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# ELLIPTIC EQUATIONS WITH SINGULARITIES: A PRIORI ANALYSIS AND NUMERICAL APPROACHES 

A Dissertation in<br>Mathematics<br>by<br>Hengguang Li

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## Abstract

Elliptic equations in a two- or three-dimensional bounded domain may have singular solutions from the non-smoothness of the domain, changes of boundary conditions, and discontinuities, singularities of the coefficients. These singularities give rise to various difficulties in the theoretical analysis and in the development of numerical algorithms for these equations. On the other hand, most of the problems arising from physics, engineering, and other applications have singularities of this form. In addition, the study on these elliptic equations leads to good understandings of other types of PDEs and systems of PDEs. This research, therefore, is not only of theoretical interest, but also of practical importance.

This dissertation includes a priori estimates (well-posedness, regularity, and Fredholm property) for these singular solutions of general elliptic equations in weighted Sobolev spaces, as well as effective finite element schemes and corresponding multigrid estimates. Applications of this theory to equations from physics and engineering will be mentioned at the end.

This self-contained work develops systematic a priori estimates in weighted Sobolev spaces in detail. It establishes the well-posedness of these equations and proves the full regularity of singular solutions between suitable weighted spaces. Besides, the Fredholm property is discussed carefully with a calculation of the index.

For the numerical methods for singular solutions, based on a priori analysis, this work constructs a sequence of finite element subspaces that recovers the optimal rate of convergence for the finite element solution. In order to efficiently solve the algebraic system of equations resulting from the finite element discretization on these finite subspaces, the method of subspace corrections and properties of weighted Sobolev spaces are used to prove the uniform convergence of the multigrid method for these singular solutions.

To illustrate wide extensions of this theory, a Schrödinger operator with singular
potentials and a degenerate operate from physics and engineering are studied in Chapter 6 and Chapter 7. It shows that similar a priori estimates and finite element algorithms work well for equations with a class of singular coefficients.

The last chapter contains a brief summary of the dissertation and plans for possible work in the future.

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## List of Symbols

$\Omega$ A bounded polygonal domain in $\mathbb{R}^{2}$, possible with curved boundaries
$\mathcal{V}$ The set of "vertices", containing geometric vertices and possible artificial vertices
$Q_{i} \quad$ The $i$-th "vertex" of the domain, $Q_{i} \in \mathcal{V}$
$\mathcal{S}_{i} \quad$ A small neighborhood of $Q_{i}, S_{i} \subset \Omega$
$H^{m}$ The usual Sobolev spaces with square-integratable weak derivatives up to order m
$\mathcal{K}_{a}^{m} \quad$ The weighted Sobolev space of order $m$ and index $a$
$\mathcal{K}_{\overrightarrow{\mathbf{a}}}^{m}$ The inhomogeneous weighted Sobolev space
$\partial^{u} \Omega$ The unfolded boundary of $\Omega$
${ }^{u} \bar{\Omega}$ The unfolded closure of $\Omega$
$\mathcal{T}$ The triangulation of $\Omega$
$\Gamma$ The interface of in the transmission problem where the coefficients jump

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## Dedication

To Jing Dai, whose constant support keeps me inspired and tireless.


## Introduction

Describing phenomena of the real world, the partial differential equation (PDE) has been one of the core topics in modern mathematics. Due to the lack of explicit formulae to express solutions, the theoretical study on PDEs mainly focuses on well-posedness (existence, uniqueness, and continuous dependence of the given data) and regularity (smoothness) of solutions. This, on the more-applied part (numerical PDEs), provides a reliable and powerful tool to approximate the continuous solution by discrete functions.

As a basic class of linear PDEs, the elliptic boundary value problem (EBVP) has been one of the starting points to study more complex, delicate PDE systems (equations of linear elasticity, Maxwell equations, Stokes equations, evolution equations, etc.), and corresponding numerical solutions. Hence, it is of fundamental importance to understand elliptic problems well both theoretically and numerically.

For EBVPs with smooth coefficients on bounded domains with a smooth boundary, plus, imposed the homogeneous Dirichlet boundary condition, prominent theory and numerical schemes have been developed, which match expectations and observations from practical applications. It has been well known, however, that the solution of an EBVP has singular behaviors when any of the conditions above is violated, which is quite normal in practice, provided that there exists a solution in a certain Sobolev space.

It has been discovered that numerical methods (finite element methods, finite difference methods, finite volume methods, spectral methods, etc.) do not always give a relatively accurate discrete approximation, especially when the solution pos-
sesses unbounded derivatives. These singularities may slow down the convergence rate of the numerical solutions, or even worse, direct numerical solutions to diverge.

A lot of work has been done on numerical treatments of singular solutions, and it remains extremely active nowadays, taking into account the lack of complete theory and of general numerical solvers. In this work, we will present results on singular solutions of EBVPs from the non-smoothness of the domain, changes of boundary conditions, and discontinuities of the coefficients. These results range from abstract spaces of functions to practical numerical schemes, solvers, with important applications in mathematical physics:

1) It provides a unified theory on well-posedness and regularity of solutions to EBVPs on general polygonal (polyhedral) domains in weighted Sobolev spaces.
2) It contributes to numerical algorithms by specifying a simple and explicit construction of finite subspaces for the finite element method (FEM), such that finite element solutions preserve the quasi-optimal convergence rate that is expected for smooth solutions.
3) As for fast solvers for the system of equations from the discretization in the special finite subspaces, it confirms the optimality of the multigrid (MG) method to handle these singular EBVPs, by estimating the convergence rate.
4) It introduces new approaches to analyze equations associated to operators with singular coefficients (Schrödinger operators, a type of degenerate operators), which have wide applications in physics.

Besides, this intact theory extends to, but not only limited to, the analysis of other equations and the development of other numerical methods.

Throughout this dissertation, we concentrate on two-dimensional domains and the FEM for the numerical method, although, in principle, similar techniques can be applied to domains of higher dimensions and other numerical methods.

### 1.1 A Brief Review

### 1.1.1 Singular Solutions of EBVPs

Classical theory [48, 52, 87, 94] on EBVPs with the homogeneous Dirichlet boundary condition indicates that the elliptic operator is Fredholm, and the solution, if
there exists one, is always smoother than the given data in Sobolev spaces, when coefficients, the boundary of the domain are smooth enough. Many good results on regularity of the solution, however, do not extend to the points where one of the following conditions appears.

1) The domain has non-smooth points (corners, edges, etc.) on the boundary.
2) The boundary conditions change types at some points, for instance, from the Dirichlet to the Neumann or vise versa.
3) The equation has jump coefficients (transmission problems).
4) The differential operator has singular or degenerate coefficients.

Well-poseness and regularity of these types of solutions have grabbed interest of many mathematicians. We here point out a seminal paper [60] by Kondratiev, which analyzed the solution with corner singularities both in weighted Sobolev spaces and in terms of singular expansions. This work triggers many new ideas, motivates deeper thoughts, even today.

For more explicit explanations and analysis of these singularities, we refer to the following: Bacuta, Nistor and Zikatanov [21], set up a general framework to analyze the solution in weighted Sobolev spaces, by proving a Poincaré-type theorem in these spaces, and making use of Fredholm properties of elliptic operators in weighted Sobolev spaces; Dauge [45], classifies corner singularities, and provides general results for the singular solution in fractional Sobolev spaces; Grisvard [55], derives the Fredholm property of elliptic operators in regular Sobolev spaces, and shows explicit functions for the singular part of the solution; Kozlov, Mazya and Rossmann [61, 62] study operator pencils for elliptic operators; Li, Mazzucato and Nistor [67], introduce the class of domains with polygonal structures, to which the a priori estimate in weighted Sobolev spaces extends, and give a clear description of the singular solutions in weighted Sobolev spaces, which come from the Neumann-Neumann vertices and singular points in transmission problems; Nazarov and Plamenevsky [80], explain properties of the index of elliptic operators as Fredholm operators; Nicaise [82], considers interface problems on polygonal domains. Numerous people have contributed to theoretical analysis of singular solutions of EBVPs. This list, therefore, has no way to be complete, while it is convenient for searching references.

### 1.1.2 Numerical Methods for Singular Solutions

With the theoretical understanding of the singular behavior in the solution, people have managed to develop various numerical algorithms to approximate the singular solution of EBVPs. In particular, several variations of the classical FEM have been successful. Based on different principles, special finite element approximation spaces are widely used for the analysis and development of finite element schemes $[5,7,13,17,21,98]$. These methods share the same property that the error between the solution and the discrete solution is distributed in some special way on every element by controlling the size of the element, such that the finite element solution approximates the solution in a good rate, although these subspaces may appear to be different. On the other hand, explicit singular expansions can also be useful for the FEM, although it is more restrictive [38]. Another alternative to speed up the convergence rate of the numerical solution without knowing regularity, is to use a posterior estimates. See [76] and references therein.

The finite element discretization results in a big system of algebraic equations. A special version of the curse of dimensionality denies most of exact solvers for such systems immediately, even though they are accurate. Thus, theory and algorithms based on iterative methods, especially MG methods [29, 57, 96, 99], dominate this field. As for systems from discretizations of singular solutions, we refer to the following: Brenner [35] analyzes the convergence rate for only partial regularity; Bramble, Pasciak, Wang and Xu [30] develope the convergence estimate without regularity assumptions for an $L^{2}$-projection based decomposition; in addition, on graded meshes, using the approximation property in [17], Yserentant [102] proves the uniform convergence of the multigrid $W$-cycle with a particular iterative method for each level for piecewise linear functions; Brannick, Li and Zikatanov [33] estimate the convergence rate of the MG $V$-cycle for standard subspace smoothers (Richardson, Jacobi, Gauss-Seidel, etc.) on graded meshes for corner singularities, and prove it converges uniformly.

For systems of PDEs (equations of linear elasticity, Maxwell equations, Stokes equations, etc.), the development of finite element schemes remains very challenging [11] even for smooth domains and for good differential operators. Thus, there are even more uncertain questions waiting for fundamental solutions for singular solutions of these equations.

### 1.2 Contents

This dissertation presents a priori estimates (well-posdness, regularity, and Fredholm property) for singular solutions of EBVPs in some weighted Sobolev spaces, as well as the analysis and development on the FEM, the MG method. More explicitly, we shall mainly deal with singular solutions for general EBVPs, from the non-smoothness of the domain, changes of boundary conditions, discontinuities of coefficients and singularities in the coefficients. The rest of the work is organized as follows.

Chapter 1 includes preliminary materials and notations that are necessary to develop out theory in chapters afterwards. We briefly recall the classical PDE theory on EBVPs with solutions of full regularity. Then, basic estimates on the convergence rate of the FEM and of the MG method on quasi-uniform meshes will be presented in Subsection 1.3.2 and Subsection 1.3.3. In the last subsection, one can find an introduction for singular solutions of different types, but mainly for corner singularities.

In Chapter 2, we discuss the techniques for estimating singular solutions by introducing a special weighted Sobolev space for corner singularities with the homogeneous Dirichlet boundary condition. It is mainly motivated by Kondratiev's work [60]. Some fundamental regularity results for general elliptic operators are proved on an infinite domain, resulting from the Mellin transform. In particular, for the Laplace operator and hence for strictly positive elliptic operators, we establish the well-posedness of the solution on the infinite domain. One will notice that the weighted Sobolev space is the natural outcome of the usual Sobolev space on the infinite domain after using the inverse of the Mellin transform. Thus, these well-posedness and regularity estimates on the infinite domain results in our main a priori estimates for corner singularities in these weighted Sobolev spaces.

Furthermore, as one can see throughout this dissertation, we can use the Mellin transform for more general singular solutions. We mention that the Mellin transform is only the starting point of the analysis, more mathematical tools are needed for the construction of the theory.

In addition, we summarize a list of lemmas to point out useful properties of the weighted Sobolev spaces. These observations will play an important role in
the development and estimates of our numerical approaches: the FEM in Chapter 4 and the MG method in Chapter 5.

Chapter 3 consists of our main a priori estimates in weighted Sobolev spaces for singularities of EBVPs from the non-smooth points on the boundary, changes of boundary conditions, and singular points on the interface of transmission problems. We consider a generation of the usual polygonal domain and define a domain with a polygonal structure. The weighed Sobolev space considered here is also a generation of the weighted Sobolev space defined in Chapter 2 tailored to different singular points.

The main contribution of this chapter is to provide estimates for the singular solutions from the vertices whose both adjacent sides are assigned Neumann boundary conditions, and from the singular points of the interface for transmission problems. The well-posedness and the Fredholm property of the singular solutions of these types in weighted Sobolev spaces, which were not quite certain in the literature, are proved for the Laplace operator, and extend to general elliptic operators. The idea is to add a specific space of smooth functions in the weighted Sobolev space to construct isomorphisms, based on a careful calculation of the index of the Fredholm operator.

This chapter provides a unified theory for general EBVPs on domains with a polygonal structure. Moreover, this theory has various applications for elliptic equations with singular coefficients, which are mentioned in Chapter 6 and Chapter 7.

In Chapter 4, we give a simple and explicit construction of a sequence of finite element subspaces for the Lagrange elements, such that the finite element solutions approximate all types of singular solutions in Chapter 3 in the quasi-optimal rate of convergence. The estimates are based on the a priori estimates from Chapter 3 and properties of weighted Sobolev space from Chapter 2. As a remarkable observation for the FEM, the smooth functions we construct for the isomorphism near Neumann-Neumann corners and singular points on the interface belong to the finite element approximation space. Therefore, the error estimates near all these singularities can be treated in a uniform way.

Chapter 5 presents the first attempt to use the theoretical results in Chapter 3 and the method of subspace corrections to estimate the convergence rate of the MG
method on graded meshes for all types of singular solutions discussed above. This analysis leads to the uniform convergence of the MG $V$-cycle on graded meshes for singular solutions with standard subspaces solvers (Richardson, weighted Jacobi, Gauss-Seidel, etc.). The MG method on pathological solutions is of theoretical and practical interest and we expect to work out more in this direction.

In the last two Chapters 6 and 7, we discuss applications of the theory from Chapter 3 and Chapter 4 to elliptic equations with singular coefficients. Chapter 6 deals with a Schrödinger type operator with singular potentials of the form $\prod\left|x-Q_{i}\right|^{-2}$, where $Q_{i}$ are points in the closure of the domain. We study a priori estimates of the solution in weighted Sobolev spaces, and correspondingly, give a robust FEM that recovers the quasi-uniform convergence rate for numerical solutions in the presence of singular solutions caused by the singular potential. In addition, our regularity estimates extend to a class of Schrödinger operators.

In Chapter 7, we focus on a priori estimates and the development of the FEM for an elliptic equation whose coefficients are degenerate on a segment of the boundary. We establish the well-posedness and regularity of the solution in weighted Sobolev spaces, and also compute the index of the spaces for the Fredholm property. One interesting finding is that the resulting triangulation in the FEM for the quasioptimal convergence rate violates the maximum angle condition, which is, however, covered by the homogeneity property of the weighted Sobolev spaces.

### 1.3 Preliminaries and Notations

In this section, we first briefly review Sobolev spaces and some related results on well-posedness and regularity of solutions for elliptic boundary value problems (EBVPs). Then, we restate useful properties of the FEM regarding the mesh generation and convergence rates of numerical solutions. Note that good convergence rates need good regularity on the solution as one of the assumptions. We also mention some estimates on the multigrid method that solves large systems of equations. Finally, we discuss corner singularities and problems they cause in numerical approximations. We let $C>0$ be a generic constant that may be different at each occurrence. The notation will follow what we define in this section throughout this dissertation.

### 1.3.1 Sobolev Spaces and the Weak Solution

Let $\Omega \subset \mathbb{R}^{2}$ be a bounded domain. The regular Sobolev spaces $H^{m}(\Omega)$ on the domain $\Omega$ for $m \in\{0,1,2, \ldots\}$ are defined as follows [48],

$$
H^{m}(\Omega)=\left\{v \in L^{2}(\Omega), \partial^{\alpha} v \in L^{2}(\Omega), \forall|\alpha| \leq m\right\}
$$

with the multi-index $\alpha=\left(\alpha_{1}, \alpha_{2}\right) \in \mathbb{Z}_{+}^{2},|\alpha|:=\alpha_{1}+\alpha_{2}$ and $\partial^{\alpha}:=\partial^{\alpha_{1}} \partial^{\alpha_{2}}$. The $H^{m}$-norm of any $v \in H^{m}(\Omega)$ is defined by

$$
\|v\|_{H^{m}(\Omega)}^{2}:=\sum_{|\alpha| \leq m}\left\|\partial^{\alpha} v\right\|_{L^{2}(\Omega)}^{2},
$$

where the $L^{2}$-norm is

$$
\|v\|_{L^{2}(\Omega)}^{2}=\int_{\Omega}|v|^{2} d x d y
$$

In addition, fractional Sobolev spaces $H^{m+s}, 0<s<1$, can be defined by interpolation [23]. For regular Sobolev spaces $H^{m}(\Omega)$ [48, 52, 87, 94], $m>0$, the space $H^{-m}(\Omega)$ is the dual space of $H_{0}^{m}(\Omega)$, where $H_{0}^{m}(\Omega)$ is defined [1] as follows,

$$
H_{0}^{m}(\Omega)=\left\{v \in H^{m}(\Omega),\left.\partial^{i} v\right|_{\partial \Omega}=0, i<m\right\}
$$

An elliptic equation associated with the elliptic operator $P$ of the divergence form on $\Omega$ can be written as

$$
\begin{equation*}
P u=-\sum_{i, j=1}^{2}\left(a^{i j}(x) u_{x_{i}}\right)_{x_{j}}+\sum_{i=1}^{2} b^{i}(x) u_{x_{i}}+c(x) u . \tag{1.1}
\end{equation*}
$$

EBVPs are equations of this type with boundary conditions. To fix ideas, we consider the following boundary value problem with a smooth boundary $\partial \Omega$,

$$
\left\{\begin{align*}
P u=f & \text { in } \Omega  \tag{1.2}\\
u=0 & \text { on } \partial \Omega
\end{align*}\right.
$$

Definition 1.3.1. We define the bilinear form $B[\cdot, \cdot]$ for the divergence form op-
erator $P$ as follows

$$
\begin{equation*}
B[u, v]:=\int_{\Omega} \sum_{i, j=1}^{2} a^{i j} u_{x_{i}} v_{x_{j}}+\sum_{i=1}^{2} b^{i}(x) u_{x_{i}} v+c(x) u v \tag{1.3}
\end{equation*}
$$

for $u, v \in H_{0}^{1}(\Omega)$. Then, we say that $u \in H_{0}^{1}(\Omega)$ is a weak solution of (1.2) if

$$
B[u, v]=(f, v),
$$

for all $v \in H_{0}^{1}(\Omega)$, where $(\cdot, \cdot)$ denotes the $L^{2}$ inner product.
With suitable assumptions on the elliptic operator $P$, the bilinear form $B[\cdot, \cdot]$ can be equivalent to the $H^{1}$ inner product on $\Omega$. In this case, the uniqueness of the solution to (1.2) follows the Lax-Milgram Theorem immediately. Moreover, based on the Rellich-Kondrachov compactness theorem, the Fredholm alternative gives more general descriptions on the kernel and range of the operator in (1.2).

A fundamental theorem on regularity of the solution in Equation (1.2) claims that the weak solution $u$ is always in $H^{m+1}(\Omega) \cap H_{0}^{1}(\Omega), m \geq 0$, as long as the given datum $f$ belongs to $H^{m-1}(\Omega)$, provided that there exists a weak solution $u \in H_{0}^{1}(\Omega)$ for (1.2), and $\partial \Omega$, the coefficients are smooth enough.

Theorem 1.3.2. For $m \geq 0$, we assume

$$
a^{i j}, b^{i}, c \in C^{m+1}(\bar{\Omega}), \quad i, j=1,2
$$

and $f \in H^{m}(\Omega)$. Suppose that $u \in H_{0}^{1}(\Omega)$ is a weak solution of the equation (1.2), and $\partial \Omega$ is $C^{m+2}$. Then, we have the estimate

$$
\|u\|_{H^{m+2}(\Omega)} \leq C\left(\|f\|_{H^{m}(\Omega)}+\|u\|_{L^{2}(\Omega)}\right),
$$

where $C$ depends only on $m, \Omega$, and the coefficients of $P$. Moreover, if $u$ is the unique solution, the above estimate can be simplified into

$$
\|u\|_{H^{m+2}(\Omega)} \leq C| | f \|_{H^{m}(\Omega)}
$$

Since no explicit formulae are available for general EBVPs, the above theo-
rem on the smoothness of the solution plays important role for development of numerical methods. However, if any of the assumptions in Theorem 3.1.1 is violated, the estimate has to be reformulated accordingly, in which case, the solution may possess singularities against the full regularity described in this theorem.

### 1.3.2 The Finite Element Method

As a powerful numerical method widely used in practical computation to solve PDEs, the FEM grabs tremendous interest from academic research as well [10, $14,34,36,39,92]$. Starting from a domain decomposition, FEM typically uses piecewise polynomials as basis functions to approximate the solution. We here show FEM schemes and estimates for elliptic equations to set up the framework and fix the notation.

In this subsection, we let $\Omega$ be a bounded two-dimensional domain with straight edges to present the basic idea behind the FEM. Generally speaking, the FEM uses functions in finite dimensional spaces to approach solutions in some Sobolev space. Thus, one major concern is how good this approximation is. People need the FE solution converges to the real solution as fast as possible, although the convergence rate can be affected by many factors.

The following definition [34] gives a classical decomposition with triangles on $\Omega$.

Definition 1.3.3. Let $\left\{\mathcal{T}^{h}\right\}, 0<h \leq 1$, be a family (triangulation) of triangles on $\Omega$ such that

$$
\max \left\{\operatorname{diam} T: T \in \mathcal{T}^{h}\right\} \leq h \operatorname{diam} \Omega
$$

Let $B_{T}$ be the largest ball contained in $T$. The family (triangulation) is said to be quasi-uniform if there is $\rho>0$ such that

$$
\min \left\{\operatorname{diam} B_{T}: T \in \mathcal{T}^{h}\right\} \geq \rho h \operatorname{diam} \Omega
$$

Let polynomials be basis functions on these triangles, and denote by $V$ the linear span of these basis functions. Then, the classical Galerkin finite element
problem of Equation (1.2) is to find $u_{h} \in V$, such that

$$
\begin{equation*}
B\left[u_{h}, v_{h}\right]=\left(f, v_{h}\right), \quad \forall v_{h} \in V \subset H_{0}^{1}(\Omega), \tag{1.4}
\end{equation*}
$$

where $B[\cdot, \cdot]$ is given by Definition 1.3.1, with $B[\cdot, \cdot]$ continuous and coercive on $H_{0}^{1}(\Omega)$. Then, the following theorem shows the possibility to estimate the error in the energy norm.

Theorem 1.3.4. (Céa) Suppose $(H,(\cdot, \cdot))$ is a Hilbert space, and $S \subset H$ is a subspace of $H$. If the bilinear form $B[\cdot, \cdot]$ is continuous and coercive on $S$ and $u$ solves (1.2), for the finite element problem (1.4) we have

$$
\left\|u-u_{h}\right\|_{S} \leq C \min _{v_{h} \in S}\left\|u-v_{h}\right\|_{S}
$$

where $C$ depends on the operator $P$ and the domain.
Céa Theorem simplifies the estimate on the global error into locating a function in $V$ that has good approximation properties. Meanwhile, Bramble-Hilbert Lemma provides a useful local estimate on $\inf _{v_{h} \in V}\left\|u-v_{h}\right\|_{H^{1}(T)}$, for any $T \in \mathcal{T}^{h}$. This lemma, however, can not extend to global estimates directly, since the special function $v_{h}$ used in the proof (generalized Taylor polynomials) may not be a function of $V$ in the global sense.

One of the remedies that can combine local estimates and global bounds is to utilize the interplant function $u_{I}(x)=u(x)$ at nodal points when the solution has good regularity (the Sobolev imbedding theorem). The interpolation $u_{I}$ is usually in the approximation space $V$, and the interplant operator $\mathcal{I}: C^{l}(\bar{T}) \rightarrow H^{m}(T)$ associated is bounded, for $m-l-1>0$. Thus, we can extend the local estimates from the Bramble-Hilbert Lemma to a global error bound by summing up the interplant error piece by piece on the triangulation, without losing the order of approximation.

Theorem 1.3.5. Suppose that the right hand side $f \in H^{m-1}(\Omega)$ in Equation (1.2) and the solution has full regularity $u \in H^{m+1}(\Omega)$. Then, the finite element solution $u_{h}$ with piecewise polynomials of degree $m$ on the quasi-uniform mesh satisfies

$$
\left\|u-u_{h}\right\|_{H^{1}(\Omega)} \leq C h^{m}|u|_{H^{m+1}(\Omega)} \leq C h^{m}| | f \|_{H^{m-1}(\Omega)},
$$

for $m \geq 1$, where $C$ does not depend on the right hand side or the level of refinements, and $h$ is the mesh size defined in Definition 1.3.3.

The estimate in Theorem 1.3.5 is called the optimal rate of convergence for the finite element solution $u_{h}$ from the usual finite element (Galerkin) method. The convergence rate above is sometimes presented in terms of dimensions of finite subspaces to reveal the computational complexity for the desired accuracy as follows,

$$
\begin{equation*}
\left\|u-u_{h}\right\|_{H^{1}(\Omega)} \leq C \operatorname{dim}(V)^{-m / 2}\|f\|_{H^{m-1}(\Omega)} \tag{1.5}
\end{equation*}
$$

Instead of the interpolation, we also mention existence of techniques on analysis of global errors for less regular solutions. See [93] for example.

### 1.3.3 Multigrid (MG) Methods

The multigrid method is one of the most efficient methods to solve systems of algebraic equations from the discretization of the FEM. Fundamental analysis for the MG method can be found, for example, in [29, 57, 96, 99]. One of the motivations for the MG method is that usual iterative methods (Jacobi, Gauss-Seidel, etc.) can quickly eliminate most of the high frequency bandwidth of errors. Thus, applications of these iterative methods on a sequence of coarse grids that match the discrete frequency of errors on each level and that represent the entire frequency range of errors, should lead to a quite efficient solver.

From the space decomposition point of view, the MG method can be considered as a subspace correction method based on a sequence of nested subspaces [100]. To be more precise, we let $P=-\Delta$ in Equation (1.2). Denote by

$$
V_{0} \subset V_{1} \subset \ldots \subset V_{j} \subset \ldots \subset V_{J} \subset H_{0}^{1}(\Omega)
$$

a sequence of finite subspaces on the corresponding quasi-uniform triangulation $\mathcal{T}_{j}$ for (1.2). Suppose the solution $u$ has full regularity. Let $A: H_{0}^{1}(\Omega) \rightarrow H^{-1}(\Omega)$ be the differential operator for Equation(1.2). The corresponding weak form is then

$$
a\left(u, v_{j}\right)=\left(A u, v_{j}\right)=\left(-\Delta u, v_{j}\right)=\left(\nabla u, \nabla v_{j}\right)=\left(f, v_{j}\right), \quad \forall v_{j} \in V_{j} .
$$

Define $Q_{j}, P_{j}: H_{0}^{1}(\Omega) \rightarrow V_{j}$ and $A_{j}: V_{j} \rightarrow V_{j}$ as orthogonal projectors and the restriction of $A$ on $V_{j}$, respectively:

$$
\begin{aligned}
\left(Q_{j} u, v_{j}\right) & =\left(u, v_{j}\right), \quad \forall u \in H_{0}^{1}(\Omega), \quad \forall u_{j}, v_{j} \in V_{j}, \\
a\left(P_{j} u, v_{j}\right) & =a\left(u, v_{j}\right), \quad\left(A u_{j}, v_{j}\right)=\left(A_{j} u_{j}, v_{j}\right) .
\end{aligned}
$$

Let $\mathcal{N}_{j}=\left\{x_{i}^{j}\right\}$ be the set of nodal points in $\mathcal{T}_{j}$. Then, the $j$ th level finite element discretization reads: Find $u_{j} \in V_{j}$, such that

$$
\begin{equation*}
A_{j} u_{j}=f_{j} \tag{1.6}
\end{equation*}
$$

where $f_{j} \in V_{j}$ satisfies $\left(f_{j}, v_{j}\right)=\left(f, v_{j}\right), \forall v_{j} \in V_{j}$.
The method of subspace corrections (MSC) reduces a multigrid process to choosing a sequence of subspaces and corresponding operators $B_{j}: V_{j} \rightarrow V_{j}$ approximating $A_{j}^{-1}, j=1, \ldots, J$. For example, in the MSC framework, the standard multigrid backslash cycle for solving (1.2) is defined by the following subspace correction scheme

$$
u_{j}^{l}=u_{j}^{l-1}+B_{j}\left(f_{j}-A_{j} u_{j}^{l-1}\right),
$$

where the operator $B_{j}: V_{j} \rightarrow V_{j}, 0 \leq j \leq J$, are recursively defined as follows [100]:

Algorithm 1.3.1. Let $R_{j} \approx A_{j}^{-1}, j>0$, denote a local relaxation method. For $j=0$, define $B_{0}=A_{0}^{-1}$. Assume that $B_{j-1}: V_{j-1} \rightarrow V_{j-1}$ is defined. Then,

1. Fine grid smoothing: For $u_{j}^{0}=0$ and $k=1,2, \cdots, n$

$$
\begin{equation*}
u_{j}^{k}=u_{j}^{k-1}+R_{j}\left(f_{j}-A_{j} u_{j}^{k-1}\right) \tag{1.7}
\end{equation*}
$$

2. Coarse grid correction: Find the corrector $e_{j-1} \in V_{j-1}$ by the iterator $B_{j-1}$

$$
e_{j-1}=B_{j-1} Q_{j-1}\left(f_{j}-A_{j} u_{j}^{n}\right)
$$

Then, $B_{j} f_{j}=u_{j}^{n}+e_{j-1}$.
Recursive application of Algorithm 1.3.1 results in an MG $V$-cycle for which
the following identity holds : $I-B_{J}^{v} A_{J}=\left(I-B_{J} A_{J}\right)^{*}\left(I-B_{J} A_{J}\right)$, where $B_{J}^{v}$ is the iterator for the MG $V$-cycle. Therefore, $B_{J} A_{J} u$ and $B_{J}^{v} A_{J} u$ represent the numerical solutions after a complete MG backslash cycle and a MG $V$-cycle, respectively. Note that the MG $W$-cycle can be formulated in a similar way. From the algorithm above, we first derive

$$
\begin{aligned}
u_{j}^{n} & =\left(I-R_{j} A_{j}\right) u_{j}^{n-1}+R_{j} A_{j} u_{j} \\
& =\left(I-R_{j} A_{j}\right)^{2} u_{j}^{n-2}-\left(I-R_{j} A_{j} u\right)^{2} u_{j}+u_{j} \\
& =-\left(I-R_{j} A_{j}\right)^{n} u_{j}+u_{j},
\end{aligned}
$$

where $u_{j}$ is the finite element solution in $V_{j}$ and $u_{j}^{n}$ is the approximation after $n$ iterations in (1.7) on the $j$ th level. Define the linear operator $T_{j}=(I-(I-$ $\left.\left.R_{j} A_{j}\right)^{n}\right) P_{j}$ and $T_{0}=P_{0}$. From this it follows that

$$
\begin{aligned}
\left(I-B_{J} A_{J}\right) u_{J} & =u_{J}-u_{J}^{n}-e_{J-1}=\left(I-T_{J}\right) u_{J}-e_{J-1} \\
& =\left(I-B_{J-1} A_{J-1}\right)\left(I-T_{J}\right) u_{J} .
\end{aligned}
$$

Application of the above identity then yields

$$
\begin{equation*}
\left(I-B_{J} A_{J}\right)=\left(I-T_{0}\right)\left(I-T_{1}\right) \cdots\left(I-T_{J}\right), \tag{1.8}
\end{equation*}
$$

which gives an opportunity to analyze the convergence rate of the MG method by estimating the norm of the operator in (1.8).

Suppose the solution $u$ of (1.2) has full regularity. Let $\|\cdot\|_{a}$ be the norm induced by the inner product $a(\cdot, \cdot)$. Then, we have the following theorem [101] that gives the contraction rate of the error for the MG $V$-cycle.

Theorem 1.3.6. Assume a symmetric relaxation method on every subspace satisfies

$$
\left(R_{j} v, v\right) \geq C h^{2}(v, v), \quad v \in V_{j}
$$

and the spectral radius $\rho\left(R_{j} A_{j}\right) \leq \omega$, for $0<\omega<2$. Then, there exist constants


Figure 1.1. Domain $\Omega$ with a re-entrant vertex $Q$.
$c_{1}, c_{2}>0$ independent of the number of refinements, such that

$$
\left\|I-B_{J} A_{J}\right\|_{a}^{2} \leq \frac{c_{1}}{c_{1}+c_{2} n},
$$

where $n$ is the number of smoothings on each level.
Based on Theorem 1.3.6, the solution from the MG $V$-cycle converges to the finite element solution uniformly, which presents a big advantage of MG methods over other numerical solvers. More details on the method of subspace corrections will be discussed in Chapter 5 .

### 1.3.4 Corner Singularities

As a class of singularities in solutions of EBVPs, corner singularities come from the non-smoothness of the boundary. Other types of singular solutions mentioned in previous sections have a similar nature. In this cases, the solution for an EBVP may not be in $H^{2}(\Omega)$, even if the right hand side $f(x) \in \mathcal{C}^{\infty}(\Omega)$. (Recall Theorem 1.3.2 for comparisons.) Note that both the FEM and the MG method requires some regularity assumptions of the solution to give good rates of convergence on quasi-uniform meshes. The presence of singularities in general destroys the optimal rate of convergence in Theorem 1.3.5 [98]. Analysis of this type of singularities can be found in $[7,45,55,67,21,60,62,80,85,44]$ and many other references.

Taking Equation (1.2) as an example, we first illustrate some fundamental understandings on corner singularities by fixing $P=-\Delta$ and $\Omega$ to be a polygonal domain with a re-entrant corner (Figure 1.1).

As in Definition 1.3.1, with integration by parts, the weak solution of (1.2) satisfies

$$
(\nabla u, \nabla v)=(f, v), \quad \forall v \in H_{0}^{1}(\Omega)
$$

Provided that $u$ has the homogeneous Dirichlet boundary condition, the Poincaré inequality indicates

$$
\|u\|_{L^{2}(\Omega)} \leq C\|\nabla u\|_{L^{2}(\Omega)}, \quad \forall u \in H_{0}^{1}(\Omega)
$$

which provides the coercivity for the bilinear form on $H_{0}^{1}(\Omega)$. Since the bilinear form is continuous on $H_{0}^{1}(\Omega)$, the Lax-Milgram Theorem guarantees the existence of a unique solution $u \in H_{0}^{1}(\Omega)$ for any $f \in H^{-1}(\Omega)$. Therefore,

$$
\Delta: H_{0}^{1}(\Omega) \rightarrow H^{-1}(\Omega)
$$

defines an isomorphism, i.e., a continuous bijection between two Banach spaces. Until this point, the result is the same as in Theorem 1.3.2. Now, we simply assume $f \in L^{2}(\Omega)$. Recall $u$ should be in $H^{2}(\Omega)$ if the domain is smooth, but it is not the case on the domain in Figure 1.1. Let $\psi \in \mathcal{C}^{\infty}(\bar{\Omega})$ be a smooth function on $\bar{\Omega}$, such that its support intersects $\partial \Omega$ only at interior points of each edge. Then, we have the following estimate for local regualrity [55].

Theorem 1.3.7. Let $u \in H_{0}^{1}(\Omega)$ be the weak solution of Equation (1.2). Then, $\psi u \in H^{2}(\Omega)$. Moreover, if $f \in H^{m}(\Omega)$ for $m \geq 0, \psi u \in H^{m+2}(\Omega)$.

This is in fact the interior regularity for the solution. Namely, the singularity only appears in the neighborhood of the singular point (corners (2-D, 3-D), edges $(3-\mathrm{D}))$ on the boundary. Therefore, with these singularities, the weak solution may be in some Sobolev space $H^{s}(\Omega)$, for $1<s \leq 2$ when $f \in L^{2}(\Omega)$. We say $u$ has full regularity if $s=2$, and partial regularity for $s<2$. Note this is always the case. Even if $f \in \mathcal{C}^{\infty}(\Omega), u$ may not be in $H^{2}(\Omega)$.

The structure of corner singularities in 2-D can be studied in a relatively explicit sense. In fact, singular functions in the neighborhood of a corner are determined by the spectrum of the resulting elliptic operator from the Mellin Transform. It
has been well known [55, 60] that for Equation (1.2) of our setting, singularities of the solution have the following behaviors near every vertex,

$$
r^{k \pi / \omega_{i}} \phi_{i}(\theta), \quad k=1,2,3, \ldots
$$

where $r, \theta$ are the local polar coordinates with the origin at the $i$ th vertex, $\omega_{i}$ denotes the interior angle of the $i$ th corner, and $\phi_{i}$ is a smooth function of $\theta$. Thus, the singular function in $H^{2}$ is actually $r^{\pi / \omega_{Q}} \phi_{Q}(\theta)$ near the re-entrant corner $Q$, for $\pi<\omega_{Q}<2 \pi$.

From the discussion above, the elliptic operator $\Delta: H^{2}(\Omega) \cap H_{0}^{1}(\Omega) \rightarrow L^{2}(\Omega)$ is injective, but no longer an isomorphism when $\Omega$ has re-entrant corners. Instead of an isomorphism, one can, however, show that $\Delta: H^{2}(\Omega) \cap H_{0}^{1}(\Omega) \rightarrow L^{2}(\Omega)$ is Fredholm with a closed range in $L^{2}(\Omega)$. Let $N \subset L^{2}(\Omega)$ be the orthogonal of the range in $L^{2}(\Omega)$. The dimension of $N$ equals the cardinality of $\left\{\lambda_{i, k}=k \pi / \omega_{i}\right.$, $\left.0<\lambda_{i, k}<1, k \in \mathbb{N}\right\}$ [55], which is contributed to by every vertex of the domain. Similar results apply to general elliptic operators and other boundary conditions.

To conclude this chapter, we mention that corner singularities may cause unexpected difficulties in numerical simulations of elliptic PDEs. One of the negative effects is that the convergence rate of the finite element solution is slowed down and was formulated by Wahlbin [98] as follows:

Theorem 1.3.8. Let $V_{n}$ be the linear finite element subspace of $H_{0}^{1}(\Omega)$ on triangles in the quasi-uniform mesh, and $h$ be the mesh size. Then, for $u \in H^{r}(\Omega) \cap H_{0}^{1}(\Omega)$ in Equation 1.2 with $1<r<2$,

$$
\min _{v_{n} \in V_{n}}\left\|u-v_{n}\right\|_{H^{1}(\Omega)} \leq C h^{r-1}\|u\|_{H^{r}(\Omega)}
$$

where $C$ dose not depend on $h$ or $u$.


## A Priori Estimates for Elliptic Operators on the Infinite Domain

In this chapter, we establish the theoretical foundation on well-posedness, regularity, and the Fredholm property associated with elliptic operators on a polygonal domain. We also point out that the same strategy in our proofs can be used for general elliptic operators and mixed boundary conditions by freezing the variables in the coefficients at the vertex, when no two adjacent edges are assigned Neumann conditions. We will mention regularity and the Fredholm property for the Neumann-Neumann problem and the transmission problem in the next chapter.

### 2.1 The Mellin Transform

For simplicity, We fix $P=-\Delta$ in Equation (1.2) and let $\Omega \in \mathbb{R}^{2}$ be a bounded polygonal domain throughout this section. Let $\left\{Q_{i}\right\}, i \geq 1$, be the set of vertices of $\Omega$. Since the solution of the equation is quite regular on any compact subset of $\Omega$ (Theorem 1.3.7) that excludes all vertices, we turn our attention to the local behavior of the solution near a vertex $Q_{i}$.

Let $\psi_{i} \in \mathcal{C}^{\infty}(\bar{\Omega}), i \geq 1$, be smooth functions with isolated supports, such that $\psi_{i}=1$ in the neighborhood of the $i$ th vertex of $\Omega$, and $0 \leq \phi_{i} \leq 1$ otherwise. Set $\psi_{0}:=1-\sum \psi_{i} \in \mathcal{C}^{\infty}(\bar{\Omega})$. We first consider the following equation on an infinite conical domain $C$ with interior angle $\omega_{i}$, such that one of its edges coincides with


Figure 2.1. The infinite conical domain $C$ with the vertex $O$ at the origin.
the $x$-axis for $x>0$ (Figure 2.1).

$$
\left\{\begin{align*}
-\Delta u=\psi_{i} f & \text { in } C  \tag{2.1}\\
u=0 & \text { on } \partial C
\end{align*}\right.
$$

Since the operator $\Delta$ is invariant under translations and rotations, (1.2) and (2.1) lead to the same equation and boundary conditions in the small neighborhood of the vertex $Q_{i}$ for $\Omega$ and in the small neighborhood of the vertex $O$ for $C$, respectively. Thus, we shall first study Equation (2.1) for the local behavior of the solution of (1.2) near the vertex $Q_{i}$. We note that the vertex $O$ of $\bar{C}$ raises similar difficulties on regularity of the solution as the vertex $Q_{i}$ of $\Omega$ in (1.2).

The resulting domain from the Mellin transform, however, will avoid the vertex. The study on operator pencils that are from the transform, turns out to be one of our key steps for the research on corner singularities.

Definition 2.1.1. The Mellin transform of a function on $C$ is defined as follows:

1) Change to polar coordinates $\left(x_{1}, x_{2}\right) \rightarrow(r, \theta)$, for $r=\sqrt{x_{1}^{2}+x_{2}^{2}}$.
2) Make the change of variables: $r=e^{t},(t=\ln r)$.
3) Apply the Fourier transform on the resulting function with respect to $t$.

We use the notation $u^{r}(r, \theta), u^{t}(t, \theta)$ to denote the corresponding functions of $u(x, y)$ after the change of variables and let $F_{i}=\psi_{i} f$. Then, the evolution of Equation (2.1) under the Mellin transform is shown below.

$$
\begin{aligned}
& -\left(\partial_{x_{1}}^{2}+\partial_{x_{2}}^{2}\right) u=F_{i} \rightarrow\left(\left(r \partial_{r}\right)^{2}+\partial_{\theta}^{2}\right) u^{r}=r^{2} F_{i}^{r} \\
& \rightarrow-\left(\partial_{t}^{2}+\partial_{\theta}^{2}\right) u^{t}=e^{2 t} F_{i}^{t} \rightarrow\left(\lambda^{2}-\partial_{\theta}^{2}\right) \hat{u}^{t}=\widehat{\left(e^{2 t} F_{i}^{t}\right)}
\end{aligned}
$$



Figure 2.2. The resulting domain $D$ from the Mellin transform on the $t-\theta$ plane.

Thus, the domain $C$ becomes the domain $D$ as in Figure 2.2. This transform gives another option to analyze the solution near the vertex on $\Omega$ in Equation (1.2) by estimating the corresponding solution on the domain $D$, after sending the original vertex to the infinity. This strategy has proved to be successful. Several basic theorems for Equations (2.1) are established in the next section.

### 2.2 A Priori Estimates

In this section, we present theorems regarding well-posedness, regularity, and the Fredholm property of the solution for the second-order elliptic equation on the infinite strip $D:(-\infty, \infty) \times\left[0, \omega_{i}\right]$ (Figure 2.2). These results, stated in terms of weighted Sobolev spaces (Definition 2.2.7), are our starting point to investigate various types of singular solutions for EBVPs.

### 2.2.1 The Infinite Domain

We consider the following equation, where $P$ is an elliptic operator from the Mellin transform on the $t-\theta$ plane, and $u=u(t, \theta)$.

$$
\left\{\begin{align*}
P u=f & \text { in } D  \tag{2.2}\\
u=0 & \text { on } \partial D .
\end{align*}\right.
$$

Then, we have an estimate on regularity of the solution $u$ of (2.2) on the infinite strip $D$.

Theorem 2.2.1. Suppose $P=a(\theta) \partial_{\theta}^{2}+b(\theta) \partial_{\theta} \partial_{t}+c(\theta) \partial_{t}^{2}+d(\theta) \partial_{\theta}+e(\theta) \partial_{t}+f(\theta)$, where the coefficients are in $\mathcal{C}^{m}([0, \alpha]), m \geq 0$. If there exists a weak solution $u \in H_{0}^{1}(D)$, for $f \in H^{m-1}(D)$, then

$$
\|u\|_{H^{m+1}(D)} \leq C\left(\|P u\|_{H^{m-1}(D)}+\|u\|_{L^{2}(D)}\right)
$$

where $C$ depends on the domain and the operator, not on $f$.
Proof. We first define partition of unity $\phi_{n}(t)$ satisfying

$$
\begin{array}{ll}
\sum_{n=-\infty}^{\infty} \phi_{n}(t)=1, \quad \phi_{n} \in \mathcal{C}_{c}^{\infty}(D) \\
\left|\partial_{t}^{i+1} \phi_{n}(t)\right|<C & \text { for } \quad i \leq m
\end{array}
$$

To be explicit, $\phi_{n}$ can be constructed in the following way for instance. Let $g_{n}(t)=$ $e^{-\frac{1}{1-(t-n)^{2}}}$ for $|t-n|<1$ and 0 otherwise. Then $g_{n}(t) \in \mathcal{C}_{c}^{\infty}(D)$. Define

$$
\phi_{n}(t)=\frac{g_{n}(t)}{\sum_{k=-\infty}^{\infty} g_{k}(t)}
$$

Thus, all $\phi_{n}$ form a partition of unity of $D$. In addition, $\sum_{k=-\infty}^{\infty} g_{k}(t) \geq C>0$ indicates the $(i+1)$-th derivative of $\phi_{n}$ is bounded above for $i \leq m$.

Denote by $S_{n}=\operatorname{supp}\left(\phi_{n}\right)$ the support of $\phi_{n}$ on $D$. Let $S_{n, \delta} \subset D$ be an open set with a smooth boundary, such that $S_{n} \subset S_{n, \delta}$ and $m\left(S_{n, \delta} \backslash S_{n}\right)<\delta$, for $\delta$ small, where $m(S)$ denotes the Lebesgue measure of the set $S$.

With the decomposition $u=\sum_{n=-\infty}^{\infty} u_{n}$, for $u_{n}=\phi_{n} u$, it can been seen that $u_{n}$ satisfies the following equation.

$$
\left\{\begin{aligned}
P u_{n} & =\phi_{n} P u+\phi_{n}^{\prime}\left(b u_{\theta}+2 c u_{t}+e u\right)+\phi_{n}^{\prime \prime} c u \text { in } S_{n, \delta} \\
u_{n} & =0 \text { on } \partial S_{n, \delta}
\end{aligned}\right.
$$

From the regularity theorem 1.3.2 for elliptic equations on domains with a smooth
boundary, we have the estimates below,

$$
\begin{aligned}
\left\|u_{n}\right\|_{H^{m+1}\left(S_{n, \delta}\right)} \leq & C\left(\left\|\phi_{n} P u+\phi_{n}^{\prime}\left(b u_{\theta}+2 c u_{t}+e u\right)+\phi_{n}^{\prime \prime} c u\right\|_{H^{m-1}\left(S_{n, \delta}\right)}\right. \\
& \left.+\left\|\phi_{n} u\right\|_{L^{2}\left(S_{n, \delta}\right)}\right) \\
\leq & C\left(\left\|\phi_{n} P u\right\|_{H^{m-1}\left(S_{n, \delta}\right)}+\left\|\phi_{n}^{\prime} b u_{\theta}\right\|_{H^{m-1}\left(S_{n, \delta}\right)}+\left\|\phi_{n}^{\prime} c u_{t}\right\|_{H^{m-1}\left(S_{n, \delta}\right)}\right. \\
& \left.+\left\|\phi_{n}^{\prime} e u\right\|_{H^{m-1}\left(S_{n, \delta}\right)}+\left\|\phi_{n}^{\prime \prime} c u\right\|_{H^{m-1}\left(S_{n, \delta}\right)}+\left\|\phi_{n} u\right\|_{L^{2}\left(S_{n, \delta}\right)}\right) \\
\leq & C\left(\|P u\|_{H^{m-1}\left(S_{n, \delta}\right)}+\|u\|_{H^{m}\left(S_{n, \delta}\right)}\right) .
\end{aligned}
$$

Thus,

$$
\sum_{n=-\infty}^{\infty}\left\|u_{n}\right\|_{H^{m+1}\left(S_{n, \delta}\right)}^{2} \leq C \sum_{n=-\infty}^{\infty}\left(\|P u\|_{H^{m-1}\left(S_{n, \delta}\right)}^{2}+\|u\|_{H^{m}\left(S_{n, \delta}\right)}^{2}\right)
$$

Note that the constant $C$ on the right hand side is different than the constant $C$ in the previous inequality. For small $\delta, S_{n, \delta}$ only intersects its adjacent neighbors, $S_{n+1, \delta}, S_{n+2, \delta}, S_{n-1, \delta}$, and $S_{n-2, \delta}$, which implies that in the summation above, no point in $D$ is counted more than three times. Therefore,

$$
\sum_{n=-\infty}^{\infty}\left\|u_{n}\right\|_{H^{m+1}\left(S_{n, \delta}\right)}^{2} \leq C\left(\|P u\|_{H^{m-1}(D)}^{2}+\|u\|_{H^{m}(D)}^{2}\right) .
$$

Since $u \in H_{0}^{1}(D)$ and $u=\sum_{n=-\infty}^{\infty} u_{n}$, we first have

$$
\left(\partial^{\alpha} u\right)^{2}=\left(\sum_{n=-\infty}^{\infty} \partial^{\alpha} u_{n}\right)^{2} \leq\left(\sum_{n=-\infty}^{\infty}\left|\partial^{\alpha} u_{n}\right|\right)^{2} \leq C \sum_{n=-\infty}^{\infty}\left|\partial^{\alpha} u_{n}\right|^{2}, \quad \forall \alpha=\left(\alpha_{1}, \alpha_{2}\right),|\alpha| \leq 2 .
$$

Moreover, since

$$
\left(\int_{D} \sum_{n=-\infty}^{\infty}\left|\partial^{\alpha} u_{n}\right|^{2}\right) \leq \sum_{n=-\infty}^{\infty}\left\|u_{n}\right\|_{H^{2}\left(S_{n, \delta}\right)}^{2} \leq C\left(\|u\|_{H^{1}(D)}^{2}+\|P u\|_{L^{2}(D)}^{2}\right),
$$

by the Lebesgue dominated convergence theorem, we obtain

$$
\|u\|_{H^{2}(D)}^{2} \leq C\left(\|P u\|_{L^{2}(D)}^{2}+\|u\|_{H^{1}(D)}^{2}\right) .
$$

In addition, from

$$
\int_{S_{n, \delta}} \nabla u \cdot \nabla u=-\int_{S_{n, \delta}} u_{n} \Delta u \leq C\|u\|_{L^{2}\left(S_{n, \delta}\right)}\|u\|_{H^{2}\left(S_{n, \delta}\right)},
$$

we note that $\|u\|_{H^{1}\left(S_{n, \delta}\right)}^{2} \leq C\left(\epsilon\|u\|_{H^{2}\left(S_{n, \delta}\right)}^{2}+\epsilon^{-1}\|u\|_{L^{2}\left(S_{n, \delta}\right)}^{2}\right)$. Thus,

$$
\sum_{n=-\infty}^{\infty}\|u\|_{H^{2}\left(S_{n, \delta}\right)}^{2} \leq C \sum_{n=-\infty}^{\infty}\left(\|P u\|_{L^{2}\left(S_{n, \delta}\right)}^{2}+\|u\|_{L^{2}\left(S_{n, \delta}\right)}^{2}\right)
$$

Combining these arguments, we can improve the above estimate by

$$
\|u\|_{H^{2}(D)}^{2} \leq C\left(\|P u\|_{L^{2}(D)}^{2}+\|u\|_{L^{2}(D)}^{2}\right)
$$

In a similar manner, we prove $u \in H^{i+1}(D)$, by accumulating the previous inequality for $i \leq m$. Therefore, we have the expected result for this theorem,

$$
\|u\|_{H^{m+1}(D)} \leq C\|P u\|_{H^{m-1}(D)}+\|u\|_{L^{2}(D)} .
$$

Remark 2.2.2. Note that if the solution $u \in H_{0}^{1}(D)$ is unique, we can ignore the $L^{2}$ norm in the estimate, such that

$$
\|u\|_{H^{m+1}(D)} \leq C\|P u\|_{H^{m-1}(D)}
$$

To avoid confusion with notation, we here fix the elliptic operator $P=-\Delta_{t}:=$ $-\partial_{t}^{2}-\partial_{\theta}^{2}$ for Equation (2.2) to illustrate well-posedness of the solution. Wellposedenss of other equation associated with strictly positive elliptic operators can be proved in the same way.

Theorem 2.2.3. $\Delta_{t}:=\partial_{t}^{2}+\partial_{\theta}^{2}: H^{m+1}(D) \cap\left\{\left.u\right|_{\theta=0, \omega_{i}}=0\right\} \rightarrow H^{m-1}(D), m \geq 0$, is an isomorphism on $D:(-\infty, \infty) \times\left[0, \omega_{i}\right]$.

Proof. We first show uniqueness of $u \in H_{0}^{1}(D)$ for $f \in H^{-1}(D)$. The weak form associated with $\Delta_{t}$ on $D$ is

$$
B[u, v]=\int_{D} \nabla u \cdot \nabla v=\int_{D} f v
$$

for any $u, v \in H_{0}^{1}(D)$.
Note that $|B[u, v]| \leq\|u\|_{H^{1}(D)}\|v\|_{H^{1}(D)}$ implies the continuity of the bilinear form. On the other hand, by the one-dimensional poincaré inequality with respect to $\theta, \int_{0}^{\omega_{i}} u_{\theta}^{2} \geq C \int_{0}^{\omega_{i}} u^{2}$, we have the coercivity $B[u, u]=\int_{D}\left(u_{t}^{2}+u_{\theta}^{2}\right) \geq C\|u\|_{H^{1}(D)}^{2}$ as well. Then, Based on Lax-Milgram Theorem, there exists a unique weak solution $u \in H_{0}^{1}(D)$ for every $f \in H^{-1}(D)$. Therefore, $\Delta_{t}$ is the case for $m=0$.

When $m \geq 1$, existence of the unique weak solution $u \in H_{0}^{1}(D)$ follows the argument above, since $H^{m-1}(D) \subset H^{-1}(D)$. Then, we prove $u \in H^{m+1}(D) \cap$ $H_{0}^{1}(D)$ by applying Theorem 2.2.1 to $\Delta_{t}$.

Therefore, $\Delta_{t}: H^{m+1}(D) \cap\left\{\left.u\right|_{\theta=0, \alpha=0}=0\right\} \rightarrow H^{m-1}(D), m \geq 0$, defines an isomorphism with

$$
\|u\|_{H^{m+1}(D)} \leq C\left\|\Delta_{t} u\right\|_{H^{m-1}(D)}
$$

We have established well-posedness and regularity properties for elliptic operators on the infinite domain $D$ with the homogeneous Dirichlet boundary condition. In addition, it will be seen that the computation of indices for the Fredholm property of weighted Sobolev spaces in the next chapter also requires understanding on the following Sobolev space with weights on $D$,

$$
\begin{equation*}
H^{m, a}(D)=\left\{u, r^{-a} \partial_{t}^{i} \partial_{\theta}^{j} u \in L^{2}(D), i+j \leq m\right\} \quad \text { for } \quad r=e^{t} \tag{2.3}
\end{equation*}
$$

Note that the dual space of $H_{0}^{1, a}(D)=H^{1, a}(D) \cap\left\{\left.u\right|_{\partial D}=0\right\}$ is $H^{-1,-a}(D)$, with respect to the pair $\langle u, v\rangle=\int_{D} u v$, for $u \in H_{0}^{1, a}(D)$ and $v \in H^{-1,-a}(D)$. Thus, we have

Lemma 2.2.4. Multiplication by $r^{a}$ gives rise to an isomorphism $H^{m, b}(D) \rightarrow$ $H^{m, a+b}(D)$, namely, $r^{a} H^{m, b}(D)=H^{m, a+b}(D)$.

Proof. Let $u=r^{a} w$ with $w \in H^{m, b}$. Then

$$
r^{-b} \partial_{t}^{i} \partial_{\theta}^{j} w \in L^{2}(D), \quad \text { for } \quad i+j \leq m
$$

Moreover,

$$
\left|r^{-a-b} \partial_{t}^{i} \partial_{\theta}^{j} u\right|=\left|r^{-a-b} \partial_{t}^{i} \partial_{\theta}^{j}\left(r^{a} w\right)\right| \leq C \sum_{k=0}^{i} r^{-b}\left|\partial_{t}^{k} \partial_{\theta}^{j} w\right| \in L^{2}(D), \quad i+j \leq m
$$

Thus, $r^{a} H^{m, b}(D)$ is continuously embedded in $H^{m, a+b}$. Since the embedding holds for any real number $a$, we complete the proof by concluding the opposite embedding

$$
H^{m, a+b}(D)=r^{a} r^{-a} H^{m, a+b}(D) \subset r^{a} H^{m, b}(D)
$$

Recall that the Fourier transform plays an important role in the Mellin (Kondratiev) transform. We here give alternative definitions of the spaces $H_{0}^{1}(D)$ and $H^{-1}(D)$ by taking the Fourier transform into account.

Lemma 2.2.5. Let $E=(-\infty, \infty) \times\left[-\omega_{i}, \omega_{i}\right]$. Define the periodic function $U$ with the period $2 \omega_{i}$ in $\theta$ on $E$, such that $U(t, \theta)=u(t, \theta)$ for $\theta \geq 0$, and $U(t, \theta)=$ $-u(t,-\theta)$ for $\theta<0$. Then,

$$
\begin{aligned}
H_{0}^{1}(D) & =\left\{\left.u\right|_{\partial D}=0, \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty}\left(1+\lambda^{2}+n^{2}\right)|\hat{U}|^{2}<\infty\right\} \\
L^{2}(D) & =\left\{u, \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty}|\hat{U}|^{2}<\infty\right\} \\
H^{-1}(D) & =\left\{u, \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty}\left(1+\lambda^{2}+n^{2}\right)^{-1}|\hat{U}|^{2}<\infty\right\}
\end{aligned}
$$

where $\hat{U}(\lambda, n)=\frac{1}{2 \sqrt{\pi \omega_{i}}} \int_{-\infty}^{\infty} \int_{-\omega_{i}}^{\omega_{i}} e^{-i \lambda t} e^{-i \frac{n \pi}{\omega_{i}}} \theta(t, \theta)$.
Proof. For $H_{0}^{1}(D)$, it is clear that $U$ is an odd function for $\theta$ on $E$ generated by $u$. Thus, $u \in H_{0}^{1}(D)$ indicates $U \in H_{0}^{1}(E)$. Then, based on the periodicity and the behavior at infinity of $U(t, \theta)$, we have

$$
\partial_{t}^{a} \widehat{\partial_{\theta}^{b} U(t, \theta)}=(i \lambda)^{a}\left(i \frac{n \pi}{\omega_{i}}\right)^{b} \hat{U}(\lambda, n), \quad a+b \leq 1 .
$$

Moreover, by Parseval's Theorem

$$
\begin{aligned}
\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} n^{2 b} \lambda^{2 a}|\hat{U}|^{2} & =\left(\frac{\omega_{i}}{\pi}\right)^{2 b}\left\|(i \lambda)^{a}\left(i \frac{n \pi}{\omega_{i}}\right)^{b} \hat{U}\right\|_{L^{2}}^{2} \\
& \leq C \int_{E}\left|\partial_{t}^{a} \partial_{\theta}^{b} U\right|^{2}<\infty
\end{aligned}
$$

Thus

$$
\sum_{n=-\infty}^{\infty} \int\left(1+n^{2}+\lambda^{2}\right)|\hat{U}|^{2} \leq C\|U\|_{H^{1}(E)}^{2}<\infty
$$

Suppose conversely $\sum_{n=-\infty}^{\infty} \int\left(1+n^{2}+\lambda^{2}\right)|\hat{U}|^{2}<\infty$. Then

$$
\begin{aligned}
\left\|(i \lambda)^{a}\left(i \frac{n \pi}{\alpha}\right)^{b} \hat{U}\right\|_{L^{2}(E)}^{2} & =\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty}\left(\frac{\pi}{\alpha}\right)^{2 b} n^{2 b} \lambda^{2 a}|\hat{U}|^{2} \\
& \leq C \sum_{n=-\infty}^{\infty} \int\left(1+n^{2}+\lambda^{2}\right)|\hat{U}|^{2}<\infty
\end{aligned}
$$

Hence $U \in H_{0}^{1}(E)$. Consequently,

$$
H_{0}^{1}(D)=\left\{\left.u\right|_{\partial D}=0, \quad \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty}\left(1+\lambda^{2}+n^{2}\right)|\hat{U}|^{2}<\infty\right\}
$$

For the space $L^{2}(D),\|U\|_{L^{2}(E)}=\|\hat{U}\|_{L^{2}}=\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty}|\hat{U}|^{2}$. Therefore $u \in L^{2}(D)$ is equivalent to $\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty}|\hat{U}|^{2}<\infty$ by the definition of $U$.

Therefore, from the argument above, we can define

$$
\begin{aligned}
\|u\|_{L^{2}(D)}^{2} & =\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty}|\hat{U}|^{2} \\
\|u\|_{H^{1}(D)}^{2} & =\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty}\left(1+\lambda^{2}+n^{2}\right)|\hat{U}|^{2}
\end{aligned}
$$

For $H^{-1}(D)$, we consider $v \in L^{2}(D)$ first. Hence $\hat{V} \in L^{2}$ and

$$
(v, u)=\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{V} \overline{\hat{U}}, \quad \forall u \in H_{0}^{1}(D)
$$

By duality, the $H^{-1}$-norm of $v$ is defined by

$$
\|v\|_{H^{-1}(D)}=\sup _{u \in H_{0}^{1}(D), u \neq 0} \frac{|(v, u)|}{\|u\|_{H^{1}(D)}} \leq\left\|\left(1+\lambda^{2}+n^{2}\right)^{-\frac{1}{2}} \hat{V}\right\|_{L^{2}} .
$$

If we replace $\hat{U}$ by $\left(1+n^{2}+\lambda^{2}\right)^{-1} \hat{V}$, since $v \in L^{2}(D)$, we also have $\Sigma_{n=-\infty}^{\infty} \int_{-\infty}^{\infty}(1+$ $\left.\lambda^{2}+n^{2}\right)|\hat{U}|^{2}<\infty$, which indicates $u \in H^{1}(D)$. Then, from the definition of the $H^{-1}$-norm,

$$
\|v\|_{H^{-1}(D)} \geq\left\|\left(1+\lambda^{2}+n^{2}\right)^{-\frac{1}{2}} \hat{V}\right\|_{L^{2}}
$$

Hence,

$$
\|v\|_{H^{-1}(D)}^{2}=\left\|\left(1+\lambda^{2}+n^{2}\right)^{-\frac{1}{2}} \hat{V}\right\|_{L^{2}}^{2}=\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty}\left(1+\lambda^{2}+n^{2}\right)^{-1}|\hat{V}|^{2}
$$

Since $H^{-1}(D)$ is the closure of $L^{2}(D)$ in $H^{-1}$-norm, we have

$$
\|v\|_{H^{-1}(D)}^{2}=\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty}\left(1+\lambda^{2}+n^{2}\right)^{-1}|\hat{V}|^{2} \quad \text { for } \quad v \in H^{-1}(D)
$$

Recall the weighted Sobolev space $H^{m, a}(D)$ in Equation (2.3). Besides wellposedness of the solution of Equation (2.2) with $P=-\Delta_{t}$, it is also important to study the solution of this equation on spaces $H^{m, a}(D)$, The index $a$ in the weighted space in fact determines the Fredholm property of the elliptic operator on a polygonal domain, which will be illustrated in Theorem 2.2.12.

Theorem 2.2.6. $\Delta_{t}: H^{m+1, a}(D) \cap\left\{\left.u\right|_{\theta=0, \omega_{i}}=0\right\} \rightarrow H^{m-1, a}(D), m \geq 0$, defines an isomorphism iff $a \neq \frac{k \pi}{\omega_{i}}$.

Proof. From Lemma 2.2.4, multiplication by $r^{a}$ leads to an isomorphism between spaces, $H^{m, a}(D)=r^{a} H^{m}(D)$. Therefore, instead of a direct study on the space $H^{m, a}(D)$, we alternatively prove the following equivalent statement.

$$
r^{-a} \Delta_{t} r^{a}: H^{m+1}(D) \cap\left\{\left.u\right|_{\theta=0, \omega_{i}}=0\right\} \rightarrow H^{m-1}(D), \quad m \geq 0
$$

is an isomorphism.
We first prove for $m=0$. For $f \in H^{-1}(D)$, the equation reads

$$
\left\{\begin{array}{l}
r^{-a} \Delta_{t} r^{a} u=\partial_{t}^{2} u+\partial_{\theta}^{2} u-2 a \partial_{t} u+a^{2} u=f \quad \text { in } \quad D \\
u(t, 0)=u\left(t, \omega_{i}\right)=0
\end{array}\right.
$$

Recall notation for the Fourier transform in Lemma 2.2.5. Then, the periodic function $U$ satisfies

$$
\left\{\begin{array}{l}
\partial_{t}^{2} U+\partial_{\theta}^{2} U-2 a \partial_{t} U+a^{2} U=F, \quad \text { in } \quad E \backslash\{\theta=0\} \\
U\left(t,-\omega_{i}\right)=U\left(t, \omega_{i}\right)=U(t, 0)=0
\end{array}\right.
$$

where $F(t, \theta)=f(t, \theta)$ for $\theta>0$ and $F(t, \theta)=\partial_{t}^{2}(-u(t,-\theta))+\partial_{\theta}^{2}(-u(t,-\theta))-$ $2 a \partial_{t}(-u(t,-\theta))+a^{2}(-u(t,-\theta))=-f(t,-\theta)$ for negative $\theta$. Apply the double Fourier transform on $t$ and $\theta$, the above equation becomes

$$
-\lambda^{2} \hat{U}-\frac{\pi^{2}}{\omega_{i}^{2}} n^{2} \hat{U}-2 i a \lambda \hat{U}+a^{2} \hat{U}=\hat{F}
$$

from which $\hat{U}$ can be expressed by

$$
\hat{U}=\frac{\hat{F}}{a^{2}-\lambda^{2}-\frac{\pi^{2}}{\omega_{i}^{2}} n^{2}-2 i a \lambda}
$$

Then,

$$
\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty}\left(1+n^{2}+\lambda^{2}\right)|\hat{U}|^{2}=\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty}\left(1+n^{2}+\lambda^{2}\right) \frac{|\hat{F}|^{2}}{\left(a^{2}-\lambda^{2}-\frac{\pi^{2}}{\omega_{i}^{2}} n^{2}\right)^{2}+4 a^{2} \lambda^{2}}
$$

Since $-U(t, \theta)=U(t,-\theta)$, we have

$$
-\hat{U}(\lambda, n)=\hat{U}(\lambda,-n)
$$

by the definition of the Fourier transform. Therefore, $\hat{U}(\lambda, 0)=0$. Based on

Lemma 2.2.5,

$$
\begin{aligned}
\|u\|_{H^{1}(D)}^{2} & =\sum_{n \neq 0} \int_{-\infty}^{\infty}\left(1+n^{2}+\lambda^{2}\right)|\hat{U}|^{2} \\
& =\sum_{n \neq 0} \int_{-\infty}^{\infty}\left(1+n^{2}+\lambda^{2}\right) \frac{|\hat{F}|^{2}}{\left(a^{2}-\lambda^{2}-\frac{\pi^{2}}{\omega_{i}^{2}} n^{2}\right)^{2}+4 a^{2} \lambda^{2}}
\end{aligned}
$$

Moreover, recall that $f \in H^{-1}(D)$ indicates $\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty}\left(1+n^{2}+\lambda^{2}\right)^{-1}|\hat{F}|^{2}<\infty$ (Lemma 2.2.5). Then, to make sure that $u$ is in $H_{0}^{1}(D)$, it suffices to show that there exists a constant $C$, such that

$$
\frac{\left(1+n^{2}+\lambda^{2}\right)^{2}}{\left(a^{2}-\lambda^{2}-\frac{\pi^{2}}{\omega_{i}^{2}} n^{2}\right)^{2}+4 a^{2} \lambda^{2}} \leq C
$$

For small $n$ and $\lambda,\left(a^{2}-\lambda^{2}-\frac{\pi^{2}}{\omega_{i}^{2}} n^{2}\right)^{2}+4 a^{2} \lambda^{2}$ is away from 0 , since $a \neq \frac{k \pi}{\omega_{i}}$ and $n \neq$ 0 . Therefore, there exists a constant $C>0$, such that $\left(a^{2}-\lambda^{2}-\frac{\pi^{2}}{\omega_{i}^{2}} n^{2}\right)^{2}+4 a^{2} \lambda^{2} \geq C$, since it is a continuous function in this case.

For large $n$ and $\lambda, \frac{\left(1+n^{2}+\lambda^{2}\right)^{2}}{\left(a^{2}-\lambda^{2}-\frac{\pi^{2}}{\omega_{i}^{2}} n^{2}\right)^{2}+4 a^{2} \lambda^{2}}$ is the ratio of two polynomials of the same degree for both $n$ and $\lambda$. To be more precise, $\frac{\left(1+n^{2}+\lambda^{2}\right)^{2}}{\left.\left(a^{2}-\lambda^{2}-\frac{\pi^{2}}{\omega_{i}^{2}}\right)^{2}\right)^{2}+4 a^{2} \lambda^{2}} \leq C\left(1+\frac{\omega_{i}^{4}}{\pi^{4}}\right)$ as $n, \lambda \rightarrow \pm \infty$.

Therefore, $\frac{\left(1+n^{2}+\lambda^{2}\right)^{2}}{\left(a^{2}-\lambda^{2}-\frac{\pi^{2}}{\omega_{i}^{2}} n^{2}\right)^{2}+4 a^{2} \lambda^{2}}$ has an upper bound for $n \neq 0, a \neq \frac{k \pi}{\omega_{i}}$, hence, $u \in H_{0}^{1}(D)$ for $f \in H^{-1}(D)$.

For $m \geq 1$, the operator $\partial_{t}^{2}+\partial_{\theta}^{2}-2 a \partial_{t}+a^{2}$ only contains constant coefficients. By Theorem 2.2.1, the $H_{0}^{1}$ weak solution $u$ associated with this operator in fact belongs to $H^{m+1}(D) \cap H_{0}^{1}(D)$ for $f \in H^{m-1}(D)$, which completes the proof.

### 2.2.2 The Weighted Sobolev Space

Based on the study of Equation (2.2) on the infinite strip $D$, we have described basic properties of the solution. Recall that the domain $D$ comes from the Mellin transform of the infinite conical domain with interior angle $\omega_{i}$ (Figure 2.2). Hence, they are merely the local well-posedness and regularity properties of the solution for Equation (1.2) on the polygonal domain $\Omega$, near the a vertex. Combination of
these local results into a uniform global description of the solution on $\Omega$ is nontrivial work. We first give an intuitive definition of the weighted Sobolev space on $\Omega$, which presents the counterpart of the space $H^{m, a}$ on $D$ near every corner. More discussions and some variations of this space will be shown in the next chapter.

Definition 2.2.7. Denote by $Q_{i}, i \geq 1$, the $i$ th vertex of $\Omega$. Let $r_{i}$ be the distance function to $Q_{i}$. Define the function $\vartheta$ on $\bar{\Omega}$, such that $\vartheta=\prod_{i} r_{i}$. Then, the weighted Sobolev space for Equation (1.2) is

$$
\mathcal{K}_{a}^{m}(\Omega)=\left\{u, \vartheta^{|\alpha|-a} \partial^{\alpha} u \in L^{2}(\Omega),|\alpha| \leq m\right\}
$$

where $\alpha=\left(\alpha_{1}, \alpha_{2}\right), \forall \alpha_{1}, \alpha_{2} \in \mathbb{Z}_{+} \cup\{0\}$, is the multi-index.
Remark 2.2.8. Let $\mathcal{K}_{-a}^{-1}(\Omega):=\left(\mathcal{K}_{a}^{1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}\right)^{\prime}$ with $L^{2}(\Omega)$ as the pivot space. It is easy to see that the space $\mathcal{K}_{a}^{m}(\Omega)$ is equivalent to $H^{m}(\Omega)$ away from the vertices. Note that near the vertex, the space $\mathcal{K}_{1+a}^{m}(\Omega)$ is derived from $H^{m, a}(D)$ of (2.3) based on the Mellin (Kondratiev) transform. Therefore, our estimates in 2.2.1, 2.2.3, and 2.2.6 have their variants in terms of $\mathcal{K}_{a}^{m}$ near vertices, respectively. Furthermore, we define the following sets of functions,

$$
\begin{aligned}
\mathcal{C}_{H, a}^{\infty}(D) & :=\left\{v \in \mathcal{C}_{0}^{\infty}(D), r^{-a} \partial_{t}^{i} \partial_{\theta}^{j} v \in L^{2}(D), \forall i, j \in \mathbb{Z}_{+} \cup\{0\}\right\} \\
\mathcal{C}_{\mathcal{K}, a}^{\infty}(\Omega) & :=\left\{v \in \mathcal{C}_{0}^{\infty}(\Omega), \vartheta^{|\alpha|-1-a} \partial^{\alpha} v \in L^{2}(\Omega), \forall \alpha_{1}, \alpha_{2} \in \mathbb{Z}_{+} \cup\{0\}\right\} .
\end{aligned}
$$

Thus, $\mathcal{C}_{H, a}^{\infty}(D)$ is a dense set of the space $H_{a}^{m}(D)$ with respect to the $H_{a}^{m}$-norm, based on the definition in (2.3). Correspondingly, we conclude that with the $\mathcal{K}_{1+a^{-}}^{m}$ norm, all functions in $\mathcal{C}_{\mathcal{K}, \Omega}^{\infty}(\Omega)$ form a dense set of $\mathcal{K}_{1+a}^{m}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}$.

With these properties, we have the following lemmas.
Lemma 2.2.9. Define $(\Delta u, v):=-(\nabla u, \nabla v)$ for $u \in \mathcal{K}_{1+a}^{1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}$, $v \in$ $\mathcal{K}_{1-a}^{1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}$. Then, the operator $\Delta: \mathcal{K}_{1+a}^{m+1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\} \rightarrow \mathcal{K}_{-1+a}^{m-1}(\Omega)$, $m \geq 0$, defines a bounded linear map.

Proof. The proof follows from a direct calculation.

Lemma 2.2.10. Define $\Delta_{a}:=\Delta: \mathcal{K}_{1+a}^{1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\} \rightarrow \mathcal{K}_{-1+a}^{-1}(\Omega)$. Let $\Delta^{*}: \mathcal{K}_{1-a}^{1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\} \rightarrow \mathcal{K}_{-1-a}^{-1}(\Omega)$ be its conjugate operator defined by

$$
(\Delta u, v)=\left(u, \Delta^{*} v\right)
$$

where $(\cdot, \cdot)$ is the $L^{2}$ inner product for $L^{2}$ functions, $\forall u \in \mathcal{K}_{1+a}^{1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}$, $\forall v \in \mathcal{K}_{1-a}^{1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}$. Then, $\Delta^{*}=\Delta_{-a}$.

Proof. For any $u^{c} \in \mathcal{C}_{\mathcal{K}, a}$ and $v^{c} \in \mathcal{C}_{\mathcal{K},-a}$, from integration by parts we have

$$
\left(\Delta u^{c}, v^{c}\right)=-\left(\nabla u^{c}, \nabla v^{c}\right)=\left(u^{c}, \Delta v^{c}\right) .
$$

Thus, it suffices to show $\forall u \in \mathcal{K}_{1+a}^{1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}, \forall v \in \mathcal{K}_{1-a}^{1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}$,

$$
(\Delta u, v)=(u, \Delta v)
$$

Note that
$(\Delta u, v)-\left(\Delta u^{c}, v^{c}\right)=\left(\Delta u, v-v^{c}\right)+\left(\Delta\left(u-u^{c}\right), v^{c}\right)=\left(\Delta u, v-v^{c}\right)+\left(u-u^{c}, \Delta^{*} v^{c}\right)$.
Since $\mathcal{C}_{\mathcal{K}, a}(\Omega)$ and $\mathcal{C}_{\mathcal{K},-a}(\Omega)$ are dense sets of $\mathcal{K}_{1+a}^{1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}$ and $\mathcal{K}_{1-a}^{1}(\Omega) \cap$ $\left\{\left.u\right|_{\partial \Omega}=0\right\}$, we can choose $u^{c}, v^{c}$, such that $\left\|u-u^{c}\right\|_{\mathcal{K}_{1+a}^{1}(\Omega)},\left\|v-v^{c}\right\|_{\mathcal{K}_{1-a}^{1}(\Omega)}<\epsilon$, for $\epsilon$ small. Note $\Delta u \in \mathcal{K}_{-1+a}^{-1}(\Omega)=\left(\mathcal{K}_{1-a}^{1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}\right)^{\prime}$ from Lemma 2.2.9. Then, $\left(\Delta u, v-v^{c}\right)$ can be arbitrarily small. For the same reason, $\left(u-u^{c}, \Delta^{*} v^{c}\right) \rightarrow 0$ also. Therefore,

$$
(\Delta u, v)-\left(\Delta u^{c}, v^{c}\right) \rightarrow 0 .
$$

The same arguments are available for $(u, \Delta v)-\left(u^{c}, \Delta v^{c}\right)$. Thus, there are sequences $u_{i}^{c} \in \mathcal{C}_{\mathcal{K}, a}(\Omega), v_{i}^{c} \in \mathcal{C}_{\mathcal{K},-a}(\Omega)$, such that

$$
(\Delta u, v)-(u, \Delta v)=(\Delta u, v)-\left(\Delta u_{i}^{c}, v_{i}^{c}\right)+\left(u_{i}^{c}, \Delta v_{i}^{c}\right)-(u, \Delta v) \rightarrow 0 .
$$

Then, we already showed

$$
(\Delta u, v)=(u, \Delta v)=\left(u, \Delta^{*} v\right)
$$

which implies $\Delta^{*}=\Delta_{-a}$.
Recall a bounded operator $A: X \rightarrow Y$ between two Banach spaces is Fredholm if the kernel of $A$ and the co-kernel $Y / A X$ are finite dimensional spaces, and $A$ has a closed range.

Lemma 2.2.11. Let $\Delta_{a}:=\Delta: \mathcal{K}_{1+a}^{1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\} \rightarrow \mathcal{K}_{-1+a}^{-1}(\Omega)$ and $X=$ $\operatorname{ker}\left(\Delta_{a}\right)$ be the kernel of $\Delta_{a}$. Then, $X \subset \mathcal{K}_{a+1}^{\infty}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}$.

Proof. For any function $u \in X, \Delta u=0 \in \mathcal{K}_{a-1}^{\infty}(\Omega)$. Then this lemma follows from the regularity estimates in Theorem 2.2.1.

The following theorem discusses the Fredholm property of the operator $P=-\Delta$ in Equation (1.2). Similar results that apply to general elliptic operators and other boundary conditions are available in Chapter 3.

Theorem 2.2.12. Suppose $\Omega$ has vertices $Q_{i}$ with the interior angles $\omega_{i}, i \geq 1$. Then, the operator $-\Delta_{a}:=-\Delta: \mathcal{K}_{a+1}^{m+1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\} \rightarrow \mathcal{K}_{a-1}^{m-1}(\Omega), m \geq 0$, is Fredholm iff $a \neq k \pi / \omega_{i}, k \neq 0$.

Proof. The proof is based on the partition of unity and the construction of operators $R_{i}$ on different components of the domain, then pasting all parts together.

Let $B(x, r)$ be the ball centered at $x$ with radius $r$. Define partition of unity $\phi_{0}, \phi_{i} \in \mathcal{C}^{\infty}(\bar{\Omega}), \phi_{0}+\sum_{i} \phi_{i}=1$ on $\bar{\Omega}, i \geq 1$, such that for $\epsilon_{2}>\epsilon_{1}>0, \phi_{i}=1$ in $B\left(Q_{i}, \epsilon_{1}\right) \cap \bar{\Omega}$, and $\phi_{i}=0$ outside of $B\left(Q_{i}, \epsilon_{2}\right)$ and of $\bar{\Omega}$. Therefore, $\phi_{0}=0$ near the vertices.

Denote by $C_{i}$ the infinite conical domain with angle $\omega_{i}$, By Remark 2.2.2, Theorem 2.2.3, and Theorem 2.2.6, the equation

$$
\left\{\begin{aligned}
-\Delta u_{i}=\phi_{i} f & \text { in } C_{i} \\
u_{i}=0 & \text { on } \partial C_{i} .
\end{aligned}\right.
$$

has a unique solution $u_{i} \in \mathcal{K}_{a+1}^{m+1}\left(C_{i}\right) \cap\left\{\left.u_{i}\right|_{\partial C_{i}}=0\right\}$ for $f \in \mathcal{K}_{a-1}^{m-1}\left(C_{i}\right)$ in each $C_{i}$, $i \geq 1, m \geq 0$, provided that $a$ is different from $k \pi / \omega_{i}, k \neq 0$.

In addition, there exists a unique solution $u_{0} \in H^{1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}$ for

$$
\left\{\begin{aligned}
-\Delta u_{0}=\phi_{0} f & \text { in } \Omega \\
u_{0}=0 & \text { on } \partial \Omega
\end{aligned}\right.
$$

if $f \in \mathcal{K}_{a-1}^{m-1}(\Omega)$, since $\mathcal{K}_{a-1}^{m-1}$ and $H^{m-1}$ are equivalent on the support of $\phi_{0} f$. Moreover, $u_{0}$ has higher interior regularity given $\phi_{0} f$ is smoother.

Thus, we paste solutions of different parts by constructing a regularizer $R_{a}$ : $\mathcal{K}_{a-1}^{m-1}(\Omega) \rightarrow \mathcal{K}_{a+1}^{m+1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}$ as follows. Let $R_{i}: \mathcal{K}_{a-1}^{m-1}\left(C_{i}\right) \rightarrow \mathcal{K}_{a+1}^{m+1}\left(C_{i}\right) \cap$ $\left\{\left.u\right|_{\partial C_{i}}=0\right\}, i \geq 1$, and $R_{0}: \mathcal{K}_{a-1}^{m-1}(\Omega) \rightarrow H^{1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}$ be linear operators, such that

$$
\begin{aligned}
& R_{i}\left(\phi_{i} f\right)=u_{i} \quad \text { in } \quad C_{i} \\
& R_{0}\left(\phi_{0} f\right)=u_{0} \quad \text { in } \quad \Omega
\end{aligned}
$$

These operators are continuous based on the regularity theorems 2.2.1, 2.2.6, and well defined by the corresponding differential operators and boundary conditions when $a \neq k \pi / \omega_{i}, k \neq 0$.

Define $\chi_{i} \in \mathcal{C}^{\infty}(\bar{\Omega}), i \geq 0$, such that $\chi_{i}=1$ on $\operatorname{supp}\left(\phi_{i}\right)$, and $\chi_{i}=0$ outside an $\epsilon$-neighborhood of $\operatorname{supp}\left(\phi_{i}\right)$ on $\bar{\Omega}$ that dose not touch any of the vertices. Note that $\chi_{0} u_{0}$ has support away from the corners. Therefore, $\chi_{0} u_{0} \in \mathcal{K}_{a+1}^{m+1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}$ when $f \in \mathcal{K}_{a-1}^{m-1}(\Omega)$. Then, for $i \geq 0$, the regularizer $R_{a}$ is defined by

$$
\begin{equation*}
u=R_{a}(f)=\sum_{i} \chi_{i} R_{i}\left(\phi_{i} f\right)=\sum_{i} \chi_{i} u_{i} \tag{2.4}
\end{equation*}
$$

such that the local behavior of $u_{i}$ near vertices is embedded in $u$, and $u$ satisfies the homogeneous Dirichlet boundary condition. Thus, $R_{a}: \mathcal{K}_{a-1}^{m-1}(\Omega) \rightarrow \mathcal{K}_{a+1}^{m+1}(\Omega) \cap$ $\left\{\left.u\right|_{\partial \Omega}=0\right\}$ definies a bounded linear operator by regularity results and by the construction. Recall $\Delta_{a}:=\Delta: \mathcal{K}_{1+a}^{1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\} \rightarrow \mathcal{K}_{-1+a}^{-1}(\Omega)$. We first show that $-\Delta_{a} R_{a}: \mathcal{K}_{a-1}^{m-1}(\Omega) \rightarrow \mathcal{K}_{a-1}^{m-1}(\Omega)$ is Fredholm.

From (2.4),

$$
-\Delta_{a} u=-\Delta_{a} R_{a}(f)=-\left(\sum_{i} \chi_{i} \Delta u_{i}+2 \sum_{i} \nabla \chi_{i} \cdot \nabla u_{i}+\sum_{i} \Delta \chi_{i} u_{i}\right)
$$

$$
=f-\left(2 \sum_{i} \nabla \chi_{i} \cdot \nabla R_{i}\left(\phi_{i} f\right)+\sum_{i} \Delta \chi_{i} R_{i}\left(\phi_{i} f\right)\right) .
$$

Define

$$
T_{a} f=2 \sum_{i} \nabla \chi_{i} \cdot \nabla R_{i}\left(\phi_{i} f\right)+\sum_{i} \Delta \chi_{i} R_{i}\left(\phi_{i} f\right) .
$$

We show the operator $T_{a}: \mathcal{K}_{a-1}^{m-1}(\Omega) \rightarrow \mathcal{K}_{a-1}^{m-1}(\Omega)$ is compact in $\mathcal{K}_{a-1}^{m-1}(\Omega)$ as follows. Let $S$ be the support of $T_{a} f$. Note $S$ excludes vertices of the domain. Hence $\mathcal{K}_{a+1}^{m}$ is equivalent to $H^{m}$ in this region, and $T_{a} f \in H^{m}(\Omega)$, vanishes outside $S$. Then, by the regularity estimates, we have $\left\|T_{a} f\right\|_{H^{m}(\Omega)} \leq C\left\|T_{a} f\right\|_{\mathcal{K}_{a+1}^{m}(\Omega)} \leq C\|u\|_{\mathcal{K}_{a+1}^{m+1}(\Omega)} \leq$ $C\left||f|_{\mathcal{K}_{a-1}^{m-1}(\Omega)}\right.$. By the Rellich-Kondrachov Theorem, for any bounded sequence $f_{i} \in$ $\mathcal{K}_{a-1}^{m-1}(\Omega), T_{a} f_{i} \in H^{m}(\Omega)$ is also a bounded sequence, and hence has a convergent sub-squence in $H^{m-1}(\Omega)\left(\mathcal{K}_{a-1}^{m-1}(\Omega)\right), m \geq 1$, since for $T_{a} f_{i}, H^{m-1}(\Omega)$ is equivalent to $\mathcal{K}_{a-1}^{m-1}(\Omega)$.

For $m=0, T_{a} f_{i}$ is bounded in the Hilbert space $L^{2}(\Omega)$. Therefore, it has a sub-sequence $\left\{T_{a} f_{k}\right\}$ that converges weakly in the sense

$$
\left(T_{a} f_{k}-T_{a} f_{k+1}, v\right)=\int_{\Omega}\left(T_{a} f_{k}-T_{a} f_{k+1}\right) v \rightarrow 0, \quad \forall v \in L^{2}(\Omega)
$$

Note $T_{a} f_{k}=0$ near corners. We can choose any $v \in \mathcal{K}_{a+1}^{0}(\Omega)$ in the equation above. This immediately shows $\left\{T_{a} f_{k}\right\}$ converges in the $\mathcal{K}_{a-1}^{-1}(\Omega)$ norm defined by

$$
\|f\|_{\mathcal{K}_{a-1}^{-1}(\Omega)}:=\sup _{\|v\|_{\mathcal{K}_{a+1}^{1}(\Omega)}^{1}=1} \int_{\Omega} f v, \quad \forall v \in \mathcal{K}_{a+1}^{1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}
$$

whenever the $L^{2}$-inner product is well defined.
Then, we write

$$
-\Delta_{a} u=-\Delta_{a} R_{a}(f)=f-T_{a} f=\left(I-T_{a}\right) f
$$

where $T_{a}: \mathcal{K}_{a-1}^{m-1}(\Omega) \rightarrow \mathcal{K}_{a-1}^{m-1}(\Omega)$ is a compact operator, which indicates $-\Delta_{a} R_{a}=$ $I-T_{a}$ is Fredholm on $\mathcal{K}_{a-1}^{m-1}(\Omega), m \geq 0$.

To complete the proof of this theorem, we next show a similar result for the operator $-R_{a}^{*} \Delta_{-a}: \mathcal{K}_{-a+1}^{1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\} \rightarrow \mathcal{K}_{-a+1}^{1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}$, where $R_{a}^{*}$ :
$\mathcal{K}_{-a-1}^{-1}(\Omega) \rightarrow \mathcal{K}_{-a+1}^{1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}$ is the conjugate operator of $R_{a}: \mathcal{K}_{a-1}^{-1}(\Omega) \rightarrow$ $\mathcal{K}_{a+1}^{1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}$ defined by

$$
\left(u, R_{a}^{*} v\right)=\left(R_{a} u, v\right)
$$

$\forall u, v$ in the domain indicated above. Therefore, $R_{a}^{*}$ is continuous.
From the discussion above, $-\Delta_{a} R_{a}=I+T_{a}: \mathcal{K}_{a-1}^{-1}(\Omega) \rightarrow \mathcal{K}_{a-1}^{-1}(\Omega)$ is Fredholm, $\forall a \neq k \pi / \omega_{i}, k \neq 0$. Therefore, its conjugate operator $R_{a}^{*}\left(-\Delta_{a}\right)^{*}=I+T_{a}^{*}$ : $\mathcal{K}_{-a+1}^{1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\} \rightarrow \mathcal{K}_{-a+1}^{1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}$ is also Fredholm with the same range for $a$ and $k$, since $T_{a}^{*}: \mathcal{K}_{-a+1}^{1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\} \rightarrow \mathcal{K}_{-a+1}^{1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}$ is a compact operator.

From Lemma 2.2.10, $\Delta^{*}=\Delta_{-a}=\Delta: \mathcal{K}_{-a+1}^{1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\} \rightarrow \mathcal{K}_{-a-1}^{-1}(\Omega)$. Then,

$$
-R_{a}^{*} \Delta_{-a}=R_{a}^{*}\left(-\Delta^{*}\right)=I-T_{a}^{*}: \mathcal{K}_{-a+1}^{1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\} \rightarrow \mathcal{K}_{-a+1}^{1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}
$$

is Fredholm $\forall a \neq k \pi / \omega_{i}, k \neq 0$. Note the index $a$ for the Fredholm property is arbitrary except for a countable set. Hence, we also have

$$
-R_{-a}^{*} \Delta_{a}: \mathcal{K}_{a+1}^{1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\} \rightarrow \mathcal{K}_{a+1}^{1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}
$$

defines a Fredholm operator. We then conclude that $\Delta: \mathcal{K}_{a+1}^{m+1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\} \rightarrow$ $\mathcal{K}_{a-1}^{m-1}(\Omega)$ is Fredholm for $m=0$.

Let $X \subset \mathcal{K}_{a+1}^{1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}, Y \subset \mathcal{K}_{a-1}^{-1}(\Omega)$ be the kernel and range of the operator $\Delta$, respectively. Then, since it is Fredholm, $\operatorname{dim}(X)<\infty, Y$ is a closed space, and $\operatorname{dim}\left(\mathcal{K}_{a-1}^{-1}(\Omega) / Y\right)<\infty$. Moreover,

$$
\Delta: \mathcal{K}_{a+1}^{1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\} / X \rightarrow Y
$$

defines an isomorphism.
Let us then consider the following operator,

$$
\Delta: \mathcal{K}_{a+1}^{m+1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\} \rightarrow \mathcal{K}_{a-1}^{m-1}(\Omega) .
$$

From Lemma 2.2.11, $X:=\operatorname{ker}\left(\Delta_{a}\right) \subset \mathcal{K}_{a+1}^{\infty}(\Omega) \subset \mathcal{K}_{a+1}^{m+1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}$, which
implies the kernel of the above operator is equal to $X$, and hence is a finite dimensional space. Note $\forall f \in \mathcal{K}_{a-1}^{m-1}(\Omega) \cap Y$, based on the Fredholm property for $m=0$, there is a solution $u \in \mathcal{K}_{a+1}^{1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}$. Furthermore, the regularity estimates near vertices and in the interior domain leads to $u \in \mathcal{K}_{a+1}^{m+1}(\Omega)$. Hence, the range of the operator $=\mathcal{K}_{a-1}^{m-1}(\Omega) \cap Y$ and $\mathcal{K}_{a-1}^{m-1}(\Omega) /\left(\mathcal{K}_{a-1}^{m-1}(\Omega) \cap Y\right)$ is finite dimensional.

Thus, it suffices to show the range $\mathcal{K}_{a-1}^{m-1}(\Omega) \cap Y$ is closed in the $\mathcal{K}_{a-1}^{m-1}$-norm to complete the proof. Note $Y$ can be characterized by $Y=\left\{u \in \mathcal{K}_{a-1}^{-1}(\Omega), \phi_{j}(u)=\right.$ $\left.0, \phi_{j} \in \operatorname{ker}\left(\mathcal{K}_{a-1}^{-1}(\Omega)^{*}\right)\right\}$, since it is the closed range of the a continuous operator. Therefore,
$\mathcal{K}_{a-1}^{m-1}(\Omega) \cap Y=\left\{u \in \mathcal{K}_{a-1}^{m-1}(\Omega), \phi_{j}(u)=0, \phi_{j} \in \mathcal{K}_{a-1}^{-1}(\Omega)^{*}\right.$ is the basis of $\left.\operatorname{ker}\left(\Delta^{*}\right)\right\}$.
Then the completion of the space $\mathcal{K}_{a-1}^{m-1}(\Omega) \cap Y$ follows from the continuity of $\phi_{i}$ on $\mathcal{K}_{a-1}^{m-1}(\Omega)$.

We have proved $\Delta: \mathcal{K}_{a+1}^{m+1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\} \rightarrow \mathcal{K}_{a-1}^{m-1}(\Omega)$ is Fredholm iff $a \neq$ $k \pi / \omega_{i}$ and $k \neq 0$. Similar proofs can be carried out for other elliptic operators and different boundary conditions.

### 2.3 Some Lemmas for Weighted Sobolev Spaces

Besides the well-posedness, regularity and the Fredholm property, the weighted Sobolev space introduced in Definition 2.2.7 has other important properties we will use throughout this work. In this section, we mainly focus on some related lemmas. More results on Sobolev spaces with weights can be found in [2, 17, 45, $60,62,61,63,84]$.

### 2.3.1 Notation

Recall that the definition of our weighted Sobolev spaces depends on the choice of a subset (of vertices) $\mathcal{V}=\left\{Q_{0}, \ldots, Q_{v}\right\} \subset \bar{\Omega}$. In the following definition of the weighted Sobolev spaces $\mathcal{K}_{a}^{m}(\Omega)$, we want to replace the weight $\vartheta$ with a smoothed version, denoted by $\rho$. The following remark explains how this is done.

Remark 2.3.1. Let us denote by $l$ the minimum of the non-zero distances from a point $Q \in \mathcal{V}$ to an edge of $\Omega$. Let

$$
\begin{equation*}
\tilde{l}:=\min (1 / 2, l / 4) \quad \text { and } \quad \mathcal{S}_{i}:=\Omega \cap B\left(Q_{i}, \tilde{l}\right) \tag{2.5}
\end{equation*}
$$

where $B\left(Q_{i}, \tilde{l}\right)$ denotes the ball centered at $Q_{i} \in \mathcal{S}$ with radius $\tilde{l}$. Note that sets $\mathcal{S}_{i}:=\Omega \cap B\left(Q_{i}, \tilde{l}\right)$ are disjoint. Also, we denote by $\mathcal{O}_{i}=\mathcal{S}_{i} / 2:=\Omega \cap B\left(Q_{i}, \tilde{l} / 2\right)$. Then, we define

$$
\begin{gather*}
\rho: \bar{\Omega} \rightarrow[0,2 \tilde{l}], \quad \rho \in \mathcal{C}^{\infty}(\bar{\Omega} \backslash \mathcal{V}) \\
\rho(x)= \begin{cases}\vartheta(x)=\left|x-Q_{i}\right| & \text { on } \mathcal{S}_{i} \text { and } \\
\rho(x) \geq \tilde{l} / 2 & \text { outside } \mathcal{S}:=\cup \mathcal{S}_{i}\end{cases} \tag{2.6}
\end{gather*}
$$

The quotients $\rho / \vartheta$ and $\vartheta / \rho$ are bounded, and we can replace $\vartheta$ with $\rho$ in all the definition of the weighted Sobolev spaces. In particular, this leads to the following equivalent definition of the weighted Sobolev spaces:

Definition 2.3.2. We define

$$
\begin{equation*}
\mathcal{K}_{a}^{m}(\Omega):=\left\{v: \Omega \rightarrow \mathbb{R}, \rho^{i+j-a} \partial_{x}^{i} \partial_{y}^{j} v \in L^{2}(\Omega), \forall i+j \leq m\right\} \tag{2.7}
\end{equation*}
$$

For any open set $G \subset \Omega$ and any $v: G \rightarrow \mathbb{R}$, we denote

$$
\|v\|_{\mathcal{K}_{a}^{m}(G)}^{2}:=\sum_{|\alpha| \leq m}\left\|\rho^{|\alpha|-a} \partial^{\alpha} v\right\|_{L^{2}(G)}^{2} .
$$

We endow $\mathcal{K}_{a}^{m}(\Omega)$ with the norm $\|v\|_{\mathcal{K}_{a}^{m}(\Omega)}^{2}$. In particular, the inner product on the Hilbert space $\mathcal{K}_{a}^{m}(\Omega)$ becomes

$$
(u, v)_{\mathcal{K}_{a}^{m}(\Omega)}=\sum_{i+j \leq m} \int_{\Omega} \rho^{2(i+j-a)}\left(\partial_{x}^{i} \partial_{y}^{j} u\right)\left(\partial_{x}^{i} \partial_{y}^{j} \bar{v}\right) d x d y
$$

The completeness of $L^{2}(\Omega)$ and standard arguments, see [48] for instance, show that the weighted Sobolev space $\mathcal{K}_{a}^{m}(\Omega)$ is complete. Note all the norms are equivalent if we replace $\rho$ by $\vartheta$.

### 2.3.2 Lemmas

For simplicity, we denote from now on $\mathcal{K}_{a}^{m}=\mathcal{K}_{a}^{m}(\Omega)$ and $H^{m}=H^{m}(\Omega)$.
In this subsection, we summarize several properties of $\mathcal{K}_{a}^{m}$ that are useful for the development of our theory in Chapter 3 and Chapter 4. As usual, we use $r$ and $\theta$ as the variables in the polar coordinates. In addition, by $A \equiv B$, we mean that there exist constants $C_{1}>0, C_{2}>0$, such that $C_{1} A \leq B \leq C_{2} A$. Meanwhile, an isomorphism of Banach spaces means a continuous bijection.

We first have the following alternative definition of the space $\mathcal{K}_{a}^{m}:=\mathcal{K}_{a}^{m}(\Omega)$. Let $\tilde{l}:=\min (1 / 2, l / 4)$ and $\mathcal{S}_{i}:=\Omega \cap B\left(Q_{i}, \tilde{l}\right)$ be as in Equation (2.5) of our previous remark.

Lemma 2.3.3. On every $\mathcal{S}_{i}:=\Omega \cap B\left(Q_{i}, \tilde{l}\right)$, we set a local polar coordinate system $(r, \theta)$, where $Q_{i}$ is sitting at the origin (and hence $\rho=r$ on each $\mathcal{S}_{i}$ ). Denote $\mathcal{O}_{i}=\mathcal{S}_{i} / 2=\Omega \cap B\left(Q_{i}, \tilde{l} / 2\right)$, as before, and $\mathcal{O}:=\cup \mathcal{O}_{i}$. Then, the weighted Sobolev space $\mathcal{K}_{a}^{m}:=\mathcal{K}_{a}^{m}(\Omega)$ is also given by

$$
\mathcal{K}_{a}^{m}=\left\{v: \Omega \rightarrow \mathbb{R}, v \in H^{m}(\Omega \backslash \mathcal{O}), \quad \text { and } r^{-a}\left(r \partial_{r}\right)^{j} \partial_{\theta}^{k} v \in L^{2}\left(\mathcal{S}_{i}\right), j+k \leq m\right\}
$$

Proof. We note that $\Omega \backslash \mathcal{O}$ is a subset of $\Omega$, whose closure is away from all $Q_{i} \in \mathcal{V}$. According to the definitions of $\rho$ and $\tilde{l}, \rho \geq \tilde{l} / 2$ on $\Omega \backslash \mathcal{O}$. Thus, the $H^{m}$-norm and the $\mathcal{K}_{a}^{m}$-norm are equivalent on this region, since $\rho$ is smooth and bounded from above and from 0.

On the region $\mathcal{S}_{i}$, the differential operators $\partial_{x}$ and $\partial_{y}$ can be written in terms of $r$ and $\theta$, the variables in the polar coordinates centered at $Q_{i}$,

$$
\begin{aligned}
& \partial_{x}=(\cos \theta) \partial_{r}-\frac{\sin \theta}{r} \partial_{\theta} \\
& \partial_{y}=(\sin \theta) \partial_{r}+\frac{\cos \theta}{r} \partial_{\theta} .
\end{aligned}
$$

Since $\rho=r$ on $\mathcal{S}_{i}$, we have

$$
\begin{aligned}
& \sum_{j+k \leq m}\left\|\rho^{j+k-a} \partial_{x}^{j} \partial_{y}^{k} u\right\|_{L^{2}\left(\mathcal{V}_{i}\right)} \\
& \quad=\sum_{j+k \leq m}\left\|r^{j+k-a}\left(\cos \theta \partial_{r}-\frac{\sin \theta}{r} \partial_{\theta}\right)^{j}\left(\sin \theta \partial_{r}+\frac{\cos \theta}{r} \partial_{\theta}\right)^{k} u\right\|_{L^{2}\left(\mathcal{V}_{i}\right)}
\end{aligned}
$$

$$
\leq \sum_{j+k \leq m}\left\|r^{-a}\left(r \partial_{r}\right)^{j} \partial_{\theta}^{k} u\right\|_{L^{2}\left(\mathcal{V}_{i}\right)}
$$

We have a similar formula expressing $r \partial_{r}$ and $\partial_{\theta}$ in terms of $r \partial_{x}$ and $r \partial_{y}$, which provides us with the opposite inequality and completes the proof.

Lemma 2.3.4. The function $\rho^{j+k-a} \partial_{x}^{j} \partial_{y}^{k} \rho^{a}$ is bounded on $\Omega$.
Proof. The function $\rho^{j+k-a} \partial_{x}^{j} \partial_{y}^{k} \rho^{a}$ is bounded on $\Omega \backslash \mathcal{S}:=\Omega \backslash\left(\cup \mathcal{S}_{i}\right)$ because $\rho \geq \tilde{l} / 2$ is smooth.

It remains to verify that our function is bounded on every $\mathcal{S}_{i}$ as well. By changing $x, y$ into $r, \theta$ in the polar coordinates centered at $Q_{i} \in \mathcal{V}$, as in Lemma 2.3.3, we have

$$
\begin{aligned}
\left|\rho^{j+k-a} \partial_{x}^{j} \partial_{y}^{k} \rho^{a}\right| & =\left|r^{j+k-a}\left(\cos \theta \partial_{r}-\frac{\sin \theta}{r} \partial_{\theta}\right)^{j}\left(\sin \theta \partial_{r}+\frac{\cos \theta}{r} \partial_{\theta}\right)^{k} r^{a}\right| \\
& \leq C\left|r^{-a} \sum_{s+t \leq j+k}\left(r \partial_{r}\right)^{s} \partial_{\theta}^{t} r^{a}\right| \\
& \leq C\left|r^{-a} \sum_{s \leq j+k}\left(r \partial_{r}\right)^{s} r^{a}\right|=C\left|\sum_{s \leq j+k} a^{s}\right|
\end{aligned}
$$

Therefore, $\rho^{j+k-a} \partial_{x}^{j} \partial_{y}^{k} \rho^{a}$ is bounded on $\Omega$, as stated.
This lemma leads to the following isomorphism between weighted Sobolev spaces.

Lemma 2.3.5. We have $\rho^{b} \mathcal{K}_{a}^{m}=\mathcal{K}_{a+b}^{m}$, where $\rho^{b} \mathcal{K}_{a}^{m}=\left\{\rho^{b} v, \forall v \in \mathcal{K}_{a}^{m}\right\}$. More precisely, the multiplication by $\rho^{b}$ defines an isomorphism $\mathcal{K}_{a}^{m} \rightarrow \mathcal{K}_{a+b}^{m}$.

Proof. Let $v \in \mathcal{K}_{a}^{m}$ and $w=\rho^{b} v$. Then $\left|\rho^{i+j-a} \partial_{x}^{i} \partial_{y}^{j} v\right| \in L^{2}$, for $i+j \leq m$. Thus, we verify $w \in \mathcal{K}_{a+b}^{m}$ by checking the inequalities below,

$$
\begin{aligned}
\left|\rho^{i+j-a-b} \partial_{x}^{i} \partial_{y}^{j} w\right| & =\left|\rho^{i+j-a-b} \sum_{s \leq i, t \leq j}\binom{i}{s}\binom{j}{t} \partial_{x}^{s} \partial_{y}^{t} \rho^{b} \partial_{x}^{i-s} \partial_{y}^{j-t} v\right| \\
& \leq C \sum_{s \leq i, t \leq j}\left|\rho^{(i+j-s-t)-a} \partial_{x}^{i-s} \partial_{y}^{j-t} v\right| \in L^{2}(\Omega)
\end{aligned}
$$

with the last inequality following from Lemma 2.3.4. Therefore, $\rho^{b} \mathcal{K}_{a}^{m}$ is continuously embedded in $\mathcal{K}_{a+b}^{m}$. In other words, the map $\rho^{b}: \mathcal{K}_{a}^{m} \rightarrow \mathcal{K}_{a+b}^{m}$ is continuous.

On the other hand, because this embedding holds for any real number $b$, we have the opposite inclusion:

$$
\mathcal{K}_{a+b}^{m}=\rho^{b} \rho^{-b} \mathcal{K}_{a+b}^{m} \subset \rho^{b} \mathcal{K}_{a}^{m}
$$

To complete the proof we also notice that the inverse of multiplication by $\rho^{b}$ is multiplication by $\rho^{-b}$, which is also continuous.

Recall that $\mathcal{S}_{i}=\Omega \cap B\left(Q_{i}, \tilde{l}\right)$ is a small neighborhood of $Q_{i} \in \mathcal{V}$. Therefore, $\rho(x, y) \leq \tilde{l}$ on $\mathcal{S}_{i}$, and we have

Lemma 2.3.6. Let $G \subset \mathcal{S}_{i}$ be an open subset of $\mathcal{S}_{i}$, such that $\rho \leq \xi \leq \tilde{l}$ on $G$. Then, for $m^{\prime} \leq m$ and $a^{\prime} \leq a$, we have
(a) $\mathcal{K}_{a}^{m} \subset \mathcal{K}_{a^{\prime}}^{m^{\prime}}:=\mathcal{K}_{a^{\prime}}^{m^{\prime}}(\Omega)$ and
(b) $\|u\|_{\mathcal{K}_{a^{\prime}}^{m{ }^{\prime}}(G)} \leq \xi^{a-a^{\prime}}\|u\|_{\mathcal{K}_{a}^{m}(G)}, \forall u \in \mathcal{K}_{a}^{m}$.

Proof. The function $\rho^{a-a^{\prime}}$ is bounded for $a \geq a^{\prime}$ and hence

$$
\sum_{j+k \leq m^{\prime}}\left\|\rho^{j+k-a^{\prime}} \partial_{x}^{i} \partial_{y}^{j} u\right\|_{L^{2}}^{2} \leq C \sum_{j+k \leq m}\left\|\rho^{j+k-a} \partial_{x}^{j} \partial_{y}^{k} u\right\|_{L^{2}}^{2}
$$

for any $u \in \mathcal{K}_{a}^{m}$ and any $m^{\prime} \leq m$. This proves (a).
Similarly, (b) follows from the following estimates on $G$,

$$
\|u\|_{\mathcal{K}_{a^{\prime}}^{m^{\prime}(G)}}^{2} \leq \xi^{2\left(a-a^{\prime}\right)} \sum_{j+k \leq m}\left\|\xi^{a^{\prime^{\prime}-a} \rho^{j+k-a^{\prime}}} \partial_{x}^{j} \partial_{y}^{k} u\right\|_{L^{2}(G)}^{2} \leq \xi^{2\left(a-a^{\prime}\right)}\|u\|_{\mathcal{K}_{a}^{m}(G)}^{2}
$$

Our proof is complete.
Let $\tilde{G} \subset \Omega$ be a compact subset of $\Omega$. Note that there exists a constant $C>0$, such that $\rho \geq C$ on $\tilde{G}$. Then, the following lemma asserts that the $H^{m}$-norm and the $\mathcal{K}_{a}^{m}$-norm are equivalent on $H^{m}(\tilde{G})$. Recall that $\tilde{l}$ and $\mathcal{S}_{i}$ were introduced in Equation (2.5).

Lemma 2.3.7. For $0<\xi \leq \tilde{l} / 4$, let $\tilde{G} \subset \Omega$ be an open subset, such that $\rho \geq \xi$ on $\tilde{G}$. Then, $\|u\|_{H^{m}(\tilde{G})} \leq M_{1}\|u\|_{\mathcal{K}_{a}^{m}(\tilde{G})}$ and $\|u\|_{\mathcal{K}_{a}^{m}(\tilde{G})} \leq M_{2}\|u\|_{H^{m}(\tilde{G})}$, for $\forall u \in$ $H^{m}(\tilde{G}) . M_{1}$ and $M_{2}$ depend on $\xi$ and $m$, but not on $u$.

Proof. We note that $\rho$ is smooth and bounded below by $\xi$ on $\tilde{G}$. Then, the proof follows from the definitions of $\mathcal{K}_{a}^{m}$ and $H^{m}$.

However $\mathcal{K}_{a}^{m}\left(\mathcal{S}_{i}\right)$ and $H^{m}\left(\mathcal{S}_{i}\right)$ are quite different.
Lemma 2.3.8. Let $G \subset \mathcal{S}_{i}$ be an open subset, such that $\rho \leq \xi \leq \tilde{l}$ on this region. Let $m \geq 0$, then
(a) $\|u\|_{H^{m}(G)} \leq \xi^{a-m}\|u\|_{\mathcal{K}_{a}^{m}(G)}$, if $a \geq m$, and
(b) $\|u\|_{\mathcal{K}_{a}^{m}(G)} \leq \xi^{-a}\|u\|_{H^{m}(G)}$, if $a \leq 0$.

Proof. The proof is based on the direct verification of the inequalities below. First, for $a \geq m$, by Lemma 2.3.6, we have

$$
\|u\|_{\mathcal{K}_{m}^{m}(G)} \leq \xi^{a-m}\|u\|_{\mathcal{K}_{a}^{m}(G)} .
$$

Therefore, from the definition of the $K_{a}^{m}$-norm, the first inequality in this lemma is obtained by

$$
\begin{aligned}
\|u\|_{H^{m}(G)}^{2} & =\sum_{j+k \leq m}\left\|\partial_{x}^{j} \partial_{y}^{k} u\right\|_{L^{2}(G)}^{2} \\
& \leq \sum_{j+k \leq m}\left\|\rho^{-m} \rho^{j+k} \partial_{x}^{j} \partial_{y}^{k} u\right\|_{L^{2}(G)}^{2} \\
& =\|u\|_{\mathcal{K}_{m}^{m}(G)}^{2} \leq \xi^{2(a-m)}\|u\|_{\mathcal{K}_{a}^{m}(G)}^{2} .
\end{aligned}
$$

For $a \leq 0$ we have

$$
\begin{aligned}
\|u\|_{\mathcal{K}_{a}^{m}(G)}^{2} & =\sum_{j+k \leq m}\left\|\rho^{-a} \rho^{j+k} \partial_{x}^{j} \partial_{y}^{k} u\right\|_{L^{2}(G)}^{2} \\
& \leq \xi^{-2 a} \sum_{j+k \leq m}\left\|\partial_{x}^{j} \partial_{y}^{k} u\right\|_{L^{2}(G)}^{2}=\xi^{-2 a}\|u\|_{H^{m}(G)}^{2},
\end{aligned}
$$

which completes the proof.
We shall need the extension of Lemma 2.3.8 to the entire domain $\Omega$, which reads as follows. Recall that we denote $H^{m}:=H^{m}(\Omega)$ and $\mathcal{K}_{a}^{m}:=\mathcal{K}_{a}^{m}(\Omega)$.

Corollary 2.3.9. For $u: \Omega \rightarrow \mathbb{R},\|u\|_{H^{m}} \leq M_{1}\|u\|_{\mathcal{K}_{m}^{m}}$ and $\|u\|_{\mathcal{K}_{a}^{m}} \leq M_{2}\|u\|_{H^{m}}$ for $a \leq 0$, where $M_{1}$ and $M_{2}$ depend on $m$ and $a$.

Proof. The proof is a direct combination of the estimates in Lemma 2.3.7 and the estimates for the neighborhoods of all $Q_{i} \in \mathcal{V}$ in Lemma 2.3.8.

In the next chapter, we will continue to discuss properties of an inhomogeneous weighted Sobolev space, which is a generalization of the weighted Sobolev space from Definition 2.3.2. It will be seen that all the lemmas from this subsection hold for those spaces. We may not repeat similar results there since we can refer to this section.

\section*{|  |
| :---: |
| Chapter | 3}

## A Priori Analysis in Weighted Sobolev Spaces

## Introduction

In this chapter we study a priori estimates (well-posedenss, regularity and the Fredholm property) for general elliptic operators on a class of domains with polygonal structures, which is a generation of the usual polygonal domains. The introduction of this class of domains, with extensions of our results developed in Chapter 2 , leads to estimates in weighted Sobolev spaces for corner singularities, NeumannNeumann, and transmission problems.

Let $\Omega \subset \mathbb{R}^{2}$ be a bounded polygonal domain, possibly with curved boundaries, cracks, or vertices touching the boundary. We consider on $\Omega$ the second order, strongly elliptic operator in divergence form $P=-\operatorname{div} A \nabla$ with piecewise smooth coefficients with jump discontinuities. More precisely, we assume that $\bar{\Omega}=\cup \bar{\Omega}_{j}$, where $\Omega_{j}$ are disjoint domais with a polygonal structure such that the interfaces $\Gamma:=\cup \partial \Omega_{j} \backslash \partial \Omega$ is a union of disjoint piecewise smooth curves. We denote by $D_{\nu}^{P}:=\sum_{i j} \nu^{i} a_{i j} \partial_{j}$ the conormal derivative associated to $P$, where $A=\left(a_{i j}\right)$ is the symmetric matrix of coefficients of $P$. Let $\partial \Omega=\partial_{D} \Omega \cup \partial_{N} \Omega$ be a disjoint union, with $\partial_{D} \Omega$ a union of closed sides of $\Omega$. We are interested in the non-homogeneous mixed boundary value problem

$$
\begin{equation*}
P u=f \text { in } \Omega, \quad D_{\nu}^{P} u=g_{N} \text { on } \partial_{N} \Omega, \quad \text { and } u=g_{D} \text { on } \partial_{D} \Omega . \tag{3.1}
\end{equation*}
$$



Figure 3.1. The domain $\Omega$ and the interface $\Gamma$
(When $\Omega$ has cracks, the boundary conditions have to be suitably interpreted using the "unfolded boundary.")

The equation $P u=f$ has to be interpreted in a weak sense and then the discontinuity of the coefficients $a^{i j}$ at the interface $\Gamma$ leads to the following "transmission conditions." Note that our interface does not ramify. We can therefore make a choice and denote $u^{+}$and $u^{-}$the non-tangential limits of $u$ at the two sides of the interface. Also, let $D_{\nu}^{P+}$ and $D_{\nu}^{P-}$ be the two conormal derivatives associated to $P$ at the two sides of the interface. Then the usual transmission conditions $u^{+}=u^{-}$ and $D_{\nu}^{P+} u=D_{\nu}^{P-} u$ at the two sides of the interface are a consequence of the weak formulation, and will always be considered as part of Equation (3.1). The more general conditions $u^{+}-u^{-}=h_{0}$ and $D_{\nu}^{P+} u-D_{\nu}^{P-} u=h_{1}$ can be treated with only minor modifications.

In the rest of this chapter, we establish regularity results for Equation (3.1) in the weighted Sobolev spaces $\mathcal{K}_{\overrightarrow{\mathbf{a}}}^{m}(\Omega)$ (Definition (3.11)). Compared with Definition 2.2.7, we here identify specific weights for each singular point, which make $P$ Fredholm following Theorem 2.2.12 and the results of Kondratiev [60] and Nicaise [82]. If no two adjacent sides are assigned Neumann boundary conditions (i.e., when there are no Neumann-Neumann corners), then we also obtain a well-posedness result for the weight parameter $\overrightarrow{\mathbf{a}}$ close to 1 . In general, we compute the index of the resulting operator, when it is Fredholm, and we use this to obtain a decomposition $u=u_{\text {reg }}+\sigma$ of the solution $u$ of (3.1) into a more regular function and a singular function. If $\Omega$ is smooth, then $\sigma$ is simply a constant. See below for a more complete description of our results and for an account of the earlier results.

One of our main motivations is the analysis of the Finite Element Method for Equation (3.1). We are especially interested in obtaining a sequence of meshes that
provides quasi-optimal rates of convergence. A construction of such a sequence of meshes will be provided in Chapter 4.

The problem of constructing sequences of meshes that provide quasi-optimal rates of convergence has received a lot of attention in the works of Apel [4], Apel and Nicaise [6], Babuška and collaborators [13, 17, 56, 18], Bacuta, Nistor, and Zikatanov [21], Costabel and Dauge [41, 42, 43], Dauge [45], Grisvard [54], Kellogg [58], Lubuma and Nicaise [72], Schatz, Sloan, and Wahlbin [90], and many others. In this chapter, we establish the a priori estimates necessary for verifying that the sequence of graded meshes that will be constructed in the next chapter yields quasioptimal rates of convergence, while also addressing here several issues that may be of interest in concrete applications. For instance, we consider cracks and higher regularity for transmission problems, issues that are relevant in practice, but have received less attention than for example the issue of singularities near corners. We also consider more systematically mixed boundary conditions. Moreover, we will show how to treat the additional theoretical difficulties caused by the NeumannNeumann corners, obtaining, in particular, that from a computational point of view, the Neumann-Neumann corners behave like the other types of corners. A priori estimates are a well established tool in Numerical Analysis [8, 9, 14, 16, 22, $34,39,57,59,92$, and also our starting point to handle elliptic equations with singular solutions.

Regularity and numerical issues for transmission problems were studied before in several papers, see Nicaise [82] and Nicaise and Sändig [84] and the references therein. Like in these two papers, we use weighted Sobolev spaces. However, our emphasis is not on singular functions, but on well-posedness results. We also obtain higher regularity results in weighted Sobolev spaces and a new well-posedness result for the case when there are Neumann-Neumann corners, combining Fredholm and index calculations with the use of the "first singular function", which is nothing but a constant in this case.

In Section 3.1, we first formulate our problem by briefly introducing the domain with a polygonal structure and the elliptic operator $P$ for the boundary value/transmission problem, Equation (3.1). Then, we define the weighted Sobolev space $\mathcal{K}_{\overrightarrow{\mathbf{a}}}^{m}(\Omega)$, and state our main results, Theorems 3.1.1 and 3.1.3, on regularity of the solution and solvability of the value/transmission problem (3.1) in weighted

Sobolev spaces. We agree that whenever transmission problems are considered, we assume that all the vertices of the domains with a polygonal structure $\Omega_{j} \subset \Omega$ are included in the set of vertices of $\Omega$.

In Section 3.2, we discuss the Fredholm property of the operator associated to Equation (3.1), provided that the coefficients are piecewise constant. For definiteness, let us consider first the case of the Laplace operator (so no interface, but mixed boundary conditions). Notice that $D_{\nu}^{P}=\partial_{\nu}$ if $P=\Delta$. We then prove that

$$
\Delta:\left\{u \in \mathcal{K}_{\overrightarrow{\mathbf{a}}+1}^{m+1}(\Omega),\left.u\right|_{\partial_{D} \Omega}=0, \partial_{\nu} u=0\right\} \rightarrow \mathcal{K}_{\overrightarrow{\mathbf{a}}-1}^{m-1}(\Omega)
$$

is an isomorphism when no adjacent sides are endowed with Neumann-Neumann boundary conditions (i.e., when there are no Neumann-Neumann corners). In general, let $\tilde{W}_{s} \in \mathcal{C}^{\infty}(\bar{\Omega})$ be the linear span of smooth functions on $\Omega$ that are constant near the vertices of $\Omega$ and satisfy the boundary conditions with $g_{D}=0$ and $g_{N}=0$. (These functions can be non-zero only close to the Neumann-Neumann vertices.) Then our isomorphism becomes

$$
\Delta: V:=\left\{u \in \mathcal{K}_{\overrightarrow{\mathbf{a}}+1}^{m+1}(\Omega),\left.u\right|_{\partial_{D} \Omega}=0, \partial_{\nu} u=0\right\}+\tilde{W}_{s} \rightarrow \mathcal{K}_{\overrightarrow{\mathbf{a}}-1}^{m-1}(\Omega)
$$

(One can replace $\tilde{W}_{s}$ with a subspace of dimension the number of NeumannNeumann corners so that the above sum becomes a direct sum.) An estimate on regularity of the solution on the domain with Neumann-Neumann vertices is given in Theorem 3.2.5. Then, we extend this result to general elliptic operators and transmission problems.

In Section 3.3, we introduce the concept of a domain with polygonal structure in order to extend our theory to include pathologies such as cracks and vertices touching a smooth face. In Section 3.4, we establish several properties of the weighted Sobolev space that are crucial for analyzing boundary value/transmission problems on domains with a polygonal structure. In Section 3.5, we use results from previous sections to prove a priori estimates (Theorem 3.1.1 and Theorem 3.1.3) for boundary value/transmission problems.

Throughout this chapter, " $x:=y$ " means that " $x$ " is defined to be equal to " $y$," as customary. Also, by $C$ we shall denote a generic constant that can be different at each occurrence.

### 3.1 Formulation of the Problem and Main Results

In this section we formulate our problem and state our three main results on the well-posedness and regularity for Equation (3.1), which deal with general smooth interfaces in the domain. Note that no-interface is only a special case in our results. We start by describing informally the class of "domains with a polygonal structure," then we define the differential operator $P$, and we state our boundary value/transmission problem associated to $P$ and a partition of our domain $\Omega$ into smaller domains with a polygonal structure. The formal definition of domains with a polygonal structure is given in Section 3.3. The three main theorems, Theorems 3.1.1, 3.1.2 and 3.1.3, will be proved in the last section, Section 3.5.

### 3.1. 1 The Domain

We now provide an informal description of the domains used in this chapter. The formal definitions will be given in Section 3.3. We consider polygonal-like domains $\Omega$ that may have cracks or vertices that touch a smooth part of the boundary. More precisely, in this chapter $\Omega$ will always denote a bounded domain with polygonal structure (Definition 3.3.2), a class of domains introduced (with a different name and a slightly different definition) in [45]. We also allow for mixed DirichletNeumann boundary conditions.

While the formal definition of "domains with a polygonal structure" will only be given in the following section, we can quickly introduce now the main ideas and motivations behind this definition. Recall that polygonal domains are not always Lipschitz domains, however, the outer normal to the boundary is well-defined except at the vertices. If cracks are present, then the outer normal is not welldefined since $\partial \bar{\Omega} \neq \partial \Omega$. In order to study cracks, each modeled as a double covering of a piecewise-smooth curve, we then need to separate the two normal directions in which we approach the boundary, see for example [45]. This distinction is also needed when we study vertices that touch the boundary (see Figure 3.2). When the cracks ramify, we need further to differentiate from which direction we approach the point of ramification (see Figure 3.3). This distinction will be


Figure 3.2. Domain with two vertices touching a side.
achieved by considering the connected neighborhoods of $B(x, r) \cap \Omega$, when $x$ is on the boundary, following Dauge [45]. Then each point $x$ on the boundary can be associated to a finite collection of points, one for each component of $B(x, r) \cap \Omega$ where $r=r_{x}$ is small enough. More precisely, $\partial^{u} \Omega=\cup_{x}\{x\} \times \pi_{0}\left(B\left(x, r_{x}\right) \cap \Omega\right)$, with $x$ ranging through $\partial \Omega$. Given the proper topological structure, this process leads to the concept of "unfolded boundary" $\partial^{u} \Omega$, which is a multiple branched covering of $\partial \Omega$, and to the concept of "unfolded closure" ${ }^{u} \bar{\Omega}:=\Omega \cup \partial^{u} \Omega$. These concepts are defined in more detail in Section 3.3.

When considering mixed boundary conditions, it is well known that singularities appear at the points where the boundary conditions change (from Dirichlet to Neumann and vise versa). These singularities are very similar in structure to the singularities that appear at the geometric vertices. It makes sense therefore to regard the points where the boundary conditions change as "artificial vertices," which together with the geometric vertices will form the set of all vertices of $\Omega$. (See Figures 3.3 and 3.2, where the real vertices are marked with a dot and artificial vertices with a cross for mixed boundary conditions.) In fact, we go even further, and we fix a finite set $\mathcal{V} \subset \partial^{u} \Omega$ such that all connected components of $\partial^{u} \Omega \backslash \mathcal{V}$ consists of smooth curves on which a unique type of boundary condition (Dirichlet or Neumann) is given. The points of $\mathcal{V}$ will be called the vertices of $\Omega$. The set $\mathcal{V}$ will include all geometric vertices and all points where the type of boundary conditions changes, but may include other points as well (these points may come from the interface or be simply some arbitrarily chosen points). This allows for a greater generality that is convenient when studying operators with singularities or discontinuities in the coefficients. Applications of our theory for operators with
singularities are discussed in Chapter 7 and Chapter 6.
The choice of $\mathcal{V}$ thus defines on $\Omega$ a structure that is not entirely determined by geometry, but depends also on the specifics of our boundary value problem and of the coefficients. This structure, in turns, when combined with the introduction of the unfolded boundary, gives rise to the concept of a "domain with a polygonal structure." Domains with a polygonal structure already appear in [75] with a slightly more restrictive definition than the one used here.

The concept of "unfolded boundary" is not necessary in the presence of simple cracks (see [75]). However, this notion allows to treat in a unified manner both cracks and vertices that touch a smooth part. These points deserve special attention. Except for a finite set, the points $x$ on $\partial \Omega$ are smooth, in the sense that in a small neighborhood of $x$, the boundary of $\Omega$ coincides with a smooth curve $\gamma$. Let $x$ be such a smooth point. Then there are two possibilities: the first is that $\Omega$ lies on exactly one side of $\gamma$ and the second one is that $\Omega$ lies on both sides of $\gamma$. In the first case, the point $x$ is covered by exactly one point $X$ in the unfolded boundary. In the second case, $x$ is covered by exactly two points in the unfolded boundary of $\Omega$, and we say that $x$ is a crack point of $\Omega$. If $x$ is smooth, we also call any unfolded point covering it smooth. Then, the set of smooth points in the unfolded bundary of $\Omega$ is still a disjoint union of smooth curves, which we refer to as the (open) sides of $\Omega$. A crack has always two sides.

Let $\{\nu\}$ be the set of outer unit normal vectors to $\partial \Omega$, more precisely, $\nu:=-\tilde{\nu}$, where $\tilde{\nu}$ is a normal vector that points towards the interior of $\Omega$. The outer normal vector $\nu$ is defined even at smooth crack points (but not at vertices) of $\Omega$ provided a direction of non-tangential approach to the boundary is specified. Hence, there is a one-to-one correspondence between the smooth points $X$ of the unfolded boundary $\partial^{u} \Omega$ and the set of outer normal unit vectors. Furthermore, we can account for the dual nature of a point where a vertex touch a smooth part of the boundary. This point will be covered by unfolded points $Y, X_{1}, X_{2}, \ldots \in \partial^{u} \Omega$ such that the $X_{j} \mathrm{~s}$ are singular and $Y$ is smooth.

For a priori estimates in weighted Sobolev spaces, we consider weights that depend on the vertex, which is a more general setting than the one considered in the last chapter. Thus to each unfolded point $Q$ covering a vertex of the domain $\Omega$, we associate a parameter $a_{Q}$. The set of all these parameters will be denoted


Figure 3.3. Domain with ramified crack.
by $\overrightarrow{\mathbf{a}}$. By $\overrightarrow{\mathbf{a}}+t$ we denote the vector obtained from $\overrightarrow{\mathbf{a}}$ by adding the real number $t$ to each component. In particular, we write $t$ instead of $\overrightarrow{\mathbf{a}}$ if all the components of $\overrightarrow{\mathbf{a}}$ are equal to $t$. This construction leads to weighted Sobolev spaces of the form $\mathcal{K}_{\overrightarrow{\mathbf{a}}}^{m}(\Omega), m \in \mathbb{Z}_{+}$, whose definition is recalled in Section 3.4.

### 3.1.2 The Equation

We consider a second order scalar differential operator with real coefficients

$$
\begin{gather*}
P: \mathcal{C}_{\mathrm{c}}^{\infty}(\Omega) \rightarrow \mathcal{C}_{\mathrm{c}}^{\infty}(\Omega) \\
P=-\sum_{i, j=1}^{3} \partial_{j} a^{i j} \partial_{i} . \tag{3.2}
\end{gather*}
$$

We assume, for simplicity, that $a^{i j}=a^{j i}$. Although our methods apply to systems as well, we here restrict our attention to scalar operators with non-constant real coefficient. (With some mild assumption on the lower order coefficients, our results extend also to operators of the form $P=-\sum_{i, j=1}^{2} \partial_{j} a^{i j} \partial_{i}+\sum_{i=1}^{2} b^{i} \partial_{i}+c$.)

In all our results, we also assume that our operator $P$ is uniformly strongly elliptic, in the sense:

$$
\begin{equation*}
\sum_{i, j=1}^{2} a^{i j}(x) \xi_{i} \xi_{j} \geq C\|\xi\|^{2} \tag{3.3}
\end{equation*}
$$

for some constant $C>0$ independent of $x \in \bar{\Omega}$ and $\xi \in \mathbb{R}^{2}$.
We also assume that we are given a decomposition

$$
\bar{\Omega}=\cup_{j=1}^{N} \bar{\Omega}_{j},
$$

where $\Omega_{j}$ are disjoint domains with a polygonal structure (so each $\Omega_{j}$ is an open set according to our conventions). We assume that all the coefficients of the differential operator $P$ are piecewise smooth in the sense that they extend to smooth functions on $\bar{\Omega}_{j}$. Therefore the possible jumps of the coefficients of $P$ will always occur on parts of $\partial \Omega_{j}$, for some $j$. For transmission problems, the polygonal structure of each $\Omega_{j}$ will be taken into account.

To formulate our problem, we also assume that the unfolded boundary $\partial^{u} \Omega$ is partitioned into two disjoint sets $\partial_{D} \Omega$ and $\partial_{N} \Omega$, with $\partial_{D} \Omega$ a union of closed sides of $\Omega$ :

$$
\partial^{u} \Omega=\partial_{N} \Omega \cup \partial_{D} \Omega, \quad \partial_{D} \Omega \cap \partial_{N} \Omega=\emptyset
$$

We assume that we are given Neumann data $g_{N} \in \mathcal{K}_{a-1 / 2}^{-1 / 2}\left(\partial_{N} \Omega\right)$. (The surjectivity of the restriction map allows us to take the Dirichlet data $g_{D}$ to be zero.)

The boundary-value problem (3.1) with $g_{D}=0$ is interpreted in a weak (or variational) sense. This is achieved using the bilinear form $B_{P}(u, v)$ defined by:

$$
\begin{equation*}
B_{P}(u, v):=\sum_{i j} \int_{\Omega} a^{i j} \partial_{i} u \partial_{j} v d x \tag{3.4}
\end{equation*}
$$

$1 \leq i, j \leq 2$, which is well-defined for any $u, v \in H^{1}(\Omega)$. Let $\nu=\left(\nu_{i}\right)$ be the outer unit normal vector to $\partial^{u} \Omega$. We also denote by $D_{\nu}^{P}$ the conormal derivative operator associated to $P$, which is defined by

$$
\begin{equation*}
\left(D_{\nu}^{P} u\right):=\sum_{i j} \nu_{i} a^{i j} \partial_{j} u \tag{3.5}
\end{equation*}
$$

The definition of $D_{\nu}^{P} u$ should be understood in the sense of the trace at the boundary. In particular, $D_{\nu}^{P} u$ is defined almost everywhere if $u$ is regular enough as a non-tangential limit, which is consistent with $\nu$ being defined only almost everywhere on $\partial^{u} \Omega$. We remark that $\nu$ is defined except at the vertices because we double the boundary at the smooth crack points.

Let $u \in H^{2}(\Omega)$ and $v \in H^{1}(\Omega)$ be real valued. Then $P, B_{P}$, and $D_{\nu}^{P}$ are related as in the ordinary Green's Theorem by:

$$
\begin{equation*}
(P u, v)_{L^{2}(\Omega)}=B_{P}(u, v)-\left(D_{\nu}^{P} u, v\right)_{L^{2}\left(\partial^{u} \Omega\right)} . \tag{3.6}
\end{equation*}
$$

In fact, it is enough to observe that a domain with polygonal structure can be partitioned into a finite number of Lipschitz domains to which Green's Theorem applies [54, 103]. This allows us to introduce the weak solution $u$ of Equation (3.1) with $g_{D}=0$ as the unique $u \in \mathcal{K}_{\overrightarrow{\mathbf{a}}+1}^{1}(\Omega)$ satisfying $u=0$ on $\partial_{D} \Omega$, and especially $\forall v \in \mathcal{K}_{\overrightarrow{\mathbf{a}}+1}^{1}$,

$$
\begin{equation*}
B_{P}(u, v)=(f, v)_{L^{2}(\Omega)}+\left(g_{N}, v\right)_{\partial_{N} \Omega} \tag{3.7}
\end{equation*}
$$

where the second paranthesis denotes the pairing between a distribution and a (suitable) function.

Denote by $\Gamma:=\left(\cup_{j}^{N} \partial \Omega_{j}\right) \backslash \partial \Omega$ the interface of our problem, namely the set where the coefficients are allowed to have jump discontinuities. We assume that $\Gamma$ is the union of finitely many disjoint piecewise smooth curves. That means that we can make a choice of sides of $\Gamma$ and denote by $u^{+}$and by $u^{-}$the two non-tangential limits of $u$ at the two sides of the interface $\Gamma$. Also, we assume that all the vertices of the domains with a polygonal structure $\Omega_{j}$ that are on the boundary of $\Omega$ were included in the set of vertices of $\Omega$. Similarly, let us denote by $D_{\nu}^{P+}$ and $D_{\nu}^{P-}$ the two conormal derivatives associated to $P$ and the two sides of the interface. Then for $u$ smooth enough, the weak form of the boundary value problem (3.1) is made precise as the mixed boundary value/interface problem

$$
\left\{\begin{array}{cl}
P u=f & \text { in } \Omega,  \tag{3.8}\\
u=g_{D} & \\
\text { on } \partial_{D} \Omega \subset \partial^{u} \Omega, \\
D_{\nu}^{P} u=g_{N} & \\
\text { on } \partial_{N} \Omega \subset \partial^{u} \Omega, \\
u^{+}=u^{-} & \\
\text {on }^{\Gamma} \\
D_{\nu}^{P+} u=D_{\nu}^{P-} u & \\
\text { on } \Gamma,
\end{array}\right.
$$

with $g_{D}=0$, where it is crucial that $\partial_{N} \Omega$ and $\partial_{D} \Omega$ are subsets of the unfolded boundary. (If $u$ is only in $\mathcal{K}_{\overrightarrow{\mathbf{a}}+1}^{1}(\Omega)$ and is hence defined in weakly in Equation (3.7), then Equation (3.8) is not defined and is not included in the formulation of the problem. The difference $D_{\nu}^{P+} u-D_{\nu}^{P-} u$ may be non-zero, but may be included as a distributional term in $f$.)

Thus the usual transmission conditions $u^{+}=u^{-}$and $D_{\nu}^{P+} u=D_{\nu}^{P-} u$ at the two sides of the interface are a consequence of the weak formulation, and will al-
ways be considered as part of Equation (3.1). The slightly more general conditions $u^{+}-u^{-}=h_{0}$ and $D_{\nu}^{P+} u-D_{\nu}^{P-} u=h_{1}$ can be treated with only minor modifications. (See [84].) More precisely, the term $h_{0}$ is treated by extension and then by substracting from $u$ the extension function (like the term $g_{D}$ ). The term $h_{1}$ is treated by introducing in the the weak formulation also the term $\int_{\Gamma} h_{1} u d s$, where $d s$ is the arc length (measure) on $\Gamma$.

Let $\mathcal{V} \subset{ }^{u} \bar{\Omega}$ be an arbitrary subset. Let $\vartheta_{Q}(x)$ be the distance from $x$ to $Q \in \mathcal{V}$, computed using paths in ${ }^{u} \bar{\Omega}$ and let

$$
\begin{equation*}
\vartheta(x)=\prod_{Q \in \mathcal{V}} \vartheta_{Q}(x) . \tag{3.9}
\end{equation*}
$$

Let $\overrightarrow{\mathbf{a}}=\left(a_{Q}\right)$ be a vector with real components indexed by $Q \in \mathcal{V}$. We shall denote $r+\overrightarrow{\mathbf{a}}=\left(r+a_{Q}\right)$ and

$$
\begin{equation*}
\vartheta^{r+\overrightarrow{\mathbf{a}}}(x):=\prod_{Q \in \mathcal{V}} \vartheta_{Q}(x)^{r+a_{Q}}(x)=\vartheta^{r}(x) \vartheta^{\overrightarrow{\mathbf{a}}}(x) . \tag{3.10}
\end{equation*}
$$

Then we define the mth weighted Sobolev space with weights by

$$
\begin{equation*}
\mathcal{K}_{\overrightarrow{\mathbf{a}}}^{m}(\Omega):=\left\{f: \Omega \rightarrow \mathbb{C}, \vartheta^{|\alpha|-\overrightarrow{\mathbf{a}}} \partial^{\alpha} f \in L^{2}(\Omega), \text { for all }|\alpha| \leq m\right\} \tag{3.11}
\end{equation*}
$$

Let

$$
\begin{equation*}
\mathcal{H}_{\overrightarrow{\mathbf{a}}}:=\left\{u \in \mathcal{K}_{1+\overrightarrow{\mathbf{a}}}^{1}(\Omega), u=0 \text { on } \partial_{D} \Omega\right\} . \tag{3.12}
\end{equation*}
$$

We say that $P$ is coercive on $\mathcal{H}_{0}$ if there exists $\theta>0$ and $\gamma \in \mathbb{R}$ such that

$$
\begin{equation*}
B_{P}(u, u) \geq \theta(\nabla u, \nabla v)_{L^{2}(\Omega)}-\gamma\left(r_{\Omega}^{-2} u, v\right)_{L^{2}(\Omega)}, \quad \text { for all } u, v \in \mathcal{H}_{0} . \tag{3.13}
\end{equation*}
$$

If the above relation is true for some $\gamma<0$, we shall say that $P$ is strictly coercive on $\mathcal{H}_{0}$ and write $P>0$. The term "strictly positive" may be used instead of "strictly coercive" in other references.

For $m=0$, the Equation (3.8) must be understood in a weak sense as follows. First, we take $g_{D}=0$ for simplicity and we require only $u^{+}=u^{-}$, because the relation $D_{\nu}^{P+} u=D_{\nu}^{P-} u$ does not make sense for $m=0$. Then, we replace the data $\left(f, g_{N}\right)$ with an element $\Phi \in \mathcal{H}_{\overrightarrow{\mathbf{a}}}^{*}$ defined by $\Phi(u)=\int_{\Omega} f u d x+\int_{\partial_{N} \Omega} g_{N} u d S(x)$.

Our solution $u$ is then given by

$$
\begin{equation*}
B_{P}(u, v)=\Phi(v) \quad \text { for all } v \in \mathcal{H}_{\overrightarrow{\mathbf{a}}} . \tag{3.14}
\end{equation*}
$$

In order to establish regularity and solvability of (3.8), besides assuming $P$ is uniformly strongly elliptic, we also use coercive estimates. Recall $P$ is scalar. Then a regularity estimate for (3.8) is given in Theorem 3.1.1. Moreover, with zero boundary conditions on $\partial_{D} \Omega, P$ is coercive on $\mathcal{H}_{0}$. If there are no NeumannNeumann vertices and the interface $\Gamma$ is smooth, then $P$ is strictly coercive on $\mathcal{H}_{0}$.

For simplicity, the main examples of the operator $P$ considered here are $P=$ $-\Delta$ and $P=-\operatorname{div}(A \nabla), A>0$. Both $-\Delta$ and $-\operatorname{div}(a \nabla)$ are coercive on $\mathcal{H}_{0}$ for any choice of $\partial_{D} \Omega$ and $\partial_{N} \Omega$. When $\partial_{N} \Omega$ contains no adjacent sides and the interface is smooth, it can been seen that these operators are strictly coercive on $\mathcal{H}_{0}$.

### 3.1.3 Estimates for Smooth Interfaces

With the above discussions, we here state three general results on regularity and well-posedness for the boundary-value problem (3.8), which are extensions of the a priori results we obtained in Chapter 2. See also [21, 19, 60, 61, 75, 80, 82, 84] for related research. By "well-posedness" we mean "existence and uniqueness of solutions and continuos dependence on the data." These general theorems 3.1.1, 3.1.2, and 3.1.3 will be proved in Section 3.5.

Recall that for transmission problems we assume that all the vertices of the sub-domains with a polygonal structure $\Omega_{j}$ that are on the boundary of $\Omega$ are included in the set of vertices of $\Omega$. We assume an interface that is the union of finitely many disjoint, piecewise smooth curves without self-intersections.

Theorem 3.1.1. Assume that $P=-\operatorname{div} A \nabla$ is a uniformly strongly elliptic, scalar operator in divergence form on $\Omega$ with piecewise smooth coefficients (Equation (3.2)). Also, assume that $u: \Omega \rightarrow \mathbb{R}$ with $\left.u\right|_{\Omega_{j}} \in \mathcal{K}_{\overrightarrow{\mathbf{a}}+1}^{1}\left(\Omega_{j}\right)$ satisfies the boundary value/transmission problem (3.1). Let $m \geq 0$, and suppose that $g_{N} \in$ $\mathcal{K}_{\overrightarrow{\mathbf{a}}-1 / 2}^{m-1 / 2}\left(\partial_{N} \Omega\right), g_{D} \in \mathcal{K}_{\overrightarrow{\mathbf{a}+1 / 2}}^{m+1 / 2}\left(\partial_{D} \Omega\right)$, and $f: \Omega \rightarrow \mathbb{R}$ is such that $\left.f\right|_{\Omega_{j}} \in \mathcal{K}_{\overrightarrow{\mathbf{a}}-1}^{m-1}\left(\Omega_{j}\right)$.

Then $\left.u\right|_{\Omega_{j}} \in \mathcal{K}_{\overrightarrow{\mathbf{a}}+1}^{m+1}\left(\Omega_{j}\right)$, for each $j$, and we have the estimate

$$
\begin{aligned}
\|u\|_{\mathcal{K}_{\overrightarrow{\mathbf{a}}+1}^{m+1}\left(\Omega_{j}\right)} \leq C\left(\sum_{j=1}^{N}\|f\|_{\mathcal{K}_{\overrightarrow{\mathbf{a}}-1}^{m-1}\left(\Omega_{j}\right)}+\left\|g_{N}\right\|_{\mathcal{K}_{\overrightarrow{\mathbf{a}}-1 / 2}^{m-1 / 2}\left(\partial_{N} \Omega\right)}+\right. \\
\left.\left\|g_{D}\right\|_{\mathcal{K}_{\overrightarrow{\mathbf{a}}+1 / 2}^{m+1 / 2}\left(\partial_{D} \Omega\right)}+\|u\|_{\mathcal{K}_{\overrightarrow{\mathbf{a}}+1}^{0}(\Omega)}\right)
\end{aligned}
$$

for a constant $C$ that is independent of $u$ and the data $f, g_{N}$, and $g_{D}$.
Recall that $\Phi=\left(f, g_{N}\right) \in \mathcal{H}_{\overrightarrow{\mathbf{a}}}^{*}$ and that the solution $u$ is given by Equation (3.14). Then we have the following basic coercivity result. (See also Theorem 3.2.5 and 3.2.7 for Neumann-Neumann and transmission problems.)

Theorem 3.1.2. Assume that $P$ is a uniformly strongly elliptic, scalar operator on $\bar{\Omega}$. Assume also that no two adjacent sides of $\Omega$ are given Neumann boundary conditions and that the interface $\Gamma$ is smooth. Then $P$ is strongly coercive on $\mathcal{H}_{0}$ and for each vertex $Q$ of $\Omega$ there exists a positive constant $\eta_{Q}$ with the following property: for any $\Phi \in \mathcal{H}_{\overrightarrow{\mathbf{a}}}^{*}$ with $\left|a_{Q}\right|<\eta_{Q}$, there exists a unique weak solution $u \in \mathcal{K}_{\overrightarrow{\mathbf{a}}+1}^{1}(\Omega), u=0$ on $\partial_{D} \Omega$ of Equation (3.8) and we have the estimate

$$
\|u\|_{\mathcal{K}_{\overrightarrow{\mathbf{a}+1}}^{1}(\Omega)} \leq C\|\Phi\|
$$

for a constant $C=C(\overrightarrow{\mathbf{a}})$ that is independent of $\Phi$.
Combining the above two theorems, we obtain the following result.
Theorem 3.1.3. Let $m \geq 1$. In addition to the assumptions of the above theorem, assume that $g_{N} \in \mathcal{K}_{\overrightarrow{\mathbf{a}}-1 / 2}^{m-1 / 2}\left(\partial_{N} \Omega\right), g_{D} \in \mathcal{K}_{\overrightarrow{\mathbf{a}}+1 / 2}^{m+1 / 2}\left(\partial_{D} \Omega\right)$, and that $f: \Omega \rightarrow \mathbb{R}$ is such that $\left.f\right|_{\Omega_{j}} \in \mathcal{K}_{\overrightarrow{\mathbf{a}}-1}^{m-1}\left(\Omega_{j}\right)$. Then the solution $u \in \mathcal{K}_{\overrightarrow{\mathbf{a}}+1}^{1}(\Omega)$ of Equation (3.8) satisfies $\left.u\right|_{\Omega_{j}} \in \mathcal{K}_{\overrightarrow{\mathbf{a}+1}}^{m+1}\left(\Omega_{j}\right)$ for all $j$ and we have the estimate

$$
\|u\|_{\mathcal{K}_{\overrightarrow{\mathbf{a}}+1}^{m+1}\left(\Omega_{j}\right)} \leq C\left(\sum_{j}\|f\|_{\mathcal{K}_{\overrightarrow{\mathbf{a}}-1}^{m-1}\left(\Omega_{j}\right)}+\left\|g_{N}\right\|_{\mathcal{K}_{\overrightarrow{\mathbf{a}}-1 / 2}^{m-1 / 2}\left(\partial_{N} \Omega\right)}+\left\|g_{D}\right\|_{\mathcal{K}_{\overrightarrow{\mathbf{a}}+1 / 2}^{m+1 / 2}\left(\partial_{D} \Omega\right)}\right)
$$

See Theorems 3.2.5 and 3.2.7 for well-posedness results that allow general interfaces and Neumann-Neumann vertices (but require an augmented domain for the operator).

If $P=-\sum_{i, j=1}^{2} \partial_{j} a^{i j} \partial_{i}+\sum_{i=1}^{2} b^{i} \partial_{i}+c$, lower order coefficients are included, then we need additional assumptions on $b^{i}$ and $c$ for the coercivity of $P$, while the regularity results still hold since low order terms contribute much less than the leading term if those coefficients satisfy certain regularity.

We note that continuous dependence of the solution on the data immediately follows from the estimate above since the boundary-value problem is linear.

The reason for the condition that $\partial_{N} \Omega$ contain no adjacent sides, is that then we can establish the weighted form of Poincaré inequality $\int_{\Omega}|\nabla u|^{2} d x \geq C \int_{\Omega} \vartheta^{-2}|u|^{2}$ for all $u \in \mathcal{H}_{0}$. (See Lemma 3.5.1 and Remark 3.3.5.) This inequality shows immediately that $P$ is strictly coercive on $\mathcal{H}_{0}:=\mathcal{K}_{1}^{1}(\Omega) \cap\left\{\left.u\right|_{\partial_{D} \Omega}=0\right\}$ if there are no Neumann-Neumann vertices and the interface is smooth. In fact, in the occurrence of either of the above cases, the solution for Equation (3.8) may not even be in $\mathcal{H}_{0}$.

Let us denote by $\tilde{P} v=\left(\left.\oplus P\right|_{\Omega_{j}}, D_{\nu}^{P}\right) v:=\left(\left.P v\right|_{\Omega_{1}}, \ldots,\left.P v\right|_{\Omega_{N}}, D_{\nu}^{P} v\right)$, decorated with various indices. As a corollary, we obtain the following isomorphism.

Corollary 3.1.4. Under the assumptions of the above two theorems and $m \geq 1$, we have that the map

$$
\begin{aligned}
& \tilde{P}_{m, \overrightarrow{\mathbf{a}}}:=\left(\left.\oplus P\right|_{\Omega_{j}}, D_{\nu}^{P}\right):\left\{u: \Omega \rightarrow \mathbb{R},\left.u\right|_{\Omega_{j}} \in \mathcal{K}_{\overrightarrow{\mathbf{a}}+1}^{m+1}\left(\Omega_{j}\right), u=0 \text { on } \partial_{D} \Omega\right. \\
&\left.u^{+}=u^{-} \text {and } D_{\nu}^{P+} u=D_{\nu}^{P-} u \text { on } \Gamma\right\} \rightarrow \oplus_{j} \mathcal{K}_{\overrightarrow{\mathbf{a}}-1}^{m-1}\left(\Omega_{j}\right) \oplus \mathcal{K}_{\overrightarrow{\mathbf{a}}-1 / 2}^{m-1 / 2}\left(\partial_{N} \Omega\right)
\end{aligned}
$$

is an isomorphism for $\left|a_{Q}\right|<\eta_{Q}$.
The above three theorems extend to the case of polyhedral domain in three dimensions using same proofs. Moreover, our treatment in weighted Sobolev spaces for Neumann problems and non-smooth interfaces will be discussed in the section. We here briefly mention that the Neumann problem in three dimensions, however, requires significantly more work than in two dimensions.

### 3.2 Estimates of Neumann Problems and Singular Transmission Problems

Before proceeding to the proofs of the previous three theorems, we include some observations for elliptic equations involving Neumann-Neumann corners and nonsmooth interfaces. For simplicity, we consider the Laplace operator (when there are no interfaces) and $-\operatorname{div} A \nabla$, with $A$ piecewise constant if there are interfaces (so on each subdomain, the operator is still a multiple of the Laplace operator).

### 3.2.1 The Laplace Operator

In the case $P=-\Delta$, it is then possible to explicitly determine the values of the constants $\eta_{Q}$ appearing in Theorems 3.1.2 and 3.1.3 for the isomorphism. In this subsection, we let $P=-\Delta$ and assume there are no interfaces, that is, $\Omega=\Omega_{1}$.

Recall that a bounded operator $A: X \rightarrow Y$ between Banach spaces is Fredholm if the kernel of $A$ (that is the space $\operatorname{ker}(A):=\{A x=0\}$ ) and $Y / A X$ are finite dimensional spaces. For a Fredholm operator $A$, the index is defined by the formula $\operatorname{ind}(A)=\operatorname{dim} \operatorname{ker}(A)-\operatorname{dim}(Y / A X)$.

Let $\mathcal{V}:=\left\{Q_{i}\right\}$ be the vertex set. We let $k \in \mathbb{Z}$ if $Q \in \mathcal{V}$ is a Neumann-Neumann vertex, $k \in \mathbb{Z} \backslash\{0\}$ if $Q \in \mathcal{V}$ is a Dirichlet-Dirichlet vertex, and $k \in 1 / 2+\mathbb{Z}$ otherwise for points where the boundary condition changes. We then let $\alpha_{Q}$ be the interior angle at $Q\left(\alpha_{Q}=2 \pi\right.$ if $Q$ is the tip of a crack $)$ and define

$$
\begin{equation*}
\Sigma_{Q}:=\left\{k \pi / \alpha_{Q}\right\}, \tag{3.15}
\end{equation*}
$$

with $k$ as explained. Let $r_{Q}$ be the distance to the vertex $Q$ and $a_{Q}$ be the component of $\overrightarrow{\mathbf{a}}$ corresponding to $Q$. Based on Lemma 2.2.4, a necessary condition for the isomorphism $-\Delta: \mathcal{K}_{\overrightarrow{\mathbf{a}}+1}^{m+1}(\Omega) \cap\left\{\left.u\right|_{\partial_{D} \Omega}=0\right\} \rightarrow \mathcal{K}_{\overrightarrow{\mathbf{a}}-1}^{m-1}(\Omega)$ is that $-r_{Q}^{-a_{Q}} \Delta r_{Q}^{a_{Q}}$ : $\mathcal{K}_{1}^{m+1}\left(C_{Q}\right)$ +original boundary conditions $\rightarrow \mathcal{K}_{-1}^{m-1}\left(C_{Q}\right)$ defines an isomorphism for each infinite conical domain $C_{Q}$ whose sides coincide with the tangent lines of the boundary $\partial \Omega$ at vertex $Q$ (Figure 2.1).

Recall the Mellin transform (Definition 2.1.1). The resulting operator pencil $P_{Q}(\tau)$ (or indicial family) associated to $-\Delta$ at $Q$ is $P_{Q}(\tau):=(\tau-\imath \epsilon)^{2}-\partial_{\theta}^{2}$, where $\imath=\sqrt{-1}$. The operator $P_{Q}(\tau)$ is defined on functions in $H^{2}\left(\left[0, \alpha_{Q}\right]\right)$ that satisfy
the given boundary conditions, and is obtained by evaluating

$$
\begin{equation*}
P_{Q}(\tau) \hat{\phi(\theta)}:=-\left(r_{Q}^{-\epsilon+\widehat{2} \widehat{r_{Q}^{\epsilon}}} \phi(\theta)\right)=\left((\tau-\imath \epsilon)^{2}-\partial_{\theta}^{2}\right) \hat{\phi(\theta)}, \tag{3.16}
\end{equation*}
$$

where $r_{Q}$ denotes the distance to $Q$ and $\hat{\phi}$ is the Fourier transform of $\phi(\theta)$ with respect to $t=\ln r_{Q}$ on the infinite strip after the Mellin transform. (See Chapter 2 for details.) Thus, since the Mellin transform is a one-to-one map, the non-trivial solution of the equation $\left.P_{Q}(\tau) \phi \hat{( } \theta\right)=0$ with the original boundary conditions will destroy the isomorphism $-r_{Q}^{-a_{Q}} \Delta r_{Q}^{a_{Q}}: \mathcal{K}_{1}^{m+1}\left(C_{Q}\right)+$ original boundary conditions $\rightarrow$ $\mathcal{K}_{-1}^{m-1}\left(C_{Q}\right)$ as discussed above. Note the eighenvalues of $-\partial_{\theta}^{2}$ with homogeneous boundary conditions (Dirichlet or Neumann) on $H^{2}\left[0, \alpha_{Q}\right]$ is given by the set $\Sigma_{Q}$ with corresponding values of $k$. Then $P(\tau)$ is invertible for all $\tau \in \mathbb{R}$, as long as $\epsilon \notin \Sigma_{Q}$.

In the following theorem we shall consider the operator

$$
\begin{equation*}
\tilde{\Delta}_{\overrightarrow{\mathbf{a}}}:=\left(\Delta, \partial_{\nu}\right): \mathcal{K}_{\overrightarrow{\mathbf{a}}+1}^{m+1}(\Omega) \cap\left\{\left.u\right|_{\partial_{D} \Omega}=0\right\} \rightarrow \mathcal{K}_{\overrightarrow{\mathbf{a}}-1}^{m-1}(\Omega) \oplus \mathcal{K}_{\overrightarrow{\mathbf{a}}-1 / 2}^{m-1 / 2}\left(\partial_{N} \Omega\right) \tag{3.17}
\end{equation*}
$$

defined first for $m \geq 1$. We can extend its definition to $m=0$ as follows. Let $\mathcal{H}_{\overrightarrow{\mathbf{a}}}:=\left\{u \in \mathcal{K}_{\overrightarrow{\mathbf{a}}+1}^{1}(\Omega), u=0\right.$ on $\left.\partial_{D} \Omega\right\}$, as before. For $m=0$

$$
\begin{equation*}
\tilde{\Delta}_{\overrightarrow{\mathbf{a}}}: \mathcal{H}_{\overrightarrow{\mathbf{a}}} \rightarrow\left(\mathcal{H}_{\overrightarrow{\mathbf{a}}}\right)^{*}, \quad\left(\tilde{\Delta}_{\overrightarrow{\mathbf{a}}} u, v\right):=-(\nabla u, \nabla v), \tag{3.18}
\end{equation*}
$$

for $u \in \mathcal{H}_{\overrightarrow{\mathbf{a}}}$ and $v \in \mathcal{H}_{-\overrightarrow{\mathbf{a}}}$ (Lemma 2.2.10). (This definition also extends to transmission problems.)

The following result is due to Theorem 2.2.12. See also Kondratiev [60], when there are no interfaces, and Nicaise [82] for the case of transmission problems.

Theorem 3.2.1. Let $P=-\Delta, m \geq 0$, and $\overrightarrow{\mathbf{a}}=\left(a_{Q}\right)$. Also, let $\tilde{\Delta}_{\overrightarrow{\mathbf{a}}}$ be the operator defined in Equations (3.17) and (3.18). Then $\tilde{\Delta}_{\overrightarrow{\mathbf{a}}}$ is Fredholm if, and only if, $a_{Q} \notin \Sigma_{Q}$. Moreover, its index is independent of $m$.

Proof. The Fredholm criterion has been proved in Theorem 2.2.12 for the homogeneous Dirichlet boundary conditions. For mixed and Neumannn-Neumann corners, the proof works exactly in the same way. That is, once there is a unique solution on the infinite strip after the Mellin transform (operator pencils avoid the set $\Sigma$
for all vertices), $\tilde{\Delta}_{\overrightarrow{\mathbf{a}}}$ is Fredholm from a regularity estimate. See also [60, 65, 91]. We now prove that the index is independent of $m$.

Indeed, if $u \in \mathcal{H}_{\overrightarrow{\mathbf{a}}}$ is such that $\Delta_{\overrightarrow{\mathbf{a}}} u=0$, Lemma 2.2.11 and Theorem 3.1.1 imply that $u \in \mathcal{K}_{\overrightarrow{\mathbf{a}}+1}^{\infty}(\Omega)$. Hence, the dimension of $\operatorname{ker}\left(\tilde{\Delta}_{\overrightarrow{\mathbf{a}}}\right)$ is independent of $m$. Note the range of the operator $\Delta: \mathcal{K}_{a+1}^{m+1}(\Omega)\left\{\left.u\right|_{\partial_{D} \Omega}=0,\left.\partial_{\nu} u\right|_{\partial_{N} \Omega}=g_{N}\right\}$ can be characterized by

$$
R(\Delta)=\left\{f \in \mathcal{K}_{a-1}^{m-1}(\Omega), \phi_{j}(f)=0, \phi_{j} \in \mathcal{K}_{a-1}^{-1}(\Omega)^{*} \text { form the basis of } \operatorname{ker}\left(\Delta^{*}\right)\right\}
$$

where $\Delta^{*}$ is the dual operator of $\Delta$ for $m=0$. Then the dimension of the co-kernel $\mathcal{K}_{a-1}^{m-1}(\Omega) / R(\Delta)=$ the dimension of $\phi_{i}$, which is independent of $m$ as well. In fact, $\phi_{i} \in \mathcal{K}_{1-a}^{\infty}(\Omega)$ from the same observation for the adjoint problem.

The reason we are interested in the case $m=0$ is that as in Lemma 2.2.10, we have

$$
\begin{equation*}
\left(\tilde{\Delta}_{\overrightarrow{\mathbf{a}}}\right)^{*}=\tilde{\Delta}_{-\overrightarrow{\mathbf{a}}} \tag{3.19}
\end{equation*}
$$

It is possible to determine the index of the operators $\tilde{\Delta}_{\overrightarrow{\mathbf{a}}}$ using Equation (3.19) and the following index calculation. We fix two multi-indices $\overrightarrow{\mathbf{a}}=\left(a_{Q}\right)$ and $\overrightarrow{\mathbf{b}}=$ $\left(b_{Q}\right)$ that yield Fredholm operators in Theorem 3.2.1 and differ at exactly one position $Q: a_{Q}<b_{Q}$ but $a_{R}=b_{R}$ if $R \neq Q$. Let us count the number of values in the set $\left(a_{Q}, b_{Q}\right) \cap \Sigma_{Q}$, with the values corresponding to $k=0$ in the definition of $\Sigma_{Q}$, Equation (3.15), counted twice (because of multiplicity, which happens only in the case of Neumann-Neumann boundary conditions). Let $N$ be the total number.

The following result can be found in [82]. See also [40, 60, 61, 77, 80].
Theorem 3.2.2. Assume the conditions of Theorem 3.2.1 are satisfied. Then

$$
\operatorname{ind}\left(\tilde{\Delta}_{\overrightarrow{\mathbf{b}}}\right)-\operatorname{ind}\left(\tilde{\Delta}_{\overrightarrow{\mathbf{a}}}\right)=-N
$$

This allows to determine the index of $\tilde{\Delta}_{\vec{a}}$. For simplicity, we do that only for $a_{Q}>0$ and small. Let $\delta_{Q}$ be the minimum values of $\Sigma_{Q} \cap(0, \infty)$. Let $\alpha_{Q} \in(0,2 \pi]$ be the angle of $\Omega$ at $Q$. Then $\delta_{Q}=\pi / \alpha_{Q}$, if both sides meeting at $Q$ are assigned the same type of boundary conditions, and by $2 \delta_{Q}=\pi / \alpha_{Q}$ otherwise.

Theorem 3.2.3. Assume the conditions of Theorem 3.2.1 are satisfied and let $N_{0}$ be the number of vertices $Q$ such that both sides adjacent to $Q$ are assigned Neumann boundary conditions. Let $m \geq 0$. Then $\tilde{\Delta}_{a}$ is Fredholm for $0<a_{Q}<\delta_{Q}$ with index

$$
\operatorname{ind}\left(\tilde{\Delta}_{\overrightarrow{\mathbf{a}}}\right)=-N_{0}
$$

Consequently, $\tilde{\Delta}_{-\overrightarrow{\mathbf{a}}}$ has index $-N_{0}$ for $0<a_{Q}<\delta_{Q}$.
For transmission problems, we shall count in $N_{0}$ also the points where the interface $\Gamma$ is not smooth.

Proof. Since the index is independent of $m$, we can assume that $m=0$. A repeated application of Theorem 3.2.2 (more precisely of its generalization for $m=0$ ) gives that $\operatorname{ind}\left(\tilde{\Delta}_{a}\right)-\operatorname{ind}\left(\tilde{\Delta}_{-a}\right)=-2 N_{0}$ (each time when we change an index from $-a$ to $a$ we lose a 2 in the index, because the value $k=0$ is counted twice). Since $\tilde{\Delta}_{-a}=\tilde{\Delta}_{a}^{*}$, we have $\operatorname{ind}\left(\tilde{\Delta}_{-a}\right)=-\operatorname{ind}\left(\tilde{\Delta}_{a}\right)$. This yields the result.

We now proceed to a more careful study of the invertibility properties of $\tilde{\Delta}_{\vec{a}}$. In particular, we shall determine the constants $\eta_{Q}$ appearing in Theorems 3.1.2 and 3.1.3.

Let us chose for each vertex $Q \in \mathcal{V}$ a function $\chi_{Q} \in \mathcal{C}^{\infty}(\bar{\Omega})$ that is constant equal to 1 in a neighborhood of $Q$ and satisfies $\partial_{\nu} \chi_{Q}=0$ on the boundary. We can chose these functions to have disjoint supports. Let $W_{s}$ be the linear span of the functions $\chi_{Q}$ that correspond to Neumann-Neumann vertices $Q$. (For transmission problems, we have to take into account also the points where the interface $\Gamma$ is not smooth.)

We shall need the following lemma (Green's formula for suitable functions).
Lemma 3.2.4. Assume all $a_{Q} \geq 0$ and $u, v \in \mathcal{K}_{\overrightarrow{\mathbf{a}}+1}^{2}(\Omega)+W_{s}$. Then

$$
(\Delta u, v)+(\nabla u, \nabla v)=\left(\partial_{\nu} u, v\right)_{\partial \Omega} .
$$

Proof. If $u$ and $v$ are actually constant close to the corners, then we can round off the corners without changing the terms of the formulas, and hence we obtain the result, since Green's formula is known for smooth domains and $u$ and $v$ are in $H^{2}$. In general, we notice that $C(u, v):=(\Delta u, v)+(\nabla u, \nabla v)=\left(\partial_{\nu} u, v\right)_{\partial \Omega}$ depends
continuously on $u$ and $v$ (this is where we use the condition $a_{Q} \geq 0$ in our proof) and we can use a density argument as in Lemma 2.2.10.

The following is the main result for the Neumann problems.
Theorem 3.2.5. Let $\overrightarrow{\mathbf{a}}=\left(a_{Q}\right)$ with $0<a_{Q}<\delta_{Q}$ and $m \geq 1$. Assume we are given Dirichlet boundary conditions on at least one side of $\Omega$. Then for any $f \in \mathcal{K}_{\overrightarrow{\mathbf{a}}-1}^{m-1}(\Omega)$ and any $g_{N} \in \mathcal{K}_{\overrightarrow{\mathbf{a}}-1 / 2}^{m-1 / 2}(\Omega)$, we can find a unique $u=u_{\mathrm{reg}}+w_{s}$, $u_{\mathrm{reg}} \in \mathcal{K}_{\overrightarrow{\mathbf{a}}+1}^{m+1}(\Omega)$, $w_{s} \in W_{s}$ satisfying $-\Delta u=f, u=0$ on $\partial_{D} \Omega$, and $\partial_{\nu} u=g_{N}$ on $\partial_{N} \Omega$. Moreover, we have the following continuity result

$$
\left\|u_{\mathrm{reg}}\right\|_{\mathcal{K}_{\overrightarrow{\mathbf{a}+1}}^{m+1}(\Omega)}+\left\|w_{s}\right\| \leq C\left(\|f\|_{\substack{\overrightarrow{\mathbf{a}}-1 \\ m-1 \\ \hline}}+\left\|g_{N}\right\|_{\mathcal{K}_{\overrightarrow{\mathbf{a}}-1 / 2}^{m-1 / 2}(\Omega)}\right)
$$

for a constant $C>0$ independent of $f$ and $g_{N}$. The same result remains true for the pure Neumann problem if we factor the constant functions, as usual.

Proof. We can reduce right away to the case when $g_{D}=0$ and $g_{N}=0$.
Let $V=\left\{u \in \mathcal{K}_{\overrightarrow{\mathbf{a}}+1}^{m+1}(\Omega),\left.u\right|_{\partial_{D} \Omega}=0,\left.\partial_{\nu} u\right|_{\partial_{N} \Omega}=0\right\}+W_{s}$. Let $\Delta_{\mathcal{K}}:=\Delta:\{u \in$ $\left.\mathcal{K}_{\overrightarrow{\mathbf{a}}+1}^{m+1}(\Omega),\left.u\right|_{\partial_{D} \Omega}=0, \partial_{\nu} u=0\right\} \rightarrow \mathcal{K}_{a-1}^{m-1}(\Omega)$, and $\Delta_{W}:=\Delta: W_{s} \rightarrow \mathcal{K}_{a-1}^{m-1}(\Omega)$. We first notice that the map

$$
\begin{equation*}
\Delta: V \rightarrow \mathcal{K}_{\overrightarrow{\mathbf{a}}-1}^{m-1}(\Omega)=\Delta_{\mathcal{K}} \oplus \Delta_{W}=\Delta_{\mathcal{K}} \oplus 0+0 \oplus \Delta_{W} \tag{3.20}
\end{equation*}
$$

is well defined and continuous $(m \geq 1)$, where $\Delta_{\mathcal{K}} \oplus 0$ is Fredholm with index $=N_{0}-N_{0}=0$, since the dimension of its kernel $W_{s}$ is $N_{0}$. Note $0 \oplus \Delta_{W}$ is a compact operator. Therefore $\Delta_{\mathcal{K}} \oplus 0+0 \oplus \Delta_{W}$ is also Fredholm with index 0 . (ind $\left(\Delta_{\mathcal{K}} \oplus 0+0 \oplus \Delta_{W}\right)=\operatorname{ind}\left(\Delta_{\mathcal{K}} \oplus 0\right)$, since there exits a Fredholm operator $R$, such that $R\left(\Delta_{\mathcal{K}} \oplus 0+0 \oplus \Delta_{W}\right)=I+A$, with a compact operator $A$, whose index is 0 .)

We shall show that this map is actually an isomorphism if there is at least one Dirichlet side. To this end, it is enough to show that this map is injective.

To this end, we shall use Green's formula $(\Delta u, v)+(\nabla u, \nabla v)=\left(\partial_{\nu} u, v\right)_{\partial \Omega}$, which applies to $u, v \in \mathcal{K}_{\overrightarrow{\mathbf{a}}+1}^{m+1}(\Omega)+W_{s}$, by Lemma 3.2.4. If $u \in V$ is such that $\Delta u=0$, then the usual energy argument applies to show that $u$ is a constant. (Use Green's formula for $v=u$ to conclude that $\nabla u=0$.)

If there is at least one Dirichlet side, the constant must be zero, and hence $u=0$. In the pure Neumann case, we get that the kernel of the map of Equation (3.20) consists of scalars. Another application of Green's formula shows that $(\Delta u, 1)=0$, which identifies the range of $\Delta$ as the functions with mean zero (in the pure Neumann case).

Assume that all components of $\overrightarrow{\mathbf{a}}$ are non-negative (we write $\overrightarrow{\mathbf{a}} \geq 0$ ). The same argument as in the above theorem also shows that $\tilde{\Delta}_{\vec{a}}$ is injective. Using also Equation (3.19), we then see that $\tilde{\Delta}_{-\overrightarrow{\mathbf{a}}}$ is surjective whenever it is Fredholm. This implies Theorem 3.1.2 for $\overrightarrow{\mathbf{a}}=0$. (Note that $\tilde{\Delta}_{0}$ is Fredholm precisely when there are no Neumann-Neumann faces. For operators of the form $-\operatorname{div} A \nabla$ with $A$ piecewise smooth, one has to assume also that the interface $\Gamma$ is smooth, or otherwise the Fredholm property for the critical weight $\overrightarrow{\mathbf{a}}=0$ is lost.)

We can now determine the constants $\eta_{Q}$ in Theorems 3.1.2 and 3.1.3, in which no adjacent Neumann boundary conditions are allowed.

Theorem 3.2.6. Assume $P=-\Delta$. Then we can take $\eta_{Q}=\delta_{Q}$ in Theorem 3.1.2. Proof. Assume that $\left|a_{Q}\right|<\eta_{Q}$. Then $\tilde{\Delta}_{\overrightarrow{\mathbf{a}}}$ is Fredholm of index zero, because the index is homotopy invariant and $\tilde{\Delta}_{\overrightarrow{\mathbf{a}}}$ is invertible when all $a_{Q}=0$, by the above discussion. The invertibility of $\tilde{\Delta}_{\vec{a}}, a_{Q}=0$ is proved in Section 3.5 , based on a Poincaré-like inequality in weighted Sobolev spaces. Assume $\tilde{\Delta}_{\overrightarrow{\mathbf{a}}} u=0$ for some $u \in \mathcal{H}_{\overrightarrow{\mathbf{a}}}$. Then the singular function expansion of $u$ close to each vertex gives that $u \in \mathcal{H}_{\overrightarrow{\mathbf{b}}}$ for all $\overrightarrow{\mathbf{b}}=\left(b_{Q}\right)$ with $0<b_{Q}<\eta_{Q}[61,82]$ (the value $\eta_{Q}$ is the exponent $s$ of the first singular function $r^{s} \phi(\theta)$, in polar coordinates centered at $Q$ ). Since $\tilde{\Delta}_{\overrightarrow{\mathbf{b}}}$ is injective for $b_{Q}>0$ (Theorem 3.2.5 for the case $w_{s}=0$ ), we obtain that $\tilde{\Delta}_{\overrightarrow{\mathbf{a}}}$ is injective for $\left|a_{Q}\right|<\eta_{Q}$. Since it is Fredholm of index zero, it must be, in fact, an isomorphism.

### 3.2.2 Transmission Problems

The results of the previous section remain valid for general operators and transmission problems with $\bar{\Omega}=\cup \bar{\Omega}{ }_{j}$, with a different (more complicated) definition of the sets $\Sigma_{Q}$. We consider only the case $P=-\operatorname{div} A \nabla u$, where $A$ is a piecewise constant function. Recall that for transmission problems we assume that all the
vertices of the domains with a polygonal structure $\Omega_{j}$ that are on the boundary of $\Omega$ are included in the set of vertices $\mathcal{V}$ of $\Omega$.

Then the set $\Sigma_{Q}$, when $Q$ is a vertex, is determined by $\{ \pm \sqrt{\lambda}\}$, where $\lambda$ ranges through the set of eigenvalues of $-\partial_{\theta} A \partial_{\theta}$ on $H^{2}\left(\left[0, \alpha_{Q}\right]\right)$ and suitable boundary conditions. For $Q$ an internal singular point, we consider the operator $-\partial_{\theta} A \partial_{\theta}$ on $H^{2}([0,2 \pi])$ with periodic boundary conditions. We still take $\eta_{Q}>0$ to be the least value in $\Sigma_{Q} \cap(0, \infty)$. It works in the same way as finding the eigenvalues of operator pencils discussed in the previous subsection.

For $m=0$ (global regularity), the operator $P$ is given by a similar formula to Equation (3.18), namely $(P u, v)=(A \nabla u, \nabla v)$. Let $A^{+}$and $A^{-}$be the limit values of $A$ at the two sides of the interface $\Gamma$. The only other value of $m$ that we can directly use is $m=1$, in which case we also have to take into account the transmission conditions $u^{+}=u^{-}$and $A^{+} \partial_{\nu}^{+} u=A^{-} \partial_{\nu}^{-} u$ at each interface. (For higher values of $m$ we get additional conditions at the interface. For $m=0$ the normal derivative at the boundary is not defined.) In view of Corollary 3.1.4, the definition then becomes

$$
\begin{align*}
& P_{\nu}:=\left(P, \partial_{\nu}\right): \mathcal{D}_{\overrightarrow{\mathbf{a}}} \rightarrow \mathcal{K}_{\overrightarrow{\mathbf{a}}-1}^{0}(\Omega) \oplus \mathcal{K}_{\overrightarrow{\mathbf{a}}-1 / 2}^{1 / 2}\left(\partial_{N} \Omega\right) \\
& \qquad \mathcal{D}_{\overrightarrow{\mathbf{a}}}:=\left\{u: \Omega \rightarrow \mathbb{R},\left.u\right|_{\Omega_{j}} \in \mathcal{K}_{\overrightarrow{\mathbf{a}}+1}^{2}\left(\Omega_{j}\right),\left.u\right|_{\partial_{D} \Omega}=0\right. \\
&  \tag{3.21}\\
& \left.\quad u^{+}=u^{-}, \text {and } A^{+} \partial_{\nu}^{+} u=A^{-} \partial_{\nu}^{-} u\right\}
\end{align*}
$$

The theorems of the previous subsection then remain true for our transmission problem (recall that we consider only locally constant coefficients), with the following changes. In Theorem 3.2.1, we take only $m=0$ or $m=1$. In Theorem 3.2.3, we again assume only $m=0$ or $m=1$ and in $N_{0}$ we also count the number of internal vertices. The proofs are as in Kondratiev's paper [60]. Theorem 3.2.2 is essentially unchanged (in particular, we continue to count twice $0 \in\left(a_{Q}, b_{Q}\right) \cap \Sigma_{Q}$ ). (So $N_{0}$ is the number of Neumann-Neumann vertices plus the number of the vertices arising at interfaces, but not on the boundary.)

We here state explicitly the form of Theorem 3.2.5, which will be needed in applications. In the following statement, $W_{s}$ is the linear span of the functions $\chi_{Q}$ with $Q$ corresponding to Neumann-Neumann vertices and to the internal vertices. We require that all the functions $\chi_{Q}$ have disjoint supports. Also, recall that for
each Neumann-Neumann vertex $Q$, the function $\chi_{Q}$ satisfies $\chi_{Q}=0$ on $\partial_{D} \Omega$ and $\partial_{\nu} \chi_{Q}=0$ on $\partial_{N} \Omega$. However, the functions $\chi_{Q}$ corresponding to internal vertices $Q$ need not satisfy any boundary conditions.

Theorem 3.2.7. Let $\overrightarrow{\mathbf{a}}=\left(a_{Q}\right)$ with $0<a_{Q}<\delta_{Q}$ and $m \geq 1$. Assume we are given Dirichlet boundary conditions on at least one side of $\Omega$. Then for any $f: \Omega \rightarrow \mathbb{R}$ such that $\left.f\right|_{\Omega_{j}} \in \mathcal{K}_{\overrightarrow{\mathbf{a}}-1}^{m-1}\left(\Omega_{j}\right)$, for all $j$, and any $g_{N} \in \mathcal{K}_{\overrightarrow{\mathbf{a}}-1 / 2}^{m-1 / 2}\left(\partial_{N} \Omega\right)$, we can find a unique $u=u_{\mathrm{reg}}+w_{s}, u_{\mathrm{reg}}: \Omega \rightarrow \mathbb{R},\left.u_{\mathrm{reg}}\right|_{\Omega_{j}} \in \mathcal{K}_{\overrightarrow{\mathrm{a}}+1}^{m+1}\left(\Omega_{j}\right)$, $w_{s} \in W_{s}$ satisfying $-\operatorname{div} A \nabla u=f, u=0$ on $\partial_{D} \Omega, \partial_{\nu} u=g_{N}$ on $\partial_{N} \Omega$, and the transmission conditions $u^{+}=u^{-}$and $A^{+} \partial_{\nu}^{+} u=A^{-} \partial_{\nu}^{-} u$ on the interface $\Gamma$. Moreover, we have the following continuity result

$$
\sum_{j}\left\|u_{\mathrm{reg}}\right\|_{\mathcal{K}_{\overrightarrow{\mathbf{a}}+1}^{m+1}\left(\Omega_{j}\right)}+\left\|w_{s}\right\| \leq C\left(\sum_{j}\|f\|_{\mathcal{K}_{\overrightarrow{\mathbf{a}}-1}^{m-1}\left(\Omega_{j}\right)}+\left\|g_{N}\right\|_{\mathcal{K}_{\overrightarrow{\mathbf{a}}-1 / 2}^{m-1 / 2}\left(\partial_{N} \Omega\right)}\right)
$$

for a constant $C>0$ independent of $f$ and $g_{N}$. The same result remains true for the pure Neumann problem if we factor the constant functions, as usual.

Proof. Assume first $m=1$. Then the same proof as that of Theorem 3.2.5 applies. (We need to restrict to the case $m=1$ to have the previous results available, which require $m=0,1$. We need then $m=1$ to be able to restrict to the boundary and apply Green's formula.) For the other values of $m$ we use the case $m=1$ to show the existence of a solution and then use the regularity result of Theorem 3.1.1.

Remark 3.2.8. Here are a few simple observations. Any norm can be used on the finite space $W_{s}$ (they are all equivalent because $W_{s}$ is finite dimensional). Also, We notice that $W_{s} \cap \mathcal{K}_{\overrightarrow{\mathbf{a}}+1}^{1}(\Omega)=0$, whenever $a_{Q}>0$ for any Neumann-Neumann vertex $Q$ or internal $Q$. The condition $a_{Q} \in\left(0, \eta_{Q}\right)$ can be relaxed to $\left|a_{Q}\right|<\eta_{Q}$ for the vertices that are either Dirichlet-Dirichlet or Dirichlet-Neumann. We can also increase $a_{Q}$, provided that we include more singular functions.

### 3.3 Domains with a Polygonal Structure

In this section, we shall introduce the class of domains $\Omega$ with a polygonal structure and define the weight function $r_{\Omega}$, as well as the space $\mathcal{C}^{\infty}(\Sigma \Omega) \subset \mathcal{C}^{\infty}(\Omega)$, which is the right space of smooth functions for our purposes. We have briefly mentioned
this type of domains in Section 3.1, and virtually made statements in these domains thereafter.

In order to deal with cracks and vertices touching a smooth face, we need to introduce a refinement of the boundary $\partial \Omega$ and of the closure $\bar{\Omega}$ to take into account the direction from which we approach the boundary. This leads to the notions of "unfolded boundary" $\partial^{u} \Omega$ and "unfolded closure" ${ }^{u} \bar{\Omega}$ of an open set $\Omega$ with a polygonal structure. These concepts were introduced by Dauge [45] in the more general framework of "corner domains."

### 3.3.1 The Unfolded Boundary and Closure

Recall that the boundary $\partial A$ of an open set $A$ in a topological space is defined by $\partial A:=\bar{A} \backslash A$, that is, the points of the closure of $A$ that are not contained in $A$. For our purposes, in order to deal for example with domains with cracks, it will be important to specify from which direction we approach the boundary. This is necessary when the domain $\Omega$ is such that $\partial \Omega \neq \partial \bar{\Omega}$, so that $\Omega$ is on both sides of parts of the boundary. (See Figure 3.3 for example.) For this reason, we now introduce the "unfolded boundary" of a domain with a simple boundary.

As usual, we denote by $B^{k}(x ; r) \subset \mathbb{R}^{k}, k=1,2$, the open unit ball of center $x$ and radius $r>0$, and with $S^{k-1}(x, r)$ the sphere centered at $x$ with radius $r$. We denote the unit ball and unit sphere centered at 0 in $\mathbb{R}^{k}$ respectively with $B^{k}$ and $S^{k-1}$. Thus $S^{0}=\{-1,1\}$ and $S^{1}$ is the unit circle. The set of connected components of an open set $\omega$ will be denoted by $\pi_{0}(\omega)$.

Definition 3.3.1. Let $\Omega \subset \mathbb{R}^{k}$ be an open set. We shall say that $\Omega$ has a simple boundary if for every $x \in \partial \Omega$ there exists $r_{x}>0$ such that the open set $B^{k}\left(x ; r_{x}\right) \cap \Omega$ has finitely many connected components and the inclusion $B^{k}(x ; r) \cap$ $\Omega \subset B^{k}\left(x ; r_{x}\right) \cap \Omega$ induces a bijection $\pi_{0}\left(B^{k}(p ; r) \cap \Omega\right) \simeq \pi_{0}\left(B^{k}\left(x ; r_{x}\right) \cap \Omega\right)$ of connected components for all $0<r<r_{x}$. If $\Omega$ has simple boundary, then we define the set

$$
\partial^{u} \Omega:=\bigcup_{x \in \partial \Omega}\left(\{x\} \times \pi_{0}\left(B^{k}\left(x ; r_{x}\right) \cap \Omega\right)\right)
$$

and call it the unfolded boundary of $\Omega$. We shall also denote by ${ }^{u} \bar{\Omega}:=\Omega \cup \partial^{u} \Omega$ the unfolded closure of $\Omega$.

It follows from the definition of the unfolded boundary that every point in $\partial^{u} \Omega$ is a pair $X=(x, U)$, where $x \in \partial \Omega$ and $U$ is one of the connected components of $B^{k}\left(x ; r_{x}\right) \cap \Omega$.

The unfolded closure ${ }^{u} \bar{\Omega}$ has a natural topology that makes $\Omega$ an open subset of ${ }^{u} \bar{\Omega}$ and induces the same topology on it. To define this topology we now introduce a basis $V(x, U, r), r>0$, for the system of neighborhoods of an arbitrary point $X=(x, U) \in \partial^{u} \Omega$. Namely, for any $0<r<r_{x}$ we define $V(x, U, r)=\{(q, W)\} \cup$ $(B(x ; r) \cap U)$ where $q \in B(x ; r) \cap \partial \Omega$ and $W \subset U$ is a connected component of $B(q ; \rho) \cap \Omega$ for $\rho$ small enough. This topology extends the natural topology on $\Omega \subset \mathbb{R}^{n}$ in the sense that, if ${ }^{u} \bar{\Omega}=\bar{\Omega}$, then the topology on ${ }^{u} \bar{\Omega}$ is the same as that on $\bar{\Omega}$.

There exists a natural map $\kappa:{ }^{u} \bar{\Omega} \rightarrow \bar{\Omega}$, which is the identity on $\Omega$ and sends a point $X=(x, U) \in \partial^{u} \Omega$ to $x$. This map is continuous and, in fact, a homeomorphism on $\Omega$. A continuous map $\phi: V \rightarrow{ }^{u} \bar{\Omega}$ will be called smooth if $\kappa \circ \phi$ is smooth.

### 3.3.2 Domains with a Polygonal Structure

We are now ready to formulate precisely the class of domains under consideration in this chapter. Here and troughtout the rest of this chapter, $r_{x}$ will always be as in Definition 3.3.1.

Definition 3.3.2. A domain with a polygonal structure $\Omega$ in a two dimensional manifold $M$ is an open subset $\Omega \subset M$ with a simple boundary together with a distinguished finite subset $\mathcal{V} \subset \partial^{u} \Omega$ of its unfolded boundary such that, for each $X=(x, U) \in \partial^{u} \Omega$, we are given a neighborhood $V_{X}$ of $x$ in $\mathbb{R}^{2}$ satisfying
(i) there is an open, non-empty interval $\omega_{X}^{\prime} \subset S^{1}, \omega_{X}^{\prime} \neq S^{1}$, and a diffeomorphism $\phi_{X}^{\prime}: V_{X} \rightarrow B^{2}:=B^{2}(0 ; 1)$ such that, in polar coordinates $(r, \theta)$, we have

$$
\phi_{X}^{\prime}\left(V_{X} \cap U\right)=\left\{(r, \theta), r \in(0,1), \theta \in \omega_{X}\right\}
$$

(ii) $\phi_{X}^{\prime}(x)=0$;
(iii) if $X:=(x, U) \notin \mathcal{V}$, then $\omega_{X}^{\prime}$ has length $\pi$ (i.e., it is a half circle).

If $X$ is a vertex, we let $\omega_{X}:=\omega_{X}^{\prime}$ and $\phi_{X}:=\phi_{X}^{\prime}$. If $X$ is not a vertex, on the other hand, then we know that $\omega_{X}^{\prime}$ has length $\pi$, and we replace $\phi_{X}^{\prime}$ with an isomorphism $\phi_{X}: V_{X} \rightarrow(-1,1) \times(-1,1)$ such that $\phi_{X}\left(V_{X} \cap U\right)=(-1,1) \times(0,1)$ and $\phi_{X}\left(V_{X} \cap \partial \Omega\right)=(-1,1) \times\{0\}$.

The set $\mathcal{V}$ is called the set of vertices of (the polygonal structure on) $\Omega$. In view of applications to FEM, which are the focus of the next chapter, we will restrict our attention to bounded domains $\Omega$.

The points in $\mathcal{V}$ are called the vertices of $\Omega$. Note that the set $\Omega$ does not determine its set of vertices and hence it does not determine its polygonal structure, because in the definition of a domain with a polygonal structure we can always increase the set $\mathcal{V}$. However, if there is a polygonal structure on $\Omega$, then the one with the minimum set of vertices is unique. These are the true vertices of $\Omega$. The other vertices of $\Omega$ will be called artificial vertices. The true vertices are the ones for which $\omega_{X}$ is not an half circle. The artificial vertices, and polygonal structures in general, are useful for the study of mixed boundary value problems, and of some elliptic operators with singular coefficients (see Chapter 6). We will call a point $X \in \partial^{u} \Omega$ smooth if $X \notin \mathcal{V}$. A point $x \in \partial \Omega$ is smooth if all points in $\kappa^{-1}(x) \subset \partial^{u} \Omega$ are smooth. If $\Omega$ is connected in a small neighborhood of a point $x \in \partial \Omega$, then we identify $x$ with its unique lift $X$ in the unfolded boundary $\partial^{u} \Omega$.

It follows from the definition above that each component of $\partial^{u} \Omega \backslash \mathcal{V}$ is a smooth curve $\gamma$ without self-intersections such that $\bar{\gamma} \subset \gamma \cup \mathcal{V}$. The curves $\gamma$ will be called the open sides of $\Omega$. For example, a smooth domain in $\mathbb{R}^{2}$ is a domain with a polygonal structure if we take $\mathcal{V}=\emptyset$ and the sides are the connected components of the boundary.

A (straight sides) polygonal domain $\mathbb{P}$ is a typical example of a domain with a polygonal structure. In this case, $\partial^{u} \mathbb{P} \equiv \partial \mathbb{P}$ and ${ }^{u} \overline{\mathbb{P}} \equiv \overline{\mathbb{P}}$. Examples of domains with polygonal structures are given in Figures 3.3 and 3.2 in Section 3.1, where the true vertices are represented by a dot whereas the artificial vertices are represented by a cross (i.e., $\times$ ).

When one vertex $x$ of a concave polygon touches the boundary (see Figure 3.4), then $x$ is covered by two unfolded points $X$ and $Y \in \partial^{u} \Omega$, where $X$ is a (true) vertex in $\mathcal{V}$, while $Y$ is a smooth point. We allow for points in $\partial \Omega$ to be covered by $k$ points in $\partial^{u} \Omega$ with $k$ arbitrary large, as in Figure 3.2. On the other hand, a


Figure 3.4. A vertex touching a smooth side. The picture actually represents ${ }^{u} \bar{\Omega}$. The map $\kappa$ identifies $X$ and $Y$.
point $x \in \partial \Omega$ that is not the image of a vertex will be covered by exactly one or two points in $\partial^{u} \Omega$ as follows. The point $x$ will be covered by one point in $\partial^{u} \Omega$ if $x \in \partial \bar{\Omega}$ (i.e., if $\Omega$ is not on both sides of its boundary near $x$ ) and by two points in $\partial^{u} \Omega$ if $x \notin \partial \bar{\Omega}$ (i.e., if $\Omega$ is on both sides of its boundary near $x$ ). The latter situation is encountered exactly if $x$ belongs to a crack in the domain.

We next define the outer normal $\nu(X)$ at each $X \in \partial^{u} \Omega \backslash \mathcal{V}$, that is, everywhere on the unfolded boundary except at the vertices. Let $X=(x, U)$ with $x \in \partial \Omega$. Since $X$ is smooth, $\partial^{u} \Omega$ is a smooth curve at $X$ with exactly two unit normal vectors. We set then $\nu(X)$ to be the unit normal not pointing into $U$. We will denote the collection of all outer unit vectors as $\partial^{\nu} \Omega$, the oriented boundary of $\Omega$. In particular, the map

$$
\nu:\left(\partial^{u} \Omega \backslash \mathcal{V}\right) \rightarrow \mathbb{R}^{2}
$$

that assigns to $X$ the unit outer normal $\nu(X)$ is well-defined, continuous, and it gives a canonical homeomorphism of

$$
\partial^{u} \Omega \backslash \mathcal{V} \approx \partial^{\nu} \Omega
$$

### 3.3.3 Smooth Functions

The notion of unfolded boundary $\partial^{u} \Omega$ and unfolded closure ${ }^{u} \bar{\Omega}$ of $\Omega$ give rise to a class $\mathcal{C}^{\infty}(\Sigma \Omega)$ of smooth functions on $\Omega$ which is more appropriate for the applications in the next chapter than the smooth functions on $\bar{\Omega}$. In particular, the canonical weight function defined in the next subsection belongs to $\mathcal{C}^{\infty}(\Sigma \Omega)$.

Let $X=(x, U) \in \partial^{u} \Omega$ be an arbitrary (unfolded) point. Recall that the neighborhood $V_{X}$ and the local diffeomorphism $\phi_{X}$ of Definition 3.3.2 are such
that $\phi_{X}\left(V_{X} \cap U\right)=\left\{(r, \theta), r \in(0,1), \theta \in \omega_{X}\right\}$. Since $\omega_{X}$ is an open interval in $S^{1}$ satisfying $\omega_{X} \neq S^{1}$, we can write $\omega_{X}=\left(e^{i t}, e^{i(t+\alpha)}\right)$, with $t \in[0,2 \pi], 0<\alpha \leq 2 \pi$. We then define $\Sigma \omega_{X}=[t, t+\alpha]$. (The case of $\alpha=2 \pi$ corresponds to $x$ being the tip of a crack.)

Definition 3.3.3. Let $f \in \mathcal{C}^{\infty}(\Omega)$. We say that $f \in \mathcal{C}^{\infty}(\Sigma \Omega)$ if, for any $X \in \partial^{u} \Omega$, the function $f \circ \phi_{X}^{-1}$, when written in polar coordinates $r, \theta$, extends to a smooth function of $(r, \theta) \in[0, \epsilon) \times \Sigma \omega_{X}$, with $\epsilon<1$.

In addition, if $U \subset \Omega$ is open and $f \in \mathcal{C}^{\infty}(U)$, we say that $f \in \mathcal{C}^{\infty}(\Sigma U)$ if it is the restriction of a function in $\mathcal{C}^{\infty}(\Sigma \Omega)$.

The definition of $\mathcal{C}^{\infty}(\Sigma U)$ depends on $\Omega$, although we do not explicitly indicate it. It is possible to show that there exists a manifold with corners $\Sigma \Omega$ such that $\mathcal{C}^{\infty}(\Sigma \Omega)$ is exactly the algebra of smooth functions on $\Sigma \Omega$, which justifies our notation. In this work, however, it is not necessary to describe $\Sigma \Omega$. From the definition it follows immediately that $\mathcal{C}^{\infty}(\Sigma \Omega)$ is an algebra and the following inclusions hold:

$$
\mathcal{C}^{\infty}(\bar{\Omega}) \subset \mathcal{C}^{\infty}(\Sigma \Omega) \subset \mathcal{C}^{\infty}(\Omega)
$$

We will call functions in $\mathcal{C}^{\infty}(\Sigma \Omega)$ smooth on $\Sigma \Omega$. For example, let us write the outer normal function $\nu$ introduced above as $\nu=\left(\nu^{1}, \nu^{2}\right)$. Then each of $\nu^{j}$ extends to a function that is smooth on $\Sigma \Omega$. Also, by definition, the coordinate functions $r(y)$ and $\theta(y)$, where $(r, \theta)$ are polar coordinates near a point of $u \bar{\Omega}$, are smooth on $\Sigma \Omega$. This is one of the main motivations for introducing the class $\mathcal{C}^{\infty}(\Sigma \Omega)$.

For transmission problems, when $\bar{\Omega}=\cup_{j} \bar{\Omega}_{j}$ for some disjoint domains $\Omega_{j}$ with a polygonal structure, we define $\mathcal{C}^{\infty}(\Sigma \Omega)=\mathcal{C}(\Omega) \cap \mathcal{C}^{\infty}\left(\Sigma \Omega_{j}\right)$. (Here $\mathcal{C}^{\infty}\left(\Sigma \Omega_{j}\right)$ is defined using the polygonal structure of $\Omega_{j}$, without any reference to $\Omega$.) That is, $\mathcal{C}^{\infty}(\Sigma \Omega)$ consists of the continuous functions on $\Omega$ whose restrictions to each $\Omega_{j}$ are in $\mathcal{C}^{\infty}\left(\Sigma \Omega_{j}\right)$. In this way, we allow for additional singularities at the vertices of $\Omega_{j}$.

### 3.3.4 The Canonical Weight Function

In the analysis of differential equations on polygonal and polyhedral domains, the distance function $\vartheta(x)$ from $x$ to the non-smooth points on the boundary plays an essential role $[3,45,60,61,75,80,83]$.

We first give a metric structure to ${ }^{u} \bar{\Omega}$ following Dauge [45]. Thus, for all $x$, $y \in \Omega$, we define $d(x, y):=\inf \ell[\gamma]$, where $\ell[\gamma]$ is the length of the path $\gamma$ and the infimum is taken over all paths in $\Omega$ joining $x$ to $y$. (We stress that the path $\gamma$ must be completely contained in $\Omega$, thus the straight segment joining $x$ and $y$ may not always work.) If $X=(x, U) \in \partial^{u} \Omega$ with $U$ small enough and $y \in \Omega$, we let

$$
d(X, y)=d(y, X):=\inf _{z \in U}(\|x-z\|+d(z, y))
$$

with $\|\cdot\|$ the usual Euclidean distance in $\mathbb{R}^{2}$. In particular, for $y \in U, d(X, y)=$ $\|x-y\|$. Finally, if $X, Y \in \partial^{u} \Omega$,

$$
\begin{equation*}
d(X, Y):=\inf _{z \in \Omega}(d(X, z)+d(z, Y)) \tag{3.22}
\end{equation*}
$$

Next we let $\vartheta:{ }^{u} \bar{\Omega} \rightarrow \mathbb{R}_{+}$be the distance to the set of vertices $\mathcal{V}$ of $\Omega$ (which includes all singular points on the boundary), that is, $\vartheta(x):=\min _{Q \in \mathcal{V}} d(x, Q)$. The function $\vartheta$ is then continuous on ${ }^{u} \bar{\Omega}$, because the distance $d$ can be used to define the topology on ${ }^{u} \bar{\Omega}$.

We now introduce a smooth function $r_{\Omega}$ that is comparable to $\vartheta$ and has the same good scaling properties with respect to dilations as $\vartheta$, but, unlike $\vartheta, r_{\Omega}$ belongs to $\mathcal{C}^{\infty}(\Sigma \Omega)$. Recall that $\vartheta(x)=\prod_{Q} d(x, Q)$ from (3.9), where $d$ is the distance defined in Equation (3.22) and $Q$ ranges through the set $\mathcal{V}$. Scaling of the norm will be used in Chapter 4 to prove quasi-optimal rates of convergence for the FEM.

Recall that every vertex $Q$ of $\Omega$ is a pair $(q, U)$, where $q \in \bar{\Omega}$ and $U$ is a connected component of $B\left(q, r_{q}\right) \cap \Omega$, for some $r_{q}>0$ small enough.

Definition 3.3.4. Let $Q=(q, U)$ range through $\mathcal{V}$ and $\chi_{Q} \in \mathcal{C}^{\infty}\left(\mathbb{R}_{+} \cup\{0\}\right)$ be a smooth function such that

$$
\begin{cases}\chi_{Q}(t)=t, & 0 \leq t \leq \operatorname{diam}(U) / 2<1 \\ \chi_{Q}(t) \geq \operatorname{diam}(U) / 4 & \operatorname{diam}(U) / 2 \leq t \leq \operatorname{diam}(U) \\ \chi_{Q}(t)=1, & t \geq \operatorname{diam}(U)\end{cases}
$$

Then the canonical weight function $r_{\Omega}$ on $\Omega$ is given by

$$
\begin{equation*}
r_{\Omega}(y)=\prod_{Q \in \mathcal{V}} \chi_{Q}(d(Q, y)), \quad y \in{ }^{u} \bar{\Omega} \tag{3.23}
\end{equation*}
$$

For any vertex $Q=(q, U) \in \mathcal{V} \subset \partial^{u} \Omega$ and $y \in U, r_{\Omega}(y)$ is simply the Euclidean distance from $y$ to $q=\kappa(Q) \in \partial \Omega$, and, furthermore, $r_{\Omega}$ is a homogeneous function of degree one with respect to dilations with center $Q$ and ratio $\lambda<1$.

The definition of $r_{\Omega}$ shows that $r_{\Omega} / \vartheta$ extends to a continuous function ${ }^{u} \bar{\Omega} \rightarrow$ $(0, \infty)$. In particular, for $\Omega$ bounded, there is a $C>0$ such that $C^{-1} \vartheta(x) \leq$ $r_{\Omega}(x) \leq C \vartheta(x)$.

Remark 3.3.5. Let $\chi \in \mathcal{C}^{\infty}(\bar{\Omega})$ be a smooth function. Assume that $\chi \geq 0$ and $\chi(Q)>0$ at the Neumann-Neumann vertices and at the vertices of the internal interfaces. Then in Theorems 3.1.2 and 3.1.3 we can drop the condition that there be no Neumann-Neumann vertices and no internal vertices if we replace $P$ with $P+\chi r_{\Omega}^{-2}$. See also [69, 75].

### 3.4 Properties of Sobolev Spaces with Weights

In this section, we establish some properties of these spaces needed for our analysis. We consider inhomogeneous spaces where the weight may depend on a particular vertex near that vertex (cf. [37, 74] for polyhedral domains). The use of inhomogeneous norms allows us to theoretically justify the use of different grading parameters at different vertices when constructing graded meshes that provide quasi-optimal rates on convergence. A general graded strategy will be introduced in the next chapter.

Below we use the standard notation $H^{s}(\Omega), s \in \mathbb{R}$, for the classical $L^{2}$-based (unweighted) Sobolev spaces, and we denote the space of all square integrable functions on compact sets of $\Omega$ by $L_{\mathrm{loc}}^{2}(\Omega)$. Also, by an isomorphism we mean a continuous linear bijection with a continuous inverse.

### 3.4.1 Differential Operators

We begin with defining an appropriate class of differential operators on $\Omega$. For proofs of the results in this section we refer to [75] (in that paper the class of domains with polygonal structure is slightly smaller, but the same proofs still apply).

We state first a preliminary lemma. We denote a point $x \in \mathbb{R}^{2}$ by $x=\left(x_{1}, x_{2}\right)$. If $\vec{\alpha}=\left(\alpha_{Q}\right)$, where $Q \in \mathcal{V}$ ranges through the set of vertices of $\Omega$, we also define

$$
r_{\Omega}(y)^{\vec{\alpha}}:=\prod_{Q \in \mathcal{V}} \chi_{Q}(d(Q, y))^{\alpha_{Q}}, \quad y \in{ }^{u} \bar{\Omega}
$$

(Recall $\chi_{Q}$ was introduced in Definition 3.3.4 and that, if $Q=(q, U) \in \mathcal{V}$, then $r_{\Omega}(y)=\|Q-y\|$ for $\left.y \in U.\right)$

Lemma 3.4.1. The functions $\partial_{x_{1}} r_{\Omega}, \partial_{x_{2}} r_{\Omega}$ are in $\mathcal{C}^{\infty}(\Sigma \Omega)$. If $u \in \mathcal{C}^{\infty}(\Sigma \Omega)$, then the functions $r_{\Omega} \partial_{x_{1}} u$, $r_{\Omega} \partial_{x_{2}} u$ are also in $\mathcal{C}^{\infty}(\Sigma \Omega)$. Finally, for $\vec{\lambda}$ a multi-exponent, $r_{\Omega}^{-\vec{\lambda}}\left(r_{\Omega} \partial_{j}\right) r_{\Omega}^{\vec{\lambda}}-r_{\Omega} \partial_{j} \in \mathcal{C}^{\infty}(\Sigma \Omega)$, where $\partial_{j}$ stands for either of $\partial_{x_{1}}, \partial_{x_{2}}$.

Next, we denote by $\operatorname{Diff}_{0}^{m}(\Omega)$ the differential operators of order $m$ on $\Omega$ linearly generated by differential operators of the form

$$
a\left(r_{\Omega} \partial\right)^{\alpha}:=a\left(r_{\Omega} \partial_{x_{1}}\right)^{\alpha_{1}}\left(r_{\Omega} \partial_{x_{2}}\right)^{\alpha_{2}}, \quad|\alpha|:=\alpha_{1}+\alpha_{2} \leq m, a \in \mathcal{C}^{\infty}(\Sigma \Omega)
$$

We let $\operatorname{Diff}_{0}^{m}(\Omega):=\mathcal{C}^{\infty}(\Sigma \Omega)$ and $\operatorname{Diff}_{0}^{\infty}(\Omega):=\bigcup_{m} \operatorname{Diff}_{0}^{m}(\Omega)$. Recall the standard multi-index notation for derivatives $\partial^{\alpha}=\partial_{x_{1}}^{\alpha_{1}} \partial_{x_{2}}^{\alpha_{2}}$, with $\alpha=\left(\alpha_{1}, \alpha_{2}\right)$.

An immediate consequence of Lemma 3.4.1 is that $\left[r_{\Omega} \partial_{j}, r_{\Omega} \partial_{k}\right] \in \operatorname{Diff}{ }_{0}^{1}(\Omega)$, from which it follows:

Proposition 3.4.2. We have $\operatorname{Diff}_{0}^{k}(\Omega) \operatorname{Diff}_{0}^{m}(\Omega) \subset \operatorname{Diff}_{0}^{k+m}(\Omega)$. Hence $\operatorname{Diff}_{0}^{\infty}(\Omega)$ is an algebra. If $P$ is a differential operator of order $m$ with smooth coefficients, then $r_{\Omega}^{m} P \in \operatorname{Diff}_{0}^{m}(\Omega)$.

### 3.4.2 Inhomogeneous Weighted Sobolev Spaces

The properties from Section 2.3 extend to the weighted Sobolev space considered here. In addition, we give other important descriptions of weighted Sobolev spaces
in this subsection. We start by recalling the definition of the Babuška-Kondratiev spaces, which are Sobolev spaces with weight $r_{\Omega}^{\overrightarrow{\mathbf{a}}}$

$$
\begin{align*}
& W_{B K}^{m, p, \overrightarrow{\mathbf{a}}}(\Omega):=\left\{u: \Omega \rightarrow \mathbb{R}, r_{\Omega}^{|\alpha|-\overrightarrow{\mathbf{a}}} \partial^{\alpha} u \in L^{p}(\Omega), \quad \text { for all }|\alpha| \leq k\right\}  \tag{3.24}\\
& \\
& \quad m \in \mathbb{Z}_{+}, 1 \leq p \leq \infty
\end{align*}
$$

where $r_{\Omega}$ is the canonical weight function introduced in the previous section. If $p=2$, we denote $\mathcal{K}_{\overrightarrow{\mathbf{a}}}^{m}(\Omega):=W^{m, 2, \overrightarrow{\mathbf{a}}}(\Omega)$. The norm on $\mathcal{K}_{\overrightarrow{\mathbf{a}}}^{m}(\Omega)$ is

$$
\|u\|_{\mathcal{K}_{\overrightarrow{\mathbf{a}}}^{m}(\Omega)}^{2}:=\sum_{|\alpha| \leq m}\left\|r_{\Omega}^{|\alpha|-\overrightarrow{\mathbf{a}}} \partial^{\alpha} u\right\|_{L^{2}(\Omega)}^{2}
$$

A standard arguments shows that $\mathcal{K}_{\overrightarrow{\mathbf{a}}}^{m}(\Omega)$ is complete (and hence a Hilbert space).
The vector $\overrightarrow{\mathbf{a}}$ is called a multi-exponent with components $a_{Q}$, one for each $Q \in \mathcal{V}$. We recall from Section 3.1 that $\overrightarrow{\mathbf{a}}+t$ denotes the vector obtained from $\overrightarrow{\mathbf{a}}$ by adding the real number $t$ to each component. In particular, we write $t$ instead of $\overrightarrow{\mathbf{a}}$ if all the components of $\overrightarrow{\mathbf{a}}$ are equal to $t$. We also recall the neighborhood $V_{X}$ and the diffeomorphism $\phi_{X}$ of Definition 3.3.2 for each $X \in \partial^{u} \Omega$.

We immediately have from the definition that:

$$
\mathcal{K}_{0}^{0}(\Omega)=L^{2}(\Omega), \quad r_{\Omega}^{\overrightarrow{\mathbf{b}}} \mathcal{K}_{\overrightarrow{\mathbf{a}}}^{m}(\Omega)=\mathcal{K}_{\overrightarrow{\mathbf{a}}+\overrightarrow{\mathbf{b}}}^{m}(\Omega)
$$

since the function $r_{\Omega}^{|\beta|-t} \partial^{\beta} r_{\Omega}^{t}$ is bounded on $\Omega$ for any multiindex $\beta$ and $t \in \mathbb{R}$.
We can define the spaces $\mathcal{K}_{\overrightarrow{\mathbf{a}}}^{-k}(\Omega), k \in \mathbb{Z}_{+}$, by duality, as usual. However, we shall need only the spaces $\mathcal{H}_{\overrightarrow{\mathbf{a}}}^{*}$, where $\mathcal{H}_{\overrightarrow{\mathbf{a}}}:=\left\{u \in \mathcal{K}_{\overrightarrow{\mathbf{a}}+1}^{1}(\Omega), u=0\right.$ on $\left.\partial_{D} \Omega\right\}$, defined in Equation (3.12).

For the analysis of the boundary value problem (3.8) and subsequently for the approximation results in the next chapter, it will be convenient to give an equivalent characterization of the spaces $\mathcal{K}_{\vec{a}}^{m}$ in terms of partitions of unity adapted to the geometry of $\Omega$. The partitions of unity considered here are the dyadic partitions of unity considered in [41, 61], and in many other papers. We refer to [75] for details on the construction as well as more references.

For each $X=(x, U) \in \partial^{u} \Omega$, we recall again the diffeomorphism $\phi_{X}:\left(V_{X} \cap U\right) \rightarrow$ $\left\{(r, \theta), r \in(0,1), \theta \in \omega_{X}\right\}$ of definition 3.3.2, and we let $\varphi: \mathbb{R} \rightarrow[0,1]$ be a smooth
function such that $\varphi=1$ on $[0,1]$ and $\varphi$ has support in $[-1,2]$. Then we consider the family of functions $\varphi_{n}(y):=\varphi\left(\log _{2} d(y, X)-n\right), y \in V_{X} \cap U$, for all values of $n$ for which these functions are supported in $V_{X} \cap U$. Here $d(y, X)$ is the distance to $X$ as in Section 3.3.4. We also consider a smooth partition of unity $\left(\xi_{k}\right)$ of $\omega_{X}$ consisting of at least two functions. We then set $\mathcal{F}_{X}:=\left\{\psi=\left(\xi_{k} \circ \theta \circ \phi_{X}\right) \varphi_{n}\right\}$. Since ${ }^{u} \bar{\Omega}$ is compact, we can take a finite subcover of $\partial^{u} \Omega$ by the open sets $V_{X}$ and we choose the subcover to include $V_{Q}$ for all $Q \in \mathcal{V}$. We let $\mathcal{F}$ be the union of all the families $\mathcal{F}_{X}$ for each $V_{X}$ in this finite subcover. We also let $U$ be an open set such that $\left(\cup_{Q \in \mathcal{V}} V_{Q}\right)^{c} \subset U \subset \bar{U} \subset{ }^{u} \bar{\Omega}$, and let $\mathcal{F}_{0}$ be a finite partition of unity on $\left(\cup_{Q \in \mathcal{V}} V_{Q}\right)^{c}$ subordinate to a $U$. One can easily show that the family $\mathcal{F}_{X}$ is countable and uniformly locally finite, since it is constructed via dilations. Hence, the countable family $\mathcal{F} \cup \mathcal{F}_{0}$ gives rise to a partition of unity in the standard way:

$$
f=\psi / g, \quad g=\sum_{\psi \in \mathcal{F} \cup \mathcal{F}_{0}} \psi
$$

With abuse of notation we will identify $f$ with $\psi$. This partition of unity has the important following properties (by refining the partition $\mathcal{F}_{0}$ if necessary). See Lemma 5.5 in [75] for a proof.

Lemma 3.4.3. There exists a constant $C>0$ such that

$$
\begin{equation*}
\left|r_{\Omega}^{|\alpha|} \partial^{\alpha} \psi(y)\right| \leq C \text { for all } y \in \Omega \text { and all } \psi \in \mathcal{F}_{\Omega} \text { or } \psi \in \mathcal{P}_{\Omega} \tag{3.25}
\end{equation*}
$$

A point $y \in \Omega$ belongs to at most $\kappa_{\Omega}$ of the supports of the functions $\psi$ in our partition of unity $\mathcal{P}_{\Omega}$. Moreover, for any $\psi \in \mathcal{P}_{\Omega}$, the support of $\psi$ contains no vertex, intersects at most one side of $\Omega$, and has diameter $\leq C r_{\Omega}(y)$, for any $y$ in the support of $\psi$ and a constant $C$ independent of $\psi$.

By construction, supp is diffeomorphic to a disk or a half disk, depending on whether supp intersects a side of $\Omega$ or not. We will call the preimage of the center of the disk the "center" of supp. We also index every $\psi \in \mathcal{P}_{\Omega}$ as follows: we write $\psi=\psi_{J}$, where $J=(j, k), j \in \mathbb{N}$ and $k=0$ or $k$ ranges over $Q \in \mathcal{V}$ depending whether supp $\subset V_{Q}$ or not.

The weighted space $\mathcal{K}_{\overrightarrow{\mathbf{a}}}^{m}(\Omega)$ will be characterized using the above partition of unity $\mathcal{P}_{\Omega}$ and dilations. For each $\psi=\psi_{J}$ in $\mathcal{P}_{\Omega}$ with support in $\Omega$, we shall denote
by $x_{J}$ it center and define $\alpha_{J}(x)=\alpha_{x_{J}}(x):=\theta_{J}\left(x-x_{J}\right)+x_{J}$ the dilation of ratio $\theta_{J}$ and center $x_{J}$, where $\theta_{J}=r_{\Omega}\left(x_{J}\right)$. Recall that $r_{\Omega}$ is comparable to $\vartheta$, the distance to the sides of $\Omega$. In $\operatorname{supp}_{J}$ intersects a side of $\Omega$, we cannot employ dilations that are isotropic in every directions. We will then use the projection onto the rescaled tangent line at $x_{J}$. Here $x_{J}=\kappa\left(X_{J}\right)$ is the again the center of $\operatorname{supp}_{J}$, where $X_{J} \in \partial^{u} \Omega \backslash \mathcal{V}$ and $\kappa: \partial^{u} \Omega \rightarrow \partial \Omega$ is the canonical projection. Since there is a well defined unit outer normal at each $X \in \partial^{u} \Omega \backslash \mathcal{V}$, there is a well-defined half-plane to $\partial \Omega$ at $x_{J}$ for each $X_{J}=\kappa^{-1}\left(x_{J}\right)$ in $\partial^{u} \Omega$ (exactly two if $x_{J}$ is a crack point and exactly one otherwise), which we denote by $\mathbb{H}_{J}$. Then $\partial \mathbb{H}_{J}$ is the tangent line to $\partial \Omega$ at $x_{J}$ and we let $\pi_{J}$ to be the canonical projection from $\partial \Omega$ to this line, which is a diffeomorphism on $\partial \Omega$ close to $x_{J}$. Let us denote by $\sigma\left(X_{J}, r\right)$ the image under this projection, an (open) segment in $T_{x_{J}} \partial \Omega$ of lentgh $2 r, r$ sufficiently small. Finally, for $z \in \sigma\left(x_{J}, r\right)$ and $t \in[0, r)$ we set

$$
\chi_{J}(z, t):=\left(\pi_{J} \circ \kappa\right)^{-1}(z)+t r_{\Omega}\left(X_{J}\right) X_{J}=\left(\pi_{j} \circ \kappa\right)^{-1}(z)-t \nu\left(X_{J}\right)
$$

The map $\phi_{J}$ is then a diffeomorphism of $\sigma\left(x_{J}, r\right) \times[0, r)$ onto a neighborhood $W_{X_{J}}$ of $X_{j}$ in ${ }^{u} \bar{\Omega}$ such that $\sigma\left(x_{J}, r\right) \times\{0\}$ maps to the unfolded boundary $\partial^{u} \Omega$ and $\sigma\left(x_{J}, r\right) \times(0, r)$ maps to $\Omega$.

We can now easily give a complete definition of the spaces $\mathcal{K}_{\vec{a}}^{m}(\Omega)$ solely in terms of localizations. We let $\mathcal{J}$ be the set of indices $J$ such that the support of $\psi_{J}$ intersects $\partial \Omega$. We can assume that the support of each of the functions $\psi_{J}$ is small enough so that it is contained in the range $W_{X_{J}}$ of the diffeomorphism $\chi_{J}$. Let

$$
\begin{equation*}
\nu_{m, \overrightarrow{\mathbf{a}}}(u)^{2}:=\sum_{J \notin \mathcal{J}} \theta_{J}^{2-2 a_{J}}\left\|\left(\psi_{J} u\right) \circ \alpha_{J}\right\|_{H^{m}\left(\mathbb{R}^{2}\right)}^{2}+\sum_{J \in \mathcal{J}} \theta_{J}^{2-2 a_{J}}\left\|\left(\psi_{J} u\right) \circ \chi_{J}^{-1}\right\|_{H^{m}\left(\mathbb{H}_{J}\right)}^{2} . \tag{3.26}
\end{equation*}
$$

Then $u \in \mathcal{K}_{a}^{m}(\Omega), m \in \mathbb{Z}$, if, and only if, $\nu_{m, \overrightarrow{\mathbf{a}}}(u)<\infty$. Moreover, $\nu_{m, \overrightarrow{\mathbf{a}}}$ defines an equivalent norm on $\mathcal{K}_{\stackrel{\rightharpoonup}{\mathbf{a}}}^{m}(\Omega)$. Above, we set $a_{J}=1$ if $J=(j, 0)$ and $a_{J}=a_{Q}$ if $J=(j, Q)$, with $a_{Q}$ a component of the multi-exponent $\overrightarrow{\mathbf{a}}$. Recall that $r_{\Omega}(x)=1$ if $x \notin V_{Q}$ for some $Q \in \mathcal{V}$.

We can also easily define spaces on the unfolded boundary of $\Omega$ as follows. Let

$$
\begin{equation*}
\mu_{s, \overrightarrow{\mathbf{a}}}(u)^{2}:=\sum_{J \in \mathcal{J}} \theta^{2-a_{J}}\left\|\left(\psi_{J} u\right) \circ \chi_{J}^{-1}\right\|_{H^{s}\left(\partial \mathbb{H}_{J}\right)}^{2} . \tag{3.27}
\end{equation*}
$$

Then $u \in \mathcal{K}_{\overrightarrow{\mathbf{a}}}^{s}\left(\partial^{u} \Omega\right), s \in \mathbb{R}$, if, and only if, $\mu_{m, \overrightarrow{\mathbf{a}}}(u)<\infty$. Moreover, $\mu_{m, \overrightarrow{\mathbf{a}}}$ defines an equivalent norm on $\mathcal{K}_{\overrightarrow{\mathbf{a}}}^{s}\left(\partial^{u} \Omega\right)$, and $\mathcal{J}$ and $a_{J}$ are defined as in (3.26). Note that there are no compatibility conditions among the different sides of $\partial^{u} \Omega$, so that it follows from the definition that the dual of $\mathcal{K}_{\overrightarrow{\mathbf{a}}}^{s}\left(\partial^{u} \Omega\right)$ with the pivot $L^{2}\left(\partial^{u} \Omega\right)$ is exactly $\mathcal{K}_{-\mathbf{a}}^{-s}\left(\partial^{u} \Omega\right)$. Furthermore, if $\partial_{D} \Omega \subset \partial^{u} \Omega$ is a union of closed sides $\gamma$ of $\Omega$, then

$$
\mathcal{K}_{\overrightarrow{\mathbf{a}}}^{m}\left(\partial \Omega_{D}\right):=\oplus_{\gamma \subset \partial \Omega_{D}} \mathcal{K}_{\overrightarrow{\mathbf{a}}}^{m}(\gamma)
$$

where $\mathcal{K}_{\overrightarrow{\mathbf{a}}}^{m}(\gamma)$ is obtained by restricting $J$ in (3.27) to all indices for which $x_{J} \in$ $\gamma$. Finally, we state a trace result, which is proved as for the usual BabuškaKondratiev spaces (see, e.g., [2]).

For the following theorem, let us notice that ${ }^{u} \bar{\Omega} \backslash \mathcal{V}$ is a smooth, non-compact manifold with boundary. Therefore the space $\mathcal{C}_{\mathrm{c}}^{\infty}\left({ }^{u} \bar{\Omega} \backslash \mathcal{V}\right)$ of smooth, compactly supported functions on ${ }^{u} \bar{\Omega} \backslash \mathcal{V}$ is well defined.

Theorem 3.4.4. The space $\mathcal{C}_{c}^{\infty}\left({ }^{u} \bar{\Omega} \backslash \mathcal{V}\right)$ is dense in $\mathcal{K}_{\vec{a}}^{m}(\Omega), m \in \mathbb{Z}_{+}$. Then the restriction to $\partial_{D} \Omega$ extends to a continuous, surjective map

$$
\left.\mathcal{K}_{\overrightarrow{\mathbf{a}}}^{m}(\Omega) \ni u \rightarrow u\right|_{\partial_{D} \Omega} \in \mathcal{K}_{\overrightarrow{\mathbf{a}}-1 / 2}^{m-1 / 2}\left(\partial_{D} \Omega\right)
$$

for $m \geq 1$.
See any of the papers $[2,3,75]$ for a proof.
We conclude this section with a result on the action of differential operators on the spaces $\mathcal{K}_{\overrightarrow{\mathbf{a}}}^{m}(\Omega)$.
Lemma 3.4.5. Let $P$ be a differential operator of order $k$ on $\mathbb{R}^{2}$ with smooth coefficients. Then $P$ defines a continuous map $P: \mathcal{K}_{\overrightarrow{\mathbf{a}}}^{m}(\Omega) \rightarrow \mathcal{K}_{\overrightarrow{\mathbf{a}}-k}^{m-k}(\Omega), m \geq$ k. Moreover, $\vec{\lambda} \rightarrow r_{\Omega}^{-\vec{\lambda}} \operatorname{Pr}_{\Omega}^{\vec{\lambda}}$ defines a continuous family of bounded operators $\mathcal{K}_{\overrightarrow{\mathbf{a}}}^{m}(\Omega) \rightarrow \mathcal{K}_{\overrightarrow{\mathbf{a}}-k}^{m-k}(\Omega)$.

This result follows by combining the characterization (3.26) of the norm via partition of unity with Proposition 3.4.2. See [2, 3, 75] for the simple details.

### 3.5 Proofs of the Three Main Theorems

In this section, we prove Theorems 3.1.1, 3.1.2, and 3.1.3. As is well known, the singularities at the points where the interface meets the boundary are similar in nature to those at the vertices [58, 84, 85]. The typical approach to the study of these singularities is using singular functions. Our approach is to use wellposedness results as in Theorem 2.2.12. The treatment of the boundary conditions and of the vertex singularities is the same also for transmission problems, so we only sketch this part, referring for more details to Chapter 2 and [3, 21, 75]. There, homogeneous Babuška-Kondratiev spaces are used, but the use of partition of unity allows us to localize all results, essentially the same proof applies to the spaces $\mathcal{K}_{\overrightarrow{\mathbf{a}}}^{m}(\Omega)$.

Let $P$ be the variable coefficient, scalar operator defined in (3.2). We recall that our aim is to obtain regularity and solvability estimates for the problem (3.8) (namely the problem $P u=f$ in $\Omega, D_{\nu}^{P} u=g_{N}$ on $\partial_{N} \Omega, u=g_{D}$ on $\partial_{D} \Omega$, and $u^{+}=u^{-}$and $D_{\nu}^{P+} u=D_{\nu}^{P-} u$ on the interface $\Gamma$ ) with $\partial_{N} \Omega$ and $\partial_{D} \Omega$ disjoint subsets of the unfolded boundary, with $\partial_{D} \Omega$ a union of closed sides of $\Omega$, and with $\Gamma:=\cup \partial \Omega_{j} \backslash \partial \Omega$. We assume that

$$
f \in \mathcal{K}_{\overrightarrow{\mathbf{a}}-1}^{m-1}(\Omega), \quad g_{D} \in \mathcal{K}_{\overrightarrow{\mathbf{a}}+1 / 2}^{m+1 / 2}\left(\partial_{D} \Omega\right), \quad \text { and } \quad g_{N} \in \mathcal{K}_{\overrightarrow{\mathbf{a}}-1 / 2}^{m-1 / 2}\left(\partial_{N} \Omega\right)
$$

Recall that the boundary value problem (3.8) is defined also for $m=0$ using the weak formulation of our boundary value problem, see Equation (3.7).

Proof. (of Theorem 3.1.1). We follow the proof in Theorem 2.2.1 using the partition of unity. This is done exactly as in Theorem 2.2.1, except for the case of interfaces. We shall hence concentrate on this case.

Let us consider Equation (3.8) with $\Omega=\mathbb{R}^{n}$ and $\Gamma=\left\{x_{n}=0\right\}$ (so no boundary, only an interface equal to $\mathbb{R}^{n-1}$ ). Let $B_{R}$ be the ball of radius $R$ centered at 0 in $\mathbb{R}^{n}$. Then we have the following regularity for interface problems. Assume $u \in H_{0}^{1}\left(B_{1}\right)$ and $\left.P u\right|_{\mathbb{R}_{ \pm}} \in H^{m-1}\left(\mathbb{R}_{ \pm}^{n}\right)$, then $\left.u\right|_{\mathbb{R}_{ \pm}} \in H^{m+1}\left(\mathbb{R}_{ \pm}^{n}\right)$, [82]. We use this estimate on the reference ball, after we also straighten the interface.

Using also the equivalent norms $\nu_{m+1, \overrightarrow{\mathbf{a}}+1}$ on $\mathcal{K}_{\overrightarrow{\mathbf{a}}+1}^{m+1}\left(\Omega_{j}\right)$ introduced in Equation (3.26), we see that we need to control the commutator $\left[P, \psi_{J}\right]$. We keep the notation
of the above paragraph. Then this can be done by induction on each half space $\mathbb{R}_{ \pm}$ since on each of these subspaces the commutator $\left[P, \psi_{J}\right]$ is a first order differential operator with smooth coefficients.

We turn now to the issue of well-posedness for the boundary value/transmission problem (3.8). First, we employ a weighted form of Poincaré's inequality to conclude that the uniformly strongly elliptic, scalar operator $P$ is strictly coercive on $\mathcal{H}_{0}$ provided $\partial_{N} \Omega$ contains no two adjacent sides. Recall that near each vertex $Q,{ }^{u} \bar{\Omega}$ is diffeomorphic to a sector of angle $0<\alpha \leq 2 \pi$. It is therefore enough to establish the weighted Poincaré's inequality that we need for such a sector, given that there are only finitely-many vertices. The proof in this case is standard [21, 69, 80].

Lemma 3.5.1. Let $\Omega \subset \mathbb{R}^{2}$ be a domain with a polygonal structure. Let $r_{\Omega}(z)$ be the canonical weight function on $\Omega$ and let $\partial_{D} \Omega$ be a non-empty closed subset of the unfolded boundary $\partial^{u} \Omega$ such that $\partial_{N} \Omega=\partial^{u} \Omega \backslash \partial_{D} \Omega$ is a union of oriented open sides of $\Omega$, no two of which are adjacent. Then there exists a constant $C_{\Omega}>0$ such that

$$
\|u\|_{\mathcal{K}_{1}^{0}(\Omega)}^{2}:=\int_{\Omega} \frac{|u(z)|^{2}}{r_{\Omega}(x)^{2}} d z \leq C_{\Omega} \int_{\Omega}|\nabla u(z)|^{2} d z
$$

for any $u \in H^{1}(\Omega)$ satisfying $u=0$ on $\partial_{D} \Omega$.
We are now ready to prove Theorems 3.1.2 and 3.1.3.
Proof. of Theorems 3.1.2 and 3.1.3. Lemma 3.5.1 above immediately implies that $-\Delta$ is coercive on $\mathcal{H}_{0}$ and strictly coercive if $\partial_{N} \Omega$ does not contain adjacent sides and the interface is smooth. (That is $B_{-\Delta}(u, u) \geq C\|u\|_{\mathcal{K}_{1}^{1}(\Omega)}$.)

Let $\mathcal{H}_{\overrightarrow{\mathbf{a}}}:=\left\{u \in \mathcal{K}_{1+\overrightarrow{\mathbf{a}}}^{1}(\Omega), u=0\right.$ on $\left.\partial \Omega\right\}$, as before. Since $P$ is in divergence form, by applying Green's theorem

$$
B_{P}(u, u)=\int_{\Omega} \sum_{i j} a^{i j}(x) \partial_{i} u(x) \partial_{j} u(x) d x \geq C_{P} \int_{\Omega}|\nabla u(x)|^{2} d x
$$

provided $u \in \mathcal{H}_{0}$, using Equation (3.3) and Lemma 3.5.1. Green's theorem applies by splitting $\Omega$ into a finite number of Lipschitz domains [54]. Combining the above estimate with the weighted Poincare's inequality, we obtain that $P$ is strictly coercive on $\mathcal{H}_{0}$. This proves the first part of Theorem (3.1.2).

Recall the maps

$$
\begin{aligned}
& \tilde{P}_{m, \overrightarrow{\mathbf{a}}}:=\left(\left.\oplus P\right|_{\Omega_{j}}, D_{\nu}^{P}\right):\left\{u: \Omega \rightarrow \mathbb{R},\left.u\right|_{\Omega_{j}} \in \mathcal{K}_{\overrightarrow{\mathbf{a}}+1}^{m+1}\left(\Omega_{j}\right), u=0 \text { on } \partial_{D} \Omega\right. \\
&\left.u^{+}=u^{-} \text {and } D_{\nu}^{P+} u=D_{\nu}^{P-} u \text { on } \Gamma\right\} \rightarrow \oplus_{j} \mathcal{K}_{\overrightarrow{\mathbf{a}}-1}^{m-1}\left(\Omega_{j}\right) \oplus \mathcal{K}_{\overrightarrow{\mathbf{a}}-1 / 2}^{m-1 / 2}\left(\partial_{N} \Omega\right)
\end{aligned}
$$

of Corollary 3.1.4. To prove the rest of the Theorems 3.1.2 and 3.1.3, we need to prove that $\tilde{P}_{m, \overrightarrow{\mathbf{a}}}$ is an isomorphism for $m \geq 0$ and $\left|a_{Q}\right|<\eta_{Q}$.

The strict coercivity of $B_{P}$ on $\mathcal{H}_{0}$ gives that $B_{P}$ satisfies the assumptions of the Lax-Milgram lemma, and hence $B_{P}^{*}: \mathcal{H}_{0} \rightarrow \mathcal{H}_{0}^{*}$ is an isomorphism, where $B_{P}^{*}(u)(v)=B_{P}(u, v)$. This is equivalent, by definition, with the fact that $\tilde{P}_{0,0}$ is an isomorphism. Hence, Theorems 3.1.2 and 3.1.3 were established for $m=0$ and $\overrightarrow{\mathbf{a}}=0$.

The extension of our results from $\overrightarrow{\mathbf{a}}=0$ to $\overrightarrow{\mathbf{a}}$ satisfying $\left|a_{Q}\right|<\eta_{Q}$ is done by continuity as follows. By Lemma 3.4.5, $r_{\Omega}^{-\overrightarrow{\mathbf{a}}} P_{m, \overrightarrow{\mathbf{a}}} r_{\Omega}^{\overrightarrow{\mathbf{a}}}$ will all act on the same space and depend continuously on $\overrightarrow{\mathbf{a}}$. Since $P_{0,0}$ is an isomorphism, we obtain that $P_{0, \overrightarrow{\mathbf{a}}}$ is an isomorphism for $\overrightarrow{\mathbf{a}}$ close to 0 . In particular, there exists $\eta_{Q}>0$ such that for $\left|a_{Q}\right|<\eta_{Q}, P_{0, \overrightarrow{\mathbf{a}}}$ is an isomorphism, by continuity. This completes the proof of Theorems 3.1.2 and 3.1.3 for $m=0$.

It only remains to prove Theorem (3.1.3) for $m \geq 1$. Indeed, Theorem 3.1.1 gives that $\tilde{P}_{m, \overrightarrow{\mathbf{a}}}$ is surjective for $\left|a_{Q}\right|<\eta_{Q}$, since it is surjective for $m=0$. Since this map is also continuous and injective (because it is injective for $m=0$ ), it is an isomorphism. Hence $\tilde{P}_{m, \overrightarrow{\mathbf{a}}},\left|a_{Q}\right|<\eta_{Q}$, is an isomorphism by the open mapping theorem.


## The Finite Element Method for Singular Solutions

A right choice of the finite approximation space is crucial for the success of the finite element method, especially when we deal with singular solutions. The purpose of this chapter is to propose a construction of suitable finite approximation spaces by using techniques in weighted Sobolev spaces. It results in a sequence of (graded) triangular meshes $\mathcal{T}_{n}$ in the domain $\Omega$ that give quasi-optimal rates of convergence (1.5) for the Finite Element approximation of the mixed boundary value/interface problem (3.8).

### 4.1 Analysis of the FEM

Throughout this chapter, we make the following conventions. Let $\mathcal{K}_{\overrightarrow{\mathbf{a}}}^{m}:=\mathcal{K}_{\overrightarrow{\mathbf{a}}}^{m}(\Omega)$ by default. We assume that the boundary of $\Omega$ and the interface $\Gamma$ are piecewise linear and we fix a constant $m \in \mathbb{N}$ corresponding to the degree of approximation. For simplicity, we also assume for the theoretical analysis that there are no cracks or vertices touching the boundary, that is that $\bar{\Omega}={ }^{u} \bar{\Omega}$.

The case when $\bar{\Omega} \not{ }^{u} \bar{\Omega}$ can be addressed by using neighborhoods and distances in the topology of ${ }^{u} \bar{\Omega}$ as described in Chapter 3. In fact, we include a numerical test on a domain with a crack in Section 4.2. In these tests, the "right" space of approximation functions consists of functions defined on ${ }^{u} \bar{\Omega}$, and not on $\bar{\Omega}$ (we need different limits according to the connected component from which we approach a
crack point). Therefore the nodes used in the implementation will include the vertices of ${ }^{u} \bar{\Omega}$, counted as many times as they appear in that set. The same remark applies to ramifying cracks, where even more points have to be considered where the crack ramifies.

### 4.1.1 Approximation Away from the Vertices

We start by discussing the simpler approximation of the solution $u$ far from the singular points. We recall that all estimates in the spaces $\mathcal{K}_{\vec{a}}^{m}$ localize to subsets of $\Omega$.

Let $\mathcal{T}$ be a mesh (or triangulation) of $\Omega$. We denote by $\tilde{S}(\mathcal{T}, m)$ the Finite Element space associated to the mesh $\mathcal{T}$. That is, $\tilde{S}(\mathcal{T}, m)$ consists of all continuous functions $\chi: \bar{\Omega} \rightarrow \mathbb{R}$ such that $\chi$ coincides with a polynomial of degree $\leq m$ on each triangle $T \in \mathcal{T}$. Eventually, we will restrict ourselves to the smaller subspace $S(\mathcal{T}, m) \subset \tilde{S}(\mathcal{T}, m)$ of functions that are zero on the Dirichlet part of the boundary $\partial_{D} \Omega$.

We denote by $u_{I}=u_{I, \mathcal{T}, m} \in \tilde{S}(\mathcal{T}, m)$ the Lagrange interpolant of $u \in \mathcal{C}(\Omega)$. We recall its definition as follows. First, given a triangle $T$, let $\left[t_{0}, t_{1}, t_{2}\right]$ be the barycentric coordinates on $T$. The nodes of the degree $m$ Lagrange triangle $T$ are the points of $T$ whose barycentric coordinates $\left[t_{0}, t_{1}, t_{2}\right]$ satisfy $m t_{j} \in \mathbb{Z}$. The degree $m$ Lagrange interpolant $u_{I, \mathcal{T}, m}$ of $u$ is the unique function $u_{I, \mathcal{T}, m} \in \tilde{S}(\mathcal{T}, m)$ such that $u=u_{I, \mathcal{T}, m}$ at the nodes of each triangle $T_{i} \in \mathcal{T}$. The shorter notation $u_{I}$ will be used when only one mesh is understood in the discussion (recall that $m$ is fixed). The interpolant $u_{I}$ has the following approximation property [14, 34, 39, 92].

Theorem 4.1.1. Let $\mathcal{T}$ be a triangulation of a polygonal domain $\mathbb{P}$. Assume that all triangles $T_{i}$ in $\mathcal{T}$ have angles $\geq \alpha$ and sides of length $\leq h$. Let $u \in H^{m+1}(\mathbb{P})$ and let $u_{I}:=u_{I, \mathcal{T}, m} \in \tilde{S}(\mathcal{T}, m)$ be the degree $m$ Lagrange interpolant of $u$. Then, there exist a constants $C(\alpha, m)>0$ independent of $u$ such that

$$
\left\|u-u_{I}\right\|_{H^{1}(\mathbb{P})} \leq C(\alpha, m) h^{m}\|u\|_{H^{m+1}(\mathbb{P})} .
$$

The following estimate for the interpolation error then follows from the equivalence of the $H^{m}$-norm and the $\mathcal{K}_{\vec{a}}^{m}$-norm on proper subsets $\Omega$. If $G$ is an open
subset of $\Omega$, we define

$$
\begin{equation*}
\mathcal{K}_{\overrightarrow{\mathbf{a}}}^{m}\left(G ; r_{\Omega}\right):=\left\{f: \Omega \rightarrow \mathbb{C}, r_{\Omega}^{|\alpha|-\overrightarrow{\mathbf{a}}} \partial^{\alpha} f \in L^{2}(G), \text { for all }|\alpha| \leq m\right\} \tag{4.1}
\end{equation*}
$$

and we let $\|u\|_{\mathcal{K}_{\overrightarrow{\mathbf{a}}}^{m}\left(G ; r_{\Omega}\right)}$ denote the corresponding norm.
Proposition 4.1.2. Fix $\alpha>0$ and $0<\xi<\tilde{l}$. Let $G \subset \Omega$ be an open subset such that $r_{\Omega}>\xi$ on $G$. Let $\mathcal{T}=\left(T_{j}\right)$ be a triangulation of $\Omega$ with angles $\geq \alpha$ and sides $\leq h$. Then for each given weight $\overrightarrow{\mathbf{a}}$, there exists $C=C(\alpha, \xi, m, \overrightarrow{\mathbf{a}})>0$ such that

$$
\left\|u-u_{I}\right\|_{\mathcal{K}_{1}^{1}\left(G ; r_{\Omega}\right)} \leq C h^{m}\|u\|_{\mathcal{K}_{\overrightarrow{\mathbf{a}}+1}^{m+1}\left(G ; r_{\Omega}\right)}, \quad \forall u \in \mathcal{K}_{\overrightarrow{\mathbf{a}}+1}^{m+1}(\Omega)
$$

The next step is to extend the above estimates to hold near the vertices. To this end, we consider the behavior of the $\mathcal{K}_{\vec{a}}^{m}$ under appropriate dilations. We choose a positive number $\tilde{l}$ such that
(i) the sets $\mathcal{S}_{i}:=\Omega \cap B\left(Q_{i}, \tilde{l}\right)$ are disjoint,
(ii) $r_{\Omega}(x)=\left|x-Q_{i}\right|$ on $\mathcal{S}_{i}$,
(iii) $r_{\Omega}(x) \geq \tilde{l} / 2$ outside the set $\mathcal{S}:=\cup \mathcal{S}_{i}$,
where $B\left(Q_{i}, \tilde{l}\right)$ is the ball centered at a vertex $Q_{i}$ with radius $\tilde{l}$. Recall we assume $r_{\Omega} \in \mathcal{C}^{\infty}(\Omega)$ as in the last chapter. We note that the space $\mathcal{K}_{\overrightarrow{\mathbf{a}}}^{m}\left(\mathcal{S}_{i} ; r_{\Omega}\right)$ depends only on the weight $a_{Q_{i}}$. Hence we will denote it simply by $\mathcal{K}_{a}^{m}\left(\mathcal{S}_{i} ; r_{\Omega}\right)$ with $a=a_{Q_{i}}$. We can actually choose $\tilde{l}$ in the same way as in Chapter 2, where we used $\rho$ instead of $r_{\Omega}$ for the weight.

For the rest of this subsection, we fix a vertex $Q=Q_{i}$, and with abuse of notation we set $\mathcal{S}:=\mathcal{S}_{i}=\Omega \cap B(Q, \tilde{l})$. We then study the local behavior with respect to dilations of a function $v \in \mathcal{K}_{\overrightarrow{\mathbf{a}}}^{m}(\Omega)$ with support in the neighborhood $\mathcal{S}$ of a vertex $Q$. Therefore, we translate the origin to agree with $Q$ and call again $(x, y)$ the new coordinates. Let $G$ be a subset of $\mathcal{S}$ such that $\xi \leq r_{\Omega}(x) \leq \tilde{l}$ on $G$. For any fixed $0<\lambda<1$, we set $G^{\prime}:=\lambda G=\{\lambda x \mid x \in G\}$. Then, we define the dilated function

$$
v_{\lambda}(x, y):=v(\lambda x, \lambda y)
$$

for all $(x, y) \in G$. We observe that since $\mathcal{S}$ is a (straight) sector, if $G \subset \mathcal{S}$ then $G^{\prime} \subset \mathcal{S}$. (This definition makes sense, since $Q$ is the origin in the new coordinate system.)

Lemma 4.1.3. Let $G \subset \mathcal{S}$ and $G^{\prime}=\lambda G, 0<\lambda<1$. Then $\left\|u_{\lambda}\right\|_{\mathcal{K}_{a}^{m}\left(G ; r_{\Omega}\right)}=$ $\lambda^{a-1}\|u\|_{\mathcal{K}_{a}^{m}\left(G^{\prime} ; r_{\Omega}\right)}$ for any $u \in \mathcal{K}_{a}^{m}\left(\mathcal{S} ; r_{\Omega}\right)$.

Proof. The proof is based on the change of variables $w=\lambda x, z=\lambda y$. Note that on both $G \subset \mathcal{S}$ and $G^{\prime} \subset \mathcal{S}, r_{\Omega}(x, y)=$ the distance to $Q$, hence $r_{\Omega}(x, y)=$ $\lambda^{-1} r_{\Omega}(w, z)$. Then,

$$
\begin{aligned}
\left\|u_{\lambda}(x, y)\right\|_{\mathcal{K}_{a}^{m}\left(G ; r_{\Omega}\right)}^{2} & =\sum_{j+k \leq m} \int_{G}\left|r_{\Omega}^{j+k-a}(x, y) \partial_{x}^{j} \partial_{y}^{k} u_{\lambda}(x, y)\right|^{2} d x d y \\
& =\sum_{j+k \leq m} \int_{G^{\prime}}\left|\lambda^{a-j-k} r_{\Omega}^{j+k-a}(w, z) \lambda^{j+k} \partial_{w}^{j} \partial_{z}^{k} u(w, z)\right|^{2} \lambda^{-2} d w d z \\
& =\lambda^{2 a-2} \sum_{j+k \leq m} \int_{G^{\prime}}\left|r_{\Omega}^{j+k-a}(w, z) \partial_{w}^{j} \partial_{z}^{k} u(w, z)\right|^{2} d w d z \\
& \left.=\lambda^{2 a-2} \sum_{j+k \leq m} \| r_{\Omega}^{j+k-a}(w, z) \partial_{w}^{j} \partial_{z}^{k} u(w, z)\right) \|_{L^{2}\left(G^{\prime} ; r_{\Omega}\right)}^{2} \\
& =\lambda^{2 a-2}\|u\|_{\mathcal{K}_{a}^{m}\left(G^{\prime} ; r_{\Omega}\right)}^{2}
\end{aligned}
$$

which completes the proof.
Lemma 4.1.3, and Proposition 4.1.2 easily give the following interpolation estimate near a vertex $Q$.

Lemma 4.1.4. Let $G^{\prime} \subset \mathcal{S}$ be a subset such that $r_{\Omega}>\xi>0$ on $G^{\prime}$. Let $\mathcal{T}$ be triangulation of $G^{\prime}$ with angles $\geq \alpha$ and sides $\leq h$. Given $u \in \mathcal{K}_{a+1}^{m+1}\left(\mathcal{S}, r_{\Omega}\right), a \geq 0$, the degree $m$ Lagrange interpolant $u_{I, \mathcal{T}}$ of $u$ satisfies

$$
\left\|u-u_{I, \mathcal{T}}\right\|_{\mathcal{K}_{1}^{1}\left(G^{\prime} ; r_{\Omega}\right)} \leq C(\kappa, \alpha, m) \xi^{a}(h / \xi)^{m}\|u\|_{\mathcal{K}_{a+1}^{m+1}\left(G^{\prime} ; r_{\Omega}\right)}
$$

with $C(\kappa, \alpha, m)$ independent of $\xi, h, a$, and $u$.
Proof. We use Lemma 4.1.3 with $\lambda=\xi / \tilde{l}$. Recall the dilation function $u_{\lambda}(x, y)=$ $u(\lambda x, \lambda y)$, and note that dilation commutes with interpolation $u_{I \lambda}=u_{\lambda I}$. Let
$u_{I}:=u_{I, \mathcal{T}}$. Thus, we can apply Proposition 4.1.2 to the region $G=\lambda^{-1} G^{\prime} \subset \mathcal{S}$,

$$
\begin{aligned}
\left\|u-u_{I}\right\|_{\mathcal{K}_{1}^{1}\left(G^{\prime} ; r_{\Omega}\right)} & =\left\|u_{\lambda}-u_{I \lambda}\right\|_{\mathcal{K}_{1}^{1}\left(G ; r_{\Omega}\right)} \\
& =\left\|u_{\lambda}-u_{\lambda I}\right\|_{\mathcal{K}_{1}^{1}\left(G ; r_{\Omega}\right)} \\
& \leq M(h / \lambda)^{m}\left\|u_{\lambda}\right\|_{\mathcal{K}_{1}^{m+1}\left(G ; r_{\Omega}\right)} \\
& =M(h \tilde{l} / \xi)^{m}\|u\|_{\mathcal{K}_{1}^{m+1}\left(G^{\prime} ; r_{\Omega}\right)} \\
& \leq C \xi^{a}(h / \xi)^{m}\|u\|_{\mathcal{K}_{1+a}^{m+1}\left(G^{\prime} ; r_{\Omega}\right)}
\end{aligned}
$$

where the last inequality is from Lemma 2.3.6.
This lemma will be used for $\xi \rightarrow 0$, while Proposition 4.1 .2 will be used with a fixed $\xi$.

### 4.1.2 Approximation Near the Vertices

We are now ready to address approximation near the singular points. To this extent, we work with the smaller Finite Element Space $S(\mathcal{T}, m)$ defined for any mesh $\mathcal{T}$ of $\Omega$ as

$$
\begin{equation*}
S(\mathcal{T}, m):=\tilde{S}(\mathcal{T}, m) \cap \mathcal{H}_{\overrightarrow{\mathbf{a}}}=\left\{\chi \in \tilde{S}(\mathcal{T}, m), \chi=0 \text { on } \partial_{D} \Omega\right\} \tag{4.2}
\end{equation*}
$$

where $\mathcal{H}_{\overrightarrow{\mathbf{a}}}=\left\{u \in \mathcal{K}_{1+\overrightarrow{\mathbf{a}}}^{1}(\Omega), u=0\right.$ on $\left.\partial_{D} \Omega\right\}$. This definition takes into account that the variational space associated to the mixed boundary value/interface prob$\operatorname{lem}(3.1)$ is $\mathcal{H}_{\overrightarrow{\mathrm{a}}}$.
Remark 4.1.5. Recall that when the interface is not smooth or there are NeummanNeumann vertices, by Theorem 3.1.2 for any $\left|a_{Q}\right|<\eta_{Q}$ the variational solution $u$ of (3.1) can be written $u=u_{\text {reg }}+w_{s}$ with $u_{\mathrm{reg}}: \Omega \rightarrow \mathbb{R},\left.u_{\mathrm{reg}}\right|_{\Omega_{j}} \in \mathcal{K}_{\overrightarrow{\mathbf{a}+1}}^{m+1}\left(\Omega_{j}\right)$, and $w_{s} \in W_{s}$. The space $W_{s}$ is the linear span of functions $\chi_{i} \in \mathcal{C}^{\infty}(\Omega)$, one for each Neumann-Neumann or interface vertex $Q_{i}$, such that $\chi_{i}$ equals 1 on $\mathcal{S}_{i}$ and satisfying $\partial^{\nu} \chi_{i}=0$ on $\partial \Omega$. For each vertex $Q$, we therefore fix $a_{Q} \in\left(0, \eta_{Q}\right)$, and we let $\epsilon=\min \left\{a_{Q}\right\}$. With this choice, we have that $u_{\text {reg }} \in H^{1+\epsilon}(\Omega) \subset \mathcal{C}(\bar{\Omega})$, so that the interpolants of $u$ can be defined directly, since $W_{s}$ consists of smooth functions. Moreover, the condition that $r_{\Omega}^{-(1+\epsilon)} u_{\text {reg }}$ be integrable in a neighborhood of each


Figure 4.1. One refinement of the triangle $T$ with vertex $Q, \kappa=l_{1} / l_{2}$
vertex shows that $u_{\text {reg }}$ must vanish at each vertex. Therefore $u(Q)=w(Q)$ for each Neumann-Neumann or interface vertex $Q$.

We now ready to introduce the mesh refinement procedure. For each vertex $Q$, we choose a number $\kappa_{Q} \in(0,1 / 2]$ and set $\kappa=\left(\kappa_{Q}\right)$.

Definition 4.1.6. Let $\mathcal{T}$ be a triangulation of $\Omega$ such that no two vertices of $\Omega$ belong to the same triangle of $\mathcal{T}$. The $\kappa$ refinement of $\mathcal{T}$, denoted by $\kappa(\mathcal{T})$ is obtained by dividing each side $A B$ of $\mathcal{T}$ in two parts as follows. If neither $A$ nor $B$ is a vertex, then we divide $A B$ into two equal parts. Otherwise, if $A$ is a vertex, we divide $A B$ into $A C$ and $C B$ such that $|A C|=\kappa_{Q}|A B|$.

This procedure will divide each triangle $T$ into four triangles. (See Figure 4.1).
Definition 4.1.7. We define by induction $\mathcal{T}_{n+1}=\kappa\left(\mathcal{T}_{n}\right)$, where the initial mesh $\mathcal{T}_{0}$ is such that every vertex of $\Omega$ is a vertex of a triangle in $\mathcal{T}_{0}$ and all sides of the interface $\Gamma$ coincide with sides in the mesh. In addition, we chose $\mathcal{T}_{0}$ such that there is no triangle that contains more than one vertex and each edge in the mesh has length $\leq \tilde{l} / 2$ (with $\tilde{l}$ chosen as in Subsection 4.1.1). See also Definition 2.3.2.

We observe that, near the vertices, this refinement coincides with the ones introduced in $[7,17,21,88]$ for the Dirichlet problem. One of the main results of this work is to show that the same type of mesh gives optimal rates of convergence for mixed boundary value and interface problems as well.

We denote by $u_{I, n}=u_{I, \mathcal{T}_{n}, m} \in S_{n}:=S\left(\mathcal{T}_{n}, m\right)$ the degree $m$ Lagrange interpolant associated to $u \in \mathcal{C}(\bar{\Omega})$ and the mesh $\mathcal{T}_{n}$ on $\Omega$. We investigate the approximation properties afforded by the triangulation $\mathcal{T}_{n}$ close to a fixed vertex $Q$. The most interesting cases are when $Q$ is either a Neumann-Neumann vertex or a vertex of the interface. We shall therefore assume that this is the case in what
follows. With abuse of notation we let $a=a_{Q}$ and $\kappa=\kappa_{Q}$ with $\kappa_{Q} \in\left(0,2^{-m / a_{Q}}\right)$. We also fix a triangle $T \in \mathcal{T}_{0}$ that has $Q$ as a vertex. Then Theorem 3.2.7 gives that the solution $u=u_{\text {reg }}+w_{s}$ with $u_{\text {reg }} \in \mathcal{K}_{a+1}^{m+1}\left(T ; r_{\Omega}\right)$ and $w_{s} \in W_{s}$ if $f \in \mathcal{K}_{\overrightarrow{\mathbf{a}}-1}^{m-1}\left(\Omega_{j}\right)$ and $T \subset \Omega_{j}$.

We next let $T_{\kappa^{n}}=\kappa^{n} T \in \mathcal{T}_{n}$ be the triangle that is similar to $T$ with ratio $\kappa^{j}$, have $Q$ as a vertex, and have all sides parallel to the sides of $T$. Then $T_{\kappa^{n}} \subset T_{\kappa^{n-1}}$ for $n \geq 1$ (with $T_{\kappa^{o}}=T$ ). Furthermore, since $\kappa<1 / 2$ and the diameter of $T$ is $\leq \tilde{l} / 2, T_{\kappa^{j}} \subset \mathcal{S}=B(Q, \tilde{l}) \cap \Omega$ for all $n \geq 0$. Recall that we assume all functions in $W_{s}$ are constant on neighborhoods of vertices. We continue to fix $T \in T_{0}$ with vertex $Q$. The following interpolation estimate holds.

Lemma 4.1.8. Let $0<\kappa=\kappa_{Q} \leq 2^{-m / a_{Q}}, 0<a=a_{Q}<\eta_{Q}$. Let $T_{\kappa^{N}}=$ $\kappa^{N} T \subset T$ be the triangle with vertex $Q$ obtained from $T$ after $N$ refinements. Let $u_{I, N}$ be the degree $m$ Lagrange interpolant of $u$ associated to $\mathcal{T}_{N}$. Then, if $u \in\left(\mathcal{K}_{a+1}^{m+1}\left(\mathcal{S} ; r_{\Omega}\right)+W_{s}\right) \cap\left\{\left.u\right|_{\partial_{D} \Omega}=0\right\}$, on $T_{\kappa^{N}} \in \mathcal{T}_{N}$

$$
\left\|u-u_{I, N}\right\|_{\mathcal{K}_{1}^{1}\left(T_{\kappa^{N}} ; r_{\Omega}\right)} \leq C 2^{-m N}\left\|u_{\mathrm{reg}}\right\|_{\mathcal{K}_{a+1}^{m+1}\left(T_{\kappa^{N}} ; r_{\Omega}\right)}
$$

where $C$ depends on $m$ and $\kappa$, but not on $N$.
Proof. By hypothesis $u=u_{\mathrm{reg}}+w$, with $u_{\mathrm{reg}} \in \mathcal{K}_{a+1}^{m+1}(\Omega)$ and $w \in W_{s}$. To simplify the notation, we let $\phi=u_{\text {reg }}$. By Remark 4.1.5, if $N$ is large enough we can assume that $w=u(Q)$ is a constant on $T_{\kappa^{N}}$. We again denote the dilated function $\phi_{\lambda}(x, y)=\phi(\lambda x, \lambda y)$, where $(x, y)$ are coordinated at $Q$ and $0<\lambda<1$. We choose $\lambda=\kappa^{N-1}$. Then, $\phi_{\lambda}(x, y) \in \mathcal{K}_{a+1}^{m+1}\left(T_{\kappa} ; r_{\Omega}\right)$ by Lemma 4.1.3. We next introduce the auxiliary function $v=\chi \phi_{\lambda}$ on $T_{\kappa}$, where $\chi: T_{\kappa} \rightarrow[0,1]$ is a smooth function that depends only on $r_{\Omega}$ and is equal to 0 in a neighborhood of $Q$, but is equal to 1 at all the nodal points different from $Q$. Consequently,

$$
\|v\|_{\mathcal{K}_{1}^{m+1}\left(T_{\kappa} ; r_{\Omega}\right)}^{2}=\left\|\chi \phi_{\lambda}\right\|_{\mathcal{K}_{1}^{m+1}\left(T_{\kappa} ; r_{\Omega}\right)}^{2} \leq C\left\|\phi_{\lambda}\right\|_{\mathcal{K}_{1}^{m+1}\left(T_{\kappa} ; r_{\Omega}\right)}^{2},
$$

where $C$ depends on $m$ and the choice of the nodal points. Moreover, since $\phi(Q)=$ 0 by Remark 4.1.5, the interpolant of $v$ if given by $v_{I}=\left(\phi_{\lambda}\right)_{I}=\left(\phi_{I}\right)_{\lambda}$ on $T_{\kappa}$. We also observe that the interpolant of $w$ on $T_{\kappa^{N}}$ is equal to $w$, because they are both
constants, and hence $u-u_{I}=\phi-\phi_{I}$. Therefore

$$
\begin{aligned}
& \left\|u-u_{I}\right\|_{\mathcal{K}_{1}^{1}\left(T_{\kappa^{N}} ; r_{\Omega}\right)}=\left\|\phi-\phi_{I}\right\|_{\mathcal{K}_{1}^{1}\left(T_{\kappa^{N}} ; r_{\Omega}\right)}=\left\|\phi_{\lambda}-\phi_{\lambda_{I}}\right\|_{\mathcal{K}_{1}^{1}\left(T_{\kappa} ; r_{\Omega}\right)} \\
& \quad=\left\|\phi_{\lambda}-v+v-\phi_{\lambda I}\right\|_{\mathcal{K}_{1}^{1}\left(T_{\kappa} ; r_{\Omega}\right)} \leq\left\|\phi_{\lambda}-v\right\|_{\mathcal{K}_{1}^{1}\left(T_{\kappa} ; r_{\Omega}\right)}+\left\|v-\phi_{\lambda I}\right\|_{\mathcal{K}_{1}^{1}\left(T_{\kappa} ; r_{\Omega}\right)} \\
& =\left\|\phi_{\lambda}-v\right\|_{\mathcal{K}_{1}^{1}\left(T_{\kappa} ; r_{\Omega}\right)}+\left\|v-v_{I}\right\|_{\mathcal{K}_{1}^{1}\left(T_{\kappa} ; r_{\Omega}\right)} \leq C_{1}\left\|\phi_{\lambda}\right\|_{\mathcal{K}_{1}^{1}\left(T_{\kappa} ; r_{\Omega}\right)}+C_{2}\|v\|_{\mathcal{K}_{1}^{m+1}\left(T_{\kappa} ; r_{\Omega}\right)} \\
& \leq C_{1}\left\|\phi_{\lambda}\right\|_{\mathcal{K}_{1}^{1}\left(T_{\kappa} ; r_{\Omega}\right)}+C_{3}\left\|\phi_{\lambda}\right\|_{\mathcal{K}_{1}^{m+1}\left(T_{\kappa} ; r_{\Omega}\right)}=C_{1}\|\phi\|_{\mathcal{K}_{1}^{1}\left(T_{\kappa^{N}} ; r_{\Omega}\right)}+C_{3}\|\phi\|_{\mathcal{K}_{1}^{m+1}\left(T_{\kappa^{N}} ; r_{\Omega}\right)} \\
& \leq C_{4} \kappa^{N a}\|\phi\|_{\mathcal{K}_{a+1}^{m+1}\left(T_{\kappa^{N}} ; r_{\Omega}\right)} \leq C 2^{-m N}\|\phi\|_{\mathcal{K}_{a+1}^{m+1}\left(T_{\kappa^{N}} ; r_{\Omega}\right)}=C 2^{-m N}\left\|u_{\mathrm{reg}}\right\|_{\mathcal{K}_{a+1}^{m+1}\left(T_{\kappa^{N}} ; r_{\Omega}\right)},
\end{aligned}
$$

which gives the desired inequality. The second and the eighth relations above are due to Lemma 4.1.3, and the sixth is due to Proposition 4.1.2.

We now combine the bounds on $T_{\kappa^{N}}$ of the previous lemma with the bounds on sets of the form $T_{\kappa^{j}} \backslash T_{\kappa^{j+1}}$ of Lemma 4.1.4 to obtain the following estimate on an arbitrary, but fixed, triangle $T \in \mathcal{T}_{0}$ that has a vertex $Q$ in common with $\Omega$ (the more difficult case not handled by Proposition 4.1.2).

Proposition 4.1.9. Let $T \in \mathcal{T}_{0}$ such that a vertex $Q$ of $T$ belongs to $\mathcal{V}$. Let $0<\kappa_{Q} \leq 2^{-m / a_{Q}}, 0<a_{Q}<\eta_{Q}$. Then there exists a constant $C>0$, such that

$$
\left\|u-u_{I, N}\right\|_{\mathcal{K}_{1}^{1}\left(T ; r_{\Omega}\right)} \leq C 2^{-m N}\left\|u_{\mathrm{reg}}\right\|_{\mathcal{K}_{a+1}^{m+1}\left(T ; r_{\Omega}\right)}
$$

for all $u=u_{\mathrm{reg}}+w$, where $w \in W_{s}$ and $u_{\mathrm{reg}} \in \mathcal{K}_{a+1}^{1}(\Omega)$ is such that $u_{\mathrm{reg}} \in$ $\mathcal{K}_{a+1}^{m+1}\left(\Omega_{j}\right)$, for all $j$.

Proof. As before, we set $\kappa_{Q}=\kappa$ and $a_{Q}=a$. As in the proof of Lemma 4.1.8, we have $u-u_{I}=u_{\text {reg }}-u_{\text {reg }, I}$.

Definition 4.1.7 shows that the mesh size of $T_{\kappa^{j-1}} \backslash T_{\kappa^{j}}$ is $\simeq \kappa^{j-1} 2^{j-1-N}$. Then, $r_{\Omega}(x) \geq \xi$ with $\xi=O\left(\kappa^{j-1}\right)$ on $T_{\kappa^{j-1}} \backslash T_{\kappa^{j}}$, so that Lemma 4.1.4 yields

$$
\begin{aligned}
\left\|u-u_{I}\right\|_{\mathcal{K}_{1}^{1}\left(T_{\kappa j} j-1 \backslash T_{\kappa} j ; r_{\Omega}\right)} & \leq C_{1} \kappa^{(j-1) a} 2^{m(j-1-N)}\left\|u_{\mathrm{reg}}\right\|_{\mathcal{K}_{a+1}^{m+1}\left(T_{\kappa j} j-1 \backslash T_{\kappa^{j}} ; r_{\Omega}\right)} \\
& \leq C_{2} 2^{-(j-1) m} 2^{-N m+(j-1) m}\left\|u_{\mathrm{reg}}\right\|_{\mathcal{K}_{a+1}^{m+1}\left(T_{\kappa} j-1 \backslash T_{\kappa j} ; r_{\Omega}\right)} \\
& =C 2^{-N m}\left\|u_{\mathrm{reg}}\right\|_{\mathcal{K}_{a+1}^{m+1}\left(T_{\kappa} j-1 \backslash T_{\kappa} j ; r_{\Omega}\right)}
\end{aligned}
$$

where $C$ depends on $\kappa$, but not on the subset $T_{\kappa^{j-1}} \backslash T_{\kappa^{j}}$. We then complete the proof of Proposition 6.3 .8 by adding up these error estimates on all the subsets
$T_{\kappa^{j-1}} \backslash T_{\kappa^{j}}, 1 \leq j \leq N$, and the estimate of the interpolation error on $T_{\kappa^{N}}$ given in Lemma 4.1.8, .

Remark 4.1.10. If $\mathbb{T}$ denotes the union of all the initial triangles that contain vertices of $\Omega$, then $\mathbb{T}$ is a neighborhood of the set of vertices in $\bar{\Omega}$. Furthermore, the interpolation error on $\mathbb{T}$ is obatined as $\left\|u-u_{I}\right\|_{\mathcal{K}_{1}^{1}\left(\mathbb{T} ; r_{\Omega}\right)} \leq C 2^{-m N}\left\|u_{\text {reg }}\right\|_{\mathcal{K}_{a+1}^{m+1}\left(\mathbb{T} ; r_{\Omega}\right)}$ by summing up the squares of the estimates in Proposition 6.3.8 over all the triangles, as long as $\kappa_{Q}$ is chosen appropriately.

We now combine all previous results to obtain a global interpolation error estimate on $\Omega$.

Theorem 4.1.11. Let $m \geq 1$ and for each vertex $Q \in \mathcal{V}$ fix $0<a_{Q}<\eta_{Q}$ and $0<\kappa_{Q}<2^{-m / a_{Q}}$. Assume that the conditions of Theorem 3.2.7 are satisfied and let $u$ be the corresponding solution problem (3.8) with $f: \Omega \rightarrow \mathbb{R}$ such that $f \in \mathcal{K}_{\overrightarrow{\mathbf{a}}-1}^{m-1}\left(\Omega_{j}\right)$ for all $j$. Let $\mathcal{T}_{n}$ be the $n$-th refinement of an intial triangulation $\mathcal{T}_{o}$ as in Definition 4.1.7. Let $S_{n}:=S_{n}\left(\mathcal{T}_{n}, m\right)$ be the associated Finite Element space given in equation (4.2) and let $u_{n}=u_{S_{n}} \in S_{n}$ be the Finite Element solution. Then there exists $C>0$ such that

$$
\left\|u-u_{n}\right\|_{\mathcal{K}_{1}^{1}(\Omega)} \leq C 2^{-m n} \sum_{j}\|f\|_{\mathcal{K}_{\overrightarrow{\mathbf{a}}-1}^{m-1}\left(\Omega_{j}\right)}
$$

Proof. Let $\mathbb{T}_{i}$ be the union of initial triangles that contain a given vertex $Q_{i}$. Recall from Theorem 3.2.7 that the solution of problem (3.8) can be written as $u=u_{\text {reg }}+w$ with $w \in W_{s}$ and $\|w\|+\sum_{j}\left\|u_{\text {reg }}\right\|_{\mathcal{K}_{\overrightarrow{\mathbf{a}}+1}^{m+1}\left(\Omega_{j}\right)} \leq C \sum_{j}\|f\|_{\mathcal{K}_{\overrightarrow{\mathbf{a}}-1}^{m-1}\left(\Omega_{j}\right)}$. Because $u-u_{I}=u_{\mathrm{reg}}-u_{\mathrm{reg}, I}$ on $\mathcal{S}_{i}$, we use the previous estimates to obtain

$$
\begin{aligned}
\left\|u-u_{n}\right\|_{\mathcal{K}_{1}^{1}(\Omega)} & \leq C\left\|u-u_{I}\right\|_{\mathcal{K}_{1}^{1}(\Omega)} \\
& \leq C \sum_{j}\left(\left\|u-u_{I}\right\|_{\mathcal{K}_{1}^{1}\left(\Omega_{j} \backslash \cup \mathbb{T}_{i} ; r_{\Omega}\right)}+\sum\left\|u_{\mathrm{reg}}-u_{\mathrm{reg}, I}\right\|_{\mathcal{K}_{1}^{1}\left(\Omega_{j} \cap \mathbb{T}_{i} ; r_{\Omega}\right)}\right) \\
& \leq C 2^{-m n} \sum_{j}\left(\|u\|_{\mathcal{K}_{1}^{m+1}\left(\Omega_{j} \backslash \cup \mathbb{T}_{i} ; r_{\Omega}\right)}+\sum_{j}\left\|u_{\mathrm{reg}}\right\|_{\mathcal{K}_{\overrightarrow{\mathbf{a}}+1}^{m+1}\left(\Omega_{j} \cap \mathbb{T}_{i} ; r_{\Omega}\right)}\right) \\
& \leq C 2^{-m n}\left(\sum_{j}\left\|u_{\mathrm{reg}}\right\|_{\mathcal{K}_{\overrightarrow{\mathbf{a}}+1}^{m+1}\left(\Omega_{j}\right)}+\|w\|\right) \leq C 2^{-m n} \sum_{j}\|f\|_{\mathcal{K}_{\overrightarrow{\mathbf{a}}-1}^{m-1}\left(\Omega_{j}\right)} .
\end{aligned}
$$

The first inequality is based on Céa's Lemma and the third inequality follows from

Propositions 4.1.2 and 6.3.8.
We can finally state the main result of this section, namely the quasi-optimal convergence rate of the Finite Element solution computed using the meshes $\mathcal{T}_{n}$.

Theorem 4.1.12. Under the notation and assumptions of Theorem 4.1.11, $u_{n}=$ $u_{S_{n}} \in S_{n}:=S\left(\mathcal{T}_{n}, m\right)$ satisfies

$$
\left\|u-u_{n}\right\|_{\mathcal{K}_{1}^{1}(\Omega)} \leq C \operatorname{dim}\left(S_{n}\right)^{-m / 2} \sum_{j}\|f\|_{\mathcal{K}_{\overrightarrow{\mathbf{a}}-1}^{m-1}\left(\Omega_{j}\right)}
$$

for a constant $C>0$ independent of $f$ and $n$.
Proof. Let again $\mathcal{T}_{n}$ be the triangulation of $\Omega$ after $n$ refinements. Then, the number of triangles is $O\left(4^{n}\right)$ given the refinement procedure of Definition 4.1.6. Therefore $\operatorname{dim}\left(S_{n}\right) \simeq 4^{n}$ so that Theorem 4.1.11 gives

$$
\left\|u-u_{n}\right\|_{\mathcal{K}_{1}^{1}(\Omega)} \leq C 2^{-n m} \sum_{j}\|f\|_{\mathcal{K}_{a-1}^{m-1}\left(\Omega_{j}\right)} \leq C \operatorname{dim}\left(S_{n}\right)^{-m / 2} \sum_{j}\|f\|_{\mathcal{K}_{a-1}^{m-1}\left(\Omega_{j}\right)} .
$$

The proof is complete.
Using that $H^{m-1}\left(\Omega_{j}\right) \subset \mathcal{K}_{\overrightarrow{\mathbf{a}}-1}^{m-1}\left(\Omega_{j}\right)$ if $a_{Q} \in(0,1)$ for all vertices $q$, we obtain the following corollary.

Corollary 4.1.13. Let $0<a_{Q} \leq \min \left\{1, \eta_{Q}\right\}$ and $0<\kappa_{Q}<2^{-m / a_{Q}}$ for each vertex $Q \in \mathcal{V}$. Then, under the hypotheses of Theorem 4.1.12,

$$
\left\|u-u_{n}\right\|_{H^{1}(\Omega)} \leq C\left\|u-u_{n}\right\|_{\mathcal{K}_{1}^{1}(\Omega)} \leq C \operatorname{dim}\left(S_{n}\right)^{-m / 2} \sum_{j}\|f\|_{H^{m-1}\left(\Omega_{j}\right)}
$$

for a constant $C>0$ independent of $f \in H^{m-1}(\Omega)$ and $n$.
Note that we do not claim that $u \in H^{m+1}(\Omega)$, which is in general not true.

### 4.2 Numerical Tests

In this section, we present numerical examples which test for the quasi-optimal rates of convergence established a priori in the previous section. The convergence
history of the Finite Element solution supports our results. The Finite Element solution $u_{n} \in S_{n}$ is defined by

$$
\begin{equation*}
a\left(u_{n}, v_{n}\right):=\sum_{i, j=1}^{2} \int_{\Omega} A^{i j} \partial_{i} u_{n} \partial_{j} v_{n} d x=\left(f, v_{n}\right), \quad \forall v_{n} \in S_{n} \tag{4.3}
\end{equation*}
$$

for the operator $P=-\operatorname{div} A \Delta$. To verify the theoretical prediction, we focus on the more challenging problem where Neumann-Neumann vertices and interfaces are present. We start by testing different configurations of mixed Dirichlet/Neumann boundary conditions, but no interface, on several different domains for the simple model problem (4.4),

$$
\left\{\begin{align*}
-\Delta u=1 & \text { in } \Omega  \tag{4.4}\\
u=0 & \text { on } \partial_{D} \Omega \\
\partial_{\nu} u=0 & \text { on } \partial_{N} \Omega
\end{align*}\right.
$$

In particular, we consider non-convex domains $\Omega$ with a crack. In this case, the optimal grading can be computed explicitly beforehand. We then perform a test for the model transmission problem

$$
\left\{\begin{align*}
-\operatorname{div}(a(x, y) \nabla u)=1 & \text { in } \Omega  \tag{4.5}\\
u=0 & \text { on } \partial \Omega
\end{align*}\right.
$$

where $a$ is a piece-wise constant function. We have run also a few tests with $m=2$, which also seem to confirm our theoretical results. However, more refinement steps seem to be necessary in this case to achieve results that are as convincing as in the case $m=1$. Thus more powerful (i.e., faster) algorithms and codes will need to be used to test the case $m=2$ completely. Below, all tests are for $m=1$. Under the consideration that corner singularities for homogeneous Dirichlet boundary conditions have been well known in the literature, we mainly concentrate on other problems that are covered by our theory.


Figure 4.2. Domain with crack: initial triangles (left); triangulation after one refinement, $\kappa=0.2$ (right)

### 4.2.1 Domains with Cracks and Neumann-Neumann Vertices

We discuss the results of two tests for the mixed boundary value problem (4.4). In the first test, we impose pure Dirichlet boundary conditions, i.e., , we take $\partial_{D} \Omega=\partial \Omega$, but on a domain with a crack. Specifically, we let $\Omega=(0,1) \times(0,1) \backslash$ $\{(x, 0.5), 0<x<0.5\}$ with the tip of the crack at the point $(0.5,0.5)$ (see Figure 4.2). The presence of the crack forces a singularity in $H^{2}$ at the tip of the crack. By the arguments in Section 4.1, any mesh grading $0<a<\eta=\pi / 2 \pi=1 / 2$ should yield quasi-optimal rates of convergence as long as the decay ratio $\kappa$ of triangles in subsequent refinements satisfies $\kappa=2^{-1 / a}<2^{-1 / \eta}=0.25$ near the crack tip. In fact, in this case the solution is $H^{2}$ away from the crack, but is only in $H^{s}$, $s<1+\eta=1.5$, near the crack (following [60]). Recall that the mesh size $h$ after $j$ refinements is $O\left(2^{j}\right)$. Thus, quasi-uniform meshes should give a convergence rate no better than $h^{0.5}$ [98].

In the second test, $\Omega$ is the non-convex domain of Figure 4.3 with a reentrant vertex $Q$. The interior angle at $Q$ is $1.65 \pi$. We impose Neumann boundary conditions on both sides adjacent to the vertex $Q$, and Dirichlet boundary conditions on other edges. Again, the solution will have a singularity in $H^{2}$ at the reentrant corner In this case, the arguments of Sections 3.2 and 4.1 imply that we can take $0<a<\eta=\pi / 1.65 \pi \approx 0.61$ for the mesh grading, and consequently, the quasioptimal rates of convergence should be recovered as long as the decay ratio $\kappa$ of triangles in subsequent refinements satisfies $\kappa=2^{-1 / a}<2^{-1 / \eta} \approx 0.32$ near $Q$.

| $j \backslash \kappa$ | $e: \kappa=0.1$ | $e: \kappa=0.2$ | $e: \kappa=0.3$ | $e: \kappa=0.4$ | $e: \kappa=0.5$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| 3 | 0.76 | 0.79 | 0.79 | 0.83 | 0.77 |
| 4 | 0.88 | 0.90 | 0.89 | 0.82 | 0.76 |
| 5 | 0.94 | 0.95 | 0.91 | 0.79 | 0.70 |
| 6 | 0.97 | 0.97 | 0.92 | 0.76 | 0.63 |
| 7 | 0.99 | 0.98 | 0.91 | 0.73 | 0.57 |
| 8 | 0.99 | 0.98 | 0.91 | 0.71 | 0.54 |
| 9 | 1.00 | 0.99 | 0.90 | 0.69 | 0.52 |

Table 4.1. Convergence history in the case of a crack domain.


Figure 4.3. Initial triangles for a Neumann-Neumann vertex $Q$ (left); the triangulation after one refinement, $\kappa=0.2$ (right).


Figure 4.4. The numerical solution for the mixed problem and a Neumann-Neumann vertex.

| $j \backslash \kappa$ | $e: \kappa=0.1$ | $e: \kappa=0.2$ | $e: \kappa=0.4$ | $e: \kappa=0.5$ |
| :--- | :--- | :--- | :--- | :--- |
| 3 | 0.91 | 0.93 | 0.95 | 0.94 |
| 4 | 0.96 | 0.97 | 0.97 | 0.96 |
| 5 | 0.98 | 0.99 | 0.98 | 0.95 |
| 6 | 0.99 | 1.00 | 0.98 | 0.93 |
| 7 | 1.00 | 1.00 | 0.97 | 0.89 |
| 8 | 1.00 | 1.00 | 0.96 | 0.84 |

Table 4.2. Convergence history in the case of a Neumann-Neumann vertex.

The convergence history for the FEM solutions in the two tests are given respectively in Table 4.1 and Table 4.2. Both tables confirm the predicted rates of convergence. The most left column in each table of this section contains the number of refinements from the initial triangulation of the domain. In each of the other columns, we list the convergence rate of the numerical solution for the problem (4.4) computed by the formula

$$
\begin{equation*}
e=\log _{2}\left(\frac{\left|u_{j-1}-u_{j}\right|_{H^{1}}}{\left|u_{j}-u_{j+1}\right|_{H^{1}}}\right), \tag{4.6}
\end{equation*}
$$

where $u_{j}$ is the Finite Element solution after $j$ mesh refinements. Therefore, since the dimension of the space $S_{n}$ grows by the factor of 4 with every refinement for linear finite element approximations, $e$ should be very close to 1 if the numerical solutions yield quasi-optimal rates of convergence, an argument strongly verified in the two tables. In Table 4.2, for example, we achieve quasi-optimal convergence rate whenever the decay ratio $\kappa<0.32$, since $e \rightarrow 1$ after a few refinements. On the other hand, if $\kappa>0.32$, the convergence rates decrease with successive refinements due to the effect of the singularity at $Q$. In fact, for $\kappa=0.5$ we expect the values of $e$ to approach 0.61 , which is the asymptotical convergence rate on quasi-uniform meshes for a function in $H^{1.61}$.

### 4.2.2 Domains with Artificial Vertices

We discuss again a test for the model mixed boundary value problem (4.4), but now we test convergence in the presence of an artificial vertex, where the boundary conditions change on a given side. We take the domain to be the unit square $\Omega=$


Figure 4.5. Domain with artificial vertex: initial triangles (left); triangulation after four refinements, $\kappa=0.2$ (right).

| $j \backslash \kappa$ | $e: \kappa=0.1$ | $e: \kappa=0.2$ | $e: \kappa=0.3$ | $e: \kappa=0.4$ | $e: \kappa=0.5$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| 3 | 0.84 | 0.87 | 0.87 | 0.84 | 0.81 |
| 4 | 0.91 | 0.93 | 0.91 | 0.85 | 0.77 |
| 5 | 0.95 | 0.95 | 0.91 | 0.82 | 0.70 |
| 6 | 0.97 | 0.96 | 0.91 | 0.78 | 0.63 |
| 7 | 0.99 | 0.97 | 0.90 | 0.75 | 0.57 |
| 8 | 0.99 | 0.98 | 0.90 | 0.72 | 0.54 |
| 9 | 1.00 | 0.98 | 0.89 | 0.70 | 0.52 |

Table 4.3. Convergence history in the case of an artificial vertex.
$(0,1) \times(0,1)$ and we impose the the mixed boundary conditions $\partial_{N} \Omega=\{(x, 0), 0<$ $x<0.5\}, \partial_{D} \Omega=\Omega \backslash \partial_{N} \Omega$ (see Figure 4.5). In this case, the solution is $H^{2}$ near all geometric vertices, as the interior angle is $\pi / 2$, but it does possess a singularity at the artificial vertex $Q=(0.5,0)$, where the boundary conditions change. Near such a vertex, the maximum mesh grading from Section 3.2 is $\eta_{Q}=0.5 \pi / \pi=0.5$. Then, quasi-optimal rates of convergence can be obtained on graded meshes if the decay ratio $\kappa$ of triangles in subsequent refinements satisfies $o<\kappa=2^{-1 / a}<$ $2^{-1 / \eta}=0.25$ near the singular point $(0.5,0)$. The optimal rate is again supported by the convergence history of the numerical solution in Table 4.3.)

### 4.2.3 Transmission Problems

We discuss finally a test for the model transmission problem (4.5), The singularities in the solution arise from jumps in the coefficient $a$ across the interface.


Figure 4.6. The transmission problem: initial triangles (left); triangulation after four refinements, $\kappa=0.2$ (right).

As discussed in Section 4.1, quasi-optimal rates of convergence can be achieved a priori by organizing triangles in the initial triangulation so that each side on the interface is a side of one of the triangles as well. We verify a posteriori that this construction yields the predicted rates of convergence. We choose the domain again to be the square $\Omega=(-1,1) \times(-1,1)$ with a single, but nonsmooth, interface $\Gamma$ as in Figure 4.6, which identifies two subdomains $\Omega_{j}, j=1,2$. We also pick the coefficient $a(x, y)$ in (4.5) of the form

$$
a(x, y)= \begin{cases}1 & \text { on } \Omega_{1} \\ 30 & \text { on } \Omega_{2}\end{cases}
$$

The large jump across the interface makes the numerical analysis more challenging. The solution of (4.5) may have singularities in $H^{2}$ at the points $Q_{1}=(-1,1)$, $Q_{2}=(1,0)$ where the interface joins the boundary, and at $Q_{3}=(0,0)$, which is a vertex for the interface (there are no singularities again in $H^{2}$ at the square geometric vertices).

Again based on the results of Sections 3.2 and 4.1, for each singular point $Q_{1}$, $i=1,2,3$, there exists a positive number $\eta_{i}$, depending on the interior angle and the coefficients, such that, if the decay rate $\kappa_{i}$ of triangles in successive refinements satifies $0<\kappa_{i}<2^{-1 / \eta_{i}}$ near each vertex $Q_{1}$, quasi-optimal rates of convergence can be obtained for the finite element solution. We observe that the solution


Figure 4.7. The numerical solution for the transmission problem.

| $j \backslash \kappa$ | $e: \kappa=0.1$ | $e: \kappa=0.2$ | $e: \kappa=0.3$ | $e: \kappa=0.4$ | $e: \kappa=0.5$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| 3 | 0.82 | 0.83 | 0.84 | 0.83 | 0.78 |
| 4 | 0.91 | 0.91 | 0.91 | 0.90 | 0.83 |
| 5 | 0.97 | 0.97 | 0.96 | 0.94 | 0.86 |
| 6 | 0.99 | 0.99 | 0.98 | 0.95 | 0.85 |
| 7 | 1.00 | 0.99 | 0.99 | 0.95 | 0.82 |
| 8 | 1.00 | 1.00 | 1.00 | 0.95 | 0.80 |

Table 4.4. Convergence history for the transmission problem
belongs to $H^{2}$ in the neighborhood of a vertex, whenever $\eta_{i} \geq 1$, and therefore, a quasi-uniform mesh near that vertex is sufficient in this case.

Instead of computing $\eta_{i}$ explicitly, as a formula is not readily available, we test different values of $\kappa_{i}<0.5$ near each singular points until we obtain values of $e$ approaching 1. This limit signals, as discussed above, that we have reached quasi-optimal rates of convergence for the numerical solution. The value of $e$ is given in equation (4.6). Once again, the convergence history in Table 4.4 strongly supports the theoretical findings. In particular, no special mesh grading is needed near the points $(-1,1)$ and $(1,0)$. Near the internal vertex $(0,0)$, however, we found the optimal grading ratio to be $\kappa_{3} \in(0.3,0.4)$, in agreement with the results of Theorem 3.2.7 and Theorem 4.1.12). Figure 4.6 shows the mesh refinement near $(0,0)$ when $\kappa=0.2$.

\section*{|  |
| :---: |
| Chapter |}

## The Multigrid Method on Graded Meshes

The multigrid (MG) method is arguably one of the most efficient techniques for solving the large systems of algebraic equations resulting from finite element discretizations of elliptic boundary value problems. Many of the known results on the convergence properties of MG methods for elliptic equations can be found in monographs and survey papers by Bramble [29], Hackbusch [57], Trottenberg, Oosterlee and Schüller [96], Xu [99] and the references therein.

Typical techniques in the analysis of the MG method need assumptions on the regularity of the solution and on the uniformity of the mesh in the FEM. We have succeeded in recovering the quasi-optimal rate of convergence in the FEM by a special construction of the finite element subspaces in Chapter 4 based on a priori estimates in weighted Sobolev spaces from Chapter 3. Note both assumptions above (regularity of the solution and uniformity of the mesh) for the MG method are violated here. Thus, the analysis of the MG method and the development of effective MG solvers require new treatments for singular solutions.

For a brief literature review, a result for the uniform convergence of the multigrid method assuming full regularity was derived by Braess and Hackbusch in [28]; in Brenner's paper [35], the analysis of the convergence rate for only partial regularity was presented; Bramble, Pasciak, Wang and Xu [30] developed the convergence estimate without regularity assumptions for an $L^{2}$-projection based decomposition. In addition, on graded meshes, using the approximation property in [17],

Yserentant [102] proved the uniform convergence of the multigrid $W$-cycle with a particular iterative method on each level for piecewise linear functions. There are also many other more classical convergence proofs that use algebraic techniques and derive convergence results based on assumptions related to, but nevertheless different from, the regularity of the underlying PDE [32, 97].

In this chapter, using a space decomposition for elliptic projections and the estimate on the weighted Sobolev space $\mathcal{K}_{a}^{m}$ (Theorem 3.1.1, for example), we prove the uniform convergence of the multigrid $V$-cycle with standard subspace smoothers (Richardson, weighted Jacobi, Gauss-Seidel, etc.) for elliptic problems with corner/mixed singularities, discretized using graded meshes and linear finite element functions from the previous chapter. To date, this type of convergence analysis has only been carried out for problems with full elliptic regularity. The result presented here establishes the uniform convergence of the MG method for problems with less regular solutions discretized using graded meshes that appropriately capture the correct behavior of the solution near the singularities. Although the main convergence theorem can be modified for elliptic problems discretized on general graded meshes, for exposition, we restrict our discussion to the graded mesh refinement strategy introduced in Definition 4.1.7. Before proceeding, we mention that, with appropriate modifications, our analysis for linear elements can also be applied to higher-order finite element methods. Also, for simplicity, we here only consider singular solutions of the Poisson equation from geometric vertices with Dirichlet boundary conditions on adjacent sides or singular solutions form changes of boundary conditions, even thought the theory extends to other singularities (Neumann-Neumann vertices, transmission problems, etc.) of general elliptic equations .

### 5.1 Preliminaries and Notation

Let $\Omega$ be a bounded polygonal domain, possibly with cracks, in $\mathbb{R}^{2}$ and consider the following prototype elliptic equation with mixed boundary conditions

$$
\left\{\begin{align*}
-\Delta u=f & \text { in } \quad \Omega  \tag{5.1}\\
u=0 & \text { on } \quad \partial_{D} \Omega \\
\partial u / \partial n=0 & \text { on } \quad \partial_{N} \Omega
\end{align*}\right.
$$

where $\partial_{D} \Omega$ and $\partial_{N} \Omega$ consist of segments of the boundary, and we assume the Neumann boundary condition is not imposed on adjacent sides of the boundary. We note that, in the Sobolev space $H^{m}$, singularities appear in the solution near vertices of the domain. Here, by vertices, we mean the points on $\bar{\Omega}$ where singularities in $H^{2}(\Omega)$ are located, namely, the geometric vertices on reentrant corners, crack points, or points with an interior angle $\theta>\pi / 2$, where the boundary conditions change.

Let $H_{D}^{1}(\Omega)=\left\{u \in H^{1}(\Omega) \mid u=0\right.$ on $\left.\partial_{D} \Omega\right\}$ be the space of $H^{1}(\Omega)$ functions with zero trace on $\partial_{D} \Omega, \mathcal{T}_{j}, 0 \leq j \leq J$, be a sequence of appropriately graded and nested triangulations of $\Omega$, and $\mathcal{M}_{j}, 0 \leq j \leq J$, be the finite element space associated with the linear Lagrange triangle [39] on $\mathcal{T}_{j}$. Note $\mathcal{M}_{n}=S_{n}$ from Chapter 4. Then,

$$
\mathcal{M}_{0} \subset \mathcal{M}_{1} \subset \ldots \subset \mathcal{M}_{j} \subset \ldots \subset \mathcal{M}_{J} \subset H_{D}^{1}(\Omega)
$$

Let $A$ be the differential operator associated with equation (5.1). Solving (5.1) amounts to finding an approximation $u_{J} \in \mathcal{M}_{J}$ such that

$$
a\left(u_{J}, v_{J}\right)=\left(A u_{J}, v_{J}\right)=\left(\nabla u_{J}, \nabla v_{J}\right)=\left(f, v_{J}\right), \quad \forall v_{J} \in \mathcal{M}_{J}
$$

Denoting by $N_{J}$ the dimension of the space $\mathcal{M}_{J}$, recall that on good graded meshes (Definition 4.1.7), one can recover the following quasi-optimal rate of convergence for the finite element approximation $u_{J} \in \mathcal{M}_{J}$ on $\mathcal{T}_{J}$ :

$$
\left\|u-u_{J}\right\|_{H^{1}(\Omega)} \leq C N_{J}^{-1 / 2}\|f\|_{L^{2}(\Omega)}
$$

The main objective of this chapter is to prove the uniform convergence of the multigrid $V$-cycle with standard subspace smoothers (Richardson, weighted Jacobi, Gauss-Seidel, etc.) and linear interpolation applied to the 2D Poisson equation discretized using piecewise linear functions on graded meshes obtained via the grading strategy introduced in the previous chapter. Moreover, we shall show that the convergence rate, $c$, of the MG $V$-cycle satisfies

$$
c \leq \frac{c_{1}}{c_{1}+c_{2} n}
$$

where $c_{1}, c_{2}$ are mesh-independent constants related to the elliptic equation and the smoother, respectively, and $n$ is the number of iterative solves on each subspace. We note that this result can also be used to estimate the efficiency of other subspace smoothers on graded meshes.

In Section 5.2, we recall specific properties of the weighted Sobolev space $\mathcal{K}_{a}^{m}(\Omega)$ for boundary value problem (5.1) that are useful for our MG analysis, and review the method of subspace corrections. Then, in Section 5.3, we prove the approximation and smoothing properties which in turn lead to our main MG convergence theorem. Section 5.4 contains numerical results of the proposed method applied to problem (5.1).

### 5.2 Weighted Sobolev Spaces and the Method of Subspace Corrections

In this section, we begin by recalling the weighted Sobolev space $\mathcal{K}_{a}^{m}(\Omega)$ and the mesh refinement strategy under consideration for recovering quasi-optimal rates of convergence of the finite element solution. Then, we present the method of subspace corrections and a technique for estimating the norm of the product of non-expansive operators.

### 5.2.1 Weighted Sobolev Spaces and Graded Meshes

It has been shown in the last chapter, that with a careful choice of the parameters in the weight, the singular behavior of the solution in Equation (5.1) can be captured
well in the weighted Sobolev spaces (Equation (3.11)). Namely, there is no loss of regularity of the solution in these spaces and the corresponding refinements of meshes are optimal in the sense of Theorem 5.2.2 below.

Let $\mathcal{V}=\left\{Q_{i}\right\}$ be the set of vertices of the domain, on which the solution has singularities in $H^{2}(\Omega)$. Recall the definition of the weighted Sobolev space in Chapter 3 and its norm. Thus, to simplify our presentation, we have summarized the following proposition.

Proposition 5.2.1. We have $|v|_{\mathcal{K}_{1}^{1}(\Omega)} \simeq|v|_{H^{1}(\Omega)},\|v\|_{\mathcal{K}_{1}^{0}(\Omega)} \geq C| | v \|_{L^{2}(\Omega)}$ and the Poincare type inequality $\|v\|_{\mathcal{K}_{1}^{0}(\Omega)} \leq C|v|_{\mathcal{K}_{1}^{1}(\Omega)}$ for $v \in \mathcal{K}_{1}^{1}(\Omega) \cap\left\{\left.v\right|_{\partial_{D} \Omega}=0\right\}$.

Here, $a \simeq b$ means there exist positive constants $C_{1}, C_{2}$, such that $C_{1} b \leq a \leq C_{2} b$ as in the previous chapter. Also, we recall the $\kappa$-refinement from Definition 4.1.7. We note that other refinements, for example, those found in $[7,17]$ also satisfy this condition, although they follow different constructions. Then, for Equation (5.1), the quasi-optimal convergence rate for the finite element solution can be stated as follows.

Theorem 5.2.2. Let $u_{j} \in \mathcal{M}_{j}$ be the finite element solution of Equation (5.1), and denote by $N_{j}$ the dimension of $\mathcal{M}_{j}$. Then, there exists a constant $B_{1}=B_{1}(\Omega, \kappa, \epsilon)$, such that

$$
\left\|u-u_{j}\right\|_{H^{1}(\Omega)} \leq B_{1} N_{j}^{-1 / 2}\|f\|_{\mathcal{K}_{\epsilon-1}^{0}(\Omega)} \leq B_{1} N_{j}^{-1 / 2}\|f\|_{L^{2}(\Omega)}
$$

for every $f \in L^{2}(\Omega)$, where $0<\epsilon<\eta<1$, and $\eta$ is the smallest positive number in all $\Sigma_{Q_{i}}$ (See Equation (3.15))., $\mathcal{M}_{j}$ is the finite element space of linear functions on the graded mesh $\mathcal{T}_{j}$, as described in the introduction.

Remark 5.2.3. For $u \notin H^{2}(\Omega)$, this theorem follows from the fact that the differential operator $A: \mathcal{K}_{1+\epsilon}^{m+1}(\Omega) \cap\left\{u=0\right.$, on $\left.\partial_{D} \Omega\right\} \rightarrow \mathcal{K}_{-1+\epsilon}^{m-1}(\Omega), m \geq 0$, in Equation (5.1), is an isomorphism between the weighted Sobolev spaces.

### 5.2.2 The Method of Subspace Corrections

In this subsection, we review the method of subspace corrections and provide an identity for estimating the norm of the product of non-expansive operators. In
addition, Lemma 5.2.4 reveals the connection between the matrix representation and operator representation of the MG method.

Let $H_{D}^{1}(\Omega)=\left\{u \in H^{1}(\Omega) \mid u=0\right.$ on $\left.\partial_{D} \Omega\right\}$ be the Hilbert space associated with Equation (5.1), $\mathcal{T}_{j}$ be the associated graded mesh, as defined in Definition 4.1.7, $\mathcal{M}_{j} \in H_{D}^{1}(\Omega)$ be the space of piecewise linear functions on $\mathcal{T}_{j}$, and $A: H_{D}^{1}(\Omega) \rightarrow$ $\left(H_{D}^{1}(\Omega)\right)^{\prime}$ be the corresponding differential operator. The weak form for (5.1) is then

$$
a(u, v)=(A u, v)=(-\Delta u, v)=(\nabla u, \nabla v)=(f, v), \quad \forall v \in H_{D}^{1}(\Omega)
$$

where the pairing $(\cdot, \cdot)$ is the inner product in $L^{2}(\Omega)$. Here, $a(\cdot, \cdot)$ is a continuous bilinear form on $H_{D}^{1}(\Omega) \times H_{D}^{1}(\Omega)$ and by the Poincare inequality is also coercive. In addition, since the $\mathcal{T}_{j}$ are nested,

$$
\mathcal{M}_{0} \subset \mathcal{M}_{1} \subset \ldots \subset \mathcal{M}_{j} \subset \ldots \subset \mathcal{M}_{J} \subset H_{D}^{1}(\Omega)
$$

Define $S_{j}, P_{j}: H_{D}^{1}(\Omega) \rightarrow \mathcal{M}_{j}$ and $A_{j}: \mathcal{M}_{j} \rightarrow \mathcal{M}_{j}$ as orthogonal projectors and the restriction of $A$ on $\mathcal{M}_{j}$, respectively:

$$
\begin{aligned}
& \left(S_{j} u, v_{j}\right)=\left(u, v_{j}\right), \quad a\left(P_{j} u, v_{j}\right)=a\left(u, v_{j}\right), \\
& \left(A u_{j}, v_{j}\right)=\left(A_{j} u_{j}, v_{j}\right), \quad \forall u \in H_{D}^{1}(\Omega), \quad \forall u_{j}, v_{j} \in \mathcal{M}_{j} .
\end{aligned}
$$

Let $\mathcal{N}_{j}=\left\{x_{i}^{j}\right\}$ be the set of nodal points in $\mathcal{T}_{j}$ and $\phi_{k}\left(x_{i}^{j}\right)=\delta_{i, k}$ be the linear finite element nodal basis function corresponding to node $x_{k}^{j}$. Then, the $j$ th level finite element discretization reads: Find $u_{j} \in \mathcal{M}_{j}$, such that

$$
\begin{equation*}
A_{j} u_{j}=f_{j} \tag{5.2}
\end{equation*}
$$

where $f_{j} \in \mathcal{M}_{j}$ satisfies $\left(f_{j}, v_{j}\right)=\left(f, v_{j}\right), \forall v_{j} \in \mathcal{M}_{j}$.
The method of subspace corrections (MSC) reduces a multigrid process to choosing a sequence of subspaces and corresponding operators $B_{j}: \mathcal{M}_{j} \rightarrow \mathcal{M}_{j}$ approximating $A_{j}^{-1}, j=1, \ldots, J$. For example, in the MSC framework, the standard multigrid backslash cycle for solving (5.2) is defined by the following subspace
correction scheme

$$
u_{j, l}=u_{j, l-1}+B_{j}\left(f_{j}-A_{j} u_{j, l-1}\right),
$$

where the operators $B_{j}: \mathcal{M}_{j} \rightarrow \mathcal{M}_{j}, 0 \leq j \leq J$, are recursively defined as follows [100].

Algorithm 5.2.1. Let $R_{j} \approx A_{j}^{-1}, j>0$, denote a local relaxation method. For $j=0$, define $B_{0}=A_{0}^{-1}$. Assume that $B_{j-1}: \mathcal{M}_{j-1} \rightarrow \mathcal{M}_{j-1}$ is defined. Then,

1. Fine grid smoothing: For $u_{j}^{0}=0$ and $k=1,2, \cdots, n$,

$$
\begin{equation*}
u_{j}^{k}=u_{j}^{k-1}+R_{j}\left(f_{j}-A_{j} u_{j}^{k-1}\right) \tag{5.3}
\end{equation*}
$$

2. Coarse grid correction: Find the corrector $e_{j-1} \in \mathcal{M}_{j-1}$ by the iterator $B_{j-1}$

$$
e_{j-1}=B_{j-1} Q_{j-1}\left(f_{j}-A_{j} u_{j}^{n}\right)
$$

Then, $B_{j} f_{j}=u_{j}^{n}+e_{j-1}$.
Recursive application of Algorithm 5.2.1 results in an MG $V$-cycle for which the following identity holds : $I-B_{J}^{v} A_{J}=\left(I-B_{J} A_{J}\right)^{*}\left(I-B_{J} A_{J}\right)$ [100], where $B_{J}^{v}$ is the iterator for the MG $V$-cycle. Direct computation gives the following useful result

$$
\begin{aligned}
u_{j}^{n} & =\left(I-R_{j} A_{j}\right) u_{j}^{n-1}+R_{j} A_{j} u_{j} \\
& =\left(I-R_{j} A_{j}\right)^{2} u_{j}^{n-2}-\left(I-R_{j} A_{j}\right)^{2} u_{j}+u_{j} \\
& =-\left(I-R_{j} A_{j}\right)^{n} u_{j}+u_{j},
\end{aligned}
$$

where $u_{j}$ is the finite element solution of (5.2) and $u_{j}^{n}$ is the approximation after $n$ iterations of (5.3) on the $j$ th level. Let $T_{j}=\left(I-\left(I-R_{j} A_{j}\right)^{n}\right) P_{j}$ be a linear operator and define $T_{0}=P_{0}$. We have the following identity:

$$
\begin{aligned}
\left(I-B_{J} A_{J}\right) u_{J} & =u_{J}-u_{J}^{n}-e_{J-1}=\left(I-T_{J}\right) u_{J}-e_{J-1} \\
& =\left(I-B_{J-1} A_{J-1} P_{J-1}\right)\left(I-T_{J}\right) u_{J},
\end{aligned}
$$

where, for $B_{J-1}=A_{J-1}^{-1}$, this becomes a two-level method. Recursive application
of this identity then yields the error propagation operator of a Multigrid V-cycle:

$$
\left(I-B_{J} A_{J}\right)=\left(I-T_{0}\right)\left(I-T_{1}\right) \cdots\left(I-T_{J}\right)
$$

To estimate the uniform convergence of the multigrid $V$-cycle, we thus need to show that

$$
\left\|I-B_{J}^{v} A_{J}\right\|_{a}=\left\|I-B_{J} A_{J}\right\|_{a}^{2} \leq c<1,
$$

where $c$ is independent of $J$ and $\|u\|_{a}^{2}=a(u, u)=(A u, u)$ on $\Omega$.
Associated with each $T_{j}$, we introduce its symmetrization

$$
\bar{T}_{j}=T_{j}+T_{j}^{*}-T_{j}^{*} T_{j},
$$

where $T_{j}^{*}$ is the adjoint operator of $T_{j}$ with respect to the inner product $a(\cdot, \cdot)$. By a well-known result found in [101], the following estimate holds

$$
\left\|I-B_{J} A_{J}\right\|_{a}^{2}=\frac{c_{0}}{1+c_{0}},
$$

where

$$
\begin{equation*}
c_{0} \leq \sup _{\|v\|_{a}=1} \sum_{j=1}^{J} a\left(\left(\bar{T}_{j}^{-1}-I\right)\left(P_{j}-P_{j-1}\right) v,\left(P_{j}-P_{j-1}\right) v\right) \tag{5.4}
\end{equation*}
$$

Now, to prove the uniform convergence of the proposed MG scheme, we must derive a uniform bound on the constant $c_{0}$.

Although the above presentation is in terms of operators, the matrix representation of the smoothing step (5.3) is often used in practice. By the matrix representation $\mathbf{R}$ of an operator $R$ on $\mathcal{M}_{j}$, we here mean that with respect to the basis $\left\{\phi_{i}\right\}_{i=1}^{N_{j}}$ of $\mathcal{M}_{j}$,

$$
R\left(\phi_{k}\right)=\sum_{i=1}^{N_{j}} \mathbf{R}_{i, k} \phi_{i},
$$

where $\mathbf{R}_{i, k}$ is the ( $i, k$ ) component of the matrix $\mathbf{R}$. Throughout the chapter, we use boldfaced letters to denote vectors and matrices.

Let $\mathbf{A}_{S}=\mathbf{D}-\mathbf{L}-\mathbf{U}$ be the stiffness matrix associated with the operator $A_{j}$,
where the matrix $\mathbf{D}$ only consists of the diagonal entries of $\mathbf{A}_{S}$, while matrices $-\mathbf{L}$ and $-\mathbf{U}$ are the strictly lower and upper triangular parts of $\mathbf{A}_{S}$, respectively. Denote by $\mathbf{R}_{M}$ the corresponding matrix of the smoother $R_{j}$ on the $j$ th level. For example, $\mathbf{R}_{M}=\mathbf{D}^{-1}$ for the Jacobi method, and $\mathbf{R}_{M}=(\mathbf{D}-\mathbf{L})^{-1}$ for the Gauss-Seidel method. In addition, let $\mathbf{u}^{l}, \mathbf{u}^{l-1}$, and $\mathbf{f}$ be the vectors containing the coordinates of $u_{j}^{l}, u_{j}^{l-1}, f_{j} \in \mathcal{M}_{j}$ on the basis $\left\{\phi_{i}\right\}_{i=1}^{N_{i}}$, namely $u_{j}^{l}=\sum_{i=1}^{N_{j}} \mathbf{u}_{i}^{l} \phi_{i}$. Then, one smoothing step for solving (5.2) on a single level $j$ in terms of matrices reads

$$
\begin{equation*}
\mathbf{u}^{l}=\mathbf{u}^{l-1}+\mathbf{R}_{M}\left(\mathbf{M f}-\mathbf{A}_{S} \mathbf{u}^{l-1}\right) \tag{5.5}
\end{equation*}
$$

where $\mathbf{M}$ is the mass matrix, and $\mathbf{M}_{i, k}=\left(\phi_{i}, \phi_{k}\right)$.
Lemma 5.2.4. Let $\mathbf{R}$ be the matrix representation of the smoother $R_{j}$ in Equation (5.3). Then,

$$
\mathbf{R}=\mathbf{R}_{M} \mathbf{M}
$$

Hence,

$$
R_{j}\left(\phi_{k}\right)=\sum_{i=1}^{N_{j}} \mathbf{R}_{i, k} \phi_{i}=\sum_{i=1}^{N_{j}}\left(\mathbf{R}_{M} \mathbf{M}\right)_{i, k} \phi_{i}
$$

and

$$
\mathbf{u}^{l}=\mathbf{u}^{l-1}+\mathbf{R}_{M}\left(\mathbf{M f}-\mathbf{A}_{S} \mathbf{u}^{l-1}\right)=\mathbf{u}^{l-1}+\mathbf{R}\left(\mathbf{f}-\mathbf{M}^{-1} \mathbf{A}_{S} \mathbf{u}^{l-1}\right)
$$

Proof. Denote by A the matrix representation of the operator $A$. Note that

$$
\left(A \phi_{i}, \phi_{k}\right)=\left(\sum_{m=1}^{N_{j}} \mathbf{A}_{m, i} \phi_{m}, \phi_{k}\right)=\left(\nabla \phi_{k}, \nabla \phi_{i}\right)=\left(\mathbf{A}_{S}\right)_{k, i}
$$

indicates $\mathbf{A}_{S}=\mathbf{M A}$. Moreover, In terms of matrices and vectors, Equation (5.3) also reads

$$
\sum_{i=1}^{N_{j}} \mathbf{u}_{i}^{l} \phi_{i}=\sum_{i=1}^{N_{j}} \mathbf{u}_{i}^{l-1} \phi_{i}+\sum_{i=1}^{N_{j}} \sum_{k=1}^{N_{j}} \mathbf{R}_{k, i} \mathbf{f}_{i} \phi_{k}-\sum_{i=1}^{N_{j}} \sum_{k=1}^{N_{j}} \sum_{m=1}^{N_{j}} \mathbf{R}_{m, k} \mathbf{A}_{k, i} \mathbf{u}_{i} \phi_{m}
$$

Then, the inner product with $\phi_{n}$ on both sides, $1 \leq n \leq N_{j}$, leads to

$$
\mathbf{M u}^{l}=\mathbf{M u}^{l-1}+\mathbf{M R f}-\mathbf{M R A u} .
$$

Multiplication by $\mathbf{M}^{-1}$ gives

$$
\mathbf{u}^{l}=\mathbf{u}^{l-1}+\mathbf{R}(\mathbf{f}-\mathbf{A} \mathbf{u})
$$

Taking into account that Equations (3) and (5) represent the same iteration, we have

$$
\mathbf{R f}=\mathbf{R}_{M} \mathbf{M f}
$$

Note the above equation holds for any $\mathbf{f} \in \mathbb{R}^{N_{j}}$. Therefore, $\mathbf{R}=\mathbf{R}_{M} \mathbf{M}$, which completes the proof.

### 5.3 Uniform Convergence of the MG Method on Graded Meshes

Next, we derive an estimate for the constant $c_{0}$ in (5.4) of Section 5.2 and then proceed to establish the main convergence theorem of the chapter. We begin by proving several lemmas that are needed in the convergence proof. For simplicity, we assume that there is only a single point $Q_{0} \in \bar{\Omega}$, for which the solution of Equation (5.1) has a singularity in $H^{2}(\Omega)$, and that a nested sequence of graded meshes has been constructed, as described in Definition 4.1.7. The same argument, however, carries over to problems on domains with multiple singularities and also for similar refinement strategies.

Denote by $\left\{T_{i}^{Q_{0}}\right\}$ all the initial triangles with the common vertex $Q_{0}$. Recall that the function $r_{\Omega}$ in the weight equals the distance to $Q_{0}$ on these triangles. Based on the process in Definition 4.1.7, after $N$ refinements, the region $\cup T_{i}^{Q_{0}}$ is partitioned into $N+1$ sub-domains (layers) $D_{n}, 0 \leq n \leq N$, whose sizes decrease by the factor $\kappa$ as they approach $Q_{0}$ (See Figure 5.1). In addition, $r_{\Omega}(x, y) \simeq \kappa^{n}$ on



Figure 5.1. Initial triangles with vertex $Q_{0}$ (left); layer $D_{0}$ and $D_{1}$ after one refinement (right), $\kappa=0.2$.
$D_{n}$ for $0 \leq n<N$ and $r_{\Omega}(x, y) \leq C \kappa^{N}$ on $D_{N}$. Meanwhile, sub-triangles (nested meshes) are generated in these layers $D_{n}, 0 \leq n \leq N$, with corresponding mesh size of order $O\left(\kappa^{n} 2^{n-N}\right)$.

Note that $\Omega=\left(\cup D_{n}\right) \cup\left(\Omega \backslash \cup D_{n}\right)$. Let $\partial D_{n}$ be the boundary of $D_{n}$. Then, we define a piecewise constant function $r_{p}(x, y)$ on $\bar{\Omega}$ as follows.

$$
r_{p}(x, y)=\left\{\begin{array}{l}
(1 / 2 \kappa)^{n} \text { on } \bar{D}_{n} \backslash \partial D_{n-1}, \quad \text { for } \quad 1<n \leq N \\
1 \text { otherwise },
\end{array}\right.
$$

where $N=J$ is the number of refinements for $\mathcal{T}_{J}$. Therefore, the restriction of $r_{p}$ on every $T_{i}^{Q_{0}} \cap D_{n}$ is a constant. Recall that $\epsilon<1$ is the parameter for $\kappa$, such that $\kappa=2^{-1 / \epsilon}$. Define the weighted inner product with respect to $r_{p}$,

$$
(u, v)_{r_{p}}=\left(r_{p} u, r_{p} v\right)=\int_{\Omega} r_{p}^{2} u v
$$

In addition, the above inner product induces the norm,

$$
\|u\|_{r_{p}}=(u, u)_{r_{p}}^{1 / 2} .
$$

Then, the following estimate holds.

## Lemma 5.3.1.

$$
\left(u_{j}-P_{j-1} u_{j}, u_{j}-P_{j-1} u_{j}\right)_{r_{p}} \leq \frac{c_{1}}{N_{j}} a\left(u_{j}-P_{j-1} u_{j}, u_{j}-P_{j-1} u_{j}\right), \quad \forall u_{j} \in \mathcal{M}_{j},
$$

where $N_{j}=O\left(2^{2 j}\right)$ is the dimension of $\mathcal{M}_{j}$.
Proof. This lemma can be proved by the duality argument as follows.
Consider the following boundary value problem

$$
\left\{\begin{aligned}
&-\Delta w=r_{p}^{2}\left(u_{j}-P_{j-1} u_{j}\right) \quad \text { in } \Omega \\
& w=0 \\
& \text { on } \quad \partial_{D} \Omega \\
& \partial w / \partial n=0 \text { on } \partial_{N} \Omega
\end{aligned}\right.
$$

Then, since $P_{j-1} w \in \mathcal{M}_{j-1}$, from the equation above, we have

$$
\begin{aligned}
\left(r_{p}\left(u_{j}-P_{j-1} u_{j}\right), r_{p}\left(u_{j}-P_{j-1} u_{j}\right)\right) & =\left(r_{p}^{2}\left(u_{j}-P_{j-1} u_{j}\right), u_{j}-P_{j-1} u_{j}\right) \\
& =\left(\nabla w, \nabla\left(u_{j}-P_{j-1} u_{j}\right)\right) \\
& =\left(\nabla\left(w-P_{j-1} w\right), \nabla\left(u_{j}-P_{j-1} u_{j}\right)\right) .
\end{aligned}
$$

We note that $\Delta w$ is a piecewise linear function on the graded triangulation $\mathcal{T}_{j}$ that is derived after $j$ refinements. From the results of Theorem 5.2.2, we conclude

$$
\begin{aligned}
\left|w-P_{j-1} w\right|_{H^{1}(\Omega)}^{2} & \leq\left(C_{1} / N_{j-1}\right)\|\Delta w\|_{\mathcal{K}_{\epsilon-1}^{0}(\Omega)}^{2} \\
& =\left(C_{1} / N_{j-1}\right)\left(\sum_{n=0}^{j}\left\|r_{\Omega}^{1-\epsilon} \Delta w\right\|_{L^{2}\left(D_{n}\right)}^{2}+\left\|r_{\Omega}^{1-\epsilon} \Delta w\right\|_{L^{2}\left(\Omega \backslash \cup D_{n}\right)}^{2}\right) \\
& \leq\left(C / N_{j-1}\right)\left(\sum_{n=0}^{j}\left\|\kappa^{n(1-\epsilon)} \Delta w\right\|_{L^{2}\left(D_{n}\right)}^{2}+\|\Delta w\|_{L^{2}\left(\Omega \backslash \cup D_{n}\right)}^{2}\right) \\
& =\left(C / N_{j-1}\right)\left(\sum_{n=0}^{j}\left\|2^{n} \kappa^{n} \Delta w\right\|_{L^{2}\left(D_{n}\right)}^{2}+\|\Delta w\|_{L^{2}\left(\Omega \backslash \cup D_{n}\right)}^{2}\right) \\
& =\left(C / N_{j-1}\right)\left(\sum_{n=0}^{j}\left\|r_{p}^{-1} \Delta w\right\|_{L^{2}\left(D_{n}\right)}^{2}+\|\Delta w\|_{L^{2}\left(\Omega \backslash \cup D_{n}\right)}^{2}\right) \\
& =\left(C / N_{j-1}\right)\left\|r_{p}^{-1} \Delta w\right\|_{L^{2}(\Omega)}^{2} .
\end{aligned}
$$

The inequalities above are based on the definition of $\kappa, r_{p}$ and related norms. Now, since, $N_{j}=O\left(N_{j-1}\right)$, combining the results above, we have

$$
\left\|u_{j}-P_{j-1} u_{j}\right\|_{r_{p}}^{2} \leq \frac{\left|w-P_{j-1} w\right|_{H^{1}}^{2}\left|u_{j}-P_{j-1} u_{j}\right|_{H^{1}}^{2}}{\left\|\left(u_{j}-P_{j-1} u_{j}\right)\right\|_{r_{p}}^{2}}
$$

$$
\begin{aligned}
& =\frac{\left|w-P_{j-1} w\right|_{H^{1}}^{2}\left|u_{j}-P_{j-1} u_{j}\right|_{H^{1}}^{2}}{\left\|r_{p}^{-1} \Delta w\right\|_{L^{2}}^{2}} \\
& \leq \frac{c_{1}}{N_{j}}\left|u_{j}-P_{j-1} u_{j}\right|_{H^{1}}^{2}=\frac{c_{1}}{N_{j}} a\left(u_{j}-P_{j-1} u_{j}, u_{j}-P_{j-1} u_{j}\right),
\end{aligned}
$$

which completes the proof.
Recall that the matrix form $\mathbf{R}_{M}$ and the matrix representation $\mathbf{R}$ of a smoother $R_{j}$ are different from Lemma 5.2.4. Then, we have the following result regarding the smoother $\bar{R}_{j}=R_{j}+R_{j}^{t}-R_{j}^{t} A_{j} R_{j}$ on $\mathcal{M}_{j}$, which is the symmetrization of $R_{j}$, where $R_{j}^{t}$ is the adjoint of $R_{j}$ with respect to $(\cdot, \cdot)$.

Lemma 5.3.2. For the subspace smoother $\bar{R}_{j}: \mathcal{M}_{j} \rightarrow \mathcal{M}_{j}$, assume there is a constant $C>0$ independent of $j$, such that the corresponding matrix form $\overline{\mathbf{R}}_{M}$ satisfies

$$
\mathbf{v}^{T} \overline{\mathbf{R}}_{M} \mathbf{v} \geq C \mathbf{v}^{T} \mathbf{v}, \quad \forall \mathbf{v} \in \mathbb{R}^{N_{j}}
$$

on every level $j$, where $N_{j}$ is the dimension of the subspace $\mathcal{M}_{j}$. Then, there exists $c_{2}>0$, also independent of the level $j$, such that the following estimate holds on each graded mesh $\mathcal{T}_{j}$,

$$
\frac{c_{2}}{N_{j}}\left(\bar{R}_{j} v, v\right) \leq\left(\bar{R}_{j} v, \bar{R}_{j} v\right)_{r_{p}}, \quad \forall v \in \mathcal{M}_{j}
$$

Proof. For any $v=\sum_{i} \mathbf{v}_{i} \phi_{i} \in \mathcal{M}_{j}$, from Lemma 5.2.4, we have

$$
\left(\bar{R}_{j} v, v\right)=\left(\sum_{m} \mathbf{v}_{m} \sum_{k}\left(\overline{\mathbf{R}}_{M} \mathbf{M}\right)_{k, m} \phi_{k}, \sum_{i} \mathbf{v}_{i} \phi_{i}\right)=\mathbf{v}^{T} \mathbf{M}^{T} \overline{\mathbf{R}}_{M} \mathbf{M} \mathbf{v}
$$

On the other hand,

$$
\begin{aligned}
\left(\bar{R}_{j} v, \bar{R}_{j} v\right)_{r_{p}} & =\left(\sum_{m} \mathbf{v}_{m} \sum_{k}\left(\overline{\mathbf{R}}_{M} \mathbf{M}\right)_{k, m} \phi_{k}, \sum_{l} \mathbf{v}_{l} \sum_{i}\left(\overline{\mathbf{R}}_{M} \mathbf{M}\right)_{i, l} \phi_{i}\right) \\
& =\mathbf{v}^{T} \mathbf{M}^{T} \overline{\mathbf{R}}_{M} \tilde{\mathbf{M}} \overline{\mathbf{R}}_{M} \mathbf{M} \mathbf{v}
\end{aligned}
$$

where $\tilde{\mathbf{M}}$ is a matrix satifying $(\tilde{\mathbf{M}})_{i, k}=\left(r_{p} \phi_{i}, r_{p} \phi_{k}\right)$. Note that both $\mathbf{M}$ and $\tilde{\mathbf{M}}$ are symmetric positive definite (SPD). Now, suppose $\left.\operatorname{supp} \phi_{i}\right) \cap D_{n} \neq \emptyset, 0 \leq n \leq j$.

Then, on supp $\phi_{i}$ ), the mesh size is $O\left(\kappa^{n} 2^{n-j}\right)$ and $r_{p} \simeq(1 / 2 \kappa)^{n}$, respectively, since $\left.\operatorname{supp} \phi_{i}\right)$ is covered by at most two adjacent layers. Thus, all the non-zero elements in $\tilde{\mathbf{M}}$ are positive and $\tilde{\mathbf{M}} \simeq 2^{-2 j} \simeq 1 / N_{j}$. To complete the proof, it is sufficient to show that there exist $C>0$, such that

$$
\mathbf{w}^{T} \overline{\mathbf{R}}_{M}^{1 / 2} \tilde{\mathbf{M}} \overline{\mathbf{R}}_{M}^{1 / 2} \mathbf{w} \geq\left(C / N_{j}\right) \mathbf{w}^{T} \mathbf{w}
$$

where $\mathbf{w}=\overline{\mathbf{R}}_{M}^{1 / 2} \mathbf{M} \mathbf{v}$.
From the condition on $\overline{\mathbf{R}}_{M}$ and the estimates on $\tilde{\mathbf{M}}$, it follows that

$$
\mathbf{w}^{T} \overline{\mathbf{R}}_{M}^{1 / 2} \tilde{\mathbf{M}} \overline{\mathbf{R}}_{M}^{1 / 2} \mathbf{w} \simeq\left(1 / N_{j}\right) \mathbf{w}^{T} \overline{\mathbf{R}}_{M} \mathbf{w} \geq\left(C / N_{j}\right) \mathbf{w}^{T} \mathbf{w}
$$

Remark 5.3.3. For our choice of graded meshes, the triangles remain shape-regular elements, that is, the minimum angles of the triangles are bounded away from 0 . Therefore, the stiffness matrix $\mathbf{A}_{S}$ has a bounded number of nonzero entries per row and each entry is of order $O(1)$. Hence, the maximum eigenvalue of $\mathbf{A}_{S}$ is bounded. For this reason, standard smoothers (Richardson, weighted Jacobi, Gauss-Seidel, etc.) satisfy Lemma 5.3.2, and $\left(\mathbf{R}_{M}\right)_{i, j}=O(1)$ as well, since they are all from part of the matrix $\mathbf{A}_{S}$. Moreover, if $\mathbf{R}_{M}$ is SPD and the spectral radius $r_{\Omega}\left(\mathbf{R}_{M} \mathbf{A}_{S}\right) \leq \omega$, for $0<\omega<1$, then based on Lemma 5.2.4,

$$
\begin{aligned}
a\left(R_{j} A_{j} v, v\right) & =\left(A_{j} R_{j} A_{j} v, v\right) \\
& =\mathbf{v}^{T} \mathbf{A}_{S} \mathbf{R}_{M} \mathbf{A}_{S} \mathbf{v} \\
& \leq \omega a(v, v)
\end{aligned}
$$

The last inequality follows from the similarity of the matrix $\mathbf{A}_{S}^{1 / 2} \mathbf{R}_{M} \mathbf{A}_{S}^{1 / 2}$ and the matrix $\mathbf{R}_{M} \mathbf{A}_{S}$. Note that the above inequality implies the spectral radius of $R_{j} A_{j} \leq \omega$, since $R_{j} A_{j}$ is symmetric with respect to $a(\cdot, \cdot)$.

We then define the following operators for the MG $V$-cycle. Recall $T_{j}$ from Section 5.2 and let $R_{j}$ denote a subspace smoother satisfying Lemma 5.3.2. Recall the symmetrization $\bar{R}_{j}$ of $R_{j}$, and assume the spectral radius $r_{\Omega}\left(\bar{R}_{j} A_{j}\right) \leq \omega$ for
$0<\omega<1$. Note that $R_{j}^{t}$ is the adjoint of $R_{j}$ with respect to $(\cdot, \cdot)$ and $T_{j}^{*}$ is the adjoint of $T_{j}$ with respect to $a(\cdot, \cdot)$. With $n$ smoothing steps, where $R_{j}$ and $R_{j}^{t}$ are applied alternatingly, the operator $G_{j}$ and $G_{j}^{*}$ are defined as follows,

$$
G_{j}=I-R_{j} A_{j}, \quad G_{j}^{*}=I-R_{j}^{t} A_{j}
$$

With this choice

$$
T_{j}=\left\{\begin{array}{l}
P_{j}-\left(G_{j}^{*} G_{j}\right)^{\frac{n}{2}} P_{j} \quad \text { for even } n \\
P_{j}-G_{j}\left(G_{j}^{*} G_{j}\right)^{\frac{n-1}{2}} P_{j} \text { for odd } n
\end{array}\right.
$$

Therefore, if we define

$$
G_{j, n}= \begin{cases}G_{j}^{*} G_{j} & \text { for even } n \\ G_{j} G_{j}^{*} & \text { for odd } n\end{cases}
$$

since $P_{j}^{2}=P_{j}$,

$$
\bar{T}_{j}=T_{j}+T_{j}^{*}-T_{j}^{*} T_{j}=\left(I-G_{j, n}^{n}\right) P_{j} .
$$

Note that $\bar{T}_{j}$ is invertible on $\mathcal{M}_{j}$, and hence $\bar{T}_{j}^{-1}$ exists.
The main result concerning the uniform convergence of the MG V-cycle for our model problem is summarized in the next theorem.

Theorem 5.3.4. On every triangulation $\mathcal{T}_{j}$, suppose that the smoother on each subspace $\mathcal{M}_{j}$ satisfies Lemma 5.3.2. Then, following the algorithm described above, we have

$$
\left\|I-B_{J} A_{J}\right\|_{a}^{2}=\frac{c_{0}}{1+c_{0}} \leq \frac{c_{1}}{c_{1}+c_{2} n}
$$

where $c_{1}$ and $c_{2}$ are constants from Lemma 5.3.1 and Lemma 5.3.2.
Proof. Recall (5.4) from Section 5.2. To estimate the constant $c_{0}$, we first consider the decomposition $v=\sum_{j} v_{j}$ for any $v \in \mathcal{M}_{J}$ with

$$
v_{j}=\left(P_{j}-P_{j-1}\right) v \in \mathcal{M}_{j}
$$

Then, Lemma 5.3.1 implies

$$
N_{j}\left(v_{j}, v_{j}\right)_{r_{p}} \leq c_{1} a\left(v_{j}, v_{j}\right)
$$

Estimating the identity of Xu and Zikatanov [101], , we have

$$
\begin{aligned}
a\left(\bar{T}_{j}^{-1}\left(I-\bar{T}_{j}\right) v_{j}, v_{j}\right) & =a\left(\left(I-G_{j, n}^{n}\right)^{-1} G_{j, n}^{n} v_{j}, v_{j}\right) \\
& =\left(\bar{R}_{j}^{-1} \bar{R}_{j} A_{j}\left(I-G_{j, n}^{n}\right)^{-1} G_{j, n}^{n} v_{j}, v_{j}\right) \\
& =\left(\bar{R}_{j}^{-1}\left(I-G_{j, n}\right)\left(I-G_{j, n}^{n}\right)^{-1} G_{j, n}^{n} v_{j}, v_{j}\right)
\end{aligned}
$$

Note that $G_{j, n}^{k}, k \leq n$, is in fact a polynomial of $\bar{R}_{j} A_{j}$. Therefore, $\bar{R}_{j}^{-1 / 2}(I-$ $\left.G_{j, n}\right) \bar{R}_{j}^{1 / 2}, \bar{R}_{j}^{-1 / 2} G_{j, n}^{n} \bar{R}^{1 / 2}$, and $\bar{R}_{j}^{-1 / 2}\left(I-G_{j, n}^{n}\right) \bar{R}_{j}^{1 / 2}$ are all polynomials of the term $\bar{R}_{j}^{1 / 2} A_{j} \bar{R}_{j}^{1 / 2}$, where $\bar{R}_{j}^{-1 / 2}\left(I-G_{j, n}^{n}\right) \bar{R}_{j}^{1 / 2}=\left(\bar{R}_{j}^{-1 / 2}\left(I-G_{j, n}^{n}\right)^{-1} \bar{R}_{j}^{1 / 2}\right)^{-1}$. Thus, it can be seen that $\bar{R}_{j}^{-1 / 2}\left(I-G_{j, n}\right) \bar{R}_{j}^{1 / 2}, \bar{R}_{j}^{-1 / 2} G_{j, n}^{n} \bar{R}^{1 / 2}, \bar{R}_{j}^{-1 / 2}\left(I-G_{j, n}^{n}\right)^{-1} \bar{R}_{j}^{1 / 2}$ commute with each other, and hence, $\bar{R}_{j}^{-1 / 2}\left(I-G_{j, n}\right)\left(I-G_{j, n}^{n}\right)^{-1} G_{j, n}^{n} \bar{R}^{1 / 2}$ is symmetric with respect to $(\cdot, \cdot)$.

Then, based on the above argument, defining $w_{j}=\bar{R}_{j}^{-1 / 2} v_{j}$, we have

$$
\begin{aligned}
a\left(\bar{T}_{j}^{-1}\left(I-\bar{T}_{j}\right) v_{j}, v_{j}\right) & =\left(\bar{R}_{j}^{-1 / 2}\left(I-G_{j, n}\right)\left(I-G_{j, n}^{n}\right)^{-1} G_{j, n}^{n} \bar{R}^{1 / 2} w_{j}, w_{j}\right) \\
& \leq \max _{t \in[0,1]}(1-t)\left(1-t^{n}\right)^{-1} t^{n}\left(\bar{R}_{j}^{-1} v_{j}, v_{j}\right) \\
& \leq \frac{1}{n}\left(\bar{R}_{j}^{-1} v_{j}, v_{j}\right) \leq \frac{N_{j}}{c_{2} n}\left(v_{j}, v_{j}\right)_{r_{p}}
\end{aligned}
$$

where the last inequality is from Lemma 5.3.2. Moreover,

$$
\sum_{j=0}^{J} a\left(\bar{T}_{j}^{-1}\left(I-\bar{T}_{j}\right) v_{j}, v_{j}\right) \leq \sum_{j=1}^{J} \frac{N_{j}}{c_{2} n}\left(v_{j}, v_{j}\right)_{r_{p}} \leq \sum_{j=0}^{J} \frac{c_{1}}{c_{2} n} a\left(v_{j}, v_{j}\right)=\frac{c_{1}}{c_{2} n} a(v, v)
$$

Therefore, $c_{0} \leq c_{1} /\left(c_{2} n\right)$ and consequently, the method of subspace corrections yields the following convergence estimate for the multigrid $V$-cycle:

$$
\left\|I-B_{J} A_{J}\right\|_{a}^{2}=\frac{c_{0}}{1+c_{0}} \leq \frac{c_{1}}{c_{1}+c_{2} n},
$$

which completes the proof.



Figure 5.2. Crack: initial triangulation (left) and the triangulation after one refinement (right), $\kappa=0.2$.

### 5.4 Numerical Illustration

This section contains numerical results for the proposed MG V-cycle applied to the 2D Poisson equation with a single corner-like singularity. The model test problem we consider here is given by

$$
\left\{\begin{align*}
-\Delta u=f & \text { in } \quad \Omega  \tag{5.6}\\
u=0 & \text { on } \quad \partial \Omega
\end{align*}\right.
$$

where the singularity occurs at the tip of the crack $\{(x, y), 0 \leq x \leq 0.5, y=0.5\}$, for $\Omega=(0,1) \times(0,1)$ as in Figure 5.2.

The MG scheme used to solve (5.6) is a standard MG V-cycle with linear interpolation. The sequence of coarse-level problems defining the MG hierarchy is obtained by re-discretizing (5.6) on the nested meshes constructed using the grading strategy described in Definition 4.1.7. The reported results are for $\mathrm{V}(1,1)-$ cycles and Gauss Seidel (GS) as a smoother. The asymptotic convergence factors are computed using $100 \mathrm{~V}(1,1)$-cycles applied to the homogeneous problem starting with an $O(1)$ random initial approximation.

The asymptotic convergence factors reported in Table 5.1 clearly demonstrate our theoretical estimates in that the they are independent of the number of refinement levels. To obtain a more complete picture of the overall effectiveness of our MG solver, we examine also storage and work-per-cycle measures. These are usually expressed in terms of operator complexity, defined as the number of nonzero
entries stored in the operators on all levels divided by the number of non-zero entries in the finest-level matrix, and grid complexity defined as the sum of the dimensions of operators over all levels divided by the dimension of the finest-level operator. The grid and, especially, the operator complexities can be viewed as proportionality constants that indicate how expensive the entire $V$-cycle is compared to performing only the finest-level relaxations of the $V$-cycle. For our test problem, the grid and operator complexities were 1.2 and 1.3 , respectively, independent of the number of levels. Considering the low grid and operator complexities the performance of the resulting MG solver applied to problem (5.6) is comparable to that of standard geometric MG applied to the Poisson equation with full regularity, i.e., without corner-like singularities; for the Poisson equation discretized on uniformly refined grids, standard MG with a GS smoother and linear interpolation yields $r_{\Omega M G} \approx .35$.

| levels | 2 | 3 | 4 | 5 | 6 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $r_{\Omega M G}(\mathrm{GS})$ | .40 | .53 | .56 | .53 | .50 |

Table 5.1. Asymptotic convergence factors $\left(r_{\Omega M G}\right)$ for the MG V $(1,1)$-cycle applied to problem (5.6) with Gauss Seidel smoother

Besides the results in this chapter, new results for the MG method for singular solutions are expected.
$\square$

## Application I: a Schrödinger Type Operator

In this chapter and Chapter 7, we will present applications of our a priori estimates in weighted Sobolev spaces and techniques in the development of the numerical schemes for singular solutions from elliptic equations with singular coefficients. Equations of this type widely appear in mathematical models of physics and engineering. Due to the lack of a unified theory, the mathematical study on these equations has been a difficult topic. We will show our theory extends to a class of Schrödinger operators and a degenerate operator, which represents our first attempt on applications in this field.

As usual, let $\Omega \subset \mathbb{R}^{2}$ be a bounded polygonal domain. We assume that a finite set of points $M \subset \bar{\Omega}$ was fixed and let $R(x):=\prod_{P_{i} \in M}\left|x-P_{i}\right|^{2}$, where $\left|x-P_{i}\right|$ denotes the Euclidean distance from $x$ to $P_{i}$. We also assume that a potential function $V$ is given such that the product $R V$ extends to a smooth function on $\bar{\Omega}$. We define

$$
\begin{equation*}
H:=-\Delta+V . \tag{6.1}
\end{equation*}
$$

The typical example we have in mind is $H=-\Delta+\delta r^{-2}$, where $\delta>0$ and $r$ is the distance to the origin $O$. In this typical example, $M$ consists of a single point, $M=\{O\}, O \in \bar{\Omega}$. The origin $O$ is not required to be a vertex of our polygonal domain $\Omega$.

We are interested in studying the Finite Element approximations of the solu-
tions to the mixed boundary value problem

$$
\left\{\begin{align*}
H u=f & \text { in } \Omega,  \tag{6.2}\\
\partial_{\nu} u=0 & \text { on } \partial_{N} \Omega, \text { and } \\
u=0 & \text { on } \partial_{D} \Omega
\end{align*}\right.
$$

for a decomposition $\partial \Omega=\partial_{D} \Omega \cup \partial_{N} \Omega$ into a region with Dirichlet and, respectively, Neumann boundary conditions.

Equations of this kind appear in Quantum mechanics in the form of Schrödinger equations with centrifugal potentials $[79,78]$ and in fluid dynamics [71]. Therefore, the study of the regularity and of the numerical approximation of the solutions of Equation (6.2) are of practical importance. Non-homogeneous boundary conditions can also be treated by reducing to homogeneous boundary conditions.

The case when $V$ is non-singular is well understood, so we mainly concentrate on the case when $V$ has non-trivial singularities of the form $\delta r^{-2}$. In this case, the usual theorems on the well posedness of elliptic boundary value problems in the usual Sobolev spaces [48, 70, 81, 94] do not apply. In fact, the solution $u$ of Equation (6.2) will have limited Sobolev regularity in the neighborhoods of the points $P_{i} \in M$ and close to the vertices or the points where the type of the boundary conditions changes. This is an issue, because, as is well known, the lack of regularity of the solution $u$ slows down the convergence rate for the numerical approximation in the Finite Element Method when quasi-uniform meshes are used. It turns out, however, that the difficulties caused by the singularities of our potential $V$ are of the same nature as the singularities caused by the vertices of $\Omega$ and by the presence of mixed boundary conditions. This observation is the starting point for the work presented in this chapter.

Many papers are devoted to the analysis of the singularities of (6.2) arising from the non-smoothness of the boundary when the potential $V$ is smooth, see for example the monographs $[27,55,62,61,80]$ and the references within. See also the research papers $[21,42,44,59,60,73,74,75,83,98]$. The numerical approximation of the solutions of (6.2) for $V=0$ was studied in a very large number of papers, including $[6,7,13,17,19,42,46,82,88]$. The case of the Schrödinger operator with a magnetic potential on a polygonal domain was studied in [26]. A good
introduction to the various methods for treating corner singularities can be found in [92].

Significantly less papers were devoted to the case of singular coefficients, nevertheless, techniques for the estimation of the finite element approximation for boundary value problems with singular coefficients can be found in the papers of Eriksson and Thomée [47], Franchi and Tesi [50], Li [66], and references there in. Also, Bespalov and Rukavishnikov [24, 89] studied the $p$-version finite element approximation in the case when $V$ has a single singularity at the origin, if the origin is a boundary point. Arroyo, Bespalov and Heuer [12] investigated the finite element method in some low-order weighted space for equations with singular coefficients of a different type than ours. These approaches to approximating solutions of partial differential equations with singular coefficients are thus seen to depend on the characters of the singularities.

In this chapter, we provide a unified numerical treatment of the difficulties caused by the singularities of the coefficients, the geometry of the domain, and the boundary conditions in the framework of weighted Sobolev spaces. These weighted Sobolev spaces are defined to take into account all the singularites (introduced by the domain, by the change of boundary conditions, and by the singularities of $V$ ). In order to introduce these weighted Sobolev spaces, let us first notice that the assumption that $R V$ extends to a smooth function on $\bar{\Omega}$ continues to be satisfied if we increase the set $\mathcal{V}$ of singular points (used to define $R:=\prod_{P_{i} \in M}\left|x-P_{i}\right|^{2}$ ). We shall assume therefore from now on that $\mathcal{V}$ also contains all vertices of $\Omega$ and all points where the boundary conditions change from Dirichlet to Neumann. Recall we used $\mathcal{V}$ to denote the vertex set of the domain in previous chapters. We here let $\mathcal{V}$ contain all geometrical vertices, points where the boundary condition changes and also singular points of the potential $V$.

Let $\vartheta(x)$ be the distance from $x$ to $\mathcal{V}$. Also, let $m \in \mathbb{N} \cup\{0\}:=\{0,1,2, \ldots\}$ and $a \in \mathbb{R}$. Then, in Definition 6.1.1, we introduce the $m$ th weighted Sobolev space on $\Omega$ with index $a$ by

$$
\begin{equation*}
\mathcal{K}_{a}^{m}(\Omega):=\left\{u: \Omega \rightarrow \mathbb{R}, \vartheta^{i+j-a} \partial_{x}^{i} \partial_{y}^{j} u \in L^{2}(\Omega), \forall i+j \leq m\right\} \tag{6.3}
\end{equation*}
$$

One can allow the index $a$ to be different at every point of $\mathcal{V}$, as in previous
chapters for instance, but we shall not pursue this simple generalization in order not to complicate the notation. The definition of the weighted Sobolev space is similar to (3.11) and Definition 2.3.2 except we add new points in the set $\mathcal{V}$.

Let us denote by $\left(v_{1}, v_{2}\right)$ the $L^{2}$-inner product of two functions $v_{1}, v_{2}$. The weak solution $u \in \mathcal{K}_{1}^{1}(\Omega) \cap\left\{u=0\right.$ on $\left.\partial_{D} \Omega\right\}$ of Equation (6.2) is defined by

$$
\begin{equation*}
a(u, v):=(\nabla u, \nabla v)+(V u, v)=(f, v), \quad \forall v \in \mathcal{K}_{1}^{1}(\Omega) \cap\left\{u=0 \text { on } \partial_{D} \Omega\right\} . \tag{6.4}
\end{equation*}
$$

(See Equation (6.16) for the case of non-homogeneous boundary conditions.) We first establish in Theorem 6.2.1 regularity results in the weighted Sobolev spaces $\mathcal{K}_{a}^{m}(\Omega)$ for the weak solution $u$ of Equation (6.2). In particular, we obtain that there is no loss of $\mathcal{K}_{a}^{m}(\Omega)$-regularity for the solution $u$. Under some additional assumptions on the potential $V$ and on the boundary conditions, we also establish the well-posedness of the boundary value problem (6.2), that is, we establish the existence of a unique solution $u$ depending continuously on the data $f$. See Babuska and Aziz [14], Bacuta, Nistor, and Zikatanov [21], Costabel and Dauge [42], Kondratiev [60], Lubuma and Nicaise [73], Mazzucato and Nistor [75], Nicaise [83] or the monographs [45, 62, 61, 80] for related results.

By analogy with the definition of the weak solution of (6.2), the discrete solution $u_{S} \in S \subset \mathcal{K}_{1}^{1}(\Omega) \cap\left\{u=0\right.$ on $\left.\partial_{D} \Omega\right\}$ of (6.2) is defined as usual by

$$
\begin{equation*}
a\left(u_{S}, v_{S}\right):=\left(\nabla u_{S}, \nabla v_{S}\right)+\left(V u_{S}, v_{S}\right)=\left(f, v_{S}\right), \quad \forall v_{S} \in S \tag{6.5}
\end{equation*}
$$

Using our well-posedness results on Equation (6.2), we shall provide a simple, explicit construction of a sequence of meshes $\mathcal{T}_{n}$ on $\Omega$, suitably graded towards the singularities $S$ that provides quasi-optimal rates of convergence for the finite element method applied to $H:=-\Delta+V$ in the following sense. Let

$$
\begin{equation*}
S_{n} \subset \mathcal{K}_{1}^{1}(\Omega) \cap\left\{u=0 \text { on } \partial_{D} \Omega\right\} . \tag{6.6}
\end{equation*}
$$

be the sequence of Finite Element Spaces consisting of continuous functions on $\Omega$ that coincide on each triangle of the mesh $\mathcal{T}_{n}$ with a degree $m$ polynomial. Let
$u_{n}:=u_{S_{n}} \in S_{n}$ be the corresponding discrete solutions. Then

$$
\begin{equation*}
\left\|u-u_{n}\right\|_{H^{1}(\Omega)} \leq C \operatorname{dim}\left(S_{n}\right)^{-m / 2}\|f\|_{H^{m-1}(\Omega)} \tag{6.7}
\end{equation*}
$$

where $f \in H^{m-1}(\Omega), m \geq 1$, is otherwise arbitrary and $C$ is a constant that depends on $\Omega$ and $m$, but not on $n$ or $f$ (we do not assume $u \in H^{m+1}(\Omega)$ ). Therefore we recover the optimal rate of convergence that is expected for smooth solutions [10, 14, 34, 92].

Our proof of the asymptotic order of convergence, Inequality (6.7), uses also a generalization of the Bramble-Hilbert Lemma to weighted Sobolev spaces on $\Omega$ and on the dilation property of our weighted Sobolev spaces.

The rest of this chapter is organized as follows. In Section 6.1, we shall define our weighted Sobolev spaces $\mathcal{K}_{a}^{m}(\Omega)$ and recall their properties from Chapter 2. Meanwhile, we shall introduce some notation that will be used throughout this chapter. In Section 6.2, we shall prove our theoretical results on the regularity and well-posedness of the solution $u$ of Equation (6.2) in the spaces $\mathcal{K}_{a}^{m}(\Omega)$. Moreover, we shall prove that, under certain additional mild assumptions on the potential $V$, the operator $H=-\Delta+V$ is Fredholm

$$
\begin{equation*}
H: \mathcal{K}_{a+1}^{m+1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega_{D}}=0\right\} \longrightarrow \mathcal{K}_{a-1}^{m-1}(\Omega) \tag{6.8}
\end{equation*}
$$

with the exception of $a$ in a certain countable subset. Moreover, for $|a|$ small enough, we show that $H$ is an isomorphism. This isomorphism is crucial for the construction of finite subspaces to obtain the quasi-optimal rates of convergence.

In Section 6.3, we will analyze the finite element solutions $u_{n}=u_{S_{n}}$ defined by the variational form, Equation (6.5), and prove that it satisfies quasi-optimal rates of convergence (Equation (6.7)).

In Section 6.4, we shall present numerical results on some model problems for different domains, where a singular point $P \in \mathcal{V}$ corresponding to the singularity of the potential $V$ is either an interior point or a boundary point. For both cases, we will compare the rates of convergence of the numerical solutions on different meshes. The convergence history will convincingly verify our theoretical prediction and demonstrate the effectiveness of our approach to construct the finite element spaces.

### 6.1 Weighted Sobolev Spaces

As explained above, it is more convenient to consider the boundary value problem (6.2) in weighted Sobolev spaces. In this section, we shall recall the weighted Sobolev space $\mathcal{K}_{a}^{m}(\Omega)$ and their needed properties from Chapter 2. More results on Sobolev spaces with weights can be found in $[2,17,45,60,62,61,63,84]$.

We follow Definition 2.3.2 in Chapter 2 to define the weighted Sobolev space for Equation (6.2). As in (6.3), the only difference we make here is to introduce more points in the "vertex" set $\mathcal{V}$, such that all singularities of the potential $V$ are also in $\mathcal{V}$.

Recall we let $l$ be the minimum of the non-zero distances from a point $Q_{i} \in \mathcal{V}$ to an edge of $\Omega$ and the distances between any two points $Q_{i}, Q_{j} \in \mathcal{V}$. Let

$$
\begin{equation*}
\tilde{l}:=\min (1 / 2, l / 4) \quad \text { and } \quad \mathcal{S}_{i}:=\Omega \cap B\left(Q_{i}, \tilde{l}\right), \tag{6.9}
\end{equation*}
$$

where $B\left(Q_{i}, \tilde{l}\right)$ denotes the ball centered at $Q_{i} \in \mathcal{V}$ with radius $\tilde{l}$. Note that sets $\mathcal{S}_{i}$ are disjoint. Then, we define $\rho: \bar{\Omega} \rightarrow[0,2 \tilde{l}]$ such that $\rho(x)=\vartheta(x)=\left|x-Q_{i}\right|$ on $\mathcal{S}_{i}$ and $\rho(x) \geq \tilde{l} / 2$ outside $\mathcal{S}:=\cup \mathcal{S}_{i}$. We can further assume that $\rho$ is smooth at all points of $\bar{\Omega}$ except at $\mathcal{V}$. Then the quotients $\rho / \vartheta$ and $\vartheta / \rho$ are bounded and we can replace $\vartheta$ with $\rho$ in all the formulas. We therefore replace $\vartheta$ with $\rho$ from now on.

This leads to the following definition of the weighted Sobolev spaces:
Definition 6.1.1. We have

$$
\begin{equation*}
\mathcal{K}_{a}^{m}(\Omega):=\left\{v: \Omega \rightarrow \mathbb{R}, \rho^{i+j-a} \partial_{x}^{i} \partial_{y}^{j} v \in L^{2}(\Omega), \forall i+j \leq m\right\} \tag{6.10}
\end{equation*}
$$

Recall the lemmas in Subsection 2.3.2, which also hold for the weighted Sobolev space defined in this chapter.

We now concentrate on the local behavior of a function $v \in \mathcal{K}_{a}^{m}=\mathcal{K}_{a}^{m}(\Omega)$ in the neighborhood of $Q_{i} \in \mathcal{V}$. The goal is to prove the dilation invariance of the norm as in Lemma 4.1.3. For the sake of simplicity, we consider a new coordinate system that is a simple translation of the old $x-y$ coordinate system, such that $Q_{i}$ is translated to the origin. Let $G \subset \mathcal{S}_{i}$ be the subset, such that $\rho \leq \xi \leq \tilde{l}$ on $G$. For $0<\lambda<1$, let $G^{\prime}:=\lambda G$. Then, we define the dilation of a function on $G$ in
the new coordinate system as follows,

$$
v_{\lambda}(x, y):=v(\lambda x, \lambda y)
$$

for all $(x, y) \in G \subset \mathcal{S}_{i}$. (This definition makes sense, since $Q_{i}$ is the origin in the new coordinate system.)

Lemma 6.1.2. Let $0<\lambda<1$ and $G \subset \mathcal{S}_{i}$ be an open subset such that $G^{\prime}:=\lambda G \subset$ $\mathcal{S}_{i}$. Then $\left\|u_{\lambda}\right\|_{\mathcal{K}_{a}^{m}(G)}=\lambda^{a-1}\|u\|_{\mathcal{K}_{a}^{m}\left(G^{\prime}\right)}$ for any $u \in \mathcal{K}_{a}^{m}\left(\mathcal{S}_{i}\right)$.

Proof. As in Lemma 4.1.3, the proof is based on the change of variables $w=\lambda x$, $z=\lambda y$. Note that on both $G \subset \mathcal{S}_{i}$ and $G^{\prime} \subset \mathcal{S}_{i}, \rho(x, y)=$ the distance from $(x, y)$ to $Q_{i}$, hence $\rho(x, y)=\lambda^{-1} \rho(w, z)$. Then,

$$
\begin{aligned}
\left\|u_{\lambda}(x, y)\right\|_{\mathcal{K}_{a}^{m}(G)}^{2} & =\sum_{j+k \leq m} \int_{G}\left|\rho^{j+k-a}(x, y) \partial_{x}^{j} \partial_{y}^{k} u_{\lambda}(x, y)\right|^{2} d x d y \\
& =\sum_{j+k \leq m} \int_{G^{\prime}}\left|\lambda^{a-j-k} \rho^{j+k-a}(w, z) \lambda^{j+k} \partial_{w}^{j} \partial_{z}^{k} u(w, z)\right|^{2} \lambda^{-2} d w d z \\
& =\lambda^{2 a-2} \sum_{j+k \leq m} \int_{G^{\prime}}\left|\rho^{j+k-a}(w, z) \partial_{w}^{j} \partial_{z}^{k} u(w, z)\right|^{2} d w d z \\
& \left.=\lambda^{2 a-2} \sum_{j+k \leq m} \| \rho^{j+k-a}(w, z) \partial_{w}^{j} \partial_{z}^{k} u(w, z)\right) \|_{L^{2}\left(G^{\prime}\right)}^{2} \\
& =\lambda^{2 a-2}\|u\|_{\mathcal{K}_{a}^{m}\left(G^{\prime}\right)}^{2}
\end{aligned}
$$

which completes the proof.
Recall the function $R(x)=\prod_{P_{i} \in M}\left|x-P_{i}\right|^{2}$ and the operator $H:=-\Delta+V$. Note the the potential function $V$ satisfies $R(x) V(x) \in \mathcal{C}^{\infty}(\Omega)$. Then, we conclude this section with the following result.

Lemma 6.1.3. The operator $H$ defines a continuous map: $\mathcal{K}_{a+1}^{m+1} \rightarrow \mathcal{K}_{a-1}^{m-1}, m \geq 1$.
Proof. We shall show that there is $C>0$, such that $\|H u\|_{\mathcal{K}_{a-1}^{m-1}} \leq C\|u\|_{\mathcal{K}_{a+1}^{m+1}}$ for all $u \in \mathcal{K}_{a+1}^{m+1}$.

On $\Omega \backslash \mathcal{S}:=\Omega \backslash\left(\cup_{i} \mathcal{S}_{i}\right), H=-\Delta+V$ is a second order differential operator with bounded coefficients, bacause $|V| \leq C_{1}<\infty$ on $\Omega \backslash \mathcal{S}$. Therefore, it defines
a bounded operator $H^{m+1}(\Omega \backslash \mathcal{V}) \rightarrow H^{m-1}(\Omega \backslash \mathcal{V})$ [48]. Lemma 2.3.7 then gives $\|H u\|_{\mathcal{K}_{a-1}^{m}(\Omega \backslash \mathcal{S})} \leq C\|u\|_{\mathcal{K}_{a+1}^{m}(\Omega \backslash \mathcal{S})}$.

On $\mathcal{S}_{i}$, we write $\left.-\Delta+V=r^{-2}\left(\left(r \partial_{r}\right)^{2}+\partial_{\theta}^{2}\right)+r^{2} V\right)$. Then Lemma 2.3.3 and the equation $r^{-\lambda}\left(r \partial_{r}\right)^{n} r^{\lambda} u=\left(r \partial_{r}+\lambda\right)^{n} u$ give $\|H u\|_{\mathcal{K}_{a-1}^{m}\left(\mathcal{S}_{i}\right)} \leq C\|u\|_{\mathcal{K}_{a+1}^{m}\left(\mathcal{S}_{i}\right)}$. Adding all the similar inequalities completes the proof.

Let us say a few words about the boundary weighted Sobolev spaces

$$
\begin{equation*}
\mathcal{K}_{a}^{m}(S) \simeq \oplus_{F \subset S} \mathcal{K}_{a}^{m}(S) \tag{6.11}
\end{equation*}
$$

where $S \subset \partial \Omega$ is a union of sides of $\Omega$. Each side can be identified with $I=[-1,1]$, so it is enough to define

$$
\begin{equation*}
\mathcal{K}_{a}^{m}(I)=\left\{\left(1-t^{2}\right)^{k-a} f^{(k)} \in L^{2}(I), k \leq m\right\} . \tag{6.12}
\end{equation*}
$$

For $s \in[0, \infty)$ we define $\mathcal{K}_{a}^{s}(I)$ by interpolation, and for $s<0$ we extend this definition by duality: $\mathcal{K}_{a}^{s}(I)=\left(\mathcal{K}_{-a}^{-s}(I)\right)^{*}$. See $[2,3,21,75,63,95]$. The usual issues with Sobolev spaces of fractional order of the form "integer $+1 / 2$ " do not arise in the case of the Sobolev spaces on the boundary (they do arise though for the ones on $\Omega$ ).

It was shown that

$$
\begin{equation*}
\left.\mathcal{K}_{a}^{m}(\Omega) \ni u \rightarrow u\right|_{\partial \Omega} \in \mathcal{K}_{a-1 / 2}^{m-1 / 2}(\partial \Omega), \quad m \geq 1 \tag{6.13}
\end{equation*}
$$

is a continuous surjective map, as in the case of smooth, bounded domains. A similar result holds for the normal derivative, yielding again a continuous and surjective map

$$
\begin{equation*}
\left.\mathcal{K}_{a}^{m}(\Omega) \ni u \rightarrow \partial_{\nu} u\right|_{\partial \Omega} \in \mathcal{K}_{a-3 / 2}^{m-3 / 2}(\partial \Omega), \quad m \geq 2 \tag{6.14}
\end{equation*}
$$

See for example $[2,3,21,63,75,95]$.

### 6.2 The Well-posedness and Regularity of the Solution

In this section, we shall study the well-posedness and the regularity of the solution of the boundary value problem (6.2) in the weighted Sobolev spaces $\mathcal{K}_{a}^{m}$.

Let us assume that the boundary of $\Omega$ was partitioned as $\partial_{N} \Omega=\partial \Omega \backslash \partial_{D} \Omega$, with $\partial_{D} \Omega$ a union of closed sides of $\Omega$. (A side of $\Omega$ is a segment $I \subset \partial \Omega$ whose end points are in $\mathcal{V}$ but whose interior contains no other points of $\mathcal{V}$.) Then we consider the following slight extension of Equation (6.2)

$$
\left\{\begin{align*}
H u=f & \text { in } \Omega,  \tag{6.15}\\
\partial_{\nu} u=g_{N} & \text { on } \partial_{N} \Omega, \text { and } \\
u=g_{D} & \text { on } \partial_{D} \Omega,
\end{align*}\right.
$$

where, we recall, $H=-\Delta+V$. First, we can assume $g_{D}=0$ by the surjectivity of the trace map. Then, with integration by parts, we actually look for the solution $u \in \mathcal{K}_{a+1}^{1}(\Omega)$ of (6.15) in the following weak sense. Namely, $u$ satisfies

$$
\begin{equation*}
a(u, v):=(\nabla u, \nabla v)+(V u, v)=(f, v)_{L^{2}(\Omega)}+\left\langle g_{N}, \bar{v}\right\rangle_{\partial_{D} \Omega} \tag{6.16}
\end{equation*}
$$

for all $v \in \mathcal{K}_{1}^{1}(\Omega) \cap\left\{u=0\right.$ on $\left.\partial_{D} \Omega\right\}$, where $\left\langle g_{N}, \bar{v}\right\rangle_{\partial_{D} \Omega}$ is the value of the distribution $g_{N}$ on the function $v$. (This is a slight extension of (6.4).)

Thus, we have the following standard regularity result.
Theorem 6.2.1. Let $M \subset \bar{\Omega}$ be our given finite set, $R(x):=\prod_{Q_{i} \in M}\left|x-Q_{i}\right|^{2}$, and $V: \bar{\Omega} \backslash M \rightarrow \mathbb{R}$ be such that $R V$ extends to a smooth function on $\bar{\Omega}$. Let $u \in \mathcal{K}_{a+1}^{1}:=\mathcal{K}_{a+1}^{1}(\Omega)$ be a solution of Equation (6.15) with $H u:=-\Delta u+V u=$ $f \in \mathcal{K}_{a-1}^{m-1}$ and with mixed boundary conditions $g_{D} \in \mathcal{K}_{a+1 / 2}^{m+1 / 2}\left(\partial_{D} \Omega\right)$ and $g_{N} \in$ $\mathcal{K}_{a-1 / 2}^{m-1 / 2}\left(\partial_{N} \Omega\right), m \geq 1$. Then $u \in \mathcal{K}_{a+1}^{m+1}$ and we have

$$
\|u\|_{\mathcal{K}_{a+1}^{m+1}} \leq C_{\mathrm{reg}}\left(\|f\|_{\mathcal{K}_{a-1}^{m-1}}+\left\|g_{D}\right\|_{\mathcal{K}_{a+1 / 2}^{m+1 / 2}\left(\partial_{D} \Omega\right)}+\left\|g_{N}\right\|_{\mathcal{K}_{a-1 / 2}^{m-1 / 2}\left(\partial_{N} \Omega\right)}+\|u\|_{\mathcal{K}_{a+1}^{0}}\right)
$$

for a constant $C_{\mathrm{reg}}>0$ independent of $f, g_{D}, g_{N}$, and $u$.
Proof. This result is standard, so we include only a sketch. Regularity is a local
property, so we may separate the behavior close to the points of $S$ where $V$ may have singularities. Close to a vertex $Q$ of $\Omega$ where $V$ is non singular, the result is known from Chapter 2 and Chapter 3. Assume, for simplicity that $Q=O$, the origin. A simple proof is obtained then by using a radial partition of unity of the form $\phi_{n}(x):=\phi_{0}\left(2^{n} x\right)$ and then applying to the functions $\phi_{n} u$ the usual regularity results for smooth domains. Details of this method can be found in Theorem 2.2.1.

It remains to deal with the behavior of $u$ near a point $Q \in M$ where $V$ is singular. If $Q$ is on the boundary or a vertex, then the proof is exactly the same as for the case when $Q$ is vertex and $V$ is non-singular at $Q$. When $Q$ is an interior point of $\Omega$, the proof is again very similar to the case when $Q$ is a vertex, if we think of this case as using periodic boundary conditions.

This regularity result for $H$ gives right away by induction the following regularity result for eigenfunctions, which we hope will be useful for the numerical determination of the eigenvalues and eigenfunctions of $H$.

Corollary 6.2.2. Assume $u \in \mathcal{K}_{a}^{0}=\rho^{a} L^{2}(\Omega), a \geq 0$, is an eigenfunction of $H$ (i. e., $H u=\lambda u$ ). Then $u \in \mathcal{K}_{a}^{m}=\mathcal{K}_{a}^{m}(\Omega)$ for any $m \in \mathbb{N}$.

Now, we shall prove the existence and uniqueness of the solution of the boundary value problem (6.2) in weighted Sobolev spaces. For this we need a few more assumptions on $V$.

Theorem 6.2.3. Let $R$ and $V$ be as in the statement of Theorem 6.2.1. Assume also that $V R \geq 0$ on $\Omega$ and that $V R(Q)>0$ if $Q \in M \cap \Omega$. Also, we assume that if $Q \in M$ separates two adjacent sides of $\Omega$ that are assigned Neumann boundary conditions, then again $V R(Q)>0$. Then there exists $\eta>0$ such that, for any $f \in \mathcal{K}_{a-1}^{m-1}, g_{D} \in \mathcal{K}_{a+1 / 2}^{m+1 / 2}\left(\partial_{D} \Omega\right)$, and $g_{N} \in \mathcal{K}_{a-1 / 2}^{m-1 / 2}\left(\partial_{N} \Omega\right), m \geq 0,|a|<\eta$, the mixed boundary value problem (6.15) has a unique solution $u \in \mathcal{K}_{a+1}^{m+1}:=\mathcal{K}_{a+1}^{m+1}(\Omega)$, which satisfies

$$
\|u\|_{\mathcal{K}_{a+1}^{m+1}} \leq C_{m, a}\left(\|f\|_{\mathcal{K}_{a-1}^{m-1}}+\left\|g_{D}\right\|_{\mathcal{K}_{a+1 / 2}^{m+1 / 2}\left(\partial_{D} \Omega\right)}+\left\|g_{N}\right\|_{\mathcal{K}_{a-1 / 2}^{m-1 / 2}\left(\partial_{N} \Omega\right)}\right)
$$

for a constant $C_{m, a}>0$ independent of $f, g_{D}$, and $g_{N}$.
Proof. We can assume $g_{D}=0$. We shall prove first the case of $m=0$. This follows from the strict positivity (or strict coercivity) of $H$ on $\mathcal{K}_{1}^{1}(\Omega) \cap\left\{u=0\right.$ on $\left.\partial_{D} \Omega\right\}$, or
more precisely of the bilinear form $a(\cdot, \cdot)$ introdced in Equations (6.4) and (6.16). We prove below this strict coercive property. We assume the functions to be real, for simplicity.

First, we need to show that $a(\cdot, \cdot)$ is continuous, that is, $a(u, v) \leq C\|u\|_{\mathcal{K}_{1}^{1}}\|v\|_{\mathcal{K}_{1}^{1}}$. Since $\rho=r$ in the neighborhood $\mathcal{S}_{i}$ of $Q_{i} \in \mathcal{V}$ and $R V$ is continuous on $\mathcal{S}_{i}$, we have $|V| \leq C \rho^{-2}$. Therefore, Cauchy-Schwarz inequality gives $a(u, v) \leq C\|u\|_{\mathcal{K}_{1}^{1}}\|v\|_{\mathcal{K}_{1}^{1}}$, for $C>0$ not depending on $u$ or $v$.

We prove strict coercivity of $a(\cdot, \cdot)$ on each $\mathcal{S}_{i}$ and on $\Omega \backslash \mathcal{S}:=\Omega \backslash\left(\cup \mathcal{S}_{i}\right)$, respectively. On the later set it is just the usual Poincaré inequality. Then, we shall verify the following inequality on every $\mathcal{S}_{i}$.

$$
\int_{\mathcal{S}_{i}}\left[\left(\partial_{x} u\right)^{2}+\left(\partial_{y} u\right)^{2}+V u^{2}\right] d x d y \geq C\|u\|_{\mathcal{K}_{1}^{1}\left(\mathcal{S}_{i}\right)}^{2}
$$

Assume first that $Q_{i}$ is a vertex with angle $\alpha_{i} \in(0,2 \pi)$. Then, $\mathcal{S}_{i}$ can be locally characterized in polar coordinates with $Q_{i}$ at the origin by

$$
\mathcal{S}_{i}=\left\{(r, \theta) \mid 0<r<\tilde{l}, 0<\theta<\alpha_{i}\right\}, \quad i \geq 0
$$

Note that $a(u, u) \geq \int_{\Omega}\left(\partial_{x} u\right)^{2}+\left(\partial_{y} u\right)^{2}$. Therefore, it suffices to show

$$
\int_{\mathcal{S}_{i}} \frac{u^{2}}{\rho^{2}} d x d y \leq C a(u, u)
$$

From the one-dimensional Poincaré inequality for $\theta$ on $\mathcal{S}_{i}$ (this is where we need the assumption that there are no adjacent Neumann sides), we have

$$
\int_{0}^{\alpha_{i}} u^{2} d \theta \leq C_{1} \int_{0}^{\alpha_{i}}\left(\partial_{\theta} u\right)^{2} d \theta
$$

By integrating in polar coordinates, we have

$$
\int_{\mathcal{S}_{i}} \frac{u^{2}}{r^{2}} d x d y=\int_{\mathcal{S}_{i}} \frac{u^{2}}{r} d r d \theta \leq C_{1} \int_{\mathcal{S}_{i}} \frac{\left(\partial_{\theta} u\right)^{2}}{r} d r d \theta
$$

Since $\int_{\mathcal{S}_{i}}\left(\partial_{x} u\right)^{2}+\left(\partial_{y} u\right)^{2} d x d y=\int_{\mathcal{S}_{i}} r\left(\partial_{r} u\right)^{2}+\frac{\left(\partial_{\theta} u\right)^{2}}{r} d r d \theta$, we get

$$
\int_{\mathcal{S}_{i}} \frac{u^{2}}{\rho^{2}} d x d y=\int_{\mathcal{S}_{i}} \frac{u^{2}}{r^{2}} d x d y \leq C \int_{\mathcal{S}_{i}}\left[\left(\partial_{x} u\right)^{2}+\left(\partial_{y} u\right)^{2}+V u^{2}\right] d x d y
$$

On the other hand, if $Q_{i} \in M \cap \Omega$ (that is, if it is an interior point) or a point separating two adjacent Neumann boundaries, we have $V \geq \delta r^{-2}, \delta>0$ (this is where we use the assumption $R V\left(Q_{i}\right)>0$ ). We can no longer use the one dimensional Poincaré inequality, but do not need it either, we rather write

$$
\int_{\mathcal{S}_{i}}\left[|\nabla u|^{2}+V u^{2}\right] d x d y \geq \delta \int_{\mathcal{S}_{i}}\left[|\nabla u|^{2}+u^{2} / r^{2}\right] d x d y=: \delta\|u\|_{\mathcal{K}_{1}^{1}\left(\mathcal{S}_{i}\right)}^{2}
$$

for $\delta>0$ small enough.
The strict coercivity of $H$ (or of the bilinear form $a$ ) on $\mathcal{K}_{1}^{1}(\Omega) \cap\left\{u=0\right.$ on $\left.\partial_{D} \Omega\right\}$ then follows by adding all these inequalities.

Let $B=\mathcal{K}_{1}^{1}(\Omega) \cap\left\{u=0\right.$ on $\left.\partial_{D} \Omega\right\}$. The Lax-Milgram Lemma then proves that $H: B \rightarrow B^{*}$ is an isomorphism, which is our result for $m=0$ and $a=0$.

We next use the continuity of the family $\rho^{\epsilon} H \rho^{-\epsilon}: B \rightarrow B^{*}$ and Lemma 2.3.5 to prove the result for $m=0$ and $|a|<\eta$, for some $\eta$ that depends only on the domain and the operator $H$. Theorem 6.2.1 shows that if the result is true for $(0, a)$, then it is true also for $(m, a)$. This completes the proof.

It is possible to explicitly determine the value of $\eta$ for Equation (6.2). Let us define $\delta_{i}=\lim _{Q \rightarrow Q_{i}}\left|Q-Q_{i}\right|^{2} V(Q)$ for any $Q_{i} \in M$. By freezing the coefficients of $H$ to $Q_{i}$, we see that the behavior of the solution $u$ on $\mathcal{S}_{i}$ is given by $-\Delta+\delta_{i} / r^{2}$. Let us denote by $\alpha_{i}$ the angle of $\Omega$ at $Q_{i}\left(\alpha_{i}=2 \pi\right.$ if $Q_{i}$ is an interior point). We let $k \in \mathbb{N}=\{1,2, \ldots\}$ if Dirichlet boundary conditions are on both sides of $Q_{i}$, $k+1 / 2 \in \mathbb{N}$ if Dirichlet/Neumann boundary conditions are on the sides of $Q_{i}$, and we let $k \in \mathbb{N} \cup\{0\}$ otherwise. We then define

$$
\begin{equation*}
\Sigma_{i}:=\left\{ \pm \sqrt{k^{2} \pi^{2} \alpha_{i}^{-2}+\delta_{i}}\right\} \tag{6.17}
\end{equation*}
$$

The operator pencil $P_{i}(\tau)$ (or indicial family) associated to $H$ at $Q_{i}$ is $P_{i}(\tau):=$
$(\tau-\imath \epsilon)^{2}-\partial_{\theta}^{2}+\delta_{i}$, and as in (3.16), is obtained by evaluating

$$
\begin{equation*}
\left(-\Delta+\delta_{i} / r^{2}\right)\left(r^{\imath \tau+\epsilon} \phi(\theta)\right)=r^{\imath \tau+\epsilon-2}\left((\tau-\imath \epsilon)^{2}-\partial_{\theta}^{2}+\delta_{i}\right) \phi(\theta) \tag{6.18}
\end{equation*}
$$

Then $P(\tau)$ is invertible for all $\tau \in \mathbb{R}$, as long as $\epsilon \notin \Sigma_{i}$. For these values of $\epsilon$, we can use the method in Chapter 2 to obtain Fredholm conditions on $\rho^{\epsilon} H \rho^{-\epsilon}$. See also $[65,91]$ and the references therein for the case of interior singularities due to the potential $V$, where no boundary is involved (in which case, a suitable pseudodifferential calculus immediately gives the desired Fredholm condition.)

Recall that a bounded operator $A: X \rightarrow Y$ between Banach spaces is Fredholm if the kernel of $A$ (that is the space $\operatorname{ker}(A):=\{A x=0\}$ ) and $Y / A X$ are finite dimensional spaces. If $A$ is unbounded but densely defined and closed, then $A$ is called Fredholm if the induced continuous operator $\Gamma(A) \rightarrow Y$, defined on the graph of $A$, is Fredholm. For a Fredholm operator $A$, we defined its index by the formula $\operatorname{ind}(A)=\operatorname{dim} \operatorname{ker}(A)-\operatorname{dim}(Y / A D(A))$ with $D(A)$ the domain of $A$ ( $D(A)=X$ precisely when $A$ is continuous).

Theorem 6.2.4. Assume that $\delta_{i} \geq 0$ and that otherwise we are under the conditions of Theorem 6.2.1. Then the boundary value problem (6.15) defines a Fredholm operator

$$
\begin{equation*}
\tilde{H}_{a}:=\left(H, \text { res }, \partial_{\nu}\right): \mathcal{K}_{a+1}^{m+1}(\Omega) \rightarrow \mathcal{K}_{a-1}^{m-1}(\Omega) \oplus \mathcal{K}_{a+1 / 2}^{m+1 / 2}\left(\partial_{D} \Omega\right) \oplus \mathcal{K}_{a-1 / 2}^{m-1 / 2}\left(\partial_{N} \Omega\right) \tag{6.19}
\end{equation*}
$$

for all $a \notin \cup_{i} \Sigma_{i}$, and with res being the restriction operator to $\partial_{D} \Omega$..
Let $\eta$ be the minimum values of $|a|, a \in \cup_{i} \Sigma_{i}$. Then $\eta>0$ exactly if the conditions of Theorem 6.2.3 are satisfiqed. Note that $\tilde{H}_{a}$ is Fredholm of index zero when $a=0$, because it is invertible. By the homotopy invariance of the index, $\tilde{H}_{a}$ is Fredholm of index zero for $|a|<\eta$. Since the domains of the operators $\tilde{H}_{a}$ are decreasing as $a$ is increasing, the kernels of the operators $\tilde{H}_{a}$ will also decrease as we $a$ increases. We therefore conclude that $\tilde{H}_{a}$ are injective for $0 \leq a<\eta$, since they have index zero, they are in fact isomorphisms. By taking the adjoint, we obtain the invertibility of $\tilde{H}_{a}$ for $-\eta<a \leq 0$ as well.

The operator $\tilde{H}_{a}$ will no longer be invertible for $|a| \geq \eta$. In that case, even if $a$ is away from the specific values above, $\tilde{H}_{a}$ is only a Fredholm operator with
a non-zero index that can be computed using the results of [60, 61, 77, 80]. The result is as follows. Let us fix $a<a^{\prime}$ and count the number of values in the set $\left(a, a^{\prime}\right) \cap \Sigma_{i}$, with the values corresponding to $k=0$ and $\delta_{i}=0$ counted twice (because of multiplicity). Let $N$ be the total number. We assume that $a, a^{\prime} \notin \cup \Sigma_{i}$. Then

$$
\begin{equation*}
\operatorname{ind}\left(\tilde{H}_{a^{\prime}}\right)-\operatorname{ind}\left(\tilde{H}_{a}\right)=-N \tag{6.20}
\end{equation*}
$$

The same result applies to the operator $H_{a}: \rho^{a} B \rightarrow \rho^{a} B^{*}$, which satisfies $H_{a}^{*}=$ $H_{-a}$. This allows to determine the index of $\tilde{H}_{a}$. This amounts to an index problem both for the Neumann and for the Schrodinger problems in the plane. See discussions in Section 3.2.

Theorem 6.2.5. conditions of Theorem 6.2.1 are satisfied and let $N$ be the number of points $Q_{i} \in S$ that do not satisfy the conditions of Theorem 6.2.3 (more precisely $\delta_{i}=R V\left(Q_{i}\right)=0$ at an interior point or at a Neumann-Neumann point). Then $\tilde{H}_{a}$ is Fredholm for $|a|<\eta, a \neq 0$, with index

$$
\operatorname{ind}\left(\tilde{H}_{ \pm a}\right)=\mp N, \quad 0<a<\eta
$$

This result holds for any complex valued $V$ such that $R V$ is smooth.

### 6.3 Estimates for the Finite Element Method

We shall study the finite element method for Equation (6.2) with assumptions on the operator $H$ as in Theorem 6.2.3. For simplicity, we assume zero Dirichlet and Neumann boundary conditions. Therefore, the corresponding weak solution is given by Equation (6.5).

For elliptic problems with smooth coefficients, various error estimates in the neighborhood of a geometric vertex of a polygonal domain can be found in [7, $21,45,86]$. A detailed discussion is written in Chapter 4. Therefore, in order to simplify the presentation, we shall turn our attention to the analysis of the finite element method close to the interior singularities of $V$. Around the other points of $\mathcal{V}$ (vertices, points where the type of the boundary conditions change), it is nevertheless understood that one of the standard gradings will be used, if needed.

Denote by $\mathcal{T}=\left\{T_{i}\right\}$ a triangulation of $\Omega$ with triangles $T_{i}$. Let $S=S(\mathcal{T}, m+1)$ be the finite element space associated to the degree $m$ Lagrange triangle, such that $S$ consists of polynomials of degree $\leq m$ on each triangle $T_{i} \in \mathcal{T}$. We are looking for a simple, explicit way to construct a class of finite element spaces $S_{n}\left(\mathcal{T}_{n}, m+1\right)$, where the numerical approximations $u_{n} \in S_{n}$ of the solution $u$ for Equation (6.2) satisfy

$$
\left\|u-u_{n}\right\|_{H^{1}} \leq C \operatorname{dim}\left(S_{n}\right)^{-m / 2}\|f\|_{H^{m-1}}, \quad \forall f \in H^{m-1}, \quad m \geq 1
$$

We shall achieve this quasi-optimal rate of convergence by considering a suitable grading close to the points of $\mathcal{V}$. The proof will be based on estimating the error in weighted Sobolev spaces. The analysis here is similar to the analysis in Chapter 4. To make this chapter as self-contained as possible, we again recall the following theorems for further estimations.

### 6.3.1 Approximation Away from the Singular Set $\mathcal{V}$

In this subsection, we approximate the solution $u$ far from the singular points, so we ignore the role of the singular set $\mathcal{V}$. In particular, although the results and constructions of this subsection are formulated for $\Omega$, often they will be used for a subpolygon $G \subset \Omega$. We first need to recall the following well-known approximation theorem [14, 34, 39, 92].

Let $\mathcal{T}=\{T\}$ be a mesh, that is a triangulation of $\Omega$ with triangles $T$. We shall denote by $\tilde{S}(\mathcal{T}, m+1)$ the Finite Element space associated to the degree $m$ Lagrange triangle. That is, $\tilde{S}(\mathcal{T}, m+1)$ consists of all continuous functions $\chi: \bar{\Omega} \rightarrow$ $\mathbb{R}$ such that $\chi$ coincides with a polynomial of degree $\leq m$ on each triangle $T \in \mathcal{T}$. (The smaller subspace $S(\mathcal{T}, m+1):=\tilde{S}(\mathcal{T}, m+1) \cap \mathcal{K}_{1}^{1}(\Omega) \cap\left\{u=0\right.$ on $\left.\partial_{D} \Omega\right\}$ will be used in our approximation results in the following subsections. It is more convenient in this subsection to use the larger subspace $\tilde{S}(\mathcal{T}, m+1)$, though. This also allows for more general results.)

We shall denote by $u_{I}=u_{I, \mathcal{T}, m+1} \in \tilde{S}(\mathcal{T}, m+1)$ the Lagrange interpolant of $u \in H^{2}(\Omega)$. First, given a triangle $T$, let $\left[t_{0}, t_{1}, t_{2}\right]$ be the barycentric coordinates on $T$. The nodes of the degree $m$ Lagrange triangle $T$ are the points of $T$ whose barycentric coordinates $\left[t_{0}, t_{1}, t_{2}\right]$ satisfy $m t_{j} \in \mathbb{Z}$. The degree $m$ Lagrange
interpolant $u_{I, \mathcal{T}, m+1}$ of $u$ is the unique function $u_{I, \mathcal{T}, m+1} \in \tilde{S}(\mathcal{T}, m+1)$ such that $u=u_{I, \mathcal{T}, m+1}$ at the nodes of each triangle $T \in \mathcal{T}$. The shorter notation $u_{I}$ will be used only when only one mesh is understood in the discussion (recall that $m$ is fixed).

Theorem 6.3.1. Suppose the bilinear form $a(\cdot, \cdot)$ for an equation is both continuous and strongly coercive on $H^{1}(D)$, where $D$ is a two-dimensional polygonal domain. Let $\tilde{S}=\tilde{S}(\mathcal{T}, m+1)$. Assume that all triangles $T_{j}$ of the triangulation $\mathcal{T}$ on the domain $D$ have angles $\geq \alpha$ and edges of length $\leq h$ and $\geq a h$. Let $u_{S} \in \tilde{S}$ and $u_{I} \in \tilde{S}$ be the finite element solution and the interpolation of the real solution, respectively. Then, there exist constants $c$ and $C_{1}=C_{1}(\alpha, m)$ such that

$$
c\left\|u-u_{S}\right\|_{H^{1}(D)} \leq\left\|u-u_{I}\right\|_{H^{1}(D)} \leq C_{1} h^{m}\|u\|_{H^{m+1}(D)}
$$

for $\forall u \in H^{m+1}(D), m \geq 1$.
The constant $c$ depends only on the bilinear form $a(\cdot, \cdot)$ by Céa's Lemma, while $C_{1}$ and $C_{2}$ are independent of the solution $u$.

Let $M:=C_{1}(\alpha) M_{1} M_{2}$, where $C_{1}(\alpha)$ is as in Theorem 6.3.1, and $M_{1}$ and $M_{2}$ are from Lemma 2.3.7. Then, we have the following estimate for the error $\left\|u-u_{I}\right\|_{\mathcal{K}_{1}^{1}(\tilde{G})}$ on a subset $\tilde{G} \subset \Omega$ that is away from any point $Q_{i} \in \mathcal{V}$.

Theorem 6.3.2. Fix $\alpha>0$ and $0<\xi<\tilde{l}$. Let $\tilde{G} \subset \Omega$ be a polygonal subset, such that $\rho>\xi$ on $\tilde{G}$. Let $\mathcal{T}=\left(T_{j}\right)$ be a triangulation of $\tilde{G}$ with angles $\geq \alpha$ and sides $\leq h$. Then

$$
\left\|u-u_{I}\right\|_{\mathcal{K}_{1}^{1}(\tilde{G})} \leq M h^{m}\|u\|_{\mathcal{K}_{1+a}^{m+1}(\tilde{G})}
$$

for $\forall u \in \mathcal{K}_{1+a}^{m+1}(\tilde{G}), m \geq 1$, where $M$ depends on $\xi$ and $\alpha$.
Proof. Note that the bilinear form $a(\cdot, \cdot)$ for Equation (6.2) is both continuous and strongly coercive on $H^{1}(\tilde{G})$. The proof of this theorem follows the arguments in Theorem 6.3.1 and the equivalence of the $H^{m}$-norm and the $\mathcal{K}_{a}^{m}$-norm in Lemma 2.3.7 on $\tilde{G}$ immediately.

Recall that we assume appropriate graded meshes have been used near the vertices of $\Omega$ where $V$ is bounded. Then, for a point $Q_{I} \in M$ where $V$ is singular with the assumptions in Theorem 6.2.3, we have the following estimates in the
small neighborhood of $Q_{I}$. Let $T_{\xi} \subset \mathcal{S}_{I}$ be a triangle with the biggest edge of length $=\xi$, and also, $Q_{I}$ is a vertex of $T_{\xi}$. Denote by $T_{\kappa \xi} \subset T_{\xi}$ the similar subtriangle of $T_{\xi}$ with the ratio of similitude $\kappa, 0<\kappa<1$, which is obtained as follows. In $T_{\xi}$, draw a line segment parallel to the edge of $T_{\xi}$ opposite to the vertex $Q_{I}$, such that the ratio of the length of this segment and the length of the opposite edge is $\kappa$. Then, $T_{\xi}$ is divided into the small triangle $T_{\kappa \xi}$ that has the common vertex $Q_{I}$ with $T_{\xi}$, and the trapezoid between the two parallel edges. This procedure leads to the below estimates near $Q_{I}$.

Theorem 6.3.3. Let $0<\kappa<1, \alpha>0$ and $T_{\xi} \subset \mathcal{S}_{i}$ be a triangle as described above. Let $\mathcal{T}=\left(T_{i}\right)$ be a triangulation of $G^{\prime}:=T_{\xi} \backslash T_{\kappa \xi}$ with angles $\geq \alpha$ and edges $\leq h$. Then

$$
\left\|u-u_{I}\right\|_{\mathcal{K}_{1}^{1}\left(G^{\prime}\right)} \leq C(\kappa) \xi^{a}(h / \xi)^{m}\|u\|_{\mathcal{K}_{1+a}^{m+1}\left(G^{\prime}\right)}
$$

for $\forall u \in \mathcal{K}_{1+a}^{m+1}\left(\mathcal{S}_{i}\right), m \geq 1, a>0$.
The proof of this theorem is the same as the proof of Lemma 4.1.4 in Chapter 4.

We shall now concentrate on the mesh refinements around an interior point where $V$ is singular.

### 6.3.2 Construction of the Finite Element Spaces

Recall that $m$ is fixed. For any mesh $\mathcal{T}$ of $\Omega$, we let

$$
\begin{aligned}
S(\mathcal{T}, m+1) & :=\tilde{S}(\mathcal{T}, m+1) \cap \mathcal{K}_{1}^{1}(\Omega) \cap\left\{u=0 \text { on } \partial_{D} \Omega\right\} \\
& =\left\{\chi \in \tilde{S}(\mathcal{T}, m+1), \chi=0 \text { on } \partial_{D} \Omega \cup \mathcal{V}\right\}
\end{aligned}
$$

The condition $\chi=0$ on $\partial_{D} \Omega$ in the above equation is due to the fact that our main variational space $\mathcal{K}_{1}^{1}(\Omega) \cap\left\{u=0\right.$ on $\left.\partial_{D} \Omega\right\}$ consists of functions that vanish on $\partial_{D} \Omega$. The condition $\chi=0$ at $\mathcal{V}$ is due to the fact that $a(\chi, \chi)<\infty$ for all $\chi \in \mathcal{K}_{1}^{1}(\Omega) \cap\left\{u=0\right.$ on $\left.\partial_{D} \Omega\right\}$.

Recall that we want to construct a sequence of meshes $\mathcal{T}_{n}$, with Finite Element spaces $S_{n}:=S\left(\mathcal{T}_{n}, m+1\right)$, such that the sequence $u_{n}:=u_{S_{n}} \in S_{n}$ of Galerkin approximations of the solution $u$ for our Schrödinger-type mixed boundary value
problem, Equation (6.2), satisfies $\left\|u-u_{n}\right\|_{\mathcal{K}_{1}^{1}} \leq C \operatorname{dim}\left(S_{n}\right)^{-m / 2}\|f\|_{H^{m-1}}$. We shall achieve this quasi-optimal rate of convergence by considering a suitable grading close to the points of $\mathcal{V}$. The proof will be based on estimating the error in weighted Sobolev spaces. At the boundary, the estimates are known from Chapter 4, so we shall now concentrate at an interior point $Q_{i}$ where $V$ is singular. The meshes $\mathcal{T}_{n}$ that are used to define the spaces $S_{n}=S\left(\mathcal{T}_{n}, m+1\right)$ will be constructed by successive refinements and will have the same number of triangles as the meshes obtained by the usual mid-point refinement.

Let $\eta=\min \left|\Sigma_{\mathrm{j}}\right|$, which satisfies (6.17). From now on, we shall assume that the right hand side $f$ of Equation (6.2) satisfies the condition $f \in H^{m-1}:=H^{m-1}(\Omega) \subset$ $\mathcal{K}_{a-1}^{m-1}:=\mathcal{K}_{a-1}^{m-1}(\Omega)$, where $0 \leq a<\min (\eta, 1), m \geq 1$. Therefore the solution $u$ of Equation (6.2) satisfies

$$
u \in \mathcal{K}_{a+1}^{m+1} \cap\left\{\left.u\right|_{\partial_{D} \Omega}=0\right\} \subset \mathcal{K}_{1}^{1}(\Omega) \cap\left\{u=0 \text { on } \partial_{D} \Omega\right\}
$$

by Theorem 6.2.3.
We now introduce our refinement procedure. Recall that the vertices of $\Omega$ and the points where the boundary conditions change are contained in $\mathcal{V}$.

Definition 6.3.4. Let $\kappa \in(0,1 / 2]$ and $\mathcal{T}$ be a triangulation of $\Omega$ such that no two vertices of $\Omega$ belong to the same triangle of $\mathcal{T}$. Then the $\kappa$ refinement of $\mathcal{T}$, denoted $\kappa(\mathcal{T})$ is obtained by dividing each edge $A B$ of $\mathcal{T}$ in two parts as follows. If neither $A$ nor $B$ is in $\mathcal{V}$, then we divide $A B$ into two equal parts. Otherwise, if say $A$ is in $\mathcal{S}$, we divide $A B$ into $A C$ and $C B$ such that $|A C|=\kappa|A B|$. This will divide each triangle of $\mathcal{T}$ into four triangles.

We now introduce our sequence of meshes. Recall that $\tilde{l}>0$ was introduced in Remark 2.3.1 and $4 \tilde{l}$ is not greater than the distance from a point $Q$ in $\mathcal{V}$ to an edge of $\Omega$ that does not contain it.

Definition 6.3.5. For a fixed $m=\{1,2, \ldots\}$, we define a sequence of meshes $\mathcal{T}_{n}$ as follows. The initial mesh $\mathcal{T}_{0}$ is such that each edge in the mesh has length $\leq \tilde{l} / 2$ and every point in $\mathcal{V}$ has to be the vertex of a triangle in the mesh. In addition, we chose $\mathcal{T}_{0}$ such that there is no triangle in $\mathcal{T}_{0}$ that contains more than one point in $\mathcal{V}$. Then we define by induction $\mathcal{T}_{n+1}=\kappa\left(\mathcal{T}_{n}\right)$ (see Definition 6.3.4). We shall


Figure 6.1. One refinement of triangle $T$ with vertex $Q_{I} \in M, \kappa=l_{1} / l_{2}$.
denote by

$$
u_{I, n}=u_{I, \mathcal{T}_{n}, m+1} \in S_{n}:=S\left(\mathcal{T}_{n}, m+1\right)
$$

the degree $m$ Lagrange interpolant associated to $u \in \mathcal{C}(\bar{\Omega})$ and the mesh $\mathcal{T}_{n}$ on $\Omega$.
Near the vertices, our refinement coincides with the ones introduced in $[7,17$, 21, 88].

We now investigate the approximation properties afforded by the triangulation $\mathcal{T}_{n}$ close to a fixed point $Q_{i} \in \mathcal{V}$. We also fix a triangle $T \in \mathcal{T}_{0}$ that has $Q_{i}$ as a vertex. Let us denote by $T_{\kappa^{j}}=\kappa^{j} T \subset T$ the small triangle belonging to $\mathcal{T}_{j}$ that is similar to $T$ with ratio $\kappa^{j}$, has $Q_{j}$ as a vertex, and has all sides parallel to the sides of $T$. Then $T_{\kappa^{j}} \subset T_{\kappa^{j-1}}$. Moreover, since $\kappa<1 / 2$ and the diameter of $T$ is $\leq \tilde{l} / 2$, we have $T_{\kappa^{j}} \subset \mathcal{S}_{i}, j \geq 1$, by the definition of $\mathcal{S}_{i}$.

Let $N$ be the level of refinement. In all the statements below, $h \simeq 2^{-N}$, in the sense that they have comparable magnitudes. In particular, we can replace $h$ with $2^{-N}$ in all the estimates below, possibly by increasing the constants. A good choice is $h=h_{0} 2^{-N}$, where $h_{0}$ is the initial mesh size.

We shall need the following general lemma.
Lemma 6.3.6. We have $\mathcal{K}_{1}^{2}:=\mathcal{K}_{1}^{2}(\Omega) \subset L^{\infty}(\Omega)$ and $\mathcal{K}_{1+a}^{2} \subset \mathcal{C}(\bar{\Omega})$, for $a>0$, and hence every function $u \in \mathcal{K}_{1+a}^{2}$ is continuous and vanishes on $\mathcal{V}$.

Proof. Let $G=T_{\kappa^{j-1}} \backslash T_{\kappa^{j}}$. The Sobolev embedding theorem gives $\|u\|_{L^{\infty}(G)} \leq$ $C\|u\|_{\mathcal{K}_{1}^{2}(G)}$. The constant can be chosen to be independent of $j$ since both norms are dilation invariant (this is obvious for the $L^{\infty}-$ norm and for the $\mathcal{K}_{1}^{2}$-norm it follows from Lemma 6.1.2). This shows that $\mathcal{K}_{1}^{2} \subset L^{\infty}(\Omega)$. The classical Sobolev embedding theorem also shows right away that $\mathcal{K}_{1}^{2} \subset \mathcal{C}(\bar{\Omega} \backslash \mathcal{V})$. The relation
$\mathcal{K}_{1+a}^{2}=\rho^{a} \mathcal{K}_{1}^{2}$ and the fact that $\rho^{a}(p) \rightarrow 0$ as $p \rightarrow Q \in \mathcal{V}$ shows that any $u \in \mathcal{K}_{1+a}^{2}$ is also continuous at every point of $\mathcal{V}$.

It follows from the above lemma that the interpolant $u_{I}$ is defined for $u \in \mathcal{K}_{1+a}^{m+1}$ and $u_{I}(Q)=0$ for all $Q \in \mathcal{V}$. See [2] for more general embedding theorems.

Lemma 6.3.7. Let $0<\kappa \leq 2^{-m / a}, 0<a<\eta$. Let us consider the small triangle $T_{\kappa^{N}}=\kappa^{N} T \subset T$ with vertex $Q_{i}$, obtained after $N$ refinements. Let $u_{I, N}$ be the degree $m$ Lagrange interpolant of $u$ associated to $\mathcal{T}_{N}$. Then, on $T_{\kappa^{N}} \in \mathcal{T}_{N}$, we have

$$
\left\|u-u_{I, N}\right\|_{\mathcal{K}_{1}^{1}\left(T_{\kappa^{N}}\right)} \leq C h^{m}\|u\|_{\mathcal{K}_{a+1}^{m+1}\left(T_{\kappa^{N}}\right)},
$$

for all $u \in \mathcal{K}_{a+1}^{m+1}\left(\mathcal{S}_{i}\right) \cap\left\{\left.u\right|_{\partial_{D} \Omega}=0\right\}, h \simeq 1 / 2^{N}$, where $C$ depends on $m$ and $\kappa$, but not on $N$.

Proof. Let us denote $u_{\lambda}(x, y)=u(\lambda x, \lambda y)$ with $Q_{i}$ as the origin. Let $\lambda=\kappa^{N}$. Then, $u_{\lambda}(x, y) \in \mathcal{K}_{a+1}^{m+1}(T)$ by Lemma 6.1.2. Let $\chi: T \rightarrow[0,1]$ be a smooth function that is equal to 0 in a neighborhood of $Q_{i}$, but is equal to 1 at all the nodal points of $T$ different from the vertex $Q_{i}$. We introduce the auxiliary function $v=\chi u_{\lambda}$ on $T$. Consequently,

$$
\|v\|_{\mathcal{K}_{1}^{m+1}(T)}^{2}=\left\|\chi u_{\lambda}\right\|_{\mathcal{K}_{1}^{m+1}(T)}^{2} \leq C\left\|u_{\lambda}\right\|_{\mathcal{K}_{1}^{m+1}(T)}^{2}
$$

where $C$ depends on $m$ and the choice of the nodal points. Moreover, since $u\left(Q_{i}\right)=$ 0 by Lemma 6.3.6, the interpolant $v_{I}=u_{\lambda I}=u_{I \lambda}$ on $T$ by the definition of $v$.

This gives

$$
\begin{aligned}
\left\|u-u_{I}\right\|_{\mathcal{K}_{1}^{1}\left(T_{\kappa^{N}}\right)} & =\left\|u_{\lambda}-v+v-u_{\lambda I}\right\|_{\mathcal{K}_{1}^{1}(T)} \\
& \leq\left\|u_{\lambda}-v\right\|_{\mathcal{K}_{1}^{1}(T)}+\left\|v-u_{\lambda I}\right\|_{\mathcal{K}_{1}^{1}(T)} \\
& =\left\|u_{\lambda}-v\right\|_{\mathcal{K}_{1}^{1}(T)}+\left\|v-v_{I}\right\|_{\mathcal{K}_{1}^{1}(T)} \\
& \leq C_{1}\left\|u_{\lambda}\right\|_{\mathcal{K}_{1}^{1}(T)}+C_{2}\|v\|_{\mathcal{K}_{1}^{m+1}(T)} \\
& \leq C_{1}\left\|u_{\lambda}\right\|_{\mathcal{K}_{1}^{1}(T)}+C_{3}\left\|u_{\lambda}\right\|_{\mathcal{K}_{1}^{m+1}(T)} \\
& =C_{1}\|u\|_{\mathcal{K}_{1}^{1}\left(T_{\kappa^{N}}\right)}+C_{3}\|u\|_{\mathcal{K}_{1}^{m+1}\left(T_{\kappa^{N}}\right)} \\
& \leq C_{4} \kappa^{N a}\|u\|_{\mathcal{K}_{a+1}^{m+1}\left(T_{\kappa^{N}}\right)} \\
& \leq C h^{m}\|u\|_{\mathcal{K}_{a+1}^{m+1}\left(T_{\kappa^{N}}\right)} .
\end{aligned}
$$

The first and the sixth relations above are due to Lemma 6.1.2; the fourth is due to Theorem 6.3.2; and the seventh is based on Lemma 2.3.6.

We now combine the estimate on $T_{\kappa^{N}}$ of the previous lemma with the estimates on the sets of the form $T_{\kappa^{j}} \backslash T_{\kappa^{j+1}}$ of Theorem 6.3.3 to obtain the following estimate on an arbitrary, but fixed, triangle $T \in \mathcal{T}_{0}$ that has a vertex in $\mathcal{V}$. We continue to fix a triangle $T$ of $\mathcal{T}_{0}$ with a vertex $Q_{i} \in \mathcal{V}$.

Proposition 6.3.8. Denote by $h \simeq 1 / 2^{N}$ the mesh size of $\mathcal{T}_{N}$ and let $0<\kappa \leq$ $2^{-m / a}, 0<a<\eta$. Then there exists a constant $C>0$, such that

$$
\left\|u-u_{I, N}\right\|_{\mathcal{K}_{1}^{1}(T)} \leq C h^{m}\|u\|_{\mathcal{K}_{a+1}^{m+1}(T)}
$$

for all $\mathcal{K}_{a+1}^{m+1}(\Omega)$.
Proof. The proof follows from the estimates on the subsets $T_{\kappa^{j-1}} \backslash T_{\kappa^{j}}, 1 \leq j \leq N$, (Theorem 6.3.3) and from the estimate on $T_{\kappa^{N}}$ (Lemma 6.3.7).

Definition 6.3.5 shows that the mesh size of $T_{\kappa^{j-1}} \backslash T_{\kappa^{j}}$, is $\simeq \kappa^{j-1} 2^{j-1-N}$. Then, using the notation in Theorem 6.3.3, we have $\xi=O\left(\kappa^{j-1}\right)$, therefore,

$$
\begin{aligned}
\left.\left\|u-u_{I}\right\|_{\mathcal{K}_{1}^{1}\left(T_{\kappa j} j-1\right.} \backslash T_{\kappa j}\right) & \left.\leq C_{1} \kappa^{(j-1) a}\left(\kappa^{j-1} 2^{j-1-N} / \kappa^{j-1}\right)^{m}\|u\|_{\mathcal{K}_{a+1}^{m+1}\left(T_{\kappa j} j-1\right.} \backslash T_{\kappa j}\right) \\
& \leq C_{2} 2^{-(j-1) m} 2^{-N m+(j-1) m}\|u\|_{\mathcal{K}_{a+1}^{m+1}\left(T_{\kappa j} j-1 \backslash T_{\kappa j}\right)} \\
& =C_{2} 2^{-N m}\|u\|_{\mathcal{K}_{a+1}^{m+1}\left(T_{\kappa j} j-1 \backslash T_{k j}\right)} \\
& \leq C h^{m}\|u\|_{\mathcal{K}_{a+1}^{m+1}\left(T_{\kappa j} j-1 \backslash T_{\kappa j}\right)},
\end{aligned}
$$

where $C$ depends on $\kappa$, but not on the subset $T_{\kappa^{j-1}} \backslash T_{\kappa^{j}}$. Since the estimate of the interpolation error on $T_{\kappa^{N}}$ has been given in Lemma 6.3.7, we complete the proof of Proposition 6.3.8 by adding up the error estimates on all the subsets $T_{\kappa^{j-1}} \backslash T_{\kappa^{j}}$, $1 \leq j \leq N$, and on $T_{\kappa^{N}}$.

Remark 6.3.9. Denote by $\mathbb{T}$ be the union of all the initial triangles that contain singular points of $V$. Then $\mathbb{T}$ is a neighborhood of $M$ in $\bar{\Omega}$. Moreover, the interpolation error on $\mathbb{T}$ also satisfies $\left\|u-u_{I}\right\|_{\mathcal{K}_{1}^{1}(\mathbb{T})} \leq C h^{m}\|u\|_{\mathcal{K}_{a+1}^{m+1}(\mathbb{T})}$ by summing up the estimates in Proposition 6.3.8 over all the triangles, as long as $\kappa$ is chosen appropriately.

Here we state our main result, namely the quasi-optimal convergence rate of the numerical solutions on our meshes.

Theorem 6.3.10. Let $0<a<\eta$ and $0<\kappa<2^{-m / a}$, with $m \geq 1$ fixed. We assume that the conditions of Theorem 6.2.3 are satisfied. Let $\mathcal{T}_{n}$ be obtained from the initial triangulation by $n$-refinements, as in Definition 6.3.5. Let $S_{n}:=$ $S_{n}\left(\mathcal{T}_{n}, m+1\right)$ be the Finite Element space given by the first equation of Subsection 6.3 .2 and $u_{n}=u_{S_{n}} \in S_{n}$ be the Finite Element solution defined by Equation (6.5). Then there exists $C>0$ such that

$$
\left\|u-u_{n}\right\|_{\mathcal{K}_{1}^{1}} \leq C h^{m}\|f\|_{\mathcal{K}_{a-1}^{m-1}}, \quad h \simeq 2^{-n}
$$

for any $f \in \mathcal{K}_{a-1}^{m-1}$.
Proof. Let $\mathbb{T}_{i}$ be the union of initial triangles that contain $Q_{i} \in \mathcal{V}$. Recall from Theorem 6.2.3 that $\|u\|_{\mathcal{K}_{a+1}^{m+1}} \leq C\|f\|_{\mathcal{K}_{a-1}^{m-1}}$. We use the previous estimates to obtain

$$
\begin{aligned}
\left\|u-u_{n}\right\|_{\mathcal{K}_{1}^{1}} & \leq C\left\|u-u_{I}\right\|_{\mathcal{K}_{1}^{1}} \\
& =C\left(\left\|u-u_{I}\right\|_{\mathcal{K}_{1}^{1}\left(\Omega \backslash \cup \mathbb{T}_{i}\right)}+\sum\left\|u-u_{I}\right\|_{\mathcal{K}_{1}^{1}\left(\mathbb{T}_{i}\right)}\right) \\
& \leq C h^{m}\left(\|u\|_{\mathcal{K}_{1}^{m+1}\left(\Omega \backslash \cup \mathbb{T}_{i}\right)}+\sum\|u\|_{\mathcal{K}_{a+1}^{m+1}\left(\mathbb{T}_{i}\right)}\right) \\
& \leq C h^{m}\|u\|_{\mathcal{K}_{a+1}^{m+1}} \leq C h^{m}\|f\|_{\mathcal{K}_{a-1}^{m-1}} .
\end{aligned}
$$

The first inequality is based on Céa's Lemma and the second inequality is based on the Theorem 6.3.2 and Preposition 6.3.8.

Then, as a direct result of the theorem above, we have the following estimate on the convergence rate of the finite element solution, which indicates that it is quasi-optimal.

Theorem 6.3.11. Using the notation and assumptions of Theorem 6.3.10, we have that $u_{n}=u_{S_{n}} \in S_{n}:=S\left(\mathcal{T}_{n}, m+1\right)$ satisfies

$$
\left\|u-u_{n}\right\|_{\mathcal{K}_{1}^{1}} \leq C \operatorname{dim}\left(S_{n}\right)^{-m / 2}\|f\|_{\mathcal{K}_{a-1}^{m-1}}
$$

for a constant independent of $f$ and $n$.

Proof. Let $\mathcal{T}_{n}$ be the triangulation of $\Omega$ after $n$ refinements. Then, the number of triangles is $O\left(4^{n}\right)$ based on the construction of triangles in different levels. Therefore, the dimension of $S, \operatorname{dim}\left(S_{n}\right) \simeq 4^{n}$, for Lagrange triangles. Thus, from Theorem 6.3.10, the following estimates are obtained,

$$
\left\|u-u_{n}\right\|_{\mathcal{K}_{1}^{1}} \leq C_{1} h^{m}\|f\|_{\mathcal{K}_{a-1}^{m-1}} \simeq C 2^{-n m}\|f\|_{\mathcal{K}_{a-1}^{m-1}} \leq C \operatorname{dim}\left(S_{n}\right)^{-m / 2}\|f\|_{\mathcal{K}_{a-1}^{m-1}} .
$$

The proof is complete.
Using that $H^{m-1} \subset \mathcal{K}_{a-1}^{m-1}$ for $a \in(0,1)$, we obtain the following corollary, under the assumptions of the above theorem.

Corollary 6.3.12. Let $0<a \leq \min \{1, \eta\}$ and $0<\kappa<2^{-m / a}$. Then

$$
\left\|u-u_{n}\right\|_{\mathcal{K}_{1}^{1}} \leq C \operatorname{dim}\left(S_{n}\right)^{-m / 2}\|f\|_{H^{m-1}}
$$

for a constant independent of $f$ and $n$.

### 6.4 Numerical Results

We present here some numerical results to illustrate the effectiveness of our mesh refinement technique. We convincingly show that our sequence of meshes achieves quasi-optimal rates of convergence. Based on the arguments in Section 6.2, the singularity of the solution near $Q_{i} \in M$ heavily depends on the parameter $\delta_{i}$. Therefore, we take equations associated to the operator

$$
H=-\Delta+\delta r^{-2}
$$

as model problems, where $r$ is the distance to the origin $O=Q_{0}$. Different values of $\delta$ as well as different positions of the origin on the domain will be considered so as to verify our theoretical prediction in all aspects. Note that in the model problems below, we have chosen $m=1$, namely, piecewise linear functions for the finite element method, since the implementation is simpler, while the results are still relevant.


Figure 6.2. Initial triangles for $\hat{\Omega}$ (left); the mesh after four refinements, $\kappa=0.2$ (right).

We first consider the following model problems on $\hat{\Omega}:=(-1,1) \times(-1,1)$, such that the origin $Q_{0}$ is an interior point of the domain. (See Figure 6.2.)

$$
\begin{align*}
& \left\{\begin{aligned}
-\Delta u+0.5 r^{-2} u=1 & \text { in } \hat{\Omega}, \\
u=0 & \text { on } \partial \hat{\Omega},
\end{aligned}\right.  \tag{6.21}\\
& \left\{\begin{aligned}
-\Delta u+2 r^{-2} u=1 & \text { in } \hat{\Omega}, \\
u=0 & \text { on } \partial \hat{\Omega}
\end{aligned}\right. \tag{6.22}
\end{align*}
$$

For the problem (6.21), the solution is not in $H^{2}$ near the origin. Then, special treatment is needed for the mesh near the origin to get the optimal rate of convergence. To be more precise, from our theory developed above (Equation (6.17)), we can take a value of $a$, such that $0<a<\eta=\sqrt{0.5} \approx 0.707$, which makes $\kappa=2^{-1 / a}<2^{-1 / \eta} \approx 0.375$. In fact, a more accurate a prior estimate [60] on the solution gives $u \in H^{s}$ for $s<1+\sqrt{0.5} \approx 1.707$. The situation in (6.22) is different, since the regularity of the solution depends on the parameter $\delta$. Based on our method, $\eta=\sqrt{2} \approx 1.414>1$, which means the solution is in $H^{2}$, and hence no graded mesh is necessary for piecewise linear functions.

Meanwhile, we also implement numerical tests on the L-shape domain $\hat{\Omega}_{1}:=$ $(-1,0) \times(-1,1) \cup[0,1) \times(0,1)$ (Figure 6.3) with mixed boundary conditions, where the origin $Q_{0}$ is the vertex of the re-entrant corner on the boundary and boundary


Figure 6.3. Initial triangles for $\hat{\Omega}_{1}$ and boundary conditions (left); the mesh after one refinement, $\kappa=0.2$ (right)
conditions change the type at $Q_{0}$. The model problems are as follows

$$
\begin{align*}
& \left\{\begin{aligned}
-\Delta u+0.15 r^{-2} u=1 & \text { in } \hat{\Omega}_{1} \\
u=0 & \text { on } \partial_{D} \hat{\Omega}_{1} \\
\partial_{v} u=0 & \text { on } \partial_{N} \hat{\Omega}_{1}
\end{aligned}\right.  \tag{6.23}\\
& \left\{\begin{aligned}
-\Delta u+1.5 r^{-2} u=1 & \text { in } \hat{\Omega}_{1} \\
u=0 & \text { on } \partial_{D} \hat{\Omega}_{1} \\
\partial_{v} u=0 & \text { on } \partial_{N} \hat{\Omega}_{1}
\end{aligned}\right. \tag{6.24}
\end{align*}
$$

where $\partial_{N} \hat{\Omega}_{1}:=\{(x, y) \mid x=0,-1<y<0\}$ and $\partial_{D} \hat{\Omega}_{1}=\partial \hat{\Omega}_{1} \backslash \partial_{N} \hat{\Omega}_{1}$. The parameter $\delta$ has a big effect on the regularity of the solution in this case as well. For (6.23), a similar a priori estimate leads to a solution $u \in H^{s}$ for $s<1+$ $\sqrt{0.15+(1 / 2)^{2}(2 / 3)^{2}} \approx 1.511$. We also use the formula (6.17) in Section 6.2 to determine $\eta=\sqrt{0.15+(1 / 2)^{2}(2 / 3)^{2}} \approx 0.511$. Thus, to recover the quasi-optimal convergence rates, we can take $\kappa=2^{-1 / \epsilon}$ for any $0<\epsilon<\eta$, which indicates $\kappa<0.258$. The value of $\eta$ in (6.24), however, is $\sqrt{1.5+(1 / 2)^{2}(2 / 3)^{2}} \approx 1.269$. Therefore, the solution is in $H^{2}$ and the numerical solutions will approximate the real solution in the quasi-optimal rate on uniform meshes near $Q_{0}$. We also note that in all the equations above, the solutions do not possess singularities in $H^{2}$ in the neighborhoods of the corners that have acute interior angles. For this reason, uniform meshes near acute corners of the domain are used in our numerical experiments.

One of the difficulties in the discretization of Equation (6.2) is to perform the numerical integration accurately. Note that the integrations involve the singular term $r^{-2}$, which is getting stronger and stronger as $r \rightarrow 0$. Regular quadrature rules for polynomials in two-dimenstions will fail, since the errors are not in a uniform order on triangles that are near the origin. Therefore, instead of quadrature rules in two-dimenstions, we integrate the corresponding function on the reference triangle in one variable first, which is analytically exact. Then, we apply the Gaussian quadrature on the one-dimenstional integral to control the error from the numerical integration. The finest mesh in our numerical tests is obtained after 10 successive refinements of the coarsest mesh and has roughly $2^{23} \approx 8 \times 10^{6}$ elements. The preconditioned conjugate gradient (PCG) method is used to solve the resulting system of algebraic equations.

Table 6.1 lists the convergence rates of the finite element solutions for equations (6.21) and (6.22), respectively, on triangulations with different values of $\kappa$. These results verify our theoretical prediction: the quasi-optimal convergence rates can be obtained as $\kappa<0.375$ for Equation (6.21); no graded meshes are needed to approximate the solution of Equation (6.22) to get the quasi-optimal rates.

The left most column in the table shows the number of the refinement levels, and $u_{j}$ represents the numerical solution on the mesh after $j$ refinements. The quantities printed out in other columns in the table are the convergence rates in the manner

$$
e=\log _{2}\left(\frac{\left|u_{j}-u_{j-1}\right|_{H_{1}^{1}}}{\left|u_{j+1}-u_{j}\right|_{H_{1}^{1}}}\right),
$$

which is quite reasonable to be the approximation of the exact convergence rate. Recall $h \equiv 1 / 2^{j}$ for the mesh after $j$ levels of refinements. Then, we see that for Equation (6.21), on appropriate graded meshes ( $\kappa<0.375$ ), the convergence rates are $h^{1}$, while on uniform meshes $(\kappa=0.5)$, the convergence rates have slowed down to $h^{0.718}$, which is very close to the theoretical rate 0.707 from our estimates above, and will get closer and closer to 0.707 . For Equation (6.22), all the convergence rates are of order $h^{1}$, which is also predicted by our theory.

In Equation (6.23) and Equation (6.24), the origin is a boundary point with mixed boundary conditions. Then the values of $\kappa$ for appropriate meshes follow another formula. For Equation (6.23), we have found that the convergence rates

| $j \backslash \kappa$ | $e: 0.1$ | $e: 0.2$ | $e: 0.3$ | $e: 0.4$ | $e: 0.5$ | $e: 0.1$ | $e: 0.2$ | $e: 0.3$ | $e: 0.4$ | $e: 0.5$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 3 | 0.903 | 0.934 | 0.941 | 0.900 | 0.815 | 0.916 | 0.937 | 0.954 | 0.960 | 0.915 |
| 4 | 0.956 | 0.972 | 0.970 | 0.919 | 0.789 | 0.968 | 0.977 | 0.983 | 0.987 | 0.960 |
| 5 | 0.982 | 0.989 | 0.982 | 0.924 | 0.764 | 0.989 | 0.992 | 0.994 | 0.996 | 0.980 |
| 6 | 0.993 | 0.995 | 0.989 | 0.926 | 0.746 | 0.996 | 0.998 | 0.998 | 0.999 | 0.990 |
| 7 | 0.998 | 0.998 | 0.992 | 0.927 | 0.733 | 0.999 | 0.999 | 1.000 | 1.000 | 0.995 |
| 8 | 0.999 | 0.999 | 0.994 | 0.927 | 0.724 | 1.000 | 1.000 | 1.000 | 1.000 | 0.997 |
| 9 | 1.000 | 1.000 | 0.996 | 0.928 | 0.718 | 1.000 | 1.000 | 1.000 | 1.000 | 0.998 |
| Convergence history for (6.21) $\left.\right\|_{\left\|u_{j}-u_{j-1}\right\|^{+1}}$ Convergence history for (6.2) |  |  |  |  |  |  |  |  |  |  |

Table 6.1. Interior singularities

| $j \backslash \kappa$ | $e: 0.1$ | $e: 0.2$ | $e: 0.3$ | $e: 0.4$ | $e: 0.5$ | $e: 0.1$ | $e: 0.2$ | $e: 0.3$ | $e: 0.4$ | $e: 0.5$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 3 | 0.885 | 0.907 | 0.896 | 0.840 | 0.754 | 0.899 | 0.920 | 0.938 | 0.947 | 0.918 |
| 4 | 0.945 | 0.953 | 0.922 | 0.828 | 0.691 | 0.961 | 0.971 | 0.978 | 0.982 | 0.958 |
| 5 | 0.973 | 0.972 | 0.926 | 0.798 | 0.626 | 0.987 | 0.991 | 0.993 | 0.994 | 0.976 |
| 6 | 0.987 | 0.981 | 0.924 | 0.767 | 0.578 | 0.996 | 0.997 | 0.998 | 0.998 | 0.985 |
| 7 | 0.994 | 0.986 | 0.921 | 0.740 | 0.548 | 0.999 | 0.999 | 0.999 | 0.999 | 0.990 |
| 8 | 0.998 | 0.990 | 0.917 | 0.720 | 0.531 | 1.000 | 1.000 | 1.000 | 1.000 | 0.994 |
| 9 | 0.999 | 0.992 | 0.914 | 0.705 | 0.521 | 1.000 | 1.000 | 1.000 | 1.000 | 0.996 |

$$
e=\log _{2}\left(\frac{\left|u_{j}-u_{j-1}\right|_{H_{1}^{1}}}{\left|u_{j+1}-u_{j}\right|_{H_{1}^{1}}}\right)
$$

Table 6.2. Boundary singularities
of the discrete solutions should be quasi-optimal $\left(h^{1}\right)$ as long as $\kappa<0.258$, which matches the numerical results in Table 6.2 perfectly. In addition, the numbers in the column for $\kappa=0.5$, Equation (6.23), are decreasing, and one can expect a convergence rate of order $h^{0.511}$ will appear at the end by the regularity of the solution. The second part of Table 6.2 implies that the convergence rates in Equation (6.24) are quasi-optimal for all $\kappa \leq 0.5$, which, once again, verifies the theory.

As a brief summary, we have tested our method on four model problems. All the results in the two tables above convincingly show that the theoretical rate of convergence can be verified in practical calculations. Therefore, for the boundary value problem (6.2), with the regularity of the solution determined in terms of weighted Sobolev spaces, the convergence rates of the numerical solutions behave like $\operatorname{dim}\left(S_{n}\right)^{-m / 2}$ on correctly graded meshes.


## Application II: an Operator Degenerate on a Segment

As the second application of our theory from Chapter 3 and Chapter 4, we consider an operator degenerate on a segment of the boundary. This type of operators appears in mathematical models of fuel cells and in fluid mechanics. Similar operators can also be found in $[31,53]$.

Let $\Omega$ be the rectangular domain $(0,1) \times(0, l)$ for $l>0$, and define the operator $\mathcal{L}_{\delta}$ as follows

$$
\mathcal{L}_{\delta}:=-\partial_{x}^{2}-\frac{\delta^{2}}{x^{2}} \partial_{y}^{2}, \quad \delta>0
$$

We then consider a class of degenerate elliptic equations on $\Omega$ corresponding to $\mathcal{L}_{\delta}$ with the Dirichlet boundary condition:

$$
\left\{\begin{array}{rll}
\mathcal{L}_{\delta} u=f & \text { in } & \Omega  \tag{7.1}\\
u=0 & \text { on } & \partial \Omega
\end{array}\right.
$$

Denote by $u$ the solution of Equation (7.1). Let $\Omega_{\xi}:=(0, \xi) \times(0, l)$ and $S_{\xi}:=$ $\Omega \backslash \Omega_{\xi}$ be subsets of $\Omega$, depending on $\xi$. Denote by $P_{r} \subset \Omega$, an arbitrary open subset containing the neighborhoods of the vertices $(1,0)$ and $(1, l)$. Then, the strong ellipticity of the operator $\mathcal{L}_{\delta}$ on $S_{\xi} \backslash P_{r}$ implies that $u \in H^{m+1}\left(S_{\xi} \backslash P_{r}\right) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}$ for any $f \in H^{m-1}(\Omega)[48,94]$.

However, it is well known that this result does not extend to the entire domain $\Omega$ in general, because of the loss of ellipticity at $x=0$ and the possible singularities
of the solution arising from the corners. In fact, it is generally impossible to find a solution $u \in H^{m+1}(\Omega)$ for large $m$, even if the given data is smooth $f \in$ $\mathcal{C}^{\infty}(\Omega)$. Various techniques have been used to investigate different degenerate elliptic equations in the papers of Boimatov [25], Felli and Schneider [49], French [51], Langlais [64], and others. In this chapter, inspired by the analysis in Chapter 3, we shall study the well-posedness and regularity of the solution of Equation (7.1) in terms of some weighted Sobolev space $\mathcal{K}_{a}^{m}$ (Definition 7.1.1). Note the the weighted Sobolev space we use in this chapter is quite different from the space in Definition 2.3.2. The well-posedness of the solution $u$ in weighted Sobolev spaces will be proved, and consequently, we shall show that there is no loss of $\mathcal{K}_{a}^{m}$-regularity for the solution of Equation (7.1).

Another main result of this chpater is regarding the numerical approximation by the finite element method (FEM). Let $S_{n}$ be a sequence of finite dimensional subspaces for the FEM. Denote by $u_{n} \in S_{n}$ the corresponding discrete solution. Then, we shall provide a simple, explicit way to construct a sequence of finite dimensional subspaces $S_{n} \subset \mathcal{K}_{1}^{1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}$, such that $u_{n}$ satisfies

$$
\left\|u-u_{n}\right\|_{H^{1}(\Omega)} \leq C \operatorname{dim}\left(S_{n}\right)^{-m / 2}\|f\|_{H^{m-1}(\Omega)}
$$

where $f \in H^{m-1}(\Omega) \cap \mathcal{K}_{-1+\epsilon}^{m-1}(\Omega)$ is arbitrary and $C$ is a constant that depends on $\Omega$ and $m$, but not $n$ or $f$. Namely, one can recover the optimal rate of convergence that is expected for smooth solutions.

In addition, according to the result of Babuska and Aziz [15], the maximum angle of the triangles in the triangulation for the FEM should be bounded away from $\pi$, such that a uniform error estimate can be obtained in the usual Sobolev spaces $H^{m}$ on each triangle. Otherwise, the energy norm of the error $\left|u-u_{n}\right|_{H^{1}}$ on $\Omega$ might be difficult to control. In our construction of subspaces $S_{n}$, however, thin triangles that violate this maximum-angle condition will appear. In fact, the maximum angle in the triangles will keep increasing with $\pi$ as the limit. We shall show that the difficulty for the estimates in this case can be overcome by a homogeneity argument in weighted Sobolev spaces.

In Section 7.1, we shall introduce our weighted Sobolev spaces $\mathcal{K}_{a}^{m}(\Omega)$ and some notations that will be used throughout this chapter. We then study the properties
of $\mathcal{K}_{a}^{m}(\Omega)$ that are important for our analysis.
In Section 7.2, we shall prove the well-posedness and regularity of the solution in $\mathcal{K}_{a}^{m}(\Omega)$. Denote by $\left(v_{1}, v_{2}\right)$ the inner product of $v_{1}, v_{2} \in H^{0}(\Omega)=L^{2}(\Omega)$. The corresponding weak solution $u \in \mathcal{K}_{1}^{1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}$ of Equation (1) is defined by

$$
a(u, v):=\left(\partial_{x} u, \partial_{x} v\right)+\delta^{2}\left(\frac{1}{x} \partial_{y} u, \frac{1}{x} \partial_{y} v\right)=\langle f, v\rangle, \quad \forall v \in \mathcal{K}_{1}^{1}(\Omega) \cap\left\{\left.v\right|_{\partial \Omega}=0\right\} .
$$

Furthermore, we will prove that, for $m \geq 0$, the operator $\mathcal{L}_{\delta}=-\partial_{x}^{2}-\frac{\delta^{2}}{x^{2}} \partial_{y}^{2}$ is an isomorphism from $\mathcal{K}_{1}^{m+1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}$ to $\mathcal{K}_{-1}^{m-1}(\Omega)$ and Fredholm $\mathcal{K}_{1+\epsilon}^{m+1}(\Omega) \cap$ $\left\{\left.u\right|_{\partial \Omega}=0\right\} \rightarrow \mathcal{K}_{-1+\epsilon}^{m-1}(\Omega)$ iff $\epsilon$ is away from a specific countable set of values.

In Section 7.3, we will analyze the numerical solution $u_{n}$ for Equation (7.1). Explicitly, we will look for $u_{n} \in S_{n}$ satisfying

$$
a\left(u_{n}, v_{n}\right)=\left\langle f, v_{n}\right\rangle, \quad \forall v_{n} \in S_{n}
$$

Denote by $\mathcal{T}=\left(T_{i}\right)$ the triangulation of $\Omega$ with triangles. Let $S=S(\mathcal{T}, m+1)$ be the finite element space associated to the degree $m$ Lagrange triangle [39], such that $S$ consists of polynomials of degree $\leq m$ on each triangle $T_{i} \in \mathcal{T}$, in which the nodes are obtained by taking points with barycentric coordinates in $m^{-1} \mathbb{Z}$. Let $u_{S} \in S$ be the numerical solution of Equation (7.1). For any continuous solution $u$, denote by $u_{I} \in S(\mathcal{T}, m+1)$ the interpolation associated to $u$, which is uniquely determined by the condition $u\left(x_{i}\right)=u_{I}\left(x_{i}\right)$ for any node. Also, a symmetric bilinear form $a(\cdot, \cdot)$ induces an equivalent norm $\|\cdot\|_{a}$ on a normed space, provided that $a(\cdot, \cdot)$ is both continuous and coercive on this space. As a result from Section 7.2, we shall show that $\|\cdot\|_{\mathcal{K}_{1}^{1}}$ and $\|\cdot\|_{a}$ are equivalent norms on $\Omega$. Therefore, based on Céa's Lemma, we have the following inequality

$$
\left\|u-u_{S}\right\|_{\mathcal{K}_{1}^{1}(\Omega)} \leq C\left\|u-u_{I}\right\|_{\mathcal{K}_{1}^{1}(\Omega)}
$$

The constant $C$ in the expression is independent of the triangulation $\mathcal{T}$ and the solution $u$. From our estimates on the interpolation error $\left\|u-u_{I}\right\|_{\mathcal{K}_{1}^{1}(\Omega)}$, we shall construct a class $\mathcal{C}(l, h, \kappa, m, \epsilon)$ of partitions $\mathcal{T}$ of $\Omega=(0,1) \times(0, l), l>0$, such
that

$$
\left\|u-u_{S}\right\|_{H^{1}(\Omega)} \leq C \operatorname{dim}(S)^{-m / 2}\|f\|_{H^{m-1}(\Omega)}, \quad \forall f \in H^{m-1}(\Omega) \cap \mathcal{K}_{-1+\epsilon}^{m-1}(\Omega)
$$

More details about the notation and proof will be given in Section 7.3.
In Section 7.4, numerical results will be presented for the operator $\mathcal{L}_{1}:=-\partial_{x}^{2}-$ $\frac{1}{x^{2}} \partial_{y}^{2}$ on $\hat{\Omega}:=(0,1) \times(0,10)$ with a smooth $f$. We will compare the rates of convergence of the numerical solutions for different mesh sizes. The convergence history will verify our theoretical prediction and demonstrate the efficiency of our technique to approximate the solution.

### 7.1 Weighted Sobolev Spaces $\mathcal{K}_{a}^{m}$

As explained above, weighted Sobolev spaces are convenient for the problem since the solution $u$ may not belong to $H^{m+1}(\Omega)$ for large $m$, even if $f \in \mathcal{C}^{\infty}(\Omega)$. In this section, we shall introduce the weighted Sobolev spaces $\mathcal{K}_{a}^{m}(\Omega)$ and establish some properties of them, which are useful for the study of the boundary value problem (7.1).

### 7.1.1 Notation

Let $X(x, y) \in \Omega$ be an arbitrary point in the domain $\Omega$. To define weighted Sobolev spaces $\mathcal{K}_{a}^{m}$ on the domain $\Omega$, we denote by $r_{1}(x, y)$ and $r_{2}(x, y)$ two smooth functions, such that $r_{1}(x, y)=$ the distance from $X(x, y)$ to $(1,0)$, if the distance $<$ $\frac{1}{4} ; r_{2}(x, y)=$ the distance from $X(x, y)$ to $(1, l)$, if the distance $<\frac{1}{4} ; \frac{1}{4} \leq r_{1}, r_{2} \leq 1$ otherwise. In addition, we require both $r_{1}(x, y)$ and $r_{2}(x, y)$ are equal to 1 if $x<\frac{1}{2}$ (Figure 7.1). The above distances are similar to the weight in Definition 2.3.2 to reflect properties of the solution of Equation (7.1) near the vertices $(1,0),(1, l)$. Denote by $\mathcal{V}=\{(1,0),(1, l),[0, y], 0 \leq y \leq l\}$ the set containing the vertices and the degenerate boundary of the domain. Then, we define the weighed Sobolev spaces $\mathcal{K}_{a}^{m}$ on $\Omega$ as follows.

Definition 7.1.1. Let $R:=\left\{x<\frac{1}{2}\right\} \cup\left\{r_{i}<\frac{1}{4}, i=1,2\right\} \subset \Omega$ be a subset of the domain. Let $\rho(x, y)$ be a positive smooth function, such that $\rho$ stands for


Figure 7.1. Domain $\Omega$ and the position of $X$.
the distance from $X(x, y)$ to the set $\mathcal{V}$ for any $X(x, y) \in R$, and $\rho$ satisfies that $\frac{1}{4} \leq \rho \leq 1$ for any point in the region $\Omega \backslash R$. Therefore, $\rho=x$ in the neighborhood of the degenerate boundary; $\rho=r_{1}$ and $\rho=r_{2}$ in the neighborhoods of $(1,0)$ and $(1, l)$ respectively. Then, for $i, j, m \in\{0,1,2, \ldots\}$, the $m$ th weighted Sobolev space is

$$
\mathcal{K}_{a}^{m}(\Omega):=\left\{v, \rho^{-a}\left(r_{1} r_{2}\right)^{i+j}\left(x \partial_{x}\right)^{i} \partial_{y}^{j} v \in L^{2}(\Omega), i+j \leq m\right\} .
$$

The $\mathcal{K}_{a}^{m}$-norm for any function $v \in \mathcal{K}_{a}^{m}(\Omega)$ is

$$
\|v\|_{\mathcal{K}_{a}^{m}(\Omega)}^{2}:=\sum_{i+j \leq m}\left\|\rho^{-a}\left(r_{1} r_{2}\right)^{i+j}\left(x \partial_{x}\right)^{i} \partial_{y}^{j} v\right\|_{L^{2}(\Omega)}^{2}
$$

In addition, we set $\Omega_{\xi}:=(0, \xi) \times(0, l) \subset \Omega$, and $S_{\xi}:=\Omega \backslash \Omega_{\xi}$ to be two particular subsets that will be used very often in the text.

Since $\rho^{-a}\left(r_{1} r_{2}\right)^{i+j}\left(x \partial_{x}\right)^{i} \partial_{y}^{j} v \in L^{2}(\Omega)$ for $\forall v \in \mathcal{K}_{a}^{m}(\Omega)$, the completeness of the space $\mathcal{K}_{a}^{m}(\Omega)$ then follows the completeness of $L^{2}(\Omega)$ and integration by parts with standard arguments in $[1,48]$. Moreover, $\mathcal{K}_{a}^{m}(\Omega)$ is a Hilbert space associated with the inner product

$$
(u, v)_{\mathcal{K}_{a}^{m}}=\sum_{i+j \leq m} \int_{\Omega}\left(\rho^{-2 a}\left(r_{1} r_{2}\right)^{2(i+j)}\left(x \partial_{x}\right)^{i} \partial_{y}^{j} u\left(x \partial_{x}\right)^{i} \partial_{y}^{j} v\right) d x d y
$$

We denote by $\mathcal{K}_{-a}^{-m}(\Omega):=\left(\mathcal{K}_{a}^{m}(\Omega) \cap\left\{\left.v\right|_{\partial \Omega}=0\right\}\right)^{\prime}$ the dual space of $\mathcal{K}_{a}^{m}(\Omega) \cap$ $\left\{\left.v\right|_{\partial \Omega}=0\right\}$ with respect to the pivot space $L^{2}(\Omega)$,

$$
\|w\|_{\mathcal{K}_{-a}^{-m}(\Omega)}:=\sup _{v \in \mathcal{K}_{a}^{m}(\Omega) \cap\left\{\left.v\right|_{\partial \Omega}=0\right\}} \frac{\left|\int_{\Omega} v w\right|}{\|v\|_{\mathcal{K}_{a}^{m}(\Omega)}}, \quad v \neq 0 .
$$

We also agree that if $\|v\|_{\mathcal{K}_{a}^{m}(\Omega)}=\infty$, then $v$ is not in $\mathcal{K}_{a}^{m}(\Omega)$.
Remark 7.1.2. In fact, near the vertices $(1,0)$ and $(1, l)$, the spaces $\mathcal{K}_{a}^{m}(\Omega)$ are the usual weighted Sobolev spaces for elliptic equations on corner singularities in previous chapters; while in the neighborhood of $x=0, \mathcal{K}_{a}^{m}(\Omega)$ can be considered as the usual weighted spaces in polar coordinates, by setting $r=x$ and $\theta=y$. Our weighted Sobolev spaces are invented in the way that is based on the property of the operator $\mathcal{L}_{\delta}$ and the geometry of the domain. For this reason, some properties of $\mathcal{K}_{a}^{m}(\Omega)$ are important for the study of the regularity of the solution and for the construction of the finite subspaces in the FEM.

Based on the definition of the weighted Sobolev space, we shall give some observations and lemmas for $\mathcal{K}_{a}^{m}(\Omega)$.

### 7.1.2 Lemmas

We here summarize several properties for the spaces $\mathcal{K}_{a}^{m}(\Omega)$ that are useful for the development of the theorems in Section 7.2 and Section 7.3. Most of the properties are derived from straightforward calculation based on the definition of $\mathcal{K}_{a}^{m}(\Omega)$ and similar to those properties we obtained in Chapter 2. For simplicity, we omit $\Omega$ in the notation $\mathcal{K}_{a}^{m}(\Omega)$ and $H^{m}(\Omega)$, which are used often below. Therefore, $\mathcal{K}_{a}^{m}=\mathcal{K}_{a}^{m}(\Omega)$ and $H^{m}=H^{m}(\Omega)$ for the rest of this chapter. Moreover, $a \simeq b$ means that there exist constants $C_{1}, C_{2}>0$, such that $C_{1} b \leq a \leq C_{2} b$. As usual, we denote by $r$ and $\theta$ the corresponding variables in the polar coordinates.

The first lemma claims an alternative definition for the space $\mathcal{K}_{a}^{m}$.
Lemma 7.1.3. Denote by $P_{x} \subset \Omega, P_{r_{1}} \subset \Omega$ and $P_{r_{2}} \subset \Omega$ the small neighborhoods of $x=0$ and the vertices $(1,0),(1, l)$ respectively. Then, for $P_{r}=P_{r_{1}} \cup P_{r_{2}}$, we have
$\mathcal{K}_{a}^{m}=\left\{u \in H_{c}^{m}(\Omega), \rho^{-a}\left(r \partial_{r}\right)^{i} \partial_{\theta}^{j} u \in L^{2}\left(P_{r}\right), \rho^{-a}\left(x \partial_{x}\right)^{i} \partial_{y}^{j} u \in L^{2}\left(P_{x}\right), \forall i+j \leq m\right\}$,
where $H_{c}^{m}(\Omega)$ is the space of functions that are in $H^{m}$ in any compact subset of $\Omega$.
Proof. On the outside of $P=P_{x} \cup P_{r}, u \in H^{m}(\Omega \backslash P)$ is equivalent to $u \in \mathcal{K}_{a}^{m}(\Omega \backslash P)$, since $\rho, r_{1}, r_{2}$ and $x$ are all bounded from above and 0 by the definition.

On the region $P_{x}$, we notice $\rho=x$ and $r_{1}=r_{2}=1$. Therefore

$$
\left\|\rho^{-a}\left(r_{1} r_{2}\right)^{i+j}\left(x \partial_{x}\right)^{i} \partial_{y}^{j} u\right\|_{L^{2}\left(P_{x}\right)}=\left\|\rho^{-a}\left(x \partial_{x}\right)^{i} \partial_{y}^{j} u\right\|_{L^{2}\left(P_{x}\right)}
$$

On $P_{r_{i}}, i=1,2$, we freeze the coefficient of $\mathcal{L}_{\delta}$ in $x$ at the vertex and change the variables $x, y$ into $r, \theta$ for the polar coordinates centered at the vertex, then

$$
\begin{aligned}
& \partial_{x}=\cos (\theta) \partial_{r}-\frac{\sin (\theta)}{r} \partial_{\theta} \\
& \partial_{y}=\sin (\theta) \partial_{r}+\frac{\cos (\theta)}{r} \partial_{\theta},
\end{aligned}
$$

where $\rho=r_{1}=r$ on $P_{r_{1}}$ and $\rho=r_{2}=r$ on $P_{r_{2}}$. The proof then follows from

$$
\begin{aligned}
& \left\|\rho^{-a} r^{i+j}\left(\cos (\theta) \partial_{r}-\frac{\sin (\theta)}{r} \partial_{\theta}\right)^{i}\left(\sin (\theta) \partial_{r}+\frac{\cos (\theta)}{r} \partial_{\theta}\right)^{j} u\right\|_{L^{2}\left(P_{r_{i}}\right)} \\
& \simeq\left\|\rho^{-a} \sum_{h+k \leq i+j}\left(r \partial_{r}\right)^{h} \partial_{\theta}^{k} u\right\|_{L^{2}\left(P_{r_{i}}\right)} .
\end{aligned}
$$

Lemma 7.1.4. The function $\rho^{-b}\left(r_{1} r_{2}\right)^{i+j}\left(x \partial_{x}\right)^{i} \partial_{y}^{j} \rho^{b}$ is bounded on $\Omega$.
Proof. On the region $P_{r}$ where $\rho=r_{1}=r$ or $\rho=r_{2}=r$, we follow the notation in Lemma 7.1.3, and change to the polar coordinates centered at the vertices. Since $\left(r \partial_{r}\right)^{k} r^{b}=b^{k} r^{b}$ and $\partial_{\theta} r=0$, we have

$$
\begin{aligned}
& \left|\rho^{-b}\left(r_{1} r_{2}\right)^{i+j}\left(x \partial_{x}\right)^{i} \partial_{y}^{j} \rho^{b}\right|=\left|r^{-b}\left(r_{1} r_{2}\right)^{i+j}\left(x \partial_{x}\right)^{i} \partial_{y}^{j} r^{b}\right| \\
& \leq C_{1}\left|r^{-b} r^{i+j} \partial_{x}^{i} \partial_{y}^{j} r^{b}\right| \\
& =C_{1}\left|r^{-b} r^{i+j}\left(\cos (\theta) \partial_{r}-\frac{\sin (\theta)}{r} \partial_{\theta}\right)^{i}\left(\sin (\theta) \partial_{r}+\frac{\cos (\theta)}{r} \partial_{\theta}\right)^{j} r^{b}\right| \\
& \leq C\left|r^{-b} \sum_{k+h \leq i+j}\left(r \partial_{r}\right)^{k} \partial_{\theta}^{h} r^{b}\right| \\
& \leq C\left|r^{-b} \sum_{k \leq i+j}\left(r \partial_{r}\right)^{k} r^{b}\right|=C\left|\sum_{k \leq i+j} b^{k}\right| .
\end{aligned}
$$

Therefore $\rho^{-b}\left(r_{1} r_{2}\right)^{i+j}\left(x \partial_{x}\right)^{i} \partial_{y}^{j} \rho^{b}$ is bounded on $P_{r}$.
On the region $P_{x}$ where $\rho=x$, we have $\left(x \partial_{x}\right)^{i} \rho^{b}=b^{i} \rho^{b}$ and $\partial_{y} \rho=0$. Thus, the proof follows

$$
\begin{aligned}
\left|\rho^{-b}\left(r_{1} r_{2}\right)^{i+j}\left(x \partial_{x}\right)^{i} \partial_{y}^{j} \rho^{b}\right| & =\left|\rho^{-b}\left(x \partial_{x}\right)^{i} \partial_{y}^{j} \rho^{b}\right| \\
& =\left|\rho^{-b}\left(x \partial_{x}\right)^{i} \rho^{b}\right|=C\left|b^{i}\right| .
\end{aligned}
$$

As for $\Omega \backslash P$, the complement of $P=P_{x} \cup P_{r}$, since $\rho$ is smooth and bounded away from 0 , the function $\rho^{-b}\left(r_{1} r_{2}\right)^{i+j}\left(x \partial_{x}\right)^{i} \partial_{y}^{j} \rho^{b}$ is bounded. Thus, the proof is completed.

## Consequently, Lemma 7.1.4 shows

Lemma 7.1.5. For the spaces $\mathcal{K}_{a}^{m}, \rho^{b} \mathcal{K}_{a}^{m}=\mathcal{K}_{a+b}^{m}$, where $\rho^{b} \mathcal{K}_{a}^{m}=\left\{\rho^{b} v, \forall v \in \mathcal{K}_{a}^{m}\right\}$. Therefore, the multiplication by $\rho^{b}$ defines an isomorphism $\mathcal{K}_{a}^{m} \rightarrow \mathcal{K}_{a+b}^{m}$.

Proof. Let $v \in \mathcal{K}_{a}^{m}$ and $w=\rho^{b} v$. Then $\left|\rho^{-a}\left(r_{1} r_{2}\right)^{i+j}\left(x \partial_{x}\right)^{i} \partial_{y}^{j} v\right| \in L^{2}$, for $i+j \leq m$. Moreover, we have

$$
\begin{aligned}
\left|\rho^{-a-b}\left(r_{1} r_{2}\right)^{i+j}\left(x \partial_{x}\right)^{i} \partial_{y}^{j} w\right| & =\left|\rho^{-a-b}\left(r_{1} r_{2}\right)^{i+j}\left(x \partial_{x}\right)^{i} \partial_{y}^{j} \rho^{b} v\right| \\
& =\rho^{-a-b}\left(r_{1} r_{2}\right)^{i+j}\left|\sum_{k \leq i, h \leq j}\left(x \partial_{x}\right)^{k} \partial_{y}^{h} \rho^{b}\left(x \partial_{x}\right)^{i-k} \partial_{y}^{j-h} v\right| \\
& \leq C \sum_{k \leq i, h \leq j}\left|\rho^{-a}\left(r_{1} r_{2}\right)^{i+j-k-h}\left(x \partial_{x}\right)^{i-k} \partial_{y}^{j-h} v\right| \in L^{2} .
\end{aligned}
$$

The last inequality is the consequence of Lemma 7.1.4. Thus, $\rho^{b} \mathcal{K}_{a}^{m}$ is continuously embedded in $\mathcal{K}_{a+b}^{m}$. On the other hand, because this embedding holds for any real number $b$, we have the opposite

$$
\mathcal{K}_{a+b}^{m}=\rho^{b} \rho^{-b} \mathcal{K}_{a+b}^{m} \subset \rho^{b} \mathcal{K}_{a}^{m}
$$

which completes the proof.
Recall that $\Omega_{\xi}=(0, \xi) \times(0, l)$. From a direct verification based on the definitions of $H^{m}$ and $\mathcal{K}_{a}^{m}$, we can also derive the following lemma.

Lemma 7.1.6. We have $\mathcal{K}_{0}^{0}=L^{2}$ and for $m^{\prime} \leq m, a^{\prime} \leq a$,

1. $\mathcal{K}_{a}^{m} \subset \mathcal{K}_{a^{\prime}}^{m^{\prime}}$
2. $\|u\|_{\mathcal{K}_{a^{\prime}}^{m^{\prime}}\left(\Omega_{\xi}\right)} \leq \xi^{a-a^{\prime}}\|u\|_{\mathcal{K}_{a}^{m}\left(\Omega_{\xi}\right)}, \forall u \in \mathcal{K}_{a}^{m}, \xi<\frac{1}{2}$.

Proof. The first argument is the result of the inequality below. $\forall u \in \mathcal{K}_{a}^{m}$ and $m^{\prime} \leq m, a^{\prime} \leq a$,

$$
\sum_{i+j \leq m^{\prime}}\left\|\rho^{-a^{\prime}}\left(r_{1} r_{2}\right)^{i+j}\left(x \partial_{x}\right)^{i} \partial_{y}^{j} u\right\|_{L^{2}}^{2} \leq C \sum_{i+j \leq m}\left\|\rho^{-a}\left(r_{1} r_{2}\right)^{i+j}\left(x \partial_{x}\right)^{i} \partial_{y}^{j} u\right\|_{L^{2}}^{2}
$$

Note that on $\Omega_{\xi}, \xi<\frac{1}{2}$, we have $\rho=x, r_{1}=r_{2}=1$. Then the second argument in this lemma follows from

$$
\begin{aligned}
\|u\|_{\mathcal{K}_{a^{\prime}}^{m}\left(\Omega_{\xi}\right)}^{2} & =\sum_{i+j \leq m^{\prime}}\left\|\rho^{-a^{\prime}}\left(r_{1} r_{2}\right)^{i+j}\left(x \partial_{x}\right)^{i} \partial_{y}^{j} u\right\|_{L^{2}\left(\Omega_{\xi}\right)}^{2} \\
& \leq \sum_{i+j \leq m}\left\|\rho^{-a^{\prime}}\left(r_{1} r_{2}\right)^{i+j}\left(x \partial_{x}\right)^{i} \partial_{y}^{j} u\right\|_{L^{2}\left(\Omega_{\xi}\right)}^{2} \\
& =\xi^{2\left(a-a^{\prime}\right)} \sum_{i+j \leq m}\left\|\xi^{a^{\prime}-a} \rho^{-a^{\prime}}\left(r_{1} r_{2}\right)^{i+j}\left(x \partial_{x}\right)^{i} \partial_{y}^{j} u\right\|_{L^{2}\left(\Omega_{\xi}\right)}^{2} \\
& \leq \xi^{2\left(a-a^{\prime}\right)} \sum_{i+j \leq m}\left\|\rho^{-a}\left(r_{1} r_{2}\right)^{i+j}\left(x \partial_{x}\right)^{i} \partial_{y}^{j} u\right\|_{L^{2}\left(\Omega_{\xi}\right)}^{2} \\
& =\xi^{2\left(a-a^{\prime}\right)}\|u\|_{\mathcal{K}_{a}^{m}\left(\Omega_{\xi}\right)}^{2} .
\end{aligned}
$$

We note that the weights defined in $\mathcal{K}_{a}^{m}$ only depend on the distances to certain parts of the boundary. From Lemma 7.1.3, we obtain that the $H^{m}$ - and $\mathcal{K}_{a}^{m}$-norm are equivalent on any compact subset of $\Omega$.

Lemma 7.1.7. Let $\xi$ be a positive number, and let $G \subset \Omega$ be an open subset such that $\rho \geq \xi$ on $G$. Then $\|u\|_{H^{m}(G)} \leq M_{1}\|u\|_{\mathcal{K}_{a}^{m}(G)}$ and $\|u\|_{\mathcal{K}_{a}^{m}(G)} \leq M_{2}\|u\|_{H^{m}(G)}$, for any $u \in H^{m}(G)$, where the constants $M_{1}$ and $M_{2}$ only depend on $\xi$ and $m$.

Proof. It follows from Definition 7.1.1 and Lemma 7.1.3
The following lemma states the relation between $H^{m}$ and $\mathcal{K}_{a}^{m}$ on $\Omega$.
Lemma 7.1.8. On $\Omega,\|u\|_{H^{m}} \leq M_{1}\|u\|_{\mathcal{K}_{m}^{m}}$, and $\|u\|_{\mathcal{K}_{a}^{m}} \leq M_{2}\|u\|_{H^{m}}$ for $a \leq 0$, where $M_{1}$ and $M_{2}$ depend on $m$ and $a$.

Proof. This lemma is basically a consequence of the definitions of those norms. Recall the definitions of $P, P_{x}$ and $P_{r}$. Then, based on $i+j \leq m$ and $\rho=x$ on $P_{x}$,

$$
\begin{aligned}
\|u\|_{\mathcal{K}_{m}^{m}\left(P_{x}\right)}^{2} & =\sum_{i+j \leq m}\left\|\rho^{-m}\left(x \partial_{x}\right)^{i} \partial_{y}^{j} u\right\|_{L^{2}\left(P_{x}\right)}^{2} \\
& \geq C \sum_{i+j \leq m}\left\|\partial_{x}^{i} \partial_{y}^{j} u\right\|_{L^{2}\left(P_{x}\right)}^{2}=C\|u\|_{H^{m}\left(P_{x}\right)}^{2} .
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\|u\|_{\mathcal{K}_{m}^{m}\left(P_{r}\right)}^{2} & \geq C \sum_{i+j \leq m}\left\|\rho^{-m} \rho^{i+j} \partial_{x}^{i} \partial_{y}^{j} u\right\|_{L^{2}\left(P_{r}\right)}^{2} \\
& \geq C \sum_{i+j \leq m}\left\|\partial_{x}^{i} \partial_{y}^{j} u\right\|_{L^{2}\left(P_{r}\right)}^{2}=C\|u\|_{H^{m}\left(P_{r}\right)}^{2}
\end{aligned}
$$

based on $i+j \leq m$ and $\rho=r_{1}$ or $r_{2}$ on $P_{r}$.
For the region $\Omega \backslash P$, Lemma 7.1 .7 shows that $\|u\|_{H^{m}(\Omega \backslash P)} \leq M_{1}\|u\|_{\mathcal{K}_{m}^{m}(\Omega \backslash P)}$, which completes the proof for the first argument in the lemma.

The second inequality can be proved in a similar way by comparing different norms on $P$ and $\Omega \backslash P$, which will be shown in Lemma 7.1.10.

Corollary 7.1.9. We have $\mathcal{K}_{m+a}^{m} \subset \rho^{a} H^{m} \subset \mathcal{K}_{a}^{m}$.
The proof is based on the isomorphism arising from the multiplication in Lemma 7.1.5 and the inequalities in Lemma 7.1.8.

The following lemma will compare $\mathcal{K}_{a}^{m}$ and $H^{m}$ near the y-axis and the vertices.
Lemma 7.1.10. Let $\xi$ be a positive number and let $G^{\prime}$ be an open subset of $\Omega$, such that $\rho<\xi$ on $G^{\prime}$. Then $\|u\|_{H^{m}\left(G^{\prime}\right)} \leq C_{1} \xi^{a-m}\|u\|_{\mathcal{K}_{a}^{m}\left(G^{\prime}\right)}$ if $a \geq m$, and $\|u\|_{\mathcal{K}_{a}^{m}\left(G^{\prime}\right)} \leq C_{2} \xi^{-a}\|u\|_{H^{m}\left(G^{\prime}\right)}$ if $a \leq 0$, where $C_{1}$ and $C_{2}$ are generic constants depending on $m$.

Proof. For $a \geq m$, we first have

$$
\|u\|_{H^{m}\left(G^{\prime}\right)} \leq C_{1}\|u\|_{\mathcal{K}_{m}^{m}\left(G^{\prime}\right)}
$$

from Lemma 7.1.8. Then, on the subregion of $G^{\prime}$ that is close to $x=0$, Lemma
7.1.6 shows

$$
\|u\|_{\mathcal{K}_{m}^{m}} \leq \xi^{a-m}\|u\|_{\mathcal{K}_{a}^{m}}
$$

For the subregion that is near one of the vertices, we have

$$
\begin{aligned}
\|u\|_{\mathcal{K}_{a}^{m}}^{2} & \geq C \sum_{i+j \leq m}\left\|\rho^{-a} \rho^{i+j} \partial_{x}^{i} \partial_{y}^{j} u\right\|_{L^{2}}^{2} \\
& \geq \xi^{2(m-a)}\left\|\partial_{x}^{i} \partial_{y}^{j} u\right\|_{L^{2}}^{2}=\xi^{2(m-a)}\|u\|_{H^{m}}^{2}
\end{aligned}
$$

The last inequality is based on $\rho<\xi$ and $i+j \leq m$ on this subregion. Therefore, $\|u\|_{H^{m}\left(G^{\prime}\right)} \leq C_{1} \xi^{a-m}\|u\|_{\mathcal{K}_{a}^{m}\left(G^{\prime}\right)}$ for $a \geq m$ by combining the estimates on different subregions of $G^{\prime}$.

For $a \leq 0$, similarly, on the subregion of $G^{\prime}$, which is close to $x=0$, because $\rho=x, a \leq 0$ and $\rho<\xi$, we have the following inequalities,

$$
\begin{aligned}
\|u\|_{\mathcal{K}_{a}^{m}}^{2} & =\sum_{i+j \leq m}\left\|\rho^{-a}\left(x \partial_{x}\right)^{i} \partial_{y}^{j} u\right\|_{L^{2}}^{2} \\
& \leq C \xi^{-2 a} \sum_{i+j \leq m}\left\|\partial_{x}^{i} \partial_{y}^{j} u\right\|_{L^{2}}^{2}=C \xi^{-2 a}\|u\|_{H^{m}}^{2}
\end{aligned}
$$

On the subregion close to one of the vertices, the inequalities are

$$
\begin{aligned}
\|u\|_{\mathcal{K}_{a}^{m}}^{2} & \leq C \sum_{i+j \leq m}\left\|\rho^{-a} \rho^{i+j} \partial_{x}^{i} \partial_{y}^{j} u\right\|_{L^{2}}^{2} \\
& \leq C \xi^{-2 a} \sum_{i+j \leq m}\left\|\partial_{x}^{i} \partial_{y}^{j} u\right\|_{L^{2}}^{2}=C \xi^{-2 a}\|u\|_{H^{m}}^{2}
\end{aligned}
$$

Therefore, for $a \leq 0,\|u\|_{\mathcal{K}_{a}^{m}\left(G^{\prime}\right)} \leq C \xi^{-a}\|u\|_{H^{m}\left(G^{\prime}\right)}$. This also provides the proof of the second inequality in Lemma 7.1.8.

We have derived several lemmas to reveal the relations between the weighted Sobolev space $\mathcal{K}_{a}^{m}$ and the regular Sobolev space $H^{m}$. They are the preliminaries for our main results in the next section. Now, we shall give an important lemma for the homogeneity of the norms of weighted Sobolev spaces $\mathcal{K}_{a}^{m}\left(\Omega_{\xi}\right)$. This is one of the main reasons that we use weighted Sobolev spaces for the analysis.

We define the dilation of a function on $\Omega_{\xi}$ first. For $0<\lambda<1$, let $G \subset \Omega_{\xi}$
be an open subset of $\Omega_{\xi}$. Also, let $v$ be a function on $G$. Then, we define the dilation function $v_{\lambda}(x, y):=v(\lambda x, y)$ for any point $(x, y) \in G^{\prime} \subset \Omega_{\xi / \lambda}$, such that $(\lambda x, y) \in G$. The relation of the norms of $v$ and its dilation $v_{\lambda}$ is given by the following lemma.

Lemma 7.1.11. Let $G \subset \Omega_{\xi} \backslash \Omega_{\lambda \xi}$ be an open subset and $u(x, y)$ be a function on $G, 0<\lambda<1, \xi / \lambda<\frac{1}{2}$. Then $\left\|u_{\lambda}\right\|_{\mathcal{K}_{a}^{m}\left(G^{\prime}\right)}^{2}=\lambda^{2 a-1}\|u\|_{\mathcal{K}_{a}^{m}(G)}^{2}$ for any $u \in \mathcal{K}_{a}^{m}(G)$. This relation also holds for $G \subset \Omega_{\lambda \xi}, \xi<\frac{1}{2}$.

Proof. The proof of the lemma follows the change of variables and the fact that $\rho=x, r_{1}=r_{2}=1$ on $G^{\prime}$ with a direct calculation. Let $w=\lambda x$, then

$$
\begin{aligned}
\left\|u_{\lambda}(x, y)\right\|_{\mathcal{K}_{a}^{m}\left(G^{\prime}\right)}^{2} & =\sum_{i+j \leq m} \int_{G^{\prime}}\left(x^{-a}\left(r_{1} r_{2}\right)^{i+j}\left(x \partial_{x}\right)^{i} \partial_{y}^{j} u_{\lambda}\right)^{2} d x d y \\
& =\sum_{i+j \leq m} \int_{G}\left(\lambda^{a} w^{-a}\left(w \partial_{w}\right)^{i} \partial_{y}^{j} u(w, y)\right)^{2} \frac{1}{\lambda} d w d y \\
& =\sum_{i+j \leq m} \int_{G} \lambda^{2 a-1}\left(w^{-a}\left(w \partial_{w}\right)^{i} \partial_{y}^{j} u(w, y)\right)^{2} d w d y \\
& =\lambda^{2 a-1} \sum_{i+j \leq m} \int_{G}\left(w^{-a}\left(w \partial_{w}\right)^{i} \partial_{y}^{j} u(w, y)\right)^{2} d w d y \\
& =\lambda^{2 a-1} \sum_{i+j \leq m}\left\|w^{-a}\left(w \partial_{w}\right)^{i} \partial_{y}^{j} u\right\|_{L^{2}(G)}^{2} \\
& =\lambda^{2 a-1}\|u\|_{\mathcal{K}_{a}^{m}(G)}^{2} .
\end{aligned}
$$

We note that the proof above can be carried out without any restriction on $G \subset$ $\Omega_{\lambda \xi}, \xi<\frac{1}{2}$ as well. Therefore, this relation for $u$ and $u_{\lambda}$ also holds on this region.

Lemma 7.1.11 is particular for the analysis of the solution near $x=0$, since one can refer to Chapter 3 and Chapter 4 for the solution around the vertices $(1,0),(1, l)$.

We now conclude this section by the following result.
Lemma 7.1.12. The operator $\mathcal{L}_{\delta}$ defines a continuous map $\mathcal{L}_{\delta}: \mathcal{K}_{a+1}^{m+1}(\Omega) \rightarrow$ $\mathcal{K}_{a-1}^{m-1}(\Omega)$.

Proof. We need to show that for $\forall u \in \mathcal{K}_{a+1}^{m+1}(\Omega),\left\|\mathcal{L}_{\delta} u\right\|_{\mathcal{K}_{a-1}^{m-1}} \leq C\|u\|_{\mathcal{K}_{a+1}^{m+1}}$ on $P_{r}, P_{x}$ and $\Omega \backslash P$, which are defined in Lemma 7.1.3.

On $\Omega \backslash P, \mathcal{L}_{\delta}=-\partial_{x}^{2}-\frac{\delta^{2}}{x^{2}} \partial_{y}^{2}$ is strongly elliptic. Therefore, it is a bounded operator $H^{m+1}(\Omega \backslash P) \rightarrow H^{m-1}(\Omega \backslash P)$ [48]. Then, the argument for this lemma follows the equivalence of the spaces $H^{m+1}(\Omega \backslash P)$ and $\mathcal{K}_{a}^{m+1}(\Omega \backslash P)$.

On $P_{r}$, let $g=r_{1} r_{2}, H^{m}=H^{m}\left(P_{r}\right)$ and $\mathcal{K}_{a}^{m}=\mathcal{K}_{a}^{m}\left(P_{r}\right)$ in the proof for simplicity. Then, based on $g \simeq \rho$, the following inequalities hold with the coefficient frozen in $x$ at 1 ,

$$
\begin{aligned}
& \left\|\mathcal{L}_{\delta} u\right\|_{\mathcal{K}_{a-1}^{m-1}} \leq C_{0} \sum_{i+j \leq m-1}\left\|\rho^{1-a} g^{i+j}\left(x \partial_{x}\right)^{i} \partial_{y}^{j}\left(-\partial_{x}^{2} u-\frac{\delta^{2}}{x^{2}} \partial_{y}^{2} u\right)\right\|_{L^{2}} \\
& =C_{0} \sum_{i+j \leq m-1}\left\|\rho^{1-a} g^{i+j} \partial_{x}^{i} \partial_{y}^{j}\left(-\partial_{x}^{2} u-\delta^{2} \partial_{y}^{2} u\right)\right\|_{L^{2}} \\
& \leq C_{1} \sum_{i+j \leq m-1}\left(\left\|\rho^{1-a} g^{i+j} \partial_{x}^{i+2} \partial_{y}^{j} u\right\|_{L^{2}}+\left\|\rho^{1-a} g^{i+j} \partial_{x}^{i} \partial_{y}^{j+2} u\right\|_{L^{2}}\right) \\
& \leq C_{2} \sum_{i+j \leq m-1}\left(\left\|\rho^{-a-1} g^{i+j+2} \partial_{x}^{i+2} \partial_{y}^{j} u\right\|_{L^{2}}+\left\|\rho^{-a-1} g^{i+j+2} \partial_{x}^{i} \partial_{y}^{j+2} u\right\|_{L^{2}}\right) \\
& \leq C\|u\|_{\mathcal{K}_{a+1}^{m+1}}
\end{aligned}
$$

On $P_{x}$, similarly, let $H^{m}=H^{m}\left(P_{x}\right)$ and $\mathcal{K}_{a}^{m}=\mathcal{K}_{a}^{m}\left(P_{x}\right)$ in the proof for simplicity. Then, based on $\rho=x, r_{1}=r_{2}=1$, we have the estimates below,

$$
\begin{aligned}
& \left\|\mathcal{L}_{\delta} u\right\|_{\mathcal{K}_{a-1}^{m-1}} \leq C_{1} \sum_{i+j \leq m-1}\left\|\rho^{1-a}\left(x \partial_{x}\right)^{i} \partial_{y}^{j}\left(-\partial_{x}^{2} u-\frac{\delta^{2}}{x^{2}} \partial_{y}^{2} u\right)\right\|_{L^{2}} \\
& \leq C_{1} \sum_{i+j \leq m-1}\left(\left\|x^{1-a}\left(x \partial_{x}\right)^{i} \partial_{x}^{2} \partial_{y}^{j} u\right\|_{L^{2}}+\left\|x^{1-a}\left(x \partial_{x}\right)^{i} \partial_{y}^{j+2} \frac{\delta^{2}}{x^{2}} u\right\|_{L^{2}}\right) \\
& \leq C_{2} \sum_{i+j \leq m-1}\left(\left\|x^{1-a}\left(\left(x \partial_{x}\right)^{i+2}-\left(x \partial_{x}\right)^{i+1}\right) x^{-2} \partial_{y}^{j} u\right\|_{L^{2}}+\right. \\
& \left.\left\|x^{-a-1}\left(-2+x \partial_{x}\right)^{i} \partial_{y}^{j+2} u\right\|_{L^{2}}\right) \\
& \leq C_{2} \sum_{i+j \leq m-1}\left(\left\|x^{-a-1}\left(-2+x \partial_{x}\right)^{i+2} \partial_{y}^{j} u\right\|_{L^{2}}+\right. \\
& \left.\left.\| x^{-a-1}\left(-2+x \partial_{x}\right)^{i+1}\right) \partial_{y}^{j} u\left\|_{L^{2}}+\right\| x^{-a-1}\left(-2+x \partial_{x}\right)^{i} \partial_{y}^{j+2} u \|_{L^{2}}\right) \\
& \leq C\|u\|_{\mathcal{K}_{a+1}^{m+1}} .
\end{aligned}
$$

We here use $x^{-a}\left(x \partial_{x}\right)^{i} x^{a} u=\left(a+x \partial_{x}\right)^{i} u$ to simplify the expression.

In the next section, we will show that this map $\mathcal{L}_{\delta}: \mathcal{K}_{a+1}^{m+1} \cap\left\{\left.u\right|_{\partial \Omega}=0\right\} \rightarrow \mathcal{K}_{a-1}^{m-1}$ is a bijection. if the index $a$ satisfies some conditions.

### 7.2 The Well-posedness and Regularity of the Solution

First, we need the following estimates on the solution of Equation (7.1). Recall that we defined the set $\mathcal{V}:=\{(1,0),(1, l),[0, y], 0 \leq y \leq l\}$. It has been shown in $[2,3,20]$ that the trace or restriction of $u \in \mathcal{K}_{a}^{m}$ on the boundary follows

$$
\left.u\right|_{\partial \Omega \backslash S} \in \mathcal{K}_{a-\frac{1}{2}}^{m-\frac{1}{2}}(\partial \Omega \backslash S)
$$

However, the estimate of the trace on $x=0$ is needed to derive the corresponding bilinear form $a(\cdot, \cdot)$ for the boundary value problem (7.1). We note that for $m \geq 1$, $\mathcal{K}_{m}^{m} \subset H^{m}$ by Lemma 7.1.8. Denote each segment of $\partial \Omega$ by $\bar{D}_{i}, i=1,2,3,4$, where $D_{i}$ is open. In particular, let $D_{1}:=(0, y), 0 \leq y \leq l$, be the corresponding open set for the degenerate boundary $x=0$. Then, the trace

$$
\left.u\right|_{D_{i}} \in H^{m-\frac{1}{2}}\left(D_{i}\right), \quad \forall u \in \mathcal{K}_{m}^{m}(\Omega)
$$

is defined [55]. Consequently, for $u \in \mathcal{K}_{1}^{1}$, the trace of $u$ is well defined in $L^{2}$ on every $D_{i}$ of the boundary $\partial \Omega$. Furthermore, we can even show that $\left.u\right|_{D_{1}}=0$ for $u \in \mathcal{K}_{1}^{1}$.

Lemma 7.2.1. For any function $u \in \mathcal{K}_{1}^{1}$, its trace on $D_{i}$ is well defined in $L^{2}$ and moreover, we have $\left.u\right|_{D_{1}}=0$. Consequently, the corresponding bilinear form for Equation (7.1) is $a(u, v)=\int_{\Omega}\left(\partial_{x} u \partial_{x} v+\frac{\delta^{2}}{x^{2}} \partial_{y} u \partial_{y} v\right) d x d y, \forall v \in \mathcal{K}_{1}^{1}(\Omega) \cap\left\{\left.v\right|_{\partial \Omega}=0\right\}$.

Proof. For $u \in \mathcal{K}_{1}^{1}$, the trace $\left.u\right|_{D_{i}}$ belongs to $H^{\frac{1}{2}}\left(D_{i}\right)$ by the arguments above, hence in $L^{2}\left(D_{i}\right)$.

Moreover, on $\Omega_{\xi}=(0, \xi) \times(0, l), \xi<1 / 2$, since $\rho=x$, we have

$$
\frac{1}{\xi^{2}} \int_{\Omega_{\xi}} u^{2} d x d y \leq \int_{\Omega_{\xi}} \frac{1}{x^{2}} u^{2} d x d y \leq C\|u\|_{\mathcal{K}_{1}^{1}\left(\Omega_{\xi}\right)}^{2}<\infty
$$

Therefore, $\int_{\Omega_{\xi}} u^{2} d x d y \rightarrow 0$ as $\xi \rightarrow 0$. Hence, the trace $\left.u\right|_{D_{1}}=0$ in $L^{2}$ by continuity. Thus, the following bilinear form is obtained by integration by parts,

$$
a(u, v)=\int_{\Omega}\left(\partial_{x} u \partial_{x} v+\frac{\delta^{2}}{x^{2}} \partial_{y} u \partial_{y} v\right) d x d y, \quad \forall v \in \mathcal{K}_{1}^{1}(\Omega) \cap\left\{\left.v\right|_{\partial \Omega}=0\right\}
$$

Now, we shall prove the existence and uniqueness of the solution of Equation (7.1) in weighted Sobolev spaces.

Theorem 7.2.2. On $\Omega$, the map $\mathcal{L}_{\delta}: \mathcal{K}_{1}^{m+1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\} \rightarrow \mathcal{K}_{-1}^{m-1}(\Omega)$ is an isomorphism, for $\delta>0, m \geq 0$. Namely, there is a unique solution $u \in$ $\mathcal{K}_{1}^{m+1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}$ for Equation (7.1) if $f \in \mathcal{K}_{-1}^{m-1}(\Omega)$.

Proof. We shall first prove it for $m=0$. From Lemma 7.2.1, we have the following weak formulation for Equation (1),
$a(u, v)=\int_{\Omega}\left(\partial_{x} u \partial_{x} v+\frac{\delta^{2}}{x^{2}} \partial_{y} u \partial_{y} v\right) d x d y=\int_{\Omega} f v d x d y, \quad \forall v \in \mathcal{K}_{1}^{1}(\Omega) \cap\left\{\left.v\right|_{\partial \Omega}=0\right\}$.
Then, we shall show the equivalence between the energy norm induced by $a(\cdot, \cdot)$ and the $\mathcal{K}_{1}^{1}$-norm $\|\cdot\|_{\mathcal{K}_{1}^{1}(\Omega)}$ to complete the proof.

Based on the definitions of $a(\cdot, \cdot)$ and the $\mathcal{K}_{1}^{1}$-norm on $\Omega$, the continuity of $a(\cdot, \cdot)$ can be verified as follows. From the Holder inequality, there exists a constant $C$, not depending on $u$ and $v$, such that $a(u, v) \leq C\|u\|_{\mathcal{K}_{1}^{1}}\|v\|_{\mathcal{K}_{1}^{1}}$. Therefore, $a(\cdot, \cdot)$ is a continuous (bounded) bilinear form on $\mathcal{K}_{1}^{1}$.

To prove the coercivity, we adopt the following notations. Let $\Omega_{\xi}=(0, \xi) \times(0, l)$ be the rectangular domain near the boundary $x=0$. Denote by $B(v, r)$ the open ball of radius $r$ centered at $v$. For any of the vertices $v_{1}=(1,0), v_{2}=(1, l)$, let $\Omega_{r_{\xi_{i}}}=\Omega \cap B\left(v_{i}, \xi\right), i=1,2$, be the corresponding conical domain, such that $\Omega_{r_{\xi_{i}}}$ can be characterized in polar coordinates by

$$
\Omega_{r_{\xi_{i}}}=\left\{(r, \theta) \mid 0<r<\xi, 0<\theta<\frac{\pi}{2}\right\} .
$$

Note that $a(u, u)$ is equivalent to $|u|_{\mathcal{K}_{1}^{1}}^{2}=\sum_{i+j=1}\left\|\rho^{-1}\left(r_{1} r_{2}\right)\left(x \partial_{x}\right)^{i} \partial_{y}^{j} u\right\|_{L^{2}}^{2}$ by their
definitions. Then, we shall first prove the following weaker inequalities for $\xi$ small,

$$
\begin{array}{r}
\int_{\Omega_{\xi}} \frac{u^{2}}{x^{2}} d x d y \leq C \int_{\Omega_{\xi}}\left(\partial_{x} u\right)^{2}+\frac{\delta^{2}}{x^{2}}\left(\partial_{y} u\right)^{2} d x d y \\
\int_{\Omega_{r_{\xi_{i}}}} \frac{u^{2}}{r^{2}} d x d y \leq C \int_{\Omega_{r_{\xi_{i}}}}\left(\partial_{x} u\right)^{2}+\delta^{2}\left(\partial_{y} u\right)^{2} d x d y
\end{array}
$$

since $\rho=x$ and $\rho=r$ on $\Omega_{\xi}$ and $\Omega_{r_{\xi_{i}}}$ respectively.
On the domain $\Omega_{\xi}$, we first have the one-dimensional Poincaré inequality for $y$,

$$
\int_{0}^{l} u^{2} d y \leq C_{1} \int_{0}^{l}\left(\partial_{y} u\right)^{2} d y
$$

By integrating with respect to $x$, we obtain

$$
\int_{\Omega_{\xi}} \frac{u^{2}}{x^{2}} d y d x \leq C_{1} \int_{\Omega_{\xi}} \frac{\left(\partial_{y} u\right)^{2}}{x^{2}} d y d x \leq C \int_{\Omega_{\xi}}\left(\partial_{x} u\right)^{2}+\frac{\delta^{2}}{x^{2}}\left(\partial_{y} u\right)^{2} d y d x
$$

where $C$ is independent of $u$.
Similarly, we have the one-dimensional Poincaré inequality for $\theta$ on $\Omega_{r_{\xi_{i}}}$,

$$
\int_{0}^{\frac{\pi}{2}} u^{2} d \theta \leq C_{1} \int_{0}^{\frac{\pi}{2}}\left(\partial_{\theta} u\right)^{2} d \theta
$$

By integrating in polar coordinates, we have

$$
\int_{\Omega_{r_{\xi_{i}}}} \frac{u^{2}}{r^{2}} d x d y=\int_{\Omega_{\xi_{\xi_{i}}}} \frac{u^{2}}{r} d r d \theta \leq C_{1} \int_{\Omega_{\xi_{\xi_{i}}}} \frac{\left(\partial_{\theta} u\right)^{2}}{r} d r d \theta
$$

Since $\int_{\Omega_{r_{\xi_{i}}}}\left(\partial_{x} u\right)^{2}+\left(\partial_{y} u\right)^{2} d x d y=\int_{\Omega_{r_{\xi_{i}}}} r\left(\partial_{r} u\right)^{2}+\frac{\left(\partial_{\theta} u\right)^{2}}{r} d r d \theta$, we now have

$$
\int_{\Omega_{r_{\xi_{i}}}} \frac{u^{2}}{r^{2}} d x d y \leq C \int_{\Omega_{r_{\xi_{i}}}}\left(\partial_{x} u\right)^{2}+\delta^{2}\left(\partial_{y} u\right)^{2} d x d y
$$

with $C$ independent of $u$.
Let $\Omega_{r_{\xi}}:=\Omega_{r_{1}} \cup \Omega_{\xi_{\xi_{2}}}$. Thus, based on the usual Poincaré inequality in $\Omega \backslash\left(\Omega_{\xi} \cup\right.$ $\Omega_{r_{\xi}}$ ) and the inequalities above, we complete the proof for the coercivity $a(u, u) \geq$ $C\|u\|_{\mathcal{K}_{1}^{1}(\Omega)}^{2}$. Then, the existence of the unique solution $u \in \mathcal{K}_{1}^{1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}$
follows the Lax-Milgram Theorem.
For $m \geq 1$, the proof follows from the results from the previous chapters, which is based on the regularity of the solution derived by the Mellin transform on an infinite domain.

As an extension from this theorem, one has the following corollary.
Corollary 7.2.3. There exists a constant $\eta>0$, depending on $\Omega$, such that

$$
\mathcal{L}_{\delta}: \mathcal{K}_{1+\epsilon}^{m+1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\} \rightarrow \mathcal{K}_{-1+\epsilon}^{m-1}(\Omega)
$$

is an isomorphism for $0<|\epsilon|<\eta$.
Proof. Denote by $\mathcal{L}_{\delta \epsilon}$ the operator defined by $\mathcal{L}_{\delta}$ but on the space $\mathcal{K}_{1+\epsilon}^{m+1} \cap\left\{\left.u\right|_{\partial \Omega}=\right.$ $0\}$. Then, from Theorem 7.2.2 and Lemma 7.1.5 (see the diagram below), the operator $\mathcal{L}_{\delta \epsilon}$ is an isomorphism if, and only if

$$
\mathcal{A}_{\delta \epsilon}:=\rho^{-\epsilon} \mathcal{L}_{\delta \epsilon} \rho^{\epsilon}: \mathcal{K}_{1}^{m+1} \cap\left\{\left.u\right|_{\partial \Omega}=0\right\} \rightarrow \mathcal{K}_{-1}^{m-1}
$$

is an isomorphism.

$$
\begin{array}{lll} 
& \mathcal{L}_{\delta \epsilon} & \\
\mathcal{K}_{1+\epsilon}^{m+1} & \longrightarrow & \mathcal{K}_{-1+\epsilon}^{m-1} \\
\rho^{\epsilon} \mid & & \downarrow \rho^{-\epsilon} \\
\mathcal{K}_{1}^{m+1} & \longrightarrow & \mathcal{K}_{-1}^{m-1}
\end{array}
$$

Then the proof follows the fact that $\mathcal{A}_{\delta \epsilon}$ is a continuous bijection as $\epsilon=0$ and the operator $\mathcal{A}_{\delta \epsilon}$ depends continuously in norm on the parameter $\epsilon$.

Remark 7.2.4. For a brief summary, we have taken the advantage of weighed spaces $\mathcal{K}_{a}^{m}$ to prove the well-posedness of the solution $u \in \mathcal{K}_{1}^{m+1} \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}$ of Equation (7.1), $\forall f \in \mathcal{K}_{-1}^{m-1}$. Furthermore, there exists some constant $\eta$, such that $\mathcal{L}_{\delta}$ is still invertible on weighted spaces that depend on $\eta$. In fact, one will find that it is important to know the exact upper bound of $\epsilon$ in the FEM. To be more precise, $\eta$ is determined by the local behavior of the solution for Equation (7.1)
near the set $\mathcal{V}=\{(1,0),(1, l),[0, y], 0 \leq y \leq l\}$. Recall $\Omega_{\xi}=(0, \xi) \times(0, l)$ and $\Omega_{r_{\xi}}=\Omega_{r_{\xi_{1}}} \cup \Omega_{r_{\xi_{2}}}$ in Theorem 7.2.2. Then, it is possible to show that $\eta_{1}=\frac{\sqrt{l^{2}+4 \delta^{2} \pi^{2}}}{2 l}$ on $\Omega_{\xi}$ for $\xi<1 / 2$. Namely, $\mathcal{A}_{\delta \epsilon}: \mathcal{K}_{1}^{m+1}\left(\Omega_{\xi}\right) \cap\left\{\left.u\right|_{y=0, l}=0\right\} \rightarrow \mathcal{K}_{-1}^{m-1}\left(\Omega_{\xi}\right)$ is invertible for $|\epsilon|<\eta_{1}=\frac{\sqrt{l^{2}+4 \delta^{2} \pi^{2}}}{2 l}$. Here, we include some arguments on operator $\mathcal{A}_{\delta \epsilon}$ and the constant $\eta$.

We focus on the region $\Omega_{\xi}=(0, \xi) \times(0, l)$ for $\xi<1 / 2$ first. The indicial family of $\mathcal{A}_{\delta \epsilon}=\rho^{-\epsilon} \mathcal{L}_{\delta \epsilon} \rho^{\epsilon}$ for $\mathcal{K}_{1}^{m+1}\left(\Omega_{\xi}\right) \cap\left\{\left.u\right|_{y=0, l}=0\right\}$ is $\left(i \tau+\epsilon+\frac{1}{2}\right)\left(i \tau+\epsilon-\frac{1}{2}\right)+\delta^{2} \partial_{y}^{2}$ acting on $H^{2}([0, l]) \cap\left\{\left.u\right|_{y=0, l}=0\right\}$. The eigenvalues of $\partial_{y}^{2}$ on $H^{2}([0, l]) \cap\left\{\left.u\right|_{y=0, l}=0\right\}$ are $-\left(\frac{k \pi}{l}\right)^{2}$ for $k \in\{1,2,3, \ldots\}$. On $\Omega_{r_{\xi_{i}}}$, the indicial family of $\mathcal{A}_{\delta \epsilon}$ for the vertex can be derived in a similar way as in Chapter 3 by the Mellin transform. Then, we have the corresponding eigenvalues that can be calculated numerically. Based on Theorem 2.2.12, the operater $\mathcal{A}_{\delta \epsilon}: \mathcal{K}_{1}^{m+1} \cap\left\{\left.u\right|_{\partial \Omega}=0\right\} \rightarrow \mathcal{K}_{-1}^{m-1}$ is Fredholm if, and only if its indicial family for the degenerate boundary $x=0$ and the indicial families for the two vertices are invertible for all $\tau \in \mathbb{R}$. This is seen to be the case for $\left(i \tau+\epsilon+\frac{1}{2}\right)\left(i \tau+\epsilon-\frac{1}{2}\right)+\delta^{2} \partial_{y}^{2}$, unless $\epsilon= \pm \frac{\sqrt{l^{2}+4 k^{2} \delta^{2} \pi^{2}}}{2 l}, k \in \mathbb{N}$. Let $\eta_{1}=\frac{\sqrt{l^{2}+4 \delta^{2} \pi^{2}}}{2 l}$. Denote by $\eta_{2}$ the smallest positive value, such that one of the indicial families of $\mathcal{A}_{\delta \epsilon}$ for the vertices is not invertible as $\epsilon=\eta_{2}$. Then, $\mathcal{A}_{\delta \epsilon}=\rho^{-\epsilon} \mathcal{L}_{\delta \epsilon} \rho^{\epsilon}: \mathcal{K}_{1}^{m+1} \cap\left\{\left.u\right|_{\partial \Omega}=0\right\} \rightarrow \mathcal{K}_{-1}^{m-1}$ is Fredholm of index 0 when $|\epsilon|<\eta=\min \left(\eta_{1}, \eta_{2}\right)$, since $\mathcal{A}_{\delta \epsilon}$ is Fredholm of index 0 as $\epsilon=0$. Moreover, we note that the kernels of the operators $\mathcal{A}_{\delta \epsilon}$ are decreasing as $\epsilon$ is increasing, we conclude that they are invertible for $0 \leq \epsilon<\eta$. By taking the adjoint, we obtain the invertibility of $\mathcal{A}_{\delta \epsilon}$ for $-\eta<\epsilon \leq 0$ as well. As the conclusion of the arguments above, we can take $\eta=\min \left(\eta_{1}, \eta_{2}\right)$, such that for $\forall|\epsilon|<\eta$, the operator $L_{\delta}: \mathcal{K}_{1+\epsilon}^{m+1} \cap\left\{\left.u\right|_{\partial \Omega}=0\right\} \rightarrow \mathcal{K}_{-1+\epsilon}^{m-1}$ is still an isomorphism.

For values of $\epsilon$ outside the range above, the operator $\mathcal{A}_{\delta \epsilon}$ will no longer be invertible. In fact, it will have a non-zero index that can be computed, for example, as in Chapter 3.

### 7.3 The Finite Element Method

In this section, we will consider the numerical approximation for the solution $u$ of Equation (7.1) by using the FEM. Denote by $\mathcal{T}=\left\{T_{i}\right\}$ a triangulation of $\Omega$ with triangles $T_{i}$. Let $S_{n}=S_{n}(\mathcal{T}, m+1)$ be the finite element space associated
to the degree $m$ Lagrange triangle such that $S_{n}$ consists of polynomials of degree $\leq m$ on each triangle $T_{i} \in \mathcal{T}$. Then, we shall try to find the optimal finite element solution $u_{n} \in S_{n}$ by an appropriate construction of the finite space. By optimal finite element solution, we here mean the finite element solution that satisfies

$$
\left\|u-u_{n}\right\|_{H^{1}(\Omega)} \leq C \operatorname{dim}\left(S_{n}\right)^{-m / 2}\|f\|_{H^{m-1}(\Omega)}, \quad \forall f \in H^{m-1}(\Omega) \cap \mathcal{K}_{-1+\epsilon}^{m-1}(\Omega)
$$

### 7.3.1 Estimate for the Interpolation

Again, to make this chapter self-contained, we repeat the following well-known approximation theorem.

Theorem 7.3.1. Suppose the bilinear form $a(\cdot, \cdot)$ for an equation is both continuous and coercive on $H^{1}$ for a star-shaped two-dimensional domain D. Let $S=S(\mathcal{T}, m+1)$. Assume that all triangles $T_{j}$ of the triangulation $\mathcal{T}$ of domain $D$ have angles $\geq \alpha$ and edges of length $\leq h$ and $\geq a h$. Namely, the triangulation is quasi-uniform. Let $u_{n} \in S$ and $u_{I} \in S$ be the finite element solution and the interpolation function respectively. Then, there exist constants $c$ and $C_{1}=C_{1}(\alpha, m)$ such that

$$
c\left\|u-u_{n}\right\|_{H^{1}(D)} \leq\left\|u-u_{I}\right\|_{H^{1}(D)} \leq C_{1} h^{m}\|u\|_{H^{m+1}(D)}
$$

$\forall u \in H^{m+1}(D), m \geq 1$.
The constant $c$ depends only on the bilinear form $a(\cdot, \cdot)$ by Céa's Lemma, while $C_{1}$ and $C_{2}$ are independent of the solution.

Let $M:=C_{1}(\alpha) M_{1} M_{2}$, where $C_{1}(\alpha)$ is as in Theorem 7.3.1, and $M_{1}$ and $M_{2}$ are from Lemma 7.1.7. Then, we have the following estimate for the error $\left\|u-u_{I}\right\|_{\mathcal{K}_{1}^{1}}$ on a subset of $\Omega$.

Theorem 7.3.2. Fix $\alpha>0$ and $0<\xi<1 / 4$. Let $P \subset \Omega$ be a star-shaped polygonal domain, such that $\rho>\xi$ on $P$. Let $\mathcal{T}=\left(T_{j}\right)$ be a triangulation of $P$ with angles $\geq \alpha$ and sides $\leq h$. Then

$$
\left\|u-u_{I}\right\|_{\mathcal{K}_{1}^{1}(P)} \leq M h^{m}\|u\|_{\mathcal{K}_{1+\epsilon}^{m+1}(P)}
$$

$\forall u \in \mathcal{K}_{1+\epsilon}^{m+1}(P)$, where $M$ depends on $\xi$ and $\alpha$.

Proof. The proof for the error estimate follows from the arguments in Theorem 7.3.1 and the equivalence of the $H^{m}$ - and the $\mathcal{K}_{a}^{m}$ - norms on $P$ immediately.

We need to point out here that the estimate in Theorem 7.3.2, for the solution $u$ of Equation (7.1) and its interpolation $u_{I}$, can not extend to the entire domain $\Omega$ in general. It is reasonable to expect singularities in $H^{m}$ of the solution, near the degenerate boundary $x=0$, since it is only in the weighted Sobolev spaces instead of the usual Sobolev spaces . Also, it is possible that we have corner singularities near the vertices away from $x=0$, depending on the parameter $\delta$ and the interior angle corresponding to the corner. In either of the cases, the constant $M$ can not be uniformly bounded, which will destroy the optimal rate of convergence. The FEM for elliptic equations with Dirichlet boundary condition near the corners has been widely studied in Chapter 4 and also see [4, 7, 21]. Thus, from this point, we concentrate on the estimate near the degenerate boundary $x=0$, since we have been quite clear for the corners. Recall $\Omega_{\xi}=(0, \xi) \times(0, l)$. Let $0<\kappa<1$. The extension of Theorem 4.2 on the thin rectangular region $\Omega_{\xi} \backslash \Omega_{\kappa \xi}$ is as follows.

Theorem 7.3.3. Let $\Omega_{\xi}=(0, \xi) \times(0, l)$ be a subset sitting in $\Omega$ with $\xi<1 / 4$. Given $0<\kappa<1$, let $\mathcal{T}=\left(T_{j}\right)$ be a triangulation for the rectangular region $U:=\Omega_{\xi} \backslash \Omega_{\kappa \xi}$ with triangles of sides $\leq h_{x}$ and $\leq h_{y}$ in the $x$-, $y$-direction respectively. Denote by $u_{I}$ the interpolant as the polynomial of degree $\leq m$ on each triangle. Then the interpolation error on $U$ is

$$
\left\|u-u_{I}\right\|_{\mathcal{K}_{1}^{1}(U)} \leq C(\kappa) \xi^{\epsilon}\left(\max \left(h_{y}, \frac{h_{x}}{2 \xi}\right)\right)^{m}\|u\|_{\mathcal{K}_{1+\epsilon}^{m+1}(U)}
$$

$\forall u \in \mathcal{K}_{1+\epsilon}^{m+1}, m \geq 1, \epsilon>0$.
Proof. Recall the dilation function $u_{\lambda}(x, y)=u(\lambda x, y)$ from Section 7.1 and note that $u_{I \lambda}=u_{\lambda I}$. Then, if we let $\lambda=2 \xi$ and $U^{\prime}=\left(\frac{\kappa}{2}, 1 / 2\right) \times(0, l)$, by Lemma 7.1.11 and Theorem 7.3.2, we have

$$
\begin{aligned}
\left\|u-u_{I}\right\|_{\mathcal{K}_{1}^{1}(U)} & =\lambda^{-\frac{1}{2}}\left\|u_{\lambda}-u_{I \lambda}\right\|_{\mathcal{K}_{1}^{1}\left(U^{\prime}\right)} \\
& =\lambda^{-\frac{1}{2}}\left\|u_{\lambda}-u_{\lambda I}\right\|_{\mathcal{K}_{1}^{1}\left(U^{\prime}\right)} \\
& \leq \lambda^{-\frac{1}{2}} M\left(\max \left(\frac{h_{x}}{\lambda}, h_{y}\right)\right)^{m}\left\|u_{\lambda}\right\|_{\mathcal{K}_{1}^{m+1}\left(U^{\prime}\right)}
\end{aligned}
$$

$$
\begin{aligned}
& =M\left(\max \left(\frac{h_{x}}{\lambda}, h_{y}\right)\right)^{m}\|u\|_{\mathcal{K}_{1}^{m+1}(U)} \\
& \leq M \xi^{\epsilon}\left(\max \left(h_{y}, \frac{h_{x}}{2 \xi}\right)\right)^{m}\|u\|_{\mathcal{K}_{1+\epsilon}^{m+1}(U)}
\end{aligned}
$$

The last inequality is based on Lemma 7.1.6.
This theorem provides us the interpolation error for $u \in \mathcal{K}_{1+\epsilon}^{m+1}$ on a rectangular strip $U$ near $x=0$. Based on our observation above, it is possible to construct a class $\mathcal{C}(l, h, \kappa, m, \epsilon)$ of partitions $\mathcal{T}$ of $\Omega$, such that the optimal convergence rate is obtained.

Before we define the class $\mathcal{C}(l, h, \kappa, m, \epsilon)$, we assume that, near the vertices $(1,0)$ and $(1, l)$, a proper graded mesh has already been chosen to recover the optimal convergence rate, which is reasonable based on our previous discussions. For this reason, we will not consider the graded mesh near the vertices in the definition below. In fact, we will use a uniform mesh near the corners to demonstrate our method to generate triangles. However, it is only for the purpose to simplify the expressions. One needs to keep in mind that the uniform mesh near the corners in the following definition will be replaced by an appropriate graded mesh generally in practical computation.

### 7.3.2 Construction of the Mesh

We shall introduce the construction of a class $\mathcal{C}(l, h, \kappa, m, \epsilon)$ of triangulations and the finite element spaces $S_{n}$ associated to it. In the class $\mathcal{C}(l, h, \kappa, m, \epsilon)$, we try to even-distribute the interpolation error, and keep the same number of triangles as in the usual mid-point triangulation.

In the notation $\mathcal{C}(l, h, \kappa, m, \epsilon)$, we denote by $h$ the size of triangles in the triangulation for $\left(\frac{1}{2}, \frac{3}{4}\right) \times(0, l) \subset \Omega ; \kappa$ is the parameter to control the decay of triangles near $x=0$. We focus only on $\Omega_{\xi}=(0, \xi) \times(0, l)$, for $\xi<\frac{1}{2}$. (Graded mesh for both of the vertices $(1,0),(1, l)$ is needed in general, but it will have the same number of triangles as in the uniform mesh.) We here define the class $\mathcal{C}(l, h, \kappa, m, \epsilon)$ for our problem.

Definition 7.3.4. For a fixed $m=\{1,2, \ldots\}, \epsilon \in(0,1], l>0, h>0$, we define $\mathcal{C}(l, h, \kappa, m, \epsilon)$ to be the following set of triangulations. First, for a positive integer


Figure 7.2. Initial mesh with three triangles(left); Triangulation after one refinement $N=1, \kappa=0.5$ (right).
$N$, choose $\kappa$, such that

$$
\kappa^{N \epsilon} \leq C(l) h^{m}
$$

Then, starting with three initial triangles sitting in $\Omega \backslash \Omega_{\frac{1}{2}}$ (Figure. 7.2), we decompose $\Omega_{\frac{1}{2}}$ into rectangular subdomains $\Omega^{0}:=\Omega_{\frac{1}{2}} \backslash \Omega_{\frac{\kappa}{2}}, \Omega^{1}:=\Omega_{\frac{\kappa}{2}} \backslash \Omega_{\frac{\kappa^{2}}{2}}, \ldots$, $\Omega^{j}:=\Omega_{\frac{\kappa^{j}}{2}} \backslash \Omega_{\frac{\kappa^{j+1}}{2}}$, for $j=0,1,2, \ldots, N-1$ and $\Omega^{N}:=\Omega_{\frac{\kappa^{N}}{2}}=\left(0, \frac{\kappa^{N}}{2}\right) \times(0, l)$. Note $h=l$ in the initial mesh. Denote by $h_{x_{j}}$ and $h_{y_{j}}$ the lengths of sides of triangles in the $x$ - and $y$-direction respectively on $\Omega^{j}$. In the $j$ th refinement, $1 \leq j<N$, we triangulate $\Omega^{j-1}$ with three triangles, such that they satisfy

$$
h_{x_{j-1}}=\kappa^{j-1} / 2-\kappa^{j} / 2, \quad \text { and } \quad h_{y_{j-1}}=l .
$$

Meanwhile, new triangles are generated in $\Omega \backslash \Omega_{\frac{\kappa^{j}-1}{2}}$, by connecting the mid-points of the old triangles. Note that there is no triangle in $\Omega^{i}, i \geq j$, yet. We simply repeat this process for $\Omega^{j}$ and $\Omega \backslash \Omega_{\frac{\kappa_{j}}{2}}$, in the next step. In the $N$ th refinement, besides generating triangles for $\Omega^{N-1}$ and $\Omega \backslash \Omega_{\frac{\kappa^{N-1}}{2}}$, we divide $\Omega^{N}$ into two triangles by the diagonal of the rectangle. Then, after $N$ refinements, the triangulation $\mathcal{T}$ is the union of the triangles in the triangulations of all the subdomains.

We now state the following property for the class $\mathcal{C}(l, h, \kappa, m, \epsilon)$ we defined above.

Theorem 7.3.5. For each $m \geq 1$, there exists a constant $C$, such that

$$
\left\|u-u_{I}\right\|_{\mathcal{K}_{1}^{1}(\Omega)} \leq C h^{m}\|u\|_{\mathcal{K}_{1+\epsilon}^{m+1}(\Omega)}
$$

for any triangulation $\mathcal{T}$ in $\mathcal{C}(l, h, \kappa, m, \epsilon)$ and $u \in \mathcal{K}_{1+\epsilon}^{m+1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}, \epsilon>0$.
The proof of this theorem needs the estimate on every $\Omega^{j}, j<N$ and $\Omega^{N}$, since it holds for $\Omega \backslash \Omega_{\frac{1}{2}}$ by our assumption. Due to the construction of the class $\mathcal{C}(l, h, \kappa, m, \epsilon)$, we present the following lemma for the last region $\Omega^{N}=\left(0, \frac{\kappa^{N}}{2}\right) \times$ ( $0, l$ ) first.
Lemma 7.3.6. On $\Omega^{N}=\left(0, \frac{\kappa^{N}}{2}\right) \times(0, l)$, from the construction of $\mathcal{C}(l, h, \kappa, m, \epsilon)$, the estimate on the error gives

$$
\left\|u-u_{I}\right\|_{\mathcal{K}_{1}^{1}\left(\Omega^{N}\right)} \leq C h^{m}\|u\|_{\mathcal{K}_{1+\epsilon}^{m+1}\left(\Omega^{N}\right)}
$$

$\forall u \in \mathcal{K}_{1+\epsilon}^{m+1}(\Omega) \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}, \epsilon>0, m \geq 1$, where $C$ depends on $m$ and $\kappa$.
Proof. The proof follows the dilation of $u$ and the introduction of an auxiliary function $v$. We define the dilation $u_{\lambda}(x, y)=u(\lambda x, y)$ for $(\lambda x, y) \in \Omega^{N}$. Let $\lambda=$ $\kappa^{N}$. Then, $u_{\lambda}(x, y) \in \mathcal{K}_{1+\epsilon}^{1+m}\left(\Omega_{\frac{1}{2}}\right)$ by Lemma 7.1.11. Meanwhile, let $\chi: \Omega_{\frac{1}{2}} \rightarrow[0,1]$ be a non-decreasing smooth function of $x$, which is equal to 0 in a neighborhood of $x=0$, but is equal to 1 at all the nodal points that do not lie on $x=0$. Then we introduce the auxiliary function $v=\chi u_{\lambda}$ on $\Omega_{\frac{1}{2}}$. Consequently, for a fixed $m$ and the corresponding nodal points in the triangulation, we have

$$
\begin{aligned}
\|v\|_{\mathcal{K}_{1}^{m+1}\left(\Omega_{\frac{1}{2}}\right)}^{2} & =\left\|\chi u_{\lambda}\right\|_{\mathcal{K}_{1}^{m+1}\left(\Omega_{\frac{1}{2}}\right)}^{2} \\
& =\sum_{i+j \leq m+1}\left\|\sum_{k \leq i} x^{-1}\left(r_{1} r_{2}\right)^{i+j}\left(x \partial_{x}\right)^{i-k} \partial_{y}^{j} u_{\lambda}\left(x \partial_{x}\right)^{k} \chi\right\|_{L^{2}\left(\Omega_{\frac{1}{2}}\right)}^{2} \\
& \leq C\left\|u_{\lambda}\right\|_{\mathcal{K}_{1}^{m+1}\left(\Omega_{\frac{1}{2}}\right)}^{2}
\end{aligned}
$$

where $C$ depends on $m$ and the function $\chi$. Moreover, one notes that the interpolation $v_{I}=u_{\lambda I}$ on $\Omega_{\frac{1}{2}}$ by the definition of $v$.

Therefore, the proof is completed by the following inequalities

$$
\left\|u-u_{I}\right\|_{\mathcal{K}_{1}^{1}\left(\Omega^{N}\right)}=\lambda^{-1 / 2}\left\|u_{\lambda}-v+v-u_{\lambda I}\right\|_{\mathcal{K}_{1}^{1}\left(\Omega_{\frac{1}{2}}\right)}
$$

$$
\begin{aligned}
& \leq \lambda^{-1 / 2}\left(\left\|u_{\lambda}-v\right\|_{\mathcal{K}_{1}^{1}\left(\Omega_{\frac{1}{2}}\right)}+\left\|v-u_{\lambda I}\right\|_{\mathcal{K}_{1}^{1}\left(\Omega_{\frac{1}{2}}\right)}\right) \\
& \leq \lambda^{-1 / 2}\left(C_{1}\left\|u_{\lambda}\right\|_{\mathcal{K}_{1}^{1}\left(\Omega_{\frac{1}{2}}\right)}+C_{2} h_{y_{N}}^{m}\|v\|_{\mathcal{K}_{1}^{m+1}\left(\Omega_{\frac{1}{2}}\right)}\right) \\
& \leq \lambda^{-1 / 2}\left(C_{1}\left\|u_{\lambda}\right\|_{\mathcal{K}_{1}^{1}\left(\Omega_{\frac{1}{2}}\right)}+C_{3} h_{y_{N}}^{m}\left\|u_{\lambda}\right\|_{\mathcal{K}_{1}^{m+1}\left(\Omega_{\frac{1}{2}}\right)}\right) \\
& =C_{1}\|u\|_{\mathcal{K}_{1}^{1}\left(\Omega^{N}\right)}+C_{3} h_{y_{N}}^{m}\|u\|_{\mathcal{K}_{1}^{m+1}\left(\Omega^{N}\right)} \\
& \leq C C_{4}\left(\frac{\kappa^{N}}{2}\right)^{\epsilon}\|u\|_{k_{1+\epsilon}^{m}\left(\Omega^{N}\right)}+C_{5} h_{y_{N}}^{m}\left(\frac{\kappa^{N}}{2}\right)^{\epsilon}\|u\|_{\mathcal{K}_{1+\epsilon}^{m+1}\left(\Omega^{N}\right)} \\
& \leq C h^{m}\|u\|_{\mathcal{K}_{1+\epsilon}^{m+1}\left(\Omega^{N}\right)} .
\end{aligned}
$$

The first and the fifth are from Lemma 7.1.11; the third and the fourth are the results of Theorem 7.3.2 and the relation between $v$ and $u_{\lambda}$; the sixth and the seventh are based on the construction of the triangulation.

We here provide the proof for Theorem 7.3.5 by summing up the estimates on every region $\Omega^{j}$ for $j \leq N$.

Proof. Since we assume the estimate is valid on $\Omega \backslash \Omega_{\frac{1}{2}}$ that contains the vertices, it is sufficient to show that the estimate still holds on $\Omega_{\frac{1}{2}}$ for completing the proof.

The basic idea is to establish the estimate $\left\|u-u_{I}\right\|_{\mathcal{K}_{1}^{1}}$ on every $\Omega^{j}$ for $j=$ $0,1,2, \ldots, N-1$ and on $\Omega^{N}$.

On every $\Omega^{j}$, based on the construction of the triangulation, $\xi=\frac{\kappa^{j}}{2}$ for Theorem 7.3.3. Then, we have

$$
\frac{h_{x_{j}}}{2 \xi} \simeq h_{y_{j}}=\frac{l}{2^{N-(j+1)}}
$$

Recall that $h$ represents the size of triangles in the region $\left(\frac{1}{2}, \frac{3}{4}\right) \times(0, l)$. Then $h=O\left(l / 2^{N}\right)$ after $N$ successive refinements. Since $\kappa^{N \epsilon} \leq C h^{m}$, by Theorem 7.3.3, we have the following for every $m$,

$$
\left\|u-u_{I}\right\|_{\mathcal{K}_{1}^{1}\left(\Omega^{j}\right)} \leq M h^{m}\|u\|_{\mathcal{K}_{1+\epsilon}^{m+1}\left(\Omega^{j}\right)}
$$

As for the last region $\Omega^{N}$, we have $\left\|u-u_{I}\right\|_{\mathcal{K}_{1}^{1}\left(\Omega^{N}\right)} \leq C h^{m}\|u\|_{\mathcal{K}_{1+\epsilon}^{m+1}\left(\Omega^{N}\right)}$ by Lemma 7.3.6. The proof of Theorem 7.3.5 then follows by adding the squares of all these norms on $\Omega^{j}$ and $\Omega^{N}$.

From this point, we assume that $0<\epsilon<1$ is chosen such that

$$
\mathcal{L}_{\delta}: \mathcal{K}_{1+\epsilon}^{m+1} \cap\left\{\left.u\right|_{\partial \Omega}=0\right\} \rightarrow \mathcal{K}_{-1+\epsilon}^{m-1}
$$

is an isomorphism, which is possible due to Corollary 7.2.3. Denote by $S$ the finite element space corresponding to the triangulation in Definition 7.3.4. Let $u_{S} \in S$ be the finite element solution of Equation (7.1). Then, we have the following estimate on $\left\|u-u_{S}\right\|_{H^{1}(\Omega)}$.

Theorem 7.3.7. Let $u \in \mathcal{K}_{1+\epsilon}^{m+1} \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}$ be the solution for Equation (7.1), $0<\epsilon<1$. Then, for each $m \geq 1$, there exists a constant $C$, such that

$$
\left\|u-u_{S}\right\|_{H_{1}^{1}(\Omega)} \leq C h^{m}\|f\|_{H^{m-1}(\Omega)}
$$

for any $\mathcal{T} \in \mathcal{C}(l, h, \kappa, m, \epsilon)$ and $\forall f \in H^{m-1}$.
Proof. We denote by $\gamma_{\delta}$ the norm of the inverse operator $\mathcal{L}_{\delta}^{-1}: \mathcal{K}_{-1+\epsilon}^{m-1} \rightarrow \mathcal{K}_{1+\epsilon}^{m+1} \cap$ $\left\{\left.u\right|_{\partial \Omega}=0\right\}$. The theorem can be proved by the following inequalities,

$$
\begin{aligned}
\left\|u-u_{V}\right\|_{H^{1}} & \leq M\left\|u-u_{V}\right\|_{\mathcal{K}_{1}^{1}} \\
& \leq C_{1} M\left\|u-u_{I}\right\|_{\mathcal{K}_{1}^{1}} \\
& \leq C_{2} M h^{m}\|u\|_{\mathcal{K}_{1+\epsilon}^{m+1}} \\
& \leq C_{2} M \gamma \gamma_{\delta} h^{m}\|f\|_{\mathcal{K}_{-1+\epsilon}^{m-1}} \\
& \leq C h^{m}\|f\|_{H^{m-1}} .
\end{aligned}
$$

The first and fifth inequalities are from Lemma 7.1.8; the second inequality is based on Céa's Lemma and the third inequality is from Theorem 7.3.5; the fourth inequality is obtained by the invertibility of the operator $\mathcal{L}_{\delta}: \mathcal{K}_{1+\epsilon}^{m+1} \cap\left\{\left.u\right|_{\partial \Omega}=\right.$ $0\} \rightarrow \mathcal{K}_{-1+\epsilon}^{m-1}$.

We have proved our theorems based on an explicit construction of the class $\mathcal{C}(l, h, \kappa, m, \epsilon)$ for $\Omega$. Our estimates on the error were expressed by $h$, the size of those triangles in $\left(\frac{1}{2}, \frac{3}{4}\right) \times(0, l)$, as in the usual quasi-uniform finite element spaces. However, since there is no uniform size for the triangles in the triangulation, it is better to formulate the estimate in terms of the dimension of the finite subspace
$S$. Based on the structure of the mesh we developed above, we attain the rate of convergence for the finite element solution $u_{S} \in S$ as follows.

Theorem 7.3.8. Let $u \in \mathcal{K}_{1+\epsilon}^{m+1} \cap\left\{\left.u\right|_{\partial \Omega}=0\right\}$ be the solution for Equation (7.1), $0<\epsilon<1$. There exists a constant $C=C(l, \kappa, h, m, \epsilon)$ for $m \geq 1$, such that

$$
\left\|u-u_{S}\right\|_{H_{1}^{1}(\Omega)} \leq C \operatorname{dim}(S)^{-m / 2}\|f\|_{H^{m-1}(\Omega)}
$$

for any partition $\mathcal{T} \in \mathcal{C}(l, h, \kappa, m, \epsilon)$ and $\forall f \in H^{m-1}(\Omega)$.
Proof. The proof is very similar to the proof of quasi-optimal rate of convergence in previous chapters. Let $\operatorname{dim}\left(S_{j}\right)$ and $\operatorname{dim}\left(S_{j-1}\right)$ be the numbers of the elements in the meshes after $j$ and $j-1$ refinements from the initial mesh respectively. Then, we have $\operatorname{dim}\left(S_{j}\right) \approx 4 \times \operatorname{dim}\left(S_{j-1}\right)+3$, hence, $\operatorname{dim}\left(S_{N}\right)=O\left(4^{N}\right)$. On the other hand, the size of the triangles in $\left(\frac{1}{2}, \frac{3}{4}\right) \times(0, l)$ satisfies $h=O\left(2^{-N}\right)$, after $N$ levels of refinement. Thus, $\operatorname{dim}(S) \approx h^{-2}$ for every $m \geq 1$. From Theorem 7.3.7, we have the following estimate in terms of the dimension of the finite subspace $S$,

$$
\left\|u-u_{S}\right\|_{H_{1}^{1}(\Omega)} \leq C \operatorname{dim}(S)^{-m / 2}\|f\|_{H^{m-1}(\Omega)}
$$

This is also the optimal convergence rate of the finite element solution expected for a smooth solution.

### 7.4 Numerical Results

We here present the numerical results to demonstrate our method to approximate the solution. The following model problem in the case $\delta=1$ and the domain $\hat{\Omega}=(0,1) \times(0,10)$ for Equation (7.1) is considered,

$$
\left\{\begin{aligned}
-\partial_{x}^{2} u-\frac{1}{x^{2}} \partial_{y}^{2} u=1 & \text { in } \hat{\Omega} \\
u=0 & \text { on } \partial \hat{\Omega}
\end{aligned}\right.
$$

We also have chosen $m=1$, namely, piecewise linear functions for the FEM, for simplicity. From the previous theorems, the solution is not automatically in $H^{2}(\hat{\Omega})$ near the degenerate boundary. In fact, we can use the same method as
in previous chapters to obtain a more accurate a priori estimate $u \in H^{s}(\hat{\Omega})$ for $s<1+\frac{\sqrt{100+4 \pi^{2}}}{20} \approx 1.59$. Note that the operator $\mathcal{L}_{1}$ is actually the Laplace operator $-\Delta$ near the vertices $(0,1),(0, l)$. The solution near those two corners, therefore, behaves like

$$
u(r, \theta)=r^{2 k} \zeta(\theta), \quad k \in \mathbb{Z}_{+}
$$

in polar coordinates [55], where the function $\zeta$ is smooth, only depending on $\theta$. For this reason, the solution has no singularity near those two vertices in $H^{2}$. The vertices $(1,0),(1, l)$ do not affect the regularity of the solution in this case. Consequently, it is not necessary to apply graded mesh there. Moreover, with a direct calculation based on our arguments in Section 7.2, we have $\eta_{1}=\frac{\sqrt{100+4 \pi^{2}}}{20}$ and $\eta_{2}=2$ on $\hat{\Omega}$. Then, one can set $0<\epsilon<\eta=\min \left(\eta_{1}, \eta_{2}\right)=\frac{\sqrt{100+4 \pi^{2}}}{20} \approx 0.59$ and take $0<\kappa=2^{-1 / \epsilon}$ for the graded mesh near the degenerate boundary $x=0$. Therefore, we have the range $\kappa<2^{-1 / 0.59} \approx 0.309$, on which the optimal rate of convergence in Theorem 7.3.8 holds for the model problem.

To construct the mesh, we start with three initial triangles (Figure. 7.3). In every step of refinement, we pick two points $\left(x_{1}, 0\right)$ and $\left(x_{2}, 10\right)$ as two vertices of the new triangle and the third vertex of the new triangle is placed at the midpiont of the base of the old triangle. Denote by $d$ the minimum distance from any point in the old triangles to $x=0$. Then, the parameter $\kappa$ controls the position of the new points, such that $x_{1}=x_{2}, \kappa=x_{1} / d$. Meanwhile, other new triangles are generated on the region that is enclosed by old triangles, by the mid-points as described in the previous section (Figure 7.3). Therefore, the new triangle that is approaching $x=0$ is specially designed to fulfill the requirement in Definition 7.3.4, while all the other new triangles are generated by connecting the mid-points of the old triangles. In the last step, the last region $\hat{\Omega}_{\frac{\kappa^{N}}{2}}$ is divided into two triangles by the diagonal of the rectangle, as described in Section 7.3.

One also notes that the triangles near the degenerate boundary are getting thinner and thinner in our construction, in which the maximum-angle condition is apparently violated. Nevertheless, the difficulty is already overcome by Theorem 7.3.3 and Theorem 7.3.5.

The finest mesh in our numerical experiments is obtained after 10 successive


Figure 7.3. Initial mesh with three triangles (left); the mesh after one refinement $\kappa=0.2$ (right).


Figure 7.4. The mesh after 5 levels of refinement for $\kappa=0.2$.
refinements of the coarsest mesh and has roughly $2^{22} \approx 4 \times 10^{6}$ elements. The preconditioned conjugate gradient (PCG) method is used to solve the resulting system of algebraic equations. We have tested several values of the parameter $\kappa$ for the model problem. The convergence rates are summarized in Table 5.1. These results convincingly show that the theoretical approximation order can be verified in practical calculations with $\kappa<0.3$.

The left most column in Table 7.1 shows the number of the refinement level, and $u_{j}$ denotes the numerical solution corresponding to the mesh after $j$ levels of refinement. The quantity printed out in other columns in the table represents the

| $j$ | $e: \kappa=0.1$ |  |  |  |  | $e: \kappa=0.2$ | $e: \kappa=0.25$ |
| :---: | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $e: \kappa=0.35$ | $e: \kappa=0.4$ | $e: \kappa=0.5$ |  |  |  |  |  |
| 2 | 0.45 | 0.46 | 0.43 | 0.35 | 0.31 | 0.22 |  |
| 3 | 0.66 | 0.63 | 0.59 | 0.48 | 0.42 | 0.31 |  |
| 4 | 0.82 | 0.78 | 0.73 | 0.61 | 0.54 | 0.40 |  |
| 5 | 0.92 | 0.88 | 0.84 | 0.71 | 0.63 | 0.47 |  |
| 6 | 0.97 | 0.94 | 0.89 | 0.76 | 0.68 | 0.52 |  |
| 7 | 0.99 | 0.96 | 0.93 | 0.79 | 0.71 | 0.55 |  |
| 8 | 1.00 | 0.98 | 0.94 | 0.81 | 0.73 | 0.56 |  |
| 9 | 1.00 | 0.99 | 0.96 | 0.82 | 0.74 | 0.57 |  |

Table 7.1. Convergence history.
convergence rate in the manner

$$
e=\log _{2}\left(\frac{\left|u_{j}-u_{j-1}\right|_{\mathcal{K}_{1}^{1}}}{\left|u_{j+1}-u_{j}\right|_{\mathcal{K}_{1}^{1}}}\right) .
$$

The quantity $e$ is not an exact convergence rate, but it turns out to be a quite reasonable approximation to it. Recall $h$ stands for the triangle size in $\left(\frac{1}{2}, \frac{3}{4}\right) \times$ $(0, l)$. We have already seen that the correctly graded refinement improves the convergence rate with a factor of about $h^{0.43}$. In fact, the improvement may be better as our theory shows that the convergence rate in the case $\kappa<0.3$ is $h^{1}$ by Theorem 7.3.5

Our theoretical prediction for the convergence rate in the case $\kappa=0.5$ is about $h^{0.59}$, and the theoretical prediction for the convergence rate in the cases $\kappa=0.1$, $\kappa=0.2$ is $h^{1}$, which is verified by Table 7.1. Moreover, one can see a big jump in the rates between $\kappa=0.25$ and $\kappa=0.35$, which strongly supports our theory for the critical number $\kappa \approx 0.3$. Thus, our numerical results completely agree with the theory we have presented in this chapter. Based on the behavior of the sequence of the numbers in every column, it is also reasonable to expect the optimal rate by doing more refinements for all $\kappa<0.3$. Therefore, we conclude from our numerical results, that, for a correct refinement, the difference between consecutive numerical solutions is decreasing like $\operatorname{dim}\left(S_{n}\right)^{-1 / 2}$, which verifies Theorem 7.3.8.

One may also notice that those numbers in each column keep increasing when $\kappa>0.3$. An explanation is that the solution consists of a singular and a regular part: $u=u_{s}+u_{r}$. The regular part $u_{r}$ dominates the behavior of the solution until
$x$ is sufficiently close to the degenerate boundary, when the singular part can be taken into account [7]. Therefore, as shown in the table, the increasing rate slows down and will be fixed at some point.

## Conclusions

In the last chapter, I will briefly summarize my Ph.D. work and give a rough direction of my future research. A Chinese proverb reads: the more one gets to know, the more one realizes he does not know. My future plans are certainly based on my current knowledge and the details may change when I become more experienced. This is why the research goal given here is a "rough" one. The big picture, however, I think will remain the same.

### 8.1 Summary

My research centers around numerical methods for partial differential equations (PDEs) with low-regularity solutions. I am interested in various aspects of this field ranging from the theoretical analysis of the PDEs, the estimates of numerical errors, to the development and implementation of numerical algorithms. Here are, more concretely, the topics I have worked on and have been included in the dissertation.

## - Partial differential equations on singular spaces.

The solution of an elliptic PDE may have singularities from the non-smoothness of the boundary, changes of boundary conditions, and jumps in the coefficients. The research I have done in this area is to establish a priori estimates (wellposedness, regularity, Fredholm properties) in weighted Sobolev spaces for general elliptic PDEs with possible singular solutions of these types (in joint work with

Mazzucato and Nistor, Chapter 3). A priori estimates in weighted Sobolev spaces represent the theoretical foundation for our numerical approach to these singular problems.

## - The finite element method (FEM).

One of the major concerns in this widely used numerical method is to construct finite element subspaces that are compatible with the target PDE and produce numerical solutions with good rates of convergence. From my recent work with Mazzucato and Nistor, a simple and explicit construction of finite element subspaces has been formulated for general elliptic PDEs with the singular solutions as mentioned above, which preserve the quasi-optimal convergence rate. In particular, the analysis of the FEM for Neumann-Neumann corners and for transmission problems (Chapter 3 and Chapter 4) is a breakthrough in this field.

## - The multigrid (MG) method.

In this area, I have focused on estimates of the convergence rate of the MG method, and on the development of efficient subspace solvers for algebraic systems of equations from the discretization of elliptic PDEs with singular solutions. I have worked with Brannick and Zikatanov [33] to address our estimates on these systems, based on the analysis in weighted Sobolev spaces and the method of subspace corrections (Chapter 5).

## - Applications to mathematical physics.

Many mathematical models in physics give rise to differential operators with singular or degenerate coefficients. Both theoretical analysis and numerical approximations to these solutions, possibly singular, require a deep mathematical understanding of the problem. My work on an operator degenerate on a segment of the boundary (Chapter 7) and the work with Nistor [69] (Chapter 6) on a modified Schrödinger operator extend our theory on singular spaces to these operators by providing concrete a priori estimates and robust finite element schemes with quasi-optimal convergence rates for the numerical solution.

## - Software development

I with Nistor developed the software package $L N G_{-} F E M$ [68] that generates user-specified graded meshes and solves elliptic PDEs on pathological domains, possibly with cracks. We have made this software available to the public for the research and education on singular solutions.

Moreover, during writing this dissertation, I, with Long Chen, are working on the superconvergence phenomenon of the finite element solution on the graded meshes proposed in Chapter 4. Because of the singularity of the solution and nonuniformity of the mesh, many counterparts of the results in the classical FEM can be considered as potential topics. Some of these topics are discussed in the next section.

### 8.2 Future Work

I have been attracted to the interplay between different fields (numerical methods and PDEs, Mathematics and Physics, Mechanics, and others). I believe that a deep understanding of the theory is the starting point for good discoveries in numerical methods.

## - Partial differential equations on singular spaces.

I am very interested in the research of singular solutions of other equations (linear elasticity, Maxwell, Stokes, Schrödinger, evolution equations, and others). I have been studying the classical theory on other PDEs, and I expect to derive further theoretical results on different differential operators associated to these equations. Foe example, different potentials (singular functions) in the Schrödinger operator may raise singular solutions of different types. It can be seen that we are able to estimate a class of these operators in weighted Sobolev spaces. This has been one of the difficulties in physics.

## - The FEM.

My main goal here is to study and design effective FEMs for PDEs with singular solutions. In particular, I have seen many applications of our work in the FEM
to equations from mathematical physics, and they have proved successful [66, 69]. I plan to continue this study and design effective finite element schemes for these PDEs, generally with singular coefficients. Also, I am very interested in the generalized finite element method (GFEM) and the adaptive finite element method (AFEM), since they give more freedom for the choice of shape functions and meshes to generate numerical solutions for singular solutions. Moreover, the FEM for other equations (linear elasticity, Maxwell, Stokes, etc.) is of great interest to me.

## - The MG method.

Besides the development of finite element schemes to obtain the quasi-optimal convergence rate for the numerical solution, I have seen other applications of our a priori estimates to numerical methods, for example, to the MG method for singular equations. Most of conventional analysis on the MG method needs certain regularity results for the solution. With our results in weighted Sobolev spaces, I would like to obtain MG methods for other low-regularity solutions (transmission problems, high-order FEMs, for example).

The mathematical research is a long-term process that needs patience, determination, and more important, consistent hard work. Equipped with motivation, dedication, and confidence from the experience of my Ph.D. study, I have and will enjoy every moment of it.

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## Vita

## Hengguang Li

The details of my childhood are inconsequential.

