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FLOER HOMOLOGY FOR ALMOST HAMILTONIAN ISOTOPIES

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Abstract

Floer homology is defined for a closed symplectic manifold which satisfies certain technical conditions, and is denoted $HF_*(M, \omega)$. Under these conditions, Seidel has introduced a homomorphism from $\pi_1(\mathrm{Ham}(M))$ to a quotient of the group of automorphisms of $HF_*(M, \omega)$.

The main goal of this thesis is to prove that if two loops of Hamiltonian diffeomorphisms are homotopic through arbitrary loops of diffeomorphisms, then the image of (the classes of) these loops agree under this homomorphism.

This can be interpreted as further evidence for what has been called the "topological rigidity of Hamiltonian loops". This phenomonon is a collection of results which indicate that the properties of a loop of Hamiltonian diffeomorphisms are more tied to the class of the loop inside of $\pi_1(\mathrm{Diff}(M))$ than one might guess. For example, on a compact manifold M, the path that a point follows under a loop of Hamiltonian diffeomorphisms of M is always contractible. In [14], further results of this type were proved using Seidel's homomorphism.

To prove this result, we extend the domain of Seidel's homomorphism to the group of all loops of diffeomorphism of M based at id which are homotopic to a Hamiltonian loop. Such a loop will be called **almost Hamiltonian**. This requires that we extend the space from which we choose to $HF_*(M, \omega)$.

The Floer homology groups are defined by choosing a (generic) pair (H, J) in $C^\infty(M \times S^1) \times \mathcal{J}(M, \omega, S^1)$. Here, $\mathcal{J}(M\omega, S^1)$ means the set of all smooth loops of ω-compatible almost complex structures. Such pairs are

called regular pairs. The Floer homology constructed using (H, J) is denoted $HF_*(M, \omega, H, J)$. If (H', J') is another such regular pair, then there is a natural isomorphism between $HF_*(M, \omega, H, J)$ and $HF_*(M, \omega, H', J')$. These isomorphisms are funtorial with respect to composition, so we can speak of $HF_*(M, \omega)$ without mention of the pair (H, J) used to define it. These isomorphisms are called the continuation isomorphisms, and are generally denoted by Φ.

Denote by G the group of all smooth loops in $\mathrm{Ham}(M)$ at id. Seidel introduces an extension of G, denoted $\widetilde{G}$. That is, there is an exact sequence of topological groups

$$1 \to \Gamma \to \widetilde{G} \to G,$$

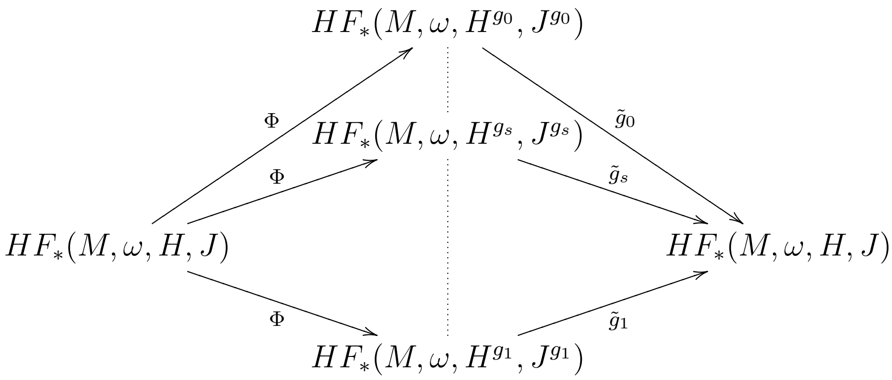
with Γ discrete. Each g in G defines a new pair (H^g, J^g), such that if $\tilde{g}$ is a lift of g to $\widetilde{G}$, then $\tilde{g}$ induces an isomorphism from $HF_*(M, \omega, H^g, J^g)$ to $HF_*(M, \omega, H, J)$. By an abuse of notation, we will still call this isomorphism $\tilde{g}$. By precomposing with the continuation isomorphisms, we obtain an automorphism of $HF_*(M, \omega, H, J)$. That is, $\tilde{g} \in \widetilde{G}$ defines an automorphism of $HF_*(M, \omega, H, J)$ by

$$HF_*(M, \omega, H, J) \xrightarrow{\Phi} HF_*(M, \omega, H^g, J^g) \xrightarrow{\tilde{g}} HF_*(M, \omega, H, J).$$

Seidel then shows that this automorphism of $HF_*(M, \omega, H, J)$ commutes with the continuation isomorphism to any other regular pair, so that $\tilde{g} \in \widetilde{G}$ induces an automorphism of $HF_*(M, \omega)$ independent of the regular pair.

Finally, Seidel proves that if $\tilde{g}_0$ and $\tilde{g}_1$ define the same element in $\pi_0(\widetilde{G})$, then they induce the same automorphism of $HF_*(M, \omega)$. We outline the intuitive argument. Suppose g_0, $g_1 \in G$ have lifts $\tilde{g}_0$, $\tilde{g}_1 \in \widetilde{G}$. This means that there is a path g_s in G connecting g_0 to g_1, and a path $\tilde{g}_s$ in $\widetilde{G}$ connecting

$\tilde{g}_0$ to $\tilde{g}_1$. Choose a regular pair (H, J), and construct $HF_*(M, \omega, H, J)$. Consider the following diagram.



Intuitively, by using $\tilde{g}_s$ to connect $\tilde{g}_0$ and $\tilde{g}_1$, the middle of the diagram can be filled in, and we can transform $HF_*(M, \omega, H^{g_0}, J^{g_0})$ into $HF_*(M, \omega, H^{g_1}, J^{g_1})$.

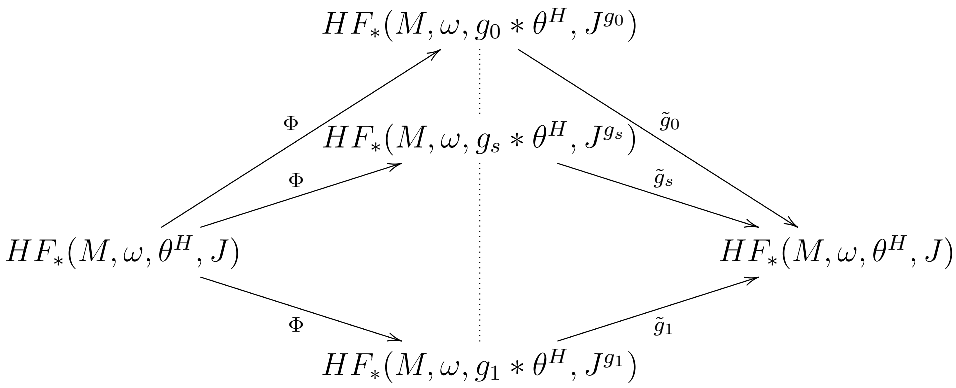
We are interested in loops in Diff(M) at id which are not Hamiltonian, but which are homotopic to a Hamiltonian loop. As given, the diagram above cannot be extended to such loops: if g is not a Hamiltonian loop, H^g does not make sense.

To overcome this difficulty, notice that the construction of the Floer homology groups $HF_*(M, \omega, H, J)$ only depends on θ^H, the Hamiltonian isotopy generated by H, and the almost complex structure J, so we are justified in writing $HF_*(M, \omega, \theta^H, J)$.

We want to rephrase the above diagram in terms of isotopies, so in section 2.1, we calculate the isotopy generated by the function H^g. We find that $\theta_t^{H^g} = g_t^{-1}\theta_t^H$. So, for θ any isotopy of M and g any loop at id in Diff(M), let $g * \theta$ be the isotopy of M given by $(g * \theta)_t := g_t^{-1}\theta_t$. In this notation, we find that

$$\theta^{H^g} = g * \theta^H.$$

Then we can rewrite the above diagram as the following.



While H^g does not make sense if g is not Hamiltonian, $g * \theta^H$ does make sense. If we can construct a Floer homology using isotopies of the form $g*\theta$, where g is an almost Hamiltonian loop in Diff(M) and θ is a Hamiltonian isotopy, then we can fill in the middle of this diagram, and use it to prove the main theorem.

Notice that if g is an almost Hamiltonian loop, and θ is a Hamiltonian isotopy, then $g*\theta$ is homotopic, relative endpoints, to a Hamiltonian isotopy. **The first step in proving the main result is to extend the space used to define $HF_*(M, \omega)$ to include generic pairs (ψ, J), where ψ is an isotopy of M which is homotopic, relative endpoints, to a Hamiltonian isotopy, and J is a smooth loop of almost complex structures with a suitably adapted compatibility condition.** (The details associated with the almost complex structures become rather technical.) This construction culminates in section 4, but the fact that it is well defined uses all of the material developed in the previous sections. We prove that these groups are naturally independent of the choice of (ψ, J) used to define them (theorem 4.16). This effectively extends our space of choices used in defining $HF_*(M, \omega)$.

Using the expanded domain of definition for $HF_*(M, \omega)$, we can extend the domain of Seidel's homomorphism to the group of almost Hamiltonian loops. By examining what happens when we homotope these loops, we prove the main result. This is given by theorem 5.3.

This thesis is organized as follows:

The first section gives a quick review of the construction of the Floer homology groups for a weakly monotone symplectic manifold, emphasizing the points that will be of concern later. We discuss the monotonicity condition, and show why it is helpful. (The condition of being monotone is a technical condition that is no longer needed to define Floer homology. It is the assumption that the founder of the theory, Andreas Floer, used in his expositions. The assumption guarantees that J-holomorphic spheres are well behaved, and simplifies many things. For weakly monotone symplectic manifolds, J-holomorphic curves are well behaved for generic J. We suspect the results hold without the assumption.) We then outline the definition of Seidel's homomorphism.

The next section rephrases Seidel's homomorphism in terms of isotopies, as is described above. Here, we calculate θ^{H^g}, the Hamiltonian isotopy generated by H^g. This section includes the proof of the fact that the Hamiltonian function H is not so important as θ^H, the isotopy it generates. More precisely, we show that if H and H' generate the same isotopy, then the groups $HF_*(M, \omega, H, J)$ and $HF_*(M, \omega, H', J)$ are actually equal by construction. Notice that this truly means *equals*, and not just isomorphic.

The third section contains the technical tools needed throughout the remainder of the thesis. In it, we outline the basic definitions and notions of Hamiltonian and almost Hamiltonian loops as described above. We also introduce the space that will be used as the expanded domain of definition for $HF_*(M, \omega)$. Because of the difficulties with ω-compatibility requirements on the almost complex structures, this domain will no longer be a cross product, but should be thought of as a bundle, with base given by the almost Hamiltonian isotopies, and the fiber over such an isotopy consisting of 1-periodic time dependent almost complex structures with a compatibility requirement that depends on the particular isotopy.

The next section gives the definition of $HF_*(M, \omega, \psi, J)$ for a generic almost Hamiltonian isotopy. As in the original construction, the definitions are not as hard as proving that the objects are well-defined. In particular, a boundary operator on a chain complex is defined by counting certain objects. The bulk of the difficulty in the construction of the Floer homology groups is in showing that the spaces in question are actually finite. (In fact, not all choices (H, J) are allowed, and we are only allowed to choose from a dense subset of all choices.) To show that our boundary operator is well-defined, we reduce to the Hamiltonian case.

The next section contains the main result. We extend the domain of Seidel's homomorphism to the group of almost Hamiltonian loops using the expanded domain of definition of $HF_*(M, \omega)$. Finally, we show that if two loops are in the same homotopy class, they have the same image under this homomorphism.

The final section consists of remarks about what has been shown, and what more might be said.

Contents

Acknowledgements

It's been quite an adventure, getting to this point. There have been highs and lows, and I am quite sure that the help of many people was instrumental in my completion of this thesis. My advisor, Professor Augusting Banyaga, inspired me to work in this particular area, and his guidance was invaluable. Many thanks also to my colleague Shantanu Dave, for listening to me ramble on about Floer homology. He understood that explaining mathematics helps tremendously in understanding mathematics. I'd like to thank my wife, Michelle, whose sacrifices don't go unnoticed. Finally, I thank my children, Brooke and Bailey; with them, I always begin my day smiling.

1 Review of Floer Homology

Floer homology was originally introduced in an attempt to prove a version of the Arnold conjecture. This conjecture states that, in the non-degenerate case, the number of fixed points of a Hamiltonian symplectomorphism of a symplectic manifold is bounded from below by the sum of the Betti numbers of the manifold. (Non-degeneracy means that the graph of the diffeomorphism inside $M \times M$ intersects the diagonal transversally.)

The idea of his proof is analogous to proving the Morse inequalities. First, the fixed points are indexed in some manner. The chain complex $CF_k(M, \omega, H)$ is defined as all formal sums of elements of the fixed point set of index k with coefficients in $\mathbb{Z}$. A boundary operator is defined on this complex. This boundary operator is defined by counting the number of certain cylinders in M.

That we can count these cylinders follows from that fact that the cylinders satisfy the same basic analytic properties as Gromov's J-holomorphic curves, introduced in [12]. The main point is that the differential equation equation defining these curves gives an elliptic (and therefore Fredholm) operator on suitable Banach spaces. Floer proves that $\partial^2 = 0$, and that the homology of this complex recovers the homology of the underlying manifold. Arnold's conjecture follows.

Floer's construction is for monotone symplectic manifolds. This is a stronger assumption than the assumption of an integral class symplectic form. In [13] and [21], this proof was extended to a larger class of manifolds.

Because we intend to extend Seidel's action, we use the his version of Floer homology, which is only a slightly modified form of that introduced in [13]. For simplicity, we will use coefficients in $\mathbb{Z}_2$, to avoid difficulties with orientation.

1.1 Symplectic Preliminaries

A symplectic manifold is a manifold M^{2n} equipped with a 2-form $\omega \in \Omega^2(M)$ such that ω is closed and non-degenerate. Any $\omega \in \Omega^2(M)$ induces a bundle map from TM to T^*M by $X \mapsto i_X\omega$. Non-degeneracy means that this bundle map is an isomorphism. This is equivalent to the requirement that ω^n is a volume form on M.

By $\mathrm{Diff}(M)$, we mean the set of all diffeomorphisms of M. This set forms a group with the product given by composition of maps. Each element of $\mathrm{Diff}(M)$ is a map $\psi : M \to M$, and so $\mathrm{Diff}(M) \subset C^\infty(M, M)$. We topologize $\mathrm{Diff}(M)$ as a subspace of $C^\infty(M, M)$ equipped with the C^∞ topology.

For (M, ω) a symplectic manifold, let

$$\mathrm{Symp}(M, \omega) = \{\psi \in \mathrm{Diff}(M) \mid \psi^*\omega = \omega\}.$$

Such diffeomorphisms will be called symplectic diffeomorphisms or symplectomorphisms. It is clear that $\mathrm{Symp}(M, \omega)$ is a subgroup of $\mathrm{Diff}(M)$, and we topologize $\mathrm{Symp}(M, \omega)$ as a subspace of $\mathrm{Diff}(M)$.

Now, consider $\mathrm{Symp}_0(M, \omega)$, the component of the identity inside the group $\mathrm{Symp}(M, \omega)$. If we let $\tilde{\omega} : TM \to T^*M$ denote the bundle isomorphism induced by ω, then each $\psi \in \mathrm{Symp}_0(M, \omega)$ corresponds to the endpoint of the flow of a time dependent "closed" vector field, i.e., a vector field of the form $\tilde{\omega}^{-1}(\alpha_t)$, where each α_t is a closed 1-form.

By considering "exact" vector fields, we obtain an important normal subgroup, called $\mathrm{Ham}(M)$. More precisely, let $H \in C^\infty(M \times \mathbb{R})$, and let $X_{H_t} = \tilde{\omega}^{-1}(dH_t)$. (This means that $\omega(X_{H_t}, \cdot) = dH_t(\cdot)$.) Integration of this vector field gives a path in $\mathrm{Diff}(M)$, denoted $\theta^H = (\theta^H_t)$, defined by the equations

$$\frac{d\theta_t^H}{dt}(p) = X_{H_t}(\theta_t^H(p)), \quad \theta_0^H = id.$$

The group of Hamiltonian diffeomorphisms of M consists of the set of all time-1 diffeomorphisms arising as in the above construction, and is denoted $\mathrm{Ham}(M)$. In fact, $\mathrm{Ham}(M)$ is a normal subgroup of $\mathrm{Symp}_0(M, \omega)$ (see [2]). We topologize $\mathrm{Ham}(M)$ as a subspace of $\mathrm{Symp}_0(M, \omega)$.

Conjecture (Arnold) In the non-degenerate case, the number of fixed points for any $\theta \in \mathrm{Ham}(M)$ is bounded from below by the sum of the Betti numbers of M.

This was first proved for $\mathbb{T}^n$ by Conley and Zehnder in [6]. Later, Floer introduced the Floer homology groups $HF_*(M, \omega)$ to prove the conjecture for certain types of symplectic manifolds, the so called **monotone** symplectic manifolds. This is essentially given in [11], using ideas formulated in [8], [9] and [10].

In [13], the definition of Floer homology was extended to **weakly monotone** symplectic manifolds. To describe these sub-classes, we need the notions of an almost complex structure on M, and Chern classes.

Definition 1.1 *An* **almost complex structure** *on a manifold M is a smooth, fiber preserving bundle map $J : TM \to TM$ such that $J^2 = -1$.*

Thus, an almost complex structure is simply a way of assigning a complex structure to each tangent space smoothly.

Usually, we restrict our attention to those almost complex structures which behave nicely with respect to the symplectic structure. Consider

$\mathbb{R}^{2n}$ equipped with its standard symplectic structure, ω_0, Riemannian metric, $\langle , \rangle$, and complex structure i. These three structures fit together in the formula $\omega_0(\cdot, i\,\cdot) = \langle \cdot, \cdot \rangle$. Since a manifold has no preferred Riemannian structure, we abstract this by simply requiring that $\omega(\cdot, J\cdot)$ gives a Riemannian metric on M.

Definition 1.2 *An almost complex structure J on a symplectic manifold (M, ω) is called ω-compatible if $\omega(\cdot, J\cdot)$ defines a Riemannian metric on M.*

Let $\mathcal{J}(M, \omega)$ denote the space of all ω-compatible almost complex structures on M. This space has a topology induced as a subspace of $\mathrm{End}(TM)$.

Proposition 1.3 *$\mathcal{J}(M, \omega)$ is non-empty and contractible.*

Proof: This is a standard fact. Given $r(\cdot, \cdot)$ any Riemannian metric on M, there is a unique automorphism $A : TM \to TM$ which satisfies $\omega(\cdot, \cdot) = r(A\cdot, \cdot)$. It is not hard to calculate that $J := (-A^2)^{\frac{1}{2}} A$ is an almost complex structure compatible with ω. Thus, $\mathcal{J}(M, \omega)$ is a deformation retract of the space of all Riemannin metrics.

□

A choice of an almost complex structure makes (TM, J) into a complex vector bundle over M, and as such, it has Chern classes, $c_i(TM, J) \in H^{2i}(M, \mathbb{Z})$ (for details about Chern classes, consult [17]).

Chern classes are defined for any complex vector bundle over a manifold M, and they are natural in the sense that for a complex bundle $V \to M$ and a smooth map $f : N \to M$, the Chern classes satisfy $c_i(f^*V) = f^*(c_i)$, where f^*V is the pullback bundle over N.

The Chern classes are defined for each choice of almost complex structure, and *a priori*, they may depend on the choice. By restricting to ω-

compatible almost complex structures, we avoid this difficulty, as the next proposition shows.

Proposition 1.4 *Let (M, ω) be a symplectic manifold, and let J^0 and J^1 be elements of $\mathcal{J}(M, \omega)$. Then the Chern classes of (TM, J^0) and (TM, J^1) agree. That is, the Chern classes are independent of the choice of almost complex structure $J \in \mathcal{J}(M, \omega)$.*

This follows from the fact that $\mathcal{J}(M, \omega)$ is contractible, and the naturality of the Chern classes.

According to this proposition, we can write $c_i(\omega)$ for $c_i(TM, J)$, where J is any ω-compatible almost complex structure. We will only have use for the first Chern class, and we will often drop the mention of ω, simply writing c_1.

Definition 1.5 *A symplectic manifold (M, ω) is called* **monotone** *if $[\omega] = \tau c_1(\omega)$ for some $\tau > 0$.*

It follows from the preceding proposition that the condition of being monotone is independent of the choice of almost complex structure. We will see that this assumption guarantees the non-existence of J-holomorphic spheres of negative Chern number, and simplifies many things. Hopefully, this assumption can be weakened with further study.

Definition 1.6 *A map $f : (N, j) \to (M, J)$ between almost complex manifolds is called (j, J)-holomorphic, or pseudo-holomorphic, if $Df \circ j = J \circ Df$.*

In the case $(N, j) = (M, J) = (\mathbb{C}, i)$, this condition reduces to the Cauchy-Riemann equations, and so generalizes the notion of holomorphic maps.

We will mostly consider pseudo-holomorphic maps with domain S^2.

Definition 1.7 *Given* $A : S^2 \to (M, \omega)$*, the* **Chern number** *of* A *is* $c_1(A) := \int_{S^2} A^* c_1 \in \mathbb{Z}$.

It is clear that the set of all Chern numbers of classes $A \in \pi_2(M)$ is a subgroup of $\mathbb{Z}$. We call the positive generator of this subgroup the **minimal Chern number** of (M, ω). That is, the minimal Chern number of (M, ω) is the smallest positive value that c_1 takes on $\pi_2(M)$. Throughout the thesis, we will write N for the minimal Chern number of (M, ω).

Endow S^2 with it's standard complex structure, j, and standard symplectic form, and choose $J \in \mathcal{J}(M, \omega)$. We will say that a map $f : (S^2, j) \to (M, J)$ is J-holomorphic if f is (j, J)-holomorphic, and we will call f a J-holomorphic curve in M. Then, given $f : (S^2, j) \to (M, \omega, J)$, Df_z is a linear map between normed linear vector spaces for each $z \in S^2$, and as such, it has a norm, $|Df_z|$.

Definition 1.8 *For* $f : (S^2, j) \to (M, \omega, J)$*, the* **energy** *of* f *is given by* $e(f) = \int_{S^2} |Df_z|^2 d\nu$*, where* ν *is the standard area form on* S^2*.*

For a proof of the following proposition, see [1].

Proposition 1.9 *Given a symplectic manifold* (M, ω)*, a choice of* $J \in \mathcal{J}(M, \omega)$*, and a map* $A : (S^2, j) \to (M, \omega, J)$*, if* A *is* J*-holomorphic, then* $\omega(A) = \frac{1}{2} e(A)$*.*

This proposition shows that whenever $A : S^2 \to (M, \omega, J)$ is a J-holomorphic map, then $\omega(A)$ is positive. If (M, ω) is monotone, this means that $c_1(A)$ is also positive, so monotonicity guarantees the non-existence of J-holomorphic spheres of negative Chern number, as claimed. The existence of such spheres obstruct the compactness of the moduli space of connecting orbits, and thus must be avoided.

We can now introduce the weakly monotone condition.

Definition 1.10 *A symplectic manifold* (M^{2n}, ω) *is called* **weakly monotone** *if, for all* $A \in \pi_2(M)$, $3 - n \leq c_1(A) < 0 \implies \omega(A) \leq 0$.

All monotone manifolds are weakly monotone. As described above, the monotonicity condition guarantees that there are no J-holomorphic spheres of negative Chern number. In the weakly monotone case, this is only generically true.

Lemma 1.11 (Hofer, Salamon) *If* (M, ω) *is a weakly monotone symplectic manifold, then there is a generic subset of* $\mathcal{J}(M, \omega)$*, such that for all* J *in this subset, there are no* J*-holomorphic spheres with negative Chern number.*

Since Floer homology is defined using generic (H, J), this will cause no difficulty. (This is not the only additional difficulty in extending Floer homology to the weakly monotone case - others will be mentioned later.)

To define his action, Seidel uses time dependent almost complex structures: we let $\mathcal{J}(M, \omega, S^1)$ denote all ω-compatible almost complex structures paramaterized by S^1. That is, $J \in \mathcal{J}(M, \omega, S^1)$ is a family (J_t) such that each J_t is an element of $\mathcal{J}(M, \omega)$, and J_t varies smoothly with t. Because of this use of time dependent almost complex structures, Seidel has to use a slightly stronger assumption than the weakly monotone assumption.

Definition 1.12 *A symplectic manifold* (M^{2n}, ω) *will be said to satisfy condition* W^+ *if, for all* $A \in \pi_2(M)$, $2 - n \leq c_1(A) < 0 \implies \omega(A) \leq 0$.

Notice that the only difference between condition W^+ and weakly monotone is that $3 - n$ is replaced by $2 - n$. Seidel introduces this slightly stronger statement because he uses time dependent almost complex structures. We need to assume condition W^+ in order to guarantee that we can choose generic *time dependent* almost complex structures for which no J_t-holomorphic sphere has negative Chern number for any fixed $t \in S^1$.

1.2 Floer Homology

Floer homology for a symplectic manifold satisfying condition W^+ is defined by choosing a 1-periodic Hamiltonian function $H \in C^\infty(M \times S^1, \mathbb{R})$ and a 1-periodic family of almost complex structures $J \in \mathcal{J}(M, \omega, S^1)$. The Hamiltonian defines X_{H_t}, a time dependent vector field on M, by the formula $i_{X_{H_t}}\omega = dH_t$. As described above, integration of this family of vector fields yields $\theta^H = (\theta_t^H)$, a path in $\mathrm{Diff}(M)$. For any fixed t, θ_t^H is a Hamiltonian diffeomorphism.

Recall that Floer homology was introduced in an attempt to prove the Arnold Conjecture, which is a statement concerning the fixed points of Hamiltonian diffeomorphisms. Let θ be a Hamiltonian diffeomorphism, and suppose there is a Hamiltonian function $H \in C^\infty(M \times S^1)$ for which the generated path in $\mathrm{Diff}(M)$ satisfies $\theta_1^H = \theta$. Then the fixed points of $\theta = \theta_1^H$ are in one-to-one correspondence with the set of maps $x : S^1 \to M$ which are integral curves of the time dependent vector field X_{H_t}.

We will define a chain complex using this set of loops in M rather than the set of fixed points of θ. Accordingly, let ΛM consist of all smooth, unbased, parameterized maps $x : S^1 \to M$. Again, we topologize ΛM as the mapping space $C^\infty(S^1, M)$ with the C^∞ topology. Let $\mathcal{L}M$ be the subspace of contractible loops. Then $\mathcal{L}M$ is topologized as a subspace of ΛM.

Notice that an element of $\mathcal{L}M$ includes the parameterization.

In Floer homology, we use $H \in C^\infty(M \times S^1)$ to define a 1-form on $\mathcal{L}M$

by the formula

$$\alpha_H(x)(\xi) = \int_0^1 \omega\big(\dot{x}(t) - X_{H_t}(x(t)), \xi(t)\big)dt.$$

We define $\mathcal{P}(H)$ as the set of zeroes of α_H. That is,

$$\mathcal{P}(H) = Zeroes(\alpha_H) = \{x \in \mathcal{L}M \mid \dot{x}(t) = X_{H_t}(x(t))\}. \quad (1)$$

Notice that each $x \in \mathcal{P}(H)$ corresponds to a fixed point of θ_1^H. In fact, each $x \in \mathcal{P}(H)$ is defined by $x(0)$, because by definition, $x(t) = \theta_t^H(x(0))$. We will actually create a chain complex using the set $\mathcal{P}(H)$ instead of the fixed point set.

To construct the Floer homology groups, we also require that the fixed point set be non-degenerate in the following sense:

Definition 1.13 *A loop $x \in \mathcal{P}(H)$ is called* **non-degenerate** *if*

$$\det\big(\mathbb{1} - D\theta_1^H(x(0))\big) \neq 0,$$

where $\theta^H = \theta_t^H$ is the Hamiltonian isotopy generated by H.

The non-degeneracy condition appears to be necessary to the Fredholm theory needed in the construction of $HF_*(M, \omega, H, J)$.

The 1-form α_H is not exact on $\mathcal{L}M$, but becomes exact when pulled back to an appropriate covering space, $\widetilde{\mathcal{L}M}$, which we introduce now.

1.2.1 Defining $\widetilde{\mathcal{L}M}$

The space $\widetilde{\mathcal{L}M}$ is constructed as first introduced in [13] : an element of $\widetilde{\mathcal{L}M}$ is an equivalence class of pairs (v, x) with $x \in \mathcal{L}M$ and $v : D^2 \to M$ satisfying $x = v|_{\partial D^2}$. To define the equivalence relation, notice that for

(v_0, x) and (v_1, x) in $C^\infty(D^2, M) \times \mathcal{L}M$ which satisfy $v_0|_{\partial D^2} = v_1|_{\partial D^2} = x$, we can define $v_0 \# v_1 : S^2 \to M$ by identifying S^2 with two copies of D^2 glued together along their boundary circle. (The map from S^2 to M constructed in this way may not be smooth, but it nevertheless defines an element of $\pi_2(M)$.)

The equivalence is defined by $(v_0, x_0) \sim (v_1, x_1) \iff x_0 = x_1$ and $\omega(v_0 \# v_1) = c_1(v_0 \# v_1) = 0$. Let $p : \widetilde{\mathcal{L}M} \to \mathcal{L}M$ be the canonical projection that takes $[v, x]$ to x. Then p is a covering map.

Let $\Gamma = \pi_2(M)/\pi_2(M)_0$ where $\pi_2(M)_0$ is the subgroup of classes A for which $\omega(A) = c_1(A) = 0$. Then Γ is the group of deck transformations for the covering $p : \widetilde{\mathcal{L}M} \to \mathcal{L}M$. Geometrically, Γ acts on $\widetilde{\mathcal{L}M}$ by "gluing in spheres". That is, for $\gamma \in \Gamma$ and $v : D^2 \to M$, we define $\gamma \# v : D^2 \to M$ by cutting out a small disc from both D^2 and S^2, and identifying the boundaries of the missing pieces. We will sometimes think of an element of Γ as a homeomorphism of $\widetilde{\mathcal{L}M}$, and sometimes think of it (via a representative) as an element of $\pi_2(M)$.

Using this covering, we define

$$\widetilde{\mathcal{P}(H)} = p^{-1}(\mathcal{P}(H)).$$

An element of $\widetilde{\mathcal{P}(H)}$ is a 1-periodic integral curve of X_{H_t}, with a choice of disc filling it out (up to an equivalence).

When we pull back to $\widetilde{\mathcal{L}M}$, $p^*\alpha_H = da_H$, where a_H is defined by

$$a_H(c) = \int_{D^2} v^*\omega - \int_0^1 H(t, x(t))dt,$$

with $c = [v, x]$. By Stoke's theorem, this is independent of the choice of representative for c. This function is called the action functional. Then

$$\widetilde{\mathcal{P}(H)} = \ p^{-1}(Zeroes(\alpha_H)) = Zeroes(p^*\alpha_H) = \mathrm{Crit}(a_H).$$

The idea is to define a chain complex using $\widetilde{\mathcal{P}(H)}$, and a boundary operator on this complex for which $\partial^2 = 0$. First, we index $\widetilde{\mathcal{P}(H)}$, which requires the non-degeneracy condition.

1.2.2 The Conley-Zehnder Index and the Chain Complex

When the set $\widetilde{\mathcal{P}(H)}$ contains only non-degenerate solutions, it can be indexed by the Conley-Zehnder index. We will have to use this later, so we give a quick review of how this index is defined. For details, see [20].

Choose any $c = [v, x] \in \widetilde{\mathcal{P}(H)}$. Then (v^*TM, ω) is a symplectic bundle over D^2. Since D^2 is contractible, there is a symplectic trivialization $\tau : (v^*TM, \omega) \to D^2 \times (\mathbb{R}^{2n}, \omega_0)$. This choice of symplectic trivialization is independent of the representative for c up to homotopy. By restricting this trivialization to the boundary circle, we obtain maps $\tau(t) : T_{x(t)}M \to \mathbb{R}^{2n}$ such that $\tau(t)^*\omega_0 = \omega_{x(t)}$ where ω_0 is the canonical symplectic form on $\mathbb{R}^{2n}$.

For a fixed $t \in S^1$, $l(t) := \tau(t)D\theta_t^H\tau(0)^{-1}$ is a symplectic, linear map from $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$. By varying t across S^1, we obtain a path of symplectic matrices.

Let $\mathrm{Sp}^*(2n)$ denote the open dense subset of $\mathrm{Sp}(2n)$ consisting of all symplectic matrices which do not have 1 as an eigenvalue. In [20], Salamon and Zehnder showed how to assign an integer to each path $\Psi : [0,1] \to \mathrm{Sp}(2n)$ which satisfies $\Psi(0) = \mathbb{1}$ and $\Psi(1) \in \mathrm{Sp}^*(2n)$. Denote this map by μ_{CZ}.

Now, for any $c = [v, x] \in \widetilde{\mathcal{P}(H)}$, if $x \in \mathcal{P}(H)$ is non-degenerate, then the path in $\mathrm{Sp}(2n)$ given by $l(t)$ defined above satisfies the conditions necessary to define $\mu_{CZ}(l(t))$. Thus, we define $\mu_H : \widetilde{\mathcal{P}(H)} \to \mathbb{Z}$ by $\mu_H(c) = \mu_{CZ}(l(t))$.

Using the Conley-Zehnder index, we set

$$\widetilde{\mathcal{P}(H)}_k = \{c \in \widetilde{\mathcal{P}(H)} \,|\, \mu_H(c) = k\}.$$

In the monotone case, Floer defined the k-chains as formal sums of elements of $\widetilde{\mathcal{P}(H)}_k$ with coefficients in $\mathbb{Z}$. For simplicity, we will use coefficients in $\mathbb{Z}_2$. In extending the theory to the weakly monotone and W^+ cases, Floer's boundary operator would not be well-defined (this will be explained later). To overcome this difficulty, we have to sacrifice the some simplicity in the chains: define $CF_k(M, \omega, H)$ as the set of all formal sums, $\sum m_c \langle c \rangle$, with $c \in \widetilde{\mathcal{P}(H)}_k$ and $m_c \in \mathbb{Z}_2$, which satisfy the condition $\{c \in \widetilde{\mathcal{P}(H)}_k \,|\, m_c \neq 0, a_H(c) > C\}$ is finite for all $C \in \mathbb{R}$. This finiteness condition will be necessary to guarantee that the boundary operator is well defined.

This set is clearly a group. We define $CF_*(M, \omega, H)$ as the graded group $\bigoplus CF_k(M, \omega, H)$. This graded group clearly has the structure of a module over $\mathbb{Z}_2$. It also has a natural structure of a module over the so-called Novikov ring of (M, ω), which we describe now.

Notice that Γ, (the group of deck transformations of the covering map $p : \widetilde{\mathcal{L}M} \to \mathcal{L}M$), acts on $CF_*(M, \omega, H)$ by $\gamma \cdot [v, x] = [\gamma(v), x]$. (Since $\gamma \in \Gamma$, it defines a homeomorphism of $\widetilde{\mathcal{L}M}$, and thus $\gamma(v)$ makes sense.) This action does not preserve the grading, and changes according to the formula

$$\mu_H(\gamma \cdot c) = \mu_H(c) - 2c_1(\gamma).$$

This formula is proved in theorem 4 of [8].

To preserve the grading, we index Γ by $\mu(\gamma) = -2c_1(\gamma)$, and we define

Λ_k as all formal sums

$$\sum_{\gamma \in \Gamma_k} m_\gamma \langle \gamma \rangle$$

such that $m_\gamma \in \mathbb{Z}_2$ and $\{\gamma \in \Gamma_k \,|\, m_\gamma \neq 0, \omega(\gamma) \leq C\}$ is finite for all $C \in \mathbb{R}$. Let $\Lambda = \bigoplus_k \Lambda_k$. Then the multiplication in Λ induced by the multiplication in $\pi_2(M)$ makes Λ into a commutative graded ring which we call the Novikov ring of (M, ω). The Γ action on $CF_*(M, \omega, H)$ extends naturally to give $CF_*(M, \omega, H)$ a graded Λ-module structure.

Remark 1 The point is that $CF_k(M, \omega, H)$ is not finitely generated as a module over $\mathbb{Z}_2$, but it is finitely generated over Λ, and its dimension is equal to the number of elements of $\mathcal{P}(H)$ with a lift to $\widetilde{\mathcal{P}(H)}_{(k \bmod (2N))}$, where N is the minimal Chern number of ω.

1.2.3 Morse Theory on $\widetilde{\mathcal{L}M}$ and the Boundary Operator

The plan is to apply Morse theory to $\widetilde{\mathcal{L}M}$ using the function a_H. In order to follow Morse theory, we want to give $\widetilde{\mathcal{L}M}$ a Riemannian metric, and consider the gradient flow of a_H. We will supply $\widetilde{\mathcal{L}M}$ with a metric by using a choice of $J \in \mathcal{J}(M, \omega, S^1)$.

Notice that $\mathcal{L}M$ is an infinite dimensional manifold, with local model space near x given by a neighborhood of the zero section inside $C^\infty(x^*TM)$, where x^*TM is the pullback bundle. The charts are given by exponentiation, using an arbitrary metric. (The size of the neighborhood depends on the choice of metric.)

A tangent vector to $x \in \mathcal{L}M$ is a smooth vector field along x. For two vectors in $T_x\mathcal{L}M$, we define

$$\langle \xi, \eta \rangle = \int_0^1 \omega\big(\xi(t), J_t \eta(t)\big) dt.$$

This is a metric whenever J_t is ω-compatible. We simply pull back this metric to $\widetilde{\mathcal{L}M}$ using the projection map p.

As in the finite dimensional case, the metric and the function a_H combine to give a vector field on $\mathcal{L}M$, called ∇a_H, by $\langle \nabla a_H, \cdot \rangle = da_H(\cdot)$. In Morse theory, one uses this vector field to define the stable and unstable manifolds of a critical point by flowing along the vector field. However, in the infinite dimensional setting, this vector field cannot be integrated to a flow, because we no longer have existence and uniqueness of solutions to partial differential equations.

In any case, it makes sense to speak of integral curves of the vector field. These integral curves can be describe using the following operator.

A curve $\tilde{u} : \mathbb{R} \to \widetilde{\mathcal{L}M}$ projects to a map $u : \mathbb{R} \times S^1 \to M$. For smooth $u \in C^\infty(\mathbb{R} \times S^1, M)$ and $(H, J) \in C^\infty(M \times S^1) \times \mathcal{J}(M, \omega, S^1)$, define $\overline{\partial}_{H,J}(u) \in C^\infty(u^*TM)$ by

$$\overline{\partial}_{H,J}(u)(s,t) = \frac{\partial u}{\partial s}(s,t) + J(u(s,t))\frac{\partial u}{\partial t}(s,t) - \nabla H_t(u(s,t)).$$

Here, ∇H_t is the gradient using the metric $r_t(\cdot, \cdot) = \omega(\cdot, J_t \cdot)$.

Proposition 1.14 *A smooth map $\tilde{u} : \mathbb{R} \to \widetilde{\mathcal{L}M}$ is an integral curve of $-\nabla a_H$ if and only if it's projection to $\mathcal{L}M$ is given by a map $u : \mathbb{R} \times S^1 \to M$ which satisfies $\overline{\partial}_{H,J}(u) = 0$.*

Proof:

Since the differential of a_H is the 1-form $p^*\alpha_H$, a simple calculation shows that $\nabla a_H(x)(t) = J_t(x(t))\dot{x}(t) - \nabla H_t(x(t))$. The integral curves satisfy $\frac{\partial u}{\partial s} + \nabla a_H(u(s, \cdot)) = 0$, so the proposition follows.

□

Given $c_-, c_+ \in \widetilde{\mathcal{P}(H)}$, the space $\mathcal{M}(c_-, c_+, H, J)$ is defined as the set of all maps $u : \mathbb{R} \times S^1 \to M$ satisfying $\overline{\partial}_{H,J}(u) = 0$, which lift to a map $\tilde{u} : \mathbb{R} \to \widetilde{\mathcal{L}M}$ such that $\lim_{s \to \pm\infty} \tilde{u}(s) = c_\pm$.

These are the "flow lines" of the (negative) gradient vector field on $\widetilde{\mathcal{L}M}$ defined by (H, J). In Morse Theory, a generic choice of metric guarantees that we can count these connecting orbits between two appropriately indexed critical points, and use this to define a boundary operator. Floer showed how to do the same in the infinite dimensional case.

Notice that in the case $H = 0$, an element of $\mathcal{M}(c_-, c_+, H, J)$ is just a J-holomorphic curve, introduced by Gromov in [12]. Floer realized that many of the ideas that Gromov introduced to deal with J-holomorphic curves apply to these perturbed curves. In particular, for a generic choice of (H, J), the spaces of connecting orbits are manifolds.

Theorem 1.15 (Floer, Hofer, Salamon) *Suppose (M, ω) satisfies condition W^+. There is a generic subset of $C^\infty(M \times S^1) \times \mathcal{J}(M, \omega, S^1)$, denoted by $\left(C^\infty(M \times S^1) \times \mathcal{J}\right)_{reg}$, such that for any pair (H, J) in $\left(C^\infty(M \times S^1) \times \mathcal{J}\right)_{reg}$, each $x \in \mathcal{P}(H)$ is non-degenerate, and for c_- and c_+ in $\widetilde{\mathcal{P}(H)}$, $\mathcal{M}(c_-, c_+, H, J)$ is a finite dimensional manifold of dimension $\mu_H(c_-) - \mu_H(c_+)$.*

Sketch of proof:

Fix $c_- = [v_-, x_-]$ and $c_+ = [v_+, x_+]$ in $\widetilde{\mathcal{P}(H)}$. Choose an arbitrary Riemannian metric on M. Let $\mathcal{B}$ be the Banach manifold of all $W^{1,p}$ maps $u : \mathbb{R} \times S^1 \to M$ with limits (in the $W^{1,p}$ sense) given by x_- and x_+. Define a bundle $\mathcal{E} \to \mathcal{B}$ by letting the fiber over $u \in \mathcal{B}$ be $L^p(u^*TM)$, the space of L^p sections of u^*TM. Then $\overline{\partial}_{H,J}$ defines a section of this Banach bundle. The parameters (H, J) can be chosen generically so that $\overline{\partial}_{H,J}$ is transverse to the zero section, and thus $\overline{\partial}_{H,J}^{-1}(0)$ is a (Banach) manifold.

Let $u : \mathbb{R} \times S^1 \to M$ be in $\mathcal{M}(c_-, c_+, H, J)$. The bundle $\mathcal{E} \to \mathcal{B}$ trivializes near $u \in \mathcal{B}$ as $W^{1,p}(u^*TM) \times L^p(u^*TM)$. The key point is that any $u' \in \mathcal{B}$ which is close enough to u is given by exponentiation of some $\xi \in W^{1,p}(u^*TM)$. Under this trivialization, the section $\overline{\partial}_{H,J}$ becomes an elliptic (and therefore Fredholm) operator, denoted D_u. By elliptic regularity, all solutions to $\overline{\partial}_{H,J}(u) = 0$ are actually smooth, so $\overline{\partial}_{H,J}^{-1}(0) = \mathcal{M}(c_-, c_+, H, J)$. There is a homeomorphism from a neighborhood of 0 in *ker* D_u to a neighborhood of 0 in $\overline{\partial}_{H,J}^{-1}(0)$. Since D_u is Fredholm, this kernel is finite dimensional.

The section $\overline{\partial}_{H,J}$ is transverse to the zero section if and only if each D_u is onto, and in this case, $\mathcal{M}(c_-, c_+, H, J)$ is actually finite dimensional, and the dimension near u is given by index(D_u). Finally, it can be shown that index(D_u) is exactly given by $\mu_H(c_-) - \mu_H(c_+)$, if u lifts to a map $\tilde{u} : \mathbb{R} \to \widetilde{\mathcal{L}M}$ with limits c_- and c_+.

□

There is a natural $\mathbb{R}$-action on $\mathcal{M}(c_-, c_+, H, J)$ via translation in the infinite direction. We let $\mathcal{M}(c_-, c_+, H, J)/\mathbb{R}$ denote the quotient manifold. If (H, J) is as in the above theorem, and if $\mu_H(c_-) - \mu_H(c_+) = 1$, then $\mathcal{M}(c_-, c_+, H, J)/\mathbb{R}$ is a 0-dimensional manifold. We want to count these spaces, so we need compactness.

Theorem 1.16 (Floer, Hofer, Salamon) *The set* $\big(C^\infty(M \times S^1) \times \mathcal{J}\big)_{reg}$ *of theorem 1.15 can be chosen such that for any* c_- *and* c_+ *in* $\widetilde{\mathcal{P}(H)}$ *with* $\mu_H(c_-) - \mu_H(c_+) = 1$, $\mathcal{M}(c_-, c_+, H, J)/\mathbb{R}$ *is compact, and thus finite.*

Sketch of proof:

This again essentially follows from the ellipticity of the linearized operators D_u, and the ability to generically avoid bubbling.

Let $u : \mathbb{R} \times S^1 \to M$ be smooth. Define the energy of such a map by

$$E(u) = \frac{1}{2} \int_0^1 \int_{-\infty}^{\infty} \Big(|\frac{\partial u}{\partial s}|^2 + |\frac{\partial u}{\partial t} - X_t^{\theta^H}(u)|^2 \Big) ds\, dt.$$

The main point is that if $\overline{\partial}_{H,J}(u) = 0$, then $E(u) < \infty$ if and only if it converges to two limit circles x_- and x_+ in $\mathcal{P}(H)$.

Now, it follows from elliptic methods that for any sequence of maps $u_j : \mathbb{R} \times S^1 \to M$ in $\mathcal{M}(c_-, c_+, H, J)$ and which have a uniform bound on $E(u_j)$, and any sequence $s_j \in \mathbb{R}$, there is a subsequence of u_j, still denoted u_j, for which the reparameterized sequence $v_j(s,t) = u_j(s - s_j, t)$ has a subsequence which converges uniformly with all derivatives on compact sets, modulo a finite set, to some limit map $u : \mathbb{R} \times S^1 \to M$. This limit map then also satisfies $\overline{\partial}_{H,J}(u) = 0$. The energy of the limit map must be less than or equal to the bound on the energy for the sequence, and therefore $\lim_{s\to\pm\infty}(u) = x^u_\pm \in \mathcal{P}(H)$. By choosing different sequences s_j, we obtain finitely many circles x_0, x_1, ... , $x_k \in \mathcal{P}(H)$, and finitely many maps $u_i : \mathbb{R} \times S^1 \to M$, $i = 1...k$ such that $\lim_{s\to-\infty} u_i(s,t) = x_{i-1}(t)$ and $\lim_{s\to\infty} u_i(s,t) = x_i(t)$. Moreover, all the maps u_i satisfy $\overline{\partial}_{H,J}(u_i) = 0$.

What remains is dealing with the finite set involved in the (weak) convergence of u_j to u. This is described by the phenomenon of bubbling, which is a fundamental part of Gromov compactness. The process can be summarized as follows. Let $\{z_1 = (s_1, t_1),\ z_2 = (s_2, t_2),\ ...,\ z_l = (s_l, t_l)\}$ be the finite set which arises in the convergence of u_j. (That is, $u_j \to u$ uniformly with all derivatives on compact sets on the domain $\mathbb{R} \times S^1 - \{z_1, z_2, ...z_l\}$.) One can can find sequences $(s_j^\nu, t_j^\nu) \in \mathbb{R} \times S^1$ for which the reparameterized maps $u_j^\nu : \mathbb{R} \times S^1 \to M$ given by $u_j^\nu(s,t) = u_j(s - s_j^\nu, t - t_j^\nu)$ converge uniformly with all derivatives on compact sets to a J_{t_ν} holomorphic sphere $v_\nu : S^2 \to M$.

Suppose $\mu_H(c_-) - \mu_H(c_+) = 1$. Then in fact, $u\#v_1\#...\#v_l$ satisfies

$$E(u) + \sum_{\nu=1}^{l} E(v_\nu) = \lim_{j\to\infty} E(u_j),$$

$$\sum_{i=1}^{k} \mu(u_i) + \sum_{j=1}^{l} 2c_1(v_j) = dim\ \mathcal{M}(c_-, c_+, H, J) = 1.$$

Since there are no J_t holomorphic sphere with negative Chern number for regular pairs (H, J), this implies that $k = 1$ and $l = 0$. That is, there is a single limit cylinder u, and no bubbling occurs.

Finally, since every $u \in \mathcal{M}(c_-, c_+, H, J)$ is an integral curve of $-\nabla a_H$, one can show that $E(u) = a_H(c_+) - a_H(c_-)$. This shows that the energy of each $u \in \mathcal{M}(c_-, c_+, H, J)$ is constant, so the theorem follows.

Notice that this is the point at which the W^+ condition is necessary.

□

This theorem was proved in the monotone case by Floer, then extended to the weakly monotone case by Hofer and Salamon.

This fact allows us to define a boundary operator on $CF_*(M, \omega, H)$ by counting the number of connecting orbits between two generators. For a regular pair (H, J) and $c_- \in \widetilde{\mathcal{P}(H)}_k$, set

$$\partial_k(H, J)(c_-) = \sum_{c_+ \in \widetilde{\mathcal{P}(H)}_{k-1}} \#\big(\mathcal{M}(c_-, c_+, H, J)/\mathbb{R}\big)\langle c_+\rangle.$$

Here, # means counting modulo 2.

There are two difficulties with this: it is not yet clear that this formal sum satisfies the finiteness condition on the chains in $CF_*(M, \omega, H)$, and we need to extend this to infinite linear combinations. Both of these problems can be solved exactly because of the finiteness condition.

For a k-chain $\kappa = \sum_{c_- \in \widetilde{\mathcal{P}(H)}_k} m_{c_-}\langle c_-\rangle$, $\partial_k(H,J)(\kappa)$ is a formal sum $\sum_{c_+ \in \widetilde{\mathcal{P}(\psi)}_{k-1}} m_{c_+}\langle c_+\rangle$, where the coefficient of $c_+ \in \widetilde{\mathcal{P}(\psi)}_{k-1}$ is given by the number of orbits (modulo time shift) connecting any $c_- \in \widetilde{\mathcal{P}(\psi)}_k$ with $m_{c_-} \neq 0$ to c_+. The above map wouldn't extend to infinite combinations if the number of such orbits could be infinite. The finiteness condition guarantees a uniform bound on the energy of all the orbits contributing to the coefficient of c_+, and thus the space of all such connecting orbits is compact as before, and thus finite.

This shows that $\partial_*(H,J)$ gives a boundary operator on $CF_*(M,\omega,H)$.

Theorem 1.17 (Floer) *Let (M,ω) be a symplectic manifold which satisfies condition W^+. The space $(C^\infty(M \times S^1) \times \mathcal{J})_{reg}$ can be chosen so that for regular pairs (H,J), the boundary operator $\partial(H,J)$ satisfies and $\partial^2 = 0$.*

Sketch of proof:

Let c_- and c_+ be such that $\mu_H(c_-) - \mu_H(c_+) = 2$. Now, the manifold $\mathcal{M}(c_-,c_+,H,J)/\mathbb{R}$ is not compact, but it is compact up to splitting of orbits. That is, we can think of $\mathcal{M}(c_-,c_+,H,J)/\mathbb{R}$ as a compact 1-dimensional manifold with boundary given by

$$\partial(\mathcal{M}(c_-,c_+,H,J)/\mathbb{R}) = \bigcup_{c_0 \in \widetilde{\mathcal{P}(H)}_{k-1}} (\mathcal{M}(c_-,c_0,H,J)/\mathbb{R}) \times (\mathcal{M}(c_0,c_+,H,J)/\mathbb{R}).$$

This depends on some general position arguments to avoid bubbling, coupled with Floer's gluing procedure.

Now, $\partial \circ \partial(c_-)$ is a formal sum of elements of $\widetilde{\mathcal{P}(H)}_{k-1}$, and the coefficient c_+ in the formal sum $\partial \circ \partial(c_-)$ is exactly the number of elements in $\partial(\mathcal{M}(c_-,c_+,H,J)/\mathbb{R})$. But this must be an even number, by the classification of compact 1-manifolds.

□

The homology of the chain complex $\big(CF_*(M,\omega,H),\partial_*(H,J)\big)$ is denoted $HF_*(M,\omega,H,J)$.

The differentials $\partial_*(H,J)$ are linear over the Novikov ring Λ, so we see that $HF_*(M,\omega,H,J)$ is also a Λ-module.

1.2.4 Continuation Isomorphisms

Finally, one can prove that these homology groups are actually independent of the choice of (regular) pair (H,J). This is done by use of the so called "continuation maps".

For two regular pairs (H^-,J^-) and (H^+,J^+), a **homotopy of regular pairs** is a function $H \in C^\infty(M\times\mathbb{R}\times S^1)$ and a $\mathbf{J} = (J_t^s) \in \mathcal{J}(M,\omega,\mathbb{R}\times S^1)$ that satisfy the following condition:

$$\begin{aligned} s \le -1 &\implies (H_t(\cdot,s),J_t^s) = (H_t^-(\cdot),J_t^-),\\ s \ge 1 &\implies (H_t(\cdot,s),J_t^s) = (H_t^+(\cdot),J_t^+). \end{aligned}$$

Given a homotopy Φ from (H^-,J^-) to (H^+,J^+) as above and a fixed $s\in\mathbb{R}$, we will usually denote by $\Phi(s)$ the pair $\big(H(\cdot,s,\cdot),J^s\big)$.

For $c_- \in \widetilde{\mathcal{P}(H^-)}$, $c_+ \in \widetilde{\mathcal{P}(H^+)}$ and Φ a homotopy of pairs from (H^-,J^-) to (H^+,J^+), the space $\mathcal{M}(c_-,c_+,\Phi)$ is defined as the space of all smooth maps $u:\mathbb{R}\times S^1\to M$ which satisfy

$$\frac{\partial u}{\partial s} + J_{s,t}(u)(\frac{\partial u}{\partial t}) - \nabla H(s,t,u) = 0,$$

and which can be lifted to a path $\tilde{u}:\mathbb{R}\to\widetilde{\mathcal{L}M}$ with limits c_-,c_+.

Theorem 1.18 (Floer) *For a generic homotopy of regular pairs, all of the spaces* $\mathcal{M}(c_-, c_+, \Phi)$ *have a finite number of elements whenever* $\mu_{H^-}(c_-) = \mu_{H^+}(c_+)$.

This is proved in the same fashion as theorem 1.16. Such a homotopy is called regular.

We define the "continuation homomorphisms" for a regular homotopy by

$$\Phi_k(H, \mathbf{J}) : CF_k(M, \omega, H^-) \to CF_k(M, \omega, H^+)$$

$$\langle c_- \rangle \mapsto \sum_{c_+ \in \widetilde{\mathcal{P}(H^+)}_k} \# \mathcal{M}^\Phi(c_-, c_+, H, \mathbf{J}) \langle c_+ \rangle, \quad c_- \in \widetilde{\mathcal{P}(H^-)}_k.$$

Again, this formula is to be extended linearly to infinite formal sums, and # means counting modulo 2. Regularity implies that # again makes sense. This is a homomorphism of chain complexes and it induces an isomorphism on the homology level. This isomorphism is denoted $\Phi_*(H, \mathbf{J})$.

Theorem 1.19 *The continuation isomorphism from* $HF_*(M, \omega, H^-, J^-)$ *to* $HF_*(M, \omega, H^+, J^+)$ *is independent of the choice of regular homotopy from* (H^-, J^-) *to* (H^+, J^+)*, and the isomorphisms are functorial with respect of composition of homotopies.*

Sketch of proof:

Given two homotopies Φ^α and Φ^β, one chooses a "homotopy of homotopies" connecting Φ^α and Φ^β. This consists of a function $H \in C^\infty(M \times [0,1] \times \mathbb{R} \times S^1)$ and an almost complex structure $J \in \mathcal{J}(M, \omega, [0,1] \times \mathbb{R} \times S^1)$. These must be such that

$$\big(H(p, 0, s, t), J_t^{0,s}\big) = \Phi^\alpha(s),$$
$$\big(H(p, 1, s, t), J_t^{1,s}\big) = \Phi^\beta(s).$$

The idea is that by fixing $r \in [0,1]$, we get a homotopy Φ^r from (H^-, J^-) to (H^+, J^+), and by varying r, we move from Φ^α to Φ^β.

Given such a homotopy of homotopies Ψ and $c_\pm \in \widetilde{\mathcal{P}(H^\pm)}$, consider the parameterized modulii space $\mathcal{M}(c_-, c_+, \Psi)$ which consists of the set of all pairs (u, r) for which $u \in \mathcal{M}(c_-, c_+, \Phi^r)$, and index $D_u = -1$. For a generic homotopy of homotopies, these spaces are compact, 0-dimensional manifolds whenever $\mu_{H^-}(c_-) = \mu_{H^+}(c_+) - 1$, and this gives a map $\Psi : CF_*(M, \omega, H^-) \to CF_{*+1}(M, \omega, H^+)$. This map is a chain homotopy between the the isomorphisms corresponding to Φ^α and Φ^β, and thus the continuations induce the same isomorphisms on homology.

□

This means that $HF_*(M, \omega)$ makes sense without mention of the choice of regular pair (H, J). In fact, the continuation homomorphisms are also linear over Λ, so we have an induced Λ-module structure on $HF_*(M, \omega)$ independent of the choice of regular pair used to define it.

The final step is in proving that $HF_*(M, \omega)$ recovers to homology of the underlying manifold. The **quantum homology** $QH_*(M, \omega)$ is the graded Λ-module given by

$$QH_k(M, \omega) = \bigoplus_{i+j=k} H_i(M, \mathbb{Z}_2) \otimes \Lambda_j.$$

Theorem 1.20 (Piunikhin, Salamon, Schwarz) *The quantum homology and Floer homology agree. That is, $HF_*(M, \omega) \backsimeq QH_*(M, \omega)$ as graded Λ-modules.*

To summarize, $H \in C^\infty(M \times S^1)$ is used to create $CF_*(M, \omega, H)$. For a generic choice of $(H, J) \in C^\infty(M \times S^1) \times \mathcal{J}(M, \omega, S^1)$, we can define a boundary operator on $CF_*(M, \omega, H)$, and we can define $HF_*(M, \omega, H, J)$ as the homology of the pair $\big(CF_*(M,\omega,H),\partial_*(H,J)\big)$. There are continuation isomorphisms between the groups defined using any choice of regular pairs, so we may speak of $HF_*(M,\omega)$ independently of the choice of regular pair used to define it.

1.3 Seidel's Action

In [21], P. Seidel introduced an action of a certain extension of $\pi_1(\mathrm{Ham}(M))$ on the Floer homology groups of (M,ω). More precisely, he introduces a topological group $\widetilde{G}$ which fits into the exact sequence

$$1 \to \Gamma \to \widetilde{G} \to G \to 1,$$

where Γ is discreet. To describe this action, we first need to introduce the extension he uses.

1.3.1 The Extension $\widetilde{G}$

Let G be the group of smooth loops in $\mathrm{Ham}(M)$ based at id. More precisely, for continuous $g : S^1 \to \mathrm{Diff}(M)$, define $\bar{g} : M \times S^1 \to M$ by $\bar{g}(p,t) = g_t(p)$. We say that g is smooth if $\bar{g}$ is smooth. Then let

$$G = \{g : S^1 \to \mathrm{Ham}(M)\,|\, g \;\text{ is }\; \text{smooth}, \;\; g_0 = g_1 = id\}.$$

We topologize G as a subspace of $C^\infty(M \times S^1, M)$ with the C^∞ topology.

Lemma 1.21 $\pi_0(G) = \pi_1(\mathrm{Ham}(M))$

Proof:

This is not immediate, because the fundamental group considers continuous loops, and G contains only smooth loops, but it follows from the fact that any continuous loop in Ham(M) can be approximated by a smooth loop, coupled with the fact that there is a neighborhood of *id* in Ham(M) such that each path component of this neighborhood is contractible .

□

Recall that ΛM consists of all parameterized loops in M, and $\mathcal{L}M$ is the subspace of contractible loops. There is a natural action of G on ΛM: for $x \in \Lambda M$ and $g \in G$, define $(g \cdot x)$ by

$$(g \cdot x)(t) = g_t(x(t)). \tag{2}$$

Proposition 1.22 $G \cdot \mathcal{L}M = \mathcal{L}M$

This is actually a consequence of theorem 1.20. We will reproduce Seidel's proof after some preliminary facts are established.

Notice that if we consider this action applied to loops which are constant in M, we obtain the folkloric fact that the image of a point under a Hamiltonian loop is contractible. It was first noticed in [4] that the image of a point under a Hamiltonian loop is homologous to zero, but contractibility is essentially a consequence of theorem 1.20.

Recall that $\widetilde{\mathcal{L}M}$ is a covering space over $\mathcal{L}M$. For $g \in G$, we have a map $g : \mathcal{L}M \to \mathcal{L}M$ given by equation 2. (See the following diagram.)

$$\begin{array}{ccc} \widetilde{\mathcal{L}M} & & \widetilde{\mathcal{L}M} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{L}M & \xrightarrow{g} & \mathcal{L}M \end{array}$$

Lemma 1.23 (Seidel) *The action of any $g \in G$ lifts to a homeomorphism of $\widetilde{\mathcal{L}M}$.*

Proof:

By the Lifting Theorem, we need only show that the action of g on $\mathcal{L}M$ preserves the set of smooth maps $S^1 \to \mathcal{L}M$ which lift to a map $S^1 \times S^1 \to \widetilde{\mathcal{L}M}$. Any such map is given by a map $A \in C^\infty(S^1 \times S^1, M)$, with $\omega(A) = c_1(A) = 0$. The image of A under the loop g is given by $(g \cdot A)(s,t) = g_t(A(s,t))$. Because g is Hamiltonian, $(g \cdot A)^*\omega = A^*(\omega)$, and thus $\omega(g \cdot A) = \omega(A)$. To prove that $c_1(g \cdot A) = 0$, notice that g induces an isomorphism of symplectic vector bundles from A^*TM to $(g \cdot A)^*TM$ by $(s,t,\xi) \to (s,t,Dg_t\xi)$. The statement then follows from naturality of the Chern classes.

□

We can now introduce the extension of G which is used in Seidel's action.

Let $\widetilde{G} \subset \big(G \times Homeo(\widetilde{\mathcal{L}M})\big)$ consist of all pairs $(g, \tilde{g})$ such that $\tilde{g}$ is a lift of the action of g on $\mathcal{L}M$ to a homeomorphism of $\widetilde{\mathcal{L}M}$.

We topologize $\widetilde{G}$ by using the C^∞ topology on G and the topology of pointwise convergence on Homeo$(\widetilde{\mathcal{L}M})$. The preceding lemma shows that $\widetilde{G}$ is non-empty. Then $\widetilde{G}$ is a topological group. The kernel of the projection $\widetilde{G} \to G$ is exactly all pairs (id, γ), with $\gamma \in \Gamma$. (Recall that Γ is the group of deck transformations of the covering $\widetilde{\mathcal{L}M} \to \mathcal{L}M$.) Thus, there is the following exact sequence of topological groups.

$$1 \to \Gamma \to \widetilde{G} \to G \to 1 \qquad (3)$$

1.3.2 The Map $\tilde{\sigma} : \widetilde{G} \to Aut(HF_*(M, \omega))$

The basic idea in Seidel's action is to choose a regular pair (H, J), and to use $g \in G$ to define a new regular pair (H^g, J^g). This new pair is defined in such a way that the action of g on $\mathcal{L}M$ gives

$$g \cdot \mathcal{P}(H^g) = \mathcal{P}(H).$$

By lifting to $(g, \tilde{g}) \in \widetilde{G}$, we see that

$$\tilde{g}(\widetilde{\mathcal{P}(H^g)}) = \widetilde{\mathcal{P}(H)}.$$

The map $\langle c \rangle \mapsto \langle \tilde{g}(c) \rangle$ induces a chain map between the complexes $\big(CF_*(M, \omega, H^g), \partial_*(H^g, J^g)\big)$ and $\big(CF_*(M, \omega, H), \partial_*(H, J)\big)$. It follows that the map $\langle c \rangle \to \langle \tilde{g}(c) \rangle$ induces a homomorphism

$$HF_*(M, \omega, H^g, J^g) \xrightarrow{\tilde{g}} HF_*(M, \omega, H, J).$$

This map is an isomorphism, with inverse induced by $(g^{-1}, \tilde{g}^{-1})$.

By precomposing with the continuation isomorphism (see section 1.2.4) from $HF_*(M, \omega, H, J)$ to $HF_*(M, \omega, H^g, J^g)$ we obtain an automorphism of $HF_*(M, \omega, H, J)$. Next, Seidel proves that this automorphism is independent of the regular pair used to define $HF_*(M, \omega)$, in the sense that the appropriate diagrams are commutative. Finally, he shows that the automorphism is well defined on $\pi_0(\widetilde{G})$. We will have need of some of the details contained inside this quick description, so we will expand the explanation a bit now.

Begin by choosing a regular pair (H, J), and let $g \in G$. Choose $K^g \in C^\infty(M \times S^1)$, a Hamiltonian function generating g.

Definition 1.24 *For H, g and K^g as above, define $H^g \in C^\infty(M \times S^1)$ by $H^g(p, t) = H(g_t(p), t) - K^g(g_t(p), t)$.*

Lemma 1.25 *For all $g \in G$ and $H \in C^\infty(M \times S^1)$, $g^* \alpha_H = \alpha^{H^g}$.*

Proof:

This is a straightforward caclulation.

□

We can use this to give a proof of proposition 1.22.

Proof of proposition 1.22:

If $g \cdot \mathcal{L}M \neq \mathcal{L}M$, then $g \cdot \mathcal{L}M$ does not intersect $\mathcal{L}M$. (This is because $\mathcal{L}M$ is a connected component of ΛM.) Suppose (H, J) is a regular pair, and H chosen small enough so that the only zeros of α_H are the critical points of H, and thus α_H has no zeros in $g \cdot \mathcal{L}M$. It then follows from 1.25 that α^{H^g} has no zeros in $\mathcal{L}M$. But this would mean that $HF_*(M, \omega, H^g, J^g) = 0$, which contradicts theorem 1.20.

□

Lemma 1.26 *For* $g \in G$, $g \cdot \mathcal{P}(H^g) = \mathcal{P}(H)$.

Proof:

Recall that the function H is used to define a 1-form on $\mathcal{L}M$, called α_H, and $\mathcal{P}(H) = Zeroes(\alpha_H)$. The lemma follows from lemma 1.25.

□

It follows from the previous lemma that for any choice of $(g, \tilde{g}) \in \widetilde{G}$, $\tilde{g}(\widetilde{\mathcal{P}(H^g)}) = \widetilde{\mathcal{P}(H)}$. However, this map does not preserve the Conley-Zehnder index. To describe the change in index, Seidel introduces a "Maslov Index" on $\widetilde{G}$. The index is defined as follows: Choose a point $c = [v, x] \in \widetilde{\mathcal{L}M}$. Trivialize the symplectic bundle (v^*TM, ω) as $D^2 \times (\mathbb{R}^{2n}, \omega_0)$. Let $A_t : (T_{x(t)}M, \omega) \to (\mathbb{R}^{2n}, \omega_0)$ be this trivialization on the boundary circle. Consider $\tilde{g}(c) = [v', g \cdot x] \in \widetilde{\mathcal{L}M}$. Trivialize $\big((v')^*TM, \omega\big)$ as $D^2 \times (\mathbb{R}^{2n}, \omega_0)$, and let $B_t : (T_{g_t(x(t))}M, \omega) \to (\mathbb{R}^{2n}, \omega_0)$ be the trivialization on the boundary, as before. Then $B_t Dg_t(A_t)^{-1}$ gives a linear, symplectic map from $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n}$, and is thus given by a symplectic matrix, denoted $l(t)$. Then $\{l(t)\}$ is a loop in Sp(2n,$\mathbb{R}$), and this loop is independent, up to homotopy, of all the choices involved, including the choice of $c \in \widetilde{\mathcal{L}M}$. Let $deg : \pi_1(\mathrm{Sp}(2n, \mathbb{R})) \to \mathbb{Z}$ be the canonical isomorphism induced by the determinant function on $U(n) \subset \mathrm{Sp}(2n, \mathbb{R})$. For details on this isomorphism, consult [1].

Definition 1.27 *Define an index map* $I : \widetilde{G} \to \mathbb{Z}$ *by* $I(g, \tilde{g}) = deg(l)$, *where* l *is the loop in* Symp$(2n)$ *defined above.*

The following is a simple lemma of Seidel:

Lemma 1.28 (Seidel) $I(g, \tilde{g})$ *depends only on* $[g, \tilde{g}] \in \pi_0(\widetilde{G})$, I *is a homomorphism, and* $I(id, \gamma) = c_1(\gamma)$.

This index is related to lemma 1.26 by the following lemma.

Lemma 1.29 (Seidel) *For* (H, J) *and* g *as above,*

$$\mu_{H^g}(c) = \mu_H(\tilde{g}(c)) - 2I(g, \tilde{g}).$$

This shows that $\langle c \rangle \mapsto \langle \tilde{g}(c) \rangle$ induces a map from $CF_k(M, \omega, H^g)$ to $CF_{k-2I(g,\tilde{g})}(M, \omega, H)$. To guarantee that this is a chain map, we have to adjust the choice of almost complex structure.

For $J \in \mathcal{J}(M, \omega, S^1)$ and $g \in G$, define $J^g \in \mathcal{J}(M, \omega, S^1)$ by

$$J_t^g = Dg_t^{-1} J_t Dg_t.$$

Seidel shows that if (H, J) is a regular pair, then so is (H^g, J^g).

Proposition 1.30 *Suppose (H, J) is a regular pair. Let c_- and c_+ be in $\widetilde{\mathcal{P}(H^g)}$. Then there is a bijection between $\mathcal{M}(c_-, c_+, H^g, J^g)/\mathbb{R}$ and $\mathcal{M}(\tilde{g}(c_-), \tilde{g}(c_+), H, J)/\mathbb{R}$.*

Proof:

Let $u : \mathbb{R} \to \widetilde{\mathcal{L}M}$ be and element of $\mathcal{M}(c_-, c_+, H^g, J^g)$. Define $v : \mathbb{R} \to \widetilde{\mathcal{L}M}$ by $v(s) = \tilde{g}(u(s))$. It is not hard to check that v is an element of $\mathcal{M}(\tilde{g}(c_-), \tilde{g}(c_+), H, J)$, and that this map is $\mathbb{R}$-equivariant.

□

Since the boundary operator consists of counting the number of elements in these spaces, it follows that $\langle c \rangle \to \langle \tilde{g}(c) \rangle$ induces a chain map from $\big(CF_*(M, \omega, H^g), \partial_*(H^g, J^g)\big)$ to $\big(CF_*(M, \omega, H), \partial_*(H, J)\big)$. As described above, this map is an isomorphism still denoted $\tilde{g}$. This gives an automorphism by $HF_*(M, \omega, H, J)$ by

$$HF_*(M, \omega, H, J) \xrightarrow{\Phi} HF_*(M, \omega, H^g, J^g) \xrightarrow{\tilde{g}} HF_*(M, \omega, H, J),$$

where Φ denotes the continuation isomorphism.

In order to show that this induces an automorphism of $HF_*(M, \omega)$ independently of the choice of (H, J), Seidel proves the following theorem.

Theorem 1.31 (Seidel) *Let (H^-, J^-) and (H^+, J^+) be regular pairs. For $(g, \tilde{g}) \in \widetilde{G}$, the following diagram commutes. (Here, Φ denotes the continuation isomorphism between the appropriate homologies.)*

$$\begin{array}{ccccc} HF_*(M,\omega,H^-,J^-) & \xrightarrow{\Phi} & HF_*(M,\omega,(H^-)^g,(J^-)^g) & \xrightarrow{\tilde{g}} & HF_*(M,\omega,H^-,J^-) \\ \Phi\downarrow & & \Phi\downarrow & & \downarrow\Phi \\ HF_*(M,\omega,H^+,J^+) & \xrightarrow{\Phi} & HF_*(M,\omega,(H^+)^g,(J^+)^g) & \xrightarrow[\tilde{g}]{} & HF_*(M,\omega,H^+,J^+) \end{array}$$

Next, Seidel shows that this map is linear over Λ. This shows that each $(g,\tilde{g}) \in \widetilde{G}$ induces an automorphism of $HF_*(M,\omega)$ as a Λ-module independent of the choice of regular pair used to define $HF_*(M,\omega)$. This automorphism is denoted $HF_*(g,\tilde{g})$. To complete the construction, Seidel proves that the automorphism is well defined on $\pi_0(\widetilde{G})$.

Theorem 1.32 (Seidel) *If $(g^0,\tilde{g}^0)$ and $(g^1,\tilde{g}^1)$ define the same element of $\pi_0(\widetilde{G})$, then $HF_*(g^0,\tilde{g}^0) = HF_*(g^1,\tilde{g}^1)$.*

This theorem means that for any regular pair (H,J), the following diagram commutes whenever $[g^0,\tilde{g}^0] = [g^1,\tilde{g}^1]$ inside $\pi_0(\widetilde{G})$.

$$\begin{array}{ccc} HF_*(M,\omega,(H^0)^{g^0},(J^0)^{g^0}) & \xrightarrow{\tilde{g}^0} & HF_*(M,\omega,H^0,J^0) \\ \Phi\downarrow & & \downarrow\Phi \\ HF_*(M,\omega,(H^1)^{g^1},(J^1)^{g^1}) & \xrightarrow[\tilde{g}^1]{} & HF_*(M,\omega,H^1,J^1) \end{array}$$

Again, Φ means the continuation isomorphisms between the appropriate homologies.

This gives a map

$$\tilde{\sigma} : \pi_0(\widetilde{G}) \to Aut(HF_*(M,\omega)),$$
$$(g,\tilde{g}) \mapsto HF_*(g,\tilde{g}).$$

We list some basic properties of this map, which are clear from the definition.

Proposition 1.33 *The map $\tilde{\sigma}$ has the following properties:*

1. $HF_*(g, \tilde{g})$ *is an automorphism of* $HF_*(M, \omega)$ *as a* Λ*-module.*

2. *If* $(g, \tilde{g}) = Id_{\tilde{G}}$*, then* $HF_*(g, \tilde{g}) = Id_{Aut(HF_*(M,\omega))}$*.*

3. *If* $(g, \tilde{g}) = (id, \gamma)$ *for some* $\gamma \in \Gamma$*, then* $HF_*(g, \tilde{g})$ *is given by multiplication by* γ*.*

4. *The map* $\tilde{\sigma}$ *is a homomorphism.*

In order to apply this to symplectic geometry, we consider the related map

$$\sigma : \pi_1(\mathrm{Ham}(M)) \to Aut(HF_*(M, \omega))/\Gamma,$$

which is well defined because of the exactness of the sequence (3). (Here, we identify $\gamma \in \Gamma$ with the automorphism of $HF_*(M, \omega)$ induced by $\langle c \rangle \mapsto \langle \gamma(c) \rangle$. Also recall that $\pi_0(G) = \pi_1(\mathrm{Ham}(M))$.)

This ends the review of Floer homology and Seidel's homomorphism.

2 Rephrasing $\tilde{\sigma}$ in terms of Isotopies

The main goal in this thesis is to prove that the homomorphism

$$\sigma : \pi_1(\mathrm{Ham}(M)) \to Aut(HF_*(M,\omega))/\Gamma$$

is well defined on the group $i_*(\pi_1(\mathrm{Ham}(M)))$, where $i : \mathrm{Ham}(M) \to \mathrm{Diff}(M)$ is the inclusion. That is, we would like to show that if two loops in $\mathrm{Ham}(M)$ are homotopic through loops in $\mathrm{Diff}(M)$, then the image of the classes of these loops agree under the homomorphisms σ. Accordingly, we are interested in the loops in $\mathrm{Diff}(M)$ which are homotopic to a Hamiltonian loop.

Definition 2.1 *A loop at id in* $\mathrm{Diff}(M)$ *will be called* **almost Hamiltonian** *is it is homotopic to a loop in* $\mathrm{Ham}(M)$.

Let $i : \mathrm{Ham}(M) \to \mathrm{Diff}(M)$ be the inclusion, and

$$i_* : \pi_1(\mathrm{Ham}(M)) \to \pi_1(\mathrm{Diff}(M))$$

be the map on fundamental groups induced by i. It is clear that g is almost Hamiltonian if and only if $[g] \in \pi_1(\mathrm{Diff}(M))$ is in the image of i_*.

We will denote the space of all almost Hamiltonian loops by D. Then D is clearly a subgroup of $\mathrm{Diff}(M)$. We equip D with the C^∞ topology.

Consider σ as a map with domain G. We will extend the domain of definition of σ to the (possibly) larger group D, and then prove this extended map is well defined on $\pi_0(D)$.

2.1 Calculating θ^{H^g}

In order to extend to domain of σ, it it instructive to calculate the Hamiltonian isotopy generated by H^g in terms of g and the isotopy generated by H.

Let $\mathcal{N}$ denote the space of all smooth isotopies of M. That is, elements of $\mathcal{N}$ are maps $\psi : [0,1] \to \mathrm{Diff}(M)$ with $\psi(0) = id$ such that the associated map $[0,1] \times M \to M$ given by $(t,p) \mapsto \psi(t)(p)$ is smooth. This space is topologized as a subspace of $C^\infty([0,1] \times M, M)$. We will generally write ψ_t for $\psi(t)$. By a slight abuse of notation, ψ_t will sometimes represent the entire isotopy.

For a detailed analysis of these spaces of isotopies, consult [2].

There is a natural action of D on $\mathcal{N}$ given by

$$(g * \psi)_t := g_t^{-1} \psi_t.$$

The reason that we include an inverse here will become clear later.

Definition 2.2 *For $\psi = (\psi_t)$ an isotopy of M, X_t^ψ is the family of vector fields on M defined by $X_t^\psi(p) = \frac{d\psi_t}{dt}(\psi_t^{-1}(p))$.*

The correspondence $\psi \leftrightarrow X_t^\psi$ is one-to-one, as each smooth 1-parameter family of vector fields determines an isotopy by integration. (Recall that M is compact.)

Notice that if θ is a Hamiltonian isotopy generated by H, then by definition, $\frac{d\theta_t}{dt}(\theta_t^{-1}(p)) = X_{H_t}$, so we see that $X_t^\theta = X_{H_t}$.

We will need the chain rule for isotopies:

Lemma 2.3 (Chain Rule) *For any isotopies $\phi = (\phi_t)$ and $\psi = (\psi_t)$ of M, the following identities hold:*

1) $X_t^{\psi\cdot\phi} = X_t^{\psi} + D\psi_t X_t^{\phi} \circ \psi_t^{-1}$
2) $X_t^{\phi^{-1}} = -D\phi_t^{-1} X_t^{\phi} \circ \phi_t$

$\square$

We will also need the following lemma.

Lemma 2.4 *For* $g \in G$, $H \in C^\infty(M \times S^1)$, *and* $K^g \in C^\infty(M \times S^1)$ *generating* g*, the following identities hold:*
1) $Dg_t^{-1} X_{H_t} \circ g_t = X_{H_t \circ g_t}$
2) $Dg_t^{-1} X_{K_t^g} \circ g_t = X_{K_t^g \circ g_t}$

Proof:

To see this, consider the following string of unravelling definitions:

$$\begin{aligned} i_{Dg_t^{-1} X_{H_t} \circ g_t} \omega(Y_p) &= \omega(X_{H_t}(g_t(p)), Dg_t Y_p) \\ &= g_t^*(i_{X_{H_t}} \omega(Y_p)) \\ &= g_t^*(dH_t)(Y_p) \\ &= d(H_t \circ g_t)(Y_p) \\ &= i_{X_{H_t \circ g_t}} \omega(Y_p) \end{aligned}$$

Since a vector field is uniquely defined by it's interior multiplication with ω, the conclusion follows. The same proof holds for the other identity.

$\square$

We can now calculate the isotopy generated by H^g. Recall that for $H \in C^\infty(M \times S^1)$, θ^H denotes the Hamiltonian isotopy of M generated by H.

Lemma 2.5 *Let $g \in G$ with generating function $K^g \in C^\infty(M \times S^1)$, and let $H \in C^\infty(M \times S^1)$. Let $H^g \in C^\infty(M \times S^1)$ be as in definition 1.24. Then $\theta^{H^g} = g * \theta^H$.*

Proof:

We need only show that $X_t^{g*\theta^H} = X_{H_t^g}$. Recall that $H_t^g = H_t \circ g_t - K_t^g \circ g_t$, and $(g * \theta^H)_t = g_t^{-1}\theta_t^H$. Then lemma 2.3 gives:

$$\begin{aligned} X_t^{g*\theta^H} &= X_t^{g^{-1}} + Dg_t^{-1} X_t^{\theta^H} \circ g_t \\ &= -Dg_t^{-1} X_t^g \circ g_t + Dg_t^{-1} X_t^{\theta^H} \circ g_t \end{aligned}$$

Now, $X_t^g = X_{K_t^g}$, and $X_t^{\theta^H} = X_{H_t}$ by definition. This, combined with lemma 2.4, gives:

$$\begin{aligned} X_t^{g*\theta^H} &= X_{H_t \circ g_t} - X_{K_t^g \circ g_t} \\ &= X_{(H_t - K_t^g) \circ g_t} \\ &= X_{H_t^g} \end{aligned}$$

This proves the proposition.

□

This is why we include an inverse in the definition of the $*$ operation.

This will allow us to rephrase Seidel's homomorphism in terms of isotopies. This rephrasing suggests a way of extending the domain of σ.

2.2 Emphasizing Isotopies

The goal of this subsection is to prove the following (nearly obvious) proposition:

Proposition 2.6 *If (H, J) and (H', J) are two regular pairs such that $\theta^H = \theta^{H'}$, then $HF_*(M, \omega, H, J) = HF_*(M, \omega, H', J)$.*

Proof:

The proof is essentially a string of trivial observations, but there are enough that it seems prudent to mention them.

As described in the first section, the construction of the Floer homology groups $HF_*(M, \omega, H, J)$ is best described using an analogy with Morse Theory. The chain complex $CF_*(M, \omega, H)$ is generated by the critical points of the action functional, indexed appropriately (with a finiteness condition). The boundary operator consists of counting "flow lines" of the the negative gradient vector field on $\widetilde{\mathcal{L}M}$ corresponding to the action functional and a metric on $\widetilde{\mathcal{L}M}$ defined by the almost complex structure J. These flow lines are solutions of the differential equation $\overline{\partial}_{H,J}(u) = 0$. Thus, to prove this proposition, we must only show that if (H, J) and (H', J) are as in proposition 2.6, (meaning they generate the same isotopy), then $\widetilde{\mathcal{P}(H)}_k = \widetilde{\mathcal{P}(H')}_k$, $\mathcal{M}(c_-, c_+, H, J) = \mathcal{M}(c_-, c_+, H', J)$, and $\overline{\partial}_{H,J} = \overline{\partial}_{H',J}$. Given these facts, proposition 2.6 is obvious, since $HF_*(M, \omega, H, J)$ depends only on these objects.

Let H and H' generate the same Hamiltonian isotopy. It then follows that $\mathcal{P}(H) = \mathcal{P}(H')$, since these sets consist of loops obtained by flowing along the corresponding isotopy. Thus, $\widetilde{\mathcal{P}(H)} = \widetilde{\mathcal{P}(H')}$. Since the Conley-Zehnder index is obtained by linearizing according to the isotopy, we see that $\widetilde{\mathcal{P}(H)}_k = \widetilde{\mathcal{P}(H')}_k$. The assumption of the proposition means that H and H' differ by a constant, so the finiteness conditions in the definitions of $CF_*(M, \omega, H)$ and $CF_*(M, \omega, H')$ agree, showing that $CF_*(M, \omega, H) = CF_*(M, \omega, H')$. The last step is to see that the boundary operators match. This is easy to see by noticing that $\overline{\partial}_{H,J} = \overline{\partial}_{H',J}$, which is true because $\nabla H = \nabla H'$.

□

Notice that this truly means equals. An isomorphism is guaranteed by the continuation maps, but this propositions says that the two groups are equal *by contruction.*

This proposition allows us to shift our attention away from the function, and toward it's generated isotopy. In fact, in later sections, we will have no function at all, and begin with an isotopy.

At each step in the construction of the Λ-module $HF_*(M,\omega,H,J)$, we can convert the dependence on the function, H, to dependence on the isotopy, θ^H.

Now that proposition 2.6 has been proved, we are justified in making the following definition.

Definition 2.7 *For $\theta = \theta^H$ a Hamiltonian isotopy and $J \in \mathcal{J}(M,\omega,S^1)$, if (H,J) is a regular, we define $HF_*(M,\omega,\theta,J)$ as $HF_*(M,\omega,H,J)$.*

2.3 Rephrasing $\tilde{\sigma}$

According to proposition 2.6, we can define $HF_*(M,\omega)$ via a generic choice of Hamiltonian isotopy and almost complex structure. We can now rephrase the map $\tilde{\sigma}$ in terms of isotopies.

Definition 2.8 *For a Hamiltonian isotopy $\theta = \theta^H$ and an almost complex structure $J \in \mathcal{J}(M,\omega,S^1)$, we will say that the pair (θ,J) is a regular pair if $(H,J) \in \big(C^\infty(M \times S^1) \times \mathcal{J})_{reg}$.*

Let (θ,J) be a regular pair, and choose $(g,\tilde{g}) \in \widetilde{G}$. Then in terms of isotopies, $(g,\tilde{g})$ induces an isomorphism from $HF_*(M,\omega,g*\theta,J^g)$ to $HF_*(M,\omega,\theta,J)$, because $\theta^{H^g} = g*\theta$. By precomposing with the continuation isomorphism from $HF_*(M,\omega,\theta,J)$ to $HF_*(M,\omega,g*\theta,J^g)$, we recover the automorphism of $HF_*(M, \omega, \theta, J)$ defined by $(g, \tilde{g})$. That is, $HF_*(g, \tilde{g})$ is the automorphism of $HF_*(M, \omega, \theta, J)$ given by

$$HF_*(M, \omega, \theta, J) \xrightarrow{\Phi} HF_*(M, \omega, g * \theta, J^g) \xrightarrow{\tilde{g}} HF_*(M, \omega, \theta, J). \quad (4)$$

This suggests a method of solving the problem at hand: if we extend the space of choices available in defining $HF_*(M, \omega)$ to include isotopies of the form $g * \theta$, where g is merely *almost Hamiltonian*, then we can simply use (4) to extend the domain of $\tilde{\sigma}$. (There are many additional details. First, we will need to adjust the compatibility requirements on the almost complex structures. Second, $(g, \tilde{g})$ is an element of $\widetilde{G}$. We will need an analogous extension of D.)

The bulk of the thesis consists of constructing $HF_*(M, \omega, \psi, J)$, where ψ is an isotopy of M of the form $g * \theta$, for some $g \in D$ and Hamiltonian isotopy θ.

3 Almost Hamiltonian Loops and Isotopies

In the previous section, we showed that the Hamiltonian function is not as important as it's generated (Hamiltonian) isotopy. The fundamental idea in the proof of the main result of the thesis is in constructing a Floer Homology for an isotopy which is not Hamiltonian, but is very close to being Hamiltonian. This section describes the notation and basic properties of these "almost Hamiltonian" loops and isotopies.

We also introduce the spaces necessary to deal with the almost complex structures. Since our isotopies may not preserve the symplectic structure, we will no longer require that our choice of almost complex structure be compatible with ω, but we will use a restriction that is related to the isotopy.

This section contains most of the technical details that are required in the construction given in section 4.

3.1 Almost Hamiltonian Loops of Diffeomorphisms

We begin by letting $\Omega\,\mathrm{Diff}_{id}(M)$ denote the group of smooth loops at id in $\mathrm{Diff}(M)$, with the multiplication given by time-wise composition.

We have already introduced the subgroups G and D, which consist of Hamiltonian loops and almost Hamiltonian loops, respectively.

There is a natural action of $\Omega\,\mathrm{Diff}_{id}(M)$ on ΛM (the space of parameterized loops in M) given by $(g \cdot x)(t) = g_t(x(t))$ for all loops $g \in \Omega\,\mathrm{Diff}_{id} M$ and $x \in \Lambda M$.

If we restrict our attention to Hamiltonian loops of diffeomorphisms, Seidel showed that contractibility of loops in M is preserved. (This is lemma 1.22.) This fact is also true for almost Hamiltonian loops:

Lemma 3.1 $D \cdot \mathcal{L}M = \mathcal{L}M$

Proof:

Let x be in $\mathcal{L}M$, and let g be an almost Hamiltonian loop. Choose $h \in G$ such that $g \sim h$. Then $g \cdot x \sim h \cdot x$, and since h is Hamiltonian, $h \cdot x$ is contractible by lemma 1.22, and thus $g \cdot x$ is contractible as well.

$\square$

Recall that in the Hamiltonian case, this fact was used to define an extension of G by proving that the action always lifts to a homeomorphism of $\widetilde{\mathcal{L}M}$. We prove the analogous theorem in the almost Hamiltonian case.

Lemma 3.2 *The action of any $g \in D$ on $\mathcal{L}M$ can be lifted to a homeomorphism of $\widetilde{\mathcal{L}M}$.*

Proof:

Apply the Lifting Theorem to the following diagram:

$$\begin{array}{ccccc} & & & & \widetilde{\mathcal{L}M} \\ & & & & \big\downarrow p \\ \widetilde{\mathcal{L}M} & \xrightarrow{p} & \mathcal{L}M & \xrightarrow{g} & \mathcal{L}M \end{array}$$

Then if the condition on the fundamental groups is satisfied, there exists a unique lift (for each choice of base points), $\tilde{g} : \widetilde{\mathcal{L}M} \to \widetilde{\mathcal{L}M}$. Keep the base points fixed, and apply the Lifting Theorem to the diagram:

$$\begin{array}{ccccc} \widetilde{\mathcal{L}M} & & & & \\ p\big\downarrow & & & & \\ \mathcal{L}M & \xleftarrow[g^{-1}]{} & \mathcal{L}M & \xleftarrow[p]{} & \widetilde{\mathcal{L}M} \end{array}$$

Then the unique lift from this diagram is exactly $\tilde{g}^{-1}$, so the lift is a homeomorphism.

So, we must show that the condition on the fundamental groups holds. That is, we must show that $\left(g_{\#}\big(p_{\#}(\pi_1(\widetilde{\mathcal{L}M}))\big)\right) \subset p_{\#}(\pi_1(\widetilde{\mathcal{L}M}))$. This translates to showing that the action of g on $\mathcal{L}M$ preserves the set of smooth maps $S^1 \to \mathcal{L}M$ which can be lifted to $\widetilde{\mathcal{L}M}$, because $\widetilde{\mathcal{L}M}$ is a connected covering, and $p_{\#}(\pi_1(\widetilde{\mathcal{L}M}))$ consists of loops in $\mathcal{L}M$ which have a lift to $\widetilde{\mathcal{L}M}$.

To this end, let $s : S^1 \to \mathcal{L}M$ be a smooth map and $\tilde{s} : S^1 \to \widetilde{\mathcal{L}M}$ a lift of s to $\widetilde{\mathcal{L}M}$. We need to show that $(g \cdot s) : S^1 \to \mathcal{L}M$ also lifts to $\widetilde{\mathcal{L}M}$. We know that $g \sim h$ where h is Hamiltonian. We also know that $(h \cdot s)$ lifts to $\widetilde{\mathcal{L}M}$, because h is Hamiltonian (Seidel proves this). Let $\widetilde{h \cdot s}$ be such a lift. Then $(\widetilde{h \cdot s})(t)$ is an equivalence class of disc in M with boundary equal to $(h \cdot s)(t)$. Because $g \sim h$, $(g \cdot s) \sim (h \cdot s)$. Each $(g \cdot s)(t)$ is a loop in M. To lift to $\widetilde{\mathcal{L}M}$, we need to smoothly specify a choice of disc in M with boundary equal to $(g \cdot s)(t)$. Fixing t, apply the given homotopy to $(g \cdot s)(t)$. This gives us a cylinder in M with boundary loops $(g \cdot s)(t)$ and $(h \cdot s)(t)$. Cap this cylinder off on the $(h \cdot s)(t)$ end using $\widetilde{h \cdot s}(t)$. Do this for each t. This is a smooth loop because the homotopy is smooth and the lift of h was smooth.

$\square$

Analogously to the Hamiltonian case, we let $\widetilde{D} \subset \big(D \times Homeo(\widetilde{\mathcal{L}M})\big)$ consist of all pairs $(g, \tilde{g})$ such that $\tilde{g}$ is a lift of the g-action to $\widetilde{\mathcal{L}M}$.

Give $\widetilde{D}$ the topology induced from the C^∞ topology on D and the topology of pointwise convergence on $\text{Homeo}(\widetilde{\mathcal{L}M})$. Then $\widetilde{D}$ is a topological group.

As before, Γ denotes the group of deck transformations of the covering $p : \widetilde{\mathcal{L}M} \to \mathcal{L}M$. We again have the following short exact sequence of topological groups:

$$1 \to \Gamma \to \widetilde{D} \to D \to 1 \tag{5}$$

3.2 Almost Hamiltonian Isotopies

Since we are shifting our attention to isotopies of M, we need to develop some notation and basic properties of the subgroups in which we are interested.

Recall that if $\psi = (\psi_t)$ is an isotopy of M, then X_t^ψ denotes the family of vector fields on M generated by ψ. Also recall that if θ is a Hamiltonian isotopy generated by H, then $X_t^\theta = X_{H_t}$.

In Floer homology, we choose a smooth 1-periodic Hamiltonian function, so that the generated isotopy is smooth, and has a 1-periodic family of vector fields. We will also restrict to the periodic case.

We let $\mathcal{K}$ denote the space of all smooth isotopies of M for which X_t^ψ is 1-periodic. (This means that $X_0^\psi = X_1^\psi$.) This space is a subgroup of the group of all isotopies.

By $\mathcal{H}$, we denote the subgroup of $\mathcal{K}$ consisting of smooth Hamiltonian isotopies of M which are generated by functions in $C^\infty(M \times S^1)$.

There is a surjection $C^\infty(M \times S^1) \to \mathcal{H}$ which takes H to θ^H. By theorems 1.15, 1.16 and 1.17, there is a dense subset,

$$(C^\infty(M \times S^1) \times \mathcal{J})_{reg} \subset \big(C^\infty(M \times S^1), \mathcal{J}(M, \omega, S^1)\big),$$

for which we can define $HF_*(M, \omega, H, J)$.

Definition 3.3 *A pair* $(\theta, J) \in \mathcal{H} \times \mathcal{J}(M, \omega, S^1)$ *will be called a* **regular pair** *if* $\theta = \theta^H$ *for some* $H \in C^\infty(M \times S^1)$ *and* $(H, J) \in (C^\infty(M \times S^1) \times \mathcal{J})_{reg}$*. (This simply means that* $HF_*(M, \omega, H, J)$ *is well-defined.) The set* *of all regular pairs will be denoted* $\big(\mathcal{H} \times \mathcal{J}\big)_{reg}$.

This definition simply converts the Hamiltonian function to it's generated isotopy, as we have been doing all along. In other words, the pair $(\theta^H, J) \in (\mathcal{H} \times \mathcal{J})$ is regular if and only if $(H, J) \in (C^\infty(M \times S^1) \times \mathcal{J})_{reg}$.

Proposition 3.4 $(\mathcal{H} \times \mathcal{J})_{reg}$ *is dense inside of* $(\mathcal{H} \times \mathcal{J})$.

Proof:

$(\mathcal{H} \times \mathcal{J})_{reg}$ is the image of a dense set under a surjective, continuous map.

□

We are interested in isotopies of M which are of the form $g * \theta$, for $g \in D$ and $\theta \in \mathcal{H}$. Any such isotopy is homotopic, relative endpoints, to a Hamiltonian isotopy. (Just homotope g to a Hamiltonian loop.)

Definition 3.5 *An isotopy* $\psi \in \mathcal{K}$ *will be called* **almost Hamiltonian** *if* ψ *is homotopic, relative endpoints, to a Hamiltonian isotopy* $\theta \in \mathcal{H}$. *The group of all almost Hamiltonian isotopies will be denoted* $\mathcal{I}$.

Notice that by definition, almost Hamiltonian isotopies are smooth and 1-periodic.

Clearly, an isotopy ψ is almost Hamiltonian if and only if $[\psi] \in \widetilde{\mathrm{Diff}_{id}(M)}$ has a representative Hamiltonian isotopy. Also, $\mathcal{I}$ is obviously a subgroup of $\Omega_{id}\,\mathrm{Diff}(M)$. We will work predominantly with the spaces $\mathcal{H}$ and $\mathcal{I}$ of smooth periodic Hamiltonian isotopies and almost Hamiltonian isotopies respectively.

3.3 Difficulties with Almost Complex Structures

We intend to define a Floer homology for an isotopy in $\mathcal{I}$. To do so, we need to choose a $J \in \mathcal{J}(M, S^1)$. In the standard setting, (when the isotopy is Hamiltonian), this choice is limited to $\mathcal{J}(M, \omega, S^1)$. That is, we must choose ω-compatible almost complex structures. When extending this to $\mathcal{I}$ (almost Hamiltonian isotopies), we are faced with the difficulty that $\psi \in \mathcal{I}$ may include diffeomorphisms which do not preserve the symplectic structure.

The solution is to consider the loop of symplectic structures defined by the isotopy. In other words, given $\psi \in \mathcal{I}$, consider the family of symplectic structures defined by $\omega_t := (\psi_t^{-1})^*\omega$. (The inverse is an annoying, but necessary technical detail.) Since $\psi \in \mathcal{I}$, we have that $\psi_1 \in \mathrm{Ham}(M)$, so $(\psi_1^{-1})^*\omega = \omega$, thus $\omega_0 = \omega_1 = \omega$, so this is actually a loop of symplectic structures.

For $\psi = \psi_t \in \mathcal{I}$, let $\mathcal{J}^\psi(M, \omega, S^1) \subset \mathcal{J}(M, S^1)$ be all time dependent almost complex structures on M such that J_t is compatible with ω_t. That is,

$$\mathcal{J}^\psi(M, \omega, S^1) = \{J \in \mathcal{J}(M, S^1) \mid J_t \in \mathcal{J}(M, (\psi_t^{-1})^*\omega)\}.$$

This space is used to create a bundle over $\mathcal{I}$ by setting the fiber over $\psi \in \mathcal{I}$ to be $\mathcal{J}^\psi(M, \omega, S^1)$. That is,

$$\mathcal{F} = \{(\psi, J) \in \mathcal{I} \times \mathcal{J}(M, S^1) \mid J \in \mathcal{J}^\psi(M, \omega, S^1)\}.$$

This space has a topology induced as a subspace of $\mathcal{I} \times \mathcal{J}(M, S^1)$.

So, in this new case, the isotopy we begin with determines a subspace of $\mathcal{J}(M, S^1)$ from which we must choose.

The $*$ operation extends to an action of D on $\mathcal{I}$ given by

$$(g * \psi)_t := g_t^{-1}\psi_t, \;\; g \in D, \;\; \psi \in \mathcal{I}.$$

There is also an action of D on $\mathcal{J}(M, S^1)$ given by

$$(g * J)_t = J_t^g := Dg_t^{-1} J_t Dg_t, \;\; g \in D, \;\; J \in \mathcal{J}(M, S^1).$$

We will to use the following two technical lemmas throughout.

Lemma 3.6 *If J is an ω-compatible almost complex structure, and ρ is any diffeomorphism of M, then $\rho^* J := D\rho^{-1} J D\rho$ is compatible with $\rho^*\omega$.*

Proof:

This is a simple calculation. Since J is ω-compatible, $r(\cdot,\cdot) := \omega(\cdot, J\cdot)$ is a Riemannian metric. So,

$$\begin{aligned}\rho^*\omega(\cdot, \rho^* J\,\cdot) &= \omega(D\rho\,\cdot, JD\rho\,\cdot)\\ &= r(D\rho\,\cdot, D\rho\,\cdot)\\ &= \rho^* r(\cdot,\cdot)\end{aligned}$$

Since r is a metric, so is $\rho^* r$.

We also need to check that that $\rho^*\omega$ is $\rho^* J$ invariant.

$$\begin{aligned}\rho^*\omega(\rho^* J\cdot, rho^* J\cdot) &= \omega(JD\rho\,\cdot, JD\rho\,\cdot)\\ &= \omega(D\rho\,\cdot, D\rho\,\cdot)\\ &= \rho^*\omega(\cdot,\cdot)\end{aligned}$$

□

Lemma 3.7 *For a given $g \in D$, $\theta \in \mathcal{H}$, and $J \in \mathcal{J}^{g*\theta}(M, \omega, S^1)$, we have that $J^{g^{-1}} \in \mathcal{J}(M, \omega, S^1)$.*

Proof:

First notice that

$$\begin{aligned}
\big((g_t^{-1}\theta_t)^{-1}\big)^*\omega(\cdot,\cdot) &= \big(\theta_t^{-1}g_t\big)^*\omega(\cdot,\cdot)\\
&= g_t^*((\theta_t^{-1})^*\omega)(\cdot,\cdot)\\
&= \big((\theta_t^{-1}\big)^*\omega)(Dg_t\cdot, Dg_t\cdot)\\
&= \omega(Dg_t\cdot, Dg_t\cdot)\\
&= g_t^*\omega(\cdot,\cdot).
\end{aligned}$$

The second to last equality follows from the fact that $\theta \in \mathcal{H}$, and thus each θ_t is a symplectic diffeomorphism.

We have that $J \in \mathcal{J}^{g*\theta}(M,\omega,S^1)$, so let $r_t(\cdot,\cdot) = \big((g_t^{-1}\theta_t)^{-1}\big)^*\omega(\cdot, J_t\cdot)$ be the Riemannian metric guaranteed by this hypothesis. Then,

$$\begin{aligned}
\omega(\cdot, J^{g^{-1}}\cdot) &= \omega(\cdot, Dg_tJ_tDg_t^{-1}\cdot)\\
&= (g_t)^*\omega(Dg_t^{-1}\cdot, J_tDg_t^{-1}\cdot)\\
&= \big((g_t^{-1}\theta_t)^{-1}\big)^*\omega(Dg_t^{-1}\cdot, J_tDg_t^{-1}\cdot)\\
&= r_t(Dg_t^{-1}\cdot, Dg_t^{-1}\cdot)\\
&= (g_t^{-1})^*r_t(\cdot,\cdot).
\end{aligned}$$

The third equality follows from the above calculation of $\big(g_t^{-1}\theta_t)^{-1}\big)^*\omega$. But r_t is a metric, thus so is $(g_t^{-1})^*r_t$. The only fact remaining to prove is that ω is $J^{g^{-1}}$ invariant. This is just a direct calculation:

$$\begin{aligned}\omega(J^{g_t^{-1}}\cdot, J^{g_t^{-1}}\cdot) &= \omega(Dg_tJ_tDg_t^{-1}\cdot, Dg_tJ_tDg_t^{-1}\cdot)\\ &= g_t^*\omega(J_tDg_t^{-1}\cdot, J_tDg_t^{-1}\cdot)\\ &= g_t^*\omega(Dg_t^{-1}\cdot, Dg_t^{-1}\cdot)\\ &= \omega(\cdot,\cdot).\end{aligned}$$

This shows that $J^{g^{-1}} \in \mathcal{J}(M,\omega,S^1)$ as promised.

□

The following proposition is fundamental to the construction given in section 4. It says that any element of $\mathcal{F}$ can be decomposed into an element of D and a Hamiltonian pair $(\theta, j) \in \mathcal{H}\times\mathcal{J}(M,\omega,S^1)$.

Proposition 3.8 $\mathcal{F} = D * (\mathcal{H}\times\mathcal{J}(M,\omega,S^1))$

Proof:

We first prove the simpler assertion that $D * \big(\mathcal{H}\times\mathcal{J}(M,\omega,S^1)\big) \subset \mathcal{F}$. Let $g \in D$, $\theta\in\mathcal{H}$, and $J \in \mathcal{J}(M,\omega,S^1)$. We need to show that $g*\theta$ is almost Hamiltonian, and that $J^g \in \mathcal{J}^{g*\theta}(M,\omega,S^1)$. Smoothly homotope g to some $h \in G$, which exists by definition of D. This defines a smooth homotopy from $g*\theta$ to $h*\theta$. But since $h\in G$, $h*\theta\in\mathcal{H}$, so $g*\theta\in\mathcal{I}$.

To show that $J^g \in \mathcal{J}^{g*\theta}(M,\omega,S^1)$, let $r_t(\cdot,\cdot)$ be the Riemannian metric given by $r_t(\cdot,\cdot) = \omega(\cdot, J_t\cdot)$. Then,

$$
\begin{aligned}
((g*\theta)_t^{-1})^*\omega(\cdot, J_t^g\cdot) &= (\theta_t^{-1}g_t)^*\omega(\cdot, Dg_t^{-1}J_tDg_t\cdot)\\
&= (g_t)^*\omega(\cdot, Dg_t^{-1}J_tDg_t\cdot)\\
&= \omega(Dg_t\cdot, J_tDg_t\cdot)\\
&= r_t(Dg_t\cdot, Dg_t\cdot)\\
&= g_t^*r_t(\cdot,\cdot)
\end{aligned}
$$

We also need to show that $(g*\theta_t)^*\omega$ is invariant under J_t^g.

$$
\begin{aligned}
((g*\theta)_t^{-1})^*\omega(J_t^g\cdot, J_t^g\cdot) &= (g_t)^*\omega(J_t^g\cdot, J_t^g\cdot)\\
&= \omega(J_tDg_t\cdot, J_tDg_t\cdot)\\
&= \omega(Dg_t\cdot, Dg_t\cdot)\\
&= (g_t)^*\omega(\cdot,\cdot)\\
&= ((g*\theta)_t^{-1})^*\omega(\cdot,\cdot).
\end{aligned}
$$

This shows that $J^g \in \mathcal{J}^{g*\theta}(M,\omega,S^1)$.

For the other inclusion, let $(\psi, J) \in \mathcal{F}$, and choose $\theta \in \mathcal{H}$ such that ψ is homotopic to θ, relative endpoints. Such a θ exists by the definition of $\mathcal{I}$.

Define $g \in \Omega\,\mathrm{Diff}_{id}(M)$ by $g_t = \theta_t\psi_t^{-1}$. Then g is contractible, so $g \in D$. But $\psi_t = g_t^{-1}\theta_t$, so $\psi = g*\theta$. Also, since $(J^{g^{-1}})^g = J$, $(\psi, J) = g*(\theta, J^{g_t^{-1}})$. By lemma 3.7, $J^{g^{-1}} \in \mathcal{J}(M,\omega,S^1)$, showing that $\mathcal{F} \subset D*(\mathcal{H}\times\mathcal{J}(M,\omega,S^1))$.

□

Remark 2 Note that a decomposition $\psi = g*\theta$ as in proposition 3.8 is not unique.

Definition 3.9 *A pair* $(\psi, J) \in \mathcal{F}$ *will be called an* **almost Hamiltonian regular pair** *if* (ψ, J) *decomposes according to proposition 3.8 as* $(\psi, J) = g * (\theta, j)$ *where* (θ, j) *is a Hamiltonian regular pair.*

We denote the space of all almost Hamiltonian regular pairs by $\mathcal{F}_{reg}$. Thus,

$$\mathcal{F}_{reg} = D * (\mathcal{H} \times \mathcal{J})_{reg}.$$

Lemma 3.10 $\mathcal{F}_{reg}$ *is a dense subset of* $\mathcal{F}$.

Proof:

Suppose (ψ, J) is an element of $\mathcal{F}$. Decompose (ψ, J) according to proposition 3.8 as $(\psi, J) = g * (\theta, j)$. Choose $(\theta^\nu, j^\nu) \in (\mathcal{H} \times \mathcal{J})_{reg}$ such that (θ^ν, j^ν) converges to (θ, j). That this is possible follows from the fact that $\left(\mathcal{H} \times \mathcal{J}\right)_{reg}$ is a dense subset of $\mathcal{H} \times \mathcal{J}(M, \omega, S^1)$ (see proposition 3.4).

Then the sequence $g * (\theta^\nu, j^\nu)$ converges to $g * (\theta, j) = (\psi, J)$. Since each (θ^ν, j^ν) is a (Hamiltonian) regular pair, $g * (\theta^\nu, j^\nu) \in \mathcal{F}_{reg}$. It follows that $\mathcal{F}_{reg}$ is a dense subset of $\mathcal{F}$.

□

3.4 The Maslov Index

Recall the Maslov index $I : \widetilde{G} \to \mathbb{Z}$. This index described the grading shift in the automorphism of $HF_*(M, \omega)$ induced by $(g, \tilde{g}) \in \widetilde{G}$. We will need to extend this index to $\widetilde{D}$.

In dealing with almost Hamiltonian loops, (i.e., the space $\widetilde{D}$), we run into a difficulty using the definition of the index described in the first section. Since our isotopies go through $\mathrm{Diff}(M)$, and not just $\mathrm{Symp}(M)$, the path of matrices obtained in the definition of the Maslov index will not be symplectic, just invertible.

We get around this difficulty by showing that if two elements of $\widetilde{G}$ can be connected by a path through $\widetilde{D}$, then their indexes agree. We then show that any element of $\widetilde{D}$ can be connected to an element of $\widetilde{G}$, and thus the index can be extended to all of $\widetilde{D}$.

Proposition 3.11 *If* $(g^0, \tilde{g}^0)$, $(g^1, \tilde{g}^1) \in \widetilde{G}$ *are such that* $[g^0, \tilde{g}^0] = [g^1, \tilde{g}^1]$ *inside* $\pi_0(\widetilde{D})$*, then* $I(g^0, \tilde{g}^0) = I(g^1, \tilde{g}^1)$*.*

Proof:

First, recall the definition if the index map I in the Hamiltonian case:

Choose a point $c = [v, x] \in \widetilde{\mathcal{L}M}$. Trivialize the symplectic bundle (v^*TM, ω) as $D^2 \times (\mathbb{R}^{2n}, \omega_0)$. Let $A_t : (T_{x(t)}M, \omega) \to (\mathbb{R}^{2n}, \omega_0)$ be this trivialization on the boundary circle. Consider $\tilde{g}(c) = [v', g \cdot x] \in \widetilde{\mathcal{L}M}$. Trivialize $\big((v')^*TM, \omega\big)$ as $D^2 \times (\mathbb{R}^{2n}, \omega_0)$, and let $B_t : (T_{g_t(x(t))}M, \omega) \to (\mathbb{R}^{2n}, \omega_0)$ be the trivialization on the boundary, as before. Then $B_t Dg_t (A_t)^{-1}$ gives a linear, symplectic map from $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n}$, and is thus given by a symplectic matrix, denoted $l(t)$. Then $\{l(t)\}$ is a loop in Sp(2n,$\mathbb{R}$), and this loop is independent, up to homotopy, of all the choices involved, including the choice of $c \in \widetilde{\mathcal{L}M}$. Let $deg : \pi_1(\text{Sp}(2n, \mathbb{R})) \to \mathbb{Z}$ be the canonical isomorphism. The index map is given by $I(g, \tilde{g}) = deg(l(t))$, and I is well defined on $\pi_0(\widetilde{G})$.

For our proof, pick any $c = [v, x] \in \widetilde{\mathcal{L}M}$. Since $[g^0, \tilde{g}^0] = [g^1, \tilde{g}^1]$ in $\pi_0(\widetilde{D})$, there is a smooth path g^s, $0 \le s \le 1$, and a smooth lift $\tilde{g}^s$, $0 \le s \le 1$, connecting $(g^0, \tilde{g}^0)$ and $(g^1, \tilde{g}^1)$. Choose this path so that $\tilde{g}^s = \tilde{g}^0$ for $s \le \frac{1}{2}$. Consider D^2 with coordinates $(s, t) = (r, \frac{\theta}{2\pi})$. (The 2π factor is just so that $(s, 0) = (s, 1)$.)

Notice that s and t are use both as parameters for diffeomorphisms and as coordinates for D^2. This is because we will assign the diffeomorphism g_t^s to the point $(s, t) \in D^2$, and it is convenient to have the same notation.

We are given $v : D^2 \to M$. Consider the pullback bundle v^*TM. We give v^*TM the structure of a symplectic fiber bundle as follows. Let ξ and η be in the fiber over the point $(s,t) \in D^2$. (That is, they are elements of $T_{v(s,t)}M$.) Give $T_{v(s,t)}M$ the symplectic structure defined by

$$\omega^g_{s,t}(\xi,\eta) := (g^s_t)^* \omega_{v(s,t)}(\xi,\eta) = \omega_{g^s_t(v(s,t))}(Dg^s_t\xi, Dg^s_t\eta).$$

This gives v^*TM the structure of a smooth, symplectic vector bundle, denoted (v^*TM, ω^g). To emphasize, $\omega^g_{s,t}$ is a symplectic form on the vector space $T_{v(s,t)}M$. (The symplectic bundle is smooth at $(0,0)$ because we choose g^s to be constant near $s = 0$.)

Since D^2 is contractible, there is a trivialization $A : (v^*TM, \omega^g) \to D^2 \times (\mathbb{R}^{2n}, \omega_0)$. Let $A_{s,t}$ be the symplectic isomorphism from $(T_{v(s,t)}M, \omega^g_{s,t})$ to $(\mathbb{R}^{2n}, \omega_0)$ given by A. For clarity:

$$A_{s,t} : (T_{v(s,t)}M, \omega^g_{s,t}) \to (\mathbb{R}^{2n}, \omega_0), \tag{6}$$

$$(A^{-1}_{s,t})^* \omega^g_{s,t} = \omega_0. \tag{7}$$

Fix $s \in [0,1]$. (Recall that s describes the radial coordinate of the disc.) Define $x^s \in \mathcal{L}M$ by $x^s(t) = v(s,t)$. Define $v^s : D^2 \to M$ by $v^s(s',t) = v(s \cdot s', t)$. Then v^s is simply the portion of v which is inside x^s, suitably reparameterized. Since $v^s(1,t) = v(s,t) = x^s(t)$, we can define $c^s \in \widetilde{\mathcal{L}M}$ by $c^s = [v^s, x^s]$.

Consider $\tilde{g}^s(c^s) \in \widetilde{\mathcal{L}M}$. Then $\tilde{g}^s(c^s) = [v', g^s \cdot x^s]$, for some $v' : D^2 \to M$. Let $B : \big((v')^*TM, \omega\big) \to D^2 \times (\mathbb{R}^{2n}, \omega_0)$ be a trivialization as before. (Note that in this case, we do not adjust the symplectic structure.) Let $B_{s,t} : \big(g^s \cdot x^s)^*TM, \omega\big) \to S^1 \times (\mathbb{R}^{2n}, \omega_0)$ be the trivialization restricted to the boundary circle. Then $B_{s,t}$ gives an isomorphism of symplectic vector spaces from $\big(T_{g_t^s(x^s(t))}M, \omega_{g_t^s(x^s(t))}\big)$ to $(\mathbb{R}^{2n}, \omega_0)$, and in particular,

$$B_{s,t} : T_{g_t^s(x^s(t))}M \to \mathbb{R}^{2n}, \tag{8}$$

$$(B_{s,t})^*\omega_0 = \omega_{g_t^s(x^s(t))}. \tag{9}$$

Finally, notice that Dg_t^s gives an isomorphism of vector spaces between $T_{x^s(t)}M$ and $T_{g_t^s(x^s(t))}M$, and

$$\big((Dg_t^s)^*(\omega_{g_t^s(x^s(t))})\big) = ((g_t^s)^*\omega)_{x^s(t)} = \big((g_t^s)^*\omega\big)_{v(s,t)} = \omega_{s,t}^g. \tag{10}$$

For $s \in [0,1]$ and $t \in S^1$, define $l_t^s : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by

$$l^s(t) = B_{s,t}Dg_t^s(x^s(t))A_{s,t}^{-1}.$$

We claim that l_t^s is a symplectic map. To see this, notice that

$$(l_t^s)^*\omega_0 = (A_{s,t}^{-1})^*\Bigg(\big(Dg_t^s\big)^*(B_{s,t}^*\omega_0)\Bigg).$$

It follows from equations 9, 10 and 7 that $(l_t^s)^*\omega_0 = \omega_0$. This shows that l_t^s is in fact symplectic.

It is clear that l_t^s is also a linear map. Thus, l_t^s is a linear, symplectic map from $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n}$. Fix $s \in [0,1]$, and allow t to vary across S^1. By considering $\{l_t\}^s$ for such s and t, we obtain a loop in $\mathrm{Sp}(2n, \mathbb{R})$.

Notice that the above procedure is very similar to that used to define $I(g, \tilde{g})$ for $(g, \tilde{g}) \in \widetilde{G}$. The main difference is that we have adjusted the symplectic form on each fiber of v^*TM. By taking $s = 0$, and using c^0 to define $I(g^0, \tilde{g}^0)$, we see that $I(g^0, \tilde{g}^0) = deg(\{l_t\}^0)$, because g^0 is a Hamiltonian loop, and so we don't have any adjustment on ω when defining $\{l^0\}_t$.

We claim that $deg(\{l_t\}^1) = I(g^1, \tilde{g}^1)$. We will use $c^1 = c = [v, x]$ to define

$I(g^1, \tilde{g}^1)$. (Recall that any point in $\widetilde{\mathcal{L}M}$ will do.) In using $c^1 = [v, x]$, we trivialize (v^*TM, ω) as $D^2 \times (\mathbb{R}^{2n}, \omega_0)$. In our above process, we trivialize the adjusted bundle (v^*TM, ω^g). We must only show that these trivializations can be chosen so that they agree on the boundary circle, as every other step in defining $\{l^s\}_t$ matches the definition of $I(g^1, \tilde{g}^1)$.

A trivialization of (v^*TM, ω) can be given by the following process. Choose an almost complex structure on each fiber of v^*TM which varies smoothly with s and t, and such that the almost complex structure corresponding to the fiber over (s,t) is compatible with $\omega_{v(s,t)}$. Given this, choose an oriented, smooth frame field on v^*TM which is orthonormal with respect to the metric induced by ω and the almost complex structures.

The same process can be used to trivialize (v^*TM, ω^g), but the almost complex structure on $T_{v(s,t)}M$ will have to be chosen so that it is compatible with $\omega^g_{s,t}$.

To accomplish this, choose any $J \in \mathcal{J}(M, \omega)$. We use J to define an almost complex structure on each fiber of v^*TM as follows. On the fiber over $(s,t) \in D^2$, (which is exactly $T_{v(s,t)}M$), supply the structure $(Dg^s_t)^{-1} Dg^0_t J (Dg^0_t)^{-1} Dg^s_t$. This defines a complex structure on the vector bundle v^*TM, and we will denote the complex structure on the fiber over (s,t) by $\bar{J}^s_t$. Then $\bar{J}^0_t = J$ for all t. It follows from lemma 3.6 that $\bar{J}^s_t$ is compatible with $\omega^g_{s,t}$ as required.

So, let J be a choice of almost complex strucure on each fiber of v^*TM which is compatible with ω, and let ζ be an oriented frame field of v^*TM which is compatible with ω, and which is orthonomal with respect to the metric induced by ω and J. Let J^g be a choice of almost complex structure on each fiber such that $J^g(s,t)$ is compatible with $\omega^g(s,t)$. Choose J^g such that $J^g(1,t) = J(1,t)$. This is possible because g^1 is a Hamiltonian loop, and thus $\omega^g_{1,t} = \omega_{v(1,t)}$. (In fact, simply use the same procedure given in the previous paragraph.)

We can modify ζ so that it becomes an oriented frame field of v^*TM which is orthonormal with respect to the metric induced by ω^g and J^g by applying the Graham-Schmidt procedure to each basis simultaneously. Call this modified frame field ζ^g. This procedure will not change $\zeta(1,t)$ for any t because $\omega^g_{1,t} = \omega_{v(1,t)}$, and $J^g(1,t) = J(1,t)$, and thus $\zeta(1,t)$ is already orthonormal for ω^g and J^g. Since $\zeta^g(1,t) = \zeta(1,t)$ for all t, the trivializations of (v^*TM, ω^g) and (v^*TM, ω) agree on the boundary circle, as desired.

By varying s across $[0,1]$, l^s_t gives a homotopy through loops in $\mathrm{Sp}(2n,\mathbb{R})$ between l^0_t and l^1_t. Since *deg* is homotopy invariant, we see that $I(g^0, \tilde{g}^0) = I(g^1, \tilde{g}^1)$, as desired.

□

Lemma 3.12 *For any $(g, \tilde{g}) \in \widetilde{D}$, there is some $(h, \tilde{h}) \in \widetilde{G}$ such that $[g, \tilde{g}] = [h, \tilde{h}]$ inside $\pi_0(\widetilde{D})$.*

Proof:

Choose a path in D from g to some $h \in G$ which exists by the definition of D. Since $[0,1]$ is contractible, the Lifting Theorem guarantees that this path has a lift to $\widetilde{D}$. Then $[g, \tilde{g}] = [h, \tilde{h}]$ inside $\pi_0(\widetilde{D})$.

□

According to these lemmas, we are justified in making the following definition.

Definition 3.13 *For $(g, \tilde{g}) \in \widetilde{D}$, let $I(g, \tilde{g}) = I(h, \tilde{h})$ for any $(h, \tilde{h}) \in \widetilde{G}$ with $[g, \tilde{g}] = [h, \tilde{h}]$ inside $\pi_0(\widetilde{D})$.*

Lemma 3.14 *$I : \widetilde{D} \to \mathbb{Z}$ is well defined on $\pi_0(\widetilde{D})$, and I is a homomorphism.*

Proof:

The fact that I is well defined on $\pi_0(\widetilde{D})$ is clear from it's definition. This combined with the fact that $I|_{\pi_0(\widetilde{G})}$ is a homomorphism (lemma 1.28) proves the second statement.

$\square$

To summarize up to this point, we have defined the group D of almost Hamiltonian loops, and it's extension $\widetilde{D}$, which allows a lift of the action to $\widetilde{\mathcal{L}M}$. There is an index map $I : \pi_0(\widetilde{D}) \to \mathbb{Z}$ which reduces to Seidel's Maslov index on $\pi_0(\widetilde{G})$.

We have introduced the space $\mathcal{F}$ which is a bundle over the space of almost Hamiltonian isotopies. The fiber over $\psi \in \mathcal{I}$ is given by $\mathcal{J}^{\psi}(M, \omega, S^1)$.

We have seen that any $(\psi, J) \in \mathcal{F})$ decomposes as $(\psi, J) = g * (\theta, j)$ for some $(\theta, j) \in (\mathcal{H} \times \mathcal{J}(M, \omega, S^1)$ and some $g \in G$, although the decomposition is not unique.

3.5 $\widetilde{\mathcal{L}M}$ Revisited

Recall that the space $\widetilde{\mathcal{L}M}$ depends on ω. The equivalence relation on $\widetilde{\mathcal{L}M}$ is $[v, x] = [v', x'] \iff x = x',\ \omega(v\#v') = c_1(v\#v') = 0$. In the original construction of the Floer homology, this is well defined for (H, J). But in our case, ω is not fixed, and J_t varies outside of $\mathcal{J}(M, \omega)$. We need to see that $\widetilde{\mathcal{L}M}$ is well defined even for $(\psi, J) \in \mathcal{F}$.

Proposition 3.15 *For $(\psi, J) \in \mathcal{F}$, $[\psi_t^* \omega] = [\omega]$ for all t, and the Chern classes satisfy $c_1(TM, J_t) = c_1(TM, J_0) = c_1(\omega)$.*

Before proving the proposition, we prove several lemmas.

Lemma 3.16 *Let* $\phi \in \mathrm{Diff}(M)$ *and* $J \in \mathcal{J}(M, \omega)$. *Then* $\phi^* J := D\phi^{-1} J D\phi$ *is an almost complex structure compatible with* $\phi^*\omega$, *i.e.,* $\phi^* J \in \mathcal{J}(M, \phi^*\omega)$.

Proof:

Because J is ω -compatible, $r(\cdot, \cdot) := \omega(\cdot, J\cdot)$ is a Riemannian metric and $\omega(J\cdot, J\cdot) = \omega(\cdot, \cdot)$. We have:

$$\begin{aligned} \phi^*\omega(\cdot, \phi^* J\cdot) &= \phi^*\omega(\cdot, D\phi^{-1} J D\phi\cdot) \\ &= \omega(D\phi\cdot, J D\phi\cdot) \\ &= r(D\phi\cdot, D\phi\cdot) \\ &= \phi^* r(\cdot, \cdot). \end{aligned}$$

Since r is a metric, so is $\phi^* r$.

Also,

$$\begin{aligned} \phi^*\omega(\phi^* J\cdot, \phi^* J\cdot) &= \phi^*\omega(D\phi^{-1} J D\phi\cdot, D\phi^{-1} J D\phi\cdot) \\ &= \omega(J D\phi\cdot, J D\phi\cdot) \\ &= \omega(D\phi\cdot, D\phi\cdot) \\ &= \phi^*\omega(\cdot, \cdot). \end{aligned}$$

This shows that $\phi^* J$ is a $\phi^*\omega$ -compatible almost complex structure.

□

Recall that $c_1(\omega)$ is defined for any symplectic structure by choosing any $J \in \mathcal{J}(M, \omega)$, and considering the complex vector bundle (TM, J), and these Chern classes are independent of the choice of J.

Lemma 3.17 *For* $\phi \in \mathrm{Diff}(M)$*,* $c_1(M, \phi^*\omega) = \phi^* c_1(M, \omega)$*.*

Proof:

By lemma 3.16, ϕ^*J is $\phi^*\omega$ -compatible, so $c_1(M, \phi^*\omega) = c_1(TM, \phi^*J)$.

Consider the following diagram:

$$\begin{array}{ccc} \phi^*(TM, J) & & (TM, J) \\ {\scriptstyle \phi^*\pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ M & \xrightarrow[\phi]{\quad} & M \end{array}$$

Here $\phi^*(TM, J)$ denotes the pulled back bundle over M induced by ϕ.

By the naturality of the Chern classes, $c_1(\phi^*(TM, J)) = \phi^* c_1(TM, J)$ (see [17]).

We need only show that $\phi^*(TM, J) \backsimeq (TM, \phi^*J)$ as complex vector bundles over M, because given this,

$$\begin{aligned} c_1(M, \phi^*\omega) &= c_1(TM, \phi^*J) \\ &= c_1(\phi^*(TM, J)) \\ &= \phi^* c_1(TM, J) \\ &= \phi^* c_1(M, J). \end{aligned}$$

To accomplish this, we exhibit a fiber preserving complex linear map between the bundles that induces the identity on the base space M.

The fiber over the point p in $(\phi^*(TM, J))$ is $\phi^*\pi^{-1}(p) = (T_{\phi(p)}M, J)$. The fiber over the point p in (TM, ϕ^*J) is $\pi^{-1}(p) = (T_pM, \phi^*J)$. Simply map $\phi^*\pi^{-1}(p)$ to $\pi^{-1}(p)$ by $D\phi_p^{-1} : T_{\phi(p)}M \to T_pM$. This is an isomorphism of (real) vector spaces because ϕ is a diffeomorphism. We must show that the map respects the complex structure on each side:

$$\begin{aligned} D\phi_p^{-1}(Jx) &= D\phi_p^{-1} J D\phi_p D\phi_p^{-1} x \\ &= \phi^* J (D\phi_p^{-1} x) \end{aligned}$$

This shows that the map between the bundles is complex linear, and it clearly induces the identity on the base, so it is an isomorphism of complex vector bundles. Using the naturality of the Chern class again gives that $c_1(\phi^*(TM, J)) = c_1(TM, \phi^* J)$. Because we discovered above that $c_1(\phi^*(TM, J)) = \phi^* c_1(TM, J)$, the lemma is proved.

□

Proof of proposition 3.15: Since each ψ_t is isotopic to the identity, the induced map on cohomology is the identity. The statement about the Chern classes follows from this fact, along with lemmas 3.16 and 3.17.

□

It follows from proposition 3.15 that for $\psi = \psi_t \in \mathcal{I}$, $\omega(A) = \psi_t^* \omega(A)$ for all $A \in \pi_2(M)$, and $c_1(A)$ is independent of J, so $\widetilde{\mathcal{L}M}$ is well defined.

4 Defining $HF_*(M, \omega, \psi, J)$

In section 2.2, we isolated the ingredients needed to construct the Λ-module $HF_*(M, \omega, H, J)$. In fact, we need only $\mathcal{P}(H)$ (along with the C-Z index), the action functional a_H, and the $\overline{\partial}_{H,J}$ operator. In this section, we define $HF_*(M, \omega, \psi, J)$ for $(\psi, J) \in \mathcal{F}_{reg}$. If (ψ, J) is Hamiltonian (meaning $\psi \in \mathcal{H}$), then $HF_*(M, \omega, \psi, J)$ reduces to $HF_*(M, \omega, H, J)$, where $H \in C^\infty(M \times S^1)$ is a generating function for ψ.

Recall that $\mathcal{I}$ is the space of 1-periodic almost Hamiltonian isotopies, and it is convenient to consider $\mathcal{F}$ as a bundle over $\mathcal{I}$, with fiber over ψ given by $\mathcal{J}^\psi(M, \omega, S^1) \subset \mathcal{J}(M, S^1)$.

4.1 Defining $\widetilde{\mathcal{P}(\psi)}$

For $\psi = (\psi_t) \in \mathcal{I}$, let $\mathcal{P}(\psi)$ consist of the contractible loops in M which arise by flowing along the isotopy ψ. That is,

$$\mathcal{P}(\psi) = \{x \in \mathcal{L}M \mid \dot{x}(t) = X_t^\psi(x(t))\}.$$

By considering the definition of $\mathcal{P}(H)$ (see section 1.2), it becomes clear that $\mathcal{P}(\psi) = \mathcal{P}(H)$ if ψ is a Hamiltonian isotopy generated by $H \in C^\infty(M \times S^1)$.

As before, it is necessary to construct $HF_*(M, \omega, \psi, J)$ using the covering space $\widetilde{\mathcal{L}M}$, so we set

$$\widetilde{\mathcal{P}(\psi)} = p^{-1}(\mathcal{P}(\psi)),$$

where $p : \widetilde{\mathcal{L}M} \to \mathcal{L}M$ is the projection.

Again, $\widetilde{\mathcal{P}(\psi)} = \widetilde{\mathcal{P}(H)}$ if ψ is Hamiltonian, generated by the Hamiltonian function H.

We wish to use the set $\widetilde{\mathcal{P}(\psi)}$ to create a chain complex, but first we must index the set.

Indexing $\widetilde{\mathcal{P}(\psi)}$ is a bit tricky. For a Hamiltonian isotopy θ, this is done using the Conley-Zehnder index $\mu_\theta : \widetilde{\mathcal{P}(\theta)} \to \mathbb{Z}$. If ψ is an almost Hamiltonian isotopy, the C-Z index does not make sense. It is obtained by linearizing ψ along the path $x(t)$ for $x \in \mathcal{P}(\psi)$, and using a symplectic trivialization to create a path of symplectic matrices. The problem in our setup is that this process will not result in a path of symplectic matrices, only invertible matrices, because ψ_t will not necessarily preserve ω.

First recall that the Conley-Zehnder index for Hamiltonian isotopies is only well-defined when the isotopy is non-degenerate. (This means that for all $x \in \mathcal{P}(\theta)$, $\det(\mathbb{1} - D\theta_1) \neq 0$.) While the non-degeneracy condition was given for Hamiltonian isotopies, it makes sense for arbitrary isotopies.

Definition 4.1 *An isotopy $\psi \in \mathcal{I}$ will be called* **non-degenerate** *if, for all $x \in \mathcal{P}(\psi)$, $\det(\mathbb{1} - D\psi_1) \neq 0$.*

We begin by defining the index for non-degenerate Hamiltonian isotopies by using it's generating function. That is, for $\theta \in \mathcal{H}$ and $c \in \mathcal{P}(H)$, let

$$\mu_\theta(c) := \mu_H(c), \;\; \theta \in \mathcal{H}, \;\; \theta = \theta^H.$$

The next theorem is the first in a series in which we decompose $\psi \in \mathcal{I}$ as $\psi = g * \theta$ according to proposition 3.8. Each time, a decomposition is chosen, and used to define some object. We then show that the construction is independent of the choice of decomposition. We will often need to choose a lift $\tilde{g}$, so that $(g, \tilde{g}) \in \widetilde{D}$.

Recall that we have extended the Maslov index to a map $I : \widetilde{D} \to \mathbb{Z}$.

Proposition 4.2 *Let $\psi = (\psi_t) \in \mathcal{I}$, and decompose as $\psi = g * \theta$ as in proposition 3.8. Suppose that ψ is a non-degenerate isotopy. Choose $\tilde{g}$ such* *that* $(g,\tilde{g}) \in \widetilde{D}$*. Then* $\mu_\psi : \widetilde{\mathcal{P}(\psi)} \to \mathbb{Z}$ *is well defined by*

$$\mu_\psi(c) = \mu_\theta(\tilde{g}(c)) - 2I(g,\tilde{g}),$$

independent of the choice of decomposition or the choice of lift $\tilde{g}$*. Moreover, if* ψ *is Hamiltonian generated by* H*, then* $\mu_\psi = \mu_H$*.*

The proof will be given after we establish some initial facts.

First notice that we haven't yet shown that $\tilde{g}(c) \in \widetilde{\mathcal{P}(\theta)}$, as this formula requires. In the Hamiltonian case, this is given by lemma 1.26. The next lemma gives the proof in the almost Hamiltonian case. Notice that lemma 1.26 is proved by considering the action functional. The proof of the next lemma is in purely topological terms.

Recall that $\mathcal{P}(\psi)$ is contained in $\mathcal{L}M$, and any $g \in D$ induces a map (still called g) from $\mathcal{L}M$ to $\mathcal{L}M$.

Lemma 4.3 *For all* $(g,\tilde{g}) \in \widetilde{D}$ *and* $\psi \in \mathcal{I}$*, the following equalities hold.*

$$g(\mathcal{P}(g * \psi)) = \mathcal{P}(\psi) \tag{11}$$

$$\tilde{g}(\widetilde{\mathcal{P}(g * \psi)}) = \widetilde{\mathcal{P}(\psi)}. \tag{12}$$

Proof:

We need only prove equation 11, because then equation 12 follows from the fact that $\tilde{g}$ lifts the action of g.

We prove that $g(\mathcal{P}(g * \psi)) \subset \mathcal{P}(\psi)$. Given $x \in \mathcal{P}(g * \psi)$, let $x^g = g \cdot x$. We need to show that $\dot{x}^g(t) = X_t^\psi\big((x^g)(t)\big)$.

Because $(g * \psi)_t = g_t^{-1}\psi_t$, lemma 2.3 gives the following two equations:

$$X_t^{g*\psi} = X_t^{g^{-1}} + Dg_t^{-1} X_t^{\psi} \circ g_t = -Dg_t^{-1} X_t^g \circ g_t + Dg_t^{-1} X_t^{\psi} \circ g_t, \tag{13}$$

$$\dot{x^g}(t) = Dg_t(\dot{x}(t)) + \frac{dg_t}{dt}(x(t)). \tag{14}$$

By definition,

$$\frac{dg_t}{dt}(x(t)) = X_t^g(g_t(x(t))). \tag{15}$$

Because $x \in \mathcal{P}(g * \psi)$,

$$\dot{x}(t) = X_t^{g*\psi}(x(t)). \tag{16}$$

Substituting equations 13, 15, and 16 into equation 14 gives

$$\begin{aligned}
\dot{x}^g(t) &= Dg_t(\dot{x}(t)) + \frac{dg_t}{dt}(x(t)) \\
&= Dg_t X_t^{g*\psi}(x(t)) + X_t^g\big(g_t(x(t))\big) \\
&= Dg_t\bigg(-Dg_t^{-1} X_t^g(g_t(x(t))) + Dg_t^{-1} X_t^{\psi}(g_t(x(t)))\bigg) + X_t^g\big(g_t(x(t))\big) \\
&= -X_t^g\big(g_t(x(t))\big) + X_t^{\psi}\big(g_t(x(t))\big) + X_t^g\big(g_t(x(t))\big) \\
&= X_t^{\psi}((x^g)(t)),
\end{aligned}$$

as desired. The same manipulations show the other inclusion.

□

Now that the formula at least makes sense, we show that the formula given in proposition 4.2 reduces to the Conley-Zehnder index in the Hamiltonian case.

Recall that given a regular pair (H, J), we have defined the pullback pair (H^g, J^g), and the C-Z index satisfies

$$\mu_{H^g}(c) = \mu_H(\tilde{g}(c)) - 2I(g, \tilde{g}). \tag{17}$$

Section 2.1 was devoted to calculating the isotopy generated by H^g. We found that

$$\theta^{H^g} = g * \theta^H. \tag{18}$$

Lemma 4.4 *For $(g, \tilde{g}) \in G$ and $\theta = \theta_t \in \mathcal{H}$, $\mu_{g*\theta}(c) = \mu_\theta(\tilde{g}(c)) - 2I(g, \tilde{g})$.*

Proof:

Recall that for $\theta \in \mathcal{H}$, μ_θ is well defined by $\mu_\theta = \mu_H$ for any $H \in C^\infty(M \times S^1)$ which generates θ. The lemma then is just the formula given in equation 17, with the functions converted to isotopies, using equation 18.

□

We also need the following lemma to prove proposition 4.2.

Lemma 4.5 *Given $\theta \in \mathcal{H}$ and $g \in D$, $g * \theta \in \mathcal{H} \iff g \in G$.*

Proof:

First, the reverse implication. Let H^θ generate θ, and H^g generate g. Define a Hamiltonian function by $H_t = H_t^\theta \circ g_t - H_t^g \circ g_t$. Then H generates $g * \theta$ by proposition 2.5.

For the forward implication, let H^θ generate θ, and $H^{g*\theta}$ generate $g * \theta$. Define a Hamiltonian function by $(K^g)_t := H_t^\theta - H_t^{g*\theta} \circ g_t^{-1}$. We claim that K^g generates g, so that g is actually Hamiltonian, and thus $g \in G$.

To see this, we need only show that $X_{K_t^g} = X_t^g$. First notice that by lemma 2.3, $X_t^{g*\theta} = X^{\theta_t} \circ g_t^{-1} - X_t^g \circ g_t^{-1}$, so $X_t^g = X_t^\theta - X_t^{g*\theta} \circ g_t^{-1}$. It then follows that

$$X_t^g = X_t^\theta - X_t^{g*\theta} \circ g_t^{-1} \tag{19}$$

Simply plug these into the definition of $X_{K_t^g}$:

$$\begin{aligned} X_{K_t^g} &= X_{(H^\theta - H^{g*\theta} \circ g^{-1})_t} \\ &= X_{H_t^\theta} - X_{(H^{g*\theta} \circ g^{-1})_t} \\ &= X_t^\theta - X^{(g*\theta_t)} \circ g_t^{-1} \\ &= X_t^g. \end{aligned}$$

Thus, g is Hamiltonian generated by K_g.

□

In subsequent sections, we will use the preceding lemma often in conjunction with proposition 3.8. It allows us to prove that many things are independent of the choice of decomposition.

Proof of proposition 4.2: Let ψ be in $\mathcal{I}$, and choose two decompositions

$$\begin{aligned} \psi &= g^0 * \theta^0 \\ &= g^1 * \theta^1 \end{aligned}$$

as in proposition 3.8. Let $\tilde{g}^0$ and $\tilde{g}^1$ be lifts of g^0 and g^1 respectively, so that $(g^0, \tilde{g}^0)$ and $(g^1, \tilde{g}^1)$ are in $\widetilde{D}$. Since $g^0 * \theta^0 = g^1 * \theta^1$, we have that $(g_t^0)^{-1}\theta_t^0 = (g_t^1)^{-1}\theta_t^1$, and so $\theta_t^0 = g_t^0(g_t^1)^{-1}\theta_t^1$. Thus, if we define a loop of diffeomorphisms of M by $\bar{g}_t = g_t^1(g_t^0)^{-1}$, we have

$$\begin{aligned}(\bar{g} * \theta^1)_t &= (\bar{g}_t)^{-1}\theta_t^1 \\ &= (g_t^1(g_t^0)^{-1})^{-1}\theta_t^1 \\ &= g_t^0(g_t^1)^{-1}\theta_t^1 \\ &= \theta_t^0.\end{aligned}$$

This means that $\bar{g} * \theta^1 = \theta^0$. But since θ^0 and θ^1 are Hamiltonian, so is $\bar{g}$, by lemma 4.5. Notice also that $\tilde{g}^1(\tilde{g}^0)^{-1}$ is a lift of the action of $\bar{g}$ to $\widetilde{\mathcal{L}M}$. Then,

$$\begin{aligned}\mu_{\theta^0}(\tilde{g}^0(c)) &= \mu_{\bar{g}*\theta^1}(\tilde{g}^0(c)) \\ &= \mu_{\theta^1}\big((\tilde{g}^1(\tilde{g}^0)^{-1}\tilde{g}^0)(c)\big) - 2I(\bar{g}, \tilde{g}^1(\tilde{g}^0)^{-1}) \\ &= \mu_{\theta^1}(\tilde{g}^1(c)) - 2\big(I(g^1, \tilde{g}^1) - I(g^0, \tilde{g}^0)\big).\end{aligned}$$

The second equality follows from the fact that $\bar{g}$ is Hamiltonian, and lemma 4.4. The third equality follows from the fact that I is a homomorphism, proved in lemma 3.14.

$\square$

This shows that the definition of the index is independent of the choices, and proves proposition 4.2. Now we may index $\widetilde{\mathcal{P}(\psi)}$ for non-degenerate $\psi \in \mathcal{I}$ by setting

$$\widetilde{\mathcal{P}(\psi)}_k = \{c \in \widetilde{\mathcal{P}(\psi)} | \mu_\psi(c) = k\}.$$

Lemma 4.6 *If $\psi = g * \theta$ is a decomposition of a non-degenerate isotopy $\psi \in \mathcal{I}$ as in proposition 3.8, then $\tilde{g}(\widetilde{\mathcal{P}(\psi)}_k) = \widetilde{\mathcal{P}(\theta)}_{k-2I(g,\tilde{g})}$.*

Proof:

By lemma 4.3, the only thing left to prove is the formula for the index change. This fact follows from the formula given in proposition 4.2.

□

4.2 The Chain Complex and Boundary Operator

We now consider a fixed $(\psi, J) \in \mathcal{F}$, and begin to define $HF_*(M, \omega, \psi, J)$. Notice that in the Hamiltonian case, the definition of the chain complex depends on the action functional, a_H. (It is part of the finiteness condition on the chains.) In the almost Hamiltonian case, there is no action functional.

To overcome this difficulty, decompose $\psi = (\psi_t)$ using proposition 3.8 as $\psi = g * \theta$, and choose $\tilde{g}$, a lift of g, so that $(g, \tilde{g})$ is in $\widetilde{D}$. In the Hamiltonian case, according to lemma 2.3 of [21], $a_{H^g} = \tilde{g}^* a_H + (\text{constant})$. Combined with proposition 2.5, this suggests that we use $\tilde{g}^* a_{H^\theta}$ as a replacement for the action functional in the definition of the chain complex, where H^θ denotes a Hamiltonian function generating θ.

We define $CF_k(M, \omega, \psi)$ as all formal sums

$$\sum_{c \in \widetilde{\mathcal{P}(\psi)}_k} m_c \langle c \rangle,$$

where $m_c \in \mathbb{Z}_2$, which satisfy the finiteness condition $\{c \in \widetilde{\mathcal{P}(\psi)}_k \,|\, m_c \neq 0,\ \tilde{g}^* a_{H^\theta}(c) > C\}$ is finite for all $C \in \mathbb{R}$.

Lemma 4.7 *$CF_k(M, \omega, \psi)$ is well defined independent of the choice of decomposition of ψ.*

Proof:

Choose two decompositions of ψ as in proposition 3.8:

$$\begin{aligned}\psi &= g^0 * \theta^0 \\ &= g^1 * \theta^1.\end{aligned}$$

Also choose lifts $\tilde{g}^0$ and $\tilde{g}^1$. Define $\bar{g} \in D$ by $\bar{g}_t = g_t^0 (g_t^1)^{-1}$. Then $\bar{g}$ is Hamiltonian by lemma 4.5. Let H^{θ^0} and H^{θ^1} generate the isotopies θ^0 and θ^1 respectively. Then $(H^{\theta^0})^{\bar{g}} = H^{\bar{g}*\theta^0} = H^{\theta^1}$ by proposition 2.5. This implies that

$$\begin{aligned}(\tilde{g}^0)^* a_{H^{\theta^0}} &= \big((\tilde{g}^0 \circ (\tilde{g}^1)^{-1}) \circ \tilde{g}^1\big)^* a_{H^{\theta^0}} \\ &= (\tilde{g}^1)^* \big((\tilde{g}^0 \circ (\tilde{g}^1)^{-1})^* a_{H^{\theta^0}}\big) \\ &= (\tilde{g}^1)^* \big(a_{(H^{\theta^0})^{\bar{g}}} + (\text{constant})\big) \\ &= (\tilde{g}^1)^* \big(a_{H^{\bar{g}*\theta^0}} + (\text{constant})\big) \\ &= (\tilde{g}^1)^* (a_{H^{\theta^1}}) + (\text{constant})\end{aligned}$$

The third equality follows from the fact that $\bar{g}$ is Hamiltonian.

It then follows that the finiteness condition is independent of the choice of decomposition.

□

We then set $CF_*(M, \omega, \psi) = \bigoplus CF_k(M, \omega, \psi)$. It is clear that for $\theta^H \in \mathcal{H}$, $CF_*(M, \omega, \theta^H)$ reduces to $CF_*(M, \omega, H)$.

As in the Hamiltonian case, $CF_*(M, \omega, \psi)$ has the structure of a Λ-module, and the dimension of $CF_k(M, \omega, \psi)$ as a module over Λ is exactly the number of elements in $\mathcal{P}(\psi)$ with index $\mu_\psi(c) = k (\text{mod } 2N)$.

Next, we define a boundary operator on $CF_*(M, \omega, \psi)$, using a choice of $J \in \mathcal{J}^\psi(M, \omega, S^1)$. The boundary operator is given in terms of the $\overline{\partial}$ operator.

Recall that for smooth $u : \mathbb{R} \times S^1 \to M$, $\overline{\partial}_{H,J}(u) \in C^\infty(u^*TM)$ is given by $\overline{\partial}_{H,J}(u)(s,t) = \frac{\partial u}{\partial s}(s,t) + J_t(u(s,t))\frac{\partial u}{\partial t}(s,t) - \nabla H_t(u(s,t))$. We would like to define an analogous operator in the almost Hamiltonian case, but we have no function H to use. To get around this difficulty, we use the following standard fact.

Lemma 4.8 *For $J \in \mathcal{J}(M, \omega)$ and $H \in C^\infty(M)$, let ∇H denote the gradient vector field of H using the metric $r(\cdot, \cdot) = \omega(\cdot, J\cdot)$. Then $\nabla H = JX_H$.*

Proof:

By definition, $dH(\cdot) = r(\nabla H, \cdot) = \omega(\nabla H, J\cdot) = \omega(J\nabla H, \cdot)$. Also by definition, $dH(\cdot) = i_{X_H}\omega(\cdot) = \omega(X_H, \cdot)$. This shows that $\omega(J\nabla H, \cdot) = \omega(X_H, \cdot)$, and so the lemma follows from the non-degeneracy of ω.

$\square$

Since $X_{H_t} = X_t^\psi$ if ψ is a Hamiltonian isotopy generated by H, we may extend the $\overline{\partial}$ operator in the following way.

For all pairs $(\psi, J) \in \mathcal{F}$ and smooth maps $u : \mathbb{R} \times S^1 \to M$, define a section $\overline{\partial}_{\psi,J}(u) \in C^\infty(u^*TM)$ by

$$\overline{\partial}_{\psi,J}(u)(s,t) = \frac{\partial u}{\partial s}(s,t) + J_t(u(s,t))\left(\frac{\partial u}{\partial t}(s,t) - X_t^\psi(u(s,t))\right). \qquad (20)$$

It follows from lemma 4.8 that if ψ is Hamiltonian, generated by H, then $\overline{\partial}_{\psi,J} = \overline{\partial}_{H,J}$.

This allows us to define the space of "connecting orbits" between two chain complex generators. For $(\psi, J) \in \mathcal{F}$ and $c_-, c_+ \in \widetilde{\mathcal{P}(\psi)}$, we define

$\mathcal{M}(c_-, c_+, \psi, J)$ as the set of all smooth maps $u : \mathbb{R} \times S^1 \to M$ such that $\overline{\partial}_{\psi,J}(u) = 0$ and there is a lift $\tilde{u} : \mathbb{R} \to \widetilde{\mathcal{L}M}$ with limits c_-, c_+.

As in the Hamiltonian case, we would like to use these spaces to define a boundary operator on $CF_*(M, \omega, H)$. In particular, if $c_\pm \in \widetilde{\mathcal{P}(\psi)}$ are such that $\mu_\psi(c_-) - \mu_\psi(c_+) = 1$, we would like to be able to count the number of elements of $\mathcal{M}(c_-, c_+, H, J)/\mathbb{R}$, where the $\mathbb{R}$-action is again translation in the s variable.

We will only be able to count these spaces for generic pairs $(\psi, J) \in \mathcal{F}$. Recall that $\mathcal{F}_{reg} = D * (\mathcal{H} \times \mathcal{J})_{reg}$, and that $g\big(\mathcal{P}(g * \theta)\big) = \mathcal{P}(\theta)$ for all $g \in D$ and $\theta \in \mathcal{H}$.

Proposition 4.9 *Let $(\theta, j) \in (\mathcal{H} \times \mathcal{J})_{reg}$, and let $(g, \tilde{g}) \in \widetilde{D}$. Given c_- and c_+ in $\widetilde{\mathcal{P}(g * \theta)}$, there is a bijection between the spaces $\mathcal{M}(c_-, c_+, g * \theta, j^g)/\mathbb{R}$ and $\mathcal{M}(\tilde{g}(c_-), \tilde{g}(c_+), \theta, j)/\mathbb{R}$.*

Proof:

For $u \in \mathcal{M}(\tilde{g}(c_-), \tilde{g}(c_+), \theta, j)$ with lift $\tilde{u} : \mathbb{R} \to \widetilde{\mathcal{L}M}$, define $\tilde{v} : \mathbb{R} \to \widetilde{\mathcal{L}M}$ by $\tilde{v}(s) = \tilde{g}^{-1}(\tilde{u}(s))$. This map clearly satisfies $\lim_{s \to \pm\infty} \tilde{v}(s) = c_\pm$. Because $u \in \mathcal{M}(\tilde{g}(c_-), \tilde{g}(c_+), \theta, j)$, $\overline{\partial}_{\theta,j}(u) = 0$. We prove $\overline{\partial}_{\psi,J}(v) = 0$.

Because $v(s,t) = g_t^{-1}(u(s,t))$, the chain rule (lemma 2.3) gives the following identities:

$$\begin{aligned}
\frac{\partial v}{\partial s}(s,t) &= Dg_t^{-1} \frac{\partial u}{\partial s}(s,t) \\
\frac{\partial v}{\partial t}(s,t) &= \frac{\partial g_t^{-1}}{\partial t}(u(s,t)) + Dg_t^{-1} \frac{\partial u}{\partial t}(s,t) \\
X_t^{g\theta}(v(s,t)) &= X_t^{g^{-1}}(v(s,t)) + Dg_t^{-1} X_t^\theta\big(g_t(v(s,t))\big).
\end{aligned}$$

Substitution into the definition of $\overline{\partial}_{g*\theta,j^g}(v)$ gives

$$\begin{aligned}
\overline{\partial}_{g*\theta,j^g}(v)(s,t) &= \frac{\partial v}{\partial s}(s,t) + j_t^g(v(s,t))\Big(\frac{\partial v}{\partial t}(s,t) - X_t^{g*\theta}(v(s,t))\Big) \\
&= Dg_t^{-1}\frac{\partial u}{\partial s}(s,t) + Dg_t^{-1} j_t Dg_t(v(s,t))\Big(Dg_t^{-1}\frac{\partial u}{\partial t}(s,t) \\
&\quad + X_t^{g^{-1}}(g_t^{-1}(u(s,t))) - Dg_t^{-1}X_t^{\theta}(u(s,t)) \\
&\quad - X_t^{g^{-1}}(g_t^{-1}(u(s,t)))\Big) \\
&= Dg_t^{-1}\Big(\frac{\partial u}{\partial s}(s,t) + j_t\big(\frac{\partial u}{\partial t}(s,t) - X_t^{\theta}(u(s,t))\big)\Big) \\
&= Dg_t^{-1}\overline{\partial}_{\theta,j}(u)(s,t).
\end{aligned}$$

Since $\overline{\partial}_{\theta,j}(u) = 0$ by assumption, $\overline{\partial}_{g*\theta,j^g}(v) = 0$ as well, proving the claim.

This map from $\mathcal{M}(\tilde{g}(c_-), \tilde{g}(c_+), \theta, j)$ to $\mathcal{M}(c_-, c_+, g*\theta, j^g)$ is injective because it is induced by a homeomorphism of $\widetilde{\mathcal{L}M}$. It has an inverse induced by the homeomorphism $\tilde{g}^{-1}$, and so is a bijection. Because the map is equivariant under the $\mathbb{R}$ action, it also induces a bijection from $\mathcal{M}(\tilde{g}(c_-), \tilde{g}(c_+), \theta, j)/\mathbb{R}$ to $\mathcal{M}(c_-, c_+, g*\theta, j^g)/\mathbb{R}$.

□

Corollary 4.10 *For* $(\psi, J) \in \mathcal{F}_{reg}$ *and* $c_-, c_+ \in \widetilde{\mathcal{P}(\psi)}$, *if* $\mu_\psi(c_-) - \mu_\psi(c_+) = 1$, *then* $\mathcal{M}(c_-, c_+, \psi, J)/\mathbb{R}$ *is finite.*

Proof:

Decompose (ψ, J) as $(\psi, J) = g*(\theta, j)$, and choose a lift $\tilde{g}$ so that $(g, \tilde{g}) \in \widetilde{D}$. Then (θ, j) is a Hamiltonian regular pair. Since $\mu_\psi(c_-) - \mu_\psi(c_+) = 1$, it follows from 4.5 that $\mu_\theta(\tilde{g}(c_-)) - \mu_\theta(\tilde{g}(c_+)) = 1$ as well. Since (θ, j) is regular, $\mathcal{M}(\tilde{g}(c_-), \tilde{g}(c_+), \theta, j)$ is finite. Thus, by proposition 4.9, $\mathcal{M}(c_-, c_+, \psi, J)$ is finite as well.

$\square$

We define a boundary operator on $CF_*(M, \omega, \psi)$ by counting these spaces. First, for $c_- \in \widetilde{\mathcal{P}(\psi)}_k$, define

$$\partial_k(\psi, J)(\langle c_- \rangle) = \sum_{c_+ \in \widetilde{\mathcal{P}(\psi)}_{k-1}} \#\big(\mathcal{M}(c_-, c_+, \psi, J)/\mathbb{R}\big)\langle c_+ \rangle. \tag{21}$$

Again, # means counting modulo 2. It is perhaps not obvious that this formal sum lies in $CF_{k-1}(M, \omega, \psi)$. In addition, to use this as our boundary operator, it must extend to all of $CF_k(M, \omega, \psi)$. This difficulty is the same as in the Hamiltonian case - for a k-chain $\kappa = \sum_{c_- \in \widetilde{\mathcal{P}(\psi)}_k} m_{c_-} \langle c_- \rangle$, $\partial_k(\kappa)$ is a formal sum $\sum_{c_+ \in \widetilde{\mathcal{P}(\psi)}_{k-1}} m_{c_+} \langle c_+ \rangle$, where the coefficient of $c_+ \in \widetilde{\mathcal{P}(\psi)}_{k-1}$ is given by the number of orbits (modulo time shift) connecting any $c_- \in \widetilde{\mathcal{P}(\psi)}_k$ with $m_{c_-} \neq 0$ to c_+. The above map wouldn't extend to $CF_k(M, \omega, \psi)$ if the number of such orbits could be infinite. In the Hamiltonian case, this is finite because the finiteness condition on the chains guarantees a uniform bound on the energy of all such connecting orbits.

Proposition 4.11 *For $(\psi, J) \in \mathcal{F}_{reg}$, the map given by equation 21 extends linearly to define a boundary operator $\partial_k(H, J) : CF_k(M, \omega, \psi) \to CF_{k-1}(M, \omega, \psi)$.*

Proof:

Let $\kappa = \sum_{c_- \in \widetilde{\mathcal{P}(\psi)}_k} m_{c_-} \langle c_- \rangle$ be a formal sum which lies in $CF_k(M, \omega, \psi)$. Decompose (ψ, J) as $(\psi, J) = g * (\theta, j)$ as in proposition 3.8, and choose a lift $\tilde{g}$ so that $(g, \tilde{g}) \in \widetilde{D}$.

Since $\tilde{g}\big(\widetilde{\mathcal{P}(\psi)}_k\big) = \widetilde{\mathcal{P}(\theta)}_{k-2I(g,\tilde{g})}$, we can apply $\tilde{g}$ to each term of κ, and obtain a formal sum of elements of $\widetilde{\mathcal{P}(\theta)}_{k-2I(g,\tilde{g})}$. We will denote this formal sum by $\tilde{g}(\kappa)$. Then $\tilde{g}(\kappa) = \sum_{c_+ \in \widetilde{\mathcal{P}(\theta)}_{k-2I(g,\tilde{g})}} m_{c_+} \langle c_+ \rangle$.

In fact, $g(\kappa)$ lies in $CF_{k-2I(g,\tilde{g})}(M,\omega,\theta)$. To see this, let $C \in \mathbb{R}$. We need to show that $\{c_+ \in \widetilde{\mathcal{P}(\theta)}_{k-2I(g,\tilde{g})} \,|\, m_{c_+} \neq 0, a_{H^\theta}(c_+) > C\}$ is finite. But this is the same as $\{\tilde{g}(c_-) \in \widetilde{\mathcal{P}(\theta)}_{k-2I(g,\tilde{g})} \,|\, m_{\tilde{g}(c_-)} \neq 0,\ a_{H^\theta}(\tilde{g}(c_-))\}$. This set can be identified with $\{c_- \in \widetilde{\mathcal{P}(\psi)}_k \,|\, m_{c_-} \neq 0, \tilde{g}^* a_H(c_-) > C\}$. This latter set is finite because κ lies in $CF_k(M,\omega,\psi)$.

As described above, $\partial_k(\psi,J)(\kappa)$ is a formal sum of elements in $\widetilde{\mathcal{P}(\psi)}_{k-1}$, and the coefficient of $c_+ \in \widetilde{\mathcal{P}(\psi)}_{k-1}$ is given by the number of orbits (modulo time shift) connecting any $c_- \in \widetilde{\mathcal{P}(\psi)}_k$ with $m_{c_-} \neq 0$ to c_+. The key point is that $\tilde{g}$ again induces a bijection between this set and the corresponding set with ψ repaced by θ. Since (θ, j) is a regular pair, and because $g(\kappa) \in CF_{k-2I(g,\tilde{g})}(M,\omega,\theta)$, this is in fact finite.

This shows that none of the coeffiecients in $\partial_k(\psi,J)(\kappa)$ is infinity, so it is actually a formal sum of elements in $\widetilde{\mathcal{P}(\psi)}_{k-1}$.

To see that $\partial_k(\psi,J)(\kappa)$ actually lies in $CF_{k-1}(M,\omega,\psi)$, we will show that $\partial_k(\psi,J)(\kappa) = \tilde{g}^{-1}\big(\partial_{k-2I(g,\tilde{g})}(\theta,j)(\tilde{g}(\kappa))\big)$. This again follows essentially from proposition 4.9, because the boundary operators are defined by counting the spaces of connecting orbits.

We have seen that $\tilde{g}$ does in fact induce a map from $CF_k(M,\omega,\psi)$ to $CF_{k-2I(g,\tilde{g})}(M,\omega,\theta)$. We show that $\tilde{g}\big(\partial_k(\psi,J)\langle c_-\rangle\big) = \partial_{k-2I(g,\tilde{g})}(\theta,j)\langle \tilde{g}(c_-)\rangle$ for all $c_- \in \widetilde{\mathcal{P}(\psi)}_k$.

By definition,

$$\begin{aligned}\tilde{g}\big(\partial_k(\psi,J)\langle c_-\rangle\big) &= \tilde{g}\big(\partial_k(g*\theta,j^g)\langle c_-\rangle\big)\\ &= \sum_{c_+ \in \widetilde{\mathcal{P}(g*\theta)}_{k-1}} \#\big(\mathcal{M}(c_-,c_+,g*\theta,j^g)/\mathbb{R}\big)\langle \tilde{g}(c_+)\rangle.\end{aligned}$$

By proposition 4.9,

$$\#\big(\mathcal{M}(c_-,c_+,g*\theta,j^g)/\mathbb{R}\big) = \#\big(\mathcal{M}(\tilde{g}(c_-),\tilde{g}(c_+),\theta,j)/\mathbb{R}\big).$$

It then follows that

$$\tilde{g}\big(\partial_k(\psi, J)\langle c_-\rangle\big) = \sum_{c_+ \in \widetilde{\mathcal{P}(g*\theta)}_{k-1}} \#\big(\mathcal{M}(\tilde{g}(c_-), \tilde{g}(c_+), \theta, j)/\mathbb{R}\big)\langle \tilde{g}(c_+)\rangle.$$

Now, by definition,

$$\partial_{k-2I(g,\tilde{g})}(\theta, j)\langle \tilde{g}(c_-)\rangle = \sum_{c' \in \widetilde{\mathcal{P}(\theta)}_{k-2I(g,\tilde{g})-1}} \#\big(\mathcal{M}(\tilde{g}(c_-), c', \theta, j)/\mathbb{R}\big)\langle c'\rangle.$$

By lemma 4.6, $\tilde{g}(\widetilde{\mathcal{P}(\psi)}_k) = \widetilde{\mathcal{P}(\theta)}_{k-2I(g,\tilde{g})}$, so this last sum can be written as

$$\sum_{c_+ \in \widetilde{\mathcal{P}(\psi)}_{k-1}} \#\big(\mathcal{M}(\tilde{g}(c_-), \tilde{g}(c_+), \theta, j)/\mathbb{R}\big)\langle \tilde{g}(c_+)\rangle,$$

and this agrees with $\tilde{g}\big(\partial_k(\psi, J)$, as claimed.

This shows that $\tilde{g}\big(\partial_k(\psi, J)\langle c_-\rangle\big) = \partial_{k-2I(g,\tilde{g})}(\theta, j)\langle \tilde{g}(c_-)\rangle$ for all $c_- \in \widetilde{\mathcal{P}(\psi)}_k$.

A similar calculation shows that the formula continues to hold for infinite sums in $CF_k(M, \omega, \psi)$. Notice that since (θ, j) is a Hamiltonian regular pair, $\partial_*(\theta, j)$ gives a well defined boundary operator on $CF_*(M, \omega, \theta)$. (This was used in the above proof.)

□

The proof of the preceding proposition contains an important fact that we separate for later use.

Proposition 4.12 *Let* $(\psi, J) \in \mathcal{F}_{reg}$, *and let* $(\psi, J) = g * (\theta, j)$ *as in proposition 3.8. Then* $\tilde{g}\big(\partial_k(\psi_t, J_t)(\kappa)\big) = \partial_{k-2I(g,\tilde{g})}(\theta, j)(\tilde{g}(\kappa))$ *for all chains*

$\kappa \in CF_k(M,\omega,\psi)$. Here, $\tilde{g}$ represents the map from $CF_*(M,\omega,\psi)$ to $CF_*(M,\omega,\theta)$ induced by $\langle c\rangle \to \langle \tilde{g}(c)\rangle$.

□

Corollary 4.13 *For $(\psi,J) \in \mathcal{F}_{reg}$, the boundary operator on $CF_*(M,\omega,\psi)$ satisfies $\partial^2 = 0$.*

Proof:

Decompose (ψ,J) as $g*(\theta,j)$, and choose $\tilde{g}$ such that $(g,\tilde{g}) \in \widetilde{D}$. Since $(\psi,J) \in \mathcal{F}_{reg}$, we have that $(\theta,j) \in (\mathcal{H}\times\mathcal{J})_{reg}$, and thus $\partial(\theta,j)^2 = 0$. So,

$$\begin{aligned}\partial_{k-1}(\psi,J)\circ\partial_k(\psi,J) &= \big(\tilde{g}^{-1}\ \ \partial_{k-1-2I(g,\tilde{g})}(\theta,j)\ \ \tilde{g}\big)\circ\big(\tilde{g}^{-1}\ \ \partial_{k-2I(g,\tilde{g})}(\theta,j)\ \ \tilde{g}\big)\\ &= \tilde{g}^{-1}\quad \partial_{k-1-2I(g,\tilde{g})}(\theta,j)\circ\partial_{k-2I(g,\tilde{g})}(\theta,j)\quad \tilde{g}\\ &= 0.\end{aligned}$$

The first equality follows from the previous proposition, and the third equality follows from the above remarks.

□

4.3 The New $HF_*(M,\omega)$

We can now define $HF_*(M,\omega,\psi,J)$ for $(\psi,J) \in \mathcal{F}_{reg}$ as the homology of the chain complex $\big(CF_*(M,\omega,\psi),\partial_*(\psi,J)\big)$.

Remark 3 Notice that the definition of $HF_*(M,\omega,\psi,J)$ doesn't depend on the decomposition $\psi = g*\theta$, but showing that the definition makes sense did use such a decomposition.

To complete the new construction of $HF_*(M,\omega)$, we will show that $HF_*(M,\omega,\psi,J)$ is naturally independent of the choice of $(\psi,J) \in \mathcal{F}_{reg}$. For notational convenience, throughout this section, we will sometimes write $HF_*(\psi, J)$ for $HF_*(M, \omega, \psi, J)$. (This is really just because commutative diagrams don't fit on the page with the extended notation.)

Before we prove the existence of these continuation isomorphisms, we show that $HF_*(M, \omega, \psi, J)$ does in fact always recover the standard Floer homology, $HF_*(M, \omega)$.

Proposition 4.14 *For $(\psi, J) \in \mathcal{F}_{reg}$, and $(g, \tilde{g}) \in \widetilde{D}$, the map $\langle c \rangle \to \langle \tilde{g}(c) \rangle$ induces an isomorphism from $HF_*(M, \omega, g * \psi, J^g)$ to $HF_*(M, \omega, \psi, J)$.*

Proof:

If follows from proposition 4.12 that multiplication by $\tilde{g}$ induces a chain map from $\big(CF_*(M, \omega, \psi, J), \partial_*(\psi, J)\big)$ to $\big(CF_*(M, \omega, \theta, j), \partial_*(\theta, j)\big)$, and as such, it induces a homomorphism on the homology level. This homomorphism is actually an isomorphism with inverse induced by multiplication by $\tilde{g}^{-1}$, so $HF_*(M, \omega, \psi, j) \backsimeq HF_*(M, \omega, \theta, j)$. But since (θ, j) is a Hamiltonian regular pair, $HF_*(M, \omega, \theta, j) \backsimeq HF_*(M, \omega)$.

$\square$

To complete the new construction of $HF_*(M, \omega)$, we need to see that $HF_*(M, \omega, \psi, J)$ is naturally independent of the choice of $(\psi, J) \in \mathcal{F}_{reg}$. Before we give the main ideas involved, notice that we have been able to bypass the usual analytic difficulties involved in Floer homology. In particular, when ψ is a Hamiltonian isotopy, corollary 4.10 follows from two basic facts about the $\overline{\partial}_{\psi,J}$ operator: suitably interpreted, it is an elliptic operator, and solutions to $\overline{\partial}_{\psi,J} = 0$ with fixed endpoints have uniformly bounded energy (once we lift to $\widetilde{\mathcal{L}M}$). It follows from the first fact that the spaces $\mathcal{M}(c_-, c_+, H, J)$ are manifolds, and it follows from the second fact that in the 0-dimensional case, these manifolds are compact. We have used a topological trick to reduce to the Hamiltonian case in order to prove corollary 4.10 when ψ is merely almost Hamiltonian.

In the standard construction, an isomorphism between the Floer homology groups $HF_*(M, \omega, H^\alpha, J^\alpha)$ and $HF_*(M, \omega, H^\beta, J^\beta)$ is given via a choice of (generic) path of pairs in $C^\infty(M \times S^1) \times \mathcal{J}(M, \omega, S^1)$ connecting (H^α, J^α) and (H^β, J^β). In our situation, this corresponds to choosing a path in $\mathcal{F}$ between two regular pairs (ψ^α, J^α) and (ψ^β, J^β).

Definition 4.15 *A* **homotopy of regular pairs** *between* (ψ^α, J^α) *and* (ψ^β, J^β) *in* $\mathcal{F}_{reg}$ *is a smooth map* $\Phi : \mathbb{R} \to \mathcal{F}$ *such that* $\Phi(s) = (\psi^\alpha, J^\alpha)$ *for* $s \leq -1$ *and* $\Phi(s) = (\psi^\beta, J^\beta)$ *for* $s \geq 1$.

Given any two regular pairs (ψ^α, J^α) and (ψ^β, J^β), there is always a homotopy between them. (This is because we can always homotope ψ^α and ψ^β to become Hamiltonian isotopies.)

Given such a homotopy, we will write $\Phi(s) = (\psi^s, J^s)$. For a homotopy Φ and a smooth map $u : \mathbb{R} \times S^1 \to M$, define a section $\overline{\partial}_\Phi(u) \in C^\infty(u^*TM)$ by

$$\overline{\partial}_\Phi(u)(s,t) = \frac{\partial u}{\partial s}(s,t) + J_t^s(u(s,t))\Big(\frac{\partial u}{\partial t}(s,t) - X_t^{\psi^s}(u(s,t))\Big). \tag{22}$$

We would like to use these spaces to define a chain map between the complexes $CF_*(M, \omega, \psi^\alpha)$ and $CF_*(M, \omega, \psi^\beta)$ by counting the solutions to this equation. For $c^\alpha \in \widetilde{\mathcal{P}(\psi^\alpha)}$ and $c^\beta \in \widetilde{\mathcal{P}(\psi^\beta)}$, define $\mathcal{M}(c^\alpha, c^\beta, \Phi)$ as the set of all smooth maps $u : \mathbb{R} \times S^1 \to M$ which satisfy $\overline{\partial}_\Phi(u) = 0$, and which lift to a map $\tilde{u} : \mathbb{R} \to \mathcal{L}M$ with limits c^α and c^β.

Equation 22 looks very similar to equation 20. The only difference is that we allow the almost complex structure and the vector field to vary with s. We showed above that solutions to equation 20 can be counted (modulo time shift). This suggests that they have the same analytic properties as the solutions to the corresponding Hamiltonian equations (that is, when the vector fields X_t^ψ are Hamiltonian).

Theorem 4.16 *For* (ψ^α, J^α) *and* (ψ^β, J^β) *in* $\mathcal{F}_{reg}$*, there exists a natural isomorphism* $\Phi^{\alpha,\beta} : HF_*(M, \omega, \psi^\alpha, J^\alpha) \to HF_*(M, \omega, \psi^\beta, J^\beta)$*. (By natural, we mean that the existence of the isomorphism depends on some choices, but the actual isomorphism is independent of the choices involved.) Moreover, these isomorphisms are functorial in the sense that* $\Phi^{\alpha,\beta} \circ \Phi^{\gamma,\alpha} = \Phi^{\gamma,\beta}$*.*

In addition, if (ψ^α, J^α) *and* (ψ^β, J^β) *are Hamiltonian regular pairs, then this isomorphism reduces to the standard continuation isomorphism.*

Proof:

Choose decompositions $(\psi^{\alpha,\beta}, J^{\alpha,\beta}) = g^{\alpha,\beta} * (\theta^{\alpha,\beta}, j^{\alpha,\beta})$ as in thereom 3.8. Make these choices so that g^α and g^β are contractible, and choose lifts $\tilde{g}^\alpha$ and $\tilde{g}^\beta$ so that $(g^\alpha, \tilde{g}^\alpha)$ and $(g^\beta, \tilde{g}^\beta)$ are in the component of the identity in $\widetilde{D}$. Choose $\bar{\Phi}$ a (Hamiltoniona) homotopy of regular pairs between $(\theta^\alpha, j^\alpha)$ and (θ^β, j^β). Choose $\tilde{g}^s$ a path in $\widetilde{D}$ which satisfies $\tilde{g}^s = \tilde{g}^\alpha$ for small enough s, and $\tilde{g}^s = \tilde{g}^\beta$ for large enough s. Define Φ a homotopy of regular pairs from (ψ^α, J^α) to (ψ^β, J^β) by $\Phi(s) = g^s * (\bar{\Phi}(s)$.

Let $c^{\alpha,\beta} \in \widetilde{\mathcal{P}(\psi^{\alpha,\beta}})$. We will show that for carefully chosen Φ, there is a bijection between $\mathcal{M}(c^\alpha, c^\beta, \Phi)$ and $\mathcal{M}(\tilde{g}^\alpha(c^\alpha), \tilde{g}^\beta(c^\beta), \bar{\Phi}))$. This is essentially a "stretching the neck" argument. First notice that if $u : \mathbb{R} \times S^1 \to M$ satisfies $\partial_{\bar{\Phi}}(u) = 0$, and if we define $v : \mathbb{R} \times S^1 \to M$ by $v(s,t) = (g_t^s)^{-1}(u(s,t))$, then $\partial_\Phi(v)(s,t) = (Dg_t^s)^{-1}\big(\frac{\partial g_t^s}{\partial s}(u(s,t))\big)$. We want to guarantee that $\partial_\Phi(v)$ is small, so that we can assign to it a true solution v' with $\partial_\Phi(v') = 0$ using the implicit function theorem. The only factor contributing to the non-vanishing of $\partial_\Phi(v)$ is the term $\frac{\partial g_t^s}{\partial s}$. We can guarantee that this term is arbitrarily small by allowing it to vary over an arbitrarily large compact interval during the homotopy.

This argument, along with the compactness of M, shows that we can find a uniform bound on $(Dg_t^s)^{-1}\big(\frac{\partial g_t^s}{\partial s}(u(s,t))\big)$, and we can make this bound arbitrary small. By applying the implicit function theorem, we see that there is a bijection between $\mathcal{M}(c^\alpha, c^\beta, \Phi)$ and $\mathcal{M}(\tilde{g}^\alpha(c^\alpha), \tilde{g}^\beta(c^\beta), \bar{\Phi})$ for such homotopies $\bar{\Phi}$. Thus, if $\mu_{\psi^\alpha}(c^\alpha) = \mu_{\psi^\beta}(c^\beta)$, then $\mathcal{M}(c^\alpha, c^\beta, \Phi)$ is finite.

We have now established that we can always find a homotopy of regular pairs Φ from (ψ^α, J^α) to (ψ^β, J^β) such that for any pair $(c^\alpha, c^\beta) \in \widetilde{\mathcal{P}(\psi^\alpha)} \times \widetilde{\mathcal{P}(\psi^\beta)}$ with $\mu_{\psi^\alpha}(c^\alpha) = \mu_{\psi^\beta}(c^\beta)$, it is the case that $\mathcal{M}(c^\alpha, c^\beta, \Phi)$ is a finite set. This defines a map $\Phi^{\alpha,\beta}$ from $CF_k(M, \omega, \psi^\alpha)$ to $CF_k(M, \omega, \psi^\beta)$ by

$$\langle c^\alpha \rangle \mapsto \sum_{c^\beta \in \widetilde{\mathcal{P}(\psi^\beta)}_k} \#\big(\mathcal{M}(c^\alpha, c^\beta, \Phi)\big)\langle c^\beta \rangle.$$

The proof that this is a chain map is exactly the same as in the Hamiltonian case. Given $c^\alpha \in \widetilde{\mathcal{P}(\psi^\alpha)}$ and $c^\beta \in \widetilde{\mathcal{P}(\psi^\beta)}$ with $\mu_{\psi^\alpha}(c^\alpha) - \mu_{\psi^\beta}(c^\beta) = 1$, consider the 1-dimensional manifold $\mathcal{M}(c^\alpha, c^\beta, \Phi)$. This manifold is not compact, but Floer's gluing procedure shows that we can consider this space as a manifold with boundary given by

$$\begin{aligned}\partial\mathcal{M}(c^\alpha, c^\beta, \Phi) = & \bigcup_{b^\alpha \in \widetilde{\mathcal{P}(\psi^\alpha)}_{\mu_{\psi^\alpha}(c^\alpha)-1}} \big(\mathcal{M}(c^\alpha, b^\alpha, \psi^\alpha, J^\alpha)/\mathbb{R}\big) \times \mathcal{M}(b^\alpha, c^\beta, \Phi) \;\cup \\ & \bigcup_{b^\beta \in \widetilde{\mathcal{P}(\psi^\beta)}_{\mu_{\psi^\alpha}(c^\alpha)}} \mathcal{M}(c^\alpha, b^\beta, \Phi) \times \big(\mathcal{M}(b^\beta, c^\beta, \psi^\beta, J^\beta)\big).\end{aligned}$$

This means that $\partial(\psi^\beta, J^\beta) \circ \Phi^{\alpha,\beta} = \Phi^{\alpha,\beta} \circ \partial(\psi^\alpha, J^\alpha)$, which means that $\Phi^{\alpha,\beta}$ is in fact a chain map, and so it induces a homomorphism on the level of homology.

This homomorphism is actually an isomorphism, with inverse induced by the homotopy $\bar{\Phi}(s) = \Phi(-s)$. (This is a homotopy from (ψ^β, J^β) to (ψ^α, J^α).)

Homotopies as above will be called regular homotopies.

Finally, we show that the continuation isomorphisms are independent of the regular homotopy used to define them.

Let Φ^0 and Φ^1 be two regular homotopies between the regular pairs (ψ^α, J^α) and (ψ^β, J^β).

Most of the ideas needed to extend this to the almost Hamiltonian case were given above. First, we know that Φ^0 and Φ^1 are of the form $\Phi^0(s) = g_0^s * \bar{\Phi}^0(s)$ and $\Phi^1(s) = g_1^s * \bar{\Phi}^1(s)$ for some homotopies $\bar{\Phi}^0$ and $\bar{\Phi}^1$ and paths $g_{1,2} : \mathbb{R} \to D$. Choose $\bar{\Psi}$ a (Hamiltonian) homotopy of homotopies from $\bar{\Phi}^0$ to $\bar{\Phi}^1$, and choose a map $g : [0,1] \times \mathbb{R} \to D$, with a lift $\tilde{g} : [0,1] \times \mathbb{R} \to \widetilde{D}$, such that each $\tilde{g}(r,s)$ is in the component of the identity.

This defines a path of homotopies connecting Φ^0 to Φ^1 by $\Psi(r)(s) = g(r,s) * \bar{\Psi}(r)(s)$.

For a fixed $r \in [0,1]$, $\overline{\partial}_{\Psi(r)}(u)$ is well defined. For $c^\alpha \in \widetilde{\mathcal{P}(\psi^\alpha)}$ and $c^\beta \in \widetilde{\mathcal{P}(\psi^\beta)}$, define $\mathcal{M}(c^\alpha, c^\beta, \Psi)$ as the set of all pairs $(u, r) \in C^\infty(\mathbb{R} \times S^1, M) \times [0,1]$ such that $\overline{\partial}_{\Psi(r)}(u) = 0$, and there exists a lift $\tilde{u} : \mathbb{R} \to \mathcal{L}M$ with limits $\lim_{s \to -\infty, +\infty} \tilde{u}(s) = c^{\alpha,\beta}$.

As above, we use a "stretching the neck" argument to show that for carefully chosen g, there is a bijection between the sets $\mathcal{M}(c^\alpha, c^\beta, \Psi)$ and $\mathcal{M}(\tilde{g}^\alpha(c^\alpha), \tilde{g}^\beta(c^\beta), \bar{\Psi})$. Thus, if $\mu_{\psi^\alpha}(c^\alpha) - \mu_{\psi^\beta}(c^\beta) = -1$, this set is finite.

Thus, we can define a map from $CF_k(M, \omega, \psi^\alpha)$ to $CF_{k+1}(M, \omega, \psi^\beta)$ by $\langle c^\alpha \rangle \mapsto \sum_{c^\beta \in \widetilde{\mathcal{P}(\psi^\beta)}_{k+1}} \#\big(\mathcal{M}(c^\alpha, c^\beta, \Psi)\big) \langle c^\beta \rangle$.

As in the Hamiltonian case, this map defines a chain homotopy between the maps induced by Φ^0 and Φ^1, proving that the induce they same isomorphism on the homology level.

□

According to this theorem, we may speak of $HF_*(M, \omega)$ independently of the choice of $(\psi, J) \in \mathcal{F}_{reg}$. In effect, we have extended the space from which we may choose to define $HF_*(M, \omega)$. Now, rather than just allowing generic pairs $(H, J) \in C^\infty(M \times S^1) \times \mathcal{J}(M, \omega, S^1)$, we allow generic pairs $(\psi, J) \in \mathcal{F}$. Next, we use this expanded version of Floer homology to extend Seidel's action of $\pi_0(\widetilde{G})$ on $HF_*(M, \omega)$ to an action of $\pi_0(\widetilde{D})$.

5 Extending the Action

We quickly recall Seidel's homomorphism from $\widetilde{G}$ to $Aut(HF_*(M,\omega))$. Begin by choosing a regular pair $(H,J) \in (C^\infty(M \times S^1) \times \mathcal{J})_{reg}$. For $(g,\tilde{g})$ in $\widetilde{G}$, let Φ be the continuation isomorphism from $HF_*(M,\omega,H,J)$ to $HF_*(M,\omega,H^g,J^g)$ (the pair (H^g,J^g) is given by definition 1.24). This defines an automorphism of $HF_*(M,\omega,H,J)$ by

$$HF_*(M,\omega,H,J) \xrightarrow[\Phi]{} HF_*(M,\omega,H^g,J^g) \xrightarrow[\tilde{g}]{} HF_*(M,\omega,H,J).$$

Here, $\tilde{g}$ means the map on homology induced by the map $\langle c\rangle \mapsto \langle \tilde{g}(c)\rangle$. This induced map is actually an isomorphism. In fact, we can speak of this action independent of the choice of regular pair. Given a different choice of regular pair, (H',J'), the following diagram commutes. (As usual, the symbol Φ represents the continuation isomorphism between the appropriate homologies.)

$$\begin{array}{ccccc} HF_*(M,\omega,H,J) & \xrightarrow[\Phi]{} & HF_*(M,\omega,H^g,J^g) & \xrightarrow[\tilde{g}]{} & HF_*(M,\omega,H,J) \\ \Phi\big\downarrow & & & & \big\downarrow\Phi \\ HF_*(M,\omega,H',J') & \xrightarrow[\Phi]{} & HF_*(M,\omega,(H')^g,(J')^g) & \xrightarrow[\tilde{g}]{} & HF_*(M,\omega,H',J') \end{array}$$

This means that $(g,\tilde{g}) \in \widetilde{G}$ induces an isomorphism of $HF_*(M,\omega)$ independent of the regular pair used to define $HF_*(M,\omega)$. In fact, this action is well defined on $\pi_0(\widetilde{G})$.

Theorem 5.1 (Seidel) *Let $(g_0,\tilde{g}_0)$ and $(g_1,\tilde{g}_1)$ be elements of $\widetilde{G}$ such that their classes inside $\pi_0(\widetilde{G})$ agree. Then $(g_0,\tilde{g}_0)$ and $(g_1,\tilde{g}_1)$ induce the same automorphisms of $HF_*(M,\omega)$. (That is, they induce the same automor-*

phism of $HF_*(M,\omega,H,J)$ *for any choice of regular pair* (H,J)*, and this automorphism commutes with the continuation isomorphism to any other regular pair.)*

$\square$

The conditions of the above theorem mean that g_0 and g_1 give the same element of $\pi_1(\mathrm{Ham}(M))$, and there is a path in $\widetilde{G}$ from $(g_0, \tilde{g}_0)$ to $(g_1, \tilde{g}_1)$.

This describes the homomorphism $\tilde{\sigma} : \pi_0(\widetilde{G}) \to Aut(HF_*(M,\omega))$ introduced by Seidel.

We want to extend this map to have domain $\widetilde{D}$. Most of the ingredients involved have already been developed. First, we move from the Hamiltonian case to the almost Hamiltonian case, and define $HF_*(M,\omega)$ via a choice of $(\psi, J) \in \mathcal{F}_{reg}$. For $(g,\tilde{g}) \in \widetilde{D}$, let Φ be the continuation isomorphism from $HF_*(M,\omega,\psi,J)$ to $HF_*(M,\omega,g*\psi,J^g)$ as in theorem 4.16. Define an automorphism of $HF_*(M,\omega,\psi,J)$ by

$$HF_*(M,\omega,\psi,J) \xrightarrow[\Phi]{} HF_*(M,\omega,g*\psi,J^g) \xrightarrow[\tilde{g}]{} HF_*(M,\omega,\psi,J).$$

It follows from proposition 2.5 that if (ψ, J) and g were Hamiltonian, (i.e., $(\psi,J) \in (\mathcal{H}\times\mathcal{J})_{reg}$ and $(g,\tilde{g}) \in \widetilde{G}$), then the above automorphism of $HF_*(M,\omega,\psi,J)$ corresponds $\tilde{\sigma}(g,\tilde{g})$.

In order to extend this to a homomorphism from $\widetilde{D}$ to $Aut(HF_*(M,\omega))$, we need only show that $(g,\tilde{g})$ induces an automorphism of $HF_*(M,\omega)$ independent of the $(\psi,J) \in \mathcal{F}_{reg}$ used to define it.

Proposition 5.2 *Let* (ψ^α, J^α) *and* (ψ^β, J^β) *be regular pairs, and let* $(g,\tilde{g})$ *be in* $\widetilde{D}$*. The following diagram commutes. (Again,* Φ *denotes the contin-*

uation isomorphisms.)

$$\begin{array}{ccccc}
HF_*(\psi^\alpha, J^\alpha) & \xrightarrow[\Phi^\alpha]{} & HF_*(g*\psi, (J^\alpha)^g) & \xrightarrow[\tilde{g}]{} & HF_*(\psi^\alpha, J^\alpha) \\
\big\downarrow \Phi & & \big\downarrow \Phi^g & & \big\downarrow \Phi \\
HF_*(\psi^\beta, J^\beta) & \xrightarrow[\Phi^\beta]{} & HF_*(g*\psi^\beta, (J^\beta)^g) & \xrightarrow[\tilde{g}]{} & HF_*(\psi^\beta, J^\beta)
\end{array}$$

Proof:

The left square of the diagram consists of the continuation isomorphisms, and so it commutes by theorem 4.16.

The fact that the right square commutes is proved in the same way as proposition 4.9. Choose a regular homotopy Φ. Define Φ^g by $\Phi^g(s) = g * (\Phi(s))$. The key point is that for $c^{\alpha,\beta} \in \widetilde{\mathcal{P}(\psi^{\alpha,\beta})}$, there is a bijection between $\mathcal{M}(c^\alpha, c^\beta, \Phi)$ and $\mathcal{M}(\tilde{g}(c^\alpha), \tilde{g}(c^\beta), \Phi^g)$. Since Φ and Φ^g consist of counting these spaces, the proposition follows.

□

Finally, we show that the above action is well defined on $\pi_0(\widetilde{D})$.

Theorem 5.3 *If* $[g_0, \tilde{g}_0] = [g_1, \tilde{g}_1]$ *in* $\pi_0(\widetilde{D})$*, then they induce the same automorphisms of* $HF_*(M, \omega)$*.*

Proof:

It is sufficient to consider the case when $(g, \tilde{g})$ is in the component of id in $\widetilde{D}$. Choose $(\psi, J) \in \mathcal{F}_{reg}$. Let Φ denote the continuation isomorphism from $HF_*(M, \omega, \psi, J)$ to $HF_*(M, \omega, g * \psi, J^g)$. We must show that $\tilde{g} \circ \Phi = Id_{HF_*(M,\omega,\psi,J)}$.

The proof is again analagous to the Hamiltonian case, with the same modifications introduced in order to proof theorem 4.16.

Let $(g_r, \tilde{g}_r)$ be a path in D connecting id to g, and let Φ be any regular homotopy from (ψ, J) to $(g^*\psi, J^g)$.

By a *deformation of homotopies* compatible with (ψ, J), g^r and Φ, we will mean a two parameter family $\Upsilon = (\psi^{r,s}, J^{r,s})$, $0 \leq r \leq 1$, $-\infty < s < \infty$, where for a fixed (r, s), $(\psi^{r,s}, J^{r,s})$ is in $\mathcal{F}$. These must satisfy the following conditions.

1) $s \leq -1 \implies \psi_t^{r,s} = g_r * \psi, \;\; J_t^{r,s} = J_t^{g_r}$,

2) $s \geq 1 \implies \psi_t^{r,s} = \psi_t, J_t^{r,s} = J_t$,

3) $\psi_t^{0,s} = \psi_t, J_t^{0,s} = J_t, \;\; \forall s \in \mathbb{R}$,

4) $(\psi^{1,s}, J^{1,s}) = \Phi(s), \;\; \forall s \in \mathbb{R}$.

Choose a deformation of homotopies $\Upsilon = (\psi^{r,s}, J^{r,s})$, and let $(r, u) \in \big([0,1] \times C^\infty(\mathbb{R} \times S^1, M)\big)$ satisfy

$$\frac{\partial u}{\partial s}(s,t) + J_t^{s,w}(u(s,t))\left(\frac{\partial u}{\partial t}(s,t) - X_t^{\psi^{s,w}}(u(s,t))\right) = 0. \tag{23}$$

We will say that such a pair converges to $c_-, c_+ \in \widetilde{\mathcal{P}(\psi)}$ if there is a smooth lift $\tilde{u} : \mathbb{R} \to \mathcal{L}M$ with limits $\lim_{s\to-\infty}(\tilde{u}(s)) = (\tilde{g}_w)^{-1}(c_-)$ and $\lim_{s\to\infty}(\tilde{u}(s)) = c_+$. The set of all such pairs is denoted by $\mathcal{M}(c_-, c_+, \Upsilon)$.

There is a deformation of homotopies such that whenever $\mu_\psi(c_+) = \mu_\psi(c_-)+1$, $\mathcal{M}(c_-, c_+, \Upsilon)$ is a finite set. This is proved exactly as in theorems 4.16. More precisely, we decompose the homotopy Φ into $\Phi(s) = h^s * (\theta^s, j^s)$ as in as in proposition 3.8, with each h^r contractibile. Denote the homotopy from $(\theta^{-\infty}, j^{-\infty})$ to $(\theta^\infty, j^\infty)$ by $\bar{\Phi}$. This gives a decomposition $(\psi, J) = h * (\theta, j)$. We choose a deformation of homotopies compatible with (θ, j), $g^r h^r$, and $\bar{\Phi}$. Since this is a Hamiltonian deformation, the finiteness holds (lemma 5.3 of [21]). As in theorem 4.16, there is a bijection from this finite set to the almost Hamiltonian set for carefully chosen decompositions.

We now proceed as in the Hamiltonian case, and use this to define a map $h : CF_*(M, \omega, \psi) \to CF_{*+1}(M, \omega, \psi)$ by $\langle c_- \rangle \mapsto \sum_{c_+} \#\mathcal{M}(c_-, c_+, \Upsilon)\langle c_+ \rangle$. Finally, it follows from lemma 5.5 in [21] that $\partial_{k+1}(\psi, J)h_k + h_{k+1}\partial_k(\psi, J) = \Phi \circ \tilde{g}^{-1} - Id_{CF_k(\psi)}$. That is, h gives a chain homotopy between $\Psi \circ \tilde{g}$ and $Id_{HF_*(\psi,J)}$, proving the theorem.

$\square$

Notice that this is not an improved proof of the analogous theorem in the Hamiltonian case, because the Hamiltonian theorem is used in proving theorem 4.16, which is used in the above proof.

6 Concluding Remarks

The methods of the proof of the main result of the thesis suggest more questions about Floer homology. In the classical case, Floer homology is defined by choosing a generic, periodic, time dependent function on M, and an almost complex structure. These choices define a function and Riemannian metric on the infinite dimensional manifold of contractible loops in M, and Floer homology can be considered as Morse theory using this function and metric. To prove the main result, we extended the available choices used in defining $HF_*(M, \omega)$. However, it appears that we lose the analogy with Morse theory when we use the extended domain. Thus, one can ask if the extended domain of definition can be shown to arise as Morse theory using some function and metric on the space of contractible loops. Even more fundamentally, exactly how far can this space of choices be extended?

There is a collection of results which seem to suggest that $\text{Ham}(M)$ is more related to the topology of M than might be guessed. For example, any homotopy class $[\psi_t] \in \pi_1(\text{Diff}(M))$ defines a map $ev_{\psi_t} : M \to \pi_1(M)$ by $ev_{\psi_t}(p)(t) = \psi_t(p)$. If ψ_t is a Hamiltonian loop, then $ev_{\psi_t} \equiv 0$. Since $\pi_1(M)$ is a topological invariant of M, this shows a deep connection between $\text{Ham}(M)$ and the topology of M. The main result of the thesis gives more evidence of this phenomenon. As of now, these rigidity results seem to be symptoms of some deep connection between $\text{Ham}(M)$ and the topology of the manifold. A long term goal is to discover the underlying connection that would explain all of the rigidity results.

These questions are intimately related to the homotopy groups of the triple $\big(\text{Diff}(M), \text{Symp}(M, \omega), \text{Ham}(M)\big)$. For the latter two, the homotopy properties of the pair $(\text{Symp}(M), \text{Ham}(M))$ are known: in dimension $n > 1$,

$\pi_n(\mathrm{Symp}(M,\omega)) = \pi_n(\mathrm{Ham}(M))$, and in dimension $n = 1$, the difference is described by the Flux homomorphisms. The rigidy phenomonon seems to be related to the homotopy properties of the pair $\big(\mathrm{Diff}(M), \mathrm{Ham}(M)\big)$. There is also still very little understanding of the homotopy properties of the pair $\big(\mathrm{Diff}(M), \mathrm{Symp}(M,\omega)\big)$, which may or may not share the same rigidity properties as the pair $\big(\mathrm{Diff}(M), \mathrm{Ham}(M)\big)$.

In the end, the thesis answers one question, but at the same time, it suggests many more questions in its place, which is as it should be.

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