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A DIRECTED CONTINUUM MODEL
OF A COLUMNAR THIN FILM

A Thesis in
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by
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Abstract

As is well known, classical continuum theories fail to adequately describe material behavior as long-range loads or interactions begin to have a significant effect on the overall behavior of the material. For example, when the presence of atoms, grain boundaries, cracks, inclusions, or pores must be considered, the material may no longer conform to the locality requirements of classical continuum theories. One particular example of such a system is a columnar thin film (CTF), which consists of regularly spaced columns (on the scale of tens of nanometers or more in diameter and one or more microns in height) attached to a substrate. This structure may experience loading conditions due to a variety of sources, including the manufacturing process or in use. As a result of the heterogeneous nature of a CTF, the film is influenced by non-local phenomena. A directed continuum theory (also known as Cosserat, micromorphic, or micropolar theory) will be used to capture the non-local behavior of the film, although the directed continuum theory is itself a local theory.

The analysis in this work begins by establishing the kinematics relationships for a discrete model, inspired by the physical structure of a CTF, and determining the discrete form of the governing equations. A Taylor series expansion of the displacement terms used in the discrete governing equations is used to obtain a continuous form of the governing equations. This work proposes a strain energy density that, following established directed continuum formulations, yields both the identical set of governing equations (found via the Taylor series) as well as the boundary conditions, i.e., a linear homogeneous boundary value problem (BVP). The BVP is analyzed to gain insight into the relationship between the behavior of the model and the input parameters. It is also solved to demonstrate the variety of deformations that may result from different boundary conditions. In addition, a non-linear discrete model is also introduced and compared with the continuum model.
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List of Symbols

$x, u(x)$ Scalar quantities are shown in standard mathematical type.

$n, \chi$ Vector quantities are shown as lower case bold face letters.

$F$ Matrices are shown as bold face capital letters.

$T, \Sigma$ Second order tensors are shown as bold upper case sans serif letters.

$K(3)$ Third order tensors are shown with the subscript (3).

$\{K(3)\}_{ijk}$ Index notation, e.g., for the $i, j, k$-th component of $K(3)$.

$u_i$ In equations dealing with the discrete system, subscripts refer to discrete nodes and not to an index.

$u, x, u'$ Derivatives with respect to position are shown in either manner.

$\dot{G}$ Derivatives with respect to time are shown with a dot.

$\hat{H}$ The hat over a symbol represents quantities associated with the body in the manifold $\mathcal{M}$, even when viewed in Euclidean space, see page 2.

$\text{sym}(A)$ The symmetric part of tensor $A$.

$\text{skw}(A)$ The skew-symmetric (or anti-symmetric) part of tensor $A$.

$\text{tr}(A)$ The trace of tensor $A$.

$\text{Im}[\lambda^-]$ The imaginary part of the term $\lambda^-$.

$\mathbb{R}$ Real space.
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Introduction

There is no doubt that the field of nanotechnology will prove the source of many of the technological developments in the 21\textsuperscript{st} century. It is therefore critical that scientists and engineers be engaged in fundamental research, which includes understanding and modeling a wide variety of nano-scale phenomena [17]. The goal of this work is to develop accurate models of micro- and nano-scale systems that possess a certain structure. This structure may be thought of as a kind of heterogeneity within the body that may also violate the locality requirements of classical continuum mechanics. For example, the state of a body at a given point may be influenced by what happens at some distance away from the immediate neighborhood surrounding the point. There are a number of different approaches, geared towards different length scales and time scales, that may be used to accomplish this stated goal.

One technique suited to the modeling of small-scale structures, molecular dynamics (MD), models the interaction between each of the atoms of the structure. The result of such an investigation gives highly detailed information in the sense that one knows the locations and velocities of every atom comprising the structure and the resulting energy properties of the entire system. Of course this knowledge is available only over a very small region (lengths on the order of 10 nanometers) and for very short time periods (picoseconds). In addition, the accuracy of the model is only as good as the potential used and MD models are computationally expensive.

Another approach seeks to derive a continuum-based solution that accurately
captures the overall behavior of the system. It may seem surprising that continuum-based solutions perform as well as they do at such small length scales. For example, it is common practice to use classical theories of elasticity and constants like the Young’s modulus for thin films less than 10 nm thick used in the formation of quantum dots via molecular beam epitaxy [126]. Ultimately, the goal of the present research effort is the development of a continuum model that accounts for the heterogeneous nature of a thin film and accurately predicts the system’s mechanical behavior.

Before proceeding with a discussion of the continuum model to be used, it is worth introducing another approach that is often used with small scale systems, the multi-scale model. Multi-scale models are intended to capture characteristics of the system across length scales. These length scales are typically grouped into various categories, such as the atomic scale (lengths on the order of nanometers), the microscopic scale (microns), the mesoscopic scale (hundreds of microns), and the macroscopic or structural scale (millimeters and larger) [71]. It is possible to discover relations between phenomena across different length scales only if such multi-scale phenomena are included in a single model [1, 124]. For example, a model that focuses only on a single feature or scale is incapable of discovering the relationship between a characteristic at the nano-scale and its affect at the macro-scale. Although the model formulated in this work is not derived from a multi-scale approach, it is possible that this type of model could be used in such a fashion in the future.

Due to the non-local and heterogeneous nature of the micro- and nano-scale systems of interest, so-called directed continuum theories will be investigated. Before describing directed continuum theories it is useful to consider classical continuum theories. In classical continuum mechanics a body $\mathcal{B}$ is said to consist of particles $\mathbf{r}$ that are mapped onto a region of Euclidean space $\mathcal{E}$. The particular mapping, called a placement, establishes the volume, configuration, and density of the body in physical space. As the body deforms in time the placement of $\mathcal{B}$ into $\mathcal{E}$ necessarily changes.

In directed continuum theories there is an additional placement of $\mathcal{B}$. This second placement, which occurs simultaneously with the classical continuum placement, maps the body into a finite dimensional, director space $\mathcal{M}$. The points of
in the director space represent a structure in addition to the structure of the body in the classical continuum sense. This additional structure is often called a microstructure or microvolume, although the prefix micro is not necessarily a reference to size. For the sake of clarity and consistency in the present work, the following language will be used. The result of the first placement (the one also used in classical continuum theories) will be referred to as the body in Euclidean space; the result of the second placement (the one associated with directed continuum theories) will be referred to as the body in the director space. The term deformation will refer to the change the body in \( \mathcal{E} \) undergoes through time, while the term evolution will refer to the change the body in \( \mathcal{M} \) undergoes through time. The structure associated with the body in \( \mathcal{M} \) is described by kinematic relationships in addition to those found in classical continuum bodies. Consequently, additional constitutive relations are required involving additional stress and strain terms that will more accurately model heterogeneous and non-local materials.

It may be helpful to discuss some of the language commonly associated with directed continuum theories. As will be seen, the body in \( \mathcal{M} \) is commonly described by a set of deformable vectors that have been called directors, hence the phrase directed continuum. (Typically, \( \mathcal{M} \) is defined as a vector space such that, in a reference configuration, the directors are given by a set of orthogonal vectors.) Directors may be functions of time and position within the body. There are a number of other names used as well: generalized, structured, oriented, polar, enhanced, and (implicit or weak) non-local theories.

As will also be seen, there are many variations of directed continuum theories. The particular type of directed continuum theory to be used in this work is the micromorphic theory, named by Eringen [32]. In such a theory, in addition to the body deforming in the classical sense, the body in director space is also allowed to evolve in a particular way. When viewed with the aid of directors, in the micromorphic theory the directors are allowed to rotate and stretch by different amounts when compared with the rotation and stretching of the deformation of the body in \( \mathcal{E} \). Other varieties of directed continuum theories include: Cosserat, higher-order, micropolar, higher strain gradient, and couple-stress theories.

The term non-local is often used in connection with directed continuum theories. This is probably due to the fact that researchers are often trying to model
non-local phenomena. In a mathematical sense there are two ways of viewing the term non-local. Bažant [7] describes the term non-local in a narrow sense (also called strong or explicit) to mean that the behavior of a point is described by an integral over a region surrounding the point. In principle, the region could extend over the entire body. A broad (also called weak or implicit) non-local model is taken to mean that derivatives with respect to position are used to model the interaction between a point and its neighboring points. As will be seen, the discrete model presented in Chapter 3 exhibits non-local behavior since a point in $E$ is affected by the neighboring points in $E$ as well as points from much further away due to the presence of the interacting columns, which are modeled by the evolution of the body in $\mathcal{M}$. Therefore, when this discrete system is modeled using a directed continuum theory, the result may be referred to as a weakly non-local theory.

The specific physical system to be considered here is known as a columnar thin film (CTF). For some time, experimentalists have been fabricating a particular type of thin film consisting of two parts: a substrate or bulk material foundation and an observable structure on the surface of the substrate on the scale of tens of nanometers to a few hundred nanometers in diameter and micron-scale in height [55, 63, 74, 100], an example of which is shown in Figure 1.1. A simplified

![Figure 1.1](image)

**Figure 1.1.** A scanning electron microscopy (SEM) image of a columnar thin film [99]. (Used with permission of the author.)
Figure 1.2. Profile of a columnar thin film. This work considers the thin film shown as a type of beam problem, with deformations in the plane of the page only. An example of a discrete model based on the CTF structure is also shown, highlighting the relationship between the components of the model and the morphology of the thin film.

drawing showing the profile of such a film is shown in Fig. 1.2. This structure resembles a grouping of regular columns or pillars that are stacked together and attached at one end to the substrate. In addition, a simplified discrete model is also shown in Fig. 1.2. A more detailed version of such a discrete model will be introduced in Chapter 3. Researchers who grow these thin films suggest that there may be many uses for them. A good summary, presented by Lakhtakia and Messier [63], includes such uses as optical sensors, optical circuits, ultra-low permittivity barriers in solid state applications, biomedical devices such as filters, and specialized reactors for chemical synthesis.

Much of the current research by experimentalists is focused on tailoring the columnar structure in various ways and with specialized features, e.g., inclined at some angle other than 90 degrees to the substrate, with varying diameter, and/or with a cork-screw type of structure. In addition it may be advantageous to etch away much of the substrate to create a very thin film with an attached nanometer-scale structure. In this work, we will consider examples with sufficiently thin substrates such that the columnar structure affects the overall deformation, even if at this point in time no such materials have actually been created in the
Figure 1.3. This model may be thought of as a bilayer thin film, with the substrate shown as Layer 1 and the columnar structure shown as Layer 2. Based on the kinematics of the model, points vertically aligned in the undeformed configuration will always remain the same distance apart. (This is shown schematically with the lines connecting points between layers.) In terms of the model, each layer is represented as a single, deforming line. The shaded regions are included for illustrative purposes and to show the differing slope of Layer 1 compared with Layer 2.

The research presented in this work is focused on modeling the mechanical properties of a thin film system, i.e., the substrate and columnar region together. It is worth mentioning the work that has been published on modeling single columns, viewing the columns as either springs or cantilevers [69,108,128]. The goal of these research efforts is to obtain an equivalent stiffness term, describing the response of a single column to a load (the load is typically perpendicular to the substrate plane). It is possible that such results could be incorporated into the directed continuum model, although at present, this aspect is not accounted for. Instead, the columns are considered to be inextensible.

Although the model has been developed by considering the structure of CTFs, it may be useful to think about the model for use with other micro- and nano-scale systems. For example, the directed continuum model may be thought of as a bilayer material, with the substrate as one layer and the interacting columnar region as another layer. An example of this is shown in Fig. 1.3. The layers differ from each other because of the different configuration of springs that constitute the substrate and the columnar structure. Although the layers are constrained to remain the same distance apart from each other, there is no requirement for continuity of slope
between layers. It may be that the model presented in this work could be used to analyze other systems such as thin films with regular pores [68], thin films with nano-voids [79], nano-scale composite materials [54], multi-layered thin films [125], substrate films used in the manufacturing of quantum dot superlattices [67], and micro-scale thin films undergoing buckling [53, 111, 123]. Therefore, the resulting model should be thought of more generally than simply as a model of a CTF and the formulation to be presented is generalizable to more complicated systems.

Having decided to develop a directed continuum theory, there are in general two different ways to proceed. One method, often termed the phenomenological approach, begins by writing constitutive or energy relations that include higher order terms, e.g., stress is written as a function of strain and the derivative of strain with respect to position. The second method, and the one adopted for this work and described in [97], begins with a discrete model consisting of known elements that is used to develop the continuum relations. (Of course one must justify the selection of the discrete model to begin with.) The advantage of this approach is that once the discrete model is given it may be simpler to obtain material properties used in the model and to deduce physical meaning from the resulting governing equations.

The continuum model is obtained by following these steps:

1. the kinematics of the continuum model are defined based on the kinematics of the discrete model,

2. strain terms for the continuum model are obtained,

3. a strain energy density is formulated that yields the desired form of the governing equations* via a Hamilton’s principle approach, and

4. boundary conditions are obtained from the Hamilton’s principle approach.

Thus, the complete BVP for the directed continuum is formulated. This will be solved for a variety of boundary conditions and material parameters.

*The desired form of the governing equations refers to those equations obtained directly from the discrete model. The resulting continuum model will also be referred to as the equivalent form of the discrete model.
Now that the basic concept of the present research has been described, an overview of the contents of this dissertation follow. A literature review in Chapter 2 presents directed continuum theories and their derivations, common uses of such theories, and a review of discrete approaches to deriving continuum theories. Sections 2.1.1, 2.1.2, and 2.4 will be most useful in following the formulation presented in Chapter 3. The remaining sections of Chapter 2 include background material, alternative formulations, and published research that may be useful for future work connected with this project.

Chapter 3 describes the complete formulation of the linear version of the CTF-based model. In Section 3.1, the actual formulation of the model begins by creating a discrete model of the system based on the physical appearance of a CTF. A nonlinear discrete system is presented in Section 3.2. The four governing Euler-Lagrange equations are formulated in Section 3.3 based on the discrete model. In Section 3.4, the equivalent directed continuum boundary value problem (BVP) is obtained. The term equivalent is taken to mean that the governing equations obtained in Section 3.3 are identical to those obtained in Section 3.4, although the boundary conditions may differ.

The resulting system of linear homogeneous ordinary differential equations is solved in Section 4.1 in Chapter 4. This solution is analyzed with different boundary conditions for the purpose of relating the inputs to the BVP (the material properties and geometry) to the resulting deformation of the film (the solution of the BVP) in Sections 4.2 and 4.3. Based on the results of Chapters 3 and 4, a number of numerical experiments are conducted on a variety of systems. The results of these analyses are presented in Chapter 5, along with some discussion of their significance. Finally, a summary and some possibilities for future research are presented in Chapter 6.
Chapter 2

Literature Review

The purpose of this section will be to introduce the reader to some of the methodology that will be used in Chapters 3 and 4 and to provide a background to the ideas contained in the present work. The micromorphic theory will be developed in Section 2.1, although as mentioned in the Introduction, Sections 2.1.1 and 2.1.2 describe everything necessary for the formulation to appear in Section 3.4. Both the historical development of directed continuum theories as well as some of the major work of the past 30 years will be presented in Section 2.2. In Section 2.3, a variety of papers are included showing the sorts of physical systems modeled using directed continuum theories. Section 2.4 provides some more specific background on the use of discrete models to motivate the development of continuum theories, which is one of the main ideas to be presented in this work as will be seen in Section 3.1.

2.1 The Micromorphic Theory

This section will provide a review of the development of the micromorphic theory that will form the basis of the work to be presented in Section 3.4. Section 2.1.1 begins by describing the kinematics of micromorphic continua and concludes by defining two different displacement gradients. The goal of Section 2.1.2 is to develop the balance of momentum, the balance of moment of momentum, and the necessary stress, strain, and constitutive relationships. The results from this section will be used to formulate the governing equations that model the behavior of a
CTF under applied end displacements. The final section here, Section 2.1.3, is not necessary for further developments in this dissertation. Rather, it is provided to complement the material presented in Section 2.1.2 based on the work of Eringen and Kafadar [35,36].

2.1.1 Displacement and Deformation Tensors

Using some of the language of Capriz [11], we begin by considering a body $\mathcal{B}$ that consists of points $\mathbf{r}$. As has already been described in Chapter 1, a directed continuum utilizes two different placements. One placement is a mapping from $\mathcal{B}$ into an Euclidean space $\mathcal{E}$ and the other placement is a mapping from $\mathcal{B}$ into a smooth, finite dimensional director space $\mathcal{M}$. The placement into $\mathcal{E}$ is consistent with a classical continuum mechanics development. A particular placement into $\mathcal{E}$, arbitrarily chosen to describe other placements, is denoted by $\mathcal{B}_\kappa$ and is called the reference configuration, see Fig. 2.1. This placement is described via a vector volume function $\chi$, such that

$$\chi: \mathcal{B} \rightarrow \mathcal{B}_\kappa, \quad \chi = \chi(\mathbf{r}) = \mathbf{p} \quad \Rightarrow \quad \mathbf{r} = \chi^{-1}(\mathbf{p}),$$

Figure 2.1. Deformation of the body in Euclidean space, $\mathcal{E}$, from $\mathcal{B}_\kappa$ to $\mathcal{B}$ with an evolution of the body in $\mathcal{M}$ described by the deformation of a single director, also shown in $\mathcal{E}$.
where $\chi(x)$ is assumed to be one-to-one and $p$ is a material point in the reference configuration, i.e., $p \in \mathcal{B}_k$.

The placement into the director space $\mathcal{M}$ may be described by

$$\hat{\Phi} : \mathcal{B} \to \mathcal{M}, \quad \hat{\Phi} = \hat{\Phi}(x),$$

(2.2)

where $\hat{\Phi}(x)$ is assumed to be one-to-one and the hat symbol appearing over a quantity indicates that the quantity is associated with the director space. In addition, $\hat{\chi}$ maps points from the director space to a reference tangent space $\mathcal{T}_k$, i.e.,

$$\hat{\chi} : \mathcal{M} \to \mathcal{T}_k, \quad \hat{\chi} = \hat{\chi}(\Phi(x)),$$

(2.3)

Therefore, based on Eqs. (2.1) and (2.3), we may write that

$$\hat{\chi} = \hat{\chi}(\Phi(\chi^{-1}(p))) = \hat{\chi}(p), \quad p \in \mathcal{B}_k.$$ (2.4)

In general, the dimension of $\mathcal{M}$ need not be related to $\mathcal{E}$ [12, 13]. In the formulation used in the present work, the dimension of the director space $\mathcal{M}$ (and the dimensions of the corresponding tangent spaces containing the directors) is always less than or equal to the dimension of $\mathcal{E}$. As a consequence of Eqs. (2.1) and (2.4), and under the assumptions regarding one-to-one mappings and the dimensionality requirements of $\mathcal{M}$ and $\mathcal{E}$, the kinematic quantities of the body placed in $\mathcal{E}$ and $\mathcal{M}$ may be viewed in the same space. This was shown in Fig. 2.1. Owing to this observation, it is possible to relate terms associated with the body in $\mathcal{E}$ directly to terms associated with the body in $\mathcal{M}$. This fact will be used to justify taking derivatives of “hat” terms with respect to differential elements associated with Euclidean space, as will happen in Eq. (2.12), and to add together a “hat” tensor with a tensor associated with $\mathcal{E}$, as will happen in Eq. (2.31).

Each point in $\mathcal{M}$ may be thought of in a number of different ways; the way these points in $\mathcal{M}$ are described is one of the differences found in the literature. For example, Rivlin [98] considers each point of $\mathcal{M}$ to be the center of mass of a collection of microelements. He assigns as many position vectors emanating from a point $Q$ in $\mathcal{M}$ as there are accompanying microelements. These position vectors are then averaged together, and the resulting average vector is what is defined to
be a director describing the microstructure at $Q$. This averaging process results in a director (vector) field over $\mathcal{M}$. Eringen and Kafadar [36] state that the common practice of introducing what may be a large number of vectors emanating from each point of $\mathcal{M}$ only helps to visualize the physical system. In fact, directors do not need to be vector-valued; one may use director fields to describe porosity, damage, orientation, etc.

In order to consider the deformation of the body in $\mathcal{E}$ denoted by $\mathcal{B}_\kappa$ and the evolution of the body in $\mathcal{M}$, it will be helpful to review the development of general directed continuum theories with the aid of a variety of well-known works [32, 35–38, 75]. The following development is largely based on the work of Eringen and Kafadar [36] and the work of Mindlin [75], although the notation used by these authors differs slightly from that presented below. The motion of the body in time through a sequence of deformed configurations is described by

$$x = f(p, t),$$  \hspace{1cm} (2.5)  

$$\hat{x} = \hat{F}(p, t)\hat{\chi},$$  \hspace{1cm} (2.6)  

where $x \in \mathcal{B}$ denotes the position at time $t$ of the points in $\mathcal{B}_\kappa$ (the deformed counterpart to $\chi$) and $\hat{x}$ is the result of the evolution of $\hat{\chi}$, as shown in Fig. 2.1. The deformation function of the body in $\mathcal{E}$ is denoted by $f$. The evolution of the body in $\mathcal{M}$ is described by a second order tensor $\hat{F}$, which is like a deformation gradient found in classical continuum mechanics ($\hat{F}$ is often called the microdeformation gradient, although the term is used for a different quantity in [75]). Note that both the deformation function $f$ and $\hat{F}$ are functions of $p \in \mathcal{B}_\kappa$ in addition to time $t$.

For the remainder of the development presented in the present work, the deformation of the body in $\mathcal{E}$ is also taken to be of the form

$$F = \text{Grad} f.$$  \hspace{1cm} (2.7)  

The derivative taken here and the derivatives taken subsequently are all with respect to position in the reference configuration, i.e., the material or Lagrangian description. When written out, the gradient will be given by “Grad” and the diver-
gence will be given by “Div”, with the upper case indicating the material reference frame. Although Eringen [35] initially develops the theory for large deformations, he quickly moves to standard linear assumptions of infinitesimal strain. This assumption will be used in the present work as well so that there will be no difference between the spatial and material derivatives.

The functions \( f \) and \( \hat{F} \hat{\chi} \) are taken to be a unique mapping from the reference configuration into the current configuration, which is the configuration at time \( t \). The inverse of the deformation of the body in \( E \) and the evolution of the body in \( M \) are assumed to exist and are of the form such that

\[
\det F > 0, \quad \det \hat{F} > 0.
\] (2.8)

In Fig. 2.1 only a single director vector is shown. One could also imagine three orthogonal unit vectors representing the reference configuration of the body in \( M \). In this case, Eq. (2.6) would be applied to the three directors. In the work to be presented in Chapter 3, there will be a single two-dimensional deformable director used.

In the work of Mindlin [75], two displacement fields are introduced: one displacement field results from the deformation of the body in \( E \), given by \( u \), and the other displacement field results from the evolution of the body in \( M \), given by \( \hat{u} \), where

\[
\begin{align*}
u &= x - \chi, \\
\hat{u} &= \hat{x} - \hat{\chi},
\end{align*}
\] (2.9, 2.10)

where the translation of \( \hat{x} \) is given by \( u \), as shown in Fig. 2.1. In his work, Mindlin considers the condition when the gradients of \( u \) and \( \hat{u} \) are small, such that the displacement gradients with respect to both the reference and deformed configurations become indistinguishable. Under this assumption the displacement gradients may be given as

\[
\begin{align*}
H &= \frac{\partial u}{\partial \chi} = \text{Grad } u, \\
\hat{H} &= \frac{\partial \hat{u}}{\partial \chi} = \text{Grad } \hat{u},
\end{align*}
\] (2.11, 2.12)
where $\mathbf{H}$ is the gradient of the displacement due to the deformation of the body in $\mathcal{E}$ and $\hat{\mathbf{H}}$ is the gradient of the displacement due to the evolution of the body in $\mathcal{M}$ as viewed in $\mathcal{E}$.* Mindlin points out that $\hat{\mathbf{H}}$ is proportional to the displacements of the directors.

Due to the infinitesimal evolution of the body in $\mathcal{M}$, the symmetric part of $\hat{\mathbf{H}}$, denoted by $\text{sym}(\hat{\mathbf{H}})$, may be considered a strain and the skew-symmetric part of $\hat{\mathbf{H}}$, denoted by $\text{skw}(\hat{\mathbf{H}})$, may be considered a rotation of the body in $\mathcal{M}$. When $\text{sym}(\hat{\mathbf{H}}) = 0$ the system is called micropolar by Eringen [33]. In this case, $\text{skw}(\hat{\mathbf{H}})$ represents an infinitesimal rotation of the now rigid directors. Since the Cosserat brothers used rigid directors, this theory is also called the Cosserat theory. Eringen [32,34] refers to the system as micromorphic when $\text{sym}(\hat{\mathbf{H}}) \neq 0$. As has already been discussed, the model described in Chapter 3 will utilize the micromorphic theory.

### 2.1.2 Formulation of Balance Laws

A directed continuum theory, as described in Section 2.1.1, includes a displacement field $\hat{\mathbf{u}}$ in addition to the displacement field $\mathbf{u}$ that is found in classical continuum theories. It should be expected that additional strain terms must be included as well. Both Mindlin and Eringen define three different, but related, strain terms. The strains defined by Eringen will be presented in Section 2.1.3. Mindlin defines a macro-strain (typically called the infinitesimal strain tensor),

$$
\mathbf{E} = \frac{1}{2}(\mathbf{H} + \mathbf{H}^T),
$$

(2.13)

a relative strain (called the relative deformation by Mindlin),

$$
\Gamma = \mathbf{H} - \hat{\mathbf{H}},
$$

(2.14)

---

*In Mindlin [75], $\{\mathbf{H}\}_{ij} = (\partial_i u_j)^T$ and $\{\hat{\mathbf{H}}\}_{ij} = (\partial_i' u'_j)^T \equiv (\psi_{ij})^T$. In Eringen [35], $\{\mathbf{H}\}_{ij} = U_{i,j}$ and $\{\hat{\mathbf{H}}\}_{ij} = \Phi_{ij}$. In addition, Eringen’s displacement vector appears as the difference between the deformed and undeformed directors such that his microdisplacement function actually equals $\mathbf{u} + \hat{\mathbf{u}}$, as given in Eqs. (2.9) and (2.10).
and the gradient of the gradient of the displacement of the body in $\mathcal{M}$ (what he calls the microdeformation gradient),

$$K_{(3)} = \text{Grad} \hat{H},$$

where the subscript $(3)$ refers to the fact that the tensor is of the third order.†

Both Mindlin and Eringen develop the necessary compatibility equations to relate terms of the strain tensors with $H$ and $\hat{H}$. Using index notation, these are given as

$$\varepsilon_{mik} \varepsilon_{nlj} \{E\}_{kl,ij} = 0,$$

$$\varepsilon_{mil} \{K_{(3)}\}_{jkl,i} = 0,$$

$$\{\{E\}_j + \{R\}_j - \{\Gamma\}_j\}_{,i} = \{\hat{H}\}_{jk,i},$$

where $\varepsilon_{ijk}$ refers to the permutation symbol and the rotation tensor is given by

$$R = \frac{1}{2}(H - H^T).$$

Since the strain terms are defined from the displacements, the compatibility equations will automatically be satisfied.

Mindlin observes that there are three stress terms that are work-conjugates of the three strain terms. The Cauchy stress $T$ is the work-conjugate of $E$, the relative stress $\Sigma$ is the work-conjugate of $\Gamma$, and the double stress $M_{(3)}$ is the work-conjugate of $K_{(3)}$,‡ where $T = T^T$, i.e., the Cauchy stress is symmetric. This is the same as writing the following relationships with the potential energy density functional $W$:

$$T = \frac{\partial W}{\partial E} = T^T, \quad \Sigma = \frac{\partial W}{\partial \Gamma}, \quad M_{(3)} = \frac{\partial W}{\partial K_{(3)}},$$

where $T$ is symmetric due to the symmetry of $E$. Using Hamilton’s principle, Mindlin derives the local forms of the balance of linear momentum and moment

---

†Mindlin [75] uses the following notation: $\{E\}_{ij} = \varepsilon_{ij}$, $\{\Gamma\}_{ij} = (\gamma_{ij})^T$, and $\{K_{(3)}\}_{ijk} = \partial_k \psi_{ji}$.

‡Mindlin [75] uses the following symbols for these three stresses: $\tau_{ij}$, $(\sigma_{ij})^T$, and $\mu_{kji}$, respectively.
of momentum in the current configuration (the spatial or Eulerian description):\textsuperscript{5}

\[
\text{Div}(T + \Sigma) + b = \rho \ddot{u},
\]

\[
\text{Div} M(3) + \Sigma + \Phi = \rho \ddot{H},
\]

where \( b \) and \( \Phi \) are called the body force and double force per unit volume, respectively, and \( \rho \) is the mass density of the body in the current configuration. Eringen and Kafadar \textsuperscript{[35,36]} provide a clearer definition of these terms, as well as the second order inertia tensor \( \bar{I} \) with units of length squared. The term \( b \) represents forces acting on points of the body \( \mathcal{B} \) due to some external action, e.g., gravity. The body force per unit volume is found in classical continuum theories. The double force per unit volume represents couples acting on points of the body due to external actions. According to Eringen and Kafadar,\textsuperscript{6} the inertia tensor \( \bar{I} \) may be thought of as the product \( \bar{I} \kappa \hat{F} \), where \( \bar{I} \kappa \) is another second order inertia tensor derived from a volume average of the directors in the reference configuration, \( \hat{X} \). The superposed dots indicate the second derivative with respect to time, i.e., the acceleration terms. For the static case the acceleration terms \( \ddot{u} \) and \( \ddot{H} \) equal zero.

If the surface force per unit area is given by the vector \( s \) and the double force per unit area is given by the tensor \( S \), then there are twelve traction boundary conditions:

\[
s = (T + \Sigma)n, \quad S = M(3)n,
\]

where \( n \) is the unit normal vector to the boundary.\textsuperscript{7}

The homogeneous quadratic potential energy density functional that is given by Mindlin is

\[
W = \frac{1}{2} C_{ijkl}\{E\}_{ij}\{E\}_{kl} + \frac{1}{2} B_{ijkl}\{\Gamma\}_{ji}\{\Gamma\}_{lk} + \frac{1}{2} A_{ijklmn}\{K(3)\}_{kji}\{K(3)\}_{nml} + D_{ijkl}\{\Gamma\}_{ji}\{K(3)\}_{mkl} + F_{ijkl}\{K(3)\}_{ji}\{E\}_{lm} + G_{ijkl}\{\Gamma\}_{ji}\{E\}_{kl}. \quad (2.24)
\]

\textsuperscript{5}Mindlin uses the derivative, which appears in the divergence operator, somewhat differently. For example, as written in Eq. (2.21), \( \{\text{Div}(T + \Sigma)\} \) equals \( \partial(\{T\}_{ij} + \{\Sigma\}_{ij})/\partial x_j \). Mindlin, in his Eq. (4.1)\textsubscript{1}, writes the equivalent term as \( \partial T_{ij} / \partial x_j \), where the subscript of the derivative differs. This is the reason that \( \Sigma \) equals \( \sigma_{ij}T \) and \( M(3) \) equals \( \mu_{kji} \) instead of \( \mu_{ijj} \).

\textsuperscript{6}Eringen calls \( \bar{I} \kappa \) the microinertia, which he writes as \( I^{KL} \), and \( \bar{I} \) the spatial microinertia, which he writes as \( I_{kji} \).

\textsuperscript{7}Mindlin used \( t_i \) for \{\text{s}\}_i \) and \( T_{ij} \) for \{\text{S}\}_{ij} \).
A similar form of the energy density will be used in Section 3.4 to determine the constitutive relations after taking advantage of the appropriate work-conjugate relations, which were introduced in Eq. (2.20). The potential energy described here is that of a simple micromorphic continuum. Based on the form of $W$ given above, the following symmetries result:

$$A_{ijklmn} = A_{lmnijk}, \quad B_{ijkl} = B_{klij}, \quad C_{ijkl} = C_{klij}. \quad (2.25)$$

Due to the symmetry of $E$ based on its definition as given in Eq. (2.13), one may also observe that

$$C_{ijkl} = C_{jikl}, \quad F_{ijklm} = F_{ijkm}, \quad G_{ijkl} = G_{ijlk}. \quad (2.26)$$

For the simple micromorphic continuum, and based on Eq. (2.20), the following constitutive equations result:

$$\{T\}_{pq} = C_{pqpj} \{E\}_{ij} + G_{ijpq} \{\Gamma\}_{ji} + F_{ijkpq} \{K(3)\}_{kji}, \quad (2.27)$$
$$\{\Sigma\}_{pq} = G_{pqij} \{E\}_{ij} + B_{ijpq} \{\Gamma\}_{ji} + D_{pqijk} \{K(3)\}_{kji}, \quad (2.28)$$
$$\{M(3)\}_{pqr} = F_{pqrij} \{E\}_{ij} + D_{ijpqr} \{\Gamma\}_{ji} + A_{pqrij} \{K(3)\}_{kji}, \quad (2.29)$$

where Mindlin observes that for an isotropic material there will be no odd order tensors, i.e., $D_{ijklm}$ and $F_{ijklm}$. (This will not be the case for the theory developed in Chapter 3.)

### 2.1.3 Additional Strain Terms

There are alternate forms of the balance equations that may be developed by using different strain terms. These may be more useful when considering nonlinear micromorphic continua. Some of these strains will be described in this section even though they will not be used in Chapter 3. For example, Eringen [35] introduces the right Cauchy-Green strain tensor and two additional strain tensors, which he calls microdeformation tensors, as follows:

$$C = F^T F, \quad \Psi = F^T \hat{F}, \quad K(3) = F^T \text{Grad} \hat{F}. \quad (2.30)$$
Under the linear theory, these three tensors may be approximated as

\[ C \approx I + H + H^T, \quad \Psi \approx I + \hat{H} + H^T, \quad K_{(3)} \approx \text{Grad} \hat{H}, \quad (2.31) \]

where \( I \) is the second order identity tensor. Eringen also defines Lagrangian and Eulerian strain tensors that are similar to those of Mindlin (in the paper, he states that although he is dealing with linear theory and there is no need to distinguish Lagrangian and Eulerian descriptions, it is done for clarity). For the Lagrangian representation, the strains obtained from Eq. (2.31) include \( E, \Gamma^*, \) and \( K_{(3)} \) defined as

\[ E = \frac{1}{2}(H + H^T), \quad \Gamma^* = H^T + \hat{H}, \quad K_{(3)} = \text{Grad} \hat{H}. \quad (2.32) \]

Eringen and Kafadar [36] suggest that an alternate set of strains should be used. These are defined as follows:

\[ C^* = F^T \hat{F}^{-T}, \quad \Psi^* = \hat{F}^T \hat{F}, \quad K^*_{(3)} = \hat{F}^{-1} \text{Grad} \hat{F}. \quad (2.33) \]

These three strain terms are related to three stress terms through work-conjugate relationships. In addition to balance of momentum and balance of moment of momentum equations, Eringen and Kafadar also formulate balance of mass, balance of microinertia, balance of energy, and balance of entropy equations. As described in the previous section, the microinertia is a second order tensor based on a volume average of the vectors \( \hat{\chi} \). In addition to developing a set of balance equations, Eringen and Kafadar include some specific examples of various deformations including rigid deformations, isochoric deformations, and different affine deformations.

### 2.2 Additional Directed Continuum Theories

This section provides additional material that may be helpful in understanding the derivations shown in Section 2.1. The material presented here is not directly used in the developments of Chapter 3. Section 2.2.1 provides a very brief history on

---

**Eringen uses the following notation:** \( \{C\}_{ij} = C_{ij}, \{\Psi\}_{ij} = \Psi_{ij}, \{K_{(3)}\}_{ijk} = \Gamma_{ijk}, \{E\}_{ij} = E_{ij}, \) and \( \{\Gamma\}_{ij} = \delta_{ij}. \)

**Eringen and Kafadar use the following notation:** \( \{C^*\}_{ij} = C_{ij}, \{\Psi^*\}_{ij} = \Psi_{ij}, \) and \( \{K^*_{(3)}\}_{ijk} = \Gamma_{ijk}. \)
the development of directed continuum theories. Alternate developments of such
theories are presented in Section 2.2.2. Although there are many more individuals
who have worked on such theories, the papers presented in this section represent
some of the most commonly referenced material. Finally, Section 2.2.3 includes
a basic review of two special cases of directed continuum theories: couple-stress
theory and higher-order strain gradient theory.

2.2.1 Early Work

The 1909 publication of the Cosserat brothers [26], Eugène and François, is con-
sidered the foundational work on directed continuum theories. They utilized such
theories to model the twist of rods and shells. As one might expect, their re-
search was also based on the work of others. For example, the work of Voigt in
1887 [122] and Duhem in 1893 [30] were used by the brothers as they developed
their ideas. Ericksen and Truesdell [31], in their introduction, provide a histori-
cal setting for the development of directed continuum theories beginning with the
work of Bernoulli and Euler.

Although certain aspects of directed continuum theories may be found in other
works before and after 1909, it is commonly held that the aforementioned work of
Ericksen and Truesdell stands at the beginning of the resurgent interest in such
theories. These authors begin by discussing the idea of a directed body described
by one vector field defining the motion of the body itself (the body in Euclidean
space) and another set of vector fields, called directors, describing the placement
of the body into $M$. As has been mentioned in Chapter 1, the body in Euclidean
space $E$ is allowed to deform and the body in the director space $M$ is allowed
to evolve according to different deformation functions. According to Ericksen and
Truesdell, the deformation of the body in $E$ leads to a strain of position and the
director deformation leads to a strain of orientation. The strain of position is the
recognized right Cauchy-Green strain tensor, as given by the first of Eqs. (2.30).
The strain of orientation captures the effect of the deformation of the directors,
where this deformation describes the evolution of the body placed into $M$. Of
course appropriate stresses are introduced for these strains.

The primary focus of Ericksen and Truesdell’s work is the static analysis of rods
and shells using the directed continuum theory that they develop. A basic idea to take from this work is that directed continuum theories begin by establishing a more descriptive kinematics for the system than is found in classical continuum theories. This necessitates additional balance laws, constitutive relations, stress terms, and strain terms.

### 2.2.2 Alternate Developments

Over the past thirty years there have been many papers written on the use and formulation of directed continuum theories. It seems that many of the papers published on this topic are associated with certain “families” of researchers. In this section some of the approaches commonly encountered in the literature will be presented. In all, five different groups of work will be discussed and the common traits they share with the development of Section 2.1 will be pointed out. These five categories include: the work of Capriz; Cosserat points, curves, and surfaces; pseudo-rigid theories; the work of Germain; and other work of interest.

#### 2.2.2.1 The work of Capriz

Many different authors have published work with or have been associated with Capriz. In early work of this kind by Capriz and Podio-Guidugli [12,13], one finds characteristics common to this family of work. These characteristics include: the use of a discrete analysis of mass points to motivate the development of continuum theories, an approach to formulating balance equations in a manner similar to Noll’s approach to classical continuum mechanics [88, 89, 119], a comparison of Capriz’s approach to other directed continuum theories, and a rather abstract and highly mathematical approach to the derivations.

In [12], Capriz and Podio-Guidugli begin with a discrete system of mass points subjected to an affine deformation. As such, the kinematics of such a system are completely described by an invertible, second order tensor that is a function of time, \( G(t) \). The overall behavior of such a system is taken as a paradigm for the behavior of each point in a continuum with microstructure.

In moving to the continuum, a complete placement of the body \( \mathcal{B} \) consists of two placements. The first placement maps \( \mathcal{B} \) into Euclidean space, \( \mathcal{E} \), and the
second maps the tangent space associated with $\mathcal{B}$ into the translation space of $\mathcal{E}$. A particular complete placement is called the reference configuration and the position of the points of the body in $\mathcal{E}$ are described by the vector field $\chi$. Using the approach of the discrete development, an arbitrary complete placement of $\mathcal{B}$ is given in terms of $\chi$ by the two fields that correspond to $x(t)$ and $G(t)$ for the discrete system, such that

$$x = x(\chi, t), \quad G = G(\chi, t) \quad \forall (\chi, t) \in \mathcal{B}_r \times \mathcal{I},$$

(2.34)

where $\mathcal{B}_r$ refers to the reference configuration as in Fig. 2.1 and $\mathcal{I}$ refers to the time interval. A tensor $W$, called the wrench (also called the gyration tensor by Eringen [32]), is introduced based on a similar formulation in the discrete case, where

$$W = \dot{G}(\chi, t)G^{-1}(\chi, t).$$

(2.35)

The authors write that the kinematical state of the system is defined by $\dot{x}$ and $W$. If $G$ is known then $W$ must also be known. From the balance laws developed by Capriz and Podio-Guidugli, it appears that it is more convenient to use the wrench tensor instead of $G$.

In their follow-up work [13], directors are introduced and discussed in terms of the tensor $G$. For the continuous case, $G$ seems to be equivalent to $\dot{F}$ that was introduced in Eq. (2.6). In addition, a macro-deformation gradient is also introduced, $F$, that describes the deformation of the body in Euclidean space defined by the vector field $x$. Many specific examples are presented in the 1980 book by Capriz [11]. A recent work by Mariano [72] provides another review of the theory as described by Capriz et al., along with more recent developments.

2.2.2.2 Cosserat points, curves, and surfaces

A number of authors have taken another approach to modeling directed continua. The basic idea is to categorize the theories based not on the type of evolution of the director field, but rather based on the dimensionality of the Euclidean space containing the body. There are four possible categories of such generalized continua, which is a term used by Green and Naghdi [49], based on the material dimension in Euclidean space: a volume, a surface, a line, and a point. For ex-
ample, a system may be nothing more than a point in Euclidean space. But the placement of the body in \( M \) may be evolving in such a way as to be described by a three-dimensional tensor that allows for rotation and stretch, as described in a variety of work by Green, Muncaster, Naghdi, and Rubin [48, 83, 102, 103, 105]. Such a system is called a Cosserat point. A Cosserat or directed curve refers to a system in Euclidean space defined by a line in \( E \) with an evolving director field (see Cohen [21]), while a Cosserat or directed surface is described by a surface in Euclidean space with an evolving director field (see Cohen and DeSilva [23] and Green and Naghdi [47]).

2.2.2.3 Pseudo-rigid theories

Independently of Muncaster [83], Cohen [22] developed an equivalent theory to the generalized continuum theory just discussed. He called this theory the pseudo-rigid theory. The terminology arises because the body is considered to be a point in \( E \) along with the placement of the body in what is called the director space. The evolution of the structure described by the director space is represented by a mapping from the director space into the translation space of \( E \). Since the point in \( E \) is incapable of deformation (being a point), while the director space is capable of evolving in time, the system is called pseudo-rigid.

The evolution of the body in director space is described by the deformation of an ellipsoid in \( E \), where the ellipsoid is defined by the eigenvalues and eigenvectors of a positive-definite, second order tensor. Cohen points out that this deformation may be equivalently described by a triad of directors so that the deformation of the ellipsoid is no different from the homogeneous deformation describing the evolution of the body in \( M \) presented in Section 2.1. To further refine their ideas, Cohen and Muncaster collaborated [24, 25] and referred to the theory of pseudo-rigid bodies as a coarse theory. The adjective “coarse” is used since they are only concerned with a motion of the center of mass, a change of orientation, and a deformation characterized by normal and shear displacements from a reference configuration of the overall body. On the other hand a fine theory would take into account the deformation of each point of the body.

of moment of momentum. According to Chapter 2 of that work, the Euler functional \( M \) “is a generalized measure of mass associated with the distribution of matter about the center of mass.” It is used to define another Euler functional, called \( E_{\lambda} \), which is obtained by placing \( M \) into Euclidean space, \( \mathcal{E} \). The \( \lambda \) subscript refers to the pair of placements, or the complete placement, that takes the point of the body into \( \mathcal{E} \) and the point of the body into a set of invertible linear maps. These maps take the director space into the translation space of \( \mathcal{E} \). The Euler tensor \( E \), defined by the Euler functional \( E_{\lambda} \), is used to define an inertia tensor \( J \), where \( J = \left( \text{tr} E \right) I - E \).

One of the main points of interest of [25] is the question of how does the pseudo-rigid body theory compare with other continuum theories. For example, if the deformation of the directors preserves distance and only allows for changes of orientation, then the pseudo-rigid theory reduces to the classical theory of a rigid body. Chapter 3 of Cohen and Muncaster’s work is entirely devoted to the issue of comparing pseudo-rigid and classical continuum theories. This work also includes a systematic study of different pseudo-rigid deformations as well as an attempt to use pseudo-rigid bodies to model observed gyroscopic motions of deformable bodies.

### 2.2.2.4 The work of Germain

Germain [46] presents a derivation of a directed continuum theory by way of the method of virtual power based on the work of d’Alembert. Each particle in a body is seen as a small continuum, such that the position of the particle is taken as the center of mass of the small, surrounding continuum. In Germain’s approach, the kinematical description of the body is given by a vector field representing two different velocities: the velocity of the particles that constitute the entire body (the body particles, denoted as \( M \) by Germain) and the velocity of the particles that constitute the continuum surrounding the particles of the body (the continuum particles, denoted as \( M' \) by Germain). The points \( M' \) form the small continuum, with center of mass at \( M \), that is labeled \( P(M) \). It is possible to write the velocity vector field of the points \( M' \) using a deformation gradient such that, using the
“hat” notation already presented in Section 2.1, one obtains the following:

\[ \hat{v} = v + L\hat{x}, \]

(2.36)

where \( \hat{v} \) is the velocity of the continuum particles, \( v \) is the velocity of the body particles, \( \hat{x} \) describes the position of the continuum particles relative to the center of mass of \( P(M) \), and \( L \) is the gradient of the relative velocities.* The use of the phrase “relative velocities” suggests that this tensor describes the motion of the particles \( M' \) relative to the center of mass \( M \). In standard linear theory, \( \text{sym}(L) \) is called the microstrain rate tensor and \( \text{skw}(L) \) is called the microrotation rate tensor. Since \( P(M) \) is assumed to be rather small, Germain views \( \hat{v} \) in terms of a Taylor series expansion with respect to \( \hat{x} \). Equation (2.36) is obtained in this manner up to the first order. Because of this, Germain refers to the theory as being of degree one.

Continuing with a first order theory, Germain defines a \textit{relative microvelocity gradient tensor} as†

\[ N = \text{Grad} \, v - L, \]

(2.37)

so that he may write an integral form of the total virtual power due to internal forces, external long-range forces, and external contact forces. The total virtual power is used to obtain Eqs. (2.21) and (2.22), although he refers to the sum \( T + \Sigma \) as the Cauchy stress tensor.‡ Four specific versions of the directed continuum theory are discussed: micropolar theory, couple-stress theory, second gradient theory, and microstructure without microstress. This last theory assumes that the microdeformations do not cause any changes to the total energy of the system.

### 2.2.2.5 Other work of interest

Relatively early in the development of directed continuum theories, Green and Rivlin [51] and Green, Naghdi, and Rivlin [50] introduced the idea of a set of

*Germain uses a different notation, such that \( \{v\}_i = U_i \), \( \{\hat{v}\}_i = U'_i \), \( \{\hat{x}\}_i = x'_i \), and \( \{L\}_{ij} = \chi_{ij} \).

†Germain uses \( \eta_{ij} \) for the tensor \( \{N\}_{ij} \).

‡Germain uses the notation \( \{T + \Sigma\}_{ij} = \tau_{ij} \), \( \{T\}_{ij} = \sigma_{ij} \) (named the intrinsic part of the stress tensor), \( \{\Sigma\}_{ij} = s_{ij} \) (named the microstress tensor), \( \{M_{(3)}\}_{ijk} = \nu_{ijk} \) (named the second microstress tensor), \( \{b\}_i = f_i \), and \( \{\Phi\}_{ij} = \Psi_{ij} \).
tensors called *multipolar displacements* as a set of kinematic variables. Multipolar displacements may be used to obtain an equivalent theory as presented by Mindlin [75], although the formulation is more abstract.

Aifantis et al. have also developed some novel approaches to modeling systems using directed continuum concepts. Aifantis also seems interested in second gradient theories and in considering the underlying ideas used to motivate these theories, e.g., whether or not a particular kinematics choice is phenomenologically based. Aifantis [2] introduces the idea of the placement of the body into the director space \( \mathcal{M} \) in a different way by looking at non-classical interactions between a body and its boundary. Triantfyllidis and Aifantis [116] also use a second gradient theory (see Section 2.2.3 for a definition of this theory) to study nonlinear material behavior.

A number of authors use variational techniques in the analysis of directed media [11,25,52,87]. For example, in the work of Nistor [87] there are two different integral functions whose variations must vanish in order for there to be solutions to the governing equations of the form of Eqs. (2.21) and (2.22) with appropriate boundary conditions. In the work of Cohen, Capriz, and Muncaster [11, 25] a Lagrangian is defined and Hamilton’s principle is used as in standard variational approaches [64].

### 2.2.3 Special Cases

Additional theories have been proposed based on relating the deformation of the directors directly to the classical deformation of the body \( \mathcal{B} \). Two such theories are presented below: the couple-stress theory and the higher-order strain gradient theory.

#### 2.2.3.1 Couple-stress theory

Couple-stress theory (also known as Koiter theory, constrained Cosserat theory, constrained rotation theory, constrained micropolar theory, and indeterminate couple-stress theory) was developed by Koiter, Mindlin, Tiersten, Toupin, and Truesdell in their work [56, 57, 78, 114, 115, 120] and used more recently by Fleck and Hutchinson [39]. Under this theory the rotational component of the general
deformation gradient is assigned to rigid directors that define the motion of the body placed in $\mathcal{M}$. The deformation of the directors is no longer independent of the deformation of the body, i.e., the directors are constrained.

For example, if a Timoshenko beam were to be considered a one-dimensional directed continuum body (specifically a micropolar or Cosserat body) then an Euler-Bernoulli beam might be thought of as being governed by the couple-stress theory. The kinematic state of a Timoshenko beam is defined by a vector field in Euclidean space that gives the shape of the beam and by a rigid director capable of rotation that defines the rotation of the cross-section of the beam. For the Euler-Bernoulli beam, a rigid but rotating director may still be used to represent the rotation of the cross-section. Of course the rotation of this director is given by the deformation of the beam itself. Specifically, it is the derivative of the displacement with respect to the horizontal position of the beam (if the displacement of the beam is given by the function $v(x)$, the rotation of the cross section is $dv(x)/dx$).

2.2.3.2 Higher-order strain gradient theory

In addition to using the rotation components of the deformation gradient of the body to define the rotation of the directors as in the couple-stress theory, one may also incorporate the stretch components of the deformation gradient as in the work of Mindlin, Eshel, Fleck, and Hutchinson [40, 76, 77]. Germain [46] points out that this theory is equivalent to a constrained micromorphic theory since the evolution of the body in $\mathcal{M}$ is identical to the deformation of the body placed in Euclidean space, $\mathcal{E}$. In terms of what has been discussed in Section 2.1.2, this means that $\mathcal{H}$ equals $H$. Based on Eqs. (2.11) and (2.14), it is apparent that $K_{(3)}$ is the gradient of the gradient of the displacement function for the body, hence the phrase second gradient. Since the work-conjugate of the second gradient of the displacement is often called the double stress, this theory is known as the double stress theory in addition to the titles of second (strain) gradient and higher-order (strain) gradient theories.
2.3 Uses of Directed Continuum Theories

There are a great variety of uses of directed continuum theories, many of which have motivated the work referenced in the previous two sections of this literature review. Although the following is not a comprehensive review of every possible use of such theories, it does represent some of the most common applications that drive research. Beginning with relatively large length scales, directed continuum theories have been used to model grid structures found in high-rise buildings [5] and granular or particulate materials [11, 80, 82, 94, 95]. A masonry wall, constructed with a regular placement of rigid body bricks, has also been modelled using Cosserat theory [73]. Brillard, Ganghoffer, and de Borst have used such methods to model the adhesive layer between two adherends [9]. It is also worth mentioning the early work of Schijve [107], who was an experimentalist trying to devise physical systems whose behavior could be modeled by directed continuum theories.

Moving to smaller length scales, directed continuum theories are used to model liquid crystals due to their inherently oriented structure [11, 66, 72]. Mariano [72] describes the use of directed continuum theories in the study of microcracks, materials with voids, multi-phase materials, and ferroelectric solids. Chambon et al. [15] use a second gradient model to study plasticity. Lakes is an experimentalist who uses directed continuum theories to study the behavior of porous media [60–62]. Trovalusci and Augusti [118] associate the displacement of the directors describing the evolution of the body in \( \mathbf{M} \) with the rigid displacement of fibers in a composite, while Lee and Stumpf [65] and Fleck and Hutchinson [39, 40] associate such an evolution with plastic deformation of a body. At a much smaller length scale Naghdi and Srinivasa [85, 86] consider the director field in terms of an atomic lattice structure.

2.4 Using Discrete Models to Formulate Continuum Models

The use of a discrete model to motivate the formulation of a continuum model is a common approach of researchers studying phenomena described by directed continuum theories. This concept was introduced in the work of Capriz and Podio-
Guidugli [12] and Masiani et al. [73]. The process of moving from a discrete model to a continuum model is not limited to directed continuum theories but is commonly used to formulate classical continuum models [10, 113] and second gradient theories [117], among others. The intention of this section of the literature review is to focus on research more closely related to the work to be presented in Sections 3.3 and 3.4.

Noor [91] describes different methods used in the analysis of large repetitive lattice structures. (The discrete model to be presented in Chapters 3 and 4 may be thought of as a repetitive lattice structure.) In the direct method, one simply takes the displacement terms for the entire discrete system and writes them in the general form

\[ f = Kx + a(x) + b, \]  

(2.38)

where the column vector \( x \) refers to the \( n \) discrete displacements, \( K \) is an \( n \times n \) matrix, \( a \) is the column vector of nonlinear terms (if there are any present), and \( b \) is a vector of constants. The vector \( f \) of length \( n \) will represent the forces or moments applied at each discrete point. If \( n \) is large, it is often beneficial to use a reduction method [90] to solve Eq. (2.38). The second method Noor describes is called the discrete field method. In this approach, the equilibrium equations are solved via a finite difference approach or are replaced with Taylor series expansions in terms of continuous displacement functions. In the substitute continuum approach, the discrete lattice is replaced with an effective continuum model. Noor requires an effective continuum model to possess five attributes: (1) equivalent energy stored in discrete and continuum models, (2) similar boundary conditions (the continuum boundary conditions must "simulate" the original boundary conditions), (3) the number of dimensions in both models must be the same, (4) local deformations must be accounted for in the continuum model, and (5) the continuum must either be described by a classical continuum model or a micropolar continuum model. Some of the language Noor uses to describe these approaches will be used in Chapter 3.
2.4.1 The Work of Triantafyllidis and Bardenhagen

Although the work of Triantafyllidis and Bardenhagen [117] does not result in a directed continuum theory of the type that will be developed in Chapter 3, their approach in finding a potential energy density functional is worth consideration. The one-dimensional discrete model used in their work is shown in Fig. 2.2. The

![Figure 2.2](image)

**Figure 2.2.** The unit cell used by Triantafyllidis and Bardenhagen [117] to model non-local phenomena with a second gradient theory. Only deformations in the horizontal direction are considered, i.e., this is a one-dimensional problem.

Authors write two different discrete equations based on the discrete kinematics of the unit cell they define. The first of these equations is a discrete form of the equilibrium equation and the second is a discrete form of the potential energy density. For both equations, they view the discrete displacements in terms of a Taylor series expansion with respect to the horizontal position, $x$. They then replace the discrete displacement terms found in the two discrete equations described above with the homogenized displacement functions. This gives them a continuous Euler-Lagrange equation in the first case and a continuous form of the potential energy density in the second case. After observing the resulting Euler-Lagrange equation from the first case, they define a potential energy density functional that would yield the identical Euler-Lagrange equation by standard variational principles. This leaves the authors with two continuous forms of the total potential energy density for their model and they note that these are not the same, although they both yield the same BVP.
2.4.2 The Work of Suiker et al.

Suiker et al. [112] begin their study into dispersion relations by establishing the kinematics of a discrete model that will be used to formulate their directed continuum. Their unit cell is a two-dimensional object consisting of nodes interacting with 6 or 8 neighboring nodes (they present two models) in a regular pattern through three different springs: a linear spring, a shear spring, and a torsional spring. The unit cell for the 7-cell hexagonal lattice model is shown in Fig. 2.3. The discrete form of the potential energy is found by adding together the contributions of each of the springs, in a similar manner to what will be presented in Section 3.1.1. For example, if the center node shown in Fig. 2.3 is the \(i\)-th node, Suiker et al. would write a discrete form of the energy based on the interaction of the six surrounding nodes with the center node. For an example, just consider the interaction of the \(i\)-th node with the node horizontally to the right of the center node, which will be called the \(i+1\)-th node. The potential energy associated with the springs connecting just these two nodes is given by

\[
U = U_{\text{linear}} + U_{\text{shear}} + U_{\text{torsional}},
\]

(2.39)

where

\[
U_{\text{linear}} = \frac{1}{2}k_{\text{linear}}(u_{i+1} - u_i)^2,
\]

(2.40)

\[
U_{\text{shear}} = \frac{1}{2}k_{\text{shear}}[v_{i+1} - v_i + l(\phi_{i+1} + \phi_i)/2]^2,
\]

(2.41)

\[
U_{\text{torsional}} = \frac{1}{2}k_{\text{torsional}}(\phi_{i+1} - \phi_i)^2.
\]

(2.42)
There are three displacements shown here for each node (node $i$ and node $i+1$): $u$ is the horizontal displacement, $v$ is the vertical displacement, and $\phi$ is the rotational displacement. The length between the two nodes in the undeformed state is given by $l$, while the various spring constants are given by the $k$-terms. Of course it would be necessary to add the contributions of the remaining five nodes to this to obtain the total discrete potential energy associated with the $i$-th node.

After obtaining the discrete form of the total potential energy, Suiker et al. calculate Lagrange’s equations in terms of $u_i$, $v_i$, and $\phi_i$. As just described with the work of Triantafyllidis and Bardenhagen [117], the discrete displacements at a node are then viewed as the order zero terms in a Taylor series expansion of a corresponding, continuous displacement function. For example, where $x$ is the coordinate in the horizontal direction and letting

$$u_i = u(x_i), \quad (2.43)$$

the horizontal displacement terms appear as*

$$u_{i+1} = u(x_i) + \frac{l}{2} \frac{d^2 u(x)}{dx^2} \bigg|_{x=x_i}, \quad (2.44)$$

By replacing $u_i$ and $u_{i+1}$ in the discrete form of Lagrange’s equations with the expressions in Eqs. (2.43) and (2.44), Suiker et al. obtain a set of partial differential equations in the continuum displacement fields, which we call the “homogenized” forms of the equations of motion. They compare these equations of motion with the equations of motion of a Cosserat (or micropolar) material, recalling that the Cosserat theory includes rigid directors capable of rotation to model the motion of the body placed in $M$. The governing equations for a Cosserat continuum are derived from Eqs. (2.21) and (2.22) and are given in many places (see [29]). They appear in [112] as follows:

$$\rho \ddot{u} = (\lambda + 2\mu)u_{,xx} + (\lambda + \mu - \kappa/2)v_{,xy} + (\mu + \kappa/2)u_{,yy} - \kappa \phi_{,y}, \quad (2.45)$$

$$\rho \ddot{v} = (\lambda + 2\mu)v_{,yy} + (\lambda + \mu - \kappa/2)u_{,xy} + (\mu + \kappa/2)v_{,xx} + \kappa \phi_{,x}, \quad (2.46)$$

$$J \ddot{\phi} = 2\gamma(\phi_{,xx} + \phi_{,yy}) - 2\kappa \phi + \kappa(u_{,y} - v_{,x}), \quad (2.47)$$

*See Eqs. (3.60)–(3.67) where these relationships appear again.
where the subscripts signify differentiation of the continuous functions with respect
to horizontal, \( x \), and vertical, \( y \), directions. The constants \( \lambda, \mu, \kappa, \) and \( \gamma \) are
constitutive coefficients (\( \lambda \) and \( \mu \) are the Lamé constants), the density is given by
\( \rho \), and the moment of inertia density is given by \( J \). The double dot indicates the
second derivative with respect to time. Vasiliev and Miroshnichenko [121] have
also used this technique in the analysis of multi-lattice structures.

\section*{2.4.3 Other Work of Interest}

In the work of Bažant [5] and Bažant and Christensen [6] the discrete system is
made up of a two-dimensional grid of Euler-Bernoulli beams. The goal of the work
is to develop a directed continuum theory to model high-rise building frames. As
in the work most recently discussed [112,121], the authors begin by writing out the
discrete form of the total potential energy and applying Taylor series expansions of
the discrete displacements in terms of the continuous displacements to generate a
potential energy functional. Applying standard variational mechanics techniques,
the potential energy is used to formulate the complete BVP that consists of three
partial differential equations and the necessary boundary conditions. The authors
make use of the work-conjugate relationships between the stress terms (normal,
shear, and couple) and the strain terms. The finite difference method is used to
obtain displacements as solutions to various loading conditions. Askar and Cak-
mak [4] used a similar approach, modeling a 2-D lattice of flexible and extensible
rods, to obtain a Cosserat continuum.

Mühlhaus et al. have also used Euler-Bernoulli beam discrete elements to gener-
ate a Cosserat continuum to model masonry columns [81]. In this work they apply
Taylor series expansions of the discrete displacements in terms of the continuous
displacement functions to discrete equations of motion in a one-dimensional
problem to yield two partial differential equations (they are partial differential
equations because of the presence of time). The authors use Fourier transforms
to determine dispersion relations for their model. In the work of Pasternak and
Mühlhaus [94], a similar procedure is followed to model a one-dimensional chain
of spherical grains attached together by linear and torsional springs. In addition
to finding a functional form of the potential energy as has already been described,
they also use an integral transformation to find an explicitly non-local Cosserat continuum version of their discrete model.

De Borst et al. [27–29] begin with the governing equations for a Cosserat continuum and discretize the model using the finite element method. They are interested in problems that exhibit significant mesh-dependence, such that finite element analysis based on classical continuum theory is of limited use. De Borst and Mühlhaus [28] present a $7 \times 7$ element stiffness matrix to relate the three normal stress, two shear stress, and two couple-stress terms to the three normal strain, two shear strain, and two microcurvature terms that are associated with Cosserat or micropolar theories in two-dimensions. Rubin [104] and Cerrolaza et al. [14] suggest ways of incorporating finite element techniques to solve nonlinear Cosserat-type problems.

Since micro-buckling problems were mentioned as possible systems to be modeled in Chapter 1, it is worth mentioning a few uses of directed continuum theories in this field of study. In one approach, a nonlinear analysis using the ideas of a Cosserat point is used to study the buckling of beams and arches [84, 106]. In an earlier work, Papamichos et al. [93] formulate a Cosserat continuum-based on a multi-layer structure made up of alternating compressible, isotropic, hypoelastic materials. Buckling loads are calculated for the Cosserat model and compared with classical results.

Modeling procedures that involve moving from a discrete model to a continuum system have been discussed. The process of going from a discrete system to a continuum is a type of homogenization. More formal homogenization methods—falling under the categories of asymptotic expansion methods, representative volume element (RVE) methods, and micro-macro homogenization methods—have been used with directed continuum theories. The asymptotic expansion approach [8, 42, 43, 127] is the most common technique used with directed continuum theories and is often used in the formulation of multi-scale models [18, 19]. Forest [41] uses the RVE approach in developing models consistent with second gradient, micropolar, and micromorphic theories. Finally, a micro-macro homogenization method (also called global-local analysis or multi-scale) is described [58] as a two part procedure more suited for large deformations and more complex placements of the body $\mathcal{B}$ into $\mathcal{M}$. The first part consists of determining constitutive relations at
a macroscopic point, often using finite element analysis. In the second part the constitute properties at the macroscale are obtained via some averaging process, e.g., this may actually require the use of asymptotic expansion methods. This technique does not yield macroscopic constitutive relations. Kouznetsova, Geers, and Brekelmans [44, 59] have used this approach with second gradient theories.
Chapter 3

Modeling a Columnar Thin Film

This chapter consists of four parts. Section 3.1 describes the creation of a discrete model based on the physical structure of a CTF. This model is assembled using five different types of springs, which may be thought of as modeling the thin film substrate, the interaction between the columns, and the interaction between the columns and the substrate of a CTF. The discrete model will then be used to obtain a system of governing algebraic equations, i.e., the discrete Lagrange’s equations. The details of a 4-spring, discrete model will be outlined in Section 3.2. In this case, horizontal and vertical components of deflection are accounted for in a single spring compared with the discrete model of Section 3.1, hence there is one less spring in the model*. In Section 3.3, a linear continuous model will be formulated based on the linear discrete model. This model will be referred to as a discrete-based continuous model. Section 3.4 will identify a form of an energy density for a continuum-based model. The governing equations obtained in Section 3.3 will also be derived as the Euler-Lagrange equations corresponding to a Lagrangian based on the energy density identified. Although this development may appear redundant, it allows for the formulation of corresponding boundary conditions. The governing equations and boundary conditions that result from this strain energy constitute the continuum BVP to be solved. Much of the formulation in Sections 3.1 and 3.4 has been described by Randow et al. [97].

*When standard assumptions of small deformations are applied to the 5-spring model, the model becomes linear. When the same assumptions are applied to the 4-spring model, the model remains nonlinear.
3.1 5-Spring Discrete Model

Discrete models are used to represent a variety of physical systems over a range of length scales. Since discrete models will be used as the building blocks of a larger structure, it is necessary to first designate some sort of representative element or unit cell. Each of the representative elements possesses both constitutive and kinematic properties. When the elements are assembled together, the material properties of the complete model are determined by the element properties and the deformation of the complete model is determined by the deformations of each of the elements. The choices made at the unit cell level will clearly impact the final continuous, or homogenized, version of the model.

3.1.1 The Discrete Form of the Strain Energy

In this work the form of the discrete model is motivated by observations of the structure of a CTF and physical intuition regarding how the columnar structure will interact with the substrate, refer to Fig. 3.1. The columns will be modeled as individual rigid bars while the substrate will be modeled as a continuous medium (although even this is discretized when the unit cell is defined). This leads to the following assumptions and simplifications:

(1) this model only considers the profile of the thin film, i.e., the displacements and rotations of the model are defined in one dimension only along the length;

(2) the substrate is viewed as a collection of Euler-Bernoulli beams each of length \( l \); and

(3) each column is assumed to be a rigid bar connected to the substrate through a linear torsional spring and connected to neighboring columns through a number of linear springs pairs.

This last assumption, regarding the use of spring pairs connecting the inextensible rods together, is the reason the model presented in this section is referred to as a 5-spring model, i.e., two springs are associated with the substrate, one torsional spring, and two springs that constitute each spring pair. The horizontal and
Figure 3.1. The discrete model of the thin film showing the labeling and the type of interaction between the elements. Note the dimensions given by $h$ and $l$. There are $n$ spring pairs, separated by a distance of $h/n$, connecting each of the rods together. The spring constants of all of the vertical springs in the spring pairs are given by $k_1$ and the spring constants of all of the horizontal springs are given by $k_2$. The torsional springs connecting the inextensible rods to the substrate are given by $k_3$. The substrate is described by two spring constants: $k_4$ is associated with axial deformation and $k_5$ is associated with bending.

Vertical deflections between points along the rigid columns are decoupled and accounted for by a horizontal and a vertical spring independently. The composition of the thin film in terms of the various springs and continuous elements is shown in Fig. 3.1, where five nodes are labeled $i-2$, $i-1$, $i$, $i+1$, and $i+2$. Figure 3.2 shows the unit cell for this model, at an arbitrary node $i$; it also shows the left and right ends of the discrete model, i.e., $i = 0$ and $i = m$, respectively. The unit cell is centered at node $i$ and consists of one discrete torsional spring associated with the column included in the unit cell as well as one-half of each of the neighboring elements. The allowable displacements for the $i$-th node are shown in Fig. 3.3. Therefore, in order to describe the $i$-th unit cell, it is necessary to define twelve discrete displacements: $u_{i-1}$, $u_i$, $u_{i+1}$, $v_{i-1}$, $v_i$, $v_{i+1}$, $\phi_{i-1}$, $\phi_i$, $\phi_{i+1}$, $\beta_{i-1}$, $\beta_i$, and $\beta_{i+1}$.

Now that the kinematics have been defined, it is possible to calculate a potential energy associated with the discrete unit cell by adding together the contributions
Figure 3.2. The unit cell of this structure is shown within the dashed box. It will consist of an entire torsional spring, with spring constant $k_3$. In addition, it will contain one-half of each neighboring substrate section (described by $k_4$ and $k_5$), linear horizontal springs ($k_2$), and linear vertical springs ($k_1$), see Fig. 3.1. Also, the left end (at $i = 0$) and the right end (at $i = m$) of the discrete model are shown. It follows that this model consists of $m + 1$ nodes, i.e., $i = 0, 1, \ldots, m - 1, m$.

Figure 3.3. There are four displacements describing the deformation of the $i$-th node from the initial position to the deformed position: displacement in the horizontal direction, $u_i$; displacement in the vertical direction, $v_i$; rotation of the substrate, $\phi_i$; and rotation of the column relative to the substrate, $\beta_i$.

of each of the elements,

$$U_{\text{discrete}} = U_0 + \sum_{i=1}^{m-1} \left[ \frac{1}{2} (U_1)_i + \frac{1}{2} (U_2)_i + (U_3)_i + \frac{1}{2} (U_4)_i + \frac{1}{2} (U_5)_i \right] + U_m, \quad (3.1)$$
where \( U_1 \) and \( U_2 \) denote the energy associated with the vertical and horizontal linear springs described by spring constants \( k_1 \) and \( k_2 \), \( U_3 \) is associated with the torsional spring given by \( k_3 \), and \( U_4 \) and \( U_5 \) are associated with the substrate spring constants, \( k_4 \) and \( k_5 \). As suggested by Fig. 3.2, there are \( m + 1 \) nodes; nodes \( i = 1 \) to \( m - 1 \) are accounted for by the summation. The energy of the left-hand half cell is denoted by \( U_0 \) and the energy of the right-hand half cell is denoted by \( U_m \).

The one-half multiples in Eq. (3.1) are present since the unit cell includes one-half of the \( k_1, k_2, k_4, \) and \( k_5 \) springs. What follows is a description of each of the terms appearing in Eq. (3.1).

The vertical and horizontal springs, comprising the spring pairs that model the interaction of the columns, are denoted by \( k_1 \) and \( k_2 \), respectively. The influence of these springs may be due to the effect of long-range interactions between the columns. From a more abstract perspective, these springs might be used to model other material properties, e.g., the material properties of a layer in a bi-material film. The corresponding strain energy terms for the \( i \)-th node will be called \((U_1)_i\) and \((U_2)_i\), such that

\[
(U_1)_i = \frac{1}{2} k_1 \sum_{j=1}^{n} \left( \{v_i - v_{i-1} - \lambda_j h [\cos(\phi_{i-1} + \beta_{i-1}) - \cos(\phi_i + \beta_i)]\}^2 
+ \{v_{i+1} - v_i - \lambda_j h [\cos(\phi_i + \beta_i) - \cos(\phi_{i+1} + \beta_{i+1})]\}^2 \right), \tag{3.2}
\]

\[
(U_2)_i = \frac{1}{2} k_2 \sum_{j=1}^{n} \left( \{u_i - u_{i-1} + \lambda_j h [\sin(\phi_{i-1} + \beta_{i-1}) - \sin(\phi_i + \beta_i)]\}^2 
+ \{u_{i+1} - u_i + \lambda_j h [\sin(\phi_i + \beta_i) - \sin(\phi_{i+1} + \beta_{i+1})]\}^2 \right), \tag{3.3}
\]

where

\[
\lambda_j \equiv \frac{j}{n}. \tag{3.4}
\]

Using the notation first introduced in Fig. 3.1, there are \( n \) spring pairs separated by the distance \( h/n \).

Next, the energy due to the torsional spring located at the \( i \)-th node and described by spring constant \( k_3 \) is given by

\[
(U_3)_i = \frac{1}{2} k_3 \beta_i^2. \tag{3.5}
\]
When $\beta_i$ equals zero, the column is perpendicular to the substrate. In this case, there is no twisting of the torsional spring and no contribution from the torsional spring to the total strain energy. The effects of the column on the substrate are transmitted to the substrate through the attachment of the rigid columns to the substrate and via the torsional springs.

The substrate is modeled as a classical Euler-Bernoulli beam (in terms of bending and shear) coupled with a classic rod (in terms of axial loads). Consequently, the longitudinal deformation is modeled with a standard linear spring with spring constant $k_4$, such that, using previously established notation,

$$ (U_4)_i = \frac{1}{2} k_4 [(u_i - u_{i-1})^2 + (u_{i+1} - u_i)^2]. \quad (3.6) $$

The $k_4$ term of the energy function given in Eq. (3.6) is equivalent to an elastic bar subjected to tension or compression when $k_4$ is taken to equal $EA/l$, where $EA$ is the equivalent stiffness of the rod. Finally, the contribution to the strain energy due to bending will be a function of displacements $v_{i-1}, v_i, v_{i+1}, \phi_{i-1}, \phi_i, \phi_{i+1}$ and material constant $k_5$, such that

$$ (U_5)_i = \frac{1}{2} k_5 \left\{ \left[ 3(v_i - v_{i-1})^2 - 3l(\phi_{i-1} + \phi_i)(v_i - v_{i-1}) + l^2(\phi_{i-1}^2 + \phi_i^2 + \phi_{i+1}^2) \right] + \left[ 3(v_{i+1} - v_i)^2 - 3l(\phi_i + \phi_{i+1})(v_{i+1} - v_i) + l^2(\phi_i^2 + \phi_i\phi_{i+1} + \phi_{i+1}^2) \right] \right\}. \quad (3.7) $$

The $k_5$ term of the energy function given in Eq. (3.7) is equivalent to that for an Euler-Bernoulli beam when $k_5$ equals $4EI/l^3$, where $EI$ is the bending stiffness of the beam and $l$ is the beam length. It is worth mentioning the other approaches discussed in Section 2.4 and how these other approaches compare with Eq. (3.7). When $\phi_i = \phi_{i+1}$ and when the material constants are equated, the potential found in [112,121] is equivalent to the substrate potential of the present work. Although a beam is continuous, here it is treated in a discrete form so that it may be combined with the other discrete components of the model.
The final terms introduced in Eq. (3.1) are $U_0$ and $U_m$, the energies associated with nodes $i = 0$ and $i = m$, see Fig. 3.2. They are given by

\[
U_0 = \frac{1}{2} k_1 \sum_{j=1}^{n} \left\{ v_1 - v_0 - \lambda_j h [\cos(\phi_0 + \beta_0) - \cos(\phi_1 + \beta_1)] \right\}^2 \\
+ \frac{1}{2} k_2 \sum_{j=1}^{n} \left\{ u_1 - u_0 + \lambda_j h [\sin(\phi_0 + \beta_0) - \sin(\phi_1 + \beta_1)] \right\}^2 \\
+ \frac{1}{2} k_3 \beta_0^2 + \frac{1}{2} k_4 (u_1 - u_0)^2 \\
+ \frac{1}{2} k_5 \left[ 3(v_1 - v_0)^2 - 3l(\phi_0 + \phi_1)(v_1 - v_0) + l^2(\phi_0^2 + \phi_0\phi_1 + \phi_1^2) \right],
\]

(3.8)

\[
U_m = \frac{1}{2} k_1 \sum_{j=1}^{n} \left\{ v_m - v_{m-1} - \lambda_j h [\cos(\phi_{m-1} + \beta_{m-1}) - \cos(\phi_m + \beta_m)] \right\}^2 \\
+ \frac{1}{2} k_2 \sum_{j=1}^{n} \left\{ u_m - u_{m-1} + \lambda_j h [\sin(\phi_{m-1} + \beta_{m-1}) - \sin(\phi_m + \beta_m)] \right\}^2 \\
+ \frac{1}{2} k_3 \beta_m^2 + \frac{1}{2} k_4 (u_m - u_{m-1})^2 + \frac{1}{2} k_5 \left[ 3(v_m - v_{m-1})^2 \\
- 3l(\phi_{m-1} + \phi_m)(v_m - v_{m-1}) + l^2(\phi_{m-1}^2 + \phi_{m-1}\phi_m + \phi_m^2) \right].
\]

(3.9)

The total strain energy of the discrete model is obtained by applying Eqs. (3.2)–(3.9) to Eq. (3.1).

Unit cells in this work are connected in a chain; unit cells in [5, 6, 112, 121] are connected in a lattice arrangement. That is, the unit cell is repeated only in the horizontal direction in this work while in the other cited work the unit cell is repeated in both horizontal and vertical directions. On the other hand, in the other references there are only three degrees of freedom at each node while in this work there are four. The most innovative aspect of the present model is the inclusion of a more complex structure in the discrete model, including rigid bars and various spring combinations joining the system together.

### 3.1.2 The Discrete Form of Lagrange’s Equations

There are a total of $4(m + 1)$ discrete displacements described by the model introduced in Section 3.1.1; there are $m + 1$ nodes and, at each node, there are four displacements, i.e., $u_i$, $v_i$, $\phi_i$, and $\beta_i$. In the static case, it is possible to obtain
Lagrange’s equations by simply taking derivatives of $U_{\text{discrete}}$, given by Eq. (3.1), with respect to each of the discrete displacements. The resulting relations are set equal to zero if there are no external loads applied. Observe that for each displacement, the form of the resulting relation is identical for all of the interior nodes $(i = 1$ to $m - 1)$, i.e.,

\[
\frac{\partial U_{\text{discrete}}}{\partial u_i} = \frac{\partial (U_2)_i}{\partial u_i} + \frac{\partial (U_4)_i}{\partial u_i} = 0, \quad i = 1, 2, \ldots, m - 1, \quad (3.10)
\]

\[
\frac{\partial U_{\text{discrete}}}{\partial v_i} = \frac{\partial (U_1)_i}{\partial v_i} + \frac{\partial (U_5)_i}{\partial v_i} = 0, \quad i = 1, 2, \ldots, m - 1, \quad (3.11)
\]

\[
\frac{\partial U_{\text{discrete}}}{\partial \phi_i} = \frac{\partial (U_2)_i}{\partial \phi_i} + \frac{\partial (U_5)_i}{\partial \phi_i} = 0, \quad i = 1, 2, \ldots, m - 1, \quad (3.12)
\]

\[
\frac{\partial U_{\text{discrete}}}{\partial \beta_i} = \frac{\partial (U_2)_i}{\partial \beta_i} + \frac{\partial (U_3)_i}{\partial \beta_i} = 0, \quad i = 1, 2, \ldots, m - 1. \quad (3.13)
\]

The discrete Lagrange’s equations for the end nodes are

\[
\frac{\partial U_{\text{discrete}}}{\partial u_0} = \frac{\partial U_0}{\partial u_0}, \quad \frac{\partial U_{\text{discrete}}}{\partial v_0} = \frac{\partial U_0}{\partial v_0}, \quad \frac{\partial U_{\text{discrete}}}{\partial \phi_0} = \frac{\partial U_0}{\partial \phi_0}, \quad \frac{\partial U_{\text{discrete}}}{\partial \beta_0} = \frac{\partial U_0}{\partial \beta_0}, \quad (3.14)
\]

\[
\frac{\partial U_{\text{discrete}}}{\partial u_m} = \frac{\partial U_m}{\partial u_m}, \quad \frac{\partial U_{\text{discrete}}}{\partial v_m} = \frac{\partial U_m}{\partial v_m}, \quad \frac{\partial U_{\text{discrete}}}{\partial \phi_m} = \frac{\partial U_m}{\partial \phi_m}, \quad \frac{\partial U_{\text{discrete}}}{\partial \beta_m} = \frac{\partial U_m}{\partial \beta_m}. \quad (3.15)
\]

In Eqs. (3.14) and (3.15), the right-hand side of each relation is viewed as prescribed. As such, Eqs. (3.14) and (3.15) may be thought of as boundary conditions and will be used in Section 3.1.3. Continuing with the formulation of the governing equations, applying Eqs. (3.2)–(3.7) to Eqs. (3.10)–(3.13) yields

\[
0 = k_2 \sum_{j=1}^{n} \{ -u_{i-1} + 2u_i - u_{i+1} + \lambda_j h[\sin(\phi_{i-1} + \beta_{i-1}) - 2\sin(\phi_i + \beta_i) + \sin(\phi_{i+1} + \beta_{i+1})] \} + k_4 (-u_{i-1} + 2u_i - u_{i+1}), \quad (3.16)
\]

\[
0 = k_1 \sum_{j=1}^{n} \{ -v_{i-1} + 2v_i - v_{i+1} - \lambda_j h[\cos(\phi_{i-1} + \beta_{i-1}) - 2\cos(\phi_i + \beta_i) + \cos(\phi_{i+1} + \beta_{i+1})] \} - \frac{3k_5}{2} [2(v_{i-1} - 2v_i + v_{i+1}) + l(\phi_{i-1} - \phi_{i+1})], \quad (3.17)
\]
0 = k_1 \sin(\phi_i + \beta_i) \sum_{j=1}^{n} \left\{ \lambda_j h (v_{i-1} - 2v_i + v_{i+1}) + (\lambda_j h)^2 \cos(\phi_i + \beta_i) + \cos(\phi_{i+1} + \beta_{i+1}) \right\} \\
+ 2 \cos(\phi_i + \beta_i) + \cos(\phi_{i+1} + \beta_{i+1}) \right\} + k_2 \sum_{j=1}^{n} \left\{ \lambda_j h (u_{i-1} - 2u_i + u_{i+1}) - (\lambda_j h)^2 \sin(\phi_i + \beta_i) \right. \\
- 2 \sin(\phi_i + \beta_i) + \sin(\phi_{i+1} + \beta_{i+1}) \right\} \right\} + k_3 \beta_i, 
\tag{3.18}

which are applied at each of the \( m - 1 \) interior nodes. When \( \lambda_j \) (see Eq. (3.4)) is introduced into Eqs. (3.16)-(3.19), these equations are seen to be convergent series of the forms

\[ \sum_{j=1}^{n} \alpha = n \alpha, \quad \sum_{j=1}^{n} \lambda_j \alpha = \left( \frac{1+n}{2} \right) \alpha, \quad \sum_{j=1}^{n} \lambda_j^2 \alpha = \left( \frac{1+3n+2n^2}{6n} \right) \alpha, \tag{3.20} \]

for any constant \( \alpha \). After applying Eqs. (3.20) to Eqs. (3.16)-(3.19), the final system of Lagrange’s equations for \( n \) spring pairs connecting the inextensible columns is

\[ 0 = k_2 \left\{ n (-u_{i-1} + 2u_i - u_{i+1}) + \left( \frac{1+n}{2} \right) h \sin(\phi_{i-1} + \beta_{i-1}) \right. \\
- 2 \sin(\phi_i + \beta_i) + \sin(\phi_{i+1} + \beta_{i+1}) \right\} + k_4 (-u_{i-1} + 2u_i - u_{i+1}), \tag{3.21} \]

\[ 0 = k_1 \left\{ n (-v_{i-1} + 2v_i - v_{i+1}) - \left( \frac{1+n}{2} \right) h \cos(\phi_{i-1} + \beta_{i-1}) - 2 \cos(\phi_i + \beta_i) \right. \]

\[ + \cos(\phi_{i+1} + \beta_{i+1}) \right\} - \frac{3k_5}{2} \left[ 2(v_{i-1} - 2v_i + 2_i+1) + l(\phi_{i-1} - \phi_{i+1}) \right], \tag{3.22} \]
\[ 0 = k_1 \sin(\phi_i + \beta_i) \left\{ \left( \frac{1 + n}{2} \right) h(v_{i-1} - 2v_i + v_{i+1}) \right. \]
\[ + \left. \left( \frac{1 + 3n + 2n^2}{6n} \right) h^2 \left[ \cos(\phi_{i-1} + \beta_{i-1}) - 2 \cos(\phi_i + \beta_i) + \cos(\phi_{i+1} + \beta_{i+1}) \right] \right\} \]
\[ + k_2 \cos(\phi_i + \beta_i) \left\{ \left( \frac{1 + n}{2} \right) h(u_{i-1} - 2u_i + u_{i+1}) \right. \]
\[ - \left. \left( \frac{1 + 3n + 2n^2}{6n} \right) h^2 \left[ \sin(\phi_{i-1} + \beta_{i-1}) - 2 \sin(\phi_i + \beta_i) + \sin(\phi_{i+1} + \beta_{i+1}) \right] \right\} \]
\[ + k_3 \beta_i. \]

Notice that if \( n = 1 \), all of the terms containing \( n \) in Eqs. (3.21)–(3.24), i.e., \( n \), \((1+n)/2\), and \((1+3n+2n^2)/6n\), equal one. It is also possible to consider the case with an infinite number of spring pairs between each inextensible rod by letting \( n \) go to \( \infty \). It is necessary to scale the spring constants \( k_1 \) and \( k_2 \) by \( n \), so that as \( n \to \infty \) the overall stiffness of the system does not become infinite. Therefore, the following will be introduced into the Lagrange’s equations:

\[ k_1 = \hat{k}_1/n, \quad k_2 = \hat{k}_2/n. \]  

(3.25)

For example, apply Eqs. (3.25) to Eq. (3.21) to obtain

\[ 0 = \hat{k}_2 \left\{ -u_{i-1} + 2u_i - u_{i+1} + \left( \frac{1 + n}{2n} \right) h \left[ \sin(\phi_{i-1} + \beta_{i-1}) \right. \right. \]
\[ - \left. \left. 2 \sin(\phi_i + \beta_i) + \sin(\phi_{i+1} + \beta_{i+1}) \right] \right\} + k_4(-u_{i-1} + 2u_i - u_{i+1}), \]  

(3.26)
such that the limit as $n \to \infty$ yields

$$0 = \hat{k}_2 \left\{ -u_{i-1} + 2u_i - u_{i+1} + \frac{h}{2} \left[ \sin(\phi_{i-1} + \beta_{i-1}) - 2 \sin(\phi_i + \beta_i) + \sin(\phi_{i+1} + \beta_{i+1}) \right] \right\} + k_4 (-u_{i-1} + 2u_i - u_{i+1}).$$

(3.27)

Following a similar procedure with Eqs. (3.22)–(3.24) yields

$$0 = \hat{k}_1 \left\{ -v_{i-1} + 2v_i - v_{i+1} + \frac{h}{2} \left[ \cos(\phi_{i-1} + \beta_{i-1}) - 2 \cos(\phi_i + \beta_i) + \cos(\phi_{i+1} + \beta_{i+1}) \right] \right\} - \frac{3k_5}{2} \left[ 2(v_{i-1} - 2v_i + v_{i+1}) + l(\phi_{i-1} - \phi_{i+1}) \right],$$

(3.28)

$$0 = \hat{k}_1 \sin(\phi_i + \beta_i) \left\{ \frac{h}{2} (v_{i-1} - 2v_i + v_{i+1}) + \frac{h^2}{3} \left[ \cos(\phi_{i-1} + \beta_{i-1}) - 2 \cos(\phi_i + \beta_i) + \cos(\phi_{i+1} + \beta_{i+1}) \right] \right\}$$

$$+ \frac{h}{2} (u_{i-1} - 2u_i + u_{i+1}) - \frac{h^2}{3} \left[ \sin(\phi_{i-1} + \beta_{i-1}) - 2 \sin(\phi_i + \beta_i) + \sin(\phi_{i+1} + \beta_{i+1}) \right]$$

$$+ k_3 \beta_i \right\} + \left\{ \frac{h}{2} (v_{i-1} - 2v_i + v_{i+1}) + \frac{h^2}{3} \left[ \cos(\phi_{i-1} + \beta_{i-1}) - 2 \cos(\phi_i + \beta_i) + \cos(\phi_{i+1} + \beta_{i+1}) \right] \right\}$$

$$+ \frac{h}{2} (u_{i-1} - 2u_i + u_{i+1}) - \frac{h^2}{3} \left[ \sin(\phi_{i-1} + \beta_{i-1}) - 2 \sin(\phi_i + \beta_i) + \sin(\phi_{i+1} + \beta_{i+1}) \right]$$

$$+ k_3 \beta_i \right\} + k_3 \beta_i.$$  

(3.29)

(3.30)

To conclude this section, Eqs. (3.21)–(3.24) are the four Lagrange’s equations for each of the $m-1$ interior nodes of the model with $n$ spring pairs connecting the rigid rods together. Equations (3.27)–(3.30) are the equivalent equations for the case with an infinite number of spring pairs with the spring constants scaled according to Eqs. (3.25).
3.1.3 The Discrete Simply Supported Film

Either Eqs. (3.21)–(3.24) or (3.27)–(3.30) lead to \(4(m - 1)\) linear equations that must be solved for the \(4(m - 1)\) interior displacement terms. These equations are all set equal to zero, since there are no external body forces or moments applied to the system. In order to actually solve a problem, it is necessary to supply boundary conditions, which are taken from Eqs. (3.14) and (3.15). The following boundary conditions of a simply supported CTF will be applied at the end nodes (at \(i = 0\) and \(m\)), see Fig. 3.4:

\[
\begin{align*}
    u_0 &= 0, \quad v_0 = 0, \quad \frac{\partial U_0}{\partial \phi_0} = 0, \quad \frac{\partial U_0}{\partial \beta_0} = 0, \\
u_m &= -l\delta_u, \quad v_m = 0, \quad \frac{\partial U_m}{\partial \phi_m} = 0, \quad \frac{\partial U_m}{\partial \beta_m} = 0.
\end{align*}
\] (3.31) (3.32)

The displacements \(u_0\), \(u_m\), \(v_0\), and \(v_m\) are specified according to Eqs. (3.31) and (3.32). In addition, the derivative relationships in these equations signify that there are no moments applied to the substrate or to the columns at the ends. Applying Eqs. (3.8) and (3.9) to the boundary conditions involving derivatives

![Figure 3.4](image)

Figure 3.4. A simply supported discrete thin film subjected to a horizontal displacement \(l\delta_u\) at the right-hand end. (The length \(l\) is the separation distance between the undeformed columns and \(\delta_u\) is a dimensionless parameter that defines the horizontal deformation.) The boundary conditions given by Eqs. (3.31) and (3.32) correspond with this figure.

with respect to \(\phi_0\), \(\beta_0\), \(\phi_m\), and \(\beta_m\) from Eqs. (3.31) and (3.32) yields

\[
0 = -k_1 \sin(\phi_0 + \beta_0) \sum_{j=1}^{n} \left\{ \lambda_j h(v_0 - v_1) + (\lambda_j h)^2 \left[ \cos(\phi_0 + \beta_0) - \cos(\phi_1 + \beta_1) \right] \right\} + k_2 \cos(\phi_0 + \beta_0) \sum_{j=1}^{n} \left\{ \lambda_j h(-u_0 + u_1) + (\lambda_j h)^2 \left[ \sin(\phi_0 + \beta_0) - \sin(\phi_1 + \beta_1) \right] \right\}
\]
\[ + \frac{k_5}{2} \left[ 3l(v_0 - v_1) + l^2(2\phi_0 + \phi_1) \right], \quad (3.33) \]

\[ 0 = -k_1 \sin(\phi_0 + \beta_0) \sum_{j=1}^{n} \left\{ \lambda_j h(v_0 - v_1) + (\lambda_j h)^2 [\cos(\phi_0 + \beta_0) - \cos(\phi_1 + \beta_1)] \right\} \]

\[ + k_2 \cos(\phi_0 + \beta_0) \sum_{j=1}^{n} \left\{ \lambda_j h(-u_0 + u_1) + (\lambda_j h)^2 [\sin(\phi_0 + \beta_0) - \sin(\phi_1 + \beta_1)] \right\} \]

\[ + k_3 \beta_0, \quad (3.34) \]

\[ 0 = k_1 \sin(\phi_m + \beta_m) \sum_{j=1}^{n} \left\{ \lambda_j h(v_{m-1} - v_m) + (\lambda_j h)^2 [\cos(\phi_{m-1} + \beta_{m-1}) \right\} \]

\[ - \cos(\phi_m + \beta_m)] \right\} - k_2 \cos(\phi_m + \beta_m) \sum_{j=1}^{n} \left\{ \lambda_j h(-u_{m-1} + u_m) \right\} \]

\[ + (\lambda_j h)^2 [\sin(\phi_{m-1} + \beta_{m-1}) - \sin(\phi_m + \beta_m)] \right\} \]

\[ + \frac{k_5}{2} \left[ 3l(v_{m-1} - v_m) + l^2(\phi_{m-1} + 2\phi_m) \right], \quad (3.35) \]

\[ 0 = k_1 \sin(\phi_m + \beta_m) \sum_{j=1}^{n} \left\{ \lambda_j h(v_{m-1} - v_m) + (\lambda_j h)^2 [\cos(\phi_{m-1} + \beta_{m-1}) \right\} \]

\[ - \cos(\phi_m + \beta_m)] \right\} - k_2 \cos(\phi_m + \beta_m) \sum_{j=1}^{n} \left\{ \lambda_j h(-u_{m-1} + u_m) \right\} \]

\[ + (\lambda_j h)^2 [\sin(\phi_{m-1} + \beta_{m-1}) - \sin(\phi_m + \beta_m)] \right\} + k_3 \beta_m. \quad (3.36) \]

In the same way that Eqs. (3.21)–(3.24) were obtained, the summations may be removed from Eqs. (3.33)–(3.36) by applying Eqs. (3.20). This leads to the following four relations for the discrete model with \( n \) spring pairs:

\[ 0 = -k_1 \sin(\phi_0 + \beta_0) \left\{ \left( \frac{1 + n}{2} \right) h(v_0 - v_1) + \left( \frac{1 + 3n + 2n^2}{6n} \right) h^2 [\cos(\phi_0 + \beta_0) \right\} \]

\[ - \cos(\phi_1 + \beta_1)] + k_2 \cos(\phi_0 + \beta_0) \left\{ \left( \frac{1 + n}{2} \right) h(-u_0 + u_1) \right\} \]

\[ + \left( \frac{1 + 3n + 2n^2}{6n} \right) h^2 [\sin(\phi_0 + \beta_0) - \sin(\phi_1 + \beta_1)] \right\} \]

\[ + \frac{k_5}{2} \left[ 3l(v_0 - v_1) + l^2(2\phi_0 + \phi_1) \right], \quad (3.37) \]

\[ 0 = -k_1 \sin(\phi_0 + \beta_0) \left\{ \left( \frac{1 + n}{2} \right) h(v_0 - v_1) + \left( \frac{1 + 3n + 2n^2}{6n} \right) h^2 [\cos(\phi_0 + \beta_0) \right\} \]
For the discrete model with an infinite number of springs, i.e., letting $n \to \infty$, Eqs. (3.25) are used to obtain

\begin{equation}
0 = k_1 \sin(\phi_m + \beta_m) \left\{ \frac{1 + n}{2} h(v_{m-1} - v_m) + \frac{1 + 3n + 2n^2}{6n} h^2 \cos(\phi_{m-1} + \beta_{m-1}) - \cos(\phi_m + \beta_m) \right\} - k_2 \cos(\phi_m + \beta_m) \left\{ \frac{1 + n}{2} h(-u_{m-1} + u_m) + \frac{1 + 3n + 2n^2}{6n} h^2 \sin(\phi_{m-1} - \beta_{m-1}) - \sin(\phi_m + \beta_m) \right\} + k_3 \beta_m, \quad (3.39)
\end{equation}

\begin{equation}
0 = -\frac{k_5}{2} [3l(v_{m-1} - v_m) + l^2(\phi_{m-1} + 2\phi_m)]
\end{equation}

For the discrete model with an infinite number of springs, i.e., letting $n \to \infty$, Eqs. (3.25) are used to obtain

\begin{equation}
0 = -\frac{k_1}{2} (v_0 - v_1) + \frac{h^2}{3} \left[ \cos(\phi_0 + \beta_0) - \cos(\phi_1 + \beta_1) \right]
\end{equation}

\begin{equation}
0 = -\frac{k_2}{2} (-u_0 + u_1) + \frac{h^2}{3} \left[ \sin(\phi_0 + \beta_0) - \sin(\phi_1 + \beta_1) \right]
\end{equation}

\begin{equation}
0 = -\frac{k_5}{2} [3l(v_0 - v_1) + l^2(2\phi_0 + \phi_1)]
\end{equation}

\begin{equation}
0 = \frac{k_1}{2} (v_{m-1} - v_m) + \frac{h^2}{3} \left[ \cos(\phi_{m-1} + \beta_{m-1}) - \cos(\phi_m + \beta_m) \right]
\end{equation}

\begin{equation}
0 = -\frac{k_2}{2} (-u_{m-1} + u_m) + \frac{h^2}{3} \left[ \sin(\phi_{m-1} - \beta_{m-1}) - \sin(\phi_m + \beta_m) \right]
\end{equation}

\begin{equation}
0 = -\frac{k_5}{2} [3l(v_{m-1} - v_m) + l^2(2\phi_{m-1} + \phi_m)]
\end{equation}
\[ + \frac{h^2}{3} [\sin(\phi_{m-1} + \beta_{m-1}) - \sin(\phi_m + \beta_m)] \right] \\
+ \frac{k_5}{2} \left[ 3l(v_{m-1} - v_m) + l^2(\phi_{m-1} + 2\phi_m) \right], \quad (3.43) \]

\[ 0 = \hat{k}_1 \sin(\phi_m + \beta_m) \left\{ \frac{h}{2}(v_{m-1} - v_m) + \frac{h^2}{3} [\cos(\phi_{m-1} + \beta_{m-1}) \right] \\
- \cos(\phi_m + \beta_m) \} - \hat{k}_2 \cos(\phi_m + \beta_m) \left\{ \frac{h}{2}(-u_{m-1} + u_m) \\
+ \frac{h^2}{3} [\sin(\phi_{m-1} + \beta_{m-1}) - \sin(\phi_m + \beta_m)] \right\} + k_3 \beta_m, \quad (3.44) \]

in the same manner that Eqs. (3.27)–(3.30) were obtained. For the simply supported CTF presented in this section, there are a total of 4\(m\) unknown displacements; there are 4\((m - 1)\) unknowns at the internal nodes and 4 unknowns \((\phi_0, \beta_0, \phi_m, \text{ and } \beta_m)\) at the boundaries. For the discrete case with \(n\) spring pairs connecting the inextensible rods together, there are 4\((m - 1)\) equations given by Eqs. (3.21)–(3.24) and there are 4 equations given by Eqs. (3.37)–(3.40). Letting \(n \rightarrow \infty\) and rescaling the spring constants \(k_1\) and \(k_2\) leads to 4\((m - 1)\) equations given by Eqs. (3.27)–(3.30) and 4 equations given by Eqs. (3.41)–(3.44).

### 3.2 4-Spring Discrete Model

The discrete model described in this section differs somewhat from the discrete model presented in Section 3.1. In the previous section, the interactions between the rigid columns were represented with pairs of springs, consisting of horizontal springs and vertical springs. The horizontal and vertical effects of deformation were decoupled to ensure that the model remained linear after the small angle functions are applied. In the current section, single linear springs are used and allowed to deform both vertically and horizontally. As will be seen, the resulting discrete energy equation will be nonlinear even when small angle assumptions are applied to remove the sine and cosine terms.

The 4-spring discrete model is shown in Fig. 3.5, with five nodes labeled \(i - 2, i - 1, i, i + 1, \text{ and } i + 2\). The unit cell for this model consists of one discrete torsional spring that models the interaction of the columns with the substrate and one-half of the neighboring springs, \(k_{1,2}, k_4, \text{ and } k_5\). The allowable displacements for the
Figure 3.5. The 4-spring discrete model of a CTF showing the labeling and the type of interaction between the elements. Note the dimensions given by $h$ and $l$. The springs connecting the inextensible rods are described by $k_{1,2}$. The torsional springs connecting the inextensible rods to the substrate are given by $k_3$. The substrate is described by two spring constants: $k_4$ is associated with axial deformation and $k_5$ is associated with bending.

$i$-th node are identical to those shown in Fig. 3.3 and are shown again in Fig. 3.6. To describe the $i$-th unit cell one must define twelve discrete displacements: $u_{i-1}$,

Figure 3.6. Figure 3.3 repeated. There are four displacements describing the deformation of the $i$-th node from the initial position to the deformed position: displacement in the horizontal direction, $u_i$; displacement in the vertical direction, $v_i$; rotation of the substrate, $\phi_i$; and rotation of the column relative to the substrate, $\beta_i$. These are the same displacement functions used throughout Chapter 3.

$u_i$, $u_{i+1}$, $v_{i-1}$, $v_i$, $v_{i+1}$, $\phi_{i-1}$, $\phi_i$, $\phi_{i+1}$, $\beta_{i-1}$, $\beta_i$, and $\beta_{i+1}$.

The discrete form of the energy is defined by adding together the contributions
of the \( m + 1 \) nodes to obtain

\[
U_{\text{NL}}^{\text{discrete}} = U_{0}^{\text{NL}} + \sum_{i=1}^{m-1} \left[ \frac{1}{2} (U_{1,2})_{i} + (U_{3})_{i} + \frac{1}{2} (U_{4})_{i} + \frac{1}{2} (U_{5})_{i} \right] + U_{m}^{\text{NL}},
\]

where the superscript “NL” refers to the nonlinearity resulting from the square roots appearing in the columnar interaction terms. The energy of the springs connecting the rigid rods is denoted by \( U_{1,2} \) and involves the spring constant \( k_{1,2} \). \( U_{3} \) is associated with the torsional spring given by \( k_{3} \), and \( U_{4} \) and \( U_{5} \) are associated with the substrate spring constants, \( k_{4} \) and \( k_{5} \). The energy of the left-hand half cell (cell 0) is denoted by \( U_{0}^{\text{NL}} \) and the energy of the right-hand half cell (cell \( m \)) is denoted by \( U_{m}^{\text{NL}} \).

The springs connecting the columns together model the long-range interactions between the columns; the corresponding energy is given by

\[
(U_{1,2})_{i} = \frac{1}{2} k_{1,2} \sum_{j=1}^{n} \left\{ \left[ l - \left( \{ l + u_{i} - u_{i-1} + \lambda_{j} h [\sin(\phi_{i-1} + \beta_{i-1}) - \sin(\phi_{i} + \beta_{i})] \}^{2} \right. \right. \\
+ \left. \left. \{ v_{i} - v_{i-1} - \lambda_{j} h [\cos(\phi_{i-1} + \beta_{i-1}) - \cos(\phi_{i} + \beta_{i})] \}^{2} \right) \right]^{1/2} \right. \\
+ \left. \left[ l - \left( \{ l + u_{i+1} - u_{i} + \lambda_{j} h [\sin(\phi_{i} + \beta_{i}) - \sin(\phi_{i+1} + \beta_{i+1})] \}^{2} \right. \right. \\
+ \left. \left. \{ v_{i+1} - v_{i} - \lambda_{j} h [\cos(\phi_{i} + \beta_{i}) - \cos(\phi_{i+1} + \beta_{i+1})] \}^{2} \right) \right]^{1/2} \},
\]

where

\[
\lambda_{j} \equiv \frac{j}{n}.
\]

The remaining energy terms given by \( (U_{3})_{i} \), \( (U_{4})_{i} \), and \( (U_{5})_{i} \) are identical to those given earlier, see Eqs. (3.5)–(3.7), and are restated here:

\[
(U_{3})_{i} = \frac{1}{2} k_{3} \beta_{i}^{2},
\]

\[
(U_{4})_{i} = \frac{1}{2} k_{4} \left[ (u_{i} - u_{i-1})^{2} + (u_{i+1} - u_{i})^{2} \right],
\]

\[
(U_{5})_{i} = \frac{1}{2} k_{5} \left\{ [3(v_{i} - v_{i-1})^{2} - 3 l(\phi_{i-1} + \phi_{i})(v_{i} - v_{i-1}) \right. \\
+ l^{2}(\phi_{i-1}^{2} + \phi_{i}^{2})] + [3(v_{i+1} - v_{i})^{2} \right. \\
+ l^{2}(\phi_{i+1}^{2} + \phi_{i}^{2})],
\]
The discrete Lagrange’s equations for the end nodes are similar to Eqs. (3.14) and are given by

\[
-3l(\phi_i + \phi_{i+1})(v_{i+1} - v_i) + l^2(\phi_i^2 + \phi_i\phi_{i+1} + \phi_{i+1}^2) \right) \right].
\]  (3.50)

The half-cell energy terms follow from Eqs. (3.46)–(3.50) and are given by

\[
U_0^{NL} = \frac{1}{2}k_{1,2} \sum_{j=1}^{n} \left[ l - \left( \{l + u_1 - u_0 + \lambda_j h [\sin(\phi_0 + \beta_0) - \sin(\phi_1 + \beta_1)] \}^2 + \{v_1 - v_0 - \lambda_j h [\cos(\phi_0 + \beta_0) - \cos(\phi_1 + \beta_1)] \}^2 \right)^{1/2} \right] + \frac{1}{2}k_3\beta_0^2 + \frac{1}{2}k_4(u_1 - u_0)^2 + \frac{1}{2}k_5 \left[ 3(v_1 - v_0)^2 - 3l(\phi_0 + \phi_1)(v_1 - v_0) + l^2(\phi_0^2 + \phi_0\phi_1 + \phi_1^2) \right],
\]  (3.51)

\[
U_m^{NL} = \frac{1}{2}k_{1,2} \sum_{j=1}^{n} \left[ l - \left( \{l + u_m - u_{m-1} + \lambda_j h [\sin(\phi_{m-1} + \beta_{m-1}) - \sin(\phi_m + \beta_m)] \}^2 + \{v_m - v_{m-1} - \lambda_j h [\cos(\phi_{m-1} + \beta_{m-1}) - \cos(\phi_m + \beta_m)] \}^2 \right)^{1/2} \right] + \frac{1}{2}k_3\beta_m^2 + \frac{1}{2}k_4(u_m - u_{m-1})^2 + \frac{1}{2}k_5 \left[ 3(v_m - v_{m-1})^2 - 3l(\phi_{m-1} + \phi_m)(v_m - v_{m-1}) + l^2(\phi_{m-1}^2 + \phi_{m-1}\phi_m + \phi_m^2) \right].
\]  (3.52)

The total strain energy for the 4-spring model is obtained by applying Eqs. (3.46)–(3.52) to (3.45).

As in Section 3.1.2, there are a total of 4(m + 1) discrete displacements to solve for. In the same manner that Eqs. (3.10)–(3.13) were obtained, the following 4(m − 1) Lagrange’s equations will be used:

\[
\frac{\partial U_{NL}^{\text{discrete}}}{\partial u_i} = \frac{\partial (U_{1,2})_i}{\partial u_i} + \frac{\partial (U_4)_i}{\partial u_i} = 0, \quad i = 1, 2, \ldots, m - 1,
\]  (3.53)

\[
\frac{\partial U_{NL}^{\text{discrete}}}{\partial v_i} = \frac{\partial (U_{1,2})_i}{\partial v_i} + \frac{\partial (U_5)_i}{\partial v_i} = 0, \quad i = 1, 2, \ldots, m - 1,
\]  (3.54)

\[
\frac{\partial U_{NL}^{\text{discrete}}}{\partial \phi_i} = \frac{\partial (U_{1,2})_i}{\partial \phi_i} + \frac{\partial (U_5)_i}{\partial \phi_i} = 0, \quad i = 1, 2, \ldots, m - 1,
\]  (3.55)

\[
\frac{\partial U_{NL}^{\text{discrete}}}{\partial \beta_i} = \frac{\partial (U_{1,2})_i}{\partial \beta_i} + \frac{\partial (U_3)_i}{\partial \beta_i} = 0, \quad i = 1, 2, \ldots, m - 1.
\]  (3.56)

The discrete Lagrange’s equations for the end nodes are similar to Eqs. (3.14)
and (3.15). For the simply supported case that was introduced in Section 3.1.3, the boundary conditions are analogous to those given by Eqs. (3.31) and (3.32):

\begin{align*}
u_0 &= 0, \quad v_0 = 0, \quad \frac{\partial U_{\text{NL}}^{\text{discrete}}}{\partial \phi_0} = \frac{\partial U_0^{\text{NL}}}{\partial \phi_0}, \quad \frac{\partial U_{\text{NL}}^{\text{discrete}}}{\partial \beta_0} = \frac{\partial U_0^{\text{NL}}}{\partial \beta_0}, \\
u_m &= -l \delta_u, \quad v_m = 0, \quad \frac{\partial U_{\text{NL}}^{\text{discrete}}}{\partial \phi_m} = \frac{\partial U_m^{\text{NL}}}{\partial \phi_m}, \quad \frac{\partial U_{\text{NL}}^{\text{discrete}}}{\partial \beta_m} = \frac{\partial U_m^{\text{NL}}}{\partial \beta_m}. \tag{3.57}
\end{align*}

Unlike in the discrete linear case, the summations involving \( n \) do not converge in the same manner that was shown in Section 3.1. In Section 5.1.1, results from the 4-spring discrete model will be presented for the case \( n = 1 \).

### 3.3 Discrete-based Continuous Model

The goal of this section is to obtain a continuous form of the governing equations and boundary conditions that were presented in Section 3.1. At the end of this section there will be four ordinary differential equations (there are four displacements for the static case, each of which is a function of the horizontal deflection) and the appropriate boundary conditions. This will be accomplished by replacing the discrete equations with Taylor series expansions in terms of continuous displacement functions. In addition, to ensure that the system of governing equations is linear, small angle assumptions will be enforced over the entire model, i.e., the standard small angle assumptions

\[ \sin(\beta_i + \phi_i) \approx \beta_i + \phi_i, \quad \cos(\beta_i + \phi_i) \approx 1, \tag{3.59} \]

will be applied to Eqs. (3.27)–(3.30) and (3.41)–(3.44). Only the case with an infinite number of spring pairs \( (n \to \infty) \) connecting the rigid columns will be considered in the remainder of this chapter. It will also be necessary to ignore any resulting terms containing products of displacement functions; removing such products is consistent with the overall linear, small deformation assumptions of the model.
3.3.1 Formulating the Governing Equations

Because of the desire to obtain a linear, second order system of governing equations (and a quadratic energy potential, as will be seen in Section 3.4), the following expansion for the displacements up to the second order will be used (recall Eq. (2.44)):

\[
\begin{align*}
    u_{i-1} &\approx u(x) - l\frac{du(x)}{dx} + \frac{l^2}{2} \frac{d^2u(x)}{dx^2}, & u_{i+1} &\approx u(x) + l\frac{du(x)}{dx} + \frac{l^2}{2} \frac{d^2u(x)}{dx^2}, \\
    u_i &\approx u(x), & v_{i-1} &\approx v(x) - l\frac{dv(x)}{dx} + \frac{l^2}{2} \frac{d^2v(x)}{dx^2}, & v_{i+1} &\approx v(x) + l\frac{dv(x)}{dx} + \frac{l^2}{2} \frac{d^2v(x)}{dx^2}, \\
    v_i &\approx v(x), & \beta_{i-1} &\approx \beta(x) - l\frac{d\beta(x)}{dx} + \frac{l^2}{2} \frac{d^2\beta(x)}{dx^2}, & \beta_{i+1} &\approx \beta(x) + l\frac{d\beta(x)}{dx} + \frac{l^2}{2} \frac{d^2\beta(x)}{dx^2}, \\
    \beta_i &\approx \beta(x), & \phi_{i-1} &\approx \phi(x) - l\frac{d\phi(x)}{dx} + \frac{l^2}{2} \frac{d^2\phi(x)}{dx^2}, & \phi_{i+1} &\approx \phi(x) + l\frac{d\phi(x)}{dx} + \frac{l^2}{2} \frac{d^2\phi(x)}{dx^2}, \\
    \phi_i &\approx \phi(x),
\end{align*}
\]

After applying the small angle assumption, Eqs. (3.59), ignoring the displacement term products as discussed, and applying Eqs. (3.60)–(3.67) to Eqs. (3.27) one obtains the following four second order, linear, homogeneous ordinary differential equations in terms of the displacement functions:

\[
\begin{align*}
    \frac{1}{2} l^2 \left\{ 2(\hat{k}_2 + k_4)u''(x) - h\hat{k}_2[\phi''(x) + \beta''(x)] \right\} &= 0, \\
    l^2 \left[ (\hat{k}_1 + 3k_5)v''(x) - 3k_5\phi'(x) \right] &= 0, \\
    -\frac{1}{6} l^2 \left[ 3h\hat{k}_2u''(x) + (3l^2k_5 - 2h^2\hat{k}_2)\phi''(x) \\
    \quad - 2h^2\hat{k}_2\beta''(x) - 18k_5v'(x) + 18k_5\phi(x) \right] &= 0, \\
    -\frac{1}{2} hl^2\hat{k}_2u''(x) + \frac{1}{3} h^2l^2\hat{k}_2\phi''(x) + \frac{1}{3} h^2l^2\hat{k}_2\beta''(x) - k_3\beta(x) &= 0.
\end{align*}
\]

Since Eqs. (3.68)–(3.71) are based on the energy terms from the internal nodes, the domain \( x \in [0, L] \) is associated with the length from \( i = 1 \) to \( i = m - 1 \) so that \( L \) equals \( (m - 2)l \). Equations (3.68)–(3.71) may be rewritten in a form such
that only one second order displacement function appears in each equation, which leads to

\[ hl^2 (\hat{k}_2 + 4k_4) u''(x) - 6k_3 \beta(x) = 0, \quad (3.72) \]
\[ (\hat{k}_1 + 3k_5) v''(x) - 3k_5 \phi'(x) = 0, \quad (3.73) \]
\[ l^4 k_5 \phi''(x) - 6l^2 k_5 v'(x) + 6l^2 k_5 \phi(x) - 2k_3 \beta(x) = 0, \quad (3.74) \]
\[ h^2 l^4 \hat{k}_2 (\hat{k}_2 + 4k_4) k_5 \phi''(x) + 6h^2 l^2 \hat{k}_2 (\hat{k}_2 + 4k_4) k_5 v'(x) - 6h^2 l^2 \hat{k}_2 (\hat{k}_2 + 4k_4) k_5 \beta(x) = 0. \quad (3.75) \]

Now that the form of the governing equations is apparent, it is helpful to nondimensionalize Eqs. (3.72)–(3.75). The characteristic length that all other length terms will be scaled by is \( l \), the separation distance between the undeformed rigid columns, leading to these substitutions:

\[ \xi \equiv x/l, \quad \tilde{u} \equiv u(\xi)/l, \quad \tilde{v} \equiv v(\xi)/l, \quad \tilde{\phi} \equiv \phi(\xi), \quad \tilde{\beta} \equiv \beta(\xi). \quad (3.76) \]

After applying Eqs. (3.76) to Eqs. (3.72)–(3.75), the governing equations may be completely nondimensionalized by dividing each one through by a non-zero constant. In particular, Eq. (3.72) is divided through by \( hl(\hat{k}_2 + 4k_4) \), Eq. (3.73) is divided through by \( \hat{k}_1 + 3k_5 \), Eq. (3.74) is divided through by \( l^2 k_5 \), and Eq. (3.75) is divided through by \( h^2 l^2 \hat{k}_2 (\hat{k}_2 + 4k_4) k_5 \). When these steps are implemented, Eqs. (3.72)–(3.75) may be written in the nondimensional form

\[ 0 = \tilde{u}''(\xi) - \kappa_1 \tilde{\beta}(\xi), \quad (3.77) \]
\[ 0 = \tilde{v}''(\xi) - \kappa_2 \tilde{\phi}'(\xi), \quad (3.78) \]
\[ 0 = \tilde{\phi}''(\xi) - 6\tilde{v}'(\xi) + 6\tilde{\phi}(\xi) - \kappa_3 \tilde{\beta}(\xi), \quad (3.79) \]
\[ 0 = \tilde{\beta}''(\xi) + 6\tilde{v}'(\xi) - 6\tilde{\phi}(\xi) + (\kappa_3 - \kappa_4) \tilde{\beta}(\xi), \quad (3.80) \]

where

\[ \kappa_1 \equiv \frac{6k_3}{h(\hat{k}_2 + 4k_4)l}, \quad (3.81) \]
\[ \kappa_2 \equiv \frac{3k_5}{\hat{k}_1 + 3k_5}. \quad (3.82) \]
\begin{align}
\kappa_3 & \equiv \frac{2k_3}{k_3 l^2}, \\
\kappa_4 & \equiv \frac{12k_3 (\hat{k}_2 + k_4)}{h^2 \hat{k}_2 (k_2 + 4k_4)}.
\end{align}

(3.83)

(3.84)

It is worth noting that, after the nondimensionalization, the governing equations are characterized by only four nondimensional material constants. Two additional nondimensional terms must be introduced to account for the total length of the model, \( L \), and the discrete column height, \( h \),

\[ \Lambda \equiv \frac{L}{l}, \quad \eta \equiv \frac{h}{l}, \]

(3.85)

such that \( \xi \in [0, \Lambda] \) (this may be introduced now that \( \Lambda \) is defined).

The constants introduced in Eqs. (3.81)–(3.85) will be used extensively and it is worth noting that all of the terms appearing in these expressions (\( \hat{k}_1, \hat{k}_2, k_3, k_4, k_5, h, l, \) and \( L \)) are greater than zero. It follows that \( \kappa_1, \kappa_2, \kappa_3, \kappa_4 > 0 \). After further consideration, it can be shown that one may introduce and summarize the following inequalities:

\[ 0 < \kappa_1 < \eta \kappa_4 / 2, \quad 0 < \kappa_2 < 1, \quad 0 < \kappa_3. \]

(3.86)

The second of these inequalities is useful for considering the effect of the classical Euler-Bernoulli constraint, \( \ddot{\phi}(\xi) = \ddot{u}(\xi) \), on this system. If the slope is constrained to equal the derivative of the beam deflection, then Eq. (3.78) leads to the conclusion that \( \kappa_2 = 1 \), which violates the requirement of the second of Eqs. (3.86). But, as \( \kappa_2 \) approaches one, one would expect to see Euler-Bernoulli beam-like behavior from the discrete-based model. (For an Euler-Bernoulli beam, one may think of \( \phi \) as a constrained generalized coordinate and the relation \( \phi = u' \) as an additional constraint equation.)

### 3.3.2 Formulating the Boundary Conditions

In addition to the governing equations, it is also necessary to consider the boundary conditions that, for the discrete model, were specified at nodes \( i = 0 \) and \( i = m \). For the left-hand boundary conditions, the following Taylor series expansions of
the displacements will be used:

\[
\begin{align*}
    u_0 &\approx u(0) - l \frac{du(0)}{dx} + \frac{l^2}{2} \frac{d^2u(0)}{dx^2}, & u_1 &= u(0), \\
v_0 &\approx v(0) - l \frac{dv(0)}{dx} + \frac{l^2}{2} \frac{d^2v(0)}{dx^2}, & v_1 &= v(0), \\
\phi_0 &\approx \phi(0) - l \frac{d\phi(0)}{dx} + \frac{l^2}{2} \frac{d^2\phi(0)}{dx^2}, & \phi_1 &= \phi(0), \\
\beta_0 &\approx \beta(0) - l \frac{d\beta(0)}{dx} + \frac{l^2}{2} \frac{d^2\beta(0)}{dx^2}, & \beta_1 &= \beta(0),
\end{align*}
\]

(3.87)–(3.90)

while for the right-hand boundary conditions, the following expansions are used:

\[
\begin{align*}
u_{m-1} &= u(L), & u_m &\approx u(L) + l \frac{du(L)}{dx} + \frac{l^2}{2} \frac{d^2u(L)}{dx^2}, \\
v_{m-1} &= v(L), & v_m &\approx v(L) + l \frac{dv(L)}{dx} + \frac{l^2}{2} \frac{d^2v(L)}{dx^2}, \\
\phi_{m-1} &= \phi(L), & \phi_m &\approx \phi(L) + l \frac{d\phi(L)}{dx} + \frac{l^2}{2} \frac{d^2\phi(L)}{dx^2}, \\
\beta_{m-1} &= \beta(L), & \beta_m &\approx \beta(L) + l \frac{d\beta(L)}{dx} + \frac{l^2}{2} \frac{d^2\beta(L)}{dx^2}.
\end{align*}
\]

(3.91)–(3.94)

It should be apparent from Eqs. (3.87)–(3.94) that the displacements at nodes \( i = 0 \) and \( m \), which exist outside the range \([0, L]\), are brought into the range via the Taylor series expansions. The discrete boundary conditions to be considered are

\[
u_0 = 0, \quad v_0 = 0, \quad u_m = -l\delta_u, \quad v_m = 0,
\]

(3.95)

along with Eqs. (3.41)–(3.44), for a total of 8 relations. To obtain the continuous form of the boundary conditions requires three steps: (1) apply the Taylor series expansions to the discrete boundary conditions, (2) apply the nondimensional relations to the resulting equations, and (3) make use of the governing equations to remove any second order terms. As an example, this procedure will be followed for the boundary condition \( u_0 = 0 \).

The first step requires applying Eq. (3.87) to the boundary condition, such that

\[
u_0 = 0 \quad \rightarrow \quad u(0) - lu'(0) + \frac{l^2}{2} u''(0) = 0.
\]

(3.96)
In the second step, this relation is nondimensionalized,

\[ l \ddot{u}(0) - l\dot{u}'(0) + l\dddot{u}'(0)/2 = 0 \rightarrow \tilde{u}(0) - \tilde{u}'(0) + \tilde{u}''(0)/2 = 0. \] (3.97)

To remove the second order term, recall the first of the nondimensionalized governing equations, given by Eq. (3.77) and rewritten here as

\[ \tilde{u}''(\xi) = \kappa_1 \tilde{\beta}(\xi), \quad \xi \in [0, \Lambda]. \] (3.98)

Applying Eq. (3.98) to (3.97) yields the continuous form of the boundary condition,

\[ \tilde{u}(0) - \tilde{u}'(0) + \kappa_1 \tilde{\beta}(0)/2 = 0. \] (3.99)

Following the same procedure, the remaining three relations from Eqs. (3.95) may be written as

\[ \tilde{v}(0) - \tilde{v}'(0) + \kappa_2 \tilde{\phi}'(0)/2 = 0, \] (3.100)

\[ \tilde{u}(\Lambda) + \tilde{u}'(\Lambda) + \kappa_1 \tilde{\beta}(\Lambda)/2 = -\delta_u, \] (3.101)

\[ \tilde{v}(\Lambda) + \tilde{v}'(\Lambda) + \kappa_2 \tilde{\phi}'(\Lambda)/2 = 0. \] (3.102)

After applying the same approach to the four natural boundary conditions given by Eqs. (3.41)–(3.44), one obtains

\[ 0 = 3\kappa_3/(6\kappa_1 - 4\eta\kappa_4)\ddot{u}'(0) - (3/2)\ddot{v}'(0) + [1 - 3\kappa_2/4 - \eta\kappa_3/(3\kappa_1 - 2\eta\kappa_4)]\tilde{\phi}'(0) \]

\[ - \eta\kappa_3/(3\kappa_1 - 2\eta\kappa_4)\tilde{\beta}'(0) + (3/2)\dot{\phi}(0) - (3\kappa_3/4)\tilde{\beta}(0), \] (3.103)

\[ 0 = 3\kappa_3/(6\kappa_1 - 4\eta\kappa_4)\ddot{u}'(0) + (3\kappa_3/2)\ddot{v}'(0) - \eta\kappa_3/(3\kappa_1 - 2\eta\kappa_4)\tilde{\phi}'(0) \]

\[ + [\kappa_3(3\kappa_1 - 2\eta(1 + \kappa_4)]/(6\kappa_1 - 4\eta\kappa_4)\tilde{\beta}'(0) - (3\kappa_3/2)\dot{\phi}(0) \]

\[ - \kappa_3(3 - \kappa_3 + \kappa_4)/4\tilde{\beta}(0), \] (3.104)

\[ 0 = 3\kappa_3/(6\kappa_1 - 4\eta\kappa_4)\ddot{u}'(\Lambda) + (3/2)\ddot{v}'(\Lambda) + [1 - 3\kappa_2/4 - \eta\kappa_3/(3\kappa_1 - 2\eta\kappa_4)]\tilde{\phi}'(\Lambda) \]

\[ - \eta\kappa_3/(3\kappa_1 - 2\eta\kappa_4)\tilde{\beta}'(\Lambda) - (3/2)\dot{\phi}(\Lambda) + (3\kappa_3/4)\tilde{\beta}(\Lambda), \] (3.105)

\[ 0 = 3\kappa_3/(6\kappa_1 - 4\eta\kappa_4)\ddot{u}'(\Lambda) - (3\kappa_3/2)\ddot{v}'(\Lambda) - \eta\kappa_3/(3\kappa_1 - 2\eta\kappa_4)\tilde{\phi}'(\Lambda) \]

\[ + [\kappa_3(3\kappa_1 - 2\eta(1 + \kappa_4)]/(6\kappa_1 - 4\eta\kappa_4)\tilde{\beta}'(\Lambda) + (3\kappa_3/2)\dot{\phi}(\Lambda) \]

\[ + \kappa_3(3 - \kappa_3 + \kappa_4)/4\tilde{\beta}(\Lambda). \] (3.106)
The BVP for the discrete-based continuous model consists of the four governing equations, Eqs. (3.77)–(3.80), and the eight boundary conditions, Eqs. (3.99)–(3.106). The difference between the discrete-based model and the directed continuum model to be presented in the following section appears in the boundary conditions. (If all of the boundary conditions are essential boundary conditions, then there is no difference between the two models.)

3.4 Formulation of the Continuum Model

The purpose of this section is to propose a directed continuum model that is equivalent* to the discrete model presented in Section 3.1. Following the outline first presented in the Introduction (Chapter 1), the directed continuum model will be obtained by

1. describing the kinematics by using the four displacement functions introduced in Section 3.4,
2. defining the strain terms based on the kinematics,
3. formulating a strain energy density $W$ that yields the same governing equations (Eqs. (3.77)–(3.80), after nondimensionalization) via a Hamilton’s principle approach, and
4. determining appropriate boundary conditions from the Hamilton’s principle approach using $W$.

The formulation to be presented follows the approach of Mindlin [75], who makes use of Hamilton’s principle† in his derivations. By obtaining an expression for $W$ that yields the same governing equations that were given by Eqs. (3.68)–(3.71), it will also be possible to obtain the boundary conditions. Since we begin this section

*When the continuum model is said to be equivalent to the discrete model, it simply means that the “homogenized” governing equations obtained from the discrete model are identical to the governing equations that result from the strain energy density proposed for the continuum model.

†Since it may not be obvious how Hamilton’s principle is applied, in Appendix A, a more obvious formulation is presented. As expected, it yields identical results to those appearing in this section.
knowing what governing equations we wish to obtain, it follows that the point of
this entire section is to obtain boundary conditions for the directed continuum
model.

The process of obtaining the complete boundary value problem outlined above
will be followed in Section 3.4.1. The formulation will parallel that of the deriva-
tion of micromorphic continua as described in Section 2.1, although the details of
Section 2.1 have been included in Section 3.4.1. In Section 3.4.2, all of the result-
ing equations (the governing equations, boundary conditions, energy, and stress
relations) will be nondimensionalized using the same approach that was used in
Section 3.3. Once the model is introduced, two different boundary conditions will
be described in Sections 3.4.3 (a simply supported thin film, Case I) and 3.4.4 (a
rigidly fixed thin film, Case II).

3.4.1 The Boundary Value Problem

The purpose of this section is to obtain a general form of the boundary value
problem that is representative of the directed continuum model of a CTF. To do
this, it will first be necessary to introduce the terms describing the kinematics of
the model (Section 3.4.1.1). The displacement gradients (Section 3.4.1.2) and the
strain terms (Section 3.4.1.3) will then be obtained. The work-conjugate equations
relating stress to strain as well as the governing equations resulting from the bal-
ance laws based on the work of Mindlin [75] will then be presented (Section 3.4.1.4).
Finally, a general form of the strain energy density will be calculated; the actual
strain energy density is found to be the one that yields the governing equations
obtained in Section 3.3 (Section 3.4.1.5). When the desired strain energy density
is found, it is possible to formulate the boundary conditions and stress relations
(Section 3.4.1.6).

3.4.1.1 The kinematics of the model

Based on the governing equations presented in Section 3.3, there will be four
independent displacement functions: \( u(x), v(x), \phi(x), \) and \( \beta(x) \), which are defined
over the domain \( x \in [0, L] \). Alternatively, these may be nondimensionalized to
yield four functions \( \tilde{u}(\xi), \tilde{v}(\xi), \tilde{\phi}(\xi) \) and \( \tilde{\beta}(\xi) \) defined over the domain \( \xi \in [0, \Lambda] \).
The kinematic displacement terms of the problem are shown in Fig. 3.7 and should be thought of in light of the ideas already discussed in Section 2.1.1. The following discussion is intended to make a connection between the development of a general description of the kinematics (Fig. 3.7(a)) and the kinematics for the CTF model (Fig. 3.7(b)).

In both cases, the body $\mathcal{B}$ consists of points $\mathbf{r}$. According to Eq. (2.1), the placement into the reference configuration ($\mathcal{B}_\kappa$) in Euclidean space $\mathcal{E}$ is accomplished by

$$\mathbf{x} = \mathbf{\chi}(\mathbf{r}) = \mathbf{p}, \quad \mathbf{p} \in \mathcal{B}_\kappa. \quad (3.107)$$

In other words, $\mathbf{\chi}$ describes the reference configuration in a classical sense and the vector field $\mathbf{x}$ describes the deformation of the body in a classical sense. The points...
\( \mathbf{p} \) in the reference configuration for the general case are given by the scalar \( x \) in the CTF case. In addition, \( \hat{\mathbf{x}} \) describes the reference configuration of the substrate and \( \mathbf{x} \) describes the deformation of the substrate, taking values in \( \mathbb{R}^2 \), where the horizontal deformation is given by \( u(x) \) and the vertical deformation is given by \( v(x) \).

The unique aspect of a directed continuum theory is the director space \( \mathcal{M} \). Due to the assumptions introduced in Section 2.1 (regarding one-to-one mapping and the dimensionality of \( \mathcal{E} \) and \( \mathcal{M} \)) and Eq. (2.4), it follows that the director (vector) field in \( \mathcal{B}_\kappa \) may be considered as a function of \( \mathbf{p} \). Equation 2.4 is rewritten as

\[
\hat{\chi} = \hat{\chi}(\Phi(\chi^{-1}(\mathbf{p}))) = \hat{\chi}(\mathbf{p}), \quad \mathbf{p} \in \mathcal{B}_\kappa.
\]

(3.108)

Therefore \( \hat{\chi} \) describes the reference configuration of the director field, while \( \hat{\mathbf{x}} \) describes its evolution. For the CTF, \( \hat{\chi} \) is given by a unit vector oriented perpendicularly to the horizontal axis, as shown in Fig. 3.7(b). The vector field \( \hat{\mathbf{x}} \) (the director field) describes two different motions: the rotation of the cross section of the thin film, which has been given by \( \phi(x) \), and the orientation of the columnar structure, which has been given by \( \beta(x) \).

### 3.4.1.2 Calculating the displacement gradients

In this section, the displacement gradient due to the classical deformation of the body (\( \mathbf{H} \)) will be calculated first, followed by the displacement gradient due to the evolution of the body in \( \mathcal{M} (\hat{\mathbf{H}}) \). The displacements \( u(x) \) and \( v(x) \) are associated with the deformation of the body in the classical sense, such that the vector function \( \mathbf{u}(x) \) is defined in terms of the scalar displacement functions as (see Fig. 3.7(b))

\[
\mathbf{u}(x) = u(x)\hat{\mathbf{e}}_1 + v(x)\hat{\mathbf{e}}_2.
\]

(3.109)

The resulting displacement gradient is defined in the standard way according to Eq. (2.11) as

\[
\mathbf{H} = \text{Grad} \mathbf{u}(x) = \begin{bmatrix} u'(x) & 0 \\ v'(x) & 0 \end{bmatrix},
\]

(3.110)
where the primes denote differentiation with respect to $x$. The right column contains zeros since $u$ and $v$ are only functions of $x$. For the model presented thus far, there is no physical meaning to derivatives with respect to any other position, i.e., other than $x$.

The next goal is to find the displacement gradient $\hat{H}$. As has already been mentioned, the director is denoted by $\hat{\chi}$ in the reference configuration and by $\hat{x}$ in the deformed configuration. The deformation of $\hat{\chi}$, whose components are taken to be the displacements $\phi(x)$ and $\beta(x)$, is associated with the evolution of the microstructure. Taking a commonly held viewpoint, according to which every point in $B$ is surrounded by a microvolume with its own coordinate system, it is assumed that the deformation within each microvolume is homogeneous. While the deformation of $\hat{\chi}$ within the microstructure is taken to be homogeneous, it is not homogeneous relative to the entire body. Therefore, it is possible to make use of the displacement gradient associated with the evolution of the microstructure, $\hat{H}$, to relate $\hat{x}$ to $\hat{\chi}$, i.e.,

$$\hat{x} = (\hat{H} + I)\hat{\chi}, \quad (3.111)$$

where $I$ is the identity matrix and $\hat{F} \equiv \hat{H} + I$, according to Eq. (2.6). The resulting gradient of the displacement due to the evolution of the body in $M$, given by Eq. (2.12), is

$$\hat{H} = \begin{bmatrix} \beta(x) & -\phi(x) \\
\phi(x) & \beta(x) \end{bmatrix}, \quad (3.112)$$

which is to say that the unit vector $\hat{\chi}$ rotates by the angle $\phi$ (where positive $\phi$ is a counter-clockwise rotation) and stretches by an amount $\beta$. The standard small angle assumptions $\sin \theta \approx \theta$ and $\cos \theta \approx 1$ for small $\theta$ have been used to obtain Eq. (3.112). The stretch and the rotation of the director vector field are given by the symmetric and skew-symmetric components of the second order tensor $\hat{H}$, such

\[\hat{H} = \begin{bmatrix} \beta(x) & -\phi(x) \\
\phi(x) & \beta(x) \end{bmatrix}, \quad (3.112)\]

where $l$ is the identity matrix and $\hat{F} \equiv \hat{H} + I$, according to Eq. (2.6). The resulting gradient of the displacement due to the evolution of the body in $M$, given by Eq. (2.12), is

\[\hat{H} = \begin{bmatrix} \beta(x) & -\phi(x) \\
\phi(x) & \beta(x) \end{bmatrix}, \quad (3.112)\]

which is to say that the unit vector $\hat{\chi}$ rotates by the angle $\phi$ (where positive $\phi$ is a counter-clockwise rotation) and stretches by an amount $\beta$. The standard small angle assumptions $\sin \theta \approx \theta$ and $\cos \theta \approx 1$ for small $\theta$ have been used to obtain Eq. (3.112). The stretch and the rotation of the director vector field are given by the symmetric and skew-symmetric components of the second order tensor $\hat{H}$, such
that

\[
\text{sym}(\hat{H}) = \begin{bmatrix}
\beta(x) & 0 \\
0 & \beta(x)
\end{bmatrix}, \quad \text{skw}(\hat{H}) = \begin{bmatrix}
0 & -\phi(x) \\
\phi(x) & 0
\end{bmatrix}.
\] (3.113)

The vector \( \hat{x} \) is seen to rotate by \( \phi \) from its reference configuration and its magnitude is given by \( 1 + \beta \). Since \( \text{sym}(\hat{H}) \) is not equal to 0, the resulting theory can be called micromorphic.

### 3.4.1.3 The strain terms

The strain terms are obtained by using the strain gradients \( H \) and \( \hat{H} \) found in the previous section. Based on the work of Mindlin [75] as presented in Eqs. (2.13)–(2.15), and using his terminology, the macro-strain is given by

\[
E = \frac{1}{2}(H + \hat{H}^T) = \begin{bmatrix}
u'(x) & v'(x)/2 \\
v'(x)/2 & 0
\end{bmatrix} \equiv \begin{bmatrix} E_{11} & E_{12} \\
E_{21} & E_{22}
\end{bmatrix},
\] (3.114)

and the relative deformation is given by

\[
\Gamma = H - \hat{H} = \begin{bmatrix}
u'(x) - \beta(x) & \phi(x) \\
v'(x) - \phi(x) & -\beta(x)
\end{bmatrix} \equiv \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\
\Gamma_{21} & \Gamma_{22}
\end{bmatrix}.
\] (3.115)

The third order microdeformation gradient \( K_{(3)} \) is given by

\[
K_{(3)} = \text{Grad} \hat{H}.
\] (3.116)

This third order tensor may be considered to be made up of one second order tensor for each spatial dimension.\(^5\) Hence for this problem, a second order microdeformation gradient is given by \( K_1 \):

\[
K_1 = \begin{bmatrix}
\beta'(x) & -\phi'(x) \\
\phi'(x) & \beta'(x)
\end{bmatrix} \equiv \begin{bmatrix} K_{111} & K_{121} \\
K_{211} & K_{221}
\end{bmatrix}.
\] (3.117)

\(^5\)In index notation, \( \{K_{(3)}\}_{ijk} \) is made up of three second order tensors: \( \{K_{(3)}\}_{ij1}, \{K_{(3)}\}_{ij2}, \) and \( \{K_{(3)}\}_{ij3} \), for the three-dimensional case. In this manner, \( \{K_1\}_{ij} \) is equivalent to \( \{K_{(3)}\}_{ij1} \).
The three strain terms, \( E \), \( \Gamma \), and \( K_1 \) have now been defined. The compatibility equations given by Eqs. (2.16)–(2.18) are satisfied by these strain tensors, as expected.

### 3.4.1.4 The work-conjugate and balance relations

It is assumed that this material is hyperelastic, i.e., there exists a strain energy density called \( W \) that may be written as a function of strains. The three stress terms that are work conjugates of the three strain terms given by Eqs. (3.114), (3.115), and (3.117) are given in the following relations, introduced earlier as Eqs. (2.20),

\[
T = \frac{\partial W}{\partial E}, \quad \Sigma = \frac{\partial W}{\partial \Gamma}, \quad M_{(3)} = \frac{\partial W}{\partial K_{(3)}}. \tag{3.118}
\]

Using Hamilton’s principle, the local form of the balance of momentum yields

\[
\text{Div} (T + \Sigma) = \text{Div} \left( \frac{\partial W}{\partial E} + \frac{\partial W}{\partial \Gamma} \right) = 0, \tag{3.119}
\]

and the local form of the balance of angular momentum yields

\[
\text{Div} M_{(3)} + \Sigma = \text{Div} \left( \frac{\partial W}{\partial K_{(3)}} \right) + \frac{\partial W}{\partial \Gamma} = 0, \tag{3.120}
\]

for the static case with no body forces. The boundary conditions resulting from such a formulation are given by Eqs. (2.23) and rewritten here as

\[
s = (T + \Sigma)n = \left( \frac{\partial W}{\partial E} + \frac{\partial W}{\partial \Gamma} \right)n, \quad S = M_{(3)}n = \frac{\partial W}{\partial K_{(3)}}n, \tag{3.121}
\]

where \( s \) is the surface force per unit area, \( S \) is the double force per unit area, and \( n \) is the unit normal vector.

### 3.4.1.5 Formulating the energy density

Considering Eqs. (3.118)–(3.121) as well as the strain tensors given by Eqs. (3.114)–(3.117), it is possible to describe the approach used to obtain the directed continuum boundary value problem in greater detail. The desired form of the governing equations are given by Eqs. (3.68)–(3.71) in Section 3.3. Therefore, it should be
possible to obtain a value of the strain energy density $W$ as a function of $E$, $\Gamma$, and $K_{(3)}$ such that Eqs. (3.119) and (3.120) will be equivalent to Eqs. (3.68)–(3.71). When the expression for $W$ is found, Eqs. (3.121) may be used to obtain the appropriate boundary conditions.

To begin, a general form of $W$ must be introduced. Using the notation of Eq. (2.24), the strain energy density is written as

$$W = \frac{1}{2} e^T C e + \frac{1}{2} \gamma^T B \gamma + \frac{1}{2} k^T A k + \gamma^T D k + k^T F e + \gamma^T G e,$$

(3.122)

where the vectors $e$, $\gamma$, and $k$ are defined as

$$e = \begin{pmatrix} E_{11} \\ E_{22} \\ E_{12} \\ E_{21} \end{pmatrix}, \quad \gamma = \begin{pmatrix} \Gamma_{11} \\ \Gamma_{22} \\ \Gamma_{12} \\ \Gamma_{21} \end{pmatrix}, \quad k = \begin{pmatrix} K_{111} \\ K_{221} \\ K_{121} \\ K_{211} \end{pmatrix}. \quad (3.123)$$

These vectors simply contain the terms found in the strain tensors in Eqs. (3.114)–(3.117). The matrices $C$, $B$, $A$, $D$, $F$, and $G$ are $4 \times 4$ and contain various degrees of symmetry based on Eqs. (2.25) and (2.26), such that

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{13} \\ C_{12} & C_{22} & C_{23} & C_{23} \\ C_{13} & C_{23} & C_{33} & C_{33} \\ C_{13} & C_{23} & C_{33} & C_{33} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{12} & B_{22} & B_{23} & B_{24} \\ B_{13} & B_{23} & B_{33} & B_{34} \\ B_{14} & B_{24} & B_{34} & B_{44} \end{bmatrix},$$

(3.124)

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{12} & A_{22} & A_{23} & A_{24} \\ A_{13} & A_{23} & A_{33} & A_{34} \\ A_{14} & A_{24} & A_{34} & A_{44} \end{bmatrix}, \quad D = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{12} & D_{22} & D_{23} & D_{24} \\ D_{13} & D_{23} & D_{33} & D_{34} \\ D_{14} & D_{24} & D_{34} & D_{44} \end{bmatrix},$$

(3.125)

$$F = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{13} \\ F_{21} & F_{22} & F_{23} & F_{23} \\ F_{31} & F_{32} & F_{33} & F_{33} \\ F_{41} & F_{42} & F_{43} & F_{43} \end{bmatrix}, \quad G = \begin{bmatrix} G_{11} & G_{12} & G_{13} & G_{13} \\ G_{21} & G_{22} & G_{23} & G_{23} \\ G_{31} & G_{32} & G_{33} & G_{33} \\ G_{41} & G_{42} & G_{43} & G_{43} \end{bmatrix}. \quad (3.126)$$

From Eqs. (3.114)–(3.117), it is apparent that there are six displacement func-
tion terms appearing in the strain tensors: \( u'(x), v'(x), \phi(x), \phi'(x), \beta(x), \) and \( \beta'(x) \). The most general quadratic homogeneous function of six terms contains 21 variables. Observing Eqs. (3.124)–(3.126), there are 66 constants. (Without the symmetry of Eqs. (2.25) and (2.26), there would be 16 times 6, or 96, constants.) Applying Eqs. (3.122)–(3.126) to (2.20) yields the following twelve stress terms:

\[
\{ \mathcal{T} \}_{11} = -(G_{11} + G_{21}) \beta(x) + (G_{31} - G_{41}) \phi(x) + (C_{11} + G_{11}) u'(x) \\
+ (C_{13} + G_{41}) v'(x) + (F_{11} + F_{21}) \beta'(x) - (F_{31} - F_{41}) \phi'(x), \\
\{ \mathcal{T} \}_{12} = -(G_{13} + G_{23}) \beta(x) + (G_{33} - G_{43}) \phi(x) + (C_{13} + G_{13}) u'(x) \\
+ (C_{33} + G_{43}) v'(x) + (F_{13} + F_{23}) \beta'(x) - (F_{33} - F_{43}) \phi'(x), \\
\{ \mathcal{T} \}_{21} = -(G_{13} + G_{23}) \beta(x) + (G_{33} - G_{43}) \phi(x) + (C_{13} + G_{13}) u'(x) \\
+ (C_{33} + G_{43}) v'(x) + (F_{13} + F_{23}) \beta'(x) - (F_{33} - F_{43}) \phi'(x), \\
\{ \mathcal{T} \}_{22} = -(G_{12} + G_{22}) \beta(x) + (G_{32} - G_{42}) \phi(x) + (C_{12} + G_{12}) u'(x) \\
+ (C_{23} + G_{42}) v'(x) + (F_{12} + F_{22}) \beta'(x) - (F_{32} - F_{42}) \phi'(x), \\
\{ \mathcal{S} \}_{11} = -(B_{11} + B_{21}) \beta(x) + (B_{13} - B_{14}) \phi(x) + (B_{11} + G_{11}) u'(x) \\
+ (B_{14} + G_{13}) v'(x) + (D_{11} + D_{12}) \beta'(x) - (D_{13} - D_{14}) \phi'(x), \\
\{ \mathcal{S} \}_{12} = -(B_{13} + B_{23}) \beta(x) + (B_{33} - B_{34}) \phi(x) + (B_{13} + G_{31}) u'(x) \\
+ (B_{34} + G_{33}) v'(x) + (D_{31} + D_{32}) \beta'(x) - (D_{33} - D_{34}) \phi'(x), \\
\{ \mathcal{S} \}_{21} = -(B_{14} + B_{24}) \beta(x) + (B_{34} - B_{44}) \phi(x) + (B_{14} + G_{41}) u'(x) \\
+ (B_{44} + G_{43}) v'(x) + (D_{41} + D_{42}) \beta'(x) - (D_{43} - D_{44}) \phi'(x), \\
\{ \mathcal{S} \}_{22} = -(B_{12} + B_{22}) \beta(x) + (B_{23} - B_{24}) \phi(x) + (B_{12} + G_{21}) u'(x) \\
+ (B_{24} + G_{23}) v'(x) + (D_{21} + D_{22}) \beta'(x) - (D_{23} - D_{24}) \phi'(x), \\
\{ \mathcal{M}_{(3)} \}_{111} = -(D_{11} + D_{21}) \beta(x) + (D_{31} - D_{41}) \phi(x) + (D_{11} + F_{11}) u'(x) \\
+ (D_{41} + F_{13}) v'(x) + (A_{11} + A_{12}) \beta'(x) - (A_{13} - A_{14}) \phi'(x), \\
\{ \mathcal{M}_{(3)} \}_{121} = -(D_{13} + D_{23}) \beta(x) + (D_{33} - D_{43}) \phi(x) + (D_{13} + F_{31}) u'(x) \\
+ (D_{43} + F_{33}) v'(x) + (A_{13} + A_{23}) \beta'(x) - (A_{33} - A_{34}) \phi'(x), \\
\{ \mathcal{M}_{(3)} \}_{211} = -(D_{14} + D_{24}) \beta(x) + (D_{34} - D_{44}) \phi(x) + (D_{14} + F_{41}) u'(x) \\
+ (D_{44} + F_{43}) v'(x) + (A_{14} + A_{24}) \beta'(x) - (A_{34} - A_{44}) \phi'(x), \\
\{ \mathcal{M}_{(3)} \}_{221} = -(D_{12} + D_{22}) \beta(x) + (D_{32} - D_{42}) \phi(x) + (D_{12} + F_{21}) u'(x)
+ (D_{42} + F_{23})v'(x) + (A_{12} + A_{22})\beta'(x) - (A_{23} - A_{24})\phi'(x), \quad (3.138)

noting that \{\mathbf{T}\}_{12} = \{\mathbf{T}\}_{21}, which is a result of the symmetry of \mathbf{E}, see Eq. (3.114).

Equation (2.21), the first of the governing equations, may be written as

\[ 0 = (\{\mathbf{T}\}_{11} + \{\Sigma\}_{11}), \quad 0 = (\{\mathbf{T}\}_{21} + \{\Sigma\}_{21}), \quad (3.139) \]

for the static case with no body or double forces per unit volume. (Since the domain of the displacement functions is one-dimensional, \{\text{Div}\mathbf{A}\}_i = \{\mathbf{A}\}_{i1,1} for \( i = 1, 2 \).) Applying Eqs. (3.127), (3.129), (3.131), and (3.133) to (3.139) leads to the following general form for two of the governing equations:

\[ 0 = -(B_{11} + B_{12} + G_{11} + G_{21})\beta'(x) + (B_{13} - B_{14} + G_{31} - G_{41})\phi'(x) \]
\[ + (B_{11} + C_{11} + 2G_{11})u''(x) + (B_{14} + C_{13} + G_{13} + G_{41})v''(x) \]
\[ + (D_{11} + D_{12} + F_{11} + F_{21})\beta''(x) - (D_{13} - D_{14} + F_{31} - F_{41})\phi''(x), \quad (3.140) \]

\[ 0 = -(B_{14} + B_{24} + G_{13} + G_{23})\beta'(x) + (B_{34} - B_{44} + G_{33} - G_{43})\phi'(x) \]
\[ + (B_{14} + C_{13} + G_{13} + G_{41})u''(x) + (B_{44} + C_{33} + 2G_{43})v''(x) \]
\[ + (D_{41} + D_{42} + F_{13} + F_{23})\beta''(x) - (D_{43} - D_{44} + F_{33} - F_{43})\phi''(x). \quad (3.141) \]

In the same manner, the remaining governing equations given by Eq. (2.22) yield four relations for the static case with no body or double forces per unit volume,

\[ 0 = \{\mathbf{M}(3)\}_{111,1} + \Sigma_{11}, \quad 0 = \{\mathbf{M}(3)\}_{211,1} + \Sigma_{21}, \quad (3.142) \]
\[ 0 = \{\mathbf{M}(3)\}_{121,1} + \Sigma_{12}, \quad 0 = \{\mathbf{M}(3)\}_{221,1} + \Sigma_{22}. \quad (3.143) \]

Equations (3.142) give the remaining two governing equations while each of the terms in the right-hand sides of Eqs. (3.143) must be set identically to zero, i.e., \{\mathbf{M}(3)\}_{121,1}, \{\mathbf{M}(3)\}_{221,1}, \Sigma_{12}, \text{and} \Sigma_{22} \text{must each equal zero. Applying Eqs. (3.131), (3.133), (3.135), and (3.137) to (3.142) yields}

\[ 0 = -(B_{11} + B_{12})\beta(x) + (B_{13} - B_{14})\phi(x) + (B_{11} + G_{11})u'(x) + (B_{14} + G_{13})v'(x) \]
\[ + (D_{12} - D_{21})\beta'(x) - (D_{13} - D_{14} - D_{31} + D_{41})\phi'(x) + (D_{11} + F_{11})u''(x) \]
\[ + (D_{41} + F_{13})v''(x) + (A_{11} + A_{12})\beta''(x) - (A_{13} - A_{14})\phi''(x), \quad (3.144) \]
\[ 0 = -(B_{14} + B_{24})\beta(x) + (B_{34} - B_{44})\phi(x) + (B_{14} + G_{11})u'(x) + (B_{44} + G_{44})v'(x) \]
\[ - (D_{14} + D_{24} - D_{41} - D_{42})\beta'(x) + (D_{34} - D_{43})\phi'(x) + (D_{14} + F_{44})u''(x) \]
\[ + (D_{44} + F_{44})v''(x) + (A_{14} + A_{24})\beta''(x) - (A_{34} - A_{44})\phi''(x), \quad (3.145) \]

and applying Eqs. (3.132), (3.134), (3.136), and (3.138) to (3.143) yields

\[ 0 = -(B_{14} + B_{24})\beta(x) + (B_{34} - B_{44})\phi(x) + (B_{14} + G_{11})u'(x) + (B_{44} + G_{44})v'(x) \]
\[ - (D_{14} + D_{24} - D_{41} - D_{42})\beta'(x) + (D_{34} - D_{43})\phi'(x) + (D_{14} + F_{44})u''(x) \]
\[ + (D_{44} + F_{44})v''(x) + (A_{14} + A_{24})\beta''(x) - (A_{34} - A_{44})\phi''(x), \quad (3.146) \]
\[ 0 = -(B_{12} + B_{22})\beta(x) + (B_{23} - B_{24})\phi(x) + (B_{12} + G_{21})u'(x) + (B_{23} + G_{23})v'(x) \]
\[ - (D_{12} - D_{21})\beta'(x) - (D_{23} - D_{24} - D_{32} + D_{42})\phi'(x) + (D_{12} + F_{21})u''(x) \]
\[ + (D_{32} + F_{23})v''(x) + (A_{12} + A_{22})\beta''(x) - (A_{23} - A_{24})\phi''(x). \quad (3.147) \]

The general form of the governing equations, given by Eqs. (3.140), (3.141), (3.144), and (3.145), will be used to obtain a strain energy for the directed continuum-based on the model introduced in Section 3.1 and further developed in Section 3.3. Since \( E_{22} \) equals zero according to Eq. (3.114), \( C_{22} \) never appears in the preceding equations. In addition, the following 12 constants cancel out automatically as a result of developing Eqs. (3.140)–(3.147): \( C_{12}, C_{23}, D_{22}, D_{33}, F_{12}, F_{22}, F_{32}, F_{42}, G_{12}, G_{22}, G_{32}, \) and \( G_{42}. \) Therefore, there are now only 53 independent constants to deal with.

Since there are only four displacement functions defined over a one-dimensional domain, there will only be four governing equations. Therefore, to ensure that each term on the right-hand side of Eqs. (3.146) and (3.147) identically equals zero, it is necessary that

\[ A_{22} = -A_{12}, \quad A_{23} = -A_{13}, \quad A_{24} = -A_{14}, \quad A_{34} = A_{33}, \quad (3.148) \]
\[ B_{22} = -B_{12}, \quad B_{23} = -B_{13}, \quad B_{24} = -B_{14}, \quad B_{34} = B_{33}, \quad (3.149) \]
\[ D_{21} = D_{12}, \quad D_{43} = D_{34}, \quad (3.150) \]
\[ F_{21} = -D_{12}, \quad F_{31} = -D_{13}, \quad F_{33} = -D_{34}. \quad (3.151) \]

\[ ^* \text{In this step, and in following steps, some terms automatically drop out from the governing equations. These terms are not explicitly removed by assigning them particular values, rather, they cancel out as a result of the form of Eqs. (3.140)–(3.147).} \]
\[ G_{21} = -B_{12}, \quad G_{23} = B_{13}, \quad G_{31} = -B_{13}, \quad G_{33} = -B_{33}, \quad (3.152) \]

and

\[ F_{23} = -D_{42} = -D_{13} - D_{24} + D_{31}, \quad D_{32} = D_{13} + D_{23} - D_{31}. \quad (3.153) \]

Equations (3.148)–(3.153) represent 20 constraints on the coefficients. In addition, four more terms simply drop out of the governing equations as a result of applying these 20 constraints: \( D_{12}, \ D_{23}, \ D_{24}, \) and \( D_{34} \). This leaves only 29 independent constants appearing in the governing equations.

Before proceeding, the desired form of the governing equations as given by Eqs. (3.68)–(3.71) are rewritten below in a sequence corresponding to Eqs. (3.140)–(3.145):

\[
0 = \frac{1}{2} l^2 \left\{ 2(\hat{k}_2 + k_4)u''(x) - h\hat{k}_2 \left[ \phi''(x) + \beta''(x) \right] \right\}, \quad (3.154)
\]
\[
0 = l^2 \left[ (\hat{k}_1 + 3k_5)v''(x) - 3k_5\phi'(x) \right], \quad (3.155)
\]
\[
0 = -\frac{1}{2} hl^2\hat{k}_2u''(x) + \frac{1}{3} h^2l^2\hat{k}_2\phi''(x) + \frac{1}{3} h^2l^2\hat{k}_2\beta''(x) - k_3\beta(x), \quad (3.156)
\]
\[
0 = -\frac{l^2}{6} \left[ 3h\hat{k}_2u''(x) + (3l^2k_5 - 2h^2\hat{k}_2)\phi''(x) - 2h^2\hat{k}_2\beta''(x) - 18k_5\psi'(x) + 18k_5\phi(x) \right]. \quad (3.157)
\]

An equivalent directed continuum model is obtained by equating Eqs. (3.140), (3.141), (3.144), and (3.145) with (3.154)–(3.157). Both Eqs. (3.140) and (3.141) contain terms involving \( \beta'(x), \ \phi'(x), \ u''(x), \ v''(x), \ \beta''(x), \) and \( \phi''(x) \). On the other hand, Eq. (3.154) only contains terms in \( u''(x), \ \beta''(x), \) and \( \phi''(x) \), while Eq. (3.155) only contains terms in \( \phi'(x) \) and \( v''(x) \). To eliminate the terms that appear in Eqs. (3.140) and (3.141) but do not appear in (3.154) and (3.155), it is necessary to set

\[
C_{13} = B_{14}, \quad F_{13} = -D_{11}, \quad F_{43} = -D_{44}, \quad (3.158)
\]
\[
G_{11} = -B_{11}, \quad G_{13} = -B_{13}, \quad G_{41} = -B_{14}. \quad (3.159)
\]

In the next step, apply the 26 coefficient relations, given by Eqs. (3.148)–(3.153),
(3.158), and (3.159), to the remaining two governing equations given by (3.144) and (3.145) to obtain an intermediate form of these two governing equations

\[
0 = -(B_{11} + B_{12})\beta(x) + (B_{13} - B_{14})\phi(x) - (D_{13} - D_{14} - D_{31} + D_{41})\phi'(x) \\
+ (D_{11} + F_{11})u''(x) + (A_{11} + A_{12})\beta''(x) - (A_{13} - A_{14})\phi''(x),
\]

(3.160)

\[
0 = (B_{13} - B_{14})\beta(x) + (B_{33} - B_{44})\phi(x) + (B_{44} + G_{43})v'(x) \\
+ (D_{13} - D_{14} - D_{31} + D_{41})\beta'(x) + (D_{14} + F_{41})u''(x) \\
- (A_{13} - A_{14})\beta''(x) - (A_{33} - A_{34})\phi''(x).
\]

(3.161)

By comparing Eq. (3.160) with (3.156), it is apparent that the terms involving \(\phi(x)\) and \(\phi'(x)\) must go to zero; by comparing Eq. (3.161) with (3.157), it is apparent that the terms involving \(\beta(x)\) and \(\beta'(x)\) must go to zero. These requirements yield the following relations:

\[
B_{14} = B_{13}, \quad D_{41} = -D_{13} + D_{14} + D_{31}.
\]

(3.162)

In addition to the 8 relations given by Eqs. (3.158), (3.159), and (3.162), 4 additional constants automatically drop out of the governing equations as a result of these substitutions: \(B_{13}, D_{13}, D_{31},\) and \(D_{44}\). Therefore, this leaves just 17 constants present in the resulting governing equations. Applying all of the constraints on the constant coefficients obtained so far, i.e., Eqs. (3.148)–(3.153), (3.158), (3.159), and (3.162), to the general form of the governing equations (Eqs. (3.140)–(3.145)) yields the following four relations:

\[
0 = -(B_{11} - C_{11})u''(x) - (D_{13} + F_{31})\phi''(x) + (D_{11} + F_{11})\beta''(x),
\]

(3.163)

\[
0 = (B_{33} + C_{33} + 2G_{33})v''(x) + (B_{33} + G_{33})\phi'(x),
\]

(3.164)

\[
0 = (D_{11} + F_{11})u''(x) - (A_{13} - A_{14})\phi''(x) + (A_{11} + A_{12})\beta''(x) - (B_{11} + B_{12})\beta(x),
\]

(3.165)

\[
0 = (D_{13} + F_{31})u''(x) - (A_{33} - A_{34})\phi''(x) + (A_{13} - A_{14})\beta''(x) + (B_{33} - B_{34})\phi(x).
\]

(3.166)

As has already been mentioned, there are 17 constants present in these governing equations. To equate Eqs. (3.163)–(3.166) to (3.154)–(3.157), the following
substitutions are made:

\[
\begin{align*}
A_{12} &= -A_{11} + h^2 l^2 \hat{k}_2 / 3V, & A_{14} &= A_{13} + h^2 l^2 \hat{k}_2 / 3V, \\
A_{44} &= A_{33} + h^2 l^2 \hat{k}_2 / 3V - l^4 k_5 / 2V, & B_{12} &= -B_{11} + k_3 / V, \\
B_{33} &= -G_{43}, & B_{44} &= -G_{43} + 3l^2 k_5 / V,
\end{align*}
\] (3.167)

where the terms are divided by the volume \( V \) to ensure that \( W \) is a strain energy density. In addition to the 10 terms that are removed as a result of Eqs. (3.167)–(3.171), the following 7 terms drop out as well: \( A_{11}, A_{13}, A_{33}, B_{11}, D_{11}, D_{14}, \) and \( G_{43} \), so that no coefficients appear in the final system of governing equations.

Although we began with 66 coefficients, the term \( C_{22} \) will never appear in the resulting energy, nor in any subsequent calculations, based solely on the kinematics. There is no need to consider this term any further so it is only necessary to consider 65 coefficients. In the course of obtaining the desired governing equations, 38 constraint relations were explicitly introduced. In addition, 27 constants simply dropped out of the formulation as a result of algebra. After applying the 38 relations to Eq. (3.122), one obtains the following form of the strain energy density:

\[ W = \left[ k_3 \beta(x)^2 / 2 + 3l^2 k_5 \phi(x)^2 / 2 + l^2 (\hat{k}_2 + k_4) u'(x)^2 / 2 - 3l^2 k_5 \phi(x) v'(x) \\
+ l^2 (\hat{k}_1 + 3k_5) v'(x)^2 / 2 - (D_{11} + 2D_{12} + D_{22}) \beta(x) \beta'(x) \\
+ (D_{13} - D_{14} + D_{23} - D_{24}) \phi(x) \beta'(x) - hl^2 \hat{k}_2 u'(x) \beta'(x) / 2 + h^2 l^2 \hat{k}_2 \beta'(x)^2 / 6 \\
+ (D_{13} - D_{14} + D_{23} - D_{24}) \phi(x) \phi'(x) - (D_{33} - 2D_{34} + D_{44}) \phi(x) \phi'(x) \\
- hl^2 \hat{k}_2 u'(x) \phi'(x) / 2 + h^2 l^2 \hat{k}_2 \beta'(x) \phi'(x) / 3 + l^2 (2h^2 \hat{k}_2 - 3l^2 k_5) \phi'(x)^2 / 12 \right] / V. \]

(3.172)

Of the 27 constants that were not explicitly removed and which could appear, only 10 remain in Eq. (3.172): \( D_{11}, D_{12}, D_{13}, D_{14}, D_{22}, D_{23}, D_{24}, D_{33}, D_{34}, \) and \( D_{44} \). These are actually grouped into three different terms in Eq. (3.172). By choosing different values of these 10 constants, one will obtain different values of the strain energy density, different stress terms, and different natural boundary conditions. But, regardless of the values of these 10 terms, one will obtain the
identical system of governing equations. These terms taken together are referred to as a null Lagrangian, since they appear in the energy but drop out when calculating the Lagrangian [116]. For the present work, $D = 0$, i.e., the entire $D$ matrix will be identically equal to zero and the energy density is given by

$$W = \left[ k_3 \beta(x)^2 / 2 + 3l^2 k_5 \phi(x)^2 / 2 + l^2 (\hat{k}_2 + k_4) u'(x)^2 / 2 - 3l^2 k_5 \phi(x)^2 / 2 + l^2 (\hat{k}_1 + 3k_5)v'(x)^2 / 2 - hl^2 \hat{k}_2 \nu'(x) \beta'(x) / 2 + h^2 l^2 \hat{k}_2 \beta'(x)^2 / 6 - hl^2 \hat{k}_2 \nu'(x) \phi'(x) / 2 + h^2 l^2 \hat{k}_2 \beta'(x) \phi'(x) / 3 + l^2 (2l \hat{k}_2 - 3l^2 k_5) \phi'(x)^2 / 12 \right] / V. \quad (3.173)$$

### 3.4.1.6 The boundary value problem

By applying the value of the energy density given by Eq. (3.173) to Eqs. (2.20), one obtains six stress terms:

$$\{T\}_{11} = \left\{ k_3 \beta(x) + l^2 (\hat{k}_2 + k_4) u'(x) - hl^2 \hat{k}_2 [\beta'(x) + \phi'(x)] / 2 \right\} / V; \quad (3.174)$$

$$\{T\}_{21} = \left[ l^2 \hat{k}_1 v'(x) \right] / V, \quad (3.175)$$

$$\{\Sigma\}_{11} = [-k_3 \beta(x)] / V, \quad (3.176)$$

$$\{\Sigma\}_{21} = \left\{ 3l^2 k_5 [v'(x) - \phi(x)] \right\} / V, \quad (3.177)$$

$$\{M_{(3)}\}_{111} = \left\{ -hl^2 \hat{k}_2 \left[ 3u'(x) - 2h [\beta'(x) + \phi'(x)] \right] / 6 \right\} / V, \quad (3.178)$$

$$\{M_{(3)}\}_{211} = \left\{ -hl^2 \hat{k}_2 \left[ u'(x) / 2 - h \beta'(x) / 3 \right] + l^2 \left[ h^2 \hat{k}_2 / 3 - l^2 k_5 / 2 \right] \phi'(x) \right\} / V. \quad (3.179)$$

Applying Eqs. (3.174)–(3.179) to (3.139) and (3.142) yields the four governing equations for the static case with no body forces or double forces per unit volume:

$$0 = \frac{1}{2} l^2 \left\{ 2(\hat{k}_2 + k_4) u''(x) - h \hat{k}_2 \left[ \phi''(x) + \beta''(x) \right] \right\}, \quad (3.180)$$

$$0 = l^2 \left[ (\hat{k}_1 + 3k_5) v''(x) - 3k_5 \phi'(x) \right], \quad (3.181)$$

$$0 = -\frac{1}{2} hl^2 \hat{k}_2 u''(x) + \frac{1}{3} h^2 l^2 \hat{k}_2 \phi''(x) + \frac{1}{3} h^2 l^2 \hat{k}_2 \beta''(x) - k_3 \beta(x), \quad (3.182)$$

$$0 = -\frac{l^2}{6} \left[ 3h \hat{k}_2 u''(x) + (3l^2 k_5 - 2h^2 \hat{k}_2) \phi''(x) - 2h^2 \hat{k}_2 \beta''(x) - 18k_5 v'(x) + 18k_5 \phi(x) \right], \quad (3.183)$$
where the \( V \) terms cancel out since each of these equations is set equal to zero.

The natural boundary conditions are given by Eqs. (2.23) and are rewritten here using index notation as

\[
\{T\}_{i1} + \{\Sigma\}_{i1} n_1 = 0, \quad \{M_{(3)}\}_{i11} n_1 = 0, \quad i = 1, 2, \tag{3.184}
\]

where \( n_1 = \pm 1 \), the surface force per unit area \( s \) equals 0, and the double force per unit area \( S \) equals 0. Applying the resulting stress terms from Eqs. (3.174)–(3.179) to Eqs. (3.184), one obtains the following natural boundary conditions:

\[
0 = \left\{ \frac{l^2 (\hat{k}_2 + k_4) u'(x)}{2} - h l^2 \hat{k}_2 [\beta'(x) + \phi'(x)] / 2 \right\} \bigg|_{x=0,L} \tag{3.185}
\]
\[
0 = l^2 \left[ (\hat{k}_1 + 3k_5)v'(x) - 3k_5 \phi(x) \right] \bigg|_{x=0,L}, \tag{3.186}
\]
\[
0 = \left( -h l^2 \hat{k}_2 \left\{ 3u'(x) - 2h [\beta'(x) + \phi'(x)] \right\} / 6 \right) \bigg|_{x=0,L}, \tag{3.187}
\]
\[
0 = \left\{ -h l^2 \hat{k}_2 \left[ u'(x)/2 - h \beta'(x)/3 \right] + l^2 \left[ h^2 \hat{k}_2/3 - l^2 k_5/2 \right] \phi'(x) \right\} \bigg|_{x=0,L}. \tag{3.188}
\]

Either the natural boundary conditions given by Eqs. (3.185)–(3.188), the essential boundary conditions given by specifying the exact value of the displacement at the boundary, or a combination of these boundary conditions will be applied based on the particular loading condition under consideration. Two different conditions will be considered in Sections 3.4.3 and 3.4.4.

### 3.4.2 Nondimensional Form of the BVP

The purpose of this section is to apply the nondimensionalization scheme introduced in Section 3.3 by Eqs. (3.81)–(3.85) to the BVP given in the previous section, beginning with the strain energy as given by Eq. (3.173):

\[
W = l^2 k_5 \left[ \frac{\kappa_3}{4} \bar{\beta}(\xi)^2 + \frac{3}{2} \bar{\phi}(\xi)^2 - \frac{3 \kappa_3 \kappa_4}{4 \kappa_1 (3 \kappa_1 - 2 \eta \kappa_4)} \bar{u}'(\xi)^2 - \frac{3 \kappa_3 \kappa_4}{4 \kappa_1 (3 \kappa_1 - 2 \eta \kappa_4)} \bar{\phi}'(\xi)^2 \right.
\]
\[
\left. + \frac{3}{2 \kappa_2} \bar{u}'(\xi)^2 + \frac{3 \kappa_3}{2 (3 \kappa_1 - 2 \eta \kappa_4)} \bar{u}'(\xi) \bar{\beta}'(\xi) - \frac{\eta \kappa_3}{2 (3 \kappa_1 - 2 \eta \kappa_4)} \bar{\beta}'(\xi)^2 \right]
\]
\[
+ \frac{3 \kappa_3}{2 (3 \kappa_1 - 2 \eta \kappa_4)} \bar{u}'(\xi) \bar{\phi}'(\xi) - \frac{\eta \kappa_3}{3 \kappa_1 - 2 \eta \kappa_4} \bar{\beta}'(\xi) \bar{\phi}'(\xi) \right).
\]
Note that $W$ still has dimensions of energy per unit volume, due to the presence of the term $l^2 k_5/V$ that is multiplied with the nondimensional term in square brackets. In a similar manner, the equations for stress, Eqs. (3.174)–(3.179), may also be nondimensionalized to give

\[
\{\mathbf{T}\}_{11} = l^2 k_5 V^{-1} \left\{ \kappa_3 \tilde{\beta} (\xi) / 2 - \frac{3\kappa_3}{2(3\kappa_1 - 2\eta\kappa_4)} [\kappa_4 \tilde{u}'(\xi) / \kappa_1 - \tilde{\beta}'(\xi) - \tilde{\phi}'(\xi)] \right\},
\]

(3.190)

\[
\{\mathbf{T}\}_{21} = -l^2 k_5 V^{-1} (3 - 3/\kappa_2) \tilde{v}'(\xi),
\]

(3.191)

\[
\{\mathbf{\Sigma}\}_{11} = -l^2 k_5 V^{-1} \kappa_3 \tilde{\beta} (\xi) / 2,
\]

(3.192)

\[
\{\mathbf{\Sigma}\}_{21} = 3l^2 k_5 V^{-1} \left[ \tilde{v}'(\xi) - \tilde{\phi}(\xi) \right]
\]

(3.193)

\[
\{\mathbf{M}\}_{111} = l^3 k_5 V^{-1} \left( \frac{\kappa_3}{2(3\kappa_1 - 2\eta\kappa_4)} \right) \left\{ 3\tilde{u}'(\xi) - 2\eta \left[ \tilde{\beta}'(\xi) + \tilde{\phi}'(\xi) \right] \right\},
\]

(3.194)

\[
\{\mathbf{M}\}_{211} = l^3 k_5 V^{-1} \left( \frac{1}{2(3\kappa_1 - 2\eta\kappa_4)} \right) \left[ 3\kappa_3 \tilde{u}'(\xi) - 2\eta\kappa_3 \tilde{\beta}'(\xi) \right.
\]

(3.195)

\[
- (3\kappa_1 + 2\eta\kappa_3 - 2\eta\kappa_4) \tilde{\phi}'(\xi)
\]

Applying the same procedure to the governing equations, Eqs. (3.180)–(3.183), yields

\[
0 = \tilde{u}''(\xi) - \kappa_1 \tilde{\beta}(\xi),
\]

(3.196)

\[
0 = \tilde{v}''(\xi) - \kappa_2 \tilde{\phi}'(\xi),
\]

(3.197)

\[
0 = \tilde{\phi}''(\xi) - 6\tilde{v}'(\xi) + 6\tilde{\phi}(\xi) - \kappa_3 \tilde{\beta}(\xi),
\]

(3.198)

\[
0 = \tilde{\beta}''(\xi) + 6\tilde{v}'(\xi) - 6\tilde{\phi}(\xi) + \left( \kappa_3 - \kappa_4 \right) \tilde{\beta}(\xi),
\]

(3.199)

which as expected, are identical to Eqs. (3.77)–(3.80). Finally, the natural boundary conditions given by Eqs. (3.185)–(3.188) may be nondimensionalized to give

\[
0 = \left[ \kappa_4 \tilde{u}'(\xi) / \kappa_1 - \tilde{\beta}'(\xi) - \tilde{\phi}'(\xi) \right] \bigg|_{\xi=0,\Lambda},
\]

(3.200)

\[
0 = \left[ \tilde{\phi}(\xi) - \tilde{v}'(\xi) / \kappa_2 \right] \bigg|_{\xi=0,\Lambda},
\]

(3.201)
\[ 0 = \left\{ -3\bar{u}'(\xi) + 2\eta \left[ \bar{\beta}'(\xi) + \bar{\phi}'(\xi) \right] \right\}_{\xi=0,\Lambda}, \tag{3.202} \]
\[ 0 = \left[ -3\kappa_3\bar{u}'(\xi) + 2\eta\kappa_3\bar{\beta}'(\xi) + (3\kappa_1 + 2\eta\kappa_3 - 2\eta\kappa_4)\bar{\phi}'(\xi) \right]_{\xi=0,\Lambda}. \tag{3.203} \]

The nondimensional BVP to be solved in Section 4 will consist of Eqs. (3.196)–(3.199) and the boundary conditions given by either Eqs. (3.200)–(3.203) or nondimensional, essential boundary conditions. (Note that Eqs. (3.202) and (3.203) differ from the natural boundary conditions formulated for the discrete-based continuous model presented in Section 3.3 and given by Eqs. (3.103)–(3.106).) The specific boundary conditions to be used in the directed continuum model will be described in the following two sections.

### 3.4.3 The Simply Supported Thin Film Model: Case I

Figure 3.8 shows a model of a thin film of nondimensional length \( \Lambda \) supported at both ends \( \xi = 0 \) and \( \xi = \Lambda \). Since the ends are pinned, the constraints consist of the following: no vertical deflection (\( \tilde{v} \)) at either end, no horizontal deflection (\( \bar{u} \)) at \( \xi = 0 \), and a horizontal deflection of \( \delta_u \) at \( \xi = \Lambda \), as shown. Therefore, the boundary conditions include four essential conditions:

\[ \tilde{u}(0) = 0, \quad \bar{u}(\Lambda) = -\delta_u, \quad \bar{v}(0) = 0, \quad \bar{v}(\Lambda) = 0, \tag{3.204} \]
and four natural boundary conditions. To obtain the natural boundary conditions, which follow from Eqs. (3.202) and (3.203), consider them at $\xi = 0$:

$$0 = -3\ddot{u}(0) + 2\eta\ddot{\beta}(0) + 2\eta\ddot{\phi}(0),$$  
(3.205)

$$0 = \kappa_3 \left[ -3\ddot{u}(0) + 2\eta\ddot{\beta}(\xi) + 2\eta\ddot{\phi}(0) \right] + (3\kappa_1 - 2\eta\kappa_4)\ddot{\phi}(0).$$  
(3.206)

Applying Eq. (3.205) to (3.206) leads to the requirement that

$$(3\kappa_1 - 2\eta\kappa_4)\ddot{\phi}'(0) = 0.$$  
(3.207)

Recalling the inequalities given by Eqs. (3.86), we make the following observations:

$$\eta\kappa_4 / 2 > \kappa_1 > 0 \rightarrow 2\eta\kappa_4 > 4\kappa_1 > 3\kappa_1 > 0,$$  
(3.208)

such that

$$3\kappa_1 - 2\eta\kappa_4 < 0.$$  
(3.209)

This means that $\ddot{\phi}(0)$ must equal zero, and based on Eq. (3.205), it follows that

$$-3\ddot{u}(0) + 2\eta\ddot{\beta}'(0) = 0.$$  

The same approach is used for the boundary at $\xi = \Lambda$ and the four natural boundary conditions are

$$\ddot{\phi}'(0) = 0, \quad \ddot{\phi}'(\Lambda) = 0,$$  
(3.210)

$$3\ddot{u}'(0) - 2\eta\ddot{\beta}'(0) = 0, \quad 3\ddot{u}'(\Lambda) - 2\eta\ddot{\beta}'(\Lambda) = 0.$$  
(3.211)

This loading condition is the directed continuum equivalent of that presented in Section 3.1.3 and shown in Fig. 3.4. This boundary condition case will be referred to as Case I in subsequent sections.

### 3.4.4 The Rigidly Fixed Thin Film Model: Case II

Figure 3.9 shows the boundary conditions subjected to the thin film model of length $\Lambda$. All eight boundary conditions, for the four displacements at each boundary, are essential boundary conditions:

$$\ddot{u}(0) = 0, \quad \ddot{u}(\Lambda) = 0, \quad \ddot{v}(0) = 0, \quad \ddot{v}(\Lambda) = \delta_v.$$  
(3.212)
Figure 3.9. A rigidly fixed continuous thin film subjected to a vertical displacement $\delta_v$. The top shaded region represents the columnar structure and the bottom represents the substrate.

Each of the boundary displacements are zero, with the exception of the nondimensionalized right-hand vertical displacement, i.e., $\tilde{v}(\Lambda) = \delta_v$. The rigidly fixed boundary condition case is referred to as Case II.

$$\tilde{\phi}(0) = 0, \quad \tilde{\phi}(\Lambda) = 0, \quad \tilde{\beta}(0) = 0, \quad \tilde{\beta}(\Lambda) = 0.$$  (3.213)
Analytical Solution and Analysis

In this section, a method to solve the BVP developed in Section 3.4 is presented. Based on this solution, a characterization scheme is proposed with the ultimate goal of relating the observed behavior of the model with the nondimensional input parameters used in the model (the $\kappa$ constants). When this is accomplished, it will be possible to finally discuss the effect the original parameters (the spring constants $k_1$, $k_2$, $k_3$, $k_4$, and $k_5$ and lengths $l$, $h$, and $L$) have on the directed continuum model. In other words, the analysis seeks to relate the input parameters to the BVP with the resulting deformations to give insight into the physical meaning of the input parameters.

Before developing the analytical solution for the directed continuum model, it is worth addressing the following question: why does this work make use of the directed continuum formulation of Section 3.4 instead of the discrete-based continuum formulation of Section 3.3? In terms of the resulting BVPs, the difference between these approaches is the difference between the boundary conditions (since the governing equations are the same). At a more fundamental level, one should consider the boundary of the model, i.e., the boundary of the domain over which the governing equations are defined, and the location of the boundary where the boundary conditions are applied, see Fig. 4.1. In the discrete-based approach, the boundary conditions of the continuum model are based on the boundary conditions of the discrete model. That is, the boundary conditions of the discrete model, applied at the boundary nodes, must be transferred to the boundary of the continuum model. As such, the discrete-based model is obtained from a different
Figure 4.1. The governing equations and the boundary conditions are obtained directly from the discrete model (via a Taylor series expansion) in the case of the discrete-based continuum model. The governing equations and boundary conditions are obtained from the strain energy density in the directed continuum model.

(longer) discrete model. In the directed continuum approach, both the governing equations and the boundary conditions come from the strain energy density and the boundary conditions have a more direct connection to what the model is actually subjected to. For example, if one fixes the vertical deflection \( \tilde{u} \) at the left-hand boundary (when \( \xi = 0 \)) for both models, the actual boundary condition will differ. In the directed continuum model, the boundary condition is rather straightforward: \( \tilde{u}(0) = \text{constant} \). In the discrete-based model, the boundary condition is of the form \( \tilde{u}(0) - \tilde{u}'(0) + \kappa_1 \tilde{\beta}(0)/2 = \text{constant} \), recall Eq. (3.99). Because the mathematical representation of the boundary conditions have a more direct connection with the physical application of the boundary conditions in the directed continuum model, the directed continuum model will be used in all subsequent sections.

### 4.1 The Solution via the Jordan Canonical Form

The purpose of this section is to demonstrate the method used to obtain an analytical solution to the system of ordinary differential equations given by Eqs. (3.196)–(3.199). As a first step, these four equations may be recast as eight first order differential equations by introducing the functions \( \bar{u}, \bar{v}, \bar{\beta}, \) and \( \bar{\phi} \), which are defined as follows:

\[
\bar{u}'(\xi) \equiv \bar{u}(\xi), \quad \bar{v}'(\xi) \equiv \bar{v}(\xi), \quad \bar{\beta}'(\xi) \equiv \bar{\beta}(\xi), \quad \bar{\phi}'(\xi) \equiv \bar{\phi}(\xi).
\]  (4.1)
Note that these relations are all nondimensional. After applying Eqs. (4.1) to (3.196)–(3.199), one obtains the following eight first order differential equations:

\[ 0 = \tilde{u}'(\xi) - \bar{u}(\xi), \quad (4.2) \]
\[ 0 = \tilde{v}'(\xi) - \bar{v}(\xi), \quad (4.3) \]
\[ 0 = \beta'\tilde{\xi}(\xi) - \bar{\beta}(\xi), \quad (4.4) \]
\[ 0 = \tilde{\phi}'(\xi) - \bar{\phi}(\xi), \quad (4.5) \]
\[ 0 = \bar{u}'(\xi) - \kappa_1\beta(\xi), \quad (4.6) \]
\[ 0 = \bar{v}'(\xi) - \kappa_2\bar{\phi}(\xi), \quad (4.7) \]
\[ 0 = \beta'(\xi) + 6\bar{v}(\xi) + (\kappa_3 - \kappa_4)\bar{\beta}(\xi) - 6\bar{\phi}(\xi), \quad (4.8) \]
\[ 0 = \bar{\phi}'(\xi) - \bar{v}(\xi) - \kappa_3\bar{\beta}(\xi) + 6\bar{\phi}(\xi). \quad (4.9) \]

Equations (4.2)–(4.9) may be written as

\[ p' - A p = 0, \quad (4.10) \]

where

\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & \kappa_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \kappa_2 & 0 & 0 & 0 \\
0 & 0 & -\kappa_3 + \kappa_4 & 6 & 0 & -6 & 0 & 0 \\
0 & 0 & \kappa_3 & -6 & 0 & 6 & 0 & 0
\end{bmatrix}, \quad (4.11)
\]

and the vector \( p \) consists of the eight displacement functions

\[
p = \{ \tilde{u}(\xi) \tilde{v}(\xi) \bar{\beta}(\xi) \bar{\phi}(\xi) \bar{u}(\xi) \bar{v}(\xi) \bar{\beta}(\xi) \bar{\phi}(\xi) \}^T. \quad (4.12)
\]

It is apparent that \( A \) is a constant coefficient matrix of the linear homogeneous system given by Eq. (4.10).

The characteristic polynomial, \( p_A(\lambda) \), of the system of equations is obtained
by calculating the following determinant:

\[ p_A(\lambda) = \det(\lambda I - A) \]
\[ = \lambda^4 \left[ \kappa_4(-6 + 6\kappa_2 - \lambda^2) + \lambda^2(6 - 6\kappa_2 + \kappa_3 + \lambda^2) \right], \]

where \( I \) is the identity matrix. The eigenvalues are obtained by setting \( p_A(\lambda) \) to zero and solving for \( \lambda \). There will be eight eigenvalues corresponding to the eighth power of \( \lambda \), which is the highest order appearing in Eq. (4.14). Four of the eigenvalues will be zero; the remaining four eigenvalues are given by \( \pm \lambda^+ \) and \( \pm \lambda^- \), where

\[ \lambda^+ = \frac{\sqrt{\kappa_4 - [\kappa_3 + 6(1 - \kappa_2)] + \sqrt{\{\kappa_4 - [\kappa_3 + 6(1 - \kappa_2)]\}^2 + 24(1 - \kappa_2)\kappa_4}}}{\sqrt{2}}, \]
\[ \lambda^- = \frac{\sqrt{\kappa_4 - [\kappa_3 + 6(1 - \kappa_2)] - \sqrt{\{\kappa_4 - [\kappa_3 + 6(1 - \kappa_2)]\}^2 + 24(1 - \kappa_2)\kappa_4}}}{\sqrt{2}}. \]

It is interesting to note that the nondimensional constant \( \kappa_1 \) does not appear in the eigenvalues. Based on the inequalities given earlier by Eqs. (3.86), it is apparent that

\[ \lambda^+, \text{Im}[\lambda^-] > 0, \]

where \( \lambda^+ \) is always a real number, \( \lambda^- \) is always an imaginary number, and \( \text{Im}[\lambda^-] \) refers to the imaginary part of \( \lambda^- \).

In order to obtain the corresponding eigenvectors, it is useful to consider another matrix that has the Jordan canonical form of \( A \), which will be called the block matrix \( J \). One may say that \( A \) and \( J \) are similar if there exists an invertible
matrix $Q$ such that $A = QJQ^{-1}$. The matrix $J$ is given by
\[
J = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\lambda^- & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda^- & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\lambda^+ & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda^+ & 0 \\
\end{bmatrix}.
\]
(4.18)

Based on what has already been given for $A$ and $J$, it follows that the $Q$ matrix is
\[
Q = \begin{bmatrix}
0 & 0 & 1 & 0 & -2\kappa_1/(\lambda^-\mu_2) & 2\kappa_1/(\lambda^-\mu_2) & -2\kappa_1/(\lambda^+\mu_1) & 2\kappa_1/(\lambda^+\mu_1) \\
1 & 0 & 0 & 0 & \kappa_2/(\lambda^-)^2 & \kappa_2/(\lambda^-)^2 & \kappa_2/(\lambda^+)^2 & \kappa_2/(\lambda^+)^2 \\
0 & 0 & 0 & 0 & -2\lambda^-/\mu_2 & 2\lambda^-/\mu_2 & -2\lambda^+/\mu_1 & 2\lambda^+/\mu_1 \\
0 & 1 & 0 & 0 & -1/\lambda^- & 1/\lambda^- & -1/\lambda^+ & 1/\lambda^+ \\
0 & 0 & 0 & 1 & 2\kappa_1/\mu_2 & 2\kappa_1/\mu_2 & 2\kappa_1/\mu_1 & 2\kappa_1/\mu_1 \\
0 & 1 & 0 & 0 & -\kappa_2/\lambda^- & \kappa_2/\lambda^- & -\kappa_2/\lambda^+ & \kappa_2/\lambda^+ \\
0 & 0 & 0 & 0 & 2(\lambda^-)^2/\mu_2 & 2(\lambda^-)^2/\mu_2 & 2(\lambda^+)^2/\mu_1 & 2(\lambda^+)^2/\mu_1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{bmatrix},
\]
(4.19)

where
\[
\mu_1 = 2\kappa_4 - 2(\lambda^+)^2,
\]
(4.20)
\[
\mu_2 = 2\kappa_4 - 2(\lambda^-)^2.
\]
(4.21)

Although not shown here, $Q$ is invertible. The matrix $J$ is in block diagonal form, that is, it may be considered to consist of six diagonal terms such that it may also
be written as

$$J = \begin{bmatrix}
J_1 & 0 & 0 & 0 & 0 & 0 \\
0 & J_2 & 0 & 0 & 0 & 0 \\
0 & 0 & J_3 & 0 & 0 & 0 \\
0 & 0 & 0 & J_4 & 0 & 0 \\
0 & 0 & 0 & 0 & J_5 & 0 \\
0 & 0 & 0 & 0 & 0 & J_6
\end{bmatrix} = \text{diag} (J_1, J_2, J_3, J_4, J_5, J_6), \quad (4.22)$$

where

$$J_1 = \begin{bmatrix}
\lambda_1 & 1 \\
0 & \lambda_1
\end{bmatrix} \quad J_2 = \begin{bmatrix}
\lambda_2 & 1 \\
0 & \lambda_2
\end{bmatrix} \quad J_3 = [\lambda_3] \quad (4.23)$$

$$J_4 = [\lambda_4] \quad J_5 = [\lambda_5] \quad J_6 = [\lambda_6], \quad (4.24)$$

and

$$\lambda_1 = \lambda_2 = 0 \quad \lambda_3 = -\lambda^- \quad \lambda_4 = \lambda^- \quad \lambda_5 = -\lambda^+ \quad \lambda_6 = \lambda^+. \quad (4.25)$$

The algebraic multiplicity for the eigenvalue zero is four since four of the eight eigenvalues have the value zero. The algebraic multiplicity for the remaining eigenvalues is one for each eigenvalue. According to standard procedures [20], it follows that

$$e^{A\xi} = Q e^{J\xi} Q^{-1}, \quad (4.26)$$

where

$$e^{J\xi} = \text{diag} (e^{J_1\xi}, e^{J_2\xi}, e^{J_3\xi}, e^{J_4\xi}, e^{J_5\xi}, e^{J_6\xi}). \quad (4.27)$$
The matrix form of $e^{J\xi}$ for this problem is given by the following:

$$
e^{J\xi} = \begin{bmatrix}
1 & \xi & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & \xi & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \xi \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\lambda-\xi} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\lambda^+\xi} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{\lambda-\xi} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{\lambda^+\xi}
\end{bmatrix}.$$  \hspace{1cm} (4.28)

A basis for $p' = A p$ is given by the matrix $P(\xi) = Q e^{J\xi}$. If the 8 columns of $P(\xi)$ are represented as vectors $P_1, P_2, \ldots, P_8$, then one may show that the solution for $p$ as it appears in Eq. (4.10) is given as

$$p = c_1 P_1 + c_2 P_2 + c_3 P_3 + c_4 P_4 + c_5 P_5 + c_6 P_6 + c_7 P_7 + c_8 P_8.$$ \hspace{1cm} (4.29)

The eight unknown constants, $c_1, c_2, \ldots, c_8$, are found by solving for the eight boundary conditions. For example, these are given by Eqs. (3.204)–(3.211) for the simply supported version of the thin film model and Eqs. (3.212) and (3.213) for the rigidly fixed boundary conditions.

Using the approach just presented, the following solution was obtained for the simply supported case (Case I) described in Section 3.4.3:

$$\tilde{u} = \frac{\delta_u}{a_1 + a_2} \left\{ a_3 \sin \left[ \text{Im}[\lambda^-](\xi - \Lambda/2) \right] + a_4 \left[ e^{(\Lambda-\xi)^+} - e^{\xi^+} \right] + a_2 \left( \frac{\Lambda/2 - \xi}{\Lambda} \right) \right\} - \frac{\delta_u}{2},$$ \hspace{1cm} (4.30)

$$\tilde{v} = \frac{\delta_u \kappa_2}{a_1 + a_2} \left\{ \frac{1}{(\lambda^+)^2} - \frac{1}{(\lambda^-)^2} + \frac{\sec(\text{Im}[\lambda^-] \Lambda/2)}{(\lambda^-)^2} \cos \left[ \text{Im}[\lambda^-](\xi - \Lambda/2) \right] - \frac{a_5}{\lambda^+} \left[ e^{(\Lambda-\xi)\lambda^+} + e^{\xi\lambda^+} \right] \right\},$$ \hspace{1cm} (4.31)

$$\tilde{\phi} = \frac{\delta_u}{a_1 + a_2} \left\{ \frac{\sec(\text{Im}[\lambda^-] \Lambda/2)}{\text{Im}[\lambda^-]} \sin \left[ \text{Im}[\lambda^-](\xi - \Lambda/2) \right] + a_5 \left[ e^{(\Lambda-\xi)\lambda^+} - e^{\xi\lambda^+} \right] \right\},$$ \hspace{1cm} (4.32)
\[ \bar{\beta} = \frac{\delta_u}{a_1 + a_2} \left\{ a_6 \sin \left[ \text{Im}[\lambda^-](\xi - \Lambda/2) \right] + a_7 \left[ e^{(\Lambda-\xi)\lambda^+} - e^{\xi\lambda^+} \right] \right\} , \] 

where

\[ a_1 = \frac{2\kappa_1}{\text{Im}[\lambda^-][(\lambda^-)^2 - \kappa^2]} \tan(\text{Im}[\lambda^-]\Lambda/2) + \frac{2\kappa_1}{\lambda^+[(\lambda^+)^2 - \kappa^2]} \frac{1 - e^{\lambda^+\lambda^+}}{1 + e^{\lambda^+\lambda^+}} , \]  

\[ a_2 = \kappa_1 \lambda^+ \left( \frac{1}{(\lambda^+)^2 - \kappa^2} - \frac{1}{(\lambda^-)^2 - \kappa^2} \right) - \frac{2\eta_\lambda}{3} \pi \lambda^+ \left( \frac{(\lambda^+)^2}{(\lambda^+)^2 - \kappa^2} - \frac{(\lambda^-)^2}{(\lambda^-)^2 - \kappa^2} \right) , \]  

\[ a_3 = \frac{-\kappa_1}{(\lambda^-)^2 - \kappa^2} \sec(\text{Im}[\lambda^-]\Lambda/2) , \]  

\[ a_4 = \frac{1}{(\lambda^+)^2 - \kappa^2} \frac{1}{1 + e^{\lambda^+\lambda^+}} , \]  

\[ a_5 = \frac{1}{\lambda^+} \frac{1}{1 + e^{\lambda^+\lambda^+}} , \]  

\[ a_6 = \frac{\text{Im}[\lambda^-] \cos(\text{Im}[\lambda^-]\Lambda/2)}{(\lambda^-)^2 - \kappa^2} , \]  

\[ a_7 = \frac{-1}{(\lambda^+)^2 - \kappa^2} \frac{1}{1 + e^{\lambda^+\lambda^+}} , \]

For the rigidly fixed boundary conditions (Case II) given in Section 3.4.4, one obtains the following solution:

\[ \tilde{u} = \delta_v \kappa_1 \left\{ \frac{b_1}{\text{Im}[\lambda^-] [\kappa_- (\lambda^-)^2]} \cos \left[ \text{Im}[\lambda^-](\xi - \Lambda/2) \right] + \frac{b_2}{\lambda^+ [\kappa_- (\lambda^+)^2]} \left[ e^{\lambda^+\xi\lambda^+} + e^{(\Lambda-\xi)\lambda^+} \right] + b_3 \right\} , \]  

\[ \tilde{v} = \delta_v \kappa_2 \left\{ \frac{-b_1}{(\lambda^-)^2} \sin \left[ \text{Im}[\lambda^-](\xi - \Lambda/2) \right] + \frac{b_2}{(\lambda^+)^2} \left[ e^{\lambda^+\xi\lambda^+} - e^{(\Lambda-\xi)\lambda^+} \right] + \frac{b_3 \kappa_4}{\kappa_2} \xi + b_4 \right\} , \]  

\[ \tilde{\phi} = \delta_v \left\{ \frac{b_1}{\text{Im}[\lambda^-]} \cos \left[ \text{Im}[\lambda^-](\xi - \Lambda/2) \right] + \frac{b_2}{\lambda^+} \left[ e^{\lambda^+\xi\lambda^+} + e^{(\Lambda-\xi)\lambda^+} \right] + b_3 \kappa_4 \right\} , \]  

\[ \tilde{\beta} = \delta_v \left\{ \frac{-\text{Im}[\lambda^-]b_1}{\kappa_4 - (\lambda^-)^2} \cos \left[ \text{Im}[\lambda^-](\xi - \Lambda/2) \right] + \frac{\lambda^+ b_2}{\kappa_4 - (\lambda^+)^2} \left[ e^{\lambda^+\xi\lambda^+} + e^{(\Lambda-\xi)\lambda^+} \right] \right\} , \]

where

\[ b^{-1}_1 = \frac{-1}{\lambda^+} \frac{[(\lambda^-)^2 - (\lambda^+)^2] \kappa_4 \lambda^+}{\text{Im}[\lambda^-] \lambda^+ [\kappa_- (\lambda^-)^2]} + \frac{2\lambda^+ \kappa_2 \tan(\Lambda \text{Im}[\lambda^-]/2)}{(\lambda^-)^2} . \]
\[
\begin{align*}
&+ \frac{2\text{Im}[\lambda^-] \kappa_2 [\kappa_4 - (\lambda^+)^2] \tanh(\Lambda \lambda^+/2)}{(\lambda^+)^2 [-\kappa_4 + (\lambda^-)^2]} \cos(\text{Im}[\lambda^-] \Lambda/2), \\
&b_2^{-1} = \frac{1 + e^{\Lambda \lambda^+}}{\text{Im}[\lambda^-]} \left\{ \frac{[(\lambda^-)^2 - (\lambda^+)^2] \kappa_4 \Lambda}{\text{Im}[\lambda^-] \lambda^+ [\kappa_4 - (\lambda^+)^2]} + \frac{2\text{Im}[\lambda^-] \kappa_2 \tanh(\Lambda \lambda^+/2)}{(\lambda^+)^2} \\
&\quad \quad + \frac{2\lambda^+ \kappa_2 \kappa_4 - (\lambda^-)^2] \tan(\Lambda \text{Im}[\lambda^-]/2)}{(\lambda^-)^2 [-\kappa_4 + (\lambda^+)^2]} \right\}, \\
&b_3^{-1} = \frac{1}{(\lambda^-)^2 - (\lambda^+)^2} \left( \kappa_4 \Lambda \left[ (\lambda^-)^2 - (\lambda^+)^2 \right] + 2\kappa_2 \left\{ \text{Im}[\lambda^-] \lambda^+ \left[ \lambda^+ \tan(\Lambda \text{Im}[\lambda^-]/2) \right. \right. \\
&\quad \quad - \text{Im}[\lambda^-] \tanh(\Lambda \lambda^+/2) \left. \right. \right. \\
&\quad \quad + \kappa_4 \left[ (\lambda^+)^2 \tan(\Lambda \text{Im}[\lambda^-]/2)/\text{Im}[\lambda^-] - (\lambda^-)^2 \tanh(\Lambda \lambda^+/2)/\lambda^+ \right] \right\} \right), \\
&b_4^{-1} = 2\kappa_2 + \left\{ \kappa_4 \Lambda (\lambda^-)^2 \left[ (\lambda^-)^2 - (\lambda^+)^2 \right] \lambda^+ \right\} \left( \text{Im}[\lambda^-] \left\{ [-\kappa_4 \\
&\quad + (\lambda^-)^2] (\lambda^+)^2 \tan(\Lambda \text{Im}[\lambda^-]/2) \\
&\quad + (\lambda^-)^2 \left[ \kappa_4 - (\lambda^+)^2 \right] \text{Im}[\lambda^-] \tanh(\Lambda \lambda^+/2) \right\} \right)^{-1}.
\end{align*}
\]

Note that in Eq. (4.42), the solution for \( \tilde{v} \) for the rigidly fixed boundary conditions, the solution includes the term \( \sin [\text{Im}[\lambda^-] (\xi - \Lambda/2)] \) and the term \( e^{\xi \lambda^+} - e^{(\Lambda - \xi) \lambda^+} \). These same terms appear in the solutions for \( \tilde{u}, \tilde{\phi}, \) and \( \tilde{\beta} \) for the simply supported boundary conditions, given by Eqs. (4.30), (4.32), and (4.33). In a similar manner, consider the solutions for \( \tilde{u}, \tilde{\phi}, \) and \( \tilde{\beta} \) for the rigidly fixed boundary conditions, given by Eqs. (4.41), (4.43), and (4.44). In these relations, the solution includes the term \( \cos [\text{Im}[\lambda^-] (\xi - \Lambda/2)] \) and the term \( e^{\xi \lambda^+} + e^{(\Lambda - \xi) \lambda^+} \). These terms appear in the solution for \( \tilde{v} \) for the simply supported boundary conditions, given by Eq. (4.31). There are two components of the solution of primary interest: an oscillatory component (which refers to the trigonometric functions) and an exponential component. Therefore, there are only two oscillatory forms and two exponential forms associated with the displacement functions for the two different boundary conditions presented.
4.2 Analysis of the Simply Supported Boundary Condition (Case I)

The purpose of this section is to analyze the solution of the simply supported thin film model given by Eqs. (4.30)–(4.33) to better understand the effects of the spring constants on the overall behavior of the model. (Recall that the simply supported boundary condition is referred to as the Case I boundary condition.) The vertical displacement $\tilde{v}$, see Eq. (4.31), is given primary attention since experimentalists are typically concerned with measuring curvature, obtained from the vertical displacement, or the study of bending, sometimes accompanied by buckling.

There are three types of terms within the braces of Eq. (4.31): a constant term, a term consisting of a cosine function of $\xi$ (which we will refer to as the oscillatory term), and an exponential term (or non-oscillatory term). The constant term, given by the difference $(\lambda^+)^{-2} - (\lambda^-)^{-2}$, is of little concern in the following analysis since it represents a rigid body motion. The relative contributions of the oscillating term and the exponential term to the overall displacement will be used to classify the behavior of a particular model. There will be three different categories of solutions associated with the proposed characterization scheme.

1. The vertical deflection may be dominated by the oscillating term.
2. The vertical deflection may be dominated by the exponential term.
3. The vertical deflection may be strongly influenced by the first two.

A characterization factor, $\zeta_I$, will be associated with these three categories. Since both the oscillating and exponential terms are multiplied by the same constant in Eq. (4.31), $\delta_u \kappa_2/(a_1 + a_2)$, it is only necessary to consider the terms within the braces when comparing the relative contribution of each type of deflection to the overall vertical deflection. In addition, since the goal of this characterization scheme is to compare the relative contributions of the oscillatory and exponential terms to the vertical deflection, only the following from Eq. (4.31) will be considered:

$$
\frac{\sec(\text{Im}[\lambda^-]\Lambda/2)}{(\lambda^-)^2} \cos(\text{Im}[\lambda^-](\xi - \Lambda/2)) - \frac{1}{(\lambda^+)^2(1 + e^{\lambda^+})}\left[ e^{(\Lambda - \xi)\lambda^+} + e^{\xi\lambda^+}\right].
$$

(4.49)
The oscillating term is given by

$$\frac{\sec(\text{Im}[\lambda^-] \Lambda/2)}{(\lambda^-)^2} \cos \left[ \text{Im}[\lambda^-](\xi - \Lambda/2) \right], \quad (4.50)$$

where $\xi \in [0, \Lambda]$. The maximum value of (4.50) occurs at $\xi = \Lambda/2$, since $\cos(0) = 1$. It follows that the maximum positive value of (4.50) is given by

$$\text{Im}[\lambda^-]^{-2} \left| \sec(\text{Im}[\lambda^-] \Lambda/2) \right|, \quad (4.51)$$

This value quantifies the contribution of the oscillatory part of the solution to the overall vertical deflection of the CTF as long as the overall length $\Lambda$ is greater than or equal to $\pi/\text{Im}[\lambda^-]$.

When $\Lambda$ is less than $\pi/\text{Im}[\lambda^-]$, then the oscillatory contribution will be given by the difference between the value of the cosine term at $\xi = \Lambda/2$ (the maximum) and at $\xi = \Lambda$ (the endpoint), i.e.,

$$\frac{\sec(\text{Im}[\lambda^-] \Lambda/2)}{(\lambda^-)^2} \left\{ \cos(0) - \cos \left[ \text{Im}[\lambda^-](\Lambda - \Lambda/2) \right] \right\} \to \text{Im}[\lambda^-]^{-2} \left[ \sec(\text{Im}[\lambda^-] \Lambda/2) - 1 \right], \quad (4.52)$$

noting that the absolute value signs are removed since $\sec(\theta) > 0$ for $0 < \theta < \pi/2$. Therefore, either the relation given by (4.51) or the relation given by (4.52) will be compared with a value that quantifies the contribution of the exponential portion of the solution to the overall thin film deflection, which must be determined next.

The function $e^{(\Lambda-\xi)\lambda^+} + e^{\xi\lambda^+}$ is non-oscillatory and symmetric (about $\xi = \Lambda/2$) with a minimum value at $\xi = \Lambda/2$ and identical, maximum values at $\xi = 0, \Lambda$. To obtain a term that may be compared with (4.51), consider the difference between the magnitude of the exponential term at $\xi = 0$ and at $\xi = \Lambda/2$, i.e.,

$$\frac{1}{(\lambda^+)^2(1 + e^{\lambda^+\Lambda})} \left[ (e^{\lambda^+\Lambda} + e^0) - (e^{\lambda^+\Lambda/2} + e^{\lambda^+\Lambda/2}) \right]. \quad (4.53)$$

After some manipulation, Eq. (4.53) reduces to

$$\left[ 1 - \text{sech}(\lambda^+\Lambda/2) \right]/(\lambda^+)^2. \quad (4.54)$$
It is now convenient to introduce the characteristic function \( \zeta_I \), which is defined using (4.52) and (4.54) or (4.51) and (4.54), as the ratio of the deflection associated with the oscillatory term relative to the deflection associated with the exponential term,

\[
\zeta_I \equiv \begin{cases} 
(\lambda^+ / \text{Im}[\lambda^-])^2 \left[ \sec(\text{Im}[\lambda^-] \Lambda/2) - 1 \right] / [1 - \text{sech}(\lambda^+ \Lambda/2)], & \Lambda < \pi / \text{Im}[\lambda^-], \\
(\lambda^+ / \text{Im}[\lambda^-])^2 \left| \text{sec}(\text{Im}[\lambda^-] \Lambda/2) \right| / [1 - \text{sech}(\lambda^+ \Lambda/2)], & \Lambda \geq \pi / \text{Im}[\lambda^-].
\end{cases}
\]

The subscript “I” refers to the boundary condition Case I. It follows that if \( \zeta_I \approx 1 \), the contributions of the oscillatory and exponential terms to the vertical deflection of the film are similar in magnitude. If \( \zeta_I \ll 1 \), then the exponential (or non-oscillatory) component is dominant; when \( \zeta_I \gg 1 \), the oscillatory component is dominant. The three different categories introduced at the beginning of this section have now been associated with the numerical value of \( \zeta_I \).

By considering the form of Eqs. (4.55), it is apparent that the behavior of a particular model will be defined by the eigenvalues and the total length of the thin film. The eigenvalues are determined from the unit cell of the discrete model (its geometry and spring constants) and the total length is based on the number of unit cells comprising the entire structure. By thinking of these two aspects of the model separately, it is apparent that only the characteristics of the unit cell define the term \((\lambda^+ / \text{Im}[\lambda^-])^2\). On the other hand, both the unit cell and the total length \( \Lambda \) appear in the ratio of secant and hyperbolic secant terms.

Before proceeding with an analysis of \((\lambda^+ / \text{Im}[\lambda^-])^2\), consider the form of the secant and hyperbolic secant terms as they appear in Eqs. (4.55) and as plotted in Fig. 4.2 (note that in Fig. 4.2(b), the inverse of \(1 - \text{sech}(\lambda^+ \Lambda/2)\) is plotted, i.e., it is the product of the functions plotted in Fig. 4.2 that appears in Eqs. (4.55)). The following four observations are worth noting.

1. The ratio \(|\sec(\text{Im}[\lambda^-] \Lambda/2)|/[1 - \text{sech}(\lambda^+ \Lambda/2)]\) is greater than one for all values of \(\lambda^+\), \(\text{Im}[\lambda^-]\), and \(\Lambda\).

2. The ratio \([\sec(\text{Im}[\lambda^-] \Lambda/2) - 1]/[1 - \text{sech}(\lambda^+ \Lambda/2)]\) may possibly be less than one only if \(\text{Im}[\lambda^-] \Lambda/2\) is less than approximately 1.047 (this value is the intersection of the dashed line in Fig. 4.2(a) with the plot of \(\sec(\text{Im}[\lambda^-] \Lambda/2) - \))
Figure 4.2. These plots are taken from terms appearing in Eqs. (4.55). Figure 4.2(a) is a plot of $\sec(\text{Im}(\lambda^-)\Lambda/2) - 1$ as a function of $\text{Im}(\lambda^-)\Lambda/2 < \pi/2$ and $|\sec(\text{Im}(\lambda^-)\Lambda/2)|$ as a function of $\text{Im}(\lambda^-)\Lambda/2 > \pi/2$. Figure 4.2(b) is a plot of $1/[1 - \text{sech}(\lambda^+\Lambda/2)]$ as a function of $\lambda^+\Lambda/2$ (that is, the inverse of $1 - \text{sech}(\lambda^+\Lambda/2)$ is plotted). A dashed line indicates that the value of the vertical axis is one.

1.)

(3) According to Fig. 4.2(a), one way to have a large value of $|\sec(\text{Im}(\lambda^-)\Lambda/2)|/[1 - \text{sech}(\lambda^+\Lambda/2)]$ is to ensure that $\text{Im}(\lambda^-)\Lambda \approx (2m + 1)\pi$, where $m$ is any integer.

(4) Another way to obtain a large value of $|\sec(\text{Im}(\lambda^-)\Lambda/2)|/[1 - \text{sech}(\lambda^+\Lambda/2)]$ is to ensure that $\lambda^+\Lambda/2$ is small, according to Fig. 4.2(b).

Based on the second observation and recalling Eqs. (4.55), it is more difficult to obtain a non-oscillatory solution, i.e., $\zeta_I \ll 1$, from the ratio of secant and hyperbolic secant terms alone. On the other hand, specific choices of the overall length $\Lambda$ will lead to large values of $\zeta_I$ (the oscillatory solutions). Finally, Fig. 4.2 suggests that much of the range of values of $\text{Im}(\lambda^-)$, $\Lambda^+$, and $\Lambda$ makes the ratio of secant and hyperbolic secant terms on the order of one, i.e., neither oscillatory nor non-oscillatory-dominant. In other words, $(\lambda^+/\text{Im}(\lambda^-))^2$ from Eqs. (4.55) should
play the key role in characterizing the behavior of the solution. Therefore, the following analysis will focus on the role of the term $(\lambda^+ / \text{Im}[\lambda^-])^2$.

Recalling the definitions of the eigenvalues $\lambda^-$ and $\lambda^+$ given by Eqs. (4.15) and (4.16), it follows that

$$
\left( \frac{\lambda^+}{\text{Im}[\lambda^-]} \right)^2 = \frac{c_1 + \sqrt{c_1^2 + c_2}}{-c_1 + \sqrt{c_1^2 + c_2}},
$$

(4.56)

where

$$
c_1 \equiv \kappa_4 - [\kappa_3 + 6(1 - \kappa_2)], \quad c_2 \equiv 24(1 - \kappa_2)\kappa_4 > 0.
$$

(4.57)

The behavior of Eq. (4.56) is dependent on the value of $c_1$, i.e.,

$$
c_1 < 0 \quad \rightarrow \quad \left( \frac{\lambda^+}{\text{Im}[\lambda^-]} \right)^2 = \frac{-1 + \sqrt{1 + c_2/c_1^2}}{1 + \sqrt{1 + c_2/c_1^2}},
$$

(4.58)

$$
c_1 = 0 \quad \rightarrow \quad \left( \frac{\lambda^+}{\text{Im}[\lambda^-]} \right)^2 = 1,
$$

(4.59)

$$
c_1 > 0 \quad \rightarrow \quad \left( \frac{\lambda^+}{\text{Im}[\lambda^-]} \right)^2 = \frac{1 + \sqrt{1 + c_2/c_1^2}}{-1 + \sqrt{1 + c_2/c_1^2}},
$$

(4.60)

such that $(\lambda^+ / \text{Im}[\lambda^-])^2$ may be plotted as a function of $c_2/c_1^2$, where

$$
\frac{c_2}{c_1^2} = \frac{24(1 - \kappa_2)\kappa_4}{\{\kappa_4 - [\kappa_3 + 6(1 - \kappa_2)]\}^2}.
$$

(4.61)

The results of Eqs. (4.58)–(4.60) lead to the plot of $(\lambda^+ / \text{Im}[\lambda^-])^2$ shown in Fig. 4.3, where there are three different curves corresponding to $c_1 < 0$, $c_1 = 0$, and $c_1 > 0$.

Based on Fig. 4.3, to obtain a large value of $(\lambda^+ / \text{Im}[\lambda^-])^2$ requires that $c_1 > 0$ and that $c_2/c_1^2$ must be small. On the other hand, to obtain a small value of $(\lambda^+ / \text{Im}[\lambda^-])^2$ requires that $c_1 < 0$ and that $c_2/c_1^2$ must be small (note that the same requirement for $c_2/c_1^2$ holds for both large and small values of $(\lambda^+ / \text{Im}[\lambda^-])^2$). With these insights, it is now possible to look at $\zeta_I$ directly in terms of $c_1$ and $c_2$. To do so, it will be necessary to choose a value for $\Lambda$. Four plots of $\zeta_I$ are shown in Fig. 4.4 for $\Lambda = 100$. In addition, a bold curve representing a plot of $(\lambda^+ / \text{Im}[\lambda^-])^2$ is shown in each figure; this function will be used to characterize the behavior of
Figure 4.3. This figure shows the value of \((\lambda^+ / \text{Im}[\lambda^-])^2\) as a function of a term defined by three of the nondimensional constants: \(\kappa_2, \kappa_3, \) and \(\kappa_4\). There are three different cases based on the relationship between \(\kappa_4 \) and \(\kappa_3 + 6(1 - \kappa_2)\), as expressed using the term \(c_1\).

The oscillatory nature of the plots is of course a result of the secant term in \(\zeta_1\).*

At this point, it is also possible to address the issue raised in Fig. 4.2(a) when \(\Lambda < \pi / \text{Im}[\lambda^-]\). Recalling that figure, it is apparent that the term \(\sec(\text{Im}[\lambda^-]\Lambda / 2) - 1\) goes to zero as \(\text{Im}[\lambda^-]\Lambda / 2\) gets very small; this suggests that \(\zeta_1\) might also get very small. But according to Fig. 4.4, in this situation \(\zeta_1\) is actually quite large. In fact, as \(\text{Im}[\lambda^-]\Lambda / 2\) gets smaller, the value of \(\zeta_1\) continues to increase. In this way, the system remains oscillatory-dominant.

Therefore, we can generally characterize the simply supported thin film system using Eq. (4.56) and Fig. 4.3. As has already been discussed, to obtain an oscillatory-dominant solution, or a large value of \((\lambda^+ / \text{Im}[\lambda^-])^2\), requires that \(c_1 > 0\) and that \(c_2 / c_1^2\) must be small. These requirements lead to the following two sets of inequalities:

\[
\kappa_4 > \kappa_3 \text{ and } \kappa_2 \approx 1 \quad \text{or} \quad \kappa_4 \gg \kappa_3 + 6(1 - \kappa_2).
\] (4.62)

The requirement that \(c_2 / c_1^2\) must be small is satisfied since either \(c_2\) itself is small (when \(\kappa_2 \approx 1\)) or \(c_1^2 \gg c_2\) (when \(\kappa_4\) is large). The requirement that \(c_1\) be greater than 0 is satisfied by either of Eqs. (4.62) by inspection.

*Based on Fig. 4.2, by choosing special values of \(\Lambda\) it is possible to obtain large values of \((\lambda^+ / \text{Im}[\lambda^-])^2\). An example of this will be presented in Section 5.1.4.
Figure 4.4. This figure shows four plots of $\zeta_1$ versus $c_1$, based on Eq. (4.55), for the overall length $\Lambda$ equal to 100 and $c_2$ equaling 0.01, 0.1, 1, and 10. Note that the presence of the term $\sec(\text{Im}[\lambda^-]/\Lambda/2) - 1$, shown in Fig. 4.2(a), appears in the plot associated with $c_2 = 0.01$. The bold curve in each figure is a plot of $(\lambda^+ / \text{Im}[\lambda^-])^2$ as a function of $c_1$.

For the non-oscillatory solution, it is necessary that $c_1 < 0$ and that $c_2/c_1^2$ must still be small. For this case, any one of the following three conditions are necessary:

$$\kappa_3 > \kappa_4 \text{ and } \kappa_2 \approx 1, \quad \kappa_3 \gg \kappa_4, \text{ or } \kappa_4 \ll 1.$$  \hspace{1cm} (4.63)

The requirement that $c_2/c_1^2$ must be small is satisfied when $c_2$ is small (when $\kappa_2 \approx 1$ or when $\kappa_4 \ll 1$) or when $c_1^2 \gg c_2$ (when $\kappa_3 \gg \kappa_4$). It is worth mentioning again that the behavior of this system (originally based on a discrete model with five material and three geometric parameters) may be characterized by a single ratio of two terms, $c_2/c_1^2$. 
4.2.1 Oscillatory-dominant Solution

Conclusions drawn from Eqs. (4.62) will be used to consider an oscillatory-dominant solution, i.e., when \((\lambda^+ / \text{Im}[\lambda^-])^2\) is large. As will be shown, in general, a stiff substrate (in terms of bending) leads to an oscillatory-dominant solution. One of the requirements for a large value of \((\lambda^+ / \text{Im}[\lambda^-])^2\) is that \(\kappa_2 \approx 1\) and \(\kappa_4 > \kappa_3\). Referring to Eq. (3.82), when \(\kappa_2 \approx 1\) it follows that

\[ k_5 \gg \hat{k}_1. \]  

(4.64)

Recalling Fig. 3.1, Eq. (4.64) may be interpreted to mean that the substrate must be much stiffer in terms of resistance to bending, characterized by \(k_5\), than the columnar part of the structure, characterized by \(\hat{k}_1\). Next we will consider the inequality \(\kappa_4 > \kappa_3\) and Eqs. (3.83) and (3.84):

\[ \kappa_4 > \kappa_3 \rightarrow k_5 > \frac{1}{6} \left( \frac{h}{l} \right)^2 \frac{\hat{k}_2 (\hat{k}_2 + 4k_4)}{\hat{k}_2 + k_4}. \]  

(4.65)

By considering extreme values for \(k_4\), it is possible to simplify the right-hand side of Eq. (4.65), so that one obtains

\[ k_5 > \frac{1}{6} \left( \frac{h}{l} \right)^2 \hat{k}_2 \text{ when } k_4 \rightarrow 0, \quad k_5 > \frac{2}{3} \left( \frac{h}{l} \right)^2 \hat{k}_2 \text{ when } k_4 \rightarrow \infty. \]  

(4.66)

We will simply make use of the stricter inequality from Eqs. (4.66) to enforce the requirement \(\kappa_4 > \kappa_3\), i.e.,

\[ k_5 > \frac{2}{3} \left( \frac{h}{l} \right)^2 \hat{k}_2. \]  

(4.67)

Equation (4.67) informs us that there is a second requirement for oscillatory solutions: the film substrate must be stiff in terms of bending, characterized by \(k_5\), when compared with the stiffness of the columnar structure in terms of axial deformation, characterized by \(\hat{k}_2\). Therefore, one way of obtaining an oscillatory-dominant solution is for the substrate to be stiffer in bending than the columnar structure is in bending and in axial deformation.

A second way to obtain a large value of \((\lambda^+ / \text{Im}[\lambda^-])^2\), and an oscillatory-
dominant solution, is to satisfy the following inequality from Eqs. (4.62):

\[ \kappa_4 \gg \kappa_3 + 6(1 - \kappa_2). \]  

(4.68)

There are two possible ways to proceed with this inequality, depending on whether \( 6(1 - \kappa_2) \) is much greater than \( \kappa_3 \) or not. If \( 6(1 - \kappa_2) \) is much greater than \( \kappa_3 \), it follows from Eqs. (3.82) and (3.83) that the torsional spring constant \( k_3 \) must be much less than both the substrate bending stiffness and the columnar vertical springs, i.e., \( k_3 \ll \hat{k}_1, k_5 \). In this situation, Eqs. (4.62) lead to the requirement that \( \kappa_4 \gg 6(1 - \kappa_2) \). Based on Eqs. (3.82) and (3.84), it follows that the torsional springs must be much stiffer than the axial columnar springs, i.e., \( k_3 \gg \hat{k}_2 \). Therefore, a second way to obtain an oscillatory-dominant solution is to have a weak torsional spring relative to the bending stiffness of the columnar structure and substrate but stiff relative to the axial stiffness of the columnar structure.

On the other hand, if \( 6(1 - \kappa_2) \) is not much greater than \( \kappa_3 \), the inequality of Eq. (4.68) suggests that \( \kappa_4 \gg \kappa_3 \). Based on the preceding discussion, this situation arises whenever the torsional spring constant is not much less than the substrate bending stiffness and/or the columnar structure vertical spring constant. Applying the same procedure that was used in Eqs. (4.65)–(4.67), it follows that the substrate bending stiffness must be much greater than the axial stiffness of the columnar structure, i.e., \( k_5 \gg \hat{k}_2 \), recall Eq. (4.67). The requirements on the magnitudes of the torsional springs and the substrate bending stiffness describe a third way to obtain an oscillatory-dominant solution.

To summarize the recent discussion, the three different approaches used to obtain an oscillatory-dominant solution are given by the following three sets of inequalities:

\[ k_5 \gg \hat{k}_1 \quad \text{and} \quad k_5 > (2/3)(h/l)^2 \hat{k}_2, \]  

(4.69)

\[ \hat{k}_1, k_5 \gg k_3 \gg \hat{k}_2, \]  

(4.70)

\[ k_5 \gg \hat{k}_2 \quad \text{and} \quad k_3 \gg \hat{k}_1 \quad \text{and/or} \quad k_3 \gg k_5, \]  

(4.71)

which are all based on the nondimensional inequalities presented in Eqs. (4.62). In each of these cases, the substrate stiffness in bending, which is characterized by
At this point it is worth considering the limiting case where the behavior of the thin film model presented thus far approaches the behavior of an Euler-Bernoulli beam. One question that may be asked in the context of this characterization scheme is whether an Euler-Bernoulli beam is described by an oscillating or non-oscillating solution. In order to proceed, it is worth finding the wavelength of the solution. Since the oscillations are due to the cosine term in Eq. (4.31), it follows that the wavelength is given by

$$\mu = \frac{2\pi}{\text{Im} [\lambda^c]}.$$  \hspace{1cm} (4.72)

If the overall length of the film $\Lambda$ is less than the wavelength $\mu$, the oscillatory-dominant vertical deformation will consist of a curve with a single maximum or minimum over the range $\Lambda$. In fact, this is what occurs in the limiting case with no columnar structure present. In such a situation the columnar springs would no longer be present, i.e., $\hat{k}_1, \hat{k}_2, k_3 \to 0$. Applying these limiting conditions to Eqs. (3.82) and (3.83) leads to $\kappa_2 \to 1$ and $\kappa_3 \to 0$. From Eqs. (4.62), it is apparent that this case is an oscillatory-dominant solution. According to Eq. (4.16), under these circumstances $\text{Im} [\lambda^c] \to 0$. Therefore, the wavelength is infinite, according to Eq. (4.72), resulting in a curve with a single maximum or minimum. Since the limiting case just presented corresponds to an Euler-Bernoulli beam, the results of this explanation are consistent with our expectations and the Euler-Bernoulli beam limiting case corresponds to an oscillatory-dominant solution. This type of behavior will be shown Examples (a)–(c) in Fig. 5.11.

### 4.2.2 Exponential-dominant Solution

As will be demonstrated in this section, an exponential (or non-oscillating) solution is generally obtained with one of three different conditions: (1) the columnar structure is axially stiff but relatively flexible in bending, (2) the columnar structure is significantly stiffer in axial deformation than the substrate is in bending, or (3) the columnar structure is de-coupled from the substrate by reducing the spring constant associated with the torsional spring in the discrete model that connects columns to substrate, $k_3$. For an exponential-dominant solution, there are three
different cases given by Eqs. (4.63) and rewritten here as

$$\kappa_3 > \kappa_4 \text{ and } \kappa_2 \approx 1, \quad \kappa_3 \gg \kappa_4, \quad \text{or} \quad \kappa_4 \ll 1. \quad (4.73)$$

In the first instance, as in the oscillatory-dominant case just discussed, it is necessary that $k_5 \gg \hat{k}_1$ as a result of the requirement that $\kappa_2 \approx 1$ (refer to Eq. (4.64)). Recalling Eqs. (4.65) and (4.66), the inequality $\kappa_3 > \kappa_4$ leads to $k_5 < (1/6)(h/l)^2 \hat{k}_2$. These two inequalities may be summarized to yield

$$\frac{1}{6} \left( \frac{h}{l} \right)^2 \hat{k}_2 > k_5 \gg \hat{k}_1. \quad (4.74)$$

That is, one way of obtaining an exponential-dominant solution is to require that the bending stiffness of the substrate (described by $k_5$) must be much greater than the bending stiffness of the columnar structure (described by $\hat{k}_1$) and that the substrate bending stiffness must be less than the axial stiffness of the columnar structure (described by $\hat{k}_2$). In other words, the axial stiffness of the columnar structure must dominate the model to obtain a non-oscillatory solution.

The second requirement given by Eqs. (4.73), $\kappa_3 \gg \kappa_4$, yields the following inequality in the same manner that was used to obtain Eq. (4.74):

$$\hat{k}_2 \gg k_5, \quad (4.75)$$

where the term $(1/6)(h/l)^2$ is removed as long as $h$ is of a similar magnitude to $l$. Therefore, if the columnar structure is significantly stiffer in axial deformation (described by $\hat{k}_2$) than the substrate is in bending (described by $k_5$), the resulting deformation will be dominated by the exponential part of the solution.

The third requirement of Eqs. (4.73), $\kappa_4 \ll 1$, along with Eq. (3.84), leads to

$$\kappa_4 = \frac{12k_3(\hat{k}_2 + k_4)}{h^2\hat{k}_2(\hat{k}_2 + 4k_4)} \approx \frac{12k_3}{h^2\hat{k}_2} \ll 1 \quad \Rightarrow \quad k_3 \ll \frac{1}{12}h^2\hat{k}_2. \quad (4.76)$$

According to Eqs. (4.76), by making the torsional spring constant $k_3$ very small, one also obtains a non-oscillatory solution. That is, reducing the coupling between the columnar structure and the substrate leads to an exponential-dominant solution.
4.2.3 Mixed Solution

If the requirements of Eqs. (4.69)–(4.71) and (4.74)–(4.76) are not satisfied, the solution will be of a mixed type. In other words, the oscillatory and non-oscillatory solutions will be of a similar magnitude. The exception will be if a particular value of \( \Lambda \) is chosen that leads to a large value of \( \zeta_I \), according to Eqs. (4.55) and Fig. 4.2.

4.3 Analysis of the Rigidly Fixed Boundary Condition (Case II)

One purpose of introducing another set of boundary conditions is to obtain a solution \( \tilde{v}(\xi) \) of a different form compared with the solution obtained and analyzed in Section 4.2. (The vertical deflection given by Eq. (4.42) will be used to classify the overall CTF behavior, as was done in the previous section.) The classification scheme proposed in Section 4.2 will be used in this section and an equivalent characterization function \( \zeta_{II} \) will be defined to describe the relative contribution of the oscillatory and exponential parts of the solution, given by Eq. (4.42). There is also a linear term present in Eq. (4.42) that is not present in Eq. (4.31). Recall that only the relative magnitudes of the oscillatory and exponential components of the vertical deflection were considered when determining \( \zeta_I \) in Section 4.2. In a similar manner, the magnitudes of the oscillatory and exponential terms in this section will not be compared with the linear term in developing the classification scheme. Only the following term from Eq. (4.42) is considered:

\[
\frac{-b_1}{(\Lambda^-)^2} \sin \left[ \text{Im}[\Lambda^-](\xi - \Lambda/2) \right] + \frac{b_2}{(\Lambda^+)^2} \left[ e^{\xi \Lambda^+} - e^{(\Lambda^-\xi)(\Lambda^+)} \right].
\] (4.77)

The oscillatory-dominant part of the solution is given by

\[
\frac{-b_1}{(\Lambda^-)^2} \sin \left[ \text{Im}[\Lambda^-](\xi - \Lambda/2) \right],
\] (4.78)

where \( \xi \in [0, \Lambda] \). As long as \( \Lambda \geq \pi/\text{Im}[\Lambda^-] \), the largest value of the sine term from (4.78) will be 1. Otherwise, the largest value will be \( \sin (\text{Im}[\Lambda^-]\Lambda/2) \), which is the value of the sine term at \( \xi = \Lambda \).
The exponential term from (4.77) is

$$\frac{b_2}{(\lambda^2)} \left[ e^{\xi \lambda^+} - e^{(\Lambda - \xi) \lambda^+} \right].$$ (4.79)

As was done earlier in developing (4.53), the contribution of the exponential part of the solution will be given by the difference in magnitudes between $\xi = 0$ and $\xi = \Lambda/2$,

$$\frac{b_2}{(\lambda^2)} \left[ (e^{\Lambda \lambda^+} - e^0) - (e^{\Lambda \lambda^+/2} - e^{\Lambda \lambda^+/2}) \right] \rightarrow \frac{b_2}{(\lambda^2)} (e^{\Lambda \lambda^+} - 1).$$ (4.80)

The ratio of the largest value of (4.78) and the distance of the peak to the center of (4.79) is called $\zeta_{\Pi}$ and, after some manipulation, has the form

$$\zeta_{\Pi} \equiv \begin{cases} \frac{\kappa_4 + \text{Im}[\lambda^-]^2}{\kappa_4 - (\lambda^+)^2} \left( \frac{\lambda^+}{\text{Im}[\lambda^-]} \right)^3 \frac{\tan(\text{Im}[\lambda^-] \Lambda/2)}{\tanh(\lambda^+ \Lambda/2)}, & \Lambda < \pi/\text{Im}[\lambda^-], \\ \frac{\kappa_4 + \text{Im}[\lambda^-]^2}{\kappa_4 - (\lambda^+)^2} \left( \frac{\lambda^+}{\text{Im}[\lambda^-]} \right)^3 \frac{\sec(\text{Im}[\lambda^-] \Lambda/2)}{\tanh(\lambda^+ \Lambda/2)}, & \Lambda \geq \pi/\text{Im}[\lambda^-]. \end{cases}$$ (4.81)

Equations (4.81) are similar to Eqs. (4.55), although there are some differences. Equation (4.55) contains the term $1/[1 - \text{sech}(\lambda^+ \Lambda/2)]$ and Eqs. (4.81) contains the term $1/\tanh(\lambda^+ \Lambda/2)$. In addition, Eqs. (4.81) contain a quotient involving $\kappa_4$, $\lambda^+$, and $\lambda^-$. As before, the magnitude of $\zeta_{\Pi}$ describes the relative contribution to the overall solution of the oscillatory and non-oscillatory components: when $\zeta_{\Pi} \ll 1$ the exponential solution is dominant, when $\zeta_{\Pi} \gg 1$ the oscillatory solution is dominant, and when $\zeta_{\Pi} \approx 1$ the oscillatory and exponential solutions are of a similar magnitude.

The analysis of $\zeta_{\Pi}$ begins by considering $|\tan(\text{Im}[\lambda^-] \Lambda/2)|$ and by considering $|\sec(\text{Im}[\lambda^-] \Lambda/2)|$ over the appropriate range given in Eq. (4.81), as shown in Fig. 4.5(a). In Fig. 4.5(b), the plot of $1/\tanh(\lambda^+ \Lambda/2)$ is shown. Note the similarities between Figs. (4.2) and (4.5). Continuing with our analysis of $\zeta_{\Pi}$, consider $(\lambda^+/\text{Im}[\lambda^-])^3$ from Eqs. (4.81); this term differs only by the order of the exponent compared with the similar term found in Eqs. (4.55). A plot of $(\lambda^+/\text{Im}[\lambda^-])^3$ is shown in Fig. 4.6. The same approach used to generate the plot in Fig. 4.3 was
Figure 4.5. These plots are taken from terms appearing in Eqs. (4.81). Figure 4.5(a) is a plot of $|\tan(\text{Im}[-\Lambda/2])|$ versus $\text{Im}[-\Lambda/2]$ over the range $(0, \pi/2)$ and $|\sec(\text{Im}[-\Lambda/2])|$ for $\text{Im}[-\Lambda/2] \geq \pi/2$. Figure 4.5(b) is a plot of $1/\tanh(\lambda^+/\Lambda/2)$ as a function of $\lambda^+/\Lambda/2$. Note that the inverse of $\tanh(\lambda^+/\Lambda/2)$ is plotted in Fig. 4.5(b).

used to generate the plot in Fig. 4.6. For example, Eq. (4.56) is replaced with

$$
\left( \frac{\lambda^+}{\text{Im}[\lambda^-]} \right)^3 = \left( \frac{c_1 + \sqrt{c_1^2 + c_2}}{-c_1 + \sqrt{c_1^2 + c_2}} \right)^{3/2},
$$

(4.82)

and $c_1$ and $c_2$ are defined as before, using Eqs. (4.57).

Based on the similarities between Figs. 4.2 and 4.5 and between Figs. 4.3 and 4.6, the conclusions that were drawn regarding the model’s behavior for Case I may also be applied, so far, to the model subjected to the boundary conditions of Case II. This should not be surprising, since we do not expect drastic changes in the effects of the input parameters on the overall behavior of the system from simply changing the boundary conditions. Of course, there is an additional term in $\zeta_{\text{II}}$ that does not appear in $\zeta_{\text{I}}$.

The following term appears in Eqs. (4.81) for $\zeta_{\text{II}}$ but not in Eqs. (4.55) for $\zeta_{\text{I}}$:

$$
\frac{\kappa_4 + \text{Im}[-\Lambda]^2}{\kappa_4 - (\lambda^+)^2}.
$$

(4.83)
\[ (\frac{\lambda^+}{\text{Im}[\lambda^-]} )^3 \]

\[ \kappa_4 > \kappa_3 + 6(1 - \kappa_2) \Leftrightarrow c_1 > 0 \]
\[ \kappa_4 = \kappa_3 + 6(1 - \kappa_2) \Leftrightarrow c_1 = 0 \]
\[ \kappa_4 < \kappa_3 + 6(1 - \kappa_2) \Leftrightarrow c_1 < 0 \]

**Figure 4.6.** This figure shows the value of \((\lambda^+/\text{Im}[\lambda^-])^3\) as a function of a term defined by three of the nondimensional constants: \(\kappa_2, \kappa_3,\) and \(\kappa_4\). There are three different cases based on the relationship between \(\kappa_4\) and \(\kappa_3 + 6(1 - \kappa_2)\), recalling that \(0 < \kappa_2 < 1\).

The ratio given by (4.83) may be written in a simplified form as

\[ \frac{1 + \sqrt{c_3 + c_4}}{1 - \sqrt{c_3 + c_4}}, \quad (4.84) \]

where

\[ c_3 = \left[ \frac{\kappa_4 - (\kappa_3 + 6(1 - \kappa_2))}{\kappa_4 + (\kappa_3 + 6(1 - \kappa_2))} \right]^2, \quad c_4 = \frac{24(1 - \kappa_2)\kappa_4}{(\kappa_4 - (\kappa_3 + 6(1 - \kappa_2)))^2}. \quad (4.85) \]

Based on the inequalities given by Eqs. (3.86), it follows that

\[ 0 < c_3 < 1, \quad 0 < c_4 \leq 1, \quad c_3 + c_4 < 1. \quad (4.86) \]

It is most important to consider cases that might otherwise exhibit exponential-dominant behavior to see if (4.83) might change the type of behavior, e.g., can the presence of (4.83) make a deformation that is non-oscillatory in Case I become oscillatory in Case II?

It is apparent that the quotient given by (4.84) becomes large as the denominator becomes very small, i.e., as \(c_3 + c_4 \approx 1\), which may be written using Eqs. (4.85) as

\[ \left[ \kappa_4 - (\kappa_3 + 6(1 - \kappa_2)) \right]^2 + 24(1 - \kappa_2)\kappa_4 - \left[ \kappa_4 + (\kappa_3 + 6(1 - \kappa_2)) \right]^2 \approx 0. \quad (4.87) \]
Equation (4.87) may be simplified to yield
\[ \kappa_3 \kappa_4 \approx 0. \] (4.88)

Based on this relation, the following limits of the ratio given by (4.83) will prove helpful:
\[
\lim_{\kappa_3 \to 0} \frac{\kappa_4 + \text{Im}[\lambda^-]^2}{\kappa_4 - (\lambda^+)^2} = \infty, \quad \lim_{\kappa_4 \to 0} \frac{\kappa_4 + \text{Im}[\lambda^-]^2}{\kappa_4 - (\lambda^+)^2} = \infty. \] (4.89)

Therefore, we will take each of these limits in turn, beginning with the case of a very small value of \( \kappa_3 \). When \( \kappa_3 \) is small, the ratio given by (4.83) is large; this suggests that \( \zeta_{II} \) might be large (a large \( \zeta_{II} \) is associated with an oscillatory-dominant system). Recall one of the oscillatory-dominant inequalities given by Eqs. (4.62)
\[ \kappa_4 \gg \kappa_3 + 6(1 - \kappa_2). \] (4.90)

When \( \kappa_3 \approx 0 \), it follows from Eq. (4.90) that a large value of \( \zeta_I \) is associated with the condition \( \kappa_4 \gg 6(1 - \kappa_2) \). Therefore, consider the case with \( \kappa_3 \approx 0 \) and \( \kappa_4 \gg 6(1 - \kappa_2) \). That is, under the Case I boundary conditions, such a model would be of a mixed type; under the Case II conditions, this model will be oscillatory-dominant.

Next, consider the case of a small value for \( \kappa_4 \) and recall one of the inequalities associated with the exponential-dominant solution for Case I and given by Eqs. (4.63):
\[ \kappa_4 \ll 1. \] (4.91)

Therefore, when \( \kappa_4 \) is very small, the Case I model is non-oscillatory while the Case II model will either be mixed or oscillatory-dominant. In conclusion, when \( \kappa_3 \) and/or \( \kappa_4 \) are very small, the Case II system will become more oscillatory when compared with the Case I system. These phenomena may be explained by considering that the torsional spring constant \( k_3 \) appears in the numerator of both \( \kappa_3 \) and \( \kappa_4 \), such that when \( k_3 \) is small, both \( \kappa_3 \) and \( \kappa_4 \) will tend to be small. The small spring constant associated with the torsional spring tends to weaken the connection between the substrate and columnar structure. In Case I, the boundary condition deformation is applied directly to the substrate and not to the columnar structure. In Case II, the columns on both ends are required to be perpendicular
to the substrate. In this way, although the torsional springs are weak, they are forced to play a role in Case II. Hence the columnar structure will influence the CTF in Case II when it would otherwise have little influence in Case I. An example illustrating this will be presented in Section 5.1.5. Other than this situation, the conclusions regarding $\zeta_I$ may also be applied to $\zeta_{II}$. 
Chapter 5

Some Numerical Results

This chapter is divided into two main sections. The first section (Section 5.1) focuses solely on the models that have been developed in Chapter 3, with insight gained from the analysis presented in Chapter 4. In the second section (Section 5.2), some possible applications to current problems are presented. It is hoped that this chapter will provide greater insight into the nature of the models developed and will lead to new applications for which the models may be used.

5.1 Results Based on Various Models

In this first section of this chapter, five different examples will be presented. Each of these examples is intended to explore in greater detail the models that have been formulated in the present work. The first example (Section 5.1.1) presents an overview of all of the models, including linear and nonlinear discrete models, the discrete-based continuous model, and the directed continuum model. The second (Section 5.1.2) and third (Section 5.1.3) examples focus on the Case I (simply supported) and Case II (rigidly fixed) boundary conditions. The fourth example shows the effect that changing the CTF length may have on the vertical deflection (Section 5.1.4). The fifth example examines the change in deflection type due solely to changing boundary conditions (Section 5.1.5).
5.1.1 Oscillatory-dominant Case—Discrete and Continuous Solutions

The purpose of this first set of examples is to examine the variety of models developed in this work (including discrete and continuous models) for a single system of material constants. Since the discrete models are defined in terms of the $k$ constants (with dimensions), and since the continuous models are defined in terms of the dimensionless $\kappa$ constants, it will be necessary to present two, equivalent systems of materials constants. (Although the constants with dimension are presented first, they are based on the nondimensional constants presented later; this explains the choice of the particular values of the $k$ terms and the number of digits reported.) For the discrete models, the constants are $k_1 = 1.6461$, $k_2 = 16$, $k_3 = 6.667$, $k_4 = 1$, $k_5 = 4.938$, $h = 0.5$, $L = 100$, and $l = 1^*$. In addition, $n = 1$ for the discrete models, i.e., there is a single spring pair for the kinematically linear discrete model (or a single spring for the kinematically nonlinear discrete model) connecting each of the neighboring rigid columns together. The system of equations used in the discrete cases have been solved using Mathematica and the FindRoot command.

For the continuous models, the equivalent constants are $\kappa_1 = 4$, $\kappa_2 = 0.9$, $\kappa_3 = 2.7$, $\kappa_4 = 17$, $\eta = 0.5$, and $\Lambda = 100$. For the directed continuum model, these values lead to a wavelength $\mu = 7.5$ and characterization factor $\zeta_1 = 61$ and $\zeta_\Pi = 1870$, depending on which boundary condition is applied. In both cases, the expected vertical deformation will be oscillatory based on the values of $\zeta_1$ and $\zeta_\Pi$.

In addition, since the wavelength is less than the overall length, we will expect to see the oscillations along the film length in the continuous models.

The first example, shown in Fig. 5.1, is based on the discrete formulation presented in Section 3.1 with boundary conditions as presented in Section 3.1.3, i.e., the discrete simply supported CTF. There are 404 linear equations that must be solved to obtain the single solution that is presented in the figure. From the discrete formulation of the total energy, it is determined that the solution presented in Fig. 5.1 results in a strain energy of 0.3208. For the discrete examples, since the

\*The numerical values of these constants must be given in a consistent set of units. For example, in the given example, $l = 1$ in, $k_1 = 1.6461$ lb/in, $k_2 = 16$ lb/in, $k_3 = 6.667$ lb·in, $k_4 = 1$ lb/in, $k_5 = 4.938$ lb/in, $h = 0.5$ in, and $L = 100$ in.
Figure 5.1. Kinematically linear discrete model with $U_{\text{discrete}} = 0.3208$.

nondimensionalization has not been applied, the results are shown in terms of the deformations $u(x)$, $v(x)$, $\phi(x)$, and $\beta(x)$.

The following five figures, Figs. 5.2–5.6, are all solutions to the kinematically nonlinear discrete model that was presented in Section 3.2. The boundary condi-

Figure 5.2. Kinematically nonlinear discrete model with $n = 1$, $U_{\text{discrete}}^{\text{NL}} = 0.02417$. This is the lowest energy solution.
there are other solutions between Figs. 5.5 and 5.6. Figure 5.6 is shown just as an example of a higher energy solution.) The approximation symbol is used in Figures 5.5 and 5.6 to reflect the greater error associated with the root finder for these examples.

In order to see the relationships among the various solutions, Fig. 5.7 is included. In this figure, the vertical deflections given by $v(x)$ for the linear discrete case and the five nonlinear discrete cases are shown. As the energy associated with the nonlinear solutions increases, the number of oscillations in the solution increases while the magnitude of the solution decreases.
In Fig. 5.8, the discrete-based continuous solution to the simply supported problem from Section 3.3 is shown. The continuous solutions have been nondimensionalized such that the results are shown in terms of \( \tilde{u}(\xi), \tilde{v}(\xi), \tilde{\phi}(\xi), \) and \( \tilde{\beta}(\xi) \). The difference between this example and the following example (using the directed continuum approach) is one of boundary conditions. The boundary conditions for the discrete-based continuous model are taken directly from the discrete model; the boundary conditions for the directed continuum model follow from the strain energy density that also yields the governing equations. The directed continuum example, developed from the formulation in Section 3.4, is shown in
Figure 5.7. Vertical displacements from previous six figures. The linear discrete model is so labeled, and the nonlinear results are labeled according to the energy values.

Figure 5.8. Discrete-based continuous model of simply supported CTF. The axes are scaled in the same manner as those appearing in Fig. 5.9.

Fig. 5.9. The results shown in Figs. 5.8 and 5.9 are based on the same simply supported boundary conditions (Case I), although the equations used to represent these conditions are not the same.

Finally, the directed continuum model is also applied to the rigidly fixed boundary conditions, with results shown in Fig. 5.10. The oscillatory nature of the solutions shown in Figs. 5.9 and 5.10 is expected, when we recall the values of ζ_I and ζ_{II} for this case. In addition, it may be worth comparing the discrete and continuous solutions. This is easily done since, although only the continuous models
Figure 5.9. Directed continuum model for simply supported (Case I) boundary conditions.

Figure 5.10. Directed continuum model for rigidly fixed (Case II) boundary conditions.

are nondimensionalized, the unit cell length \( l \) equals one for the discrete models. Recalling Eqs. (3.76), it follows that for this case, \( \tilde{u} = u, \tilde{v} = v, \) and \( \xi = x \). In all cases, \( \tilde{\phi} \equiv \phi \) and \( \tilde{\beta} \equiv \beta \). Therefore, we may simply compare any of the results in this section, for both discrete and continuous models.

The discrete model, as shown in Fig. 5.1, is not at all oscillatory, but the values of the deformations are of a similar magnitude to the values of the continuous models, as shown in Figs. 5.8 and 5.9. In some sense, the linear discrete model gives an average of the linear continuous models. On the other hand, the nonlinear discrete models are less directly related to the continuous models, which are based
on the linear discrete model. In any case, the nonlinear discrete model is still constrained to the linear regime in two particular ways.

(1) Although the horizontal and vertical deformation effects on the columnar structure are not decoupled as they are in the linear case, the horizontal and vertical deformation remains decoupled in the substrate component of the model. That is, the effect of bending on the substrate has no effect on the axial deformation of the substrate.

(2) The springs, which may be thought of as harmonic potentials, are linear.

In any event, the higher energy metastable states (e.g., see Fig. 5.6) appear to more closely resemble the results from the continuous linear models (e.g., see Fig. 5.9).

### 5.1.2 Simply Supported (Case I) Boundary Conditions

In this section, the simply supported boundary condition will be applied to the directed continuum theory of Section 3.4. The material constants and geometry for the 13 examples to be presented are shown in Table 5.1. With the exception

<table>
<thead>
<tr>
<th>Ex.</th>
<th>$\kappa_1$</th>
<th>$\kappa_2$</th>
<th>$\kappa_3$</th>
<th>$\kappa_4$</th>
<th>$\eta$</th>
<th>$\Lambda$</th>
<th>$\mu$</th>
<th>$\zeta_I$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>4</td>
<td>0.99995</td>
<td>2.7</td>
<td>30</td>
<td>1.5</td>
<td>100</td>
<td>346</td>
<td>5.2×10^4</td>
</tr>
<tr>
<td>(b)</td>
<td>4</td>
<td>0.9998</td>
<td>2.7</td>
<td>24</td>
<td>1.5</td>
<td>100</td>
<td>171</td>
<td>6.0×10^4</td>
</tr>
<tr>
<td>(c)</td>
<td>4</td>
<td>0.99945</td>
<td>2.7</td>
<td>17</td>
<td>1.5</td>
<td>100</td>
<td>100</td>
<td>3.7×10^3</td>
</tr>
<tr>
<td>(d)</td>
<td>4</td>
<td>0.998</td>
<td>2.7</td>
<td>12</td>
<td>1.5</td>
<td>100</td>
<td>50.5</td>
<td>6.0×10^2</td>
</tr>
<tr>
<td>(e)</td>
<td>4</td>
<td>0.99</td>
<td>7</td>
<td>17</td>
<td>1</td>
<td>100</td>
<td>19.7</td>
<td>1.0×10^2</td>
</tr>
<tr>
<td>(f)</td>
<td>4</td>
<td>0.9</td>
<td>2.7</td>
<td>17</td>
<td>0.5</td>
<td>100</td>
<td>7.5</td>
<td>6.1×10^1</td>
</tr>
<tr>
<td>(g)</td>
<td>4</td>
<td>0.9</td>
<td>2.7</td>
<td>11.5</td>
<td>0.5</td>
<td>100</td>
<td>7.2</td>
<td>1.2×10^1</td>
</tr>
<tr>
<td>(h)</td>
<td>4</td>
<td>0.9</td>
<td>3</td>
<td>3.4</td>
<td>0.5</td>
<td>100</td>
<td>5.1</td>
<td>1.5×10^0</td>
</tr>
<tr>
<td>(i)</td>
<td>4</td>
<td>0.95</td>
<td>11</td>
<td>3.4</td>
<td>0.5</td>
<td>100</td>
<td>2.2</td>
<td>1.7×10^-2</td>
</tr>
<tr>
<td>(j)</td>
<td>0.2</td>
<td>0.999</td>
<td>2</td>
<td>0.2</td>
<td>2.5</td>
<td>100</td>
<td>4.7</td>
<td>2.3×10^-3</td>
</tr>
<tr>
<td>(k)</td>
<td>0.0003</td>
<td>0.5</td>
<td>0.002</td>
<td>0.005</td>
<td>1</td>
<td>100</td>
<td>3.6</td>
<td>7.5×10^-3</td>
</tr>
<tr>
<td>(l)</td>
<td>0.7</td>
<td>0.56</td>
<td>500</td>
<td>2</td>
<td>1</td>
<td>100</td>
<td>0.3</td>
<td>2.3×10^-5</td>
</tr>
<tr>
<td>(m)</td>
<td>4</td>
<td>0.999</td>
<td>2</td>
<td>0.2</td>
<td>2.5</td>
<td>100</td>
<td>4.7</td>
<td>2.3×10^-3</td>
</tr>
</tbody>
</table>
of Examples (a) and (m), the various examples are shown in order of value of $\zeta_I$, from largest to smallest. These 13 examples are presented in Fig. 5.11 in a different manner than was done in the previous section, Section 5.1.1. Instead of showing

$$\begin{align*}
(a) & \quad \zeta_I = 1.35 \times 10^5 \\
(b) & \quad \zeta_I = 5.95 \times 10^4 \\
(c) & \quad \zeta_I = 3.65 \times 10^3 \\
(d) & \quad \zeta_I = 602 \\
(e) & \quad \zeta_I = 101 \\
(f) & \quad \zeta_I = 60.7 \\
(g) & \quad \zeta_I = 11.8 \\
(h) & \quad \zeta_I = 1.49 \\
i & \quad \zeta_I = 0.0165 \\
j & \quad \zeta_I = 2.27 \times 10^{-3} \\
k & \quad \zeta_I = 7.51 \times 10^{-3} \\
l & \quad \zeta_I = 2.27 \times 10^{-5} \\
m & \quad \zeta_I = 2.27 \times 10^{-3}
\end{align*}$$

*Figure 5.11. Directed continuum model subjected to Case I (simply supported) boundary conditions.*

each deformation plotted separately, the results of Fig. 5.11 are plotted to give the overall behavior of the film. The lower region represents the substrate of the CTF
in the continuum model: the center curve is based on $\tilde{u}$ and $\tilde{v}$, while the bottom curve is also a function of $\tilde{\phi}$. The top region represents the columnar structure and the top curve is a function of all four deformations: $\tilde{u}$, $\tilde{v}$, $\tilde{\phi}$, and $\tilde{\beta}$. Figure 5.12 is included to make this clear. The same type of plot will be used in subsequent sections. It should also be noted that the deformations have been exaggerated to make them more obvious.

Examples (a)–(f) show oscillatory-dominant vertical deflections. In Examples (a)–(c), the wavelength $\mu$ is greater than or equal to the overall length $\Lambda$, such that no oscillatory behavior is actually observed (this was discussed in Section 4.2.1). In these examples, the substrate is stiffer than the columnar structure, consistent with the discussion in Section 4.2.1. Examples (g) and (h) represent a mixed type of solution, where the oscillatory and exponential components are of a similar magnitude. Examples (i)–(m) represent a non-oscillatory solution. In Example (k), the torsional spring connecting the rigid rods to the substrate is not very stiff, i.e., $\kappa_4 \ll 1$ according to Section 4.2.2 and Eq. (4.76). In the other exponential solutions, the axial stiffness of the columnar structure is large, i.e., $\kappa_3 > \kappa_4$.

Although Example (m) is non-oscillatory, it differs from the other non-oscillatory examples in that the resulting deformation is similar to Example (a). (If these were beams, the deformation of (a) and (m) would be considered to be due to a positive bending moment.) At this point it may be worth considering the oscillatory and non-oscillatory components of the vertical deflection for Examples (a) and (m) in greater detail, as shown in Fig. 5.13. It is apparent from this figure that the
Figure 5.13. Comparison of constant, oscillating, and non-oscillating (exponential) components of solutions for $\tilde{v}(\xi)$ for Examples (a) and (m) presented in Fig. 5.11.

The non-oscillating component of Example (a) and the oscillating component of (m) do not significantly contribute to the overall deformation of the film. On the other hand, the oscillating component of Example (a) and the non-oscillating component of (m) look very similar.

Example (k) is particularly interesting as the columnar structure appears to bend out from the substrate in Fig. 5.11. To see the deformation in greater detail, consider Fig. 5.14. Typically, the substrate rotation $\tilde{\phi}(\xi)$ is greater than the rotation of the rigid rods $\tilde{\beta}(\xi)$. But due to the small spring constant associated with the torsional spring ($k_3$), it should not be surprising that $\tilde{\beta}(\xi)$ is relatively large. In the other examples presented in this section, $\tilde{\beta}(\xi)$ is small, meaning that the rigid
rods are nearly perpendicular to the substrate. This is true even for Example (l), although looking at Fig. 5.11, the columnar structure at the ends does not appear to be perpendicular to the substrate. This apparent contradiction is explained when one recalls that the rotation of the substrate does not necessarily equal the slope of the vertical deformation.

5.1.3 Rigidly Fixed (Case II) Boundary Conditions

The ten examples shown in this section are taken from the directed continuum model with rigidly fixed boundary conditions. Table 5.2 describes the material and geometric input parameters to the model. As in the previous section, the examples are shown in decreasing order of $\zeta_{II}$ in Fig. 5.15. Examples (a)–(f) are oscillatory-dominant based on their values of $\zeta_{II}$ and Examples (i) and (j) are exponential-dominant. For the oscillatory-dominant examples, the substrate is stiffer in bending than the columnar structure; in the non-oscillatory solutions, the columnar structure is stiffer in axial loading (recall Section 4.3). Only in Example (a) is the wavelength greater than the overall length, so the other solutions given by Examples (b)–(f) are oscillatory since their wavelengths are less than the overall length.

Table 5.2. The geometric and material constants for the ten examples described in this section and shown in Fig. 5.15. The wavelength of the oscillatory part of the solution, $\mu$, and the value used to classify the system behavior, $\zeta_{II}$, are also included. For the Case II boundary condition, the right-hand side is raised by the amount $\delta_v = 1$.

<table>
<thead>
<tr>
<th>Example</th>
<th>$\kappa_1$</th>
<th>$\kappa_2$</th>
<th>$\kappa_3$</th>
<th>$\kappa_4$</th>
<th>$\eta$</th>
<th>$\Lambda$</th>
<th>$\mu$</th>
<th>$\zeta_{II}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>2.5</td>
<td>0.9999</td>
<td>8</td>
<td>50</td>
<td>1</td>
<td>100</td>
<td>235</td>
<td>3.7×10^8</td>
</tr>
<tr>
<td>(b)</td>
<td>2.5</td>
<td>0.995</td>
<td>8</td>
<td>50</td>
<td>1</td>
<td>100</td>
<td>33</td>
<td>2.5×10^5</td>
</tr>
<tr>
<td>(c)</td>
<td>2.5</td>
<td>0.9999</td>
<td>8</td>
<td>10</td>
<td>1</td>
<td>100</td>
<td>114</td>
<td>2.4×10^4</td>
</tr>
<tr>
<td>(d)</td>
<td>2.5</td>
<td>0.9</td>
<td>8</td>
<td>50</td>
<td>1</td>
<td>100</td>
<td>7.4</td>
<td>1.4×10^4</td>
</tr>
<tr>
<td>(e)</td>
<td>2.5</td>
<td>0.5</td>
<td>8</td>
<td>50</td>
<td>1</td>
<td>100</td>
<td>3.3</td>
<td>3.2×10^2</td>
</tr>
<tr>
<td>(f)</td>
<td>2.5</td>
<td>0.995</td>
<td>8</td>
<td>10</td>
<td>1</td>
<td>100</td>
<td>16.7</td>
<td>7.4×10^1</td>
</tr>
<tr>
<td>(g)</td>
<td>2.5</td>
<td>0.5</td>
<td>8</td>
<td>7.5</td>
<td>1</td>
<td>100</td>
<td>2.4</td>
<td>1.8×10^1</td>
</tr>
<tr>
<td>(h)</td>
<td>2.5</td>
<td>0.5</td>
<td>14</td>
<td>7.5</td>
<td>1</td>
<td>100</td>
<td>1.9</td>
<td>2.6×10^{-1}</td>
</tr>
<tr>
<td>(i)</td>
<td>2.5</td>
<td>0.9</td>
<td>14</td>
<td>7.5</td>
<td>1</td>
<td>100</td>
<td>2.3</td>
<td>5.0×10^{-2}</td>
</tr>
<tr>
<td>(j)</td>
<td>2.5</td>
<td>0.999</td>
<td>14</td>
<td>7.5</td>
<td>1</td>
<td>100</td>
<td>2.5</td>
<td>1.8×10^{-4}</td>
</tr>
</tbody>
</table>

5.1.3 Rigidly Fixed (Case II) Boundary Conditions

The ten examples shown in this section are taken from the directed continuum model with rigidly fixed boundary conditions. Table 5.2 describes the material and geometric input parameters to the model. As in the previous section, the examples are shown in decreasing order of $\zeta_{II}$ in Fig. 5.15. Examples (a)–(f) are oscillatory-dominant based on their values of $\zeta_{II}$ and Examples (i) and (j) are exponential-dominant. For the oscillatory-dominant examples, the substrate is stiffer in bending than the columnar structure; in the non-oscillatory solutions, the columnar structure is stiffer in axial loading (recall Section 4.3). Only in Example (a) is the wavelength greater than the overall length, so the other solutions given by Examples (b)–(f) are oscillatory since their wavelengths are less than the overall length.
As in the previous section, an oscillatory and a non-oscillatory example exhibit similar behavior, see Examples (a) and (j). To see how this occurs, consider the plots shown in Fig. 5.16. The oscillating component of Example (a) is much larger than the non-oscillating component, while the non-oscillating component of Example (j) is much larger than the oscillating component. For the Case II boundary condition, the vertical deflection is somewhat overwhelmed by the prescribed vertical deflection.
5.1.4 Example Showing Effect of Changes to $\Lambda$

The purpose of this section is to consider the effect that changing the overall length may have on the behavior of the system. Recalling Fig. 4.2(a), if $\text{Im}[\lambda^{-}\Lambda]$ is close to $(2m+1)\pi$ then small changes in $\Lambda$ will cause significant changes in $\zeta_{\Pi}$. The input parameters for a directed continuum model of an example of this type are given in Table 5.3. In Example (b), the magnitudes of the oscillatory and exponential components of the solution are essentially the same, while in Example (a), the oscillatory component is larger. This is apparent in the results that are shown in Fig. 5.17. That is, without making any changes to the unit cell but simply by changing the number of unit cells in the model, the model’s results may change significantly.

Table 5.3. The geometric and material constants for the two examples described in this section. The wavelength of the oscillatory part of the solution, $\mu$, and the value used to classify the system behavior, $\zeta_{\Pi}$, are also included. For the Case II boundary condition, the right-hand side is raised by the amount $\delta_v = 1$.

<table>
<thead>
<tr>
<th>Example</th>
<th>$\kappa_1$</th>
<th>$\kappa_2$</th>
<th>$\kappa_3$</th>
<th>$\kappa_4$</th>
<th>$\eta$</th>
<th>$\Lambda$</th>
<th>$\mu$</th>
<th>$\zeta_{\Pi}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>2.5</td>
<td>0.5</td>
<td>8</td>
<td>7.5</td>
<td>1</td>
<td>100</td>
<td>2.4</td>
<td>18</td>
</tr>
<tr>
<td>(b)</td>
<td>2.5</td>
<td>0.5</td>
<td>8</td>
<td>7.5</td>
<td>1</td>
<td>101</td>
<td>2.4</td>
<td>1.2</td>
</tr>
</tbody>
</table>
Figure 5.17. Directed continuum model with Case II boundary conditions. The purpose of these examples is to show the effect that changing the overall length $\Lambda$ has on the overall behavior of the film. Example (a) shown above is equivalent to Example (g) from Section 5.1.3.

5.1.5 Changing Deformation Type by Changing Boundary Condition

As was discussed in Section 4.3, there are instances when the deformation type as characterized by $\zeta_I$ and $\zeta_{II}$ will differ for identical input parameters. This phenomenon was encountered when $\kappa_3$ and/or $\kappa_4$ are very small. Three examples illustrating this are identified as Examples (a)–(c) in Table 5.4. In Example (a), for the simply-supported boundary condition, $\zeta_I$ characterizes the deformation as being mixed (both oscillating and non-oscillating solutions are of a comparable magnitude). But for the rigidly fixed boundary condition, the deformation is oscillatory-dominant. For Examples (b) and (c) the Case I boundary condition leads to non-oscillatory solutions. The Case II boundary condition leads to a mixed solution for Example (b) and an oscillatory solution for Example (c). Plots showing the oscillating and non-oscillating components of the vertical deflection for these three examples are shown in Fig. 5.18. Based on the analysis of Chapter 4,

Table 5.4. The parameters for three examples described in Section 5.1.5. Deformation characterization factors for Cases I and II are given.

<table>
<thead>
<tr>
<th>Ex.</th>
<th>$\kappa_1$</th>
<th>$\kappa_2$</th>
<th>$\kappa_3$</th>
<th>$\kappa_4$</th>
<th>$\eta$</th>
<th>$\Lambda$</th>
<th>$\mu$</th>
<th>$\zeta_I$</th>
<th>$\zeta_{II}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>0.2</td>
<td>0.5</td>
<td>0.0001</td>
<td>1</td>
<td>0.5</td>
<td>100</td>
<td>3.6</td>
<td>$1.6 \times 10^0$</td>
<td>$1.5 \times 10^5$</td>
</tr>
<tr>
<td>(b)</td>
<td>0.000001</td>
<td>0.5</td>
<td>0.1</td>
<td>0.00001</td>
<td>0.5</td>
<td>100</td>
<td>3.6</td>
<td>$2.6 \times 10^{-4}$</td>
<td>$3.4 \times 10^{-1}$</td>
</tr>
<tr>
<td>(c)</td>
<td>0.0003</td>
<td>0.5</td>
<td>0.002</td>
<td>0.005</td>
<td>1</td>
<td>100</td>
<td>3.6</td>
<td>$7.5 \times 10^{-3}$</td>
<td>$2.6 \times 10^2$</td>
</tr>
</tbody>
</table>
Figure 5.18. Comparison of oscillating and non-oscillating (bold) components of vertical deformation for Case I (simply supported) and Case II (rigidly fixed) boundary conditions. Case I (a) is mixed and Case II (a) is oscillatory-dominant. Cases I (b) and (c) are non-oscillatory; Case II (b) is mixed and Case II (c) is oscillatory.

these conditions are the only ones that cause changes in the behavior of the model based on changing the boundary conditions between Cases I and II.

5.2 Possible Applications

In this section of the results, possible uses of the directed continuum model based on a CTF will be offered. The issue of varying curvature throughout the thin film will be discussed in Section 5.2.1. Then the possibilities of using the directed continuum approach in the analysis of micro-buckling of thin films will be presented in Section 5.2.2. Finally, it is suggested that the there may be some use of directed continuum theories in modeling substrates used in the formation of quantum dot superlattices (man-made assemblies of quantum dots) in Section 5.2.3.
5.2.1 Application to Thin Film with Varying Curvature

Often, a stress associated with a thin film is determined experimentally by obtaining a single-valued measure of curvature. This is typically accomplished by focusing a laser along the surface of a thin film and measuring the change in the angle of deflection of the laser onto a detector device [70, 96]. The relationship between a single value of curvature and a stress is usually expressed via the Stoney equation [92], which is given as

\[
\sigma_f = \frac{\kappa E_s d_s^2}{6 (1 - \nu_s) d_f},
\]

(5.1)

where \(E_s\) is the elastic modulus of the substrate, \(d_s\) is the thickness of the substrate, \(\nu_s\) is the Poisson’s ration of the substrate, \(d_f\) is the thickness of the film, \(\sigma_f\) is the stress in the film, and \(\kappa\) is the experimentally measured curvature. The Euler-Bernoulli beam theory gives a single value of curvature for a beam governed by the relation \(v'''(x) = 0\), where the displacement \(v\) is given as a function of position \(x\) along the length of the beam. If a thin film were to be modeled using such a theory, there will be a constant \(\kappa\) to be used with Eq. (5.1).

On the other hand, if the columnar structure of a CTF is modeled using the directed continuum theory as presented here, it may be difficult to talk about a single curvature measurement for the entire film. Although the system as described by the Case I boundary conditions may be adequately modeled using the Stoney equation, consider a case where any single value of \(\kappa\) measured in an experiment would not represent the actual deflection of the thin film. This is apparent when one applies the following parameters to the continuum model with simply supported boundary conditions: \(\kappa_1 = 5.9, \kappa_2 = 0.85, \kappa_3 = 500, \kappa_4 = 10, \eta = 1, \Lambda = 100,\) and \(\delta_u = 2\). This leads to the characterization factor \(\zeta_I = 10^{-5}\) and wavelength \(\mu = 0.3\).

Some of the resulting deformation functions are plotted in Fig. 5.19. In addition to the plot of \(\tilde{v}(\xi)\) for this example, the constant curvature fit of the vertical deflection is also plotted. That is, if the directed continuum theory gave the actual deformation of the film and if the Stoney equation (constant \(\kappa\)) approach were being used to analyze the actual film, then the Stoney equation approach would treat the actual deformation as if it were the deformation given by the constant
\[ \tilde{\phi}(\xi) \]

\[ \tilde{v}'(\xi) \]

\[ \tilde{\phi}(\xi) \]

\[ \text{constant } \kappa \text{ fit} \]

\[ \text{directed continuum} \]

**Figure 5.19.** The deformation functions \( \tilde{v} \), \( \tilde{v}' \), and \( \tilde{\phi} \) for the simply supported (Case I) continuum model presented. In addition, the constant curvature (\( \kappa \)) fit of the vertical deflection is shown.

\( \kappa \) fit. In the case of constant curvature, it would follow that \( \tilde{v}'(\xi) \) would equal \( \tilde{\phi}(\xi) \). The right-hand plot of Fig. 5.19 shows that, although these two functions are similar, they are not equal to one another.

In order to make this point clearer, it is possible to calculate the curvature from the continuum model with the standard equation

\[ \kappa = \frac{\tilde{v}''(\xi)}{[1 + \tilde{v}'(\xi)^2]^{3/2}}. \]  

(5.2)

A plot of the calculated curvature and the constant curvature is shown in Fig. 5.20. One consequence of these results is that the directed continuum model describes a thin film with a varying stress state while the Stoney equation approach yields a uniform stress state. These results also suggest that an estimate of the stress given by Eq. (5.1) may not be representative of the actual physical system. Experimen-
eral observations of thin films with varying curvatures are discussed by Rosakis, et al. [101]. The purpose of that work is to demonstrate an experimental technique that is capable of measuring variations in curvature for a variety of systems, including thin film Al-coated Si wafers, SiO$_2$Si$_3$N$_4$/SiO$_2$ membranes, and thin film Cr-coated Si wafers.

5.2.2 Application to Micro-buckling

The continuum model formulated in the present work may provide some insights into the study of micro-buckling thin films\(^\dagger\). In the work of Stafford et al. [111], one of the aspects of the analysis presented is to consider relationships between the wavelengths of the buckled thin films and the material properties of the thin films. (The films are analyzed using classical theories and are described using elastic moduli and Poisson’s ratios.) Consider the five examples described in Table 5.5 and shown in Fig. 5.21. The different values of $\kappa_2$ and $\kappa_3$ are obtained by considering

<table>
<thead>
<tr>
<th>Example</th>
<th>$\kappa_1$</th>
<th>$\kappa_2$</th>
<th>$\kappa_3$</th>
<th>$\kappa_4$</th>
<th>$\eta$</th>
<th>$\Lambda$</th>
<th>$\mu$</th>
<th>$\zeta_I$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>4</td>
<td>0.9044</td>
<td>8.56</td>
<td>11.5</td>
<td>0.6</td>
<td>100</td>
<td>5</td>
<td>8.6</td>
</tr>
<tr>
<td>(b)</td>
<td>4</td>
<td>0.9406</td>
<td>5.116</td>
<td>11.5</td>
<td>0.6</td>
<td>100</td>
<td>8</td>
<td>8800</td>
</tr>
<tr>
<td>(c)</td>
<td>4</td>
<td>0.9655</td>
<td>2.892</td>
<td>11.5</td>
<td>0.6</td>
<td>100</td>
<td>12</td>
<td>63</td>
</tr>
<tr>
<td>(d)</td>
<td>4</td>
<td>0.9783</td>
<td>1.7995</td>
<td>11.5</td>
<td>0.6</td>
<td>100</td>
<td>16</td>
<td>89</td>
</tr>
<tr>
<td>(e)</td>
<td>4</td>
<td>0.9894</td>
<td>0.8652</td>
<td>11.5</td>
<td>0.6</td>
<td>100</td>
<td>24</td>
<td>180</td>
</tr>
</tbody>
</table>

how they were originally defined by Eqs. (3.82) and (3.83) and rewritten as

$$
\kappa_2 \equiv \frac{3k_5}{k_1 + 3k_5}, \quad \kappa_3 \equiv \frac{2k_3}{k_5l^2},
$$

(5.3)

where $k_5$ only appears in the relations for $\kappa_2$ and $\kappa_3$. Recall that $k_5$ is associated with the bending stiffness of the substrate. Using Eq. (4.72), it is possible relate the

\(^\dagger\)It should be noted that the use of the term buckling is not meant to imply any sort of instability. Rather, in order to be consistent with the terminology of the work cited in this section, buckling refers only to the oscillatory deformation that has been observed in the thin films.
Figure 5.21. Directed continuum model subjected to Case I boundary conditions. The purpose of these examples is to consider the effect of the substrate on the wavelength $\mu$ of the oscillations of the deformed film.

The wavelength $\mu$ to the substrate stiffness $k_5$; this has been plotted in Fig. 5.22. The labeled points along the curve correspond to the examples shown in Fig. 5.21.

Figure 5.22. A plot showing the relation between the wavelength $\mu$ and the substrate spring constant $k_5$, which is associated with bending stiffness. The labeled points correspond to the five examples shown in Fig. 5.21.

The plot shown in Fig. 5.22 is similar to the kinds of experimentally and analytically determined relations presented in the literature, e.g. [111]. An example of a buckled thin film is shown in Fig. 5.23. This multi-layer film was subjected to a compressive load to cause buckling.

5.2.3 Application to Self-Assembly of QDs

In the assembly of quantum dot superlattices, it is desirable for the deposited atoms forming the quantum dots to be in as regular an array as possible. To this end, so called strain engineering techniques are used whereby the substrate is strained in such a manner as to cause the deposited atoms to preferentially form in
controlled and regular arrays (this is also referred to as self-assembly) [16, 45]. As it turns out, the substrates, which are often thought of as films, typically deform in an oscillatory manner reminiscent of the oscillatory-dominant solutions presented in this work. For example, in Leroy et al. [67], the wavelength of a particular substrate is on the order of 135 nm and the amplitude is on the order of 0.4 nm. One approach used to model this phenomenon is to consider the substrate to be subjected to a hydrostatic strain that results in a sinusoidal surface profile [110]. In all of the works cited, classical elasticity is used to model the substrate upon which the atoms are being deposited. It is possible that the directed continuum approach could be used to more accurately model the substrate behavior.
Chapter 6

Summary and Future Work

The work presented thus far has included the formulation of two discrete models, a 5-spring model and a 4-spring model, based on observable physical attributes of a microscale CTF. In addition, two different continuous models have been formulated based on the linearized 5-spring discrete model: a discrete-based continuous model and a directed continuum model. The directed continuum model consists of a BVP with appropriate boundary conditions. It differs from the directed continuum models found in the literature in that a more complicated structure has been included in the original discrete model. The structure includes rigid bars, axial springs, torsional springs, and combinations of linear springs. The resulting BVP has also been analyzed to obtain a numerical quantity ($\zeta_I$ and $\zeta_{II}$) that characterizes the general behavior of the vertical deformation of the film (identical procedures may be used to quantify any of the deformation functions that result from the model). Finally, a variety of numerical examples were presented that describe the model’s behavior and suggest ways in which the model may be used for real world problems.

The substrate and the columnar structure of the model are not identical, although they do share some similarities. For example, for the linear models, both sections are defined with two spring constants, i.e., $\hat{k}_1$ and $\hat{k}_2$ for the columnar structure and $k_4$ and $k_5$ for the substrate. In addition, $\hat{k}_2$ and $k_4$ are associated with axial deformations, while $\hat{k}_1$ and $k_5$ are associated with bending. From the analysis of Chapter 4, it was determined that there are three main types of solutions: oscillatory, non-oscillatory, and mixed. The oscillatory solutions occur
when the bending stiffness of the substrate, described by $k_5$, dominates. The non-oscillatory solution occurs when the axial stiffness of the columnar structure, described by $\hat{k}_2$, dominates. (The non-oscillatory case also occurs when the coupling between the substrate and columnar structure is relaxed by reducing the value of $k_3$, the torsional spring connecting the columns to the substrate.) It was also shown that at both extremes (the oscillatory and non-oscillatory extremes), the solutions exhibit similar behavior in terms of overall deformation. This is another example of the similarities between the substrate and columnar parts of the model.

For the solution to be non-oscillatory, the columnar structure should not “see” a stiff substrate. This occurs either if the columnar structure is itself stiffer than the substrate or if the stiffness of the connecting torsional springs is reduced. One effect of the presence of the torsional springs is that the columnar part of the model and the substrate part of the model are not perfectly bonded together. In other words, there may be a discontinuity in the slope between substrate and columnar regions since $\tilde{\beta}(\xi)$ is not required to be zero.

The numerical results presented demonstrate the variety of possible physical responses due to simple deformations of the film when using a directed continuum theory. It is suggested that heterogeneous materials necessitate the use of higher-order theories to accurately capture their physical behavior. Examples of this nature, including films with varying curvature, micro-buckling, and strained substrates used in the self-assembly of quantum dots, have been suggested. To make a direct connection between the theory and one of these examples, it will be necessary to use experimental studies. As nano-scale materials are further investigated and the need to model their mechanical responses becomes greater, micropolar, micromorphic, and Cosserat-type theories may well contribute to the engineering community’s approaches to modeling in the future.

The directed continuum model, as formulated thus far, may be used to analyze a variety of heterogeneous systems. In addition, it will serve as the basis from which other, more complicated, models may be proposed. For example, the insights gained into the behavior of this linear continuum model will aid in the formulation of nonlinear models. The nonlinearity may be due to any or all of the following sources: kinematic nonlinearity (as has been introduced in the kinemat-
ically nonlinear discrete model), nonlinearity accounting for large deformations, and nonlinear potentials (or nonlinear springs). In addition, the present model may also be useful in formulating a BVP consisting of partial differential equations. This may be as a result of including kinetic effects or of increasing the spatial dimensionality of the model (or both).

When the models become more complicated, and particularly when they become nonlinear, the approach used in the present work to first define a strain energy density may no longer be helpful. In any event, the current model may serve an important role in ensuring that the behavior of any proposed nonlinear model has some connection to the formulated continuum model. That is, the directed continuum model will serve as a benchmark to check the results in limiting conditions of a nonlinear model.

With more complicated directed continuum models, it will be necessary to resort to alternate solution methods, i.e., other than analytical solutions. As has already been discussed in Chapter 2, finite element and homogenization approaches have proven useful for Cosserat-type models.

In conclusion, there are many opportunities for extending the proposed model to account for more complicated types of observed behavior and more complicated physical systems. In addition, there is a need to verify the applicability of the current model to the suggested physical systems via experimentation. Future research into this topic should focus on (1) ensuring that the derived models accurately describe the physical behavior of the real-world systems under consideration and (2) ensuring that the development of the models is consistent with established approaches of engineering, physics, and mathematics. It is hoped that this work will prove useful to that end.
Appendix A

General Form of Hamiltonian Formulation

The purpose of this appendix is to obtain the governing equations from a strain energy density in a more obvious formulation using Hamilton’s principle. Although Hamilton’s principle is used in the formulation presented in Section 3.4, it may not be obvious since the approach of Mindlin is followed [75] (Mindlin bases his formulation on Hamilton’s principle too). The strain energy density given by Eq. (3.173) is used to write an energy density per unit length, \( \bar{W} \), i.e.,

\[
\bar{W} = \left[ \frac{k_3 \beta(x)^2}{2} + 3l^2k_5\phi(x)^2}{2} + \frac{\hat{l}^2(\hat{k}_2 + k_4)u'(x)^2}{2} - 3l^2k_5\phi(x)u'(x) \right] (A.1)
+ \left[ \frac{\hat{l}^2(\hat{k}_1 + 3k_5)v'(x)^2}{2} - hl^2\hat{k}_2u'(x)\beta'(x)/2 + h^2l^2\hat{k}_2\beta'(x)^2/6 \right. \\
- hl^2\hat{k}_2u'(x)\phi'(x)/2 + h^2l^2\hat{k}_2\beta'(x)\phi'(x)/3 + l^2(2h^2\hat{k}_2 - 3l^2k_5)\phi'(x)^2/12]/L \\
= \bar{W} (u'(x), v'(x), \phi(x), \phi'(x), \beta(x), \beta'(x)) (A.2)
= WA, (A.3)
\]

noting that the six functions \( u'(x), v'(x), \phi(x), \phi'(x), \beta(x), \) and \( \beta'(x) \) are found in the strain tensors given by Eqs. (3.114), (3.115), and (3.117) and the cross-sectional area is given by \( A \). (The volume \( V \) equals \( AL \).) The total potential energy based on \( \bar{W} \) to be minimized with Hamilton’s principle, \( U \), is found by integrating \( \bar{W} \)
over the length $L$ of the beam,

$$U = \int_0^L \bar{W}(u', v', \phi, \phi', \theta, \theta'; x) \, dx,$$  

(A.4)

where $0 \leq x \leq L$. To minimize the definite integral expression for the potential energy it is necessary to look at the variation of the energy with respect to each of the displacement fields, i.e.,

$$\delta U = \delta U \bigg|_u + \delta U \bigg|_v + \delta U \bigg|_\beta + \delta U \bigg|_\phi.$$  

(A.5)

For brevity, only the details of the variation with respect to $u$ will be shown here. Integration by parts is used in obtaining the following relation:

$$\delta U \bigg|_u = \int_0^L \frac{\partial \bar{W}}{\partial u} \delta u \, dx + \int_0^L \frac{\partial \bar{W}}{\partial u'} \delta u' \, dx$$  

(A.6)

$$= \int_0^L \frac{\partial \bar{W}}{\partial u} \delta u \, dx + \frac{\partial \bar{W}}{\partial u} \bigg|_0^L \delta u - \int_0^L \frac{\partial \bar{W}}{\partial x} \delta u' \, dx$$  

(A.7)

$$= \int_0^L \left( \frac{\partial \bar{W}}{\partial u} - \frac{\partial}{\partial x} \frac{\partial \bar{W}}{\partial u'} \right) \delta u \, dx + \frac{\partial \bar{W}}{\partial u'} \bigg|_{x=L} \delta u(L) - \frac{\partial \bar{W}}{\partial u'} \bigg|_{x=0} \delta u(0).$$  

(A.8)

The remaining three variations are obtained similarly, which gives,

$$\delta U \bigg|_v = \int_0^L \left( \frac{\partial \bar{W}}{\partial v} - \frac{\partial}{\partial x} \frac{\partial \bar{W}}{\partial v'} \right) \delta v \, dx + \frac{\partial \bar{W}}{\partial v'} \bigg|_{x=L} \delta v(L) - \frac{\partial \bar{W}}{\partial v'} \bigg|_{x=0} \delta v(0),$$  

(A.9)

$$\delta U \bigg|_\beta = \int_0^L \left( \frac{\partial \bar{W}}{\partial \beta} - \frac{\partial}{\partial x} \frac{\partial \bar{W}}{\partial \beta'} \right) \delta \beta \, dx + \frac{\partial \bar{W}}{\partial \beta'} \bigg|_{x=L} \delta \beta(L) - \frac{\partial \bar{W}}{\partial \beta'} \bigg|_{x=0} \delta \beta(0),$$  

(A.10)

$$\delta U \bigg|_\phi = \int_0^L \left( \frac{\partial \bar{W}}{\partial \phi} - \frac{\partial}{\partial x} \frac{\partial \bar{W}}{\partial \phi'} \right) \delta \phi \, dx + \frac{\partial \bar{W}}{\partial \phi'} \bigg|_{x=L} \delta \phi(L) - \frac{\partial \bar{W}}{\partial \phi'} \bigg|_{x=0} \delta \phi(0).$$  

(A.11)

For the variation of $U$ to be equal to zero, the integrands on the right-hand sides of Eqs. (A.8)-(A.11) must equal zero. For an arbitrary variation in $u$ this means that the term in parenthesis in Eq. (A.8) must identically equal zero. Applying Eq. (A.1) to Eq. (A.8), one obtains the following differential equation:

$$-l^2 \left( \hat{k}_2 + k_4 \right) u''(x) + hl^2 \hat{k}_2 \beta''(x)/2 + hl^2 \hat{k}_2 \phi''(x)/2 = 0,$$  

(A.12)
which is the same as Eq. (3.68) if one were to multiply both sides by negative one. Likewise, considering arbitrary variations in $v$, $\beta$, and $\phi$, it is necessary to apply Eq. (A.1) to Eqs. (A.9)–(A.11) to obtain Eqs. (3.69)–(3.71). In this manner, the governing equations obtained in Chapter 3 are also obtained here.

The next step is to determine the natural boundary conditions from $\bar{W}$; in Section 3.4.1.5, these are given by Eqs. (3.185)–(3.188). If there are no externally applied loads, then either displacements must be specified at a boundary (essential boundary conditions) or the derivatives appearing in the second and third terms of Eqs. (A.8)–(A.11) must be zero (natural boundary conditions). For example, the natural boundary conditions based on the variation with respect to $u$ require that $\partial \bar{W}/\partial u'$ must equal zero at the boundary, i.e.,

$$
\frac{\partial \bar{W}}{\partial u'} = l^2(\hat{k}_2 + k_4)u' - l^2 h \hat{k}_2 (\phi' + \beta')/2 = 0.
$$

Equation (A.13) is identical to Eq. (3.185). In the same manner, applying $\bar{W}$ from Eq. (A.1) to the appropriate derivative terms found in Eqs. (A.9)–(A.11) yields the natural boundary conditions given in Eqs. (3.186)–(3.188).
Nontechnical Abstract

One of the main tasks of engineers and scientists is to develop accurate models of materials or structures observed in reality. Such models are essential for researchers studying how best to use a particular material or structure in a given application. Ideally, such models can even give insights into how something behaves before such behavior is even observed in an experiment. Different approaches used in modeling have different advantages and disadvantages. For example, some models take into account highly detailed information about the nature of matter at an atomic scale. The trade-off for such detailed knowledge is that we are only able to model relatively small amounts of matter for small periods of time. On the other extreme, classical continuum models are capable of describing the behavior of large objects over long periods of time. The trade-off for this type of knowledge is that we must greatly simplify our understanding of reality. For example, in classical continuum theories we consider material to be completely uniform, effectively ignoring the fact that atoms, crystal grain boundaries, or any small cracks exist. Even so, classical continuum theories are used in many engineering applications every day throughout the world.

One area where classical continuum theories may not adequately predict a material’s behavior occurs when our material or structure is very small and atomic interactions must be explicitly accounted for, e.g., the nanoscale. There is no doubt that the field of nanotechnology will prove the source of many of the technological developments in the 21st century. It is therefore critical that scientists and engineers be engaged in fundamental research, which includes understanding and
modeling a wide variety of nanoscale phenomena.

The goal of this research effort is to propose a continuum model that is capable of accounting for long-range atomic interactions. This proposed continuum model cannot be classical since we must account for long-range interactions. On the other hand, we want to take advantage of the benefits of a continuum approach, namely the ability to model large systems over long periods of time. The model being proposed is actually based on work nearly 100 years old that was published in 1909 by brothers Eugène and François Cosserat. In a classical continuum model, each point of a material in the model is thought of in a straightforward manner as a point of the material in the real world. That is, when an object stretches in the real world, the points of the model comprising the body likewise stretch. In a Cosserat-type of model, each point contains additional information; in principle, each point may contain any information the modeler desires. Since the Cosserat-type of model contains more information than a classical continuum model, it is capable of modeling much more complicated properties. In this work, a particular nano-scale structure serves as the motivation for a nano-scale Cosserat-type continuum model.

The primary result of this research is the development of a model based on the structure of a nano-scale thin film. Included in this development is a scheme that characterizes the behavior of the film and accounts for variations in the behavior. A number of different possible uses for such a theory are suggested (including thin film buckling and thin films with varying curvature) and results from the model are compared with results published by experimentalists. It is hoped that this model will serve as a benchmark for future models incorporating nonlinear interactions, nonlinear motions, and time.
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