selection in the second Math 6 lesson, Harold applied his mathematical knowledge and reasoning in ways he did not in the first lesson. Harold’s description of his thinking reflected careful thought about his goals for the students and how the tasks might have supported the development of the key understandings in light of their existing ones. He developed a task that was experientially real, problematic, and accessible to students as a result. The data from Harold’s unpacking of the learning goals and his selection of tasks provided further evidence of the explanatory power of pedagogical content reasoning. Not only did the data further establish the importance of pedagogical content reasoning in the planning of instruction, it also provided insight into the role pedagogical reasoning potentially plays in trying to understand the differences in instruction. The
more thoroughly a teacher considered the existing understandings of students and the
mathematical understandings he sought to develop in the construction of lead tasks, the more the
task was likely to foster the mathematical activity to support the development of those
understandings. The successful completion of the tasks in the second lesson juxtaposed with the
struggles in the first lesson provided powerful evidence of the difference this kind of application
of mathematical knowledge and reasoning can have on the mathematical activity of students.

**Susan’s Reasoning During the Planning of Instruction**

The careful unpacking of learning goals exhibited by Jackie and Harold and the selection
and sequencing of tasks demonstrated by Jackie throughout her three lessons and by Harold
during the second Math 6 lesson provided evidence of the nature of the role mathematical
reasoning played in the planning of instruction. In contrast to the work of Harold and Jackie and
as discussed in the previous chapter, differences emerged in the nature of the planning of
instruction observed in the work with Susan and Duncan. These differences can be understood
in terms of the nature of Susan’s and Duncan’s application of their mathematical knowledge and
reasoning to the unpacking of the learning goals and the sequencing and selection of tasks.

The first contrast with the work of Jackie and Harold can readily be seen in the
description of the learning goals for students on the part of Susan. When articulating the goal of
the first lesson of the sequence (this lesson was observed being taught to two different sections),
Susan gave a nonspecific response.

Susan: Exposure to rational expressions; trying to understand the definition of a rational
function and to see how it fits in initially, an introduction to how it fits into our study of
functions.

[Susan, Interview #1, Lines 16–18]

When asked for further elaboration and specification, Susan replied,
Interviewer: What more specifically do you want students to understand about rational functions and how they connect to… [S interrupts]

Susan: So specifically, the fact that it involves a ratio, that it is a ratio of algebraic expressions, not just integers. That from our previous understandings, that comes with some complications, such as zero denominators… So I did hope that they would see connections with the power functions that we have studied and the connections with asymptotes, the characteristics of functions like end-behavior, and it would be nice if we also had some time to get into the manipulation to be able to determine if an expression, just because it is written as a ratio of polynomials is necessarily a rational expression or not. In other words, does it simplify to something that is rational or not.  

[Susan, Interview #1, Lines 24–39]

The broad nature of the expressed learning goals represented important features of a rich, connected understanding of rational functions—the definition, the restrictions, the graphical behavior, and the connections to other functions. The selection and sequencing of tasks certainly reflected these goals for students. Susan presented tasks to students that directed their mathematical activity towards each of these areas. The first lead task asked the students to describe domain, range, end-behaviors, and connections to previously studied functions. Another lead task asked the students to consider the definition and determine if the functions were rational functions or not (according to Susan’s interpretation of the definition).

In the construction of lead tasks Susan also considered the existing understandings of students and sought to engage those understandings to support the development of understanding of aspects of rational functions. She asked students to connect the definition of rational functions to the definition of rational numbers. She asked students to make an explicit connection between the first rational function and the negative odd power functions of the form, $f(x) = (h(x))^{-t}$ $t$ is an odd whole number—a function they had studied previously. She drew parallels between the long division of rational expressions and the long division of whole numbers. Each of these
elements of her task selection involved a consideration of the existing understandings of students.

It is also clear that Susan possessed a great deal of common content knowledge of rational functions. She understood that some restrictions produced holes and others produced asymptotes. She understood the graphical, numerical, and algebraic behavior of rational functions at those restrictions and over the full domain. She understood various characteristics of rational functions such as domain, range, restrictions, asymptotic behavior and end-behavior. She knew the parallels between long division for rational expressions and long division for whole numbers and how to use it to rewrite rational functions to determine long-run behavior. She demonstrated these understandings through her instruction, her completion of examples, and her responses to student questions.

Susan also demonstrated some key elements of specialized content knowledge and knowledge of content and students. During the two-lesson sequence, the tasks she provided for students involved many aspects of rational functions that reflected her knowledge base. She understood the value of multiple representations and included numerical, graphical, and algebraic representations in her work with students. She asked students to examine the numerical behavior of the function around a restriction. She asked students to consider the graphical behavior of the function at each restriction as well as approaching vertical and horizontal asymptotes. She also understood the confusion students often experience with the algebraic behavior of a rational function at a hole and at an asymptote and she included tasks highlighting those algebraic differences. She understood the difficulty students typically had
with long division of rational expressions and she designed her instruction to support the
collection to the existing understanding of long division with whole numbers.

However, in spite of these positive qualities, Susan’s selection and sequencing of lead
tasks did not reflect a careful unpacking of the learning goals and development of a learning
progression. It is unclear from Susan’s description of her goals what specific understandings
she wanted to develop in her students. Susan identified the major topics in the unit, but did not
provide a carefully sequenced image of her goals for students and the component understandings
required to attain those goals. For example, in her description, she referenced the idea that a
ratio of algebraic expressions involves “some complications, such as zero denominators.” She
did not express a specifically articulated description of what the complications were that she
wanted students to understand. There was limited evidence to suggest that Susan had carefully
considered what conceptions students needed to develop in order not only to find restrictions and
identify holes and asymptotes but also to understand what holes and asymptotes were, how they
were different, and how they manifested themselves in multiple representations of rational
functions. This broad-based nature of the articulation of learning goals did not seem to support
the selection or construction of tasks that were problematic and able to foster the mathematical
activity of students in ways that could support the development of specific understandings.

For example, even though she did not specifically express understanding the numerical
behavior of a rational function at a restriction as one of these complications, she engaged
students in multiple investigations of it. One such investigation was incorporated into the first
lead task, as discussed in the previous chapter, when Susan constructed the worksheet that served
as the lead task for the first observed lesson. The worksheet asked students to complete a table
of values, to describe some characteristics of the function (domain, range, end-behavior), and to connect this function to a previously studied function. None of these tasks were presented in a problematic way that would foster the development of any specific understandings. The input values on the table were not provided in a way that supported the consideration of the numerical behavior of the function around the restriction. She did not include values close to the restriction, and instead, she had to ask the students to add input values on the table and directed their attention to the behavior of the function at the restriction. She acknowledged this lack of careful attention to the input values on the table during the class. The investigation of the numerical values was quickly abandoned as Susan directed their attention to other aspects of the lead task such as the domain and range of the function. Later in the lesson, she engaged in another exploration of the numerical behavior around a restriction. The exploration was almost exclusively teacher-directed and involved very little input from students. As discussed previously, Susan asked and answered her own questions providing little opportunity for students to engage in goal-directed activity designed to meet the identified goals. It was clear from both of these instances that Susan understood the mathematics of the numerical behavior and appreciated the potential the numerical exploration held for helping students understand the behavior of the function at the restriction. Yet, the tasks were not focused on the development of a specific set of understandings about the numerical behavior of the functions around the restrictions.

A more specific articulation of the learning goals might have included the types of understandings associated with the numerical behavior of a rational function at a restriction causing a hole in the graph, at a restriction causing an asymptote, and at other asymptotes.
Unlike Susan’s initial task that involved a rational function with no hole in the graph, a consideration of these component understandings of rational functions might have led to a task with restrictions leading to a hole in the graph and an asymptote or multiple functions demonstrating the range of possibilities for restrictions and the impact on the graph. The inclusion of an equation of a function whose graph included a hole could have served as a contrast to the rational function with an asymptote as they examined the numerical behavior potentially creating some cognitive dissonance to be resolved. Susan’s lack of specificity in her articulation of the component understandings seemed to lead to a lack of intentionality about selecting tasks to develop the component understandings.

Several other elements of Susan’s instruction suggested limited reasoning about the mathematics of the learning goals and the mathematics of the tasks that seemed to shape her selection of tasks. Later in the class when Susan began to examine the differences in the algebraic representations of functions with holes and those with asymptotes, she provided students with the textbook’s definition of rational functions:

\[ f(x) = \frac{a(x)}{b(x)} \text{ where } a(x) \text{ and } b(x) \text{ are polynomials in } x \]

We assume that \( b(x) \) is not the zero polynomial.

According to Susan’s interpretation of the definition, a function that was expressed using a rational expression that reduced to an expression with a constant in the denominator was not a rational function. Essentially, she interpreted the reference to the zero polynomial as a reference to a degree-zero polynomial. This demonstrated a misunderstanding on her part and a lack of careful thought about the definition of a rational function—the mathematics under study. Further
evidence of the lack of careful reasoning about the mathematics was revealed during the subsequent class discussion when a student asked, “Wait, are you saying that a rational function has to have restrictions?” [Susan, Ob #1, Lines 461–462]. After a brief class discussion, Susan responded, “Yes, it does have to have restrictions” [Susan, Ob #1, Line 467] with no qualification regarding imaginary roots of the denominator. More careful unpacking of the mathematics under study might have avoided these inadequate interpretations of the mathematics.

After a brief discussion about the definition, Susan directed student attention to a question about whether or not a function was a rational function. As described previously, Susan asked students to consider: Why is \( \frac{x^2+2x-3}{x+3} \) not a rational function? [Susan Ob #1, Line 376]. Due to the inadequate student response, Susan was highly directive in her instruction. She led students to simplify the expression to generate an expression that was algebraically equivalent. She asserted that the expressions were not exactly equivalent. She then directed students to consider the restrictions on \( x \) in the original function and in the simplified version. After a similar approach with another rational expression that did not simplify to a linear function, Susan led students in an exploration of the behavior of the graphs around the restrictions. Even though Susan did not describe it this way, the nature of the work of the class suggested that the purpose of the question about whether or not a function was a rational function was to help students understand the algebraic and graphical representations and their differences among functions with holes and those with asymptotes. The mathematical reasoning involved in the implementation of this task and the associated tasks will be explored in the next section.
However, several dimensions of this task selection suggested an inadequate application of mathematical knowledge and reasoning on the part of Susan relative to the selection of this task. First, while the task was problematic, students were asked to determine whether the function was a rational function. As such, it focused student attention on the definition rather than on the algebraic differences between functions with restrictions that produced holes and those that produced asymptotes. Second, Susan attempted to introduce the concept of equivalence as it related to functions. Yet, she made no distinction between algebraic equivalence of expressions and functional equivalence. In other words, the rational expression and its simplified form were algebraically equivalent. However as functions, the two did not represent the same set of ordered pairs. She also limited the discussion about the non-equivalence of the two expressions to the differences in the restrictions and provided no opportunity for students to consider how to make the two expressions equivalent by identifying the restrictions. Third, the response of the students to this question about whether something was a rational function was limited. It appeared that the task did not sufficiently engage the existing understandings of the students in the completion of the task. Their limited response seemed to force Susan to engage in a series of highly directed exchanges.

Each of these elements of this lesson sequence suggested that Susan had not carefully considered the mathematics of the learning goal, the mathematics of the students, and the mathematical path from one to the other. A more careful unpacking of the mathematics might have avoided the error in the interpretation of the definition. Without that interpretation, the question about whether an expression was a rational function would not have made sense. A more careful unpacking of the learning goal might have generated more specificity and clarity.
about the goals for the lesson such as the understanding of the numerical, algebraic, and
graphical differences between rational functions with restrictions that produce holes and those
that produce asymptotes. A more specific articulation of the component understandings for the
learning goal might have informed the selection of tasks designed to support of the development
of those understandings by providing students with problematic scenarios designed to motivate
the mathematical activity of students—mathematical activity that would foster the development
of the desired understandings.

Further evidence of the lack of sufficient application of mathematical knowledge and
reasoning in the planning of instruction was found in the third observed lesson. As previously
discussed, Susan attempted to draw a parallel for the students between the algorithm for
numerical long division and the one for polynomial long division. To some degree, this task
selection involved a consideration of the existing understandings of students, the mathematics of
the learning goal (understanding of polynomial long division), and the mathematics of the task as
a way to get from one to the other. However, throughout this portion of the lesson, students
expressed confusion over the subtle differences in the “goes into” step of each long division
process. The emphasis Susan placed on the two processes being “identical” and “exactly the
same” suggested that she had not carefully unpacked the mathematics of the polynomial long
division algorithm. As discussed in the last chapter, Susan repeatedly referenced an
approximation a student made in which she rounded up to estimate the partial quotient.
However, the polynomial long division is more of a truncation process when estimating each
partial quotient in polynomial long division. Even when the researcher asked if there could be a
different set of numbers that would have helped students see the parallel more readily, Susan was
not able to identify values that would have rounded down and paralleled the truncation. These imprecise parallels Susan drew with the two processes suggested that she had not carefully unpacked the mathematics of the polynomial long division algorithm and this weakness in the application of her mathematical reasoning seemed to result in unresolved confusion on the part of students.

In the case of Susan and in contrast to the work of Harold and Jackie, Susan’s planning of instruction revealed a nonspecific identification of the learning goals and limited articulation of the component understandings of rational functions. These differences could be explained by the limited evidence of the application of mathematical reasoning to the unpacking of the learning goals. While Susan chose tasks with a consideration of the existing understandings of students and the broad goals she had articulated for students, the lack of specified, component understandings impacted the nature of the task selection. They did not present students with problematic scenarios that directed their mathematical activity in ways that could lead to the development of the particular understandings Susan identified. As a result, the lead tasks chosen were not carefully sequenced and did not support the learning goals to the greatest possible degree.

**Duncan’s Reasoning During the Planning of Instruction**

The application of mathematical reasoning in the planning of instruction for Duncan revealed further evidence of the potential explanatory power of the construct of pedagogical reasoning. Throughout the three lessons, Duncan demonstrated a solid mathematical knowledge of parametric equations. He chose examples that spanned the full range of possible procedural demands on students and integrated review of previously studied functions thoughtfully and
creatively. He also demonstrated numerous aspects of his knowledge of content and teaching and content and students. He recognized and used the graphing calculator as a learning tool for students and demonstrated his own facility with the device. He anticipated the aspects of the content with which students were likely to have difficulty with and chose examples that would introduce them to parametric equations in accessible ways. With a focus of his instruction that was more procedural and introductory in nature, the selection of tasks for the lessons involved a consideration of the range of examples that could represent the breadth of the representational and procedural differences the students might encounter. It also involved a consideration of the existing understanding of students in that specific tasks were chosen to review previously studied content. However, the analysis of the unpacking of the learning goals and the selection and sequencing of tasks revealed a limited reasoning about the concepts under study and the conceptions of the students. The lack of attention to these underlying concepts provided limited support to the selection or construction of tasks that were problematic and directed the mathematical activity of the students in goal-directed ways. Instead, the tasks directed the activity of the students in procedural ways.

In the first lesson, Duncan focused on the parametric and rectangular graphs of three different sets of quantities. His learning goal for the lesson was limited to exposing students to these graphs. He expressed this goal on multiple occasions throughout the class and during the stimulated recall interview. When asked about the first example choice, Duncan shared this perspective.

Duncan: I wanted just an introduction into looking at horizontal and vertical components separately, and then how we can put those together. And I think I did that in all three of my examples. Which was kind of the theme I wanted so they could see things separately and what they are like together.
As his response illustrated, Duncan’s consideration of what students needed to understand about parametric equations began with a simple notion of looking at things separately and putting them together. It did not progress much beyond this notion in the first lesson. The tasks he chose for the first lesson reflected this goal. When asked about his reason for selecting the second set of parametric equations relating the populations of foxes and rabbits, Duncan highlighted three factors. First, he expressed his desire to review the parameters and characteristics of trigonometric functions. Second, he valued the opportunity for students to model natural phenomena. Third, he described wanting another example of looking at them separately and looking at them together.

D (to the interviewer): Here’s a good example of what these things look like separately and then putting it together, I think they were surprised that it was elliptical. I think I heard some kids say it was the circle of life kind of thing. Essentially, that is what it is although it is not circular. But that is the purpose to getting them to see, yea, we can model something that happens in nature, pretty easily mathematically, and then from this mathematical model, we can actually look at how these two animals are related to each other and what that looks like graphically.

Again, Duncan’s expressed purpose was that he wanted the students to see “what these things look like separately and then putting it together.” Even though he referenced a consideration of “how these two animals are related to each other,” he asked no questions of students asking them to think in terms of the modeling of the relationship between the quantities. Additionally, he gave limited explanations that connected the graphical representation of the parametric equations with the relationship between the two quantities being modeled. The tasks given to students were largely teacher-directed and required limited thinking beyond the application of procedures, the manipulation of the graphing calculator, and the interpretation of trigonometric functions.
When Duncan first introduced the parametric equations for the populations of foxes and rabbits, he reviewed parameters of the functions and what part of the graph they affected. The researcher asked about the purpose he had in going into this detail.

Interviewer: Other than pure review, was there any other reason to review these trig functions, the parameters for these trig functions.
Duncan: No.

[Duncan, Ob #1, Lines 248–251]

Duncan was clear that review was the purpose of his inclusion of the trigonometric functions and his discussion of their characteristics as represented in the equations. Yet, as the class discussion evolved, students began to notice some possible connection between the amplitudes of the parametric equations and the shape of the rectangular graph. If the amplitudes were the same, it was going to be a circle, and if they were different, it was going to be an ellipse. While Duncan acknowledged those connections as they emerged, his description of the limited purpose (for review only) of the discussion of the characteristics suggested that he did not anticipate them in the planning of instruction. In the interview, he was asked directly about why he discussed the parameters of the sinusoids and left them on the board.

Interviewer: Were their other purposes in having them there other than review and having them for the windows—to help you determine the windows?
Duncan: Not so much, but the kids did make a connection when they asked about…when we got to the elliptical shape and we put them together, the kids started to realize, oh, if it was circular instead of elliptical, the amplitudes would be the same. I didn’t expect, I wasn’t thinking that they were going to make that connection but they did so I was glad that they were still up on the board.

[Duncan, Ob #1, Lines 283–292]

Furthermore, he left the consideration of the connection at a procedural level.

Duncan: Now, quick question: What if the amplitudes were the same? What if they were both 500?
Student: You would get a circle
Duncan: A circle. Which leads me right to my next example. A little segue. You like that?

[Daniel, Ob #1, Lines 636–652]

He asked the students the question, received a correct response, and moved directly into the next example without any discussion about why this might be the case. The level of the goals for the lesson and the expressed purposes of the examples he chose suggested a limited consideration of the mathematical concepts underlying parametric equations and conceptions students need to develop to understand them. Not surprisingly, Duncan’s consideration of the mathematics of the tasks he gave to students seemed to match the consideration he gave to the mathematics of the learning goals as the tasks chosen seemed to accomplish his goals. While Duncan demonstrated some strength of content knowledge about parametric and trigonometric functions, little evidence existed that he applied that reasoning to the planning of this instruction beyond a procedural level. The planning of this lesson appeared to involve primarily the identification of parametric functions with sufficient variety—one was in tabular form, one modeled populations and produced an ellipse, and one modeled circular motion. The tasks were not problematic nor did they direct the mathematical activity of students in ways that would foster the development of the underlying concepts.

Limited evidence existed for the application of mathematical reasoning in the planning of the second observed lesson as well. In that lesson, Duncan’s primary focus was on getting students to construct parametric equations to represent a given situation including ones in which the circle or ellipse needed to “start” at a particular point. Duncan used the term start repeatedly, and he delayed making any reference to a connection between this “starting point” and the values of x and y at t = 0 until late in the lesson. This connection would seem to be an essential element
of an understanding of the use of phase shifts to model situations accurately. Instead, students seemed to proceed with a trial and error approach initially followed by a teacher-led explanation of the use of a phase shift.

Duncan: There are a couple of ways to do this, but I like the idea of giving you guys a table and at time zero, I want to be at the position (0,12). [D constructs a table on the board with three columns: \( t \), \( x \), and \( y \) and fills in the first line with 0, 0, and 12]. If I am centered at the origin, I want to be straight up at twelve so you guys have to figure out how that is going to work.

When asked about this table representation, Duncan replied:

Duncan: None of them are taking my advice. Even when I was looking at them today, they’re not…I want them just to be able to think about…You know the question is do we use sine or cosine [pointing to the parametric equations for \( x(t) \) and \( y(t) \)]. They are used to thinking of \( x \) in terms of cosine and \( y \) in terms of sine, but they told me to write them that way. So I wanted them to say why and if somebody said, “I just tried it that way on my calculator and it worked.” Well, why did it work? Can you think about those values? But you know, they were not latching onto that idea, that concept.

After Duncan suggested to the students that they determine the starting point by thinking about manipulating the function to \( t = 0 \) to correspond to \( y = 12 \), he noted that “none of them are taking my advice.” He also acknowledged his desire to have students be able to explain why their approaches worked. Clearly, Duncan understood the mathematical relationships underlying the starting point. He understood how to manipulate the trigonometric functions in multiple ways to “start” at different points, and he expressed a desire for students to explain why a particular approach worked. He also recognized the natural tendency of students to focus more on trial and error approaches or the switching of the sine and cosine functions to accomplish some phase shifts. However, there was limited evidence, outside of this one reference to the conceptual underpinnings of the phase shift, that he applied that knowledge to the selection of tasks.
designed to help students develop an understanding of these underlying concepts and their relationships. While he presented a problematic scenario to students, the task did not direct the mathematical activity of the students in ways that supported the use of $t = 0$ in the determination of the appropriate parameters for the model. He simply asked the students to produce the function with a particular starting point and allowed them to use their calculators to perform trials until they found a solution. With the focus on producing the result, the mathematical activity of the students was limited largely to trial-and-error methods.

These excerpts from the observed lessons of Duncan revealed the limited application of mathematical reasoning to his planning of instruction. For Duncan, the learning goals for the lesson were introductory and procedural in nature. The selection of tasks involved identifying tasks that incorporated a review of previously learned content, but those existing understandings were not used to develop new understandings. In the first lesson, the tasks did not present any problematic scenarios for the students to resolve that would have fostered the development of some of the underlying concepts of parametric equations. In the case of the second observed lesson, the mathematical knowledge and reasoning of the teacher applied to the planning of instruction seemed to be limited to the selection of warm-up tasks asking the students to produce a graph with a different “starting point” and the clock problem which asked the students to model the motion beginning at a defined time. The tasks were problematic but did not foster the mathematical activity of the students in ways that would support the development of a conceptual understanding of the starting point. Duncan seemed to draw on his extensive accumulated knowledge of the mathematical background of the students, the previously studied functions and equations of the class, and the breadth of examples he understood students could
encounter in problems involving parametric equations. With more careful unpacking of the learning goals and the identification of component understandings, tasks could have more readily been selected that would have presented problematic scenarios to students designed to foster that mathematical activity that would support the development of the underlying concepts. The selection of tasks suggests, there was limited evidence of mathematical reasoning applied to this planning either in the unpacking of the mathematics of the learning goal, the mathematics of the students or the mathematics of the tasks.

**Summary of Evidence of Mathematical Reasoning During the Planning of Instruction**

Of the teachers in this study, the data from the observations and interviews with Jackie and Harold provide the most compelling evidence of the role of mathematical reasoning in the planning of instruction—namely, the unique nature of the interplay between the mathematical reasoning and pedagogical decisions as teachers maintain a triune focus on the mathematics of the students, the mathematics of the learning goal and the potential mathematical path from one to the other. The careful unpacking of the learning goals and the corresponding selection and sequencing of tasks provided evidence of the extent of the pedagogical content reasoning those teachers applied to the planning of instruction. They reasoned from the anticipated understandings of students as they considered the ways the students would approach the given tasks, what conclusions they may draw, and how those efforts would support the learning goals for the lesson sequence.

The data also provided support for the notion that differences in the application of pedagogical content reasoning during instruction can explain the differences in the nature of the instruction at critical instructional moments for the teachers in the study. The degree to which
Jackie and Harold unpacked the mathematics of the learning goal differed significantly from the work of Susan and Duncan. Jackie and Harold used those unpacked goals in the selection and sequencing of tasks in ways the limited unpacking of Susan and Duncan could not support. With the clear and specific goals for students in mind, Jackie and Harold demonstrated an ability to develop tasks and sequence those tasks in ways that met those clearly specified goals. Their specificity afforded these teachers the opportunity to target the development of more specific understandings with each task. They were able to structure those tasks as problematic—creating goals within the tasks to direct the mathematical activity of students in ways that fostered the development of the desired understandings.

For Susan, the chosen tasks and their sequence reflected broader goals. Those goals, while representing important aspects of an understanding of rational functions, were not as well specified. Particularly in the first observed lesson, the tasks were not designed and sequenced to develop specific understandings leading to instruction; they lacked focus and seemed to offer limited assurances that these students developed the understandings for which Susan strived. Likewise for Duncan, he chose tasks that reflected the introductory and procedural nature of the goals he had for students. Without the careful unpacking of the conceptual understandings underlying these procedural tasks, he had no basis for selecting or designing tasks to support the development of the component understandings and connecting those to existing understandings. This evidence suggested the more extensive the pedagogical reasoning at the critical instructional moments in the planning of instruction the more closely aligned the planning of instruction would be with the image of instruction derived from the literature and described in chapter 2. An analysis of the use of mathematical knowledge and reasoning in the planning of instruction for
Susan and Duncan revealed less extensive application of mathematical reasoning in the planning of instruction and seemed to serve as an explanatory factor for the differences in the planning of instruction outlined in the previous chapter.

**The Importance of Pedagogical Content Reasoning During the Implementation of Tasks**

Other data demonstrating the existence and importance of pedagogical mathematical reasoning were found throughout the classroom and interview data for the teachers in the study. Exploring the use of mathematical knowledge and reasoning during instruction at critical instructional moments involved a focused look at the teacher’s interpretation of student responses and responses to student responses to the given tasks. Through an analysis of the data at the critical instructional moments, four types of situations involving the interpretation of and responses to student responses were identified:

- A student was not able to provide a significant response,
- A student response was incomplete, inadequate, or inaccurate,
- A student gave an alternative but accurate response,
- A student demonstrated apparent readiness for a new idea.

The observed responses of teachers to these situations and their expressed thinking during the stimulated recall interviews not only revealed the active application of mathematical knowledge and reasoning during instruction, but also suggested the role that this pedagogical reasoning can play in potentially explaining the differences in the instructional approaches of the teachers in the study. Designing instructional interventions for students who expressed their thinking in these types of situations required the teacher to actively reason about the mathematics of the student as
expressed or demonstrated, the mathematics of the task or learning goal, and the potential mathematical path from one to the other that an associated task might encourage.

**Jackie’s Reasoning During the Implementation of Tasks**

In Jackie’s teaching at critical instructional moments, several examples of her application of mathematical reasoning were present in the data. Three different types of situations emerged that required her to apply her mathematical reasoning in the moment: situations in which a student provided an alternative, but valid response, situations in which a student provided an inaccurate or incomplete response, and situations in which a student demonstrated apparent readiness for a new idea. An analysis of the pedagogical content reasoning in these moments follows.

Several instances of a student providing an alternative, but valid response occurred in the data. An excerpt from the first observed lesson illustrated her responses to students in those situations. In the first lesson, Jackie elicited student ideas about how to find the missing measurements in a kite. The students had been given a kite with the two non-end angles marked as right angles, one angle marked as 54 degrees, and one leg of four as shown in Figure 5.3.

![Figure 5.3. The first lead task presented by Jackie](Jackie Ob #1, Line 24).
They were asked to find all measurements of angles, sides, and diagonals. In the stimulated recall interview, the researcher asked Jackie about an exchange with a student.

Interviewer: At this point, I think it was Marlee, that suggested that you draw BD.
Jackie: I didn’t want to draw BD yet, that’s why I put the new plastic on top [the teacher laid a clean overhead sheet on top of her work to do the work suggested by the student] but I didn’t want to discourage her either so I followed her train of thought and then I took that away so that we could come back to the more logical sequence which was to find this diagonal [teacher points to the vertex diagonal, AC] first and then use it to get the other angles.
Interviewer: Now, you pursued her line of reasoning to the bitter end.
Jackie: Pretty much.
Interviewer: So tell me, when she said, “Draw BD”, tell me a little bit about how you thought things might unfold.
Jackie: What I was thinking was, “Oh rats!” cause I wanted that to come later. But then I could…well we kind of hammered away at the perpendicular nature of the diagonals so I could see that maybe she was going there, creating right triangles, which I am all over. I want to do that. So that is why I wanted to pursue it. 

[Jackie Ob #1, Lines 113–129]

In this excerpt, Jackie indicated that the suggestion from Marlee to draw BD was not her preferred method for this problem. She acknowledged considering the effect on the student if she chose not to pursue it, weighing the importance of one of her broader goals for her class—empowerment of the students. During class she placed a clean overhead sheet on top of the diagram so that she could pursue Marlee’s line of reasoning and come back to her diagram in its current form after doing so. From her reflection on the moment, Jackie also understood the potentially productive nature of the approach, and acknowledged some mathematical benefits of pursuing work with the second diagonal.

This episode required Jackie to apply pedagogical content reasoning in several ways. While this student’s approach was not Jackie’s preferred path for the solution to the problem or the path that she would prefer students take, she certainly knew that this was a possible response.
from a student. As soon as the student suggested, “Draw BD,” Jackie anticipated the direction in which she thought the student would go and concluded that direction would be fruitful (re-emphasizing the right triangles). In this way, she could see the mathematical path to a solution. To do so, she had to reason mathematically using the student’s thinking to determine the instructional move—pursue or not pursue—weighing her learning goals for the students and determining a path to accomplish both. This exchange and Jackie’s reported thinking about it suggested that construction of her response to the student’s response of, “Draw BD,” involved more than a recall of accumulated knowledge. It required reasoning about the mathematics of the task, the mathematics of the student, her learning goals for students, and the potential path to accomplishing those goals.

Other instances in which students provided an alternative, but valid approach also occurred in the data. On several occasions during the three lessons, the students suggested using the laws of sines or cosines to find the sides or angles in a right triangle. Again, Jackie found herself weighing competing goals for students. On the one hand she embraced alternative strategies and encouraged students to understand their validity and how they connected and compared with other valid strategies. On the other hand, her learning progression for the three lessons depended on students focusing their attention on using the right triangle trigonometric ratios to ultimately see the association between the coordinates of a point on the plane and the angle a vector made with the positive x-axis. Jackie developed three different responses to students in an effort to affirm the validity of the method while attempting to generate a preference for using the trigonometric ratios.
In the first instance, a student had suggested they use the law of sines and the law of cosines to determine the side lengths in the kite problem. Jackie elicited other ideas from the class until someone suggested using the trigonometric ratios.

Sue: I would just use sine, cosine and tangent.
Jackie: Marlee, can I just do that?
Marlee: Yea.
Jackie: Why?
Marlee: Cause its basically the same thing.
Jackie: Kinda
Student: It’s a right triangle.
Jackie: So would you have gotten the right answer?
Marlee: Yes
Jackie: Would you have done way too much work?
Marlee: Yes.

As was typical during instruction, Jackie asked the students for their methods and often asked other students to consider the validity of those methods. This approach encouraged students to continue thinking about their approach. In this case, Susan used it as a way to ensure that her preferred method was considered by the students. Once both approaches were a part of the class discussion, Jackie asked the student who first suggested using the laws of sines and cosines to evaluate the use of the trigonometric ratios and assess the amount of work required by each method. The goal was to direct student attention to the use of the trigonometric ratios in right triangles—one of the key elements of her learning progression.

In the second observed lesson, Jackie focused the work of the class on a vector addition problem in which students were asked to determine the magnitude and angle of the resultant vector. They had found the coordinates of the endpoint of the resultant vector and Jackie began asking them about how they could find the magnitude and angle. One student suggested drawing a segment parallel to one of the vectors to form a parallelogram. Another student suggested
dropping a perpendicular. Once again, both methods were valid, but dropping the perpendicular supported the learning progression Jackie had identified. As Jackie pressed the students to explain the mathematical tools they would use to solve the problem in each case, the question once again came down to whether it was better to use the laws of sines and cosines or the trigonometric ratios.

Student: The law of cosines is probably easiest
Jackie: Seriously? Here’s what we are going to do: We are going to do a race. I want…
Before we race. What is the length of this blue line? (see Figure 5.4)
Student: 5.64

Figure 5.4. The working diagram from the vector addition problem

Jackie: What is the length of the line AB? (labeling A and B)
Student: 12.5
Jackie: All right, so, I want this side of the room to use sine, cosine and tangent ratios. Let me draw your triangle (teacher draws the two triangles below). This side of the room I want to use law of sines and cosines. Let’s just see who gets there first.

[Jackie, Ob #2, Lines 406–415]
When a student suggested that the law of cosines was easiest, Jackie responded, “Seriously?” suggesting surprise at the suggestion. She designed the race task in the moment to reinforce the notion that the use of the trigonometric ratios was the more efficient mathematical method.

A third occurrence of a student choosing to use the laws of sines and cosines in a right triangle occurred in the third observed lesson. After students worked independently on a task similar to the vector problem above, Jackie asked a student to share her solution. The student used the law of sines. Once again, Jackie affirmed the validity of this method, but reinforced the validity and preference for the use of the trigonometric ratios in a right triangle. In this case, Jackie used the student responses to show the equivalency of the two methods pointing out that the sine of 90 degrees is 1.

In all three of these instances, Jackie was faced with competing goals. She valued alternative methods and sought to empower students as mathematical thinkers and problem-solvers. If she rejected their alternative method, she risked undermining that empowerment. However, her learning progression for the three-lesson sequence depended on students thinking in terms of the right triangle trigonometric ratios, not the laws of sines and cosines. These competing goals for students required Jackie to apply her mathematical knowledge and reasoning to develop tasks that reinforced her broader learning goals for students while supporting the learning progression. In the first two instances, she demonstrated the greater efficiency in the use of the trigonometric ratios when compared to the application of the laws of sines and cosines. In the third instance, she demonstrated the equivalence of the two methods which demonstrated the extra steps involved. In all three instances, she worked with the expressed understandings of
students and considered the mathematics of the task and her learning goals as she sought to develop a mathematical path from those existing understandings to the desired ones.

The second type of situation that arose requiring Jackie to apply her mathematical reasoning in the moment involved students providing inadequate, incomplete, and inaccurate responses. Two particular instances stood out in the data. The first occurred in the second observed lesson as the class discussed one of the vector problems Jackie had situated on the coordinate plane. Jackie reminded students of the task that had been partially completed the previous day.

Jackie: You were going to find out for me how far to the right and up I needed to get on the coordinate plane to make the 70 degree angle work for a measure of 6. Right? So, what did you do?...Remember, we drew a triangle that was similar to it (Teacher draws a right triangle with a 70 degree angle) and we said what is the relationship to the sides and observed that we could use the trigonometric properties, sine, cosine, and tangent and talk about the ratio of the sides.

The problem involved a vector of length 6 at an angle of 70 degrees. Jackie reminded the class that they had dropped a perpendicular to form the right triangle. A student commented, “I didn’t know about the right triangle so I just put a different angle there” [Jackie, Ob #2, Line 14]. Jackie recognized the student’s confusion and responded by probing and ultimately designing a task to address the student’s inadequate conception.

Jackie: Didn’t I? If you are on the coordinate plane, and you are moving to the right and then up, does there have to be a right angle?
Student: no
Jackie: Really? If you are plotting an ordered pair (teacher sketches a set of coordinate axes) and I would like to plot the ordered pair, (2,3), how do I do it?
Student: Go over 2 and up 3.
Jackie: Plotting the points, over 2 and up three (going on a diagonal line).
Student: Straight up
Jackie: Straight up, so what kind of angle?
Student: Right angle.
Jackie: So there is a right angle here. So even though it wasn’t stated, because you are on the coordinate plane

[Jackie, Ob #2, Lines 66–77]

In this excerpt, Jackie first attempted to resolve the student’s confusion by reminding him that the vector was situated on a coordinate plane and asking, “does there have to be a right angle?”

When the student maintained his conception and said no in response, Jackie asked him to plot an ordered pair. She plotted the points on the board as he told her to go over 2 and up 3. Once she began going up on a diagonal line, the student corrected her and ultimately corrected his own misconception.

In this exchange, Jackie constructed two responses in the moment. The first clarified that the student indeed had not applied his conception of the coordinate plane accurately to this problem. The second involved a task that required the student to apply what he understood about the coordinate plane in a way that resolved the confusion over the implied presence of the right angle. In doing so, Jackie had to develop an understanding of the missing conception of the student and design a task that would allow the student to resolve that missing piece. In other words, she had to unpack the mathematics of the student and construct a mathematical path from those expressed and anticipated understandings to the more robust understandings that were required for this vector problem. That required mathematical reasoning about each the mathematics of the student and the mathematics of the task.

A second instance of a student providing an inadequate, incomplete, and inaccurate response is also worth discussing. As the students were working with Jackie to complete the vector task—the problem involved a vector of length 6 at an angle of 70 degrees—discussed previously, a student expressed confusion over the use of the trigonometric ratios.
Student: I have a question about the cosine thing. When you are talking about the adjacent side over the hypotenuse, what if you were trying to find like the 90 degree angle or the side opposite that?

[Jackie, Ob #2, Lines 115–118]

The excerpts from the subsequent class discussion and the stimulated recall interviews revealed that Jackie did not initially understand the student’s question or confusion. However, the series of exchanges demonstrated Jackie’s efforts to understand student thinking and construct responses to develop more robust understandings.

In her first attempt to resolve the student’s confusion, Jackie introduced a different problem.

Jackie: I think I understand your question. Let me do another example and see if I do. (teacher gets a blank overhead and draws a right triangle, as shown in Figure 5.5).

![Figure 5.5. Jackie’s task created to address student question](image)

Jackie: Suppose I don’t know this [pointing to the hypotenuse]. Let’s call this $c$ and keep our 70 degree angle. Suppose I do know this side and suppose it is 8. Could I use sine, cosine or tangent to get $c$?

[Jackie, Ob #2, Lines 128–132]

Jackie constructed this right triangle with the missing hypotenuse based on her understanding of the student’s understanding at the time. As she stated in the stimulated recall interview, “I thought she was asking, ‘if I had a right triangle, not a parallelogram, how would I find the
missing sides’ ” [Jackie, Ob #2, Lines 150–151]. When the student did not quickly provide an accurate response to her question about using sine, cosine, and tangent, Jackie directed student attention to the general approach they had taken with problems throughout the course.

Jackie: Okay, here’s what we have been doing, really all year. In a diagram like this, we have been trying to glean facts from this and put them in some equation that states a true relationship among the facts, right? Some of the facts we know and some we don’t. So, following that same pattern, I want an equation that we have established as valid that has 8, 70, and c because if I had that equation, I could solve for c.

[Jackie, Ob #2, Lines 142–146]

The student realized that the task was not exactly what she was asking about and restated her question to Jackie.

Student: I was thinking what if you had something like cosine 90 equals and you would have to have the adjacent over the hypotenuse. How would you do that because the c would be the hypotenuse and the opposite.

Jackie: Yea, I see what you mean. So maybe I have to have a point of view that is not based at angle C. I really don’t always have to be staring across at the side I want. I could be standing here or here (teacher points to the vertices of the non-right angles).

[Jackie, Ob #2, Lines 176–181]

Again, Jackie made an attempt to understand the student’s thinking and to address her confusion by directing her attention to the idea that you do not have to be “staring across at the side I want.” A student offered an equation using the trigonometric ratio that could have been solved for c in the Figure 5.5 and again, Jackie probes the student to determine whether the confusion has been resolved.

Student: What you do is sine 70 equals 8 over c.
Jackie: [teacher writes the equation and directs question to the student with the initial confusion] Is that what you had a question about?
Student: No, but, if you only have the 90 degree angle and sine 90 equals 1 so c over c the hypotenuse.
Jackie: Okay, let me do that. (teacher draws a right triangle with a hypotenuse of c and leg of 8) Can I find c?

[Jackie, Ob #2, Lines 182–187]
When the student indicated that her confusion remained unresolved, Jackie constructed another
task based on what she understood the student’s thinking to be. Finally, a fellow student
explained the student’s confusion.

Student: I think I see what she is saying.
Jackie: Tell me.
Student: She’s saying since sine 90 is one, it would be sine 90 equals c over c because the
c would be opposite and the hypotenuse.
Jackie: Ohho. Those ratios only work when you are standing at one of the not right
angles. The law of sines and cosines, you could be standing anywhere. Generally, we
don’t even use those when we have a right angle. But if I wanted to use the opposite
over the hypotenuse, I am talking about angle A or angle B, I am not talking about
angle C. I just totally misunderstood your question. That was your question too,
right? I can put myself anywhere on the triangle I want and I am going to stand on a
vertex that is not the vertex of a right angle.

[Jackie, Ob #2, Lines 190–200]

The student described the confusion as a question about how to apply the definition of sine if you
were using the right angle given that the hypotenuse would also be the opposite side. Jackie put
her explanation in terms of the ratios only working “when you are standing at one of the not right
angles.” The student indicated that this explanation resolved her confusion.

Throughout this lesson segment, Jackie made repeated attempts to develop a task in
response to her conceptions of the student’s thinking, the mathematically productive conceptions
required to solve the problem, and the potential of the task to highlight the salient features of the
mathematics needed to resolve the confusion. Even though Jackie did not understand the
student’s confusion until the end, the episode revealed her repeated efforts to understand student
conceptions and to develop tasks to address any weaknesses in those conceptions. To do so
required Jackie to reason mathematically about the student’s conceptions that were not
adequately expressed and to construct tasks designed to address the student’s confusion and
develop the types of productive understandings required by the task and ultimately, would
support the learning goal for the lesson. Once again, Jackie’s thinking revealed a triune focus on the mathematics of the student, the mathematics of the task or learning goal, and the potential mathematical path from one to the other. Such thinking required an active application of mathematical knowledge and reasoning to design the instructional intervention—in other words, pedagogical content reasoning.

The third type of situation that arose requiring Jackie to apply her mathematical reasoning in the moment involved situations in which students demonstrated an apparent readiness for a new idea. In light of the careful unpacking of the mathematics of the learning goal and the well-reasoned selection and sequencing of tasks, Jackie seemed to successfully interpret student thinking and introduce modified or new associated tasks to introduce new dimensions of the ideas under study. The most compelling example occurred in the third observed lesson as Jackie introduced students to the idea the three lesson sequence was designed to develop—the association between the coordinates of a point on a ray and the angle the ray forms with the \( x \)-axis. A description of the progression of the lesson is followed by a discussion of the pedagogical content reasoning involved in delivering this instruction.

This portion of the lesson began with a review of the vector problem introduced the previous day and discussed earlier in this section. Students were initially asked to find the coordinates of the endpoint of the resultant vector for a vector of magnitude six at an angle of seventy degrees and a vector of length eight along the positive \( x \)-axis. The students readily found the coordinates as this task represented the culmination and synthesis of much of their work for the two previous days as they worked with parallelograms and triangles using triangle trigonometry in problems on the coordinate plane involving vectors. Jackie introduced the next
iteration of the problem by simplifying the diagram of the vectors and parallelogram with which they had just been working.

Jackie: Now I want us to look at this triangle that I am going to call ABC in a minute (teacher traces the triangle with a squiggly line). And I want you to get the angle here (see Figure 5.6).

![Figure 5.6](image)

**Figure 5.6. The original task showing the triangle of interest, ΔABC**

Student: The angle?
Jackie: So, I am going to trace this (teacher overlays a transparency to trace triangle outlined with squiggly line in Figure 5.4; The trace is shown in Figure 5.7). There is a lot of stuff on here that we do not need. We need this 12.6 and 3.9. So what is the length of this side?

![Figure 5.7](image)

**Figure 5.7. The traced version of ΔABC from Figure 5.6**

Student: x
Jackie: 12.6 And this side?
Student: 3.9
Jackie: Can you get me this angle? Do it. Write an equation.

[Jackie, Ob #3, Lines 182–194]
As described in this passage, the task involved finding the angle and Jackie simplified diagram to eliminate the distractions of the now unrelated portions of the diagram. She could have introduced an entirely new task, but instead she kept it connected to the current work of the students by simplifying the existing diagram. The choice of introducing this associated task in this way had several effects. First and most obviously, the simplification of the diagram made the problem itself relatively simple and familiar to the students. They were readily able to produce the angle using the trigonometric ratio. Second, the simplification of the parallelogram diagram seemed to create a natural connection to the work of the previous 2 days. As mentioned, the original task involved a synthesis of everything they had been working on the past 2 days. While it was unclear exactly what role this played for students, some dimension of this connection seemed important. Third, the diagram simplification seemed to focus the students’ attention on the aspects of the task that were most salient for the next step in the learning progression.

From this simplified diagram, the students determined the angle and the magnitude of that resultant vector. With the mathematical calculation completed, Jackie introduced an alternate version of the task using the same numbers while slightly altering the task.

Jackie: I want to try an experiment. What if there was no triangle (teacher draws the Figure 5.8), and I wanted to find alpha.
In this version, Jackie used the same vector but redrew the diagram without the triangle. Again, the slight modification seemed to serve multiple purposes. First, the diagram, without a triangle, presumably made sense to the students because the original problem involved vectors. Essentially, this ray represented a vector drawn to the point (12.6, 3.9), and by connecting the situation to vectors, Jackie seemed to remind the students of the concept of angles measured with the positive $x$-axis. Second, the diagram itself showed only an angle and the coordinates of the endpoint—the two quantities Jackie sought to help students associate. Finally and most importantly, the mathematical calculation of the angle is identical to the previous problem and Jackie used that parallel calculation to support the association of the coordinates with the angle, a point she specifically referenced in a re-statement of the task.

Jackie: I want to make myself clear. I want you to look at this lovely equation (teacher boxes in the equation involving tangent from the previous problem) and I want you to tell me if there is a way to get that equation up here (pointing to the simplified diagram) without creating an imaginary triangle?

[Jackie, Ob #3, Lines 355–359]
Jackie directed student attention to the equation involving the tangent ratio from the triangle version of the problem and asked them to re-produce that equation without thinking of a triangle. This prompt led to the students developing the formula associating the coordinates with the angle.

\[ \tan \alpha = \frac{y - \text{coordinate}}{x - \text{coordinate}} \]  

[Jackie, Ob #3, Line 379]

Jackie reinforced the association by asking, “If you think of it that way, do you need the triangle at all?” [Jackie, Ob #3, Line 380].

Jackie knew that she wanted to introduce students to the association of the coordinates to the angle during this lesson. As discussed previously, she believed this desired conceptualization laid a foundation for understanding the unit circle, trigonometric functions, and polar coordinates. She developed a sequence of tasks, each derived from the previous task, designed, as Jackie described it, to “move very gradually away from that triangle” [Jackie, Ob #3, Lines 272–273]. This sequencing of tasks was deliberate. It evolved from Jackie’s careful consideration of what she anticipated to be the existing understandings to students. It progressed in incremental ways as Jackie selected or designed tasks to connect with previous tasks while also holding the potential to develop the desired association between the coordinates and the angle. It required simultaneous attention to mathematical development and pedagogical decisions. Mathematically, Jackie had to anticipate how her students would engage in completing each task and design each task to motivate productive mathematical activity. Pedagogically, Jackie had to consider how to structure and connect the tasks in ways that would support the mathematical development choosing to use simplified portions of the diagrams and
the same numbers from the previous tasks. These approaches demonstrated the application of mathematical knowledge and reasoning to the implementation of tasks in both the interpretation of student responses and the construction of responses to those student responses.

This excerpt from the third observed lesson also demonstrated the effectiveness of the sequence of tasks. Not only were students able to generate the formula representing the association between the coordinates and the angle, they demonstrated their understanding of two distinct implications of this new association. After Jackie reminded the students that angles are simply a measure of a rotation and she explicitly stated that they were now going to define every angle as one that starts at the $x$-axis, she asked the students what that meant about the size of angles. They were almost universally clear that they could be bigger than 180 degrees. When Jackie gave them the coordinates $(5, -3)$ and asked them to find the angle, they accepted the notion that angles could be negative with almost no discussion and explained that the negative meant that “it is going downwards” [Jackie, Ob #3, Line 441]. The ease with which the students drew these conclusions and demonstrated their understanding of these implications of the new definition of angle provided ample evidence of the success of the lesson in developing their understanding of this association between angles and the coordinates. That success reflected Jackie’s effective interpretation of the student’s readiness for these new ideas and the thoughtful construction of incrementally sequenced tasks to support the development of those understandings. Each of these dimensions of her delivery of instruction required the active application of her mathematical knowledge and reasoning—pedagogical content reasoning—to the implementation of tasks.
As just discussed, three types of situations requiring the interpretation of student responses and the construction of responses to student responses appeared in the data from the observed lessons of Jackie: situations in which a student response was incomplete, inadequate, or inaccurate, situations in which a student gave an alternative but accurate response, and situations in which a student demonstrated apparent readiness for a new idea. In each of these situations, Jackie demonstrated a thoughtful application of pedagogical content reasoning during instruction. When a student offered an alternative but valid approach, she demonstrated her ability to recognize the mathematical validity and determine an instructional approach that would support the attainment of her instructional goals even when they were somewhat in conflict. When students offered incomplete, inaccurate, or inadequate responses, Jackie demonstrated her commitment to understanding the mathematics of the student and designing an instructional intervention to support the development of more robust understandings. When students demonstrated a readiness for the next step in the learning progression, Jackie designed well-sequenced and incrementally progressive tasks to support the development of the desired understandings. Each of these situations required Jackie to maintain a triune focus on the mathematics of the student, the mathematics of the learning goal and the potential path from one to the other. This focus required mathematical reasoning to make instructional decisions.

Harold’s Reasoning During the Implementation of Tasks

Like Jackie’s approach, Harold’s approach to the classroom in large measure determined the types of situations requiring mathematical reasoning he was likely to encounter. However, his tendency to focus on narrow indicators of understanding and the precise use of language did not afford students many opportunities to offer alternative approaches that were acceptable to
him. Rather, Harold’s emphasis on the narrow indicators of understanding led to more situations in which a student was not able to provide a significant or acceptable response or ones in which a student response was incomplete, inadequate, or inaccurate. While the types of situations that arose were different for Harold and Jackie, these situations, as was the case for Jackie, placed demands on the mathematical knowledge and reasoning of Harold as he sought to understand student thinking and construct responses to student responses.

As discussed in the previous chapter, Harold’s narrow indicators of understanding often curtailed his efforts to understand the conceptions of students. In these cases, Harold’s application of his mathematical knowledge and reasoning was focused on identifying what was wrong with a student’s thinking relative to the mathematics of the learning goal and constructing a response to re-emphasize his path to getting there. In other cases, Harold maintained a triune focus on the mathematics of the student, the mathematics of the learning goal, and the mathematical path from one to the other as he constructed responses to student responses. Examples of each of these are discussed in this section.

Harold’s reflections during the stimulated recall interviews and the responses to student responses in the first Math 6 lesson informed the understanding of his use of mathematical reasoning in situations in which students were not able to provide a significant response. As discussed, Harold presented students with a task intended to lay a foundation for the division of fractions. The initial lead task involved representing $a \div b$ as $a \cdot \frac{1}{b}$. A summary of the sequence of associated tasks provided in the previous chapter is also provided here. Each task was constructed in response to the difficulty the students demonstrated in expressing the specific, component understanding Harold hoped to help students develop.
As discussed in the previous section, Harold gave several indications in the class as well as in the stimulated recall interviews that the difficulty of students demonstrating their understanding of this initial component understanding was unanticipated. The unanticipated nature of the difficulty suggests that the use of these associated tasks was unanticipated and required him to construct those tasks in the moment. Such a construction required some level of mathematical reasoning about the nature of the anticipated or expressed understandings of students and the mathematics of the learning goal as reflected in the component understanding he sought to develop. The work of the class related to these tasks and Harold’s commentary on it from the stimulated recall interview served to support this claim.

As discussed in the section on his responses to student responses, the two primary features of the pattern in Harold’s responses to students’ responses were his facility with multiple representations and the shifts in structure and complexity represented by each new associated task. Each of these associated tasks required a consideration of the demonstrated understandings
of the students and the conceptions Harold targeted for development. While many of the student responses in this section provided Harold with little additional insight into their thinking, Harold expressed his understanding of their understanding on several occasions. That understanding was informed by their inability to construct a suitable response (from Harold’s perspective) to each of the tasks presented to them, except for one. As the data suggests, the construction of these associated tasks was based in large measure on this understanding of their understanding.

After the students were unable to provide a significant response to the lead task, Harold provided them with an associated task using whole numbers.

Harold: What is another way to do 6 divided by 2 [H writes the expression shown in Figure 5.9 on the board]. You know it is three, you know what it should equal, but I am not asking that. I am asking what is another way to express it.

![Figure 5.9. The original statement of the task](image)

Student: Six times 1 over 2
Harold: [H writes the student response, shown in Figure 5.10 on the board]

![Figure 5.10. The completion of the task offered by a student](image)

Harold: Is that true?  

This example could represent drawing on his knowledge of content and teaching (Ball, 2008) and what he understood about the students’ ability to understand concrete examples with simple numbers more readily than abstract ones. If this were the case, the creation of a numerical
example for this context suggested an application of that knowledge in the moment. In the discussion about this move to a numerical example, Harold described what he thought the students understood at this point in the lesson.

Interviewer: Did you think they all understand that [pointing to figure 5.10]? Harold: [Long Pause]. I did. But, I don’t think they understood it as six halves added together. I think they understood it as half of 6. I think they are pretty good with dividing something by 2 is the same as taking half of it. I am convinced they are good with that, but I have really not done much work with taking six halves and adding them together because frankly I thought it was easier and I didn’t think it would be a problem. But clearly, it was.

[Harold Ob #1, Lines 176–184]

This discussion suggested careful thought about the existing understandings of students as Harold presented this task and suggested that the numerical example was chosen as a result of this consideration of the understandings of these particular students. In other words, he reasoned from his understanding of their understanding to develop a mathematical example that could support the advancement of their understanding along the learning progression he outlined.

However, it is important to note that he made a number of assumptions about the nature of the student conceptions since their responses to the task provided little evidence of those conceptions.

In spite of the apparent progress towards the learning goal from the numerical example, the students provided a lukewarm response to the re-presentation of the algebraic model. In response, Harold asked, “What’s a fourth of twelve?” [Harold, Ob #1, Line 261]. When asked about the choice of this task, Harold explained the purpose of this task and how it could support the development of their understanding.

Harold: I was pretty sure that they were good with this. I was pretty sure they were okay with taking a fourth of twelve. I didn’t think they would get stuck on this. So, if they can take a fourth of twelve, we have gone over the commutative property, again, they
are pretty good with that, so we should be able to move from this to twelve quarters. No problem. Get three wholes, no problem. I am still thinking we are going to be able to recover this. So I am still trying to recover.  

[Harold, Ob #1, Lines 265–273]

In this description of his thinking, Harold indicated confidence in their ability to complete this task and described a mathematical path from their existing understanding to “twelve quarters” and then to “three wholes.” Such a path would have completed the model of the relationship representing the first component understanding of his learning progression: \( a \div b \) as \( a \cdot \frac{1}{b} \). This description reflected the mathematical reasoning that went into the construction of this associated task—his reasoning about the mathematics of the student (what they already seemed to understand), the mathematics of the learning goal (the component understanding he wanted them to model and internalize), and the mathematical path from one to the other (the way he anticipated students completing the task).

The students were again unable to provide an acceptable response and this difficulty led to a sequence of his highly teacher-directed sequence of exchanges:

— Harold asked in a directed way, “If we want to express this in terms of the definition of multiplication, could we have twelve groups of a fourth?” [Harold, Ob #1, Lines 310–311].

— Harold then asked, “What would twelve groups of a fourth look like?” [Harold, Ob #1, Line 313].

— Harold modeled one-fourth for them with a bar model with one-fourth shaded, and then asks, “Let’s say we have twelve of these. What’s that equal?” [Harold, Ob #1, Line 339].
— He answered the question for them saying, “Twelve quarters” [Harold, Ob #1, Line 342], and

— He proceeded to draw more versions of the model for one-fourth and say, “If we have twelve of these, what’s that equal?” [Harold, Ob #1, Lines 447–450].

This portion of the lesson illustrated the difficulty Harold had in engaging students to think about the situation in the way(s) that supported the development of the component understandings he identified in his learning progression. This difficulty can be explained in terms of some of Harold’s reasoning about the mathematics of the students. Their difficulty would suggest that Harold’s assessment of their understanding was that it was more robust than it actually was. He based his assessment primarily on the assumptions he made about the nature of their existing conceptions rather than from their responses to tasks or expressions of their thinking about the tasks. In addition and as discussed previously, the tasks up to this point in the lesson were devoid of context and therefore limited the ways in which they could engage their existing understandings. This factor also seemed to be a case of overestimating the quality of their existing understandings and their ability to reason about tasks in the abstract.

When asked about his choice of representation of the task that asked students to model one-fourth of 12, Harold explained in more detail what he was looking for and alluded to productive and unproductive conceptions demonstrated by students.

Harold: I wanted them to count the number of quarters they have and this is how many wholes I have. They have to recognize that that is the question we are talking about. I have twelve quarters of a whole and I am multiplying it, then the product is the total number of wholes that I have. And a bunch of them ended up, you know, once we finally got there, basically drawing a line or a circle around four of them and saying, “One whole, two wholes, three wholes.” In fact, one young lady did exactly that. But I was getting so many twelve forty-eighths that I knew there was something wrong.
Interviewer:  So, what were they doing to get to twelve forty-eighths?

Harold:  They were making the model, but they were only using the model to get equivalent fractions. They were only kind of doing what they had done before.

[Harold, Ob #1, Lines 395–409]

This passage reflected a deeper consideration of student thinking than was present during the other parts of the lesson. Not only did he articulate a well-reasoned solution path—thinking of how many wholes and realizing that the product will give you the number of wholes—but he also made observations about their work with the task. He noticed many were getting 12 forty-eighths, and he reasoned that “they were using the model to get equivalent fractions.” In Harold’s assessment, the visual model he chose for the task, “one-fourth of twelve” contributed to the student confusion during the lesson. In response to this consideration, he moved to the use money as the context of the task and a quarter to represent the fourth.

This is basically the model we used to represent that one out of four is the same as two out of eight and is the same as three out of twelve, etc. And sure enough, that is how the vast majority of them approached it. And that is kind of why I went to the actual quarter. I have twelve of these, how much do I got. That’s why I went to that model later.

[Harold, Ob #1, Lines 380–385]

Harold expressed a keen understanding of student thinking on this problem and the role the chosen model played in supporting that. He recognized the error in their thinking and designed a new task to directly address the weakness in their approach. These conceptions were not anticipated prior to the class or the use of the example. Thus, Harold’s understanding of them had to come in the moment and his response to them—his decision to use the quarter as a model—came out of an active, mathematical consideration of his understanding of their conceptions and a well-reasoned path to get from those conceptions to the desired understandings.
These examples from the first Math 6 lesson revealed Harold’s application of his mathematical knowledge and reasoning in situations in which a student was not able to provide a significant response. The other type of situation that occurred prominently in the data involved a student or students providing responses that were incomplete, inadequate, or inaccurate. Two episodes from Harold’s Algebra I classes stood out in the data.

The first centered around a student’s response to the question, “What makes something a solution to a system?” [Harold, Ob #2, Lines 55–56]. The student stated that a solution to a system was something that “makes both equations true” [Harold, Ob #2, Line 126]. Recognizing the limited applicability of this response and the inadequacy of it for the solution to this system of inequalities, Harold directed attention to those inadequacies directly, “We don’t have two and they aren’t equations so we know your definition can’t make sense, right?” [Harold, Ob #2, Line 139]. In response, the student provided an accurate definition that applied only to inequalities, and Harold responded,

Harold: Could you have an equation thrown in here?
Student: Yes, you could
Harold: Do you guys know the word I use to describe equations and inequalities?
Student: Statements
Harold: Statements. I use it very deliberately. By saying statements, you can cover whether it is an equation or an inequality.

[Harold, Ob #2, Lines 148–153]

Harold introduced the word, “statements” so that the working definition of what it means to solve a system had the broadest possible applicability. To reinforce this broad applicability and develop the understanding of it in students, Harold created the following associated task.

Harold: Here, let me show you something. I could give you this system (Harold wrote the system shown in Figure 5.11 on the board):
Harold: And add this to it (Harold added the two equations shown in Figure 5.12)

Several elements are of interest here. First, Harold was able to produce an example, in the moment, of a system of equations and inequalities that reinforced his broad approach to the definitions and strategies for solving systems of equations. He created a counterexample of sorts to demonstrate the inadequacy of the student’s limited conception of a solution to a system that also reinforced the broad applicability of his focus on solutions.

Second, even though he did not carefully reason to create an example that had a solution, he reasoned in the moment to recognize that this system did not have a solution. He also
understood that even without a solution, it was still going to support his broader goals. He claimed that he realized “right after I wrote it” [Harold, Ob #2, Line 196] that there was not a solution but that he was “okay with that” [Harold, Ob #2, Line 198]. Further evidence of his sound mathematical reasoning in the moment was found in the question posed to students, “What is the only possible solution to it?” [Harold, Ob #2, Line 183]. With this question, he was able to turn the unintended “no solution” into an instructional moment that supported his goal of developing a broad understanding of solutions to systems.

Third, due to some ambiguity in how he wrote \( y = 4 \), some students interpreted that as \( y = x \). When asked, he immediately turned this interpretation into another example for the class to consider. Again, Harold recognized on the spot that the system would still have at most one solution as he asked, “If that was \( y = x \), what’s the only possible solution?” [Harold, Ob #2, Line 208]. This example emphasized his focus throughout the lesson on solutions—a more conceptually rich approach to solving a system of equations or inequalities—rather than on a process of solving a system by graphing. It also demonstrated his in-the-moment reasoning about the mathematics of the task and the mathematics of his learning goal as he developed this associated task to support the development of more robust conceptions of solutions to systems.

A final example of Harold’s responses to mathematically inadequate responses can be found in the discussion regarding how to shade the graph to represent the solutions to an inequality. While Harold emphasized the pointwise analysis to determine shading throughout the discussion, students offered various alternative short cuts. One student offered the idea that if the inequality symbol is a less-than symbol, then you shade below.
Harold: Let me show you the danger in memorizing that “if it is less than, you shade below.” Let’s suppose I wrote this equation [he meant inequality] backwards. [H writes the inequality and shown in Figure 5.13].

![Figure 5.13. The inequality in reverse order](image)

Harold: Same equation. Is this the exact same inequality, sorry, not equation?
Student: No.
Student: Yep.
Student: No.
Student: Yes.
Student: Yes.
Harold: It’s written in reverse, right? I have seen students memorize that when it is greater than, you shade above. So guess what they did on this exact same equation. Sorry inequality.
Student: Shaded above.
Harold: Guys, you can’t memorize that stuff. As with everything this year, you gotta understand why it’s true.

[Harold, Ob #3, Lines 806–821]

The choice of counterexample seemed to be particularly compelling in light of the fact that the students knew the appropriate shading, could easily recognize the relationship as the same, and that the rule, “shade below,” would not yield the same result. Once again, Harold demonstrated his in-the-moment, mathematical reasoning in response to a student who provided an mathematically inadequate response. In light of the revealed understandings of a student through his hypothesis about when to shade below the line and Harold’s understanding about the mathematical relationships he wanted the students to understand, he produced this counter-example to the student hypothesis that directed the student’s attention to the inadequacy of his thinking.
These examples demonstrated Harold’s ability to construct or modify tasks in response to student responses that were mathematically inadequate. When that ability was coupled with a careful, mathematical consideration of the existing understandings of students and the tasks were constructed with those understandings in mind, Harold seemed to effectively advance the understandings of students. His responses cannot be completely accounted for as recall of accumulated knowledge or application of a pedagogical strategy. Rather, they reflect an active application of mathematical reasoning to a triune focus on the mathematics of the student, the mathematics of the learning goal and the mathematical path from one to the other.

Susan’s Reasoning During the Implementation of Tasks

In contrast to the data from Jackie and Harold, Susan’s pedagogical content reasoning during the implementation of instruction did not always produce instruction that was aligned with the image of instruction presented in chapter 2. Some pedagogical content reasoning was may have been masked by the inconsistency observed in Susan’s elicitation of student thinking, her interpretation of student responses, and her responses to student responses. As discussed previously, students offered responses to tasks, asked questions when confused, and made conjectures about relationships at various times throughout the three lessons. However, those responses often left an incomplete picture of their thinking, and Susan frequently chose not to ask students to provide additional explanation, justification, elaboration or clarification. In many cases, she provided the additional mathematical clarity herself, and in other cases, she left a limited student response or conception unexplored.

The implementation of one of the lead tasks from the first observed lesson provided several rich examples of the opportunities Susan had to apply her mathematical knowledge and
reasoning during instruction and the role her mathematical reasoning could have played. This episode from the first observed lesson provided evidence of a number of elements of the extent and nature of the mathematical reasoning Susan applied during instruction. Three of the four types of situations that required the application of mathematical knowledge and reasoning on the part of the teacher appeared in the data:

- situations in which a student was unable to provide a significant response,
- situations in which a student response was incomplete, inadequate, or inaccurate, and
- situations in which a student demonstrated a readiness for a new idea.

These situations manifested themselves throughout this 25-minute portion of the lesson. Susan asked a number of questions of students that they had limited basis for answering, creating situations in which a student was not able to provide a significant response. Student thinking was expressed, often incompletely, inadequately, or inaccurately, but not fully explored. Students offered potentially useful comments, observations, or questions that went unused. They offered hypotheses that went unresolved. In short, the mathematics of the students was left largely unpacked, the mathematics of the learning goal was left largely unexplored, and the mathematical path from the perceived existing understandings to desired understandings was often left unconsidered.

After introducing the definition of a rational function (and Susan’s interpretation of it), Susan presented students with two rational functions, each being the reciprocal of the other and asked the students to explain whether each was a rational function. The questions were introduced and discussed separately.
Susan described the purpose of the task in the stimulated recall interview.

Susan: The notion that if the original ratio is something that simplifies so that the denominator is constant, then you no longer have a rational expression. That was the point of that question. We looked at the definition of a rational function and I wanted them to see that it doesn’t just mean a polynomial divided by a polynomial, you have to look more closely.

Susan expressed that she designed the task only to apply the definition and for students to know that it was not as simple as looking at the function to see that it was a ratio of polynomials. As such, the task was accessible to students. Under Susan’s interpretation of the definition, the students simply had to factor the polynomials, cancel the common factors, and see if the expression was still a ratio of polynomials with a nonconstant denominator. However, the discussion of this task led into a range of other related understandings—the notion of functional equivalency versus algebraic equivalency, the algebraic, graphic, and numeric manifestations of holes and asymptotes, and the identification of restrictions that caused holes and those that caused asymptotes. As discussed in the previous section, Susan neglected to fully unpack the mathematics of this task and this lack of application of her mathematical reasoning manifested itself in the implementation of the task in a number of ways.

Susan’s initial presentation of the problem involved the suggestion that the students simplify the rational expression by canceling the common factor.

Susan: Oh, so what is this equivalent to [referring to the left side of the equals sign in Figure 5.14]?  
Student: x-1
Susan: Except it isn’t exactly equivalent to it. It isn’t exactly equivalent to it. [pause]
Why? Why is this [pointing to \( x-1 \)] not exactly equivalent to what we started with?
Student: [student initiated an inaudible response but did not finish]
Susan: Think about this: are there any restrictions to what \( x \) can equal [pointing to the original function]?

Several elements of this initial engagement with the task were noteworthy. After a student offered a successful factorization of the numerator and suggested canceling the common factor, Susan used the word “equivalent” when asking the students to produce the simplified version. She immediately asserted, “Except it isn’t exactly equivalent to it,” asked the students “why?” and then she interrupted a student response to suggest they think about the restrictions on \( x \).

In this exchange, Susan introduced the notion of functional equivalence in a situation in which there was algebraic equivalence of the two expressions. While her interactions with students in this excerpt did not suggest it, Susan expressed an understanding for this application of equivalence in the stimulated recall interview.

Susan: …but if I just looked at this original equation and said, “How many solutions are there? Okay, an infinite number.” But we do know that two of those solutions don’t exist. So our domain is everything except that. But if we look at the simplified form, what ordered pairs satisfy that. Well, we get one more, even though it is an infinite set, that satisfies this [pointing to the simplified function] that doesn’t satisfy this [pointing to the original function] and that is why they are not equivalent.

Susan adeptly described the essential elements of the mathematical relationship between these two related functions and the ordered pairs they represented. Yet, she did not apply that mathematical understanding to the implementation of the task. Her suggestion that the
expressions were equivalent and not equivalent presented a level of cognitive dissonance to the students. That discontinuity in the use of the word, “equivalent,” could have motivated the mathematical activity of the students in ways that would have supported the development of an understanding of the algebraic manifestations of restrictions that produce holes and the difference in those manifestations with restrictions that produce asymptotes. However, the discontinuity went largely unresolved during the observed classes.

Instead, in the subsequent exchange, Susan elicited the restrictions from the students, noted that there were no restrictions on the linear expression, $x-1$, and asserted that the two functions were not equivalent.

Susan: Are there any restrictions on this [pointing to the expression, $x-1$]? [pause, no responses] None. That is a linear expression. Okay, so they are not exactly equivalent. This simplifies to a linear function. That’s why we say it is not a rational function.

[Susan, Ob #1, Lines 575–577]

Susan elicited the identification of the restrictions on the original function from students. The remainder of the mathematical reasoning was provided by Susan. With this response, she did not offer the students an opportunity to resolve the dual notions of equivalence. She could have presented to students an associated task to support the application of the concept to functions or to help students differentiate between the two notions of equivalence. A task asking the students to directly compare and contrast the graphs of the original function and the simplified version could have served such a purpose. In short, the lead task for this portion of the class was introduced with a limited, expressed purpose, limited unpacking of the mathematics required to complete the task and limited consideration of associated tasks designed to support the development of the required understandings. The implementation of the task involved missed
opportunities to elicit expressions of student thinking and to motivate the need for the mathematical activity that could support the development of a richer, more connected understanding of the algebraic and graphical representations of the function.

Questions about the extent and nature of the mathematical reasoning Susan applied during instruction continued to arise as the data from this section of the class was analyzed. When exploring the second rational function to determine whether it was rational, Susan asked the students whether they expected to get a vertical asymptote at each of the restrictions.

Teacher (to class): Will we get vertical asymptotes in both of those places on the graph? [pause] We know \( x \) cannot be one and \( x \) cannot be negative three. Will we get vertical asymptotes in both of those places? [long pause]
   Interviewer: So you asked them if they would get a vertical asymptote in both places. Did you expect them to be able to answer that?
   Susan: No.
   Interviewer: What were you expecting them to do with that?
   Susan: I was half expecting them to say, “Sure you will,” in which case I would say, “Check it out graphically,” or “Just use your calculator and check it out.”

[Susan, Ob #1, Lines 597–605]

When the researcher asked Susan what she expected of students, she indicated that she did not expect them to be able to answer that question but thought that they might assume the answer was yes. The acknowledgement that she did not expect the students to be able to answer the question suggested that Susan had given limited thought to the construction of the task and its accessibility to students. Nevertheless, the associated task offered students the opportunity to explore the nature of the behavior of the graph at each of the restrictions. In that sense, it was a potentially useful task for fostering the kind of mathematical activity that would support these more connected understandings. However, the task did not direct the mathematical activity of
the students sufficiently to offer this kind of opportunity. During the class, Susan got no
response from the students so she suggested they use their calculator to graph it.

Susan: Try it with your calculator. Let me repeat the question. We are starting with this
expression. We are going to treat it like a function—a rational function. This
one. My question is, we know that x cannot be one and x cannot be negative
three. Check out the graph right now.

The direction to graph the function stimulated some mathematical thinking on the part of the
student though the unstructured nature of it offered limited opportunity for the students to focus
on the development of particular understandings.

After giving students some time to graph the two versions of the function on their
calculators, three student responses were offered presumably to the question about whether there
will be an asymptote at both restrictions, but even the question he was attempting to answer was
a bit unclear.

Sam: That first one we do.
Susan: Yes [pointing to the original expression]
Sam: Wouldn’t it only be one since you don’t have that other factor in there?
Susan: So you think only one would be the restriction?
Sam: Cause we are using the simplified version of the…
Susan: Yes, if I am using the simplified version, I get a different function. Call that g(x)
[Teacher labels g(x) in Figure 5.15].

Figure 5.15. The simplified function labeled as g(x).

Susan: Which is rational [referring to g(x)], but there is only one restriction associated
with that. [To the class] So what are you seeing? [Susan switches the projection to the
graphing calculator].
The first comment from the student, “That first one we do,” was ambiguous. The student might have been asserting that at the first restriction Susan listed, \(x=1\), there was an asymptote. Alternatively, he might have been asserting that at the first restriction numerically, \(x = -3\), there was an asymptote, or he might have been suggesting that for the original, unsimplified function, there were asymptotes at both restrictions. The reference was unclear, yet Susan responded with a “yes,” though her interpretation of the student’s response is unclear. Her response suggested that she did not recognize any alternative interpretations of the student’s response, which suggested an absence of careful mathematical reasoning about the mathematics of the student and the mathematics of the task.

The second comment in the excerpt also warranted some follow-up. “Wouldn’t it only be one since you don’t have that other factor in there?” If the student was still answering the question about whether there would be an asymptote at each restriction, then this student response suggested the student was looking at the simplified version of the function and concluding that there was only one asymptote. While this interpretation would have suggested an incomplete understanding, the student could have been thinking in productive and potentially useful ways for the development of an understanding of restrictions that produce holes and those that produce asymptotes. Assuming some additional probing could have confirmed the student’s thinking, the teacher could have used this student’s thinking to direct attention to the differences between what the simplified version of the graph suggested about the asymptotes and restrictions and what the unsimplified version suggested.

Instead, Susan’s probing involved asking the student whether he thought “only one would be the restriction?” This question did not force the student to make a careful distinction between
values of the variable that represent restrictions and those that produce asymptotes. The phrasing of this question also suggested Susan thought the student was referring to restrictions, not asymptotes. However, it remained unclear because she did not afford the student the chance to fully respond before interrupting him to suggest that he was correct if they were looking at the simplified version and treating it like a different function. While this point was potentially off target with reference to what the student was asserting, it was nevertheless a potentially productive one. It directed the student’s attention to an important consideration—that if you treat the simplified version of the function like a different function, it was going to have some differences with the original function. These differences and attention to them could have served to support the understanding of the algebraic and graphic manifestations of the differences between restrictions that produced holes and those that produced asymptotes. However, Susan curtailed the discussion and shifted back to asking the class what they were seeing when they graph the function. This shift left this portion of the discussion disconnected from the ensuing investigation of the numerical behavior of the function at the two restrictions.

More mathematical reasoning about how a student might approach the task and what conceptions she might have wanted students to develop might have led her to structure the task in ways that would have directed the mathematical activity of the students to a greater extent. She might have constructed a task in which students were asked to compare the graphs of the original form of the function and the simplified form. Such a task could have helped students consider what is the same about the functions and what is different. The comparison could have informed students about how the simplified form provided information about the graph of the original function. However, in the absence of such a task, the mathematical activity of the
students appeared unfocused and their responses were somewhat ambiguous making it difficult for Susan to respond in ways that supported the development of rich, connected understandings.

This entire exchange demonstrated a limited effort on Susan’s part to understand the mathematics of the student. The observations, comments and questions offered by the student revealed some elements of the student’s thinking that were potentially productive. However, Susan’s apparent interpretation of the student’s thinking suggested less than full and careful consideration of the mathematics of the learning goal (understanding the algebraic, graphical, and numerical differences between restrictions that produce holes and those that produce asymptotes) and the mathematics of the student (his conceptions and their potentially productive aspects). Her response to the student responses reflected a limited consideration of the mathematical path from the expressed understandings of students and the desired understandings inherent in the learning goal.

The class discussion around the examination of the graph of the function to answer the question about the asymptotes continued without pursuing a deeper consideration of this student’s thinking about the number of asymptotes and at what values of \( x \) they occurred. Instead, Susan led the students through an investigation of the numerical and graphical behavior of the function at the two restrictions. While this activity held promise for supporting the connections among representations of the rational function, it was largely teacher-directed. Susan showed the students the differences in the numerical behavior and told them what to conclude about those differences.

Susan: When I ask the calculator to tell me what \( y \) goes with an \( x \) of 1, what happens?
Student: There’s nothing there.
Susan: Nothing. Ah, but what if I move ever so slightly to the right? I let the calculator decide how much I would move to the right. If I move just a little tiny bit, [T enters
In this excerpt, Susan explained how to investigate the behavior—“move ever so slightly to the right” and “slightly to the left”—and what to conclude, “So, I see that vertical asymptote.”

Susan followed an almost identical pattern in looking at the numerical values of the function at the other restriction, $x = -3$.

Susan: When I move a little to the left of negative three, I get a $y$-value. Those two points are pretty darn close to each other. I am not seeing that asymptotic behavior [$T$ is doing the chicken motion with her arms] going on there. Do you know what is going on there? I get a hole in the graph.

Again, Susan provided the mathematical conclusion for the students without giving them a chance to consider what the numerical behavior implied about the graph and table of values. Without responses from students, it was impossible for Susan to know the nature of their understanding and the episode placed limited demands on Susan’s mathematical reasoning about those understandings. The highly teacher-led investigations required only that Susan reason about the problem itself and give some consideration to constructing an explanation that the students could potentially follow.

Immediately following this teacher-led investigation, Susan posed the following question to students:

Teacher [to the class]: Okay, so [$T$ moves back to the worksheet] sometimes you’ll get holes, sometimes you’ll get asymptotes. How can you tell which? [pause] How can you tell when you will get a hole and when you will get an asymptote?

Interviewer: What about at that moment? How can you tell which you are going to get? A) do you think they knew a hole in the graph and when that was caused…

Susan: [interrupting] No, not at all, that was the point.

Interviewer: And then so what did you expect them to do with this question?
Susan: [pause, in a quiet voice] Um, explore. [5-second pause] You know it is ridiculous to think that with a sample of size one that someone is going to come up with a conjecture that might be valid.

[Susan, Ob #1, Lines 697–707]

Again, Susan posed a question that, as she acknowledged, she did not expect them to be able to answer. Such an approach suggested a limited consideration of the perceived existing understandings of the students and a careless construction of this associated task. The previous investigation offered Susan no insight into the existing conceptions of students about holes and asymptotes. The potentially productive, expressed understandings of Sam in the prior exchange were left unexplored and unused. This task asking students about the restrictions that produce holes and asymptotes, as presented, offered little direction for the mathematical activity of the students and limited engagement of existing understandings. The pause before answering the interviewer’s question about her expectations and the quiet, almost timid way in which Susan responded suggested that “explore” might have been an idea offered in the stimulated recall interview to explain her choices rather than something truly considered at the time the task was given. The statement following the long pause suggested doubt in her approach (or in at least her stated approach) in getting students to explore to discover the relationship. Each of these elements revealed limited mathematical reasoning in the design of the associated tasks and in the implementation of these tasks.

One final exchange in this section of the first observed lesson reinforced the limited nature of Susan’s application of her knowledge and reasoning in the implementation of instruction. The question Susan posed about determining when restrictions produced asymptotes and when they produced holes led to a number of student hypotheses. Susan’s response to these hypotheses revealed a limited application of her mathematical reasoning in the construction of
her responses to these student responses. A discussion of her response to the first hypothesis follows.

The first student hypothesis immediately followed the question Susan posed to the students.

Susan: Okay, so [T moves back to the worksheet] sometimes you’ll get holes, sometimes you’ll get asymptotes. How can you tell which? [pause] How can you tell when you will get a hole and when you will get an asymptote?
Student: When you have more than one restriction
Susan: What happened for this original problem? [T refers to the first function shown in Figure 5.16]

![Figure 5.16. The original rational function and its simplification](image)

Susan, Ob #1, Lines 697–717]

The student proposed a response to her question, “When you have more than one restriction,” but the proposition is incomplete. Did he mean that when there was more than one restriction, we would get a hole and an asymptote? Did he mean that when you have more than one restriction you would get a hole? Or did he mean something slightly different. It was unclear because Susan did not elicit more clarification from the student. The lack of elicitation followed by a suggested counterexample suggested that Susan assumed that she understood what the student meant. Such a scenario would imply that Susan had not reasoned carefully about the student’s hypothesis and the multiple interpretations possible.

When Susan asked the student to think about the first problem, “What happened for this original problem?” she could have been offering a counterexample to the student’s hypothesis or ignoring the student’s hypothesis all together and suggesting an examination of the first problem could provide some insights into how to determine when you get a hole and when you get an
asymptote. If Susan believed she was offering a counterexample, then clearly she did not reason carefully about her response to the student’s response. Since the first problem involved only one restriction, it was not a counterexample to the hypothesis. If Susan was ignoring the hypothesis and suggesting the students examine the first task for insights into answering the question, then she was neglecting to reason about the potential mathematical path from the understandings of the students to the desired understandings she hoped to develop. In either interpretation, Susan’s response suggested a limited unpacking of the mathematics of the student, the mathematics of the associated task, and the potential mathematical path from the expressed understandings of the student to the desired understandings.

Further evidence of the limited nature of her application of mathematical reasoning in the moment was found in the last portion of her response to the student. After directing the student to look at the original problem, Susan almost immediately discontinued that consideration and offered a different example.

Susan: We are getting ahead of ourselves because we looked at this graphically too much. So what might you think would determine a hole versus an asymptote? Let me have you just consider, [writing on board] \(x + 1\) divided by \(x + 1\) squared…eh, no, let’s not make it…\(x + 1\) times \(x - 2\).

\[
f(x) = \frac{x + 1}{(x + 1)(x + 2)}
\]

After directing student attention away from the original problem, she repeated the question and constructed a different example. Again, it was not a counterexample to any of the possible interpretations of the student’s hypothesis, “When you have more than one restriction.” The example had two restrictions, one produced a hole and one produced an asymptote. However, if Susan was simply introducing another example to investigate and determine when a restriction
produced a hole and when it produced an asymptote, it was unclear why she would have abandoned the consideration of the original problem. Unfortunately, Susan was not asked about her thinking in the stimulated recall interview, so the reason behind her decisions remains unclear.

As the new example was explored, a student pointed out that the rational expression could be simplified. Susan acknowledged the validity of that simplification and once again posed a question about the equivalence of the functions directing students to the restrictions. After a brief exchange discussing the simplification, the following exchange occurred.

Susan: I don’t really want to simplify it, because they are not equivalent.
Student: Then why do it?
Susan: Well, we are studying this rational function and characteristics of it. Okay?

The student’s question regarding the reason behind the simplification was an important one. It seemed to represent one of the essential, component understandings for an algebraic understanding of restrictions that produce holes and those that produce asymptotes. To understand this distinction, a student must understand how the simplification informed the determination of holes and asymptotes. The student’s question suggested that this component understanding was missing. Susan did not recognize the significance of this question in the moment nor did she respond to it in a way that would support the development of this key understanding. Instead, the nature of Susan’s response again suggested a lack of careful reasoning about the mathematics of the student, the mathematics of the learning goal, and the potential mathematical path from one to the other.

Throughout the lesson segment, Susan’s interpretation of and construction of responses to responses indicated numerous limitations in the pedagogical content reasoning of Susan—
limitations in understanding the mathematics of the students, considering the mathematics of the learning goal, and constructing associated tasks designed to support the development of the component understandings. It began with questions asking students to apply the definition of rational functions (as Susan interpreted it) to two different functions—the stated purpose of the initial tasks. Yet, the class discussion evolved into a consideration of restrictions that produce holes and those that produce asymptotes. This aspect of the work of the class could have been anticipated had Susan unpacked the mathematics of the lead task more thoroughly. In light of the limited, stated goal of the lead task, it seemed safe to view this portion of the lesson as largely unplanned, requiring in-the-moment application of the mathematical knowledge and reasoning during instruction. Numerous opportunities to understand and develop the thinking of students emerged in this portion of the data. However, Susan’s responses to the responses of students suggested a pattern of limited application of mathematical knowledge and reasoning in ways that would support the development of rich conceptions of the numeric, algebraic, and graphic representations of domain restrictions in rational functions.

These limitations are of particular interest. Susan clearly possessed a full appreciation for the mathematics of rational functions. She demonstrated some careful thought about the nuances of the nature of the equivalency between the original rational function and the simplified version of it. She recognized the paramount importance of these differences in equivalence and directed student attention to it. She also appropriately focused student attention on the restrictions of each version of the function. With this depth of understanding about the mathematics and her awareness of the typically problematic nature for students, she seems to demonstrate a robust specialized content knowledge and knowledge of content and students. However, in spite of this
breadth of knowledge, her instruction did not appear to elicit the kind of mathematical activity on the part of students that would foster the development of these types of understandings in them. Accounting for this difference represents an important facet of this study and the limited ways she applied her mathematical reasoning during instruction seemed to explain the differences.

**Duncan’s Reasoning During the Implementation of Tasks**

As discussed previously, the dominant pattern in Duncan’s instruction was teacher-directed dialogue with limited elicitation of student thinking. This structure provided few opportunities to understand how Duncan applied his mathematical reasoning as he interpreted student responses and constructed responses to student responses. A few notable exceptions were found during the three-lesson sequence.

An exchange from the first observed lesson demonstrated Duncan’s approach to a situation in which a student expressed confusion about the work of the class. In the exchange, Duncan had to interpret and respond to a question from a student about the meaning of the parametric and rectangular graphs of some of his examples. The question came during the example involving the populations of foxes and the rabbits represented by two sinusoidal functions. After a lengthy class discussion about the graphs of these parametric equations, a student asked, “I don’t understand what this is?” [Duncan, Ob #1, Line 652]. During the stimulated recall interview, the researcher asked Duncan about what he believed she was asking about.

**Interviewer:** Do you think her question was about the \( R \) versus \( T \) and the \( F \) versus \( T \) or about the \( R \) vs. \( F \)?

**Duncan:** I think it was both. I think she’s comfortable with \( R \) versus \( F \) cause that is really \( x \) versus \( y \) in her mind…When she asked me the question, I got the impression that she did not completely understand why is the time piece necessary.

[Duncan, Ob #1, Lines 698–709]
Duncan’s response to her question involved a real-world example that Duncan indicated he developed on the spot. He described looking at the growth of a person over time suggesting that you could look at the vertical and horizontal change. He had interpreted her confusion to be about the inclusion of time to model this phenomenon and Duncan constructed and provided another example to her that he believed might address that confusion.

The student followed Duncan’s example with a couple of follow up questions.

Student: Is it always going to be an ellipse?
Duncan: No, just in this example, it happened to be an ellipse. We can get all kinds of different shapes.
Student: So could they just not be related?
Duncan: Yea, I could say the relationship between you getting taller and how dark the green grass is outside. There is no relationship between those two things.

The student expressed confusion about the rectangular graph constructed from the parametrics, “Is it always going to be an ellipse?” and then followed it up with a question, “So could they just not be related?” Again, Duncan responded with another real-world example created on the spot in response to the question. The construction of each of these examples represented Duncan’s use of pedagogical content reasoning to respond to a student’s response.

However, the lesson segment also illustrates Duncan’s efforts to interpret the student’s understanding. In his initial assessment of the student’s question provided in the stimulated recall interview, Duncan believed she was “comfortable with $R$ versus $F$. Yet, her follow-up questions involved the $R$ versus $F$ graph and the relationship it represented. Even with those follow-up questions, Duncan did not understand her first expressed confusion, “I don’t understand what this is?” to be about the relationship between $R$ versus $F$. It was not until the
stimulated recall interview that Duncan made further sense of the situation after being prompted by the interviewer.

Duncan: You know what, I think you are right because remember her follow up question was so we are trying to relate two things and I said well your height versus the color of the grass outside, they aren’t related. So I think she was thinking about the rabbits and foxes together.

[Duncan, Ob #1, Lines 720–724]

In this excerpt, he expressed his newly constructed understanding that she might have been asking about the $R$ versus $F$ graph and relationship. His initial interpretation of her confusion informed his response, but a deeper reasoning about her confusion and a learning goal focused on the development of an understanding of the relationships the parametric and their rectangular counterparts represent could have led to responses to this student that could have fostered those understandings.

An exchange during the second observed lesson demonstrated Duncan’s approach to a situation in which the student offered an alternative, but valid solution. As discussed in the previous chapter, the students were working on the clock problem, which asked them to develop the parametric equations to model the movement of the clock between 12 and 1 o’clock. The students were given time to develop their parametric equations, and one student was asked to present his equations to the class. His approach to the “starting point” dilemma was to use the sine function with the horizontal motion and the cosine function with the vertical motion. He provided the following equations to the class:

$$x(t) = 12 \sin t \quad y(t) = 12 \cos t$$

[Duncan, Ob #2, Line 315]
In response to this student’s work, Duncan affirmed the validity of the approach and suggested the students consider an alternative but more powerful approach of using phase shifts.

Duncan: There are a couple of ways to do this, but I like the idea of giving you guys a table and at time zero, I want to be at the position (0,12). [D constructs a table on the board with three columns: t, x, and y and fills in the first line with 0, 0, and 12]. If I am centered at the origin, I want to be straight up at twelve so you guys have to figure out how that is going to work. [Pointing to the equations provided by Alvin on the board, line 315] That does work for these equations, so Alvin is good.

In this response to the alternative but valid work, Duncan reframed the task by connecting the values of t, x, and y at the “starting point” with the considerations students needed to give in order to produce parametric equations that would model the motion of the hands of the clock. Duncan provided students with this more conceptually oriented way to think about the original task, but did not provide a motivation for the students to engage the task in this way. He affirmed Alvin’s approach without suggesting there were limitations to it or providing students with a task that would have required them to use the conceptually oriented approach.

When Duncan observed that students continued to approach the task for the hour hand the way Alvin approached it for the minute hand, he reframed the task again. In this instance, he structured the task to force the students to model the motion in the way he wanted them to do so.

Duncan: Okay, now, my second question is… I thought you guys would do it this way [D points to the set of equations just discussed and provided by Alvin above, Line 315]. You would switch sine and cosine to get it to start at noon. But, if I tell you that you have to do it so that x has to be cosine and y has to be sine, see if you can do this…and I am going to give you a hint. [D writes the two sets of equations below on the board].

\[
\begin{align*}
\text{minutes} & \quad x(t) = 12 \cos(  ) \\
& \quad y(t) = 12 \sin(  ) \\
\text{hours} & \quad x(t) = 9 \cos(  ) \\
& \quad y(t) = 9 \sin(  )
\end{align*}
\]

Duncan: You are going to have to add something to the parentheses. And the same thing down here. You are going to have to do this for minutes and for hours.
In this presentation of the reframed task, Duncan directly insisted that the student model the
motion by using a cosine function to model the horizontal movement and a sine function to
model the vertical movement. While this approach accomplished his goal of getting students to
model the motion in this more broadly useful way, Duncan did not promote an understanding on
the part of students of why this approach was preferred. The situation required Duncan to
consider the existing understandings of the student and their demonstrated responses to the
original task. It involved a consideration of the mathematics related to his learning goal—
specifically the more broadly applicable approach using phase shifts. However, the
mathematical reasoning applied to this instructional decision seemed to be limited to these two
dimensions. The task constructed involved a rather arbitrary (from the student’s perspective)
motivation for working towards a solution to the task using phase shifts—the teacher’s
preference. A more extensive application of mathematical reasoning to the construction of the
reframed task might have motivated the work and connected it more powerfully to their
understanding of the approach involving the use of a sine function to model horizontal motion
and a cosine function to model vertical motion (the original student approach). This aspect of his
reasoning about the construction of a response to a student response seemed limited in this
instance.

A third example, which occurred during the third observed lesson, provided an
opportunity to examine Duncan’s approach to working student propositions. Duncan presented a
lead task to students in which they were asked to use a given set of parametric equations to write
a single, rectangular equation. The first problem provided students with the equations, \( x = \frac{3t}{4} \) and \( y = 2t - 1 \).

Duncan: So here’s my question. I want it just in terms of x and y. That’s what rectangular means. I don’t want it in terms of t, in terms of time. Anybody have a suggestion?

As discussed previously, Duncan received three different suggested approaches from students that placed demands on his ability to reason through those approaches to determine a potentially productive response.

In the first response, a student suggested, “Set t to 0x and make sure x is in the equation” [Duncan, Ob #3, Line 171]. In the stimulated recall interview, Duncan acknowledged that he was not sure what the student meant. Rather than probing for more clarity, Duncan seemed to determine that the student’s thinking could not be used productively to advance the understanding of the class. Duncan offered a dismissive response and called on another student. That student suggested a potentially productive response.

Student: Is it kind of like when you have a system of equations when you put one of them in terms of one of the variables and then put it in the other equation?

Duncan: Perhaps. Does that seem like a good idea? Tom, is that what you were thinking? Bill, is that what you were thinking?

In this instance, Duncan recognized this approach as a useful approach but continued to elicit input from students. In response, a student suggested yet another approach.

Student: I was just thinking use the inverse (inaudible) and set t equal to zero.

Duncan: Well if you set \( t = 0 \), then that is just going to give you one value. For instance, if \( t \) is zero, it is going to give me zero here [D points to the equation for \( x \)] and negative one here [D points to the equation for \( y \)]. That will give me a point, and I can get a few values. That is not a bad idea.
The student suggested finding the values for $x$ and $y$ for $t = 0$. Duncan followed through on the student’s suggestion and emphasized that the approach was going to yield a point. He also hinted at the potentially useful way this approach could be applied to the task. Duncan explained his thinking about this student’s response and Duncan’s response to him during the stimulated recall interview.

Interviewer: So this idea… you got the idea from a student about “Is it like a system of equations?...put one in terms of the other… and then you got this idea about…
Duncan: picking a point
Interviewer: Picking a point or a value for $t$... you ended up pursuing that for a second instead of continuing with ‘it is like solving a system. So, talk about that just a little bit.
Duncan: My thought was that if they were going to pick a value for $t$, that they may try to plot a few points in terms of $x$ and $y$, okay let $t$ be one, let $t$ be two, and create a table of values and then from there they might be able to recognize a pattern in those inputs and outputs and then from there come up with an equation. But I got the impression that once I said you will get one and negative one, then they weren’t sure what to do with that. I didn’t hear anybody say, ‘well, let’s get another point. Let’s get three or four points.’

[Duncan, Ob #3, Lines 186–202]

Duncan explained his extensive reasoning about the student’s proposed approach. He recognized the potential value of the approach if it involved finding multiple points and trying to recognize a pattern. His response to students directed their attention to this aspect of the approach when he said, “and I can get a few values.” However, he concluded that the students “weren’t sure what to do with that.” This exchange, both in class and in the stimulated recall interview, demonstrated an effective application of mathematical reasoning to understanding the proposed approach of the student, recognizing its potential value relative to the completion of the task, suggesting an extension of the task to assess the student’s ability to understand it as a potentially productive approach.
Instead of a student response that suggested a pursuit of this alternative approach, a student returned the discussion towards approaching the parametric equations like a system.

Student: Just make it equal to $t$ and plug that in. So with the $y$ equation, you would add one and basically isolate $t$ and get $y$ where you had all of the other stuff and then…Add one to both sides. And then divide both sides by two. [D writes the steps on the board].

Even though this approach offered the most viable approach in general, the specific application of it by this student still involved the application of mathematical reasoning by Duncan in the construction of responses. Duncan recognized that the proposed solution, although valid, involved more complicated algebraic manipulation. He encouraged the student to work through that more complicated approach before exposing them to the more direct way of solving the $x(t)$ equation for $t$ and plugging it into the equation for $y(t)$.

Interviewer: You mentioned earlier that you were glad that somebody chose this way [The interviewer refers to the solving the $y$ equation for $t$]. Is this something you frequently do, you show alternative ways?
Duncan: If students choose a technique or a way I didn’t expect, I’ll run with it. I want kids to see alternative or different approaches even if it is wrong because sometimes you can get some good discussion out of it. I knew when John said, well let’s do it the way he did it, that we would get the correct answer and his algebra was pretty good for the most part. Then I wanted the kids to see, well wow, there is a more direct way. And thankfully, before I even had a chance to show it, Karen said, ‘Hey, we can do that more efficiently.’

In this excerpt, Duncan expressed his understanding of the complexity of the approach in advance of the class actually working through it. This understanding derived from his mental construction of algebraic work involved in the substitution—his mathematical reasoning about the proposed approach. He also knew he could motivate the need to consider the algebraic manipulation before deciding which equation to solve for $t$ by showing the students the
alternative method after seeing this one. The exchange demonstrates multiple ways Duncan applied mathematical reasoning in the moment to interpret student responses and to construct responses to student responses.

Even though Duncan’s instruction involved a large amount of teacher-directed dialogue and his goals for instruction for the observed lessons were more introductory and procedural in nature, these examples demonstrate his application of mathematical reasoning during instruction when student responses were elicited and used to shape instruction. The examples also provide additional evidence of the existence of pedagogical content reasoning and the role it potentially plays in the implementation of instruction. Duncan applied mathematical reasoning in the implementation of instruction as he constructed examples to address a student’s question or confusion. He applied his mathematical reasoning to understand alternative, but valid approaches, their implications, and the appropriate instructional response to them. And he applied his mathematical reasoning as he identified more powerful ways of approaching tasks and developed associated tasks designed to foster the student use of those more powerful approaches.

**Summary of Evidence of Mathematical Reasoning During Instruction.**

Collectively, the data from the classroom observations and stimulated recall interviews reinforced the importance of the role of mathematical reasoning during the planning and delivery of instruction. It also supported the potential power of pedagogical content reasoning in explaining the instructional differences among the four teachers in the study at critical instructional moments.
As was the case in the classroom and interview data from Jackie and Harold, evidence existed to suggest that the differences in the nature of instruction among the four teachers in the study can be explained to some extent by the nature of the application of mathematical knowledge and reasoning at critical instructional moments.

Throughout her work with students, Jackie maintained a triune focus on the mathematics of the student, the mathematics of the task or learning goal, and the potential mathematical path from one to the other. The data suggested that she applied her mathematical knowledge and reasoning to this focus as she interpreted student thinking and constructed responses to student responses that supported the attainment of her learning goals, even when those were in apparent conflict. She used student thinking to shape her instruction as she responded to alternative, but valid approaches, inadequate or inaccurate student conceptions, and demonstrated readiness for new ideas. Each situation required an active application of her mathematical reasoning to the interpretation of student responses and the construction of responses to student responses.

Harold’s efforts to maintain the triune focus on the mathematics of the student, the mathematics of the task or learning goal, and the potential path from one to the other were not consistently maintained. On the one hand, his narrow indicators of understanding and rigid approaches to tasks often curtailed his efforts to understand the conceptions of students and to reason about the tasks in the ways they would. He thought about student conceptions of the mathematics, but he often listened for a particular expression of those conceptions or made too many assumptions about those expressions of understanding. He reasoned about the mathematics of the task, but often only in terms of the narrow path to a solution he defined. He
reasoned about the mathematical path from the assumed, existing understandings of students to the desired understandings related to the learning goal, but the path remained rigidly defined.

On the other hand, Harold often demonstrated a careful consideration of the conceptions of students and designed responses to student responses accordingly. During the first Math 6 lesson, he constructed series of associated tasks that used multiple representations and varied in structure and complexity in response to situations in which students could not provide a useful mathematical response. The lead tasks for the second Math 6 lesson, as discussed in the previous section, were constructed in response to the work of the class during the first lesson. The flexibility Harold demonstrated in those responses seemed to be a function of his active application of mathematical reasoning to understand student thinking and construct tasks designed to use those perceived existing understandings to support the development of the desired ones. Harold’s responses to inadequate or inaccurate student responses also revealed an active application of mathematical reasoning to the interpretation of and construction of responses to student responses. He demonstrated his ability to reason to the logical conclusion of student hypotheses and produce counterexamples to those propositions that directed student attention to the salient features of their mathematics while supporting the development of rich, connected understandings.

The data from the classroom observations of Susan suggested limitations in her application of mathematical reasoning to understanding and interpreting student conceptions. The tendency to make assumptions about student responses, rather than to explore student thinking more thoroughly, often contributed to responses to student responses that were highly teacher directed and involved persistent, unresolved confusion on the part of students. There
also seemed to be a pattern of under-developed consideration of the complexity of the tasks and learning goals. These differences can be understood in terms of the limited depth in the reasoning about the mathematics of the students, the mathematics of the learning goals and the potential path from one to the other in the interpretation of student responses and the construction of responses to student responses.

The Nature of Pedagogical Content Reasoning (PCR) for Mathematics Instruction

For this study, an image of instruction based on a review of empirical and theoretical literature was developed. That image of instruction identified the importance of clearly identified learning goals and well-chosen tasks—tasks that were accessible and experientially real to students while also motivating productive mathematical activity. It emphasized the power of implementing tasks in ways that required the students to express their thinking and make the mathematical decisions in the completion of tasks. That kind of instruction required a triune focus on the mathematics of the student, the mathematics of the learning goal, and the mathematical path from one to the other. This study was designed to focus on understanding the demands on the mathematical knowledge of teachers and its use during the planning and implementation of instruction as teachers sought to maintain this triune attention. The construct of pedagogical content reasoning emerged as a potential way to describe the nature of these demands and the use of mathematical knowledge in practice.

Out of that image of instruction, critical instructional moments were identified as the unit of study for this research. Four types of critical instructional moments were initially identified as important moments in the planning and delivery of instruction. They were derived from a
theoretical perspective and refined through the analysis of the data. Those moments occurred in
the planning and delivery of instruction and were identified as the identification of learning
goals, the selection of tasks, the elicitation and interpretation of student responses, and the
construction of responses to student responses. The analysis of the classroom observations and
stimulated recall interviews centered on these moments and involved trying to understand the
nature of instruction at these critical instructional moments and the use of mathematical
knowledge and reasoning during these moments.

Based on the analysis of the nature of instruction and the consideration of the use of
mathematical knowledge and reasoning to make instructional decisions, differences emerged in
the instructional practice at critical instructional moments among the four teachers in the study.
As discussed in the previous section, these differences could not be fully explained through the
constructs associated with the mathematical and pedagogical knowledge of teachers. More
specifically, maintaining a dynamic consideration of the mathematics of the student, the
mathematics of the task or learning goal, and the potential mathematical path from one to the
other seemed to require that teachers actively apply their mathematical knowledge and reasoning
in the planning and delivery of instruction. When the data showed this type of reasoning being
applied to the planning and delivery of instruction, the nature of instruction more closely
reflected the image of instruction derived from the literature and outlined in chapter 2.
Furthermore, when deficits in the nature of instruction relative to this image were noted, those
deficits could be linked with limited applications of mathematical knowledge and reasoning.
The discussion in this section summarizes the observed differences an application of pedagogical content reasoning at critical instructional moments and the observed impact on instructional practices at those moments.

**Pedagogical Content Reasoning in the Planning of Instruction**

As observed in this study, pedagogical content reasoning in the planning of mathematics instruction involves the active application of mathematical knowledge and reasoning to the identification and unpacking of learning goals and the selection and sequencing of tasks. In the planning stages, translating the learning goals into a form that supports the selection and sequencing of tasks requires an unpacking of the learning goal into its component understandings. At times, a teacher might have previously identified, or had identified for him or her, these component understandings of the learning goal. More often, this unpacking requires an active application of mathematical reasoning to work backwards from the learning goal to construct the set of mathematical conceptions that are essential for the attainment of the learning goal. Once a learning goal is unpacked into component understandings, a teacher can use these component understandings to select or design well-chosen lead tasks but only with a careful consideration of the perceived or anticipated existing understandings of students in mind. To be well-chosen, tasks must be accessible to students using their existing understandings and successfully engage those existing understandings in goal-directed mathematical activity. The tasks must be sequenced to motivate the mathematical activity of the students in ways that support the development of the component understandings and the attainment of the learning goal. In short, the teacher must anticipate how a student might engage his or her existing
understandings (as perceived) as he or she envisions the mathematical activity the task or sequence of tasks might elicit from the students. He or she could thus determine whether that anticipated activity supports the attainment of the learning goal. In this way, the planning of instruction involves a triune focus on the mathematics of the student, the mathematics of the learning goal(s), and the potential path from one to the other. The use of mathematical knowledge at the critical instructional moments of the identification of learning goals and the selection of tasks is explicated further in the following discussion. The work of a teacher drawing on existing accumulated understandings is contrasted with the work of a teacher applying pedagogical content reasoning in these moments.

The use of mathematical knowledge and PCR during the identification of learning goals. Goals for a lesson sequence can readily be identified by a teacher through his or her accumulated mathematical knowledge. Drawing on existing content knowledge can allow a teacher to identify the elements of a topic that are important for a student to know and be able to do. Susan’s goals for students during her two-lesson sequence on rational functions provide an example of this. When she was asked to specifically identify her goals for students, she provided the following response.

So specifically, the fact that it involves a ratio, that it is a ratio of algebraic expressions, not just integers. That from our previous understandings, that comes with some complications, such as zero denominators. And also, we just finished polynomials, our overview of polynomials. And actually, for some of them when we studied quadratics, they had some difficulty with manipulating quadratic expressions so I thought there would be for some, more comfort level, than others in recognizing those connections within those polynomials. So I did hope that they would see connections with the power functions that we have studied and the connections with asymptotes, the characteristics of functions like end-behavior, and it would be nice if we also had some time to get into the manipulation to be able to determine if an expression, just because it is written as a ratio of polynomials is necessarily a rational expression or not. In other words, does it simplify to something that is rational or not.
The broad and nonspecific nature of these understandings suggest that Susan identified these goals based on her accumulated understandings of rational functions. There is no evidence to suggest that these goals were identified through some process of applying her mathematical reasoning to unpacking the goal of understanding rational functions.

In contrast however, through the application of PCR, goals can become unpacked in ways that focus the attention of the teacher on the conceptions students need to possess or develop in order to understand the mathematical idea in a given lesson sequence. This identification of the component understandings and the resulting heightened focus on them it brings supports the work of the teacher at each of the other critical instructional moments—the selection and sequencing of tasks, the interpretation of student responses, and the construction of responses to student responses—in ways that a less unpacked set of learning goals is not able to do.

This pattern was observed in contrasting the work of Susan with the work of Harold in his Math 6 lessons. As previously discussed, Susan operated with a comprehensive set of goals that were not unpacked. The tasks she selected and implemented with students, the ways she interpreted student thinking, and her responses to student responses served these broader goals but appeared to lack a focus on the development of specific, component understandings. In contrast, Harold had unpacked his learning goal of understanding the division of fractions quite extensively and he used those identified understandings to design specific tasks, identify inadequate student conceptions, and construct responses designed to use the perceived existing understandings of students to support the development of more robust conceptions.
The use of mathematical knowledge and PCR during the selection and sequencing of lead tasks. Like the identification of learning goals, the selection of lead tasks designed to support the attainment of the learning goals can involve the teacher simply drawing on the accumulated, existing mathematical knowledge about the topic and the other dimensions of the mathematical knowledge for teaching such as specialized content knowledge which might involve the range of examples, representations, and problems involving this mathematical idea. It might involve drawing on an existing knowledge of content and students that includes an understanding of how students develop an understanding of this topic and what misconceptions might typically emerge or a knowledge of content and teaching that includes an understanding of various approaches to developing the desired understandings.

Duncan’s selection and sequencing of tasks in his teaching revealed such a pattern. The examples he chose throughout the three lessons involved a broad range of aspects of parametric equations and incorporated a review of many previously studied functions. As discussed previously, the set of problems selected for students to complete during the third observed lesson illustrated these qualities. The goal for this segment of the lesson involved developing the students’ skills of students in expressing a relationship described by parametric equations with a single, rectangular equation.

Duncan: Now today, let me pull up what we are going to do today. [D goes to computer].
Okay, here are some equations [D projects the following equations on the board].

4. \( x = \frac{3t}{4}, y = 2t - 1 \)
5. \( x = 2t, y = \frac{4}{t} \)
6. \( x = t^2 - 3, y = t^2 + 1 \)
7. \( x = ln(t), y = e^{m(t)} \)
8. \( x = sec(\theta), y = cos(\theta) \)
Duncan described the reasoning behind the choice of examples in this way.

Duncan: I purposely chose the first two to be... well the first one to be a very simple way in that you could isolate \( t \) in either equation... And the next one, I wanted to do a little bit of review with exponential, you know, let’s look at exponential functions, how do we undo that, logarithms... Six, seven, and eight were purposefully written so that you did not have to solve for the parameter. There was an easier way to eliminate the parameter than actually solving for \( t \)... And then nine and ten was just once again a review of how I could take a circle or an ellipse and put it back into rectangular form hoping that they would recognize that those two equations were those two shapes.

The sample problems Duncan chose throughout his three lessons reflected this kind of thinking. He understood the scope of different situations the students might encounter in the problem sets and included examples that embody those differences. He also demonstrated a pattern of integrating review of previously studied functions and relationships into the sample problems designed to demonstrate and build new skills. These choices appear largely based in his existing, accumulated understandings of mathematics and the content under study rather than an active application of mathematical reasoning to task selection.

In contrast, however, through the application of PCR, tasks become selected (and possibly designed) and sequenced in ways that enhance the opportunities students have to develop rich, connected understandings. The application of PCR supports teachers’ efforts to select tasks that target the development of specific component understandings related to the learning goals. Jackie’s choice of the parallelogram task and its initial presentation represents such an effort. When she was introducing the first lead task in the sequence designed to support the development of an understanding of the association between the coordinates of points in the plane with the angle formed with the positive x-axis, she began with a lead task showing two
different parallelograms with sides of lengths 5 and 9 and different sized included angles as shown in Figure 5.17.

![Figure 5.17. Jackie's initial presentation of the parallelogram lead task](image)

When asked why she chose this task, Jackie shared her thinking.

Jackie: Oh, because I am working up to vectors. So, we have already done, “Find the diagonal of a parallelogram.” Just to use the Law of Cosines and Sines. But in my ulterior motive kind of way, doing it here and then going into vectors, first I want them to tell me the diagonal is not the same for every parallelogram with the same length of sides—it depends on the angle… So what we are trying to do is draw a new conclusion from the diagonal. So we got this length and this angle.

[Jackie, Ob #1, Lines 475–494]

Jackie acknowledges that the students have already worked this type of problem before, but she wanted them to “draw a new conclusion.” In particular, she wanted students to recognize that the angle depends on the diagonal length—an initial conceptualization of the relationship between the coordinates of the points on the plane and the angle. The application of PCR to the selection and sequencing of tasks for Jackie involves considering the anticipated, existing understandings of students, the ways a student with those understandings might reason about a task, and the conceptions that mathematical work might foster in relation to the learning goal.

The application of PCR also supports teachers’ efforts to engage the existing understandings of students to develop new, more robust ones as was the case for Susan’s choice.
of the long division task and Harold’s second Math 6 lesson. Susan’s goal for her students was to understand polynomial long division—how to do it and how to use it to rewrite rational functions. She identified the existing understandings of long division of whole numbers and attempted to help students draw parallels between the two algorithms. The task was chosen with this specific effort in mind and involved her reasoning about these parallels. While there were some previously identified shortcomings in this effort, it nevertheless illustrated the point.

Furthermore, Susan’s long division task also reinforced the usefulness of PCR in the selection of tasks even without an extensive unpacking of the learning goals for the lesson.

Harold’s design and sequencing of lead tasks for his second Math 6 lesson also provided an example of the application of PCR to select tasks designed to foster the use of existing understandings to develop new ones. After considering the ways in which the students engaged in the mathematical activity of the first class, Harold designed two problematic tasks to engage students in mathematical activity that could have potentially led to the development of the desired, component understanding of the division of fractions. For Harold and Susan, the consideration of the existing understandings of students combined with an active application of mathematical reasoning supported their selection and sequencing of tasks in ways that students would use existing understandings in the construction of new ones. This approach stands in contrast to approaches like the one by Duncan previously described in which the primary consideration of existing understandings seemed to involve the review of previously studied concepts and procedures.

Another apparent difference with task selection and the application of PCR involved the selection of problematic tasks. The contrast between the lead task chosen by Harold in his first
Math 6 lesson and the lead tasks for his second Math 6 lesson illustrated the difference. As discussed previously, Harold asked students to model the first component understanding of the division of fractions algebraically in the first lesson. In the second lesson, he developed a real-world task involving the modeling of the distribution of money to four people. This task involved the same mathematical relationship as the lead task for the first lesson. For lead task selection for the first Math 6 lesson, Harold did not anticipate the difficulty of the students in completing the task. Therefore, he gave limited consideration to designing the task in ways that engaged the existing understandings of students and motivated their mathematical activity in ways that would support the development of the desired, component understanding. For the second Math 6 lesson, not only did the tasks engage those existing understandings, but the tasks presented a meaningful context to the students and a mathematical dilemma to resolve. The task selection emerged from the mathematical reasoning Harold did about the mathematics of the student, the mathematics of the learning goal, and the potential path from one to the other.

Finally, the application of PCR to the selection and sequencing of tasks supports teachers’ efforts to incrementally develop the component understandings. The lead tasks of Jackie during the three-lesson sequence involved a series of mathematically similar tasks with slight variations. The parallelogram task described previously, was already mathematical familiar to the students. When Jackie shifted the focus to vectors, she used the same setup as the parallelogram problem. When she shifted the context to the coordinate plane, she used the same parallelogram setup. When she shifted the work of the class to consider the relationship between the coordinates of the point and the angle, she traced the triangle from the parallelogram problem and then traced only the vector onto a new coordinate plane. These incremental shifts seemed to
originate from her attention to the existing understandings of students and how she might use those to build new understandings—what Jackie described as looking at a familiar diagram and “drawing a new conclusion.”

Likewise, the two lead tasks in Harold’s second Math 6 lesson demonstrated a similar approach. After a successful completion of the first task, Harold introduced a slight modification in the second. Students were asked to model the first task, and after successful completion of that task, the second task was introduced.

Task 1: Model this situation: $12 needs to be distributed evenly among four brothers.  
[Harold, Ob #4, Line 14]

Task 2: Model this situation: She has $12 in quarters. She has all of these quarters, but she is giving each kid a quarter until she runs out of money.  
[Harold, Ob #4, Lines 25–26]

The shift from dollar bills to quarters was an incremental one, but changed the modeling of the task in mathematically important ways. Harold used those differences to reinforce the component understanding of the division of fractions he tried to develop during these two observed lessons.

Both Jackie and Harold selected or designed lead tasks that involved the same mathematical relationships but with a shift in context, the presented students with the opportunity to develop new understandings. Not only did these sequences remove some of the mathematical burden from the completion of the new task (since the mathematics was similar) but they also allowed the students to use those newly reinforced understandings to support the development of new understandings connected to their emerging ones. This type of selection and sequencing of tasks emerged from the teachers’ active application of mathematical reasoning to this critical instructional moment.
**Pedagogical Content Reasoning in the Delivery of Instruction**

As observed in this study, pedagogical content reasoning during the delivery of instruction involves an active application of mathematical knowledge and reasoning to the interpretation of student responses and the construction of responses to student responses. During the implementation of tasks, interpretation of student responses often involved developing an understanding of the student’s conception of the mathematics under study based on expressed understandings in responses to tasks, questions, and problems. With that conception of the student’s thinking in mind, the teacher must determine whether the students’ perceived thinking represented a productive step towards the learning goal, if it could be used as a productive step, or if it would be unproductive relative to the learning goal to pursue it. Once this determination is made, the teacher must construct an appropriate response in light of the learning goal. The response could take many forms—asking a question, requiring a justification, offering a counterexample, modifying the task, directing student attention to a salient feature of the task, and so on. When the thinking of a student resembles thinking a teacher has previously encountered, it is possible to construct a response from a repertoire of examples and responses. However, more often, a teacher must actively construct the response in the moment based on his or her triune focus on the existing understandings of the student, the learning goal, and a potential progression from one to the other. This process of actively applying mathematical reasoning to interpret and construct responses to student responses is the essence of pedagogical content reasoning.

The use of mathematical knowledge and PCR with the interpretation of student responses. Like the work of the teacher at other critical instructional moments, the interpretation
of student responses can involve simply a drawing on the existing, accumulated mathematical understandings of the teacher. Student responses could be understood and interpreted as similar to responses a teacher has experienced previously or to conceptions a teacher understands students typically develop about the content under study. A student might offer an alternative approach or pose a hypothesis familiar to the teacher, and the teacher might already understand the veracity of the student’s thinking. In these cases, a teacher might be drawing on his or her knowledge of content and students. The student responses could also be understood as being aligned with the teacher’s understanding of the concepts which would require a drawing on the content (common or specialized) knowledge of the teacher. Whatever the case, the teacher could conceivably make sense of the student response by simply drawing on his or her existing knowledge of mathematics and students.

Through the application of PCR, the interpretation of student responses becomes more focused on the teacher developing a conception of the student’s understanding rather than categorizing it as one of the ways of thinking the teacher has observed or come to understand previously. It might involve following a student hypothesis to its logical conclusion in order to understand its implications or it might involve reconciling an alternative approach with more standard approaches to determine its validity as was the case with Duncan during the third observed lesson. During that lesson, when a student offered a proposed solution approach to converting a pair of parametric equations to a single rectangular one, Duncan had to assess the viability of the approach. The student proposed setting \( t = 0 \). Duncan explained his reasoning.

Duncan: My thought was that if they were going to pick a value for \( t \), that they may try to plot a few points in terms of \( x \) and \( y \), okay let \( t \) be one, let \( t \) be two, and create a table of values and then from there they might be able to recognize a pattern in those inputs and outputs and then from there come up with an equation. But I got the impression
that once I said you will get one and negative one, then they weren’t sure what to do with that. I didn’t hear anybody say, ‘well, let’s get another point. Let’s get three or four points.’

[Duncan, Ob #3, Lines 186–202]

Not only did Duncan describe his extensive reasoning to determine if the student’s approach is correct, he also described his interpretation of the nonresponsiveness of students. He used all of these to determine the next instructional move. Like the application of PCR to the identification of learning goals, the application of PCR applied to the interpretation of student responses supports the work of the teacher in other critical instructional moments—most prominently in the construction of responses to student responses. This excerpt from Duncan illustrated this point.

The interpretation of student responses demonstrated by Jackie and Harold throughout their work with students represented the efforts of those teachers to build conceptions of the students’ conceptions. As previously discussed, excerpts such as these suggested that the active application of mathematical reasoning to build conceptions of student conceptions occurred during the lessons. An excerpt from each teacher is provided.

Interviewer: Did you think they all understand that?
Harold: [Long Pause]. I did. But, I don’t think they understood it as six halves added together. I think they understood it as half of six. I think they are pretty good with dividing something by two is the same as taking half of it. I am convinced they are good with that, but I have really not done much work with taking six halves and adding them together because frankly I thought it was easier and I didn’t think it would be a problem. But clearly, it was.

[Harold Ob #1, Lines 176–184]

In this excerpt, Harold specifically described what he believed the students understood and how they were conceiving of the $6 \times \frac{1}{2}$. He also alluded to the assumptions he made about the nature of their understanding previously. Jackie also revealed the depth of her active interpretation of student thinking and its implications. During the review of the kite problem
from the first observed lesson, she discussed her thinking about the student’s suggestion to draw a diagonal.

Interviewer: So tell me, when she said, “Draw BD”, tell me a little bit about how you thought things might unfold.
Jackie: What I was thinking was, “Oh rats!” cause I wanted that to come later. But then I could…well we kind of hammered away at the perpendicular nature of the diagonals so I could see that maybe she was going there, creating right triangles, which I am all over. I want to do that. So that is why I wanted to pursue it.

[Jackie, Ob #1, Lines 123–129]

Jackie described an initial response revealing her recognition that the student’s proposed approach did not match Jackie’s intended approach. She then explained how she anticipated how she could have “hammered away at the perpendicular nature of the diagonals.” Jackie’s expressed thinking represents her efforts to reason through the implications of a student’s response and make an instructional decision accordingly.

When the interpretation of student conceptions is coupled with extensively unpacked learning goals, teachers are able to distinguish the productive conceptions from the unproductive ones. The work of Jackie and Harold (both classes) demonstrate this dynamic. With the extensive unpacking of learning goals, Jackie was able to identify and interpret student responses that supported the learning progression and those that she needed to shape and direct in a different direction. One such example occurred in the first observed lesson. Jackie had presented the parallelogram task and asked students to find the longer diagonal.

Student: Can you draw like the opposite diagonal?”
Jackie: The other diagonal? We could, except I don’t want that one. I have a reason, but we could find that one.
Student: Wouldn’t that be easier, though?
Jackie: Well, yes.

[Jackie, Ob #1, Line 525].
Because Jackie had a specifically articulated learning progression and a task designed to foster the development of a particular component understanding, the interpretation of the student response and the determination whether it was a productive line of reasoning was readily made by Jackie. When the interpretation of student conceptions is also combined with tasks that are selected and sequenced in ways that are intentionally designed to develop rich, connected understandings, then power of the conceptions of student conceptions becomes more evident and easier to apply to the construction of responses to student responses.

**The use of mathematical knowledge and PCR during the construction of responses to student responses.** Responding to student responses can also simply involve the teacher drawing on the existing, accumulated knowledge of mathematics and mathematics teaching and learning. To respond to a familiar student hypothesis, a teacher could select a counter-example from the repertoire of examples or present the logical implications of a student’s proposition based on what he or she has experienced previously. Based on an interpretation of a student’s conception, a teacher might select an associated task from his or her repertoire that holds potential to advance a student’s thinking. These approaches would involve a teacher drawing on his or her existing, accumulated mathematical and pedagogical understandings.

As observed in the data for this study, through the application of PCR, the construction of responses to student responses involves active consideration of the existing and demonstrated conceptions of students in the selection or construction of associated tasks and other responses to student responses. This application manifested itself in several ways through the data.

At times, the application of PCR involved identifying or constructing tasks that vary in the structure and complexity, like the tasks of Harold in the first Math 6 lesson, in an effort to
engage the existing understandings of students to construct new ones. As discussed previously, in response to student difficulties, Harold constructed a series of associated tasks varying in the nature of representation (numerical, symbolic, and visual), the complexity of the numbers used (whole numbers and fractions). He sequenced them according to the students’ response— reducing complexity until the students were successful in completing the task and then increasing complexity or changing the representation as he anticipated they would encounter success. These adjustments to instruction in response to student responses required an active application of mathematical reasoning to the mathematics of the student, the mathematics of the learning goal, and the construction and selection of associated tasks.

At other times, the application of PCR to the construction of responses to student responses involved constructing tasks to support the attainment of multiple learning goals even when those goals might be in conflict as was the case with Jackie’s responses to students proposing an alternative method that did not support the lesson-level learning goals. As discussed previously, in multiple instances, students offered alternative, but valid approaches that did not support the lesson level learning goals. Jackie had to design responses to those student responses that met those dual and conflicting goals.

Sometimes, it might involve directing student attention to the salient mathematical feature of mathematics through an associated task or constructing a relevant example as Duncan did in response to student questions about what parametric equations represent. In the moment, Duncan used real-world relationships in response to student questions and to reinforce meaning—specifically the height over time and the height and green-ness of grass examples in the first observed lesson. It might involve constructing an associated task designed to
specifically direct the mathematical activity of the students as was the case with Duncan and his response to students who were having difficulty approaching the “start” position with anything but a trial-and-error approach. He designed a task to force them to use phase shifts. Sometimes those associated tasks might take the form of counterexamples to student hypotheses or propositions as Harold did in constructing counterexamples involving systems of equations and inequalities. When a student suggested that an inequality with a less than symbol should be shaded below the line, Harold constructed a counterexample on the spot to reinforce to students the pitfalls in memorizing such an approach.

Finally, the application of PCR to the construction of student responses led to the use of a student’s response as a basis of inquiry for the entire class. Several examples of this type of use of PCR appeared in the data. During the first observed lesson, a student in Susan’s class asked, “Wait, are you saying that the rational function has to have restrictions?” [Susan, Ob #1, Line 622]. Jackie used an errant student response to launch a whole-class inquiry into the way the angle between two parallel lines changed as the vertex of the angle moved along the line. A student in Harold’s second Algebra I class did not recognize the significance of Harold’s use of the word “statement” when referring to the equations or inequalities in a system. Harold turned that difficulty into a whole class consideration of a system of equations and inequalities—something they had not seen previously.

**Summary**

Pedagogical content reasoning played a role in shaping the nature of instruction at each of the critical instructional moments. The data from this study suggests that the impact of PCR on the nature of instruction is distinct from ways a teacher’s mathematical and pedagogical
knowledge shapes instruction. This section provided several examples from the data in an effort to draw some of those distinctions more clearly. Through the application of PCR and with a triune focus on the mathematics of the student, the mathematics of the learning goal, and the mathematical path from one to the other, the identification of learning goals became extensively unpacked and the attention of the teacher became more focused on the essential component understandings which support the attainment of the learning goal. Applied to the selection and sequencing of tasks, PCR supported the selection or construction of tasks designed to develop specific component understandings, to engage the existing understandings of students to develop new ones, and to incrementally develop the component understandings. When a teacher was interpreting student responses, PCR supported the development of the teacher’s conceptions of the students’ conceptions, allowed a teacher to follow a student hypothesis or proposition to a logical conclusion, and helped him or her distinguish the productive conceptions from the unproductive ones. And finally, during the construction of responses to student responses, the application of PCR facilitated the use of a student’s response as a springboard for inquiry and supported the selection or construction of associated tasks that varied in structure and complexity as appropriate for students, supported the attainment of multiple learning goals, directed student attention to the salient mathematical features, and directed the mathematical activity of the students in ways that fostered the attainment of the learning goals.
Chapter 6:
Contributions, Limitations, and Future Research
Contributions

Researchers in mathematics education and related fields have continued to make progress in recent years in their efforts to understand the nature and extent of the mathematical knowledge for teaching and its impact on classroom instruction, and ultimately, student learning. This study complements and supplements a number of those efforts by focusing on the use of knowledge and reasoning in practice.

The work of researchers involving the development of useful conceptualizations of the domain of the mathematical knowledge for teaching (Ball, et al., 2008; Hill, Ball, et al., 2008; Hill, et al., 2004; McCrory, et al., 2012) has laid a foundation for a careful examination of the multidimensional nature of the domain but calls for further conceptualization of the domains of teacher knowledge persist. The work of Ball, Hill and their colleagues has been particularly focused on developing assessments to measure various domains of mathematical knowledge for teaching (MKT) but has failed to establish empirically some of the delineations in mathematical knowledge for teaching such as the difference between specialized content knowledge and common content knowledge. Researchers have used these assessments and others like them in an effort to establish links between student achievement and teachers’ mathematical knowledge for teaching (Ball, et al., 2008; Baumert, et al., 2010; Campbell et al., 2014; Shechtman, et al., 2010). While the work of Ball, Hill and their colleagues has suggested a correlation between MKT and student achievement as well as between MKT and the mathematical quality of instruction (Charalambous & Hill, 2012; Hill, Blunk, et al., 2008; Hill & Charalambous, 2012; Hill, Rowan, & Ball, 2005), the results of other studies are not as compelling. Wilhelm (2014) found a significant relationship between MKT and the maintenance of cognitive demand during
task implementation but did not find a significant or consistent relationship between MKT and task selection. Shechtman and her colleagues (2010) found weak links between MKT of teachers and the mathematical quality of instruction and only one of the three different studies they conducted revealed a correlation between MKT and student achievement. Baumert and his colleagues (2010) found that more of the variance in student achievement (39%) was explained by measures of pedagogical content knowledge than was explained by measures of content knowledge (4.6%). Stein and Kaufman (2010) found no correlation between teacher capacity (a measure that included teacher mathematical knowledge) and the quality of implementation of reformed curricula. These mixed findings prompted researchers to examine other dimensions of mathematical and pedagogical knowledge such as the mediating and interactive effects of curriculum (Charalambous & Hill, 2012; Hill & Charalambous, 2012; Stein & Kaufman, 2010) and to consider the impact of mathematical and pedagogical knowledge on specific elements of instruction such as teachers’ classroom decision-making (Shechtman, et al., 2010), the unpacking of learning goals (Morris, et al., 2009) and the selection and maintenance of cognitively demanding tasks (Charalambous, 2010; Stein & Kaufman, 2010; Wilhelm, 2014).

Many of the studies in the MKT literature seek to establish links between the accumulated mathematical and pedagogical knowledge of teachers and instructional practice. As evidenced by the mixed results of the studies, the relationship is a complex one. A few studies have sought to understand the nature of the application of teachers’ knowledge during the planning and implementation of instruction. This study sought to add to the field’s developing understanding of the impact of the mathematical understandings of teachers on classroom
practice by examining the use of mathematical knowledge and mathematical reasoning during instruction.

It is important to note the recent work of Heid, Wilson, and their colleagues (in press) as it represents a comprehensive effort to conceptualize the nature of mathematical knowledge and its use for secondary mathematics teachers. The MUST framework (Mathematical Understandings for Secondary Teaching) includes three major components of these understandings: Mathematical proficiency, mathematical activity, and the mathematical context of teaching. Like the work of this present study, the framework, grounded in secondary mathematics teaching practice, incorporates the dynamic application of mathematical knowledge and reasoning into the understandings that they have identified that secondary teachers need. This dynamic application of mathematical knowledge was identified by the construct of pedagogical content reasoning in this present study.

**The Construct of Critical Instructional Moments**

In order to study the in-the-moment decision-making of teachers, it was necessary to focus on particular moments in the planning and delivery of instruction—moments that matter. The construct of *critical instructional moments* provided that focus. These moments were initially identified and delineated through an analysis of theoretical and empirical studies. During the initial phases of data analysis, these delineations were refined so that critical instructional moments could be identified and used as focal points for the study of the use of mathematical knowledge in the planning and delivery of instruction. The researcher focused on understanding the demands on and use of the mathematical knowledge and reasoning of teachers in these moments: *the identification and unpacking of learning goals, the selection and*
development of lead tasks, the elicitation and interpretation of student responses, and the construction of responses to student responses.

The construct of critical instructional moments and the refinements to their definitions and delineations represent one of the potentially useful contributions of this study. Other researchers have used related or similar qualities of instruction as they assessed the relationship between the mathematical knowledge of teachers and instructional practice. For example, Baumert and his colleagues (2010) examined four dimensions of instructional quality—the provision of cognitively activating learning opportunities, the curricular level of tasks, the individual learning support, and classroom management. Stein and Kaufman (2010) defined high quality lessons as those in which high cognitive demand is maintained, student thinking is attended to, student responses are used to move the class towards the mathematical goals, and mathematical reasoning is expected on the part of the students. Ball, Hill, and their colleagues identified the mathematical quality of instruction by focusing on mathematical errors of the teacher, responses to students (both appropriate and inappropriate), the richness of the mathematics discussed and presented in class, the mathematical language used by the teacher, and the connectedness of the classroom activity to mathematics. Like the present study, these researchers’ elements of instructional quality included a consideration of the tasks and the implementation of those tasks in the classroom. Like the present study, they involved a focus on the actions of the teacher during instruction, a consideration of student thinking, and the instructional practices that produce rich, connected understandings.

However, unlike these approaches, critical instructional moments were identified as sites to be studied—moments during the planning and implementation of instruction at which teachers
were likely to use their knowledge and reasoning. The teacher actions and instructional
decisions at these moments could support the development of rich, connected understandings on
the part of students or they might not. Critical instructional moments do not define the quality of
instruction, but the teachers’ instructional decisions at those moments do. Thus, critical
instructional moments can serve as a unit of analysis in subsequent studies as researchers
continue to seek a deeper understanding of the relationship between teacher quality, teacher
training and development, instructional practice, and student outcomes. With the potential to
play such an important role in the emerging understandings of the mathematics educators and
researchers, continued refinement of the construct would be warranted. Are there other critical
instructional moments? Does there need to be greater specification of these critical instructional
moments? Are these moments the most critical? Critical instructional moments can serve to
reduce the complexity of the work of the teacher in the classroom to focus the attention of the
researcher on understanding instructional decisions at those moments and the impact of those
decisions on student outcomes.

Some of this work of refining the construct has already been initiated albeit
unintentionally. For example, Van Zoest, Leatham, and their colleagues have developed a
framework for identifying mathematically significant pedagogical openings (MOST) (Van Zoest,
Leatham, Peterson, & Stockero, 2013). The MOST framework identifies three characteristics of
these significant pedagogical openings: observable student thinking, significant mathematics
relative to the students, the goals for the course, and the discipline of mathematics, as well as
pedagogical openings that could lead to advancement towards the instructional goals. There are
several ways in which this work intersects with the construct of critical instructional moments and the work of this present study.

First, the MOST framework serves as a tool to classify a particular type of student response. While some student responses do not meet the criteria of a MOST moment, they all require some type of interpretation. The MOST framework provides a tool for identifying which ones might be most interesting as a researcher and useful as a teacher. In this way, the framework extends the work of this study and allows for a distinction to be made among the student responses requiring interpretation and potentially would be most deserving of the construction of a response.

A second point of intersection with the present study involves the nature of the process of identifying a MOST moment. For a teacher, identifying MOST moments would seem to be a critical element of instruction that maintains a triune focus on the mathematics of the student, the mathematics of the learning goal, and the potential path from one to the other. Identifying these moments requires the kind of in-the-moment reasoning examined and observed in this study. In this way, the work of these researchers affirms the importance of understanding the demands on teacher knowledge and reasoning presented by moments like these MOST moments while also providing a tool for identification of these moments during instruction.

The Construct of Pedagogical Content Reasoning

Identifying critical instructional moments in the planning and delivery of instruction served to focus the primary work of the study—understanding the nature of the instructional decisions at these moments and the demands placed on the mathematical knowledge and reasoning of the teacher during these moments. The analysis of the work of each of the four
teachers in this study at critical instructional moments revealed differences in their approaches at each of the four critical instructional moments.

Jackie clearly articulated a learning progression involving unpacked learning goals along with tasks designed and sequenced in ways that seemed to foster the development of the component understandings of the learning goal. Throughout her classes, she elicited student thinking and designed instruction in response to that thinking—providing tasks, counterexamples, questions, and other responses that made the salient features of the mathematics visible while supporting the learning goals.

Harold also expressed extensively unpacked learning goals for students and designed and sequenced tasks with those learning goals and the existing understandings of students in mind. However, he was not always successful in identifying tasks that activated the existing understandings of students and supported the attainment of his learning goals. Nevertheless, during implementation of tasks, Harold consistently maintained the level of cognitive demand and attended to student thinking in most instances. He structured the activity of the students by emphasizing representations, modeling internal dialogue, and using relational definitions that captured the essence of the concept rather than the instrumental use.

While Susan identified the full range of topics students needed to know to understand rational functions, her goals for students and unpacking of the mathematics of those goals were largely underspecified. This lack of specificity and articulation inhibited her selection and sequencing of tasks. While the lead tasks she chose offered opportunities for students to meet her learning goals, they did not reflect a clear sense of progression of ideas and understandings. Susan attended to the existing understandings of students in the selection of lead tasks and
attempted to use those existing understandings to support the development of new ones. However, the lead tasks were not designed to be particularly problematic to students in ways that would motivate the mathematical activity related to the learning goals. This characteristic seemed to force Susan into instructional situations in which she had to provide more of the explanation, clarification, and elaboration during instruction. Susan also attended to student thinking during instruction and often allowed it to shape instruction. However, Susan’s responses to student responses involved a limited use of associated tasks designed to bridge the conceptual gap between the expressed understandings of students and the mathematics of the learning goal.

Duncan’s learning goals were procedural and introductory in nature. He articulated no component or prerequisite understandings, but he designed tasks to incorporate a wide variety of previously studied functions for the purposes of review rather than engaging the existing understandings of students in ways that supported new understandings. While Duncan’s instruction was well organized, it typically revolved around the completion of procedurally oriented tasks. Those tasks were not problematic for students but they were accessible using the existing understandings and they supported his learning goals for students. Because the mathematical activity was largely procedural in the observed lessons, Duncan directed the student activity to a large degree, and the dialogue in class rarely focused on student conceptions.

These observed differences in instruction at critical instructional moments occurred along several dimensions for each critical instructional moment and can be summarized as follows. With respect to the articulation of learning goals and the unpacking of those goals into component understandings, differences were observed in the identification of component
understandings essential to achieving the learning goal, in the sequencing of those component understandings into a progression of understandings, and in the translation of learning goals to lesson goals. With respect to the nature of lead task selection, differences were observed in the consideration of the existing understandings of students, in the problematic nature of the task, in the goal directed nature of the task, and in the opportunity the task provided for students to scaffold new understandings to existing ones. With respect to the elicitation and interpretation of student responses, differences were observed in the extent to which student thinking was elicited, in the extent to which students or the teacher provided the mathematical explanations, clarifications, elaborations, or justifications, in the tolerance of the teacher for imprecision of thought or language, and in the tendency to focus on what was right about a student’s thinking. And finally, with respect to the responses to student responses, differences were observed in the use of associated tasks, in the nature of the direction of student attention to the salient features of the task, student thinking, or a mathematical idea, and in the degree to which student thinking was used as a springboard for inquiry.

The results of this study suggest that the accumulated mathematical knowledge for teaching in all its forms does not fully explain the mathematical activity of the teacher during instruction resulting in these differences. Rather, these differences are the result of the active application of mathematical knowledge and reasoning to the work of teaching. The process of making mathematical sense of a classroom situation and determining an instructional path during the planning or delivery of instruction involved in-the-moment reasoning about the mathematics of the classroom and the accumulated mathematical understandings of the teacher. It involved drawing mathematical and pedagogical conclusions in order to understand student thinking and
to design and implement instruction to support the development of rich, connected understandings of mathematics. For this reason, this active application of mathematical knowledge and reasoning is identified as *pedagogical content reasoning* and evidence of the role it played in the planning and delivery of instruction appeared throughout the data.

Pedagogical content reasoning is involved in the unpacking of learning goals and the identification of component understandings that support the learning goal. Unless a teacher has previously established the component understandings of a particular topic, the process of unpacking a learning goal involves asking the question, “What does a student have to understand in order to fully understand this idea?” Answering this question requires an analysis of the mathematics of that particular learning goal. Identifying the component understandings could often be simply drawing on specialized content knowledge. However, when a teacher reasons mathematically about the learning goal in the context of a particular group of students, the lesson goals become unpacked in a way that focuses the attention of the teacher on the specific conceptions that support the attainment of the learning goals. This heightened focus and more detailed attention supports the work of the teacher during other critical instructional moments—the selection and sequencing of tasks, the interpretation of student responses, and the construction of responses to student responses.

Pedagogical content reasoning is also involved when tasks are developed or selected and sequenced with a consideration of the mathematics of the student and the mathematics of the learning goal. To evaluate the accessibility of a task, a teacher must anticipate how a student might reason using his or her presumed existing understandings and ways of working to engage with the task. “How would I expect the student to respond to this task?” “What is the likely
solution path he or she will take?” This type of evaluation requires a teacher to draw conclusions about the observations he or she has made of students that reveal some elements of their existing understandings. It involves reasoning mathematically using those perceived existing understandings to solve the problem and anticipating what understandings a student might develop through the completion of the task. These anticipated understandings are then used to determine an appropriate and potentially productive sequencing of tasks designed to develop the component understandings and ultimately achieve the learning goal. The selection and sequencing of tasks undoubtedly draws deeply on the accumulated mathematical and pedagogical knowledge of a teacher and the repertoire of examples, representations, tasks, and other components of that knowledge base. However, since these considerations are so contextualized for a particular group of students, it is difficult to imagine that the selection and sequencing of tasks can occur simply by drawing on accumulated mathematical or pedagogical knowledge. The data suggests there is another dimension to this process of the design of instruction—pedagogical content reasoning—as a teacher considers a triune focus on the mathematics of the student, the mathematics of the learning goal, and the potential path from one to the other. When pedagogical content reasoning is applied to the selection and sequencing of tasks, the opportunities for students to develop rich, connected understandings are enhanced. Tasks are selected to target the development of specific, component understandings related to the learning goals. The existing understandings of students are used not just to review previously studied content, but they are engaged and used to support the development of new understandings. And finally, when pedagogical content reasoning is applied to this critical
instructional moment, tasks are selected and sequenced to support the incremental development of the component understandings.

During the implementation of instruction, an active application of mathematical knowledge and reasoning was also readily seen when a teacher carefully considered student thinking and constructed instructional responses to the responses of students. When a student responds to a task in some way—asking a question, providing a solution, proposing a hypothesis, and so on—a teacher must interpret that response, make inferences about the student's emerging conceptions, and assess the quality of those conceptions relative to the learning goals of the lesson. At times, this interpretation involves a straightforward drawing on the accumulated mathematical and pedagogical knowledge of a teacher. A student response could be familiar to the teacher in some way—a typical conception, a common error, or an alternative approach—and a teacher simply needs to consider how this idea aligns with those previously encountered in order to make sense of it. However, when pedagogical content reasoning is applied, the interpretation of student responses becomes more focused on the teacher building an conception of the student's conception in the specific context of the experience for that student. At times, a student response might be unanticipated and unfamiliar to a teacher. In those situations, the pedagogical content reasoning applied to the interpretation of student responses might involve the teacher determining the validity of a proposed solution method. It might involve reasoning to the logical conclusion of a student hypothesis to determine validity or a potential counterexample. As was the case in other critical instructional moments, the application of pedagogical content reasoning to the interpretation supports the work of the teacher in the construction of responses to student responses.
Finally, pedagogical content reasoning was observed in the construction of responses to student responses as well. Based on the response of a student and the inferences a teacher might make about the existing conceptions of the student, a teacher might design a task, develop a question or counterexample, or provide some other type of representation to direct a student’s attention to the salient features of the task, the mathematics of the student, or the mathematics of the task or learning goal. In order to offer a response to a student response that supports the development of student understanding related to the learning goal, the construction of the response requires a triune focus on the mathematics of the student, the mathematics of the learning goal, and the potential mathematical path from one to the other. In addition, any task selected or designed must be accessible to students and motivate the mathematical activity of the student so that he or she can form more productive conceptions relative to the learning goal. This process requires more than drawing on accumulated knowledge. Rather, it involves an active reasoning about how a student might use his or her existing understandings in the completion of the task and how that anticipated mathematical work might support the development of more productive conceptions relative to the learning goal. It involves reasoning about the mathematics of the possible tasks to select one that offers an appropriate opportunity for a student to advance his or her understanding. The active application pedagogical content reasoning to the construction of responses to student responses enables teachers to use student responses as springboards for inquiry. It supports the selection and design of associated tasks that vary in structure and complexity in ways that direct student attention to the salient mathematical features, that support the attainment of multiple learning goals and that maintain cognitive demand. It involves mathematical reasoning about the potential responses of students,
about the possible tasks, and about the potential mathematical development a task might promote as a teacher determines an appropriate instructional move.

**The Explanatory Power of PCR**

The data from this study provided ample evidence to suggest that the construct of pedagogical content reasoning holds some explanatory power. It seems to capture an element of the process of planning and delivering instruction that cannot fully be explained by current conceptualizations of the mathematical knowledge for teaching. However, the data also suggests another dimension to the explanatory power of the construct—a possible correspondence between the degree of alignment of a teacher’s instructional practice with the image of instruction and the extent and consistency of the application of pedagogical content reasoning to the design and delivery of instruction.

When the observed instruction of the four teachers in the study was compared to the image of instruction derived from the literature in chapter two, varying degrees of alignment with that image of instruction emerged. Jackie’s instructional practice aligned most consistently with that image as the tasks presented to students were problematic, accessible, and offered the students the opportunity to develop the desired, component understandings through their work on the task. She implemented those tasks in ways that maintained the student’s position as sense-maker and decision-maker, and she directed students to reflect on their work in ways that fostered the development of new understandings.

Harold’s degree of alignment with the image of instruction wavered somewhat. Harold often presented tasks to students that were problematic, yet they were not always accessible to students. In some cases, the tasks were not presented in a way that engaged the existing
understandings of students in goal-directed activity designed to promote the development of new understandings. When implementing the tasks, Harold maintained a high level of demand on students and demonstrated his ability to reframe tasks in multiple ways. Like Jackie, he kept the student as the sense-maker and decision-maker and designed tasks in response to the expressed understandings of students.

Susan’s instruction also contained several elements that aligned well with the image of instruction but many elements that did not. Susan’s goals for students were comprehensive and broad, but the conceptions of students that would support the attainment of those goals were not specifically articulated. Most tasks she presented to students were accessible to students as she gave consideration to the existing understandings of students in the selection of tasks. However, some tasks were not designed in ways that directed the mathematical activity of the students in support of the attainment of the learning goals. In essence, they did not present a problematic situation that engaged the existing understandings of students to resolve some dissonance or to answer a question. During the implementation of these tasks, Susan often resorted to teacher-provided explanations, justifications, clarifications, and elaborations. This choice often undermined the position of the student as sense-maker and decision-maker.

Duncan’s goals for students were procedural and introductory in nature. As such, the lead tasks he selected for students aligned with those goals. The selected tasks provided opportunities for students to use previously learned content but did not foster the use of those existing understandings to resolve a dilemma that could lead to a conceptual advance. They were more illustrative of procedures they were to implement rather than problematic for students. His implementation of these tasks typically involved student work on a task and a group
presentation of the solution. His interpretation of and responses to student responses demonstrated his command of the content. In a few instances, student responses required him to develop representations to address their confusion or to consider alternative solution methods in ways that supported his learning goals. His responses to student responses most often involved clarification of the application of a routine.

These differences in the nature of the alignment of instruction with the image of instruction seemed to correspond to differences in the extent to which each teacher engaged in mathematical reasoning in the planning and delivery of instruction—in other words, pedagogical content reasoning. Not only did Jackie’s instruction most closely match the image of instruction, but she also exhibited the most extensive and consistent pedagogical content reasoning at each critical instructional moment. For the other teachers, an apparent correlation existed between those instances in which their instructional practice aligned with the image of instruction and their application of pedagogical content reasoning during that critical instructional moment. When their instructional practice was less aligned, limited evidence existed of their application of mathematical reasoning at the critical instructional moment. These apparent links warrant future investigation.

**Implications**

The possible implications of this study are multifaceted. The constructs of critical instructional moments and pedagogical content reasoning offer researchers and mathematics educators useful tools for research, professional development, and teacher training.
The identification of these moments was grounded in the existing literature and refined through this study. If critical instructional moments are important pivot points for instruction, then the instructional decisions of a teacher at those moments will not only shape instruction in significant ways, but in doing so, will affect the opportunities students have to internalize important mathematical understanding. Identifying these moments and understanding instructional practice at these moments becomes vitally important for a number of reasons. For research efforts, a viable and effective articulation of critical instructional moments affords researchers a useful tool for focusing their work around these moments. It supports research to answer questions such as: What instructional approaches at these moments yield the strongest student outcomes? What influences instructional decisions at these moments? What professional development or teacher training efforts impact instructional decisions at these moments? For teacher development efforts, a clear articulation and identification of critical instructional moments can focus their efforts on the instructional practice in those moments and developing the skills and understandings of teachers to make the most of those moments.

Through the analysis of the demands on the mathematical knowledge of teachers at these critical instructional moments, this study established the importance of mathematical reasoning during instruction and established the construct of pedagogical content reasoning as a way to capture the dual nature of the reasoning. Efforts to conceptualize the domains of mathematical knowledge for teaching provide clarity about the diverse nature of the mathematical knowledge needed for teaching and many of the ways in which that knowledge intersects with pedagogical knowledge. By situating this study in the classroom, the research revealed the importance of an active application of that knowledge base during instruction and the role it played during critical
instructional moments. As such, pedagogical content reasoning becomes a potentially important skill to be developed in teachers. It is not enough to build the common content knowledge and specialized content knowledge of teachers. Professional developers and teacher educators need to develop the skill of teachers in using those forms of knowledge—reasoning mathematically to understand the mathematics of the student and breaking down the mathematics of the learning goal and constructing a mathematical path from one to the other.

Pedagogical content reasoning also adds another potentially important dimension to the conceptualizations of mathematical knowledge for teaching as a vehicle through which some forms of mathematical and pedagogical knowledge can be developed. For example, the application of PCR to the unpacking of learning goals seems to serve to build the specialized content knowledge of the teacher. The unpacking of the learning goal essentially involves answering the question, “What mathematical ideas and relationships does a student need to understand and how does he or she need to be conceiving of these mathematical ideas and relationships to understand this mathematical concept or topic?” It generates a set of component understandings for the topic. Presumably, the experience a teacher has in doing this unpacking relative to a particular topic affords the teacher the opportunity to draw on this work in the future. In other words, this work would not necessarily have to be done each time a teacher teaches a topic. Thus, pedagogical reasoning applied to the identification of learning goals would seem to be a mechanism for deepening the specialized content knowledge of the teacher.

The application of PCR to the selection and sequencing of tasks also holds potential for building the specialized content knowledge of the teacher. As tasks are identified or constructed to develop specific component understandings, to use the existing understandings of students to
develop new ones, and sequenced in ways that incrementally develop those understandings, the specialized content knowledge of the teacher is developed. These tasks and their sequencing add to the repertoire of examples and representations and ways of presenting mathematical content for teaching potentially enhancing the teacher’s specialized content knowledge or knowledge of content and teaching.

The application of PCR to the interpretation of and construction of responses to student responses would seem to work in concert in the potential development of the teacher’s knowledge of content and students and knowledge of content and teaching. As teachers build conceptions of student conceptions, those understandings of student ways of thinking of the topic under study can be added to the accumulated knowledge of content and students. As students respond to those tasks and through reflection on those responses, teachers can potentially deepen their understanding of how best to develop the component mathematical understandings the tasks were presumably designed to develop. This deepened understanding reflects an enhanced knowledge of content and teaching.

The model of teacher knowledge in action shown in Figure 2.2 in chapter 2 hypothesized this knowledge-building aspect of pedagogical content reasoning. As a teacher considers the observations from the classroom and draws on the knowledge and belief base to make instructional decisions, the reflection on the instructional interventions and the observed responses of students affords the teacher the opportunity to add to his or her existing knowledge base. This opportunity is represented by the arrow drawn from the pedagogical content reasoning to the accumulated knowledge, beliefs, and conceptions.
This theoretical understanding seemed to be supported by the data. When a teacher unpacks a learning goal, he or she may not have to do so again. The decompressed mathematics may be internalized and become a part of the specialized content knowledge of the teacher. As a teacher works to understand a student response mathematically, the experience is added to the accumulated understandings of common student conceptions. As tasks are developed to foster a particular set of understandings, they are added to the repertoire of examples or problems a teacher has at his or her disposal. As he or she reflects on the nature of instructional interventions and the responses of students, those experiences can get internalized as one of the various forms of mathematical and pedagogical knowledge. However, it is important to note that without a measure of the domains of mathematical knowledge, firm conclusions about this relationship are yet to be drawn.

A third important dimension of pedagogical content reasoning also emerged from the study and has implications for the field. The data suggest an important relationship could exist between the pedagogical reasoning of the teacher and instructional practice aligned with the image of instruction. If this image of instruction, based on theoretical and empirical studies, constitutes instructional approaches that foster rich, connected understandings on the part of students, then understanding the nature of this relationship between pedagogical content reasoning and these instructional practices represents an important pursuit due to the potential power of this finding. Developing a teacher’s ability to apply pedagogical content reasoning at critical instructional moments becomes paramount to changing practice to advance student outcomes.
Limitations

There are several elements of this study that warrant more careful consideration and suggest potential limitations of the conclusions. First and foremost, this study was limited in scale. While the work of the four teachers in this study at over 250 critical instructional moments was analyzed, the study only examined the practice of four teachers. The differences among the observed instructional practice of those four teachers meant that only a portion of these critical instructional moments involve instruction aligned with the image of instruction for this study. The contrasts were useful but did not provide the study did not provide a broad enough sample of teachers necessary to bring more confidence to the observations, analysis, and conclusions.

A second limitation involved the scope of the study. This study included no assessments of the mathematical or pedagogical knowledge of the teachers in the study. Work on the development of broad assessments of the mathematical knowledge for teaching secondary mathematics are yet to be fully developed. The work of the researchers who developed the MUST framework (Heid, Wilson, in press) could serve as a powerful foundation from which to develop these broad assessments. A second option for the assessment of the mathematical and pedagogical knowledge of the teachers in this study could have involved focusing the study on the teaching of a particular mathematical topic, using (or developing) an assessment of the knowledge of the participants of that topic, and observing lessons and analyzing data only from teaching on that topic. It was a conscious choice of the researcher to let the mathematical content be a function of the observed lessons and avoid examining the use of mathematical knowledge and reasoning as they taught a particular mathematical topic. The researcher wanted
to examine any potential ways of using mathematical knowledge that cut across all content domains and the focus on a particular topic could have limited conclusions in a different way by tying them to a particular mathematical topic.

Steps were taken in the design of the study to mitigate this limitation. The selection of highly experienced teachers with advanced degrees in mathematics or mathematics education was designed to assure some level of confidence in the assumption that they had a rich knowledge base from which to draw and that limitations observed in their application of mathematical knowledge and reasoning at critical instructional moments would not be limited by the depth and breadth of their knowledge. The analysis of the data suggested that this was indeed the case, but nevertheless, the study did not include an assessment of the mathematical and pedagogical knowledge of the participants and this dimension of the study represents a limitation to the application of the results.

A third limitation of the study involved the degree to which the findings of this study depended on some of the underlying assumptions made by the researcher. Three of those assumptions are highlighted here.

The first of those assumptions that served to shape the study in significant ways involved assumptions about critical instructional moments. This construct was developed for this study out of the theoretical and empirical literature that supported the identification of these pivot points for the nature of instruction. While this review of the literature provided a solid rationale for the identification of these moments, it is important to acknowledge that these moments may not represent a comprehensive list of the important moments of instruction. The distinctions made among these moments shaped the study in significant ways. The results of the study are
limited by the degree to which those research-based assumptions about which moments are
critical represent the range of critical moments of instruction.

The second assumption involved the image of instruction derived from the theoretical and
empirical literature and described in chapter 2. For this study, it was necessary to identify
instructional practices at the critical instructional moments for which there was some basis to
believe that these practices would support the development of rich, connected understandings on
the part of the students. The practices served as lens through which the data was analyzed to
provide some assurance that the researcher focused on the use of mathematical knowledge and
reasoning when teachers were engaging in teaching practices aligned with our goal of developing
rich, connected understandings in students. It was not of interest to the researcher to deepen our
knowledge of the use of mathematical knowledge and reasoning during the implementation of
teaching practices that were not aligned with the development of this type of understanding.
That focus would not seem to serve the most immediate needs of the field in understanding how
to train and develop math teachers to provide the kind of instruction that will have the most
powerful impact on our students and their understanding. Nevertheless, a focus on a particular
image of instruction brings with it a risk of a natural bias seeping into the research. The
mathematics education community has not demonstrated widespread agreement about
instructional practice even though patterns seem to be emerging. The image of instruction
described in this study represented an attempt by this researcher to build a model of instruction
from these theoretical and empirical studies but it is limited to this researcher’s review of a
particular set of research studies.
A third assumption underlying the results of this study involved the use of stimulated recall interviews. The researcher believed that this research method provided the best opportunity to understanding the thinking of the teacher at the critical instructional moments. However, the method is not without its limitations. As discussed previously and as others have suggested (Yinger, 1986), it is difficult to know whether the responses of the teacher during these stimulated recall interviews accurately reflect the thinking in the moment during instruction or a postlesson reflection about the teaching, or an in-the-moment response of the teacher during the interview. The researcher attempted to mitigate these potential limitations at the beginning of the interviews by emphasizing the need for the teacher to distinguish between responses that reflected thinking during the lesson from thinking after the fact. He also worked to triangulate conclusions about the thinking of the teacher using the background data, the pre-observation data, and other episodes and responses during the lesson and stimulated recall interview. Some specific references and distinctions made by teachers during the interviews reflected some awareness on their part of the need to draw these distinctions. Other data suggested that teachers were not as careful about making these distinctions as the research needed. Nevertheless and even with these limitations, the use of the stimulated recall interview represented the best option for accessing the teacher’s input on their instructional decisions and the reasoning behind them.

**Future Research**

While these limitations suggest that conclusions should be tepidly drawn, the reality of the evidence should not be overlooked. Patterns in the data were discernible and potentially
provide useful insights for the mathematics education community. At the same time, the findings raised a number of questions warranting further study.

**Further Work with Critical Instructional Moments**

Critical instructional moments are grounded in a large body of existing research and the delineations among them emerged through this study. Their identification can support future work of researchers efforts to understand the impact of mathematical and pedagogical knowledge on classroom practice, the role a dynamic application of that knowledge base plays, and ultimately connect the use of mathematical knowledge and reasoning to student outcomes. The identification of these moments can also serve teacher educators and professional developers as they focus the attention of teachers on particular moments of instruction and make efforts to support the development of powerful approaches in these moments.

However, several aspects warrant further analysis. Are critical instructional moments really critical? These moments were identified as moments that matter in the process of planning and implementing instruction but are they the only ones? Do they matter more than other moments during the planning and implementation of instruction? Studies involving the refinement of the identification of these pivot points during the instructional process are needed to solidify these definitions and delineations. At the same time, these moments are closely associated with the image of instruction derived from existing theoretical and empirical literature. While image of instruction derived from existing theoretical and empirical research, those elements and approaches warrant a continual review in order to further this research on two fronts. First, the critical instructional moments will necessarily be linked to the instructional practices that foster rich, connected understandings. If the image of instruction changes, the
critical moments might also. Second, if a relationship exists between pedagogical content reasoning and the image of instruction, then a robust image of instruction is essential for understanding a meaningful relationship.

**Further Work with Pedagogical Content Reasoning**

This study was not about what a teacher knows as much as it is about how he or she uses that knowledge in practice. The findings suggested that pedagogical content reasoning is a potentially useful construct for understanding how mathematical and pedagogical knowledge gets put into practice. While teachers clearly draw on their existing mathematical and pedagogical knowledge during the design and delivery of instruction, there is a more dynamic process at work that seems to be captured by this construct. The findings also suggested that some instructional differences among teachers could be explained by differences in the nature of their application of mathematical reasoning to the design and delivery of instruction. However, the limitation of the study discussed previously suggests the need for further work on the refinement of this study.

As discussed in the Limitations sections, this work did not include a careful assessment of the accumulated understandings of each teacher. Any advancement of this work and any potential understanding the construct of pedagogical content reasoning offers mathematics education will depend on future work that accounts for the existing understandings of teachers in specific ways. Studies that focus on a particular content area would allow researchers to test common and specialized content knowledge of teachers prior to instruction and more readily determine what knowledge might have been used and/or constructed in the moment and in what ways the teacher drew on existing knowledge. With the accompanying assessment of a teacher’s
mathematical knowledge, careful analysis could be completed to separate use of accumulated knowledge and pedagogical content reasoning at critical instructional moments. Such a study would confirm and extend the results of this study and support the exploration of the relationship between the mathematical knowledge of the teacher, the application of mathematical reasoning during the design and delivery of instruction, and the instructional practices of the teacher.

A second important area of future research with respect to the construct of pedagogical content reasoning would involve understanding the suspected correlation between the application of pedagogical content reasoning and the alignment of instructional practice with the image of instruction. Does this relationship exist? How strong is it? This type of research, if the suspected correlation is confirmed, would provide remarkable clarity for mathematics researchers and teacher educators allowing a greater focus on the development of pedagogical content reasoning to produce instructional reform. It would also lead to other important investigations. For example, what accounts for differences in pedagogical content reasoning? How does one develop pedagogical content reasoning?

**Conclusion**

In spite of the limited scope of this study, the findings provide an important glimpse into areas of great importance for mathematics education improvement efforts. Understanding the nature of mathematical knowledge for teaching has captured the attention of many researchers in recent years. The results of those efforts suggest the need for continued refinement of the conceptualizations of the domains of mathematical knowledge for teaching. This study worked from the conceptualizations of Ball, Hill and their colleagues to examine dimensions of the
mathematical and pedagogical knowledge of teachers that might be underrepresented in their work. The results of this study suggest that refinements to our conceptualizations may need to incorporate mathematical and pedagogical reasoning into the domains of accumulated knowledge. The recent work on the mathematical understanding for secondary teaching (MUST) framework (Heid, Wilson, in press) reflects the same realization that our conceptualizations should include a more dynamic component that accounts for the mathematical activity of the teacher. While a teacher cannot reason about knowledge he or she does not possess, he or she cannot apply his or her knowledge effectively without considering the context in which that knowledge is being applied. That application requires an active use of mathematical reasoning to make sense of the mathematics of the student in the context of the learning goals in order to make instructional decisions.
Appendix A: Reformed Teaching Observation Protocol (RTOP)

Daiyo Sawada          Michael Piburn
_External Evaluator_ _Internal Evaluator_

and

Kathleen Falconer, Jeff Turley, Russell Benford and Irene Bloom
_Evaluation Facilitation Group (EFG)_

Technical Report No. IN00-1
_Arizona Collaborative for Excellence in the Preparation of Teachers_
_Arizona State University_

I. BACKGROUND INFORMATION

Name of teacher ____________________  Announced Observation? __________
Location of class __________________________________________________________ (district, school, room)
Years of Teaching____________________  Teaching Certification ______________
Subject observed____________________  Grade level _________________________
Observer __________________________  Date of observation _________________
Start time __________________________  End time __________________________

II. CONTEXTUAL BACKGROUND AND ACTIVITIES

In the space provided below please give a brief description of the lesson observed, the classroom setting in which the lesson took place (space, seating arrangements, etc.), and any relevant details about the students (number, gender, ethnicity) and teacher that you think are important. Use diagrams if they seem appropriate.
Record here events which may help in documenting the ratings.

<table>
<thead>
<tr>
<th>Time</th>
<th>Description of Event</th>
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### III. LESSON DESIGN AND IMPLEMENTATION

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<tr>
<td>1)</td>
<td>The instructional strategies and activities respected students’ prior</td>
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<td>knowledge and the preconceptions inherent therein.</td>
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<td>2)</td>
<td>The lesson was designed to engage students as members of a learning</td>
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<td>community.</td>
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<td>3)</td>
<td>In this lesson, student exploration preceded formal presentation.</td>
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<td>4)</td>
<td>This lesson encouraged students to seek and value alternative modes of</td>
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<td>investigation or of problem solving.</td>
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<td>5)</td>
<td>The focus and direction of the lesson was often determined by ideas</td>
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<td>originating with students.</td>
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### IV. CONTENT

#### Propositional knowledge

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<td>6)</td>
<td>The lesson involved fundamental concepts of the subject.</td>
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<td>7)</td>
<td>The lesson promoted strongly coherent conceptual understanding.</td>
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<td>8)</td>
<td>The teacher had a solid grasp of the subject matter content inherent in</td>
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<td>the lesson.</td>
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<td>9)</td>
<td>Elements of abstraction (i.e., symbolic representations, theory building)</td>
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<td>were encouraged when it was important to do so.</td>
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<td>10)</td>
<td>Connections with other content disciplines and/or real world phenomena</td>
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<td>were explored and valued.</td>
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#### Procedural Knowledge

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<tr>
<td>11)</td>
<td>Students used a variety of means (models, drawings, graphs, concrete</td>
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<td>materials, manipulatives, etc.) to represent phenomena.</td>
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<td>12)</td>
<td>Students made predictions, estimations and/or hypotheses and devised</td>
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<td>means for testing them.</td>
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<td>13)</td>
<td>Students were actively engaged in thought-provoking activity that</td>
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<td>often involved the critical assessment of procedures.</td>
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<td>14)</td>
<td>Students were reflective about their learning.</td>
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<td>15)</td>
<td>Intellectual rigor, constructive criticism, and the challenging of ideas</td>
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<td>were valued.</td>
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Record here events which may help in documenting the ratings.

<table>
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<tr>
<th>Time</th>
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## V. CLASSROOM CULTURE

### Communicative Interactions

<table>
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<tr>
<th>16) Students were involved in the communication of their ideas to others using a variety of means and media.</th>
<th>Never Occurred</th>
<th>Very Descriptive</th>
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<th>17) The teacher’s questions triggered divergent modes of thinking.</th>
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<th>18) There was a high proportion of student talk and a significant amount of it occurred between and among students.</th>
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<th>19) Student questions and comments often determined the focus and direction of classroom discourse.</th>
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<th>20) There was a climate of respect for what others had to say.</th>
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### Student/Teacher Relationships

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<th>21) Active participation of students was encouraged and valued.</th>
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<th>22) Students were encouraged to generate conjectures, alternative solution strategies, and ways of interpreting evidence.</th>
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<th>23) In general the teacher was patient with students.</th>
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<th>24) The teacher acted as a resource person, working to support and enhance student investigations.</th>
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<th>25) The metaphor “teacher as listener” was very characteristic of this classroom.</th>
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Additional comments you may wish to make about this lesson.

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2000 Revision

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Appendix B: Background Interview Guide

The interview will capture a variety of background data:

- Educational background including degree, major, relevant coursework in mathematics, psychology, and education
- Teaching experience including years of teaching, courses and grade levels taught, schools
- Professional development work focusing on work the teacher identifies as significant
- School profile data including average class size, college matriculation rate, and average score on math portion of the SAT or ACT.

The interview will examine the views of the teacher about mathematics, teaching, students, and learning through the following questions:

What is mathematics?

How do you convey that understanding of mathematics to students through your class, the activities, and your teaching?

What are your goals for students? What understandings are most important for your students to walk out of your class with?

What are some of the ways you accomplish those goals?

How do you decide what kind of activities to use with students?

How do you plan instruction? How might your plans differ from what is actually accomplished during a given class?
Appendix C: Pre-Observation Interview Guide

Pre-observation conference:
1. What are your goals for the lesson?

2. What are the activities in which you will engage the students to meet those goals?

3. How do you expect them to respond?

4. What difficulties do you expect them to encounter and how will you handle those situations?
Appendix D: Stimulated Recall Interview Guide

During the preliminary analysis of the observation, the researcher will design questions to elicit teacher reasoning during interview that will be contextualized to fit the situation. The following questions represent the types of questions that will be used:

a. Tell me about this part of the lesson.

b. With reference to a teacher’s action...
   i. What was your intent?
   ii. Did you consciously think that?
   iii. Is that different from how you think about it now?
   iv. How did you come to decide to do that?
   v. What did you hope to gain from that choice of problem, representation, question, example...?
   vi. Have you seen that before or did you have to think about it on the spot?
   vii. Would you handle that situation differently if it came up in another class?
   viii. Why did you choose to do that when you did?

c. With reference to student work, responses, questions...
   i. How did you interpret the student’s response at the time?
   ii. What, if anything, was important about it?
   iii. What did you think the student understood at this point?
   iv. Do you remember making a conscious observation about the student’s understanding at the time?
   v. Is that different from how you interpret the situation now?
   vi. Why did you think that then and now?
   vii. Why did you choose to address (or ignore) this response?
   viii. What do you think other students understood about this student’s work that was made visible to them?
Appendix E: Lead Tasks for Duncan

Lead Tasks from Observation #1

**Lead Task:** Students were asked to graph a set of parametric relationships defined by the table of values. (Lines 27–134)

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**Task Presentation**

Lines 68–78 (excerpts):
D: What is the independent variable?
D: And the independent variable when we graph it goes where: horizontally or vertically
D: So the first thing we are going to do is graph $x$ vs. time which means we are looking at just the first two columns so we want to plot just the values in those first two columns

Line 117–22 (excerpts):
D: Alright, you guys do $y$ of $t$
D: And what do you suppose this is going to represent?
Student: The change in the $y$

Lines 129–134 (excerpts):
D: What is more interesting, I think, is that we looked at how it moved horizontally on the first graph, how it moved vertically on the second graph but we look at it’s movement together- horizontal and vertical together. So, what I am asking you to do here (draws coordinate axes and labels them $x$ and $y$), is do $x$-$y$, just like we normally do over 4 seconds. You guys do this one. Put it together.
Lead Task: Students were asked to graph a set of parametric equations on their calculators that modeled the populations of foxes and rabbits. The equations were given (Lines 196–224)

\[ R(t) = -500 \sin \left( \frac{\pi}{6} t \right) + 1000 \]
\[ F(t) = 50 \cos \left( \frac{\pi}{6} t \right) + 150 \]

Task presentation:
D: Back to the days of trig. Let’s see how much you guys remember. Okay, first thing. Alright, this is going to be rabbits and foxes. Alright, quick question, if we look at rabbit populations and fox populations over time, do you think they are related?
Student: No.
Student: Yes.
Student: (inaudible)
Student: Cause foxes eat rabbits.
D: Cause foxes eat rabbits. So we know, foxes eat rabbits. So, let me ask you intuitively for a minute, if the rabbit population’s climbing (motions upward with hand) what do you suppose the fox population is going to do?
Student: Decline
Student: (multiple responses speaking over top of one another)
D: David?
Student: Increase
D: why?
Student: Food.
D: There’s more food. So, if the rabbit population is going up, here comes the fox population and as the fox population goes up, what do you think is going to start to happen to the rabbit population?
Student: Go down
D: It is going to go down (motions a curving downward with hand). And as that food source goes down, what do you think is going to happen to the fox population?
Student: Go down
Student: Level off
D: Go down, level off perhaps and then go down. And do you guys see how they could cycle back and forth? (motioning with hands up and down—sinusoidally). What type of equation have we seen like this before?
Student: Sine and cosine
**Lead Task:** Students were asked to graph a set of parametric equations on their calculators that modeled the motion of the London Eye. The equations were given (Lines 754–774):

\[
\begin{align*}
\xi(t) &= 225 \sin \left( \frac{\pi}{15} t \right) \\
\eta(t) &= 225 \sin \left( \frac{\pi}{15} (t - 7.5) \right) + 225
\end{align*}
\]

**Task presentation:**
D: Alright, next example. This is probably going to be the last one. Anybody been to London? Anybody been up... Did you say you had? Did you go on the London Eye, the Ferris wheel? Do you remember how big it was? Was it fast? Pretty slow.
Student: About 4 mph?
D: I don’t know how fast it, but we could figure it out. It’s actually 30 minutes (teacher writes 30 min/revolution and 450 ft diameter on the board). This is called the London Eye. It’s a huge Ferris Wheel. 450 feet in diameter. You get on at the bottom and it’s going slow enough where you can actually just step off of it and other people can just step onto it. And it takes 30 minutes to go around.
Student: One time?
D: One time. It’s pretty slow. It gives everyone a chance to see London. And you are 450 feet off the ground which would be roughly 45 stories, 50 stories when you are at the top, so you have a pretty good view of everything. Now, I am going to give you a couple of equations.
**Lead Tasks from Observation #2**

**Lead Task:** Students were asked to find the parametric equations of a circle and an ellipse with given characteristics.  
(Lines 7–20)  
What we are going to do first is just a quick review from yesterday. Two problems. [D shows two problems on the screen.]

3. Find the parametric equations that represent a circle, centered at (-2,3), radius of 4 and starts at the point (-6,3).
4. Find the parametric equations that represent an ellipse whose major vertices are (4,10) and (4,0) and whose minor vertices are (2,5) and (-6,5).

**Task presentation:**
Duncan: Do the first one, just do it on your calculator. Once you think you have it, raise your hand, I want to take a look at it. Again, radian mode, parametric mode, that’s what your calculators should be in. You pick an appropriate window. I am not going to tell you the window. But I want you to show me the parametric equations that give me that circle, centered at that point with a radius of 4 and starts at (-6,3).

**Lead Task:** Students were asked to model a race between two people with one person starting six seconds later than the other and finishing at the same time. The modeling was to be done parametrically and on the graphing calculators.  
(Lines 183–229)  
Duncan: Okay, we have two people running 100 meters. The first person takes off running at a constant rate the whole time. It takes them 12 seconds to run the race. So they are pretty fast, but not super fast. The second person waits until the first person is 50 meters ahead and they take off running and catch them. So the second person is a burner. He can fly. It would be a world record, so they may be on a bike or in a car or something. I want to show you how we can look at this on our calculators.

**Task presentation:**
Duncan: Now I want to show you one more thing we haven’t done and you are going to need your calculator. But before we do this, everybody go to mode on your calculator. It is about halfway down. Yours is probably highlighted as sequential, it should be simultaneous. Keep yourself in radian mode. Keep yourself in parametric mode. Now, here’s the idea. So first thing, do we have any horizontal distance we are talking about? 
Student: 100 meters  
Duncan: Yes. Do we have any vertical distance we are talking about? 
Student: No
Duncan: No. So what I would like to do, I would like to see both runners. So if you think about it, they are running in a straight line, a horizontal line. What is the equation of a horizontal line?

Student: \(y = mx + b\)

Duncan: Horizontal. No, okay, I am okay with that. What would \(m\) be...if it is horizontal?

Student: zero

Duncan: zero. It is really just \(y = b\), \(y\) equals a number. Okay? So here is what we are going to do. [D begins to write on board] I am kind of setting you up on this one. I am giving you a bit of extra information so we can do the next problem I want. [continues writing] Now we are going to do parametric equations for both runners. Now what I want you to do is to pick just a couple of random numbers for \(y\). [D writes \(y(t) = 4\) under Runner 1 and \(y(t) = 8\) under Runner 2.] What does \(y = 4\) look like?

Lead Task: Students were asked to model the motion of the hands of the clock between noon and one o’clock. (Lines 483–541)

Duncan: Because here is the next problem. This is the one I am interested in for today. This is the focus of my lesson for today. [D shows the problem on the overhead]

_A circular clock has a minute hand that is 12 inches long and an hour hand that is 9 inches long. Use parametric equations to model the movement of both hands as they move from 12 o’clock to 1 o’clock._

Task presentation:

Duncan: We are going to do groups of three randomly selected. [D goes through a group selection process with students] Okay, John, give me a number, 1–18... [and so on] Get in your groups. Now, before we start. If you look here, think about this clock that is right above my head. We are going to start at 12 o’clock. The long hand which is called the minute hand, for some of you who have forgotten, will go all the way around the circle in one hour, but the shorter hand, the hour hand is only going to go from 12 to 1. So you are going to give me parametric equations for two circles that are different radii. I told you what they are. But you’ve got to figure out because one circle is going all the way around while the other is only make part of a circle. See if you can figure it out. Get with your partners. [Students working in groups; D moving around the room].
Lead Tasks from Observation #3

**Lead Task:** Students are asked to convert several sets of parametric equations into rectangular form.
(Lines 3–162)

Duncan: Now today, let me pull up what we are going to do today. [D goes to computer]. Okay, here are some equations [D projects the following equations on the board].

4. \( x = \frac{3t}{4}, y = 2t - 1 \)
5. \( x = 2^t, y = \frac{4}{t} \)
6. \( x = t^2 - 3, y = t^2 + 1 \)
7. \( x = ln(t), y = e^{ln(t)} \)
8. \( x = \sec(\theta), y = \cos(\theta) \)
9. \( x = 2\cos(\beta), y = 2\sin(\beta) \)

**Task presentation:**
Duncan: Now, I want to do a couple of these with you and then I am going to ask you to do a couple on your own and that means you can talk to somebody as you do this. Here’s what we are going to do today. I am going to show you how to convert from parametric equations, which those are, to rectangular. Now, I am not sure I have used the word rectangular much in describing functions, but you know when you guys graph, isn’t our graph paper rectangular? Well, it is actually squares, but squares are special rectangles. So when I say rectangular, I am talking about an x-y axis. So we do things in terms of x and y. That’s what you guys have been doing ever since you guys have been graphing functions. Since you started doing stuff in terms of x and y. But there’s other…there’s polar graph which we will talk about next week which is actually circular graph paper. And you can do some things on that.
Appendix F: Lead Tasks for Harold

Lead Tasks from Observation #1

**Lead Task:** Students were asked to construct a model one of the component understandings for the division of fractions.
(Lines 1–96)

Harold: What’s another way to write a divided by b?

**Task presentation:**
Harold: Guys, I want you to know that all you are doing at this point is writing the same thing in a different way. Are we okay with that? Whereas here [pointing to the first equation in the figure] these are two very different things. What we are saying is that these two are always equal. There’s a big difference. You guys understand that, right? We had to basically prove this yesterday, yeah? Do we need to prove this [pointing to the second equation]? This means that [pointing to the two sides of the equation]. Does everybody understand the difference?
Student: yep.
Harold: I want you to tell me something that is equal to both of these that is different. I’ll give you a hint, that’s not division.

**Associated tasks:**
- “What is another way to do six divided by 2?” [Harold Ob #1, Lines 161]
- “So how would you express this idea with variables?” [Harold Ob #1, Line 220]
- “What’s a fourth of twelve?” [Harold Ob #1, Line 261].
- “What would twelve groups of a fourth look like?” [Harold Ob #1, Line 313]
- “[Drawing a rectangle] Here’s a whole…[Dividing it into four parts and shading one part] Here’s one-fourth of a whole, right?...Let’s say we have twelve of these. What’s that equal?” [Harold Ob #1, Lines 333–339]
- “On your own sheet of paper, model twelve times a fourth. Model it and figure out what it equals.” [Harold Ob #1, Lines 459–460]
- “[Holding up a quarter coin] This is a fourth of a dollar. How can I use these to model this [pointing to twelve times one-fourth]? [Harold Ob #1, Lines 545–546]
- “I want you to multiply three times two using dollars. Model that now.” [Harold Ob #1, Line 603]
Lead Tasks from Observation #2

**Lead Task:** Students are asked to solve a system of linear inequalities.
(Lines 2–178)

Harold: We will spend the rest of class time trying to finish up solving this system of inequalities. That’s all of the class time we are spending on the quiz. If it is ready now, please turn it in. If not, Monday is okay. Put everything away. Work in your group to solve this system.

---

**Task presentation:**

Harold: Tom, what does it mean to solve a system?
Student: We went over this yesterday. To show all of the solutions to the system.
Harold: Let me ask something real quick. What do you guys think of the word, “Show?”
Student: To represent.
Student: To define.
Harold: What do you guys think of the word, “Show?” I like the word, “Show” especially for this, because what is the only way to find all of the solutions?
Student: To show them
Harold: You would have to represent them and what is the only way to represent them in this sort of system?
Harold: You have to show them graphically. I like the word show. Keep going Tom. You said it and I kind of cut you off. I think you said quite well.
Student: To show…what did I say?
Harold: To show all of the solutions to the system.
Harold: What makes something a solution a solution to a system, Allen?
Student: To make both equations true
Harold: Now you can memorize that it makes both equations true. Are you going to sound silly?
Student: Yes.
Harold: Try again. We don’t have two and they aren’t equations so we know your definition can’t make sense, right?
Student: To show the solution we would show all (inaudible) inequalities…
Harold: Could you have an equation thrown in here?
Student: Yes, you could
Harold: Do you guys know the word I use to describe equations and inequalities?
Student: Statements
Harold: Statements. I use it very deliberately. By saying statements, you can cover whether it is an equation or an inequality.
Harold: Would it be like me to maybe thrown in a couple of equations?
Student: Yea, it would.

**Associated tasks:**
Harold: Here, let me show you something. I could give you this system…And add this to it…What is the only possible solution to it?
**Lead Tasks from Observation #3**

**Lead Task:** Students are asked to solve a system of inequalities—the same system they worked on the previous class.

(Lines 4–136)

![System of Inequalities]

**Task presentation:**

Harold: Does everybody remember what we are trying to do? [Pause] Matt, what are we trying to do? [Pause] What’s the problem asking us to do?

Student: [inaudible]

Harold: John, what are we trying to do?

Student: Solve the system

Harold: What’s that mean?

Student: Find the values. [H changes his expression] Find the value or values [H gestures as to ask for more]

Harold: [to the class] Alright guys, so far we got two strikes on this question. Joni, what are we trying to do?

Student: Use a calculator to help

Harold: What are we trying to do on this problem, Joni?

Student: Well, I am going to get one number to make all of those true.

Harold: All of what true?

Student: All of the equations…well, two numbers a y and x

Harold: That do what?

Student: That make all of the statements true up there.
Lead Tasks from Observation #4

**Lead Task:** Students are asked to model a situation in which a quantity of money is distributed among four people.

(Lines 3–127)

Harold: Let’s do [H writes on the board] “$12 needs to be distributed evenly among four brothers.” My sister has three boys, but let’s say she has four of them. The four boys need to distribute $12 evenly among themselves.

**Task presentation:**

Harold: I want you to model, and if you want you can think of it use the model to answer the question but what are we focusing on? What’s our objective? If I am giving you a question or problem, is your goal to get the answer?

Student: No

Harold: What’s your goal?

Student: (in unison) to understand

Harold: Very good. But do I want you to answer the question?

Student: Yes.

Harold: I am going to leave it somewhat open ended. In what ways do I want you to model it?

Student: Graph paper regions, lines, I mean number lines

Student: Pictures

Student: Circles

Harold: Let’s take all of those and put those under a visual model. A drawing. So give me just one drawing of it. You can use number lines, regions, but just give me one visual model. What other models are there? What other ways are there to represent it?

Student: Regions

Harold: That’s a visual model

Student: (varied indiscriminate responses)

Harold: Could you represent this algebraically?

Student: Yes

Harold: Do so. In this case it’s numerically, because in this case are we going to have any variables.

Harold: So does everybody understand the situation? My sister, she has 12 dollars. She gives it to her four sons. They have to distribute it evenly among themselves. Does that make sense? So do they get 12 dollars each?

Student: No.

Harold: Very good. When you think you got it, let me know. Both numerical and visual model. You can do either first. Just make sure they coincide with each other.
**Lead Task:** Students are asked to consider a variation of the previous situation in which the money is in quarters, not dollars, and produce a numerical statement that represents the situation.
(Lines 274–378)

**Task presentation:**
Harold: So, I am going to do it a little bit differently. I want you guys to come up with the numerical statement for the way I divide it up. Does that make sense? So I am going to show you the model, and you come up with the numerical statement. Same problem but I am going to do it totally differently. Ready? The way I am going to do it is let’s say that my sister, instead of having 12 dollar bills, let’s say she has a bunch of quarters. So all of the quarters are grouped into dollars.
Student: So it would be 48 quarters.
Harold: Nice. So she’s got these 48 quarters but they are all grouped into dollars. So what she is doing is that she is giving one quarter from each dollar to each kid. So here’s what it looks like. Here are her four quarters.

Harold: Each quarter is going to a different kid

Harold: And she’s doing that until she runs out of money. So it’s kind of like she doesn’t know she has 12 dollars. Instead, she has all of these quarters, but she is giving each kid a quarter until she runs out of money. Does that make sense?
Student: She has to have an even amount of coins.
Harold: She’s still got 12 dollars in quarters, she just doesn’t know it.
Student: How many kids does she have?
Harold: Still four. And does everybody understand that is why she is giving a quarter each?
Student: She’s got all of these quarters. She knows she’s got some number of dollars. So, hey, an even way to split it up—and if you guys were in groups of four this would be a little better—but let’s say that I wanted to split it evenly. I am not sure how much money I
have, but I know I need to split it evenly, so I’ll give a quarter to each of you. Right?
I’ll go all the way around. Questions on that?
Student: Wait. How many kids are there though?
Harold: Same number as before. Same amount of money. So I haven’t changed the
problem. What have I changed?
Student: The model and the numeric
Harold: [pointing to the model] Is the model different?
Student: Each kid will get 12 quarters.
Harold: I am doing this, twelve times. [H extends model]

Student: (inaudible)
Harold: Is that true? Good question. I want you guys to… by the way, how much money
would each kid get?
Student: They would each get 12 quarters.
Harold: It’s still gonna be three bucks, right? So each kid is going to get 12 quarters. Cause
again, one quarter per dollar.
Student: Yea
Harold: So everybody understands the model and why it would work? Write it numerically.
So, everybody recognizes that your model matched up with your numerical expression.
Right? You had 12 things divided into four groups and you got three things in each
group. I want you to model this method of distributing the money.
Student: You just did.
Harold: Oh sorry, I want you to write the numerical expression for it. State something
numerically that matches up with this.
Appendix G: Lead Tasks for Jackie

Lead Tasks from Observation #1

**Lead Task:** Students were asked to find all of the missing lengths and angles of the given kite.  
(Lines 21–26)

Task presentation:
Jackie: Meanwhile, let’s look at the bonus from yesterday.  (Teacher draws a kite with various measurements labeled.  The problem asked students to find all missing lengths and angles.) So here’s the deal.  Find all measurements: sides, angles and diagonals.  Give me any piece of information you know.
Lead Task: Students are asked to find the length of a diagonal of a parallelogram given two sides and the included angle. (Lines 467–510)

Task presentation:
Jackie: Alright, a new thing. The other day we looked at a parallelogram. (Teacher draws two parallelograms on the overhead with sides 5 and 9). I have two parallelograms whose sides are 5 and 9. Do they have the same length diagonal? (teacher draws the longer diagonal in each parallelogram)

Student: No.
Jackie: Why?
Student: Because that angle measurement changes?
Jackie: I am asking you.
Student: It does.
Jackie: It does. It seems to. What if it was a rectangle? (teacher draws only a right triangle with legs 5 and 9) Which one has the longer diagonal?
Student: The middle one.
Student: They look the same.
Student: It’s because the opposite angle is going to be across from the bigger side or something like that. I mean the larger angle is going to be across from the larger side.
Jackie: Yeah
Student: The angles are getting bigger or smaller so the sides have to change.
Jackie: So if I wanted to find the length of the diagonal, what would I have to know?

**Lead Task:** Students are asked to draw parallels between this geometric problem and a vector problem with forces.

**Task presentation:**
Jackie: Now how is this useful. How many parallelograms do we need the diagonal, the measure of in this way? Maybe not so many, right? But, imagine this. We are trying to move a piano. You and I. I am assuming you are stronger than I am. We are standing here (teacher points to the vertex of the parallelogram originally formed by the sides labeled 5 and 9), where the piano is. It is not a big piano, so we are really not able to get where we can both be behind it pushing at the same angle. We are trying, but inadvertently, you are pushing that way and I am pushing this way. The angle between what we are pushing is 35 degrees. I am able to push with a force of 5 pounds. You being stronger are pushing with a force of 9 pounds. So I am here and you are here and we are pushing the piano.
Student: Are we trying to push it up a hill?
Jackie: No, we are not, just across the room?
Student: Can we use an incline to maybe make the force a little more strenuous?
Jackie: Let’s just go with this. Guess what AC represents.

**Lead Task:** Students are asked to determine the speed and direction of a swimmer given the speed and direction of the swimming and the current.

Jackie: [Using the diagram produced in the task presentation below] Let’s say the current is 3 miles per hour, not an unusual current in the ocean and let’s say you can swim two miles per hour. What is your speed in the actual water?

**Task presentation:**
Jackie: Some of you are swimmers. Suppose you swim out into the ocean. Here’s the beach and you swim out into the ocean (teacher draws an arrow perpendicular to the
“beach”). Here’s your chair and beach towel. (teacher draws a box at the endpoint of the arrow on the “beach”).

Jackie: What inevitably happens when you turn around and look for your beach towel?
Student: You have drifted away.
Jackie: Yeah, you inadvertently ended up swimming there. (teacher draws a dotted line from the starting point on the “beach” forming an approximately 45 degree angle with the swimming path).
Jackie: Why?
Student: Because of the current.
Jackie: Because of the current. Here is what you were doing. (teacher traces the arrow) and here is what the current was doing (teacher draws an arrow parallel to the “beach” from the endpoint of the arrow and to the endpoint of the dotted line).

Jackie: And the diagonal is where you actually ended up.
Lead Tasks from Observation #2

**Lead Task:** Students were asked to find the coordinates of the endpoint of a resultant vector given two vectors and the angle between them. (Lines 4–34)

Jackie: This is what I am really interested in (teacher draws the line from the origin to the point (12.5, 5.63)). What is the length of that? And what is the angle? I am just curious if putting all of this on the coordinate plane made that easier.

![Graph](image)

**Task presentation:**
Jackie: I had vectors of 6 and 10.5 and an angle of 70 and I wanted to put it on the coordinate plane. You were going to find out for me how far to the right and up I needed to get on the coordinate plane to make the 70 degree angle work for a measure of 6. Right? So, what did you do?

![Graph](image)

Jackie: There are a lot of ways to do it, what did you do? We didn’t get to finish it in class yesterday, so I said finish it for homework. Remember, we drew a triangle that was similar to it (Teacher draws a right triangle with a 70 degree angle) and we said what is the relationship to the sides and observed that we could use the trigonometric properties, sine, cosine, and tangent and talk about the ratio of the sides.
### Lead Tasks from Observation #3

**Lead Task:** Students are asked to find the coordinates of the endpoint of the diagonal of a parallelogram given the two sides and the angle between them.

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<th>Task presentation:</th>
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<tr>
<td>Jackie: I kind of want to go back to where we were the other day. So this is a different situation, but we had some vectors and we put the vectors on the coordinate plane. Do you remember that? And then we found the ordered pair for the end of the vector, where we would have our arrow, up here. And then we used that ordered pair to get the angle of direction. So we are calling this angle alpha (teacher labels the angle formed by the vector and the horizontal axis). So let’s catch up with this diagram. We have a vector of 6 and one of 8 and I want to get that ordered pair. Can you remember how we did that?</td>
</tr>
</tbody>
</table>
**Lead Task:** Students are asked to find the angle the diagonal makes with the positive x-axis and the length of the diagonal.
(Lines 183–201)

Jackie: Now I want us to look at this triangle that I am going to call ABC in a minute (teacher traces the triangle with a squiggly line). And I want you to get the angle here.

**Task presentation:**
Jackie: So, I am going to trace this (teacher overlays a transparency to trace triangle). There is a lot of stuff on here that we do not need. We need this 12.6 and 3.9. So what is the length of this side?

Student: x
Jackie: 12.6 And this side?
Student: 3.9
Jackie: Can you get me this angle? Do it. Write an equation.
Student: Which angle?
Jackie: This angle right here (pointing to the angle formed by the vector and the x-axis).
Student: okay, that one.
Jackie: Does it help to look at it without all of this other junk? Just sort of isolating that triangle?
Student: yes
Jackie: Good. Cause you can do that for yourself by just copying something out.
**Lead Task:** Students are asked to develop a way to get the angle formed by a vector and the positive x-axis given the coordinates of the endpoint of the vector without thinking about a triangle.
(Lines 248–340)

Jackie: Excellent. Let’s see. I want to try an experiment. What if there was no triangle (teacher draws the following diagram on the overhead) And I wanted to find alpha.

![Diagram](image)

**Task presentation:**
Student: You could make a triangle.
Jackie: I don’t want to make a triangle. I am tired of making triangles. What did Marlee do that she could have done without a triangle?
Student: Well, if you had a triangle, you could tell what the vertices would be because of the coordinates.
Jackie: What do you mean because of the coordinates?
Student: Well, you know that one side is 12.6 and the other side is going to be 3.9…
Jackie: So you are still thinking about a triangle.
Student: yes.
Jackie: So if there was a triangle there, those would be the right sides, but I don’t know that you have to draw it.
Student: Yea.
Student: That’s assuming we have a right triangle.
Jackie: Well we have control over that because our imaginary triangle is whatever we want, right? Okay, could we take this a step further. I want to make myself clear. I want you to look at this lovely equation (teacher boxes in the equation involving tangent from the previous problem) and I want you to tell me if there is a way to get that equation up here (pointing to the simplified diagram) without creating an imaginary triangle?
Appendix H: Lead Tasks for Susan

Lead Tasks from Observation #1

<table>
<thead>
<tr>
<th>Lead Task: Students were asked to recall the definition of rational numbers and make some connections to rational functions. (Lines 1–32)</th>
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<tbody>
<tr>
<td>Susan: What is a rational number?</td>
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</table>

<table>
<thead>
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<th>Task presentation:</th>
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<tr>
<td>Susan: Today, we are going to study something called rational functions. Where do you think that word comes from, “rational.” You’ve seen it before I think, in terms of numbers.</td>
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<tr>
<td>Student: Rations like when you ration something.</td>
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<td>Susan: That’s true. I am sure in history class you’ve seen some ramifications of having to ration things like gas.</td>
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<tr>
<td>Student: Isn’t it a number that, like, doesn’t go on or repeat decimals.</td>
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<tr>
<td>Susan: Does NOT go on? So, when you say a number that doesn’t end, do you mean a decimal number?</td>
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<tr>
<td>Susan: What are rational numbers? Hint: there is a root word in there [T underlines “ratio” in “rational”]</td>
</tr>
</tbody>
</table>
Lead Task: Students are asked to complete the worksheet which asks them to fill in a table of values, describe domain, range, end-behavior, and to answer a few related questions. (Lines 141–179)

Susan: Today we are going to extend our notion of what is rational to algebra, not just numbers. We are going to look at ratios of polynomials. So I am starting you off on today’s lesson with this worksheet.

For \( f(x) = \frac{1}{x-3} \), fill in the table of values (when necessary, round to two decimal places) and then answer the questions to the right.

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<th>( x )</th>
<th>( f(x) = \frac{1}{x-3} )</th>
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What is the domain of \( f(x) \)?

What is the right-end behavior of \( f(x) \)?

What is the left-end behavior of \( f(x) \)?

What is the range of \( f(x) \)?

What connections are there to a previously studied function?

Task presentation:
Susan: Fill in the blank with your calculator so it will make our work a lot more efficient. What should we put in the calculator? Some of you who already started, tell me what
you’ve done.
Student: whatever [inaudible] I plug in for x.
Susan: Do I write the expression every time? Is there a more efficient way? [T is working on a projected TI emulator]
Student: [inaudible]
Susan: Make a table from what? What else do we need to put it before we actually put it in the table?
Student: y equals
Susan: Okay, so we put in 1 divided by the quantity x – 3. Are those parentheses necessary?
Student: Yes
Susan: So that there is no confusion that we are dividing by all of x minus three. And then you mentioned that we could get these values from a table. How would we do that?
Student: Do second graph
Susan: Second graph [T completes the operation on the calculator and a table pops up]

Susan: Oh, my default here…What about “table set” though? Some of these numbers are there. Since these numbers are not increasing by a constant amount [referring to the numbers in the table on the worksheet] on the table I am trying to fill out, they are not increasing by a constant amount, there is a nice shortcut way I can do this. [T shows the “table set” screen] It doesn’t matter what I put in for “TblStart” or delta-table, if we go to Independent variable…Ask. Choose that. I would like you to work through this first page right now. Filling in numbers and answering questions. Don’t move to the back side right now.

**Lead Task:** Students are asked to determine if a function (actually a rational, polynomial expression) is really a rational function. They are asked to explain.
(Lines 360–383)

Why is \( \frac{x^2 + 2x - 3}{x+3} \) not a rational function?
Task presentation:
Susan: Okay, so, why are we looking at all of this? [goes back to the questions from the handout]. If you consider what a rational function is, this is a definition straight from your book.

A rational function is a function that can be put in the form: \[ f(x) = \frac{a(x)}{b(x)} \]

where \(a(x)\) and \(b(x)\) are polynomials in \(x\)

We assume that \(b(x)\) is not the zero polynomial.

Susan: We have a ratio of polynomials in \(x\) where the numerator and denominator are polynomials in \(x\) and note that it says that we assume that \(b(x)\), and that’s the denominator, is not the zero polynomial. The zero polynomial? They mean the degree zero. The degree zero is what?

Student: one

Susan: It might be one if I have one times \(x\) to the zero.

Student: or just zero

Susan: if I have zero times \(x\) to the zero. This [pointing to the phrase] is just meaning that it is not a constant. Alright, first question, let me give you a minute to think about it. To save you some time, I copied some of these on your worksheet. So the question about the first one on the backside is: Why is that not a rational function? [long pause]

Lead Task: Students are asked to determine if a function (actually a rational, polynomial expression) is really a rational function. They are asked to explain.

(Lines 425–435)

Is \(\frac{x+3}{x^2+2x-3}\) a rational function? Why?

Task presentation:
Susan: What about this one? Those are the same polynomials, so we know we can write this one as \(x\) plus three over \(x\) plus three times \(x\) minus one. Which means I can still divide out \(x\) plus three, so what do we have now? We have the same thing as before?

Yes? No?

Student: One over \(x\) minus 1

Susan: [as she writes] one over \(x\) minus 1. So is that a rational function? [long pause]

Student: Yes.

Susan: Yes, it is. We have a polynomial of degree zero in the numerator and a polynomial of degree one in the denominator.
**Lead Task:** Students are asked to determine if we will get vertical asymptotes at both points of domain restriction for the previous function.  
(Lines 443–459)

Susan: Will we get vertical asymptotes in both of those places on the graph?

**Task presentation:**

Susan: We know \( x \) cannot be one and \( x \) cannot be negative three. Will we get vertical asymptotes in both of those places? [long pause] Try it with your calculator. Let me repeat the question. We are starting with this expression. We are going to treat it like a function—a rational function. This one. My question is, we know that \( x \) cannot be one and \( x \) cannot be negative three. Check out the graph right now.
### Lead Tasks from Observation #2

<table>
<thead>
<tr>
<th>Lead Task:</th>
<th>Students were asked to recall the definition of rational numbers and make some connections to rational functions.</th>
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| Task presentation: | Susan: [Susan writes the following numbers on the board and labels them rational: \(1.3\), \(\sqrt{4}\), 1.2] These are rational numbers. [Susan writes the following number on the board and labels it as irrational: \(\sqrt{3}\)] We have established that this is one and one third which makes it rational because I can write it as, here’s a clue...That can be written as four over three [referring to 1.3 repeating]. This can be written as two over one [referring to the square root of four]. This can be written as...what fraction? [referring to 1.2] But, by re-writing the numbers this way [T points to each fractional representation], we are emphasizing the fact that they are rational numbers. So, what’s the connection? |
Lead Task: Students are asked to complete the worksheet which asks them to fill in a table of values, describe domain, range, end-behavior, and to answer a few related questions. (Lines 48–70)

For \( f(x) = \frac{1}{x-3} \), fill in the table of values (when necessary, round to two decimal places) and then answer the questions to the right.

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What is the right-end behavior of \( f(x) \)?

What is the left-end behavior of \( f(x) \)?

What is the range of \( f(x) \)?

What connections are there to a previously studied function?

Task presentation:
Susan: Okay, but today, we are going to expand your thinking to the algebraic of rational, rational functions. [passing out the worksheet] I produced this worksheet rather quickly and I did not setup the table in the most advantageous way. So get out your calculators. The first thing I have asked you to do is to come up with a table of values. What is the most convenient way for you to get a large number of values in a table of values from a function in your calculator?
Lead Task: Students are asked to determine if a function (actually a rational, polynomial expression) is really a rational function. They are asked to explain. (Lines 211–245)

Why is \( \frac{x^2 + 2x - 3}{x+3} \) not a rational function?

Task presentation:

Susan: Okay, on the second page, if I had copied all that I wanted to go over with you, it would have taken too much paper, so I did not. [T brings up some additional work on the screen.] This definition of a rational function is straight from your textbook. A rational function is a function that can be put in the form:

\[
f(x) = \frac{a(x)}{b(x)} \quad \text{where } a(x) \text{ and } b(x) \text{ are polynomials in } x\]

We assume that \( b(x) \) is not the zero polynomial.

Susan: It is connected with rational numbers, how? How is this connected to rational
Student: because it’s dealing with a ratio
Susan: Exactly. It’s still dealing with a ratio. Only instead of a ratio of integers, we have a ratio of…
Student: Polynomials
Susan: Polynomials. Yup. The notation is saying a of x and b of x are polynomials. B of x is not the zero polynomial. Now we know we can’t have a zero denominator. But what happens if we have a degree-zero polynomial? What is a degree zero polynomial?
Student: one
Susan: One is an example of a degree-zero polynomial. Any other examples? [pause] I am feeling exotic today. I think my example will be 12.7362 times the square root of 17 divided by 18,413 and a half.
Student: What was the question again?
Susan: An example of a degree-zero polynomial. In other words, where does the degree of a polynomial come from again?
Student: The variable
Susan: The variable. Specifically, what about the variable?
Student: The exponent.
Susan: The exponent. The highest powered term. So if there is no variable, then the power has to be zero, but the coefficient could be anything. Okay, so is the example of one as a zero degree polynomial? Yes, because it is the same as one times x to the zero. So we want genuine polynomials here. We don’t want the denominator to be a degree zero polynomial.
Susan: So my question is, and you have this fraction written down on your paper. My question is why is that not a rational function? Why does that not fit the definition of a rational function? It looks like one. We’ve got a polynomial divided by a polynomial.

**Lead Task:** Students are asked to determine if a function (actually a rational, polynomial expression) is really a rational function. They are asked to explain. 
(Lines 310–339)

Is \( \frac{x+5}{x^2+2x-3} \) a rational function? Why?

**Task presentation:**
Susan: What about the next example? Is that a rational function? [long pause] It bears a lot of similarity to the one above. What do you think, Devin?
Lead Task: Students are asked to simplify some complex algebraic fractions.  
(Lines 341–348)

Task presentation:
Susan: Some of you may be a bit rusty in manipulating this. I have three examples [T refers to the bottom part of the worksheet] that we would like to get in the form of numerator simplified, denominator simplified, in other words, looking like a rational expression. This [pointing to the third complex fraction] one is sort of the bonus. Right now, try to get the first two. The goal is to get an expression where we have one polynomial in the numerator and one in the denominator. You may discuss your methods with people sitting near you.

Lead Task: Students are asked to consider what causes a function to have vertical asymptote.  
(Lines 357–382)

Task presentation:
Susan: Vertical asymptotes, you’ve seen, are often features of rational functions. What causes…in other words if you look at it algebraically, what’s our clue that we might have a vertical asymptote?
**Lead Tasks from Observation #3**

**Lead Task:** Students are asked to complete a long division problem without a calculator.
(Lines 17–62)

**Task presentation:**
Susan: Alright, this is what we are going to do today. I am going to over what division means in Algebra as compared to what it meant in arithmetic. Then we are going to look again at some of what we talked about yesterday. I did not do a very good job of shoring all of that up for you yesterday. And, extend it just a little bit. So, essentially, what we will be doing today will be all of the rest of what you need to know from this unit. Do I expect you to be proficient with it? No, not at all, but I do encourage you to stop me if you get confused about anything that we will be doing. You will be taking lots of notes. We are going to be looking at lots of sample problems and at the end, we will talk about those things that I want you to know from this unit. And when we meet again, we will just be reinforcing all of what we have talked about.

Susan: You might recall that you learned how to do long division a long time ago. The way I learned it, was a four-step algorithm. So if we start with just an arithmetic problem, we might write it as a divided by b. What’s the name that we give to the answer to a division problem?
Student: quotient
Susan: Quotient. Very good. So, I’ll say that the answer to that is some value that is a quotient. [T writes on the board]

Long Division of Integers: a 4 step algorithm

\[
\frac{a}{b} = q
\]

Susan: But sometimes it gets messy because sometimes we say the numbers don’t divide evenly so we end up with what?
Student: remainder
Susan: Good, you are really up on your vocabulary.
Student: What did he say?
Susan: Remainder. We have some options as to how we write that remainder. One way we
could write it is as the remainder divided by the divisor. That would look like that. [T writes]

\[
\frac{a}{b} = q + \frac{r}{b}
\]

Student: Why would you do that?
Susan: Plus the remainder. Suppose I want to divide five by two. What do you get?
Student: 2.5
Susan: Or if I write it as a mixed number, two and…
Student: a half.
Susan: A half. I took that remainder of one and placed it over the divisor of two. Suppose I have seven thirds. If I carry out that division, I get how much? [While pointing to the expression above] Seven divided by three equals how much?
Student: Two and a third
Susan: Two and a third. Notice how I write that [pointing to the right side of the equation] two plus remainder of one divided by that divisor, three. The problems obviously can get much more complicated. Suppose I have 17—I would like you to do this without a calculator, I know for some of you that is not an easy thing—three thousand eight hundred and twenty. Go ahead and do that. You have been doing it for years, but most of the time lately, you probably have been doing it with a calculator.

**Lead Task:** Students are asked to apply the same long-division process to a rational expression.
(Lines 118–128)

Susan: Now we are going to apply the same process, but to algebra. So, I want to look at a division with polynomials. Same four step process. Suppose I have:
**Task presentation:**
Susan: The first thing we are going to do is set it up as a long division. So what goes on the outside?
Student: x plus 2
Susan: x plus 2, that’s called the divisor. We are dividing into the dividend. Which is?
Student: Three x squared minus five x plus 4. [T writes]

Susan: Now we are ready to start the process.

**Lead Task:** Use polynomial long division to understand the end-behavior of rational functions.
(Lines 477–481)

**Task presentation:**
Susan: Alright, why are we doing this? We are going to look at what is called long-run behavior for rational functions. This division is pretty straight forward compared to some of the examples we just did. Here, I am doing long division to see another way of expressing this function.


Van Zoest, L. R., Leatham, K., Peterson, B., & Stockero, S. L. (2013). *Conceptualizing mathematically significant pedagogical openings to build on student thinking*. Paper


Vita
David B. Perkinson

Education
Ph.D., Curriculum and Instruction (Mathematics Education), The Pennsylvania State University, 2015
  *Dissertation: The Mathematical Reasoning of Teachers During the Design and Delivery of Instruction.*
M.A., Mathematics Education, University of North Carolina at Charlotte, 2000
B.S., Mathematics and Philosophy, St. Andrews Presbyterian College, 1988

Awards and Fellowships
O’Herron Teaching Award, Charlotte Country Day School, 1997-2002
Mid-Atlantic Center for Learning and Teaching Fellowship, 2002-2005
NAIS Aspiring Heads Fellowship, 2013-2014

Publications and Presentations
*Transforming your Teaching*, National Council of Teachers of Mathematics Annual Conference, St. Louis, MO: 2006
*Introduction to Lesson Study*, Charlotte Country Day School, One-day Workshop, 2007

Experience
Mathematics Teacher (Grades 8-12), McCallie School, Chattanooga, TN, 1988-1994
Mathematics and Physics Teacher (Grades 9-12), Charlotte Country Day School, Charlotte, NC 1994-2002
Head of Middle School, Spartanburg Day School, Spartanburg, SC, 2005-2012
Head of Upper School, Episcopal High School, Baton Rouge, LA, 2012-Present